Topological invariants of gapped quantum lattice systems

Thesis by
Nikita A. Sopenko

In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2023
Defended May 15, 2023

Nikita A. Sopenko
ORCID: 0000-0002-8479-1924

To my parents.

ACKNOWLEDGEMENTS

I'm indebted to my advisor Anton Kapustin not only for his scientific wisdom and his ideas that drastically affected my way of thinking but also for his constant support and encouragement even during hard times. I was lucky to have the best advisor I could possibly imagine.

I'm grateful to my collaborators Adam Artymowicz and Bowen Yang for their help and the numerous stimulating discussions I had the pleasure to participate in.

I'd like to thank Alexei Kitaev for useful conversations and for sharing his ideas. Most of the thesis is inspired by his pioneering work on the subject. I also want to thank Sergei Gukov for helping me to settle in at Caltech and his guidance throughout the early stages of my PhD.

Many thanks to Yoshiko Ogata, Bruno Nachtergaele, Karl-Hermann Neeb and Michael Levin for their comments and suggestions on the work that became part of this thesis.

Last, but not least, I'd like to thank my friends and colleagues with whom I enjoyed discussing physics, mathematics and other matters: Alexander Avdoshkin, Alex Bols, Yu-An Chen, Mykola Dedushenko, Alexandre Perozim de Faveri, David Frenklakh, Roman Geiko, Andrei Grekov, Angus Gruen, Tamir Hemo, Yakov Kononov, Alexey Milekhin, Sunghyuk Park, Fedor Popov, Blazej Ruba, Mykola Semenyakin, Lev Spodyneiko, Ilya Vilkoviskiy.

ABSTRACT

In the first part of the thesis, a systematic way to construct topological invariants of gapped states of quantum lattices systems is proposed. It provides a generalization of the Berry phase and its equivariant analogue to systems with locality in arbitrary dimensions. For a smooth family of gapped ground states in d dimensions, it gives a closed $(d+2)$-form on the parameter space which generalizes the curvature of the Berry connection. Its cohomology class is a topological invariant of the family. When the family is equivariant under the action of a compact Lie group G, topological invariants take values in the equivariant cohomology of the parameter space. These invariants unify and generalize the Hall conductance and the Thouless pump. We prove quantization properties of the invariants for low-dimensional invertible systems.

In the second part, we discuss the properties of the invariant associated with the Hall conductance for 2d lattice systems with $U(1)$-symmetry. We define anyonic states associated with the flux insertions and relate their statistics to this invariant. We also provide the construction of states realizing chiral topological order with a non-trivial value of this invariant. The construction is based on the data of a unitary regular vertex operator algebra.

PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Nikita Sopenko. "Chiral topologically ordered states on a lattice from vertex operator algebras". In: *arXiv preprint arXiv:2301.08697* (2023). arXiv: 2301.08697.

[2] Anton Kapustin and Nikita Sopenko. "Local Noether theorem for quantum lattice systems and topological invariants of gapped states". In: *Journal of Mathematical Physics* 63.9 (2022), p. 091903. DOI: 10.1063/5.0085964.
All authors contributed equally.

[3] Anton Kapustin, Nikita Sopenko, and Bowen Yang. "A classification of invertible phases of bosonic quantum lattice systems in one dimension". In: *Journal of Mathematical Physics* 62.8 (Aug. 2021), p. 081901. DOI: 10.1063/5.0055996.
All authors contributed equally.

[4] Anton Kapustin and Nikita Sopenko. "Hall conductance and the statistics of flux insertions in gapped interacting lattice systems". In: *Journal of Mathematical Physics* 61.10 (2020), p. 101901. DOI: 10.1063/5.0022944.
All authors contributed equally.

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Chapter 1

INTRODUCTION

One of the most significant discoveries in theoretical physics over the last four decades is the existence of topological order. It arises in quantum matter at zero temperature with a non-vanishing energy gap for local excitations, examples of which include topological insulators and fractional quantum Hall liquids. It was realized that some properties of such systems are robust against perturbations that do not close the gap, leading to the emergence of new phases of matter known as topological or gapped phases. They do not fit into the traditional Landau paradigm and provide a playground for fascinating phenomena of topological nature. These include fractional statistics of quasi-particles and universal anomalous transport of energy and current, making topological phases of matter an exciting area of study for physicists.

It is natural to ask how to characterize and classify topological phases of quantum matter at zero temperature. Since systems with the same gapped ground state can be deformed into each other without phase transitions, all the information about the phase must be encoded in the entanglement pattern of the ground-state wave function [64]. It is, however, remarkably hard to extract this information from the knowledge of the wave-function directly since the universal properties are encoded in a rather non-local way. This non-locality manifests itself in the existence of topological excitations and/or topological transport and implies the impossibility to disentangle the ground state by a locality-preserving adiabatic evolution.

Many classifications schemes of topological phases of matter have been proposed over the years. Phases of gapped non-interacting fermions can be classified using K-theory, as pointed out in the pioneering works [57, 56, 43]. However, these methods do not generalize to interacting systems. It is believed that at least some class of interacting gapped phases can be captured by topological quantum field theories (TQFT). While TQFT had a great impact on our intuition about the behavior of topologically ordered systems and provides a guess for what sort of universal information characterizes them, it is remarkably hard to give a microscopic justification for this belief. In particular, it is mysterious how the underlying geometry if the field theoretic description can emerge from something microscopic like particles and spins.

To pose the problem mathematically, one needs to choose a suitable microscopic model for “matter". As it is difficult to deal with quantum systems of many particles with Coulomb-like potential in full generality, one often considers some effective but still microscopic lattice systems with arbitrary rapidly-decaying interactions[1]. One may hope that once the physical and mathematical principles behind the classification of topological phases of such such systems are understood, they can also be applied to real matter that consists of atoms.

In this thesis, we apply methods of quantum statistical mechanics and operator algebra to gapped states of quantum lattice systems. We give a mathematical definition of various topological invariants of such states and prove some of their properties.

In Chapter 2, we define a class of models we are working with and introduce a formalism that will be used in the reminder of the work. We discuss how densities and currents of physical quantities can be defined on the lattice and prove a version of a Noether theorem. We also give a mathematical formalization of the notion of a topological phase by introducing a special class of automorphisms of the algebra of observables.

In Chapter 3, we provide the first application of our formalism. We describe a systematic way to produce topological invariants of gapped states or smooth families of gapped states in the same topological phase originating from Berry curvature. They give a generalization of Berry classes to the many-body setting and for d-dimensional lattice systems take values in the degree-$(d + 2)$ de Rham cohomology of the parameter space. Higher Berry classes allow one to probe the topology of the space of gapped states and to detect high-codimension critical loci in the phase diagram where the gap closes [30]. In the case of invertible quantum lattice systems, higher Berry classes were first discussed by Kitaev [41]; for general gapped systems explicit formulas for them were written down in [38] using a choice of a family of Hamiltonians. Our construction makes it clear that higher Berry classes depend only on the family of states, not the Hamiltonians used to define them. We also introduce equivariant version of Berry classes for systems with a Lie group symmetry and show that they unify and generalize Hall conductance and Thouless pump invariants. Our main technical tools are the Lieb-Robinson bound in the form proved in [50, 47] and the linear integral transforms introduced in [25, 40, 55].

Chapter 4 is devoted to the analysis of invertible states of one-dimensional systems. We prove that all invertible states are in the trivial phase. We also show, that higher

[1]Long-range Coulomb-like interactions are often screened

Berry classes for families of such states are quantized.

In Chapter 5 we give a more detailed analysis of 2d gapped states with $U(1)$-symmetry. We relate the equivariant Berry class σ defined in Chapter 3 to the Hall conductance and show how one can explicitly construct states representing anyons that correspond to vortices. Using the properties of these vortices, we give a proof that for invertible systems the numerical value of σ is quantized and takes values in $2\mathbb{Z}$ or $\mathbb{Z}$ depending on whether the systems is bosonic or fermionic.

The last Chapter 6 aims to introduce a new class of states of 2d lattice systems which describe chiral topological order. The construction is based on the data of a unitary regular vertex operator algebra and provides a realization of the topological order associated with this vertex algebra. We provide various evidence for that, but our account in this chapter will be more expository.

Notation

- $\mathscr{F}_\infty$ - the Fréchet space of functions $\mathbb{R}_{r\geq 0} \to \mathbb{R}$ which decay at infinity faster than any power of r and topologized by a sequence of norms
$$\|f\|_\alpha := \sup_{r\geq 0}(1+r)^\alpha |f(r)|,\ \alpha \in \mathbb{N}_0.$$
A sequence f_k, $k \in \mathbb{N}$, converges to 0 iff $\|f_k\|_\alpha \to 0$ for all α.

- $\mathscr{F}_\infty^+$ - subset of monotonically decreasing non-negative functions from $\mathscr{F}_\infty$.

- $\theta : \mathbb{R} \to \mathbb{R}$ - the indicator function of $[0, \infty)$.

- The diameter of a subset $X \subset \mathbb{R}^d$ is defined by
$$\mathrm{diam}(X) := \sup_{x,y\in X} |x - y|.$$

- The distance between any subsets X and Y is defined by
$$\mathrm{dist}(X, Y) := \inf_{x\in X, y\in Y} |x - y|.$$

- χ_Y - the indicator function of $Y \subset \mathbb{R}^d$, i.e., $\chi_Y(x) = 1$ if $x \in Y$ and $\chi_Y(x) = 0$ if $x \notin Y$.

- The complement of a subset X is denoted X^c.

- An open ball of radius r with the center at $x \in \mathbb{R}^d$ is defined by $B_x(r) := \{y \in \mathbb{R}^d : |x - y| < r\}$. Its complement in $\mathbb{R}^d$ is denoted $B^c_x(r)$.

- $\mathrm{Fin}(\Lambda)$ - the set of finite subsets of a set Λ. It is a directed set with respect to inclusion.

- For a Hilbert space $\mathcal{H}$, by $\mathcal{B}(\mathcal{H})$ we denote the C^*-algebra of bounded operators on $\mathcal{H}$.

- For a state ψ of a C^*-algebra $\mathscr{A}$, we denote the evaluation of ψ on $\mathcal{A} \in \mathscr{A}$ by $\langle \mathcal{A} \rangle_\psi$.

- For a spin system, the tracial state (or the infinite temperature state) is denoted $\langle \cdot \rangle_\infty$.

- The GNS representation of a state ψ is denoted $(\pi_\psi, \mathcal{H}_\psi, |\psi\rangle)$.

- The conditional expectation value of $\mathcal{A} \in \mathscr{A}_\ell$ on $\mathscr{A}_X$ is denoted $\mathcal{A}|_X$.

Chapter 2

QUANTUM SPIN SYSTEMS

The goal of this preliminary chapter is to define the class of models we are working with and introduce the terminology. We assume that the reader is familiar with some basics of quantum statistical mechanics (see, e.g., [11, 12]) and only give proper description of non-standard notions and facts.

In Section 2.1, we introduce C^*-algebras of quasi-local observables $\mathscr{A}$ for lattice systems. We also define a certain dense subalgebra $\mathscr{A}_{a\ell} \subset \mathscr{A}$ of observables with rapidly-decaying localization which we call the algebra of *almost local observables*. This subalgebra is not complete with respect to the norm topology of $\mathscr{A}$, but it is complete with respect to a topology defined by a non-decreasing family of norms labeled by a non-negative integer. Such topological vector spaces are known as (graded) Fréchet spaces. The central role played by $\mathscr{A}_{a\ell}$ and its relatives necessitates a systematic use of calculus in Fréchet spaces as described, for example, in [24].

In Section 2.2, we define a differential graded Fréchet-Lie algebra (DGFLA) associated with a quantum spin system which encodes all local[1] Hamiltonians, their densities, currents, and their higher analogs. We prove that the cohomology of this DGFLA vanishes. This kinematic result implies, among other things, that any local continuous symmetry of a local lattice Hamiltonian gives rise to a local conserved current. Since the relation between symmetries and local conservation laws is usually referred to as the Noether theorem, we call the DGFLA attached to a lattice system the *local Noether complex*. The current corresponding to a symmetry is not unique, but the ambiguity can be completely described. Non-uniqueness of the energy current, in particular, is the cause of many complications in the theory of heat transport, see, e.g., [16]. The effect of these ambiguities on the standard Kubo-Greenwood formulas for transport coefficients has been studied in [7, 39]. Hopefully, our results help clarify such issues, at least for lattice systems.

In Section 2.3, we define a certain class of automorphisms of the algebra that corresponds to the evolution of the system performed by a Hamiltonian with rapidly decaying interactions. The main use of such automorphisms is to defined an

[1] In the sense that interactions are either finite-range or decay faster than any power of the distance.

equivalence relation on gapped states of lattice systems that we describe in Section 2.4 via quasi-adiabatic evolution in a sense of [28].

2.1 Algebra of observables

Lattices

Definition 2.1. *We say that a subset* $\Lambda \subset \mathbb{R}^d$ *is uniformly discrete if*

$$\epsilon := \inf_{\substack{j,k \in \Lambda \\ j \neq k}} |j-k| > 0. \tag{2.1}$$

Definition 2.2. *We say that a subset* $\Lambda \subset \mathbb{R}^d$ *is relatively dense if*

$$\mathbf{\Delta} := \sup_{x \in \mathbb{R}^d} \inf_{j \in \Lambda} |x-j| < \infty. \tag{2.2}$$

Definition 2.3. *A Delone subset* $\Lambda \subset \mathbb{R}^d$ *is a countable subset that is uniformly discrete and relatively dense.*

In what follows we consider Delone subsets with the metric on $\mathbb{R}^d$ being rescaled so that $\mathbf{\Delta} < 1/2$. Then every open ball of diameter 1 has a nonempty intersection with Λ, and the cardinality of the intersection is upper bounded by $C_d \epsilon^d$ for some C_d depending only on d. Elements of Λ are called sites.

Functions

We will denote by $\mathscr{F}_\infty$ the space of functions $\mathbb{R}_{r\geq 0} \to \mathbb{R}$ which decay at infinity faster than any power of r. We denote the subset of monotonically decreasing non-negative functions by $\mathscr{F}_\infty^+$. The space $\mathscr{F}_\infty$ is a Fréchet space topologized by a sequence of norms (see Appendix A.1 for a brief reminder on seminorms and the topologies on vector spaces defined by countable families of seminorms)

$$\|f\|_\alpha := \sup_{r \geq 0} (1+r)^\alpha |f(r)|, \ \alpha \in \mathbb{N}_0. \tag{2.3}$$

A sequence f_k, $k \in \mathbb{N}$, converges to 0 iff $\|f_k\|_\alpha \to 0$ for all α.

Finally, $\theta : \mathbb{R} \to \mathbb{R}$ will denote the indicator function of $[0, \infty)$.

Quasi-local algebra

To a finite set Γ and a collection of separable Hilbert spaces $\{\mathcal{V}_j\}_{j \in \Gamma}$, one can associate the Hilbert space $\mathcal{V}_\Gamma = \bigotimes_{j\in\Gamma} \mathcal{V}_j$ and the C^*-algebra $\mathscr{A}_\Gamma = \mathcal{B}(\bigotimes_{j \in \Gamma} \mathcal{V}_j)$. By convention, $\mathscr{A}_\emptyset := \mathbb{C}$.

Let S be a (possibly-infinite) set, and let $\{\mathcal{V}_j\}_{j\in S}$ be a collection of separable Hilbert spaces. The set $\mathrm{Fin}(S)$ of finite subsets of S is a directed set with the order given by the inclusion. For for a pair $\Gamma, \Gamma' \in \mathrm{Fin}(S)$ with $\Gamma \subseteq \Gamma'$, there is a canonical homomorphism of $*$-algebras $e_{\Gamma\Gamma'} : \mathscr{A}_\Gamma \to \mathscr{A}_{\Gamma'}$ given by $a \to a \otimes 1_{\mathcal{V}_{\Gamma'\setminus\Gamma}}$. The data $\big(\{\mathscr{A}_\Gamma\}_{\Gamma\in\mathrm{Fin}(S)}, \{e_{\Gamma\Gamma'}\}_{\Gamma,\Gamma'\in\mathrm{Fin}(S)}\big)$ defines the direct system of $*$-algebras. We define the $*$-algebra $\mathscr{A}_\ell$ as the direct limit $\mathscr{A}_\ell := \varinjlim_{\Gamma} \mathscr{A}_\Gamma$ and call it *the algebra of local observables*. For $\mathcal{A} \in \mathscr{A}_\ell$, the minimal set Γ such that $\mathcal{A}$ belongs to $\mathscr{A}_\Gamma$ is called the *support* of $\mathcal{A}$. We define the C^*-algebra $\mathscr{A}$ as the norm completion of $\mathscr{A}_\ell$ and call it *the algebra of quasi-local observables*.

Let $\Lambda \subset \mathbb{R}^d$ be a Delone subset. By a (bosonic) *lattice system* on Λ, we mean a collection of separable Hilbert spaces $\{\mathcal{V}_j\}_{j\in\Lambda}$ and the associated algebras of local and quasi-local observables. By the abuse of notation, we denote lattice systems by $\mathscr{A}$ assuming that it contains the information about the lattice Λ and $\{\mathcal{V}_j\}_{j\in\Lambda}$. By a (bosonic) *spin system*, we mean a bosonic lattice system with uniformly bounded $\{\dim \mathcal{V}_j\}_{j\in\Lambda}$. The maximal value of $\dim \mathcal{V}_j$ is called the *maximal spin* of a spin system. If we replace the Hilbert spaces $\{\mathcal{V}_j\}_j$ with $\mathbb{Z}/2\mathbb{Z}$ graded Hilbert spaces and tensor products with $\mathbb{Z}/2\mathbb{Z}$ graded tensor products, we get a definition of *fermionic lattice systems* or *fermionic spin systems*.

Remark 2.1. *In the following, except for Chapter 6 or unless otherwise specified, for simplicity we consider only bosonic spin systems. Most results can be straightforwardly generalized to the case of fermionic spin systems, and some modifications needed for that are mentioned explicitly in the text. We believe that many results can also be generalized to the case of lattice systems with infinite-dimensional on-site Hilbert spaces.*

We say that a lattice system has (internal) G-symmetry for a compact Lie group G, if each $\mathcal{V}_j$ is equipped with a unitary representation of G. Every such representation can be written as direct sum of irreducible unitary representations each of which is finite-dimensional.

There is a natural monoidal structure on systems. Given two systems $\mathscr{A}$ and $\mathscr{A}'$ on $\mathbb{R}^d$ we can consider *the stack* or the *composition* of two systems denoted $\mathscr{A} \otimes \mathscr{A}'$ with the lattice being the union of two lattices, and Hilbert spaces for a site $j \in \Lambda \cup \Lambda'$ being $\mathcal{V}_j \otimes \mathcal{V}'_j$ if $j \in \Lambda \cap \Lambda'$ and $\mathcal{V}_j$ (respectively, $\mathcal{V}'_j$) if $j \in \Lambda$ and $j \notin \Lambda'$ (respectively, if $j \in \Lambda'$ and $j \notin \Lambda$).

Let $\mathscr{A}$ be a spin system on a lattice Λ. For any $j \in \Lambda$ we let $\Pi_j : \mathscr{A}_\ell \to \mathscr{A}_\ell$ be the averaging over local unitaries supported at j, i.e., for any $\mathcal{B} \in \mathscr{A}_\ell$ we let $\Pi_j(\mathcal{B}) = \int U\mathcal{B}U^* d\mu_j(U)$, where $d\mu_j(U)$ is the Haar measure on the group of unitary elements of $\mathscr{A}_j$ normalized so that $\int 1\, d\mu_j(U) = 1$. It is well-known that Π_j is positive and does not increase the norm. The maps Π_j for different j commute, therefore for any $\Gamma \in \mathrm{Fin}(\Lambda)$ we can define $\Pi_\Gamma := \prod_{j\in\Gamma} \Pi_j$. For $X \subset \mathbb{R}^d$ we let

$$\Pi_X := \varinjlim_{\Gamma\subset X} \Pi_\Gamma. \tag{2.4}$$

We will use without comment various obvious properties of the averaging maps Π_X, for example $\Pi_X \circ \Pi_Y = \Pi_{X\cup Y}$ for any $X, Y \subset \mathbb{R}^d$. The following simple estimate is also sometimes useful:

$$\|\mathcal{A} - \Pi_X(\mathcal{A})\| \leq \sup_{\mathcal{B}\in\mathscr{A}_X} \frac{\|[\mathcal{A},\mathcal{B}]\|}{\|\mathcal{B}\|}. \tag{2.5}$$

Note that $\Pi_\Lambda = \varinjlim_{\Gamma} \Pi_\Gamma$ maps any $\mathcal{B} \in \mathscr{A}_\ell$ to the center of $\mathscr{A}_\ell$ and thus can be thought of as a state (i.e., a positive normalized linear function) on $\mathscr{A}_\ell$. We will denote it $\langle\cdot\rangle_\infty$ since it is the infinite temperature state. This state can also be defined as the unique tracial state on $\mathscr{A}_\ell$. More generally, for any $X \subset \mathbb{R}^d$ we define the conditional expectation value $\mathscr{A}_\ell \to \mathscr{A}_X$ by

$$\mathcal{A} \mapsto \mathcal{A}|_X := \Pi_{X^c}(\mathcal{A}). \tag{2.6}$$

Conditional expectation value does not increase the norm.

Almost local algebras

The algebra $\mathscr{A}$ of quasi-local observable of a lattice system has a rather rough dependence on the geometry of the ambient space. For example, it does not capture the metric or even the dimensionality of the space.

More sensitive algebras can be produced by a completion of the algebra $\mathscr{A}_\ell$ in the norm or a family of norms which depends on the geometry. One way to produce such a norm is to choose a decaying function $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and a point $x \in \mathbb{R}$. Then

$$\|\mathcal{A}\|_{h,x} := \|\mathcal{A}\| + \sup_r \left(h(r) \inf_{\mathcal{B}\in\mathscr{A}_{B_x(r)}} \|\mathcal{A} - \mathcal{B}\| \right) \tag{2.7}$$

is a norm. There is no canonical choice of h or x.

The central role of this manuscript is played by a certain subalgebra $\mathscr{A}_{a\ell} \subset \mathscr{A}$ of quasi-local observables which decay rapidly at infinity. Roughly speaking, $\mathscr{A}_{a\ell}$

consists of quasi-local observables which on a ball of radius r can be approximated by local observables with an $\mathcal{O}(r^{-\infty})$ error. Such an algebra corresponds to the completion in the norms of the form above with (f, x) being of the form $((1+r)^{\alpha}, j)$ for $\alpha \in \mathbb{N}$ and fixed $j \in \Lambda$.

To define $\mathscr{A}_{a\ell}$ we first introduce some notation. For any $\mathcal{A} \in \mathscr{A}$, $x \in \mathbb{R}^d$, and $r \geq 0$ let

$$f_x(\mathcal{A}, r) := \inf_{\mathcal{B} \in \mathscr{A}_{B_x(r)}} \|\mathcal{A} - \mathcal{B}\|. \tag{2.8}$$

This is a monotonically decreasing non-negative function of r which takes values in $[0, \|\mathcal{A}\|]$, approaches zero as $r \to \infty$, and for any $x \in \mathbb{R}^d$ and any $r \geq 0$ satisfies the triangle inequality:

$$f_x(\mathcal{A}_1 + \mathcal{A}_2, r) \leq f_x(\mathcal{A}_1, r) + f_x(\mathcal{A}_2, r). \tag{2.9}$$

Thus for a fixed $x \in \mathbb{R}^d$ and fixed $r \geq 0$ $f_x(\cdot, r)$ is a seminorm on $\mathscr{A}$. Note for future use that if $\Gamma \in \mathrm{Fin}(\Lambda)$ and $\mathcal{A} \in \mathscr{A}_\Gamma$, then $f_x(\mathcal{A}, r) = 0$ for any $r > \mathrm{diam}(\{x\} \cup \Gamma)$, and thus $f_x(\mathcal{A}, r) \leq \|\mathcal{A}\| \theta(r - \mathrm{diam}(\{x\} \cup \Gamma))$.

Given $b(r) \in \mathscr{F}_\infty$ we will say that an observable $\mathcal{A} \in \mathscr{A}$ is *b-localized* at x if $f_x(\mathcal{A}, r) \leq b(r)$ for all r. Note that for such an observable $f_x(\mathcal{A}, r) \in \mathscr{F}^+_\infty$.

For $j \in \Lambda$, let us define the norms

$$\|\mathcal{A}\|'_{j,\alpha} := \|\mathcal{A}\| + \sup_r (1+r)^\alpha f_j(\mathcal{A}, r), \ \alpha \in \mathbb{N}. \tag{2.10}$$

Lemma 2.1. *Let $\mathscr{A}_{al}$ be a $*$-subalgebra $\mathscr{A}_{al} \subset \mathscr{A}$, and let us fix $j \in \Lambda$. The following characterizations of $\mathscr{A}_{a\ell}$ are all equivalent:*

(1) $\mathscr{A}_{a\ell}$ is a subspace of $\mathscr{A}$ defined by the condition $\|\mathcal{A}\|'_{j,\alpha} < \infty$ for all $\alpha \in \mathbb{N}$.

(2) $\mathscr{A}_{a\ell}$ consists of elements $\mathcal{A} \in \mathscr{A}$ which are b-localized at j for some $b(r) \in \mathscr{F}^+_\infty$.

(3) $\mathscr{A}_{a\ell}$ is the completion of the algebra $\mathscr{A}_\ell$ with respect to the norms $\|\cdot\|'_{j,\alpha}$, $\alpha \in \mathbb{N}$.

$\mathscr{A}_{a\ell}$ thus defined does not depend on j.

Proof. The implication (1) $\Rightarrow$ (2) follows directly from the definition of the norms $\|\cdot\|'_{j,\alpha}$. The definition of the completion with respect to norms implies (3) $\Rightarrow$ (1).

To show that (2) implies (3), let $\{\mathcal{A}^{(n)}\}$ be a sequence of observables $\mathcal{A}^{(n)} \in \mathscr{A}_{B_j(n)}$ for $n \in \mathbb{N}$ for which the infimum in the definition of $f_j(\mathcal{A}, n)$ is reached: $f_j(\mathcal{A}, n) = \|\mathcal{A} - \mathcal{A}^{(n)}\|$. We have

$$\|\mathcal{A} - \mathcal{A}^{(n)}\|'_{j,\alpha} = \|\mathcal{A} - \mathcal{A}^{(n)}\| + \sup_r (1+r)^\alpha f_j(\mathcal{A} - \mathcal{A}^{(n)}, r) \leq$$
$$\leq f_j(\mathcal{A}, n) + \sup_{r \geq n} (1+r)^\alpha f_j(\mathcal{A}, r). \quad (2.11)$$

Therefore $\{\mathcal{A}^{(n)}\}$ converges in the norms $\|\cdot\|'_{j,\alpha}$ to $\mathcal{A}$.

Independence of j is manifest in the characterization (2). □

Definition 2.4. *The space of almost local observables $\mathscr{A}_{a\ell}$ is the $*$-subalgebra satisfying the equivalent characterizations of Lemma 2.1.*

As recalled in Appendix A.1, one can use the family of norms $\|\cdot\|'_{j,\alpha}$ for a fixed $j \in \Lambda$ to define a topology on $\mathscr{A}_{a\ell}$. In Appendix A.2 we show that this topology on $\mathscr{A}_{a\ell}$ does not depend on j and turns $\mathscr{A}_{a\ell}$ into a *Fréchet algebra*.

Lie algebras

The algebras $\mathscr{A}_\Gamma$, $\mathscr{A}_\ell$ define the Lie algebras with the bracket being the commutator. The elements of the quotient by the ideal generated by the identity can be identified with the equivalence classes of observables $[\mathcal{A}]$ under $\mathcal{A} \sim \mathcal{A} + \lambda$ for $\lambda \in \mathbb{C}$. We denote by $\mathfrak{d}_\Gamma$, $\mathfrak{d}_l$ the real subalgebras of these quotients, elements of which correspond to equivalence classes $[\mathcal{A}]$ of skew-self-adjoint observables $\mathcal{A}^* = -\mathcal{A}$. We also define the Lie algebra $\mathfrak{d}_{al}$ as the completion of $\mathfrak{d}_l$ with respect to the norms

$$\|\mathsf{a}\|_{j,\alpha} := \sup_r (1+r)^\alpha f_j(\mathsf{a}, r) \quad (2.12)$$

where

$$f_x(\mathsf{a}, r) := \inf_{\mathcal{B} \in \mathscr{A}_{B_x(r)}} \|\mathcal{A} - \mathcal{B}\| \quad (2.13)$$

and $\mathcal{A} \in \mathscr{A}_{a\ell}$ is any lift of $\mathsf{a} \in \mathfrak{d}_l$. Since the bracket is continuous, $\mathfrak{d}_{al}$ is a *Fréchet-Lie algebra*. Elements of $\mathfrak{d}_{al}$ define derivations of the algebra $\mathscr{A}$, which preserve $\mathscr{A}_{a\ell}$. For $\mathsf{a} \in \mathfrak{d}_{al}$, we denote the value of the corresponding map $\mathsf{a} : \mathscr{A} \to \mathscr{A}$ on $\mathcal{A} \in \mathscr{A}$ by $\mathsf{a}(\mathcal{A})$.

When all on-site Hilbert spaces are finite-dimensional, we can identify $\mathfrak{d}_\Gamma$, $\mathfrak{d}_l$, $\mathfrak{d}_{al}$ with the Lie algebra of traceless skew-self-adjoint elements of $\mathscr{A}_\Gamma$, $\mathscr{A}_\ell$, $\mathscr{A}_{a\ell}$, respectively. For $\mathcal{A} \in \mathscr{A}_{a\ell}$, sometimes we write $\mathcal{A} \in \mathfrak{d}_{al}$ if $\mathcal{A} = -\mathcal{A}^*$ and $\langle \mathcal{A} \rangle_\infty = 0$. Note that for any $\mathcal{A} \in \mathfrak{d}_l, \mathfrak{d}_{al}$ we have $\|\mathcal{A}\| \geq f_x(\mathcal{A}, 0) \geq \|\mathcal{A}\|/2$ for any $x \in \mathbb{R}^d$. Therefore, upon restriction to traceless anti-self-adjoint observables one can replace the norms (2.10) with an equivalent set of norms

$$\|\mathcal{A}\|_{j,\alpha} := \sup_r (1+r)^\alpha f_j(\mathcal{A}, r), \ \alpha \in \mathbb{N}_0. \tag{2.14}$$

In this work we will mostly work with traceless anti-self-adjoint observables and then will use the norms (2.14). Two other equivalent sets of norms are defined in Appendix A.4.

2.2 Hamiltonians and currents

Interactions

A Hamiltonian for an infinite spin system is an unbounded densely-defined real derivation of the algebra $\mathscr{A}$. The Hamiltonians of physical interest usually have the form

$$\mathcal{A} \mapsto \sum_{\Gamma \in \mathrm{Fin}(\Lambda)} \Phi_\Gamma(\mathcal{A}) \tag{2.15}$$

where $\Phi_\Gamma \in \mathfrak{d}_\Gamma$ satisfies $\Phi_\Gamma^* = -\Phi_\Gamma$. Typically one also assumes that $\sup_\Gamma \|\Phi_\Gamma\| < \infty$. In the mathematical physics literature the function $\Phi : \mathrm{Fin}(\Lambda) \to \mathfrak{d}_l$, $\Gamma \mapsto \Phi_\Gamma$, is known as an interaction.

The domain of such a derivation depends on how rapidly $\|\Phi_\Gamma\|$ decays with the size of Γ. As a minimum, it should be defined everywhere on $\mathscr{A}_\ell$. Further, it should be possible to exponentiate a physically sensible derivation to an automorphism of $\mathscr{A}$ to defined a dynamics of a spin system. Before we can describe suitable decay conditions, however, we need to deal with the fact that the map from interactions to derivations is many-to-one and that there is a large “gauge freedom" in choosing the function Φ for a given derivation. Any decay condition on Φ should respect this freedom. Unfortunately, it is not straightforward to describe this “gauge freedom.” To rectify the situation, one may impose a suitable “gauge condition” on Φ so that a derivation determines Φ uniquely. One natural condition is to demand that for any proper inclusion $\Gamma' \subset \Gamma$ and any $\mathcal{A} \in \mathfrak{d}_{\Gamma'}$ one has $\langle \Phi_\Gamma \mathcal{A} \rangle_\infty = 0$. This condition implies that Φ_Γ is not localized on any proper subset of Γ. Its advantage is that it does not depend on any choices. But this gauge condition is difficult to work with when the interaction has exponential or slower than exponential decay because the number of finite subsets of $B_j(r) \cap \Lambda$ grows as e^{Cr^d}. Later we will describe a convenient but non-canonical “gauge condition” on Φ.

Another drawback of describing a Hamiltonian via an interaction Φ is that there is no natural notion of energy density (and therefore also of energy current). As an alternative, we may study derivations of the form

$$\delta_{\mathsf{h}} : \mathcal{A} \mapsto \sum_{j \in \Lambda} [\mathsf{h}_j, \mathcal{A}], \tag{2.16}$$

where each h_j is a traceless anti-self-adjoint observable which in some sense is localized in the neighborhood of the site j and can be interpreted as i times the energy density on this site. The gauge freedom is present in this approach as well, since the observables h_j are not uniquely determined by the derivation δ_{h}. However, it is fairly straightforward to parameterize this gauge freedom and to define a class of derivations which is mathematically natural and is large enough to describe lattice systems with rapidly decaying interactions.

For that, we introduce a certain chain complex, which we call *the complex of currents*. We first describe this complex for finite-range interactions, and then show how to complete it to a complex of rapidly decaying currents using a family of norms, in the same way as the algebra $\mathscr{A}_\ell$ can be completed to $\mathscr{A}_{a\ell}$.

The complex of finite-range currents

Definition of the complex

For any non-negative integer q we define a uniformly local (UL) q-chain as a skew-symmetric function $\mathsf{a} : \Lambda^{q+1} \to \mathfrak{d}_l$ for which there are constants $C > 0$ and $R > 0$ such that for any $\{j_0, j_1, ..., j_q\} \subset \Lambda$ and any $a \in \{0, 1, ..., q\}$ we have $\mathsf{a}_{j_0 \dots j_q} \in \mathfrak{d}_{B_{j_a}(R)}$ and $\|\mathsf{a}_{j_0 \dots j_q}\| \leq C$. The smallest possible value of R is called the range of a. If the distance between any j_a and j_b is greater than $2R$, then $\mathsf{a}_{j_0 \dots j_q} = 0$. The space of UL q-chains will be denoted $C_q(\mathfrak{d}_l)$. The boundary operator $\partial_q : C_q(\mathfrak{d}_l) \to C_{q-1}(\mathfrak{d}_l)$ has the form

$$\left(\partial_q \mathsf{a}\right)_{j_1 \dots j_q} := \sum_{j_0 \in \Lambda} \mathsf{a}_{j_0 j_1 \dots j_q}. \tag{2.17}$$

Since $\mathsf{a}_{j_0 j_1 \dots j_q}$ is skew-symmetric in indices, we have $\partial_q \circ \partial_{q+1} = 0$ for all q. Note that to any UL 0-chain $\mathsf{a} : j \mapsto \mathsf{a}_j$ one can attach a derivation of $\mathscr{A}_\ell$

$$\delta_{\mathsf{a}} : \mathcal{A} \mapsto \sum_{j \in \Lambda} \mathsf{a}_j(\mathcal{A}). \tag{2.18}$$

We will call derivations of this form UL derivations. Physically, they correspond to finite-range Hamiltonians. It is easy to see that a UL derivation δ_{a} is not affected if we add to a an exact 0-chain. We will show that this is the only gauge freedom associated to UL derivations. That is, we will show that the space of UL derivations is isomorphic to the zeroth homology $H_0(\mathfrak{d}_l)$ of the complex

$$\dots \xrightarrow{\partial_2} C_1(\mathfrak{d}_l) \xrightarrow{\partial_1} C_0(\mathfrak{d}_l) \to 0. \tag{2.19}$$

Homology

To compute the homology of $(C_\bullet(\mathfrak{d}_l), \partial)$, we introduce the following definition.

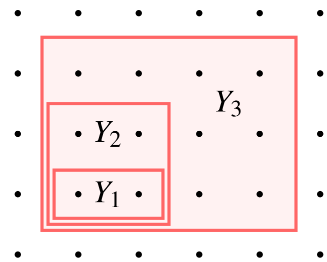


Figure 2.1: Let $\Lambda \subset \mathbb{R}^2$ be the lattice $(\mathbb{Z}+\frac{1}{2})^2$ with two-dimensional on-site Hilbert spaces. Let $\sigma^i_{(x,y)}$ be the Pauli matrix observables on site (x,y). The brick decomposition of $\mathcal{A} = \sigma^z_{(\frac{3}{2},\frac{3}{2})}\sigma^x_{(\frac{5}{2},\frac{3}{2})} + \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^y_{(\frac{5}{2},\frac{5}{2})} + \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^z_{(\frac{3}{2},\frac{5}{2})}\sigma^z_{(\frac{9}{2},\frac{7}{2})} + \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^y_{(\frac{9}{2},\frac{5}{2})}\sigma^z_{(\frac{7}{2},\frac{7}{2})}$ has three terms $\mathcal{A}^{Y_1} = \sigma^z_{(\frac{3}{2},\frac{3}{2})}\sigma^x_{(\frac{5}{2},\frac{3}{2})}$, $\mathcal{A}^{Y_2} = \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^y_{(\frac{5}{2},\frac{5}{2})}$, $\mathcal{A}^{Y_3} = \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^z_{(\frac{3}{2},\frac{5}{2})}\sigma^z_{(\frac{9}{2},\frac{7}{2})} + \sigma^x_{(\frac{3}{2},\frac{3}{2})}\sigma^y_{(\frac{9}{2},\frac{5}{2})}\sigma^z_{(\frac{7}{2},\frac{7}{2})}$ for bricks Y_1, Y_2, Y_3 shown on the picture.

Definition 2.5. *A brick is a subset of $\mathbb{R}^d$ of the form $\{(x_1,\ldots,x_d) : n_i \le x_i < m_i,\ i = 1,\ldots,d\}$, where n_i and m_i are integers satisfying $n_i < m_i$. The empty brick is the empty subset.*

We denote the set of all bricks in $\mathbb{R}^d$ together with the empty brick by $\mathbb{B}_d$. $\mathbb{B}_d$ is a poset with respect to inclusion.[2] Any finite subset of Λ is contained in some brick.

Recall that for any $Y \subset \mathbb{R}^d$ we let $\mathfrak{d}_Y = \mathfrak{d}_l \cap \mathscr{A}_Y$. Clearly, if $Z \subset Y$, then $\mathfrak{d}_Z \subset \mathfrak{d}_Y$. For any $Y \in \mathbb{B}_d$ we define $\mathfrak{d}^Y$ to be the orthogonal complement of the subspace

$$\sum_{\substack{Z\in\mathbb{B}_d \\ Z\subsetneq Y}} \mathfrak{d}_Z \tag{2.20}$$

in $\mathfrak{d}_Y$ with respect to the inner product $\langle \mathcal{A}, \mathcal{B}\rangle := \langle \mathcal{A}^*\mathcal{B}\rangle_\infty$. Elements of $\mathfrak{d}^Y$ are anti-self-adjoint traceless local observables which are localized on Y but not on any brick which is a proper subset of Y. It is easy to see that for any $Y \in \mathbb{B}_d$ we have a direct sum decomposition $\mathfrak{d}_Y = \bigoplus_{Z\in\mathbb{B}_d, Z\subseteq Y} \mathfrak{d}^Z$ and $\mathfrak{d}_l = \bigoplus_{Y\in\mathbb{B}_d} \mathfrak{d}^Y$. For any $\mathcal{A} \in \mathfrak{d}_l$ we will denote by $\mathcal{A}^Y$ its component in $\mathfrak{d}^Y$. For an example of a brick decomposition see Fig. 2.1. Clearly, if $\mathcal{A} \in \mathfrak{d}_X$, then $\mathcal{A}^Y = 0$ whenever $Y \cap X = \emptyset$. Additional properties of the brick expansion can be found in Appendix A.3.

Let $h_q : C_q(\mathfrak{d}_l) \to C_{q+1}(\mathfrak{d}_l)$ be a map defined by

$$h_q(\mathsf{a})_{j_0\ldots j_{q+1}} = \sum_{Y\in\mathbb{B}_d}\sum_{k=0}^{q+1}(-1)^k \frac{\chi_Y(j_k)}{|Y\cap\Lambda|}\mathsf{a}^Y_{j_0\ldots\widehat{j_k}\ldots j_{q+1}}, \tag{2.21}$$

[2] In fact it is a lattice. That is, every two elements of $\mathbb{B}_d$ have a join and a meet in $\mathbb{B}_d$.

where $\widehat{j_k}$ denotes the omission of j_k. Since the sum over Y is finite, h_q is well-defined. Note that $h_{q-1} \circ \partial_q + \partial_{q+1} \circ h_q = \mathrm{id}$ for any $q > 0$. Therefore we have the following

Theorem 2.1. *$H_q(\mathfrak{d}_l) = 0$ for all $q > 0$.*

To describe the homology in degree zero, we introduce the following definition.

Definition 2.6. *Let $\mathfrak{D}_l$ be the space of bounded functions $\mathsf{A} : \mathbb{B}_d \to \mathfrak{d}_l$, $Y \mapsto \mathsf{A}^Y$, such that*

- $\mathsf{A}^Y \in \mathfrak{d}^Y$,
- $\mathsf{A}^Y = 0$ *for all Y of sufficiently large diameter.*

We also extend the definition of ∂_q and h_q by introducing maps $\partial_0 : C_0(\mathfrak{d}_l) \to \mathfrak{D}_l$ and $h_{-1} : \mathfrak{D}_l \to C_0(\mathfrak{d}_l)$ defined by

$$(\partial_0 \mathsf{a})^Y := \sum_{j \in \Lambda} \mathsf{a}_j^Y, \tag{2.22}$$

$$h_{-1}(\mathsf{A})_j := \sum_{Y \in \mathbb{B}_d} \frac{\chi_j(Y)}{|Y \cap \Lambda|} \mathsf{A}^Y. \tag{2.23}$$

Note that the sums on the r.h.s. of both expressions are finite and thus well-defined. We have

$$h_{q-1} \circ \partial_q + \partial_{q+1} \circ h_q = \mathrm{id}, \quad q \geq 0, \tag{2.24}$$

$$\partial_0 \circ h_{-1} = \mathrm{id}. \tag{2.25}$$

Theorem 2.2. *$H_0(\mathfrak{d}_l)$ is isomorphic to $\mathfrak{D}_l$ and to the space of UL derivations.*

Proof. To any UL 0-chain a we attach a function $Y \mapsto \mathsf{A}^Y = \sum_j \mathsf{a}_j^Y$. The sum over j is finite. It is easy to see that this function belongs to $\mathfrak{D}_l$. Furthermore, it is trivially checked that if the 0-chain a is exact, then the corresponding function vanishes. Thus we get a map $\rho_l : H_0(\mathfrak{d}_l) \to \mathfrak{D}_l$.

This map is surjective because a right inverse exists: to an element A of $\mathfrak{D}_l$ one can attach a UL 0-chain $h_{-1}(\mathsf{A})$. To prove injectivity, suppose that a UL 0-chain a satisfies $\sum_j \mathsf{a}_j^Y = 0$ for all $Y \in \mathbb{B}_d$. For any $j, k \in \Lambda$ let

$$\mathsf{b}_{jk} = \sum_{Y \in \mathbb{B}_d} \frac{\chi_j(Y)}{|Y \cap \Lambda|} \mathsf{a}_k^Y - \sum_{Y \in \mathbb{B}_d} \frac{\chi_k(Y)}{|Y \cap \Lambda|} \mathsf{a}_j^Y. \tag{2.26}$$

It is straightforward to check that the collection of observables b_{jk} defines a UL 1-chain, and that $\partial\mathsf{b} = \mathsf{a}$. Thus the map ρ_l is an isomorphism.

By definition of UL derivations, the map from $H_0(\mathfrak{d}_l)$ to UL derivations defined by (2.16) is surjective. To prove injectivity, suppose $\delta_{\mathsf{a}} = 0$ for some UL 0-chain a. Then for any $\mathcal{A} \in \mathscr{A}_\ell$ we have

$$\sum_{Y \in \mathbb{B}_d} [\mathsf{A}^Y, \mathcal{A}] = 0, \tag{2.27}$$

where $\mathsf{A}^Y = \sum_j \mathsf{a}_j^Y$ is an element of $\mathfrak{d}^Y$. Let us pick an arbitrary brick $Z \in \mathbb{B}_d$. Then for any $\mathcal{A} \in \mathscr{A}_Z$ the sum in (2.27) truncates to those Y which have a nonempty intersection with Z. Thus if we define a traceless local observable $\mathcal{B}$ by

$$\mathcal{B} = \sum_{Y \in \mathbb{B}_d, Y \cap Z \neq \emptyset} \mathsf{A}^Y, \tag{2.28}$$

then $\mathcal{B} = 1_Z \otimes \tilde{\mathcal{B}}$, where $\tilde{\mathcal{B}} \in \mathscr{A}_{\overline{Z}}$. Therefore $\mathcal{B}^X = 0$ for any brick X such that $X \subseteq Z$. On the other hand, from the definition of $\mathcal{B}$ we have that for such bricks $\mathcal{B}^X = \mathsf{A}^X$. Since Z was arbitrary, we conclude that $\mathsf{A}^Y = 0$ for all $Y \in \mathbb{B}_d$. □

Thus, the augmented complex

$$\ldots \xrightarrow{\partial_2} C_1(\mathfrak{d}_l) \xrightarrow{\partial_1} C_0(\mathfrak{d}_l) \xrightarrow{\partial_0} \mathfrak{D}_l \to 0 \tag{2.29}$$

is contractible with a contracting homotopy h_q, $q \geq -1$. We call it the *uniformly local Noether complex*. It is graded by integers $q \geq -1$.

For any $\mathsf{F} \in \mathfrak{D}_l$ we denote the action of the corresponding UL derivation on $\mathcal{A} \in \mathfrak{d}_l$ by $\mathsf{F}(\mathcal{A})$. Explicitly, $\mathsf{F}(\mathcal{A}) = \sum_Y [\mathsf{F}^Y, \mathcal{A}]$.

Brackets

An important property of the space of UL derivations $\mathfrak{D}_l$ is that it has the structure of a Lie algebra. This is easiest to see if we identify it with $H_0(\mathfrak{d}_l)$. For given $\mathsf{F}, \mathsf{G} \in \mathfrak{D}_l$ and $\mathsf{f}, \mathsf{g} \in C_0(\mathfrak{d}_l)$ such that $\mathsf{F} = \partial\mathsf{f}$ and $\mathsf{G} = \partial\mathsf{g}$, the Lie bracket can be defined by

$$\{\mathsf{F}, \mathsf{G}\} := \partial([\mathsf{f}, \mathsf{g}\}) \tag{2.30}$$

where the components of a UL 0-chain $[\mathsf{f}, \mathsf{g}\}$ are defined as a finite sum

$$[\mathsf{f}, \mathsf{g}\}_j := \sum_{k \in \Lambda} [\mathsf{f}_k, \mathsf{g}_j]. \tag{2.31}$$

The bracket $[\cdot,\cdot\}$ on 0-chains is not skew-symmetric and awkward to work with. But one can express it through a more natural structure which exists on the augmented complex $C_\bullet(\mathfrak{d}_l) \to \mathfrak{D}_l$: the structure of a 1-shifted dg-Lie algebra. This means that there is a degree 1 bracket $\{\cdot,\cdot\}$ on the augmented complex which is graded-skew-symmetric:

$$\{\mathsf{f},\mathsf{g}\} = -(-1)^{(|\mathsf{f}|+1)(|\mathsf{g}|+1)}\{\mathsf{g},\mathsf{f}\}, \tag{2.32}$$

satisfies the graded Jacobi identity:

$$(-1)^{(|\mathsf{f}|+1)(|\mathsf{h}|+1)}\{\mathsf{f},\{\mathsf{g},\mathsf{h}\}\} + (-1)^{(|\mathsf{g}|+1)(|\mathsf{f}|+1)}\{\mathsf{g},\{\mathsf{h},\mathsf{f}\}\} + \\ + (-1)^{(|\mathsf{h}+1)(|\mathsf{g}|+1)}\{\mathsf{h},\{\mathsf{f},\mathsf{g}\}\} = 0, \tag{2.33}$$

and the graded Leibniz rule:

$$\partial\{\mathsf{f},\mathsf{g}\} = \{\partial\mathsf{f},\mathsf{g}\} + (-1)^{|\mathsf{f}|+1}\{\mathsf{f},\partial\mathsf{g}\}. \tag{2.34}$$

Here $|\cdot|$ denotes the degree of a chain. For $\mathsf{f} \in C_p(\mathfrak{d}_l)$, $\mathsf{g} \in C_q(\mathfrak{d}_l)$, $\mathsf{F} \in \mathfrak{D}_l$ the bracket is defined by

$$\{\mathsf{f},\mathsf{g}\}_{j_0\dots j_{p+q+1}} := \frac{1}{p!q!}[\mathsf{f}_{j_0\dots j_p},\mathsf{g}_{j_{p+1}\dots j_{p+q+1}}] + \text{(signed permutations)}, \tag{2.35}$$

$$\{\mathsf{F},\mathsf{g}\}_{j_0\dots j_q} := \mathsf{F}\left(\mathsf{g}_{j_0\dots j_q}\right), \tag{2.36}$$

while the bracket of two UL derivations is defined to be their Lie bracket eq. (2.30). Then for any two UL 0-chains f,g we can write

$$[\mathsf{f},\mathsf{g}\} = \{\partial\mathsf{f},\mathsf{g}\}. \tag{2.37}$$

The non-skew-symmetric bracket $[\cdot,\cdot\}$ on 0-chains is an example of a "derived bracket" [6].

There is an injective Lie algebra homomorphism from $\mathfrak{d}_l$ to the Lie algebra of UL derivations $\mathfrak{D}_l$ which sends $\mathcal{B} \in \mathfrak{d}_l$ to the derivation $\mathcal{A} \mapsto [\mathcal{B},\mathcal{A}]$. One can describe the image of this homomorphism more intrinsically by making the following definition.

Definition 2.7. *An element* $\mathsf{A} \in \mathfrak{D}_l$ *is called summable if* $\mathsf{A}^Y \neq 0$ *only for finitely many bricks.*

Physically, summable UL derivations correspond to interactions which are localized at a point. Obviously, summable UL derivations form a Lie sub-algebra of $\mathfrak{D}_l$. In the following we identify it with $\mathfrak{d}_l$.

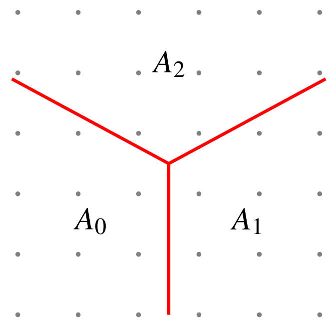


Figure 2.2: An example of a conical partition of $\mathbb{R}^2$.

Integration

If we interpret a 0-chain $\mathsf{b} \in C_0(\mathfrak{d}_l)$ as a density of energy or some other physical quantity, then it is natural to define energy in a region A as a derivation $\mathsf{b}_A \in \mathfrak{D}_l$ which acts on $\mathcal{A} \in \mathfrak{d}_l$ by $\mathsf{b}_A(\mathcal{A}) = \sum_{j\in A}[\mathsf{b}_j, \mathcal{A}]$. Equivalently, $\mathsf{b}^Y_A = \sum_{j\in A} \mathsf{b}^Y_j$. Generalizing this, for any $\mathsf{b} \in C_q(\mathfrak{d}_l)$ the contraction of b with regions $A_0, ..., A_q \subset \mathbb{R}^d$ is a derivation $\mathsf{b}_{A_0...A_q} \in \mathfrak{D}_l$ defined by

$$\mathsf{b}^Y_{A_0...A_q} = \sum_{j_k\in A_k,\, k=0,...,q} \mathsf{b}^Y_{j_0...j_q}. \tag{2.38}$$

We may interpret $\mathsf{b}_{A_0...A_q}$ as an "integral" of b over $A_0, ..., A_q$. Since chains are antisymmetric in j_a, without loss of generality we can assume that the regions are non-intersecting.

Note that if all the regions $A_0, ..., A_q$ are infinite, then in general the derivation $\mathsf{b}_{A_0...A_q}$ is not summable. For the contraction to be a summable derivation (that is, an element of $\mathfrak{d}_l$), one needs to choose the regions $A_0, \dots, A_q$ with some care. For our purposes the following set of regions will suffice. Let us pick a point $p \in \mathbb{R}^d$ and a triangulation of S^{d-1} as a boundary of a d-simplex. Let $\sigma_0, \dots, \sigma_d$ be its open $(d-1)$-simplices and let A_a, $a = 0, \dots, d$, be an open subset of $\mathbb{R}^d$ which in polar coordinates has the form $\mathbb{R}_+ \times \sigma_a$. We will say that A_a is a conical region with base σ_a and apex p. More generally, for a fixed p and a fixed triangulation of S^{d-1} into $d+1$ simplices we say that an open set A_a is an eventually conical region with apex p and base σ_a if outside of a ball $B_p(r)$ it coincides with $\mathbb{R}_+ \times \sigma_a$. We say that an ordered partition $(A_0, ..., A_d)$ of $\mathbb{R}^d$ with Λ being in the interior is a *conical partition*, if $A_0, ..., A_d$ are eventually conical regions with an apex p and bases $\sigma_0, ..., \sigma_d$ (see Fig. 2.2). To any conical partition $(A_0, ..., A_d)$ and a UL d-chain b one can attach an element of $\mathfrak{d}_l$:

$$\mathsf{b}_{A_0...A_d} = \sum_{j_k\in A_k,\, k=0,...,d} \mathsf{b}_{j_0...j_d} \tag{2.39}$$

which is the contraction of b with $A_0, \ldots, A_d$. This sum is finite. Note that the expression (2.39) does not depend on the ordering of the simplices σ_a provided they correspond to a fixed orientation of S^{d-1} and changes sign when the orientation is flipped. A version of Stokes' theorem holds: $(\partial \mathsf{c})_{A_0 \ldots A_d} = 0$ for any $\mathsf{c} \in C_{d+1}(\mathfrak{d}_l)$.

Remark 2.2. *With any conical partition $(A_0, ..., A_d)$ we can also associate an integrated version of the complex eq. (2.29) of Čech type. Let Δ^k be the set of $(d-k-1)$-simplices of the triangulation of S^{d-1} as a boundary of a d-simplex. For any $\sigma \in \Delta^k$ there is an associated set of cones $\{A_{i_0}, ..., A_{i_k}\}$. We denote the image of the contraction of $C_k(\mathfrak{d}_l)$ with $\{A_{i_0}, ..., A_{i_k}\}$ by $\mathfrak{D}_l^{(\sigma)}$. Each $\mathfrak{D}_l^{(\sigma)}$ is a Lie subalgebra of $\mathfrak{D}_l$. Then we have an exact sequence*

$$\mathfrak{d}_l \to \bigoplus_{\sigma \in \Delta^{d-1}} \mathfrak{D}_l^{(\sigma)} \to \bigoplus_{\sigma \in \Delta^{d-2}} \mathfrak{D}_l^{(\sigma)} \to \ldots \to \bigoplus_{\sigma \in \Delta^0} \mathfrak{D}_l^{(\sigma)} \to \mathfrak{D}_l \tag{2.40}$$

with the differential being a sum of signed injections.

The complex of rapidly decaying currents

In the same way as $\mathfrak{d}_l$ can be completed to $\mathfrak{d}_{al}$ using a family of norms, the space $C_q(\mathfrak{d}_l)$, $q \in \mathbb{N}_0$, of UL q-chains can be completed using the norms

$$\|\mathsf{a}\|_\alpha := \sup_r (1+r)^\alpha f(\mathsf{a}, r) = \sup_{a \in \{0,1,\ldots,q\}} \sup_{j_0,\ldots,j_q \in \Lambda} \|\mathsf{a}_{j_0 \ldots j_q}\|_{j_a,\alpha}, \quad \alpha \in \mathbb{N}_0 \tag{2.41}$$

where

$$f(\mathsf{a}, r) := \sup_{a \in \{0,1,\ldots,q\}} \sup_{j_0,\ldots,j_q \in \Lambda} f_{j_a}(\mathsf{a}_{j_0 \ldots j_q}, r) \tag{2.42}$$

is defined for any $\mathsf{a} \in C_q(\mathfrak{d}_l)$. We call the completed space the space of uniformly almost local (UAL) q-chains and denote it by $C_q(\mathfrak{d}_{al})$. This means that any element $\mathsf{a} \in C_q(\mathfrak{d}_{al})$ can be represented by a sequence $\{\mathsf{a}^{(n)}\}$, $n \in \mathbb{N}$ of UL q-chains $\mathsf{a}^{(n)} \in C_q(\mathfrak{d}_l)$ such that for any $\alpha \in \mathbb{N}_0$ $\|\mathsf{a}^{(n)} - \mathsf{a}^{(m)}\|_\alpha$ can be made arbitrarily small by taking arbitrary sufficiently large n, m.

Lemma 2.2. *The following characterizations of UAL chains are all equivalent:*

(1) *A skew-symmetric function $\mathsf{a} : \Lambda^{q+1} \to \mathfrak{d}_{al}$ defines an element of $C_q(\mathfrak{d}_{al})$ if $\|\mathsf{a}\|_\alpha < \infty$ for any $\alpha \in \mathbb{N}$.*

(2) *A skew-symmetric function $\mathsf{a} : \Lambda^{q+1} \to \mathfrak{d}_{al}$ defines an element of $C_q(\mathfrak{d}_{al})$ if there is a function $b(r) \in \mathscr{F}_\infty$ such that for any $j_0, ..., j_q$ the observable $\mathsf{a}_{j_0 \ldots j_q}$ is b-localized at j_a for any $a \in \{0, 1, ..., q\}$.*

(3) $C_q(\mathfrak{d}_{al})$ *is the completion of* $C_q(\mathfrak{d}_l)$ *with respect to the norms* $\|\cdot\|_\alpha$.

Proof. As in Lemma 2.1 the implication (3) ⇒ (1) is straightforward.

If all the norms are finite, the function $f(\mathsf{a}, r)$ can be upper-bounded by an element of $\mathscr{F}_\infty^+$. Therefore (1) implies (2).

It is left to show (2) ⇒ (3). Let $\{\mathsf{a}^{(n)}\}$ be a sequence of elements of $C_q(\mathfrak{d}_l)$ such that $\mathsf{a}^{(n)}_{j_0\ldots j_q}$ is a best possible approximation of $\mathsf{a}_{j_0\ldots j_q}$ by a traceless observable on $\mathfrak{d}_{B_{j_0}(n)\cap B_{j_1}(n)\cap\ldots\cap B_{j_q}(n)}$. Lemma A.1 implies

$$\|\mathsf{a}_{j_0\ldots j_q} - \mathsf{a}^{(n)}_{j_0\ldots j_q}\| \leq 2(2q+1)b(n). \tag{2.43}$$

Therefore for any $a \in \{0, \ldots, q\}$ one has

$$f_{j_a}\left(\mathsf{a}_{j_0\ldots j_q} - \mathsf{a}^{(n)}_{j_0\ldots j_q}, r\right) \leq 2(2q+1)\min(b(r), b(n)) \tag{2.44}$$

and therefore

$$\|\mathsf{a} - \mathsf{a}^{(n)}\|_\alpha \leq 2(2q+1)\sup_{r\geq n}(1+r)^\alpha b(r). \tag{2.45}$$

Thus the sequence $\{\mathsf{a}^{(n)}\}$ converges in the norms $\|\cdot\|_\alpha$ to a. Therefore (2) ⇒ (3). □

We use the family of norms $\|\cdot\|_\alpha$ to define a topology on $C_q(\mathfrak{d}_{al})$ as discussed in Appendix A.1. Characterization (3) implies that this topology turns $C_q(\mathfrak{d}_{al})$ into a Fréchet space. Two other equivalent families of norms are described in Appendix A.4.

Similarly, the space $\mathfrak{D}_l$ can be completed using the norms

$$\|\mathsf{F}\|^{br}_\alpha := \sup_{Y\in\mathbb{B}_d}(1+\mathrm{diam}(Y))^\alpha \|\mathsf{F}^Y\| < \infty, \quad \forall\alpha\in\mathbb{N}_0. \tag{2.46}$$

The resulting space is denoted by $\mathfrak{D}_{al}$, and the resulting space of derivations of $\mathfrak{d}_{al}$ is called the space of UAL derivations.

In Appendix A.4 we show that the boundary map ∂, the contracting homotopy h, the bracket $\{\cdot,\cdot\}$, and the contraction maps on $C_\bullet(\mathfrak{d}_l)$ can be extended to maps on $C_\bullet(\mathfrak{d}_{al})$ continuous in the Fréchet topology. Therefore we have the complex

$$\ldots \xrightarrow{\partial_2} C_1(\mathfrak{d}_{al}) \xrightarrow{\partial_1} C_0(\mathfrak{d}_{al}) \to 0 \tag{2.47}$$

and the corresponding augmented exact complex

$$\ldots \xrightarrow{\partial_2} C_1(\mathfrak{d}_{al}) \xrightarrow{\partial_1} C_0(\mathfrak{d}_{al}) \xrightarrow{\partial_0} \mathfrak{D}_{al} \to 0 \tag{2.48}$$

with the structure of a 1-shifted dg-Fréchet-Lie algebra. We call the latter the *uniformly almost local Noether complex* or simply *Noether complex* and denote it $\mathcal{N}_\bullet$. It is graded by integers $q \geq -1$. The existence of the extension of the contracting homotopy h implies

Theorem 2.3 (Local Noether theorem)**.** *The Noether complex $\mathcal{N}_\bullet$ is exact.*

Remark 2.3. *The space of UAL derivations $\mathfrak{D}_{al}$ can be regarded as a subspace of the space of interactions satisfying the following "gauge condition": $\Phi_\Gamma \in \mathfrak{d}^Y$ if $\Gamma = Y \cap \Lambda$ for a brick $Y \in \mathbb{B}_d$ and $\Phi_\Gamma = 0$ otherwise.*

It is convenient for the following sections to define localization criteria for derivations from $\mathfrak{D}_{al}$. For $f \in \mathscr{F}^+_\infty$, we say that F is *f-local* if $\|\mathsf{F}^Y\| \leq f(\mathrm{diam}(Y))$. For a region A and $f \in \mathscr{F}^+_\infty$, we say that F is *f-localized on A*, if $\|\mathsf{F}^Y\| \leq f(\sup_{x\in Y} \mathrm{dist}(x, A))$. We say that F is *almost localized on A*, if it is f-localized for some $f \in \mathscr{F}^+_\infty$.

There is an injective Lie algebra homomorphism from $\mathfrak{d}_{al}$ to the Lie algebra of UAL derivations which sends $\mathcal{B} \in \mathfrak{d}_{al}$ to the derivation $\mathcal{A} \mapsto [\mathcal{B}, \mathcal{A}]$. One can describe this sub-space more intrinsically by making the following definition.

Definition 2.8. *An element $\mathsf{F} \in \mathfrak{D}_{al}$ is called summable if the infinite sum $\sum_Y \mathsf{F}^Y$ is absolutely convergent in the Fréchet topology of $\mathfrak{d}_{al}$.*

The image of $\mathfrak{d}_{al}$ under the embedding into $\mathfrak{D}_{al}$ consists precisely of summable elements of $\mathfrak{D}_{al}$. Physically, summable UAL derivations correspond to interactions which are approximately localized at a point. Summable UAL derivations obviously form a Lie sub-algebra of $\mathfrak{D}_{al}$. In fact, it is easy to see that they form an ideal.

Finally, the contraction of any $\mathsf{b} \in C_d(\mathfrak{d}_{al})$ with a conical partition $(A_0, ..., A_d)$ of $\mathbb{R}^d$ gives a summable element of $\mathfrak{D}_{al}$, as Prop. A.10 shows. Moreover the corresponding map $C_d(\mathfrak{d}_{al}) \to \mathfrak{d}_{al}$ is continuous. As in the Remark 2.2, there is an almost local version of the integrated complex eq. (2.40).

Relation to energy and charge currents

As mentioned in the introduction, currents on a lattice can be defined using the language of chains. Consider a lattice system with finite-range interactions. The dynamics for such a system is described by a Hamiltonian that can be regarded (after multiplication by i) as a UL derivation H. The Hamiltonian density can be defined as a UL 0-chain h such that $\mathsf{H} = \partial \mathsf{h}$. Obviously, for a fixed H the 0-chain h is far from unique. The ambiguity can be fully characterized, since by Theorem 2.2 any two choices of h differ by a boundary ∂ of a 1-chain. Once h is fixed, we can define an energy current $\mathsf{j}^E : \Lambda \times \Lambda \to \mathscr{A}_\ell$ as a solution of the equation

$$\sum_k [\mathsf{h}_k, \mathsf{h}_j] = -\sum_k \mathsf{j}^E_{kj}. \tag{2.49}$$

The observable j^E_{kj} represents the energy flow from site j to site k. It is natural to require j^E_{jk} to be traceless, so that there is no energy flow between sites in the infinite-temperature state. Since h is finite-range, it is also natural to require j^E_{jk} to vanish whenever j and k are sufficiently far apart, and to be localized near j and k. Thus j^E is a UL 1-chain. The equation (2.49) can be written using the algebraic operations on the UL Noether complex:

$$\{\mathsf{H}, \mathsf{h}\} = -\partial \mathsf{j}^E. \tag{2.50}$$

Note that this equation for j^E is guaranteed to have a solution because by the properties of the shifted Lie bracket the l.h.s. is closed, $\partial\{\mathsf{H}, \mathsf{h}\} = \{\mathsf{H}, \mathsf{H}\} = 0$, and thus exact. Similarly, a Hamiltonian H with rapidly decaying interactions can be regarded as a self-adjoint UAL 0-chain h such that $\mathsf{H} = \partial \mathsf{h}$, while an energy current is defined to be a self-adjoint UAL 1-chain j^E solving the equation (2.50). Both in the UL and UAL cases, the equation has an obvious solution:

$$\mathsf{j}^E_{kj} = -[\mathsf{h}_k, \mathsf{h}_j], \tag{2.51}$$

which can be written using the operations on the Noether complexes as

$$\mathsf{j}^E = -\frac{1}{2}\{\mathsf{h}, \mathsf{h}\}. \tag{2.52}$$

Triviality of $H_1(\mathfrak{d}_l)$ and $H_1(\mathfrak{d}_{al})$ ensures that any other solution differs from (2.52) by an exact 1-chain.

Similarly, a continuous one-parameter symmetry of a lattice system is encoded into a charge density which, after multiplication by i, can be viewed as a 0-chain q (usually assumed to be uniformly local). The Hamiltonian H is said to be q-invariant if the derivations corresponding to q and H commute, $\{\partial \mathsf{q}, \mathsf{H}\} = 0$. If the symmetry group is compact, using the average over the group action we can always make sure that the Hamiltonian density h satisfies $\{\partial \mathsf{q}, \mathsf{h}\} = 0$. A current for the symmetry generated by q is a UL or UAL 1-chain j solving the equation

$$\sum_k [\mathsf{h}_k, \mathsf{q}_j] = -\sum_k \mathsf{j}_{kj}. \tag{2.53}$$

This equation can also be written using index-free notation:

$$\{\mathsf{H}, \mathsf{q}\} = -\partial \mathsf{j}. \tag{2.54}$$

A solution always exists because the l.h.s. is closed. Any two solutions differ by an exact 1-chain. If $\{\partial \mathsf{q}, \mathsf{h}\} = 0$, one can write an explicit solution

$$\mathsf{j} = -\{\mathsf{h}, \mathsf{q}\}. \tag{2.55}$$

From the above discussion, it is clear that all densities and currents have ambiguities. However they can be fully characterized, and one expects that physical quantities, such as transport coefficients in linear response theory, are not affected by this "gauge freedom." Transformation properties of Kubo formulas under such re-definitions of the Hamiltonian density have been analyzed in [7, 39].

2.3 Locally generated automorphisms

To describe an evolution of a lattice system in the Heisenberg picture, we need to specify a family of automorphisms $\mathrm{Aut}(\mathscr{A})$ of the algebra $\mathscr{A}$. In this section we describe how such families can be produced by integrating the derivations in $\mathfrak{D}_{al}$ or continuous one-parameter families of derivations. Such an evolution can be interpreted as an evolution by the Hamiltonian that corresponds to the derivation.

Let $C([0,1], \mathfrak{D}_{al})$ be the Fréchet space of $\mathfrak{D}_{al}$-valued functions on the interval $[0,1]$ (see Appendix A for a brief discussion of functions valued in Fréchet spaces). In Appendix B.2 we show that for any $\mathsf{G} \in C([0,1], \mathfrak{D}_{al})$ there is a unique one-parameter family of automorphisms $\alpha_{\mathsf{G}} : [0,1] \to \mathrm{Aut}(\mathscr{A})$, the value of which on $s \in [0,1]$ we denote by $\alpha_{\mathsf{G}}^{(s)}$, such that for any $\mathcal{A} \in \mathscr{A}_{a\ell}$ the curve $s \mapsto \alpha_{\mathsf{G}}^{(s)}(\mathcal{A})$ is continuously differentiable and solves the differential equation

$$\frac{d\alpha_{\mathsf{G}}^{(s)}(\mathcal{A})}{ds} = \alpha_{\mathsf{G}}^{(s)}(\mathsf{G}(s)(\mathcal{A})) \tag{2.56}$$

with the initial condition $\alpha_{\mathsf{G}}^{(0)} = \mathrm{id}$. We call such one-parameter families *locally generated paths* (LGP). One may also regard G as a component of a continuous $\mathfrak{D}_{al}$-valued 1-form $\mathsf{G}ds$ on $[0,1]$. In what follow we will not distinguish between this 1-form and the function G and for any continuous $\mathfrak{D}_{al}$-valued 1-form F denote by α_{F} a unique one-parameter family of automorphisms $\alpha_{\mathsf{F}} : [0,1] \to \mathrm{Aut}(\mathscr{A}_{a\ell})$ defined by

$$d\alpha_{\mathsf{F}}^{(s)}(\mathcal{A}) = \alpha_{\mathsf{F}}^{(s)}(\mathsf{F}(s)(\mathcal{A})). \tag{2.57}$$

We also write $\alpha_{\mathsf{F}}^{(s)}$ for a family of automorphisms generated by a constant UAL derivation $\mathsf{F} \in \mathfrak{D}_{al}$.

The map $\mathsf{G} \mapsto \alpha_{\mathsf{G}}$ from continuous 1-parameter families of UAL derivations to LGPs is clearly 1-1. This allows us to identify the set of LGPs of unit time with the Fréchet space $C([0,1],\mathfrak{D}_{al})$ and thus make the former into a Fréchet manifold. The set of such LGPs also has a group structure. The composition $\alpha_{\mathsf{F}} \circ \alpha_{\mathsf{G}}$ of two LGPs is an LGP generated by $\mathsf{G}(s) + (\alpha_{\mathsf{G}}^{(s)})^{-1}(\mathsf{F}(s))$. The inverse α_{F}^{-1} is an LGP generated by $-(\alpha_{\mathsf{F}}^{(s)})^{-1}(\mathsf{F}(s))$. By Prop. B.2, both the composition and the inverse are smooth maps of Fréchet spaces and thus the set of LGPs of unit time is a Fréchet-Lie group. Its Lie algebra is $C([0,1],\mathfrak{D}_{al})$ with the Lie bracket

$$\{\mathsf{F},\mathsf{G}\}(s) = \int_0^s (\{\mathsf{F}(u),\mathsf{G}(s)\} - \{\mathsf{G}(u),\mathsf{F}(s)\})\, du. \tag{2.58}$$

Definition 2.9. *We say that an automorphism $\beta \in \mathrm{Aut}(\mathscr{A})$ is locally generated (LGA) if there exist $\mathsf{F} \in C([0,1],\mathfrak{D}_{al})$, such that $\beta = \alpha_{\mathsf{F}}^{(1)}$.*

Remark 2.4. *It is plausible that the group of LGAs is a Fréchet-Lie group integrating the Lie algebra of UAL derivations $\mathfrak{D}_{al}$. The group of LGPs is supposed to be the group of based continuous paths in the group of LGAs.*

The action of LGAs and LGPs on observables can be extended to an action on UAL derivations and chains in a straightforward way:

$$\left(\alpha_{\mathsf{G}}^{(s)}(\mathsf{a})\right)_{j_0\ldots j_q} = \alpha_{\mathsf{G}}^{(s)}\left(\mathsf{a}_{j_0\ldots j_q}\right), \quad \mathsf{a} \in C_q(\mathfrak{d}_{al}), \tag{2.59}$$

$$\left(\alpha_{\mathsf{G}}^{(s)}(\mathsf{A})\right)^Y = \sum_{Z\in\mathbb{B}_d} \left(\alpha_{\mathsf{G}}^{(s)}\left(\mathsf{A}^Z\right)\right)^Y, \quad \mathsf{A} \in \mathfrak{D}_{al}. \tag{2.60}$$

By Proposition B.2, this action is jointly continuous and smooth.

We say that an LGA β is *f-local* for some $f \in \mathscr{F}_\infty^+$ if there exist f-local $\mathsf{F} \in C([0,1],\mathfrak{D}_{al})$ such that $\beta = \alpha_{\mathsf{F}}^{(1)}$. We say that it is *$f$-localized* on a region $A \subset \mathbb{R}^d$ if F is f-localized on A. We say that it is *almost localized* on a region $A \subset \mathbb{R}^d$ if it is f-localized on A for some $f \in \mathscr{F}_\infty^+$.

Remark 2.5. *Suppose $\mathsf{F} \in C([0,1],\mathfrak{D}_{al})$ is f-localized on a finite region so that $\mathsf{F}(s) = \mathrm{ad}_{\mathcal{A}(s)}$ for some traceless $\mathcal{A}(s) \in \mathscr{A}_{a\ell}$. There is a canonical family of unitaries $\mathcal{U}(s)$, such that $\alpha_{\mathsf{F}}^{(s)} = \mathrm{Ad}_{\mathcal{U}(s)}$. It is defined by the solution of the equation $\frac{d}{d\phi}\mathcal{U}(s) = \mathcal{U}(s)\mathcal{A}(s)$.*

Let $\mathcal{M}$ be a finite-dimensional manifold. We say that a family of LGPs of unit time $\beta_m = \alpha_{\mathsf{F}_m}$, $m \in \mathcal{M}$, is smooth, if the corresponding map $\mathsf{F} : \mathcal{M} \to C([0,1], \mathfrak{D}_{al})$ is smooth (smooth here means that derivatives of all orders exist, see Appendix A for a further discussion). This is equivalent to saying that $\mathsf{G}(s,m)$ is jointly continuous in s and m and infinitely differentiable in m. As explained in Remark B.1, to any such family of LGPs one can assign a smooth 1-form $\omega_\beta \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$ satisfying $d\left(\beta^{(1)}(\mathcal{A})\right) = \beta^{(1)}\left(\omega_\beta(\mathcal{A})\right)$ for any $\mathcal{A} \in \mathfrak{d}_{al}$. The 1-form ω_β is flat, i.e., $d\omega_\beta + \frac{1}{2}\{\omega_\beta, \omega_\beta\} = 0$.

Similarly, we can define smooth families of LGAs.

Definition 2.10. *A smooth family of LGAs β over $\mathcal{M}$ is a family of LGAs for which there exists $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$ such that for any smooth path $\gamma : [0,1] \to \mathcal{M}$ one has $\beta_{\gamma(s)} = \beta_{\gamma_0} \circ \alpha^{(s)}_{\gamma^*\mathsf{G}}$.*

Note that for a smooth family of LGAs β the corresponding G is unique and satisfies $d\mathsf{G} + \frac{1}{2}\{\mathsf{G}, \mathsf{G}\} = 0$.

2.4 States

Recall that a state of a unital C^*-algebra $\mathscr{A}$ is a positive linear functional $\psi : \mathscr{A} \to \mathbb{C}$ such that $\psi(1) = 1$, while a pure state is an extremal point of the convex set of states.

Notation 2.1. *We prefer to use a non-standard notation and denote the evaluation $\psi(\mathcal{A})$ of a state ψ on a given observable $\mathcal{A} \in \mathscr{A}$ by $\langle \mathcal{A} \rangle_\psi$ as it is common in the physics literature.*

With every state one can associate a representation that is canonical up to isomorphism using Gelfand–Naimark–Segal construction. We denote it by $(\pi_\psi, \mathcal{H}_\psi, |\psi\rangle)$ where $\mathcal{H}_\psi$ is a Hilbert space, $|\psi\rangle \in \mathcal{H}_\psi$ is a cyclic vector, $\pi_\psi : \mathcal{A} \to \mathcal{B}(\mathcal{H}_\psi)$, and call it a GNS representation. We have $\langle \mathcal{A} \rangle_\psi = \langle \psi | \pi_\psi(\mathcal{A}) | \psi \rangle$.

States of lattice systems

We say that a state ψ of a lattice system $\mathscr{A}$ is a product state, if for any $\mathcal{A}, \mathcal{B} \in \mathscr{A}_\ell$ with disjoint supports we have $\langle \mathcal{A}\mathcal{B} \rangle_\psi = \langle \mathcal{A} \rangle_\psi \langle \mathcal{B} \rangle_\psi$. We say that ψ is factorized, if it is a pure product state. For a region A, we let $\psi|_A$ be the state on $\mathscr{A}_A$ obtained by the restriction of ψ to the subalgebra $\mathscr{A}_A \subset \mathscr{A}$. A spin system $\mathscr{A}$ has a unique tracial state (or infinite temperature state) denoted $\langle \cdot \rangle_\infty$ defined by the property $\langle \mathcal{A}\mathcal{B} \rangle_\infty = \langle \mathcal{B}\mathcal{A} \rangle_\infty$. We define the average $\langle \mathsf{a} \rangle_\psi$ of an element of $\mathfrak{d}_{al}$ to be the average $\langle \mathcal{A} \rangle_\psi$ of its (unique) traceless representative $\mathcal{A} \in \mathscr{A}_{a\ell}$, $\mathsf{a} = \mathrm{ad}_{\mathcal{A}}$.

For any state ψ and an automorphism $\alpha \in \mathrm{Aut}(\mathscr{A})$, one can define a new state $\psi' = \psi \circ \alpha$ given by $\langle \mathcal{A} \rangle_{\psi'} := \langle \alpha(\mathcal{A}) \rangle_{\psi}$. We say that a derivation F *does note excite* the state ψ or *preserves* the state ψ, if $\langle \mathsf{F}(\mathcal{A}) \rangle_{\psi} = 0$ for any $\mathcal{A} \in \mathscr{A}_{\ell}$. For an observable $\mathcal{A} \in \mathscr{A}$, we say that $\mathcal{A}$ does not excite the state, if the same holds for the derivation $\mathrm{ad}_{\mathcal{A}}(\,\cdot\,) := [\mathcal{A}, \,\cdot\,]$. Similarly, for an automorphism $\alpha \in \mathrm{Aut}(\mathscr{A})$, the same notion means $\psi \circ \alpha = \psi$.

In this work we are mostly interested in states of spin systems at zero temperature, and therefore all states on the whole system will be pure. A pure state ψ is said to be a ground state of a Hamiltonian $\mathsf{H} \in \mathfrak{D}_{al}$ if for any $\mathcal{A} \in \mathscr{A}_{a\ell}$ one has $-i\langle \mathcal{A}^* \mathsf{H}(\mathcal{A}) \rangle_{\psi} \geq 0$. Any such state is necessarily invariant under the 1-parameter group of automorphisms generated by H [12].

Gapped states

One of the main objectives of the present work is the analysis of topological properties of the class of gapped states defined as follows.

Definition 2.11. *A pure state ψ is said to be a gapped ground state of $\mathsf{H} \in \mathfrak{D}_{al}$ with a gap greater or equal than $\Delta > 0$ if $-i\langle \mathcal{A}^* \mathsf{H}(\mathcal{A}) \rangle_{\psi} \geq \Delta \left(\langle \mathcal{A}^* \mathcal{A} \rangle_{\psi} - |\langle \mathcal{A} \rangle_{\psi}|^2 \right)$ for any $\mathcal{A} \in \mathscr{A}_{a\ell}$. If ψ satisfies the above condition for some unspecified choice of $\mathsf{H} \in \mathfrak{D}_{al}$ and $\Delta > 0$, we will say that ψ is gapped. We will also say that a derivation $\mathsf{H} \in \mathfrak{D}_{al}$ is gapped if there exists a state ψ which is a gapped ground state for H (such a state need not be unique).*

Warning 2.1. *The notion of a gapped state depends on the class of Hamiltonians one allows. States which are gapped for one class of Hamiltonians may not be gapped for another class with a faster decay of interactions. In this work, unless specified explicitly, by gapped state we always mean gapped for the Hamiltonians from $\mathfrak{D}_{al}$.*

Remark 2.6. *If ψ is a gapped ground state of $\mathsf{H} \in \mathfrak{D}_{al}$, then ψ is the only vector state in the GNS representation of ψ which is the ground state of H. Further, if $\hat{H}$ is the generator of the 1-parameter group of automorphisms $\alpha_{\mathsf{H}}(s)$ in the GNS representation of ψ, then $\hat{H}$ is a positive operator annihilating the vacuum vector, and its spectrum on the orthogonal complement of the vacuum vector is contained in $[\Delta, +\infty)$.*

Topological phases

Definition 2.12. *We say that two states ψ and ψ' of a lattice system $\mathscr{A}$ are LGA-equivalent, if there is an LGA $\alpha \in \mathrm{Aut}(\mathscr{A})$, such that $\psi = \psi' \circ \alpha$.*

Given states ψ and ψ' of lattice systems $\mathscr{A}$ and $\mathscr{A}'$, respectively, one has a natural state of the composite systems $\mathscr{A} \otimes \mathscr{A}'$ denoted $\psi \otimes \psi'$.

Definition 2.13. *We say that two states ψ and ψ' of possibly different lattice systems $\mathscr{A}$ and $\mathscr{A}'$ are stably LGA-equivalent if there are lattice systems $\tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}}'$ satisfying $\mathscr{A} \otimes \mathscr{A} \cong \mathscr{A}' \otimes \tilde{\mathscr{A}}'$, a factorized states ψ_0 of $\tilde{\mathscr{A}}$, a factorized states ψ'_0 of $\tilde{\mathscr{A}}'$ such that $\psi \otimes \psi_0$ is LGA-equivalent to $\psi' \otimes \psi'_0$.*

Being in the same phase defines an equivalence relation on states of lattice systems. If two states are stably LGA-equivalent we say that they are in the same *topological phase* or simply *phase*. A monoidal structure on states of spin systems induces a monoid structure the set of phases.

Warning 2.2. *While it is reasonable to say that two stably LGA-equivalent states model physical systems in the same topological phase, we do not intend to say that it is a complete definition, as there might be two states which are not stably LGA-equivalent, but describe systems which are equivalent from physical perspective.*

In the following by *topological invariants* of states we mean properties which are the same for any two stably LGA-equivalent states.

The notion of a topological phase can be generalized to the case of systems with a Lie group symmetry G in a straightforward way. We say that two G-invariant states are in the same *G-invariant phase*, if they are stably LGA-equivalent and the corresponding LGA can be generated by a G-invariant path of UAL derivations.

Invertible phases

There is a natural class of states representing the invertible part of the monoid of phases introduced by Kitaev [41] that forms an abelian group.

Definition 2.14. *Let ψ be a pure state of a lattice system $\mathscr{A}$. We say that ψ is invertible, if there is another system $\mathscr{A}'$ with a pure state ψ and an LGA $\alpha \in \mathrm{Aut}(\mathscr{A} \otimes \mathscr{A}')$, such that the state $(\psi \otimes \psi') \circ \alpha$ is factorized. In this case, we say that the state ψ' is an inverse of ψ.*

Given a state $\psi\otimes\psi'$ of a composite system $\mathscr{A}\otimes\mathscr{A}'$ and an observable $\mathcal{A}\in\mathscr{A}\otimes\mathscr{A}'$, one can define a partial average $\langle\mathcal{A}\rangle_{\psi'}\in\mathscr{A}$ that on observables of the form $\mathcal{O}\otimes\mathcal{O}'\in\mathscr{A}\otimes\mathscr{A}'$ is given by $\langle\mathcal{O}\otimes\mathcal{O}'\rangle_{\psi'}=\langle\mathcal{O}'\rangle_{\psi'}\mathcal{O}$. The value on general observables is obtained by the linear extension. Note that $\|\langle\mathcal{A}\rangle_{\psi'}\|\leq\|\mathcal{A}\|$. In particular, for $\mathcal{A}\in\mathscr{A}_{a\ell}$ we have

$$f_j(\langle\mathcal{A}\rangle_{\psi'},r)=\inf_{\mathcal{B}\in\mathscr{A}_{B_j(r)}}\|\mathcal{B}-\langle\mathcal{A}\rangle_{\psi'}\|\leq\inf_{\mathcal{B}\in(\mathscr{A}\otimes\mathscr{A}')_{B_j(r)}}\|\langle\mathcal{B}-\mathcal{A}\rangle_{\psi'}\|\leq$$
$$\leq\inf_{\mathcal{B}\in(\mathscr{A}\otimes\mathscr{A}')_{B_j(r)}}\|\mathcal{B}-\mathcal{A}\|=f_j(\mathcal{A},r). \quad (2.61)$$

Hence, if $\mathcal{A}$ is g-localized at j, then so is $\langle\mathcal{A}\rangle_{\psi'}$. Using this fact we can extend the partial averaging to $C_q(\mathfrak{d}_{al})$, and since any $\mathsf{F}\in\mathfrak{D}_{al}$ can be represented as $\partial\mathsf{f}$ for some $\mathsf{f}\in C_0(\mathfrak{d}_{al})$, we can also extend the partial averaging to $\mathfrak{D}_{al}$.

Proposition 2.1. *An invertible state ψ is gapped.*

Proof. Let ψ' be a inverse of ψ, and let α be an LGA of the composite system, such that $\Psi_0=(\psi\otimes\psi')\circ\alpha$ is factorized. Let us choose UAL Hamiltonian F for the composite system such that Ψ_0 is a gapped ground state for F with a gap greater than $\Delta>0$ (we can choose F to be $\partial\mathsf{f}$ for an on-site 0-chain f). Then $(\psi\otimes\psi')$ is a gapped ground state for $\alpha(\mathsf{F})$. Let H be a UAL Hamiltonian obtained from $\alpha(\mathsf{F})$ by partial averaging over ψ'. Then for any $\mathcal{A}\in\mathscr{A}$ we have

$$-i\langle\mathcal{A}^*\mathsf{H}(\mathcal{A})\rangle_{\psi}=-i\langle\alpha(\mathcal{B}^*)\alpha(\mathsf{F}(\mathcal{B}))\rangle_{\psi\otimes\psi'}=-i\langle\mathcal{B}^*\mathsf{F}(\mathcal{B})\rangle_{\Psi_0}\geq$$
$$\geq\Delta\left(\langle\mathcal{B}^*\mathcal{B}\rangle_{\Psi_0}-|\langle\mathcal{B}\rangle_{\Psi_0}|^2\right)=\Delta\left(\langle\mathcal{A}^*\mathcal{A}\rangle_{\psi}-|\langle\mathcal{A}\rangle_{\psi}|^2\right) \quad (2.62)$$

where $\mathcal{B}=\alpha^{-1}(\mathcal{A}\otimes 1)$. Thus, ψ is a gapped ground state for H with a gap greater than $\Delta>0$. □

Chapter 3

BERRY CLASSES FOR GAPPED STATES OF SPIN SYSTEMS

A bosonic system without any locality can be modeled by the algebra of bounded operators on a Hilbert space $\mathcal{H}$. Pure states of the system correspond to self-adjoint rank-1 projectors P in $\mathcal{B}(\mathcal{H})$ that defines a line in the Hilbert space. All such projectors can be continuously (in the natural topology) deformed into each other, that can be interpreted as the absence of non-trivial gapped phases of such systems. However, the space of all such projectors is not contractible as it has non-trivial second homotopy group. In particular, given a smooth family of self-adjoint projectors (that defines a smooth family of states) parameterized by a smooth compact manifold $\mathcal{M}$, we can construct a 2-form $f = \mathrm{Tr}(PdPdP) \in \Omega^2(\mathcal{M}, i\mathbb{R})$ that gives the celebrated Berry curvature. A non-triviality of the cohomology class $[f/(2\pi i)] \in H^2(\mathcal{M}, \mathbb{Z})$, which we call the Berry class, gives an obstruction to the contractibility of such a family.

One may wonder if analogous invariants exist for many-body systems with locality. As the Berry curvature is extensive with respect to the system size, its value diverges in the thermodynamic limit and it is unclear how to make sense of it for a microscopic model without imposing any translational symmetry[1].

The physics heuristic, based on the assumption that at long distances a gapped system can be effectively described by a field theory, suggests that with every smooth family of systems, one can associate a de Rham cohomology class taking values in $H^{d+2}(\mathcal{M}, \mathbb{R})$. Indeed, for a $(d+1)$-dimensional effective field theory on a manifold Σ, one can write down a topological term in the effective action that has the form $\int_B f^*\omega$, where ω is a closed $(d+2)$-form on the space of parameters $\mathcal{X}$, B is a manifold such that $\partial B = \Sigma$ and f is a map $f : B \to \mathcal{X}$. If the systems have a Lie group symmetry, one can also use gauge fields to construct topological terms. In particular, one can write down the Chern-Simons topological terms for any even d. All such topological terms are characterized by the equivariant cohomology $H_G^{d+2}(\mathcal{M}, \mathbb{R})$ which may be non-trivial even if $\mathcal{M}$ is a point. In particular, by Chern-Weil theory, the space of Chern-Simons forms of degree $2k+1$ is $H^{2k+2}(BG, \mathbb{R})$ that agrees with $H_G^4(\mathrm{pt}, \mathbb{R})$.

[1]If the lattice has a translational symmetry with the fundamental domain of a finite volume (e.g., $\mathbb{Z}^d$ symmetry of a cubic lattice on $\mathbb{R}^d$) and if each state of the family is invariant under it, one can talk about the Berry curvature per unit cell.

The goal of this chapter is to define topological invariants of families of gapped states of spin system (possibly with an internal symmetry described by a connected Lie group G) which take values in $H^{d+2}_G(\mathcal{M},\mathbb{R})$ and are obtained by solving a suitable Maurer-Cartan equation. We call them *G-equivariant Berry classes*. We show that the usual Berry class, the Hall conductance, and the Thouless pump invariants are all particular instances of this invariant. More generally, when $\mathcal{M} = \mathrm{pt}$, the equivariant cohomology can be identified with the space of G-invariant polynomials on the Lie algebra $\mathfrak{g}$ of G. Thus, for any even d, our construction attaches a G-invariant polynomial of degree $(d+2)/2$ to a G-invariant gapped state in d dimensions. This is in agreement with the TQFT approach, which predicts that topological invariants of such systems should take values in the space of Chern-Simons forms of degree $d+1$. It is well known that the latter arise from invariant polynomials on the Lie algebra.

We start with formulating a condition on a state that allows us to construct the invariant in Section 3.1. It can be stated as an exactness of the subcomplex of the Noether complex that consists of derivations and chains preserving the state. We show that it is satisfied for any gapped state. In Section 3.2, we give a definition of a smooth family of states based on the idea of quasi-adiabatic evolution [28] and introduce a differential graded Fréchet-Lie algebra of differential forms taking values in derivations and chains associated with it. We use this algebra to define higher Berry classes and their equivariant analogues in Sections 3.3 and 3.4, respectively. We describe various examples at the end of the chapter.

3.1 Complexes associated to gapped states

Given a pure state ψ, one can study the algebras of observables, Hamiltonians or currents which preserve it. Recall that a UAL derivation F does not excite ψ if for any $\mathcal{A} \in \mathfrak{d}_{al}$ one has $\langle \mathsf{F}(\mathcal{A})\rangle_\psi = 0$. Such derivations form a Lie sub-algebra in $\mathfrak{D}_{al}$ which we denote $\mathfrak{D}^\psi_{al}$. Similarly, the Lie sub-algebra $\mathfrak{d}^\psi_{al} \subset \mathfrak{d}_{al}$ consists of $\mathcal{B} \in \mathfrak{d}_{al}$ such that $\langle [\mathcal{B},\mathcal{A}]\rangle_\psi = 0$ for all $\mathcal{A} \in \mathfrak{d}_{al}$. We also define a subcomplex $C_\bullet(\mathfrak{d}^\psi_{al}) \hookrightarrow C_\bullet(\mathfrak{d}_{al})$ of chains that do not excite ψ and the complex

$$\dots \xrightarrow{\partial_2} C_1(\mathfrak{d}^\psi_{al}) \xrightarrow{\partial_1} C_0(\mathfrak{d}^\psi_{al}) \xrightarrow{\partial_0} \mathfrak{D}^\psi_{al} \to 0 \tag{3.1}$$

of chains and derivations that do not excite ψ. The latter is a sub-complex of the UAL Noether complex $\mathcal{N}_\bullet$. We will denote it $\mathcal{N}^\psi_\bullet$. The space of k-cycles of $\mathcal{N}^\psi$ will be denoted $Z_k(\mathcal{N}^\psi)$. As usual, the k-th homology group of the complex is the quotient of the space of k-cycles by the subspace of k-boundaries.

Our first goal of this chapter is to prove the following:

Theorem 3.1 (Local Noether theorem for gapped states)**.** *Let ψ be a gapped state of a spin system. Then the homology $H_\bullet(\mathcal{N}^\psi)$ is trivial. Moreover, there is a continuous map $h_k^\psi : Z_k(\mathcal{N}^\psi) \to \mathcal{N}_{k+1}^\psi$ such that $\partial_{k+1} \circ h_k^\psi = \mathrm{Id}$.*

As we will see later, the exactness of the complex of state-preserving currents provides a natural way to construct invariants of states and families of states.

To analyze the homology of complex $\mathcal{N}^\psi$ we will make use of certain linear maps defined by means of integral transforms. For any $\mathsf{H} \in \mathfrak{D}_{al}$ and a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(t) = \mathcal{O}(|t|^{-\infty})$ we define a map $\mathscr{I}_{\mathsf{H},f} : \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$ by

$$\mathscr{I}_{\mathsf{H},f}(\cdot) := \int_{-\infty}^{+\infty} f(t)\alpha_{\mathsf{H}}^{(t)}(\cdot)dt. \tag{3.2}$$

It is shown in Appendix A.5 that $\mathscr{I}_{\mathsf{H},f}$ is well-defined and continuous. More precisely, for any a-localized $\mathcal{A} \in \mathfrak{d}_{al}$ we have

$$\|\mathscr{I}_{\mathsf{H},f}(\mathcal{A})\|_{j,\alpha} \leq C_\alpha \|\mathcal{A}\|_{j,\alpha}, \quad \alpha \in \mathbb{N}_0 \tag{3.3}$$

where $C_\alpha > 0$ depends on H, f and a. This estimate implies that $\mathscr{I}_{\mathsf{H},f}$ is continuous but is stronger than continuity because C_α does not depend on j. In other words, the map $\mathscr{I}_{\mathsf{H},f}$ is equicontinuous w. r. to a family of metrics on $\mathfrak{d}_{al}$ labeled by j. This ensures that $\mathscr{I}_{\mathsf{H},f}$ extend to continuous chain maps $\mathscr{I}_{\mathsf{H},f} : \mathcal{N}_\bullet \to \mathcal{N}_\bullet$. On k-chains with $k \geq 0$ it is defined by

$$\mathscr{I}_{\mathsf{H},f}(\mathsf{a})_{j_0 \ldots j_k} = \mathscr{I}_{\mathsf{H},f}(\mathsf{a}_{j_0 \ldots j_k}), \tag{3.4}$$

while on derivations it is defined by

$$\mathscr{I}_{\mathsf{H},f}(\mathsf{A})^Y = \sum_Z \left(\mathscr{I}_{\mathsf{H},f}(\mathsf{A}^Z)\right)^Y. \tag{3.5}$$

If $\alpha_{\mathsf{H}}^{(t)}$ preserves a state ψ, then $\mathscr{I}_{\mathsf{H},f}$ preserves the subspace $\mathfrak{d}_{al}^\psi$ and the subcomplex $\mathcal{N}^\psi$. Indeed, for any $\mathcal{A}, \mathcal{B} \in \mathfrak{d}_{al}$ we have

$$\langle [\mathscr{I}_{\mathsf{H},f}(\mathcal{A}), \mathcal{B}] \rangle_\psi = \int_{-\infty}^{+\infty} f(t) \langle [\mathcal{A}, \alpha_{\mathsf{H}}^{(-t)}(\mathcal{B})] \rangle_\psi dt. \tag{3.6}$$

Therefore if $\mathcal{A} \in \mathfrak{d}_{al}^\psi$, then $\mathscr{I}_{\mathsf{H},f}(\mathcal{A}) \in \mathfrak{d}_{al}^\psi$.

If ψ is a gapped ground state of H, then with a clever choice of f a stronger result holds.

Lemma 3.1. *Let ψ be a gapped ground state of $\mathsf{H} \in \mathfrak{D}_{al}$ with a gap greater or equal than $\Delta > 0$. Let $w_\Delta(t)$ be an even continuous function $w_\Delta(t) = \mathcal{O}(|t|^{-\infty})$ satisfying $\int w_\Delta(t)e^{-i\omega t}dt = 0$ for $|\omega| > \Delta'$ for some $0 < \Delta' < \Delta$ and $\int w_\Delta(t)dt = 1$.* [2] *Then for any $\mathcal{A} \in \mathfrak{d}_{al}$ and $\mathcal{B} \in \mathfrak{d}_{al}$ we have*

$$\langle \mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A})\mathcal{B}\rangle_\psi = \langle \mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A})\rangle_\psi \langle \mathcal{B}\rangle_\psi, \tag{3.7}$$

and thus $\mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A}) \in \mathfrak{d}^\psi_{al}$ and $\mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{N}_\bullet) \subseteq \mathcal{N}^\psi_\bullet$.

Proof. Let π_ψ be the GNS representation corresponding to ψ and dP_ω, $\omega \in \mathbb{R}$, be the projection-valued measure on $\mathbb{R}$ corresponding to the self-adjoint operator $\hat{H}$. Then

$$\begin{aligned}\langle \mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A})\mathcal{B}\rangle_\psi &= \int_{-\infty}^{+\infty} d\omega \int_{-\infty}^{+\infty} dt\, w_\Delta(t)e^{-i\omega t}\langle 0|\pi_\psi(\mathcal{A})dP_\omega \pi_\psi(\mathcal{B})|0\rangle = \\ &= \int_{-\Delta'}^{+\Delta'} d\omega \int_{-\infty}^{+\infty} dt\, w_\Delta(t)e^{-i\omega t}\langle 0|\pi_\psi(\mathcal{A})dP_\omega \pi_\psi(\mathcal{B})|0\rangle = \\ &= \int_{-\infty}^{+\infty} dt\, w_\Delta(t)\langle \mathcal{A}\rangle_\psi \langle \mathcal{B}\rangle_\psi = \langle \mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A})\rangle_\psi \langle \mathcal{B}\rangle_\psi.\end{aligned} \tag{3.8}$$

□

Using this lemma, one easily obtains the following result.

Lemma 3.2. *For any $\mathsf{H} \in \mathfrak{D}_{al}$ with a gapped ground state ψ there exists $\mathsf{h}^\psi \in C_0(\mathfrak{d}^\psi_{al})$ such that $\mathsf{H} = \partial \mathsf{h}^\psi$.*

Proof. Let $w_\Delta = \mathcal{O}(|t|^{-\infty})$ be a function as in the statement of Lemma 3.1, and suppose $\mathsf{H} = \partial \mathsf{h}$ for some $\mathsf{h} \in C_0(\mathfrak{d}_{al})$. Let $\mathsf{h}^\psi = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathsf{h})$. Then for any $\mathcal{A} \in \mathscr{A}_{a\ell}$ we have

$$\partial \mathsf{h}^\psi(\mathcal{A}) = \int_{-\infty}^{+\infty} w_\Delta(t)\alpha^{(t)}_{\mathsf{H}}\mathsf{H}(\alpha^{(-t)}_{\mathsf{H}}(\mathcal{A}))dt = \int_{-\infty}^{+\infty} w_\Delta(t)\mathsf{H}(\mathcal{A})dt = \mathsf{H}(\mathcal{A}). \tag{3.9}$$

□

Remark 3.1. *The construction of h^ψ by means of the map $\mathscr{I}_{\mathsf{H},w_\Delta}$ is due to Kitaev [40].*

Another interesting choice for f is described in the next lemma.

[2] Such functions exist [26, 4]. See Lemma 2.3 from [4] for an explicit example.

Lemma 3.3. *Let ψ be a gapped ground state of $\mathsf{H} \in \mathfrak{D}_{al}$. Let $W_\Delta(t) = \mathcal{O}(|t|^{-\infty})$ be an odd piecewise-continuous function defined for $t > 0$ by $W_\Delta(|t|) = -\int_{|t|}^{\infty} w_\Delta(s)ds$. Then for any $\mathcal{A} \in \mathfrak{d}_{al}$ we have*

$$\mathcal{A} - \mathscr{I}_{\mathsf{H},W_\Delta}(\mathsf{H}(\mathcal{A})) = \mathscr{I}_{\mathsf{H},w_\Delta}(\mathcal{A}). \tag{3.10}$$

Proof. Straightforward computation. □

Remark 3.2. *The map $\mathscr{I}_{\mathsf{H},W_\Delta}$ first appeared in [55].*

In what follows we will use a shorthand t^ψ for the chain map $\mathscr{I}_{\mathsf{H},w_\Delta} : \mathcal{N}_\bullet \to \mathcal{N}_\bullet$ with w_Δ chosen as in Lemma 3.1 and a shorthand $\mathcal{J}^\psi$ for the chain map $\mathscr{I}_{\mathsf{H},W_\Delta} : \mathcal{N}_\bullet \to \mathcal{N}_\bullet$ with W_Δ chosen as in Lemma 3.3. These chain maps preserve the subcomplex $\mathcal{N}^\psi_\bullet$. In addition t^ψ maps $\mathcal{N}_\bullet$ to $\mathcal{N}^\psi_\bullet$. Finally, they satisfy an identity

$$\mathsf{a} - \mathcal{J}^\psi(\{\mathsf{H},\mathsf{a}\}) = t^\psi(\mathsf{a}), \quad \forall \mathsf{a} \in \mathcal{N}_\bullet. \tag{3.11}$$

We are now ready to prove Theorem 3.1.

Proof of Theorem 3.1. Let $\mathsf{H} \in \mathfrak{D}_{al}$ be a gapped Hamiltonian for ψ. By Lemma 3.2, there exists $\mathsf{h}^\psi \in C_0(\mathfrak{d}^\psi_{al})$ such that $\partial \mathsf{h}^\psi = \mathsf{H}$. For any $\mathsf{f} \in \mathcal{N}_\bullet$, let

$$s^\psi(\mathsf{f}) = \mathcal{J}^\psi\left(\{\mathsf{h}^\psi, \mathsf{f}\}\right). \tag{3.12}$$

s^ψ is a continuous linear map $\mathcal{N}_k \to \mathcal{N}_{k+1}$ which maps $\mathcal{N}^\psi_k$ to $\mathcal{N}^\psi_{k+1}$. Now suppose $\mathsf{a} \in Z_k(\mathcal{N}^\psi)$. By Prop. A.7 $\mathsf{a} = \partial\mathsf{b}$, where $\mathsf{b} = h(\mathsf{a}) \in \mathcal{N}_{k+1}$ is a continuous linear function of a. The identity (3.11) implies

$$\mathsf{a} = \partial t^\psi(\mathsf{b}) + \partial s^\psi(\mathsf{a}) = \partial(t^\psi \circ h(\mathsf{a}) + s^\psi(\mathsf{a})). \tag{3.13}$$

Therefore we can set $h^\psi = t^\psi \circ h + s^\psi$. □

Remark 3.3. *The vanishing of $H_\bullet(\mathcal{N}^\psi)$ can also be explained as follows. Eq. (3.11) implies that s^ψ induces a contracting homotopy on the quotient complex $\mathcal{N}_\bullet/\mathcal{N}^\psi_\bullet$, therefore the homology of $\mathcal{N}_\bullet/\mathcal{N}^\psi_\bullet$ is trivial. The homology of the UAL Noether complex is also trivial. Therefore the long exact sequence of homology groups (in the category of vector spaces) corresponding to the short exact sequence*

$$0 \to \mathcal{N}^\psi_\bullet \to \mathcal{N}_\bullet \to \mathcal{N}_\bullet/\mathcal{N}^\psi_\bullet \to 0 \tag{3.14}$$

implies that $H_\bullet(\mathcal{N}^\psi)$ is trivial. However, for our purposes it is important to know that a continuous linear map h^ψ inverting ∂ exists.

In the following sections we will define invariants of gapped states and smooth families of gapped states under LGA-equivalence. The definition depends exclusively on the existence of a derivation $\mathsf{H} \in \mathfrak{D}^{\psi}_{al}$ and a map $\mathcal{J}^{\psi}$ with the properties described above[3].

3.2 Smooth families of states of spin systems

Let $\mathcal{M}$ be a compact connected smooth manifold. Consider a family of states ψ parameterized by $\mathcal{M}$. We denote a state at the point $m \in \mathcal{M}$ by $\psi_m : \mathscr{A} \to \mathbb{C}$. The averaging over states defines a function $\langle \cdot \rangle_{\psi} : \mathcal{M} \to \mathbb{C}$.

Definition 3.1. *A smooth family of states ψ over $\mathcal{M}$ is a family of states for which there exists $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$ such that for any smooth path $\gamma : [0,1] \to \mathcal{M}$ one has $\psi_{\gamma(s)} = \psi_{\gamma(0)} \circ \alpha^{(s)}_{\gamma^*\mathsf{G}}$.*

Remark 3.4. *As explained in Appendix A.5, the result of A. Moon and Y. Ogata [48] implies that for a smooth family of gapped UL Hamiltonians $\mathsf{H} \in \Omega^0(\mathcal{M}, \mathfrak{D}_{al})$ with a unique gapped ground state ψ_m $\forall m \in \mathcal{M}$ and such that the function $\mathcal{M} \mapsto \langle \mathcal{A} \rangle_{\psi}$ is smooth for all $\mathcal{A} \in \mathscr{A}_{a\ell}$, the family ψ is smooth in the sense of Definition 3.1.*

Proposition 3.1. *A family of states ψ is smooth if and only if for any observable $\mathcal{A} \in \mathscr{A}_{a\ell}$ the function $\mathcal{M} \mapsto \langle \mathcal{A} \rangle_{\psi}$ is smooth and satisfies*

$$d\langle \mathcal{A} \rangle_{\psi} = \langle \mathsf{G}(\mathcal{A}) \rangle_{\psi} \tag{3.15}$$

for some $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$. That is, a family of states ψ is smooth iff it is parallel with respect to the connection $d + \mathsf{G}$ on the trivial vector bundle with fiber $\mathscr{A}_{a\ell}$.

Proof. Pick a point $m_0 \in \mathcal{M}$. For any $m \in \mathcal{M}$ there is a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = m_0$, $\gamma(1) = m$, and by assumption $\langle \mathcal{A} \rangle_{\psi_{\gamma(s)}} = \langle \alpha^{(s)}_{\gamma^*\mathsf{G}}(\mathcal{A}) \rangle_{\psi_{m_0}}$. By Prop. B.1 this function is smooth and its derivative at $s = 1$ is $\langle \gamma^*(1)\mathsf{G}(\mathcal{A}) \rangle_{\psi_m}$. This implies (3.15). Conversely, eq. (3.15) implies that for any such path γ and any $\mathcal{A} \in \mathscr{A}_{a\ell}$ the function $s \mapsto \langle (\alpha^{(s)}_{\gamma^*\mathsf{G}})^{-1}(\mathcal{A}) \rangle_{\psi_{\gamma(s)}}$ is constant. Therefore $\psi_{\gamma(s)} = \psi_{\gamma(0)} \circ \alpha^{(s)}_{\gamma^*\mathsf{G}}$. □

Corollary 3.1. *Let $\{\mathcal{U}^{(\mathsf{a})}\}$ be an open cover of $\mathcal{M}$. A family of states ψ over $\mathcal{M}$ is smooth if and only if it is smooth over each element of the cover.*

[3]The existence of $\mathcal{J}^{\psi}$ roughly means that any restriction of an arbitrary state-preserving Hamiltonian to a region (that in general does not preserve the state) can be modified at the boundary of the region, so that the resulting Hamiltonian still preserves the state. A similar property has been considered in [3].

Proof. The only if direction is obvious. Now suppose one is given a $\mathfrak{D}_{al}$-valued 1-form $\mathsf{G}^{(\mathsf{a})}$ on each $\mathcal{U}^{(\mathsf{a})}$ such that ψ is parallel with respect to each of them. Then we can construct G on $\mathcal{M}$ with respect to which ψ is parallel using a partition of unity for some open cover subordinate to $\{\mathcal{U}^{(\mathsf{a})}\}$. □

If every element of the cover $\{\mathcal{U}^{(\mathsf{a})}\}$ is smoothly contractible, one can also describe a smooth family of states using locally-defined smooth families of LGPs.

Proposition 3.2. *Let $\{\mathcal{U}^{(\mathsf{a})}\}$ be a finite cover such that each element is smoothly contractible. A family $\psi_m, m \in \mathcal{M}$, is smooth if and only if there is a state ψ_0 and smooth families $(\beta^{(\mathsf{a})})_m$, $m \in \mathcal{U}^{(\mathsf{a})}$, of LGPs such that $\psi_m = \psi_0 \circ (\beta^{(\mathsf{a})})^{(1)}_m$ for any $m \in \mathcal{U}^{(\mathsf{a})}$ and any* a.

Proof. Suppose ψ_m is parallel with respect to $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$. We pick a point $m_0 \in \mathcal{M}$ and let $\psi_0 = \psi_{m_0}$. For every a let $\gamma^{(\mathsf{a})} : \mathcal{U}^{(\mathsf{a})} \times [0,1] \to \mathcal{M}$ be a smooth homotopy between the map of $\mathcal{U}^{(\mathsf{a})}$ to m_0 and the identity map. We regard it as a smooth family of paths in $\mathcal{M}$ based at m_0 and labeled by m and let $(\beta^{(\mathsf{a})})_m := \alpha_{\gamma_m^{(\mathsf{a})*}\mathsf{G}}$. In the opposite direction, given families of LGPs $\beta^{(\mathsf{a})}$ we can define $\mathsf{G} = \sum_{\mathsf{a}} f^{(\mathsf{a})}\omega^{(\mathsf{a})}$, where $\omega^{(\mathsf{a})} := \omega_{\beta^{(\mathsf{a})}}$ and $f^{(\mathsf{a})}$ is a partition of unity. Since the family of states ψ is parallel with respect to the connection $d + \omega^{(\mathsf{a})}$ on each $\mathcal{U}^{(\mathsf{a})}$, it is also parallel with respect to $d + \mathsf{G}$. □

Given a smooth family of states $\psi_m, m \in \mathcal{M}$, one may ask whether it is possible to choose a globally defined smooth family of LGPs β_m, $m \in \mathcal{M}$, such that $\psi_m = \psi_0 \circ \beta_m$. Since the space of LGPs is smoothly contractible, this is possible only if the family of states is smoothly homotopic to a constant family. For $d = 0$, when the algebra $\mathscr{A}_{a\ell}$ is simply the algebra of operators on a finite-dimensional Hilbert space $\mathcal{H}$, it is well known that not all families of states are homotopic to a constant family. Indeed, for $d = 0$ a smooth family of states is the same as a smooth line bundle $\mathcal{L}$ over $\mathcal{M}$. If the 1st Chern class of this line bundle is non-trivial, the family is not homotopic to a constant one. If ψ_m is a ground state of a family of gapped Hamiltonians parameterized by $\mathcal{M}$, it is the cohomology class of the Berry curvature which provides an obstruction for the existence of a globally defined family of LGPs. Below we will construct similar obstructions ("higher Berry classes") for smooth families of gapped states with $d > 0$. This includes smooth families of ground states of gapped finite-range Hamiltonians.

To construct these obstructions we will use a bi-complex of differential forms taking values in the complex $\mathcal{N} = \left(C_\bullet(\mathfrak{d}_{al}) \xrightarrow{\partial} \mathfrak{D}_{al}\right)$:

$$\ldots \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_1(\mathfrak{d}_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_0(\mathfrak{d}_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, \mathfrak{D}_{al}) \xrightarrow{\partial} 0, \tag{3.16}$$

with the second differential being the de Rham differential d. Itt will be convenient to use a cohomological rather than homological grading on $\mathcal{N}$ and accordingly denote its degree k-component by $\mathcal{N}^k$. Then d has bi-degree $(1,0)$, while ∂ has bi-degree $(0,1)$. We will also shift the grading on $\mathcal{N}$ so that $\mathfrak{D}_{al}$ sits in degree 0 and the bracket has degree 0. Then $\mathcal{N}^\bullet$ becomes a defferential graded Fréchet-Liet algebra (DGFLA).

The bi-complex (3.16) is a graded tensor product of the DGFLA $\mathcal{N}^\bullet$ and the supercommutative algebra $\Omega^\bullet(\mathcal{M}, \mathbb{R})$. Therefore it has a natural graded bracket which makes it into a graded Lie algebra. Explicitly, the bracket of decomposable elements $\mathsf{g} \otimes \phi$ and $\mathsf{h} \otimes \theta$, $\mathsf{g}, \mathsf{h} \in \mathcal{N}^\bullet$, $\phi, \theta \in \Omega^\bullet(\mathcal{M}, \mathbb{R})$, is

$$[\mathsf{g} \otimes \phi, \mathsf{h} \otimes \theta] = (-1)^{|\phi||\mathsf{h}|}[\mathsf{g}, \mathsf{h}] \otimes (\phi \wedge \theta). \tag{3.17}$$

The space $\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet)$ equipped with the total differential $d + \partial$ and the bracket is a DGFLA.

When $\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet)$ is regarded as a complex with respect to ∂, it has a sub-complex consisting of chains and derivations preserving ψ pointwise on $\mathcal{M}$:

$$\ldots \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_1(\mathfrak{d}^\psi_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, C_0(\mathfrak{d}^\psi_{al})) \xrightarrow{\partial} \Omega^\bullet(\mathcal{M}, \mathfrak{D}^\psi_{al}) \xrightarrow{\partial} 0. \tag{3.18}$$

We will denote it $\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet_\psi)$. This sub-complex is not preserved by d. But if the family $(\mathcal{M}, \psi)$ is parallel with respect to $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$, then it is preserved by a "covariant differential" $D := d + \{\mathsf{G}, \cdot\}$. Indeed, for any $\mathsf{b} \in \Omega^m(\mathcal{M}, \mathcal{N}^q_\psi)$ and any $\mathcal{A} \in \mathscr{A}_{a\ell}$ we have

$$\langle\{D\mathsf{b}, \mathcal{A}\}\rangle_\psi = \langle D\{\mathsf{b}, \mathcal{A}\}\rangle_\psi + (-1)^{q+m}\langle\{\mathsf{b}, D\mathcal{A}\}\rangle_\psi = d\langle\{\mathsf{b}, \mathcal{A}\}\rangle_\psi = 0. \tag{3.19}$$

D has bi-degree $(1,0)$ and supercommutes with ∂ provided we flip the sign of ∂ on odd-degree forms. It also satisfies $D^2 = \{\mathsf{F}, \cdot\}$, where the "curvature" $\mathsf{F} := d\mathsf{G} + \frac{1}{2}\{\mathsf{G}, \mathsf{G}\} \in \Omega^2(\mathcal{M}, \mathfrak{D}_{al})$ is covariantly constant: $D\mathsf{F} = 0$. In addition,

$$\langle \mathsf{F}(\mathcal{A})\rangle_\psi = \langle D^2\mathcal{A}\rangle_\psi = d^2\langle\mathcal{A}\rangle_\psi = 0. \tag{3.20}$$

Therefore $\mathsf{F} \in \Omega^2(\mathcal{M}, \mathfrak{D}^\psi_{al})$.

Remark 3.5. *The above properties of D and F mean that $\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet_\psi)$ equipped with $D + \partial$ is a curved differential graded Lie algebra (CDGLA) with curvature F, while the inclusion of $(\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet_\psi), D + \partial)$ into $(\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet), d + \partial)$ is a morphism of CDGLAs.*

Theorem 3.2. *Let $(\mathcal{M}, \psi)$ be a smooth family of gapped states parameterized by a compact connected manifold $\mathcal{M}$. Then the complex (3.18) is exact with respect to ∂. Moreover, there is a continuous linear map $h^\psi_\mathcal{M}$ from $\Omega^\bullet(\mathcal{M}, Z^k(\mathcal{N}_\psi))$ to $\Omega^\bullet(\mathcal{M}, \mathcal{N}^{k-1}_\psi)$ which is right inverse to ∂.*

Remark 3.6. *Since all states in a smooth family are related by LGAs, to check whether a family is gapped it is sufficient to check whether any particular state in the family is gapped.*

Proof of Theorem 3.2. Let us fix a finite smoothly contractible cover $\{\mathcal{U}^{(\mathsf{a})}\}$ and families of LGPs $\beta^{(\mathsf{a})}$, such that $\psi_m = \psi_0 \circ (\beta^{(\mathsf{a})})^{(1)}_m$ for some ψ_0. Since ψ_0 is gapped, the complex (3.18) becomes exact if we replace the family ψ_m with the constant family ψ_0. The corresponding right inverse to ∂ is the map h^{ψ_0} from Theorem 3.1. Then the right inverse to ∂ for the restriction of (3.18) to $\mathcal{U}^{(\mathsf{a})}$ is $h^\psi_{\mathcal{U}^{(\mathsf{a})}} = (\beta^{(\mathsf{a})})^{(1)} \circ h^{\psi_0} \circ ((\beta^{(\mathsf{a})})^{(1)})^{-1}$. Then picking a partition of unity $f^{(\mathsf{a})}$ subordinate to the cover we get a right inverse to ∂ on the whole $\mathcal{M}$ by letting $h^\psi_\mathcal{M} = \sum_\mathsf{a} f^{(\mathsf{a})} h^\psi_{\mathcal{U}^{(\mathsf{a})}}$.

□

3.3 Higher Berry classes

For $d = 0$, we have $\mathfrak{D}_{al} = \mathfrak{d}_{al}$. A smooth family of pure states can be identified with a smooth family of self-adjoint rank-1 projectors P in $\mathscr{A}_{a\ell}$ parameterized by $\mathcal{M}$. The gapped condition is vacuous for $d = 0$. Then F is a closed 2-form on $\mathcal{M}$ with values in $\mathfrak{D}_{al}$ which commutes with P. If one defines a 2-form $f = \langle \mathsf{F} \rangle_\psi = \mathrm{Tr} P\mathsf{F}$, then f is closed and purely imaginary. If we identify G with the adiabatic connection arising from a family of gapped Hamiltonians for which ψ is the family of ground states, then f is the curvature of the Berry connection.

For $d > 0$ the expression $\langle \mathsf{F} \rangle_\psi$ does not make sense since the derivation F need not be summable (i.e., the Berry curvature is divergent in the thermodynamic limit). Instead, following [41, 37], we will define descendants of F and use them to construct an element of $\Omega^{d+2}(\mathcal{M}, \mathfrak{d}^\psi_{al})$ whose average will be a closed $(d + 2)$-form on $\mathcal{M}$.

Let $\mathbf{d} = d + \partial$. If $(\mathcal{M}, \psi)$ is parallel with respect to $D = d + \mathsf{G}$, then using Theorem 3.2 we can recursively build $\mathsf{g}^{(n)} \in \Omega^{n+2}(\mathcal{M}, \mathcal{N}_\psi^{-n-1})$, $n \geq 0$, such that $\mathsf{G}^\bullet := \mathsf{G} + \sum_{n=0}^{\infty} \mathsf{g}^{(n)}$ satisfies the Maurer-Cartan (MC) equation

$$\mathbf{d}\mathsf{G}^\bullet + \frac{1}{2}\{\mathsf{G}^\bullet, \mathsf{G}^\bullet\} = 0. \tag{3.21}$$

Indeed, the MC equation is equivalent to the following system of equations:

$$\mathsf{F} = -\partial \mathsf{g}^{(0)}, \tag{3.22}$$

$$D\mathsf{g}^{(n)} + \frac{1}{2}\sum_{k=0}^{n-1}\left\{\mathsf{g}^{(k)}, \mathsf{g}^{(n-1-k)}\right\} = -\partial \mathsf{g}^{(n+1)}, \qquad n = 0, 1, 2, \ldots \tag{3.23}$$

It is straightforward to check that if the first n equations are satisfied, then the left-hand side of the $(n+1)$-st equation is annihilated by ∂. Therefore by Theorem 3.2 a solution for $\mathsf{g}^{(n+1)}$ exists.

Any solution to the MC equation defines a differential $\mathbf{D} = \mathbf{d} + \mathsf{G}^\bullet$ on $\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet)$ which preserves $\Omega^\bullet(\mathcal{M}, \mathcal{N}_\psi^\bullet)$ and satisfies $\mathbf{D}^2 = 0$. The differential $\mathbf{D}$ turns $\Omega^\bullet(\mathcal{M}, \mathcal{N}_\psi^\bullet)$ into a DGFLA. Note also that

$$d\langle \mathsf{g}^{(n)}\rangle_\psi = -\langle \partial \mathsf{g}^{(n+1)}\rangle_\psi, \tag{3.24}$$

where we have used $\langle\{\mathsf{g}^{(k)}, \cdot\}\rangle_\psi = 0$.

For a d-dimensional system and any conical partition $(A_0, ..., A_d)$ of $\mathbb{R}^d$ we have an evaluation operation $\langle(\,\cdot\,)_{A_0 \ldots A_d}\rangle : \Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet) \to \Omega^\bullet(\mathcal{M}, i\mathbb{R})$ that takes value $\langle \mathsf{a}_{A_0 \ldots A_d}\rangle_\psi$ if $\mathsf{a} \in \Omega^\bullet(\mathcal{M}, \mathcal{N}^{-d-1})$ and is 0 otherwise.

Let

$$f := \langle \mathsf{G}^\bullet_{A_0 A_1 \ldots A_d}\rangle_\psi \in \Omega^{d+2}(\mathcal{M}, i\mathbb{R}). \tag{3.25}$$

We have

$$df = d\langle \mathsf{g}^{(d)}_{A_0 A_1 \ldots A_d}\rangle_\psi = -\langle(\partial \mathsf{g}^{(d+1)})_{A_0 A_1 \ldots A_d}\rangle_\psi = 0. \tag{3.26}$$

Therefore f defines a de Rham cohomology class $[f] \in H^{d+2}(\mathcal{M}, i\mathbb{R})$. Clearly, f is the same for all orderings of $(A_0, ..., A_d)$ which correspond to the same orientation of bases on S^{d-1}. Note also that f is locally computable: if we truncate the sum defining $\mathsf{g}^{(d)}_{A_0 A_1 \ldots A_d}$ by replacing each A_a with $A_a \cap B_p(r)$, this will modify f only by an $\mathcal{O}(r^{-\infty})$ quantity.

One and the same smooth family of states can be parallel with respect to many different connections $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$, and for a fixed connection G, the solution of the descent equation (3.21) is also far from unique. We will now show that the class $[f]$ is not affected by these choices and defines an invariant of $(\mathcal{M}, \psi)$.

Theorem 3.3. *The class* $[f] \in H^{d+2}(\mathcal{M}, i\mathbb{R})$ *defines an invariant of the family of states* $(\mathcal{M}, \psi)$*. In particular, it does not depend on the choice of* $\mathsf{G}^\bullet$ *or a conical partition* $(A_0, ..., A_d)$ *(provided it corresponds to some fixed orientation of* S^{d-1}*).*

Proof. Suppose we have changed the regions $A_0, \ldots, A_d$ by reassigning a finite region $A_0 \to A_0 + B$ and $A_1 \to A_1 - B$. Then

$$\Delta f = \langle \mathsf{g}^{(d)}_{B(A_0+A_1)A_2\ldots A_d} \rangle_\psi = -\langle \partial \mathsf{g}^{(d)}_{BA_2\ldots A_d} \rangle_\psi = d\langle \mathsf{g}^{(d-1)}_{BA_2\ldots A_d} \rangle_\psi \tag{3.27}$$

Thus $[f]$ does not depend on the choice of $A_0, \ldots, A_d$ for a fixed triangulation of S^{d-1}. The same argument combined with local computability implies that $[f]$ does not depend on the choice of the triangulation of S^{d-1}.

Let us fix a map $h^\psi_\mathcal{M}$ as in Theorem 3.2. Using $h^\psi_\mathcal{M}$, for any two choices G,$\tilde{\mathsf{G}}$ of the connection on a given family of states $(\mathcal{M}, \psi)$ we can construct solutions $\mathsf{g}^\bullet$, $\tilde{\mathsf{g}}^\bullet$ of the MC equation eq. (3.21). Let h and $\tilde{\mathsf{h}}$ be the elements of $\Omega^1(\mathcal{M}, C_0(\mathfrak{d}_{al}))$ such that $\mathsf{G} = \partial \mathsf{h}$ and $\tilde{\mathsf{G}} = \partial \tilde{\mathsf{h}}$. Since $\langle [\mathsf{G} - \tilde{\mathsf{G}}, \cdot] \rangle = 0$, we have $\mathsf{G} - \tilde{\mathsf{G}} = \partial \mathsf{k}$ for some $\mathsf{k} \in \Omega^1(\mathcal{M}, C_0(\mathfrak{d}^\psi_{al}))$. For a half-space A defined by $x > 0$ for some linear coordinate x on $\mathbb{R}^d$ we let $\hat{\mathsf{h}}_j = \mathsf{h}_j + \chi_A(j)\mathsf{k}_j$. Then $\hat{\mathsf{G}} = \partial \hat{\mathsf{h}}$ is a connection which preserves the family $(\mathcal{M}, \psi)$ and interpolates between G for $x \ll 0$ and $\tilde{\mathsf{G}}$ for $x \gg 0$. Starting with $\hat{\mathsf{G}}$ and using $h^\psi_\mathcal{M}$ we can construct a solution $\hat{\mathsf{g}}^\bullet$ of the descent equation that interpolates between $\mathsf{g}^\bullet$ and $\tilde{\mathsf{g}}^\bullet$. The corresponding class $[\hat{f}] \in H^{d+2}(\mathcal{M}, \mathbb{Z})$ is independent of the choice of conical partition, and therefore by choosing the apex of a conical partition with sufficiently large and negative x (resp. sufficiently large and positive x) we can make it arbitrary close to the class $[f]$ corresponding to $\mathsf{G}^\bullet$ (resp. the class $[\tilde{f}]$ corresponding to $\tilde{\mathsf{G}}^\bullet$). Hence $[f] = [\hat{f}] = [\tilde{f}]$.

It is left to show that for a fixed conical partition $(A_0, ..., A_d)$ and a fixed G the class $[f]$ does not depend on the choice of the solution of the MC equation eq. (3.21). Let $\mathsf{g}^\bullet$ be any solution of the MC equation, and let $\mathsf{g}^\bullet_k$ be a solution obtained from G, $\mathsf{g}^{(0)}$, ...,$\mathsf{g}^{(k-1)}$ using $h^\psi_\mathcal{M}$. Let $[f]$ and $[f_k]$ be the corresponding classes. Note that $[f_0]$ depends only on G and $h^\psi_\mathcal{M}$. Similarly to the argument from the previous paragraph, since $\partial(\mathsf{g}^{(k)}_{k+1} - \mathsf{g}^{(k)}_k) = 0$ for any k we can construct $\hat{\mathsf{g}}^\bullet_{k,k+1}$ that interpolates between $\mathsf{g}^\bullet_k$ and $\mathsf{g}^\bullet_{k+1}$. Hence we have $[f_k] = [f_{k+1}]$. By definition $[f] = [f_{d+1}]$, and therefore $[f] = [f_0]$, which does not depend on the choice of $\mathsf{g}^\bullet$. □

Remark 3.7. *For families of invertible states, it is expected that the class* $[f/(2\pi i)] \in H^{d+2}(\mathcal{M}, i\mathbb{R})$ *can be refined. In particular, it lies in the image of* $H^{d+2}(\mathcal{M}, i\mathbb{Z}) \to H^{d+2}(\mathcal{M}, i\mathbb{R})$ *for systems of dimension* $d \leq 2$*. We give proof of this for 1d spin systems in Section 4.2.*

3.4 Equivariant higher Berry classes

Let G be a compact connected Lie group with the Lie algebra $\mathfrak{g}$. In this section we define equivariant higher Berry classes for gapped states and their smooth families in the presence of G-symmetry. They generalize the Hall conductance of $U(1)$-invariant gapped 2d states defined in [34] and the Thouless pump [61] and its analogs in higher dimensions [37].

We assume that each on-site Hilbert space $\mathcal{V}_j$ is acted upon by a unitary representation of G. Let $\gamma : G \to \mathrm{Aut}(\mathscr{A})$ be the corresponding homomorphism. We denote the derivations corresponding to the infinitesimal generators of the symmetry by $\mathsf{Q} \in \mathfrak{D}_{al} \otimes \mathfrak{g}^*$, that is on G we have $d\gamma^{(g)}(\,\cdot\,) = \gamma^{(g)}(\mathsf{Q}^{(g)}_{\theta}(\,\cdot\,))$, where $\theta = g^{-1}dg \in \Omega^1(G, \mathfrak{g})$ is the Maurer-Cartan form.

Let $\mathcal{M}$ be a manifold with a smooth G-action $L_g : \mathcal{M} \to \mathcal{M}$, $g \in G$. Let $v \in \Gamma(T\mathcal{M}) \otimes \mathfrak{g}^*$ be the vector fields corresponding to this action. If we choose a basis r^a for $\mathfrak{g}^*$, then we can write $v = \sum_a v_a \otimes r^a$, where v_a is a vector field on $\mathcal{M}$. There is an induced G-action on $\Omega^\bullet(\mathcal{M}, \mathbb{R})$ via the pullback $\omega \mapsto L^*_{g^{-1}}\omega$. Infinitesimally, it is given by $\mathcal{L}_v = \sum_a \mathcal{L}_{v^a} \otimes r^a$ where $\mathcal{L}_{v_a} = \iota_{v_a} d + d\iota_{v_a}$ is the Lie derivative along v_a.

Let $\mathrm{Sym}^\bullet\,\mathfrak{g}^*$ be the ring of polynomial functions on $\mathfrak{g}$. We regard it as a graded vector space, with linear functions on $\mathfrak{g}$ sitting in degree 2. G acts on it via the co-adjoint action Ad^*_g. To define the Cartan model for the equivariant cohomology $H^\bullet_G(\mathcal{M}, \mathbb{R})$ we consider the graded vector space $\Omega^\bullet(\mathcal{M}, \mathbb{R}) \otimes \mathrm{Sym}^\bullet\,\mathfrak{g}^*$ equipped with degree-1 maps $d \otimes 1$, $\iota_v = \sum_a \iota_{v^a} \otimes r^a$, and a degree-2 map $\mathcal{L}_v = \iota_v d + d\iota_v$. Then the complex of the Cartan model $\Omega^\bullet_G(\mathcal{M}, \mathbb{R})$ is defined to be $(\Omega^\bullet(\mathcal{M}, \mathbb{R}) \otimes \mathrm{Sym}^\bullet\,\mathfrak{g}^*)^G$, where $(\,\cdot\,)^G$ denoted the G-invariant part, with the differential being $d_C = d + \iota_v$. Note that $d_C^2 = \mathcal{L}_v$ vanishes when restricted to $\Omega^\bullet_G(\mathcal{M}, \mathbb{R})$.

Let $(\mathcal{M}, \psi)$ be a smooth family of gapped states over a compact connected manifold $\mathcal{M}$.

Definition 3.2. *A family $(\mathcal{M}, \psi)$ is called G-equivariant if $\psi_m \circ \gamma^{(g)} = \psi_{L_{g^{-1}}(m)}$ for all $m \in \mathcal{M}$ and all $g \in G$.*

We can define equivariant analogs of differential forms taking values in observables, derivations and chains. They are annihilated by $\mathcal{L}_v - \mathsf{Q}$. There is an averaging operation that projects spaces of such differential forms to their G-equivariant subspaces:

$$(\,\cdot\,)^G = \int_G d\mu_G(g)\, \gamma^{(g)}(L^*_g(\mathrm{Ad}^*_{g^{-1}}(\,\cdot\,))), \tag{3.28}$$

where μ_G is the Haar measure with the normalization $\int_G d\mu_G(g) = 1$. The existence of this operation ensures that G-equivariant versions of the complexes (3.16) and (3.18) are exact.

Given a smooth G-equivariant family of gapped states $(\mathcal{M}, \psi)$, we consider the complex of equivariant differential forms $\Omega_G^\bullet(\mathcal{M}, \mathcal{N}^\bullet)$ which is the totalization of $(\Omega^\bullet(\mathcal{M}, \mathcal{N}^\bullet) \otimes \mathrm{Sym}^\bullet \mathfrak{g}^*)^G$ with elements of $\mathfrak{g}^*$ being assigned degree 2. Let $\mathbf{d}_C = d_C + \partial$. Then using the G-equivariant version of Theorem 3.2 we can recursively build

$$\mathsf{g}^{(n)} \in \bigoplus_{0 \leq 2k \leq n+2} (\Omega_G^{n+2-2k}(\mathcal{M}, \mathcal{N}_\psi^{-n-1}) \otimes \mathrm{Sym}^k \mathfrak{g}^*)^G, \quad n \geq 0,$$

such that $\mathsf{G}^\bullet := \mathsf{G} + \sum_{n=0}^\infty \mathsf{g}^{(n)}$ satisfies

$$\mathbf{d}_C \mathsf{G}^\bullet + \frac{1}{2}\{\mathsf{G}^\bullet, \mathsf{G}^\bullet\} + \mathsf{Q} = 0 \tag{3.29}$$

so that for $\mathbf{D}_C = \mathbf{d}_C + \mathsf{G}^\bullet$ we have $\mathbf{D}_C^2 = \mathcal{L}_v - \mathsf{Q}$ that vanishes on $\Omega_G^\bullet(\mathcal{M}, \mathcal{N}^\bullet)$.

Similarly to the previous subsection, we can define the evaluation operation $\langle (\,\cdot\,)_{A_0 \dots A_d} \rangle : \Omega_G^\bullet(\mathcal{M}, \mathcal{N}^\bullet) \to \Omega_G^\bullet(\mathcal{M}, i\mathbb{R})$. For a conical partition $(A_0, \dots, A_d)$ we define an equivariant differential form

$$f = \langle \mathsf{G}^\bullet_{A_0 \dots A_d} \rangle_\psi \in \Omega_G^\bullet(\mathcal{M}, i\mathbb{R}) \tag{3.30}$$

which is closed with respect to d_C. In general, unlike in the non-equivariant case, this form is not homogeneous when regarded as an ordinary form. The form f represents a class of total degree $d + 2$ in the equivariant cohomology:

$$[f] = [\langle \mathsf{G}^\bullet_{A_0 \dots A_d} \rangle_\psi] \in H_G^\bullet(\mathcal{M}, i\mathbb{R}). \tag{3.31}$$

In the same way as in Theorem 3.3 one can show that this class is independent of the choice of $(A_0, \dots, A_d)$ (provided it corresponds to some fixed orientation of S^{d-1}) and $\mathsf{G}^\bullet$, and therefore defines an invariant of the family $(\mathcal{M}, \psi)$.

Example 3.1 (Hall conductance)**.** *Let $d = 2$, $\mathcal{M} = pt$ and $G = U(1)$. The corresponding state ψ must be $U(1)$-invariant. For $G = U(1)$ we can identify $(\mathrm{Sym}^{k+1} \mathfrak{g}^*)^G \cong \mathbb{R}$ via the isomorphism that sends the minimal integral element $t^{\otimes(k+1)}$ to 1. It is easy to see from the Maurer-Cartan equation eq. (3.29) that $\mathsf{G}^\bullet$ only has components of even chain degree. Let $\mathsf{m}^{(2k)}$ be the component of $\mathsf{G}^\bullet$ of* *chain degree 2k. The lowest two components of eq. (3.29) are*

$$\mathsf{Q} = -\partial \mathsf{m}^{(0)}, \tag{3.32}$$

$$\frac{1}{2}\{\mathsf{m}^{(0)}, \mathsf{m}^{(0)}\} = -\partial \mathsf{m}^{(2)}. \tag{3.33}$$

In more detail, eq. (3.32) says that the derivation Q *has the form*

$$\mathsf{Q}(\mathcal{A}) = -\sum_{j\in\Lambda}[\mathsf{m}^{(0)}_j, \mathcal{A}], \tag{3.34}$$

where for all $j \in \Lambda$ *the observable* $\mathsf{m}^{(0)}_j \in \mathfrak{d}^{\psi}_{al}$ *is* $U(1)$*-invariant and does not excite* ψ*. Eq. (3.33) reads in components:*

$$[\mathsf{m}^{(0)}_j, \mathsf{m}^{(0)}_k] = -\sum_{l\in\Lambda}\mathsf{m}^{(2)}_{ljk}, \tag{3.35}$$

where the observables $\mathsf{m}^{(2)}_{ljk} \in \mathfrak{d}^{\psi}_{al}$ *are* $U(1)$*-invariant and do not excite* ψ*. The contraction of* $\mathsf{m}^{(2)}$ *with a conical partition* A_0, A_1, A_2 *defines an invariant*

$$\sigma^{(2)} := 4\pi i\langle \mathsf{m}^{(2)}_{A_0A_1A_2}\rangle_\psi \in H^4_{U(1)}(pt, \mathbb{R}) \cong \mathbb{R}. \tag{3.36}$$

In Section 5.1 we discuss this invariant in more detail. We show that for ground states of gapped Hamiltonians in two dimensions $(\sigma^{(2)}/2\pi)$ *coincides with the Hall conductance and prove that for invertible bosonic states* $\sigma \in 2\mathbb{Z}$*, while for invertible fermionic states* $\sigma \in \mathbb{Z}$*.*

Example 3.2 (Non-abelian Hall conductance)**.** *The case of* $d = 2$*,* $\mathcal{M} = pt$ *and arbitrary Lie group* G *is similar to example above, but now* $\mathsf{m}^{(0)} \in \left(C_0(\mathfrak{d}^{\psi}_{al}) \otimes \mathfrak{g}^*\right)^G$*, satisfying* $\mathsf{Q} = -\partial \mathsf{m}^{(0)}$*, and* $\mathsf{m}^{(2)} \in \left(C_2(\mathfrak{d}^{\psi}_{al}) \otimes Sym^2\mathfrak{g}^*\right)^G$ *solving*

$$[\mathsf{m}^{(0)}_j, \mathsf{m}^{(0)}_k] = -\sum_{l\in\Lambda}\mathsf{m}^{(2)}_{ljk}. \tag{3.37}$$

The contraction of $\mathsf{m}^{(2)}$ *with a conical partition* A_0, A_1, A_2 *defines an invariant*

$$\sigma^{(2)} := 4\pi i\langle \mathsf{m}^{(2)}_{A_0A_1A_2}\rangle_\psi \in H^4_G(pt, \mathbb{R}) \cong (Sym^2\mathfrak{g}^*)^G. \tag{3.38}$$

For a simple Lie algebra $\mathfrak{g}$ *we have* $(Sym^2\mathfrak{g}^*)^G \cong \mathbb{R}$*. This agrees with the expectation that the response of a gapped 2d system with a symmetry G to a background gauge field is described by a 3d Chern-Simons action whose coefficient (level) is a topological invariant.*

Example 3.3 (Higher-dimensional generalizations of the Hall conductance)**.** *More generally, let us take $\mathcal{M} = pt$ but consider an arbitrary d and an arbitrary compact Lie group G. Let ψ be a G-invariant gapped state. As before, $\mathsf{G}^\bullet$ only has components of even chain degree. Let $\mathsf{m}^{(2k)}$ be the component of $\mathsf{G}^\bullet$ of chain degree $2k$. For $d = 2k$ the contraction with $A_0, ..., A_d$ defines an invariant*

$$i\langle \mathsf{m}^{(2k)}_{A_0\ldots A_{2k}}\rangle_\psi \in (\mathrm{Sym}^{k+1}\,\mathfrak{g}^*)^G \cong H^{2k+2}_G(pt,\mathbb{R}). \tag{3.39}$$

Example 3.4 (Thouless pump and its generalizations)**.** *Let $G = U(1)$ act trivially on $\mathcal{M}$ and ψ be a smooth family of $U(1)$-invariant states parameterized by $\mathcal{M}$. We may identify $\mathfrak{g}^* \cong \mathbb{R}$. Let $\mathsf{t}^{(k)} \in (\Omega^k(\mathcal{M}, \mathcal{N}^{-k-1}_\psi) \otimes \mathfrak{g}^*)^G$ be the component of $\mathsf{G}^\bullet$ of chain degree k and form degree k, $k = 0, 1, \ldots$. The Maurer-Cartan equation (3.29) implies an infinite number of equations for $\mathsf{t}^{(k)}$, the first three of which look as follows:*

$$\mathsf{Q} = -\partial \mathsf{t}^{(0)}, \tag{3.40}$$

$$D\mathsf{t}^{(0)} = -\partial \mathsf{t}^{(1)}, \tag{3.41}$$

$$D\mathsf{t}^{(1)} + \{\mathsf{g}^{(0)}, \mathsf{t}^{(0)}\} = -\partial \mathsf{t}^{(2)}. \tag{3.42}$$

For $d = 1$ and $\mathcal{M} = S^1$ the invariant $\int_{S^1}\langle \mathsf{t}^{(1)}_{A_0A_1}\rangle_\psi$ computes the charge pumped through any point of the system in the process of the adiabatic evolution along a loop S^1 in the parameter space.

For $d = k$ and $\alpha \in H_k(\mathcal{M},\mathbb{Z})$ the invariant $\int_\alpha \langle \mathsf{t}^{(k)}_{A_0A_1\ldots A_k}\rangle_\psi$ is a higher-dimensional generalization of the Thouless pump introduced in [38].

Finally, let us discuss an example of a family of 1d states with a nonzero non-equivariant higher Berry class. (Other examples of families with a nonzero higher Berry class appeared in [63].)

Example 3.5. *The family is associated with a compact connected Lie group G and a G-invariant 2d state in the trivial phase but with a nonzero non-abelian Hall conductance (such 2d states are known as Symmetry Protected Topological states).*

Let us consider a two-dimensional lattice system $(\Lambda, \mathscr{A})$ with an action of G as defined in Section 3.4. Since $\mathsf{Q} = \partial \mathsf{q}$ for $\mathsf{q} \in C_0(\mathfrak{d}_{al}) \otimes \mathfrak{g}^*$, for any $\Gamma \subset \mathbb{R}^2$ we can define an action of G generated by $\mathsf{Q}|_\Gamma = \mathsf{q}_\Gamma$. We denote the corresponding homomorphism by $\gamma_\Gamma : G \to Aut(\mathscr{A})$. We denote the left-invariant Maurer-Cartan form by $\theta := g^{-1}\,dg \in \Omega^1(G,\mathfrak{g})$.

Let ψ_0 be a factorized pure state. Let β be an LGP such that the state $\omega = \psi_0 \circ \beta^{(1)}$ is also G-invariant, with a G-action defined by $\mathsf{Q} \in (\mathfrak{D}_{al} \otimes \mathfrak{g}^*)^G$. The equivariant Berry class eq. (3.38) for ω defines an invariant quadratic form $\sigma^{(2)}$ on $\mathfrak{g}$. This form is the non-abelian analog of the Hall conductance.

Let (A_0, A_1, A_2) be a conical partition. We define a smooth family of states $(\mathcal{M}, \psi)$ with $\mathcal{M} = G$, such that at a point $g \in G$ we have $\psi_g = \psi_0 \circ \beta^{(1)} \circ \gamma^{(g)}_{A_0} \circ (\beta^{(1)})^{-1}$. Note that though the state is defined on a two-dimensional lattice, far from the boundary of A_0 it is almost factorized. Hence we may call it "quasi-one-dimensional."

We have

$$d\langle \mathcal{A} \rangle_\psi = \langle \{\beta^{(1)}(\gamma^{-1}_{A_0}(\mathsf{q}_{A_0})), \mathcal{A}\} \rangle_\psi. \tag{3.43}$$

Since $\mathsf{Q} \in (\mathfrak{D}^\omega_{al} \otimes \mathfrak{g}^*)^G$, there are G-invariant $\tilde{\mathsf{q}} \in C_0(\mathfrak{d}^\omega_{al}) \otimes \mathfrak{g}^*$ and $\mathsf{k} \in C_1(\mathfrak{d}_{al}) \otimes \mathfrak{g}^*$ such that $\mathsf{q} = \tilde{\mathsf{q}} - \partial \mathsf{k}$. Therefore $(\mathcal{M}, \psi)$ is a family of quasi-one-dimensional states parallel with respect to $\mathsf{G} = \beta^{(1)}(\gamma^{-1}_{A_0}(\mathsf{k}_{A_0 \overline{A}_0}(\theta))) \in \Omega^1(G, \mathfrak{D}_{al})$.

We have

$$\begin{aligned}
&\gamma_{A_0} \circ (\beta^{(1)})^{-1} \left(d\mathsf{G} + \frac{1}{2}\{\mathsf{G}, \mathsf{G}\} \right) = \\
&\qquad = \{\mathsf{q}_{A_0}(\theta), \mathsf{k}_{A_0\overline{A}_0}(\theta)\} + \mathsf{k}_{A_0\overline{A}_0}(d\theta) + \frac{1}{2}\{\mathsf{k}_{A_0\overline{A}_0}(\theta), \mathsf{k}_{A_0\overline{A}_0}(\theta)\} = \\
&\qquad = -\frac{1}{2}\{\tilde{\mathsf{q}}_{\overline{A}_0}(\theta), \tilde{\mathsf{q}}_{A_0}(\theta)\} = -\frac{1}{2}\{\tilde{\mathsf{q}}_{A_1}(\theta), \tilde{\mathsf{q}}_{A_0}(\theta)\} - \frac{1}{2}\{\tilde{\mathsf{q}}_{A_2}(\theta), \tilde{\mathsf{q}}_{A_0}(\theta)\}.
\end{aligned} \tag{3.44}$$

Hence the contraction of the Berry class $[f] \in H^3(\mathcal{M}, i\mathbb{R})$ for a family $(\mathcal{M}, \psi)$ with $[\mathcal{M}]$ is given by

$$\begin{aligned}
\langle [f], [\mathcal{M}] \rangle &= \int_G f = \\
&= \frac{1}{2}\langle \int_G \{\mathsf{q}_{A_0}(\theta) - \mathsf{k}_{A_0\overline{A}_0}(\theta), \{\tilde{\mathsf{q}}_{A_1}(\theta), \tilde{\mathsf{q}}_{A_0}(\theta)\}\} \rangle_\omega = \\
&= -\frac{1}{4}\langle \int_G \{\tilde{\mathsf{q}}_{A_1}(\theta), \{\tilde{\mathsf{q}}_{A_0}(\theta), \tilde{\mathsf{q}}_{A_0}(\theta)\}\} \rangle_\omega = \\
&= -\frac{1}{6}\langle \int_G \{\tilde{\mathsf{q}}_{A_1}(\theta), \tilde{\mathsf{q}}_{A_0}(d\theta)\} \rangle_\omega = \frac{1}{12\pi i} \int_G \langle \theta, d\theta \rangle_{\sigma^{(2)}}.
\end{aligned} \tag{3.45}$$

Note that all derivations inside the averages $\langle \cdot \rangle_\omega$ belong to the subspace of summable derivations $\mathfrak{d}^\omega_{al}$, and therefore the average is well-defined. If $\sigma^{(2)}$ does not vanish, the Berry class $[f]$ is non-trivial.

Chapter 4

INVERTIBLE STATES OF 1D SPIN SYSTEMS

This chapter is devoted to the analysis of invertible states of one-dimensional spin systems. In Section 4.1 we prove that all invertible states are in the trivial phase, while in Section 4.2, we prove quantization of Berry classes for families of such states.

4.1 A classification of invertible states of 1d spin systems

In this section we show that any invertible state of a 1d bosonic[1] spin system is stably LGA-equivalent to a factorized state. For simplicity of the exposition, we assume that the lattice of the spin system is $\Lambda \cong \mathbb{Z} \subset \mathbb{R}$. A general case can be treated similarly. By the origin of a half-line on $\mathbb{R}$ we mean its boundary point.

We will say that a pure 1d state has bounded entanglement entropy if the entanglement entropies of all intervals $[j,k] \subset \mathbb{Z}$ are uniformly bounded.

Remark 4.1. *It was shown by Matsui [46] that if ψ has bounded entanglement entropy then it has the split property: the von Neumann algebras $\mathcal{M}_A = \Pi_\psi(\mathscr{A}_A)''$ and $\mathcal{M}_{\bar{A}} = \Pi_\psi(\mathscr{A}_{\bar{A}})''$ for a half-line A (and the GNS representation $(\pi_\psi, \mathcal{H}_\psi)$) are Type I von Neumann algebras. Since they are each other's commutants and generate $\mathcal{B}(\mathcal{H}_\psi)$, they must be Type I factors. Thus $\mathcal{M}_A \simeq \mathcal{B}(\mathcal{H}_A)$ for some Hilbert space $\mathcal{H}_A$, and the restriction $\psi|_A$ to a half-line A can be described by a density matrix ρ_A on $\mathcal{H}_A$. In fact, [46] shows that $\mathcal{H}_A$ can be identified with the GNS Hilbert space of one of the Schmidt vector states of ψ, which are all unitarily equivalent.*

Lemma 4.1. *Invertible 1d states have bounded entanglement entropy.*

Proof. Suppose we have a state ψ obtained from a factorized state ψ_0 by conjugations with almost local unitaries $\mathcal{U}_j$ and $\mathcal{U}_k$ which are g-localized at sites j and k, respectively, for some $g \in \mathscr{F}_\infty^+$. We have

$$|\langle \mathcal{U}_j \mathcal{U}_k \mathcal{A}_l \mathcal{U}_k^* \mathcal{U}_j^* \rangle_{\psi_0} - \langle \mathcal{A}_l \rangle_{\psi_0}| \leq (g(|j-l|) + g(|k-l|)) \, \|\mathcal{A}_l\|. \tag{4.1}$$

By Fannes' inequality [19], the entropy of the site l in the state ψ is bounded by $h(|j-l|) + h(|k-l|)$ for some $h \in \mathscr{F}_\infty^+$ that depends only on g and the maximal

[1]This is not true in the case of fermionic systems: Kitaev chain provides a counter-example.

spin. Therefore such a state has a uniform bound on the entanglement entropy of any interval $[j,k]$. Since any LGA can be written as a composition of conjugations with unitaries almost localized at j and k and LGAs which are generated by derivations strictly localized inside $[j,k]$ and outside $[j,k]$, and since the stack of an invertible state with its inverse is LGA-equivalent to a factorized state, the same is true for any invertible state. □

Let ψ be a possibly mixed state on a half-line A which is f-close (see Definition C.1) at the origin of A to a pure factorized state ψ_0 on A. By Corollary 2.6.11 of [11], ψ is normal in the GNS representation of ψ_0 and can be described by a density matrix.

Lemma 4.2. *Let ψ be a pure 1d state on $\mathscr{A}$. Suppose ψ is f-close at* 0 *to a pure factorized state ω^+ on* $[0,+\infty)$ *and is f-close at* 0 *to a pure factorized state ω^- on* $(-\infty,0)$. *Then it is g-close at 0 on the whole line to $\omega^+\otimes\omega^-$ for some $g\in\mathscr{F}^+_\infty$ which depends only on f and the maximal spin.*

Proof. Since the states are split, we can describe them using density matrices on appropriate Hilbert spaces. The decomposition of $\mathbb{Z}$ into the union $(-\infty,-n)\sqcup\Gamma_n\sqcup(n,+\infty)$ gives rise to a tensor product decomposition $\mathcal{H}=\mathcal{H}^-_n\otimes\mathcal{H}_{\Gamma_n}\otimes\mathcal{H}^+_n$. Let ψ^+_n and ω^+_n be restrictions of ψ and ω^+ to $(n,+\infty)$, and ψ^-_n and ω^-_n be restrictions of ψ and ω^- to $(-\infty,-n)$. Let $\rho^\pm_n$ and $\sigma^\pm_n$ be the corresponding density matrices. Let ψ_n be a restriction of ψ to $(-\infty,-n)\cup(n,+\infty)$ with the corresponding density matrix ρ_n. We have

$$\|\rho^\pm_n-\sigma^\pm_n\|_1\le f(n). \tag{4.2}$$

Since trace norm is multiplicative under tensor product, we have

$$\|(\rho^-_n\otimes\rho^+_n)-(\sigma^-_n\otimes\sigma^+_n)\|_1\le\|\rho^-_n-\sigma^-_n\|_1+\|\rho^+_n-\sigma^+_n\|_1\le 2f(n). \tag{4.3}$$

On the other hand, Fannes' inequality implies that for sufficiently large n the entropy of $\rho^\pm_n$ is upper-bounded by $h(n)$, where $h\in\mathscr{F}^+_\infty$ depends only on f and the maximal spin. Therefore mutual informations $I(\rho^-_n:\rho^+_n)$ are also upper-bounded by $h(n)$, and the quantum Pinsker inequality implies

$$\|\rho_n-(\rho^-_n\otimes\rho^+_n)\|_1\le 2\sqrt{h(n)}. \tag{4.4}$$

Combining this with eq. (4.3), we get

$$\|\rho_n-(\sigma^-_n\otimes\sigma^+_n)\|_1\le 2f(n)+2\sqrt{h(n)}. \tag{4.5}$$

□

We say that a set of (ordered) eigenvalues $\{\lambda_j\}$ has *g-decay* for some $g = \mathscr{F}^+_\infty$ if $\varepsilon(k) \leq g(\log(k))$, where $\varepsilon(k) = \sum_{j=k+1}^{\infty} \lambda_j$.

Lemma 4.3. *Let ψ be a state on a half-line A which is f-close at the origin of A to a pure factorized state ψ_0 for $f \in \mathscr{F}^+_\infty$. Then its density matrix (in the GNS Hilbert space of this factorized state) has eigenvalues with g-decay for some $g \in \mathscr{F}^+_\infty$ that depends only on f and the maximal spin. Conversely, for any density matrix on a half-line A (in the GNS Hilbert space of a pure factorized state) whose eigenvalues have g-decay there is a state on that half-line which has the same eigenvalues and is f-close at the origin of A to this pure factorized state. Furthermore, one can choose $f \in \mathscr{F}^+_\infty$ so that it depends only on g and the maximal spin.*

Proof. Without loss of generality, we assume $A = \mathbb{R}_{\geq 0}$. Suppose ψ is f-close to a pure factorized state at 0. It can be purified on the whole line (e.g., in a system consisting of the given system on a half-line and its reflected copy on the other half-line). Moreover, by Lemma 4.2 we can choose this pure state to be f'-close at 0 to a pure factorized state on the whole line for some $f' \in \mathscr{F}^+_\infty$ that depends only on f. By Proposition C.2, it can be produced from a pure factorized state on the whole line by a unitary observable $\mathcal{U}$ which is h-localized at 0 for some $h \in \mathscr{F}^+_\infty$ which depends only on f'. Let $|\psi_0\rangle$ be a GNS vector for the corresponding factorized state ψ_0. By Lemma A.1 of [35], there is $h' \in \mathscr{F}^+_\infty$ such that for any $r > 0$ there is a unitary observable $\mathcal{U}^{(r)} \in \mathscr{A}_\ell$ with the support on $\Gamma_r = B_r(0)$ such that

$$\|\pi_{\psi_0}(\mathcal{U})|\psi_0\rangle - \pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle\| \leq h'(r). \tag{4.6}$$

On the other hand we have

$$\|\pi_{\psi_0}(\mathcal{U})|\psi_0\rangle - \pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle\| \geq \|\rho - \rho^{(r)}\|_1 \tag{4.7}$$

where ρ is the density matrix for ψ and $\rho^{(r)}$ is the density matrix for $\Pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle$ on A. The tracial distance between any two density matrices ρ and ρ' can be bounded from below in terms of their eigenvalues [60]:

$$\|\rho - \rho'\|_1 \geq \sum_{j=1}^{\infty} |\lambda_j(\rho) - \lambda_j(\rho')|, \tag{4.8}$$

where the eigenvalues λ_i are ordered in decreasing order. Applying this to ρ and $\rho^{(r)}$ and noting that $\rho^{(r)}$ has rank at most $\dim \mathcal{H}_{A\cap\Gamma_r}$, we get

$$\|\rho - \rho^{(r)}\|_1 \geq \varepsilon(\dim \mathcal{H}_{A\cap\Gamma_r}). \tag{4.9}$$

Combining (4.6), (4.7) and (4.9) we get

$$\varepsilon(\dim \mathcal{H}_{A\cap\Gamma_r}) \leq h'(r). \tag{4.10}$$

Since $\dim \mathcal{H}_{A\cap\Gamma_r}$ is upper-bounded by $\exp(cr)$ for some positive constant c, we have $\varepsilon(k) \leq g(\log(k))$ for some $g \in \mathscr{F}^+_\infty$ which depends only on f and c.

Conversely, suppose we are given a density matrix on a half-line A with eigenvalues $\lambda_1 \geq \lambda_2 \geq \ldots$. We may assume that the dimension of $\mathscr{A}_A$ is infinite, since otherwise the statement is obviously true. Pick any pure factorized state ψ_0 on A and choose a basis in each on-site Hilbert space $\mathcal{V}_j$, $j \in A$, such that for all j the first basis vector gives the state $\psi_0|_{\mathscr{A}_j}$. This gives a lexicographic basis $|n\rangle$, $n \in \mathbb{N}$, in the GNS Hilbert space of ψ_0. By our assumption on the boundness of $d_j = \dim\ \mathcal{V}_j$, there is a positive constant c such that for any r and any $n < e^{cr}$ the vector state $|n\rangle\langle n|$ coincides with ψ_0 outside of Γ_r. Therefore the state $\sum_{n=1}^{\infty} \lambda_n |n\rangle\langle n|$ is f-close at 0 to ψ_0, where $f(r) = g(cr)$. □

By Lemma 4.3 any restriction $\psi|_A$ of a state ψ with g-decay of Schmidt coefficients to a half-line A can be purified by a state on $\bar{A}$, which is f-close at the origin of A to a pure factorized state for some $f \in \mathscr{F}^+_\infty$ that depends on g only. We call such state a *truncation* of ψ to A. Clearly, if ω is a truncation of ψ to A, then $\omega|_A = \psi|_A$. The following lemma shows that truncations exist for all invertible states.

Lemma 4.4. *Let ψ be an invertible state with an inverse ψ', such that that $\Psi = \psi \otimes \psi'$ can be produced by an f-local LGA β from a pure factorized state $\Psi_0 = \psi_0 \otimes \psi'_0$. Then ψ has g-decay of Schmidt coefficients for some $g = \mathscr{F}^+_\infty$ that depends only on f.*

Proof. By Lemma 4.1 and the results of [46], Ψ, ψ, and ψ' have the split property. Therefore for any half-line $A \subset \mathbb{Z}$ the corresponding GNS Hilbert spaces factorize into Hilbert spaces for A and Hilbert spaces for $\bar{A}$, and the restrictions $\psi|_A$, $\psi'|_A$, $\Psi|_A$ can be described by density matrices ρ_A, ρ'_A and P_A in the Hilbert spaces for A [46]. The restriction of $\Psi \circ \beta$ on $\bar{A}$ is f-close at the origin of A to a pure factorized state, and therefore by Lemma 4.3 the density matrix P_A has g-decay of Schmidt coefficients for some $g \in \mathscr{F}^+_\infty$ that depends on f. Since $P_A = \rho_A \otimes \rho'_A$, the same is true for ρ_A and ρ'_A.

□

Lemma 4.5. *Any truncation of an invertible state ψ to any half-line is in a trivial phase.*

Proof. Let ψ be an invertible state on $\mathscr{A}$ with an inverse ψ' such that that $\Psi = \psi \otimes \psi'$ can be produced by an LGA $(\alpha^{(1)}_{\mathsf{F}})^{-1}$ with f-local $\mathsf{F} \in C([0,1], \mathfrak{D}_{al})$ from a pure factorized state $\Psi_0 = \psi_0 \otimes \psi'_0$.

For $k \in \mathbb{Z}$, let ϕ_k be a truncation of ψ to $[k, \infty)$, and let ϕ'_k be a truncation of ψ' to $[k, \infty)$. Since by Lemma 4.2 $(\phi_k \otimes \phi'_k) \circ \alpha_{\mathsf{F}|_{[k,\infty)}}$ is g-close at k to $\psi_0 \otimes \psi'_0$ for some $g \in \mathscr{F}^+_\infty$ that depends on f only, Proposition C.2 implies that ϕ_k is invertible with the inverse ϕ'_k. Let $\tilde{\phi}_k$ be a pure state on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$, where $\mathscr{A}^{(1)}$ and $\mathscr{A}^{(2)}$ are two copies of $\mathscr{A}$, with the following two properties: (1) its restriction to $(-\infty, k)$ coincides with the factorized pure state $(\psi_0 \otimes \psi_0)|_{(-\infty,k)}$; (2) its restriction to $A = [k, +\infty)$ is a purification of $\psi|_A$ on $\mathscr{A}^{(1)}_A$ by some state on $\mathscr{A}^{(2)}_A$ which is f'-close at k to a factorized state for some $f' \in \mathscr{F}^+_\infty$ that depends on f only. The existence of such a state follows from Lemma 4.4. Similarly, we can define a state $\tilde{\phi}'_k$ on $\mathscr{A}'^{(1)} \otimes \mathscr{A}'^{(2)}$ which is an inverse of $\tilde{\phi}_k$. We let $\alpha^{(1)}_{\tilde{\mathsf{G}}}$ be an LGA that maps $\tilde{\phi}_k \otimes \tilde{\phi}'_k$ to $\psi_0 \otimes \psi_0 \otimes \psi'_0 \otimes \psi'_0$. $\tilde{\mathsf{G}}$ can be chosen to be vanishing on $(-\infty, k)$ and g'-local with $g' \in \mathscr{F}^+_\infty$ depending on g only.

Let us first show that the state $\psi_0 \otimes \phi_k \otimes \psi'_0 \otimes \psi_0$ can be transformed into $\tilde{\phi}_k \otimes \psi'_0 \otimes \psi_0$ by applying a certain LGA β, then an LGA h_2-localized at k (for some $h_2 \in \mathscr{F}^+_\infty$), and finally the inverse of β. The sequence of steps is shown schematically in Fig. 1, where it is also indicated that β is a composition of two LGAs described in more detail below.

Equivalently, we can apply β to both states and then show that the resulting states are related by an LGA h_2-localized at k. β is a composition of two LGAs. The first one has the form $\mathrm{Id} \otimes \mathrm{Id} \otimes \alpha^{(1)}_{\mathsf{G}}$, where $\alpha^{(1)}_{\mathsf{G}}$ maps $\psi'_0 \otimes \psi_0$ to $\phi'_k \otimes \phi_k$, see Fig. 1. The second one has the form $\mathrm{Id} \otimes \alpha^{(1)}_{\mathsf{F}|_{[k+1,\infty)}} \otimes \mathrm{Id}$. The product of these two LGAs maps the two states of interest to the states $\Xi_k \otimes \phi_k$ and $\tilde{\Xi}_k \otimes \phi_k$, where both Ξ_k and $\tilde{\Xi}$ are h_1-close at k to the same pure factorized state on $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}'$. Here $h_1 \in \mathscr{F}^+_\infty$ which depends only on f. By Proposition C.2 Ξ_k and $\tilde{\Xi}_k$ are related by a conjugation with a unitary observable which is h_2-localized at k for some $h_2 \in \mathscr{F}^+_\infty$ depending only on f.

Let us fix $L \in \mathbb{N}$. Since the state $\tilde{\phi}'_k$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$ is invertible, by Lemma 4.4 its restriction to $(-\infty, k+L)$ has $\tilde{g}$-decay of Schmidt coefficients with $\tilde{g}$ depending only on f. Also, since the restriction of the state $\tilde{\phi}'_k$ to $(-\infty, k)$ is factorized, the nonzero Schmidt coeffients are the same as for the state $\tilde{\phi}_k|_{[k,k+L)}$. In particular, the number of nonzero Schmidt coefficients does not exceed the dimension of $\mathscr{A}_{[k,k+L)} \otimes \mathscr{A}_{[k,k+L)}$.

Let us tensor $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$ with $\mathscr{A}^{(3)} \otimes \mathscr{A}^{(4)}$. By the proof of Lemma 4.3 one can find a state χ_k on $\mathscr{A}^{(1)}_{[k,k+L)} \otimes \mathscr{A}^{(2)}_{[k,k+L)} \otimes \mathscr{A}^{(3)}_{[k,k+L)} \otimes \mathscr{A}^{(4)}_{[k,k+L)}$ which is a purification of $\tilde{\phi}_k|_{[k,k+L)}$ on $\mathscr{A}^{(1)}_{[k,k+L)} \otimes \mathscr{A}^{(2)}_{[k,k+L)}$ and such that its restriction to $\mathscr{A}^{(3)}_{[k,k+L)} \otimes \mathscr{A}^{(4)}_{[k,k+L)}$ is h-close to a factorized state at $k+L$ for some $h \in \mathscr{F}^+_\infty$ that depends on f only. The state χ_k can be produced from $(\psi_0 \otimes \psi_0 \otimes \psi_0 \otimes \psi_0)|_{[k,k+L)}$ by a conjugation with a unitary $\mathcal{V}^{(0)} \in \mathscr{A}^{(1)}_{[k,k+L)} \otimes \mathscr{A}^{(2)}_{[k,k+L)} \otimes \mathscr{A}^{(3)}_{[k,k+L)} \otimes \mathscr{A}^{(4)}_{[k,k+L)}$.

Consider the states $\psi_0|_{[k,\infty)} \otimes \psi_0|_{[k,\infty)} \otimes \tilde{\phi}_k|_{[k,\infty)}$ and $\chi_k \otimes (\psi_0|_{[k+L,\infty)} \otimes \psi_0|_{[k+L,\infty)} \otimes \tilde{\phi}_{k+L})$ on the algebra $\mathscr{A}^{(3)}_{[k,\infty)} \otimes \mathscr{A}^{(4)}_{[k,\infty)} \otimes \mathscr{A}^{(1)}_{[k,\infty)} \otimes \mathscr{A}^{(2)}_{[k,\infty)}$ (see Fig. 4.2). They are stably related by an almost local unitary which is h'-localized at $k+L$, with $h' \in \mathscr{F}^+_\infty$ which depends only on f. Indeed, we can first tensor both states with $\psi'_0 \otimes \psi'_0 \otimes \psi_0 \otimes \psi_0$ restricted to $[k,+\infty)$, then produce on these ancillas the state $\tilde{\phi}'_k \otimes \tilde{\phi}_k$ with a g'-local LGA, and apply $\alpha^{(1)}_{\tilde{\mathsf{G}}|_{(-\infty,k+L)}} \circ \alpha^{(1)}_{\tilde{\mathsf{G}}|_{[k+L,\infty)}}$ acting on the tensor product of the original states and $\tilde{\phi}'_k$ in an obvious way. In the same way as in the previous paragraph one can argue that these states are related by an almost local at $k+L$ unitary. This implies that the original states are also related by almost local at $k+L$ unitary $\mathcal{U}^{(0)}$.

We have shown that the states $\tilde{\phi}_k$ and $\tilde{\phi}_{k+L}$ are related (after tensoring with a total of six copies of factorized states ψ_0 and ψ'_0) by a conjugation with an almost local unitary $\mathcal{U}^{(0)}$ followed by a conjugation with a strictly local unitary $\mathcal{V}^{(0)}$. Similarly, we can construct such unitaries $\mathcal{U}^{(n)}$, $\mathcal{V}^{(n)}$ relating stabilizations of $\tilde{\phi}_{k+nL}$ and $\tilde{\phi}_{k+(n+1)L}$. By Lemma B.6 we can choose L such that an ordered product of conjugations with $\prod_{n=0}^{\infty} \mathcal{U}^{(n)}$ is an LGA. Since $\mathcal{V}^{(n)}$ commute with $\mathcal{U}^{(n')}$ for $n < n'$, an ordered product $\prod_{n=0}^{\infty} \mathcal{V}^{(n)}\mathcal{U}^{(n)}$ is equal to $\prod_{m=0}^{\infty} \mathcal{V}^{(m)} \prod_{n=0}^{\infty} \mathcal{U}^{(n)}$ and therefore is also an LGA. By construction it relates $\tilde{\phi}_k$ to a factorized state, and therefore ϕ_k is in the trivial phase.

□

Theorem 4.1. *Any invertible bosonic 1d state ψ of a spin system is in the trivial phase.*

Proof. Let ψ' be an inverse state for ψ, and let $(\alpha^{(1)}_{\mathsf{F}})^{-1}$ be an LGA with f-local F that produces $(\psi \otimes \psi')$ from a pure factorized state $(\psi_0 \otimes \psi'_0)$. It enough to show that the state $\psi_0 \otimes \psi \otimes \psi'_0 \otimes \psi_0$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)} \otimes \mathscr{A}^{(4)}$ is in the trivial phase.

Let A be a half-line $[0,\infty)$. Since $\mathscr{A}^{(1)}_A$ is identical to $\mathscr{A}^{(2)}_A$ there is a pure state on $\mathscr{A}^{(1)}_{\bar{A}} \otimes \mathscr{A}^{(2)}_A$ identical to ψ. Let ω_- its truncation to $\bar{A}$. Similarly, let ω_+ be a truncation to A of a pure state on $\mathscr{A}^{(2)}_{\bar{A}} \otimes \mathscr{A}^{(1)}_A$ identical to ψ. By Lemma 4.5, the state $(\omega_- \otimes \omega_+) \otimes \psi'_0 \otimes \psi_0$ is in the trivial phase. Therefore it is enough to show that

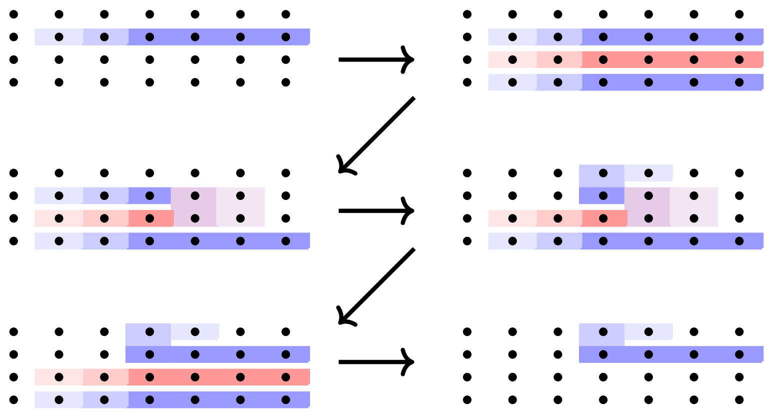


Figure 4.1: The sequence of steps that transforms $\psi_0 \otimes \phi_k \otimes \psi_0' \otimes \psi_0$ into $\tilde{\phi}_k \otimes \psi_0' \otimes \psi_0$. The regions shaded in blue denote the entangled parts of ψ, while the regions shaded in red denote the entangled parts of ψ'. The regions where the state is close to a factorized state are schematically indicated by a faded shading.

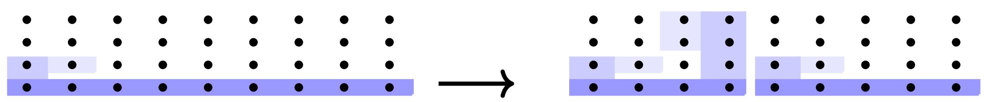


Figure 4.2: States $\tilde{\phi}_k$ and $\chi_k \otimes (\psi_0|_{k+L,\infty} \otimes \psi_0|_{k+L,\infty} \otimes \tilde{\phi}_{k+L})$

$(\omega_- \otimes \omega_+) \otimes \psi_0' \otimes \psi_0$ and $\psi_0 \otimes \psi \otimes \psi_0' \otimes \psi_0$ are related by an LGA. In fact, as we show below, such an LGA can be chosen to be almost localized at 0.

First, we apply to both states an LGA which acts only on the last two factors and produces $\psi' \otimes \psi$ out of $\psi_0' \otimes \psi_0$. This gives us $\psi_0 \otimes \psi \otimes \psi' \otimes \psi$ and $(\omega_- \otimes \omega_+) \otimes \psi' \otimes \psi$.

Second, we apply a composition of $\alpha^{(1)}_{\mathsf{F}|_{\bar{A}}}$ and $\alpha^{(1)}_{\mathsf{F}|_A}$ on $\mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$ to states $(\omega_- \otimes \omega_+) \otimes \psi'$ and $\psi_0 \otimes \psi \otimes \psi'$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$. This transformation is shown schematically in Fig. 4.3. By Lemma 4.2 this gives two states which are both g-close at 0 to a pure factorized state $\psi_0 \otimes \psi_0 \otimes \psi_0'$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$, for some $g \in \mathscr{F}^+_\infty$. By Proposition C.2 these states are related by a conjugation with an almost local unitary. Applying the first two steps backwards we conclude that the two original states are related by an LGA almost localized at 0.

□

4.2 Quantization of the Berry class for 1d invertible states.

Let ψ be a smooth family of zero-dimensional states parameterized by a smooth compact manifold $\mathcal{M}$. In this case the Berry class $[f] \in H^2(\mathcal{M}, i\mathbb{R})$ defined in Chapter 3 is quantized, i.e., $[f/(2\pi i)]$ belongs to the image on the canonical map

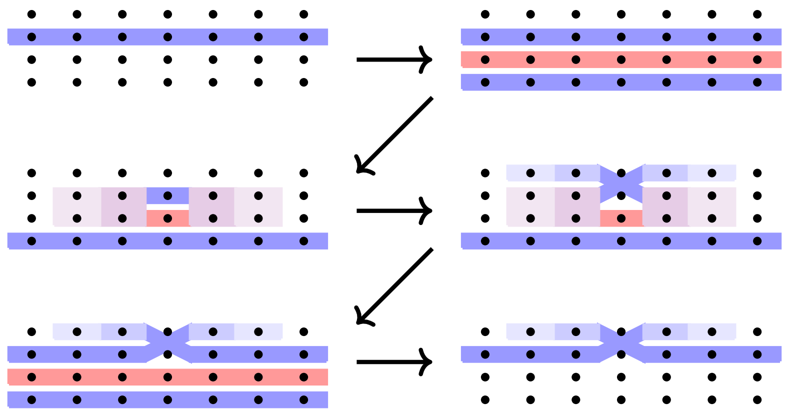


Figure 4.3: The sequence of steps that transforms $\psi_0 \otimes \psi \otimes \psi_0' \otimes \psi_0$ into $(\omega_- \otimes \omega_+) \otimes \psi_0' \otimes \psi_0$. The coloring in the same as in Fig. 4.1.

$H^2(\mathcal{M}, i\mathbb{Z}) \to H^2(\mathcal{M}, i\mathbb{R})$ and therefore

$$\frac{1}{2\pi i} \int_\Sigma f \in \mathbb{Z} \tag{4.11}$$

for any closed two-dimensional submanifold $\Sigma \subset \mathcal{M}$. One can deduce it by contemplating f as a curvature of a 1-form gauge field. Mathematically, such a gauge field describes a connection on the canonical line bundle over $\mathcal{M}$. If $\{U^{(\mathsf{a})}\}_{\mathsf{a}\in I}$ is good[2] open cover of $\mathcal{M}$, then a connection can be defined by a pair $(h^{(\mathsf{ab})} \in C^\infty(U^{(\mathsf{ab})}, U(1)), a^{(\mathsf{a})} \in \Omega^1(U^{(\mathsf{a})}, i\mathbb{R}))$ satisfying

$$h^{(\mathsf{ab})} h^{(\mathsf{bc})} = h^{(\mathsf{ac})}, \tag{4.12}$$

$$a^{(\mathsf{a})} - a^{(\mathsf{b})} = h^{(\mathsf{ab})-1} dh^{(\mathsf{ab})}, \tag{4.13}$$

where $U^{(\mathsf{ab})} = U^{(\mathsf{a})} \cap U^{(\mathsf{b})}$. The existence of such a pair with $f = da^{(\mathsf{a})}$ implies quantization of $[f/(2\pi i)]$.

In this section, we prove the quantization of the Berry class $[f] \in H^3(\mathcal{M}, i\mathbb{R})$ for a family of invertible states ψ of 1d spin systems parameterized by a smooth manifold $\mathcal{M}$, i.e., we show that $[f/(2\pi i)]$ belongs to the image on the canonical map $H^3(\mathcal{M}, i\mathbb{Z}) \to H^3(\mathcal{M}, i\mathbb{R})$ and therefore

$$\frac{1}{2\pi i} \int_\Sigma f \in \mathbb{Z} \tag{4.14}$$

for any closed three-dimensional submanifold $\Sigma \subset \mathcal{M}$. By analogy with the zero-dimensional case, we contemplate f as a curvature of a 2-form gauge field.

[2] By good we mean that all charts and all their non-empty intersections are diffeomorphic to an open ball.

Mathematically, such a gauge field corresponds a connection on a line bundle gerbe over $\mathcal{M}$ that can be defined by a triple $(h^{(\mathsf{abc})} \in C^\infty(U^{(\mathsf{abc})}, U(1)), a^{(\mathsf{ab})} \in \Omega^1(U^{(\mathsf{ab})}, i\mathbb{R}), b^{(\mathsf{a})} \in \Omega^2(U^{(\mathsf{a})}, i\mathbb{R}))$ satisfying

$$h^{(\mathsf{abc})} h^{(\mathsf{acd})} = h^{(\mathsf{abd})} h^{(\mathsf{bcd})}, \tag{4.15}$$

$$a^{(\mathsf{ab})} - a^{(\mathsf{ac})} + a^{(\mathsf{bc})} = h^{(\mathsf{abc})-1} dh^{(\mathsf{abc})}, \tag{4.16}$$

$$b^{(\mathsf{a})} - b^{(\mathsf{b})} = da^{(\mathsf{ab})}, \tag{4.17}$$

where $U^{(\mathsf{abc})} = U^{(\mathsf{a})} \cap U^{(\mathsf{b})} \cap U^{(\mathsf{c})}$. The existence of such a triple with $f = db^{(\mathsf{a})}$ implies quantization of $[f/(2\pi i)]$.

Remark 4.2. *Let $\mathcal{O}^\times$ be the sheaf of smooth $U(1)$-valued functions on $\mathcal{M}$ and Ω^n be the sheaf of smooth purely imaginary n-forms on $\mathcal{M}$. Then the triple $(h^{(\mathsf{abc})}, a^{(\mathsf{ab})}, b^{(\mathsf{a})})$ can be interpreted as a degree-2 Deligne-Beilinson 2-cocycle is a 2-cocycle of the totalization of the Čech cochain complex with coefficients in the complex of sheaves*

$$[\mathcal{O}^\times \xrightarrow{d \log} \Omega^1 \xrightarrow{d} \Omega^2]. \tag{4.18}$$

Deligne-Beilinson cocycle

Let ψ be a smooth family of invertible states of 1d spin systems over a compact smooth manifold $\mathcal{M}$. In the proof of quantization of $[f/(2\pi i)]$, without loss of generality we can assume that each state of the family is LGA-equivalent to a factorized state ψ_0. Indeed, we can always stack ψ with a constant family of states ψ' given by an inverse of any state of the family ψ. The resulting family $\psi \otimes \psi'$ has the same Berry class $[f]$.

We pick a partition (A_0, A_1) of $\mathbb{R}$ into half-lines and a connection $\mathsf{G}^\bullet$ that contains $(\mathsf{G}, \mathsf{g}^{(0)}, \mathsf{g}^{(1)})$ and defines the 3-form $f = \langle \mathsf{g}^{(1)}_{A_0 A_1} \rangle_\psi$.

Let us choose smooth families of LGAs $\beta^{(\mathsf{a})}, \beta_0^{(\mathsf{a})}, \beta_1^{(\mathsf{a})}$ over $U^{(\mathsf{a})}$ such that $\beta^{(\mathsf{a})} := \beta_0^{(\mathsf{a})} \circ \beta_1^{(\mathsf{a})}$, $\psi_0 = \psi \circ \beta^{(\mathsf{a})}$ with $\beta_0^{(\mathsf{a})}, \beta_1^{(\mathsf{a})}$ being almost localized on A_0 and A_1, respectively. Since $\psi = \psi_0 \circ \beta^{(\mathsf{a})-1}$, we can also choose smooth families of uniformly almost localized at the apex of (A_0, A_1) unitaries $\mathcal{U}^{(\mathsf{ab})} \in C^\infty(U^{(\mathsf{ab})}, \mathscr{A}_{a\ell})$ such that $\langle \mathcal{U}^{(\mathsf{ab})-1} d\mathcal{U}^{(\mathsf{ab})} \rangle_\infty = 0$ and $\psi = \psi \circ \beta^{(\mathsf{ab})}$ for $\beta^{(\mathsf{ab})} = \mathrm{Ad}_{\mathcal{U}^{(\mathsf{ab})}} \circ \beta_1^{(\mathsf{a})} \circ (\beta_1^{(\mathsf{b})})^{-1}$. We define smooth families of unitaries $\mathcal{U}^{(\mathsf{abc})} = \beta^{(\mathsf{ab})}(\mathcal{U}^{(\mathsf{bc})}) \mathcal{U}^{(\mathsf{ab})} (\mathcal{U}^{(\mathsf{ac})})^{-1}$ over $U^{(\mathsf{abc})}$. Since $\mathcal{U}^{(\mathsf{abc})}$ preserves the family of states, its average $\langle \mathcal{U}^{(\mathsf{abc})} \rangle_\psi$ defines a smooth $U(1)$-valued function. On $U^{(\mathsf{abcd})}$ we have

$$\mathcal{U}^{(\mathsf{abc})} \mathcal{U}^{(\mathsf{acd})} = \beta^{(\mathsf{ab})}(\mathcal{U}^{(\mathsf{bcd})}) \mathcal{U}^{(\mathsf{abd})} \tag{4.19}$$

and therefore

$$\langle \mathcal{U}^{(\mathsf{abc})}\rangle_\psi \langle \mathcal{U}^{(\mathsf{acd})}\rangle_\psi = \langle \mathcal{U}^{(\mathsf{abd})}\rangle_\psi \langle \mathcal{U}^{(\mathsf{bcd})}\rangle_\psi . \tag{4.20}$$

Thus, $\langle \mathcal{U}^{(\mathsf{abc})}\rangle_\psi$ defines a Čech 2-cocycle and a cohomology class in $H^2(\mathcal{M}, \mathcal{O}^\times) \cong H^3(\mathcal{M}, \mathbb{Z})$.

We define $\mathsf{B}^{(\mathsf{ab})} = \beta^{(\mathsf{ab})-1} d\beta^{(\mathsf{ab})} \in \Omega^1(U^{(\mathsf{ab})}, \mathfrak{D}^\psi_{al})$. By local Noether theorem for gapped states we can choose $\mathsf{A}^{(\mathsf{a})} \in \Omega^1(U^{(\mathsf{a})}, \mathfrak{D}_{al})$ that differs from $\mathsf{G}|_{A_0} - \beta^{(\mathsf{a})}(\beta^{(\mathsf{a})-1} d\beta^{(\mathsf{a})})|_{A_1}$ by a smooth family $\Omega^1(U^{(\mathsf{a})}, \mathfrak{d}_{al})$ uniformly almost localized at the apex of (A_0, A_1) and therefore interpolates between G far from A_1 and $-\beta^{(\mathsf{a})}(\beta^{(\mathsf{a})-1} d\beta^{(\mathsf{a})})$ far from A_0. Note that $\mathsf{B}^{(\mathsf{ab})}$ interpolates between 0 and $(\mathsf{A}^{(\mathsf{b})} - \beta^{(\mathsf{ab})-1}(\mathsf{A}^{(\mathsf{a})}))$.

By setting[3]

$$h^{(\mathsf{abc})} := \langle \mathcal{U}^{(\mathsf{abc})}\rangle_\psi , \tag{4.21}$$

$$a^{(\mathsf{ab})} = \langle (\mathsf{B}^{(\mathsf{ab})} - \mathsf{A}^{(\mathsf{b})} + \beta^{(\mathsf{ab})-1}(\mathsf{A}^{(\mathsf{a})}))\rangle_\psi , \tag{4.22}$$

$$b^{(\mathsf{a})} = \langle (d\mathsf{A}^{(\mathsf{a})} + \frac{1}{2}\{\mathsf{A}^{(\mathsf{a})}, \mathsf{A}^{(\mathsf{a})}\} - \mathsf{g}^{(0)}_{A_0})\rangle_\psi , \tag{4.23}$$

with

$$h^{(\mathsf{abc})-1} dh^{(\mathsf{abc})} = \langle (\beta^{(\mathsf{bc})-1}(\mathsf{B}^{(\mathsf{ab})}) - \mathsf{B}^{(\mathsf{ac})} + \mathsf{B}^{(\mathsf{bc})})\rangle , \tag{4.24}$$

$$da^{(\mathsf{ab})} = \langle (\beta^{(\mathsf{ab})-1}(d\mathsf{A}^{(\mathsf{a})} + \frac{1}{2}\{\mathsf{A}^{(\mathsf{a})}, \mathsf{A}^{(\mathsf{a})}\}) - (d\mathsf{A}^{(\mathsf{b})} + \frac{1}{2}\{\mathsf{A}^{(\mathsf{b})}, \mathsf{A}^{(\mathsf{b})}\}))\rangle , \tag{4.25}$$

$$db^{(\mathsf{a})} = -\langle D\mathsf{g}^{(0)}_{A_0}\rangle = -\langle \mathsf{g}^{(1)}_{A_0 A_1}\rangle = -f , \tag{4.26}$$

we get the triple $(h^{(\mathsf{abc})}, a^{(\mathsf{ab})}, b^{(\mathsf{a})})$ satisfying the relations eq. (4.15) and $-f = db^{(\mathsf{a})}$. That implies quantization of the Berry class $[f/(2\pi i)]$.

Remark 4.3. *In contrast to the zero-dimensional case, the construction of a connection and its curvature f on a line bundle gerbe involves various choices and therefore is far from being canonical. Nevertheless, the cohomology class in $H^3(\mathcal{M}, \mathbb{Z})$ that defines the isomorphism class of line bundle gerbes is unambiguous as it is mapped to the Berry class.*

Remark 4.4. *Note that if there is a smooth family of LGAs β over $\mathcal{M}$ such that $\psi_0 = \psi \circ \beta$, then the Berry class vanishes. Therefore non-triviality of $[f/(2\pi i)]$ gives an obstruction to the existence of such β.*

[3] Note that all derivations inside the averages below are almost localized at a point.

Chapter 5

HALL CONDUCTANCE AS THE TOPOLOGICAL INVARIANT

Quantization of the Hall conductance of insulating two-dimensional materials at low temperatures is one of the most remarkable phenomena in condensed matter physics. Starting with the seminal work of Laughlin [44], many theoretical explanations of this phenomenon have been proposed which vary in their assumptions and degree of rigor. For systems of non-interacting fermions with either an energy gap or a mobility gap there are several proofs that zero-temperature Hall conductance σ_{Hall} times $2\pi\hbar/e^2$ is an integer in the infinite-volume limit [62, 9, 10, 8, 1]. In particular, Laughlin's flux-insertion argument was made rigorous in this setting by Avron, Seiler and Simon [1]. The case of interacting systems is more involved, since quantization of Hall conductance generally holds only in the absence of topological order. Much progress can be made using the relation between σ_{Hall} and the curvature of the Berry connection of the system compactified on a large torus. This line of work originated with Avron and Seiler [2] and culminated in the proof by Hastings and Michalakis that for a gapped system on a large torus with a non-degenerate ground state the difference between $2\pi\hbar\sigma_{Hall}/e^2$ and the nearest integer is almost-exponentially small in the size of the system L, as defined in [29] (see also [5] for a simplified proof).

In this chapter, we apply the formalism described above to the case of 2d spin systems with $U(1)$ symmetry. Since it allows to work directly in the thermodynamic limit rather than on a particular compact surface, it gives a way to define the Hall conductance as an invariant of a phase that can be computed locally. As was already mentioned in Example 3.1, this topological invariant is a particular case of an equivariant higher Berry class σ which is defined for any state with acyclic Noether complex $\mathcal{N}^{\psi}_{\bullet}$ that makes it manifestly independent of a parent gapped Hamiltonian for the state. We confirm this claim in Section 5.1.

We further show that for states for which this invariant is defined, there is a natural way to produce new states which can be interpreted as vortices or magnetic flux insertions. Such states may represent states from different superselection sectors and correspond to abelian anyons. We compute the charge and the statistic of such anyons and relate it to σ in Section 5.2 providing a formalization of the argument from [59]. Using these results, we give proof of the quantization $\sigma \in \mathbb{Z}$ for invertible states in Section 5.3. For bosonic invertible spin systems, our methods allow us to show $\sigma \in 2\mathbb{Z}$ which is stronger than the result from [29, 5].

5.1 $U(1)$-equivariant Berry class as a Hall conductance

In this section we consider 2d spin systems with on-site $U(1)$ symmetry. We denote the automorphism that corresponds to a group element $g = e^{i\phi}$ by $\gamma^{(\phi)} \in \mathrm{Aut}(\mathscr{A})$. It is generated by a constant derivation $\mathsf{Q} \in \mathfrak{D}_{al}$. As we require $U(1)$ action to be on-site, Q is given by $\partial\mathsf{q}$ for a 0-chain $\mathsf{q} \in C_0(\mathfrak{d}_{al})$ such that $\mathsf{q}_j \in \mathfrak{d}_j$. We can interpret q_j as the operator of the charge on site j. For a region A, we let $\gamma_A^{(\phi)}$ be the automorphism generated by a constant derivation q_A. It corresponds to the action of the symmetry restricted to the region A.

In the following we often use conical partitions of the 2d plane (as defined in Section 2.2) and call them tripartitions.

Definition 5.1. *Let $\mathscr{A}$ be a 2d spin system with on-site $U(1)$ symmetry. We say that a pure $U(1)$-invariant state ψ of $\mathscr{A}$ has no local spontaneous symmetry breaking if there exist $U(1)$-invariant $\mathsf{k} \in C_1(\mathfrak{d}_{al})$ such that $\tilde{\mathsf{q}} := \mathsf{q} - \partial\mathsf{k}$ does not excite ψ.*

Informally, a state ψ has no local spontaneous symmetry breaking if the action of the charge operator inside a region on this state can be compensated by the action of a UAL Hamiltonian almost localized on the boundary of this region. Indeed, for any region A, since $\tilde{\mathsf{q}}_A$ does not excite the state, the action of q_A can be compensated by the action of $\mathsf{k}_{A\bar{A}}$.

Remark 5.1. *In the Definition 5.1, we can equivalently require the existence of $U(1)$-invariant $\tilde{\mathsf{q}} \in C_0(\mathfrak{d}^{\psi}_{al})$ satisfying $\partial\tilde{\mathsf{q}} = \mathsf{Q}$ as the existence of k would follow from local Noether theorem 2.3.*

Remark 5.2. *Note that if ψ has no local spontaneous symmetry breaking, then so does any state $\psi \circ \beta$ for a $U(1)$-invariant LGA β. Indeed, for the state $\psi \circ \beta$ we can use $\beta^{-1}(\tilde{\mathsf{q}})$ instead of $\tilde{\mathsf{q}}$.*

Example 5.1. *Any gapped $U(1)$ invariant state has no local spontaneous symmetry breaking as the existence of $\tilde{\mathsf{q}}$ follows from local Noether theorem for gapped states 3.1. More explicitly, given a choice of a gapped $U(1)$-invariant Hamiltonian $\mathsf{H} \in \mathfrak{D}_{al}$ we can always choose $U(1)$-invariant $\mathsf{h} \in C_0(\mathfrak{d}_{al})$ such that $\mathsf{H} = \partial\mathsf{h}$. Then we can set k to be $\mathscr{J}_{\mathsf{H},W_\Delta}(-\mathsf{j})$ where $\mathscr{J}_{\mathsf{H},W_\Delta}$ is the map from Lemma 3.3 and $\mathsf{j} := -\{\mathsf{h},\mathsf{q}\}$ is the electric current.*

For a state with no local spontaneous symmetry breaking ψ, one can define the following topological invariant:

Proposition 5.1. *Let ψ be a pure $U(1)$-invariant state with no local spontaneous symmetry breaking. Choose k and a tripartition (A_0, A_1, A_2). Then*

$$\sigma := 2\pi i \langle \{\mathsf{k}, \partial \mathsf{k}\}_{A_0 A_1 A_2} \rangle_\psi \tag{5.1}$$

does not depend on the choice of the tripartition, the choice of k, and is the same for any two states related by a $U(1)$-invariant UAL evolution.

Proof. Note that $\partial \langle \{\mathsf{k}, \partial \mathsf{k}\} \rangle_\psi = \langle \{\mathsf{q} - \tilde{\mathsf{q}}, \mathsf{q} - \tilde{\mathsf{q}}\} \rangle_\psi = 0$, where we have used that $\{\mathsf{q}, \mathsf{q}\} = 0$ and $\langle \{\tilde{\mathsf{q}}, \cdot\} \rangle_\psi = 0$. The conservation of the 2-current $\langle \{\mathsf{k}, \partial \mathsf{k}\} \rangle_\psi$ implies that the contraction $\langle \{\mathsf{k}, \partial \mathsf{k}\}_{A_0 A_1 A_2} \rangle_\psi$ does not depend on the tripartition.

Let X and Y be the right and the upper half-plane, respectively. As σ does not depend on the choice of the tripartition, we have $\sigma = 2\pi i \langle [\mathsf{k}_{X X^c}, \mathsf{k}_{Y Y^c}] \rangle_\psi$. Suppose we have a different choice of the current k that we denote k'. We have $\langle [\mathsf{k}_{X X^c}, \mathsf{k}_{Y Y^c}] \rangle_\psi = \langle [\mathsf{q}_X, \mathsf{k}_{Y Y^c}] \rangle_\psi = \langle [\mathsf{k}'_{X X^c}, \mathsf{k}_{Y Y^c}] \rangle_\psi = \langle [\mathsf{k}'_{X X^c}, \mathsf{q}_Y] \rangle_\psi = \langle [\mathsf{k}'_{X X^c}, \mathsf{k}'_{Y Y^c}] \rangle_\psi$. Therefore σ does not depend on the choice of k.

We can always split a $U(1)$-invariant derivation $\mathsf{F} = \mathsf{F}_+ + \mathsf{F}_-$ into $U(1)$-invariant derivations F_+, F_- almost localized on an upper and a lower half-plane, respectively. Therefore, it is enough to show the invariance of σ under a $U(1)$-invariant UAL evolution that is almost localized on a half-plane X. Let β be the corresponding LGA. We can compute σ for the state $\psi \circ \beta$ using a tripartition with the apex in an opposite half-plane X^c far away from X, where its value coincides with the value of σ for the state ψ. Since σ does not depend on the tripartition, it is the same for states ψ and $\psi \circ \beta$. □

Remark 5.3. *If ψ is a gapped $U(1)$-invariant state, then σ coincides with the equivariant Berry class from Example 3.1. Indeed, we have $\langle \{\mathsf{k}, \partial \mathsf{k}\}_{A_0 A_1 A_2} \rangle_\psi = \langle \{\mathsf{k}, \mathsf{q} - \tilde{\mathsf{q}}\}_{A_0 A_1 A_2} \rangle_\psi = \langle \{\mathsf{k}, \mathsf{q} + \tilde{\mathsf{q}}\}_{A_0 A_1 A_2} \rangle_\psi$ while $\partial \{\mathsf{k}, \mathsf{q} + \tilde{\mathsf{q}}\} = -\{\tilde{\mathsf{q}}, \tilde{\mathsf{q}}\} = 2\, \partial \mathsf{m}^{(2)}$. Thus, we have $\sigma = 4\pi i \langle \mathsf{m}^{(2)}_{A_0 A_1 A_2} \rangle_\psi = -4\pi i \langle [\tilde{\mathsf{q}}_{A_0}, \tilde{\mathsf{q}}_{A_1}] \rangle_\psi$.*

Let us show that for gapped states, $\sigma/2\pi$ is nothing but the zero-temperature Hall conductance. Suppose ψ is a ground state of a gapped $U(1)$-invariant Hamiltonian H. As noted in Example 5.1, we can set $\mathsf{k} = \mathscr{I}_{\mathsf{H}, W_\Delta}(\{\mathsf{h}, \mathsf{q}\})$ where $W_\Delta(t)$ is the function from Lemma 3.3 and h is a $U(1)$-invariant 0-chain such that $\mathsf{H} = \partial \mathsf{h}$. Let

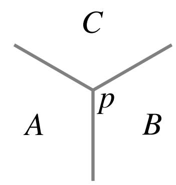


Figure 5.1: A tripartition (A, B, C) with the apex p.

$J_{jk} := i\,\pi_\psi(\mathsf{j}_{jk})$ be the images of the components of the electric current in the GNS representation of ψ. We denote the operator that corresponds to the Hamiltonian by H. Let X and Y be the right and the upper half-planes, respectively. The Hall conductance is given by the Kubo formula [54]

$$\sigma_{Hall} = \sum_{j\in X}\sum_{k\in X^c}\sum_{l\in Y}\sum_{m\in Y^c} i\langle 0|J_{jk}(1-P)\frac{1}{H^2}(1-P)J_{lm}|0\rangle - (X\leftrightarrow Y) \tag{5.2}$$

where $|0\rangle$ is a cyclic vector for the GNS representation and $P = |0\rangle\langle 0|$. Although the regions X, X^c, Y, Y^c are non-compact, the quadruple sum is absolutely convergent and thus well-defined. To see this, we note that for any two observables $\mathcal{A}$ and $\mathcal{B}$ we have

$$\begin{aligned}\langle 0|\pi_\psi(\mathcal{A})(1-P)\frac{1}{H^2}(1-P)\pi_\psi(\mathcal{B})|0\rangle =& \\ = \langle \mathscr{I}_{\mathsf{H},W_\Delta}(\mathcal{A})\mathscr{I}_{\mathsf{H},W_\Delta}(\mathcal{B})\rangle_\psi - \langle \mathscr{I}_{\mathsf{H},W_\Delta}(\mathcal{A})\rangle_\psi \langle \mathscr{I}_{\mathsf{H},W_\Delta}(\mathcal{B})\rangle_\psi.&\end{aligned} \tag{5.3}$$

Therefore we can rewrite the formula for the Hall conductance in terms of correlators of almost local observables $\mathsf{k}_{jk} = -\mathscr{I}_{\mathsf{H},W_\Delta}(\mathsf{j}_{jk})$:

$$\sigma_{Hall} = \sum_{j\in X}\sum_{k\in X^c}\sum_{l\in Y}\sum_{m\in Y^c} i\langle \mathsf{k}_{jk}\mathsf{k}_{lm}\rangle - (X\leftrightarrow Y) = i\langle[\mathsf{k}_{XX^c}, \mathsf{k}_{YY^c}]\rangle_\psi = \sigma/(2\pi). \tag{5.4}$$

5.2 Vortices and their statistics

Given a $U(1)$-invariant state ψ with no local spontaneous symmetry breaking, a choice of $\mathsf{k} \in C_1(\mathfrak{d}_{al})$ and a point $p \in \mathbb{R}^2$, one can produce a natural class of states that can be interpreted as states of a vortex (unit of magnetic flux) at p. To define it, let us make a choice of a tripartition (A, B, C) with the apex at p (see Fig. 5.2a). Let $\upsilon^{(\phi)}_{ABC}$ be the path of LGAs generated by $(\mathsf{q}_A - \mathsf{k}_{AB})$. The states of interest have the form $\psi \circ \upsilon^{(2\pi)}_{ABC}$.

Remark 5.4. *Alternatively, we can consider the path of LGAs $\tilde{\upsilon}^{(\phi)}_{ABC} = \upsilon^{(\phi)}_{ABC} \circ \gamma^{(-\phi)}_A$ that is generated by $\gamma^{(\phi)}_A(\mathsf{k}_{BA})$ which is almost localized on the path AB as $\tilde{\upsilon}^{(2\pi)}_{ABC} = \upsilon^{(2\pi)}_{ABC}$.*

Proposition 5.2. *Two vortex states at the same point p but with different choices of tripartitions (A, B, C) are related by an almost local unitary.*

Lemma 5.1. *Let $\mathsf{X} \in \mathfrak{D}_{al}$ and $\mathsf{Y} \in \mathfrak{D}_{al}$ such that $[\mathsf{X}, \mathsf{Y}] \in \mathfrak{d}_{al}$. Then the automorphism $\alpha^{(t)}_{\mathsf{X}} \circ \alpha^{(t)}_{\mathsf{Y}} \circ (\alpha^{(t)}_{\mathsf{X+Y}})^{-1}$ is a conjugation with an almost local unitary.*

Proof. We have

$$\frac{d}{dt}\left(\alpha^{(t)}_{\mathsf{X}} \circ \alpha^{(t)}_{\mathsf{Y}} \circ (\alpha^{(t)}_{\mathsf{X+Y}})^{-1}\right) = \int_0^1 ds \ \left(\alpha^{(t)}_{\mathsf{X}} \circ \alpha^{(s)}_{\mathsf{Y}} \circ [\mathsf{X}, \mathsf{Y}] \circ \alpha^{(t-s)}_{\mathsf{Y}} \circ (\alpha^{(t)}_{\mathsf{X+Y}})^{-1}\right). \tag{5.5}$$

Therefore, $\left(\alpha^{(t)}_{\mathsf{X}} \circ \alpha^{(t)}_{\mathsf{Y}} \circ (\alpha^{(t)}_{\mathsf{X+Y}})^{-1}\right)$ is generated by $\int_0^1 ds \ \alpha^{(t)}_{\mathsf{X+Y}}(\alpha^{(s-t)}_{\mathsf{Y}}([\mathsf{X}, \mathsf{Y}])) \in \mathfrak{d}_{al}$. □

Proof of Proposition 5.2. We can change the paths BC, CA, and AB by reassigning a cone-like region E from one cone of the tripartition to another (so that the resulting tripartition is still well-defined). By Lemma 5.1, in order to show that υ_{ABC} changes by composition with $\mathrm{Ad}_{\mathcal{U}}$ for some almost local unitary $\mathcal{U}$, it is enough to show that the change X of the generating derivation $\mathsf{Y} = \mathsf{q}_A - \mathsf{k}_{AB}$ is such that $[\mathsf{X}, \mathsf{Y}] \in \mathfrak{d}_{al}$ and $\psi \circ \alpha^{(2\pi)}_{\mathsf{X}}$ can be produced from ψ by conjugation with an almost local unitary.

If we vary BC this way, we have $\mathsf{X} = \mathsf{k}_{EA}$ that is almost localized at the origin. If we vary AC, then $\mathsf{X} = (\mathsf{q}_E - \mathsf{k}_{EB})$. We have $\mathsf{k}_{EB} \in \mathfrak{d}_{al}$ and $[\mathsf{q}_E, \mathsf{q}_A - \mathsf{k}_{AB}] \in \mathfrak{d}_{al}$, while q_E generates the identity automorphism at $\phi = 2\pi$. Finally, if we vary AB, then a change of the generating derivation is given by $(\tilde{\mathsf{q}}_E - \mathsf{k}_{CE})$. We have $\mathsf{k}_{CE} \in \mathfrak{d}_{al}$ and $[\tilde{\mathsf{q}}_E, \mathsf{q}_A - \mathsf{k}_{AB}] = [\tilde{\mathsf{q}}_E, \tilde{\mathsf{q}}_A] + [\tilde{\mathsf{q}}_E, \mathsf{k}_{AC}] \in \mathfrak{d}_{al}$, while $\tilde{\mathsf{q}}_E$ generates the automorphism that preserves ψ. □

Remark 5.5. *If the state ψ is gapped, one can also relate vortex states for different choices of k. Suppose we have another choice of k which we denote k'. Since $\partial(\mathsf{k}' - \mathsf{k}) \in C_0(\mathfrak{d}^{\psi}_{al})$, by local Noether theorem for gapped states 3.1 there is $\tilde{\mathsf{k}} \in C_1(\mathfrak{d}_{al})$, such that $\partial(\mathsf{k}' - \mathsf{k}) = \partial\tilde{\mathsf{k}}$. By local Noether theorem 2.3, there is $\mathsf{n} \in C_2(\mathfrak{d}^{\psi}_{al})$ such that $\mathsf{k}' = \mathsf{k} + \tilde{\mathsf{k}} + \partial\mathsf{n}$. One can split the cone A into two sub-cones A' and A'' so that $(\mathsf{q}_A - \mathsf{k}_{AB} - \tilde{\mathsf{k}}_{AB}) = (\tilde{\mathsf{q}}_{A'} + \mathsf{k}_{AC}) + (\tilde{\mathsf{q}}_{A''} - \tilde{\mathsf{k}}_{AB})$ and $[\tilde{\mathsf{q}}_{A'} + \mathsf{k}_{AC}, \tilde{\mathsf{q}}_{A''} - \tilde{\mathsf{k}}_{AB}] \in \mathfrak{d}_{al}$. Since $(\tilde{\mathsf{q}}_{A''} - \tilde{\mathsf{k}}_{AB})$ generates automorphism that preserves ψ, by Lemma 5.1 and Proposition 5.2 modification of k by $\tilde{\mathsf{k}}$ produces the vortex states related to the original one by an almost local unitary. The lemma also implies that the same is true for modification of k by $\partial\mathsf{n}$ as it changes k_{AB} by $\mathsf{n}_{ABC} \in \mathfrak{d}_{al}$.*

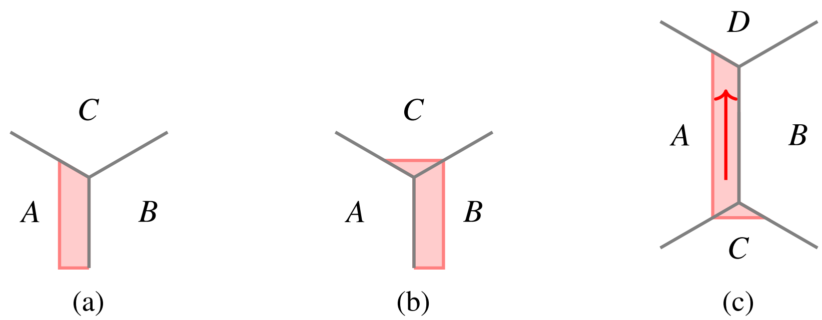


Figure 5.2: Creation, annihilation and transport of vortices. The shaded region covers sites, for which q_j are involved.

The Proposition 5.2 has an important consequence. Let $\mathcal{A}$ be an observable almost localized at a point x at distance r from the apex p of a tripartition (A, B, C). Let us choose a cone Σ with the apex at p and not containing x. We can choose another tripartition (A', B', C') obtained by a rotation around the apex so that the path $A'B'$ is inside Σ. Then using Proposition 5.2, we get

$$\langle \mathcal{A} \rangle_{\psi \circ \upsilon^{(2\pi)}_{ABC}} = \langle \mathcal{U} \mathcal{A} \mathcal{U}^{-1} \rangle_{\psi \circ \upsilon^{(2\pi)}_{A'B'C'}} + \mathcal{O}(r^{-\infty}) = \\ = \langle \mathcal{A} \rangle_{\psi \circ \upsilon^{(2\pi)}_{A'B'C'}} + \mathcal{O}(r^{-\infty}) = \langle \mathcal{A} \rangle_{\psi} + \mathcal{O}(r^{-\infty}). \quad (5.6)$$

This implies that almost local observables localized in any cone with the apex at p and far from p cannot detect the presence of a vortex at p. (This statement, however, might not be true if we consider local observables localized on a ring around p. In this case one cannot deform the paths such that there is no intersection between the ring and these paths.) In particular, this implies that for a gapped state ψ and any choice of the corresponding gapped $\mathsf{H} \in \mathfrak{D}_{al}$, the state $\psi \circ \upsilon^{(2\pi)}_{ABC}$ has finite energy.

We can also compute the charge of the vortex. The following proposition shows that it is given by σ.

Proposition 5.3. *Let $\tilde{\psi} = \psi \circ \upsilon^{(2\pi)}_{ABC}$ be a vortex state at p. Let $D = B_p(r)$. Then $\langle \mathsf{q}_D \rangle_{\tilde{\psi}} - \langle \mathsf{q}_D \rangle_{\psi} = -i\sigma + \mathcal{O}(r^{-\infty})$.*

Proof. We have

$$\alpha^{(\phi)}_{\mathsf{q}_A - \mathsf{k}_{AB}}(\mathsf{q}_D) - \mathsf{q}_D = \int_0^{\phi} \alpha^{(s)}_{\mathsf{q}_A - \mathsf{k}_{AB}}([\mathsf{q}_A - \mathsf{k}_{AB}, \mathsf{q}_D]) ds = \int_0^{\phi} \alpha^{(s)}_{\tilde{\mathsf{q}}_A + \mathsf{k}_{AC}}([-\mathsf{k}_{AB}, \mathsf{q}_D]) ds = \\ = \int_0^{\phi} \alpha^{(s)}_{\tilde{\mathsf{q}}_A}([\mathsf{q}_D, \mathsf{k}_{AB}]) ds + \mathcal{O}(r^{-\infty}) = \int_0^{\phi} \alpha^{(s)}_{\tilde{\mathsf{q}}_A}([\mathsf{q}_D, \mathsf{k}_{AB}]) ds + \mathcal{O}(r^{-\infty}). \quad (5.7)$$

Therefore,

$$\langle \mathsf{q}_D\rangle_{\tilde{\psi}} - \langle \mathsf{q}_D\rangle_{\psi} = \langle \alpha^{(2\pi)}_{\mathsf{q}_A-\mathsf{k}_{AB}}(\mathsf{q}_D) - \mathsf{q}_D\rangle_{\psi} =$$
$$= \langle \int_0^{2\pi} \alpha^{(s)}_{\tilde{\mathsf{q}}_A}([\mathsf{q}_D,\mathsf{k}_{AB}])ds\rangle_{\psi} + \mathcal{O}(r^{-\infty}) = \int_0^{2\pi} [\mathsf{k}_{DD^c},\mathsf{k}_{AB}]ds + \mathcal{O}(r^{-\infty}) = -i\sigma. \tag{5.8}$$

□

Similarly, one can define a family of LGAs $\bar{\upsilon}^{(\phi)}_{ABC}$ generated by $(\mathsf{q}_B + \mathsf{q}_C - \mathsf{k}_{BA})$ with the same properties (see Fig. 5.2b) and the state $\psi \circ \bar{\upsilon}^{(2\pi)}_{ABC}$. Note that $\upsilon^{(\phi)}_{ABC} \circ \bar{\upsilon}^{(\phi)}_{ABC} \circ \gamma^{(-\phi)} = \mathrm{Id}$ as

$$\frac{d}{d\phi}\left(\upsilon^{(\phi)}_{ABC} \circ \bar{\upsilon}^{(\phi)}_{ABC} \circ \gamma^{(-\phi)}\right) =$$
$$= \upsilon^{(\phi)}_{ABC} \circ (\mathsf{q}_A + \mathsf{q}_B + \mathsf{q}_C) \circ \bar{\upsilon}^{(\phi)}_{ABC} \circ \gamma^{(-\phi)} - \upsilon^{(\phi)}_{ABC} \circ \bar{\upsilon}^{(\phi)}_{ABC} \circ \mathsf{Q} \circ \gamma^{(-\phi)} = 0. \tag{5.9}$$

Therefore $\psi \circ \upsilon^{(2\pi)}_{ABC} \circ \bar{\upsilon}^{(2\pi)}_{ABC} = \psi$. It is natural to interpret this state as an anti-vortex. By applying automorphisms $\upsilon^{(2\pi)}$ and $\bar{\upsilon}^{(2\pi)}$ at different points and choosing the paths AB so that they do not overlap, one can create several vortices and anti-vortices at different locations.

In general, a vortex state cannot be produced by an action of an almost local observable (or even a quasi-local observable) on the ground state. However, one can create a vortex-anti-vortex pair by acting on the ground state with an LGA that is f-localized on a finite interval for f that depends on the localization of k (and therefore is a conjugation with an almost local unitary). For example, suppose one wants to create a vortex at the point ABD and an anti-vortex at BCA (see Fig. 5.2c). This can be accomplished using an automorphism generated by $\gamma^{(\phi)}_{A+C}(\mathsf{k}_{BA})$. By the Remark 2.5, there is a canonical path of almost local unitaries that corresponds to the path of LGAs $\alpha^{(\phi)}_{\gamma_{A+C}(\mathsf{k}_{BA})}$.

To study the transport of vortices along paths, we will first define the set of vortex configurations and paths of interest. We only consider configurations with a finite number of vortices, so the set of initial and final positions of vortices is always finite. These positions are vertices of a trivalent graph, some of whose edges connect the vertices and some go off to infinity. The paths needed to define vortex states are paths on this graph. For simplicity we will assume that the graph is a tree. The edges of the graph need not be straight lines or segments, but we need to assume that they

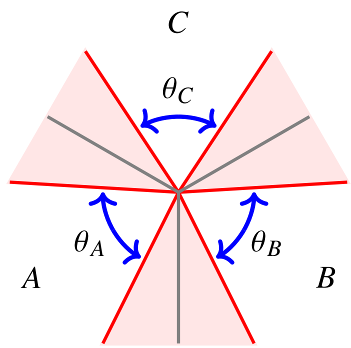


Figure 5.3: Admissible paths AB, BC, and CA meeting at the point ABC.

do not come close to each other. One way to achieve this is the following recursive procedure. Let us fix an angle θ_c and a vertex p with adjacent regions A,B,C (see Fig.5.3). Removing p will cause the graph to fall into three components each of which is itself a tree. We require that each component is contained in an eventually conical region with the apex at p such that the angles between adjacent boundaries of different regions are greater than θ_c. Then for each component, we take the vertex connected to p as the basepoint and repeat the procedure. Any trivalent graph which satisfies these requirements will be called admissible.

Lemma 5.2. *Let (A,B,C) be an admissible tripartition of the plane. Then the path of LGAs $\alpha^{(\phi)}_{\tilde{\mathsf{q}}_A} \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_B} \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_C} \circ \gamma^{(-\phi)}$ is generated by $\mathsf{p} \in C([0,2\pi], \mathfrak{d}^{\psi}_{al})$ that is f-localized at the apex of (A,B,C) for some f that depends on the localization of $\tilde{\mathsf{q}}$ only and $\langle \mathsf{p}(\phi)\rangle_\psi = i\phi\,\sigma/(4\pi)$.*

Proof. We have

$$\frac{d}{d\phi}\left(\alpha^{(\phi)}_{\tilde{\mathsf{q}}_A} \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_B} \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_C} \circ \gamma^{(-\phi)}\right) = \alpha^{(\phi)}_{\tilde{\mathsf{q}}_A} \circ (\tilde{\mathsf{q}}_{A+B}) \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_B} \circ (\tilde{\mathsf{q}}_C - \mathsf{Q}) \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_C} \circ \gamma^{(-\phi)} =$$
$$= \int_0^\phi ds\left(\alpha^{(\phi)}_{\tilde{\mathsf{q}}_A} \circ \alpha^{(s)}_{\tilde{\mathsf{q}}_B} \circ [\tilde{\mathsf{q}}_A, \tilde{\mathsf{q}}_B] \circ \alpha^{(\phi-s)}_{\tilde{\mathsf{q}}_B} \circ \alpha^{(\phi)}_{\tilde{\mathsf{q}}_C} \circ \gamma^{(-\phi)}\right). \quad (5.10)$$

Therefore the path of LGAs is generated by $\mathsf{p} \in C([0,2\pi], \mathfrak{d}_{al})$ where

$$\mathsf{p}(\phi) = \int_0^\phi ds\, \gamma^{(\phi)}(\alpha^{(-\phi)}_{\tilde{\mathsf{q}}_C}(\alpha^{(s-\phi)}_{\tilde{\mathsf{q}}_B}([\tilde{\mathsf{q}}_A, \tilde{\mathsf{q}}_B]))). \quad (5.11)$$

Since all automorphisms and derivations preserve ψ, we have $\langle \mathsf{p}(\phi)\rangle_\psi = \phi\langle[\tilde{\mathsf{q}}_A, \tilde{\mathsf{q}}_B]\rangle_\psi = i\phi\,\sigma/(4\pi)$. □

Let us consider a processes shown on Fig. 5.4. We assume that the graph formed by the lines and vertices is admissible and the distances between triple points are

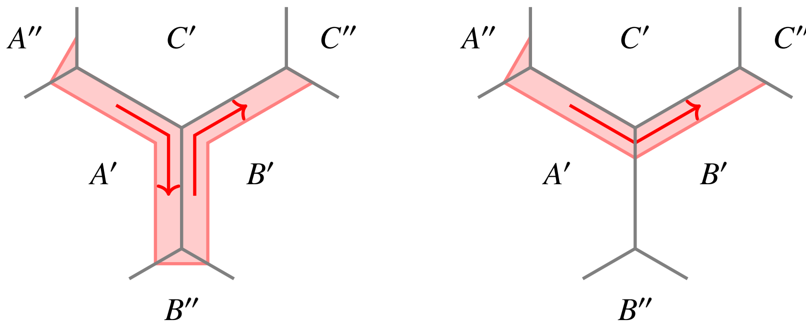


Figure 5.4: Transport of vortices.

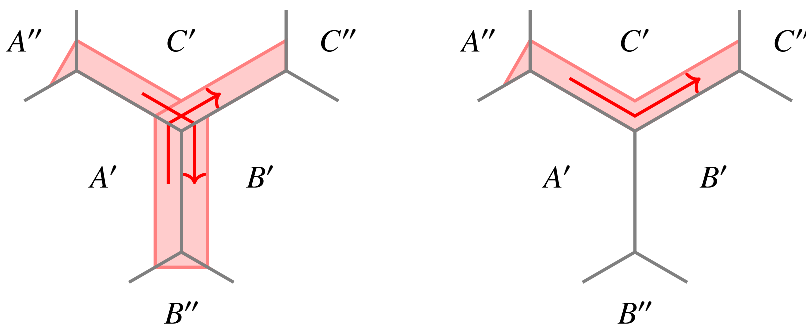


Figure 5.5: Transport of vortices along intersecting paths.

bigger than some L. In the first process we consequently produces two vortex-anti-vortex pairs so that the vortex of the first pair is annihilated by the anti-vortex of the second. We denote the corresponding unitaries performing the transport by $\mathcal{U}_A$ and $\mathcal{U}_B$. In the second process, we create a vortex-anti-vortex pair directly. The corresponding unitary is denoted $\mathcal{U}_{AB}$. Combining eq. (5.9), the proof of Lemma 5.2 and applying it to the path of unitaries $\mathcal{U}_A\gamma_A^{(\phi)}(\mathcal{U}_B\gamma_B^{(\phi)}(\mathcal{U}_{AB}^*))$, we get $\langle\mathcal{U}_B\mathcal{U}_A\mathcal{U}_{AB}^*\rangle_\psi = e^{i\pi\sigma/2} + \mathcal{O}(L^{-\infty})$.

We see that vortex-transport operators, for large paths $L \to \infty$ without "sharp" turns, compose in the expected way except for a phase $e^{\pi i\sigma/2}$. Similarly, one can show that if one transports a vortex so that shaded regions intersect (see Fig. 5.5), one gets a phase $e^{-\pi i\sigma/2}$.

One can create vortices and anti-vortices at the corners of a rectangle using transport operators 12 and 34 (see Fig. 5.6) and then annihilate them in a different order by applying transport operators 23 followed by 41. After the application of the operator 23 one gets the inverse of the application of the operator 41 times a phase factor $e^{\pi i\sigma/2}e^{\pi i\sigma/2}$. Therefore the net result of this operation is multiplication of the vacuum vector by $e^{\pi i\sigma}$ plus $\mathcal{O}(L^{-\infty})$ corrections.

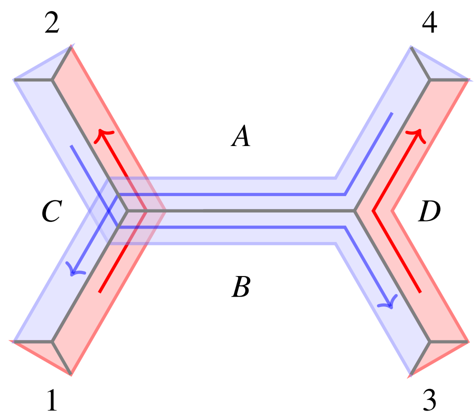


Figure 5.6: One first creates vortex/anti-vortex pairs by operators 12 and 34 (shaded in red), and then annihilates them by first applying operator 23, and then 41 (shaded in blue).

5.3 Quantization for invertible phases

In general, one cannot create a single vortex by applying some almost local or even quasi-local observable to the ground state vector, i.e., the single-vortex state and the ground state can belong to different superselection sectors. However, invertible systems are special in this regard.

Proposition 5.4. *Let ψ be an invertible $U(1)$-invariant state of a spin system $\mathscr{A}$ with $U(1)$ symmetry. Choose a corresponding k and a tripartition (A, B, C) with the apex at p. Then a vortex state $\tilde{\psi} = \psi \circ \upsilon^{(2\pi)}_{ABC}$ can be produced from ψ by a conjugation with an almost local unitary.*

Proof. Let $(\overline{\mathscr{A}}, \bar{\psi})$ be the inverse of $(\mathscr{A}, \psi)$, and let β be an LGA such that $(\psi\otimes\bar{\psi})\circ\beta = \Psi_0$ for a factorized state Ψ_0 on $\mathscr{A} \otimes \overline{\mathscr{A}}$.

Since $\upsilon^{(2\pi)}_{ABC}$ is f-localized on the path AB for some $f \in \mathscr{F}^+_\infty$ that depends on the localization of k only, the restriction of the state $(\tilde{\psi} \otimes \bar{\psi}) \circ \beta$ to C is g-close at p to $\Psi_0|_C$ for some $g \in \mathscr{F}^+_\infty$ depending on f. Choose a cone C'' inside C and a tripartition (A', B', C') with the same apex such that the path $A'B'$ is contained inside C''. Let $\tilde{\psi}' = \psi \circ \upsilon^{(2\pi)}_{A'B'C'}$. Since $\upsilon^{(2\pi)}_{A'B'C'}$ is f'-localized on the path AB for some $f' \in \mathscr{F}^+_\infty$ that depends on the localization of k only, the restriction of the state $(\tilde{\psi}' \otimes \bar{\psi}) \circ \beta$ to C^c is g'-close at p to $\Psi_0|_{C^c}$ for some $g' \in \mathscr{F}^+_\infty$ depending on f'. By Proposition 5.2, states $\tilde{\psi}$ and $\tilde{\psi}'$ are related by an almost local unitary. Therefore, the restrictions of the state $(\tilde{\psi} \otimes \bar{\psi}) \circ \beta$ to C and to C^c are g''-close at p to $\Psi_0|_C$ and $\Psi_0|_{C^c}$, respectively, for some $g'' \in \mathscr{F}^+_\infty$. By the same proof as in Lemma 4.2, we conclude that $(\tilde{\psi} \otimes \bar{\psi}) \circ \beta$

is h-close at p to Ψ_0 for some $h \in \mathscr{F}_\infty^+$, and therefore, by Proposition C.2 there is a unitary $\mathcal{V}_{ABC} \in \mathscr{A}_{a\ell}$, such that $(\tilde{\psi} \otimes \psi') \circ \beta = \Psi_0 \circ \mathrm{Ad}_{\mathcal{V}_{ABC}}$. □

Remark 5.6. *For an admissible tripartition (A, B, C), the localization of the unitary $\mathcal{V}_{ABC}$ that produces a vortex state depends on the localization of* k *only.*

The fact that a vortex can be produced by an almost local unitary implies that it has an integer charge as an operator representing Q in the GNS representation has an integer spectrum. Together with Proposition 5.3, it gives

Corollary 5.1. *Let ψ be an invertible $U(1)$-invariant state of a spin system $\mathscr{A}$. Then $\sigma \in \mathbb{Z}$.*

The relation between the statistics of vortices and the invariant σ, allows us to prove a stronger result.

Theorem 5.1. *Let ψ be an invertible $U(1)$-invariant state of a spin system $\mathscr{A}$. Then $\sigma \in \mathbb{Z}$. Moreover, if the system is bosonic, then $\sigma \in 2\mathbb{Z}$.*

Proof. Firstly, we consider the case of a bosonic spin system.

By Proposition 5.4 and Remark 5.6, for a fixed choice of k, there is $f \in \mathscr{F}_\infty^+$ such that any vortex state constructed using an admissible graph can be produced by an almost local unitary which is f-localized at the point of the vortex.

Let us consider the partition of the plane depicted on Fig. 5.7 with segments connecting triple points having length L. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be almost local unitary observables creating vortices at points 1 and 2 as shown on Fig. 5.7a and Fig. 5.7b, let $\mathcal{W}_{\bar{1}2}$ be a transport operator shown on Fig. 5.7c, and let $|0\rangle$ be the cyclic vector in the GNS representation $(\pi_\psi, \mathcal{H}_\psi)$ of ψ. Then, using the results from Section 5.2, we have

$$\begin{aligned}|0\rangle &= \pi_\psi\left((\mathcal{V}_2^{-1}\mathcal{V}_1)(\mathcal{V}_2\mathcal{V}_1^{-1})\right)|0\rangle + \mathcal{O}(L^{-\infty}) = \\ &= e^{-\pi i\sigma/2}\pi_\psi\left(\mathcal{V}_2^{-1}\mathcal{V}_1\mathcal{W}_{\bar{1}2}\right)|0\rangle + \mathcal{O}(L^{-\infty}) = e^{-\pi i\sigma}|0\rangle + \mathcal{O}(L^{-\infty}). \quad (5.12)\end{aligned}$$

As L can be chosen to be arbitrarily large, we have $\sigma \in 2\mathbb{Z}$.

For fermionic systems the arguments are the same, but the unitaries $\mathcal{V}_{1,2}$ relating the vortex states to the ground state can either (both) preserve or flip fermionic parity. In the former case, the unitaries have even fermionic parity, and the same arguments as

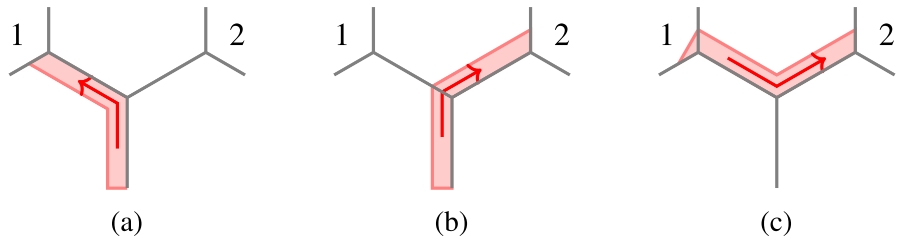


Figure 5.7: The processes corresponding to $\mathcal{V}_1$, $\mathcal{V}_2$ and $\mathcal{W}_{\bar{1}2}$.

above show that $\sigma \in 2\mathbb{Z}$. In the latter case, they have odd fermionic parity, and thus operators creating vortices at widely separated points anti-commute up to $\mathcal{O}(L^{-\infty})$ terms. The above argument then shows that σ is an odd integer. Thus vortices are bosons or fermions depending on whether σ is even or odd, in agreement with [59].

$\square$

Chapter 6

STATES OF 2D LATTICE SYSTEMS FROM VERTEX ALGEBRAS

It is believed that topological phases of matter in two dimensions can be classified by $(2+1)$-dimensional unitary topological quantum field theories (TQFT) and that the whole information about this TQFT is encoded in the entanglement structure of the ground state. In particular, for any such topological field theory, there should be a pure state of a lattice model on $\mathbb{R}^2$ that has the corresponding topological order.

There is a big class of non-chiral topological orders that can be realized by tensor network states defined using the Turaev-Viro construction. The string-net models introduced by M. Levin and X.-G. Wen [45, 23, 14] provide parent commuting projector Hamiltonians for such states. The situation is very different for chiral topological order as only a few chiral interacting topological models have been solved exactly [40]. One of the main features of such states is the non-existence of a boundary with short-range correlations. When the boundary modes of the system with such a ground state can be described by a conformal field theory, this obstruction manifests itself in the central charge of the chiral Virasoro algebra.

It was suggested a long time ago that the wave function of electrons for the ground state of a fractional quantum hall system is related to some vertex operator algebra [49], generalizing Laughlin's wave function. Similar ideas have been used to propose various candidates for chiral topologically ordered states of lattice systems [33, 58, 53]. While some expected properties of such states can be checked numerically, it is difficult to verify analytically that they are representatives of the correct topological phase. To the best of our knowledge, no general proposal exists for an arbitrary unitary TQFT. Even a lattice state for the simplest non-trivial invertible bosonic phase, known as Kitaev's E_8 phase, has not been described explicitly.

In this chapter, we propose class of states of lattice systems with infinite-dimensional on-site Hilbert spaces[1] associated with any simple unitary regular vertex operator algebra. We show that the states from this class are well-defined in the thermodynamic limit and have exponentially decaying correlations. We argue that they have the topological order associated with the vertex operator algebra and propose a way to

[1] The entanglement of each site in the construction is finite. The states can be well approximated by replacing the infinite-dimensional spaces with finite-dimensional ones of large enough dimension.

insert anyons. In particular, we obtain a lattice realization of a fractional quantum Hall state with such a topological order[2]. We argue that our states belong to the class for which flux insertions and the topological invariant associated with the Hall conductance can be defined in a similar way as in Chapter 5, and show that the latter coincides with the expected value. The usage of infinite-dimensional on-site Hilbert spaces allows us to use the full power of conformal invariance and get analytic results by relating state averages of observables of a lattice system to correlators of a certain auxiliary tensor network statistical model that was recently introduced in [13]. It also allows us to map the problems of computing invariants of states and determining the properties of topological excitations to evaluations of correlation functions in CFT which can be done exactly. We emphasize that our chiral states are not tensor network states. However, we show that they can be obtained as a limit of a family of non-chiral tensor network states together with a decoupled complex conjugated copy.

Our exposition would be more informal in this chapter compared to the previous ones. We give a precise description of the class of states and show that it is well-defined, but we only sketch the proofs of various properties. A more detailed account needs a generalization of the methods from previous chapters to the case of lattice systems with infinite-dimensional on-site Hilbert spaces and will appear elsewhere.

6.1 General construction

Preliminaries

Vertex operator algebra

Let V be a simple unitary regular vertex operator algebra with the vacuum vector $|0\rangle \in \mathsf{V}$ and vertex operators $Y(\cdot, z) : \mathsf{V} \to \operatorname{End} \mathsf{V}[[z^{\pm 1}]]$. The energy-momentum tensor (the vertex operator for the conformal element) is denoted $T(z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$ with the central charge c. Let $\operatorname{Rep}(\mathsf{V})$ be the category of representations of V, which has the structure of a unitary modular tensor category [31]. We denote the finite set of simple objects of $\operatorname{Rep}(\mathsf{V})$ by I with the trivial object denoted $a = \mathbf{1}$ and with the dual label being $\bar{a}$. We let $\mathsf{V}^{(a)}$ be the corresponding modules, and let $\mathcal{V}^{(a)}$ be the corresponding Hilbert spaces. We choose an orthonormal basis $e_m^{(a)}$ in $\mathsf{V}^{(a)}$ and the corresponding basis $|e_m^{(a)}\rangle$ in $\mathcal{V}^{(a)}$ labeled by integers $m \in \mathbb{N}_0$ such that $|e_m^{(a)}\rangle$ are eigenvectors of L_0 with the weight $h^{(a)}(m)$ and $h^{(a)}(m) \leq h^{(a)}(m')$ if $m < m'$. When we omit the label a, we mean $a = \mathbf{1}$.

[2] For an integer quantum Hall phase, a commuting projector Hamiltonian model has been proposed in [17].

By the state-operator correspondence, any vector $|e_m\rangle \in \mathcal{V}$ on the boundary of the unit disk D can be obtained by the insertion of $Y(e_m, 0)$. More generally, if we have a holomorphic embedding of the unit disk $\varphi : D \to \mathbb{C}$, in order to obtain a state $|e_m\rangle$ on the boundary of the closure of the image, we need to insert (see, e.g., Section 5.4 of [20])

$$\mathcal{O}_\varphi(e_m) := Y(R_\varphi e_m, \varphi(0)) \tag{6.1}$$

where

$$R_\varphi = v_0^{L_0} \exp\left(\sum_{m=1}^{\infty} v_m L_m\right) = \exp\left(\sum_{m=1}^{\infty} \frac{v_m}{v_0^m} L_m\right) v_0^{L_0} \tag{6.2}$$

and $v_n \in \mathbb{C}$ are given by the solution of

$$v_0 \exp\left(\sum_{m=1}^{\infty} v_m z^{m+1} \partial_z\right) z = \varphi(z) - \varphi(0). \tag{6.3}$$

We also introduce $\langle \alpha, \varphi|$ as the covector that is parallel to $\langle 0|\mathcal{O}_\varphi(\alpha)^\dagger$ and normalized such that $\langle \alpha, \varphi|\mathcal{O}_\varphi(\alpha)|0\rangle = 1$.

Evaluation maps

Let Γ be a finite set, and let $\mathcal{V}_\Gamma := \bigotimes_{j\in\Gamma} \mathcal{V}_j$ where $\mathcal{V}_j \cong \mathcal{V}$. Let $\varphi_\Gamma := \{\varphi_j\}_{j\in\Gamma}$ be a collection of holomorphic embeddings $\varphi_j : D \to \mathbb{C}$ of open unit disks with disjoint images, which closures we denote by B_j. The vector $|\Psi_\Gamma\rangle \in \mathcal{V}_\Gamma$ is defined by

$$\langle \Psi_\Gamma | e_{m_\Gamma}\rangle = \langle \prod_{j\in\Gamma} \mathcal{O}_{\varphi_j}(e_{m_j})\rangle \tag{6.4}$$

where $|e_{m_\Gamma}\rangle := \bigotimes_{j\in\Gamma} |e_{m_j}\rangle \in \mathcal{V}_\Gamma$. This vector is well-defined since $\langle \Psi_\Gamma|\Psi_\Gamma\rangle$ is given by the partition function on the Riemann surface obtained by gluing two copies of $\mathbb{C}\backslash\{B_j\}_{j\in\Gamma}$ along the boundaries of B_j (in the sector with only the states of the trivial module running through the tubes connecting the holes) and therefore is finite. We denote the corresponding pure state on the algebra $\mathcal{B}(\mathcal{V}_\Gamma)$ of bounded operators on $\mathcal{V}_\Gamma$ by $\Psi_\Gamma : \mathcal{B}(\mathcal{V}_\Gamma) \to \mathbb{C}$. We can interpret $\mathcal{V}_\Gamma$ as the Hilbert space for chiral modes living on the boundaries of B_j.

Let Γ_* be a pointed set $\Gamma \sqcup \{*\}$ obtained by adjoining a new element $*$, and let $\mathcal{V}_{\Gamma_*} := \mathcal{V} \otimes \mathcal{V}_\Gamma$. If, in addition to the data from the previous paragraph, we have a holomorphic embedding $\varphi_* : D \to \mathbb{C}$ such that its image contains all $\{B_j\}_{j\in\Gamma}$, we can define a vector $|\Psi_{\Gamma_*}\rangle \in \mathcal{V}_{\Gamma_*}$ by

$$\langle \Psi_{\Gamma_*}|e_{m_*} e_{m_\Gamma}\rangle = \langle e_{m_*}, \varphi_*| \prod_{j\in\Gamma} \mathcal{O}_{\varphi_j}(e_{m_j})\rangle. \tag{6.5}$$

This vector defines a natural pure state Ψ_{Γ_*} of modes on the boundary of Σ that is the closure of $(\operatorname{Im}\varphi_*)\backslash\left(\bigcup_{j\in\Gamma} B_j\right)$.

Conformal field theory

When we say CFT, we always mean the diagonal (or Cardy case [22]) conformal field theory associated with V. We denote the anti-holomorphic sector by $\overline{\mathsf{V}}$. The Hilbert space on a circle is $\mathcal{H}^{\mathrm{CFT}} = \bigoplus_{a\in I}\left(\mathcal{V}^{(a)}\otimes\overline{\mathcal{V}}^{(a)}\right)$. The topological line defects and the elementary conformal boundary conditions [15] for the diagonal CFT are labeled by elements of I. We denote the Hilbert space of states on the interval with the left boundary condition a and the right boundary condition b by $\mathcal{H}^{(ab)}$. We choose a basis $|e_m^{(ab)}\rangle$, $m\in\mathbb{N}_0$ ordered by weights. We also use $\hat{\mathcal{H}} := \bigoplus_{a,b\in M}\mathcal{H}^{(ab)}$.

An elementary conformal boundary condition on the outer part of the disk of radius $r<1$ corresponds to the state on the circle

$$|a\rangle = r^{-c/6}\sum_{b\in I}\frac{S_{ab}}{\sqrt{S_{\mathbf{1}b}}}\sum_{m=0}^{\infty} r^{2h^{(b)}(m)}|e_m^{(b)}\rangle\otimes|\overline{e_m^{(b)}}\rangle \in \mathcal{H}^{\mathrm{CFT}} \tag{6.6}$$

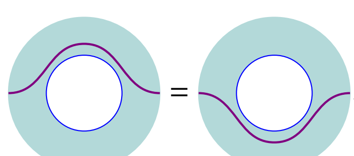
where $r^{-c/6}$ is the Liouville factor for the disk, and S_{ab} are the components of the S matrix. Their linear combinations

$$|b\rangle\!\rangle = (S_{\mathbf{1}b})^{1/2}\sum_{a\in I} S_{ba}^{*}|a\rangle = r^{-c/6}\sum_{m=0}^{\infty} r^{2h^{(b)}(m)}|e_m^{(b)}\rangle\otimes|\overline{e_m^{(b)}}\rangle \tag{6.7}$$

are called the *Ishibashi states*. We extensively use the linear combination of elementary boundary conditions (or topological line defects) that corresponds to the Ishibashi state of the vacuum

$$|\mathbf{1}\rangle\!\rangle = (S_{\mathbf{11}})^{1/2}\sum_{a\in I} S_{\mathbf{1}a}|a\rangle = r^{-c/6}\sum_{m=0}^{\infty} r^{2h(m)}|e_m\rangle\otimes|\overline{e_m}\rangle. \tag{6.8}$$

We call it *cloaking linear combination* following [13], where its various properties are discussed. In particular, any topological line defect can freely pass through a hole with the cloaking linear combination of boundary conditions without changing the correlation functions

 (6.9)

The discussion about the state-operator correspondence can be generalized to bulk and boundary field insertions of the diagonal CFT. For a holomorphic embedding

$f : D \to \mathbb{C}$ of the unit disk, we denote the bulk field insertion at $f(0)$ that produces $|\alpha\rangle \otimes |\bar{\alpha}\rangle \in \mathcal{V}^{(a)} \otimes \overline{\mathcal{V}}^{(a)}$ on the boundary of the image by $\mathcal{O}^{(a)}_f(\alpha \otimes \bar{\alpha})$. For f that maps the interval $[-1, 0]$ to the boundary with the elementary boundary condition a, the interval $[0, 1]$ to the boundary with the elementary boundary condition b and the upper half-disk to the bulk, the boundary field insertion at $f(0)$ that produces an open state vector $|\alpha\rangle \in \mathcal{H}^{(ab)}$ on the image of the upper half-circle is denoted $\mathcal{O}^{(ab)}_f(\alpha)$.

Chiral state

Let Λ be the two-dimensional lattice which we identify with $\mathbb{Z}^2 \subset \mathbb{C}$. The elements of the lattice are called sites, while the segments connecting two neighboring sites are called edges. The dual lattice is denoted $\Lambda^\vee$. The location of $j \in \Lambda$ or $j \in \Lambda^\vee$ on the complex plane is denoted $z_j \in \mathbb{C}$. For a finite subset $\Gamma \subset \Lambda$ we let the algebra of observables of the finite lattice system on Γ be $\mathscr{A}_\Gamma := \bigotimes_{j\in\Gamma} \mathcal{B}(\mathcal{V}_j)$. When V is fermionic, we use the graded tensor product. In the thermodynamic limit, we define the algebra of quasi-local observables $\mathscr{A}$ on Λ, which is the norm completion of the algebra of local observables $\mathscr{A}_\ell$ defined by the canonical direct limit $\mathscr{A}_\ell := \varinjlim_{\Gamma} \mathscr{A}_\Gamma$.

We consider $\Gamma \subset \Lambda$ that are the sets of sites inside the square $\{z \in \mathbb{C} : |\operatorname{Re}(z)|, |\operatorname{Im}(z)| < N + 1/2\}$ for some $N \in \mathbb{N}$. As explained in Section 6.1, to each such subset we can associate vectors $|\Psi_\Gamma\rangle \in \mathcal{V}_\Gamma$ and $|\Psi_{\Gamma_*}\rangle \in \mathcal{V}_{\Gamma_*}$ if we choose a collection of holomorphic embeddings $\varphi_{\Gamma_*} := \{\varphi_j\}_{j\in\Gamma_*}$ of the unit disk. To fix this data, we choose a function $f(z)$ that maps the unit disk into the interior of the square with vertices at $(\pm 1 \pm i)\varepsilon$ for $0 < \varepsilon < 1/2$ and let $\varphi_j(z) := f(z) + z_j$. We also fix some φ_* such that $\operatorname{Im}\varphi_*$ is given by the square from the first sentence of this paragraph. The resulting surface Σ is the square $\operatorname{Im}\varphi_*$ with a collection of holes at each site of Γ (see Fig. 6.1).

In the following, for convenience, we choose a particular one-parameter family of functions $f(z)$ defined in eq. (D.6) with the parameter $\tau > 0$ that characterizes the size of the holes. When $\tau \to 0$, the holes almost touch each other, while when τ is large, the holes are very small. To emphasize the dependence on τ, we write Σ_τ, $\Psi_{\tau,\Gamma}$ and Ψ_{τ,Γ_*}

The norm $\langle\Psi_{\tau,\Gamma_*}|\Psi_{\tau,\Gamma_*}\rangle$ is given by the partition function on the Riemann surface obtained by gluing two copies of Σ_τ in the sector with only the states of the trivial module running through the tubes connecting the holes. Alternatively, it is given by the partition function of CFT on Σ_τ with the cloaking linear combinations of

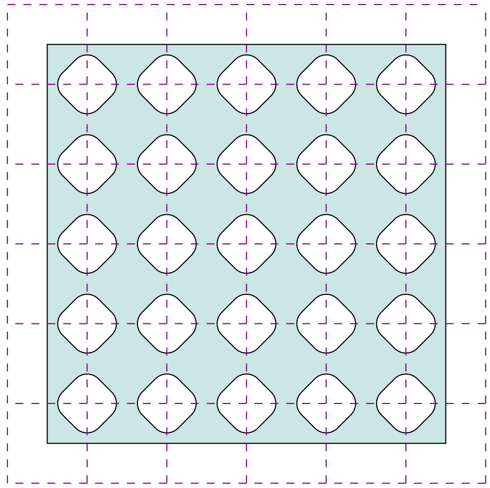


Figure 6.1: The surface Σ_τ and its partition into elementary blocks.

elementary boundary conditions. It can be computed by cutting the resulting surface along the edges of Λ and gluing using the usual rules (see Fig. 6.1). Such a decomposition defines a tensor network statistical model that is similar to the one introduced in [13]. Note that when the holes are large, the necks of the elementary blocks are very thin, and therefore the states with non-zero weight running through the necks are suppressed.

We claim that the direct limit of the restriction of the state Ψ_{τ,Γ_*} to $\mathscr{A}_\Gamma$ over $\Gamma \subset \Lambda$ exists and defines a pure state Ψ_τ on $\mathscr{A}$. In Appendix D.3, we argue that it is true at least for sufficiently small but finite τ using the cluster expansion, though we believe that it is true for any τ. We also show that the state Ψ_τ has exponential decay of correlations of local observables.

Remark 6.1. *While we have fixed the lattice to be a square lattice and the shape of the holes, we believe that construction works for any (not necessarily ordered) lattice and holes around sites that homogeneously fill the plane so that the widths of the necks of all elementary blocks are uniformly bounded. In particular, for the cluster expansion to work, we only need all the holes to be relatively close to each other so that all the necks are sufficiently thin.*

Remark 6.2. *We can make the correlation length arbitrarily small by choosing a small enough value of τ. When τ grows, the holes shrink, and each site becomes more and more disentangled. However, the correlation length grows with τ since the necks of the elementary blocks become thicker and the states running through the necks are becoming less and less suppressed. Therefore, decreasing the size of the* *holes homogeneously does not allow to disentangle the sites without destroying the locality.*

Remark 6.3. *The states $\Psi_{\tau,\Gamma}$ and Ψ_{τ,Γ_*} almost coincide in the bulk, and therefore we can also get Ψ_τ by taking the thermodynamic limit of $\Psi_{\tau,\Gamma}$. The advantage of working with Ψ_{τ,Γ_*} over $\Psi_{\tau,\Gamma}$ is that the former has decaying correlations for any two distant observables of $\mathscr{A}_\Gamma$, while the latter has relatively large correlations between distant observables near the boundary (as the decay of such correlations is only polynomial). The entanglement with the modes living in the auxiliary factor $\mathcal{V}$ in $\mathcal{V}_{\Gamma_*}$ effectively localizes the boundary correlations.*

Remark 6.4. *One can analogously define states on a finite lattice modeling the system on an arbitrary Riemann surface. In this case, one generally has a non-trivial space of conformal blocks, and each basis element gives its own state.*

In the remainder of the chapter, we give arguments in support of the fact that Ψ_τ has the topological order associated with the vertex algebra V.

Local perturbations

One natural way to modify the state Ψ_τ locally is to insert a vertex operator into the defining correlator eq.(6.5) for Ψ_{Γ_*} (such that the resulting vector in $\mathcal{V}_{\Gamma_*}$ is non-zero) and take the thermodynamic limit. Such an insertion corresponds to a change of an element of the tensor network statistical model that computes the averages of observables in the state Ψ_τ. The analysis from Appendix D.3 implies that the change of the expectation values of observables outside a large disk around the insertion exponentially decays with the size of the disk. Using this, one can argue that the modified state can be obtained from Ψ_τ by an almost local[3] unitary observable (see Appendix C).

A local variation of the moduli of the Riemann surface Σ_τ (e.g., changing the size of a hole) corresponds to the insertion of a certain combination of vertex operators from the Virasoro subalgebra, and therefore can also be performed by an almost local unitary observable. The localization of the observable (i.e., the rate of decay of the "tails") depends on the correlation length. Suppose a global variation of the moduli is composed of local changes which keep the correlation length bounded. Using

[3] By almost local, we mean that the observable can be approximated by a local one with the error (in the operator norm) that decays rapidly (faster than any power) with the diameter of the support (see [36] for a precise characterization).

unitaries for local changes, one can construct a finite time Hamiltonian evolution, with the Hamiltonian being a sum of almost local terms, that performs the global change. Therefore, states related by such a global variation are in the same topological phase. In particular, the states Ψ_τ are in the same phase for different values of τ as long as the correlation length stays finite.

One can also modify the state by entangling it with ancillas. If we describe the initial ancillary state as a pure state of the trivial module living on the boundary of a decoupled disk, one can create a “wormhole" between this disk and Σ_τ (i.e., cutting a small disk out of each surface and gluing the boundaries) by inserting a certain combination of vertex operators as only the states of the trivial module are running through this wormhole. Hence, the modified state can be obtained from the original one by an almost local unitary which entangles the sites of the lattice with the ancilla. By inverting the process, we can disentangle any single site of the lattice system. Note, however, that in this way, we can not disentangle all the spins of the system keeping the correlation length finite (see Remark 6.2).

Anyons

One can also modify the state Ψ_τ by inserting non-trivial modules of V. A choice of a holomorphic embedding of the open unit disk $\upsilon : D \to \Sigma_\tau$ and a vector $\alpha^{(a)} \in \mathsf{V}^{(a)}$ gives the map $\mathcal{V}^{(a)} \otimes \mathcal{V}_\Gamma \to \mathbb{C}$ that evaluates the amplitude on the Riemann surface with fixed state vectors $|\alpha^{(a)}\rangle \in \mathcal{V}^{(a)}$, $|e^{(a)}_{m_*}\rangle \in \mathcal{V}^{(a)}$, $\{|e_{m_j}\rangle \in \mathcal{V}_j\}_{j\in\Gamma}$ on the boundaries of the closures of the images of υ, φ_*, $\{\varphi_j\}_{j\in\Gamma}$, respectively. Therefore υ and $\alpha^{(a)}$ define $|\Xi^{(a)}_{\tau,\Gamma_*}\rangle \in \mathcal{V}^{(a)}_{\Gamma_*} := \mathcal{V}^{(a)} \otimes \mathcal{V}_\Gamma$. We interpret the thermodynamic limit $\Xi^{(a)}_\tau$ of the corresponding state on $\mathscr{A}_\Gamma$ as a state of the anyon of type a localized on sites in the vicinity of the image of υ. More generally, one can insert several anyons by choosing a collection of embeddings of open unit disks into Σ_τ, a collection of elements of the modules, and an element of the corresponding space of conformal blocks. The determination of braiding properties is reduced to the analysis of the correlation functions of CFT.

By the arguments from the previous subsection, one can move anyons (or create particle-antiparticle pairs) on the lattice using a Hamiltonian evolution, with the Hamiltonian being a sum of almost local terms localized along the path. In particular, a non-trivial anyon inside a large ring can be detected by an observable of the algebra that performs a transport of an anyon with non-trivial braiding around the ring. It implies that $\Xi^{(a)}_\tau$ is not related to Ψ_τ by an almost local unitary observable.

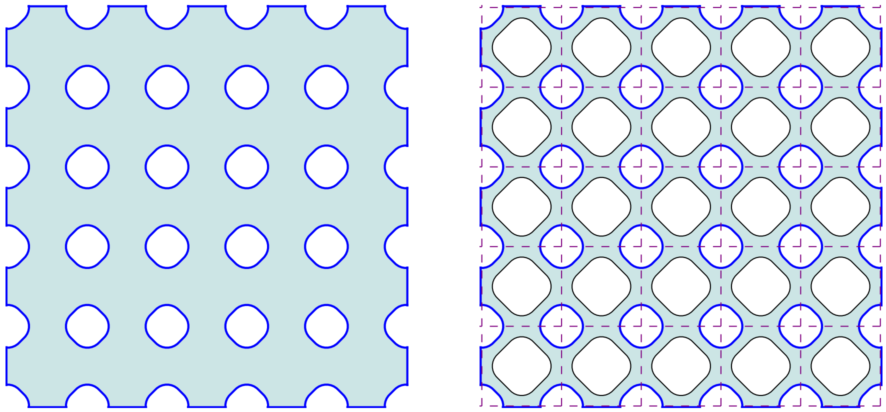


Figure 6.2: The surfaces $\tilde{\Sigma}_\sigma$ (on the left) and $\Sigma_{\tau,\sigma}$ (on the right).

Doubled state and invertibility

While the averages in the state Ψ_τ can be computed using an auxiliary tensor network statistical model, the state itself is not defined as a tensor network state. However, there is a natural family of tensor network states $\Phi_\tau(\sigma)$, $\sigma \in (0,\infty)$ of the doubled system, such that in the limit $\sigma \to \infty$ one gets $\Psi_\tau \otimes \overline{\Psi}_\tau$, where $\overline{\Psi}_\tau$ is the state obtained by complex conjugation.

Let us first define $\Phi_\tau(\sigma)$ using the correlation functions of CFT operators. Let $\tilde{\Sigma}_\sigma$ be the surface obtained by taking the closure of $\operatorname{Im}\varphi_*$ after removing the images of $\tilde{f}(z) + z_k$ for each $k \in \Lambda^\vee$ (see Fig. 6.2), where $\tilde{f}(z)$ is given by $f(z)$ with τ replaced by σ. We define $|\Phi_{\tau,\Gamma}(\sigma)\rangle \in \mathcal{V}_\Gamma \otimes \overline{\mathcal{V}}_\Gamma$ by

$$\langle \Phi_{\tau,\Gamma}(\sigma)|e_{m_\Gamma}\bar{e}_{m_\Gamma}\rangle = \langle \prod_{j\in\Gamma} \mathcal{O}_{\varphi_j}(e_{m_j}\otimes \bar{e}_{m_j})\rangle^{\mathrm{CFT}}_{\tilde{\Sigma}_\sigma} \tag{6.10}$$

with the cloaking linear combination of boundary conditions along the boundary of $\tilde{\Sigma}_\sigma$. Alternatively, one can define the components of $|\Phi_{\tau,\Gamma}(\sigma)\rangle$ as a correlator of the vertex algebra on a Riemann surface obtained by gluing two copies of $\tilde{\Sigma}_\sigma$ in the sector with only the states of the trivial module running through the holes.

We claim that the direct limit of the state $\Phi_{\tau,\Gamma}(\sigma)$ over $\Gamma \subset \Lambda$ exists and defines a pure state $\Phi_\tau(\sigma)$ on $\mathscr{A} \otimes \mathscr{A}$ with exponential decay of correlations. In the same way as for Ψ_τ, one can show that this claim holds at least for small enough τ using the cluster expansion.

To define it as a tensor network state, let $\Sigma_{\tau,\sigma}$ be the surface obtained by removing the images of φ_Γ from $\tilde{\Sigma}_\sigma$ (see Fig. 6.2). We can cut the surface $\Sigma_{\tau,\sigma}$ along the edges of the lattice $\Lambda^\vee$ to decompose it into elementary blocks. Each block gives a map $\mathcal{V} \otimes \overline{\mathcal{V}} \otimes \hat{\mathcal{H}}^{\otimes 4} \to \mathbb{C}$ that defines a tensor network state with the physical space being $\mathcal{V} \otimes \overline{\mathcal{V}}$ and with the auxiliary Hilbert space being $\hat{\mathcal{H}}$ on each leg. In Appendix D.2, we explain how one can get an expression for the components of the map in terms of the CFT correlation functions on the unit disk.

The parameter τ gives an upper bound on the correlation length of the states $\Phi_\tau(\sigma)$ for any σ, that follows from the auxiliary statistical model obtained by cutting along the edges of Λ in the same way as in Section 6.1. When σ is large, the presence of the holes near the sites of the dual lattice is negligible, and the state $\Phi_\tau(\sigma)$ is locally close to the tensor product of pure states Ψ_τ and $\overline{\Psi}_\tau$. On the contrary, when $\sigma \to 0$, the holes are large, and the individual spins become more and more disentangled. One can produce $\Phi_\tau(\sigma)$ from $\Psi_\tau \otimes \overline{\Psi}_\tau$ by creating "wormholes" (with only the states of the trivial module running though it) at each site of the dual lattice. As argued in Section 6.1, each wormhole can be created by an almost local unitary. The localization of such unitaries would not be affected if some wormholes have already been created. Hence, one can compose such creation processes to argue that both states are in the same topological phase. The same argument does not work if one tries to disentangle the state $\Phi_\tau(\sigma)$ by disconnecting the necks of the tensor network since the states of non-trivial modules are running through them.

The situation is special when V is a holomorphic vertex operator algebra, i.e., $I = \{\mathbf{1}\}$. In that case, there is only a single boundary condition, and only one state running through the legs of the tensor network is not suppressed in the limit $\sigma \to 0$. The state $\Phi_\tau(\sigma)$ can be completely disentangled into a factorized state, and since $\Phi_\tau(\sigma)$ is in the same phase as $\Psi_\tau \otimes \overline{\Psi}_\tau$, the state Ψ_τ is invertible with the inverse being $\overline{\Psi}_\tau$. We conjecture that the states Ψ_τ for holomorphic vertex algebras with different values of c are in different invertible phase.

Remark 6.5. *It is believed that invertible bosonic phases in two dimensions are classified by $c \in 8\mathbb{Z}$ [42]. Our construction produces a candidate for a state in a non-trivial invertible phase for any holomorphic vertex operator algebra, including a representative of Kitaev's E_8 phase, which is believed to be a generator of all invertible phases. It would be desirable to show that any holomorphic vertex operator algebra gives a state in the same topological phase as a stack of several E_8 states.*

Remark 6.6. *While in this note we do not provide explicit parent Hamiltonians, we want to point out that given a parent Hamiltonian H for a tensor network state $\Phi_\tau(\sigma)$ we can construct a Hamiltonian for Ψ_τ by applying to H a uniformly almost local unitary evolution that produces $\Phi_\tau(\sigma)$ from $\Psi_\tau \otimes \overline{\Psi}_\tau$ and taking the partial average of the result over $\overline{\Psi}_\tau$. In particular, it provides a parent gapped Hamiltonian*

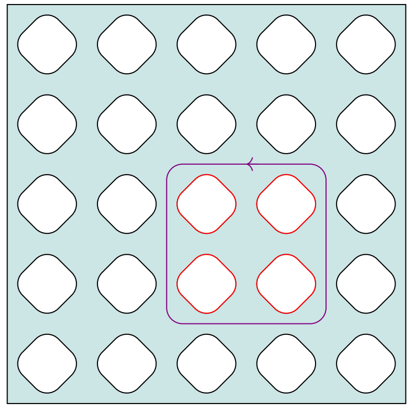


Figure 6.3: The action of the $U(1)$ charge on a region A can be implemented by inserting $\int_{\partial A} J(z)\frac{dz}{2\pi i}$ into the defining correlator eq. (6.4).

in the invertible case (since $\Phi_\tau(\sigma)$ can be produced from a factorized state by a uniformly almost local unitary Hamiltonian evolution and hence gapped).

Internal Lie-group symmetry

Suppose V has a $U(1)$-symmetry at level k. Let $J(z)$ be the corresponding current with the operator product expansion

$$J(z)J(w) = \frac{k}{(z-w)^2} + \ldots \tag{6.11}$$

Our lattice model associated with V naturally has the on-site $U(1)$-symmetry that corresponds to the $U(1)$ action on the vacuum module. The state Ψ_τ is $U(1)$-invariant. For a region A, let Q_A be the generator of $U(1)$ action on sites inside A. We have

$$\langle \Psi_{\tau,\Gamma}|Q_A|e_{m_\Gamma}\rangle = \langle\left(\oint_{\partial A}\frac{dz}{2\pi i}J(z)\right)\prod_{j\in\Gamma}\mathcal{O}_{\varphi_j}(e_{m_j})\rangle. \tag{6.12}$$

As argued in Appendix C, there is a self-adjoint almost local observable $K(z)\in\mathscr{A}$ such that the difference between the state vector for the state affected by the insertion of $J(z)$ and the vector $K(z)|\Psi_\tau\rangle$ is proportional to $|\Psi_\tau\rangle$. This observable can be chosen to be $U(1)$-invariant using averaging over $U(1)$ action. Since the action of Q_A corresponds to the insertion of the integral of $J(z)$ along ∂A, it can be compensated by the integral of $K(z)$ over ∂A, which corresponds to a uniformly almost local Hamiltonian localized on ∂A.

Using $K(z)$, in a way similar to Chapter 5, we can define an analog of a $U(1)$-equivariant Berry class of spin systems σ for our lattice system that is given by

$$\sigma = 2\pi i\langle\left[\int_{\mathrm{Im}z=0}\frac{dz}{2\pi i},\int_{\mathrm{Re}w=0}\frac{dw}{2\pi i}\right]J(z)J(w)\rangle^{\mathrm{CFT}}_{\Sigma_\tau}. \tag{6.13}$$

Only the singular term contributes to the integral. Since

$$\left[\int_{\mathrm{Im}z=0}\frac{dz}{2\pi i},\int_{\mathrm{Re}w=0}\frac{dw}{2\pi i}\right]\frac{1}{(z-w)^2}=\frac{1}{2\pi i}, \tag{6.14}$$

we have $\sigma = k$.

One can similarly consider the case of V with G-symmetry for a compact Lie group G. The analog of a G-equivariant Berry class of spin systems for the state Ψ_τ can be identified with the level of the G-current subalgebra.

6.2 Examples

Free fermions

The vertex operator algebra of a single Weyl fermion is generated by the fields

$$\psi(z) = \sum_{n\in\mathbb{Z}} \psi_{-1/2-n} z^n, \tag{6.15}$$

$$\bar{\psi}(z) = \sum_{n\in\mathbb{Z}} \bar{\psi}_{-1/2-n} z^n \tag{6.16}$$

with $\{\bar{\psi}_n, \psi_m\} = \delta_{m+n,0}$, $\{\bar{\psi}_n, \bar{\psi}_m\} = \{\psi_n, \psi_m\} = 0$ and the operator product expansion (OPE)

$$\bar{\psi}(z)\psi(w) = \frac{1}{z-w} + \ldots \tag{6.17}$$

Each site of the lattice system has infinitely many fermionic modes labeled by $r \in \mathbb{Z} + 1/2$. The entanglement of modes in the state Ψ_τ decays with $|r|$ so that modes with large $|r|$ are almost disentangled.

We have $U(1)$ currents

$$J(z) = \sum_{n\in\mathbb{Z}} z^n J_{-n-1} =: \bar{\psi}(z)\psi(z) : \tag{6.18}$$

and the corresponding on-site $U(1)$ symmetry action for the lattice model. The class σ from eq. (6.13) of Ψ_τ is given by the level of the $U(1)$-current algebra that, in our case, is equal to 1.

The lattice state Ψ_τ is free, i.e., the average of any observable can be expressed through the two-point functions of the fermionic creation and annihilation operators using Wick's theorem. These two-point functions define a projector in the single particle Hilbert space that fully characterizes the many-body free state. The index of the projector determines whether the state is in a non-trivial phase [9, 1, 40]. The doubled free state always has the $U(1)$ symmetry that swaps the fermionic modes, and its σ is given by the index of the projector of the original system. Therefore the index of the projector for Ψ_τ for a single Weyl fermion is 2.

The vertex algebra of a single Majorana-Weyl fermion is generated by

$$\chi(z) = \sum_{n\in\mathbb{Z}} \chi_{-1/2-n} z^n \tag{6.19}$$

with $\{\chi_n, \chi_m\} = \delta_{m+n,0}$ and the OPE

$$\chi(z)\chi(w) = \frac{1}{z-w} + ... \tag{6.20}$$

Its double gives the Weyl fermion, and therefore the index of the projector of the state corresponding to this vertex algebra is 1.

The states $\Phi_\tau(\sigma)$ are also free and have the trivial index. They give a path between a factorized state and $\overline{\Psi}_\tau \otimes \Psi_\tau$ that shows invertibility of Ψ_τ.

Free bosons

With every finite rank lattice L equipped with a positive symmetric bilinear form $K : L \times L \to \mathbb{Z}$, one can associate the vertex operator algebra of free periodic bosons with the OPE

$$\phi^a(z)\phi^b(w) = -K^{ab} \log(z-w) + ... \tag{6.21}$$

Depending on whether the lattice is even or odd, we can generate examples of chiral states of bosonic and fermionic lattice systems.

E_8 state

The simplest non-trivial free bosonic system that has no anyons is given by free periodic bosons ϕ^a, $a = 1, ..., 8$ with

$$\langle \phi^a(z)\phi^b(w)\rangle = -(C^{-1})^{ab} \log(z-w) \tag{6.22}$$

where $(C^{-1})^{ab}$ is the inverse of the Cartan matrix of E_8. The corresponding state Ψ_τ is invertible, as discussed in Section 6.1.

$U(1)_m$ state

Let us now consider the topologically ordered states produced by a single periodic boson with

$$\langle \phi(z)\phi(w)\rangle = -\frac{1}{m} \log(z-w) \tag{6.23}$$

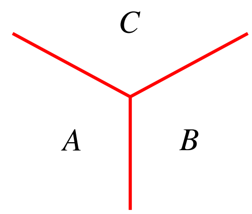


Figure 6.4: The partition of the plane into three cone-like regions.

for $m \in 2\mathbb{Z}$. The simple modules are labeled by $p = 0, \ldots, m-1$. The module with label p corresponds to the field $e^{ip\phi}$.

The system has $U(1)$ symmetry $J(z) = i\partial\phi$ at level $k = 1/m$

$$J(z)J(w) = \frac{1/m}{(z-w)^2} + \ldots \tag{6.24}$$

Therefore the resulting state realizes a fractional quantum Hall topological order with $\sigma = 1/m$.

Using $K(z)$ from Section 6.1, we can define the analogs of vortex states from Section 5.2. For a tripartition (A, B, C), let K_{AB} be the derivation that corresponds to the integral of $K(z)$ along the path AB. The action of $(Q_A - K_{AB})$ on Ψ_τ corresponds to the insertion of $\int_{AB} \frac{dz}{2\pi i} J(z)$ where the integral is performed along the half-line AB. Since $J(z) = i\partial\phi$, the state with the unit of flux and the state obtained by the insertion of $e^{i\phi}$ into the apex of the tripartition are related by an almost local unitary. Thus, both states represent an anyon with fractional charge $\sigma = 1/m$.

Appendix A

TOPOLOGY OF ALGEBRAS OF OBSERVABLES

A.1 Fréchet spaces

A seminorm on a real or complex vector space V is a map $V \to \mathbb{R}$, $v \mapsto \|v\|$ such that $\|v\| \geq 0$ for all $v \in V$, $\|v + v'\| \leq \|v\| + \|v'\|$ for all $v, v' \in V$, and $\|cv\| = |c|\|v\|$ for all $v \in V$ and all scalars c. A seminorm is a norm if $\|v\| = 0$ implies $v = 0$.

A Fréchet space is a complete Hausdorff topological vector space whose topology is determined by a countable family of seminorms $\|\cdot\|_\alpha$, $\alpha \in \mathbb{N}_0$. A base of neighborhoods of zero for such a topology consists of sets

$$U_{(\alpha_1,\varepsilon_1)\dots(\alpha_n,\varepsilon_n)} = \{v \in V : \|v\|_{\alpha_i} < \varepsilon_i,\ i = 1, \dots, n\}, \tag{A.1}$$

where $n \in \mathbb{N}$, $\alpha_i \in \mathbb{N}_0$, and $\varepsilon_i > 0$. Any finite-dimensional Euclidean vector space is a special case where all the seminorms happen to be the same and equal to the Euclidean norm.

In this work we will be often dealing with a situation where the seminorms satisfy $\|\cdot\|_0 \leq \|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$. One calls such Fréchet spaces graded Fréchet spaces. Then the sets $U_{\alpha,\varepsilon} = \{v \in V : \|v\|_\alpha < \varepsilon\}$, $\alpha \in \mathbb{N}_0$, also form a base of neighborhoods of zero. A linear map $f : V \to V'$ between graded Fréchet spaces is continuous iff for any $\alpha \in \mathbb{N}_0$ there is a $\beta \in \mathbb{N}_0$ and a constant C_α such that $\|f(v)\|_\alpha \leq C_\alpha \|v\|_\beta$. The Cartesian product of two graded Fréchet spaces V, V' is also a graded Fréchet space, with the seminorms $\|(v, v')\|_\alpha = \|v\|_\alpha + \|v'\|'_\alpha$.

Different families of seminorms on V may define the same topology; in that case one says that the families are equivalent. A family of seminorms $\|\cdot\|'_\beta$, $\beta \in \mathbb{N}_0$ is equivalent to a family $\|\cdot\|_\alpha$, $\alpha \in \mathbb{N}_0$, if for any β there is an α and a constant C_β such that $\|\cdot\|'_\beta \leq C_\beta \|\cdot\|_\alpha$, and vice versa, for any α there is an β and a constant C'_α such that $\|\cdot\|_\alpha \leq C'_\alpha \|\cdot\|'_\beta$.

If X is a compact topological space and V is a graded Fréchet space, then the space $C(X, V)$ of continuous V-valued functions on X is also a graded Fréchet space. The corresponding family of seminorms is $\|f\|_\alpha = \sup_{x \in X} \|f(x)\|_\alpha$, $\alpha \in \mathbb{N}_0$. In the case when $X = [a, b] \subset \mathbb{R}$ elements of $C([a, b], V)$ are called continuous curves in V. Most basic rules of calculus (such as the existence of integrals of continuous functions, the Fundamental Theorem of Calculus, the Mean Value Theorem, the continuous dependence of integrals of continuous functions on parameters, etc.) hold in the setting of continuous functions on regions in $\mathbb{R}^n$ valued in a Fréchet space V, see [24] for a review.

If X is a non-compact topological space which is a union of compact subsets $K_0 \subset K_1 \subset \ldots$ such that $K_i \subseteq \mathrm{int}(K_{i+1})$ for all i, then $C(X, V)$ is a Fréchet space whose topology can be defined using seminorms $\|f\|_{n,\alpha} = \sup_{x \in K_n} \|f(x)\|_\alpha$, $n, \alpha \in \mathbb{N}_0$. This topology is independent of the choice of the compact subsets K_i. Similar seminorms are also useful when defining Fréchet topology on spaces of smooth functions and differential forms on manifolds. Let $\mathcal{M}$ be a compact manifold of dimension M and V be a graded Fréchet space. A function $f : \mathcal{M} \to V$ is called smooth iff derivatives of all orders exist and are continuous. In particular, a smooth function $f : [a, b] \to V$ is called a smooth curve in V. The space of smooth V-valued functions on $\mathcal{M}$ is denoted $C^\infty(\mathcal{M}, V)$. One can define a Fréchet topology on $C^\infty(\mathcal{M}, V)$ as follows. First, we choose an atlas $\left\{\left(U^{(\mathsf{a})}, \kappa^{(\mathsf{a})} : U^{(\mathsf{a})} \to \mathbb{R}^M\right)\right\}$ for $\mathcal{M}$. Then we define a family of seminorms labeled by a chart index a, a compact subset $K_n^{(\mathsf{a})} \subset U^{(\mathsf{a})}$, $k \in \mathbb{N}_0$, and $\alpha \in \mathbb{N}_0$:

$$\|f\|_{\mathsf{a},n,k,\alpha} = \max_{\mathsf{a}} \max_{|I| \leq k} \sup_{m \in K_n^{(\mathsf{a})}} \|\partial_I f(m)\|_\alpha. \tag{A.2}$$

Here $I = \{i_1, \ldots, i_M\} \in \mathbb{N}_0^M$, is a multi-index and $|I| = \sum_{q=1}^M i_q$. Similarly, one can define a Fréchet topology on the space of smooth p-forms $\Omega^p(\mathcal{M}, V)$ by regarding the restriction of a p-form ω to $U^{(\mathsf{a})}$ as a collection of $\binom{M}{p}$ V-valued functions. One can show that the topologies thus defined do not depend on the choice of the atlas.

We will also need the notion of a possibly nonlinear smooth map from a Fréchet space W to a Fréchet space V. One says that $f : W \to V$ is continuously differentiable if the directional derivative

$$df(w, \Delta w) = \lim_{t \to 0} \frac{f(w + t\Delta w) - f(w)}{t} \tag{A.3}$$

exists and is a continuous function on $W \times W$. Iterating this definition, one says that a function $f : W \to V$ is smooth if directional derivatives of all orders exist and are continuous functions on $W \times W \times \ldots$. In particular, continuous linear maps are smooth. So are continuous bilinear maps from $W \times W'$ to V. Composition of smooth maps is smooth, other basic rules of calculus also hold true [24]. We will use these notions to construct certain smooth maps from a manifold $\mathcal{M}$ to a Fréchet space V as compositions of smooth maps from $\mathcal{M}$ to a Fréchet space W and smooth maps from W to V.

A.2 Algebra of almost local observables

Fix $j \in \Lambda$. Since $\|\cdot\|'_{j,\alpha}$ are norms and $\mathscr{A}_{a\ell}$ is complete in these norms, the topology induced by $\|\cdot\|'_{j,\alpha}$ makes $\mathscr{A}_{a\ell}$ into a complete Hausdorff space, and therefore into a Fréchet space.

Proposition A.1. *$\mathscr{A}_{a\ell}$ is a Fréchet algebra.*

Proof. For any $\mathcal{A}, \mathcal{A}' \in \mathscr{A}_{a\ell}$ and all $r \geq 0$ one has

$$f_j(\mathcal{A}\mathcal{A}', r) \leq f_j(\mathcal{A}, r) f_j(\mathcal{A}', r) + \|\mathcal{A}\| f_j(\mathcal{A}', r) + \|\mathcal{A}'\| f_j(\mathcal{A}, r) \leq \\ \leq \frac{3}{2}\left(\|\mathcal{A}\| f_j(\mathcal{A}', r) + \|\mathcal{A}'\| f_j(\mathcal{A}, r)\right). \quad \text{(A.4)}$$

Therefore for any $\alpha \in \mathbb{N}$ one has

$$\|\mathcal{A}\mathcal{A}'\|'_{j,\alpha} \leq \frac{3}{2}\|\mathcal{A}\|'_{j,\alpha}\|\mathcal{A}'\|'_{j,\alpha} \quad \text{(A.5)}$$

that implies joint continuity of the multiplication. □

Proposition A.2. *The topology on $\mathscr{A}_{a\ell}$ defined by the norms $\|\cdot\|'_{j,\alpha}$ for a fixed $j \in \Lambda$ is independent of the choice of j.*

Proof. For any $j, k \in \Lambda$ with $R = |j - k|$ we have $f_j(\mathcal{A}, r + R) \leq f_k(\mathcal{A}, r)$. Hence

$$\|\mathcal{A}\|'_{j,\alpha} \leq \|\mathcal{A}\| + \sup_r (1 + r + R)^\alpha f_k(\mathcal{A}, r) \leq (1 + R)^\alpha \|\mathcal{A}\|'_{k,\alpha}. \quad \text{(A.6)}$$

Therefore the families of norms $\{\|\cdot\|'_{j,\alpha}\}$ and $\{\|\cdot\|'_{k,\alpha}\}$ are equivalent. □

Lemma A.1. *Let $j, k \in \Lambda$. Suppose an observable $\mathcal{A} \in \mathscr{A}$ satisfies $f_j(\mathcal{A}, r) \leq \varepsilon_1$ and $f_k(\mathcal{A}, r) \leq \varepsilon_2$. Let $\mathcal{B} \in \mathscr{A}_{B_j(r) \cap B_j(r)}$ be a best possible approximation of $\mathcal{A}$ on $B_j(r) \cap B_k(r)$. Then*

$$\|\mathcal{A} - \mathcal{B}\| \leq \varepsilon_1 + \varepsilon_2 + \min(\varepsilon_1, \varepsilon_2). \quad \text{(A.7)}$$

Proof. Let $\mathcal{A}^{(1)}$ and $\mathcal{A}^{(2)}$ be best possible approximations of $\mathcal{A}$ on $B_j := B_j(r)$ and $B_k := B_k(r)$, respectively, and let $\mathcal{B}^{(1)}$ (resp. $\mathcal{B}^{(2)}$) be a best possible approximation of $\mathcal{A}^{(1)}$ (resp. $\mathcal{A}^{(2)}$) on $B_{jk} := B_j(r) \cap B_k(r)$. Then

$$\|\mathcal{A} - \mathcal{B}\| \leq \|\mathcal{A} - \mathcal{B}^{(1)}\| \leq \|\mathcal{A} - \mathcal{A}^{(1)}\| + \|\mathcal{A}^{(1)} - \mathcal{B}^{(1)}\| \leq \\ \leq \varepsilon_1 + \|\Pi_{B_k \setminus B_{jk}}(\mathcal{A}^{(1)} - \mathcal{A}^{(2)})\| \leq 2\varepsilon_1 + \varepsilon_2. \quad \text{(A.8)}$$

□

A.3 Brick expansion

A brick in $\mathbb{R}^d$ is a subset of $\mathbb{R}^d$ of the form $\{(x_1,\ldots,x_d) : n_i \le x_i < m_i,\ i = 1,\ldots,d\}$, where n_i and m_i are integers satisfying $n_i < m_i$. The empty subset is also regarded as a brick. A unit brick is a brick with $m_i = n_i + 1$ for all i. The intersection of any two bricks is a brick. The set of all bricks in $\mathbb{R}^d$ (including the empty set) is denoted $\mathbb{B}_d$. It is a poset with a partial order given by inclusion. This poset has a lower bound (the empty set) and is locally finite (i.e., for any two bricks $Y, Y' \in \mathbb{B}_d$ the set $\{Z \in \mathbb{B}_d, Y \le Z \le Y'\}$ is finite).

In Section 2.2, we defined the subspace $\mathfrak{d}^Y \subset \mathfrak{d}_Y$ as an orthogonal complement of

$$\sum_{\substack{Z\in\mathbb{B}_d\\ Z\subsetneq Y}} \mathfrak{d}_Z \tag{A.9}$$

with respect to the inner product $\langle \mathcal{A}, \mathcal{B}\rangle = \langle \mathcal{A}^*\mathcal{B}\rangle_\infty$. One can give a more explicit description of this subspace using a *Pauli basis* of $\mathscr{A}_\ell$. This is a basis obtained by choosing an orthonormal self-adjoint basis $\mathcal{E}^k_j$, $k = 0,\ldots,d_j^2 - 1$, $d_j := \dim \mathcal{V}_j$ for each $\mathscr{A}_j$ (where the inner product is the same as above) so that $\mathcal{E}^0_j$ is the identity element in $\mathscr{A}_j$. The resulting basis elements of $\mathscr{A}_\ell$ can be labeled by functions $\nu : \Lambda \to \mathbb{N}_0$ with $\nu(j) < d_j^2$ which vanish outside of a finite set. The identity element in $\mathscr{A}_\ell$ corresponds to ν being identically zero. If we denote by $\mathrm{supp}(\nu) \in \mathrm{Fin}(\Lambda)$ the support of ν, then a basis for $\mathfrak{d}_Y$ consists of those $\mathcal{E}_\nu$ for which $\mathrm{supp}(\nu)$ is nonempty and $\mathrm{supp}(\nu) \subseteq Y$. A basis for $\mathfrak{d}^Y$ consists of those $\mathcal{E}_\nu$ for which, in addition, $\mathrm{supp}(\nu) \not\subseteq Z$ for any brick $Z \subsetneq Y$. Note that since $\mathbf{\Delta} < 1/2$, every unit brick (and therefore also every nonempty brick) contains at least one point of Λ. Therefore $\mathfrak{d}^Y$ is nonzero for every nonempty brick Y. For any $\mathcal{A} \in \mathfrak{d}_l$ we have

$$\mathcal{A}|_Y = \sum_{Y'\in\mathbb{B}_d, Y'\subseteq Y} \mathcal{A}^{Y'}. \tag{A.10}$$

Recall [32] that to any locally finite poset P which is bounded from below one can attach its Möbius function $\mu_P : P \times P \to \mathbb{Z}$. This function has the following property. Let $f : P \to V$ be any function with values in a vector space V. Let us define another function $g : P \to V$ by $g(y) = \sum_{z\le y} f(z)$. Then f can be expressed through g by

$$f(y) = \sum_{z\le y} \mu_P(z,y) g(z). \tag{A.11}$$

The function $\mu_P(z,y)$ is uniquely defined by this property if we demand $\mu(z,y) = 0$ for $z \nleq y$. The Möbius functions is multiplicative under Cartesian product: if P, Q

are locally finite posets bounded from below, and $P \times Q$ is given the obvious partial order, then $\mu_{P\times Q} = \mu_P \cdot \mu_Q$.

Proposition A.3. *For any $\mathcal{A} \in \mathfrak{d}_l$ we have $\|\mathcal{A}^Y\| \leq 4^d \|\mathcal{A}\|$.*

Proof. Note first that the poset $\mathbb{B}_d$ is the Cartesian product of d copies of $\mathbb{B}_1$. The Möbius function of $\mathbb{B}_1$ is easily computed:

$$\mu_{\mathbb{B}_1}([n', m'), [n, m)) = \begin{cases} (-1)^{(n'-n)+(m-m')}, \text{ if } (n'-n), (m-m') \in \{0, 1\}, \\ 0, \quad \text{otherwise.} \end{cases} \tag{A.12}$$

Therefore $\mu_{\mathbb{B}_d}(Y', Y) \in \{0, 1, -1\}$ for all $Y', Y \in \mathbb{B}_d$ and is nonzero if and only if the integers n_i, m_i and n'_i, m'_i defining Y and Y' satisfy $m_i - m'_i \in \{0, 1\}$ and $n'_i - n_i \in \{0, 1\}$ for all i. Applying the inversion formula to (A.10) we get

$$\mathcal{A}^Y = \sum_{Y' \in \mathbb{B}_d, Y' \subseteq Y} \mu_d(Y', Y) \mathcal{A}|_{Y'}. \tag{A.13}$$

Since the sum on the r.h.s. contains exactly 4^d terms, and since $\|\mathcal{A}|_Y\| \leq \|\mathcal{A}\|$, we get the desired estimate. □

Similarly, if for any $\mathcal{A} \in \mathfrak{d}_{al}$ we define $\mathcal{A}^Y$ to be the $\mathfrak{d}^Y$ component of $\mathcal{A}|_Y \in \mathfrak{d}_Y$, then we have the following estimate.

Proposition A.4. *For any $a \in \mathscr{F}^+_\infty$ and any $\mathcal{A} \in \mathfrak{d}_{al}$ which is a-localized at $j \in \Lambda$ we have*

$$\|\mathcal{A}^Y\| \leq b(\text{diam}(\{j\} \cup Y)) \tag{A.14}$$

where

$$b(r) = 2^{2d+1} a(\max(0, r/(2\sqrt{d}) - 2)). \tag{A.15}$$

Proof. Let $J = (J_1, \ldots, J_d)$ be a point of $\mathbb{Z}^d$ such that $|J - j| < 1/2$ (such a point exists because we normalized the metric so that $\mathbf{\Delta} < 1/2$). Suppose a nonempty brick $Y \in \mathbb{B}_d$ is defined by integers $n_i < m_i, i = 1, \ldots, d$. Let $K = \max\{|m_1 - J_1|, ..., |m_d - J_d|, |n_1 - J_1|, ..., |n_d - J_d|\}$. Let Z be the brick $[J_1 - K, J_1 + K) \times \ldots \times [J_d - K, J_d + K)$ and let $Z' = [J_1 - K + 1, J_1 + K - 1) \times \ldots \times [J_d - K + 1, J_d + K - 1)$. Obviously, Z contains both Y and Z'. It is also easy to see that $B_j(r) \subset Z$ for all $r < K - 1/2$, and $B_j(r) \subset Z'$ for all $r < K - 3/2$. Note also that Y contains a unit brick which does not intersect Z', and thus Y contains at least one point of Λ which is not in Z'.

Since $\mathrm{diam}(\{j\} \cup Y) \leq 2K\sqrt{d}$, for $K < 2$ we get $\mathrm{diam}(\{j\} \cup Y) < 4\sqrt{d}$, and thus $b(\mathrm{diam}(\{j\} \cup Y)) = 2^{2d+1}a(0)$. Note also that $\|\mathcal{A}|_Y\| \leq \|\mathcal{A}\| \leq 2a(0)$. Therefore by Prop. A.3 we have $\|\mathcal{A}^Y\| \leq 2^{2d+1}a(0)$ and the condition (A.14) is satisfied.

It remains to consider the case $K \geq 2$. For any $r > 0$ let $\mathcal{B}^{(r)}$ be a best possible approximation of $\mathcal{A}$ on a ball $B_j(r)$. Since $B_j(K-2) \subset Z'$, $\mathcal{B}^{(K-2)}|_Y$ is supported on the brick $Y \cap Z'$. Since Y contains points of Λ which are not in $Y \cap Z'$, $\left(\mathcal{B}^{(K-2)}\right)^Y = 0$ and thus $\mathcal{A}^Y = (\mathcal{A} - \mathcal{B}^{(K-2)})^Y$. Therefore by Prop. A.3 we get $\|\mathcal{A}^Y\| \leq 2^{2d+1}a(K-2)$. Since $\mathrm{diam}(\{j\} \cup Y) \leq 2K\sqrt{d}$, the condition (A.14) is satisfied. □

Corollary A.1. *For any $\mathcal{A} \in \mathscr{A}_{a\ell}$ the sum $\sum_{Y \in \mathbb{B}_d} \mathcal{A}^Y$ Fréchet-converges to $\mathcal{A}$.*

Proof. By the above lemma, $\|\mathcal{A}^Y\|_{j,\alpha} \leq (1+r)^\alpha b(r)$, where $r = \mathrm{diam}(\{j\} \cup Y)$ and $b(r) \in \mathscr{F}^+_\infty$. This proves convergence. By eq. (A.10), the sum is $\mathcal{A}$. □

A.4 Chains

Other families of norms

In the body of the work, $\mathfrak{d}_{al}$ was defined as a completion of $\mathfrak{d}_l$ with respect to a family of norms $\|\cdot\|_{j,\alpha}$ where $j \in \Lambda$ was fixed and α ranged over $\mathbb{N}_0$. Sometimes it is useful to consider two other natural families of norms on $\mathfrak{d}_l$ labeled by the same data:

$$\|\mathcal{A}\|^{cev}_{j,\alpha} := \sup_r (1+r)^\alpha \|\mathcal{A} - \mathcal{A}|_{B_j(r)}\|. \tag{A.16}$$

$$\|\mathcal{A}\|^{br}_{j,\alpha} := \sup_{Y \in \mathbb{B}_d} (1 + \mathrm{diam}(\{j\} \cup Y))^\alpha \|\mathcal{A}^Y\|. \tag{A.17}$$

For a fixed j, all three family of norms are non-decreasing with α.

Proposition A.5. *The families of norms $\{\|\cdot\|_{j,\alpha}\}$, $\{\|\cdot\|^{cev.}_{j,\alpha}\}$, and $\{\|\cdot\|^{br}_{j,\alpha}\}$ define the same topology on $\mathfrak{d}_l$, and thus give rise to the same completion $\mathfrak{d}_{al}$.*

Proof. To show the equivalence of two non-decreasing families of norms, we need to show that each norm from the first family is upper-bounded by a multiple of a norm from the second family, and vice versa.

It follows directly from the definition that $\|\mathcal{A}\|_{j,\alpha} \leq \|\mathcal{A}\|^{cev}_{j,\alpha}$ for all $\alpha \in \mathbb{N}_0$ and all $\mathcal{A} \in \mathfrak{d}_l$.

Prop. A.4 implies

$$\|\mathcal{A}^Y\| \leq 2^{2d+1} f_j\left(\mathcal{A}, \max\left(0, \frac{1}{2\sqrt{d}}\mathrm{diam}(\{j\} \cup Y) - 2\right)\right). \tag{A.18}$$

Hence $\|\mathcal{A}\|^{br}_{j,\alpha} \le C_\alpha \|\mathcal{A}\|_{j,\alpha}$ for some $C_\alpha > 0$.

Finally, let us show that $\|\mathcal{A}\|^{cev}_{j,\alpha}$ is upper-bounded by a multiple of $\|\mathcal{A}\|^{br}_{j,\alpha+2d+1}$. For any observable $\mathcal{A} \in \mathfrak{d}_l$ and any brick $Y \subset B_j(r)$ we have $\mathcal{A}^Y|_{B_j(r)} = \mathcal{A}^Y$. Therefore

$$\mathcal{A} - \mathcal{A}|_{B_j(r)} = \sum_{Y \not\subseteq B_j(r)} \left(\mathcal{A}^Y - \mathcal{A}^Y|_{B_j(r)} \right), \tag{A.19}$$

and thus

$$\|\mathcal{A} - \mathcal{A}|_{B_j(r)}\| \le 2 \sum_{Y \not\subseteq B_j(r)} \|\mathcal{A}^Y\|. \tag{A.20}$$

Since for any $Y \not\subseteq B_j(r)$ we have $\mathrm{diam}(\{j\} \cup Y) \ge r$, we get

$$\|\mathcal{A}\|^{cev}_{j,\alpha} \le 2C_d \|\mathcal{A}\|^{br}_{j,\alpha+2d+1}, \tag{A.21}$$

where we have used

$$\sum_{Y \in \mathbb{B}_d} (1 + \mathrm{diam}(Y \cup \{j\}))^{-(2d+1)} \le C_d \tag{A.22}$$

for some constant C_d that depends on d only. □

Similarly, in addition to $\{\|\cdot\|_\alpha\}$ on $C_q(\mathfrak{d}_{al})$ we can introduce families of norms:

$$\|\mathsf{a}\|^{cev}_\alpha = \sup_{a \in \{0,1,\dots,q\}} \sup_{j_0,\dots,j_q \in \Lambda} \|\mathsf{a}_{j_0 \dots j_q}\|^{cev}_{j_a,\alpha}, \tag{A.23}$$

$$\|\mathsf{a}\|^{br}_\alpha = \sup_{a \in \{0,1,\dots,q\}} \sup_{j_0,\dots,j_q \in \Lambda} \|\mathsf{a}_{j_0 \dots j_q}\|^{br}_{j_a,\alpha}. \tag{A.24}$$

Prop. A.5 implies that all these families of norms are equivalent with the following dominance relations

$$\|\mathsf{a}\|_\alpha \le \|\mathsf{a}\|^{cev}_\alpha, \quad \|\mathsf{a}\|^{cev}_\alpha \le C \|\mathsf{a}\|^{br}_{\alpha+2d+1}, \quad \|\mathsf{a}\|^{br}_\alpha \le C_\alpha \|\mathsf{a}\|_\alpha. \tag{A.25}$$

Continuity of chain maps

Proposition A.6. *The boundary operator $\partial_q : C_q(\mathfrak{d}_{al}) \to C_{q-1}(\mathfrak{d}_{al})$ for $q \ge 0$ is well defined and continuous.*

Proof. For $q = 0$, let $\mathsf{a} \in C_0(\mathfrak{d}_{al})$. Then

$$\|\partial \mathsf{a}\|^{br}_\alpha \le \sup_{Y \in \mathbb{B}_d} \sum_{j \in \Lambda} (1 + \mathrm{diam}(Y \cup \{j\}))^\alpha \|\mathsf{a}^Y_j\| \le C \|\mathsf{a}\|^{br}_{\alpha+d+1} \tag{A.26}$$

where we have used

$$\sum_{j\in\Lambda}(1+\mathrm{diam}(Y\cup\{j\}))^{-(d+1)} \le C \tag{A.27}$$

for some constant C that depends on the lattice only.

Similarly, for $q>0$, let $\mathsf{a}\in C_q(\mathfrak{d}_{al})$. Then

$$\begin{aligned}\|\partial\mathsf{a}\|^{br}_{\alpha} &\le \\ &\le \sup_{Y\in\mathbb{B}_d}\sup_{a\in\{1,\ldots,q\}}\sup_{j_1,\ldots,j_q\in\Lambda}\sum_{j_0\in\Lambda}(1+\mathrm{diam}(Y\cup\{j_a\}))^{\alpha}\|\mathsf{a}^Y_{j_0\ldots j_q}\| \le \\ &\le C'\|\mathsf{a}\|^{br}_{\alpha+d+1}\end{aligned} \tag{A.28}$$

for some constant C' that depends on the lattice only. Thus, the map ∂_q is well-defined and continuous for any $q\ge 0$. □

Recall that on UL chains we have a map $h_q : C_q(\mathfrak{d}_l)\to C_{q+1}(\mathfrak{d}_l)$ for $q\ge -1$ defined by

$$h_q(\mathsf{a})_{j_0\ldots j_{q+1}} = \sum_{Y\in\mathbb{B}_d}\sum_{k=0}^{q+1}(-1)^k\frac{\chi_Y(j_k)}{|Y\cap\Lambda|}\mathsf{a}^Y_{j_0\ldots\widehat{j_k}\ldots j_{q+1}} \tag{A.29}$$

for $q\ge 0$ and

$$h_{-1}(\mathsf{A})_{j_0} = \sum_{Y\in\mathbb{B}_d}\frac{\chi_Y(j_0)}{|Y\cap\Lambda|}\mathsf{A}^Y \tag{A.30}$$

for $q=-1$, that gives a contracting homotopy for the augmented complex $C_\bullet(\mathfrak{d}_l)\to\mathfrak{D}_l$, i.e., $h_{q-1}\circ\partial_q+\partial_{q+1}\circ h_q=\mathrm{id}$ and $\partial_0\circ h_{-1}=\mathrm{id}$.

Proposition A.7. *The map h_q extends to a continuous linear map $h_q : C_q(\mathfrak{d}_{al})\to C_{q+1}(\mathfrak{d}_{al})$, that gives a contracting homotopy for the augmented complex $C_\bullet(\mathfrak{d}_{al})\to\mathfrak{D}_{al}$.*

Proof. For $\mathsf{a}\in C_q(\mathfrak{d}_{al})$ with $q\ge 0$ we have

$$\|h_q(\mathsf{a})\|^{br}_{\alpha}\le(q+2)\|\mathsf{a}\|^{br}_{\alpha}. \tag{A.31}$$

Similarly, for $\mathsf{A}\in\mathfrak{D}_{al}$

$$\|h_{-1}(\mathsf{A})\|^{br}_{\alpha}\le\|\mathsf{A}\|^{br}_{\alpha}. \tag{A.32}$$

□

Proposition A.8. *The bracket $\{\cdot,\cdot\}: C_p(\mathfrak{d}_{al})\times C_q(\mathfrak{d}_{al})\to C_{p+q+1}(\mathfrak{d}_{al})$ for $p,q\ge -1$ is well defined and jointly continuous.*

Proof. Let $\mathcal{A}, \mathcal{B} \in \mathfrak{d}_{al}$ and $j, k \in \Lambda$. By the definition eq. (2.8) we can choose $\mathcal{A}_n \in \mathscr{A}_{B_j(n)}$, such that $\|\mathcal{A} - \mathcal{A}_n\| \leq f_j(\mathcal{A}, n)$. Then for $\mathcal{A}^{(n)} = \mathcal{A}_{n+1} - \mathcal{A}_n$ we have

$$\mathcal{A} = \sum_{n \in \mathbb{N}_0} \mathcal{A}^{(n)}, \tag{A.33}$$

with $\mathcal{A}^{(n)} \in \mathscr{A}_{B_j(n+1)}$ and $\| \sum_{m \geq n}^{\infty} \mathcal{A}^{(m)} \| \leq f_j(\mathcal{A}, n)$. Similarly we define $\mathcal{B}^{(n)}$ so that $\| \sum_{m \geq n}^{\infty} \mathcal{B}^{(m)} \| \leq f_k(\mathcal{B}, n)$. Let $\mathcal{C}^{(n,m)} = [\mathcal{A}^{(n)}, \mathcal{B}^{(m)}]$. Clearly, $\mathcal{C}^{(n,m)} = 0$ if $|j - k| \geq n + m + 2$. On the other hand, for $|j - k| < n + m + 2$ the observable $\mathcal{C}^{(n,m)}$ is localized on a ball of radius $2(m + n + 2)$ centered at j and has a norm bounded from above by $8 f_j(\mathcal{A}, n) f_k(\mathcal{B}, m)$. Then we have an estimate

$$\sup_r (1 + r)^\alpha f_j(\mathcal{C}^{(n,m)}, r) \leq 8(1 + 2(m + n + 2))^\alpha f_j(\mathcal{A}, n) f_k(\mathcal{B}, m). \tag{A.34}$$

Hence

$$\|[\mathcal{A}, \mathcal{B}]\|_{j,\alpha} \leq 8 \sum_{n,m \in \mathbb{N}_0} (1 + 2(m + n + 2))^\alpha f_j(\mathcal{A}, n) f_k(\mathcal{B}, m). \tag{A.35}$$

Moreover, for any $\alpha \geq 0$ we have an estimate

$$\sum_{n \in \mathbb{N}_0} (n + 1)^\alpha f_j(\mathcal{A}, n) \leq \|\mathcal{A}\|_{j,\alpha+2} \sum_{n \in \mathbb{N}} 1/(n + 1)^2. \tag{A.36}$$

Therefore

$$\|[\mathcal{A}, \mathcal{B}]\|_{j,\alpha} \leq C_\alpha \|\mathcal{A}\|_{j,\alpha+2} \|\mathcal{B}\|_{k,\alpha+2} \tag{A.37}$$

where C_α is some constant depending on α only.

Similarly, for any $\mathsf{a} \in C_{p \geq 0}(\mathfrak{d}_{al})$ and $\mathsf{b} \in C_{q \geq 0}(\mathfrak{d}_{al})$ we have

$$\|\{\mathsf{a}, \mathsf{b}\}\|_\alpha \leq C_{\alpha,p,q} \|\mathsf{a}\|_{\alpha+2} \|\mathsf{b}\|_{\alpha+2} \tag{A.38}$$

for some constant $C_{\alpha,p,q}$.

For $\mathsf{A} \in \mathfrak{D}_{al}$ and $\mathsf{b} \in C_q(\mathfrak{d}_{al})$ using eq. (A.28) we have

$$\|\{\mathsf{A}, \mathsf{b}\}\|_\alpha^{br} = \|[\partial(h_{-1}(\mathsf{A})), \mathsf{b}]\|_\alpha^{br} \leq C_{\alpha,q} \|\mathsf{A}\|_{\alpha+d+3}^{br} \|\mathsf{b}\|_{\alpha+d+3} \tag{A.39}$$

for some constant $C_{\alpha,q}$, where h_{-1} is a contracting homotopy from Proposition A.7.

Finally, for $\mathsf{A}, \mathsf{B} \in \mathfrak{D}_{al}$ we have

$$\|\{\mathsf{A}, \mathsf{B}\}\|_\alpha^{br} = \|\partial\{\mathsf{A}, h_{-1}(\mathsf{B})\}\|_\alpha^{br} \leq C_\alpha \|\mathsf{A}\|_{\alpha+2d+4}^{br} \|\mathsf{B}\|_{\alpha+2d+4}^{br}. \tag{A.40}$$

Thus the map $\{\cdot, \cdot\}$ is well-defined and continuous.

□

Proposition A.9. *The contraction* $\mathsf{b}_{A_0\ldots A_q} \in \mathfrak{D}_{al}$ *of* $\mathsf{b} \in C_q(\mathfrak{d}_{al})$ *with regions* $A_0, \ldots, A_q$ *is a well-defined and continuous.*

Proof. By the same argument as in the proof of the Proposition A.6, we have

$$\|\mathsf{b}_{A_0}\|^{br.}_{\alpha} \leq C\|\mathsf{b}\|^{br.}_{\alpha+d+1} \tag{A.41}$$

where $\mathsf{b}_{A_0} \in C_{q-1}(\mathfrak{d}_{al})$ is a (possibly partial) contraction defined by

$$(\mathsf{b}_{A_0})^{Y}_{j_1\ldots j_q} := \sum_{j_0\in A_0} \mathsf{b}^{Y}_{j_0\ldots j_q}. \tag{A.42}$$

Therefore

$$\|\mathsf{b}_{A_0\ldots A_q}\|^{br.}_{\alpha} \leq C^{q+1}\|\mathsf{b}\|^{br.}_{\alpha+(q+1)(d+1)}. \tag{A.43}$$

□

Proposition A.10. *For any conical partition* $(A_0, \ldots, A_d)$ *of* $\mathbb{R}^d$ *the contraction* $(\cdot)_{A_0\ldots A_d}$ *is en element of* $\mathfrak{d}_{al}$ *and defines a linear continuous map* $(\cdot)_{A_0\ldots A_d} : C_d(\mathfrak{d}_{al}) \to \mathfrak{d}_{al}$.

Proof. Without loss of generality, we can assume that all A_a are conical regions, since otherwise the map $(\cdot)_{A_0\ldots A_d}$ differs from a contraction with a conical partition by a manifestly continuous linear map.

Let $p \in \mathbb{R}^d$ be the apex of $(A_0, \ldots, A_d)$. Note that the number of tuples $\{j_0, \ldots, j_d\}$ such that $j_a \in A_a$ and $|j_a - p| \leq R$ is less than $CR^{d(d+1)}$ for some constant C. Also not that when at least one j_a belongs to $B^c_p(R)$ we have $\mathrm{diam}(\{j_0, \ldots, j_q\}) \leq C'R$ for some constant C'. In the latter case by Lemma A.1 we have $\|\mathsf{b}_{j_0\ldots j_d}\| \leq 3f(\mathsf{b}, C'R/2)$ for any $\mathsf{b} \in C_d(\mathfrak{d}_{al})$.

Hence for any $\mathsf{b} \in C_d(\mathfrak{d}_{al})$ we have

$$f_p(\mathsf{b}_{A_0,\ldots,A_d}, r) \leq C(r/2)^{d(d+1)} f(\mathsf{b}, r/2) + \\ + \sum_{n=0}^{\infty} C''(1+n+r/2)^{d(d+1)} f(\mathsf{b}, C'(n+r/2)/2) \tag{A.44}$$

for some constant C'', that implies

$$\|\mathsf{b}_{A_0,\ldots,A_d}\|_{p,\alpha} \leq C_\alpha \|\mathsf{b}\|_{\alpha+d(d+1)+2}. \tag{A.45}$$

□

A.5 Ground states of gapped Hamiltonians

For any $\mathsf{H} \in \mathfrak{D}_{al}$ and a piecewise-continuous function $f : \mathbb{R} \to \mathbb{R}$ satisfying $f(t) = \mathcal{O}(|t|^{-\infty})$ let $\mathscr{I}_{\mathsf{H},f} : \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$ be the map

$$\mathscr{I}_{\mathsf{H},f}(\cdot) := \int_{-\infty}^{+\infty} f(t)\alpha_{\mathsf{H}}^{(t)}(\cdot)dt. \tag{A.46}$$

Lemma A.2. *The map $\mathscr{I}_{\mathsf{H},f}$ is a well-defined continuous map.*

Proof. Let us choose $h \in \mathscr{F}_\infty^+$ such that

$$\sup_{j\in\Lambda} \sum_{\substack{Y\ni j \\ \mathrm{diam}(Y)\geq r}} \|\mathsf{H}^Y\| \leq h(r). \tag{A.47}$$

By Theorem 2.1 from [47], for any $\mathcal{A} \in \mathscr{A}_{B_j(R)}$ and $\mathcal{B} \in \mathscr{A}_{B_j^c(R+r)}$ with $r > 1$ and any $0 < \sigma < 1$ we have

$$\frac{\|[\alpha_{\mathsf{H}}(t)(\mathcal{A}), \mathcal{B}]\|}{\|\mathcal{A}\|\|\mathcal{B}\|} \leq C_1 R^d e^{vt - r^{1-\sigma}} + C_2 t R^d (1+r)^d h(r^\sigma) + \\ + C_3 t e^{vt-r^{1-\sigma}} R^{2d} r^{\sigma+d} h(r^\sigma) \tag{A.48}$$

for some constants C_1, C_2, C_3, v independent of j, t, r, R.

Let $t_0 = r^{1-\sigma}/2v$. Then

$$\begin{aligned} \frac{\|[\mathscr{I}_{\mathsf{H},f}(\mathcal{A}), \mathcal{B}]\|}{\|\mathcal{A}\|\|\mathcal{B}\|} &\leq \\ &\leq 2\left(\int_0^{t_0} |f(t)| \frac{\|[\alpha_{\mathsf{H}}(t)(\mathcal{A}), \mathcal{B}]\|}{\|\mathcal{A}\|\|\mathcal{B}\|} dt\right) + 4\int_{t_0}^{\infty} |f(t)|dt \leq \\ &\leq 2C' R^{2d}(C_1 t_0 e^{vt_0 - r^{1-\sigma}} + C_2 t_0^2 (1+r)^d h(r^\sigma) + \\ &+ C_3 t_0^2 e^{vt_0 - r^{1-\sigma}} r^{\sigma+d} h(r^\sigma)) + 4\int_{t_0}^{\infty} |f(t)|dt =: g(R, R+r) \end{aligned} \tag{A.49}$$

for some constant C'. Since $g(r/2, r) \in \mathscr{F}_\infty$, in the same way as in eq. (B.8), we get an estimate for any $\mathcal{A} \in \mathscr{A}_\ell$

$$f_j(\mathscr{I}_{\mathsf{H},f}(\mathcal{A}), r) \leq f_j(\mathcal{A}, r/2) + 2\|\mathcal{A}\| g(r/2, r) + 2 f_j(\mathcal{A}, r/2) g(r/2, r), \tag{A.50}$$

which implies

$$\|\mathscr{I}_{\mathsf{H},f}(\mathcal{A})\|'_{j,\alpha} \leq C_{g,\alpha} \|\mathcal{A}\|'_{j,\alpha}, \quad \alpha \in \mathbb{N}_0 \tag{A.51}$$

for some $C_{g,\alpha} > 0$. Therefore $\mathscr{I}_{\mathsf{H},f} : \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$ is well-defined and continuous.

□

Remark A.1. *The version of the Lieb-Robinson bounds proved in [4] is sufficient to prove the existence of the map $\mathscr{I}_{\mathsf{H},f}$ for UL Hamiltonians or Hamiltonians with exponential decay, but not for arbitrary UAL Hamiltonians. It was pointed to us by Bruno Nachtergaele that the case of UAL Hamiltonians can be dealt with using the improved Lieb-Robinson bounds from [47, 18]. Further implications of these improved bounds for gapped Hamiltonians are studied in [51].*

Using the result of [48] one can show that a smooth family of gapped UL Hamiltonians under certain additional assumptions defines a smooth family of gapped states in the sense of Definition 3.1 (though we expect that a similar result should hold for a smooth family of gapped UAL Hamiltonians):

Proposition A.11. *Let $\mathcal{M}$ be a compact manifold, and let H be a $\mathfrak{D}_l$-valued function which is smooth when regarded as $\mathfrak{D}_{al}$-valued function. Suppose for any $m \in \mathcal{M}$ the derivation H_m is gapped with a unique ground state ψ_m. Suppose also that for any $\mathcal{A} \in \mathscr{A}_{a\ell}$ the average $\langle \mathcal{A} \rangle_{\psi_m}$ is a smooth function on $\mathcal{M}$. Then ψ is a smooth family of gapped states.*

Proof. Since $\mathcal{M}$ is compact, there exists $\Delta > 0$ which bounds from below the gap of H_m for any $m \in \mathcal{M}$.

Let us define $\mathsf{G} \in \Omega^1(\mathcal{M}, \mathfrak{D}_{al})$ by $\mathsf{G} = -\mathscr{I}_{\mathsf{H},W_\Delta}(d\mathsf{H})$, where $W_\Delta(t) = \mathcal{O}(|t|^{-\infty})$ is an odd function such that $\int W_\Delta(t) e^{-i\omega t} dt = \frac{i}{\omega}$ for $|\omega| > \Delta'$ for some $0 < \Delta' < \Delta$ (see Lemma 3.3).

For any smooth path $p : [0, 1] \to \mathcal{M}$ the family $p^*\mathsf{H}$ satisfies the conditions of the Theorem 1.3 from [48] that guarantees $p^*\psi(s) = p^*\psi(0) \circ \alpha^{(s)}_{p^*\mathsf{G}}$.

□

Appendix B

SOME CONSEQUENCES OF THE LIEB-ROBINSON BOUND

B.1 Reproducing functions

We say that $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is reproducing for Λ, if

$$C_f := \sup_{j,k\in\Lambda} \sum_{l\in\Lambda} \frac{f(|j-l|)f(|l-k|)}{f(|j-k|)} < \infty. \tag{B.1}$$

Note that $1/(1+r)^\nu$ is reproducing for any $\Lambda \subset \mathbb{R}^d$ if $\nu > d$, but not every $f \in \mathscr{F}_\infty^+$ is reproducing.

Lemma B.1. *For any $f \in \mathscr{F}_\infty^+$ there is $\tilde{f} \in \mathscr{F}_\infty^+$ that upper-bounds f and $A > 0$ such that $\tilde{f}(r)\tilde{f}(s)/\tilde{f}(r+s) \leq A$ for all $r, s \in \mathbb{R}_{\geq 0}$.*

Proof. Without loss of generality we can assume $f(0) = 1/2$.

Let $h(r) := -(\log f(r))$. This is a monotonically increasing positive function. If $h(r) \geq Cr + D$ for some $C > 0$, the function $\tilde{f}(r) = e^{-Cr-D}$ satisfies the required conditions (with $A = e^{-D}$). Otherwise, let $\tilde{h}(r) = r \inf_{0\leq s\leq r}(h(s)/s)$ and $\tilde{f} = e^{-\tilde{h}(r)}$. It is easy to check that $\tilde{f}$ satisfies the required conditions. □

Lemma B.2. *Any $f \in \mathscr{F}_\infty^+$ can be upper-bounded by $\tilde{f} \in \mathscr{F}_\infty^+$ which is reproducing for Λ.*

Proof. By Lemma B.1 f can be upper-bounded by $f' \in \mathscr{F}_\infty^+$ such that for some $A > 0$ we have $f'(r)f'(s) \leq Af'(r+s)$ for any $r, s \in \mathbb{R}_{\geq 0}$. Let $\tilde{f}(r) = B\sqrt{f'(r)}/(1+r)^{d+1}$ with $B = \sqrt{\|f'\|_{2d+2}}$. Then $\tilde{f}(r) \geq f'(r)$ for all $r \geq 0$ and

$$\begin{aligned}\sum_{l\in\Lambda} \tilde{f}(|j-l|)\tilde{f}(|l-k|) = B^2 \sum_{l\in\Lambda} \frac{\sqrt{f'(|j-l|)}}{(1+|j-l|)^{d+1}} \frac{\sqrt{f'(|l-k|)}}{(1+|l-k|)^{d+1}} \leq \\ \leq B^2 \sum_{l\in\Lambda} \frac{A^{1/2}\sqrt{f'(|j-k|)}}{(1+|j-l|)^{d+1}(1+|l-k|)^{d+1}} \leq \\ \leq C'A^{1/2}B^2 \frac{\sqrt{f'(|j-k|)}}{(1+|j-k|)^{d+1}} = C'A^{1/2}B\tilde{f}(|j-k|)\end{aligned} \tag{B.2}$$

where C' is some constant that depends on the lattice Λ only. □

Lemma B.3. *For any sequence $\{f_n\}$, $n \in \mathbb{N}$ of functions $f_n \in \mathscr{F}^+_\infty$ converging to $f \in \mathscr{F}^+_\infty$, there is a reproducing for Λ function $g \in \mathscr{F}^+_\infty$ that upper-bounds f and f_n for any $n \in \mathbb{N}$.*

Proof. Let $\tilde{g}(r) := \sup_{n\in\mathbb{N}} f_n(r)$. We have an estimate

$$\|\tilde{g}\|_\alpha \leq \sup_{n\in\mathbb{N}} \|f_n - f\|_\alpha + \|f\|_\alpha. \tag{B.3}$$

Since f_n converges to f, this implies $\tilde{g} \in \mathscr{F}_\infty$ and thus can be upper-bounded by a function from $\mathscr{F}^+_\infty$. Hence by Lemma B.2 there exists $g \in \mathscr{F}^+_\infty$ which is reproducing and upper-bounds $\tilde{g}$ and f. □

B.2 Locally generated automorphisms

Proposition B.1. *For any $\mathsf{G} \in C([0,1], \mathfrak{D}_{al})$, there is a family of automorphisms $\alpha_\mathsf{G} : [0,1] \to \mathrm{Aut}(\mathscr{A})$ such that $\forall \mathcal{A} \in \mathscr{A}_{a\ell}$ and $\forall s \in [0,1]$ we have $\alpha^{(s)}_\mathsf{G}(\mathcal{A}) \in \mathscr{A}_{a\ell}$ and the function $\alpha_\mathsf{G}(\mathcal{A}) : [0,1] \to \mathscr{A}_{a\ell}, s \mapsto \alpha^{(s)}_\mathsf{G}(\mathcal{A})$ is continuously differentiable and satisfies*

$$\frac{d\alpha^{(s)}_\mathsf{G}(\mathcal{A})}{ds} = \alpha^{(s)}_\mathsf{G}(\mathsf{G}(s)(\mathcal{A})). \tag{B.4}$$

Proof. To show this we invoke the version of the Lieb-Robinson bound from [27, 50, 52] for an interaction defined in terms of G^Y, $Y \in \mathbb{B}_d$.

Let $h \in \mathscr{F}^+_\infty$ be the function $h(r) = \sup_s \sup_{Y:\mathrm{diam}(Y)\geq r} \|\mathsf{G}^Y(s)\|$. We can choose $g \in \mathscr{F}^+_\infty$ (e.g., we can take $A(h(r))^\alpha$ for some constants A and $0 < \alpha < 1$) such that

$$\sup_{j,k\in\Lambda} \sup_s \sum_{\substack{Y\in\mathbb{B}_d \\ Y\ni j,k}} \frac{\|\mathsf{G}^Y(s)\|}{g(|j-k|)} \leq 1. \tag{B.5}$$

Moreover, by Lemma B.2 g can be chosen to be reproducing, with a reproducing constant $C_g > 0$.

Let $\{\Gamma_n\}$ be an exhausting sequence of bricks. Let $\mathsf{G}^{(n)} = \mathsf{G}|_{\Gamma_n}$. It is easy to see that for any $\mathcal{A} \in \mathscr{A}_\ell$ the sequence of $\mathscr{A}_{a\ell}$-valued functions $\{\mathsf{G}^{(n)}(s)(\mathcal{A})\}$ converges to $\mathsf{G}(s)(\mathcal{A})$ in the Fréchet topology of $C([0,1], \mathscr{A}_{a\ell})$.

Let $\alpha_{\mathsf{G}^{(n)}}$ be the family of automorphisms of $\mathscr{A}_{\Gamma_n}$ defined by

$$\frac{d\alpha^{(s)}_{\mathsf{G}^{(n)}}(\mathcal{A})}{ds} = \alpha^{(s)}_{\mathsf{G}^{(n)}}(\mathsf{G}^{(n)}(s)(\mathcal{A})). \tag{B.6}$$

and the initial condition $\alpha^{(0)}_{\mathsf{G}^{(n)}} = \mathrm{id}$. It is well-known that such an automorphism exists and is unique (it is the holonomy of the parallel transport with respect to the connection $\frac{d}{ds} + \mathsf{G}^{(n)}$ on a trivial bundle with fiber $\mathscr{A}_{\Gamma_n}$). We extend $\alpha_{\mathsf{G}^{(n)}}$ to the whole $\mathscr{A}_\ell$ in the obvious way. Theorem 2.1 from [4] (or more precisely, its version for time-dependent interactions from [52]) then guarantees the following estimate

$$\frac{\|[\alpha^{(s)}_{\mathsf{G}^{(n)}}(\mathcal{A}),\mathcal{B}]\|}{\|\mathcal{A}\|\|\mathcal{B}\|} \leq \frac{2}{C_g}(e^{2C_g s} - \theta(R-r)) \sum_{\substack{j\in B_j(r)\\ k\in B^c_j(R)}} g(|j-k|) =: 2h(r,R) \quad \text{(B.7)}$$

for any $\mathcal{A} \in \mathscr{A}_{B_j(r)}$ and $\mathcal{B} \in \mathscr{A}_{B^c_j(R)}$, while Theorem 2.2 in [50] proves the existence of the limit $\lim_{n\to\infty} \alpha^{(s)}_{\mathsf{G}^{(n)}} =: \alpha^{(s)}_{\mathsf{G}} \in \mathrm{Aut}(\mathscr{A})$ satisfying the same estimate. We have an estimate

$$\begin{aligned} f_j(\alpha^{(s)}_{\mathsf{G}}(\mathcal{A}),r) &\leq f_j(\alpha^{(s)}_{\mathsf{G}}(\mathcal{A}^{(r/2)}),r) + \|\mathcal{A} - \mathcal{A}^{(r/2)}\| \leq \\ &\leq \|\alpha^{(s)}_{\mathsf{G}}(\mathcal{A}^{(r/2)}) - (\alpha^{(s)}_{\mathsf{G}}(\mathcal{A}^{(r/2)}))|_{B_j(r)}\| + f_j(\mathcal{A},r/2) \leq \\ &\leq 2\|\mathcal{A}^{(r/2)}\|h(r/2,r) + f_j(\mathcal{A},r/2) \leq \\ &\leq f_j(\mathcal{A},r/2) + 2\|\mathcal{A}\|h(r/2,r) + 2f_j(\mathcal{A},r/2)h(r/2,r), \end{aligned} \quad \text{(B.8)}$$

where $\mathcal{A}^{(r/2)}$ is a best possible approximation of $\mathcal{A}$ on $B_j(r/2)$ and we used (2.5) to go from the second to the third line. This implies

$$\|\alpha^{(s)}_{\mathsf{G}}(\mathcal{A})\|'_{j,\beta} \leq C_{g,\beta}\|\mathcal{A}\|'_{j,\beta}, \quad \beta \in \mathbb{N}_0 \quad \text{(B.9)}$$

for some $C_{g,\beta} > 0$. Therefore $\alpha^{(s)}_{\mathsf{G}}(\mathcal{A}) \in \mathscr{A}_{a\ell}$ for any $\mathcal{A} \in \mathscr{A}_{a\ell}$.

Let $\mathcal{A} \in \mathscr{A}_\ell$. Eq. (B.6) implies

$$\alpha^{(s+\Delta s)}_{\mathsf{G}^{(n)}}(\mathcal{A}) - \alpha^{(s)}_{\mathsf{G}^{(n)}}(\mathcal{A}) = \int_s^{s+\Delta s} \alpha^{(u)}_{\mathsf{G}^{(n)}} \left(\mathsf{G}^{(n)}(u)(\mathcal{A})\right) du. \quad \text{(B.10)}$$

Since according to Theorem 2.2 of [50] $\alpha^{(s)}_{\mathsf{G}^{(n)}}(\mathcal{A})$ converges in norm to its $n \to \infty$ limit uniformly in s on any compact subset of $\mathbb{R}$, we may exchange the limit $n \to \infty$ and integration and get

$$\alpha^{(s+\Delta s)}_{\mathsf{G}}(\mathcal{A}) - \alpha^{(s)}_{\mathsf{G}}(\mathcal{A}) = \int_s^{s+\Delta s} \alpha^{(u)}_{\mathsf{G}} \left(\mathsf{G}(u)(\mathcal{A})\right) du. \quad \text{(B.11)}$$

To deduce this for general $\mathcal{A} \in \mathscr{A}_{a\ell}$, we choose a sequence of local observables $\mathcal{A}^{(n)}$ converging to $\mathcal{A}$ in the Fréchet topology and use the uniform convergence of $\mathsf{G}(s)(\mathcal{A}^{(n)})$ to $\mathsf{G}(s)(\mathcal{A})$ on $[0,1]$ and Prop. A.8 to show that (B.11) holds for $\mathcal{A} \in \mathscr{A}_{a\ell}$.

□

Lemma B.4. *The map* $\alpha^{(1)} : C([0,1],\mathfrak{D}_{al}) \times \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$, $(\mathsf{G},\mathcal{A}) \mapsto \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})$ *is continuous.*

Proof. First, let us show continuity in G. Let $\{\Delta\mathsf{G}_n(s)\}$, $n \in \mathbb{N}_0$, be a sequence in $C([0,1],\mathfrak{D}_{al})$ converging to 0. Note that by Lemma B.3 we can find $g \in \mathscr{F}^+_\infty$, such that for any n eq. (B.5) holds for G replaced with $\mathsf{G}+\Delta\mathsf{G}_n$. Therefore eq. (B.8) implies

$$\|\alpha^{(u)}_{\mathsf{G}+\Delta\mathsf{G}_n}(\mathcal{A})\|'_{j,\alpha} \leq B_\alpha \|\mathcal{A}\|'_{j,\alpha} \tag{B.12}$$

for any $u \in [0,1]$ and any $n \in \mathbb{N}_0$ and some constants $B_\alpha > 0$ depending on g only. Hence

$$\begin{aligned} &\|\alpha^{(1)}_{\mathsf{G}+\Delta\mathsf{G}_n}(\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\|'_{j,\alpha} = \\ &\qquad = \|\int_0^1 du \frac{d}{du}\left(\alpha^{(u)}_{\mathsf{G}+\Delta\mathsf{G}_n}(\alpha^{(u)-1}_{\mathsf{G}}(\alpha^{(1)}_{\mathsf{G}}(\mathcal{A})))\right)\|'_{j,\alpha} \leq \\ &\qquad \leq B_\alpha \int_0^1 du \|(\Delta\mathsf{G}_n(u))\left(\alpha^{(u)-1}_{\mathsf{G}}(\alpha^{(1)}_{\mathsf{G}}(\mathcal{A}))\right)\|'_{j,\alpha} \leq \\ &\qquad\qquad \leq \tilde{B}_\alpha \|\mathcal{A}\|'_{j,\alpha+d+3} \|\Delta\mathsf{G}_n\|^{br}_{\alpha+d+3} \end{aligned} \tag{B.13}$$

for some constant $\tilde{B}_\alpha$. Here we have used the same estimate as in (A.39). Since the r.h.s. converges to zero as $n \to \infty$, this proves continuity in G.

To show joint continuity, we similarly choose g for a converging sequence $(\mathsf{G}_n, \mathcal{A}_n) \to (\mathsf{G},\mathcal{A})$. Then

$$\begin{aligned} &\|\alpha^{(1)}_{\mathsf{G}+\Delta\mathsf{G}}(\mathcal{A}+\Delta\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\|'_{j,\alpha} \leq \|\alpha^{(1)}_{\mathsf{G}+\Delta\mathsf{G}}(\Delta\mathcal{A})\|'_{j,\alpha} + \\ &\qquad + \|\alpha^{(1)}_{\mathsf{G}+\Delta\mathsf{G}}(\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\|'_{j,\alpha} \leq B_\alpha \|\Delta\mathcal{A}\|'_{j,\alpha} + \\ &\qquad\qquad + \|\alpha^{(1)}_{\mathsf{G}+\Delta\mathsf{G}}(\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\|'_{j,\alpha}. \end{aligned} \tag{B.14}$$

□

Corollary B.1. *The map* $\alpha : C([0,1],\mathfrak{D}_{al}) \times [0,1] \times \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$, $(\mathsf{G}, s, \mathcal{A}) \mapsto \alpha_{\mathsf{G}}(s)(\mathcal{A})$ *is continuous.*

Proof. Consider the "rescaling map" $\lambda : C([0,1],\mathfrak{D}_{al}) \times [0,1] \to C([0,1],\mathfrak{D}_{al})$, $(\mathsf{G}, s) \mapsto \lambda(\mathsf{G}, s)(u) = s\mathsf{G}(su)$. It is easy to check that this map is continuous. It is also straightforward to check that $\alpha(\mathsf{G}, s, \mathcal{A}) = \alpha^{(1)}(\lambda(\mathsf{G}, s), \mathcal{A})$. Therefore by Lemma B.4 the map α is continuous. □

Proposition B.2. *The map* $\alpha^{(1)} : C([0,1], \mathfrak{D}_{al}) \times \mathscr{A}_{a\ell} \to \mathscr{A}_{a\ell}$ *defined by* $(\mathsf{G}, \mathcal{A}) \mapsto \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})$ *is smooth.*

Proof. $\alpha^{(1)}_{\mathsf{G}}(\mathcal{A})$ is linear in $\mathcal{A}$ and by Lemma B.4 is jointly continuous in G and $\mathcal{A}$. Therefore it is sufficient to show that it is a smooth function of G. As in the proof of Lemma B.4, we write

$$\alpha^{(1)}_{\mathsf{G}+t\Delta\mathsf{G}}(\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A}) = \\ = \int_0^1 \frac{d}{du}\left[\alpha^{(u)}_{\mathsf{G}+t\Delta\mathsf{G}} \circ \alpha^{(u)-1}_{\mathsf{G}} \circ \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\right] du = \\ = t\int_0^1 \alpha^{(u)}_{\mathsf{G}+t\Delta\mathsf{G}}\left(\Delta\mathsf{G}(u)\left(\alpha^{(u)-1}_{\mathsf{G}} \circ \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\right)\right) du. \tag{B.15}$$

Using Cor. B.1, we get

$$\lim_{t\to 0} \frac{\alpha^{(1)}_{\mathsf{G}+t\Delta\mathsf{G}}(\mathcal{A}) - \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})}{t} = \int_0^1 \alpha^{(u)}_{\mathsf{G}}\left(\Delta\mathsf{G}(u)\left(\alpha^{(u)-1}_{\mathsf{G}} \circ \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\right)\right) du. \tag{B.16}$$

This shows that the directional derivative of $\alpha^{(1)}$ with respect to G exists. Moreover, by Cor. B.1 and Prop. A.8 the derivative is continuous. Iterating the argument, we infer that $\alpha^{(1)}_{\mathsf{G}}(\mathcal{A})$ is a smooth function of G. □

Remark B.1. *It follows from the above computation that if* G *is a smooth* $C([0,1], \mathfrak{D}_{al})$*-valued function on a manifold* $\mathcal{M}$*, then* $\alpha^{(1)}_{\mathsf{G}}(\mathcal{A})$ *is a smooth* $\mathscr{A}_{a\ell}$*-valued function on* $\mathcal{M}$ *whose* $d_{\mathcal{M}}$*-derivative is given by*

$$d_{\mathcal{M}}\alpha^{(1)}_{\mathsf{G}}(\mathcal{A}) = \int_0^1 \alpha^{(u)}_{\mathsf{G}}\left(d_{\mathcal{M}}\mathsf{G}(u)\left((\alpha^{(u)}_{\mathsf{G}})^{-1} \circ \alpha^{(1)}_{\mathsf{G}}(\mathcal{A})\right)\right) du. \tag{B.17}$$

Somewhat schematically, we can also write

$$(\alpha^{(1)}_{\mathsf{G}})^{-1} \circ d_{\mathcal{M}}\alpha^{(1)}_{\mathsf{G}} = \int_0^1 \left((\alpha^{(1)}_{\mathsf{G}})^{-1} \circ \alpha^{(u)}_{\mathsf{G}}\right)(d_{\mathcal{M}}\mathsf{G}(u))du. \tag{B.18}$$

This formula is schematic because $\alpha^{(1)}_{\mathsf{G}}$ *is a function on* $\mathcal{M}$ *valued in automorphisms of* $\mathscr{A}_{a\ell}$*, and we do not introduce any topology on the set of automorphisms. The proper interpretation of this formula is as follows. Note that the r.h.s. of eq. (B.18) is an element of* $\Omega^1(\mathcal{M}, \mathfrak{D}_{al})$*. Let us denote it* ω_{G}*. Then for any* $\mathcal{B} \in \mathscr{A}_{a\ell}$ *the 1-form* $(\alpha^{(1)}_{\mathsf{G}})^{-1} \circ d_{\mathcal{M}}\alpha^{(1)}_{\mathsf{G}}(\mathcal{B}) \in \Omega^1(\mathcal{M}, \mathscr{A}_{a\ell})$ *is equal to* $\omega_{\mathsf{G}}(\mathcal{B})$*. More generally, if* $\mathcal{B}$ *is a smooth* $\mathscr{A}_{a\ell}$*-valued function on* $\mathcal{M}$*, then*

$$d_{\mathcal{M}}\alpha^{(1)}_{\mathsf{G}}(\mathcal{B}) = \alpha^{(1)}_{\mathsf{G}}\left(d_{\mathcal{M}}\mathcal{B} + \omega_{\mathsf{G}}(\mathcal{B})\right). \tag{B.19}$$

This implies that the covariant differential $d_{\mathcal{M}} + \omega_{\mathsf{G}}(\,\cdot\,)$ *on the trivial bundle with fiber* $\mathscr{A}_{a\ell}$ *is flat, i.e.,* $d_{\mathcal{M}}\omega_{\mathsf{G}} + \frac{1}{2}\{\omega_{\mathsf{G}}, \omega_{\mathsf{G}}\} = 0$.

B.3 Discrete Lieb-Robinson bound

Lemma B.5. *Given observables $\{\mathcal{A}_i\}_{i=0}^{\infty}$ such that $\sum_i \|\mathcal{A}_i - 1\|$ converges, $\prod_i \mathcal{A}_i = \mathcal{A}_0\mathcal{A}_1 \ldots$ exists as an observable.*

Proof. Let $\mathcal{C}_n = \prod_{i=0}^{n} \mathcal{A}_i$ and $\mathcal{B}_i = \mathcal{A}_{i+1} - 1$. Then

$$\mathcal{C}_n = \sum_{i=0}^{n-1} \mathcal{C}_i \mathcal{B}_i + \mathcal{C}_0. \tag{B.20}$$

This implies $\|\mathcal{C}_n\| \leq \sum_{i=0}^{n-1} \|\mathcal{B}_i\|\|\mathcal{C}_i\| + \|\mathcal{C}_0\|$, which by Gronwall's lemma implies

$$\|\mathcal{C}_n\| \leq \|\mathcal{C}_0\| \exp\left(\sum_{i=0}^{n-1} \|\mathcal{B}_i\|\right) \leq \|\mathcal{C}_0\| \exp\left(\sum_{i=0}^{\infty} \|\mathcal{B}_i\|\right) < \infty.$$

Thus the right hand side of equation (B.20) converges absolutely. □

Corollary B.2. *Given strictly local unitaries $\mathcal{V}_k$ on $[(-k - 1/2)L, (k + 1/2)L]$ with $\|\mathcal{V}_k - 1\| \leq h((k + 1/2)L)$ for some $h \in \mathscr{F}_\infty^+$, the product $\mathcal{V}_0\mathcal{V}_1\mathcal{V}_2\ldots$ exists. Furthermore, it is f-localized at 0 for some $f \in \mathscr{F}_\infty^+$ determined by h.*

Proof. As $h(r) = \mathcal{O}(r^{-\infty})$ and $\|\mathcal{V}_k - 1\| \leq h((k+1/2)L)$, $\sum_k \|\mathcal{V}_k - 1\| \leq \sum_k h((k+1/2)L)$ converges. Therefore, the product $\mathcal{C}_\infty = \mathcal{V}_0\mathcal{V}_1\mathcal{V}_2 \ldots$ exists by the lemma above. Let $\mathcal{C}_n = \mathcal{V}_0\mathcal{V}_1\mathcal{V}_2 \ldots \mathcal{V}_n$ and $\mathcal{B}_i = \mathcal{V}_{i+1} - 1$. Then

$$\mathcal{C}_n = \sum_{i=0}^{n-1} \mathcal{C}_i \mathcal{B}_i + \mathcal{C}_0. \tag{B.21}$$

For any $\mathcal{A} \in \mathscr{A}_j$,

$$\begin{aligned} \|[\mathcal{C}_\infty, \mathcal{A}]\| = \|[\sum_{i=0}^{\infty} \mathcal{C}_i \mathcal{B}_i + \mathcal{C}_0, \mathcal{A}]\| &\leq \sum_{(i+\frac{1}{2})L > j} \|[\mathcal{C}_i \mathcal{B}_i, \mathcal{A}]\| \\ &\leq 2\|\mathcal{A}\| \sum_{(i+\frac{1}{2})L > j} \|\mathcal{B}_i\| \leq 2\|\mathcal{A}\| \sum_{(i+\frac{1}{2})L > j} h((i + 1/2)L). \end{aligned} \tag{B.22}$$

Thus, we may let $f(r) = \sum_{s>r} h(s) = \mathcal{O}(r^{-\infty})$. □

Lemma B.6. *Let Λ be a 1-dimensional lattice with sites $j \in \mathbb{Z} \subset \mathbb{R}$. For any $f \in \mathscr{F}_\infty^+$ there is $L \in \mathbb{N}$ such that any ordered composition $\overrightarrow{\prod_{n=-\infty}^{\infty}} \beta^{(n)}$ of LGAs $\beta^{(n)}$ which are f-localized at $j = nL$ for $n \in \mathbb{Z}$ is an LGA.*

Proof. First, note that it is enough to show this for $\overrightarrow{\prod_{n=0}^{\infty}}\beta^{(n)}$. Second, to prove the latter it is enough to show that with an appropriate choice of L for any f-localized at 0 observable $\mathcal{A}$ the observable $(\overleftarrow{\prod_{n=1}^{N}}\beta^{(n)}(\mathcal{A})$ is almost localized at 0 with localization depending on f only (in particular, independent of N), and that as $N \to \infty$ it converges in the norm to some element of $\mathscr{A}$.

Let $\mathcal{U}^{(n)} = e^{i\mathcal{B}^{(n)}}$ be a unitary that corresponds to $\beta^{(n)}$. It can be represented as a product $(\mathcal{V}_0^{(n)}\mathcal{V}_1^{(n)}\mathcal{V}_2^{(n)}...)$ of strictly local unitaries $\mathcal{V}_k^{(n)}$ on $B_n((k+\frac{1}{2})L) := [(n-k-\frac{1}{2})L, (n+k+\frac{1}{2})L]$, so that $\|\mathcal{V}_k^{(n)} - 1\| \leq h((k+\frac{1}{2})L)$ for some $h \in \mathscr{F}_\infty^+$ that depends on f only. This is achieved by letting $(\mathcal{V}_0^{(n)}\mathcal{V}_1^{(n)}...\mathcal{V}_k^{(n)}) = e^{i\mathcal{B}|_{B_n((k+\frac{1}{2})L)}}$.

Since conjugation of a strictly local observable $\mathcal{A}$ with a unitary $\mathcal{U}$ strictly local in the localization set of $\mathcal{A}$ does not change the property $\|\mathcal{A} - 1\| < \varepsilon$ and preserves the localization set, we can rearrange unitaries $\mathcal{V}_k^{(n)}$ in the product

$$(\mathcal{V}_0^{(1)}\mathcal{V}_1^{(1)}\mathcal{V}_2^{(1)}...)(\mathcal{V}_0^{(2)}\mathcal{V}_1^{(2)}\mathcal{V}_2^{(2)}...)...(\mathcal{V}_0^{(N)}\mathcal{V}_1^{(N)}\mathcal{V}_2^{(N)}...) \tag{B.23}$$

in the following order:

$$(\tilde{\mathcal{V}}_0^{(1)})(\tilde{\mathcal{V}}_0^{(2)}\tilde{\mathcal{V}}_1^{(1)})(\tilde{\mathcal{V}}_0^{(3)}\tilde{\mathcal{V}}_1^{(2)}\tilde{\mathcal{V}}_2^{(1)})...(\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)})..., \tag{B.24}$$

where $\tilde{\mathcal{V}}_k^{(n)}$ is obtained from $\mathcal{V}_k^{(n)}$ by conjugation with $\mathcal{V}_l^{(m)}$ with m, l satisfying $n+1 \leq m \leq n+k$ and $0 \leq l \leq n+k-m$. Importantly, $\tilde{\mathcal{V}}_k^{(n)}$ is strictly local on the same interval as $\mathcal{V}_k^{(n)}$ and still satisfies $\|\tilde{\mathcal{V}}_k^{(n)} - 1\| \leq h((k+\frac{1}{2})L)$. The infinite product eq. (B.24) is a well-defined almost-local observable by Corollary B.2. Indeed, $\|\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)} - 1\| \leq \sum_{i=1}^{n}\|\tilde{\mathcal{V}}_{n-i}^{(i)} - 1\|$ by repeatedly applying inequality $\|\mathcal{A}\mathcal{B} - 1 - \mathcal{A} + \mathcal{A}\| \leq \|\mathcal{A}\|\|\mathcal{B} - 1\| + \|\mathcal{A} - 1\|$. Since for any fixed N we have $\mathcal{V}_k^{(n)} = 1$ for $n > N$, $\|\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)} - 1\| \leq \sum_{i=1}^{N}\|\tilde{\mathcal{V}}_{n-i}^{(i)} - 1\| \leq \sum_{i=1}^{N} h((n-i+\frac{1}{2})L)$ satisfies the assumption of Corollary B.2. After this rearrangement the infinite product eq. (B.24) still converges to the same unitary observable as eq. (B.23).

Let $\tilde{\mathcal{U}}^{(n)} = \tilde{\mathcal{V}}_0^{(n)}...\tilde{\mathcal{V}}_{n-1}^{(1)}$. We can represent $\mathcal{A} = \sum_{p=0}^{\infty}\mathcal{A}_p$ with $\sum_{p=0}^{n}\mathcal{A}_p = \mathcal{A}|_{B_0(n+1/2)}$. Let $\mathcal{A}_p^{(0)} := \mathcal{A}_p$, $\mathcal{A}_p^{(n)} := \sum_{k=0}^{n-1}\tilde{\mathcal{U}}^{(n)*}[\mathcal{A}_p^{(k)}, \tilde{\mathcal{U}}^{(n)}]$. Note that $\mathcal{A}_p^{(n)} \in \mathscr{A}_{B_0(p+\frac{1}{2})}$ for $n \leq p$, and $\mathcal{A}_p^{(n)} \in \mathscr{A}_{B_0(n+\frac{1}{2})}$ for $n > p$. Therefore we have

$$\|\mathcal{A}_0^{(n)}\| = \sum_{k=0}^{n-1}\|[\mathcal{A}_0^{(k)}, \tilde{\mathcal{U}}^{(n)}]\| \leq \sum_{k=0}^{n-1}\sum_{l>\frac{n-k-1}{2}}^{n-1}\|[\mathcal{A}_0^{(k)}, \tilde{\mathcal{V}}_l^{(n-l)}]\| \leq$$

$$\leq \sum_{k=0}^{n-1}\sum_{l>\frac{n-k-1}{2}}^{\infty}2\|\mathcal{A}_0^{(k)}\|h((l+\frac{1}{2})L) \leq 2\sum_{k=0}^{n-1}\|\mathcal{A}_0^{(k)}\|g_{n-k} \tag{B.25}$$

where $g_n := g(nL/2)$ for $g(n) := \sum_{l\geq n}^{\infty} h(l)$.

By Lemma B.2, any function $g \in \mathscr{F}_\infty^+$ can be upper-bounded by a reproducing function $\tilde{g} \in \mathscr{F}_\infty^+$. We can further upper-bound $\tilde{g}(r)$ by a reproducing function $g'(r) = A\tilde{g}(r)^\alpha/r^\nu \in \mathscr{F}_\infty^+$ for some constants A, $0 < \alpha < 1$ and $\nu > d$. Since $1/r^\nu$ is also reproducing, we have

$$A^2 \sum_{k=1}^{n-1} \frac{\tilde{g}(kL/2)^\alpha \tilde{g}((n-k)L/2)^\alpha}{(kL/2)^\nu((n-k)L/2)^\nu} < \frac{C}{L^\nu} A \frac{\tilde{g}(nL/2)^\alpha}{(nL/2)^\nu}, \tag{B.26}$$

and therefore for $g'_n := g'(nL/2) \geq g_n$ we have $\sum_{k=1}^{n-1} g'_k g'_{n-k} < (C/L^\nu) g'_n$ for some constant C. By changing L one can make $(C/L^\nu) < 1/2$. Therefore one can choose L so that for $a_n = 2ng'_n$ we have

$$\begin{aligned} &2(g_n \cdot 1 + g_{n-1}a_1 + g_{n-2}a_2 + \ldots + g_1 a_{n-1}) \leq \\ &\quad \leq 2(g'_n \cdot 1 + 2g'_{n-1}g'_1 + 4g'_{n-2}g'_2 + \ldots + 2(n-1)g'_1 g'_{n-1}) \leq 2ng'_n = a_n. \end{aligned} \tag{B.27}$$

Together with eq. (B.25) this implies that $\|\mathcal{A}_0^{(n)}\|/\|\mathcal{A}_0\|$ can be upper-bounded by $a_n = \mathcal{O}(n^{-\infty})$, and the sequence $\sum_{k=0}^n \mathcal{A}_0^{(k)}$ converges in the norm to some element, which is almost localized at 0. By construction the localization depends on f only.

In the same way one can estimate the norms of $\mathcal{A}_p^{(n+p)}$ for $n, p > 0$ and bound $\|\mathcal{A}_p^{(n+p)}\|/\|\mathcal{A}_p\|$ by a sequence $a_n = \mathcal{O}(n^{-\infty})$. Together with $\|\mathcal{A}_p\| = \|\sum_{q=0}^p \mathcal{A}_p^{(q)}\|$ that ensures convergence of $\sum_{p=0}^\infty \sum_{n=0}^\infty \mathcal{A}_p^{(n)}$ to some almost localized at 0 observable with localization depending on f only.

□

Appendix C

LOCAL PERTURBATIONS

Definition C.1. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-increasing function. We say that two states $\psi, \psi' : \mathscr{A}_\Lambda \to \mathbb{C}$ of a d-dimensional lattice system $\mathscr{A}_\Lambda$ are f-close at $x \in \mathbb{R}^d$ if for any $\mathscr{A} \in \mathscr{A}_{B_r^c(x)}$ and $r > 0$ we have*

$$|\langle \mathcal{A} \rangle_{\psi'} - \langle \mathcal{A} \rangle_{\psi}| \leq f(r) \|\mathcal{A}\|. \tag{C.1}$$

If there exit $C, a > 0$ and $x \in \mathbb{R}^d$ such that the states are f-close at x for $f(r) = Ce^{-ar}$, we say that two states are exponentially close.

Proposition C.1. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-increasing function, such that* $\lim_{r\to\infty} f(r) = 0$. *If two pure states $\psi, \psi' : \mathscr{A} \to \mathbb{C}$ of a lattice system $\mathscr{A}_\Lambda$ are f-close at x for some $x \in \mathbb{R}^d$, then there is a unitary element $\mathcal{U} \in \mathscr{A}_\Lambda$ such that* $\psi = \psi' \circ \mathrm{Ad}_{\mathcal{U}}$.

Proof. By Corollary 2.6.11 [11], both states are unitarily equivalent. Hence, they are vector states in the same Hilbert space that can be chosen to be space of the GNS representation associated with ψ. Let us choose the corresponding state vectors $|\psi\rangle$ and $|\psi'\rangle$. By Kadison transitivity theorem, one vector can be produced from another by a unitary element $\mathcal{U}$ of $\mathscr{A}$. □

Corollary C.1. *The theorem also implies that there is a self-adjoint observable $\mathcal{K} \in \mathscr{A}$ such that $|\psi'\rangle - \mathcal{K}|\psi\rangle$ is proportional to $|\psi\rangle$.*

The goal of this section is study the situations in which the observable $\mathcal{U}$ from Proposition C.1 can be chosen to be almost local.

We start with recalling well-known facts about the fidelity $F(\rho, \sigma) := (\|\sqrt{\rho}\sqrt{\sigma}\|_1)^2$ of quantum states on a separable Hilbert space described by density matrices ρ and σ. Firstly, the fidelity satisfies Fuchs van de Graaf inequalities

$$1 - \sqrt{F(\rho, \sigma)} \leq \frac{1}{2}\|\rho - \sigma\|_1 \leq \sqrt{1 - F(\rho, \sigma)}. \tag{C.2}$$

Secondly, by Uhlmann's theorem, one can identify the fidelity $F(\rho, \sigma)$ with the the maximal overlap $|\langle \chi_\rho | \chi_\sigma \rangle|^2$ of unit vectors $|\chi_\rho\rangle, |\chi_\sigma\rangle \in \mathcal{H} \otimes \mathcal{H}'$ for some $\mathcal{H}' \cong \mathcal{H}$ representing purifications of ρ and σ on $\mathcal{H} \otimes \mathcal{H}'$.

Proposition C.2. *Let ψ be a pure state on $\mathscr{A}$ which is f-close at the origin to a factorized state ψ_0 for some $f(r) = \mathcal{O}(r^{-\infty})$. Then ψ and ψ_0 are unitarily equivalent and one can be produced from the other by a conjugation with a unitary $\mathcal{U} \in \mathscr{A}_{a\ell}$ with localization that depend on f only.*

Proof. Unitary equivalence of states ψ and ψ_0 follows from Proposition C.1. Let $(\Pi_0, \mathcal{H}_0, |0\rangle)$ be the GNS data for ψ_0. The state ψ is a vector state corresponding to $|\psi\rangle \in \mathcal{H}_0$. Let $\mathcal{V}_n$ be a subspace of $\mathcal{H}_0$ spanned by vectors which can be produced from $|0\rangle$ by an observable localized on Γ_n. Note that $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{V}_3 \subset ....$

Let n_0 be such that $f(n_0) < 1/2$. Let us temporarily fix $n \geq n_0$ and not indicate it explicitly. Let us estimate the angle between the vector $|\psi\rangle$ and the subspace $\mathcal{V} = \mathcal{V}_n$. The Hilbert space $\mathcal{H}_0$ is isomorphic to $\mathcal{H}_\Gamma \otimes \mathcal{H}_{\bar{\Gamma}}$, where the Hilbert spaces $\mathcal{H}_\Gamma$ and $\mathcal{H}_{\bar{\Gamma}}$ carry representations of $\mathscr{A}_\Gamma$ and $\mathscr{A}_{\bar{\Gamma}}$, respectively. The restrictions of vector states ψ and ψ_0 to $\mathscr{A}_{\bar{\Gamma}}$ can be described by density matrices ρ and ρ_0 on $\mathcal{H}_{\bar{\Gamma}}$. The density matrix ρ_0 is pure, but ρ is mixed, in general. We have

$$\|\rho - \rho_0\|_1 \leq \varepsilon \tag{C.3}$$

where $\varepsilon = f(n) \in (0, 1)$. Fuchs–van de Graaf inequality implies that for fidelity we have

$$F(\rho, \rho_0) := \|(\rho)^{1/2}(\rho_0)^{1/2}\|_1 \geq 1 - \frac{\varepsilon}{2}. \tag{C.4}$$

Let

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |\eta_i\rangle \otimes |\xi_i\rangle \tag{C.5}$$

be the Schmidt decomposition of $|\psi\rangle$. Here $N \leq \dim\mathcal{H}_\Gamma$, $|\eta_i\rangle$, $i = 1, \ldots, N$, are orthonormal vectors in $\mathcal{H}_\Gamma$, $|\xi_i\rangle$, $i = 1, \ldots, N$, are orthonormal vectors in $\mathcal{H}_{\bar{\Gamma}}$, and λ_i, $i = 1, \ldots, N$, are positive numbers satisfying $\sum_i \lambda_i = 1$. Since $|0\rangle$ is factorized, its Schmidt decomposition contains only a single term:

$$|0\rangle = |0_\Gamma\rangle \otimes |0_{\bar{\Gamma}}\rangle. \tag{C.6}$$

Let $a_i = \langle \xi_i | 0_{\bar{\Gamma}} \rangle$. The fidelity of ρ and ρ_0 can be expressed in terms of λ_i and a_i:

$$F(\rho, \rho_0) = \left(\sum_i \lambda_i |a_i|^2 \right)^{1/2}. \tag{C.7}$$

We define $|v\rangle = \sum_i \sqrt{\lambda_i} a_i^* |\eta_i\rangle$ and let

$$|\chi\rangle = \left(\sum_k \lambda_k |a_k|^2\right)^{-1/2} |v\rangle \otimes |0_{\bar{\Gamma}}\rangle. \tag{C.8}$$

Then it is easy to see that

$$|\langle\psi|\chi\rangle| = \left(\sum_i \lambda_i |a_i|^2\right)^{1/2} = F(\rho, \rho_0) \geq 1 - \frac{\varepsilon}{2}. \tag{C.9}$$

Since $\varepsilon < 1/2$, $|\psi\rangle$ is not orthogonal to the subspace $\mathcal{V}$.

For any $n \geq n_0$ let $|\chi_n\rangle \in \mathcal{V}_n$ be as above (geometrically, it is the normalized projection of $|\psi\rangle$ to $\mathcal{V}_n$). The estimate (C.9) implies

$$|\langle\chi_n|\chi_{n+1}\rangle| \geq 1 - 2\varepsilon_n, \tag{C.10}$$

where $\varepsilon_n = f(n)$. Let $\mathcal{U}_{n_0} = e^{i\mathcal{G}_{n_0}}$ be a unitary localized on Γ_{n_0} that implements a rotation of $|0\rangle$ to $|\chi_{n_0}\rangle$ with $\|\mathcal{G}_{n_0}\| \leq \pi$. We can also choose unitary observables $\mathcal{U}_n$ for $n \geq n_0$ localized on Γ_{n+1} and satisfying $\|1 - \mathcal{U}_n\| \leq (4\varepsilon_n)^{1/2}$ which implement rotations of $|\chi_n\rangle$ to $|\chi_{n+1}\rangle$, and which therefore can be written as $\mathcal{U}_n = e^{i\mathcal{G}_n}$ for an observable $\mathcal{G}_n$ local on Γ_{n+1} with $\|\mathcal{G}_n\| \leq 2(2\varepsilon_n)^{1/2}$. The ordered product of all such unitaries over $n \geq n_0$ can be written as $\mathcal{U} = e^{i\mathcal{G}}$. By construction, this unitary maps $|0\rangle$ to $|\psi\rangle$. Moreover, since $\|\mathcal{G}_n\| \leq 2(2f(n))^{1/2}$ for $n \geq n_0$, $\mathcal{G}$ is g-localized for some MDP function $g(r) = \mathcal{O}(r^{-\infty})$ that only depends on $f(r)$, and $\|\mathcal{G}\| \leq \sum_{n=n_0}^{\infty} 2(2f(n))^{1/2} + \pi$, a quantity that also depends only on $f(r)$.

□

To generalize the result for states which are not necessarily factorized at infinity, we need some additional requirements.

Definition C.2. *Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a non-increasing function. We say that a state $\psi : \mathscr{A} \to \mathbb{C}$ of a lattice system $\mathscr{A}_\Lambda$ has an f-decay of mutual correlations, if for any finite disjoint subsets $X, Y \subset \Lambda$ and any $\mathcal{O} \in \mathscr{A}_{X \cup Y}$, we have*

$$|\langle\mathcal{O}\rangle_\psi - \langle\mathcal{O}\rangle_{\psi|_X \otimes \psi|_Y}| \leq f(r)\|\mathcal{O}\|, \tag{C.11}$$

where r is the distance between X and Y. If there exists $f \in \mathcal{O}(r^{-\infty})$, such that ψ has f-decay, we say that ψ has a rapid decay of mutual correlations. If there exist $C, a > 0$ such that ψ has f-decay for $f = Ce^{-ar}$, we say that ψ has an exponential decay of mutual correlations.

Remark C.1. *The conditions of the Definition C.2 implies what is usually meant by rapid or exponential decay of correlations, as the latter correspond to the special case* $\mathcal{O} = \mathcal{O}_X\mathcal{O}_Y$ *for* $\mathcal{O}_X \in \mathscr{A}_X$, $\mathcal{O}_Y \in \mathscr{A}_Y$. *In general, it is a stronger condition.*

Lemma C.1. *Let* $\mathcal{H}_A$, $\mathcal{H}_B$, $\mathcal{H}_C$ *be separable Hilbert spaces, and let* $\varepsilon, \alpha, \delta \in \mathbb{R}_{>0}$. *Let* $|\chi\rangle$ *and* $|\chi'\rangle$ *be unit vectors on* $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$, *such that 1)* $\||\chi'\rangle - |\chi\rangle\| \leq \sqrt{\varepsilon}$, *2)* $\|\rho'_{BC} - \rho_{BC}\|_1 \leq \alpha$, *3)* $\|\rho_{AC} - \rho_A \otimes \rho_C\|_1 \leq \delta^2$ *and* $\|\rho'_{AC} - \rho'_A \otimes \rho'_C\|_1 \leq \delta^2$, *where* ρ_X *and* ρ'_X *are density matrices for the restrictions of* $|\chi\rangle\langle\chi|$ *and* $|\chi'\rangle\langle\chi'|$ *to* $X \in \{A, B, C, AB, BC, AC\}$, *respectively. Then there is a unitary* $\mathcal{U}_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ *such that* $\|\mathcal{U}_{AB} - 1\| \leq \sqrt{\varepsilon} + \delta + \sqrt{2\delta + \alpha}$ *and* $\||\chi'\rangle - \mathcal{U}_{AB}|\chi\rangle\| \leq \delta + \sqrt{2\delta + \alpha}$.

Proof. In the proof, we implicitly use that $\||\phi'\rangle - |\phi\rangle\| \leq \sqrt{\epsilon}$ is equivalent to $\mathrm{Re}\langle\phi'|\phi\rangle \geq 1 - \frac{\epsilon}{2}$ for any unit vectors $|\phi'\rangle$, $|\phi\rangle$ in a Hilbert space.

By the first Fuchs van de Graaf inequality and Uhlmann's theorem for ρ_{AC} and $\rho_A \otimes \rho_C$, there is a purification $|\tilde{\chi}\rangle$ of $\rho_A \otimes \rho_C$ such that $|\langle\tilde{\chi}|\chi\rangle| \geq (1 - \delta^2/2)$ and $\||\tilde{\chi}\rangle - |\chi\rangle\| \leq \delta$. Let $\tilde{\rho}_X$ be the density matrix for the restriction of $|\tilde{\chi}\rangle\langle\tilde{\chi}|$ to X. By the second Fuchs van de Graaf inequality we have $\|\tilde{\rho}_{BC} - \rho_{BC}\|_1 \leq 2\delta$ and therefore $\|\rho'_{BC} - \tilde{\rho}_{BC}\|_1 \leq 2\delta + \alpha$. By the first Fuchs van de Graaf inequality and Uhlmann's theorem, there is a vector $|\tilde{\chi}'\rangle$ that purifies $\tilde{\rho}_{BC}$ such that $\langle\chi'|\tilde{\chi}'\rangle \in \mathbb{R}$ and $\langle\chi'|\tilde{\chi}'\rangle \geq 1 - (2\delta + \alpha)/2$. Hence, we have

$$\||\tilde{\chi}'\rangle - |\tilde{\chi}\rangle\| \leq \sqrt{2\delta + \alpha} + \sqrt{\varepsilon} + \delta. \tag{C.12}$$

The states $|\tilde{\chi}\rangle\langle\tilde{\chi}|$ and $|\tilde{\chi}'\rangle\langle\tilde{\chi}'|$ have no correlations between A and C and their restriction to BC coincide. Therefore, there is a unitary $\mathcal{U}_{AB} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ such that $\mathcal{U}_{AB}|\tilde{\chi}\rangle = |\tilde{\chi}'\rangle$. Moreover, eq. (C.12) implies that we can satisfy $\|\mathcal{U}_{AB} - 1\| \leq \sqrt{\varepsilon} + \delta + \sqrt{2\delta + \alpha}$. We also have $\|\mathcal{U}_{AB}|\chi\rangle - |\chi'\rangle\| \leq \||\chi\rangle - |\tilde{\chi}\rangle\| + \||\tilde{\chi}'\rangle - |\chi'\rangle\| \leq \delta + \sqrt{2\delta + \alpha}$. □

Proposition C.3. *Let* ψ *and* ψ' *be pure states of a lattice system* $\mathscr{A}$. *Suppose both states have* f*-decay of mutual correlations and are* f*-close to each other at 0. Then there is an almost local unitary* $\mathcal{U} \in \mathscr{A}_{a\ell}$, *such that* $\psi = \psi' \circ \mathrm{Ad}_{\mathcal{U}}$, *and* $\mathcal{U}$ *is* g*-localized at 0 with* g *depending on* f *only. Moreover, if* f *is rapidly decaying (exponentially decaying), then so is* g.

Proof. By Proposition C.1, the states ψ and ψ' are unitarily equivalent. Let us choose the Hilbert space $\mathcal{H}$ for the GNS representation of one of the states, and let us choose state vectors $|\psi\rangle, |\psi'\rangle \in \mathcal{H}$ representing ψ and ψ', respectively.

Let $\{r_n\}_{n\in\mathbb{N}_0}$ be an increasing sequence of real numbers, $A_n := B_{r_{n-1}}(0)$, $B_n := B^c_{r_{n-1}}(0) \cap B_{r_n}(0)$, $C_n := B^c_{r_n}(0)$, $\delta_n := \sqrt{f(r_n - r_{n-1})}$, $\alpha_n := f(r_n)$, $\sqrt{\varepsilon_n} := \delta_{n-1} + \sqrt{2\delta_{n-1} + \alpha_{n-1}}$ for $n > 1$ and $\sqrt{\varepsilon_{n=1}} := \||\psi'\rangle - |\psi\rangle\|$. We iteratively apply Lemma C.1 for every n to get a sequence of unitaries $\mathcal{U}_n \in \mathscr{A}_{B_{r_n}(0)}$ such that $\|\mathcal{U}_n - 1\| \leq \sqrt{\varepsilon_n} + \sqrt{\varepsilon_{n+1}}$ and $\||\psi'\rangle - \mathcal{U}^{(n)}|\psi\rangle\| \leq \sqrt{\varepsilon_{n+1}}$, where $\mathcal{U}^{(n)} := \mathcal{U}_n ... \mathcal{U}_1 \in \mathscr{A}_{r_n}$.

We can choose $\{r_n\}_{n\in\mathbb{N}_0}$ such that the sequence $\{\sqrt{\varepsilon_n}\}_{n\in\mathbb{N}_0}$ converges to 0. For example, one can take $r_n = n^2 R$ for some $R \in \mathbb{R}$. Since $\|\mathcal{U}_n - \mathcal{U}_{n-1}\| \leq \sqrt{\varepsilon_n} + \sqrt{\varepsilon_{n+1}}$, $\{\mathcal{U}_n\}_{n\in\mathbb{N}_0}$ converges in the norm to some $\mathcal{U} \in \mathscr{A}_\Lambda$. Moreover, if f is superpolynomially (or exponentially) decaying, then $\mathcal{U}$ is almost local (or exponentially local). □

Lemma C.2. *Let $|\chi\rangle$ be a unit vector on $\mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C$ and $\mathcal{A} \in \mathcal{B}(\mathcal{H}_A)$ such that 1) $\|\rho_{AC} - \rho_A \otimes \rho_C\| \leq \delta^2$, 2) $\|\mathcal{A}\| = 1$ and $\|\mathcal{A}|\chi\rangle\| \leq \varepsilon$. Then there is a constant $c \in \mathbb{C}$ and a self-adjoint $\mathcal{K} \in \mathcal{B}(\mathcal{H}_{BC})$ such that $\|c + \mathcal{K}\| \leq 2(\varepsilon + \delta)$ and $\|(c + \mathcal{K} - \mathcal{A})|\chi\rangle\| \leq \delta(1 + 2\varepsilon + 2\delta)$.*

Proof. By the first Fuchs van de Graaf inequality and Uhlmann's theorem for ρ_{AC} and $\rho_A \otimes \rho_C$, there is a purification $|\tilde{\chi}\rangle$ of $\rho_A \otimes \rho_C$ such that $|\langle\tilde{\chi}|\chi\rangle| \geq (1 - \delta^2/2)$ and $\||\tilde{\chi}\rangle - |\chi\rangle\| \leq \delta$. Then $\|\mathcal{A}|\tilde{\chi}\rangle\| \leq \varepsilon + \delta$. Since $|\tilde{\chi}\rangle\langle\tilde{\chi}|$ has no correlations between A and C, there is $c \in \mathbb{C}$ and a self-adjoint $\mathcal{K} \in \mathcal{B}(\mathcal{H}_A \otimes \mathcal{H}_B)$ with $|c|, \|\mathcal{K}\| \leq (\varepsilon + \delta)$ such that $\mathcal{A}|\tilde{\chi}\rangle = (c + \mathcal{K})|\tilde{\chi}\rangle$. We have $\|c + \mathcal{K}\| \leq 2(\varepsilon + \delta)$ and $\|(c + \mathcal{K} - \mathcal{A})|\chi\rangle\| \leq \|(c + \mathcal{K} - \mathcal{A})\|\||\tilde{\chi}\rangle - |\chi\rangle\| \leq \delta(1 + 2\varepsilon + 2\delta)$. □

Proposition C.4. *Let ψ and ψ' be pure states of a lattice system $\mathscr{A}$. Suppose both states have f-decay of correlations and are f-close to each other at 0. Then there is an almost local unitary $\mathcal{U} \in \mathscr{A}_{a\ell}$, such that $\psi = \psi' \circ \mathrm{Ad}_{\mathcal{U}}$, and $\mathcal{U}$ is g-localized at 0 with g depending on f only. Moreover, if f is rapidly decaying (exponentially decaying), then so is g.*

Proof. Since $|\psi'\rangle = \mathcal{U}|\psi\rangle$, and since we can represent $\mathcal{U}$ as a sum of local observables with norms rapidly decaying with the size of the support, to show the existence of $\mathcal{K}$, it is enough to consider the case $|\psi'\rangle = \mathcal{A}|\psi\rangle$ for a local observable $\mathcal{A}$.

Assume the support of $\mathcal{A}$ is inside D_r. Suppose we have found $c_r \in \mathbb{C}$ and a self-adjoint $\mathcal{K}_r$ with the support on D_r such that $\|(\mathcal{A} - c_r - \mathcal{K}_r)|\psi\rangle\| \leq \varepsilon\|(\mathcal{A} - c_r - \mathcal{K}_r)\|$ for some ε. Let $\delta = Ce^{-\alpha r}$. Lemma C.2 allows us to find $c_{3r} \in \mathbb{C}$ and a self-adjoint $\mathcal{K}_{3r}$ with the support on D_{3r} such that $\|c_{3r} + \mathcal{K}_{3r}\| \leq 2(\varepsilon + \delta)\|(\mathcal{A} - c_r - \mathcal{K}_r)\|$ and $\|(\mathcal{A} - c_r - \mathcal{K}_r - c_{3r} - \mathcal{K}_{3r})|\psi\rangle\| \leq \delta(1 + 2\varepsilon + 2\delta)\|(\mathcal{A} - c_r - \mathcal{K}_r)\|$.

We can iterate the procedure, in the same way as for unitaries above, to find $c^{(n)}$ and $\mathcal{K}^{(n)}$ on D_{R_n} such that $\|c^{(n)} + \mathcal{K}^{(n)}\|$ and $\|(\mathcal{A} - \sum_{l=0}^{n}(c^{(l)} + \mathcal{K}^{(l)}))|\psi\rangle\|$ decay exponentially with R_n. The sum of $\mathcal{K}^{(n)}$ gives a desired $\mathcal{K}$. □

Appendix D

TENSOR NETWORKS FROM CFT

D.1 Weierstrass elliptic function

Let Λ be the lattice $n_1\omega_1+n_2\omega_2, n_1, n_2 \in \mathbb{Z}$ generated by $\omega_1, \omega_2 \in \mathbb{C}$. The Weierstrass elliptic function is defined by

$$\wp_{\omega_1,\omega_2}(z) = \wp_\Lambda(z) := \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right), \tag{D.1}$$

$$\wp'_{\omega_1,\omega_2}(z) = \wp'_\Lambda(z) := -\sum_{\lambda\in\Lambda} \left(\frac{2}{(z-\lambda)^3}\right). \tag{D.2}$$

Any elliptic function is a rational function of $\wp_\Lambda(z)$ and $\wp'_\Lambda(z)$. Some useful constants are $e_1 = \wp_\Lambda(\omega_1/2)$, $e_2 = \wp_\Lambda(\omega_2/2)$, $e_3 = \wp_\Lambda((\omega_1+\omega_2)/2)$ and $g_2 = -4(e_1e_2 + e_2e_3 + e_1e_3) = 60G_4$, $g_3 = 4e_1e_2e_3 = 140G_6$, where

$$G_k = \sum_{\lambda\in\Lambda\setminus\{0\}} \lambda^{-k}. \tag{D.3}$$

We have

$$\wp'_\Lambda(z)^2 = 4\wp_\Lambda(z)^3 - g_2\wp_\Lambda(z) - g_3 = 4(\wp_\Lambda(z)-e_1)(\wp_\Lambda(z)-e_2)(\wp_\Lambda(z)-e_3). \tag{D.4}$$

The inverse of $\wp_\Lambda(z)$ is given by

$$u_\Lambda(w) := -\int_w^\infty \frac{dy}{\sqrt{4y^3 - g_2y - g_3}} = -\int_w^\infty \frac{dy}{\sqrt{4(y-e_1)(y-e_2)(y-e_3)}}. \tag{D.5}$$

D.2 The shape of the holes and the basic tensor

Let us set $n = 4$. Consider cylinders with the coordinate $w \sim w+1$ and cuts along the half-lines $[\frac{k}{n}, \frac{k}{n} - i\infty]$, $k = 0, 1, 2, 3$. We assign such cylinders to sites of the square lattice, such that for a given site, each cut corresponds to one of the four adjacent edges, and glue them along the cuts corresponding to the same edge to produce a Riemann surface. Let $u_\Lambda(z)$ be the inverse of the Weierstrass function $\wp_\Lambda$ for the lattice Λ and $e_1 = \wp_\Lambda((1+i)/2)$. The function $u_\Lambda(e_1e^{-4\pi iw})$ maps the Riemann surface to the plane such that the circles with constant $\operatorname{Im} w$ are mapped to contours around sites of Λ or $\Lambda^\vee$ depending on weather $\operatorname{Im} w > 0$ or $\operatorname{Im} w < 0$. The size of the region inside the contour is controlled by $|\operatorname{Im} w|$.

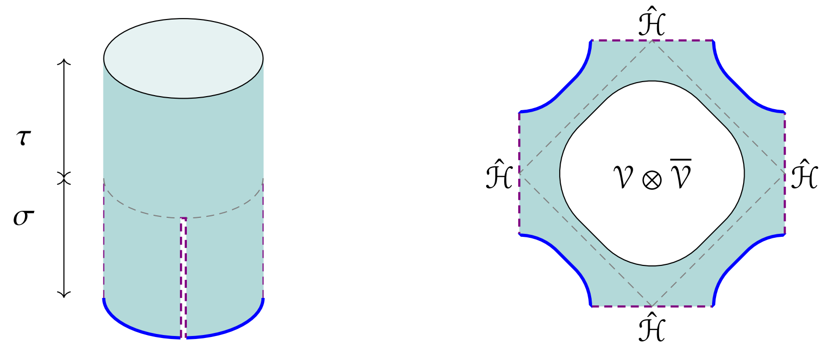


Figure D.1: The cylinder and its image under the map $u_\Lambda(e_1 e^{-4\pi i w})$ to the unit square. The region $(-\sigma) \leq \operatorname{Im} w \leq \tau$ gives the elementary block.

We set the function $f(z)$ from Section 6.1 to be

$$f(z) = u_\Lambda(e_1 z^{-2} e^{4\pi\tau}) \tag{D.6}$$

so that the image of the Riemann surface after removal of $\operatorname{Im} w > \tau$ and $\operatorname{Im} w < (-\sigma)$ defines $\Sigma_{\tau,\sigma}$ where $\tau, \sigma > 0$. The image of the region $(-\sigma) \leq \operatorname{Im} w \leq \tau$ on a single cylinder defines the elementary block (Fig. D.1). It is convenient to work with such $\Sigma_{\tau,\sigma}$ since the parameters τ and σ have a simple geometric interpretation on the cylinder. In the limit $\sigma \to \infty$ we get the surface Σ_τ.

If we only remove $\operatorname{Im} w < (-\sigma)$, the cylinder defines the map $\langle T_\sigma| : \hat{\mathcal{H}}^{\otimes 4} \to \mathbb{C}$ that is the amplitude of the elementary block (with filled inner hole) with fixed states on the necks. It vanishes on basis elements $|e^{(a_1 b_1)}_{m_1} \ldots e^{(a_n b_n)}_{m_n}\rangle$ that do not satisfy $b_k = a_{k+1}$. To express it through correlation functions of CFT on the unit disk[1], one should find a holomorphic map $w \to \xi(w)$ of the elementary block into the unit disk as shown in Fig. D.2 and functions $g_0(z), \ldots, g_{n-1}(z)$ such that g_k maps the upper half of the unit disk into the white region near $e^{\frac{2\pi i k}{n}}$. Then

$$\langle T_\sigma | e^{(a_1 a_2)}_{m_1} \ldots e^{(a_n a_1)}_{m_n}\rangle = e^{A_L(\sigma,\tau)} (S_{\mathbf{11}})^{3/2} \Big\langle \Big(\prod_{k=1}^{n} \mathcal{O}^{(a_k a_{k+1})}_{g_k} (e^{(a_k a_{k+1})}_{m_k}) \Big) \Big\rangle^{\mathrm{CFT}}_{D} \tag{D.7}$$

where e^{A_L} is the overall Liouville factor that does not depend on the boundary conditions or states and is not relevant in the following. We can take

$$\xi(w) = \left(\frac{1 + \sqrt{\cosh^2(n\pi\sigma)\tanh^2(n\pi i(w + i\sigma)) - \sinh^2(n\pi\sigma)}}{\cosh(n\pi\sigma)(1 - \tanh(n\pi i(w + i\sigma)))} \right)^{1/n}, \tag{D.8}$$

[1]See Section 6 of [13] for a similar computation.

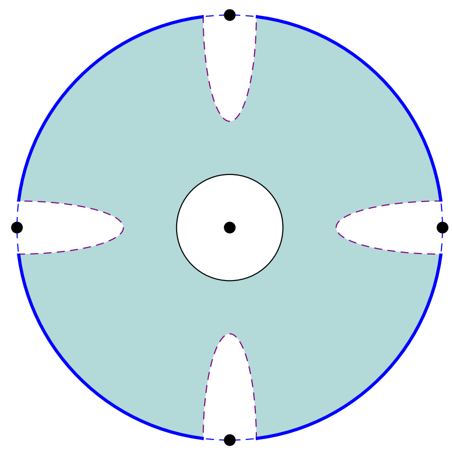


Figure D.2: The images of the maps g_0, g_1, g_2, g_3, h are shown in white with the black dots being the images of 0.

$$g_k(z) = e^{\frac{2\pi i k}{n}} \left(\frac{i - \tanh(n\pi\sigma/2)\, z}{i + \tanh(n\pi\sigma/2)\, z} \right)^{1/n}. \tag{D.9}$$

Note that when $\sigma \to 0$, the components $\langle T_\sigma | e^{(a_1 a_2)}_{m_1} \ldots e^{(a_n a_1)}_{m_n} \rangle$ are suppressed by $\tanh(n\pi\sigma/2)$ to the power of the total weight of basis elements, as can be seen from the rotation map eq. (6.2). That allows us to get an upper bound on the components. More precisely, for any β we can find σ_0, such that for any $0 < \sigma < \sigma_0$ we have

$$\frac{|\langle T_\sigma | e^{(a_1 a_2)}_{m_1} \ldots e^{(a_n a_1)}_{m_n} \rangle|}{|\langle T_\sigma | e^{(\mathbf{11})}_{0} \ldots e^{(\mathbf{11})}_{0} \rangle|} \leq \exp\left(-\beta \sum_{k=1}^{n} h^{(a_k a_{k+1})}(m_k) \right). \tag{D.10}$$

The map $\langle T_\tau |$ defines the statistical model for the computation of averages in the state Ψ_τ as we discuss in Appendix D.3.

When τ is finite, the cylinder defines the map $\langle T_{\tau,\sigma} | : \mathcal{V} \otimes \overline{\mathcal{V}} \otimes \hat{\mathcal{H}}^{\otimes 4} \to \mathbb{C}$. To express it through correlation functions of CFT on the unit disk, in addition to g_k, we need the function $h(z) = \xi(\frac{1}{2\pi i} \log(z e^{-2\pi\tau}))$ which maps the unit disk to the white region in the center in Fig. (D.2). We have

$$\langle T_{\tau,\sigma} | \alpha\bar{\alpha} e^{(a_1 a_2)}_{m_1} \ldots e^{(a_n a_1)}_{m_n} \rangle = e^{A_L(\sigma,\tau)} (S_{\mathbf{11}})^{3/2} \langle \left(\prod_{k=1}^{n} \mathcal{O}^{(a_k a_{k+1})}_{g_k} (e^{(a_k a_{k+1})}_{m_k}) \right) \mathcal{O}_h(\alpha \otimes \bar{\alpha}) \rangle^{\mathrm{CFT}}_D \tag{D.11}$$

where $|\alpha\bar{\alpha}\rangle \in \mathcal{V} \otimes \overline{\mathcal{V}}$. The tensor network state defined using the map $\langle T_{\tau,\sigma}|$ gives the state $\Phi_\tau(\sigma)$.

Remark D.1. *We have discussed only the maps* $\hat{\mathcal{H}}^{\otimes 4} \to \mathbb{C}$ *and* $\mathcal{V} \otimes \overline{\mathcal{V}} \otimes \hat{\mathcal{H}}^{\otimes 4} \to \mathbb{C}$ *corresponding to the blocks in the interior of the network. The maps corresponding* *to blocks on the boundary can be defined by contracting the outward legs with some elements of* $\hat{\mathcal{H}}$*, corresponding to attaching the boundaries, in a straightforward way.*

D.3 Cluster expansion for the auxiliary tensor network

In this section, we first recall standard facts about the cluster expansion for statistical models, also known as the polymer expansion. We follow Ch. 5 of [21]. We then apply it to tensor networks coming from our construction. We will use it to prove that the states Ψ_τ defined in the main text are well-defined in the thermodynamic limit, at least for a small enough value of τ, and that the correlators decay exponentially with the distance.

General setup

Let Γ be a set equipped with a function $w : \Gamma \to \mathbb{C}$ and a binary relation that contains the diagonal. We call elements of Γ polymers; $w(\gamma)$ is the weight of a polymer $\gamma \in \Gamma$. If (γ_1, γ_2) does not belong to the binary relation, we write $\gamma_1 \cap \gamma_2 = \emptyset$ and say that the polymers do not intersect. Otherwise, we write $\gamma_1 \cap \gamma_2 \neq \emptyset$. A subset of polymers $\Gamma' \subset \Gamma$ is called disconnected if all its elements do not intersect with each other. The partition function is defined by

$$Z = \sum_{\Gamma'} \prod_{\gamma \in \Gamma'} w(\gamma) \tag{D.12}$$

where the sum is over all disconnected subsets $\Gamma' \subset \Gamma$.

We call a collection X of elements of Γ a cluster if the graph of intersections[2] $G(X)$ of X is connected. The index of a cluster X is defined to be

$$I(X) := \sum_{H} (-1)^{|E(H)|} \tag{D.13}$$

where the sum is over all connected subgraphs H of $G(X)$ and $|E(H)|$ is the number of edges in H. If a collection X is not a cluster, we set $I(X) = 0$. Then, formally, we have

$$\log Z = \sum_{n \geq 1} \sum_{\gamma_1 \in \Gamma} \cdots \sum_{\gamma_n \in \Gamma} \frac{1}{n!} I(\{\gamma_1, \ldots, \gamma_n\}) \prod_{i=1}^{n} w(\gamma_i) =$$
$$= \sum_{X \in C} \frac{I(X)}{\prod_{\gamma \in \Gamma} n_X(\gamma)!} \prod_{\gamma \in X} w(\gamma) =: \sum_{X \in C} \mathcal{W}(X) \tag{D.14}$$

[2] The graph of intersections is the graph with vertices being the elements of X and edges being pairs $\{\gamma_1, \gamma_2\}$ such that $\gamma_1 \cap \gamma_2 \neq \emptyset$.

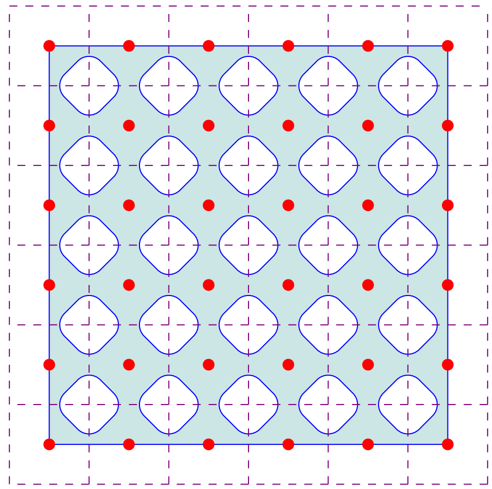


Figure D.3: Each red point is the vertex of V_G. The edges E_G are not shown, but they connect two neighboring red points. A square containing some vertex v corresponds to the piece of the surface that defines the vector $|T^{(v)}\rangle \in \otimes_{e\ni v}\hat{\mathcal{H}}$.

where $n_X(\gamma)$ is the number of times the element γ appears in X and C is the set of clusters. This expansion is convergent provided a certain sufficient condition is satisfied. We will use the following criterion (Theorem 5.4 from [21]). Suppose there is a function $|\cdot| : \Gamma \to \mathbb{R}_{>0}$ such that for any $\gamma \in \Gamma$ we have

$$\frac{1}{|\gamma|}\sum_{\gamma':\gamma\cap\gamma'\neq\emptyset}|w(\gamma')|e^{|\gamma'|} \leq 1 \tag{D.15}$$

and

$$\sum_{\gamma'}|w(\gamma')|e^{|\gamma'|} < \infty. \tag{D.16}$$

Then for any $\gamma_1 \in \Gamma$ we have

$$1 + \sum_{n\geq 2}\sum_{\gamma_2}\dots\sum_{\gamma_n}\frac{|I(\{\gamma_1,\dots,\gamma_n\})|}{(n-1)!}\prod_{i=2}^{n}|w(\gamma_i)| \leq e^{|\gamma_1|}. \tag{D.17}$$

In particular, it guarantees the convergence of the cluster expansion eq. (D.14).

CFT tensor network

To compute the averages of observables in the state Ψ_{τ,Γ_*}, we introduce an auxiliary statistical model.

Let V, E, F be the set of vertices, edges, and faces of the infinite square grid defined by the dual lattice $\Lambda^\vee$. We choose the same orientation on each edge and face. Let G be the subgraph of size $(2N)\times(2N)$ as show in Fig. (D.3). The corresponding sets of vertices, edges, and faces are denoted by V_G, E_G, F_G, respectively. Let us cut the surface Σ_τ into simple blocks, as shown in Fig. D.3. Each block containing a vertex $v \in V_G$ defines a map $\langle T^{(v)}| : \bigotimes_{e \ni v} \hat{\mathcal{H}} \to \mathbb{C}$ where $e \in E_G$. The components of the map can be computed as explained in Appendix D.2. The space of states of the model is $\mathcal{H}_G = \bigotimes_{e \in E_G} \hat{\mathcal{H}}_e$, where $\hat{\mathcal{H}}_e \cong \hat{\mathcal{H}}$ is the space associated with each edge of E_G. Using a natural pairing on each $\hat{\mathcal{H}}_e$, the contracted maps $\langle T^{(v)}|$ compute the partition function of the CFT on Σ_τ with the cloaking combination of elementary boundary conditions. We label the basis elements of $\mathcal{H}_G$ by a function $P : F \to I$ that is constant on $f \notin F_G$, and a function $J : E \to \mathbb{N}_0$ with $J(e) = 0$ for $e \notin E_G$. A pair (P, J) is called a configuration. The function P defines the choice of elementary boundary conditions on each boundary, while the function J defines the corresponding vectors in $\hat{\mathcal{H}}$. Hence, each configuration (P, J) defines a basis element $|v, P, J\rangle \in \otimes_{e \ni v} \hat{\mathcal{H}}_e$ for each vertex $v \in V_G$. We define the weight $W(P, J)$ of a configuration (P, J) by the contraction of $\langle T^{(v)}|v, P, J\rangle\langle v, P, J|$. The partition function of the model is given by

$$Z = \sum_{P,J} W(P, J). \tag{D.18}$$

Let A be a connected set of faces on the lattice Λ, and let $S = \{s_1, ..., s_k\}$ be the set of edges of E_G intersecting the boundary of A (see Fig. D.4). The contraction of maps $\langle T^{(v)}|$ for v that belong to faces of A defines a map $\langle T_A| : \bigotimes_{s \in S} \hat{\mathcal{H}}_s \to \mathbb{C}$ or the corresponding vector $|T_A\rangle \in \bigotimes_{s \in S} \hat{\mathcal{H}}_s$. Similarly, we can define $\langle T_{\bar{A}}|$ for the complementary set $\bar{A} := F_G \backslash A$. The partition function is given by $\langle T_{\bar{A}}|T_A\rangle$

Any local observable $\mathcal{A} \in \mathscr{A}$ with the support inside A gives another canonical map $\langle O(\mathcal{A})| : \bigotimes_{s \in S} \hat{\mathcal{H}}_s \to \mathbb{C}$. Note that $\langle O(1)| = \langle T_A|$. The average of the observable $\mathcal{A}$ can be expressed by

$$\langle \mathcal{A} \rangle_{\Psi_{\tau,\Gamma_*}} = \frac{\langle T_{\bar{A}}|O(\mathcal{A})\rangle}{\langle T_{\bar{A}}|T_A\rangle}. \tag{D.19}$$

Note that $\||O(\mathcal{A})\rangle\| \leq \|\mathcal{A}\|\||T_A\rangle\|$.

In the following, we analyze the model in the regime of small τ. When τ is small, the system energetically prefers the configurations with constant P and $J(e) = 0$, and we can perform the cluster expansion around this point. We show that there is τ_0 such that for any $0 < \tau < \tau_0$ the cluster expansion converges, and the correlations decay exponentially. We use the fact that for any β there is τ_β such that for $\tau < \tau_\beta$

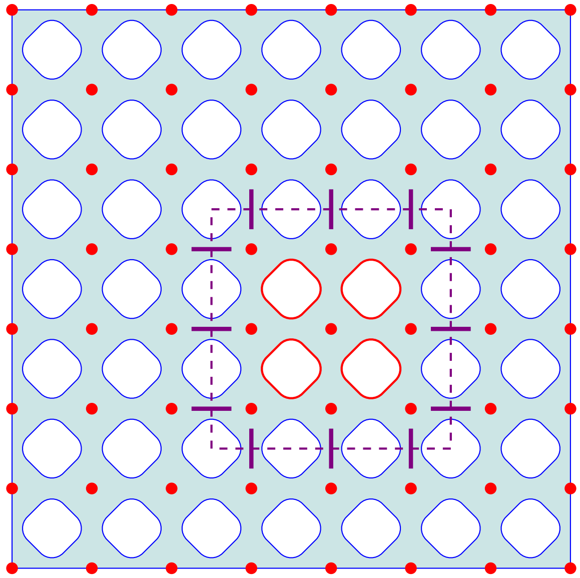


Figure D.4: A consists of four faces forming a square. The edges of S are parallel to violet segments.

the following bound holds

$$\frac{|\langle T^{(v)}|e_{m_1}^{(a_1a_2)}\ldots e_{m_n}^{(a_na_1)}\rangle|}{|\langle T^{(v)}|e_0^{(aa)}\ldots e_0^{(aa)}\rangle|} \leq \exp\left(-\beta\sum_{k=1}^{n} h^{(a_ka_{k+1})}(m_k)\right) \tag{D.20}$$

as argued in Appendix D.2.

We first analyze the model for a holomorphic V, so that $I = \{\mathbf{1}\}$. Then we discuss the modifications for general V.

Single vacuum

Suppose V is holomorphic. In this case, there is only one sector $a = \mathbf{1}$, and the configurations are labeled only by functions J. We omit P in this subsection. We also renormalize all $\langle T^{(v)}|$ so that the lowest weight components are equal to 1. For a configuration J, we say that an edge e is activated if $J(e) > 0$. We can apply the cluster expansion with polymers being connected sets of edges and with the weight of a polymer being the sum of $W(J)$ over configurations J such that the set of activated edges coincides with the polymer.

Let $|\gamma|$ be the number of edges in γ. Then for any $e \in E_G$ we have

$$\sum_{\gamma \ni e} |w(\gamma)| e^{|\gamma|} \leq \sum_{l\geq 1} n_l e^l \left(\sum_{m=1}^{\infty} e^{-2\beta h(m)}\right)^l \tag{D.21}$$

where n_l is the number of connected subsets of E_G intersecting e of cardinality l. The sum in the brackets is related to the character of V, and for regular V with modular Rep(V) can be made arbitrarily small by varying β. Since[3] $n_l \leq 4^{2l}$, for a large enough β the cluster expansion convergence criteria eq. (D.15) and eq. (D.16) are satisfied for any N. Moreover, by eq.(D.17), for large enough β we have

$$\sum_{X:\mathrm{supp}(X)\ni e} |\mathcal{W}(X)| e^{c|X|} \leq 1 \tag{D.22}$$

and therefore

$$\sum_{\substack{X:|X|\geq R \\ \mathrm{supp}(X)\ni e}} |\mathcal{W}(X)| \leq e^{-cR} \tag{D.23}$$

for any $e \in E_G$ and $R > 0$, where $\mathcal{W}(X)$ is the weight of the cluster defined in eq. (D.14), $|X|$ is the number of edges appearing in the polymers of X and c is some constant.

The averages eq. (D.19) correspond to the contraction with $(\langle T_{\bar{A}}|T_A\rangle)^{-1}|T_{\bar{A}}\rangle$. Let us write $|T_{\bar{A}}^{(N)}\rangle$ in this paragraph to indicate the dependence on N. We can use the cluster expansion to compute the overlap of normalized vectors for consecutive N

$$\frac{\langle T_{\bar{A}}^{(N+1)}|T_{\bar{A}}^{(N)}\rangle\langle T_{\bar{A}}^{(N)}|T_{\bar{A}}^{(N+1)}\rangle}{\langle T_{\bar{A}}^{(N+1)}|T_{\bar{A}}^{(N+1)}\rangle\langle T_{\bar{A}}^{(N)}|T_{\bar{A}}^{(N)}\rangle} \tag{D.24}$$

with the polymers and clusters living in the tensor networks obtained by gluing two complements of A of the original networks along the edges of S. Note that the clusters which do not intersect simultaneously the boundary and S cancel. The remaining clusters have at least as many edges as the distance between A and the boundary, and eq.(D.23) implies that their contribution decays exponentially with N. Similarly, the norms of $(\langle T_{\bar{A}}^{(N)}|T_A\rangle)^{-1}\langle T_{\bar{A}}^{(N)}|$ are upper bounded, and since they have unit overlap with $|T_A\rangle$, the thermodynamic limit $N \to \infty$ exists. Therefore the state Ψ_τ is well-defined.

Suppose now we have two observables $\mathcal{A} \in \mathcal{B}(\mathcal{V}_A)$ and $\mathcal{B} \in \mathcal{B}(\mathcal{V}_B)$ localized in disjoint collections of plaquettes A and B with the corresponding sets of edges S_A and S_B intersecting the boundaries of the regions. To find a bound on $|\langle\mathcal{A}\mathcal{B}\rangle_{\Psi_\tau} - \langle\mathcal{A}\rangle_{\Psi_\tau}\langle\mathcal{B}\rangle_{\Psi_\tau}|$, we can compute the normalized overlap of vectors $|T_{\overline{AB}}\rangle$ and $|T_{\bar{A}}T_{\bar{B}}\rangle :=$

[3] That can be argued as follows. For each connected subset of edges and a vertex v that belongs to it, there is a path of length $2l$ that starts at v and crosses each edge exactly twice. The number of all paths of length $2l$ starting at some fixed vertex of e and crossing e is bounded by 4^{2l}.

$|T_{\bar{A}}\rangle \otimes |T_{\bar{B}}\rangle$ corresponding to the contraction of $|T^{(v)}\rangle$ in the complement of the corresponding regions. The cluster expansion of

$$\frac{\langle T_{\overline{AB}}|T_{\bar{A}}T_{\bar{B}}\rangle\langle T_{\bar{A}}T_{\bar{B}}|T_{\overline{AB}}\rangle}{\langle T_{\bar{A}}T_{\bar{B}}|T_{\bar{A}}T_{\bar{B}}\rangle\langle T_{\overline{AB}}|T_{\overline{AB}}\rangle} \tag{D.25}$$

on the corresponding glued tensor networks has the contribution only from clusters intersecting both A and B. They have at least as many edges as the shortest distance R between A and B. Therefore using eq.(D.23) one can get

$$\|\rho_{AB} - \rho_A \otimes \rho_B\|_1 \leq C\min(|S_A|, |S_B|)\|\mathcal{A}\|\|\mathcal{B}\|e^{-\alpha R} \tag{D.26}$$

for some constants C, α and for density matrices ρ_A, ρ_B, ρ_{AB} corresponding to the restrictions of the state Ψ_τ to the supports A and B of the observables $\mathcal{A}$ and $\mathcal{B}$, respectively. In particular, we have

$$|\langle\mathcal{A}\mathcal{B}\rangle_{\Psi_\tau} - \langle\mathcal{A}\rangle_{\Psi_\tau}\langle\mathcal{B}\rangle_{\Psi_\tau}| \leq C\min(|S_A|, |S_B|)\|\mathcal{A}\|\|\mathcal{B}\|e^{-\alpha R}. \tag{D.27}$$

Thus correlations between observable in the state Ψ_τ have exponential decay. Similarly, if one modifies the state Ψ_τ by a local insertion of vertex operators (as in Section 6.1), the change of the expectation values of observables decays exponentially with the distance between the insertions and the support of the observable.

Multiple vacua

For a non-holomorphic V, the polymers would be again the subsets of connected edges of E_G. For a configuration (P, J), an edge e is activated (i.e., the state running through the corresponding neck has a non-zero weight) if $J(e) > 0$ or if P has different values on the adjacent faces. The weight of a polymer is given by the sum of $W(P, J)$ over all configurations (P, J) which set of activated edges coincides with the polymer. Note that P of the configurations contributing to the weight has the same value in each connected component of the partition of the plane defined by the polymer. For a given P, we can replace the elementary boundary condition around each hole (defined by P) with the trivial elementary boundary condition and the appropriate topological defect around the hole. For the computation of the weight of a given polymer γ, we can reconnect all the topological defects in deactivated necks as the states running through those necks must have zero weight. Each reconnection of defects with label a gives a factor $(S_{\mathbf{1}a}/S_{\mathbf{11}})^{-1}$. The resulting configuration has contractible topological defects near the vertices inside each connected component (which give a factor $(S_{\mathbf{1}a}/S_{\mathbf{11}})$) and topological defects running along the edges of the polymer. That allows to show that the contribution of a collection of non-intersecting polymers to the partition function is given by the product of weights. We have a bound

$$|w(\gamma)| \leq \left(\sum_{a,b} \sum_{m:h^{(ab)}(m)>0} e^{-\beta h^{(ab)}(m)} \right)^{|\gamma|} \tag{D.28}$$

that can be made arbitrarily small by varying β, and therefore, in the same way as for holomorphic V, for a large enough β the cluster expansion convergence criteria eq. (D.15) and eq. (D.16) are satisfied for any N.

We can organize the cluster expansion in a similar way to show that, at least for large enough β (or small enough τ), the averages of observables $\langle \mathcal{A} \rangle_{\Psi_\tau}$ are well-defined in the thermodynamic limit, and the correlators of two local observables decay exponentially with the distance between their supports. Modifications of the state Ψ_τ by a local insertion of vertex operators can also be treated similarly, while for insertions of non-trivial modules (see Section 6.1) the arguments are no longer applicable since it may not be possible to contract the topological defects after the reconnections. That is consistent with the fact that the averages of observables on an annulus might be affected by an insertion of an anyon into the center even if the annulus is very large.

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