RAYLEIGH'S PROBLEM AT LOW MACH NUMBER ACCORDING TO THE KINETIC THEORY OF GASES

Thesis by

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ABSTRACT

Rayleigh's problem of an infinite flat plate set into uniform motion impulsively in its own plane is studied by using Grad's equations and boundary conditions developed from the kinetic theory of gases. For a heat insulated plate and a small impulsive velocity (low Mach number), only tangential shear stress and velocity and energy (heat) flow parallel to the plate are generated, while the pressure, density, and temperature of the gas remain unchanged. Moreover, no normal velocity, normal stress, or normal energy flow is developed. Near the start of the motion the flow behaves like a "free-molecule flow", and all physical quantities are analytic functions of the flow parameters and time. The results obtained for "large time", however, add to the growing lack of confidence in the Burnett-type series expansions in powers of mean free path. Although such expansions are obtained here, they are poorly convergent and inappropriate to the problem. To replace these unsatisfactory solutions, approximate closed-form solutions valid for all values of the time are developed, which agree with the free-molecule values for small time and the classical Rayleigh solution for large time. This technique may be useful in studying more general flow problems within the framework of the kinetic theory of gases.

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a	speed of sound
a_o	speed of sound in the undisturbed gas
$a_i^{(n)}$	Hermite coefficient
C_f	skin friction coefficient
$\vec{c}$	intrinsic velocity $= \vec{\xi} - \vec{u}$
c_i	component of intrinsic velocity $\vec{c}$
C_p	specific heat at constant pressure
C_v	specific heat at constant volume
$F(\vec{\xi},\vec{x},t)$	number molecular distribution function
$f(\vec{\xi},\vec{x},t)$	mass molecular distribution function $= mF(\vec{\xi},\vec{x},t)$
$f^{(0)}(\vec{x},t)$	local Maxwellian distribution function $= \frac{\rho}{(2\pi RT)^{\frac{3}{2}}} e^{-\frac{C^2}{2RT}}$
$\mathcal{H}_i^{(n)}$	Hermite polynomials of order n
K	constant in the Rayleigh analogy $x = KUt$
L	mean free path
M	Mach number $= U/a_o$
m	mass of a molecule
p	pressure of gas
p_{ij}	stress tensor
p_{xx}	normal stress in x direction
p_{yy}	normal stress in y direction
p_{xy}	shear stress
$Q(t,y)$	any physical quantity
$\bar{Q}(s,y)$	Laplace transform of $Q(t,y)$

q_i	energy (heat) flux
q_x	energy (heat) flux in x direction
q_y	energy (heat) flux in y direction
R	gas constant
Re	Reynolds number $= \rho_o U^2 t/\mu_o$
r	perpendicular from first molecule to relative path of second before collision (Fig. 2)
r	modulus of the complex variable s (Fig. 4)
s	variable in Laplace transformation
T	temperature of gas
T_w	temperature of the plate
T	temperature of the surface of no-slip
t	time
U	impulsive velocity of plate in its own plane
u	velocity of gas in the x direction
$\vec{u}$	gas velocity vector
v	velocity of gas in the y direction
$\vec{v}$	dimensionless intrinsic velocity $= \vec{c}/\sqrt{RT}$
X_i	component of external forces per unit mass
$\vec{x}$	space vectors
x_i	Cartesian coordinates $i = 1, 2, 3$
x	coordinate parallel to the plate
y	coordinate normal to the plate

α	fraction of incident gas molecules which is specularly reflected from the plate
α'	energy accommodation coefficient
γ	ratio of specific heats $= \frac{c_p}{c_v}$
δ_{ij}	Kronecker's delta
ϵ	polar angle (Fig. 2)
ϵ'	a small parameter $= \frac{M\sqrt{\gamma}}{\sqrt{Re}}$
ζ	distance of surface of no-slip below the flat plate (Fig. 1)
θ	argument of the complex variable s
λ	coefficient of heat conduction
μ	coefficient of viscosity
ν	kinematic coefficient of viscosity $= \frac{\mu}{\rho}$
$\vec{\xi}$	molecular velocity vector
ξ_i	component of molecular velocity vector $\vec{\xi}$
ρ	density of gas
σ	tangential momentum reflection coefficient
σ'	normal momentum reflection coefficient
τ	average time between successive collisions of a molecule
τ^*	average duration of a collision
ω	index of viscosity variation with temperature

Subscript $_o$ indicates quantities in the undisturbed gas

Prime ' indicates perturbation quantities

I. INTRODUCTION

The problem of an infinite flat plate set into uniform motion in its own plane in an "incompressible" viscous fluid was first treated by Rayleigh.[1] For a compressible fluid the problem has been examined for "small" times by Howarth[2], for "large" times by Van Dyke[3], and, recently, for "intermediate" times by Stewartson[4]. (Here the time is measured in terms of the parameter a^2t/ν.)* Howarth[2] questioned the significance of his solution because of the extremely short time interval during which the pressure on the plate surface decays to a fraction of its value immediately after the start of the motion. For standard conditions of initial temperature and pressure, the plate pressure falls to one-half of its maximum value in 5×10^{-10} seconds, and to one-tenth of this value in 10^{-8} seconds, as compared with an average time τ between successive collisions of a molecule of about 10^{-10} seconds. Over such short time intervals there are only a few collisions per molecule, and the gas can no longer be treated on the basis of the Navier-Stokes equations. In fact, for $t/\tau = O(1)$ one would expect that the situation corresponds more closely to "free-molecule" flow, where the collisions between the gas molecules and the plate surface are much more important than the collisions between molecules in the gas.

These considerations suggest that Rayleigh's problem should be attacked by means of the kinetic theory of gases in order to determine

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* The quantity $(a^2t/\nu)^{\frac{1}{2}}$ is proportional to the ratio of the speed of sound to the rate of diffusion of vorticity. Alternatively, the parameter $a^2t/\nu = Re/M^2$ is proportional to the ratio of the elapsed time from the start of the motion to the average time between successive collisions of a molecule, τ $(Re/M_o^2 = 4/3 \ t/\tau)$.

the initial history of the important physical quantities and the manner in which these quantities approach the behavior predicted by the Navier-Stokes equations for $t/\tau >> 1$. In this way the whole time-history is treated within a single theoretical framework. Of course, the present problem is highly idealized because of the absence of a leading edge, but the results obtained for the well-defined Rayleigh problem should provide a qualitative understanding of molecular effects in more complex cases, just as Rayleigh's original solution contributed to the understanding of boundary layer flows and the concept of similarity at high Reynolds numbers. For example, when the free stream Mach number of a steady flow is very high, the intense heating of the gas near the surface plays the most important role in creating pressure disturbances propagating away from the surface. In this case, a certain analogy exists between Rayleigh's problem and the development of the steady flow downstream of the leading edge of a thin, flat plate. One of the eventual tasks of the present study is to clarify this analogy and to distinguish carefully between molecular effects and viscous effects that would ordinarily be expected at low local Reynolds numbers within the Navier-Stokes framework. This particular example is cited also to emphasize that this study is not confined to the special case of a highly-rarefied gas, where the mean free path is always large compared to any significant physical dimension. We are mainly interested in flows of low or moderate density, in which the lateral dimensions of the flow field are initially of the order of the local mean free path and eventually grow to be large compared with the mean free path.

Previous studies of molecular effects in Rayleigh's problem have dealt mainly with the "first-order" influence of velocity-slip and temperature-jump at the plate surface. The Rayleigh problem in slip-flow was first investigated by Schaaf[5]. The classical heat conduction equation governing the velocity component parallel to the plate for a fluid of constant properties was used. Instead of the usual boundary condition that the gas moves with the plate, Schaaf imposed the "slip" boundary condition. He solved this system for the tangential velocity component and obtained the skin friction by differentiation. The Rayleigh problem was then related to the steady boundary layer flow over a semi-infinite plate by the transformation $x = KUt$, where x is the distance from the leading edge along the plate in the boundary layer flow, U and t are, respectively, the impulsive plate velocity and time in the unsteady Rayleigh problem, and K is a numerical constant. If one accepts the Burnett equations as the approximate system in the slip-flow region, then Schaaf's approach is well justified for the Rayleigh problem in slip-flow at low Mach number. In this case, the non-linear higher order Burnett terms in the expression for the viscous shearing stress may be neglected, and hence the momentum equation in the direction parallel to the plate remains exactly the same as that used by Rayleigh. The only difference then lies in the boundary conditions. Although Schaaf's analysis is valid only in the slip-flow regime, he nevertheless extrapolated his results over the whole range. For free molecule flow he obtains a finite skin friction twice the correct value. The same remark applies also to the slip velocity, which would be equal to the plate velocity when extrapolated from Schaaf's result to free molecule flow, while the correct value is half of the plate velocity for a diffusively-reflecting surface. Since the momentum transferred from the plate to the gas differs by a factor of two in the two cases, so does the skin friction coefficient.

The trend of the variation of skin friction coefficient with the parameter $\sqrt{Re}/M$, predicted by Schaaf's theory, is followed surprisingly well by the experiments of Sherman[6], who measured the drag force on a number of plates in steady flow under the following conditions: (1) Mach number between 2.3 and 2.9, and Reynolds number between 30 and 800 per inch; (2) Mach number between 2.7 and 3.6, and Reynolds number between 55 and 1850 per inch; and (3) Mach numbers of 0.2 and 0.6, and Reynolds number between 15 and 480 per inch. However, no direct comparison between theory and experiment can be made because Schaaf's analysis does not take into account the effects of the interaction between the shock wave and the viscous layer in supersonic flow, and the effect of the finite length of the plate.

Mirels[7] extended Schaaf's work to the case of variable fluid properties. By transforming coordinates and assuming $\rho\mu =$ constant, he was able to solve for the gas velocity component parallel to the plate separately from other unknown quantities. The Rayleigh problem was then related to the steady boundary layer flow over a flat plate. Mirels also extrapolated his results over the whole flow regime and like Schaaf obtained twice the correct values of slip velocity and skin friction for free molecule flow. Mirels' work seems less satisfactory than that of Schaaf. If the impulsive velocity of the plate in Rayleigh's problem is large enough to warrant consideration of compressibility, then the conventional Navier-Stokes equations would be inadequate in describing the problem in the slip-flow.

An analysis similar to Schaaf's was made by Kane[8] for the heat transfer from a flat plate with a temperature-jump boundary condition. The energy equation of the Navier-Stokes' framework was linearized by neglecting dissipation and regarding the fluid properties as constant. Thus, the linearized energy equation with the temperature-jump boundary condition corresponds to the linearized momentum equation with the velocity slip boundary condition in Schaaf's analysis. The same method of solution was employed, and the heat transfer obtained. Kane's result would also be valid only for low Mach number flow, since in general the energy equation can not be linearized. This heat transfer analysis was extended later to cylinders, spheres[9], and cones[10].

By dealing directly with the steady, uniform flow over a semi-infinite flat plate, Maslen[11] obtained the second approximation to the laminar boundary layer in slip flow. He assumed that the velocities and the thermodynamic properties of the fluid may be expanded in powers of the small quantity $\epsilon' = (M\sqrt{\gamma})/(\sqrt{Re})$, which is essentially the ratio of the molecular mean free path to the boundary layer thickness. By introducing these expansions into the Burnett equations and also into the velocity-slip and the temperature-jump boundary conditions, one finds that the conventional boundary layer equations with velocity slip and temperature jump boundary conditions are correct to the order of ϵ'. The Burnett terms give contributions of the order of ϵ'^2 and higher. The second approximation, correct to the order of ϵ', indicates a decrease in heat transfer and for supersonic flow an increase in skin friction. For subsonic flow there is no first order shear effect. The change in heat transfer is caused by the slip, but the effect on the skin friction is connected with the interaction between the boundary layer and the external flow. Maslen also obtained the incompressible Rayleigh solution (Appendix D, Reference 11) and found the same subsonic features: namely, zero first-order shear and a decrease in heat transfer.

Independently of Maslen, Nonweiler[12] also investigated the laminar boundary layer in slip flow at low Mach numbers. He obtained the boundary layer equations with the velocity slip and the temperature jump boundary conditions as the governing system correct up to the order of 1/Re. This boundary layer problem with slip was transformed to one without slip referred to a "plane of no slip", which is at a distance ζ below the flat plate. (See Fig. 1.)

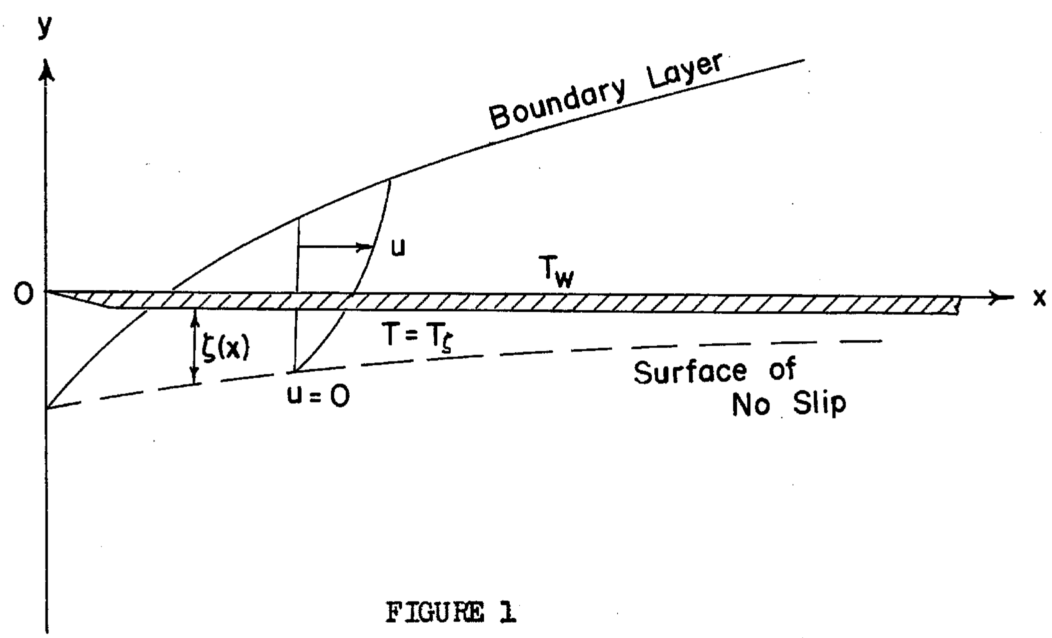

FIGURE 1

However, the temperature of the gas at the plane of no-slip, T_{ξ}, is different from the temperature of the gas at the actual surface, T_w. In general, at $y = -\xi$, $u = 0$, $T_{\xi} \neq T_w$. For zero heat transfer in low Mach number flow, it was found that $T_{\xi} = T_w$. In this case, Nonweiler found a skin friction increase due to slip as a product of dp/dx and $M/\sqrt{Re}$. If the pressure gradient dp/dx is zero, then slip has no first order shear effect. The effect of slip alone is solely responsible for a heat transfer decrease. Nonweiler also found no effect of temperature jump on skin friction in the cases he studied.

All the results discussed above, as restricted by the framework used, are applicable only to the border regime between continuum and slip flow and could not be expected to hold over the whole domain. Two approaches have been employed to investigate molecular effects in the flow over a flat plate: namely, the study of the Rayleigh problem and the study of the steady boundary layer flow. In the latter case, the problem is not well-defined because of the presence of the leading edge, near which the molecular effects are most important. At any rate, one thing is clear: the ratio $M/\sqrt{Re}$ is the fundamentally-important flow parameter. However, in the boundary layer approach the assumption that the various quantities could be expanded in powers of this parameter is questionable, and the conclusion that the velocity slip has no immediate effect on the skin friction is rather unsatisfactory.

Near the start of the Rayleigh motion, as mentioned before, one expects the flow to resemble a "free-molecule" flow. In the usual free-molecule flow analysis the incident molecules are supposed to be characterized by the Maxwell distribution function based on conditions "far" from the surface. In Rayleigh's problem this distribution function will be modified as soon as a sufficient number of the first reflected molecules have had time to collide with the new incident molecules. Therefore, the behavior of the flow quantities in the initial stages should be described by means of a suitable expansion procedure in terms of the parameter $t/\tau \sim at/L \sim Re/M^2$. Apparently the idea of improving upon the free molecule flow values by seeking solutions of the Maxwell-Boltzmann equation in terms of power series in 1/L originated with Jaffé. Keller[13] applied Jaffé's method to the problem of determining the drag on a moving body as well as to the problem of free expansion, but only the first approximation is given explicitly. The general expressions thus obtained for the drag on a convex body in the free molecule flow are the same as those obtained by Heinemann[14] and others. Heinemann followed the physical approach of obtaining the drag on arbitrary convex bodies in free-molecule flow by considering the momentum imparted by the gas molecules to the body. The usual assumption of monatomic gas and Maxwellian molecular velocity distribution far from the body was made, and formulas for the drag of a flat plate, a sphere, a cylinder, and an ellipsoid were given. For the particular cases of a flat plate moving perpendicular to its plane, Heinemann also obtained a second approximation for the surface pressure to take account of those molecules that collide once in the gas.

It will be interesting to see if solutions of Rayleigh's problem obtained in the present study take the form of series expansions in the parameter t/τ for small times. Conceivably such solutions developed from the free molecule flow regime, combined with the <u>proper</u> solutions for large time, may cover most of the range of interest.

Perhaps the most comprehensive attempt to obtain a general approximate solution of the Maxwell-Boltzmann equation is that of H. Grad[15]. In Grad's method the molecular distribution function $f(\vec{\xi}, \vec{x}, t)$ is expanded in terms of Hermite polynomials depending only on $\vec{\xi}$, where $\vec{\xi}$ is the molecular velocity vector, $\vec{x}$ is the space vector, and t is the time. The first term of this series expansion is the Maxwell distribution function based on <u>local</u> quantities. The coefficients of the higher terms are taken as the new state variables, which are functions of $\vec{x}$, t, and these coefficients are identified with the "moments" of the distribution function with respect to $\vec{\xi}$. The first moment gives the flux density of molecules, while the higher moments are essentially the normal and shear stresses and the heat flux quantities, which are now regarded as separate dependent variables, not explicitly related to the derivatives of $\vec{\xi}$ and T (absolute temperature) with respect to $\vec{x}$. By stopping the expansion after a finite number of terms and substituting the Hermite polynomial series for $f(\vec{\xi}, \vec{x}, t)$ into the Maxwell-Boltzmann equation, a system of partial differential equations for pressure, density, velocity, stresses, and heat flux quantities of the gas is obtained. The usual equations expressing conservation of mass, momentum, and energy are always included in the general set. This system of equations should govern all possible flow regimes from free molecule flow to the inviscid or Euler flow.

Grad's boundary conditions at a solid surface are derived by introducing a single "reflection parameter", α, which is similar to the Maxwell-Smoluchowski concepts of reflection coefficient and energy accommodation coefficient. This parameter denotes the fraction of molecules that is reflected specularly. Considering the special case of a perfectly reflecting wall for which $\alpha = 1$, Grad found three of the moments are required to vanish. By working on these three moments for the general case of $0 \leq \alpha \leq 1$, he obtained three boundary conditions. However, Grad's mathematical argument of deriving these boundary conditions may be interpreted physically by applying the conservation laws of mass, momentum, and energy. These boundary conditions are therefore re-derived in Section II.B from a physical point of view.

Because of the uncertainty over the actual physical reflection process, it is thought sufficient at the present time to take the tangential momentum reflection coefficient σ and the energy accommodation coefficient α' as identical and the normal momentum reflection coefficient σ' introduced by Schaaf and Bell[16] as unity. These assumptions are equivalent to Grad's introduction of a single reflection parameter α.

The equality $\sigma = \alpha'$ is correct only if the reflection process is actually as simple as Maxwell supposed; i.e., a certain fraction $1 - \sigma$ of the incident molecules is reflected like light rays, while the rest are reflected diffusely with a Maxwell distribution corresponding to the surface temperature T_w. Hurlbut's[17] molecular beam experiments on the scattering of air and nitrogen from solid surfaces (steel, aluminum, glass) show that σ is very close to unity. Hurlbut remarks that these results do not imply that almost all molecules are reflected diffusely, but that probably a certain fraction are reflected elastically from tilt planes with a random orientation over the surface. This explanation may account for the fact that many investigators have found α' to be significantly less than unity while $\sigma \cong 1$. Once a sound theoretical foundation is established, it might be possible to obtain α' and σ indirectly by independent measurements of surface pressure and heat transfer rate (or surface temperature), just as the ordinary coefficient of viscosity is obtained from the flow rate in Poiseuille motion.

For very violent processes in which the gas is far from the equilibrium state, Grad's formulation may be inadequate, or even incorrect, but for a wide class of flow problems the contracted thirteen-moment method is probably sufficient. In each case, a comparison of the magnitude of the neglected higher moments with those retained must be made a posteriori in order to test the validity of the approximation. Grad's method has been applied to the plane shock wave[18] and to the classical problem of the steady flow between two concentric rotating cylinders[19]. A recent extension[20] of Grad's method to "rough sphere" molecules to include rotational degrees of freedom and relaxation processes would seem to be premature, as long as the theoretical foundations of the simpler model are not yet firmly established.

In Grad's work[18] on the steady plane shock wave, it was found that the solution breaks down for shocks stronger than M = 1.65, which could be taken as the limit of validity of Grad's thirteen-moment equations. The Navier-Stokes solution for the shock profile begins to deviate from the thirteen-moment result at about M = 1.2. According to Grad, then, this point of M = 1.2 shock strength marks the limit of applicability of the Navier-Stokes equations, although formally the shock solution exists for infinite strength shock in the Navier-Stokes framework.* On the other hand, Gilbarg[21] concluded there is nothing in

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* Of course, relaxation effects would have to be taken into accaunt.

the shock wave study so far to suggest that the classical continuum theory is not fully adequate to describe the flow. The recent experiment of Sherman[22] begins with M = 1.72, while Grad's theory ceases to hold beyond M = 1.65. Consequently, no direct comparison could be made there. In short, the question of which system best describes the structure of a shock wave is still in dispute and has to be settled by further research, both theoretical and experimental.

From the above considerations, it is safe to say: (1) Grad's thirteen-moment equations are sufficiently flexible to describe most flows in which there is no shock wave stronger than that of normal oncoming Mach number M = 1.65. This last difficulty is certainly avoided in the present study; (2) Although the Navier-Stokes framework yields a formal solution of the shock wave problem even for infinitely strong shocks, the Navier-Stokes' solution for Rayleigh's problem breaks down completely in the initial stage of the flow, giving infinite skin friction. Therefore, the classical equations of Navier-Stokes are definitely not applicable to the study of the initial phases of Rayleigh's problem.

In the present investigation, Grad's thirteen-moment method is applied to Rayleigh's problem. The boundary conditions are restated for the case of a moving surface, and the equivalent assumptions $\sigma = \alpha'$ and $\sigma' = 1$ are retained. At this stage the original restriction to a monatomic gas is thought not to be too severe. In general, the problem is highly non-linear; therefore, this phase of the study deals with the case in which the plate velocity is small compared with the speed of sound in the gas before the start of the motion. The differential equations and boundary conditions are linearized, and momentum and energy are uncoupled. In other words, the problem is purposely simplified to the point where the molecular effects associated with a purely-tangential motion should be clearly discernible. Solutions obtained in this case should provide a basis for attacking Rayleigh's problem with arbitrary plate velocity, where the pressure, temperature, and velocity fields are interrelated and the problem is more complicated.

II. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

A. Maxwell-Boltzmann Equation and Grad's Method of Solution*

The basic equation of the kinetic theory of gases is the well-known Maxwell-Boltzmann integro-differential equation:

$$\frac{\partial F}{\partial t} + \xi_i \frac{\partial F}{\partial x_i} = \int |\vec{\xi}_1 - \vec{\xi}\,| \,(F'F_1' - FF_1)\, r\,dr\,d\epsilon\, d\vec{\xi}_1 \tag{2.1}$$

where

t = time

x_i = Cartesian coordinates, $i = 1, 2, 3$

ξ_i = component of molecular velocity vector $\vec{\xi}$

$F(\vec{\xi},\vec{x},t)$ = molecular distribution function, so that $F(\vec{\xi},\vec{x},t)\,d\vec{x}\,d\vec{\xi}$ = number of molecules in the physical volume $d\vec{x}$ around $\vec{x}$ having velocities within $d\vec{\xi}$ of $\vec{\xi}$.

$|\vec{\xi}_1 - \vec{\xi}\,|$ = relative speed between two molecules before collision

r, ϵ = polar coordinates in plane through first molecule perpendicular to initial trajectory of second molecule (See Fig. 2.)

r = perpendicular from first molecule to relative path of second before collision

ϵ = polar angle

subscript 1 refers to the second molecule which is colliding with the first

prime ' refers to quantities after collision

repeated indices denote summation

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* For details, see Reference 15.

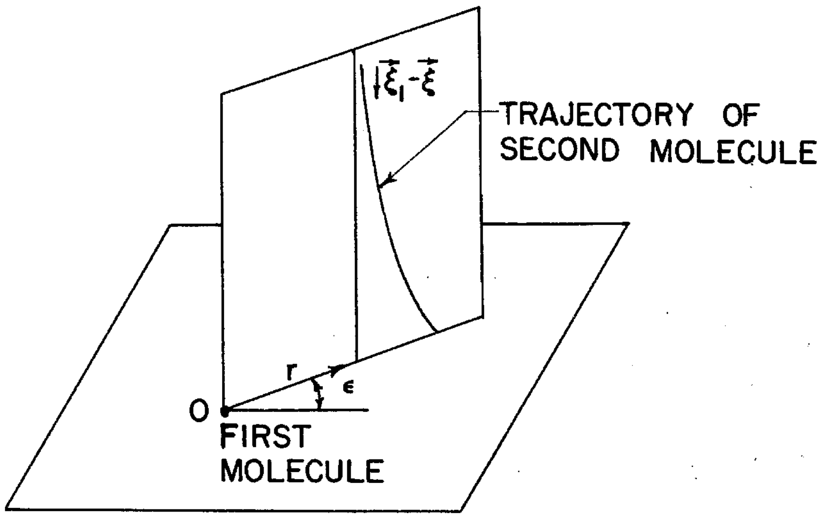

FIGURE 2

The left-hand side of Eq. (2.1) represents the net rate of increase in the number of molecules in the elementary volume $dx_1 \cdot dx_2 \cdot dx_3$ caused by change with time and by flux through all six faces. The right-hand side is the gain in number of molecules entering the velocity range $d\xi_1 \cdot d\xi_2 \cdot d\xi_3$, less those leaving due to the effect of collision between molecules.

In the derivation of the Maxwell-Boltzmann equation (2.1) the following assumptions are involved:

(1) Point Molecules Physically, this statement implies a monatomic gas for which the molecular distribution function may be written as a function of $\vec{\xi}$, $\vec{x}$, and t alone. Also, the ratio of specific heats and the Prandtl number are, respectively,

$$\gamma = \frac{c_p}{c_v} = \frac{5}{3}$$

$$P_r = \frac{c_p \mu}{\lambda} = \frac{2}{3} \tag{2.2}$$

(2) Complete Collision This assumption states that there exists a time interval dt, which is large compared to the average duration of a collision, τ^*, but small compared to the average time between collisions, τ,

$$\tau^* << dt << \tau$$

The usual binary collision assumption states merely that τ^* is small compared to τ. The more stringent condition here allows incomplete collisions to be ignored ($dt >> \tau^*$), and at the same time the possible interference of a third molecule to be ignored too ($dt << \tau$). The condition $\tau^* << \tau$ is equivalent to the condition that the size of a molecule is much smaller than the mean free path L, which restricts the consideration to dilute gases.

(3) Slowly Varying Molecular Distribution Function This restricts the distribution function to be essentially constant over a distance comparable to the size of a molecule but does not restrict the variation in the distribution function over distances comparable to the mean free path.

(4) Molecular Chaos Any special way of choosing a group of molecules has no influence on the distribution of velocities.

It is also to be noted that the effect of the external forces on the distribution function $X_i(\partial F/\partial \xi_i)$, where X_i is the component of external forces per unit mass, has been neglected on the left-hand side of Eq. (2.1).

It is convenient to introduce the mass density

$$f(\vec{\xi}, \vec{x}, t) = m\, F(\vec{\xi}, \vec{x}, t) \tag{2.3}$$

m being the mass of a molecule. Then, the various moments may be defined as follows:

$$\rho(\vec{x}, t) = \int f(\vec{\xi}, \vec{x}, t)\, d\vec{\xi} \tag{2.4}$$

where the integral is extended over the whole three-dimensional velocity space. Obviously, ρ is the average mass density. Also

$$\vec{u}(\vec{x}, t) = \frac{1}{\rho}\int \vec{\xi} f \, d\vec{\xi} \tag{2.5}$$

is the macroscopic flow velocity. For higher moments, which depend on the molecular velocity relative to the mass flow velocity, the intrinsic velocity

$$\vec{c}(\vec{\xi}, \vec{x}, t) = \vec{\xi} - \vec{u}(\vec{x}, t) \tag{2.6}$$

is introduced, so that

$$\begin{aligned} P_{ij}(\vec{x}, t) &= \int c_i c_j f \, d\vec{\xi} \\ &= p\, \delta_{ij} + p_{ij} \end{aligned} \tag{2.7}$$

where

p = hydrostatic pressure

$\delta_{ij} = 1 \qquad i = j$

$\quad\; = 0 \qquad i \neq j$

with

$$P_{ii} = 3p \qquad p_{ii} = 0 \tag{2.8}$$

and

$$S_{ijk}(\vec{x}, t) = \int c_i c_j c_k f \, d\vec{\xi} \tag{2.9}$$

with

$$S_{ijj} = S_i \tag{2.10}$$

The temperature is defined as proportional to the mean kinetic energy measured with respect to the flow velocity $\vec{u}$. For a monatomic gas, the kinetic energy per unit mass is 3/2 RT, where R is the gas constant. An element of kinetic energy is $\frac{1}{2}c^2 f \, d\vec{x} \, d\vec{\xi}$; therefore,

$$\begin{aligned} \frac{3}{2} RT &= \frac{1}{\rho} \int \frac{1}{2} c^2 f \, d\vec{\xi} \\ &= \frac{3}{2} \frac{p}{\rho} \end{aligned}$$

by Eqs. (2.7) and (2.8), or

$$p = \rho RT \tag{2.11}$$

which is the equation of state.

The physical meaning of the second moment, P_{ij}, is the stress tensor due to molecular motion, since $c_i \, c_j \, f \, d\vec{x} \, d\vec{\xi}$ is the rate at which the j-component momentum is transferred in the i-direction. Similarly, the contracted third moment $S_i = 2 \int c_i \frac{1}{2} c^2 f \, d\vec{\xi}$ is twice the rate of transfer of kinetic energy per unit area with respect to the mean motion, $\vec{u}$, i.e., twice the heat flow. From Eqs. (2.9) and (2.10) one has

$$S_i(\vec{x}, t) = \int c_i c^2 f \, d\vec{\xi} = 2 q_i(\vec{x}, t) \tag{2.12}$$

Hereafter, the more common quantity q_i will be used throughout instead of the original quantity S_i introduced by Grad.

Multiplying both sides of the Maxwell-Boltzmann equation (2.1) by 1, $\vec{\xi}$, and ξ^2 , respectively, and integrating, one obtains

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho u_i\right) = 0 \tag{2.13}$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial p_{ij}}{\partial x_j} = 0 \tag{2.14}$$

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial x_i}(u_i p) + \frac{2}{3} p \frac{\partial u_i}{\partial x_i} + \frac{2}{3} p_{ij}\frac{\partial u_i}{\partial x_j} + \frac{2}{3}\frac{\partial q_i}{\partial x_i} = 0 \tag{2.15}$$

in which the definitions of the moments have been employed. Immediately, one recognizes Eqs. (2.13)-(2.15) to be the conservation equations of mass, momentum, and energy for a monatomic gas.

By assuming a special form of the molecular distribution function, dependent upon only temperature, pressure (or density), flow velocity, and their gradients in the fluid, the stresses p_{ij} and the heat fluxes q_i may be expressed in terms of these quantities. Thus, the system of conservation equations (2.13)-(2.15) becomes a determinate one. Included within this scheme are the Euler equations, the Navier-Stokes equations, and the Burnett equations. All these equations are, in some sense, successive approximations to the Maxwell-Boltzmann equation.

A new method of solving the Maxwell-Boltzmann equation has been developed by Grad[15]. The method is supposed to give more general solutions than that of Hilbert, Enskog, and Chapman. The essential feature of Grad's method is that the departure of the distribution function from Maxwellian is taken to be linearly dependent on stresses and heat fluxes, which are not assumed to depend explicitly on temperature, pressure, mean velocity, and their gradients.

An expression for the molecular distribution function is assumed, of the form

$$f(\vec{\xi},\vec{x},t) = f^{(0)}(\vec{x},t)\left\{1+\sum_{n=1}^{\infty}\frac{1}{n!}\,a_i^{(n)}(\vec{x},t)\,\mathcal{H}_i^{(n)}(\vec{\xi})\right\} \tag{2.16}$$

where

$$f^{(0)}(\vec{x},t) = \frac{\rho}{(2\pi RT)^{\frac{3}{2}}}\,e^{-\frac{C^2}{2RT}} \tag{2.17}$$

= local Maxwellian distribution, not exactly the equilibrium distribution, since $\vec{c}(\vec{u})$ and T can be functions of $\vec{x}$ and t

$$a_i^{(n)} = \frac{1}{\rho}\int f\,\mathcal{H}_i^{(n)}\,d\vec{\xi} \tag{2.18}$$

= Hermite coefficients, which are essentially state variables (Cf. Eq. (2.21).)

$\mathcal{H}_i^{(n)}$ = Hermite polynomials of order n, which are known functions of $\vec{\xi}$

$$\begin{cases} \mathcal{H}_i^{(1)} = v_i \\ \mathcal{H}_i^{(2)} = v_i v_j - \delta_{ij} \\ \mathcal{H}_{ijk}^{(3)} = v_i v_j v_k - (v_i\delta_{jk} + v_j\delta_{ik} + v_k\delta_{ij}),\ \text{etc.} \end{cases} \tag{2.19}$$

in which $\vec{v}$ is the dimensionless intrinsic velocity

$$\vec{C} = \vec{v}\sqrt{RT} \tag{2.20}$$

where $\vec{c}$ is defined by Eq. (2.6). Substituting Eq. (2.19) into Eq. (2.18)

$$\begin{aligned} a_i^{(1)} &= 0 \\ a_{ij}^{(2)} &= \frac{p_{ij}}{p} \\ a_{ijk}^{(3)} &= \frac{2q_{ijk}}{p\sqrt{RT}} \end{aligned} \tag{2.21}$$

The first significant coefficients are the stresses $a_{ij} = p_{ij}/p$ and the heat flow $a_{ijj}^{(3)} = \dfrac{2q_i}{p\sqrt{RT}}$.

By taking a sufficiently large number of terms in the Hermite expansion (2.16), Grad believes it is possible to approximate f to any desired order. However, it is assumed that the gas is sufficiently close to equilibrium for the distribution function f to be approximated by three terms.

From Eqs. (2.19)-(2.21), a simple expansion of Eq. (2.16) is

$$f = f^{(0)}\left\{1 + \frac{p_{ij}}{2pRT}c_i c_j - \frac{q_i c_i}{pRT}\left(1 - \frac{c^2}{5RT}\right)\right\} \tag{2.22}$$

It is interesting to compare expansion (2.22) with Enskog's solution to the Maxwell-Boltzmann equation for slowly-varying flow[23]

$$\begin{aligned} f &= f^{(0)}\left\{1 + \Phi(\vec{\xi}, \vec{x})\right\} \\ &= f^{(0)}\left\{1 + B_0 + mB_i\xi_i + mB_4 c^2 + \psi_1 + \psi_2 + \psi_3 + \psi_4\right\} \end{aligned}$$

where B_0, B_1, ..., B_4 are constants determined by the following five conditions

$$\int f^{(0)}\Phi\, d\vec{\xi} = 0$$

$$\int f^{(0)}\xi_i\,\Phi\, d\vec{\xi} = 0 \qquad i = 1, 2, 3$$

$$\int f^{(0)}\xi_i\,\xi_i\, d\vec{\xi} = 0$$

ξ_i = component of molecular velocity vector $\vec{\xi}$

$f^{(0)}$ = Maxwellian distribution function

m = mass of a gas molecule

$\vec{c}$ = intrinsic velocity = $\vec{\xi} - \vec{u}$

$\vec{u}$ = flow velocity

$c^2 = (\xi_1 - u_1)^2 + (\xi_2 - u_2)^2 + (\xi_3 - u_3)^2$

$$\psi_1 = -\xi_1 \left\{ \frac{1}{\rho}\frac{\partial \rho}{\partial x_1} - \frac{1}{T}\frac{\partial T}{\partial x_1}\left(\frac{3}{2} - \frac{c^2}{2RT}\right)\frac{\partial T}{\partial x_1} \right\} \phi(c)$$

$$\psi_4 = \left\{ (\xi_1^2 - \tfrac{1}{3}c^2)\frac{\partial u_1}{\partial x_1} + (\xi_2^2 - \tfrac{1}{3}c^2)\frac{\partial u_2}{\partial x_2} + (\xi_3^2 - \tfrac{1}{3}c^2)\frac{\partial u_3}{\partial x_3} \right.$$
$$\left. + \xi_1\xi_2\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) + \xi_2\xi_3\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) + \xi_3\xi_1\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) \right\} \phi_1(c)$$

ψ_2, ψ_3 are obtained from ψ_1 by changing x_1 into x_2, x_3, respectively, and ξ_1 into ξ_2, ξ_3.

$\phi(c)$, $\phi_1(c)$ are functions of c and the constants of gas only

As already mentioned, Enskog's form of the molecular distribution function depends only on temperature, density (or pressure), flow velocity, and their gradients. The time dependence of Enskog's solution is through that of temperature, density (or pressure), and flow velocity of the gas. However, in Grad's expansion (2.16), the Hermite coefficients $a_i^{(n)}$ vary with time directly as well as temperature, pressure, and flow velocity. Arbitrary initial values may be assigned to p_{ij} and q_i (Cf. Eq. (2.21)) and subsequent values determined from the partial differential equations (2.23).

If one replaces p_{ij} and q_i in expansion (2.22) by the following Navier-Stokes' approximation:

$$p_{ij} = -\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{2}{3}\mu\,\delta_{ij}\frac{\partial u_k}{\partial x_k}$$

$$q_i = -\lambda\frac{\partial T}{\partial x_i}$$

where μ and λ are the coefficients of viscosity and heat conductivity, respectively, then it is seen that Grad's expansion (2.22) will take the form of Enskog's solution.

The expansion (2.22) is valid for all flows in which the macroscopic variables do not vary by a large amount over a mean free path. The variation allowed is sufficient to include the study of shock waves of medium strength, particularly shocks whose normal component of the on-coming Mach number is less than 1.65.* In addition, the basic assumptions for the Boltzmann equation are always imposed.

By substituting expansion (2.22) into Eq. (2.1) and recalling Eq. (2.3)f = mF, one obtains the Grad equations, adding the conservation equations (2.13) to (2.15) and the equation of state (2.11) for completeness

$$
\left\{
\begin{aligned}
& \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_r}(\rho u_r) = 0 \\
& \frac{\partial u_i}{\partial t} + u_r \frac{\partial u_i}{\partial x_r} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} + \frac{1}{\rho}\frac{\partial p_{ir}}{\partial x_r} = 0 \\
& \frac{\partial p}{\partial t} + \frac{\partial}{\partial x_r}(u_r p) + \frac{2}{3} p \frac{\partial u_r}{\partial x_r} + \frac{2}{3} p_{ir}\frac{\partial u_i}{\partial x_r} + \frac{2}{3}\frac{\partial q_r}{\partial x_r} = 0 \\
& p = \rho R T \\
& \frac{\partial p_{ij}}{\partial t} + \frac{\partial}{\partial x_r}(u_r p_{ij}) + \frac{2}{5}\left(\frac{\partial q_i}{\partial x_j} + \frac{\partial q_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial q_r}{\partial x_r}\right) + p_{ir}\frac{\partial u_j}{\partial x_r} + p_{jr}\frac{\partial u_i}{\partial x_r} \\
& \qquad - \frac{2}{3}\delta_{ij} p_{rs}\frac{\partial u_r}{\partial x_s} + p\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial u_r}{\partial x_r}\right) + \frac{p}{\mu} p_{ij} = 0 \\
& \frac{\partial q_i}{\partial t} + \frac{\partial}{\partial x_r}(u_r q_i) + \frac{7}{5} q_r \frac{\partial u_i}{\partial x_r} + \frac{2}{5} q_r \frac{\partial u_r}{\partial x_i} + \frac{2}{5} q_i \frac{\partial u_r}{\partial x_r} + RT\frac{\partial p_{ir}}{\partial x_r} \\
& \qquad + \frac{7}{2} p_{ir}\frac{\partial RT}{\partial x_r} - \frac{p_{ir}}{\rho}\left(\frac{\partial p}{\partial x_r} + \frac{\partial p_{rs}}{\partial x_s}\right) + \frac{5}{2} p \frac{\partial RT}{\partial x_i} + \frac{2}{3}\frac{p}{\mu} q_i = 0
\end{aligned}
\right. \tag{2.23}
$$

The last two equations above are the same as Eq. (5.18) of

---

* See Introduction, page 11, or Reference 18.

Reference 15, except that S_i has been replaced by $2q_i$, as in Eq. (2.12); and B_1/m by $RT/6\mu$, as in Eq. (5.30) of Reference 15.

In Grad's system, the stresses and the heat fluxes are regarded as new dependent variables governed by additional equations rather than being expressed explicitly in terms of the state variables and their gradients. These additional equations, i.e., the last two of Eq. (2.23), which make the system determinate, may be called, respectively, the stress and the heat equations. It is to be noted that the Euler equations, the Navier-Stokes equations, and the Burnett equations may be deduced from the Grad equations.* Consequently, the Grad equations cover a larger part of the manifold of the solutions to the Boltzmann equation than the others do.

B. Boundary Conditions in Grad's Method

Together with the partial differential equations, Grad also gives the general boundary conditions. Considering the boundary perpendicular to the x_1- direction and the condition that a certain fraction α of the incident molecules is specularly reflected and that the remaining molecules are absorbed by the wall and re-emitted with a Maxwellian distribution at the temperature of the wall, T_w, one writes the boundary condition as

$$f^{+}(\xi_1, \xi_2, \xi_3) = \alpha f^{-}(-\xi_1, \xi_2, \xi_3) + k e^{-\frac{\xi^2}{2RT_w}} \qquad \xi_1 > 0 \qquad (2.24)$$

where

---

* H. T. Yang: "Reduction of Grad Thirteen-Moment Equations to Burnett Equations for Slip Flow", (unpublished), GALCIT, 1954.

in which
$$f(\vec{\xi}) = f^{+}(\vec{\xi}) + f^{-}(\vec{\xi})$$
$$f^{+}(\vec{\xi}) = 0 \text{ for } \xi_1 < 0$$
$$f^{-}(\vec{\xi}) = 0 \text{ for } \xi_1 > 0$$

and k is determined by the condition that the wall does not collect molecules, i.e., $u_1 = 0$. (See Fig. 3)

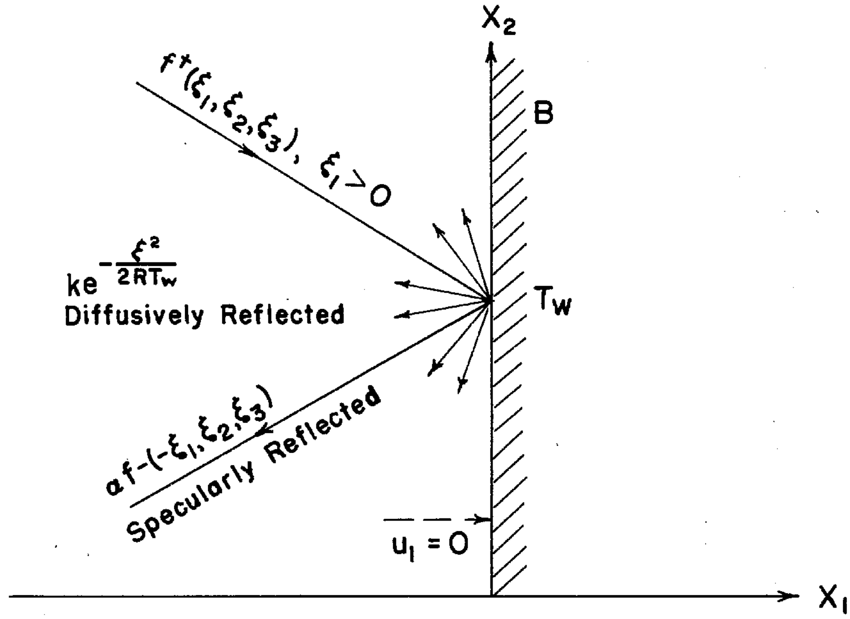

FIGURE 3

The boundary conditions were originally derived by Grad on a mathematical ground of symmetry consideration in the specularly reflected case. For better understanding, these boundary conditions will be re-derived here from a physical viewpoint based on conservation laws.

For the present third approximation, the molecular distribution function is given by expansion (2.22). In the two dimensional case, one has

$$
\begin{aligned}
u_3 &= 0 \\
p_{33} &= 0 \qquad p_{13} = 0 \qquad p_{23} = 0 \qquad\qquad (2.25)\\
q_3 &= 0 \\
p_{11} + p_{22} &= 0 \qquad \text{from Eq. (2.8)}
\end{aligned}
$$

Writing out expansion (2.22) and using relations (2.25), we have the molecular distribution function for the incident molecules,

$$
f^{+}(\xi_1,\xi_2,\xi_3) = \frac{\rho}{(2\pi RT)^{\frac{3}{2}}}\, e^{-\frac{(\xi_1-u_1)^2+(\xi_2-u_2)^2+\xi_3^2}{2RT}}
$$

$$
\times\Bigg[1+\frac{p_{11}(\xi_1-u_1)^2}{2pRT}-\frac{p_{11}(\xi_2-u_2)^2}{2pRT}+\frac{p_{12}(\xi_1-u_1)(\xi_2-u_2)}{pRT}
$$

$$
-\frac{q_1(\xi_1-u_1)}{pRT}-\frac{q_2(\xi_2-u_2)}{pRT}+\frac{q_1(\xi_1-u_1)^3}{5p(RT)^2}+\frac{q_1(\xi_1-u_1)(\xi_2-u_2)^2}{5p(RT)^2}+\frac{q_1(\xi_1-u_1)\xi_3^2}{5p(RT)^2} \qquad (2.26)
$$

$$
+\frac{q_2(\xi_1-u_1)^2(\xi_2-u_2)}{5p(RT)^2}+\frac{q_2(\xi_2-u_2)^3}{5p(RT)^2}+\frac{q_2(\xi_2-u_2)\xi_3^2}{5p(RT)^2}\Bigg]
$$

The molecular distribution function for the specularly reflected molecules is obtained by replacing ξ_1 by $-\xi_1$, in expression (2.26). The Maxwellian distribution function at the wall temperature is, by Eq. (2.17),

$$
f_w = k\, e^{-\frac{\xi_1^2+\xi_2^2+\xi_3^2}{2RT_w}} \qquad (2.27)
$$

where

$$
k = \frac{\rho_w}{(2\pi RT_w)^{\frac{3}{2}}}
$$

The mass of molecules striking the unit area of the wall is

$$
m^{+} = \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{0}^{\infty} \xi_1 f^{+}(\xi_1,\xi_2,\xi_3)\, d\xi_1
$$

Substituting Eq. (2.26) in the above expression and carrying out the integration, we get, with $u_1 = 0$ at the wall

$$m^+ = \frac{p+\frac{1}{2}p_{11}}{\sqrt{2\pi RT}} \tag{2.28}$$

Similarly, the mass of molecules leaving the unit area of wall in specular reflection is

$$\begin{aligned} \alpha m^- &= \alpha \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{0} \xi_1 f^-(-\xi_1, \xi_2, \xi_3)\, d\xi_1 \\ &= \alpha \frac{p+\frac{1}{2}p_{11}}{\sqrt{2\pi RT}} \end{aligned} \tag{2.29}$$

The mass of molecules leaving in Maxwellian reflection is, by Eq. (2.27),

$$\begin{aligned} k m_w &= k \int_{-\infty}^{\infty} e^{-\frac{\xi_3^2}{2RT_w}} d\xi_3 \int_{-\infty}^{\infty} e^{-\frac{\xi_2^2}{2RT_w}} d\xi_2 \int_{-\infty}^{0} \xi_1 e^{-\frac{\xi_1^2}{2RT_w}} d\xi_1 \\ &= k\, 2\pi (RT_w)^2 \end{aligned} \tag{2.30}$$

By conservation of mass, the mass of molecules striking the wall should be equal to that leaving. Equating expression (2.28) to the sum of expressions (2.29) and (2.30), one has

$$\frac{p+\frac{1}{2}p_{11}}{\sqrt{2\pi RT}} = \alpha \frac{p+\frac{1}{2}p_{11}}{\sqrt{2\pi RT}} + k\, 2\pi (RT_w)^2$$

Hence,

$$k = \frac{1-\alpha}{2\pi (RT_w)^2} \frac{p+\frac{1}{2}p_{11}}{\sqrt{2\pi RT}} \tag{2.31}$$

The tangential momentum brought to the unit area of the wall by the incident molecules is

$$\begin{aligned} M^+ &= \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} (\xi_2 - u_2)\, d\xi_2 \int_{0}^{\infty} (\xi_1 - u_1) f^+(\xi_1, \xi_2, \xi_3)\, d\xi_1 \\ &= \frac{p_{12}}{2} + \frac{q_2}{5\sqrt{2\pi RT}} \end{aligned} \tag{2.32}$$

The tangential momentum carried away from the unit area of the wall by the specularly reflected molecules is

$$
\begin{aligned}
\alpha M^{-} &= \alpha \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} (\xi_2 - u_2)\, d\xi_2 \int_{-\infty}^{0} (\xi_1 - u_1) f^{-}(-\xi_1, \xi_2, \xi_3)\, d\xi_1 \\
&= \alpha \left(-\frac{p_{12}}{2} + \frac{q_2}{5\sqrt{2\pi RT}} \right)
\end{aligned}
\tag{2.33}
$$

and that carried by the molecules emitted diffusely from the surface is

$$
\begin{aligned}
k\, M_w &= k \int_{-\infty}^{\infty} e^{-\frac{\xi_3^2}{2RT_w}}\, d\xi_3 \int_{-\infty}^{\infty} (\xi_2 - u_2)\, e^{-\frac{\xi_2^2}{2RT_w}}\, d\xi_2 \int_{-\infty}^{0} (\xi_1 - u_1)\, e^{-\frac{\xi_1^2}{2RT_w}}\, d\xi_1 \\
&= -k\, 2\pi (RT_w)^2 u_2
\end{aligned}
\tag{2.34}
$$

By conservation of momentum, we have

$$
M^{+} = \alpha M^{-} + k\, M_w
$$

or

$$
\frac{p_{12}}{2} + \frac{q_1}{5\sqrt{2\pi RT}} = \alpha \left(-\frac{p_{12}}{2} + \frac{q_1}{5\sqrt{2\pi RT}} \right) - k\, 2\pi (RT_w)^2 u_2
$$

Putting in k as given by Eq. (2.31) and simplifying, we obtain

$$
\frac{p_{12}}{p} + \frac{2(1-\alpha) u_2}{(1+\alpha)(2\pi RT)^{\frac{1}{2}}} \left(1 + \frac{p_{11}}{2p} \right) + \frac{2(1-\alpha) q_2}{5(1+\alpha) p \sqrt{2\pi RT}} = 0
\tag{2.35}
$$

The kinetic energy with respect to the mean motion, $\vec{u}$, brought to the unit area of the wall by the incident molecules is

$$
\begin{aligned}
E^{+} &= \frac{1}{2} \int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{0}^{\infty} (\xi_1 - u_1) \left[(\xi_1 - u_1)^2 + (\xi_2 - u_2)^2 + \xi_3^2 \right] f^{+}(\xi_1, \xi_2, \xi_3)\, d\xi_1 \\
&= \frac{q_1}{2} + \left(2p + \frac{3}{2} p_{11} \right) \left(\frac{RT}{2\pi} \right)^{\frac{1}{2}}
\end{aligned}
\tag{2.36}
$$

The kinetic energy with respect to the mean motion carried away from the unit area of the wall by molecules in specular reflection is

$$\alpha E^- = \frac{1}{2}\int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{0} (\xi_1 - u_1)\left[(\xi_1-u_1)^2 + (\xi_2-u_2)^2 + \xi_3^2\right] f^-(-\xi_1,\xi_2,\xi_3)\, d\xi_1$$
$$= \alpha\left[-\frac{q_1}{2} + \left(2p + \frac{3}{2}p_{11}\right)\left(\frac{RT}{2\pi}\right)^{\frac{1}{2}}\right] \qquad (2.37)$$

and that carried away by molecules in Maxwellian reflection is

$$kE_w = \frac{1}{2}\int_{-\infty}^{\infty} d\xi_3 \int_{-\infty}^{\infty} d\xi_2 \int_{-\infty}^{0} (\xi_1 - u_1)\left[(\xi_1-u_1)^2 + (\xi_2-u_2)^2 + \xi_3^2\right] e^{-\frac{\xi_1^2+\xi_2^2+\xi_3^2}{2RT_w}}\, d\xi_1$$
$$= k\pi (RT_w)^2 (4RT_w + u_2^2) \qquad (2.38)$$

By conservation of energy, we have

$$E^+ = \alpha E^- + kE_w$$

Using Eqs. (2.36), (2.37), and (2.38)

$$\frac{q_1}{2} + \left(2p + \frac{3}{2}p_{11}\right)\left(\frac{RT}{2\pi}\right)^{\frac{1}{2}} = -\alpha\frac{q_1}{2} + \alpha\left(2p + \frac{3}{2}p_{11}\right)\left(\frac{RT}{2\pi}\right)^{\frac{1}{2}}$$
$$+ k\pi (RT_w)^2 (4RT_w + u_2^2)$$

Putting in k as given by Eq. (2.31) and simplifying, we get

$$\left(\frac{2\pi}{RT}\right)^{\frac{1}{2}}\frac{q_1}{p} + \frac{4(1-\alpha)}{1+\alpha}\left[1 - \frac{T_w}{T} + \frac{p_{11}}{2p}\left(\frac{3}{2} - \frac{T_w}{T}\right) - \left(1 + \frac{p_{11}}{2p}\right)\frac{\left(\frac{u_2}{2}\right)^2}{RT}\right] = 0 \qquad (2.39)$$

It is to be noted that the boundary condition $u_1 = 0$ has been used throughout in the foregoing calculation.

To sum up, the appropriate boundary conditions for the Grad equations (2.23) are the following:

$$u_1 = 0 \qquad (2.40)$$

$$\frac{p_{12}}{p} + \frac{2(1-\alpha)u_2}{(1+\alpha)(2\pi RT)^{\frac{1}{2}}}\left(1 + \frac{p_{11}}{2p}\right) + \frac{2(1-\alpha)q_2}{5(1+\alpha)p\,(2\pi RT)^{\frac{1}{2}}} = 0 \qquad (2.41)$$

$$\left(\frac{2\pi}{RT}\right)^{\frac{1}{2}}\frac{q_1}{p}+\frac{4(1-\alpha)}{1+\alpha}\left[1-\frac{T_w}{T}+\frac{p_{11}}{2p}\left(\frac{3}{2}-\frac{T_w}{T}\right)-\left(1+\frac{p_{11}}{2p}\right)\frac{\left(\frac{u_2}{2}\right)^2}{RT}\right]=0 \qquad (2.42)$$

The last two boundary conditions are the same as Eqs. (6.30) and (6.31) in Reference 15, as originally derived by Grad, with S_i replaced by $2q_i$. However, it is to be noted that in Reference 15 the factor $\frac{1}{2}$ was missing in the last term of the last equation. The first boundary condition states that the wall does not collect or emit gas molecules. The second one states that a tangential stress or a tangential heat flow at the boundary is associated with a tangential slip velocity of the stream. The third one states, roughly, that a normal heat flow is associated with a temperature jump between the wall and the adjacent gas.

For flows not too far from the equilibrium state,

$$p_{11} << p, \; T_w \cong T, \; u \cong 0$$

the last two boundary conditions simplify to

$$\frac{p_{12}}{p}+\frac{2(1-\alpha)u_2}{(1+\alpha)(2\pi RT)^{\frac{1}{2}}}+\frac{2(1-\alpha)q_2}{5(1+\alpha)p(2\pi RT)^{\frac{1}{2}}}=0 \qquad (2.43)$$

$$\left(\frac{2\pi}{RT}\right)^{\frac{1}{2}}\frac{q_1}{p}+\frac{4(1-\alpha)}{1+\alpha}\left(1-\frac{T_w}{T}+\frac{p_{11}}{4p}\right)=0 \qquad (2.44)$$

which are the boundary conditions (6.30) and (6.31) in Reference 15.

III. RAYLEIGH'S PROBLEM FOR THE CASE OF SMALL IMPULSIVE VELOCITY

A. Formulation of Rayleigh's Problem in General

Rayleigh's problem consists of the study of the flow above an infinite flat plate suddenly accelerated from rest to uniform motion with velocity, U. There are two independent variables: namely, the time, t, and the coordinate normal to the plate, y. There is no variation of flow properties along the plate, viz., $\partial(\ )/\partial x = 0$. The dependent variables are the density ρ, the pressure p (or the temperature T as related by the equation of state (2.11)), the velocity components, u and v, the stresses p_{xx}, p_{xy}, and p_{yy}, and the heat fluxes q_x and q_y, making a total of nine. For Rayleigh's problem, the system of Grad equations (2.23) becomes:

Continuity:

$$\frac{\partial \rho}{\partial t} + v\frac{\partial \rho}{\partial y} + \rho\frac{\partial v}{\partial y} = 0 \tag{3.1}$$

Momentum:

$$\frac{\partial u}{\partial t} + v\frac{\partial u}{\partial y} + \frac{1}{\rho}\frac{\partial p_{xy}}{\partial y} = 0 \tag{3.2}$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial y} + \frac{1}{\rho}\frac{\partial p}{\partial y} + \frac{1}{\rho}\frac{\partial p_{yy}}{\partial y} = 0 \tag{3.3}$$

Energy:

$$\frac{\partial p}{\partial t} + v\frac{\partial p}{\partial y} + \frac{2}{3}p_{xy}\frac{\partial u}{\partial y} + \left(\frac{5}{3}p + \frac{2}{3}p_{yy}\right)\frac{\partial v}{\partial y} + \frac{2}{3}\frac{\partial q_y}{\partial y} = 0 \tag{3.4}$$

State:

$$p = \rho R T \tag{3.5}$$

Stress:

$$\frac{\partial p_{xx}}{\partial t}+v\frac{\partial p_{xx}}{\partial y}+\frac{4}{3}p_{xy}\frac{\partial u}{\partial y}+\left(p_{xx}-\frac{2}{3}p_{yy}-\frac{2}{3}p\right)\frac{\partial v}{\partial y}-\frac{4}{15}\frac{\partial q_y}{\partial y}+\frac{p}{\mu}p_{xx}=0 \tag{3.6}$$

$$\frac{\partial p_{xy}}{\partial t}+v\frac{\partial p_{xy}}{\partial y}+\left(p+p_{yy}\right)\frac{\partial u}{\partial y}+2p_{xy}\frac{\partial v}{\partial y}+\frac{2}{5}\frac{\partial q_x}{\partial y}+\frac{p}{\mu}p_{xy}=0 \tag{3.7}$$

$$\frac{\partial p_{yy}}{\partial t}+v\frac{\partial p_{yy}}{\partial y}+\left(\frac{4}{3}p+\frac{7}{3}p_{yy}\right)\frac{\partial v}{\partial y}-\frac{2}{3}p_{xy}\frac{\partial u}{\partial y}+\frac{8}{15}\frac{\partial q_y}{\partial y}+\frac{p}{\mu}p_{yy}=0 \tag{3.8}$$

Heat:

$$\frac{\partial q_x}{\partial t}+v\frac{\partial q_x}{\partial y}+\frac{7}{5}q_y\frac{\partial u}{\partial y}+\frac{7}{5}q_x\frac{\partial v}{\partial y}+\frac{5}{2}\frac{p_{xy}}{\rho}\frac{\partial p}{\partial y}-\frac{7}{2}\frac{p\,p_{xx}}{\rho^2}\frac{\partial \rho}{\partial y}+\left(\frac{p}{\rho}-\frac{p_{xx}}{\rho}\right)\frac{\partial p_{xy}}{\partial y}-\frac{p_{xy}}{\rho}\frac{\partial p_{yy}}{\partial y}+\frac{2}{3}\frac{p}{\mu}q_x=0 \tag{3.9}$$

$$\frac{\partial q_y}{\partial t}+v\frac{\partial q_y}{\partial y}+\frac{16}{5}q_y\frac{\partial v}{\partial y}+\frac{2}{5}q_x\frac{\partial u}{\partial y}+\frac{5}{2}\left(\frac{p}{\rho}+\frac{p_{yy}}{\rho}\right)\frac{\partial p}{\partial y}-\left(\frac{5}{2}\frac{p^2}{\rho^2}+\frac{7}{2}\frac{p\,p_{yy}}{\rho^2}\right)\frac{\partial \rho}{\partial y}+\left(\frac{p}{\rho}-\frac{p_{yy}}{\rho}\right)\frac{\partial p_{yy}}{\partial y}-\frac{p_{xy}}{\rho}\frac{\partial p_{xy}}{\partial y}+\frac{2}{3}\frac{p}{\mu}q_y=0 \tag{3.10}$$

The initial conditions for the gas field originally at rest are as follows, with the subscript 0 denoting undisturbed quantities:

$$\begin{aligned}
u(0,y) &= 0\\
v(0,y) &= 0\\
\rho(0,y) &= \rho_0\\
p(0,y) &= p_0\\
(\;T(0,y) &= T_0\;)\\
p_{xx}(0,y) &= 0\\
p_{xy}(0,y) &= 0\\
p_{yy}(0,y) &= 0\\
q_x(0,y) &= 0\\
q_y(0,y) &= 0
\end{aligned} \tag{3.11}$$

At the surface of the plate y = 0, the boundary conditions

(2.40) to (2.42) become

$$v(t,0) = 0 \tag{3.12}$$

$$\frac{p_{xy}(t)}{p(t)} + \frac{2(1-\alpha)}{(1+\alpha)(2\pi R)^{\frac{1}{2}}} \frac{u(t)-U}{[T(t)]^{\frac{1}{2}}}\left(1+\frac{p_{yy}(t)}{2p(t)}\right) + \frac{2(1-\alpha)}{5(1+\alpha)(2\pi R)^{\frac{1}{2}}} \frac{q_x(t)}{p(t)[T(t)]^{\frac{1}{2}}} = 0 \tag{3.13}$$

$$\left(\frac{2\pi}{R}\right)^{\frac{1}{2}} \frac{1}{[T(t)]^{\frac{1}{2}}} \frac{q_y(t)}{p(t)} + \frac{4(1-\alpha)}{1+\alpha}\left[1 - \frac{T_w}{T(t)} + \frac{p_{yy}(t)}{2p(t)}\left(\frac{3}{2} - \frac{T_w}{T(t)}\right) - \left(1+\frac{p_{yy}(t)}{2p(t)}\right)\frac{\{u(t)-U\}^2}{4RT(t)}\right] = 0 \tag{3.14}$$

where α = fraction of the incident molecules specularly reflected from the plate

U = impulsive velocity of the plate

T_w = temperature of the plate

At $y = \infty$, all the dependent variables are required to be finite.

B. Linearization of Equations of Motion and Boundary Conditions

The system of partial differential equations (3.1) to (3.10) for Rayleigh's problem is highly non-linear and, hence, extremely difficult to solve. However, if we limit ourselves to the special case where the plate impulsive velocity, U, is small, and where the temperature difference between the plate and the undisturbed gas, $T_w - T_o$, is small, then the equations as well as the boundary conditions are linearized. The study of the simplified problem should throw some light on the general problem of arbitrary impulsive velocity.

Under the assumptions of small impulsive velocity and small temperature difference, one may write

$$
\begin{aligned}
u &= u' \\
v &= v' \\
\rho &= \rho_o + \rho' \\
p &= p_o + p' \\
T &= T_o + T' \\
p_{xx} &= p_{xx}' \\
p_{xy} &= p_{xy}' \\
p_{yy} &= p_{yy}' \\
q_x &= q_x' \\
q_y &= q_y'
\end{aligned}
\tag{3.15}
$$

where the primed quantities denote small perturbations in the undisturbed quantities. The coefficient of viscosity depends on the temperature only:

$$\frac{\mu}{\mu_0} = \left(\frac{T}{T_o}\right)^{\omega}$$

Hence,

$$\mu = \mu_o\left(1 + \frac{T'}{T_o}\right)^{\omega} = \mu_o + \mu' \tag{3.16}$$

Now, substitute expressions (3.15) and (3.16) in Eqs. (3.1) to (3.10). The system of partial differential equations becomes

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial v'}{\partial y} = 0 \tag{3.17}$$

$$\frac{\partial u'}{\partial t} + \frac{1}{\rho_o} \frac{\partial p_{xy}'}{\partial y} = 0 \tag{3.18}$$

$$\frac{\partial v'}{\partial t} + \frac{1}{\rho_o} \frac{\partial p'}{\partial y} + \frac{1}{\rho_o} \frac{\partial p_{yy}'}{\partial y} = 0 \tag{3.19}$$

$$\frac{\partial p'}{\partial t} + \frac{5}{3} p_0 \frac{\partial v'}{\partial y} + \frac{2}{3} \frac{\partial q_y'}{\partial y} = 0 \tag{3.20}$$

$$\frac{\partial p_{xx}'}{\partial t} - \frac{2}{3} p_0 \frac{\partial v'}{\partial y} - \frac{4}{15} \frac{\partial q_y'}{\partial y} + \frac{p_0}{\mu_0} p_{xx}' = 0 \tag{3.21}$$

$$\frac{\partial p_{xy}'}{\partial t} + p_0 \frac{\partial u'}{\partial y} + \frac{2}{5} \frac{\partial q_x'}{\partial y} + \frac{p_0}{\mu_0} p_{xy}' = 0 \tag{3.22}$$

$$\frac{\partial p_{yy}'}{\partial t} + \frac{4}{3} p_0 \frac{\partial v'}{\partial y} + \frac{8}{15} \frac{\partial q_y'}{\partial y} + \frac{p_0}{\mu_0} p_{yy}' = 0 \tag{3.23}$$

$$\frac{\partial q_x'}{\partial t} + \frac{p_0}{\rho_0} \frac{\partial p_{xy}'}{\partial y} + \frac{2}{3} \frac{p_0}{\mu_0} q_x' = 0 \tag{3.24}$$

$$\frac{\partial q_y'}{\partial t} + \frac{5}{2} \frac{p_0}{\rho_0} \frac{\partial p'}{\partial y} - \frac{5}{2} \frac{p_0^2}{\rho_0^2} \frac{\partial \rho'}{\partial y} + \frac{p_0}{\rho_0} \frac{\partial p_{yy}'}{\partial y} + \frac{2}{3} \frac{p_0}{\mu_0} q_y' = 0 \tag{3.25}$$

and the equation of state (3.5) becomes

$$p' = R\,(\rho_0 T' + T^0 \rho') \tag{3.26}$$

or

$$T' = \frac{1}{R\rho_0} p' - \frac{T_0}{\rho_0} \rho' \tag{3.27}$$

The initial conditions are, from (3.11)

$$Q'(0, y) = 0 \tag{3.28}$$

where Q(t,y) is any physical quantity. The above conditions state merely that there is no initial disturbance.

The boundary conditions (3.12) to (3.14) are linearized as

$$v'(t, 0) = 0 \tag{3.29}$$

$$\frac{p_{xy}'(t,0)}{p_0} + \frac{2(1-\alpha)}{(1+\alpha)(2\pi R)^{\frac{1}{2}}} \frac{u'(t,0)-U}{[T_0]^{\frac{1}{2}}} + \frac{2(1-\alpha)}{5(1+\alpha)(2\pi R)^{\frac{1}{2}}} \frac{q_x'(t,0)}{p_0[T_0]^{\frac{1}{2}}} = 0 \tag{3.30}$$

$$\left(\frac{2\pi}{RT_0}\right)^{\frac{1}{2}} \frac{q_y'(t,0)}{p_0} + \frac{4(1-\alpha)}{1+\alpha}\left[1 - \frac{T_w}{T_0} + \frac{p_{yy}'(t,0)}{4p_0}\right] = 0 \tag{3.31}$$

and these boundary conditions take the same form as (2.43) and (2.44) for flows not too far from equilibrium.

It is also interesting to note that Eqs. (3.18), (3.22), and (3.24) with boundary condition (3.30) completely determine the quantities induced parallel to the plate motion: namely, the tangential flow velocity, u; the tangential shearing stress p_{xy}', and the tangential heat flow q_x'. The other quantities like normal flow velocity v', density ρ ', pressure p', temperature T', normal stresses p_{xx}', p_{yy}', and normal heat flow q_y' are governed by Eqs. (3.17), (3.19), (3.20), (3.21), (3.23), (3.25), and the boundary conditions (3.29) and (3.31). In other words, momentum and energy are not coupled. The dynamic effects of the plate impulsive velocity, U, and the thermal effects of the temperature difference between the plate and the undisturbed gas, $T_w - T_o$, may be treated separately and then superposed.

The characteristics of the linearized system (3.17) to (3.25) are found by the vanishing of the determinant (3.32)

$$
\begin{vmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_0} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{\rho_0} & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\rho_0} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{5}{3}p_0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{2}{3} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{2}{3}p_0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{4}{15} \\
0 & 0 & 0 & 0 & 0 & p_0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{2}{5} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{4}{3}p_0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{8}{15} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p_0}{\rho_0} & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -\frac{5}{2}\frac{p_0^2}{\rho_0^2} & 0 & \frac{5}{2}\frac{p_0}{\rho_0} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{p_0}{\rho_0} & 0 & 0 & 1 & 0 \\
dt & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & dt & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & dt & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & dt & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dy & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dy & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dy & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dy & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & dt & dy
\end{vmatrix} = 0 \tag{3.32}
$$

From the above, we obtain

$$
\begin{aligned}
&\frac{dy}{dt} = 0 \\
&\frac{dy}{dt} = \pm\sqrt{\frac{7}{5}\frac{p_0}{\rho_0}} = \pm\frac{\sqrt{21}}{5}a_0
\end{aligned} \tag{3.33}
$$

where

$a_o = \sqrt{\gamma R T_o} = \sqrt{\frac{5}{3} R T_o}$ is the isentropic speed of sound in the undisturbed monatomic gas

and

$$\frac{dy}{dt} = 0, 0$$

$$\frac{dy}{dt} = \pm\sqrt{\frac{13 \pm \sqrt{94}}{5} \frac{p_o}{\rho_o}} = \pm\frac{\sqrt{3(13 \pm \sqrt{94}}}{5} a_o \tag{3.34}$$

The solution dy/dt = 0 means that the particle path is one characteristic. The solutions (3.33) and (3.34) represent characteristic directions at "soundspeeds" dy/dt, different from the isentropic soundspeed which is the characteristic slope from the Euler equations. All the characteristics obtained here are identical to those given by Grad in Reference 15, except that due to linearization, the present characteristics are straight lines and are known in advance. It is to be noted that the characteristics (3.33) are associated with the tangential quantities u', p_{xy}', and q_x' dependent upon the impulsive velocity of the plate U, while the characteristics (3.34) are associated with the rest of quantities dependent upon the temperature difference $T_w - T_o$.

C. Laplace Transform Solutions of the Linearized Rayleigh Problem

To solve the linearized equations (3.17) to (3.25), we apply the Laplace transform defined by[24,25]

$$\int_0^\infty e^{-st} Q(t,y)\, dt = \overline{Q}(s,y) \tag{3.35}$$

where

Q(t,y) is a function defined for all positive values of the variable t s is the transform variable

$\bar{Q}(s,y)$ is called the Laplace Transform of $Q(t,y)$.

By definition of (3.35), we have the following:

$$\int_0^\infty e^{-st} \frac{\partial Q(t,y)}{\partial t} dt = e^{-st} Q(t,y)\Big|_0^\infty + s\int_0^\infty e^{-st} Q(t,y)\,dt \qquad (3.36)$$

$$= -Q(0,y) + s\bar{Q}(s,y)$$

$$\int_0^\infty e^{-st} \frac{\partial Q(t,y)}{\partial y} dt = \frac{\partial}{\partial y}\int_0^\infty e^{-st} Q(t,y)\,dt = \frac{d}{dy}\bar{Q}(s,y) \qquad (3.37)$$

and for Q_o being a constant

$$\int_0^\infty e^{-st} Q_o\,dt = Q_o \frac{1}{s} \qquad (3.38)$$

By using Eqs. (3.35) to (3.38) and the initial conditions (3.28), we transform the partial differential equations (3.17) to (3.25) into the following system of ordinary differential equations:

$$\frac{d\overline{v'}}{dy} = -\frac{1}{\rho_o} s\,\overline{\rho'} \qquad (3.39)$$

$$\frac{d\overline{p'_{xy}}}{dy} = -\rho_o s\,\overline{u'} \qquad (3.40)$$

$$\frac{d\overline{p'}}{dy} + \frac{d\overline{p'_{yy}}}{dy} = -\rho_o s\,\overline{v'} \qquad (3.41)$$

$$\frac{5}{3}\frac{d\overline{v'}}{dy} + \frac{2}{3}\frac{1}{p_o}\frac{d\overline{q'_y}}{dy} = -\frac{1}{p_o} s\,\overline{p'} \qquad (3.42)$$

$$\frac{5}{3}\frac{d\overline{v'}}{dy} + \frac{2}{3}\frac{1}{p_o}\frac{d\overline{q'_y}}{dy} = \frac{5}{2}\left(\frac{1}{p_o}s + \frac{1}{\mu_o}\right)\overline{p'_{xx}} \qquad (3.43)$$

$$\frac{d\overline{u'}}{dy} \qquad + \frac{2}{5}\frac{1}{p_o}\frac{d\overline{q'_x}}{dy} = -\left(\frac{1}{p_o}s + \frac{1}{\mu_o}\right)\overline{p'_{xy}} \tag{3.44}$$

$$\frac{5}{3}\frac{d\overline{v'}}{dy} \qquad + \frac{2}{3}\frac{1}{p_o}\frac{d\overline{q'_y}}{dy} = -\frac{5}{4}\left(\frac{1}{p_o}s + \frac{1}{\mu_o}\right)\overline{p'_{yy}} \tag{3.45}$$

$$\frac{d\overline{p'_{xy}}}{dy} = -\left(\frac{\rho_o}{p_o}s + \frac{2}{3}\frac{\rho_o}{\mu_o}\right)\overline{q'_x} \tag{3.46}$$

$$\frac{5}{2}\frac{d\overline{p'}}{dy} - \frac{5}{2}\frac{p_o}{\rho_o}\frac{d\overline{\rho'}}{dy} \qquad + \frac{d\overline{p'_{yy}}}{dy} = -\left(\frac{\rho_o}{p_o}s + \frac{2}{3}\frac{\rho_o}{\mu_o}\right)\overline{q'_y} \tag{3.47}$$

The equation of state (3.27) transforms into

$$\overline{T'} = \frac{1}{R\rho_o}\overline{p'} - \frac{T_o}{\rho_o}\overline{\rho'} \tag{3.48}$$

The boundary conditions (3.29) to (3.31) transform to

$$\overline{v'}(s,0) = 0 \tag{3.49}$$

$$\frac{\overline{p'_{xy}}(s,0)}{p_o} + \frac{2(1-\alpha)}{(1+\alpha)(2\pi R T_o)^{\frac{1}{2}}}\left[\overline{u'}(s,0) - U\frac{1}{s} + \frac{1}{5}\frac{\overline{q'_x}(s,0)}{p_o}\right] = 0 \tag{3.50}$$

$$\left(\frac{2\pi}{RT_o}\right)^{\frac{1}{2}}\frac{\overline{q'_y}(s,0)}{p_o} + \frac{4(1-\alpha)}{1+\alpha}\left[\left(1 - \frac{T_w}{T_o}\right)\frac{1}{s} + \frac{\overline{p'_{yy}}(s,0)}{4p_o}\right] = 0 \tag{3.51}$$

From Eqs. (3.40) and (3.46),

$$\overline{u'} = -\frac{1}{\rho_o}\frac{1}{s}\frac{d\overline{p'_{xy}}}{dy} \tag{3.52}$$

$$\overline{q'_x} = -\frac{1}{\frac{\rho_o}{p_o}s + \frac{2}{3}\frac{\rho_o}{\mu_o}}\frac{d\overline{p'_{xy}}}{dy} \tag{3.53}$$

Substituting Eqs. (3.52) and (3.53) into Eq. (3.44),

$$-\frac{1}{\rho_o}\frac{1}{s}\frac{d^2\overline{p'_{xy}}}{dy^2}-\frac{2}{5}\frac{1}{p_o}\frac{1}{\frac{\rho_o}{p_o}s+\frac{2}{3}\frac{p_o}{\mu_o}}\frac{d^2\overline{p'_{xy}}}{dy^2} = -\left(\frac{1}{p_o}s+\frac{1}{\mu_o}\right)\overline{p'_{xy}}$$

or

$$\frac{d^2\overline{p'_{xy}}}{dy^2}-5\frac{\rho_o}{p_o}\frac{s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}\overline{p'_{xy}} = 0 \tag{3.54}$$

Since $\overline{p_{xy}'}$ should not be infinite at $y = \infty$, we have

$$\overline{p'_{xy}} = A(s)\,e^{-f(s)y} \tag{3.55}$$

where

$$f(s) = \frac{5}{a_o}\sqrt{\frac{s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{3\left(21s+10\frac{p_o}{\mu_o}\right)}} \tag{3.56}$$

A(s) is an arbitrary function to be determined from the boundary condition (3.50), and the radical is taken to be real and positive. From Eqs. (3.52) and (3.53)

$$\overline{u'} = A(s)\frac{1}{\rho_o}\frac{1}{s}\sqrt{\frac{5\frac{\rho_o}{p_o}s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}}\,e^{-f(s)y} \tag{3.57}$$

$$\overline{q'_x} = A(s)\frac{3\frac{p_o}{\rho_o}}{3s+2\frac{p_o}{\mu_o}}\sqrt{\frac{5\frac{\rho_o}{p_o}s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}}\,e^{-f(s)y} \tag{3.58}$$

To evaluate A(s), we substitute Eqs. (3.55), (3.57), and (3.58), into the boundary condition (3.50)

$$A(s)\left\{\frac{(1+\alpha)(2\pi RT_o)^{\frac{1}{2}}}{2(1-\alpha)}\frac{1}{p_o}+\frac{1}{\rho_o}\frac{1}{s}\sqrt{\frac{5\frac{\rho_o}{p_o}s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}}+\frac{1}{5}\frac{1}{\rho_o}\frac{3}{3s+2\frac{p_o}{\mu_o}}\sqrt{\frac{5\frac{\rho_o}{p_o}s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}}\right\}=\frac{U}{s}$$

Hence,

$$A(s)=p_oM_o\frac{1}{\frac{1+\alpha}{1-\alpha}\left(\frac{3\pi}{10}\right)^{\frac{1}{2}}s+\left(1+\frac{3}{5}\frac{s}{3s+2\frac{p_o}{\mu_o}}\right)\sqrt{\frac{3s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{21s+10\frac{p_o}{\mu_o}}}} \tag{3.59}$$

where

$M_o = U/a_o$ = Mach number of the moving plate

U = impulsive velocity of the plate

$a_o = \sqrt{\frac{5}{3}\frac{p_o}{\rho_o}}$ = isentropic speed of sound of the undisturbed fluid

Substituting the expression (3.59) into Eqs. (3.55), (3.57), and (3.58) we find the transformed tangential stress, velocity, and heat flow as follows:

$$\overline{p_{xy}'} = p_o \frac{U}{a_o} \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}\,s + \left(\frac{18}{5}s + 2\frac{p_o}{\mu_o}\right)\sqrt{\frac{3s\left(s+\frac{p_o}{\mu_o}\right)}{\left(3s+2\frac{p_o}{\mu_o}\right)\left(21s+10\frac{p_o}{\mu_o}\right)}}}\, e^{-f(s)y} \qquad (3.60)$$

$$\overline{u'} = U \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}\,s\sqrt{\frac{s\left(21s+10\frac{p_o}{\mu_o}\right)}{3\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}} + s\left(1+\frac{3}{5}\frac{s}{3s+2\frac{p_o}{\mu_o}}\right)}\, e^{-f(s)y} \qquad (3.61)$$

$$\overline{q_x'} = p_o U \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{30}}\,s\sqrt{\frac{\left(21s+10\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{3s\left(s+\frac{p_o}{\mu_o}\right)}} + \frac{2}{3}\left(\frac{9}{5}s+\frac{p_o}{\mu_o}\right)}\, e^{-f(s)y} \qquad (3.62)$$

where

$$f(s) = \frac{5}{a_o}\sqrt{\frac{s\left(s+\frac{p_o}{\mu_o}\right)\left(3s+2\frac{p_o}{\mu_o}\right)}{3\left(21s+10\frac{p_o}{\mu_o}\right)}} \qquad (3.56)$$

The rest of the transformed quantities, such as normal velocity $\overline{v'}$, density $\overline{\rho'}$, pressure $\overline{p'}$, temperature $\overline{T'}$, normal stresses $\overline{p_{xx}'}$, $\overline{p_{yy}'}$, and normal heat flux $\overline{q_y'}$, are given by Eqs. (A.17) to (A.23) in the Appendix.

It is interesting to note once again from expressions (3.60) to (3.62) and (A.17) to (A.23) that the effect of the impulsive velocity U of the flat plate and the effect of the temperature difference $T_w - T_o$ between the plate and the undisturbed gas are separated by linearization. Consequently, the Rayleigh's problem of an infinite flat plate suddenly accelerated to uniform motion with a small impulsive velocity and a small temperature difference is split into two parts:

(1) The impulsive velocity U gives rise to the tangential quantities, viz.,shearing stress p_{xy}, tangential velocity u, and heat flow along the plate q_x, as obtained in Eqs. (3.60) to (3.62).

(2) The temperature difference $T_w - T_o$ produces the rest of the disturbances: namely, pressure p', normal stresses p_{xx}, p_{yy}, vertical velocity v, density ρ' (temperature T'), and heat flow normal to the plate q_y, as found in Eqs. (A.17) to (A.23).

This separation of effects is made possible mainly because the energy dissipation terms, such as p_{xy} $(\partial u/\partial y)$ in the energy equation (3.4) and $\left\{u(t) - U\right\}^2$ in the boundary condition (3.14), were neglected in the linearizing procedure. When the impulsive velocity, as well as the temperature difference, is large, the situation would certainly be different.

Our main concern in the present problem is the effect due to impulsive velocity U. Physically, neglecting the temperature difference corresponds to the case of a heat insulated plate, for which the condition $q_y\big|_{y=o} = 0$ requires $T_w = T_o$ [Cf. Eq. (A.23) and Eq. (A.16) which gives $B(s) = 0$ for $T_w = T_o$.] In the following sections, Rayleigh's problem of an insulated flat plate at low Mach number will be investigated in detail.

We may foresee the nature of the solutions by examining Eqs. (3.60) to (3.62). For small time, most of the contributions to the quantities p_{xy}, u, and q_x would come from $\bar{p}_{xy}$, $\bar{u}$, and $\bar{q}_x$, respectively, when s is large. From Eqs. (3.60) to (3.62) we see that when s is large, $\bar{p}_{xy}$, $\bar{u}$, and $\bar{q}_x$ are all of the form $\frac{1}{s} e^{-\frac{5}{\sqrt{21}} \frac{y}{\alpha_0} s}$, which gives step-functions in $\left(t - \frac{5}{\sqrt{21}} \frac{y}{\alpha_0}\right)$. For large time, the contributions to p_{xy}, u, and q_x would come from $\bar{p}_{xy}$, $\bar{u}$, and $\bar{q}_x$ when s is small, which are of the form $\frac{1}{\sqrt{s}} e^{-\frac{y}{\sqrt{\mu_0}} \sqrt{s}}$, $\frac{1}{s} e^{-\frac{y}{\sqrt{\mu_0}} \sqrt{s}}$, $e^{-\frac{y}{\sqrt{\mu_0}} \sqrt{s}}$, respectively. The corresponding inversion functions are diffusive in character. These distinct features of the solutions for small and large values of the time are just what one would have expected: namely, during the initial stage of the motion, the quantities p_{xy}, u, and q_x are impulsively developed as step functions in y and the region of main influence grows linearly with time; for later times, these quantities are diffused out smoothly in the whole space by viscosity, and the viscous layer grows as $\sqrt{t}$.

IV. THE HEAT INSULATED PLATE

For a heat insulated plate, there is no heat transfer between the plate and the adjacent gas. Symbolically

$$q_y(t,0) \equiv q_y(t,y)\Big|_{y=0} = 0 \qquad \text{for all } t. \tag{4.1}$$

Taking the Laplace transform as defined in Eq. (3.35)

$$\bar{q}_y(s,0) = 0 \qquad \text{for all } s. \tag{4.2}$$

From expression (A.23) in the Appendix, we see that the condition (4.2) requires

$$T_w = T_o \tag{4.3}$$

i.e., the temperature of the flat plate is the same as that of the undisturbed gas.

Therefore, for Rayleigh's problem of an insulated infinite flat plate suddenly accelerated to uniform motion with a small velocity U, we have from expressions (A.17) to (A.23) with $1 - (T_w/T_o) = 0$,

$$p' = 0, \quad p_{yy}' = 0, \quad p_{xx}' = 0, \quad v' = 0, \quad \rho' = 0, \quad T' = 0, \quad q_y' = 0$$

Thus, Eq. (3.15) becomes

$$\begin{aligned} u &= u' \\ v &= 0 \\ \rho &= \rho_o \\ p &= p_o \\ T &= T_o = T_w \\ p_{xx} &= 0 \end{aligned}$$

(cont. on next page)

$$
\begin{aligned}
p_{xy} &= p_{xy}' \\
p_{yy} &= 0 \\
q_x &= q_x' \\
q_y &= 0
\end{aligned}
\tag{4.4}
$$

i.e., the density, the pressure, and the temperature of the gas remain unchanged; no normal velocity, normal stress, or normal heat flow is developed. The impulsive velocity gives rise to tangential velocity, tangential stress, and tangential energy (heat) flux only.

However, it is important to note that the solutions for the tangential quantities p_{xy}, u, and q_x so obtained apply whether the plate is insulated or not, because of the separability of effects in the linearized problem. For a non-insulated plate, in addition to p_{xy}, u, and q_x, we superpose the other quantities p', p_{yy}, p_{xx}, v, ρ ', T', and q_y obtained by inverting Eqs. (A.17) to (A.23).

A. Discussion of the Transformed Quantities in the s-Plane

In expressions (4.4) the non-vanishing perturbation quantities u, p_{xy}, and q_x are to be obtained by inverting Eqs. (3.60) to (3.62)

$$
\bar{p}_{xy}(s,y) = p_0 \frac{U}{a_0} \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} s + \left(\frac{18}{5}s + 2\frac{p_0}{\mu_0}\right)\sqrt{\dfrac{3s\left(s+\frac{p_0}{\mu_0}\right)}{\left(3s+2\frac{p_0}{\mu_0}\right)\left(21s+10\frac{p_0}{\mu_0}\right)}}} e^{-f(s)y} \tag{3.60}
$$

$$
\bar{u}(s,y) = U \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}\, s \sqrt{\dfrac{s\left(21s+10\frac{p_0}{\mu_0}\right)}{3\left(s+\frac{p_0}{\mu_0}\right)\left(3s+2\frac{p_0}{\mu_0}\right)}} + s\left(1+\frac{3}{5}\,\frac{s}{3s+2\frac{p_0}{\mu_0}}\right)} e^{-f(s)y} \tag{3.61}
$$

$$\bar{q}_x(s,y) = p_o U \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{30}}\, s \sqrt{\frac{(21s+10\frac{p_o}{\mu_o})(3s+2\frac{p_o}{\mu_o})}{3s(s+\frac{p_o}{\mu_o})}} + \frac{2}{3}\left(\frac{9}{5}s + \frac{p_o}{\mu_o}\right)}\, e^{-f(s)y} \qquad (3.62)$$

where

$$f(s) = \frac{5}{a_o}\sqrt{\frac{s(s+\frac{p_o}{\mu_o})(3s+2\frac{p_o}{\mu_o})}{3(21s+10\frac{p_o}{\mu_o})}} \qquad (3.56)$$

The shear stress p_{xy}, gas velocity u, tangential energy transfer q_x are given by the inversion integral[24]:

$$p_{xy}(t,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}\, \bar{p}_{xy}(s,y)\, ds \qquad (4.5)$$

where c is a constant greater than the real part of all the singularities of $\bar{p}_{xy}$ (s,y), and similar integrals for u and q_x.

From expressions (3.60) to (3.62), we see that the functions $\bar{p}_{xy}$, $\bar{u}$, and $\bar{q}_x$ all have branch points at

$$s = -\frac{p_o}{\mu_o},\ -\frac{2}{3}\frac{p_o}{\mu_o},\ -\frac{10}{21}\frac{p_o}{\mu_o},\ 0 \qquad (4.6)$$

and also poles at roots of each denominator. (For $\alpha = 0$, the roots of the denominator in Eq. (3.60) are 0, $-.492(p_o/\mu_o)$, $-.649(p_o/\mu_o)$.)

All three denominators are essentially positive for real positive s and will not vanish. This property is demonstrated by representing the complex variable in polar coordinates as

$$s = re^{i\theta} \qquad \text{where} \qquad |\theta| < \pi/2 \qquad (4.7)$$

and the other three factors in the radical accordingly as shown in Fig. 4.

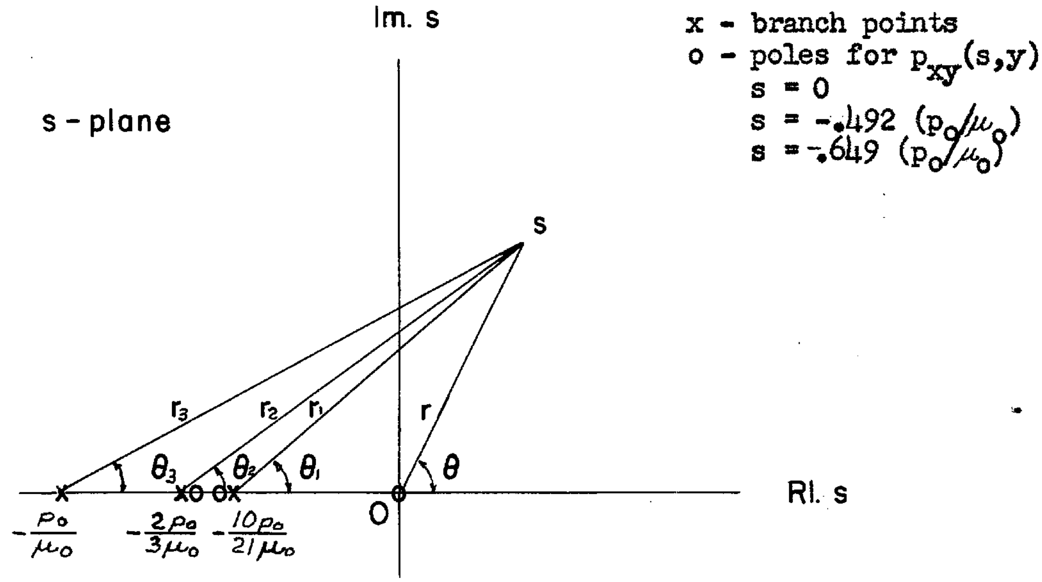

FIGURE 4

The radical in the denominator of the function $\overline{p_{xy}}$ may thus be written as

$$\sqrt{\frac{3s\left(s+\frac{p_o}{\mu_o}\right)}{\left(3s+2\frac{p_o}{\mu_o}\right)\left(21s+10\frac{p_o}{\mu_o}\right)}} = \frac{1}{\sqrt{21}}\sqrt{\frac{r r_3}{r_2 r_1}}\, e^{i\frac{1}{2}(\theta+\theta_3-\theta_2-\theta_1)} \tag{4.8}$$

with notations shown in Fig. 4. Now

$$\begin{aligned} \left|\theta+\theta_3-\theta_2-\theta_1\right| &= \left|\theta-\theta_1-(\theta_2-\theta_3)\right| \\ &< \left|\theta-\theta_1\right| + \left|\theta_2-\theta_3\right| \\ &< |\theta| + |\theta_2| \\ &< |\theta| + |\theta| = 2\theta \end{aligned}$$

But $|\theta| < \pi/2$ for s having positive real part

$$\tfrac{1}{2}\ |\ \theta + \theta_3 - \theta_2 - \theta_1\ | < |\theta| < \pi/2$$

Hence the radical (4.8) has positive real part and the denominator of the function $\overline{p_{xy}}$ given by Eq. (3.60) will not vanish for any s with positive real part. In other words, the function $\overline{p}_{xy}$ has no poles in the half-plane Rl s > 0. Likewise, we show the same for $\overline{u}$ and $\overline{q}_x$. To sum up, we have the following statement:

> All the three functions $\overline{p}_{xy}$, $\overline{u}$, and $\overline{q}_x$ given by expressions (3.60) to (3.61) have no singularities in the half-plane Rl s > 0. (4.9)

Physically, this must be so; otherwise, the quantities p_{xy}, u, and q_x inverted from Integral (4.5) will diverge with time.

However, we are not going to apply the inversion integral (4.5), since the complete solutions are not only laborious to obtain, but also inconvenient to use. Instead, we shall find solutions suitable for small and large values of the time, which together would cover most of the flow regime, and also approximate solutions which apply to the whole regime.

From expressions (3.60) to (3.62) we see that the function $\overline{p}_{xy}(s,y)$, $\overline{u}(s,y)$, and $\overline{q}_x(s,y)$ have singularities only in the finite s-plane and are regular for $|s|$ sufficiently large. We may then expand these functions in powers of 1/s and invert them term by term to obtain solutions $p_{xy}(t,y)$, u(t,y), and $q_x(t,y)$ for small values of the time as in Heaviside's series expansion. For large values of the time, one expands these functions in powers of s and then inverts.[25]

To obtain approximate solutions for all values of the time, we try to approximate the functions $\overline{p}_{xy}$, $\overline{u}$, and $\overline{q}_x$ of Eqs. (3.60) to (3.62) by simpler functions that can be easily inverted. These simpler functions are chosen in such a way that the regular behavior at $s = \infty$ and the singular behavior at $s = 0$ are retained, while the singularities with moduli between 0 and ∞ are collapsed into a singularity simpler in nature. By inverting these simpler functions, we are able to obtain approximate solutions which approach the exact solutions for small values and large values of the time and which satisfactorily approximate the exact solutions for intermediate values of the time. The detailed discussions of the different approximations to the transforms will be taken up in the next section.

B. Comparison of Approximations to the Transforms

In this section, we shall compare the exact expressions (3.60) to (3.62) with their different approximations, which are series expansions in powers of $1/s$, in powers of s, and a simple approximate form for all values of s, corresponding respectively to solutions $p_{xy}(t,y)$, $u(t,y)$, and $q_x(t,y)$ for small values of the time, for large values of the time, and approximate solutions for all values of the time. For simplicity, we consider in detail only the expression (3.60) for $\overline{p}_{xy}$ in the special case of $y = 0$ and $\alpha = 0$, i.e., the shear stress at a diffusively reflecting wall, which interests us most. But the argument applies also to the other two expressions (3.61) and (3.62) as well as to the general case of $y \neq 0$ and $\alpha \neq 0$.

From expression (3.60) with $y = 0$ and $\alpha = 0$, we have

$$\frac{\overline{p_{xy}}(s,0)}{\mu_0 M_0} = \frac{1}{\sqrt{\frac{3\pi}{10}\frac{\mu_0}{p_0}s} + \left(\frac{18}{5}\frac{\mu_0}{p_0}s+2\right)\sqrt{\frac{3\frac{\mu_0}{p_0}s\left(\frac{\mu_0}{p_0}s+1\right)}{\left(3\frac{\mu_0}{p_0}s+2\right)\left(21\frac{\mu_0}{p_0}s+10\right)}}} \qquad (4.10)$$

The numerical values of $\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0}$ vs. $\frac{\mu_0}{p_0}s$ are computed from Eq. (4.10) and tabulated in Table 1. By expanding the function $\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0}$ given by Eq. (4.10) in powers of 1/s, we have

$$\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0} = \frac{1}{\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}}\left[\frac{p_0}{\mu_0}\frac{1}{s} - \frac{\frac{61}{21\sqrt{35}}}{\sqrt{\frac{\pi}{2}}+\frac{6}{\sqrt{35}}}\left(\frac{p_0}{\mu_0}\frac{1}{s}\right)^2 + \frac{\frac{421}{588}\sqrt{\frac{\pi}{70}}+\frac{11,231}{30,870}}{\left(\sqrt{\frac{\pi}{2}}+\frac{6}{\sqrt{35}}\right)^2}\left(\frac{p_0}{\mu_0}\frac{1}{s}\right)^3 + \cdots\right]$$

$$= 0.569\left[\frac{p_0}{\mu_0}\frac{1}{s} - .247\left(\frac{p_0}{\mu_0}\frac{1}{s}\right)^2 + .0785\left(\frac{p_0}{\mu_0}\frac{1}{s}\right)^3 + \cdots\right] \qquad (4.11)$$

Eq. (4.11) is an approximation to Eq. (4.10) for large values of s. The numerical calculations are entered in Table 1.

In order to obtain solutions for the shear stress p_{xy} for t/τ or $\frac{a^2 t}{\nu} >> 1$, we shall make use of the method of asymptotic expansions given in Reference 25.* According to this approach, an asymptotic expansion of Q(t) for large t can be deduced from the behavior of $\overline{Q}(s)$ near its singularity with the algebraically largest real part. In our case the singularity with largest real part lies at s = 0. If $\overline{Q}(s)$ can be expanded near the origin as

$$\overline{Q}(s) = \sum_{n=0}^{\infty} a_n s^{n-1} + \sum_{n=0}^{\infty} b_n s^{n-\frac{1}{2}} \qquad (4.12)$$

then Q(t) has the asymptotic expansion for large t

$$Q(t) \sim a_0 + \frac{1}{\pi}\sum_{n=0}^{\infty}(-)^n b_n \Gamma(n+\tfrac{1}{2})\, t^{-n-\frac{1}{2}} \qquad (4.13)$$

Expanding the function $\overline{p}_{xy}(s,0)$ given by Eq. (4.10) in the form of Eq. (4.12), we have

---

* For detailed discussion of the conditions under which the expansion is valid and the proof of the general result, see § 126, pp. 279-282, Reference 25.

$$\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0} = \frac{1}{\sqrt{\frac{3}{5}}\sqrt{\frac{\mu_0}{p_0}s}}\left[1-\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mu_0}{p_0}s}+\left(\frac{\pi}{2}-\frac{1}{2}\right)\frac{\mu_0}{p_0}s-\sqrt{\frac{\pi}{2}}\left(\frac{\pi}{2}-1\right)\left(\frac{\mu_0}{p_0}s\right)^{\frac{3}{2}}+\left(\frac{3571}{200}+\frac{3\pi}{4}\right)\left(\frac{\mu_0}{p_0}s\right)^{2}+\right]$$

$$= 1.291\sqrt{\frac{p_0}{\mu_0}\frac{1}{s}}\left[1-1.253\sqrt{\frac{\mu_0}{p_0}s}+1.071\frac{\mu_0}{p_0}s-.715\left(\frac{\mu_0}{p_0}s\right)^{\frac{3}{2}}+20.21\left(\frac{\mu_0}{p_0}s\right)^{2}-\cdots\right] \qquad (4.14)$$

Eq. (4.14) is an approximation to Eq. (4.10) for small values of s. The numerical values are entered in Table 1. From Eq. (4.14) or Table 1, it is clear that the convergence of the series (4.14) becomes poor when $(\mu_0/p_0)\,s \geqslant .01$, or $\sqrt{(\mu_0/p_0)s} \geqslant .1$, which corresponds to $\sqrt{(p_0/\mu_0)t} = \sqrt{3/5}\,(\sqrt{Re})/M_0 \leqslant 0(10)$. In other words, the asymptotic series in the brackets of Eq. (4.14) is useful only in a narrow region around the origin. This result implies that the Burnett-type expansion of the solution for $p_{xy}(t,0)$ is inappropriate for $\sqrt{Re}/M_0 < 10$, or $t/\tau < 100$ (see Section IV.E.1). Therefore, it is necessary to find approximate solutions which will properly bridge the gap between the initial free molecule flow and the later classical Rayleigh motion.

First we have to find a simple approximate expression to Eq. (4.10) for all values of s. For s large, it is seen from Eq. (4.11) that the dominant term for $\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0}$ is $\frac{1}{\left(\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)\frac{\mu_0}{p_0}s}$. For s small, Eq. (4.14) shows that the dominant term is $\frac{1}{\sqrt{\frac{3}{5}}\sqrt{\frac{\mu_0}{p_0}s}}$. Therefore, the simplest approximation is as follows:

$$\frac{\overline{p}_{xy}(s,0)}{\mu_0 M_0} = \frac{1}{\left(\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)\frac{\mu_0}{p_0}s+\sqrt{\frac{3}{5}}\sqrt{\frac{\mu_0}{p_0}s}} \qquad (4.15)$$

for all values of s. Of course, the approximation is very good for the small and large values of s. The numerical calculation is carried out and tabulated in Table 1, which is also plotted as Fig. 5.

From Table 1 and Fig. 5, it is clear that the approximation (4.15) to the exact expression (4.10) is satisfactory. Eq. (4.15) gives values lower than those given by Eq. (4.10), but the maximum deviation which occurs in the range $.5 < \left|\frac{\mu_o}{p_o}s\right| < .6$, is less than 20%. The function $\frac{\overline{p_{xy}}(s,o)}{\mu_o M_o}$ given by Eq. (4.10) is regular for s ranging from infinity down to the singularity having largest modulus, which is of the order $\left|\frac{\mu_o}{p_o}s\right| = O(1)$, so the expansion (4.11) is useful for $\infty \leq \left|\frac{\mu_o}{p_o}s\right| \leq 1$. For small s, the expansion is valid for a domain between 0 and the singularity having smallest modulus, which is close to the origin, so the range of validity is limited. Based on these observations, we shall obtain solutions for small values of the time by inverting expansion (4.11) and solutions for large values of the time by inverting approximate expression (4.15) to replace the unsatisfactory expansion (4.14).

C. Solutions Suitable for Small Values of the Time

By expanding $\overline{p}_{xy}$ given by Eq. (3.60) in powers of 1/s, we obtain

$$\overline{p_{xy}}(s,y) = p_o \frac{U}{a_o} \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}} e^{-\frac{125}{42\sqrt{21}}\frac{p_o}{\mu_o}\frac{y}{a_o}} \left[\frac{1}{s} - \left(\frac{\frac{61}{21\sqrt{35}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}} + \frac{6}{\sqrt{35}}} \frac{p_o}{\mu_o} - .139\frac{p_o^2}{\mu_o^2}\frac{y}{a_o} \right) \frac{1}{s^2} \right. \tag{4.16}$$

$$\left. + \left\{ \frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{421}{588} + .364}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}} + \frac{6}{\sqrt{35}}\right)^2} \frac{p_o^2}{\mu_o^2} - \left(\frac{.068{,}1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}} + \frac{6}{\sqrt{35}}} - .056{,}7 \right) \frac{p_o^3}{\mu_o^3}\frac{y}{a_o} + .009{,}64\frac{p_o^4}{\mu_o^4}\frac{y^2}{a_o^2} \right\} \frac{1}{s^3} + O\left(\frac{1}{s^4}\right) \right] e^{-\frac{5}{\sqrt{21}}\frac{y}{a_o}s}$$

By inverting* and expanding the exponent, we have

---

* See Table of Transforms, Formulas 61, 62, and 63, p. 298, Reference 24.

$$p_{xy}(t,y)=p_0\frac{U}{a_0}\,\frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}}\left[1-\left\{\frac{\frac{61}{21\sqrt{35}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}+\frac{6}{\sqrt{35}}}\,\frac{p_0}{\mu_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)+\frac{125}{42\sqrt{21}}\frac{p_0}{\mu_0}\frac{y}{a_0}\right\}\right.$$

$$\left.+\left\{\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{421}{588}+.364}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}+\frac{6}{\sqrt{35}}\right)^2}\,\frac{p_0^2}{\mu_0^2}\,\frac{\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)^2}{2}+\left(\frac{.318}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}+\frac{6}{\sqrt{35}}}+.139\right)\frac{p_0^2}{\mu_0^2}\frac{y}{a_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)+.211\frac{p_0^2}{\mu_0^2}\frac{y^2}{a_0^2}\right\}\right] \tag{4.17}$$

$$+O(t^3) \qquad t>\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

$$=0 \qquad 0<t<\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

Similarly, expanding the functions $\overline{u}$, $\overline{q_x}$ given by Eqs. (3.61) and (3.62) in powers of 1/s and inverting term by term, we obtain

$$u(t,y)=U\frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\left[1+\left\{\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{5\pi}{14}}\,\frac{5}{6}+\frac{2}{15}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\,\frac{p_0}{\mu_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)-\frac{125}{42\sqrt{21}}\frac{p_0}{\mu_0}\frac{y}{a_0}\right\}\right.$$

$$+\left\{\left(\frac{\left[\frac{1+\alpha}{1-\alpha}\sqrt{\frac{5\pi}{14}}\,\frac{5}{6}+\frac{2}{15}\right]^2}{\left[\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}\right]^2}-\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{1{,}699}{504}+\frac{4}{45}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\right)\frac{p_0^2}{\mu_0^2}\frac{\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)^2}{2}+\left(.139-\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{5\pi}{14}}\,\frac{5}{6}+\frac{2}{15}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\,\frac{125}{42\sqrt{21}}\right)\frac{p_0^2}{\mu_0^2}\frac{y}{a_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)\right. \tag{4.18}$$

$$\left.\left.+.211\frac{p_0^2}{\mu_0^2}\frac{y^2}{a_0^2}\right\}\right]+O(t^3) \qquad t>\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

$$=0 \qquad 0<t<\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

$$q_x(t,y)=p_0U\frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\left[1-\left\{\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{1}{2}+\frac{2}{3}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\,\frac{p_0}{\mu_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)+\frac{125}{42\sqrt{21}}\frac{p_0}{\mu_0}\frac{y}{a_0}\right\}\right. \tag{4.19}$$

$$+\left\{\left(\frac{\left[\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{1}{2}+\frac{2}{3}\right]^2}{\left[\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}\right]^2}-\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{299}{504}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\right)\frac{p_0^2}{\mu_0^2}\frac{\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)^2}{2}+\left(\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{70}}\,\frac{1}{2}+\frac{2}{3}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\,\frac{125}{42\sqrt{21}}+.139\right)\frac{p_0^2}{\mu_0^2}\frac{y}{a_0}\left(t-\frac{5}{\sqrt{21}}\frac{y}{a_0}\right)\right.$$

$$\left.\left.+.211\frac{p_0^2}{\mu_0^2}\frac{y^2}{a_0^2}\right\}\right]+O(t^3) \qquad t>\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

$$=0 \qquad 0<t<\frac{5}{\sqrt{21}}\frac{y}{a_0}$$

For a fixed small value of the time, the distribution of tangential stress, p_{xy}, tangential energy flow, q_x, and tangential gas velocity, u, normal to the plate surface is represented in Figs. 6 and 7, where the numbers 1, 2, and 3 designate the number of terms taken in evaluating the functions.

From Eqs. (4.17) to (4.19), the parameter in the small time solution is

$$\frac{p_o t}{\mu_o} = \frac{3}{5}\frac{\rho_o U^2 t}{\mu_o}\frac{5}{3}\frac{p_o}{\rho_o}\frac{1}{U^2} = \frac{3}{5}\frac{Re}{M_o^2} \tag{4.20}$$

where

Re = instantaneous Reynolds number = $(\rho_o U^2 t)/\mu_o$

M_o = Mach number = U/a_o = $U/\sqrt{(5/3)(p_o/\rho_o)}$

For small values of the time, or $\frac{p_o t}{\mu_o} \sim \frac{a_o^2 t}{\nu} \sim \frac{t}{\tau} < 1$, the solution in series expansions of this parameter is indeed satisfactory.

From Figs. 6 and 7, we see that when a heat insulated infinite flat plate is suddenly accelerated into uniform motion with a small impulsive velocity U, the gas far from the plate is not at all disturbed at first, because the gas molecules there are not aware of the plate motion yet. Only the gas close enough to the plate is affected by the motion. Within a thin layer $y < \frac{\sqrt{21}}{5} a_o t$, where $t = O(\tau)$, i.e., t is of the order of the time between two successive collisions of a gas molecule, the gas as a whole moves with the plate at a fraction of the impulsive velocity U as shown in Fig. 7. Also the tangential stress and the tangential energy flux are developed as shown in Fig. 6. This thin layer grows linearly with time at the rate of $(\sqrt{21}/5)a_o$, which is one of the characteristic slopes dy/dt found in Eq. (3.33)

A discussion of some of the other physical aspects of these solutions is given in Section V.

D. Solutions for Large Values of the Time

The formal solutions for the quantities p_{xy}, u, and q_x at large times are obtained by expanding $\overline{p}_{xy}$, $\overline{u}$, and $\overline{q}_x$ given by Eqs. (3.60) to (3.61) in the form of Eq. (4.12), and writing out the asymptotic solutions in the form of Eq. (4.13). Thus,

$$\overline{p_{xy}}(s,y)=p_0\frac{U}{a_0}\frac{1}{\sqrt{\frac{3}{5}}\sqrt{\frac{p_0}{\mu_0}}}\frac{1}{\sqrt{s}}\frac{1}{1+\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mu_0}{p_0}}\sqrt{s}+\frac{1}{2}\frac{\mu_0}{p_0}s-\frac{3{,}521}{200}\frac{\mu_0^2}{p_0^2}s^2+O(s^3)}e^{-\frac{y}{\sqrt{\nu_0}}\sqrt{s}-\frac{1}{5}\frac{\mu_0}{p_0}\frac{y}{\sqrt{\nu_0}}s\sqrt{s}+O(s^{\frac{5}{2}})} \tag{4.21}$$

and

$$\begin{aligned}p_{xy}(t,y)\sim p_0\frac{U}{a_0}\frac{1}{\sqrt{\pi}}\frac{1}{\sqrt{\frac{3}{5}}\sqrt{\frac{p_0}{\mu_0}t}}\Bigg[&1-\left\{\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{\pi}{4}-\frac{1}{4}\right)\frac{\mu_0}{p_0}+\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{1}{2}\sqrt{\frac{\mu_0}{p_0}}\frac{y}{\sqrt{\nu_0}}+\frac{1}{4}\frac{y^2}{\nu_0}\right\}\frac{1}{t}\\&+\left\{\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{9\pi}{16}+\frac{10{,}713}{800}\right)\frac{\mu_0^2}{p_0^2}+\left(\frac{(1+\alpha)^3}{(1-\alpha)^3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{3}{4}-\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{3}{5}\right)\left(\frac{\mu_0}{p_0}\right)^{\frac{3}{2}}\frac{y}{\sqrt{\nu_0}}+\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{3\pi}{16}-\frac{3}{80}\right)\frac{\mu_0}{p_0}\frac{y^2}{\nu_0}+\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{1}{8}\sqrt{\frac{\mu_0}{p_0}}\frac{y^3}{\nu_0^{\frac{3}{2}}}+\frac{1}{32}\frac{y^4}{\nu_0^2}\right\}\frac{1}{t^2}\\&+O\left(\frac{1}{t^3}\right)\Bigg]\end{aligned} \tag{4.22}$$

Similarly, we obtain

$$\begin{aligned}u(t,y)\sim U\Bigg\{1-\frac{1}{\sqrt{\pi t}}\Bigg[&\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mu_0}{p_0}}+\frac{y}{\sqrt{\nu_0}}-\left\{\left(\frac{(1+\alpha)^3}{(1-\alpha)^3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}-\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{1}{5}\right)\left(\frac{\mu_0}{p_0}\right)^{\frac{3}{2}}+\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{\pi}{4}-\frac{1}{20}\right)\frac{\mu_0}{p_0}\frac{y}{\sqrt{\nu_0}}\right.\\&\left.+\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mu_0}{p_0}}\frac{1}{4}\frac{y^2}{\nu_0}+\frac{1}{12}\frac{y^3}{\nu_0^{\frac{3}{2}}}\right\}\frac{1}{t}+\left\{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{153}{200}-\frac{(1+\alpha)^3}{(1-\alpha)^3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{9}{20}\right)\left(\frac{\mu_0}{p_0}\right)^{\frac{5}{2}}-\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{\pi}{2}\frac{33}{40}+\frac{87}{400}\right)\frac{\mu_0^2}{p_0^2}\frac{y}{\sqrt{\nu_0}}\right.\\&\left.+\frac{(1+\alpha)^3}{(1-\alpha)^3}\left(\frac{\pi}{2}\right)^{\frac{3}{2}}\frac{3}{8}\left(\frac{\mu_0}{p_0}\right)^{\frac{3}{2}}\frac{y^2}{\nu_0}+\left(\frac{(1+\alpha)^2}{(1-\alpha)^2}\frac{\pi}{16}+\frac{3}{80}\right)\frac{\mu_0}{p_0}\frac{y^3}{\nu_0^{\frac{3}{2}}}+\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{1}{32}\sqrt{\frac{\mu_0}{p_0}}\frac{y^4}{\nu_0^2}+\frac{1}{160}\frac{y^5}{\nu_0^{\frac{5}{2}}}\right\}\frac{1}{t^2}+O\left(\frac{1}{t^3}\right)\Bigg]\Bigg\}\end{aligned} \tag{4.23}$$

$$\begin{aligned}q_x(t,y)\sim p_0U\frac{3}{4}\frac{1}{\sqrt{2}}\frac{1}{\left(\frac{p_0}{\mu_0}t\right)^{\frac{3}{2}}}\Bigg[&\frac{1+\alpha}{1-\alpha}+\left\{\left(\frac{1+\alpha}{1-\alpha}\frac{69}{20}-\frac{(1+\alpha)^3}{(1-\alpha)^3}\frac{3\pi}{4}\right)\frac{\mu_0}{p_0}+\left(\frac{12}{5}\sqrt{\frac{2}{\pi}}-\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\frac{3}{2}\right)\sqrt{\frac{\mu_0}{p_0}}\frac{y}{\sqrt{\nu_0}}-\sqrt{\frac{2}{\pi}}\sqrt{\frac{p_0}{\mu_0}}\frac{1}{4}\frac{y^3}{\nu_0^{\frac{3}{2}}}\right\}\frac{1}{t}\\&+O\left(\frac{1}{t^2}\right)\Bigg]\end{aligned} \tag{4.24}$$

where ν_0 = coefficient of kinematic viscosity = μ_0/ρ_0.

As we have seen in the previous section the expansions for small s such as Eq. (4.21) are good approximations for $0 < |(\mu_o/p_o)\ s| < .01$. In other words, solutions (4.22) to (4.24) are convergent only for sufficiently large values of the time, i.e., $t/\tau > O(100)$. These solutions are included here for comparison with the approximate solutions to be found below.

In the previous section an approximate expression for $\bar{p}_{xy}(s,0)$ (Eq. (4.15)) was found by combining the dominant terms for large and small s. Evidently the same procedure can be employed for $\bar{p}_{xy}(s,y)$, provided that the quantity f(s) in the exponential term $e^{-f(s)y}$ (Eqs. (3.60) and (3.56)) is approximated by the dominant terms $-\left(\frac{5}{\sqrt{21}}\frac{s}{a_o}+\frac{\sqrt{s}}{\sqrt{\nu_o}}\right)$ for large and small s, i.e.,

$$\bar{p}_{xy}(s,y)=p_o\frac{U}{a_o}\frac{1}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)s+\sqrt{\frac{3}{5}}\sqrt{\frac{p_o}{\mu_o}}\sqrt{s}}\,e^{-\frac{5}{\sqrt{21}}\frac{y}{a_o}s-\frac{y}{\sqrt{\nu_o}}\sqrt{s}} \tag{4.25}$$

By inverting* expression (4.25), we obtain

$$p_{xy}(t,y)=p_o\frac{U}{a_o}\frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}}\,e^{\frac{\sqrt{\frac{3}{5}}\sqrt{\frac{p_o}{\mu_o}}\frac{y}{\sqrt{\nu_o}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}}}\,e^{\frac{\frac{3}{5}\frac{p_o}{\mu_o}t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_o t}\right)}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)^2}}\,\mathrm{erfc}\left(\frac{\sqrt{\frac{3}{5}\frac{p_o}{\mu_o}t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_o t}\right)}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}}+\frac{y}{2\sqrt{\nu_o t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_o t}\right)}}\right),\quad t>\frac{5}{\sqrt{21}}\frac{y}{a_o}$$

$$= 0 \qquad 0<t<\frac{5}{\sqrt{21}}\frac{y}{a_o} \tag{4.26}$$

This closed-form solution for the shear stress p_{xy} is not only the proper solution for large values of the time but also the approximate solution for all values of the time. Letting $t \rightarrow 0$ and bearing in mind $y = O(t)$, we have

$$\lim_{t\to 0} p_{xy}(t,y)=p_o\frac{U}{a_o}\frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}} \qquad t>\frac{5}{\sqrt{21}}\frac{y}{a_o}$$

$$= 0 \qquad 0<t<\frac{5}{\sqrt{21}}\frac{y}{a_o} \tag{4.27}$$

This expression for $p_{xy}(t,y)$ is the first term of the small time solution (4.16), which gives the free-molecule shear stress based on Grad's

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* See Table of Operations, Formula 12, p. 294 and Table of Transforms, Formula 87, p. 300, Reference 24.

distribution function (2.22). By letting $t \to \infty$ and making use of the asymptotic formula* for the complementary error function

$$\operatorname{erfc.} x \sim \frac{e^{-x^2}}{x\sqrt{\pi}}\left[1 - \frac{2!}{1!(2x)^2} + \frac{4!}{2!(2x)^4} - \frac{6!}{3!(2x)^6} + \cdots\right]$$

we obtain

$$\lim_{t\to\infty} p_{xy}(t,y) = p_o \frac{U}{a_o} \frac{1}{\sqrt{\frac{3}{5}\frac{p_o}{\mu_o}t}} \; e^{-\frac{y^2}{4\nu_o t}}$$

or, by Eq. (4.20)

$$\lim_{t\to\infty} p_{xy}(t,y) = \frac{\rho_o U^2}{\sqrt{\pi}} \frac{1}{\sqrt{Re}} \, e^{-\frac{y^2}{4\nu_o t}} \qquad \text{for all } y \qquad (4.28)$$

which is, of course, the classical Rayleigh result.[1]

In other words, the closed form approximate solution (4.26) gives the exact description of the gas behavior for both extremely small and extremely large values of the time. In the initial phase, the flow, impulsively started, is a free-molecule one, and the thin layer grows linearly with time. During the later stages, the flow is diffusive in nature with the viscous layer growing as $\sqrt{t}$ and may be treated as a continuum, as Rayleigh first did. For intermediate values of the time, Eq. (4.26) gives an approximate description of the flow field, which is essentially a diffusive layer bounded by a wave-front travelling at the speed $(\sqrt{21}/5)a_o$. In Fig. 8, the distribution of shear stress p_{xy} with respect to y for a fixed intermediate time is shown. For sufficiently large values of time, for which $y = O(\sqrt{t})$, we may rewrite Eq. (4.26) as

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* See, for instance, Dwight, H. B.: Table of Integrals, 2nd Ed., p. 129, The MacMillan Company, New York, 1947.

$$p_{xy}(t,y) = p_0 \frac{U}{a_0} \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}} e^{\frac{\sqrt{\frac{3}{5}}\sqrt{\frac{p_0}{\mu_0}}\frac{y}{\sqrt{\nu_0}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}}} e^{\frac{\frac{3}{5}\frac{p_0}{\mu_0}t}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}\right)^2}} \operatorname{erfc.}\left(\frac{\sqrt{\frac{3}{5}\frac{p_0}{\mu_0}t}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}} + \frac{y}{2\sqrt{\nu_0 t}}\right) \qquad (4.29)$$

which is again represented in Fig. 9. It is clear that for sufficiently large values of the time, the shear stress distribution is close to Rayleigh's result.

In a similar manner, the proper large time solution for the tangential energy flux q_x is found to be

$$\begin{aligned} q_x(t,y) = p_0 U \frac{1}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}} + \frac{6}{5}} \frac{1}{a-b} \Bigg[& a e^{a\frac{y}{\sqrt{\nu_0}}} e^{a^2 t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)} \operatorname{erfc.}\left(a\sqrt{t} + \frac{y}{2\sqrt{\nu_0 t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)}}\right) \\ & - b e^{b\frac{y}{\sqrt{\nu_0}}} e^{b^2 t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)} \operatorname{erfc.}\left(b\sqrt{t} + \frac{y}{2\sqrt{\nu_0 t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)}}\right)\Bigg], \qquad t > \frac{5}{\sqrt{21}}\frac{y}{a_0} \\ = 0 \qquad & \qquad 0 < t < \frac{5}{\sqrt{21}}\frac{y}{a_0} \end{aligned} \qquad (4.30)$$

where

$$\begin{matrix} a \\ b \end{matrix} = \frac{\sqrt{\frac{p_0}{\mu_0}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}} + \frac{6}{5}} \left[\frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\,\frac{1}{3} \pm \sqrt{\left(\frac{1+\alpha}{1-\alpha}\right)^2 \frac{\pi}{18} - \frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}\,\frac{2}{3} - \frac{4}{5}}\right]$$

Like Eq. (4.26), Eq. (4.30) is also an approximate solution of the quantity q_x for all values of the time. The discussion of the properties of $p_{xy}(t,y)$ also applies to $q_x(t,y)$.

The situation for the gas velocity $u(t,y)$ is a little different. The dominant terms in the expansions for large and small s are, respectively [Eq. (3.61)]:

$$\bar{u}(s,y) \cong U \frac{1}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}} + \frac{6}{5}\right)s - \left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}\frac{25}{42} + \frac{2}{15}\right)\frac{p_0}{\mu_0}} e^{-\frac{5}{\sqrt{21}}\frac{y}{a_0}s} \qquad \text{s large} \qquad (4.31)$$

$$\bar{u}(s,y) \cong U \frac{1}{s + \frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\sqrt{\frac{\mu_0}{p_0}}\, s\sqrt{s}} e^{-\frac{y}{\sqrt{\nu_0}}\sqrt{s}} \qquad \text{s small} \qquad (4.32)$$

Here the leading terms in both denominators are of the order s, and hence no approximate expression for all s, like Eq. (4.25), can be obtained for the transform $\bar{u}(s,y)$. Therefore, for the velocity of the gas, we shall obtain a closed-form solution valid only for large values of the time rather than for all values of the time. It is obtained by inverting* (4.32).

$$u(t,y) = U\left[\operatorname{erfc}\left(\frac{y}{2\sqrt{\nu_0 t}}\right) - e^{\frac{1-\alpha}{1+\alpha}\sqrt{\frac{2}{\pi}}\sqrt{\frac{\rho_0}{p_0}}\frac{p_0}{\mu_0}y}\, e^{\left(\frac{1-\alpha}{1+\alpha}\right)^2\frac{2}{\pi}\frac{p_0}{\mu_0}t}\operatorname{erfc}\left(\frac{1-\alpha}{1+\alpha}\sqrt{\frac{2}{\pi}}\sqrt{\frac{p_0}{\mu_0}t} + \frac{y}{2\sqrt{\nu_0 t}}\right)\right],\ t \text{ large} \qquad (4.33)$$

However, for the special case y = 0, it is possible to approximate $U(1/s) - \bar{u}(s,0)$ for all values of s, since the dominant terms there are of different orders. The velocity slip $U - u(t,0)$ obtained by inverting $U(1/s) - \bar{u}(s,0)$ will be given in Section IV.E.2.

E. The Skin Friction, the Slip Velocity, and the Tangential Energy (Heat) Flux of the Gas Adjacent to the Heat Insulated Plate

In this section, we are going to study the quantities which interest us most, viz., the skin friction, the slip velocity, and the tangential energy (heat) flux of the gas adjacent to the plate. All these quantities are obtained by setting y = 0 in the expressions for p_{xy}, u, and q_x found in the previous section.

So far we have not specified the surface condition of the plate; in other words, the parameter α is rather arbitrary, and it may vary from $\alpha = 1$ for specular reflection to $\alpha = 0$ for diffusive reflection. In the numerical calculation that follows, we consider the case $\alpha = 0$,

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* See Table of Transforms Formula 86, p. 300, Reference 24.

viz., the gas molecules impinging on the surface of the plate are temporarily absorbed and later re-emitted in a random fashion, such that they give up all their momentum to the plate.

1. The Skin Friction Coefficient

The instantaneous skin friction coefficient is defined as

$$C_f = \frac{P_{xy}(t,0)}{\frac{1}{2}\rho_o U^2} \tag{4.34}$$

By putting y = 0 and $\alpha = 0$ in Eqs. (4.17), (4.22), and (4.26) and using definitions (4.34) and (4.20), we obtain the following expressions for the skin friction coefficient:

$$M_o C_f = 0.683\left[1 - 0.090\,6\frac{Re}{M_o^2} + 0.008\,78\frac{Re^2}{M_o^4}\right] + O\left(\frac{Re^3}{M_o^6}\right) \qquad \text{t small} \tag{4.35}$$

$$M_o C_f = 1.128\frac{M_o}{Re^{\frac{1}{2}}}\left[1 - 0.892\frac{M_o^2}{Re} + 42.1\frac{M_o^4}{Re^2}\right] + O\left(\frac{M_o^7}{Re^{\frac{7}{2}}}\right) \qquad \text{t large} \tag{4.36}$$

$$M_o C_f \cong 0.683\, e^{\left(0.342\frac{\sqrt{Re}}{M_o}\right)^2} \operatorname{erfc}\left(0.342\frac{\sqrt{Re}}{M_o}\right) \qquad \text{all t} \tag{4.37}$$

The free molecule value of the skin friction coefficient based on Grad's distribution function is obtained by letting t $\longrightarrow$ 0 in either Eq. (4.35) or Eq. (4.36) or by substituting Eq. (4.27) into Eq. (4.34)

$$M_o C_f = \frac{\frac{6}{5}}{\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}} = 0.683 \qquad t \longrightarrow 0 \tag{4.38}$$

which is to be compared with the conventional free molecule skin friction coefficient based on Maxwellian distribution

$$M_o C_f = \sqrt{\frac{2}{\pi}\frac{1}{\gamma}} = \sqrt{\frac{2}{\pi}\frac{3}{5}} = 0.618 \tag{4.39}$$

The deviation is due to the different molecular distribution functions used, which will be discussed in Section IV.E.2. The Rayleigh result is obtained by letting $t \to \infty$ in either Eq. (4.36) or Eq. (4.37) or by substituting Eq. (4.28) into Eq. (4.34)

$$M_o C_f = \frac{2}{\sqrt{\pi}} \frac{M_o}{Re^{\frac{1}{2}}} = 1.128 \frac{M_o}{Re^{\frac{1}{2}}} \qquad t \to \infty \tag{4.40}$$

The results of the numerical calculation of the skin friction coefficient based on Eqs. (4.35) to (4.37) are tabulated in Table 2 and are also plotted in Fig. 10.

From Fig. 10, we may make the following statements:

(1) The series solution (4.35) in powers of the parameter $(Re/M_o^2) \sim t/\tau$ is indeed quite accurate for small values of the time $(t/\tau \leq 1)$. As more and more of the reflected molecules collide with the incident molecules the thickness of the layer influenced by the moving plate grows (see also Fig. 6) and the skin-friction decreases with time.

(2) On the other hand, the solution for large time in inverse powers of $(Re/M_o^2) \sim t/\tau$ is of very limited usefulness. When $M_o^2/Re = 0.01$ the second term in the expression for C_f (Eq. (4.36)) is only one per cent of the first term, but the third term is already 40 per cent as large as the second. Clearly, the Burnett-type expansion is inappropriate, at least for Rayleigh's problem.

(3) The approximate solution (4.37) presented here is not only the appropriate solution for large values of the time, but is also a satisfactory solution for all times. This approximate solution approaches the exact free molecule solution and the exact Rayleigh solution at the

respective limits and gives the right trend of the skin friction, which is enveloped by the free molecule and the Rayleigh values. It is to be noted that the approximate solution (4.37) might slightly underestimate the skin friction coefficient.

2. Slip Velocity

The slip velocity is defined as the difference between the velocity of the flat plate and the velocity of the gas adjacent to the plate, U − u(t,0). The Laplace transform of this quantity is U(1/s) − $\bar{u}$(s,0). From Eqs. (4.31) and (4.32), we obtain the approximation

$$U\frac{1}{s}-\bar{u}(s,0)=U\frac{1}{\dfrac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{1}{5}}\,s+\frac{1-\alpha}{1+\alpha}\sqrt{\frac{2}{\pi}}\sqrt{\frac{p_0}{\mu_0}}\sqrt{s}} \tag{4.41}$$

In the denominator of Eq. (4.41) the first term is the dominant term for large s or small t, and the second is the dominant term for small s or large t. These two terms are not of the same order, so the approximation is possible for the whole s range, and its inversion would be valid for all times.

By inverting* Eq. (4.41), we have

$$U-u(t,0)=U\frac{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{1}{5}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\,e^{\frac{\left(\frac{1-\alpha}{1+\alpha}\sqrt{\frac{2}{\pi}}\frac{1}{5}+\sqrt{\frac{7}{5}}\right)^2}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}\right)^2}\frac{p_0}{\mu_0}t}\,\mathrm{erfc.}\left(\frac{\frac{1-\alpha}{1+\alpha}\sqrt{\frac{2}{\pi}}\frac{1}{5}+\sqrt{\frac{7}{5}}}{\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}}+\frac{6}{5}}\sqrt{\frac{p_0}{\mu_0}t}\right) \tag{4.42}$$

Putting y = 0 and $\alpha = 0$ in Eqs. (4.18) and (4.23), $\alpha = 0$ in Eq. (4.42), and using Eq. (4.20) we obtain the following solutions for the slip velocity:

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* See Table of Transforms, Formula 43, p. 297, Reference 24.

$$1-\frac{u(t,0)}{U}=0.627-0.084,7\,\frac{Re}{M_o^2}+0.010,5\,\frac{Re^2}{M_o^4}+O\left(\frac{Re^3}{M_o^6}\right) \qquad \text{t small} \qquad (4.43)$$

$$1-\frac{u(t,0)}{U}=0.913\,\frac{M_o}{Re^{\frac{1}{2}}}-2.086\,\frac{M_o^3}{Re^{\frac{3}{2}}}+0.147\,\frac{M_o^5}{Re^{\frac{5}{2}}}+O\left(\frac{M_o^7}{Re^{\frac{7}{2}}}\right) \qquad \text{t large} \qquad (4.44)$$

$$1-\frac{u(t,0)}{U}\cong 0.627\,e^{\left(.388\frac{\sqrt{Re}}{M_o}\right)^2}\,\mathrm{erfc.}\left(.388\,\frac{\sqrt{Re}}{M_o}\right) \qquad \text{all t} \qquad (4.45)$$

The results of the numerical calculations are tabulated in Table 3 and are plotted in Fig. 11.

From Fig. 11, we see that the velocity slip at zero time is 63% rather than the conventional 50% for free molecule flow based on the Maxwellian distribution far from the plate surface. According to Grad's approximate solution of the Maxwell-Boltzmann equation, the distribution function near the plate surface deviates from Maxwellian just as soon as a finite stress and energy flux are generated (Cf. Eq. (2.22)). In other words, in this scheme the distribution function experiences a "jump" at $t = 0^+$, and this fact accounts for the difference between the present results and those of conventional free-molecule flow. (See also Section V.)

According to Fig. 11, the velocity slip is less than 10% when $\sqrt{Re}/M_o \geq 10$, and less than 1% when $\sqrt{Re}/M_o \geq 100$. For $\sqrt{Re}/M_o > 100$, the no-slip boundary condition certainly applies. The approximate formula (4.45) is seen to be somewhat less accurate than that for the skin friction.

3. Energy Flux Parallel to the Plate Surface

By putting $y = 0$ and $\alpha = 0$ in Eqs. (4.19), (4.24), and (4.30) and using Eq. (4.20) we obtain the following solutions for the tangential energy flux.

$$\frac{q_x(t,0)}{p_0 U} = 0.373\left[1 - 0.1728\frac{Re}{M_0^2} + 0.006{,}49\frac{Re^2}{M_0^4}\right] + O\left(\frac{Re^3}{M_0^6}\right) \qquad t \text{ small} \qquad (4.46)$$

$$\frac{q_x(t,0)}{p_0 U} = 1.141\frac{M_0^3}{Re^{\frac{3}{2}}} + 2.080\frac{M_0^5}{Re^{\frac{5}{2}}} + O\left(\frac{M_0^7}{Re^{\frac{7}{2}}}\right) \qquad t \text{ large} \qquad (4.47)$$

$$\frac{q_x(t,0)}{p_0 U} \cong -i\,0.394\,e^{-0.120\frac{Re}{M_0^2}}\left\{(.155{,}7 + i.474)\,e^{i0.088{,}2\frac{Re}{M_0^2}}\,\mathrm{erfc}\left[(.1205 + i.366)\frac{\sqrt{Re}}{M_0}\right]\right.$$
$$\left. - (.155{,}7 - i.474)\,e^{-i0.088{,}2\frac{Re}{M_0^2}}\,\mathrm{erfc}\left[(.120{,}5 - i.366)\frac{\sqrt{Re}}{M_0}\right]\right\} \qquad \text{all } t \qquad (4.48)$$

In using Eq. (4.48), the computation of error functions with complex arguments is involved. Computation of this kind may be facilitated by the use of Reference 26. However, we shall not carry out the involved computation here. Instead, we shall investigate the limiting values of Eq. (4.48) and give a much simpler interpolation formula.

From Eq. (4.48) we have

$$\lim_{t\to 0}\frac{q_x(t,0)}{p_0 U} = -i\,0.394(i.948) = 0.373$$

$$\lim_{t\to\infty}\frac{q_x(t,0)}{p_0 U} = 0$$

These are the leading terms of Eqs. (4.46) and (4.47), respectively.

Now Eq. (4.48) is obtained by putting $y = 0$ and $\alpha = 0$ in Eq. (4.30), which is in turn inverted from approximate transforms to Eq. (3.62). In that approximate transform, there were three terms in the denominator: namely,

$$\bar{q}_x(s,y) = \frac{1}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}} + \frac{6}{5}\right)s + \frac{2}{3}\frac{p_o}{\mu_o} + \frac{1+\alpha}{1-\alpha}\sqrt{\frac{\pi}{2}}\,\frac{2}{3}\sqrt{\frac{p_o}{\mu_o}}\sqrt{s}}\; e^{-\frac{5}{\sqrt{21}}\frac{y}{a_o}s - \frac{y}{\sqrt{\mu_o}}\sqrt{s}}$$

The first term in the denominator is the dominant term for large s, and the last two terms for small s. The third term is included to make certain that $q_x(t,y)$ should be a function of $\sqrt{t}$ for large times (small s). For purposes of numerical calculation it is sufficient to take only one term for small s. Thus,

$$\bar{q}_x(s,0) \cong \frac{1}{\left(\frac{1+\alpha}{1-\alpha}\sqrt{\frac{7\pi}{10}} + \frac{6}{5}\right)s + \frac{2}{3}\frac{p_o}{\mu_o}}$$

Putting $\alpha = 0$ and inverting*, one has

$$q_x(t,o) \cong 0.373\, e^{-0.149\frac{Re}{M_o^2}} \tag{4.49}$$

We shall use Eq. (4.49) instead of Eq. (4.48) for our computation. The results are tabulated in Table 4 and plotted in Fig. 12. Fig. 12 shows there is an appreciable amount of energy (or heat) flux along the plate during the initial stage of the motion, although $\partial T/\partial x = 0$. At later times, $(\sqrt{Re}/M_o) \geq 100$, the tangential heat flow is practically zero. Eq. (4.49) approximates the tangential energy flux suprisingly well for all values of the time. For large values of the time, Eq. (4.47) shows that the quantity q_x is a high order of infinitesimal, namely $q_x = 0(1/t^{\frac{3}{2}})$.

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* See Table of Transforms, Formula 8, p. 295, Reference 24.

F. Comparison with Schaaf's Solution for Wall Shear Stress $p_{xy}(t,0)$

The closed-form approximate solution for the shear stress is given by Eq. (4.26). For the special case of $\alpha = 0$, we have

$$p_{xy}(t,y) = p_0 \frac{U}{a_0} \frac{1}{\sqrt{\frac{3\pi}{10}+\frac{6\sqrt{21}}{35}}}\, e^{\frac{\sqrt{\frac{3}{5}}\sqrt{\frac{p_0}{\mu_0}}\frac{y}{\sqrt{\nu_0}}}{\sqrt{\frac{3\pi}{10}+\frac{6\sqrt{21}}{35}}}}\, e^{\frac{\frac{3}{5}\frac{p_0}{\mu_0}t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)}{\left(\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)^2}}\, \mathrm{erfc}\left(\frac{\sqrt{\frac{3}{5}\frac{p_0}{\mu_0}t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)}}{\sqrt{\frac{3\pi}{10}+\frac{6\sqrt{21}}{35}}} + \frac{y}{2\sqrt{\nu_0 t\left(1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}\right)}}\right), \quad t > \frac{5}{\sqrt{21}}\frac{y}{a_0} \tag{4.50}$$

$$= 0 \qquad\qquad 0 < t < \frac{5}{\sqrt{21}}\frac{y}{a_0}$$

from which the wall shear stress is

$$p_{xy}(t,0) = p_0 \frac{U}{a_0} \frac{1}{\sqrt{\frac{3\pi}{10}+\frac{6\sqrt{21}}{35}}}\, e^{\frac{\frac{3}{5}\frac{p_0}{\mu_0}t}{\left(\sqrt{\frac{3\pi}{10}}+\frac{6\sqrt{21}}{35}\right)^2}}\, \mathrm{erfc}\left(\frac{\sqrt{\frac{3}{5}\frac{p_0}{\mu_0}t}}{\sqrt{\frac{3\pi}{10}+\frac{6\sqrt{21}}{35}}}\right) \qquad \text{all } t \tag{4.51}$$

The corresponding quantities obtained by Schaaf[5], using the linearized Navier-Stokes equation and a slip boundary condition, are, respectively,

$$p_{xy}(t,y) = \frac{\mu_0 U}{L'}\, e^{\frac{y}{L'}}\, e^{\frac{\nu_0}{L'^2}t}\, \mathrm{erfc}\left(\frac{\sqrt{\nu_0 t}}{L'} + \frac{y}{2\sqrt{\nu_0 t}}\right) \tag{4.52}$$

$$p_{xy}(t,0) = \frac{\mu_0 U}{L'}\, e^{\frac{\nu_0}{L'^2}t}\, \mathrm{erfc}\left(\frac{\sqrt{\nu_0 t}}{L'}\right) \tag{4.53}$$

where L' is the mean free path used in Reference 5. The expressions for $p_{xy}(t,0)$ in Eqs. (4.51) and (4.53) are of the same form, but the numerical constants are different. However, the expression (4.50) and (4.52) for $p_{xy}(t,y)$ away from the surface are very different. Our solution of Grad equations contains the inherent wave characteristics in the term $1-\frac{5}{\sqrt{21}}\frac{y}{a_0 t}$ while Schaaf's result (4.52) is diffusive in nature. Of course, for sufficiently large time, $p_{xy}(t,y)$ given by

Eq. (4.50) is also diffusive, as shown in Eq. (4.29). In other words, near the start of the motion the shear stress satisfies a wave equation of the form $\frac{\partial^2 p_{xy}}{\partial t^2} = b^2 \frac{\partial^2 p_{xy}}{\partial y^2}$, where b is a constant, and not the Navier-Stokes equation $\frac{\partial p_{xy}}{\partial t} = \nu_0 \frac{\partial^2 p_{xy}}{\partial y^2}$. At the plate surface, however, the difference in character between the two solutions disappears, and only the numerical constants are different.

In Grad's framework $\frac{\mu_0}{L} = \frac{5}{16}\sqrt{\frac{2}{\pi}}\sqrt{p_0 \rho_0}$. Therefore, if we arbitrarily define an "effective mean free path" so that

$$L_{eff} = \frac{\sqrt{\frac{3\pi}{10}} + \frac{6\sqrt{21}}{35}}{\sqrt{\frac{3\pi}{10}}} \frac{16}{5\pi} L' = 1.843L' = 1.809L ,$$

and solve the linearized Navier-Stokes equations $p_{xy} = -\mu_0 \frac{\partial u}{\partial y}$ and $\frac{\partial u}{\partial t} = \nu_0 \frac{\partial^2 u}{\partial y^2}$, with the slip boundary condition

$$U - u(t,0) = - L_{eff} \left(\frac{\partial u}{\partial y} \right)_{y=0} ,$$

we obtain an expression for $p_{xy}(t,0)$ identical to Eq. (4.51).

At the present time it is not clear whether the result obtained by the purely formal process is coincidental and restricted to Rayleigh's problem, or is suggestive for more general flow problems.

V. DISCUSSION AND CONCLUSION

According to the solution of Rayleigh's problem by means of Grad's thirteen-moment approximation, a time interval corresponding to 50-100 collisions per molecule is required to transform the initially-discontinuous, step-function distribution of gas velocity into the classical Rayleigh distribution with a negligibly small "slip" at the surface.* As anticipated from qualitative physical considerations (Introduction) the flow behaves initially like a free-molecule flow, and all physical quantities are analytic functions of the parameter t/τ, or Re/M_o^2. Although the solutions for $t/\tau = O(1)$ seem to be satisfactory in most respects, the persistence of a sharp wave-front carrying discontinuities in u, p_{xy}, and q_x is somewhat surprising. This behavior appears to be inherent in Grad's method. One might expect that some of the fastest molecules near the plate surface would always be able to travel ahead of the wave, so that the wave front would be smoothed out. (Of course steep gradients would be expected around some "average" wave-front.) For this reason (and others) it is desirable to investigate Rayleigh's problem with the aid of the linearized Maxwell-Boltzmann equation, as Wang Chang and Uhlenbeck[27] have done for some simple steady flows. Naturally this study will be guided by the results of the present investigation.

In spite of this criticism of Grad's method, the present results suggest that other simple but fundamental non-steady flow problems should be explored by means of the linearized Grad equations. In

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* This time interval is about 10 times longer than the transit time across a steady, plane shock wave of moderate strength in a monatomic gas.

addition, the character of the solutions obtained for $t/\tau = O(1)$ encourages us to believe that Rayleigh's problem with arbitrary impulsive velocity (high Mach number) can be attacked by means of series expansions in t/τ for "small time", combined with solutions of the Navier-Stokes equations for large time.

In this connection, it is desirable to derive boundary conditions from a physical point of view. The conservation laws of mass, momentum, and energy will be employed. Also separate momentum reflection and energy accommodation coefficients will be introduced. Furthermore, in the energy consideration, the frame of reference will be fixed on the boundary itself rather than moving with the average flow velocity $\vec{u}$, so that the new energy flux q_i can be identified with the more familiar heat transfer rate.

The character of the present solutions of Rayleigh's problem for large time adds to the growing lack of confidence in the Chapman-Enskog-Burnett method of successive approximations. Unlike Wang Chang and Uhlenbeck*, we were able to obtain asymptotic series expansions in powers of mean free path (more precisely, M_o^2/Re), but just when the deviations from Rayleigh's solution become interesting the convergence of these series is so poor that they are practically useless. Grad's method avoids this difficulty, because the explicit relation between the

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* For the problem of heat conduction between two infinite parallel flat plates, and also for the planar Couette flow, the solutions obtained by Wang Chang and Uhlenbeck are not analytic in the limit of small mean free path ($d/L \to \infty$) where "d" is spacing between plates. This behavior may be connected with the particular geometry involved. Even in the Knudsen limit ($d/L \to 0$) they experienced convergence difficulties, for which they offer a simple physical explanation, but these difficulties did not appear in the Couette flow between two rotating cylinders.

distribution function and the stress and energy flux quantities in no way implies a series expansion of the Burnett type. By utilizing this "freedom" we are able to obtain approximate solutions of Rayleigh's problem for low Mach number that are valid for all values of the time. It remains to exploit this technique for more general flow problems.

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APPENDIX

LAPLACE TRANSFORMS OF v', ρ', p', T', p_{xx}', p_{yy}', and q_y'

From Eqs. (3.42), (3.43), and (3.45)

$$\frac{5}{3}\frac{d\overline{v'}}{dy} + \frac{2}{3}\frac{1}{p_0}\frac{d\overline{q_y'}}{dy} = -\frac{1}{p_0}\, s\,\overline{p'}$$
$$= \frac{5}{2}\left(\frac{1}{p_0}s + \frac{1}{\mu_0}\right)\overline{p_{xx}'}$$
$$= -\frac{5}{4}\left(\frac{1}{p_0}s + \frac{1}{\mu_0}\right)\overline{p_{yy}'}$$

Hence,

$$\overline{p_{yy}'} = \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\,\overline{p'} \tag{A.1}$$

$$\overline{p_{xx}'} = -\frac{1}{2}\overline{p_{yy}'} \qquad \text{or} \qquad p_{xx} = -\frac{1}{2}p_{yy} \tag{A.2}$$

From Eqs. (3.41), (3.39), and (4.47)

$$\overline{v'} = -\frac{1}{\rho_0}\,\frac{1}{s}\left(1 + \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\right)\frac{d\overline{p'}}{dy} \tag{A.3}$$

$$\overline{\rho'} = -\rho_0\,\frac{1}{s}\,\frac{d\overline{v'}}{dy} = \frac{1}{s^2}\left(1 + \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\right)\frac{d^2\overline{p'}}{dy^2} \tag{A.4}$$

$$\overline{q_y'} = -\frac{p_0}{\rho_0}\,\frac{1}{s+\frac{2}{3}\frac{p_0}{\mu_0}}\left[\left(\frac{5}{2} + \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\right)\frac{d\overline{p'}}{dy} - \frac{5}{2}\frac{p_0}{\rho_0}\frac{1}{s^2}\left(1 + \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\right)\frac{d^3\overline{p'}}{dy^3}\right] \tag{A.5}$$

Substituting expressions (A.3), (A.4), and (A.5) into Eq. (3.42) and simplifying

$$\left(\frac{9}{5}s + \frac{p_0}{\mu_0}\right)\frac{d^4\overline{p'}}{dy^4} - \frac{\rho_0}{p_0}\,s\left(\frac{78}{25}s^2 + \frac{16}{5}\frac{p_0}{\mu_0}s + \frac{2}{3}\frac{p_0^2}{\mu_0^2}\right)\frac{d^2\overline{p'}}{dy^2} + \frac{3}{5}\,\frac{\rho_0^2}{p_0^2}\,s^3\left(s^2 + \frac{5}{3}\frac{p_0}{\mu_0}s + \frac{2}{3}\frac{p_0^2}{\mu_0^2}\right) = 0 \tag{A.6}$$

Keeping in mind that $\overline{p'}$ should vanish at $y = \infty$, we have

$$\overline{p'} = B(s)\,e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y} + C(s)\,e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y} \tag{A.7}$$

where

$$f_1(s) = \frac{78}{25}s^2 + \frac{16}{5}\frac{P_0}{\mu_0}s + \frac{2}{3}\frac{P_0^2}{\mu_0^2} \tag{A.8}$$

$$f_2(s) = \sqrt{\frac{3384}{625}s^4 + \frac{1296}{125}\frac{P_0}{\mu_0}s^3 + \frac{188}{25}\frac{P_0^2}{\mu_0^2}s^2 + \frac{8}{3}\frac{P_0^3}{\mu_0^3}s^3 + \frac{4}{9}\frac{P_0^4}{\mu_0^4}} \tag{A.9}$$

and B(s) and C(s) are arbitrary functions to be determined from the boundary conditions (3.49) and (3.51). Differentiate Eq. (A.7)

$$\frac{d\overline{p'}}{dy} = -B(s)\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\;e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y}$$
$$-C(s)\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\;e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y}$$

By Eq. (A.3), the boundary condition (3.49) $\overline{v'}(s,0) = 0$ requires

$$\left.\frac{d\overline{p'}}{dy}\right|_{y=0} = 0$$

i.e.,

$$C(s) = -B(s)\sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}$$

Then expression (A.7) becomes

$$\overline{p'} = B(s)\left[e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y} - \sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}\;e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y}\right] \tag{A.10}$$

Differentiating Eq. (A.10) with respect to y successively, we have

$$\frac{d\overline{p'}}{dy} = -B(s)\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\left[e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y} - e^{-\sqrt{\frac{\rho_0}{P_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{P_0}{\mu_0}}}\,y}\right] \tag{A.11}$$

$$\frac{d^2\bar{p}'}{dy^2} = B(s)\frac{\rho_o}{p_o}\frac{\frac{1}{2}s}{\frac{9}{5}s+\frac{p_o}{\mu_o}}\left[(f_1(s)+f_2(s))\,e^{-\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\,y} - \sqrt{f_1^2(s)-f_2^2(s)}\;e^{-\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\,y}\right] \tag{A.12}$$

$$\frac{d^3\bar{p}'}{dy^3} = -B(s)\frac{\rho_o}{p_o}\frac{\frac{1}{2}s}{\frac{9}{5}s+\frac{p_o}{\mu_o}}\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\left[(f_1(s)+f_2(s))\,e^{-\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\,y}\right.$$

$$\left. - (f_1(s)-f_2(s))\,e^{-\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\,y}\right] \tag{A.13}$$

From Eqs. (A.1) and (A.5) we have, using Eqs. (A.11) and (A.13),

$$\bar{p}'_{yy}\Big|_{y=0} = B(s)\;\frac{4}{5}\;\frac{s}{s+\frac{p_o}{\mu_o}}\left[1-\sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}\right] \tag{A.14}$$

$$\bar{q}'_y\Big|_{y=0} = -B(s)\frac{5}{2}\frac{p_o}{\rho_o}\frac{1}{s\left(s+\frac{p_o}{\mu_o}\right)\left(s+\frac{2}{3}\frac{p_o}{\mu_o}\right)}\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\;f_2(s) \tag{A.15}$$

Substituting Eqs. (A.14) and (A.15) in the boundary condition (3.51)

$$\frac{1+\alpha}{4(1-\alpha)}\left(\frac{2\pi}{RT_o}\right)^{\frac{1}{2}}B(s)\frac{5}{2}\frac{1}{\rho_o}\frac{1}{s\left(s+\frac{p_o}{\mu_o}\right)\left(s+\frac{2}{3}\frac{p_o}{\mu_o}\right)}\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\;f_2(s)$$

$$+B(s)\;\frac{1}{5}\;\frac{s}{s+\frac{p_o}{\mu_o}}\left(\sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}-1\right) = \left(1-\frac{T_w}{T_o}\right)\frac{1}{s}$$

Hence,

$$B(s) = \left(1-\frac{T_w}{T_o}\right)\frac{1}{s}\left(s+\frac{p_o}{\mu_o}\right)\left[\frac{5}{8}\frac{1+\alpha}{1-\alpha}\left(\frac{2\pi}{RT_o}\right)^{\frac{1}{2}}\frac{1}{\rho_o}\frac{1}{s\left(s+\frac{2}{3}\frac{p_o}{\mu_o}\right)}\sqrt{\frac{\rho_o}{p_o}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_o}{\mu_o}}}\,f_2(s)+\frac{s}{5}\left(\sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}-1\right)\right]^{-1} \tag{A.16}$$

From Eqs. (A.10), (A.1), (A.2), (A.3), (A.4), and (A.5), we then have

$$\overline{p'} = B(s)\left[e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y} - \sqrt{\frac{f_1(s)+f_2(s)}{f_1(s)-f_2(s)}}\; e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y}\right] \tag{A.17}$$

$$\overline{p'_{yy}} = \frac{4}{5}\,\frac{s}{s+\frac{p_0}{\mu_0}}\,\overline{p'} \tag{A.18}$$

$$\overline{p'_{xx}} = -\frac{1}{2}\,\overline{p'_{yy}} \tag{A.19}$$

$$\begin{aligned}\overline{v'} = {} & B(s)\,\frac{1}{\rho_0}\,\frac{1}{s+\frac{p_0}{\mu_0}}\sqrt{\frac{1}{2}\frac{\rho_0}{p_0}\left(\frac{9}{5}+\frac{p_0}{\mu_0}\frac{1}{s}\right)\left(f_1(s)+f_2(s)\right)} \\ & \times\left[e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y} - e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y}\right]\end{aligned} \tag{A.20}$$

$$\begin{aligned}\overline{\rho'} = {} & B(s)\frac{1}{2}\frac{\rho_0}{p_0}\,\frac{1}{s\left(s+\frac{p_0}{\mu_0}\right)}\Bigg[\left(f_1(s)+f_2(s)\right) e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y} \\ & -\sqrt{f_1^2(s)-f_2^2(s)}\; e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y}\Bigg]\end{aligned} \tag{A.21}$$

$$\overline{T'} = \frac{1}{R\rho_0}\,\overline{p'} - \frac{T_0}{\rho_0}\,\overline{\rho'} \tag{A.22}$$

$$\begin{aligned}\overline{q'_y} = {} & -B(s)\frac{p_0}{\rho_0}\,\frac{1}{\left(s+\frac{p_0}{\mu_0}\right)\left(s+\frac{2}{3}\frac{p_0}{\mu_0}\right)}\sqrt{\frac{\rho_0}{p_0}\,\frac{s}{2}\,\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\Bigg[\left(\frac{33}{10}s+\frac{5}{2}\frac{p_0}{\mu_0}\right)\left(e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y} - e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y}\right) \\ & -\frac{5}{4}\frac{1}{s}\left\{\left(f_1(s)+f_2(s)\right)e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)+f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y} - \left(f_1(s)-f_2(s)\right)e^{-\sqrt{\frac{\rho_0}{p_0}\frac{s}{2}\frac{f_1(s)-f_2(s)}{\frac{9}{5}s+\frac{p_0}{\mu_0}}}\,y}\right\}\Bigg]\end{aligned} \tag{A.23}$$

where $f_1(s)$, $f_2(s)$, and $B(s)$ are given by Eqs. (A.8), (A.9), and (A.16), respectively.

TABLE 1

COMPARISON OF LAPLACE TRANSFORM SOLUTIONS

FOR SURFACE SHEAR STRESS $\frac{\overline{p_{xy}}(s,0)}{\mu_0 M_0}$

$\frac{\mu_0}{P_0}s$	Original Eq.(4.10)	Large s Eq.(4.11)	Small s Eq.(4.14)	Approximate Eq.(4.15)
0	∞		∞	∞
.0001	127.3		127.5	126.2
.001	39.34		39.25	38.09
.01	11.42		11.45	10.52
.05			4.37	
.1	2.82	36.30	3.63	2.378
.2		4.913	4.04	
.4	1.032			.839
.5	.863	.933	9.95	.701
.6	.744			.605
1	.483	.472	26.2	.395
3	.178	.176		.151
5	.1093	.1086		.0951
10	.0558	.0556		.0498
50	.0113	.0113		.0107
100	.0057	.0057		.0054
1,000	.00057	.00057		.00056
10,000	.000057	.000057		.000057
∞	0	0		0

TABLE 2

TIME HISTORY OF SKIN FRICTION COEFFICIENT $M_o\ C_f$ IN RAYLEIGH'S PROBLEM

$\sqrt{Re}/M_o$	Free Molecule Eq. (4.38)	Rayleigh Eq. (4.40)	small time Eq. (4.35)	large time Eq. (4.36)	Approximate Eq. (4.37)
0	.683		.683		.683
.01			.683		.681
.1			.683		.658
.316			.677		.607
.5			.668		.569
1			.627	47.6	.483
1.414			.583		.427
2		.564	.532	1.923	.364
3			.612	.534	.287
4				.313	.236
5			2.886	.232	.198
8				.141	.134
10		.113		.112	.109
100				.0113	.011
1,000		.00113		.0011	.0011
10,000				.00011	.00011

TABLE 3

TIME HISTORY OF SLIP VELOCITY

1 - u(t,0)/U IN RAYLEIGH'S PROBLEM

$\sqrt{Re}/M_o$	small time Eq. (4.43)	large time Eq. (4.44)	Approximate Eq. (4.45)
0	.627		.627
.01	.627		.625
.1	.626		.601
.316	.619		.549
.5	.607		.511
1	.553	-1.313	.425
1.414	.500		.371
2	.456	.200	.312
3	.712	.228	.242
4		.196	.197
5		.166	.165
8		.110	.110
10		.089	.089
100		.009	.009
1,000		.0009	.0009
10,000		.000009	.000009
∞		0	0

TABLE 4

TIME HISTORY OF TANGENTIAL ENERGY (HEAT) FLUX

$q_x(t,0)\ /p_o\ U$ IN RAYLEIGH'S PROBLEM

$\sqrt{Re}/M_o$	small time Eq. (4.46)	large time Eq. (4.47)	Approximate Eq. (4.49)
0	.373		.373
.01	.373		.373
.1	.372		.372
.316	.366		.367
.5	.357		.359
1	.311	3.49	.321
1.414	.254		.277
2	.154	.242	.205
3	-.010805	.0508	.097
4		.0199	.034
5		.0098	.0089
8		.00229	.00003
10		.00116	.000000
100		.000001	.000000
1,000		.000000	.000000
10,000		.000000	.000000
∞		0	0

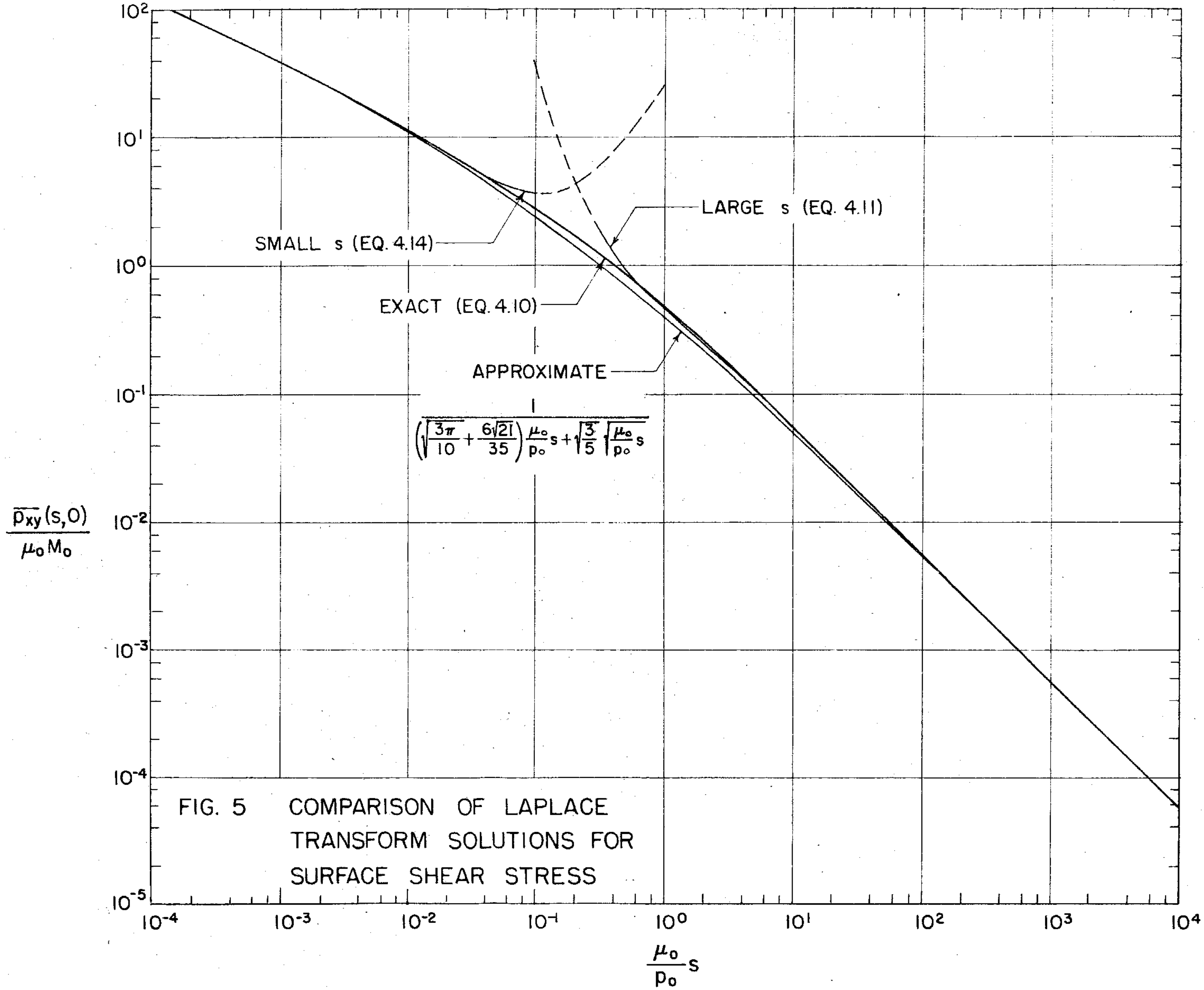

FIG. 5 COMPARISON OF LAPLACE TRANSFORM SOLUTIONS FOR SURFACE SHEAR STRESS

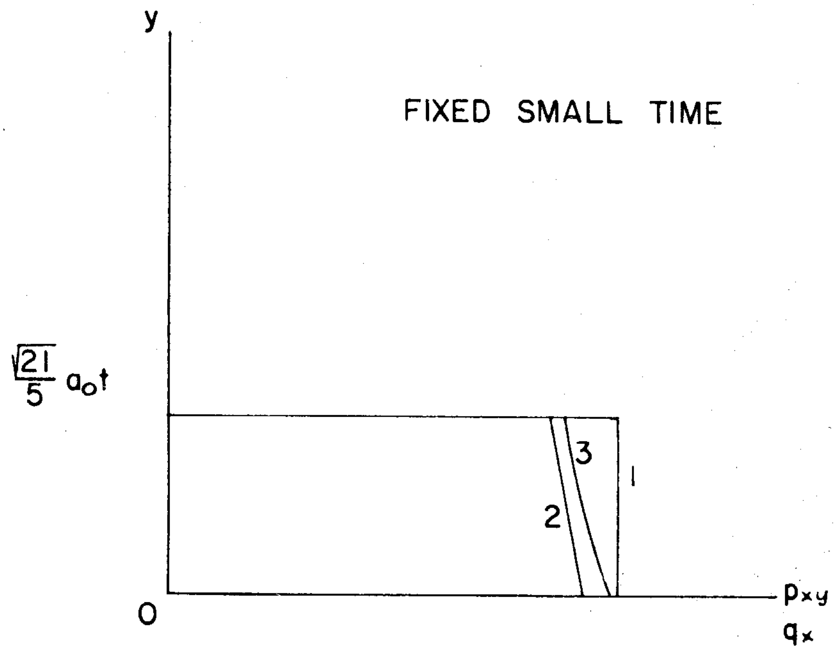

FIGURE 6

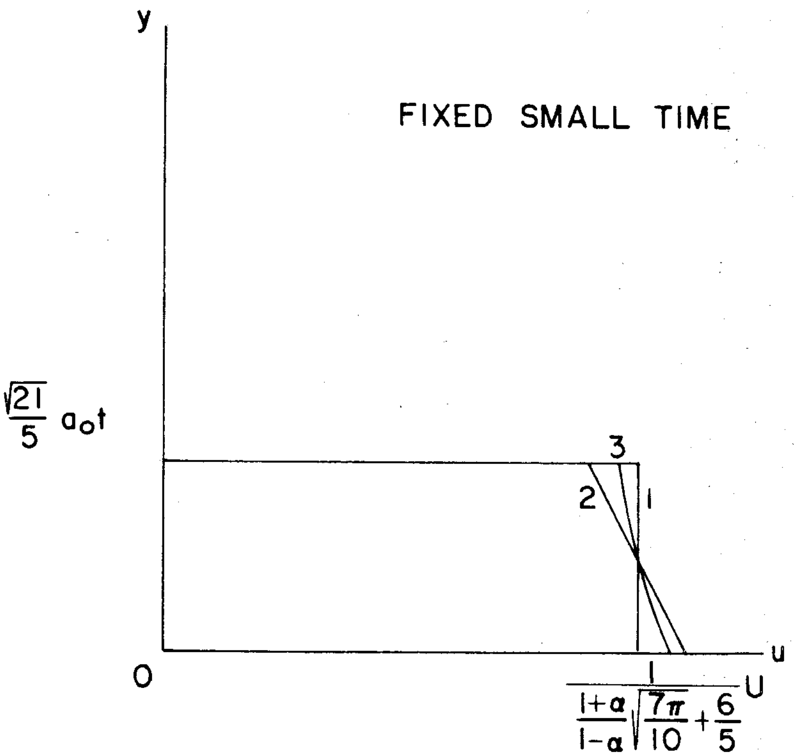

FIGURE 7

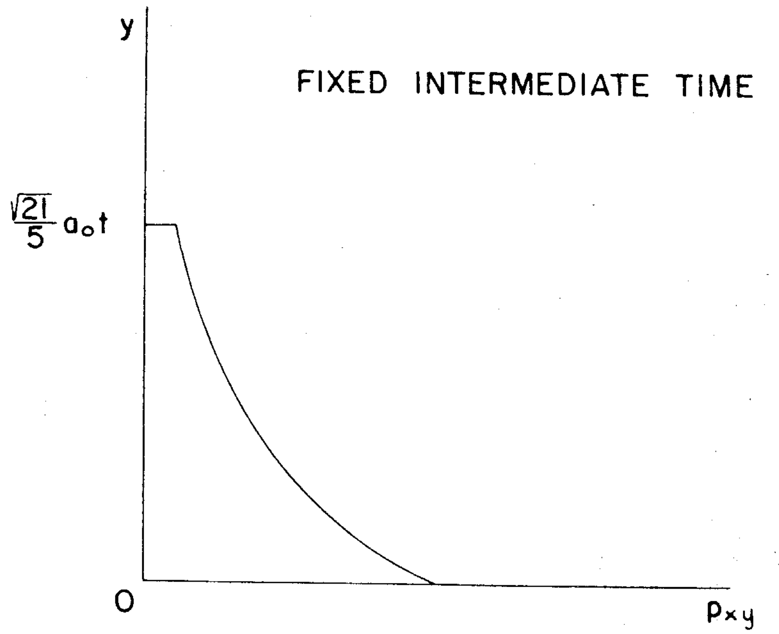

FIGURE 8

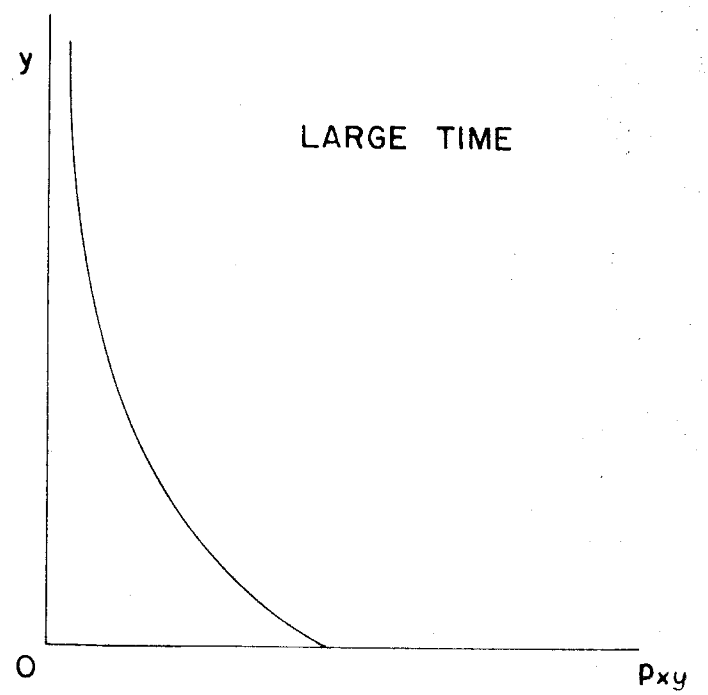

FIGURE 9

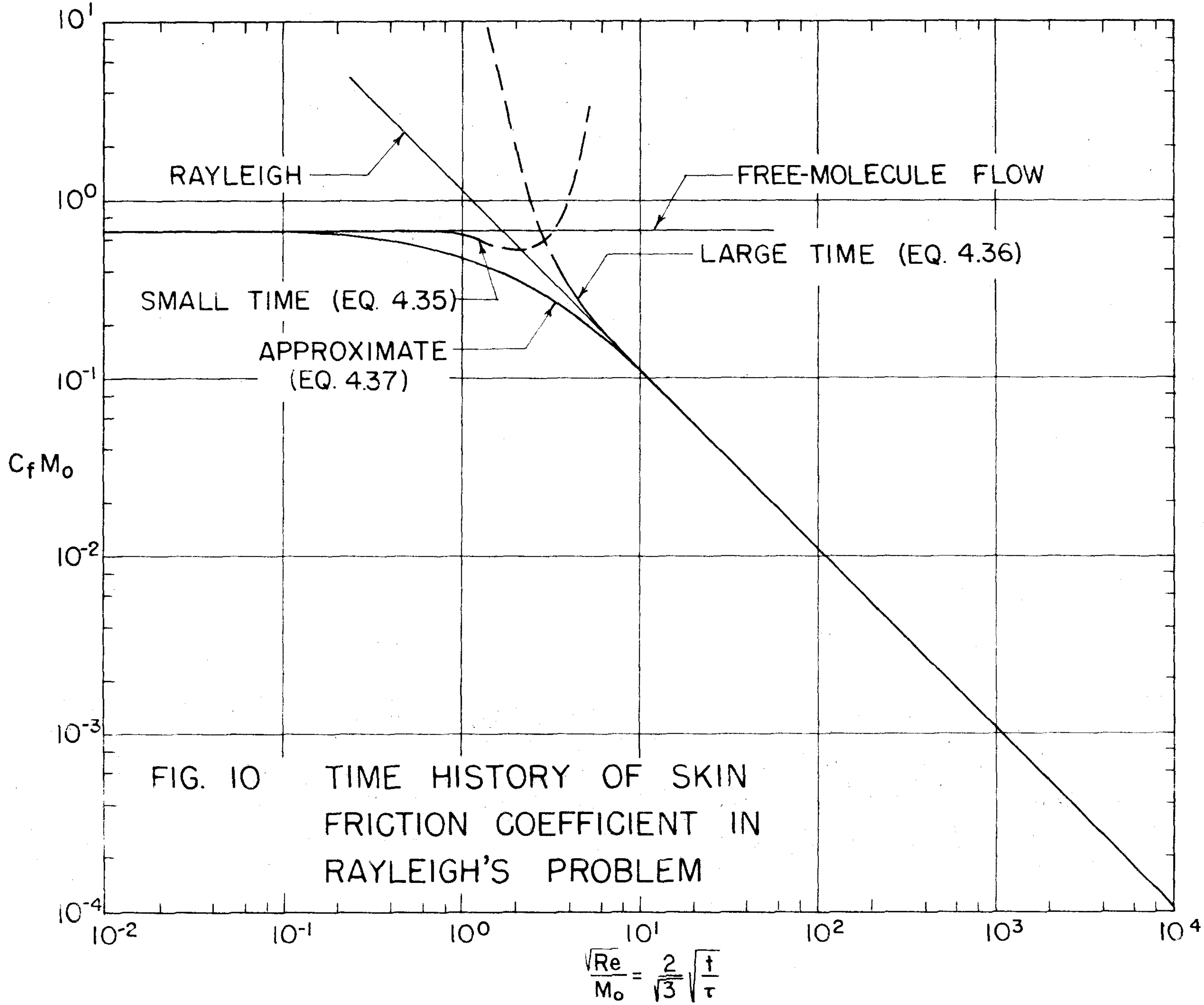

FIG. 10 TIME HISTORY OF SKIN FRICTION COEFFICIENT IN RAYLEIGH'S PROBLEM

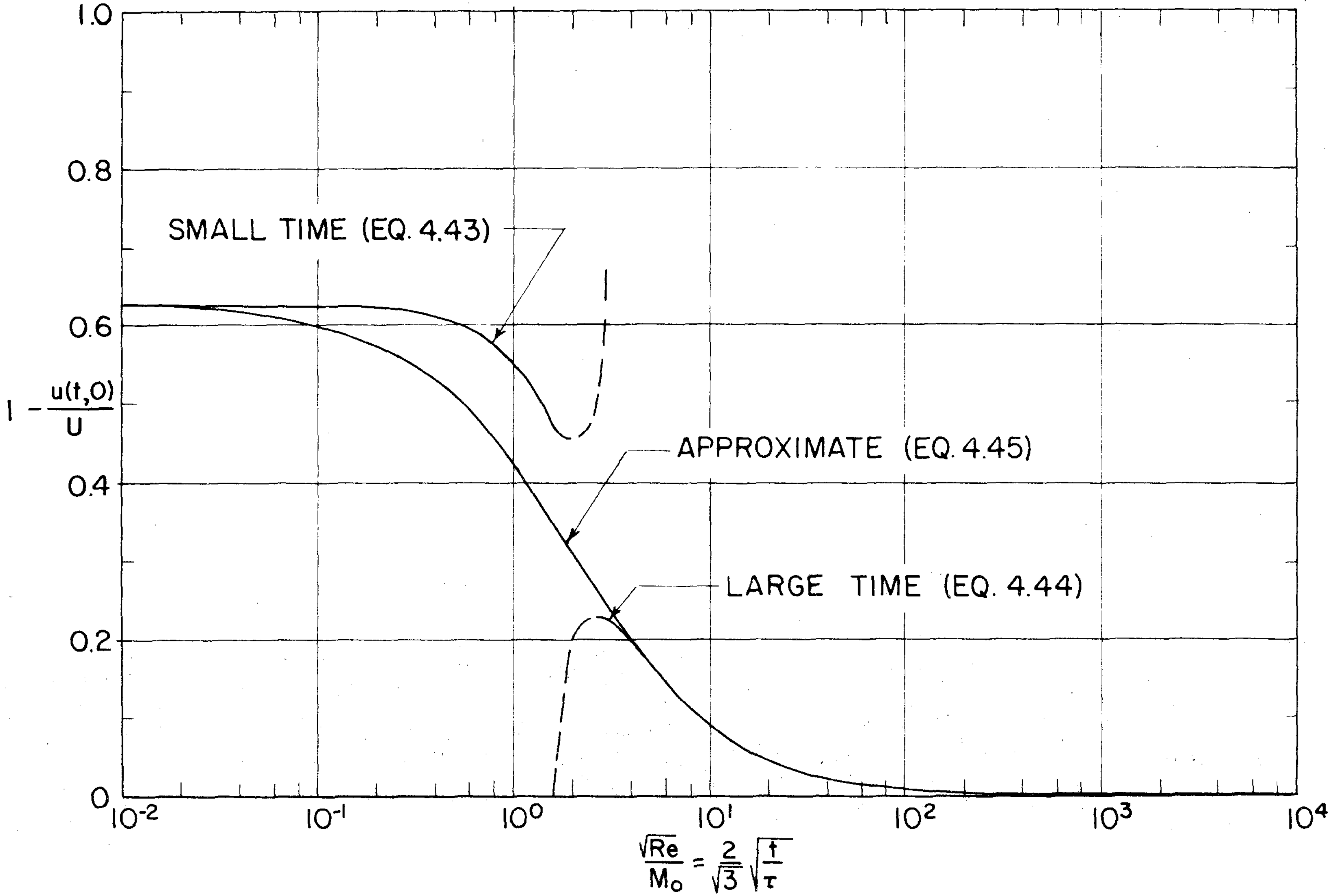

FIG. 11 TIME HISTORY OF SLIP VELOCITY IN THE RAYLEIGH PROBLEM

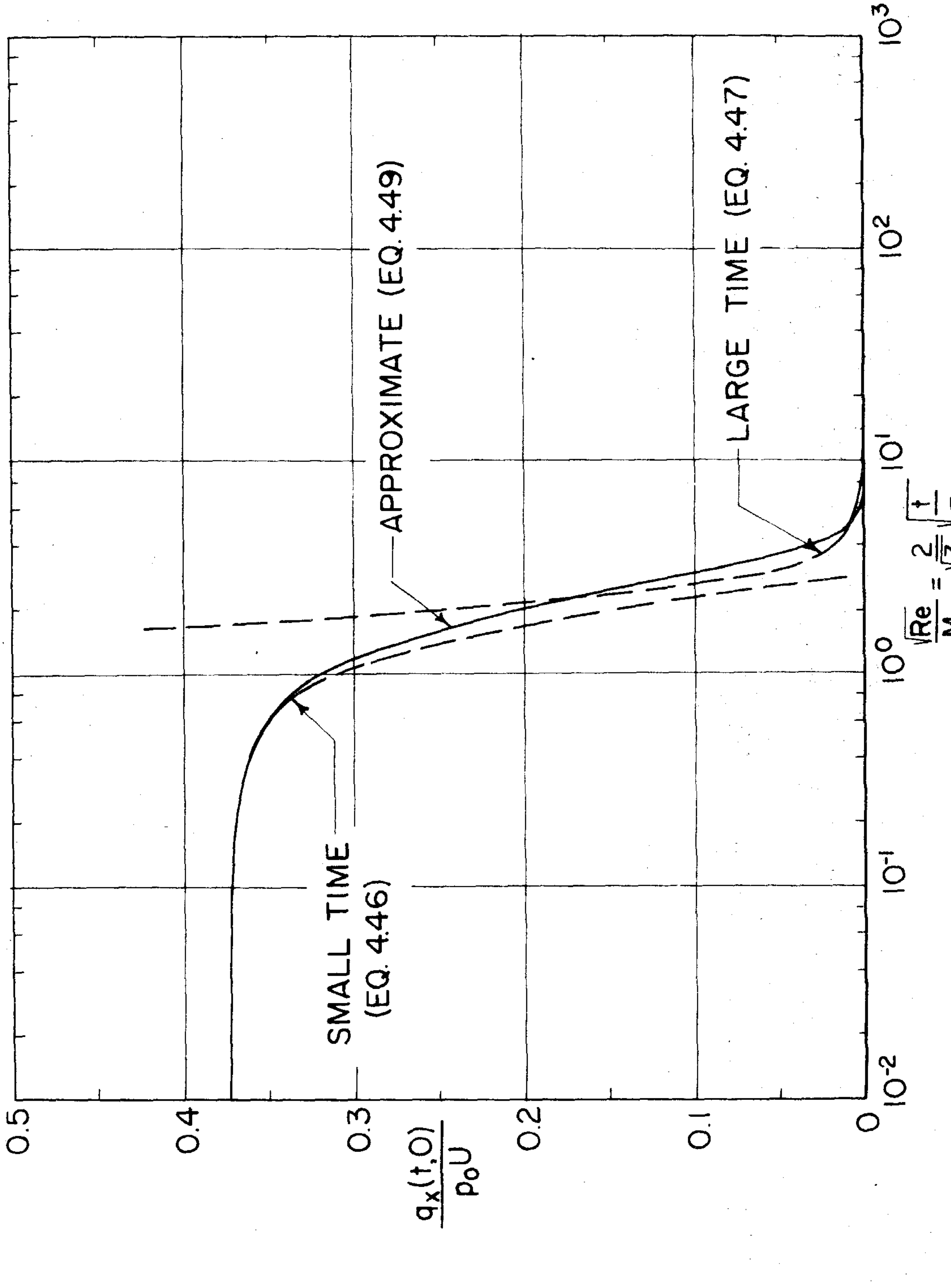

FIG. 12 TIME HISTORY OF TANGENTIAL ENERGY (HEAT) FLUX IN THE RAYLEIGH PROBLEM