Ultraviolet Scalar Unification and Gravitational Radiation Reaction from Quantum Field Theory

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ABSTRACT

Since its inception, the development of quantum field theory has been driven by a desire to describe nature at the highest energy scales, where both relativistic and quantum mechanical aspects of matter and radiation are manifest. The theory has been wildly successful in this respect, giving rise to the standard model of particle physics, as well as quantum cosmologies of the early universe. The applications of quantum field theory are not, however, restricted to high-energy physics. The theory is just as spectacular in the infrared as it is in the ultraviolet, and it serves as a mathematical nexus for physical processes spanning the energy spectrum. We will investigate two such connections in this work. In the first part, we will relate the analytic structure of the scattering amplitudes of scalar quantum field theories in the far infrared to unifying symmetries of their actions in the ultraviolet. In the sequel, we will study a system of two massive scalar fields coupled to the spacetime metric. Identifying the classical limit with the gravitational infrared, we will sift the classical dynamics of binary gravitational inspiral from the scattering amplitudes of canonical quantum gravity.

In Chapter 1, we argue that symmetry and unification can emerge as byproducts of certain physical constraints on dynamical scattering. To accomplish this we parameterize a general Lorentz invariant, four-dimensional theory of massless and massive scalar fields coupled via arbitrary local interactions. Assuming perturbative unitarity and an Adler zero condition, we prove that any finite spectrum of massless and massive modes will necessarily unify at high energies into multiplets of a linearized symmetry. Certain generators of the symmetry algebra can be derived explicitly in terms of the spectrum and three-particle interactions. Furthermore, our assumptions imply that the coset space is symmetric.

In Chapter 2, we introduce the gravitational inspiral problem. In Chapter 3, we review the Hamiltonian formulation of binary dynamics and outline the extraction of the effective Hamiltonian from scattering amplitudes. We then augment the equations of motion with a gravitational radiation reaction force, thus incorporating dissipation. In Chapter 4, we extend the setup of Kosower, Maybee, and O'Connell (KMOC), which expresses classical observables in terms of scattering amplitudes, to study the evolution of angular momentum during two-body scattering in gravity and electromagnetism. From the associated scattering amplitudes, we explicitly compute the total radiated angular momentum through the third Post-Minkowskian

order (3PM or $O(G^3)$) in general relativity and the third Post-Coulombic order (3PC or $O(\alpha^3)$) in electromagnetism.

In Chapter 5, starting with the classical expressions for radiated energy and angular momentum, we compute the corresponding instantaneous radiative fluxes in isotropic gauge. We then use these fluxes to derive the relative radiation reaction force associated to a gravitational binary system at third post-Minkowskian order, or $O(G^3)$, by imposing flux balance. Together with the conservative Hamiltonian, this force provides a complete equation of motion for the relative degree of freedom of a (bound or unbound) gravitational binary at $O(G^3)$. Finally, in Chapter 6 we compare our results with the post-Newtonian literature, finding agreement to G^3, 3PN (relative) order.

PUBLISHED CONTENT AND CONTRIBUTIONS

Part I of this work has been adapted from [1], in which C. Cheung and Z.M. derive unifying ultraviolet symmetries of scalar theories from their infrared structure, under the assumption of perturbative unitarity. C. Cheung proposed the connection between symmetries and soft theorems, proved that the assumptions of perturbative unitarity, locality, and a finite spectrum imply renormalizability, and advised Z.M. throughout the project. Z.M. proved that these assumptions, together with the Adler zero condition, imply ultraviolet symmetries of the action. Z.M. also constructed the generators of these symmetries from scattering amplitudes, and proved that the associated coset space is symmetric.

Part II derives from the (as yet unpublished) manuscript [2] and Z.M.'s research notes. Chapter 4, its attendant appendices, and parts of Chapter 2 have been adapted from [2]. In that work, E. Herrmann, Z.M., J. Parra-Martinez, and M. Ruf extend the KMOC approach to the calculation of total angular momentum radiated during classical two-body scattering and use the result to derive the complete 3PM relative equation of motion for a gravitational binary. Z.M. derived a distributional representation of the radiative angular momentum integrand and showed that problematic distributional terms could be converted to well-behaved differential operators. All authors worked to elucidate the role played by disconnected amplitudes in angular momentum radiation. E. Herrmann, J. Parra-Martinez, and M. Ruf simplified the integrand considerably by reformulating it in terms of the radiation kernel, constructed the 3PM integrands, and performed the loop integrals. Meanwhile, Z.M. computed the instantaneous fluxes from radiated energy and angular momentum, extracted the 3PM relative radiation reaction force from them, and validated the results by gauge-matching PN fluxes from literature.

[1] C. Cheung and Z. Moss, "Symmetry and Unification from Soft Theorems and Unitarity," *JHEP* **05** (2021) 161, `arXiv:2012.13076 [hep-th]`.

[2] E. Herrmann, Z. Moss, J. Parra-Martinez, and M. Ruf, "Scattering Amplitudes and Dissipative Binary Dynamics." (in prep.).

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Chapter 1

SYMMETRY AND UNIFICATION FROM SOFT THEOREMS AND UNITARITY

1.1 Introduction

It is often said that symmetry is beautiful. However, this view presumes that symmetry is a luxury with which a theory may or may not be blessed. This intuition fails when symmetry is *required* by consistency and a less symmetric theory is simply impossible. A classic example of this is gauge symmetry, which is not fundamental but rather mandated by more primitive principles such as Poincaré invariance and locality. Indeed, it is well-known that by bootstrapping the scattering amplitudes of self-interacting massless vector particles directly from these underlying assumptions, one can derive all the properties of the gluon without the aid of gauge invariance (see [1–3] and references therein). In this approach notions such as charge conservation and the Lie algebra structure emerge from these more basic concepts.

On the other hand, none of these arguments apply to global symmetries since they are not redundancies of description. So it is possible in principle for a theory to be imbued with more versus less global symmetry in a well-defined sense. It then seems genuinely remarkable when a global symmetry, spontaneously broken at long distances, is miraculously and intricately restored at short distances. From this perspective, theories are consecrated with unifying symmetries in the deep ultraviolet which are muddied at low energies due to the noninvariance of the vacuum. Aspects of the emergent pion degrees of freedom, e.g. their spectrum [4–6], nonlinearly realized symmetries [7, 8], and soft behavior [9, 10] are then taken to be derived properties of this underlying high-energy symmetry.

In this paper we pursue the exact opposite logic. Ultraviolet unification and symmetry restoration are not assumed. Instead, we will derive them as *consistency conditions* implied by an alternative set of physical principles naturally defined in the broken phase. For our setup we study a general Lorentz invariant, four-dimensional theory of scalar fields with arbitrary masses and interactions. Here the spectrum of the theory is defined to include *all* ultraviolet degrees of freedom and the interactions may be higher dimension or derivatively coupled. We then impose four physical criteria:

- **Perturbative Unitarity.** The dynamics are perturbative and unitary.
- **Locality.** The interactions are polynomials in derivatives.
- **Finite Spectrum.** The number of degrees of freedom is finite.
- **Soft Theorems.** All amplitudes vanish in the soft limit of a massless mode.

The first condition holds in any weakly coupled theory to which perturbation theory applies. The second condition is a technical assumption imposed so that the tree amplitudes are rational functions of momenta. It is violated if the theory has nonlocal interactions. The third statement is required so that high-energy limits of amplitudes are well-defined. The last and strongest assumption is a variation of the Adler zero condition [9, 10] in which we demand that all tree-level amplitudes vanish when a massless degree of freedom is taken soft.

We will prove that these four conditions imply that the massless and massive degrees of freedom necessarily unify into a multiplet which, in the high-energy limit, transforms *linearly* under a global symmetry. The generators of the ultraviolet symmetry can actually be expressed explicitly in terms of the mass spectrum and three-particle interactions. As one would expect, the ultraviolet theory need not be unique. When our assumptions do not fix all parameters in the theory, those that remain simply label the allowed space of possible ultraviolet completions consistent with a given spectrum. Interestingly, we find that our four criteria imply that the coset space of the symmetry breaking is symmetric. Note that our approach differs from prior efforts on bootstrapping the massless sector alone, either from the Adler zero [11–16] without imposing unitarity, or from alternative ultraviolet considerations [17].

The very simplest theory satisfying the above four criteria—a theory of one massless and one massive particle—is easy to understand. The only scalar potential consistent with our assumptions is the “wine-bottle” potential familiar from spontaneous symmetry breaking. In particular, the form of the low-energy basin of the potential is dictated by the Adler zero while the shape of the rest of the potential is uniquely fixed by perturbative unitarity.

The remainder of this paper is structured as follows. In Sec. 1.2 we define a general theory of massless and massive scalars coupled through arbitrary local interactions. We review how to extract the high-energy behavior of an amplitude using a minimal basis of kinematic invariants before generalizing this notion to off-shell operators. We then present a proof that the assumptions of perturbative unitarity, locality, and a finite spectrum imply the existence of a field basis in which the Lagrangian takes a renormalizable form. Afterwards, in Sec. 1.3 we impose an Adler zero condition on four- and five- particle scattering amplitudes, deriving a set of highly nontrivial constraints on the interactions of the theory. These in turn imply the existence of unbroken and broken symmetry generators which act as linear and affine transformations that mix the massless and massive states. The broken generators are constructed explicitly from the cubic couplings and masses of the constrained Lagrangian, while their commutators produce a subalgebra of the unbroken generators, thus establishing that the coset space of broken symmetries is necessarily symmetric. It is then straightforward to show that in the high-energy limit these symmetries form a subgroup of special orthogonal rotations under which all fields transform linearly. In Sec. 1.4, we demonstrate the connection between Adler zeros and symmetries explicitly through an example theory involving a single massive state and an arbitrary number of massless modes. We then present our conclusions and future directions in Sec. 1.5.

1.2 Perturbative Unitarity

Consider a Lorentz invariant, four-dimensional theory of interacting scalar fields. Here we define the spectrum so as to include all degrees of freedom in the theory, heavy or light. Furthermore, we assume that these states are finite in number so it is possible to take a high-energy limit that exceeds all the physical mass thresholds in the theory. Since unitarity forbids ghost modes even in the linearized theory, we are required to assume a quadratic dispersion relation, i.e. $p^2 = m^2$.

At the nonlinear level we allow for an *a priori* arbitrary set of local interactions which are unbounded in the number of external fields. However, we also assume that at any given number of external fields the interactions are at most polynomial in the external momenta[1]. The purpose of the latter condition is to forbid nonlocal interactions, which necessarily entail an infinite train of higher derivative corrections[2]. While nonlocalities generically arise when integrating out fields, this is not permitted here since our spectrum is defined to include all heavy and light degrees of freedom. No

[1]For simplicity we do not consider here terms involving the Levi-Civita tensor, although this would be an interesting avenue for future analysis.

[2]For this reason our conclusions will not be applicable to any theory of extended objects such as string theory.

particles have been implicitly integrated out.

Amplitudes at High Energy

An N-particle scattering amplitude $\mathcal{A}_N$ is a function of the momenta $\{p_a\}$ and flavor indices $\{I_a\}$ of the external legs, where $1 \leq a \leq N$. Naively, the high-energy behavior of $\mathcal{A}_N$ is obtained by rescaling $p_a \to z p_a$ for large z. However, this operation is not self-consistent because it does not preserve the on-shell condition for massive particles, $p_a^2 = m_{I_a}^2$, and fails to account for the fact that certain combinations of momentum, e.g. $\sum_{a=1}^N p_a = 0$, do not actually scale as z.

Nevertheless it is trivial to extract the high-energy behavior on-shell, provided we first reduce to a minimal basis of kinematic invariants (see Appendix B of [3]). *A priori*, $\mathcal{A}_N$ depends on $N(N+1)/2$ invariants of the form $p_a \cdot p_b$ for $1 \leq a, b \leq N$. We can enforce total momentum conservation by substituting all invariants involving one of the momenta in terms of the others, leaving just $N(N-1)/2$. The on-shell conditions then impose N linear relations among the remaining invariants, giving a final tally of $N(N-3)/2$ independent objects. For our explicit calculations we choose for this "minimal kinematic basis" the set[3]

$$\{\, p_a \cdot p_b \,\} \quad \text{where} \quad 1 \leq a \leq N-3 \text{ and } a < b \leq N-1. \tag{1.1}$$

Note that we have eliminated $p_{N-2} \cdot p_{N-1}$ using the on-shell condition $p_N^2 = m_{I_N}^2$.

To take the high-energy limit, we transform the minimal kinematic basis by

$$p_a \cdot p_b \to z^2\, p_a \cdot p_b, \tag{1.2}$$

for large z, which probes high energies without leaving the on-shell surface. As discussed in App. 1.A, the assumptions of perturbative unitarity, locality, finite mass spectrum, and Adler zeros for massless particles in four spacetime dimensions imply that the N-particle tree-level scattering amplitudes must satisfy the scaling bound[18]

$$\mathcal{A}_N(z \to \infty) \lesssim z^{4-N}. \tag{1.3}$$

More precisely, the modulus of any N-particle scattering amplitude is strictly bounded at large z by $|\mathcal{A}_N| \leq q|z|^r$ for some $q > 0$ and $r \leq 4 - N$. As we will see shortly, this condition strongly constrains the allowed form of the Lagrangian.

[3]Complications involving four-dimensional Gram determinant identities can be ignored since we are interested in at most five-particle scattering, which still depends on four linearly independent momentum vectors even after accounting for momentum conservation.

Note that this inequality is exactly saturated in any Lorentz invariant four-dimensional quantum field theory with dimensionless couplings, e.g. Yang-Mills theory or scalar ϕ^4 theory. Furthermore, we emphasize that Eq. (1.3) is a necessary but not sufficient condition for perturbative unitarity. For instance, tree-level amplitudes in quantum electrodynamics satisfy Eq. (1.3) but will violate perturbative unitarity at energies near the Landau pole.

Lagrangians at High Energy

High-energy behavior is even harder to discern at the level of the Lagrangian. As a simple example, consider the cubic operator $\phi\,\partial^\mu\phi\,\partial_\mu\phi$, which naively implies $\mathcal{A}_3 \sim \mathcal{O}(z^2)$ scaling for three-particle scattering. Instead, $\mathcal{A}_3 \sim \mathcal{O}(z^0)$ because all the invariants $p_a \cdot p_b$ can expressed in terms of external masses, i.e. the minimal kinematic basis is empty. As another example, the quartic operator $\phi^2\partial^\mu\phi\,\partial_\mu\phi$ naively implies $\mathcal{A}_4 \sim \mathcal{O}(z^2)$ but instead $\mathcal{A}_4 \sim s+t+u \sim \mathcal{O}(z^0)$ due to momentum conservation, on-shell conditions, and Bose symmetry.

To circumvent this annoyance we make high-energy scaling manifest at the level of the Lagrangian by defining a "minimal operator basis" in analogy with the the minimal kinematic basis for amplitudes. Consider a general off-shell N-particle operator, $F_N(\partial_1, \partial_2, \cdots, \partial_N)\phi_{I_1}\phi_{I_2}\cdots\phi_{I_N}$, where ∂_a denotes a derivative acting on the field ϕ_{I_a} with $1 \le a \le N$. Since we have assumed that all interactions are polynomially bounded in derivatives, F_N is a polynomial in its arguments.

If a subset of the fields are the same flavor, and thus indistinguishable, we explicitly symmetrize F_N on their corresponding labels. In parallel with the amplitudes approach, F_N depends *a priori* on $N(N+1)/2$ invariants $\partial_a\partial_b$ for $1 \le a, b \le N$. To derive an analog of the minimal kinematic basis for amplitudes in Eq. (1.1), we use integration by parts to shuffle all derivatives acting on ϕ_{I_N} onto the other fields, thus eliminating ∂_N by the momentum conservation constraint. By performing a field redefinition we can effectively set $\partial_a^2 = -m_{I_a}^2$ for $1 \le a \le N-1$ at the level of the action, modulo contact terms which have more than N fields and can thus be absorbed into the definition of higher order operators[4]. Here we also eliminate $\partial_{N-2}\partial_{N-1}$, which via integration by parts can be related to $\partial_N^2 = -m_{I_N}^2$.

It is then mechanical to construct the minimal operator basis, first starting with

[4]Concretely, the field transformation $\phi \to \phi + \delta\phi$ will induce the variation of the action, $S \to S + \delta S$ where $\delta S = -\int \delta\phi[(\Box + m^2)\phi + \cdots]$. Here $\delta\phi$ starts at quadratic order in fields and the ellipses denote higher order terms. By an appropriate choice of $\delta\phi$, any term in the action proportional to $\Box\phi$ can be substituted for $-m^2\phi$.

operators with the fewest number of fields, and then working our way up. In the case of F_3, all derivatives of fields can be converted, via integration by parts, to D'Alembertian operators acting on products of the other fields. Each D'Alembertian can then be eliminated in favor of an m^2 factor using a field redefinition. For example, an operator like $\partial^\mu \phi_1 \partial_\mu \phi_2 \, \phi_3$ is equivalent upon integration by parts to $(\phi_1\phi_2\Box\phi_3 - \Box\phi_1\phi_2\phi_3 - \phi_1\Box\phi_2\phi_3)/2$, which on equations of motion is equivalent to the bare potential term $\phi_1\phi_2\phi_3$ modulo terms quartic or higher in the field. So in the minimal operator basis, $\phi_1\phi_2\phi_3$ is the only allowed object at cubic order in the fields. As advertised, it exactly manifests the correct $O(z^0)$ high-energy scaling required of any three-particle amplitude. We proceed to apply the same procedure to F_4, F_5, etc., yielding a Lagrangian for which all operators correctly manifest their high-energy behavior.

Deriving Renormalizability

We have shown that in the minimal operator basis, the only possible cubic operator is a potential term. Hence, *any* scalar Lagrangian can be put into the form[5]

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \phi_I \partial_\mu \phi_I - \frac{1}{2} m_I^2 \phi_I \phi_I - \frac{1}{3!}\alpha_{IJK}\phi_I\phi_J\phi_K + \cdots , \tag{1.4}$$

where the ellipses denote all possible interactions at quartic order in fields or higher and involving an arbitrary but finite number of derivatives.

To study the theory at quartic order in the fields we analyze the four-particle scattering amplitude computed from the Feynman diagrams corresponding to the Lagrangian in Eq. (1.4). In the high-energy limit we obtain $\mathcal{A}_4(z) = \mathcal{A}_4^{\text{cont}}(z) + \mathcal{A}_4^{\text{fact}}(z)$, where we have distinguished between "contact" contributions from the quartic vertex and the s-, t-, and u-channel "factorization" diagrams coming from the cubic vertices. Since the cubic operators are derivative-free, the factorization term is simply $\mathcal{A}_4^{\text{fact}}(z) \sim O(z^{-2})$. From the perturbative unitarity bound in Eq. (1.3) we know that $\mathcal{A}_4(z) \lesssim O(z^0)$, which then implies that $\mathcal{A}_4^{\text{cont}}(z) \lesssim O(z^0)$. Since we have assumed that all interactions are polynomials in the momenta, we deduce that the quartic vertex is a constant.

The same logic applies at quintic order in the fields. Splitting the amplitude as $\mathcal{A}_5(z) = \mathcal{A}_5^{\text{cont}}(z) + \mathcal{A}_5^{\text{fact}}(z)$, we see that the contribution from all factorization diagrams must scale *at most* as $\mathcal{A}_5^{\text{fact}}(z) \sim O(z^{-2})$ since the cubic and quartic vertices are both constants. Perturbative unitarity then implies that $\mathcal{A}_5(z) \lesssim O(z^{-1})$,

[5]Throughout, all repeated indices are taken to be summed over unless otherwise specified.

which, because of our assumption of locality is impossible unless $\mathcal{A}_5^{\rm cont}(z) = 0$, so the quintic contact term must vanish. Since we are working in a minimal operator basis it then follows that the quintic Lagrangian operator must vanish completely. At sextic order and higher, the scaling bound is increasingly negative and the same reasoning applies iteratively. Hence, all Lagrangian operators are absent except for those at quartic or lower orders.

In summary, we have shown that under the assumptions of perturbative unitarity, locality, and a finite mass spectrum, the Lagrangian for any Lorentz invariant, four-dimensional theory of scalars is equivalent—up to a field redefinition that leaves all scattering amplitudes invariant—to the well-known renormalizable form

$$\mathcal{L} = -\frac{1}{2}\partial^\mu\phi_I\partial_\mu\phi_I - \frac{1}{2}m_I^2\phi_I\phi_I - \frac{1}{3!}\alpha_{IJK}\phi_I\phi_J\phi_K - \frac{1}{4!}\beta_{IJKL}\phi_I\phi_J\phi_K\phi_L, \quad (1.5)$$

where α_{IJK} and β_{IJKL} are symmetric on all indices. This argument rules out the possibility of a perturbatively unitary theory with derivatively coupled interactions, so we learn e.g. that there is no variation of the nonlinear sigma model that is well-behaved at high energies without additional modes. Here we emphasize again that our argument relies crucially on the assumption of polynomial boundedness.

1.3 Soft Theorems and Unification

We are now ready to derive explicit constraints on the Lagrangian in Eq. (1.5) derived from the Adler zero conditions. As we will see, the resulting constraints will directly imply the existence of a symmetry connecting the massless and massive degrees of freedom. Past work has applied similar logic to the pion sector alone [11, 19] as well to more exotic theories such as Dirac-Born-Infeld theory, the Galileon [12–15], and more recently even the Navier-Stokes equation [20].

Adler Zero Constraints

As discussed earlier, we assume an Adler zero condition that mandates the vanishing of all scattering amplitudes in the limit where any massless particle is taken soft. For simplicity we refer to all massless fields as pions and all massive fields as sigmas.

Without loss of generality, consider an N-particle tree-level scattering amplitude in which leg 1 is a soft pion, so $p_1 \to zp_1$ for small z. Note that here it is crucial that the minimal kinematic basis defined in Eq. (1.1) eliminates redundant kinematic invariants involving legs $N-1$ and N but does not interfere with leg 1. In the minimal kinematic basis, this soft scaling sends

$$p_1 \cdot p_a \to z\, p_1 \cdot p_a, \quad (1.6)$$

for all a. The soft-deformed amplitude has a series expansion $\mathcal{A}_N(z) \sim z^{-1}+z^0+\cdots$, where the leading $O(z^{-1})$ term is a soft pole and the subleading $O(z^0)$ term is regular. The Adler zero condition implies that the N-particle tree-level scattering amplitude satisfies

$$\mathcal{A}_N(z \to 0) = 0, \tag{1.7}$$

so the leading and subleading terms are both zero. Moreover, the coefficients of these terms are themselves complicated rational functions of the minimal kinematic basis of invariants, so each Adler zero condition actually dictates multiple constraints on the couplings of the theory.

In what follows, we compute the Adler zero constraints for four- and five-particle tree-level scattering amplitudes. In principle, additional constraints can arise at six-particle scattering and higher but they will not be necessary for the arguments in this paper.

Four-Particle Amplitude

Using the Feynman diagrams defined from the Lagrangian in Eq. (1.5), we compute the soft-deformed four-particle scattering amplitude,

$$\begin{aligned}\mathcal{A}_4(z) = &-\sum_N \frac{\alpha_{IJN}\alpha_{NKL}}{(m_J^2 - m_N^2) + 2z(p_1\cdot p_2)} - \sum_N \frac{\alpha_{IKN}\alpha_{NJL}}{(m_K^2 - m_N^2) + 2z(p_1\cdot p_3)} \\ &- \sum_N \frac{\alpha_{ILN}\alpha_{NJK}}{(m_L^2 - m_N^2) - 2z(p_1\cdot p_2 + p_1\cdot p_3)} - \beta_{IJKL},\end{aligned} \tag{1.8}$$

where I is a pion flavor index, J, K, L are flavor indices of any type, and N runs over all possible flavors of the exchanged scalar. We then take $z \to 0$ to extract the soft limit.

The Adler zero implies that the leading $O(z^{-1})$ soft pole in $\mathcal{A}_4(z)$ is zero, so

$$\begin{aligned}\sum_{N|m_N^2=m_J^2} \frac{\alpha_{IJN}\alpha_{NKL}}{2z(p_1\cdot p_2)} + \sum_{N|m_N^2=m_K^2} \frac{\alpha_{IKN}\alpha_{NJL}}{2z(p_1\cdot p_3)} \\ - \sum_{N|m_N^2=m_L^2} \frac{\alpha_{ILN}\alpha_{NJK}}{2z(p_1\cdot p_2 + p_1\cdot p_3)} = 0,\end{aligned} \tag{1.9}$$

where the sums are restricted to the subset of intermediate states which are mass-degenerate with an external state. The above expression must vanish for all possible kinematics. We can multiply through by a common denominator so that the left-hand side becomes a polynomial in $(p_1\cdot p_2)^2$, $(p_1\cdot p_2)(p_1\cdot p_3)$, and $(p_1\cdot p_3)^2$.

Since these are all independent kinematic invariants, their coefficients separately vanish, and are in fact equivalent under permutations to

$$\sum_{N|m_N^2=m_J^2} \alpha_{IJN}\alpha_{NKL} = 0. \tag{1.10}$$

For the choice $I = L$ and $J = K$, this reduces to the condition

$$\sum_{N|m_N^2=m_J^2} (\alpha_{IJN})^2 = 0, \tag{1.11}$$

so each term in the sum vanishes independently because the coupling constants are real. Recalling that I indexes a massless pion, we relabel indices to obtain our final condition,

$$\alpha_{IJK} = 0 \quad \text{for} \quad m_I^2 = 0 \quad \text{and} \quad m_J^2 = m_K^2. \tag{1.12}$$

Said another way, in this field basis the Adler zero *forbids* any cubic interaction between a pion and any two particles of the same mass. In hindsight this is obvious because the $\mathcal{O}(z^{-1})$ soft pole arises when the exchanged particle is mass-degenerate with an external state.

Meanwhile, the subleading $\mathcal{O}(z^0)$ term in $\mathcal{A}_4(z)$ also vanishes,

$$\sum_{N|m_N^2\neq m_J^2} \frac{\alpha_{IJN}\alpha_{NKL}}{m_J^2 - m_N^2} + \sum_{N|m_N^2\neq m_K^2} \frac{\alpha_{IKN}\alpha_{NJL}}{m_K^2 - m_N^2} + \sum_{N|m_N^2\neq m_L^2} \frac{\alpha_{ILN}\alpha_{NJK}}{m_L^2 - m_N^2} + \beta_{IJKL} = 0, \tag{1.13}$$

yielding another nontrivial constraint on the couplings and masses of the theory.

Five-Particle Amplitude

The calculation of the soft-deformed five-particle amplitude $\mathcal{A}_5(z)$ is straightforward but the resulting expression is quite complicated so we will not display it here. Again we observe $\mathcal{O}(z^{-1})$ soft poles which arise when exchanged states are mass-degenerate with external states, so these terms vanish on the the condition of Eq. (1.12). Meanwhile, the Adler zero condition enforces the vanishing of the $\mathcal{O}(z^0)$ term of $\mathcal{A}_5(z)$,

$$\begin{aligned} &\sum_{N|m_N^2\neq m_J^2} \frac{\alpha_{IJN}\beta_{NKLM}}{m_J^2 - m_N^2} + \sum_{N|m_N^2\neq m_K^2} \frac{\alpha_{IKN}\beta_{NJLM}}{m_K^2 - m_N^2} \\ &+ \sum_{N|m_N^2\neq m_L^2} \frac{\alpha_{ILN}\beta_{NJKM}}{m_L^2 - m_N^2} + \sum_{N|m_N^2\neq m_M^2} \frac{\alpha_{IMN}\beta_{NJKL}}{m_M^2 - m_N^2} = 0, \end{aligned} \tag{1.14}$$

where again I is a pion flavor index and J, K, L, M are flavor indices of any type. Here we have presented a simplified version of the constraint by taking the limit $p_2 \cdot p_3, p_2 \cdot p_4 \to \infty$, which is permitted since these are independent kinematic invariants and the Adler zero applies to any on-shell configuration.

Symmetry Constraints

Before determining the relationship between soft theorems and symmetry it will be helpful to know beforehand what to look for—that is, the mechanical sense in which spontaneously broken global symmetries constrain couplings and masses in the broken phase. With these constraints in hand we will then show how they coincide exactly with those derived from the Adler zero.

Spontaneous Symmetry Breaking

Typically it is assumed that in the unbroken phase there is a multiplet Φ_I transforming in a linear representation of the symmetry group G, so

$$\Phi_I \to \Phi_I + W_{IJ}\Phi_J, \tag{1.15}$$

where W_{IJ} is a generator of G. However, in the broken phase the fields acquire vacuum expectation values $v_I = \langle \Phi_I \rangle$ which are invariant only under a subgroup $H \subset G$. Expanding about true vacuum, $\Phi_I = v_I + \phi_I$, we find that the field fluctuations ϕ_I transform as

$$\phi_I \to \phi_I + W_{IJ}\phi_J + W_{IJ}v_J, \tag{1.16}$$

which is the composition of a linear transformation and a constant shift.

All transformations are classified according to whether or not they leave the vacuum invariant. If $W_{IJ}v_J = 0$ then the transformation is linear and corresponds to an unbroken generator in H that we denote by $\mathcal{T}$. If $W_{IJ}v_J \neq 0$ then the transformation is affine, i.e. is a composition of both a linear and shift component, and corresponds to a broken generator in G/H that we denote by $\mathcal{X}$. The unbroken and broken generators $\mathcal{T}$ and $\mathcal{X}$ act on the fields as

$$\phi_I \to \phi_I + \mathcal{T}\phi_I \qquad \text{for} \qquad \mathcal{T}\phi_I = T_{IJ}\phi_J \tag{1.17}$$

$$\phi_I \to \phi_I + \mathcal{X}\phi_I \qquad \text{for} \qquad \mathcal{X}\phi_I = X_{IJ}\phi_J + \lambda_I, \tag{1.18}$$

where $T_{IJ}v_J = 0$ and $X_{IJ}v_J \neq 0$, and the shift is $\lambda_I = X_{IJ}v_J$. So the unbroken generators $\mathcal{T}$ are realized as linear transformations while the broken generators

$\mathcal{X}$ are realized as affine transformations. On occasion it will be convenient to express the unbroken and broken generators explicitly in terms of the linear and shift components of their corresponding transformations, so we will sometimes write $\mathcal{T} = (T_{IJ}, 0)$ and $\mathcal{X} = (X_{IJ}, \lambda_I)$.

Linear Symmetries

Under the linear symmetry in Eq. (1.17), the Lagrangian in Eq. (1.5) transforms as

$$\begin{aligned}\mathcal{L} \to \mathcal{L} &- T_{IJ}(\partial^\mu \phi_I \partial_\mu \phi_J + m_I^2 \phi_I \phi_J) \\ &- \frac{1}{2}\alpha_{IJN} T_{NK} \phi_I \phi_J \phi_K - \frac{1}{3!}\beta_{IJKN} T_{NK} \phi_I \phi_J \phi_K \phi_L. \end{aligned} \tag{1.19}$$

For this to be a symmetry of the action, all additional terms on the right-hand side must vanish, implying a set of nontrivial constraints,

$$T_{IJ} + T_{JI} = 0 \tag{1.20}$$

$$T_{IJ} m_I^2 + T_{JI} m_J^2 = 0 \tag{1.21}$$

$$\sum_N (\alpha_{IJN} T_{NK} + \alpha_{IKN} T_{NJ} + \alpha_{JKN} T_{NI}) = 0 \tag{1.22}$$

$$\sum_N (\beta_{IJKN} T_{NL} + \beta_{IJLN} T_{NK} + \beta_{IKLN} T_{NJ} + \beta_{JKLN} T_{NI}) = 0. \tag{1.23}$$

Here, repeated indices are *not* summed unless accompanied by an explicit summation, and we have manifestly symmetrized on indices due to Bose symmetry.

Eq. (1.20) implies that T_{IJ} is antisymmetric, which is obvious because any linear symmetry of the Lagrangian in Eq. (1.5) has no explicit momentum dependence and thus must be a symmetry of the kinetic term and potential term independently. Since the kinetic term $\partial^\mu \phi_I \partial_\mu \phi_I$ is invariant under special orthogonal rotations, T_{IJ} must be antisymmetric. Meanwhile, Eq. (1.21) implies that T_{IJ} only acts nontrivially on states with equal mass, $m_I = m_J$. This is again obvious since an unbroken symmetry generator should only act on a subspace of mass-degenerate states.

Affine Symmetries

Under the affine symmetry defined in Eq. (1.17), the Lagrangian in Eq. (1.5) transforms as

$$\mathcal{L} \to \mathcal{L} - m_I^2 \lambda_I \phi_I - X_{IJ} \partial^\mu \phi_I \partial_\mu \phi_J - (m_I^2 X_{IJ} + \frac{1}{2}\alpha_{IJK}\lambda_K)\phi_I \phi_J \tag{1.24}$$

$$- \frac{1}{2}(\alpha_{IJN} X_{NK} + \frac{1}{3}\beta_{IJKL}\lambda_L)\phi_I \phi_J \phi_K - \frac{1}{3!}\beta_{IJKN} X_{NK} \phi_I \phi_J \phi_K \phi_L, \tag{1.25}$$

implying the following constraints,

$$m_I^2 \lambda_I = 0 \tag{1.26}$$

$$X_{IJ} + X_{JI} = 0 \tag{1.27}$$

$$X_{IJ} m_I^2 + X_{JI} m_J^2 + \sum_K \alpha_{IJK} \lambda_K = 0 \tag{1.28}$$

$$\sum_N (\alpha_{IJN} X_{NK} + \alpha_{IKN} X_{NJ} + \alpha_{JKN} X_{NI}) + \sum_L \beta_{IJKL} \lambda_L = 0 \tag{1.29}$$

$$\sum_N (\beta_{IJKN} X_{NL} + \beta_{IJLN} X_{NK} + \beta_{IKLN} X_{NJ} + \beta_{JKLN} X_{NI}) = 0, \tag{1.30}$$

where, once again, repeated indices are *not* summed unless accompanied by an explicit summation. Eq. (1.26) implies that the components of the vector λ_I are nonzero only when $m_I = 0$. Thus λ_I only has support on the subspace of massless pion fields. This is of course required of any nonlinearly realized symmetry. Also, as before Eq. (1.27) implies that X_{IJ} is an antisymmetric generator of the special orthogonal group.

We emphasize that the affine constraints in Eqs. (1.26)-(1.30) are satisfied by X_{IJ} even after shifting arbitrarily by any T_{IJ} which happens to satisfy the linear constraints in Eqs. (1.20)-(1.23). This is expected because $\mathcal{X} = (X_{IJ}, \lambda_I)$ is a generator of the coset space G/H and is thus only defined modulo the addition of a generator $\mathcal{T} = (T_{IJ}, 0)$ of the unbroken group H.

Finally, let us note that it is straightforward but tedious to prove closure of the symmetry algebra. In particular, taking any combination of commutators of unbroken generators $\mathcal{T}$ satisfying Eqs. (1.20)-(1.23) or broken generators $\mathcal{X}$ satisfying Eqs. (1.26)-(1.30), we obtain new generators that also satisfy these constraints.

Symmetry from Soft Theorems

At last, we are equipped to demonstrate how symmetry emerges from soft theorems and perturbative unitarity. Here we make no direct reference to vacuum expectation values or spontaneous symmetry breaking. Instead, we simply show that the Adler zero conditions derived in Eqs. (1.12)-(1.14) imply the existence of affine transformations that *precisely* satisfy the conditions in Eqs. (1.26)-(1.30) required of any generator of a spontaneously broken symmetry. In fact, we will be able to constructively derive explicit formulas for *all* the broken generators $\mathcal{X} = (X_{IJ}, \lambda_I)$ and for a subalgebra of unbroken generators $\mathcal{T} = (T_{IJ}, 0)$. We will also learn that the coset space G/H is symmetric. Lastly, we comment on the restoration of these symmetries at high energy, establishing unification as a consequence of unitarity and Adler zeros.

Constructing Generators

First, consider the Adler zero condition for a pion labelled by the flavor vector λ_I. Since these states are massless, we have that $m_I = 0$ and so Eq. (1.26) holds trivially. Second, let us *define* the broken generators to be $\mathcal{X} = (X_{IJ}, \lambda_I)$ such that

$$X_{IJ} = \left\{ \begin{array}{ccc} -\frac{1}{m_I^2 - m_J^2} \sum_K \alpha_{IJK} \lambda_K & , & m_I^2 \neq m_J^2 \\ \\ 0 & , & m_I^2 = m_J^2 \end{array} \right\}. \tag{1.31}$$

Here X_{IJ} is manifestly antisymmetric, thus satisfying Eq. (1.27). Furthermore, by construction X_{IJ} satisfies Eq. (1.28) for $m_I^2 \neq m_J^2$. Third, Eq. (1.12) trivially implies Eq. (1.28) for $m_I^2 = m_J^2$. Fourth, by contracting the free pion index in Eq. (1.13) and (1.14) with λ_I we immediately obtain Eq. (1.29) and (1.30). Thus, we have shown that X_{IJ} defined in Eq. (1.31) satisfies all the requirements expected of a broken symmetry generator. For later convenience, let us also define

$$X_{IJ} = \sum_K X_{IJ,K} \lambda_K, \tag{1.32}$$

where the index after the comma in $X_{IJ,K}$ is implicitly projected down to the pion subspace since that is where λ_I has nonzero support. On the other hand, the indices before the comma are general and can have support on both the pion and sigma subspaces. Thus, each pion field direction maps to some broken generator $X_{IJ,K}$.

It is now possible to derive formulas for some of the unbroken symmetry generators. Consider an amplitude for which legs 1 and 2 are pion fields with flavor indices I, J and legs 3 and 4 have arbitrary flavor indices K, L. In this instance we have the choice of taking the soft limit of either leg 1 or leg 2. This pair of soft limits corresponds to the four-particle $\mathcal{O}(z^0)$ Adler zero constraint from Eq. (1.13), together with the same condition with I and J swapped. The difference of these equations yields a new constraint on a commutator [21],

$$(m_K^2 - m_L^2) \sum_N \left(X_{KN,I} X_{NL,J} - X_{KN,J} X_{NL,I} \right) = 0, \tag{1.33}$$

with indices not implicitly summed. Note that even though legs 1 and 2 are bosons, the difference between constraints is not trivially zero since each constraint is derived from a different kinematic region in which either leg 1 or leg 2 is soft.

Next, we contract the commutator appearing in this constraint with an arbitrary antisymmetric tensor λ_{IJ} which only has support on the subspace of pions. Relabeling indices, we then obtain an expression for an unbroken generator $\mathcal{T} = (T_{IJ}, 0)$ where

$$T_{IJ} = \sum_{K,L} T_{IJ,KL}\lambda_{KL} \qquad \text{where} \qquad T_{IJ,KL} = X_{IN,K}X_{NJ,L} - X_{JN,K}X_{NI,L}. \quad (1.34)$$

As before, the indices after the comma are implicitly projected down to the pion subspace and the indices before the comma are general. Note that in general, the T_{IJ} constructed from Eq. (1.34) need not span the full set of unbroken generators. From this definition of T_{IJ} and Eq. (1.33) we see that Eq. (1.20) and Eq. (1.21) are automatically satisfied. This implies that T_{IJ} is a generator of the special orthogonal group which only connects fields of equal mass. Last but not least, by inserting Eq. (1.34) into Eq. (1.22) and Eq. (1.23), we obtain Eq. (1.13) and Eq. (1.14) after a bit of algebra, verifying that T_{IJ} is an unbroken symmetry generator.

Structure of the Symmetry Algebra

Armed with explicit formulas for certain symmetry generators in Eq. (1.31) and Eq. (1.34), we are now able to deduce some interesting facts about the symmetry algebra.

First, we have actually rederived a version of Goldstone's theorem [5] which says that there is a bijective mapping from the space of pions to the space of broken generators (X_{IJ}, λ_I). Injectivity holds because the Adler zero constraint is parameterized by the flavor of the soft pion, λ_I, which is both the shift vector λ_I and the vector used to construct X_{IJ} via Eq. (1.31). Any pair of distinct pion states gives rise to distinct shifts, and therefore distinct broken generators. To establish surjectivity, observe that the number of independent shift vectors λ_I cannot exceed the dimension of the space of pions—otherwise one of these vectors would have to act on a massive mode, which is forbidden by Eq. (1.26). Moreover, even if a pair of broken generators $\mathcal{X} = (X_{IJ}, \lambda_I)$ and $\mathcal{X}' = (X'_{IJ}, \lambda'_I)$ share the same shift vector, $\lambda_I = \lambda'_I$, they must still label the same element of G/H because they differ only by a linear generator $\mathcal{T} = \mathcal{X} - \mathcal{X}' = (X_{IJ} - X'_{IJ}, 0)$. It follows that the broken generators $\mathcal{X}$ constructed from Eq. (1.31) span the *full space* G/H of broken generators.

Second, our Adler zero constraints actually imply that the coset space G/H is symmetric. For a symmetric space *there exists* a basis in which the unbroken and broken generators satisfy commutation relations of the schematic form,

$$[\mathcal{T}, \mathcal{T}] \sim \mathcal{T}, \qquad [\mathcal{T}, \mathcal{X}] \sim \mathcal{X}, \qquad [\mathcal{X}, \mathcal{X}] \sim \mathcal{T}. \quad (1.35)$$

The first equation says that the unbroken generators form a subalgebra while the second equation says that the broken generators furnish a linear representation of the unbroken symmetry. The third equation is the only nontrivial condition.

Remarkably, the generators we have defined in Eq. (1.31) are precisely in a basis that manifests Eq. (1.35) automatically. This was not guaranteed, since any broken generator $\mathcal{X} = (X_{IJ}, \lambda_I)$ is defined modulo addition by any unbroken generator $\mathcal{T} = (T_{IJ}, 0)$. As it turns out, our particular broken generators are in a special basis in which they are "mass off-diagonal", i.e. $X_{IJ} = 0$ when $m_I^2 = m_J^2$. While we do not have explicit formulas for all unbroken generators, we still know they are "mass on-diagonal", i.e. $T_{IJ} = 0$ when $m_I^2 \neq m_J^2$ since any preserved symmetry must leave the spectrum invariant.

With this knowledge let us compute the action of the commutator $[\mathcal{T}, \mathcal{X}]$ on the fields,

$$[\mathcal{T}, \mathcal{X}]\phi_I = [T, X]_{IJ}\phi_J + T_{IJ}\lambda_J = \mathcal{X}'\phi_I \quad \text{where} \quad \mathcal{X}' = ([T, X]_{IJ}, T\lambda_I). \tag{1.36}$$

Since T_{IJ} and X_{IJ} are mass on-diagonal and off-diagonal, respectively, we know that their commutator $[T, X]_{IJ}$ is mass off-diagonal. Since only the broken generators are mass off-diagonal, this implies that the resulting generator is broken, so $[\mathcal{T}, \mathcal{X}] = \mathcal{X}'$. On the other hand, the commutator $[\mathcal{X}, \mathcal{X}']$ acts on the fields as

$$[\mathcal{X}, \mathcal{X}']\phi_I = [X, X']_{IJ}\phi_J + X_{IJ}\lambda'_J - X'_{IJ}\lambda_J = \mathcal{T}'\phi_I \quad \text{where} \quad \mathcal{T}' = ([X, X']_{IJ}, 0). \tag{1.37}$$

Since X_{IJ} and X'_{IJ} are both mass off-diagonal and λ_I and λ'_I reside in the subspace of pion fields, the shift component $X_{IJ}\lambda'_J - X'_{IJ}\lambda_J$ only has support on the subspace of sigma fields. However, we know from Eq. (1.26) that any shift component of an affine symmetry must act solely on the pion subspace, so the shift must vanish. Any symmetry without a shift is linear by definition, so we know that the commutator must produce an unbroken generator, and thus $[\mathcal{X}, \mathcal{X}'] = \mathcal{T}'$.

The fact that the coset space is symmetric implies that there is an automorphism of the algebra that sends $\mathcal{T} \to \mathcal{T}$ and $\mathcal{X} \to -\mathcal{X}$. In fact, this automorphism simply flips the sign of all transformations between states ϕ_I and ϕ_J for $m_I^2 \neq m_J^2$. Since $\mathcal{T}$ and $\mathcal{X}$ are mass on-diagonal and off-diagonal, respectively, under this automorphism they will be even and odd, respectively. The converse of our result is also known, where the Adler zero fails for theories where the coset space is not symmetric [22]. The importance of a symmetric coset for constructing the theory of purely pions from Adler zeros was also emphasized in [11].

Symmetry Restoration

In summary, we have shown that perturbative unitarity together with the Adler zero condition on four- and five-particle scattering implies the existence of an underlying global symmetry. This symmetry is encoded in unbroken and broken generators that act on the fields as linear and affine field-space transformations.

Given that we have derived the phenomenon of symmetry breaking, it is perhaps not so surprising that we can also derive symmetry restoration. For high-energy scattering, all dimensionful parameters become negligible relative to the momenta governing the process. Consequently in this limit we are allowed to drop all dimensionful parameters such as masses m_I, cubic couplings α_{IJK}, and shift parameters λ_I of the affine symmetry transformation in Eq. (1.17). On the other hand, the quartic couplings β_{IJKL} and the linear components of the symmetry generators, T_{IJ} and X_{IJ}, persist because they are dimensionless. Hence, these dimensionless parameters encode a symmetry of the high-energy theory under which all fields transform *linearly*. We have also shown that T_{IJ} and X_{IJ} span a subspace of generators of the special orthogonal group of rotations on all the fields, as expected. This establishes our final claim: for any perturbatively unitary, Lorentz invariant, four-dimensional theory with a finite spectrum of locally interacting scalars, the Adler zero condition implies the existence of an ultraviolet symmetry that unifies the massless and massive states.

1.4 Linear Sigma Model Example

It will be instructive to study the implications of our results in a concrete example. Consider a perturbatively unitary theory describing a single massive sigma field σ and several massless pion fields π_i for $1 \leq i \leq N-1$. In what follows, we show how the Adler zero constraints from Eq. (1.12), (1.13), and (1.14) dictate a completely unique Lagrangian corresponding to a linear sigma model with the spontaneous symmetry breaking pattern $SO(N) \to SO(N-1)$.

Since Eq. (1.12) forbids pion interactions with states of equal mass, there is no cubic interaction involving an odd number of pions, so $\alpha_{ijk} = \alpha_{i\sigma\sigma} = 0$ but $\alpha_{ij\sigma} \neq 0$,

$\alpha_{\sigma\sigma\sigma} \neq 0$. Inserting these zeros into Eq. (1.13), we obtain

$$\beta_{ijkl} = \frac{1}{m_\sigma^2}\left(\alpha_{ij\sigma}\alpha_{kl\sigma} + \alpha_{ik\sigma}\alpha_{jl\sigma} + \alpha_{il\sigma}\alpha_{jk\sigma}\right) \tag{1.38}$$

$$\beta_{ij\sigma\sigma} = \frac{1}{m_\sigma^2}\left(\alpha_{ij\sigma}\alpha_{\sigma\sigma\sigma} - 2\sum_n \alpha_{in\sigma}\alpha_{jn\sigma}\right) \tag{1.39}$$

$$\beta_{ijk\sigma} = \beta_{i\sigma\sigma\sigma} = 0, \tag{1.40}$$

while plugging into Eq. (1.14) yields

$$\sum_n \beta_{ijkn}\alpha_{nl\sigma} = \beta_{ij\sigma\sigma}\alpha_{kl\sigma} + \beta_{ik\sigma\sigma}\alpha_{jl\sigma} + \beta_{jk\sigma\sigma}\alpha_{il\sigma} \tag{1.41}$$

$$\beta_{\sigma\sigma\sigma\sigma}\alpha_{ij\sigma} = 3\sum_k \beta_{ik\sigma\sigma}\alpha_{jk\sigma}. \tag{1.42}$$

To solve these equations we go to a simplified field basis. Since the cubic coupling in the Lagrangian is $\alpha_{ij\sigma}\pi_i\pi_j\sigma$ we can perform an $SO(N-1)$ rotation on the pion fields in order to diagonalize them, effectively setting $\alpha_{ij\sigma} = \alpha_i\delta_{ij}$ with no summation implied.

By combining the first lines of Eq. (1.38) and Eq. (1.41) we can eliminate β_{ijkl} to give an equation for $\beta_{ij\sigma\sigma}$ in terms of $\alpha_{ij\sigma}$. Plugging $\beta_{ij\sigma\sigma}$ into the second line of Eq. (1.38) and setting $i = j$ we obtain the relation

$$\alpha_i(3\alpha_i - \alpha_{\sigma\sigma\sigma}) = 0, \tag{1.43}$$

implying that either $\alpha_i = 0$ or $\alpha_i = \alpha_{\sigma\sigma\sigma}/3$. For the former case, the pion π_i is a free massless scalar decoupled from the rest of the theory. We exclude this scenario without loss of generality since it corresponds to a subcase of our original setup where there is a free spectator pion. Therefore, we assume the latter case, $\alpha_i = \alpha_{\sigma\sigma\sigma}/3$. Substituting the second line of Eq. (1.38) into the second line of Eq. (1.41) eliminates $\beta_{ik\sigma\sigma}$, producing the equation $\beta_{\sigma\sigma\sigma\sigma} = \alpha_{\sigma\sigma\sigma}^2/(3m_\sigma^2)$. All couplings are thus expressed in terms of m_σ, $\alpha_{\sigma\sigma\sigma}$, and various combinations of Kronecker deltas.

Plugging these couplings into Eq. (1.5), we obtain

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \pi_i \partial_\mu \pi_i - \frac{1}{2}\partial^\mu \sigma \partial_\mu \sigma - \frac{1}{2} m_\sigma^2 \sigma^2 \\ - \frac{a_{\sigma\sigma\sigma}}{6}\sigma\left(\pi_i\pi_i + \sigma^2\right) - \frac{a_{\sigma\sigma\sigma}^2}{72 m_\sigma^2}\left(\pi_i\pi_i + \sigma^2\right)^2, \tag{1.44}$$

so the Lagrangian is fixed entirely by the mass m_σ and cubic coupling $a_{\sigma\sigma\sigma}$. We can also see that the Lagrangian is equivalent to that of the linear sigma model by defining a multiplet of fields $\Phi_I = (\pi_i, v + \sigma)$, so Eq. (1.44) becomes

$$\mathcal{L} = -\frac{1}{2}\partial^\mu \Phi_I \partial_\mu \Phi_I - \frac{\lambda}{4}(\Phi_I \Phi_I - v^2)^2, \tag{1.45}$$

where the vacuum expectation value and quartic couplings are

$$v = \frac{3 m_\sigma^2}{a_{\sigma\sigma\sigma}} \qquad \text{and} \qquad \lambda = \frac{a_{\sigma\sigma\sigma}^2}{18 m_\sigma^2}. \tag{1.46}$$

We thus learn that there is a one-to-one mapping between the Lagrangian parameters of the ultraviolet theory in the unbroken phase and physically observable quantities in the broken phase. The resulting theory is a linear sigma model with the spontaneous symmetry breaking pattern $SO(N) \to SO(N-1)$.

Eq. (1.31) and (1.34) provide explicit expressions for all the generators of $SO(N)$ in the broken phase. For the broken symmetry directions, we obtain

$$X_{i\sigma,j} = -X_{\sigma i,j} = \frac{\alpha_{ij\sigma}}{m_\sigma^2}, \tag{1.47}$$

which are the $N-1$ matrices which realize the affine transformations. As explained earlier, these generators are in bijective correspondence with the pion fields. Meanwhile, commutators of these affine transformations yield

$$T_{ij,kl} = X_{i\sigma,k} X_{\sigma j,l} - X_{i\sigma,l} X_{\sigma j,k} \propto \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}, \tag{1.48}$$

which are $(N-1)(N-2)/2$ matrices which realize the linear transformations corresponding to the unbroken symmetry $SO(N-1)$. All together, the generators $T_{ij,kl}$ and $X_{i\sigma,j}$ form the full set of $N(N-1)/2$ antisymmetric generators of the original $SO(N)$ symmetry.

We have thus proven that the unique perturbatively unitary theory of a single massive sigma field coupled to $N-1$ massless pion fields is the $SO(N) \to SO(N-1)$ linear sigma model. We emphasize that the *uniqueness* of the Lagrangian, as well as the explicit expression of *all* generators in terms of couplings and masses is special.

For a more general spectrum, the broken generators can always be determined but the same may not be true for the unbroken generators. In this case it may be that only a subset of Lagrangian parameters can be fixed. Nevertheless, we are always able to deduce the existence of an underlying symmetry that unifies the massless and massive degrees of freedom.

1.5 Conclusions

Can symmetry be mandatory rather than optional? In this paper we have argued yes, at least for a perturbatively unitary, Lorentz invariant, four-dimensional scalar theory with a finite spectrum, local interactions, and Adler zero conditions on the massless degrees of freedom.

First, we have shown that perturbative unitarity and locality enforce a strict bound on scattering amplitudes at arbitrarily high energy. In turn, this implies that the space of theories under consideration can, upon an appropriate field redefinition, always be described by a renormalizable Lagrangian. This is of course a familiar statement, but one derived here through scattering amplitudes. Second, we have demonstrated that the Adler zero constraints on massless states directly imply the existence of an underlying set of unbroken and broken symmetry generators. The latter are in bijective correspondence with the pion degrees of freedom and can be derived explicitly in terms of the couplings and masses. Furthermore, they manifestly transform the massless and massive degrees of freedom amongst each other. At high energies, the spectrum unifies into multiplets which linearly realize some subgroup of the special orthogonal group.

The present work leaves a number of avenues for future exploration. For example, it would be interesting to apply our Adler zero conditions to systematically "bootstrap" theories starting with no input other than the spectrum. We have already done this for the linear sigma model, but more generally the space of theories sculpted by our constraints should give a well-defined notion of the parameter space of allowed ultraviolet completions.

Related to this is the question of whether one can constructively derive the vacuum expectation values of the ultraviolet theory directly from the affine symmetry transformations found in the broken phase. This task is not obviously possible. In fact, it may be impossible if there exists even a single theory exhibiting affine symmetries which do not arise purely from linear symmetries in the presence of vacuum expectation values. It would be interesting to explore this option in the future.

Another natural direction along which to extend our results is higher spin, e.g. with the addition of fermions. In this case it should be possible to study Yukawa theories, as well as supersymmetric extensions like the Wess-Zumino model. For the case of gauge bosons, the goal would be to derive the most general possible Higgs mechanism consistent with an input spectrum. In this last scenario all pions are eaten and thus unphysical, so the Adler zero condition will be unnecessary and perturbative unitarity alone will be sufficient. This avenue was in fact pursed long ago in a number of seminal works [18, 23], however with additional assumptions, e.g. the structure of the electroweak sector or the existence of a Stueckelberg mechanism. More recently, progress has been made towards a purely on-shell description of the electroweak sector using unitarity bounds [24], [25]. Interesting constraints on the geometry of extra-dimensional ultraviolet completions have also been derived from unitarization of higher spin scattering [26, 27].

APPENDIX

1.A Scaling Bounds from Perturbative Unitarity

In this appendix we derive a general bound on the high-energy behavior of tree-level scattering amplitudes. Our general approach is similar in spirit to that of [18] except we will be more explicit with our assumptions and various caveats to the argument. As we have throughout the paper, we assume a perturbatively unitary theory with a finite spectrum of particles with interactions that are at most polynomial in derivatives. It will also be important that all massless particles are pions which exhibit Adler zeros.

To begin, let us define some nomenclature. For any z-dependent function $f(z)$, we define a "scaling order", $\langle f(z)\rangle = r$, to be the minimum integer exponent r such that $|f(z\to\infty)| \le q|z|^r$ for some $q>0$. The purpose of this section is to derive a D-dimensional generalization of the scaling bound in Eq. (1.3). Here we can interpret z as a deformation parameter for the kinematic invariants as defined in Eq. (1.2). However, we can also think of it as *any* parameterization of the on-shell kinematics such that energies or momenta scale at most linearly with z and where $z\to\infty$ corresponds to high-energy, fixed angle scattering[6].

We can bound the absolute value of the four-particle scattering amplitude by

$$\begin{aligned}
|\mathcal{A}_{1,2\to1,2}| &\ge \operatorname{Im}\mathcal{A}_{1,2\to1,2}\\
&= \frac{1}{2}\sum_X \int \left(\prod_{i=3}^{N_X} \frac{d^{D-1}p_i}{(2\pi)^{D-1}}\frac{1}{2E_i}\right)(2\pi)^D\delta^D(p_1+p_2-p_X)|\mathcal{A}_{1,2\to X}|^2\\
&\ge \frac{1}{2}\int \left(\prod_{i=3}^{N} \frac{d^{D-1}p_i}{(2\pi)^{D-1}}\frac{1}{2E_i}\right)(2\pi)^D\delta^D(p_1+p_2-p_X)|\mathcal{A}_{1,2\to3,\cdots,N}|^2.
\end{aligned} \tag{1.49}$$

In this inequality, we have used the optical theorem, which is a consequence of unitarity. The symbol X labels all possible intermediate states with N_X particles and total momentum p_X. In the second line, we have exploited the fact that the

[6] To sidestep caveats involving logarithmic running and high-energy Landau poles we assume that the large z limit is such that the momenta are larger but not *exponentially* larger than all the masses in the spectrum.

optical theorem involves a sum of strictly positive definite terms (the exclusive cross-sections).

This inequality implies an inequality of the scaling orders of the amplitudes appearing on the left- and right-hand sides. Roughly, the amplitudes on the right cannot grow too quickly with the energy parameter z, or the inequality will be violated for some sufficiently large z, contradicting our assumption of unitarity. Strictly speaking, this inequality applies to the full amplitudes, including trees and all loop corrections. However, the assumption of perturbativity tells us that these amplitudes must be dominated by their tree components in the $z \to \infty$ limit, so the scaling order of each amplitude must be determined by the scaling order of its tree component. This can be seen explicitly using the definition of scaling order, but the derivation is straightforward and unilluminating, so we will omit it. It follows that any inequality of amplitude scaling orders implied by the inequality Eq. (1.49) is also an inequality of the scaling orders of their tree components.

In the high-energy limit, all masses can be neglected and the large z dependence of all quantities is given by dimensional analysis. Eq. (1.49) then implies a bound on D-dimensional, N-particle tree amplitudes,

$$\langle \mathcal{A}_4 \rangle \geq (N-2)(D-2) - D + 2\langle \mathcal{A}_N \rangle. \tag{1.50}$$

Specializing to the case $N = 4$, we see that $\langle \mathcal{A}_4 \rangle \leq 4 - D$, where we have assumed that the four-particle amplitude has the same high-energy scaling in the forward and fixed-angle regimes. We will return to this assumption later. Inserting the four-particle inequality back into the N-particle inequality we obtain

$$\langle \mathcal{A}_N(z) \rangle \leq N + D - \frac{ND}{2}. \tag{1.51}$$

It is amusing to study this bound for various choices of dimension D. For $D = 2$, we find $\langle \mathcal{A}_N(z) \rangle \leq 2$ for all N, which is why two-dimensional theories can be renormalizable with arbitrarily high order interaction vertices. In contrast, for $D = 6$ the bound is $\langle \mathcal{A}_N(z) \rangle \leq 6 - 2N$, so the $N = 3$ cubic vertex is the only local interaction allowed which is consistent with perturbative unitarity. For $D > 6$ there are simply no allowed interacting theories whatsoever. Last but not least, restricting to the case for $D = 4$, we obtain

$$\langle \mathcal{A}_N(z) \rangle \leq 4 - N, \tag{1.52}$$

thus deriving the scaling bound given in Eq. (1.3).

Let us briefly discuss some subtleties in the above argument. First of all, by applying the optical theorem we have assumed that the forward limit is finite. This is not true in general, e.g. in the presence of massless particles exchanged in the t-channel. However, we have from the start assumed that all of the massless degrees of freedom are pions which exhibit Adler zeros. Consequently, Eq. (1.12) implies that a pion cannot couple to two particles of equal mass, so it does not contribute via t-channel exchange to the forward amplitude, and there is no forward singularity at tree level. More generally, one might worry about loop-level infrared divergences following from multi-particle exchanges of pions. In our case, it is possible to regulate any infrared divergences in these theories with physical mass regulators, e.g. from small masses for the pions induced by explicit symmetry breaking. Consequently, the forward limit is finite, which implies that the inclusive cross-section appearing in the optical theorem, Eq. (1.49), is as well. The scaling order of the inclusive cross section is then determined by the finite, regulator-independent parts of the exclusive cross sections, which are represented by the expression appearing in the second line of Eq. (1.49).

Second, as noted earlier, the argument above assumes that the four-particle scattering amplitude has the same scaling behavior at high energies irrespective of whether scattering is forward or fixed-angle. This condition fails in the specific case that the quartic interaction is vanishing and there is a t-channel exchange of a massive particle. In this case the left-hand side of Eq. (1.50) is not $\langle \mathcal{A}_4 \rangle$, but rather zero. In general spacetime dimensions this leads to a slightly different scaling bound than Eq. (1.51) but this difference evaporates in the case of interest, $D = 4$. Furthermore, our earlier statements about allowed interactions in $D = 2$, $D = 6$, and $D > 6$ still hold in this case.

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Part II

Gravitational Radiation

Chapter 2

INTRODUCTION

In 2015, almost a century after Einstein's prediction [1], the LIGO collaboration made the first direct detection of gravitational waves (GWs). The first signal was emitted from a binary black hole inspiral process [2] with subsequent detections including neutron star events [3]. These measurements represent the advent of gravitational wave astronomy, opening a completely new window into the workings of our universe. With roughly 100 events detected to date, we are only at the beginning of this scientific journey. The pace is set to accelerate with new experiments expected to come online in the next decade. In the intermediate future, we expect an increase in sensitivity of up to two orders of magnitude over current detectors. This would allow us to probe the astrophysics of compact binary objects and possibly even to constrain new models of fundamental physics (see e.g. [4] and references therein). In order to maximize the experimental capabilities of these gravitational-wave observatories, it is imperative that theorists provide precise gravitational waveform models which can be compared to measurement. Such models are especially relevant for accurate parameter estimation of the binary constituents.

Predictions of gravitational waveforms and calculations of classical gravitational-wave observables have traditionally been performed by members of the classical General Relativity (GR) community. Their efforts have produced a number of impressive results, both analytical and numerical. Typically, one splits the gravitational inspiral process into three stages: the inspiral, merger, and ringdown. The inspiral stage involves widely separated constituents which move slowly in a weak graviataional field, losing energy and angular momentum in the form of gravitational radiation. In this regime, the binary dynamics involve widely separated physical scales, where the characteristic radius $R \sim Gm$ of the compact objects (with total mass m) is much smaller than their mutual separation r,

$$\frac{Gm}{r} \ll 1, \tag{2.1}$$

and G denotes Newton's constant. Furthermore, the virial theorem implies that the velocities of the constituents (in units where the speed of light $c = 1$) are of the same order as the ratio R/r, and therefore $v^2 \sim \frac{Gm}{r} \ll 1$. This hierarchy of scales enables a perturbative expansion of Einstein's equations in the traditional Post-Newtonian

(PN) scheme (discussed in detail in Section 3.1). At the end of the inspiral stage, the binary constituents *merge* into a single object, radiating vast amounts of energy into the gravitational field. During the merger, gravitational fields are strong and velocities approach the speed of light. Perturbation theory thus breaks down, and we must rely on supercomputer simulations of numerical GR to describe this process. Finally, the resulting object *rings down* by the emission of quasi-normal modes and settles into its equilibrium state. This part of the process is described by black hole perturbation theory [5, 6].

In this work, we are indirectly concerned with the first stage of the inspiral process as it relates to the classic two-body GR problem involving two widely separated compact binary constituents. In the Post-Newtonian approximation scheme, the current state-of-the-art has produced a complete expression for the Hamiltonian at 4PN order [7–9] and partial results for the dynamics of spinless black holes at 5 and 6 PN [10]. Results for spinning astrophysical objects [11–15], their tidal deformability [16–19], and even absorption effects [20, 21] are also known to a lesser extent. The next generation of gravitational-wave detectors is expected to be sensitive to 6PN corrections to the gravitational waveforms, which has invigorated theoretical efforts. Recently, theorists have begun applying particle-physics technologies, such as Effective Field Theory (EFT) [22], to high-precision classical calculations in gravitational waves, further accelerating progress in the area.

In a different direction, T. Damour suggested that crucial insights into the classical binary inspiral problem can be gleaned from scattering processes involving the binary constituents. When the binary constituents follow an unbound, hyperbolic orbit, the viral theorem is no longer generally valid. Relative velocities may become arbitrarily close to the speed of light (especially at periapsis), causing the PN expansion to break down. In fact, even in the case of *bound* orbits, the assumption that $v << 1$ until the last few cycles of the inspiral phase is only generally applicable to binaries with constituents of *comparable masses*. By contrast, intermediate-mass-ratio inspirals (IMRI) and extreme-mass-ratio inspirals (EMRI) may spend thousands or millions of cycles in the relativistic regime, where the PN expansion cannot accurately model the associated waveforms [23]. The proposed space-based interferometer LISA is expected to be sensitive to signals from IMRI and EMRI, motivating the development of perturbative approaches which remain accurate when $v \sim 1$ [23, 24].

Weak-field, or Post-Minkowskian (PM) expansion is one such approach. Although it has been known to physicists since at least 1957 [25], it has recently garnered renewed attention for this reason. An increasing number of particle physicists and scattering amplitude experts, due in part to a direct call-to-action from Damour [26], have entered the field of precision gravitational-wave science. They have brought decades of experience organizing and streamlining perturbative computations in the collider physics setting to the post-Minkowskian formulation of the classical two-body problem. A number of ideas rooted in the formal structure of quantum scattering amplitudes, such as color-kinematics duality and the double-copy relationship between gauge and gravity theory [27, 28], the efficient use of EFT methods [29, 30], Eikonal exponentiation [31–33], and sophisticated multi-loop integration technologies including reverse unitarity [34–37], differential equations [38, 39], and integration-by-parts reduction [40, 41] quickly produced state-of-the-art results for the conservative two-body potential in the PM approximation. In parallel, completely new schemes based on quantum mechanics and quantum field theory led to the observable-centric approach of Kosower, Maybee, and O'Connell (KMOC) to classical physics [42]. This framework is particularly attractive because it consistently includes both potential and radiation effects. In certain cases, results obtained for hyperbolic orbits can be translated directly to the relevant inspiral (elliptic or circular) orbits either via the EFT potential or by analytic continuation of observables [43] via the boundary-to-bound map.

In this work, we extend the KMOC framework to an additional classical observable: the total radiative angular momentum loss during a two-body scattering process. The definition of angular momentum in general relativity involves subtleties well known within the mathematical GR community, where they have been related to the asymptotic symmetry structure of spacetime, BMS translations, and the choice of Bondi frames [44]. Within the last decade, Strominger and collaborators have established a relationship between the asymptotic symmetries of Minkowski space and the soft theorems of scattering amplitudes [45, 46]. It is therefore not unexpected that zero-energy ("soft") particles (gravitons or photons) play an important role in our amplitudes-based computations. As Weinberg pointed out in the 1960's, soft-particles couple universally [47], which allows them to be exponentiated and factored out of the scattering process to all orders. We therefore find it convenient to consider separately the soft and hard (finite-energy) contributions to classical quantities. The soft contribution to the angular momentum loss has already been considered in [48]. In our computations, we employ modern multi-loop computation tools, such as generalized unitarity [49–51] and the method of regions [52] for the classical expansion in conjunction with integration-by-parts relations and differential equations methods for the evaluation of loop integrals. Luckily, up to two-loop order, all relevant master integrals have already been evaluated, e.g. [53]. Our results for the angular momentum loss agree with [54].

The radiated angular momentum and energy-momentum are valuable observable proxies for gravitational waveform generation. Up to certain complications related to the "tail effect", these classical observables can be directly analytically continued from unbound to bound orbits [43]. More generally, bound and scattering dynamics are connected by the effective equations of motion for the classical binary system. Working in the Hamiltonian framework, one can encode the conservative dynamics in an effective Hamiltonian for the two-body system, eliminating (or "integrating out") field degrees of freedom [55]. By introducing a dissipative "driving force" to the Hamiltonian equations of motion (the *radiation reaction force*), one can also include the dissipative effects of gravitational-wave radiation on the binary system. A PM approximation to the effective Hamiltonian has been derived from the radial action in terms of scattering amplitudes [56–58]. Similarly, a PM approximation to the radiation reaction force can be derived from the radiated energy and angular momentum [54].

We will begin by reviewing conservative and dissipative aspects of gravitational binary dynamics, as well as their connection to gravitational scattering amplitudes in Chapter 3. In Chapter 4, we derive a KMOC-like formula for the radiative angular momentum loss and evaluate all contributions in General Relativity through order G^3. As a consistency check of our setup, we recompute the radiated linear momentum, finding full agreement with the literature [53, 59]. In Chapter 5, we extract expressions for the instantaneous energy and angular-momentum fluxes to 3PM order from the corresponding radiative loss observables. Using these fluxes, we extend the 2PM calculation of the relative radiation reaction force $\boldsymbol{F}_{\mathrm{rr}}$ from [54] to 3PM order.

Together with the 3PM effective Hamiltonian computed in [56], the 3PM radiation reaction force furnishes, for the first time, complete 3PM equations of motion for the relative binary degree of freedom,

$$\dot{\boldsymbol{r}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}} + \boldsymbol{F}_{\mathrm{rr}}, \tag{2.2}$$

where the 3PM relative Hamiltonian from [56],

$$\mathcal{H}[\vec{c}, m_1, m_2](\boldsymbol{r}, \boldsymbol{p}) = \sqrt{m_1^2 + |\boldsymbol{p}|^2} + \sqrt{m_2^2 + |\boldsymbol{p}|^2} + \sum_{n=1}^{\infty} \frac{c_n(|\boldsymbol{p}|^2)}{m^n} \left(\frac{Gm}{|\boldsymbol{r}|}\right)^n, \tag{2.3}$$

is given in terms of instantaneous coefficients c_1, c_2, c_3, which have been reproduced in Eq. (3.38), and the new ingredient, $\boldsymbol{F}_{\rm rr}$ to 3PM order, can be written in polar coordinates as

$$\boldsymbol{F}_{\rm rr} = F_r \mathbf{e}_r + \frac{F_\phi}{r} \mathbf{e}_\phi, \tag{2.4}$$

where the radial coefficient is given by

$$\begin{aligned} F_r =& \left(\frac{Gm}{r}\right)^2 \left(\frac{1}{r p_r}\right) \\ &\times \left(\frac{J}{r}\right)^2 \left(A_r^{(2)} + B_r^{(2)} \log\left(\frac{1+\sigma}{2}\right) + C_r^{(2)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \\ &+ \left(\frac{Gm}{r}\right)^3 \left(\frac{1}{r p_r}\right) \\ &\times \Bigg[m^2 \left(A_{r,1}^{(3)} + B_{r,1}^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_{r,1}^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \\ &+ \left(\frac{J}{r}\right)^2 \left(A_{r,2}^{(3)} + B_{r,2}^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_{r,2}^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \Bigg], \end{aligned} \tag{2.5}$$

with coefficients A_r, B_r, C_r defined in Eqs. (5.36), (5.37). The angular component is $(F_\phi/r)\boldsymbol{e}_\phi$, where

$$\begin{aligned} F_\phi =& \left(\frac{Gm}{r}\right)^2 \left(\frac{J}{r}\right) \left(A_\phi^{(2)} + B_\phi^{(2)} \log\left(\frac{1+\sigma}{2}\right) + C_\phi^{(2)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \\ &+ \left(\frac{Gm}{r}\right)^3 \left(\frac{J}{r}\right) \left(A_\phi^{(3)} + B_\phi^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_\phi^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right), \end{aligned} \tag{2.6}$$

with coefficients A_J, B_J, C_J defined in Eqs. (5.39) and (5.40). We conclude in Chapter 6 by cross-checking our results against post-Newtonian expansions of the instantaneous fluxes from the literature through 3PN (relative) order.

Chapter 3

GRAVITATIONAL BINARY DYNAMICS

3.1 Introduction to Gravitational Binary Dynamics

In principle, the complete classical binary dynamics of point-like, non-spinning masses can be deduced from an action containing the Einstein-Hilbert action (describing the gravitational field) minimally coupled to a pair of massive world-line actions (describing the particles), as in

$$S[x_a, g] = -\int m_1\sqrt{-g_{\mu\nu}(x_1)dx_1^\mu dx_1^\nu} - \int m_2\sqrt{-g_{\mu\nu}(x_2)dx_2^\mu dx_2^\nu} + S_{\mathrm{EH}}[g_{\mu\nu}]. \tag{3.1}$$

The associated Euler-Lagrange equations are a coupled, non-linear system containing the geodesic equations for the world-lines $x_a^\mu(\tau)$ (where τ parameterizes time) and the Einstein field equations with the distributional world-line stress-energy tensors $T_a^{\mu\nu}$ as sources. The field equations may then be solved perturbatively for $g^{\mu\nu} = \eta^{\mu\nu} + h^{\mu\nu}$, where $h^{\mu\nu}$ is a function of the world-lines $x^\mu(\tau)$ and their derivatives and $\eta^{\mu\nu}$ denotes the background Minkowski metric. Inserting the resulting metric into the geodesic equations yields equations of motion for the massive particles depending only on the $x^{\mu\nu}$ and their derivatives, thus reducing the problem to classical particle mechanics [55].

The resulting equations are complicated by two properties of the gravitational field. First, because the binary system will generate gravitational radiation, the particle dy-

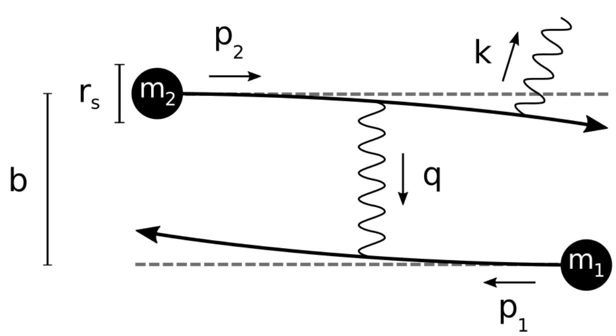

Figure 3.1.1: A diagram of unbound binary scattering. Binary constituents of masses m_1 and m_2, initially separated by impact parameter b, exchange momentum q and radiate momentum k into gravitational waves.

namics will not be conservative. Energy, momentum, and angular momentum will be radiated away from the binary system to future null infinity. This radiative dissipation frustrates an exact embedding of the equations of motion into a Lagrangian or Hamiltonian system without introducing additional degrees of freedom. However, we will see that we may formulate an effective Hamiltonian and augment Hamilton's equations of motion with an effective *radiation reaction force* acting on the massive particles and encoding this dissipation. Second, even with a radiation reaction force controlling dissipation, the effective Hamiltonian will not be local in time [58]. Just as charges may be transferred from the binary system into gravitational radiation, they may also return to the binary system at later times, introducing to the effective Hamiltonian potential a dependence on the particle dynamics at earlier times[1]. The first complication, dissipation through the radiation reaction force, is the focus of the present work. Luckily, we will avoid the second because the tail effects are only present at higher perturbative orders than those we will consider here. Before we begin our calculations, we must say more about our perturbation scheme.

Post-Minkowskian and Post-Newtonian Approximation

We will make use of two weak-field approximations at various stages of our calculation. The "post-Minkowskian", or PM approximation is a straightforward perturbation of the spacetime metric against a Minkowski background,

$$g^{\mu\nu} = \eta^{\mu\nu} + \alpha h^{\mu\nu}, \tag{3.2}$$

where $\alpha \ll 1$ (not to be confused with the electromagnetic fine-structure constant appearing in Chapter 4) is a dimensionless parameter controlling the relative size of terms of order h^n in an expansion about the η metric. Importantly, this approximation is fully relativistic at all orders. We treat GR effects as corrections to special relativity, but do not make any assumptions about the sizes of particle velocities relative to the speed of light. In the case of two-particle interactions, the size of the metric perturbation h, and therefore the strength of the gravitational field, is controlled by the distance between the two particles. The further apart they remain, the weaker their gravitational interaction becomes. The dimensionless parameter α is thus given as the ratio of two length scales: the effective Schwarzchild radius R_s

[1]Temporal nonlocality is by no means unique to gravitational dynamics. Even in the electromagnetic case, the presence of moving charges leads to the Liénard-Wiechert potentials, which depend on particle velocities evaluated at appropriately *retarded* times. These times are, however, simple functions of the distances between the charges and the speed of light. The form of the nonlocality in the gravitational case is much more complicated, owing to the self-interactions of the metric which give rise to "tail effects" [58, 60–64]

of the binary system (of total mass $m = m_1 + m_2$) and the distance r between the particles.

$$\frac{R_s}{r} = \frac{2Gm}{r} \sim \frac{Gm}{r} = \alpha. \tag{3.3}$$

The validity of the PM approximation requires that $R_s \ll r$ *throughout* the evolution of the binary. In the case of unbound orbits (hyperbolic scattering), the distance of closest approach, or *periapsis*, is controlled by the initial transverse separation, or *impact parameter* b (see 3.1). If we demand $R_s \ll b$, then expansion in $\alpha = Gm/b$ is justified. Binaries in bound orbits, on the other hand, draw closer together as they lose energy and momentum to gravitational radiation. If ℓ denotes the periapsis of a bound orbit which is initially approximately elliptical, then expansion in $\alpha = Gm/\ell$ is justified in the early phases of the inspiralling orbit. Sooner or later, the particles will come close enough that the weak-field assumption ceases to apply.

We will also employ the "post-Newtonian", or *PN* approximation. This is essentially the simultaneous application of the PM approximation and a non-relativistic (NR) approximation, in which particle velocities ($v = p/E$) are assumed to be small relative to the speed of light. This assumption introduces another dimensionless parameter v/c[2]. We work in units where $c = 1$, so this parameter is simply v, and we will assume $v \ll 1$. Given $\alpha, v \ll 1$, we can perform two-variable Taylor expansions of arbitrary quantities. The PN expansion collects subsets of terms in the simultaneous PM/NR expansion by introducing a power-counting parameter η (not to be confused with the Minkowski metric) and ascribing η scalings to α and v. In particular, $v \propto \eta$ and $\alpha \propto \eta^2$. If these powers seem arbitrary, recall that the virial theorem implies proportionality between the (time averaged) kinetic and potential energies of a stable system. In the context of Newtonian gravity, that is (up to an overall mass factor) $v^2 \propto Gm/r$, which yields the desired η scalings.

The PN approximation is attractive because the two-parameter expansion greatly simplifies intermediate expressions. This is especially true of relativistic integrals, where complicated integrands involving factors of $\sqrt{p^2 + m^2}$ are reduced to rational functions by the $p/m \ll 1$ NR expansion. The drawback of the PN approximation becomes apparent in highly eccentric bound orbits, where initially small velocities may approach the speed of light at periapsis. Hyperbolic scattering shares this pathology. In these cases, we must retreat to the PM expansion, which is applicable at all velocities.

[2]For sufficiently small velocities, $v \sim p/m$, so that $v/c \sim p/(mc) \ll 1$. This is the ratio we will actually use when we perform PN expansions later on.

Post-Newtonian approximation is a mature technique, and has been used to compute many quantities of interest in gravitational binary dynamics to staggering precision. Owing to greater technical difficulty, especially in integration, Post-Minkowskian approximations have not enjoyed the same popularity. Recently, however, PM expansions have been obtained for certain quantities of binary gravitational systems, including an effective Hamiltonian to 3PM[3] [56] and partial results for the 4PM Hamiltonian [58].

	0PN	1PN	2PN	3PN	4PN	5PN	6PN
1PM	G	Gv^2	Gv^4	Gv^6	Gv^8	Gv^{10}	Gv^{12}
2PM		G^2	G^2v^2	G^2v^4	G^2v^6	G^2v^8	G^2v^{10}
3PM			G^3	G^3v^2	G^3v^4	G^3v^6	G^3v^8
4PM				G^4	G^4v^2	G^4v^4	G^4v^6

Table 3.1.1: Terms in the simultaneous PM/PN expansions through 4PM and 6PN orders.

Conservative and Dissipative Effects in the Binary Equations of Motion

The dynamics of a gravitational binary system (bound or unbound) may be separated into conservative and dissipative parts by first introducing an effective external force and treating it as a driving force in the equations of motion,

$$M_{ab}\ddot{\boldsymbol{x}}_b(t) = \boldsymbol{F}_{\text{eff},a}[\boldsymbol{x}_1, \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2](t), \tag{3.4}$$

where $a, b \in \{1, 2\}$ denote particle 1 or 2, and $M_{ab} = \delta_{a1}\delta_{b1}m_1 + \delta_{a2}\delta_{b2}m_2$ is the mass matrix. Critically, the effective force $\boldsymbol{F}_{\text{eff}}$ is not simply a function of the particle positions $\boldsymbol{x}(t)$ and velocities $\dot{\boldsymbol{x}}(t)$ at some instant t. Rather, it has *functional* dependence on $\boldsymbol{x}$ and $\dot{\boldsymbol{x}}$ as functions of time. This is a manifestation of the temporal nonlocality of forces mediated by fields with finite group velocities.

We will split the effective force into a conservative contribution $\boldsymbol{F}_{\text{cons}}$ and a dissipative or *radiation reaction* contribution $\boldsymbol{F}_{\text{rr}}$. We then write the equation of motion as

$$M_{ab}\ddot{\boldsymbol{x}}_b(t) - \boldsymbol{F}_{\text{cons},a} = \boldsymbol{F}_{\text{rr},a}. \tag{3.5}$$

[3] nPM/nPN order is shorthand for "n^{th}" post-Minowskian/Newtonian order. An nPM expansion contains all terms of order up to and including G^n (α^n), while an nPN expansion runs through $\eta^{2(n+1)}$ (0PN corresponding to the familiar mv^2, Gm/r terms from Newtonian mechanics).

At any PM order, $\boldsymbol{F}_{\text{cons}}$ may be encoded in an effective Hamiltonian potential (though it may not be local in time) so that the entire left-hand side of Eq. (3.5) can be recovered from the associated Hamiltonian equations. The dissipative radiation reaction force cannot be embedded in a potential, local or otherwise, and will remain explicitly in the equations of motion as a driving force. At G^3 order, to which we will restrict our amplitudes calculations, we may ignore all temporally nonlocal effects, so that $\boldsymbol{F}_{\text{rr}}$ is a function only of the instantaneous positions and velocities, with no explicit time dependence.

As is usually the case in orbital mechanics, it is advantageous to separate relative and center-of-mass degrees of freedom by transforming coordinates to the center-of-momentum (COM) frame, where $\boldsymbol{p}_1 + \boldsymbol{p}_2 = 0$. The natural coordinates in this frame are the relative separation $\boldsymbol{r}$ and the COM displacement $\boldsymbol{R}$. The transformations relating $\boldsymbol{r}, \boldsymbol{R}$ and $\boldsymbol{p_1}, \boldsymbol{p_2}$ are generally complicated by relativistic effects and the acceleration of the COM frame due to the radiation reaction force, also known as the *recoil*. We will ultimately compute directly in the COM coordinates, and thus we will not need explicit expressions for these transformations. Even so, we have found it useful to compare our COM coordinates to the simplest example from classical orbital mechanics to build intuition.

In the classic case of a non-relativistic binary system interacting through a conservative force depending only on the relative displacement, the COM transformation trivializes the dynamics of the center of momentum,

$$m_a \ddot{\boldsymbol{x}}_a = \boldsymbol{F}(\boldsymbol{x_1}, \boldsymbol{x_2}) \quad \rightarrow \quad m\ddot{\boldsymbol{R}} = 0, \quad \mu \ddot{\boldsymbol{r}} = \boldsymbol{F}(\boldsymbol{r}). \tag{3.6}$$

where $m = m_1 + m_2$ is the total mass, and $\mu = m_1 m_2/(m_1 + m_2)$ is the reduced mass. The vanishing of the COM force is a consequence of (1) Poincaré invariance of the equations of motion and (2) the assumption that the force depends on particle positions, but not velocities. Translation invariance immediately implies that $\boldsymbol{F}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \boldsymbol{F}(\boldsymbol{x}_2 - \boldsymbol{x}_1)$, and rotational invariance further stipulates $\boldsymbol{F}(\boldsymbol{x}_2 - \boldsymbol{x}_1) = \boldsymbol{F}(|\boldsymbol{x}_2 - \boldsymbol{x}_1|)$. The force $\boldsymbol{F}_{\text{cons}}$ is conservative and local-in-time. If it is also independent of velocities, we may construct a Poincaré invariant Lagrangian $\mathcal{L}$ such that

$$\boldsymbol{F}_{\text{cons,a}}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\partial \mathcal{L}}{\partial \boldsymbol{x}_a}. \tag{3.7}$$

A global coordinate translation $\boldsymbol{x}_a \rightarrow \boldsymbol{x}_a + \boldsymbol{\xi}$ in the COM coordinates takes the form $\boldsymbol{R} \rightarrow \boldsymbol{\xi}$, $\boldsymbol{r} \rightarrow \boldsymbol{r}$. Thus, translation invariance implies independence of $\mathcal{L}$ from R in the COM frame, ergo

$$\boldsymbol{F}_{\text{cons},R} = \frac{\partial \mathcal{L}}{\partial \boldsymbol{R}} = 0. \tag{3.8}$$

The actual binary equations of motion are complicated by the fact that the conservative and radiation-reaction forces depend on both positions and velocities. It is true that translational invariance implies

$$\boldsymbol{F}_{\text{a}}(\boldsymbol{x}_1, \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2) = \boldsymbol{F}_{\text{a}}(\boldsymbol{x}_2 - \boldsymbol{x}_2, \dot{\boldsymbol{x}}_1, \dot{\boldsymbol{x}}_2) \tag{3.9}$$

in both cases. However, because boost invariance here is Lorentzian, not Galilean, it is not the case that velocity dependence is restricted to the difference $\dot{\boldsymbol{x}}_2 - \dot{\boldsymbol{x}}_1$. Consequently, both forces are independent of $\boldsymbol{R}$, but not $\dot{\boldsymbol{R}}$. Rotational invariance then restricts the coordinate dependence of the force to the inner products of $\boldsymbol{r}$, $\dot{\boldsymbol{R}}$, $\dot{\boldsymbol{r}}$,

$$\boldsymbol{F}_{\text{a}}(\boldsymbol{R}, \boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) = \boldsymbol{F}_{\text{a}}\left(|\boldsymbol{r}|, |\dot{\boldsymbol{R}}|, |\dot{\boldsymbol{r}}|, (\boldsymbol{r} \cdot \dot{\boldsymbol{R}}), (\boldsymbol{r} \cdot \dot{\boldsymbol{r}}), (\dot{\boldsymbol{R}} \cdot \dot{\boldsymbol{r}})\right). \tag{3.10}$$

However, the independence of the forces from the *coordinate* $\boldsymbol{R}$ no longer implies that the forces have no component along the *coordinate vector* $\partial/\partial \boldsymbol{R}$. This is seen easily in the conservative contribution, where the Lagrangian expression for the component of $\boldsymbol{F}_{\text{cons}}$ along this vector is no longer simply $\partial \mathcal{L}/\partial \boldsymbol{R}$, but contains $\partial/\partial \dot{\boldsymbol{R}}$ dependence. It follows that $\ddot{\boldsymbol{R}}$ will not generally vanish[4].

Applying these observations to our forces in the COM frame yields

$$\begin{aligned} m\ddot{\boldsymbol{R}} - \boldsymbol{F}^{COM}_{\text{cons}}(\boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) &= \boldsymbol{F}^{COM}_{\text{rr}}(\boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) \\ \mu\ddot{\boldsymbol{r}} - \boldsymbol{F}^{rel}_{\text{cons}}(\boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) &= \boldsymbol{F}^{rel}_{\text{rr}}(\boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}). \end{aligned} \tag{3.11}$$

This isn't a trivial simplification, but it is a far cry from a one-body equation of motion. As it turns out, the "mixing" of the COM and relative degrees of freedom in the arguments of the forces does not appear at G^3 order [65]. That is,

$$\boldsymbol{F}^{rel/COM}(\boldsymbol{r}, \dot{\boldsymbol{R}}, \dot{\boldsymbol{r}}) = \boldsymbol{F}^{rel/COM}_{G^3}(\boldsymbol{r}, \dot{\boldsymbol{r}}) + O(G^4). \tag{3.12}$$

Through G^3 order, then, the equations of motion for the binary system decouple[5],

$$m\ddot{\boldsymbol{R}} - \boldsymbol{F}^{COM}_{\text{cons}}(\dot{\boldsymbol{R}}) = \boldsymbol{F}^{COM}_{\text{rr}}(\dot{\boldsymbol{R}}) \tag{3.13}$$

$$\mu\ddot{\boldsymbol{r}} - \boldsymbol{F}^{rel}_{\text{cons}}(\boldsymbol{r}, \dot{\boldsymbol{r}}) = \boldsymbol{F}^{rel}_{\text{rr}}(\boldsymbol{r}, \dot{\boldsymbol{r}}). \tag{3.14}$$

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[4] Consider the case of a binary system moving through some isotropic goo. The drag force will clearly retard both COM and relative motion in (non-linear, but monotonically increasing) proportion to COM and relative velocities, respectively.

[5] In fact, the conservative COM force component vanishes completely, but we have kept the expression here for clarity.

Ultimately, we will evaluate the radiation reaction forces by comparing them to total radiative energy losses. By working in the initial center-of-mass frame, we may ignore the *recoil* force $F_{\mathrm{rr}}^{COM}(\dot{\boldsymbol{R}})$ and attribute radiative losses of energy and angular momentum entirely to the relative radiation reaction force $F_{\mathrm{rr}}^{\mathrm{rel}}$. Of course, the recoil force will deflect the COM trajectory at higher PM orders, so that the actual instantaneous COM frame is accelerated relative to the initial COM frame during the scattering process. Physically speaking, it is exactly this deflection which causes the mixing of relative and COM degrees of freedom in the full equations of motion. At G^3 order, however, we may safely ignore the COM degrees of freedom. Thus, in the second line of Eq. (3.13), we have our equation of local-in-time, dissipative one-body dynamics where both the conservative and dissipative forces depend only on relative displacement and velocity.

Hamiltonian Formulation

We chose to introduce the binary dynamics in the language of Newtonian mechanics for the sake of conceptual clarity. Actual calculations are generally performed in either Lagrangian or Hamiltonian formalism. Most of the recent work extracting PM dynamics from scattering amplitudes has used the Hamiltonian approach. We will follow suit. Given an expression for the effective Hamiltonian $\mathcal{H}(\boldsymbol{q}_a, \boldsymbol{p}_a)$ in terms of the positions and momenta of the binary constituents in an arbitrary coordinate system, one can construct a canonical transformation to COM coordinates $(\boldsymbol{q}_a, \boldsymbol{p}_a) \to (\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{p}, \boldsymbol{P})$, where $\boldsymbol{p}$ and $\boldsymbol{P}$ are the momenta canonically conjugate to $\boldsymbol{r}$ and $\boldsymbol{R}$, respectively[55].

The corresponding Hamiltonian equations are then equivalent to the equations (3.1), with the dissipative forces $F_{\mathrm{rr}}^{\mathrm{rel/COM}}$ set to zero. Dissipative effects cannot easily be included by modifying the Hamiltonian itself without introducing auxiliary degrees of freedom. Instead, following [66], we augment the Hamiltonian equations for $(\dot{\boldsymbol{p}}, \dot{\boldsymbol{P}})$ derived from $\mathcal{H}$ directly with the radiation reaction force, *viz.*

$$(\dot{\boldsymbol{r}}, \dot{\boldsymbol{R}}) = \frac{\partial \mathcal{H}}{\partial(\boldsymbol{p}, \boldsymbol{P})}, \quad (\dot{\boldsymbol{p}}, \dot{\boldsymbol{P}}) = -\frac{\partial \mathcal{H}}{\partial(\boldsymbol{r}, \boldsymbol{R})} + (\boldsymbol{F}_{\mathrm{rr}}^{\mathrm{rel}}, \boldsymbol{F}_{\mathrm{rr}}^{\mathrm{COM}})(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{p}, \boldsymbol{P}) \tag{3.15}$$

The appearance of the force in the $\dot{p}, \dot{P}$ equation, but not the $\dot{r}, \dot{R}$ equation corresponds to the definition of force as the rate-of-change of momentum. Not surprisingly, the relative and COM contributions to both the effective Hamiltonian $\mathcal{H}$ and the radiation-reaction force $\boldsymbol{F}_{\mathrm{rr}}$ decouple through G^3 order [65], such that neither

$\mathcal{H}^{\text{rel}}$ nor $\boldsymbol{F}_{\text{rr}}^{\text{rel}}$ depend on the COM phase-space coordinates $(\boldsymbol{R}, \boldsymbol{P})$, and *vice versa*,

$$\mathcal{H}(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{p}, \boldsymbol{P}) = \mathcal{H}_{\text{rel}}(\boldsymbol{r}, \boldsymbol{p}) + \mathcal{H}^{\text{COM}}(\boldsymbol{R}, \boldsymbol{P}) \tag{3.16}$$
$$(\boldsymbol{F}_{\text{rr}}^{\text{rel}}, \boldsymbol{F}_{\text{rr}}^{\text{COM}})(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{p}, \boldsymbol{P}) = (\boldsymbol{F}_{\text{rr}}^{\text{rel}}(\boldsymbol{r}, \boldsymbol{p}), \boldsymbol{F}_{\text{rr}}^{\text{COM}}(\boldsymbol{R}, \boldsymbol{P})).$$

Henceforth, we will focus only on the relative degrees of freedom, dealing with the Hamiltonian equations of a single effective body,

$$\dot{\boldsymbol{r}} = \frac{\partial \mathcal{H}}{\partial \boldsymbol{p}}, \quad \dot{\boldsymbol{p}} = -\frac{\partial \mathcal{H}}{\partial \boldsymbol{r}} + \boldsymbol{F}_{\text{rr}}, \tag{3.17}$$

where we have dropped the explicit rel tags. Throughout the rest of this work, we will assume that $\mathcal{H}$ and $\boldsymbol{F}_{\text{rr}}$ refer to the relative effective Hamiltonian and the relative, dissipative radiation reaction force, respectively.

3.2 Conservative Binary Dynamics

Through G^3 order, the complete relative dynamics are encoded in $\mathcal{H}$ and $\boldsymbol{F}_{\text{rr}}$ by way of Eq. (3.17). Both quantities can be extracted from scattering amplitudes. We will first review the extraction of the classical effective Hamiltonian from the scattering amplitudes of perturbative quantum gravity. Two approaches to this task have found considerable success in extending the state-of-the-art post-Minkowskian approximations to $\mathcal{H}$. The first makes use of an effective field theory of interacting massive scalar particles to match an interaction potential directly to the scattering amplitudes of the gravitational theory described by the action from Eq. (3.1). This EFT technique was used to extract the G^2 (1PM) correction to the Newtonian Hamiltonian[67], and later the G^3 (2PM) correction[56, 68]. The second approach exploits a gauge-invariant "action-amplitude" relationship between the two-to-two gravitational scattering amplitudes of massive scalars and the *radial action* to derive a classical expresion for this action. The effective Hamiltonian may then be constructed by matching a post-Minkowskian ansatz to the classical radial action. This method yielded the local-in-time part of the G^4 effective Hamiltonian[57] and the radial action I_R including tail effects at G^4 order[58].

Both approaches operate on the principle that gravitational scattering amplitudes, and gauge-invariant observables constructed from them, contain information about classical gravitational dynamics which can be extracted by taking the classical limit. This apparently circuitous approach is advantageous because it leverages sophisticated computational tools which have been developed for the construction and integration of loop integrands. As we will see, PM order is directly related to loop order, so these tools are perfectly suited to high-precision PM approximation.

The action-amplitude calculation of the effective Hamiltonian is closely related to our calculation of the radiation reaction force, so we will take the time to review it here before proceeding to the dissipative dynamics. Both calculations proceed in four phases

1. Identify a gauge-invariant quantity which can be expressed in terms of gravitational scattering amplitudes.
2. Determine which amplitude topologies (either off-shell diagrams or on-shell cuts) contribute to the gauge-invariant quantity in the classical limit, and discard all others.
3. Apply automated integrand construction and loop integration techniques to the surviving topologies, and take their classical limit to obtain the classical gauge-invariant quantity.
4. Construct an ansatz for the classical expression of interest (e.g. the effective Hamiltonian or the radiation reaction force), derive the classical gauge-invariant quantity in terms of this ansatz, and match the result to the result of the previous step order-by-order in G.

The Radial Action from the Conservative Amplitude

In the conservative case, the gauge-invariant quantity is the radial action

$$I_R(J) = 2\int_{r_{\rm min}}^{\infty} dr\, p_r, \tag{3.18}$$

where $p_r = \boldsymbol{p}\cdot\boldsymbol{r}/r$ is the radial component of the relative momentum, and the integral is taken over the entire scattering trajectory, from asymptotic past to asymptotic future. The argument, J, is the magnitude of the initial angular momentum. The authors of[57] elaborate on the gauge-invariant equality

$$i\mathcal{M}^{\rm cons}(\boldsymbol{q}) = 4E|\boldsymbol{p}|\mu^{-2\epsilon}\int d^{D-2}\boldsymbol{b}\, e^{i\boldsymbol{b}\cdot\boldsymbol{q}}\left(e^{iI_r(J)} - 1\right) \tag{3.19}$$

in [69]. Let's pause to unpack Eq. (3.19). The *conservative amplitude* $\mathcal{M}^{\rm cons}$ is the (fully quantum) scattering amplitude associated to the two-to-two hyperbolic scattering process depicted in Figure 3.1 of Section 3.1. The total momentum transfer during the scattering process is denoted q^μ, which in the CM frame is purely spatial: $q^i = -\boldsymbol{q}^i$. The angular momentum, $J = |\boldsymbol{J}|$ is given in terms of the impact parameter $\boldsymbol{b}$ and the initial CM relative three-momentum $\boldsymbol{p}_\infty$ as $J = |\boldsymbol{b}||\boldsymbol{p}_\infty|$.

The total energy in the CM frame is denoted E, and given in terms of $m_1, m_2, |\boldsymbol{p}|$. For reference, these kinematic quantities are summarized in Table 6.A of Appendix 6.A. The integral over impact parameter $\boldsymbol{b}$ corresponds to an integral over all possible angular momenta of the binary system. μ is the renormalization scale, and ϵ is the dimensional regulation parameter.

Quantum, Classical, and Superclassical Contributions

The second step towards the effective Hamiltonian involves isolating the classical dynamics encoded in Eq. (3.19). In principle, one could compute the entire amplitude up to the desired order in G, solve for the (quantum) radial action by expanding Eq. (3.19) in powers of G, and finally take the classical limit $\hbar \to 0$. Retaining fully quantum expressions all the way through the calculation of the amplitude is problematic because it requires one to construct and integrate loop integrands which ultimately vanish in the classical limit. In fact, the computational bottleneck in this amplitudes approach is in loop integrand construction and integration, even with state-of-the-art software. One must discard quantum contributions to the amplitude as early as possible, even before performing any loop integrals.

To understand the decomposition of amplitudes into classical and quantum contributions, we will first couch the classical limit in terms of a separation of physical energy scales in the quantum scattering problem[6]. In this language, it becomes clear that the same techniques used in effective field theory to isolate the low energy physics of an interacting system can be applied to gravitational amplitudes to isolate classical physics. What are the scales inherent to our scattering problem? In the classical theory, there are two: the impact parameter, b, and the Schwarzchild radius of the total binary mass, $r_s = 2Gm \sim Gm$[7], where, as usual, we work in units where $c = 1$. As we discussed in Section 3.1, the post-Minkowskian approximation is a weak-field expansion in the *ratio* of these two distance scales: $Gm/b \ll 1$.

In the quantum theory, the additional distance scale entering the problem is the Compton wavelength associated to the total binary mass, $\lambda = \hbar/m$. Just as the post-Minkowskian approximation arose from the assumption that the scale at which gravitation becomes strongly non-linear (r_s) is much smaller than the minimal separation of the two particles (order b), so the semi-classical approximation arises

[6]For a thorough review, see [42] and [56]

[7]One might worry that the impact parameter should be compared to the *reduced* mass, not the total mass. Because we only require that the impact parameter be much larger than the Schwarzchild scale, $b \gg Gm$ implies $b \gg G\mu$, as $m > \mu$.

from the assumption that the scale at which quantum mechanical corrections to classical GR become relevant is much smaller than all other scales in the problem. That is to say, $\lambda/b = \hbar/bm \ll 1$ and $\lambda/r_s = \hbar/Gm^2 \ll 1$. Of course, one ratio can always be expressed in terms of the other two,

$$\frac{\lambda}{b} = \frac{\lambda}{r_s}\frac{r_s}{b}, \tag{3.20}$$

establishing the hierarchy of scales shown in Table 3.2 These three length scales

	Compton λ		Schwarzchild r_s		Impact b
Length	$\hbar/m$	$\ll$	Gm	$\ll$	b
Energy	m	$\gg$	$\hbar/Gm$	$\gg$	$\hbar/b$

Table 3.2.1: The hierarchy of scales associated to the classical limit of the weak-field (PM) approximation to binary graviational dynamics.

are useful for thinking visually about the scattering process. However, in an actual scattering experiment, one has control over *just two* independent variables relevant to this hierarchy: m and b. There are also just two independent ratios controlling the hierarchy. The first ratio, α, controls the PM expansion and depends on both the impact parameter b and the total binary mass m,

$$\alpha = \left(\frac{r_s}{b}\right) \sim \left(\frac{Gm}{b}\right) \ll 1. \tag{3.21}$$

The second ratio, which we will call β, is controlled entirely by the total mass m,

$$\beta = \left(\frac{\lambda}{r_s}\right) \sim \frac{\hbar/bm}{Gm/b} = \left(\frac{m_P}{m}\right)^2 \ll 1, \tag{3.22}$$

where m_P is the Planck mass. We may now express our three scales as

$$b, \qquad r_s = \alpha b, \qquad \lambda = \alpha\beta b. \tag{3.23}$$

When $\alpha < 1$, β controls the semi-classical expansion. Because we work with momentum-space amplitudes, it is useful to think instead about the dual mass (energy) scales

$$m_b = \frac{1}{b} \quad m_s = \frac{1}{r_s} \quad m = \frac{1}{\lambda}, \tag{3.24}$$

in units where $\hbar = 1$. In terms of α, β,

$$m_b = \alpha\beta m, \quad m_s = \beta m, \quad m. \tag{3.25}$$

The mass hierarchy $m_b \ll m_s \ll m$ describing the weak-field semi-classical regime thus corresponds simply to $\alpha, \beta \ll 1$. The classical limit is taken by sending β to zero. For all binary systems of astrophysical interest, e.g. inspiraling pairs of solar-mass black holes, the total mass is much larger than the Planck mass, so we are justified in assuming $\beta \ll 1$.

As a brief aside, just as "expansion in G" is shorthand for "expansion in the ratio α", so the assertion "the classical limit, where $\hbar \to 0$" is shorthand for "the classical limit, where $\hbar/Gm^2 \to 0$ and $\hbar/bm \to 0$, or $\beta \to 0$. We cannot really change $\hbar$ or G, but because $\alpha \propto G$ and $\beta \propto \hbar$, these constants can be used to track powers of α, β implicitly.

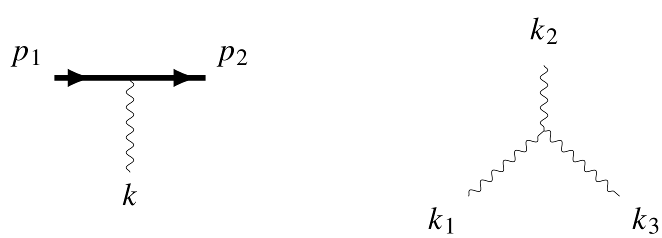
With our scales under control, we can be more precise about the expansion of the conservative amplitude. Using the same normalization as [56], the quantum action describing a pair of massive scalar fields ϕ_i of masses m_i, $i \in \{1,2\}$ minimally coupled to the graviton in D spacetime dimensions is given by

$$S_{\text{GR}} = \int d^D x \sqrt{-g} \left[-\frac{1}{16\pi G} R + \frac{1}{2} \sum_{i \in \{1,2\}} \left(D^\mu \phi_i D_\mu \phi_i - m_i^2 \phi_i^2 \right) \right], \tag{3.26}$$

where R is the Ricci scalar, g denotes the metric determinant, and D^μ is the covariant derivative. The minus sign in front of the Ricci scalar corresponds to our choice of a mostly-minus metric signature. The post-Minkowskian approximation introduces a weak perturbation $h^{\mu\nu}$ against a Minkowski background $\eta^{\mu\nu}$,

$$g^{\mu\nu} = \eta^{\mu\nu} + \kappa h^{\mu\nu}, \tag{3.27}$$

where $\kappa = \sqrt{32\pi G}$ is chosen so that the expansion of S_{GR} in $h^{\mu\nu}$ yields a canonically normalized kinetic term. That factors of $\sqrt{G}$ are carried with the field h is convenient, because it simplifies the power-counting of scale ratios in Feynman diagrams. The action S_{GR} is a "two-derivative" action, in the sense that all interaction operators involving any number of h fields contain exactly two partial derivatives. At cubic order, there are two vertices, shown in Eq. (3.28).

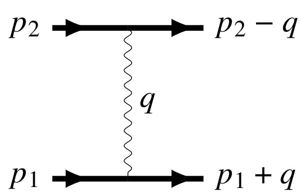


(3.28)

At higher orders, more gravitons appear at the vertex, but the number of scalars is always either zero or two. Recalling the overall $2/\kappa^2$ factor in the action, any vertex involving m particles (either scalars or gravitons) must scale as

$$V_m \sim \kappa^{m-2} P^2 \sim G^{(m-2)/2} P^2. \tag{3.29}$$

The P simply indicates a generic momentum scale. Generally, P will be a polynomial of Mandelstam invariants separately symmetric in the permutations of scalar momenta and graviton momenta. To power-count in the scale ratios, we need to consider specific diagram topologies. Because we are interested in *conservative* amplitudes, we will only consider topologies with a pair of scalars (ϕ_1, ϕ_2) in both the in and out states. Because the scalar particles are distinguishable, the only possible tree-level diagram is the t-channel,



(3.30)

The invariants appearing at the two vertices are $p_i^2 = m_i^2$, $p_i \cdot q$, and q^2. The momentum transfer q in momentum space corresponds to an impact parameter b in position space, so we may assume $b \sim 1/|q|$. Therefore, the three terms appearing in each vertex factor have scalings $\kappa p_i^2 \sim G^{1/2} m^2$, $\kappa p_i \cdot q \sim G^{1/2} m/b$, and $\kappa q^2 \sim G^{1/2} b^2$. Combining the vertices with the $1/q^2$ propagator and substituting $q = m_b = \alpha\beta m$, $(Gm)^{-1} \sim m_s = \beta m$ gives terms with scalings

$$\mathcal{M}^{\text{cons}}_{\text{tree}}(q) \sim m^2 \frac{(1 + \alpha\beta + \alpha^2\beta^2)^2}{\beta q^2}. \tag{3.31}$$

We have left the $1/q^2$ explicit because $\mathcal{M}(b)$ is related to $\mathcal{M}(q)$ by a transverse, two dimensional spatial Fourier transform against wavefunctions which restrict $q \sim 1/b$ (see [42]). The factor $d^2 q$ has the inverse scaling of the $1/q^2$ pole, so that $\mathcal{M}^{\text{cons}}(b)$ has an overall scaling $(1/\beta)(1 + \alpha\beta + \alpha^2\beta^2)^2$. The lowest order term in β scales as β^{-1}. The bare amplitude appears in observables dressed with a momentum factor scaling as $\alpha\beta$, so the leading term does, in fact, correspond to the classical contribution.

At higher loop orders, even larger negative powers of β will appear which are not cured in the observable by an overall factor of β but rather by cancellation. Such terms are generally referred to as "superclassical", and though they appear generically at all loop orders in the amplitudes, they do not contribute to the classical quantities extracted from them. In the context of the effective Hamiltonian, the irrelevance of such terms to classical quantities can be seen in two ways. Taking the EFT approach discussed in[67] and[56], the scalar-graviton amplitude $\mathcal{M}^{\text{cons}}$ is matched to a pure-scalar amplitude with an interaction potential encoding the effective Hamiltonian *in the classical limit*. Because fully quantum amplitudes appear on both sides of the matching equation, both yield β^{-p} superclassical terms in the semi-classical limit. Indeed, the same terms appear in both amplitudes, and cancel out of the matching equation. The remaining terms either scale as $\beta^{p>-1}$, in which case they are purely quantum corrections, and vanish in the $\beta \to 0$ limit, or they scale as β^{-1}, and are valid classical terms (thanks to the overall factor of β).

In the gauge-invariant observable approach[58], the radial action is related to the log of the conservative amplitude by Eq. (3.19). This mapping eliminates superclassical terms, leaving an expression for I_R which is nonsingular in the classical limit. We will extract the radiation reaction force from observables with the same property of classical regularity. Consequently, all contributions to the amplitude with superclassical ($\beta^{p<-1}$) *or* quantum ($\beta^{p>0}$) scalings appearing *at tree level* may be thrown away as soon as they are encountered. This analysis is straightforward for tree diagrams because all propagator momenta are uniquely determined by external kinematics, so the $\alpha - \beta$ expansion boils down to: first, identifying whether the Mandelstam invariants built from the momenta entering a given vertex are of order m^2 (p_1^2, p_2^2), order $m\alpha\beta$ ($p_1 \cdot q, p_2 \cdot q$), or order $m\alpha^2\beta^2$ (q^2); second, expanding the vertex kinematic factor as a polynomial in α, β; third, dressing the expression with propagator denominators (themselves polynomials in Mandelstam invariants, and therefore β; and finally, expanding the entire rational expression in β and keeping β^{-1} terms.

It is far less obvious how one should proceed at loop level, where loop momenta are integrated over the full, off-shell space $k^\mu \in \mathbb{R}^{\mathbb{D}}$. In different parts (or, suggestively, *regions*) of momentum space, the invariants $k \cdot p_i, k \cdot q, k^2$ will scale with different powers of α and β. Once all integrals have been performed, $\mathcal{M}^{\text{cons}}$ will once again become a function only of p_1, p_2, q, and expansion can proceed as it did in the tree level case (though the function will no longer be rational). The difficulties of loop integration, however, compel us to simplify the integrands as much as possible before doing any integrals. What we would really like to do, then, is to divide the full domain of loop momentum integration into *regions* where the Mandelstam invariants are well-separated in scale, and expand the integrands (which, remember, are rational functions of these invariants), in powers of the small ratios of these separated scales. This approach is known as the *method of regions*. We have relegated the details of this method to Appendix 6.B, where we illustrate the main ideas through a simple quantum-classical example integral[8].

The mass scales of our scattering problem are depicted in Figure 3.2. The three energy scales q, m_s, m are used to define three energy regions by choosing two arbitrary *separator* scales Λ_{PM} and Λ_Q, where $q < \Lambda_{PM} < m_s$ and $m_s < \Lambda_Q < m$. From low to high energies, these regions describe the classical weak-field dynamics (region I), the classical strong-field dynamics (region II), and the quantum strong-field dynamics (region III). the separators Λ_{PM} and Λ_Q are fictitious in the sense that

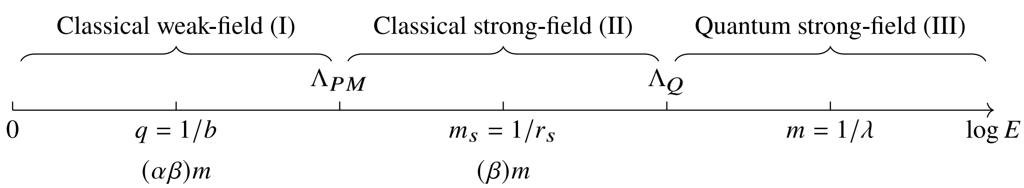


Figure 3.2.1: The energy regions of the binary scattering problem. Λ_{PM} and Λ_Q are the separators delimiting weak/strong-field and classical/quantum regions, respectively.

they drop out of all integrals in the final step, but they are very useful for dividing the energy (or more generally, momentum) space into integration regions containing each *physical* scale q, m_s, and m. The key feature of the integrals after expansion in a particular scale separation, e.g. $q \ll m$, is that their integrands decompose into Laurent series in the scale ratio $(q/m) \propto \alpha\beta$. We can then very quickly discard all quantum and superclassical terms, vastly simplifying the actual integration step.

For the sake of clarity, we will borrow an example integral from Appendix 6.B. Consider separating the following one-dimensional energy integral into classical and quantum regions,

$$I = \int_0^\infty dE \frac{E}{(E^2+q^2)(E^2+m^2)} = \left(\int_0^{\Lambda_Q} + \int_{\Lambda_Q}^\infty\right) dE \frac{E}{(E^2+q^2)(E^2+m^2)}. \tag{3.32}$$

Consulting the scale diagram in Figure 3.2 and noting that the energy scale is logarithmic, it is clear that the condition $E < \Lambda_Q$ implies $E \ll m$, and conversely

[8]For a more detailed overview, see[52].

$E > \Lambda_Q$ implies $E \gg q$, provided q and m are sufficiently well separated and assuming we have chosen Λ_Q to "split the logarithmic difference". We may therefore expand the denominators of Eq. (3.32) either in $E \ll m$ or $E \gg q$, depending on the region in question. We find

$$\begin{aligned} I &= \int_0^{\Lambda_Q} dE \frac{E}{(E^2+q^2)m^2}\left(1 - \frac{E^2}{m^2} + \frac{E^4}{m^4} + \cdots\right) \\ &+ \int_{\Lambda_Q}^{\infty} dE \frac{E}{(E^2+m^2)E^2}\left(1 - \frac{q^2}{E^2} + \frac{q^4}{E^4} + \cdots\right). \end{aligned} \tag{3.33}$$

Because $E < \Lambda_Q \ll m$ in the first integral and $E > \Lambda_Q \gg q$ in the second, we may rescale the integration variable, converting the Laurent series in E with dimensionful coefficients to Laurent series in a dimensionless variable u with *dimensionless scale ratios* as coefficients,

$$\begin{aligned} I &= \int_0^{1} du \frac{u}{(u^2+1)m^2}\left(1 - \frac{q^2}{m^2}u^2 + \frac{q^4}{m^4}u^4 + \cdots\right) \\ &+ \int_{1}^{\infty} du \frac{u}{(u^2+1)u^2m^2}\left(1 - \frac{q^2}{m^2}u^{-2} + \frac{q^4}{m^4}u^{-4} + \cdots\right). \end{aligned} \tag{3.34}$$

Because $0 < u < 1$ in the first integral and $1 < u < \infty$ in the second, the rate of convergence of the Laurent series in both integrands is controlled entirely by the ratio of scales q/m. This is exactly what we wanted! The problem with loop integrals was their dependence on mixed ratios of energy scales determined by external kinematic data (b, m, and the constants $G, \hbar$) and "dummy" energy scales determined by loop momenta, e.g. $k \cdot p_1$, which role is played by E in in our simplistic example integral. Is E/m large or small? Under the integral, this ratio takes every value between zero and infinity, precluding any decomposition of the integrand into classical, quantum, and superclassical pieces. Once we have expressed the loop integrands exclusively in dimensionless dummy variables and ratios of fixed scales, the relative sizes of terms in the regions expansion are again meaningfully "classical", "quantum", or "superclassical". Returning to Eq. (3.34), we can immediately discard everything except the constant 1 terms in the Laurent expansions, because $q/m = \alpha\beta \to 0$ in the classical limit. In fact, many of the diagram topologies contributing to $\mathcal{M}^{\text{class}}$ at a given order in α can be shown, after such an expansion-in-regions, to contain so many factors of q/m that *all* of the terms in the q/m series appearing in the integrand are purely quantum. These diagrams may be discarded at the very first stage of the calculation. Not only do their loop integrals not need to be evaluated, but the rational expressions corresponding to their integrands do not even need to be constructed, preventing a great deal of wasted effort. We will not enumerate the surviving topologies here, as they are described in detail in references[56, 58, 67] in the context of $\mathcal{M}^{\text{cons}}$, as well as in[53, 59] in the context of radiative dissipation.

Hopefully it is now somewhat clearer how classical contributions to $\mathcal{M}^{\text{cons}}$ are isolated by counting powers of the ratio β; but what about α? Once we have classical expressions for the integrand, how do we gather terms of equivalent PM (α) order? Furthermore, how many loops do we need to include to ensure that the classical limit will contain all relevant contributions up to some specified PM order? It is clear from the relation $q = \alpha\beta m$ that the post-Minkowskian and semi-classical expansions are correlated. One might worry, *prima facie*, that higher orders in α will always entail higher orders in β, which would suggest that our approach could never return classical terms at sufficiently high PM (α) orders. This is thankfully not the case. It is the third energy scale, $m_s = 1/r_s = 1/Gm = \beta m$ which saves the day.

On one hand, powers of $(\alpha\beta)$ accumulate in the numerator due to the invariant factors p_i^2, $p_i \cdot q$, and q^2 emanating from the interaction vertices (as in the tree amplitude from Eq. (3.31)). These invariants also appear in the denominator in the form of propagator factors. The net result is a rational function of m, $(\alpha\beta)$, and reduced invariants (e.g. $(p_i \cdot q)/(m_i|q|)$). The additional scale m_s enters through the *dimensionful coupling* $G \propto \kappa^2 \propto 1/(\beta m^2)$. Every vertex factor involving m particles (scalars or gravitons) is dressed with a factor $\kappa^{m-2} \propto \beta^{-(m-2)/2} m^{-(m-2)}$, as in Eq.(3.29). The coupling factors thus decrease the β scaling of n-PM (α^n) terms from the power β^n coming from Mandelstam invariants.

Using textbook arguments to relate loop order to a sum over vertex multiplicities, we find

$$\ell = \frac{1}{2}\left[(2 - N_{\text{ext}}) + \sum_{m=3}^{\infty}(m-2)V_m\right], \tag{3.35}$$

where ℓ is the loop order of a given diagram, N_{ext} is the number of external legs (always four, in the conservative case), m runs over the possible vertex multiplicities, and V_m denotes the number of vertices of each multiplicity appearing in the diagram. Note that this is *total* multiplicity, which is the sum of scalar and graviton multiplicies. The summand factor $(m-2)$ should be familiar from Eq.(3.29), where it gives the power in κ associated to a vertex of multiplicity m. The sum of $(m-2)V_m$ appearing in Eq.(3.35) is simply the total power of κ appearing in the diagram. The total power $\kappa^{2n} \propto G^n$ is then

$$n = \ell + 1. \tag{3.36}$$

An ℓ-loop diagram is thus proportional to $G^{\ell+1} \propto \beta^{-(\ell+1)}$, and so terms proportional to $(\alpha\beta)^{\ell-2}$ from the Mandelstam invariants are scaled down to $\alpha^{\ell-2}\beta^{-3}$. Finally, including the $(\alpha\beta)^2$ factor from inverse fourier transformation to impact parameter space and the $(\alpha\beta)$ momentum factor appearing alongside the amplitude in the observable, such terms scale as $\alpha^{\ell+1}\beta^0$, denoting a classical, $(\ell+1)$PM (or $G^{\ell+1}$) contribution to the observable. For a given ℓ, terms in the amplitude of order $\alpha^{p>\ell-2}$ will have positive powers of β in the observable, and are purely quantum. Terms of order $\alpha^{p<\ell-2}$ will have negative powers of β in the observable, and are thus superclassical. To summarize, classical nPM (G^n) contributions to the observable come exclusively from $(\ell = n-1)$-loop diagrams in the amplitude.

Equipped finally with precise rules relating kinematic scaling and loop order in the amplitude to semi-classical and post-Minkowskian order in the observable, we can identify and expand the relevant topologies, and subsequently construct a minimal, classical integrand. The third step on our journey to classical dynamics requires efficiently evaluating the remaining loop integrals. Because we will devote a great deal of attention to the integration step in our discussion of the dissipative calculation, we will not discuss integration here.

Effective Hamiltonian from the Radial Action

The last step of the conservative calculation requires us to convert one classical quantity (the radial action) into another (the effective Hamiltonian). The most direct approach, which we will take in our calculation of the radiation reaction force, is simply to construct an ansatz for the PM-expanded Hamiltonian with undetermined coefficients, derive the Hamiltonian equations of motion from this ansatz, and use them to compute the radial action integral. Equating the result to the expression for I_R we derived from $\mathcal{M}^{\text{cons}}$ yields an equation which can be solved by fixing the coefficients of the Hamiltonian ansatz. In particular,

$$\mathcal{H}[\vec{c}, m_1, m_2](\boldsymbol{r}, \boldsymbol{p}) = \sqrt{m_1^2 + |\boldsymbol{p}|^2} + \sqrt{m_2^2 + |\boldsymbol{p}|^2} + \sum_{n=1}^{\infty} \frac{c_n(|\boldsymbol{p}|^2)}{m^n} \left(\frac{Gm}{|\boldsymbol{r}|}\right)^n, \tag{3.37}$$

where $\vec{c}$ are the undetermined coefficients, and $\boldsymbol{r}, \boldsymbol{p}$ are three-vectors denoting the instantaneous relative position and momentum of the binary in the CM frame. In [56], the authors perform PM expansions in powers of G/r, so the factor of m^{-n} appearing below the coefficients c_n in Eq. (3.37) has been added so that the expansion is explicitly in powers of Gm/r. In the same work, Bern et al. compute the coefficients c_n through 3PM order in Eq.(10.10). For convenience, we reproduce them here verbatim

$$c_1 = \frac{\nu^2 m^2}{\gamma^2 \xi}\left(1 - 2\sigma^2\right), \tag{3.38}$$

$$c_2 = \frac{\nu^2 m^3}{\gamma^2 \xi}\left[\frac{3}{4}\left(1 - 5\sigma^2\right) - \frac{4\nu\sigma\left(1-2\sigma^2\right)}{\gamma\xi} - \frac{\nu^2(1-\xi)\left(1-2\sigma^2\right)^2}{2\gamma^3\xi^2}\right], \tag{3.39}$$

$$\begin{aligned} c_3 = \frac{\nu^2 m^4}{\gamma^2 \xi}\Bigg[&\frac{1}{12}\left(3 - 6\nu + 206\nu\sigma - 54\sigma^2 + 108\nu\sigma^2 + 4\nu\sigma^3\right) \\ &- \frac{4\nu\left(3 + 12\sigma^2 - 4\sigma^4\right)\operatorname{arcsinh}\sqrt{\frac{\sigma-1}{2}}}{\sqrt{\sigma^2-1}} - \frac{3\nu\gamma\left(1-2\sigma^2\right)\left(1-5\sigma^2\right)}{2(1+\gamma)(1+\sigma)} \\ &- \frac{3\nu\sigma\left(7-20\sigma^2\right)}{2\gamma\xi} + \frac{2\nu^3(3-4\xi)\sigma\left(1-2\sigma^2\right)^2}{\gamma^4\xi^3} \\ &- \frac{\nu^2\left(3 + 8\gamma - 3\xi - 15\sigma^2 - 80\gamma\sigma^2 + 15\xi\sigma^2\right)\left(1-2\sigma^2\right)}{4\gamma^3\xi^2} \\ &+ \frac{\nu^4(1-2\xi)\left(1-2\sigma^2\right)^3}{2\gamma^6\xi^4}\Bigg]. \end{aligned} \tag{3.40}$$

That the Hamiltonian depends only on $|\boldsymbol{p}|^2$ and $|\boldsymbol{r}| = r$ amounts to the choice of *isotropic gauge*[56] in a classical GR calculation of the radial action. The radial action is gauge-invariant, so one could, in principle, construct a Hamiltonian ansatz with $p_r = \boldsymbol{p}\cdot\boldsymbol{r}/|r|$ and θ dependence. The point is that the choice to compute a gauge invariant classical observable allows one to extract the Hamiltonian in whichever gauge yields the simplest expressions.

Using Hamilton's equations of motion, we can work out the trajectory $(\boldsymbol{r}, \boldsymbol{p})(t)$ along which to integrate the radial momentum p_r to compute I_R. Conveniently, we do not actually need complete expressions for the trajectories as functions of time in order to do the integral. We only need p_r as a function of r. We connect p_r to the isotropic gauge Hamiltonian using the conservation of angular momentum,

$$J = |b||p_\infty| = r\sqrt{|\boldsymbol{p}|^2 - p_r^2} \qquad \Longrightarrow \qquad |p_r|(r) = \sqrt{|\boldsymbol{p}|^2(r) - J^2/r^2}. \tag{3.41}$$

where $|p_\infty|$ is the magnitude of the initial relative CM momentum in the asymptotic past. We only need to compute $|\boldsymbol{p}|^2(r)$ using $\mathcal{H}$ to the same PM order as our approximation of $\mathcal{M}^{\text{cons}}$. To simplify notation and avoid any confusion, we will adopt the convention laid out in Appendix 6.A of distinguishing *initial* scattering quantities from *instantaneous* quantities by dressing the former with an over-bar. For example, the value of the instantaneous three-momentum $\boldsymbol{p}(t)$ in the asymptotic past is denoted $\bar{\boldsymbol{p}}$, and its magnitude $|p_\infty|$ becomes $|\bar{\boldsymbol{p}}|$, or simply $\bar{p}$. Conservation of energy then implies $\bar{\mathcal{H}} = \mathcal{H}(t)$ for all times t. Inserting our ansatz from Eq.(3.37), conservation reads

$$\sqrt{m_1^2 + |\bar{\boldsymbol{p}}|^2} + \sqrt{m_2^2 + |\bar{\boldsymbol{p}}|^2} = \sqrt{m_1^2 + |\boldsymbol{p}|^2(t)} + \sqrt{m_2^2 + |\boldsymbol{p}|^2(t)} + \sum_{n=1}^{\infty} \frac{c_n(|\boldsymbol{p}|^2(t))}{m^n} \left(\frac{Gm}{r(t)}\right)^n . \tag{3.42}$$

The phase-space variables on the right-hand-side are functions of time, but because we are specifying initial conditions, we can find their solutions along the associated trajectory as functions of radius alone, $(r, |\boldsymbol{p}|^2(r))$. We construct yet another PM ansatz for the momentum-squared,

$$|\boldsymbol{p}|^2(r) = |\bar{\boldsymbol{p}}|^2 + \sum_{n=1}^{\infty} \frac{P_n}{m^n} \left(\frac{Gm}{r}\right)^n , \tag{3.43}$$

where the coefficients P_n are constants of the motion, depending only on the masses m_1, m_2, the impact parameter b, the initial momentum-squared $|\bar{\boldsymbol{p}}|^2$ and the undetermined Hamiltonian coefficients $\vec{c}$. We then insert Eq.(3.43) into the conservation equation Eq.(3.42), re-expand in Gm/r, and solve for $|\bar{\boldsymbol{p}}|^2$ as a function of radius r[9]. The only other precursor to the integration from Eq.(3.18) is the periapsis r_{min}. This can be computed by constructing a PM ansatz for r_{min} and inserting it into the relation $p_r(r_{\text{min}}) = 0$. Inserting these PM expressions into Eq.(3.18) and performing the integration term-by-term yields the desired expression: a PM expansion

$$I_R^{\textbf{ansatz}} = \sum_{n=0}^{\infty} B_n \left(\frac{Gm}{b}\right)^n , \tag{3.44}$$

where B_n are coefficients depending only on initial quantities and coefficients $\vec{c}$, and factors of r have been converted to factors of b during radial integration. Equating $I_R^{\textbf{ansatz}}$ to the I_R obtained from the conservative amplitude order-by-order in $\alpha = Gm/b$ produces a system of equations in $\vec{c}$. Solving these equations for $\vec{c}$ and inserting the coefficients into the Hamiltonian ansatz Eq.(3.37) completes the calculation of the effective binary Hamiltonian from gravitational scattering amplitudes.

[9] The coefficients c_n are *functions* of $|\boldsymbol{p}|^2$, which is itself a function of r. When we say we solve for the coefficients c_n, we mean that we expand them as Taylor series in α about $\alpha = 0 \leftrightarrow |\boldsymbol{p}|^2 = |\bar{\boldsymbol{p}}|^2$ and solve for their Taylor coefficients in terms of initial data.

So many interacting quantities are simultaneously PM expanded that it can be hard to hold this matching calculation in the head. Luckily, all these steps are carefully explained in [56]; particularly in Section 11. As mentioned earlier, the state-of-the-art has produced I_R to $4PM$ order [56], as well as $\mathcal{H}$ completely to $3PM$ order [56] and, modulo tail effects, to $4PM$ order [58]. Now that we have seen how conservative classical dynamics are extracted from gravitational scattering amplitudes, we can apply similar techniques to the dissipative dynamics originating in gravitational radiation.

3.3 Dissipative Binary Dynamics

In Section 3.1, we saw that the dissipative effects of gravitational radiation on a binary system can be modeled by including a radiation reaction force, $\boldsymbol{F}_{\text{rr}}$ as a "driving term" to the conservative equations of motion. Through G^3 order in the PM expansion, the relative binary dynamics decouple from the dynamics of the center-of-momentum, permitting the reduction of the relative equations of motion to those of a single effective body, presented in Hamiltonian formalism in Eq. (3.17). In Section 3.2, we saw that the effective Hamiltonian $\mathcal{H}$ can be extracted from gravitational scattering amplitudes by relating it to a gauge-invariant classical quantity I_R which can be recovered from amplitudes in the classical limit. We will perform an analogous calculation to extract $\boldsymbol{F}_{\text{rr}}$.

Our first task is to relate $\mathcal{H}$ and $\boldsymbol{F}_{\text{rr}}$ to gauge-invariant observables of gravitational scattering which encode the dissipative effects of gravitational radiation, just as I_R encoded conservative gravitational dynamics. The addition of $\boldsymbol{F}_{\text{rr}}$ to the Hamiltonian equations Eq. (3.17) signals an irreversible exchange of relative momentum between the binary system and the radiative modes of the gravitational field. These modes carry this momentum off to future null infinity, never to return to the binary system. The gauge-invariant observables we are after are thus

$$\Delta P^{\mu}_{\text{bin}} \quad \text{and} \quad \Delta J^{\mu\nu}_{\text{bin}}, \tag{3.45}$$

the *total losses* of binary momenta to gravitational radiation throughout the scattering process.

In the conservative case, an instantaneous quantity ($\mathcal{H}$) was related to an asymptotic quantity (I_R) by integrating the radial momentum p_r over the entire scattering trajectory. Here, the instantaneous force $\boldsymbol{F}_{\text{rr}}$ is related to the total losses by integrating instantaneous binary momentum dissipation over the same. The instantaneous dissipation is usually given in terms of the instantaneous flux of linear or angular momentum current in the gravitational field. These quantities are related by the *flux balance equation*.

Let Q represent an arbitrary Noether charge (either P or J in our case). Let $Q_{\text{bin}}(t)$ denote the portion of this charge associated to the binary system at some time t. Let $\mathcal{J}_Q$ represent the corresponding Noether current of the gravitational field. Then, let $\mathcal{F}_Q(t)$ denote the integral of $\mathcal{J}_Q$ across a Cauchy surface intersecting the binary trajectory at time t. Given a foliation of the spacetime into such Cauchy surfaces, the flux $\mathcal{F}_Q(t)$ may be defined for all times. Integrating the flux from asymptotic past to asymptotic future yields the total charge gained by the gravitational field, ΔQ_{rad}. Because the full theory (involving binary constituents and gravity) is, by assumption, conservative, the total charge gained by the gravitational field must be equal and opposite to the total charge lost from the binary system, ΔQ_{bin}. In particular,

$$\int_{-\infty}^{\infty} dt\, \mathcal{F}_Q(t) = \Delta Q_{\text{rad}} = -\Delta Q_{\text{bin}}. \tag{3.46}$$

Equation (3.46) suggests a more direct correspondence between the instantaneous flux $\mathcal{F}_Q(t)$ and the instantaneous loss, or *dissipation* of charge from the binary, $\dot{Q}_{\text{bin}}$,

$$\dot{Q}_{\text{bin}} = -\mathcal{F}_Q. \tag{3.47}$$

Equation (3.47) is usually referred to as the *flux balance equation*. In fact, the instantaneous flux $\mathcal{F}_Q$ is only defined up to the addition of a *Schott term* arising from gravitational gauge freedom. These terms do not affect the total losses because they contribute nothing to the flux integral appearing in Eq. (3.46). However, the flux balance equation introduces this gauge ambiguity into the dissipation $\dot{Q}_{\text{bin}}$, and thus into the radiation reaction force. This isn't a problem, as the Hamiltonian is also a gauge-dependent quantity. We won't need to worry about gauge dependence until we compare our (isotropic-gauge) fluxes to PN expansions of ADM-gauge fluxes from the literature in Chapter 6. We will delay a detailed discussion of the Schott terms until then.

The flux balance equation allows us to represent the force $\boldsymbol{F}_{\text{rr}}$ directly in terms of gravitational momentum fluxes [54],

$$\mathcal{F}_E = -\dot{\boldsymbol{r}} \cdot \boldsymbol{F}_{\text{rr}}, \qquad \mathcal{F}_J = -F_\phi, \tag{3.48}$$

where $F_{\text{rr},\phi}$ is the component of the radiation reaction force along the angular direction $\hat{\phi}$ in a polar coordinate system on the scattering plane. Critically, Eq. (3.48) only involves the energy flux $\mathcal{F}_E$, which is the timelike projection of the linear momentum flux $\mathcal{F}_P^\mu$, and the projection $\mathcal{F}_J$ of the angular momentum flux $\mathcal{F}_J^{\mu\nu}$ onto the scattering plane. These projections are all that is required to compute the *relative* radiation-reaction force. The other flux components are necessary to compute the center-of-momentum force $\boldsymbol{F}_{\rm rr}^{\rm COM}$, but we will not perform that calculation here.

These flux projections will be extracted from the corresponding projections of the total radiated momenta,

$$\Delta E_{\rm rad}\,, \quad \Delta J_{\rm rad} \tag{3.49}$$

through the integral in Eq. (3.46). Because the fluxes/dissipations and momentum losses/radiated momenta are equivalent up to a minus sign, it doesn't really matter which pair we use to extract the radiation-reaction force, so long as we are consistent about the sign. We will choose to work with $\mathcal{F}_Q$ and $\Delta Q_{\rm rad}$ henceforth, and will simply refer to the latter as ΔQ when there is no chance of confusion.

The total radiated energy ΔE was obtained to 3PM order by Herrmann et al. in [53, 59]. We will compute ΔJ (and in fact, the whole expression $\Delta J^{\mu\nu}$) to 3PM order in Chapter 4. $\Delta J^{\mu\nu}$ was computed by Manohar et al. in [54] using a somewhat different approach, and we find agreement between our expressions. These authors also computed the 2PM relative radiation reaction force in the same work, though they did not derive explicit expressions for the instantaneous fluxes. Picking up where they left off, we will use Equation (3.46) to recover $\mathcal{F}_E$ and $\mathcal{F}_J$ at 3PM order from ΔE and ΔJ in Chapter 5. We then use Eq. (3.48) to compute $\boldsymbol{F}_{\rm rr}$ to 3PM order. Combining this force with the effective Hamiltonian from [56] produces the complete relative equation of motion for gravitational binary systems at 3PM order. We detail this equation at the close of Chapter 5. We conclude in Chapter 6 by cross-checking our results against post-Newtonian expansions of the instantaneous fluxes from literature.

Chapter 4

RADIATED ANGULAR MOMENTUM FROM SCATTERING AMPLITUDES

4.1 Observables, Symmetries, and Form Factors

Let us begin by describing the scattering process in detail. We will consider the scattering of two classical spinless point particles with masses m_1 and m_2, with respective initial four-velocities u_1^μ and u_2^μ, as well as impact parameters b_1^μ and b_2^μ relative to some origin. We are interested in the total linear and angular momentum radiated in the form of electromagnetic and/or gravitational waves in the process, which we denote by ΔP^μ[1] and $\Delta J^{\mu\nu}$ respectively.

These quantities are constrained by the symmetries of the problem. Lorentz covariance requires that the radiated momenta are linear combinations of the p_i^μ and b_i^μ or their anti-symmetric product, with coefficients given by scalar functions ("form factors") of the Lorentz invariants

$$m_i, \quad \sigma = u_1 \cdot u_2, \quad u_i \cdot (b_1 + b_2), \quad |\Delta b| = \sqrt{(b_1 - b_2)^2}, \quad (b_1 + b_2)^2\,, \tag{4.1}$$

where $\Delta b = b_1 - b_2$, and $u_i^\mu = p_i^\mu / m_i$ are the classical four-velocities. Note that we only need to consider the combination $u_i \cdot (b_1 + b_2)$, because in the classical limit the b_i are orthogonal to u_i, i.e. $u_i \cdot b_i = 0$. Furthermore, a spacetime translation $x^\mu \longrightarrow x^\mu + a^\mu$ yields the following transformation of the angular momentum

$$\Delta J^{\mu\nu} \longrightarrow \Delta J^{\mu\nu} + a^{[\mu} \Delta P^{\nu]} \tag{4.2}$$

and leaves ΔP^μ invariant. This forbids dependence on $u_i \cdot (b_1 + b_2)$ and $(b_1 + b_2)^2$. Thus the transformation properties under the full Poincaré algebra suggest the following form-factor decomposition of the radiated momentum and angular momentum[2]

$$\Delta P^\mu = \tilde{A}_1\, \check{p}_1^\mu + \tilde{A}_2\, \check{p}_2^\mu + \tilde{B} \Delta b^\mu\,, \tag{4.3}$$

$$\Delta J^{\mu\nu} = \frac{1}{2}\left(b_1^{[\mu} + b_2^{[\mu}\right) \Delta P^{\nu]} + \frac{1}{2} \Delta b^{[\mu} \left(\tilde{C}_1\, \check{p}_1^{\nu]} - \tilde{C}_2\, \check{p}_2^{\nu]}\right) + \frac{1}{2} \tilde{D}\, \check{p}_1^{[\mu} \check{p}_2^{\nu]}\,. \tag{4.4}$$

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[1] N.b. The total radiated linear momentum is represented by R^μ, not ΔP^μ, in [42].

[2] Our choice of form factor parameterization is slightly different to that in Ref. [54]. As we will see, our choice manifests the fact that the observables are polynomials in the masses.

where we have introduced for convenience the dual vectors

$$\check{p}_1^\mu = \frac{(p_1 \cdot p_2) p_2^\mu - p_2^2\, p_1^\mu}{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}\,, \quad \check{p}_2^\mu = \frac{(p_1 \cdot p_2) p_1^\mu - p_1^2\, p_2^\mu}{(p_1 \cdot p_2)^2 - p_1^2 p_2^2}\,, \tag{4.5}$$

which satisfy $p_i \cdot \check{p}_j = \delta_{ij}$.[3]

All form factors are functions only of the masses, m_i, the Lorentz factor, σ, and the magnitude of the relative impact parameter Δb. In addition to Poincaré covariance, the process is symmetric under the exchange of both particles, which maps

$$\{m_1, u_1^\mu, b_1^\mu\} \leftrightarrow \{m_2, u_2^\mu, b_2^\mu\}\,, \tag{4.6}$$

and leaves ΔP^μ and $\Delta J^{\mu\nu}$ invariant. This involution yields the relations

$$\begin{aligned} \tilde{A}_1(m_1, m_2) &= \tilde{A}_2(m_2, m_1)\,, \quad & \tilde{B}(m_1, m_2) &= -\tilde{B}(m_2, m_1)\,, \\ \tilde{C}_1(m_1, m_2) &= \tilde{C}_2(m_2, m_1)\,, \quad & \tilde{D}(m_1, m_2) &= -\tilde{D}(m_2, m_1)\,. \end{aligned} \tag{4.7}$$

4.2 Linear and Angular Momenta in the KMOC Framework

Our task is to compute the total angular momentum $\Delta J^{\mu\nu}$ radiated during classical two-body scattering, though the tools we will use are applicable to any classical observable O. In fact, we will not directly compute the classical O at all, but will instead take the KMOC approach [42] and study a quantum scattering process which reduces to the classical process in the appropriate limit. In the quantum system, we identify a (Hermitian) operator $\mathbb{O}$ corresponding to the classical quantity O. We compute the expectation of the difference of the operators $\mathbb{O}$ in the initial and final states, $\langle \Delta \mathbb{O} \rangle$, and then recover ΔO in the classical limit.

The quantum system consists of two massive scalar fields corresponding to the two compact objects, as well as a massless field which mediates the scattering. We will consider both gravitational and electromagnetic mediators. In the context of quantum field theory, the total linear and angular momentum charge operators $\mathbb{P}^\mu$ and $\mathbb{J}^{\mu\nu}$ can be derived directly from the action coupling the massive fields and the massless mediators.

From the definition of the S-matrix,

$$\begin{aligned} \langle \Delta \mathbb{O} \rangle &= \langle \text{out} | \mathbb{O} | \text{out} \rangle - \langle \text{in} | \mathbb{O} | \text{in} \rangle \\ &= \langle \text{in} | \left[S^\dagger \mathbb{O} S - \mathbb{O} \right] | \text{in} \rangle. \end{aligned} \tag{4.8}$$

[3]Note that the combination $p_{\text{rel}}^\mu \sim \check{p}_2^\mu - \check{p}_1^\mu$ is orthogonal to the total incoming momentum $P^\mu = p_1^\mu + p_2^\mu$. Thus the former is proportional the relative momentum in the center-of-momentum frame.

This expression transforms under the action of the Poincaré group on the Fock space as

$$\langle \text{in}| \left[S^\dagger \mathbb{O} S - \mathbb{O}\right] |\text{in}\rangle \to \langle \text{in}| \left[S^\dagger e^{i\mathbb{G}} \mathbb{O} e^{-i\mathbb{G}} S - e^{i\mathbb{G}} \mathbb{O} e^{-i\mathbb{G}}\right] |\text{in}\rangle, \tag{4.9}$$

where $\mathbb{G}$ belongs to the Fock space representation of the Poincaré algebra. Using the brackets of this algebra,

$$[\mathbb{P}^\mu, \mathbb{P}^\nu] = 0 \tag{4.10}$$

$$[\mathbb{J}^{\mu\nu}, \mathbb{P}^\rho] = i\eta^{\mu\rho}\, \mathbb{P}^\nu - i\eta^{\nu\rho}\, \mathbb{P}^\mu \tag{4.11}$$

$$[\mathbb{J}^{\mu\nu}, \mathbb{J}^{\rho\sigma}] = i\eta^{\mu\rho}\mathbb{J}^{\nu\sigma} - i\eta^{\mu\sigma}\mathbb{J}^{\nu\rho} - i\eta^{\nu\rho}\mathbb{J}^{\mu\sigma} + i\eta^{\nu\sigma}\mathbb{J}^{\mu\rho}, \tag{4.12}$$

it is straightforward to show that the operators $\mathbb{P}^\mu, \mathbb{J}^{\mu\nu}$ transform exactly as their classical analogs $P^\mu, J^{\mu\nu}$. It follows that the expectations $\langle \Delta\mathbb{O}\rangle$ of these operators admit the same form factor decompositions as the classical quantities $P^\mu, J^{\mu\nu}$ (see Eq. (4.3)). We will use these decompositions extensively to simplify our calculations.

Returning to the expectation expression Eq. (4.8), we expand $S = 1 + iT$, where the T operator describes the non-trivial part of the scattering. Recalling that the unitarity of S implies

$$S\,S^\dagger = 1 \quad \leftrightarrow \quad iT^\dagger - iT = T\,T^\dagger, \tag{4.13}$$

we find

$$\langle \Delta O\rangle = i\langle \text{in}|[\mathbb{O}, T]|\text{in}\rangle + \langle \text{in}|T^\dagger[\mathbb{O}, T]|\text{in}\rangle, \tag{4.14}$$

where we have used Eq. (4.13) to eliminate $T^\dagger$ in the first term of the right-hand side of Eq. (4.14). The second term can be simplified by inserting a resolution of the Fock space identity between the $T^\dagger$ and the commutator, yielding

$$\sum\!\!\!\!\!\!\int_\chi \langle \text{in}|T^\dagger|\chi\rangle\langle\chi|[\mathbb{O}, T]|\text{in}\rangle, \tag{4.15}$$

where $|\chi\rangle$ is summed/integrated over all Fock basis states. Thus, to express Eq. (4.14) purely in terms of amplitudes, we need only to express states $\mathbb{O}T|\text{in}\rangle$ in terms of Fock states. When $\mathbb{O}$ represents a linear or angular momentum observable, one can derive a relation

$$\mathbb{O}T|\text{in}\rangle = \sum\!\!\!\!\!\!\int_\chi |\chi\rangle \mathrm{O}\left[\langle\chi|T|\text{in}\rangle\right], \tag{4.16}$$

where O is a linear differential operator in momentum and polarization variables, acting on the expression $\langle\chi|T|\text{in}\rangle$. In the case of the radiated linear momentum, the differential operator reduces to multiplication by a momentum eigenvalue,

$$\mathbb{P}^{\mu}|p\rangle = p^{\mu}\,|p\rangle\,. \tag{4.17}$$

The derivation of the differential operator in the case of angular momentum is considerably more complicated, so we will only present the results here. Let ξ denote an arbitary four-vector. If we define

$$J^{\mu\nu}_{\xi} = i\,\xi^{[\mu}\partial^{\nu]}_{\xi} = i\xi^{\mu}\frac{\partial}{\partial\xi^{\nu}} - i\xi^{\nu}\frac{\partial}{\partial\xi^{\mu}}\,, \tag{4.18}$$

using the convention that antisymmetrization is understood without a combinatoric factor of 1/2, i.e. $a^{[\mu}b^{\nu]} = a^{\mu}b^{\nu} - a^{\nu}b^{\nu}$, we find

$$\text{scalar:}\qquad \mathbb{J}^{\mu\nu}T|\psi\rangle = \int d\Phi(p)\,|p\rangle J^{\mu\nu}_{p}\langle p|T|\psi\rangle\,, \tag{4.19}$$

$$\text{vector:}\qquad \mathbb{J}^{\mu\nu}T|\psi\rangle = \sum_{\lambda\in\text{pol.}}\int d\Phi(p)\,|p,\lambda\rangle(J^{\mu\nu}_{p} + J^{\mu\nu}_{\epsilon})\langle p,\lambda|T|\psi\rangle\,, \tag{4.20}$$

$$\text{graviton:}\qquad \mathbb{J}^{\mu\nu}T|\psi\rangle = \sum_{\lambda\in\text{pol.}}\int d\Phi(p)\,|p,\lambda\rangle(J^{\mu\nu}_{p} + J^{\mu\nu}_{\epsilon} + J^{\mu\nu}_{\tilde{\epsilon}})\langle p,\lambda|T|\psi\rangle\,. \tag{4.21}$$

The single particle states $|p,\lambda\rangle$ are labeled by their momenta and, where appropriate, by physical polarizations λ. For spinning messenger particles, the angular momentum operator contains contributions from spin degrees of freedom which we can write in terms of anti-symmetrized derivatives of polarization vectors ϵ^{μ}. Critically, the momentum-space derivatives in J_p are understood *not* to act on the polarizations $\epsilon(p)$. In the case of gravity, we represent the spin-2 polarization tensor without loss of generality as a symmetric traceless tensor product of spin-1 polarization vectors $\epsilon^{\mu\nu} = \frac{1}{\sqrt{2}}(\epsilon^{\mu}\,\widetilde{\epsilon}^{\nu} + \epsilon^{\nu}\,\widetilde{\epsilon}^{\mu})$, where $\epsilon\cdot\widetilde{\epsilon} = 0$.

The Classical Limit

With equations (4.14), (4.16) and the differential operators (4.19), (4.20), (4.21), we have specified everything except the initial state of the scattering process. The $|\text{in}\rangle$ state is given as a superposition of plane-wave states

$$|\text{in}\rangle = \int d\Phi_2(p_1,p_2)\,\phi_1(p_1)\phi_2(p_2)\,e^{i(p_1\cdot b_1 + p_2\cdot b_2)}\,|p_1,p_2\rangle\,, \tag{4.22}$$

$$\langle\text{in}| = \int d\Phi_2(p'_1,p'_2)\,\phi^*_1(p'_1)\phi^*_2(p'_2)\,e^{-i(p'_1\cdot b_1 + p'_2\cdot b_2)}\,\langle p'_1,p'_2|\,, \tag{4.23}$$

which are offset by the impact parameters b_1 and b_2 with respect to the origin. In the KMOC framework, the momentum-space wavefunctions $\phi_i(p_i)$ are chosen to describe massive particles in the classical limit and thus peak around the classical momenta $p_i^{\text{cl.}} = m_i u_i$. The n-particle phase-space integration measure is defined as

$$d\Phi_n(p_1, \dots, p_n) = d\Phi(p_1) \cdots d\Phi(p_n)\,, \tag{4.24}$$

where the usual single-particle phase space is

$$d\Phi(p_i) = \int \hat{d}^D p_i\, \hat{\delta}^{(+)}(p_i^2 - m_i^2)\,, \qquad \text{with} \quad \hat{\delta}^{(+)}(p_i^2 - m_i^2) = \hat{\delta}(p_i^2 - m_i^2)\,\theta(p_i^0)\,, \tag{4.25}$$

and the 'hat' notation cleans up factors of 2π

$$\hat{d}^n x \equiv \frac{d^n x}{(2\pi)^n}\,, \qquad \hat{\delta}^{(n)}(x) = (2\pi)^n\,\delta(x)\,. \tag{4.26}$$

The matrix elements of Eq. (4.14) can be calculated in terms of momentum eigenstates $|p_i\rangle$ using the definition of the 'in' states in Eq. (4.22). For instance, the first term in the final form of Eq. (4.14) is given by

$$\begin{aligned}\langle \text{in}|[\mathbb{O}, T]|\text{in}\rangle = \int d\Phi_2(p_1, p_2)\, d\Phi_2(p_1', p_2')\, \phi_1(p_1)\phi_2(p_2)\phi_1^*(p_1')\phi_2^*(p_2') \\ e^{i(p_1 - p_1')\cdot b_1 + i(p_2 - p_2')\cdot b_2}\, \langle p_1', p_2'|[\mathbb{O}, T]|p_1, p_2\rangle\end{aligned} \tag{4.27}$$

We may then apply (4.16) to express the expectation in terms of a differential operator,

$$\langle p_1' p_2'|[\mathbb{O}, T]|p_1, p_2\rangle = \mathrm{O}\left[\langle p_1' p_2'|T|p_1, p_2\rangle\right] \tag{4.28}$$

Note that the matrix element $\langle p_1', p_2'|T|p_1, p_2\rangle$ is a distribution and includes the overall momentum-conserving delta function. The exact form of O depends on the operator under consideration, and sometimes it is useful to rewrite the operator by integrating by parts.

To focus on classical contributions, it is convenient to introduce small momentum transfers $q_i = p_i - p_i' \sim \mathcal{O}(\hbar)$ that allow us to simplify equations in the $\hbar \to 0$ limit. In particular, the wavefunctions $\phi_i^*(p_i) = \phi_i^*(p_i' + q_i) \simeq \phi_i^*(p_i')$. Dropping the primes for notational compactness then allows us to rewrite Eq. (4.27) as

$$\begin{aligned}\langle \text{in}|[\mathbb{O}, T]|\text{in}\rangle \simeq & \int d\Phi_2(p_1, p_2)|\phi_1(p_1)|^2|\phi_2(p_2)|^2 \times \\ & \times \int d\Phi_2(p_1{+}q_1, p_2{+}q_2) e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\langle p_1, p_2|[\mathbb{O}, T]|p_1{+}q_1, p_2{+}q_2\rangle \\ \equiv & \left\langle \int d\Phi_2(p_1{+}q_1, p_2{+}q_2) e^{i(q_1\cdot b_1 + q_2\cdot b_2)}\, \mathrm{O}\,\langle p_1, p_2|T|p_1{+}q_1, p_2{+}q_2\rangle \right\rangle\end{aligned} \tag{4.29}$$

where we define the average quantity

$$\left\langle f(p_1,p_2)\right\rangle \equiv \int d\Phi_2(p_1,p_2)\,|\phi_1(p_1)|^2|\phi_2(p_2)|^2 f(p_1,p_2) \tag{4.30}$$

which sets the heavy particle momenta p_i to their classical values $p_i^\mu = m_i u_i^\mu$ in $f(p_1,p_2)$. We can further modify the (differential) operator O to $\widetilde{\text{O}}$ in order to pull it out of the $p_i + q_i$ phase-space integrals

$$\begin{aligned}\langle \text{in}|[\mathbb{O},T]|\text{in}\rangle &\simeq \left\langle \widetilde{\text{O}}\int d\Phi_2(p_1{+}q_1,p_2{+}q_2)e^{i(q_1\cdot b_1+q_2\cdot b_2)}\langle p_1,p_2|T|p_1{+}q_1,p_2{+}q_2\rangle\right\rangle \\ &\equiv \left\langle \widetilde{\text{O}}\mathcal{R}_\emptyset(p_i,b_i)\right\rangle\end{aligned} \tag{4.31}$$

For later convenience, we define a "radiation kernel" without messenger particles; $\mathcal{R}_\emptyset$. Generally, the operator $\widetilde{\text{O}}$ is a differential operator depending on b_i, p_i and the corresponding derivatives ∂_{b_i}, ∂_{p_i}.

To evaluate the second term in Eq. (4.14), we insert a complete set of states

$$\langle \text{in}|T^\dagger[\mathbb{O},T]|\text{in}\rangle = \sum\!\!\!\!\!\!\int_X \int d\Phi_2(\widetilde{p}_1,\widetilde{p}_2)\,\langle \text{in}|T^\dagger|\widetilde{p}_1,\widetilde{p}_2,X\rangle\,\langle \widetilde{p}_1,\widetilde{p}_2,X|[\mathbb{O},T]|\text{in}\rangle\,, \tag{4.32}$$

where we explicitly split off the contribution from the two heavy scalar particles $\widetilde{p}_1$ and $\widetilde{p}_2$ from the remaining contributions, schematically denoted by the summation over X (which includes the phase space integrals and polarization sums over the additional states). In the cases we are interested in, these contributions will come from the exchange of massless 'messenger' particles. In the classical limit, the number of heavy states is conserved and we do not have e.g. black hole pair production. Generically, X denotes multi-particle states, but, as a special case, it also includes zero messenger states, i.e. $X = \emptyset$ and $k_X = 0$. Each matrix element in Eq. (4.32) can be evaluated separately. The first one is simply related to a scattering amplitude

$$\langle \text{in}|T^\dagger|\widetilde{p}_1,\widetilde{p}_2,X\rangle = \int d\Phi_2(p_1',p_2')\,\phi_1^*(p_1')\phi_2^*(p_2')e^{-i(p_1'\cdot b_1+p_2'\cdot b_2)}\langle p_1'p_2'|T^\dagger|\widetilde{p}_1\widetilde{p}_2X\rangle, \tag{4.33}$$

where, again, we change variables from p_i' to momentum transfers $q_i' = p_i' - \widetilde{p}_i$ and expand the wavefunctions around the classical momenta about which they peak to find

$$\langle \text{in}|T^\dagger|\widetilde{p}_1\widetilde{p}_2X\rangle \simeq \phi_1^*(\widetilde{p}_1)\phi_2^*(\widetilde{p}_2)\, e^{-i(\widetilde{p}_1\cdot b_1+\widetilde{p}_2\cdot b_2)} \int d\Phi_2(\widetilde{p}_1+q_1',\, \widetilde{p}_2+q_2')e^{-i(q_1'\cdot b_1+q_2'\cdot b_2)} \times \langle \widetilde{p}_1+q_1', \widetilde{p}_2+q_2'|T^\dagger|\widetilde{p}_1, \widetilde{p}_2, X\rangle \tag{4.34}$$

where we have dropped a 'quantum contribution' to the external wavefunctions in the second equality $\phi(\widetilde{p}_i + q_i') \simeq \phi(\widetilde{p}_i)$. (As explained in [42], any momentum transfer q_i is regarded as $\mathcal{O}(\hbar)$ when taking the classical limit.) We can define a *radiation kernel* as follows

$$\mathcal{R}_X(\widetilde{p}_i, b_i) = \int d\Phi_2(\widetilde{p}_1+q_1, \widetilde{p}_2+q_2)e^{+i(q_1\cdot b_1+q_2\cdot b_2)} \langle \widetilde{p}_1, \widetilde{p}_2, X|T|\widetilde{p}_1+q_1, \widetilde{p}_2+q_2\rangle\,. \tag{4.35}$$

When X is empty, to leading order in the q-expansion this simply becomes the Fourier transform of the amplitude

$$\begin{aligned}\mathcal{R}_\emptyset(\widetilde{p}_i, b_i) &\simeq \mathcal{M}(\widetilde{p}_i, b_i)\\ &= \int \hat{d}^D q\, \hat{\delta}(2\widetilde{p}_1\cdot q)\hat{\delta}(-2\widetilde{p}_2\cdot q)e^{iq\cdot(b_1-b_2)}\mathcal{M}(\widetilde{p}_1+q, \widetilde{p}_2-q \to \widetilde{p}_1, \widetilde{p}_2)\,.\end{aligned}$$

In general, we have to be careful with subleading in q contributions from the delta functions which can interfere with classically singular terms of the amplitude. In the following, we distinguish $\mathcal{R}_\emptyset(\widetilde{p}_i, b_i)$ and $\mathcal{M}(\widetilde{p}_i, b_i)$. In terms of this radiation kernel we find that the matrix element Eq. (4.34) is

$$\langle \text{in}|T^\dagger|\widetilde{p}_1\widetilde{p}_2X\rangle \simeq \phi_1^*(\widetilde{p}_1)\phi_2^*(\widetilde{p}_2)\, e^{-i(\widetilde{p}_1\cdot b_1+\widetilde{p}_2\cdot b_2)}\mathcal{R}_X^*(\widetilde{p}_i, b_i)\,. \tag{4.36}$$

The discussion of the matrix element $\langle \widetilde{p}_1, \widetilde{p}_2X|[\mathbb{O}, T]|\text{in}\rangle$ follows along the lines of the discussion that led to Eq. (4.32) with the appropriate replacements to take into account the plane wave states from the completeness relation

$$\langle \widetilde{p}_1, \widetilde{p}_2, X|[\mathbb{O}, T]|\text{in}\rangle = \int d\Phi_2(p_1, p_2)\, \phi_1(p_1)\phi_2(p_2)e^{i(p_1\cdot b_1+p_2\cdot b_2)} \times \langle \widetilde{p}_1, \widetilde{p}_2, X|[\mathbb{O}, T]|p_1, p_2\rangle\,, \tag{4.37}$$

where we can again write the operator as acting on the matrix element

$$\begin{aligned}\langle \widetilde{p}_1, \widetilde{p}_2, X|[\mathbb{O}, T]|\text{in}\rangle &= \int d\Phi_2(p_1, p_2)\, \phi_1(p_1)\phi_2(p_2)e^{i(p_1\cdot b_1+p_2\cdot b_2)} \\ &\qquad \times \mathrm{O}\left[\langle \widetilde{p}_1, \widetilde{p}_2, X|T|p_1, p_2\rangle\right] \\ &\simeq \phi_1(\widetilde{p}_1)\phi_2(\widetilde{p}_2)e^{i(\widetilde{p}_1\cdot b_1+\widetilde{p}_2\cdot b_2)}\widetilde{\mathrm{O}}\left[\mathcal{R}_X(\widetilde{p}_i, b_i)\right]\,,\end{aligned} \tag{4.38}$$

and we have neglected quantum contributions from the external wavefunctions. As before, $\widetilde{\mathrm{O}}$ is determined by O in terms of some formal manipulations that are explained below for the specific examples under consideration. Combining Eqs. (4.36) and (4.38), we find that Eq. (4.32) evaluates to

$$\langle \mathrm{in}|T^\dagger[\mathbb{O},T]|\mathrm{in}\rangle = \left\langle\!\!\left\langle \sum\!\!\!\!\!\!\int_X \mathcal{R}^*_X(\widetilde{p}_i,b_i)\,\widetilde{\mathrm{O}}\,\mathcal{R}_X(\widetilde{p}_i,b_i) \right\rangle\!\!\right\rangle. \tag{4.39}$$

We finally arrive at a simple equation for the change of the classical observable ΔO in terms of the radiation kernels $\mathcal{R}$

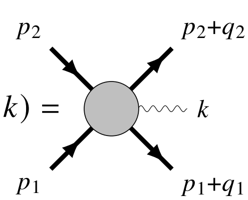
$$\Delta O = \left\langle\!\!\left\langle \mathrm{i}\,\widetilde{\mathrm{O}}\mathcal{R}_\emptyset(p_i,b_i) + \sum\!\!\!\!\!\!\int_X \mathcal{R}^*_X(p_i,b_i)\,\widetilde{\mathrm{O}}\,\mathcal{R}_X(p_i,b_i) \right\rangle\!\!\right\rangle. \tag{4.40}$$

In the last term we renamed integration variables $\widetilde{p}_i \to p_i$. Let us also state an **explicit warning** about the momenta appearing in the problem. Inside the classical localization $\langle\!\langle \cdots \rangle\!\rangle$, the momenta p_i are not yet localized to their classical values and differ by deflections of order $O(q_i)$. For this reason, at first, these p_i, unlike the classical momenta, are not yet orthogonal to the impact parameters b_i. Only at the very end of the computation, when all (differential) operators have acted, can we set all momenta to the classical values which satisfy the expected orthogonality conditions. This is particularly relevant when subleading terms in the classical expansion become important.

4.3 Scattering Amplitudes

As we have seen, scattering amplitudes are the fundamental building blocks of the KMOC framework, so we collect those amplitudes relevant to our calculation here. The four-scalar, one-graviton, tree-level amplitude in GR relevant for our $O(G^3)$ computations is given by

$$\mathcal{M}_5^{\mathrm{GR}}(p_1,p_2\to p_1{+}q_1,p_2{+}q_2,k) =$$



$$= \left[\frac{n_1}{q_1^2 q_2^2} + \frac{n_2}{(u_1\cdot k)q_1^2} + \frac{n_3}{(u_1\cdot k)^2 q_1^2} + 1\leftrightarrow 2\right], \tag{4.41}$$

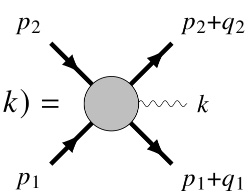
where for GR, using the simplification from Ref. [70],

$$\begin{aligned} n_1 = & -8m_1^2m_2^2\,(q_1\cdot u_2\epsilon\cdot u_1 - q_2\cdot u_1\epsilon\cdot u_2 + \sigma\epsilon\cdot q_2) \\ & \qquad\qquad \times (q_1\cdot u_2\bar\epsilon\cdot u_1 - q_2\cdot u_1\bar\epsilon\cdot u_2 + \sigma\bar\epsilon\cdot q_2) \\ & + \frac{8m_1^2m_2^2}{D_s-2}\epsilon\cdot q_2\bar\epsilon\cdot q_2\,, \\ n_2 = & -2m_2^2m_1^2\left(\sigma^2 - \frac{1}{D_s-2}\right)q_2^2\epsilon\cdot u_2\bar\epsilon\cdot u_2\,, \\ n_3 = & \, 4m_1^2m_2^2\sigma\,(\sigma\epsilon\cdot u_2\bar\epsilon\cdot q_2 + \sigma\epsilon\cdot q_2\bar\epsilon\cdot u_2 - 2q_2\cdot u_1\epsilon\cdot u_2\bar\epsilon\cdot u_2) \\ & - 4\frac{m_1^2m_2^2}{D_s-2}\,(\epsilon\cdot u_2\bar\epsilon\cdot q_2 + \epsilon\cdot q_2\bar\epsilon\cdot u_2)\ , \end{aligned} \tag{4.42}$$

in agreement with e.g. Ref. [54] .

The five-point amplitude for scalar electrodynamics is, expanded to classical order

$$\mathcal{M}_5^{\text{EM}}(p_1,p_2,q_1,q_2,k) =$$



$$= -\frac{4Q_1^2Q_2}{q_2^2}\left[\frac{p_1\cdot F\cdot p_2}{(p_1\cdot k)} + (p_1\cdot p_2)\frac{p_1\cdot F\cdot q_2}{(p_1\cdot k)^2}\right] + 1\leftrightarrow 2 \tag{4.43}$$

where $a\cdot F\cdot b = (a\cdot\epsilon_1)(k\cdot b) - (a\cdot k)(\epsilon_1\cdot b)$, in agreement with e.g. Ref. [42].

4.4 Classical Observables from Scattering Amplitudes

Radiated Momentum

Some of the simplest classical observables already considered in the original work of Kosower, Maybee, and O'Connell are the linear momentum losses either of the individual compact astrophysical objects (the impulses) or of the complete binary system due to emission of gravitational waves (total radiated momentum). These observables have been evaluated in the KMOC formalism up to $\mathcal{O}(G^3)$ in GR and maximally supersymmetric gravity [53, 59] and to $\mathcal{O}(\alpha^3)$ in (scalar) QED [71], either finding full agreement or having been confirmed by other methods [72]. The momentum losses are relatively simple in the KMOC framework, because the relevant quantum operators are diagonal in the plane-wave states. For example, in the case of total radiated linear momentum, $\mathbb{P}^\mu|p\rangle = p^\mu\,|p\rangle$.

Let us first derive a formula for the linear momentum ΔP^μ radiated away to future null infinity by the messengers. By assumption, there are no messenger particles in the initial state, so this observable only receives a *real emission contribution* from the cut piece of the KMOC kernel

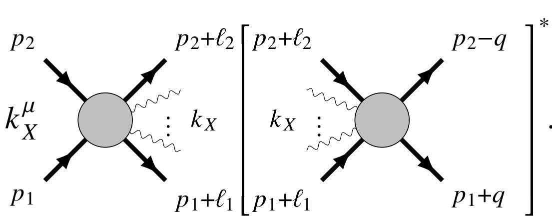
$$\Delta P^\mu = \langle \text{in}|T^\dagger \mathbb{P}^\mu T|\text{in}\rangle\,. \tag{4.44}$$

From [42], the expectation value of $\mathbb{P}^\mu$ is simply

$$\begin{aligned}\Delta P^\mu &= \left\langle \sum\!\!\!\!\!\!\int_X \mathcal{R}_X^*(p_i,b_i)\, k_X^\mu\, \mathcal{R}_X(p_i,b_i)\right\rangle \\ &= \left\langle \mathrm{i}\int \hat{d}^D q\, \hat{\delta}(2p_1\cdot q+q^2)\hat{\delta}(-2p_2\cdot q+q^2)e^{-\mathrm{i}q\cdot\Delta b}\,\mathcal{I}_P^\mu\right\rangle\end{aligned} \tag{4.45}$$

where we have used $k_X^\mu = \sum_{i\in X} k_i^\mu$ and defined the radiated momentum kernel

$$\mathcal{I}_P^\mu = -\mathrm{i}\sum\!\!\!\!\!\!\int_X \int d\tilde{\Phi}_2\, k_X^\mu \quad [\text{diagram}] \,. \tag{4.46}$$



Radiated Angular Momentum

We will now present the first calculation of the radiated angular momentum in the extended KMOC framework. The angular momentum loss is more subtle than the linear momentum loss discussed in the previous section. Heuristically, this subtlety can be understood from the fact that orbital angular momentum behaves as

$\mathbb{L}^{\mu\nu} \sim \mathbb{X}^\mu \mathbb{P}^\nu - \mathbb{P}^\mu \mathbb{X}^\nu$, where $\mathbb{X}$ is not diagonal in the momentum basis, but rather acts as a differential operator in p-space. As we will see, this gives rise to various complications, owing to the distributional nature of scattering amplitudes.

As in the linear momentum calculations, we can evaluate the radiated angular momentum

$$\Delta J^{\mu\nu} = \langle \text{in}|T^\dagger \mathbb{J}^{\mu\nu} T|\text{in}\rangle = \left\langle \sum\!\!\!\!\!\!\!\int_X \mathcal{R}^*_X(p_i, b_i)\, J^{\mu\nu}_X\, \mathcal{R}_X(p_i, b_i) \right\rangle , \tag{4.47}$$

in terms of the radiation kernels $\mathcal{R}_X$ and slightly abuse notation to introduce

$$J^{\mu\nu}_X = \sum_{i\in X} \left[J^{\mu\nu}_{k_i} + J^{\mu\nu}_{\epsilon_i} \right] . \tag{4.48}$$

Note that under a shift $b_i \to b_i + a$, the kernel $\mathcal{R}_X(p_i, b_i)$ in Eq. (4.35) transforms as

$$\mathcal{R}_X(p_i, b_i + a) = e^{\mathrm{i}a\cdot k_X}\, \mathcal{R}_X(p_i, b_i), \tag{4.49}$$

such that the Poincaré transformation of the expression is easy to check

$$\Delta J^{\mu\nu} \to \Delta J^{\mu\nu} + a^{[\mu} \left\langle \sum\!\!\!\!\!\!\!\int_X \mathcal{R}^*_X(p_i, b_i)\, k^{\nu]}_X\, \mathcal{R}_X(p_i, b_i) \right\rangle = \Delta J^{\mu\nu} + a^{[\mu} P^{\nu]} . \tag{4.50}$$

The expression for the angular momentum loss from Eq. (4.47), although very compact, is not in a form that can be easily integrated. We will occasionally need to revert these kernels to their momentum space expressions before performing any integration steps.

Static Contribution to Radiated Angular Momentum

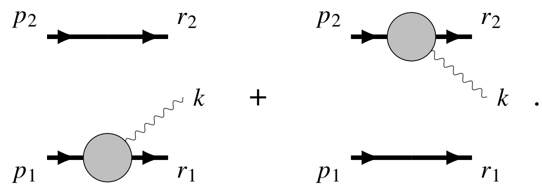
The calculation of classical observables outlined above requires consideration of all final states that can possibly be produced by the scattering process. Technically, the final states appear when we introduce into our formalism a resolution of the identity,

$$\mathbb{O}T|p_1 p_2\rangle = \sum\!\!\!\!\!\!\!\int d\Phi_2(r_1, r_2) d\Phi_n(k_1, \cdots, k_n) \mathbb{O}|r_1 r_2 k_1 \cdots k_n\rangle\langle r_1 r_2 k_1 \cdots k_n|T|p_1 p_2\rangle . \tag{4.51}$$

These intermediate states involve one of each matter particle [4] and any number of mediator particles. Crucially, the integral over the phase space of the final states enforces the positivity of all of their energies via the theta-function in Eq. (4.25).

[4] We are ultimately interested in the classical limit where the heavy states represent e.g. black holes in GR and can not be created or destroyed in the scattering process.

Somewhat surprisingly, the leading non-trivial contributions arise from *disconnected* contributions to the matrix element $\langle r_1 r_2 k|T|p_1 p_2\rangle$. In particular,



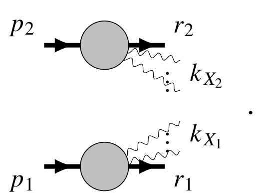
(4.52)

It is easy to see that in real kinematics, the phase space for these processes has support only for static gravitons, $k^\mu = 0$. If p_1 and k are on-shell in the first term above, for example, the on-shell condition for r_1 becomes

$$r_1^2 - m_1^2 = (p_1 + k)^2 - m_1^2 = 2p_1 \cdot k = 0\,. \tag{4.53}$$

The momentum p_1 is timelike, so we are free to boost to the rest frame of this particle. On the other hand, k is lightlike, so Eq. (4.53) can only be satisfied if $\omega_k = 0$. From the on-shell condition for the messenger, $k^2 = 0$, we learn that the spatial components must also vanish, i.e. $k^\mu = 0$.

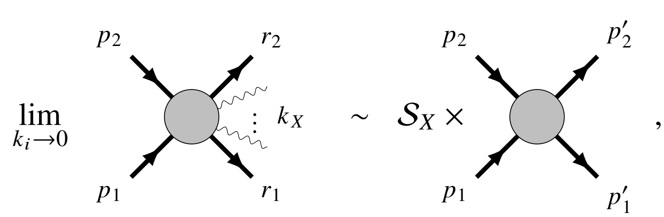
At higher orders, there are similar disconnected terms involving multiple emissions from one or both of the massive lines, without any momentum transfer between the two sides. Schematically,



(4.54)

In this case, the positivity of the energies of the emitted particles implies that the integrand only has support when all emitted mediators are exactly soft.

Finally, connected matrix elements also receive contributions from exactly soft gravitons which can be understood by appealing to the universality of the soft limits. In particular, the soft theorems of Weinberg et.al. [47] tell us that the matrix elements factorize into products of a hard scattering matrix and a soft factor. The emission of soft spin-s messengers is given by



(4.55)

where we define the leading multi-particle soft factor

$$\mathcal{S}_X = \prod_{k_i \in k_X} \sum_m \frac{(\varepsilon_i \cdot p_m)^s}{p_m \cdot k_i - i\epsilon} \,. \tag{4.56}$$

Here, the sum runs over all four hard particles. Technically this includes contributions from gravitons with small momenta, but not necessarily zero frequency. However, given that the momentum transfer $p_i + p_i' = \pm q$ is small, this also contains a contribution localized at zero frequency, due to the familiar identity

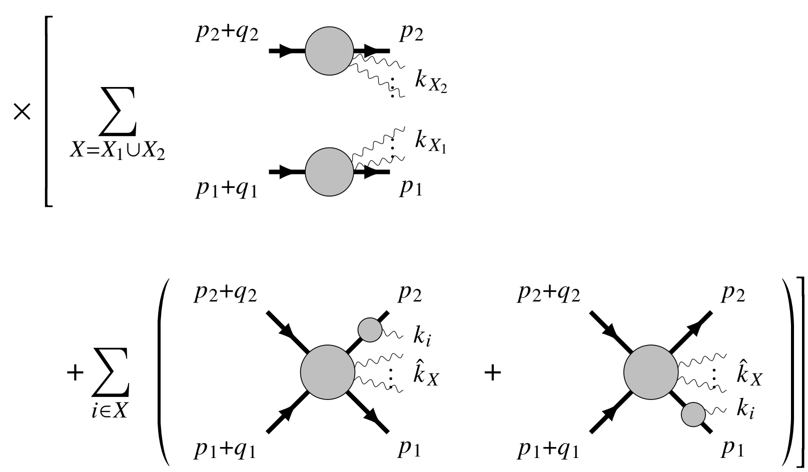
$$\begin{aligned} \frac{(\varepsilon_i \cdot p_i)^s}{p_i \cdot k_j - i\epsilon} + \frac{(\varepsilon_i \cdot p_i')^s}{p_i' \cdot k_j - i\epsilon} &\sim (\varepsilon_i \cdot p_i)^s \left[\frac{1}{p_i \cdot k_j - i\epsilon} - \frac{1}{p_i' \cdot k_j + i\epsilon} \right] \\ &= (\varepsilon_i \cdot p_i)^s \, \mathrm{i} \, \hat{\delta}(p_i \cdot k_j). \end{aligned} \tag{4.57}$$

For this reason, it is useful to separate the radiation kernel into static and non-static (dynamical) contributions

$$\mathcal{R}_X(p_i, b_i) = \mathcal{R}_X^{\mathrm{st}}(p_i, b_i) + \mathcal{R}_X^{\mathrm{dyn}}(p_i, b_i) \,. \tag{4.58}$$

Here $\mathcal{R}^{\mathrm{st}}$ only has support where at least one graviton is static, that is

$$\mathcal{R}_X^{\mathrm{st}}(p_i, b_i) = \int d\Phi_2(p_1{+}q_1, p_2{+}q_2) e^{+i(q_1\cdot b_1 + q_2 \cdot b_2)}$$

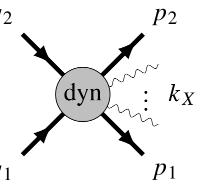


(4.59)

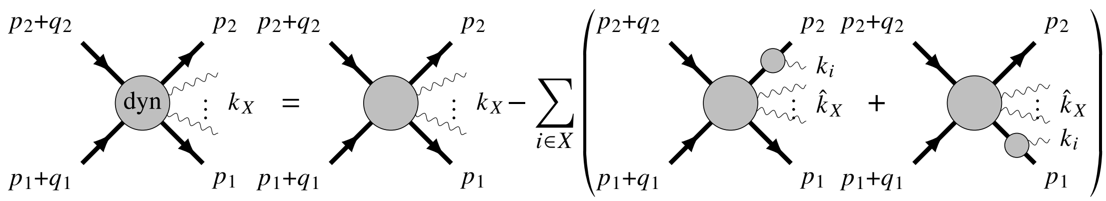
where all exposed legs are on-shell, and the momentum $\hat{k}_X$ denotes the net messenger momentum k_X excluding k_i. In addition to the final-state emissions shown in the second line, this kernel also receives contributions from initial-state emissions, though they haven't been depicted here in the interest of compactness. The static kernel thus includes all disconnected contributions to the amplitude, as well as static contributions from the connected amplitude, which arise from on-shell three-point emission in the classical limit.

On the other hand, by construction, $\mathcal{R}^{\text{dyn}}$ only receives non-static (dynamical) messenger contributions,

$$\mathcal{R}_X^{\text{dyn}}(p_i, b_i) = \int d\Phi_2(p_1{+}q_1, p_2{+}q_2)e^{+i(q_1\cdot b_1+q_2\cdot b_2)} \tag{4.60}$$



where we have formally defined the dynamical amplitude by subtracting all static contributions



(4.61)

These static contributions can be easily obtained by ignoring the $i\epsilon$ prescription in all matter propagators that do not participate in a loop.

Let us now analyze the structure of the leading static contributions, which arise from the exchange of a single static mode. The static radiation kernel takes the form

$$\mathcal{R}_X^{\text{s}}(p_i, b_i) = \int d\Phi_2(p_1{+}q_1, p_2{+}q_2)e^{+i(q_1\cdot b_1+q_2\cdot b_2)}. \tag{4.62}$$

Given the exponential form of the soft S-matrix, it is easy to calculate its contribution to the angular impulse, for mediators of spin s

$$(\Delta J_1^{\mu\nu})_{\text{soft}}^{(s)} = \frac{1}{2}\sum_{m,n}\frac{\mathcal{K}^{(s)}}{(m_n m_m)^{s-1}}\sum_{\text{pols}}\int d\Phi(k)\frac{(\varepsilon\cdot p_m)^s}{p_m\cdot k + i\epsilon}\sum_{i=1,1'} J_{p_i}^{\mu\nu}\frac{(\varepsilon\cdot p_n)^s}{p_n\cdot k - i\epsilon} + \text{c.c} \tag{4.63}$$

where $\mathcal{K}^{(s)}$ is a factor carrying the dependence on the couplings. For a scalar, photon and graviton it is respectively

$$\mathcal{K}^{(0)} = \frac{g_n g_m}{m_n m_m}\,, \quad \mathcal{K}^{(1)} = 4\pi\alpha q_n q_m\,, \quad \mathcal{K}^{(2)} = 8\pi G m_n m_m\,. \tag{4.64}$$

This can be computed by pulling out the differential operator from the integral,

$$(\Delta J_1^{\mu\nu})_{\text{soft}}^{(s)} = \sum_{n=1,1'}\sum_{m\neq n}\mathcal{K}^{(s)} J_{p_n}^{\mu\nu} I_{mn}^s = \sum_{n=1,1'}\sum_{m\neq n}\mathcal{K}^{(s)}\frac{p_n^{[\mu}p_m^{\nu]}}{m_m\, m_n}\frac{dI_{mn}^{(s)}(\sigma_{mn})}{d\sigma_{mn}} \tag{4.65}$$

where the integrals for messengers of each spin are

$$I^{(0)}_{mn} = \frac{1}{2}\int d\Phi(k)\frac{m_m m_n}{(p_m\cdot k + i\epsilon)(p_n\cdot k - i\epsilon)} - \text{c.c.} \tag{4.66}$$

$$I^{(1)}_{mn} = \sum_{\text{pols}} \frac{(\varepsilon\cdot p_m)(\varepsilon\cdot p_n)}{m_m m_n} I^{(0)}_{mn} = \sigma_{mn} I^{(0)}_{mn} \tag{4.67}$$

$$I^{(2)}_{mn} = \sum_{\text{pols}} \frac{(\varepsilon\cdot p_m)^2(\varepsilon\cdot p_n)^2}{m_m^2 m_n^2} I^{(0)}_{mn} - \text{c.c} = \left(\sigma^2_{mn} - \tfrac{1}{2}\right) I^{(0)}_{mn} \tag{4.68}$$

Thus we just have to compute the imaginary part of the scalar integral in Eq. (4.66). Note that such an imaginary part can only arise when one of the propagators goes on-shell, i.e.,

$$\frac{1}{p_m\cdot k - i\epsilon} = \text{PV} \tag{4.69}$$

where 'PV' denotes the principal value. As explained above, such delta functions only have support on exactly zero frequency gravitons. The integral is computed in Appendix 6.C with the result

$$I^{(0)}_{mn} = \text{Im}\left[\int d\Phi(k)\frac{m_m m_n}{(p_m\cdot k + i\epsilon)(p_n\cdot k - i\epsilon)}\right] = \frac{i}{16\pi}\eta_{mn}\frac{\text{arccosh}(\sigma_{mn})}{\sqrt{\sigma^2_{mn}-1}}, \tag{4.70}$$

where

$$\eta_{mn} = \begin{cases} 0 & \text{if } n \text{ and } m \text{ are both ingoing or outgoing} \\ 1 & \text{if } n \text{ is ingoing and } m \text{ is outgoing} \\ -1 & \text{if } n \text{ is outgoing and } m \text{ is ingoing} \end{cases} \tag{4.71}$$

Note that $I^{(0)}_{mn} = -I^{(0)}_{nm}$.

Putting everything together we find

$$(\Delta J_1^{\mu\nu})^{(s)}_{\text{soft}} = -\frac{1}{16\pi}\sum_{n=1,1'}\sum_{m\neq n}\mathcal{K}^{(s)}\eta_{mn}\frac{p_n^{[\mu}p_m^{\nu]}}{m_m m_n}\mathcal{I}^{(s)}(\sigma_{mn}) \tag{4.72}$$

where

$$\mathcal{I}^{(0)}(\sigma) = \frac{d}{d\sigma}\left[\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right] = \frac{1}{\sigma^2-1} - \frac{\sigma}{\sigma^2-1}\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \tag{4.73}$$

$$\mathcal{I}^{(1)}(\sigma) = \frac{d}{d\sigma}\left[\frac{\sigma\,\text{arccosh}(\sigma)}{\sqrt{\sigma^2_{mn}-1}}\right] = \frac{\sigma}{\sigma^2-1} - \frac{1}{\sigma^2-1}\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \tag{4.74}$$

$$\mathcal{I}^{(2)}(\sigma) = \frac{d}{d\sigma}\left[\left(\sigma^2-\tfrac{1}{2}\right)\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\right] = \frac{2\sigma^2-1}{2(\sigma^2-1)} + \frac{(2\sigma^2-3)\sigma}{2(\sigma^2-1)}\frac{\text{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \tag{4.75}$$

For later reference we note that

$$\mathcal{I}^{(0)}(1) = -\frac{1}{3}, \quad \mathcal{I}^{(1)}(1) = \frac{2}{3}, \quad \mathcal{I}^{(2)}(1) = \frac{11}{6}, \tag{4.76}$$

Recall that these expressions are given in terms of the hard *classical momenta*, where the outgoing momenta differ from the incoming ones by the classical impulses Δp_i, and not by the quantum momentum transfer q, i.e.,

$$p_1'^{\mu} = p_1^{\mu} + \Delta p_1^{\mu}, \quad p_2'^{\mu} = p_2^{\mu} + \Delta p_2^{\mu} \tag{4.77}$$

which in turn implies $\sigma_{12} = \sigma_{1'2'} = \sigma$, as well as

$$\begin{aligned} \sigma_{11'} &= 1 + \frac{p_1 \cdot \Delta p_1}{m_1^2}, & \sigma_{22'} &= 1 + \frac{p_2 \cdot \Delta p_2}{m_2^2} \\ \sigma_{12'} &= \sigma + \frac{p_1 \cdot \Delta p_2}{m_1 m_2}, & \sigma_{21'} &= \sigma + \frac{p_2 \cdot \Delta p_1}{m_1 m_2} \end{aligned} \tag{4.78}$$

Thus, to extract the contribution to some order in perturbation theory, we use the relations Eq. (4.78) and expand Δp_i to the desired order. Putting everything together, we write the soft graviton contribution (the $s = 2$ case) to the angular impulse as

$$\begin{aligned} (\Delta J_1^{\mu\nu})^{(s)}_{\text{soft}} = G\, p_1^{[\mu} \Delta p_1^{\nu]} \mathcal{I}^{(s)}(\sigma_{1'1}) \\ + \frac{G}{2} p_1^{[\mu} \Delta p_2^{\nu]} \mathcal{I}^{(s)}(\sigma_{12'}) \\ + \frac{G}{2} p_2^{[\mu} \Delta p_1^{\nu]} \mathcal{I}^{(s)}(\sigma_{21'}) \\ + \frac{G}{2} p_1^{[\mu} p_2^{\nu]} \left(\mathcal{I}^{(s)}(\sigma_{12'}) - \mathcal{I}^{(s)}(\sigma_{21'}) \right) \end{aligned} \tag{4.79}$$

with which we compute the soft contribution to the radiated angular momentum

$$\begin{aligned} (\Delta J^{\mu\nu})^{(s)}_{\text{soft}} &= -(\Delta J_1^{\mu\nu})^{(s)}_{\text{soft}} - (\Delta J_2^{\mu\nu})^{(s)}_{\text{soft}} \\ &= -G\, p_1^{[\mu} \Delta p_1^{\nu]} \mathcal{I}^{(s)}(\sigma_{1'1}) - G\, p_2^{[\mu} \Delta p_2^{\nu]} \mathcal{I}^{(s)}(\sigma_{2'2}) \\ &\quad - G p_1^{[\mu} \Delta p_2^{\nu]} \mathcal{I}^{(s)}(\sigma_{12'}) - G p_2^{[\mu} \Delta p_1^{\nu]} \mathcal{I}^{(s)}(\sigma_{21'}) \\ &\quad - G p_1^{[\mu} p_2^{\nu]} \left(\mathcal{I}^{(s)}(\sigma_{12'}) - \mathcal{I}^{(s)}(\sigma_{21'}) \right). \end{aligned} \tag{4.80}$$

Dynamic Contributions to Classical Observables

In contrast to the various contributions from zero-energy messenger particles, the finite-energy contributions to classical observables are contained only in connected contributions of dynamical messenger modes to the radiation kernels $\mathcal{R}_X$ in Eq. (4.40), where leading soft emissions are subtracted from the amplitudes inside the radiation kernel (see Eq. (4.61)). These contributions are more familiar from earlier computations of the classical linear impulse and the radiated linear momentum.

We now consider the finite energy graviton contributions to the radiated angular momentum in Eq. (4.47). For this observable, $\widetilde{\mathrm{O}} = J_X^{\mu\nu}$, and $J_X^{\mu\nu}\mathcal{R}_\emptyset(p_i, b_i) = 0$ as there are no messenger particles in the initial state. We now act with the angular momentum operator $J_X^{\mu\nu}$ on kernels involving at least one messenger. When distributing the operator according to the Leibniz rule, we will encounter contributions where the operator acts on the on-shell delta functions from the kernel. Such terms can be taken care of using integration-by-parts

$$\frac{\partial\delta(2p_1\cdot(k_X-q_2))}{\partial k_i^\mu} = 2p_{1,\mu}\delta'(2p_1\cdot(k_X-q_2)) = -u_{1,\mu}\,\check{u}_1\cdot\frac{\partial}{\partial q_2}\delta(2p_1\cdot(k_X-q_2))\,. \tag{4.81}$$

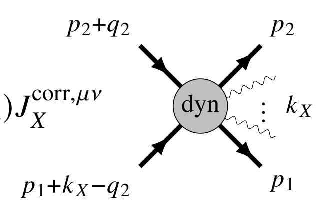
The use of the checked variables,

$$\check{u}_1 = \frac{\sigma u_2 - u_1}{\sigma^2-1}\,, \qquad \check{u}_2 = \frac{\sigma u_1 - u_2}{\sigma^2-1}\,, \tag{4.82}$$

(with $\sigma = p_1\cdot p_2/(m_1m_2)$ and $u_i = p_i/m_i$) ensures that integration by parts does not induce new derivatives of these delta functions. After getting rid of these distributional derivatives, the action of J_{k_X} on the momentum kernel can be written in a compact form,

$$J_X^{\mu\nu}\mathcal{R}_X^{\mathrm{dyn}}(p_i,b_i) = b_1^{[\mu}k_X^{\nu]}\mathcal{R}_X^{\mathrm{dyn}}(p_i,b_i) \tag{4.83}$$

$$+\, e^{\mathrm{i}k_X\cdot b_1}\int \hat{\mathrm{d}}^D q_2 e^{-\mathrm{i}\Delta b\cdot q_2}\hat{\delta}(2p_1\cdot(k_X-q_2))\hat{\delta}(2p_2\cdot q_2)J_X^{\mathrm{corr},\mu\nu}$$



,

where we introduced a corrected operator that acts directly on the amplitude

$$J_X^{\mathrm{corr},\mu\nu} = J_X^{\mu\nu} + \mathrm{i}k_X^{[\mu}u_1^{\nu]}\,\check{u}_1\cdot\partial_{q_2}\,. \tag{4.84}$$

With the help of the corrected operator in Eq. (4.84), we can now compute the connected dynamical contribution to the radiated angular momentum starting from Eq. (4.47), which involves a second dynamical radiation kernel.

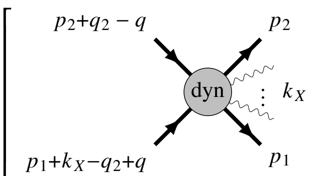
We solve momentum conservation in the right radiation kernel in a similar fashion to the left kernel $q_1' = k_X - q_2'$. Finally, in order to write the expression in a more familiar form, we change variables $q_2' \to q = q_2 - q_2'$. The integral can then be put in the form

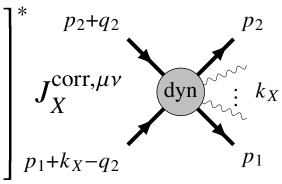
$$\Delta J^{\mu\nu}_{\text{dyn}} = \left\langle \int \hat{\mathrm{d}}^D q \, e^{-\mathrm{i}q\cdot\Delta b} \delta(2p_1\cdot q)\delta(-2p_2\cdot q) \mathcal{I}^{\mu\nu}_{\text{dyn},J}(q,p_i,b_i) \right\rangle , \tag{4.85}$$

where the *angular momentum kernel* of the interference of dynamical amplitudes $\mathcal{I}^{\mu\nu}_{\text{dyn},J}$ is

$$\mathcal{I}^{\mu\nu}_{\text{dyn},J} = b_1^{[\mu}\mathcal{I}_R^{\nu]} + \sum\!\!\!\!\!\!\int_X \int \hat{\mathrm{d}}^D q_2 \, \hat{\delta}(2p_1\cdot(k_X - q_2))\, \hat{\delta}(2p_2\cdot q_2)$$

$$\times \left[\quad \right]^* J_X^{\text{corr},\mu\nu} \tag{4.86}$$





and $\mathcal{I}_R^{\nu}$ is the radiated momentum kernel defined in Eq. (4.46). The fact that the form of the angular momentum kernel derived in Eq. (4.86) is not symmetric in the particle labels 1,2 is a remnant of our choice of parameterization of loop momenta. By contrast to the radiated momentum kernel, the radiated angular momentum kernel is defined only up to a surface term of the form

$$\partial_q^{[\mu}\left[e^{-\mathrm{i}q\cdot\Delta b}\mathcal{I}^{\nu]}\right] , \tag{4.87}$$

subject to the particular solution of momentum conservation in the right kernel. The leading order contribution to the hard radiated angular momentum kernel $\mathcal{I}^{\mu\nu}_{\text{dyn},J}$ starts at order $\mathcal{O}(G^3)$ in perturbation theory where we have tree-level five-particle amplitudes entering the integrals and a single messenger particle being exchanged.

4.5 Results

Radiated Momentum

The radiated momentum was first computed in [59] and starts at $\mathcal{O}(G^3)$

$$\Delta P^{\mu}_{(2),\mathrm{GR}} = \frac{G^3 m_1^2 m_2^2}{|\Delta b|^3} \left(m_1 \check{p}_1^{\mu} + m_2 \check{p}_2^{\mu}\right) \mathcal{E}(\sigma)\,, \tag{4.88}$$

where we have introduced the functions

$$\frac{\mathcal{E}(\sigma)}{\pi} = f_1(\sigma) + f_2(\sigma)\log\left(\frac{\sigma+1}{2}\right) + f_3(\sigma)\frac{\sigma\,\mathrm{arccosh}(\sigma)}{2\sqrt{\sigma^2-1}} \tag{4.89}$$

$$\begin{aligned} f_1(\sigma) &= \frac{210\sigma^6-552\sigma^5+339\sigma^4-912\sigma^3+3148\sigma^2-3336\sigma+1151}{48\left(\sigma^2-1\right)^{3/2}}\,, \\ f_2(\sigma) &= -\frac{35\sigma^4+60\sigma^3-150\sigma^2+76\sigma-5}{8\sqrt{\sigma^2-1}}\,, \\ f_3(\sigma) &= \frac{\left(2\sigma^2-3\right)\left(35\sigma^4-30\sigma^2+11\right)}{8(\sigma^2-1)^{3/2}}\,. \end{aligned} \tag{4.90}$$

In scalar QED, the analogous result is [71]

$$\begin{aligned} \Delta P^{\mu}_{(2),\mathrm{QED}} = \frac{\alpha^3 q_1^2 q_2^2 \pi}{|b|^3}\Bigg[&\frac{q_1 q_2}{m_1 m_2}\frac{u_1^{\mu}+u_2^{\mu}}{\sigma+1}\left\{\frac{\left(3\sigma^2+1\right)\frac{\mathrm{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} - \left(3\sigma^3-4\sigma^2+9\sigma-4\right)}{4(\sigma^2-1)^{3/2}}\right\} \\ &+ \frac{\left(3\sigma^2+1\right)\left(m_2^2 q_1^2 u_1^{\mu} + m_1^2 q_2^2 u_2^{\mu}\right)}{12 m_1^2 m_2^2 \sqrt{\sigma^2-1}}\Bigg]\,. \end{aligned} \tag{4.91}$$

Radiated Angular Momentum

In our computation of angular momentum loss up to $\mathcal{O}(G^3)$, starting with Eq. (4.80), we can use the fact that $\Delta p_2^{\mu} = -\Delta p_1^{\mu}$ due to the absence of linear momentum loss to simplify the equations

$$\begin{aligned} (\Delta J^{\mu\nu})^{(s)}_{\mathrm{soft}}\Big|_{\text{up to } G^3} = -G\,\Delta p_1^{[\mu}\Big(&p_1^{\nu]}\left[\mathcal{I}^{(s)}(\sigma_{12'})-\mathcal{I}^{(s)}(\sigma_{1'1})\right] \\ &-p_2^{\nu]}\left[\mathcal{I}^{(s)}(\sigma_{21'})-\mathcal{I}^{(s)}(\sigma_{2'2})\right]\Big) \end{aligned} \tag{4.92}$$

$$(\Delta J^{\mu\nu})_{\mathrm{soft}}\Big|_{G^2} = -G\,\Delta p_1^{[\mu}\left(p_1^{\nu]} - p_2^{\nu]}\right)\mathcal{I}(\sigma) \tag{4.93}$$

where $\sigma = \sigma_{12} > 1$ and $\mathcal{I}(\sigma)$ is familiar from e.g. Damour [73], yielding, at G^3 order

$$(\Delta J^{\mu\nu})^{(s)}_{\text{soft}}\Big|_{G^3} = -G\,(\Delta p_1^{[\mu})^{(1)}(p_1^{\nu]} - p_2^{\nu]})\mathcal{I}(\sigma) \\ - G\,(\Delta p_1^{[\mu})^{(0)}\left(\frac{p_1^{\nu]} - p_2^{\nu]}}{m_1 m_2}(\partial_\sigma \mathcal{I}^{(s)}(\sigma)) + \left(\frac{p_1^{\nu]}}{m_1^2} - \frac{p_2^{\nu]}}{m_2^2}\right)\partial_\sigma \mathcal{I}^{(s)}(1)\right). \tag{4.94}$$

The first line yields

$$-\frac{G^3 m_1 m_2 (m_1+m_2)}{|\Delta b|^3}\frac{3\pi}{4}\frac{5\sigma^2-1}{\sqrt{\sigma^2-1}}\mathcal{I}(\sigma)\Delta b^{[\mu}(p_1^{\nu]} - p_2^{\nu]}) \tag{4.95}$$

$$+\frac{G^3 m_1^3 m_2^3}{|\Delta b|^2}\frac{2(2\sigma^2-1)^2}{(\sigma^2-1)^2}(m_1^2 - m_2^2)\mathcal{I}(\sigma) p_1^{[\mu} p_2^{\nu]} \tag{4.96}$$

Finally, we proceed to perturbatively evaluate the angular momentum kernel from Eq.(4.86) to order $\mathcal{O}(G^3)$. This means we only need to consider the exchange of a single messenger. We thus expand the tree-level amplitudes in the soft region, thereby canceling the terms diverging as $\hbar \to 0$ [5]. We then perform integration-by-parts reduction [40, 41] using Kira 2.0 [74] to the cut master integrals derived in Refs. [53, 59]. In the process of reduction, IR divergences cancel [6]. We refrain from giving the the value for the angular momentum kernel because, as noted above, this object is not well-defined. After performing the remaining (transverse Fourier) integral, we obtain the hard-graviton contribution to the radiated angular momentum at $\mathcal{O}(G^3)$, of the form (4.4), which is characterized by the form factors

$$\widetilde{G}_1^{\text{Hard}} = -\widetilde{G}_2^{\text{Hard}} = \left(\frac{G}{b}\right)^3 m_1^2 m_2^2 \mathcal{B}(\sigma) + \mathcal{O}(G^4)\,, \tag{4.97}$$

$$\widetilde{H}_{12}^{\text{Hard}} = \mathcal{O}(G^4)\,. \tag{4.98}$$

The function $\mathcal{B}(\sigma)$ is expressed in terms of the basis of transcendental functions

[5]These contributions correspond to cuts which cannot be satisfied for finite-energy gravitons.

[6]The spin and orbital pieces in the gauge corresponding to the tree amplitude in (4.41) are found to be individually IR divergent, underscoring the fact that a separation of the angular momentum into spin and orbital contributions is physically meaningless.

found in the radiated linear momentum [53, 59] and is explicitly given by

$$\frac{\mathcal{B}(\sigma)}{\pi} = h_1 + h_2 \log\left(\frac{\sigma+1}{2}\right) + h_3 \frac{\sigma \operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}}\,, \tag{4.99}$$

$$h_1 = \frac{-210\sigma^7 + 882\sigma^6 - 69\sigma^5 - 3399\sigma^4 + 6792\sigma^3 - 6928\sigma^2 + 2943\sigma - 203}{96\left(\sigma^2-1\right)^{5/2}}\,, \tag{4.100}$$

$$h_2 = -\frac{175\sigma^5 - 335\sigma^4 - 490\sigma^3 + 290\sigma^2 + 539\sigma - 243}{16\left(\sigma^2-1\right)^{3/2}}\,, \tag{4.101}$$

$$h_3 = \frac{\left(2\sigma^2-3\right)\left(175\sigma^5 - 275\sigma^4 - 310\sigma^3 + 318\sigma^2 + 87\sigma - 59\right)}{32\left(\sigma^2-1\right)^{5/2}}\,. \tag{4.102}$$

As noted above, the form factors and consequently the radiated angular momentum are manifestly polynomial in the masses. We checked that our results agree with Ref. [54] to all orders in velocity up to the sign of the term proportional to Δb.

In electromagnetism, we find

$$\begin{aligned} G_1^{\text{Hard}} = \frac{\alpha^3}{b^3}\frac{\pi q_1^2 q_2^2}{m_1(\sigma^2-1)^{5/2}}\Bigg[&\frac{q_1 q_2}{m_1 m_2}(\sigma^2+2\sigma-1)\frac{\operatorname{arccosh}(\sigma)}{\sqrt{\sigma^2-1}} \\ &+\frac{q_1^2}{m_1^2}\frac{1}{12}\left(3\sigma^6+3\sigma^4-7\sigma^2+1\right) + \frac{q_1 q_2}{m_1 m_2}\frac{1}{12}\left(\sigma^3-4\sigma^2-\sigma+2\right) \\ &+\frac{q_2^2}{m_2^2}\frac{2}{3}\sigma(\sigma^2-1)\Bigg] + \mathcal{O}(\alpha^4) + \frac{E_1 E_{\text{rad}}}{E}\,, \end{aligned} \tag{4.103}$$

Chapter 5

INSTANTANEOUS FLUXES AND THE RADIATION REACTION FORCE

5.1 Energy and Angular Momentum Fluxes from Radiated Charges

In Chapter 3, we saw that total radiated charges are related to instantaneous radiative fluxes by the equation

$$\int_{-\infty}^{\infty} dt\, \mathcal{F}_Q(t) = \Delta Q_{\rm rad}, \tag{5.1}$$

where Q stands either for E or J, and that the flux balance principle,

$$\dot{Q}_{\rm bin} = -\mathcal{F}_Q, \tag{5.2}$$

relates the rate of change of the binary charge to the instantaneous flux. Because the relative radiation reaction force is a function of $\dot{Q}_{\rm bin}$, it can be related to the corresponding fluxes,

$$\mathcal{F}_E = -\dot{\boldsymbol{r}} \cdot \boldsymbol{F}^{\rm rel}_{\rm rr}, \qquad \mathcal{F}_J = -F^{\rm rel}_{\phi}. \tag{5.3}$$

We will extract the reaction force from the radiative losses in two steps. First, we will recover expressions for the instantaneous fluxes from the total losses by constructing ansätze for these fluxes, integrating them along the scattering trajectory using the 3PM Hamiltonian equations (with known 3PM Hamiltonian taken from [56]), and equating the results to $\Delta E_{\rm rad}$ and $\Delta J_{\rm rad}$. This part of the calculation is analogous to the extraction of the Hamiltonian from the radial action discussed in Section 3.2. In this case, the total radiative losses play the role of the gauge invariant observable, and we are integrating over (angular) momentum *fluxes* instead of the radial momentum. The second step, converting the instantaneous fluxes to the radiation reaction force is almost trivial, as it simply requires us to invert Eq. (5.3).

Flux Matching

Following the conventions used by Manohar et al. [54] to compute the 2PM force, we will work in center-of-momentum frame and polar coordinates, where r and ϕ are the relative distance and polar angle on the scattering plane, respectively. The corresponding canonical momentum is $\mathbf{p} = p_r \mathbf{e}_r + p_\phi \mathbf{e}_\phi$. The radiation reaction force is

$$\boldsymbol{F}_{\rm rr} = F_r \mathbf{e}_r + \frac{F_\phi}{r} \mathbf{e}_\phi, \tag{5.4}$$

where F_r and F_ϕ denote the components of this force in the radial and angular directions, respectively. In these coordinates, the force-flux relation of Eq. (5.3) becomes

$$\mathcal{F}_J = -F_\phi, \qquad \mathcal{F}_E = -\left(\dot{r}F_r + \dot{\phi}F_\phi\right). \tag{5.5}$$

Given perturbative (PM) ansästze for the fluxes, we will integrate them along the scattering trajectory and match them to the total radiated quantities ΔE and ΔJ;

$$\int_{-\infty}^{\infty} dt \mathcal{F}_Q(t) = \Delta Q. \tag{5.6}$$

This time integral can be converted to a radial integral[1] using the relation $dt = dr/\dot{r}$ and the Hamiltonian equation

$$\dot{r} = 2p_r \frac{\partial \mathcal{H}(r, p^2)}{\partial p^2}, \tag{5.7}$$

where $p^2 = |\mathbf{p}|^2$. Explicitly,

$$\Delta Q = 2\int_{r_{\rm min}}^{\infty} \frac{dr}{2p_r(r)\frac{\partial \mathcal{H}(r,p^2)}{\partial p^2}} \mathcal{F}_Q(r, p^2). \tag{5.8}$$

As in Section 3.2, the conservative Hamiltonian is given in terms of $p^2(r)$ and a PM-expanded potential,

$$\mathcal{H}(r, p^2(r)) = \sqrt{p^2(r) + m_1^2} + \sqrt{p^2(r) + m_2^2} + \sum_{n=1}^{\infty} \frac{c_n(p^2(r))}{m^n}\left(\frac{Gm}{r}\right)^n. \tag{5.9}$$

The coefficients c_n are computed in [56] to 3PM order, and have been reproduced in Eq. (3.38) for convenience. The momentum-squared is given by

$$p^2(r) = \bar{p}^2 + \sum_{n=1}^{\infty} \frac{\bar{P}_n}{m^n}\left(\frac{Gm}{r}\right)^n. \tag{5.10}$$

The coefficients $\bar{P}_n$ are also computed to 3PM order in [56][2]. There is a subtle, but critical distinction between the two expansions shown in Eqs.(5.9) and (5.10). The

[1]The radial integral in (5.8) does not give an expression for the energy loss accurate to all PM orders. We have assumed in its derivation that the gravitational interaction of the black holes can be described by a *conservative* Hamiltonian, thus neglecting the effects of dissipation of linear and angular momentum into gravitational radiation. In other words, we are computing the fluxes of radiated momenta along a conservative trajectory, using those fluxes to compute the radiation reaction force felt by the black holes along this trajectory, but subsequently ignoring the deflection from the conservative trajectory caused by this force. The deflection can, in principle, be computed order-by-order in the PM expansion using the results for $F_{\rm rr}$ derived at previous orders, but it does not correct the fluxes at 3PM, so we will ignore it entirely.

[2]The factors of m^{-n} appearing in (5.9) and (5.10) simply account for the choice made in [56] to expand in G/r instead of Gm/r. It is a matter of taste whether or not to absorb the masses into the coefficients. We will do so in our expansions of radiated charges, instantaneous fluxes, and radiation-reaction force $\boldsymbol{F}_{\rm rr}$ so that our PM expansions are explicitly power series of Gm/r.

coefficients c_n of the potential expansion in (5.9) are functions of the *instantaneous* momentum-squared $p^2(r)$, and therefore implicitly functions of r themselves. The coefficients $\bar{P}_n$ of the p^2 expansion in (5.10), by contrast, are functions only of the *asymptotic* momentum-squared $\bar{p}^2 = p_\infty^2$. Such initial, asymptotic kinematic quantities will be distinguished by an over-bar (see Appendix 6.A). The radial integrand in (5.8) depends on $\mathcal{H}$ through the factor $\partial\mathcal{H}/\partial p^2$. When we compute its PM expansion, we must take care to substitute the expressions $c_n(p^2(r))$ and $\sqrt{p^2(r) + m_a^2}$ with their own PM expansions induced by the series (5.10).

This brings us to the construction of the ansätze for the energy and angular momentum fluxes. By counting mass dimensions, we have for the energy flux

$$\mathcal{F}_E(r) = \sum_{n=3}^{\infty} \mathcal{F}_E^{(n)} \left(\frac{m}{r}\right) \left(\frac{Gm}{r}\right)^n . \tag{5.11}$$

This expansion has stripped off the *explicit* dependence on r from the coefficients $\mathcal{F}_E^{(n)}$, which can only depend on the momentum $\boldsymbol{p}$ (or derived kinematic quantities). What isn't obvious is whether $\mathcal{F}_E^{(n)}$ should depend on the *instantaneous* momentum $\boldsymbol{p}(r)$ as in the expansion of the Hamiltonian (5.9) or the *asymptotic* momentum $\bar{\boldsymbol{p}}$ as in the expansion of the momentum-squared variable (5.10). Instantaneous coefficients are the more useful choice for trajectory calculations, and were used to derive the 2PM radiation reaction force in the gravitational context in [54]. On the other hand, Porto et al. work with asymptotic coefficients in their derivations of instantaneous energy fluxes at 3PM and 4PM in [75–77]. We will work with instantaneous coefficients, $\mathcal{F}_E^{(n)}(r, p^2)$ and $\mathcal{F}_J^{(n)}(r, p^2)$, the latter belonging to the angular momentum flux ansatz,

$$\mathcal{F}_J(r) = \sum_{n=2}^{\infty} \mathcal{F}_J^{(n)} \left(\frac{\bar{J}}{r}\right) \left(\frac{Gm}{r}\right)^n , \tag{5.12}$$

where the first nonzero contribution appears at G^2 order, instead of G^3 order as in Eq. (5.11), and $\bar{J}$ denotes the initial scattering-plane angular momentum, $\bar{J} = b|\bar{p}|$.

To ease comparison with the results of Porto et al., we will borrow their notation [75] and write the PM expansions of the radiated charges as

$$\Delta Q = \sum_n \bar{j}^{-n} \Delta Q_{\text{hyp}}^{(n)} \tag{5.13}$$

where

$$\Delta Q_{\text{hyp}}^{(n)} = \bar{j}^n \Delta Q^{(n)}, \tag{5.14}$$

$\bar{j} = \bar{J}/(Gm^2\nu)$ is the dimensionless (initial) angular momentum, and $\Delta Q^{(n)}$ is the term in ΔQ proportional to $(Gm/b)^n$[3].

We first extract the 3PM instantaneous energy flux coefficient, $\mathcal{F}_E^{(3)}$ from the 3PM radiated energy given in [59]. Our coefficient agrees with the asymptotic coefficient $\bar{\mathcal{F}}_E^{(3)}$ given in [76] (where it is called $\bar{\mathcal{F}}_E^{(0)}$ according to its *relative* PM index). Next, we present a new contribution to the radiative flux literature: a calculation of $\mathcal{F}_J$ at 3PM order, which we will also express in terms of instantaneous coefficients. We have also computed the 4PM energy flux and compared our results to those of Porto et al. in Appendix 6.D as a sanity check of our integration procedure.

Instantaneous 3PM Energy Flux

The instantaneous-coefficient energy flux ansatz is given by Eq. (5.11), where $\mathcal{F}_E^{(n)}$ is a function of the instantaneous momentum-squared $p^2(r)$, as opposed to the asymptotic $\bar{p}^2$. Consulting equation (5.8) and noting that the PM expansion of the total radiated energy Δ_E begins at G^3 order, it is clear that that we only need expressions for p_r and $\partial H/\partial p^2$ to leading (G^0) order to compute $\mathcal{F}_E^{(3)}$. The leading p_r term is easily recovered from the p^2 expansion in (5.10),

$$p_r = \sqrt{p^2 - \bar{J}^2/r^2} = \sqrt{\bar{p}^2 - \bar{J}^2/r^2}\left(\frac{Gm}{r}\right)^0 + O(G^1), \tag{5.15}$$

The Hamiltonian derivative is given at leading order by

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial p^2} &= \overline{\frac{\partial\mathcal{H}}{\partial p^2}}\left(\frac{Gm}{r}\right)^0 + O(G^1)\\ &= \overline{\frac{\partial E}{\partial p^2}}\left(\frac{Gm}{r}\right)^0 + O(G^1)\end{aligned} \tag{5.16}$$

where, recall, $E = E_1 + E_2$ is the instantaneous total energy of the two binary constituents. We have also slightly abused the overbar notation to indicate the evaluation of expressions involving instantaneous kinematic quanitites at their initial, asymptotic values.

Inserting the ansatz (5.11) and the leading order expressions (5.15),(5.16) into the matching equation (5.8) and equating the G^3 terms on the right and left yields the

[3]Porto et al. use *relative* PM indexing, so every expression we index with a superscript $^{(n)}$, they index with the superscript $^{(n-n^\star)}$, where $n^\star$ is the leading order of the expansion. For example, our $\Delta E_{\text{hyp}}^{(3)}$ corresponds to their $\Delta E_{\text{hyp}}^{(0)}$, because the radiated energy begins at G^3 order, and our $\Delta J_{\text{hyp}}^{(2)}$ corresponds to their $\Delta J_{\text{hyp}}^{(0)}$, because the radiated angular momentum begins at G^2 order.

3PM coefficient

$$\mathcal{F}_E^{(3)} = \frac{2\gamma\nu\Delta E_{\text{hyp}}^{(3)}}{m\pi\xi(\sigma^2 - 1)}. \tag{5.17}$$

Inserting the explicit expression for $\Delta E_{\text{hyp}}^{(3)}$ in terms of the kinematic variables tabulated in Appendix 6.A yields

$$\mathcal{F}_E^{(3)} = A_E^{(3)} + B_E^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_E^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2 - 1}}, \tag{5.18}$$

where

$$A_E^{(3)} = \frac{\nu^3(1151 - 3336\sigma + 3148\sigma^2 - 912\sigma^3 + 339\sigma^4 - 552\sigma^5 + 210\sigma^6)}{24\gamma^3\xi(-1+\sigma^2)}$$

$$B_E^{(3)} = \frac{-\nu^3(-5 + 76\sigma - 150\sigma^2 + 60\sigma^3 + 35\sigma^4)}{4\gamma^3\xi}$$

$$C_E^{(3)} = \frac{\nu^3\sigma(-33 + 112\sigma^2 - 165\sigma^4 + 70\sigma^6)}{4\gamma^3\xi(-1+\sigma^2)}. \tag{5.19}$$

These coefficients are available in the ancillary file.

Equation (5.18) is the correct expression for the instantaneous 3PM energy flux coefficient, but in calculating it we have performed a sleight of hand. Up to this point, we have made a great deal of noise about the distinction between instantaneous and asymptotic coefficients. We have chosen to use instantaneous coefficients in our ansatz, and within Eq. (5.8) this makes sense, because the integral runs over the radii at each instant of the trajectory. Why, then, isn't there a bar over the total radiated charge ΔQ appearing on the left-hand side of the matching equation? After integration, all r dependence has been traded for b dependence, so this quantity can only depend on asymptotic kinematic quantities. However, the PM coefficients $E_{\text{hyp}}^{(3)}$ of ΔQ appear in the expression (5.18) for $\mathcal{F}_E^{(3)}$, which we have just claimed is given in terms of *instantaneous* coefficients!

In fact, the radiated charge and its PM coefficients are understood to be functions of *instantaneous* kinematic variables in both Eq. (5.8) and Eq. (5.18). The PM expansion of ΔQ is in powers of Gm/b, not Gm/r, so these instantaneous kinematic variables are simply p^2 and its various derived quantities listed in Appendix 6.A. Of course, in the asymptotic past (as $r \to \infty$), $p^2 = \bar{p}^2$ (see Eq. (5.10)), so the instantaneous kinematic variables are all equal to the asymptotic variables on the left-hand side of Eq. (5.8).

Crucially, evaluating the function $E^{(3)}_{\text{hyp}}(b, p^2)$ at $p = p(t)$ anywhere but the $t \to \infty$ limit will return *the wrong answer* for the 3PM radiated energy. Instead, we should think of $E^{(3)}_{\text{hyp}}(b, p^2)$ as a function of instantaneous kinematics which has been *lifted* from its asymptotic value $E^{(3)}_{\text{hyp}}(b, \bar{p}^2)$. When we compute $\mathcal{F}^{(3)}_E(r, p^2)$ using Eq. (5.8), we are asking:

What function of r and the instantaneous kinematics, when antidifferentiated with respect to time through its radial dependence, as in Eq. (5.8), and then evaluated at $t \to \pm\infty$ will yield a difference in these two evaluations (the indefinite integral) equal to $\Delta E(b, \bar{p}^2)$ at G^3 order?

It should come as no surprise that the answer involves the lifted instantaneous function $E^{(3)}_{\text{hyp}}(b, p^2)$ (or indeed, any of its derivatives, which we encounter at sub-leading orders). Returning to the expression $\mathcal{F}^{(3)}_E$ from Eq. (5.18), we can easily compare the result with the 3PM coefficient from (3.15) of [75]. Technically, their coefficient is the *asymptotic* coefficient $\bar{\mathcal{F}}^{(3)}_E(r, p^2)$. In general, comparing fluxes with asymptotic coefficients to fluxes with instantaneous coefficients requires substituting the PM expansion for $p^2(r)$ in terms of $\bar{p}^2$ from Eq. (5.10) into all instantaneous coefficients, expanding these coefficients in powers of Gm/r, and then collecting like powers of Gm/r whose coefficients now depend only on $\bar{p}^2$. However, because $\mathcal{F}^{(3)}_E$ is the *leading* coefficient of the energy flux, doing so would produce an asymptotic flux expansion where the leading asymptotic coefficient $\overline{\mathcal{F}}^{(3)}_E(r, \bar{p}^2)$ only receives contributions from the leading term in the $p^2(r)$ expansion (5.10). To leading order, $p^2(r) = \bar{p}^2$, so we would be left with

$$\bar{\mathcal{F}}^{(3)}_E(r, \bar{p}^2) = \mathcal{F}^{(3)}_E(r, p^2)|_{p^2 \to \bar{p}^2}. \tag{5.20}$$

in other words, at leading order, the flux coefficients are equivalent. Indeed, our expression for $\mathcal{F}^{(3)}_E$ is equivalent to the expression from (3.15) of [75]. Tedious though this exercise may have been, it has (hopefully) inured us to the subtleties of perturbative flux extraction, clearing the way for our calculation of the instantaneous angular momentum flux to 3PM order.

Instantaneous 3PM Angular Momentum Flux

The instantaneous $\mathcal{F}_J$ ansatz (5.12), reproduced here for convenience, reads

$$\mathcal{F}_J(r) = \sum_{n=2}^{\infty} \mathcal{F}_J^{(n)} \left(\frac{\bar{J}}{r}\right) \left(\frac{Gm}{r}\right)^n . \tag{5.21}$$

This flux begins at G^2 order, so the G^3 coefficient corresponds to the *sub-leading* order, by contrast to the G^3 energy flux coefficient. Consequently, we now need to substitute expressions for $p^2(r)$ and $\partial\mathcal{H}/\partial p^2$ at the first sub-leading order into the matching equation (5.8),

$$p_r = \sqrt{p^2 - \bar{J}^2/r^2} = \sqrt{\bar{p}^2 - \bar{J}^2/r^2} \left(\frac{Gm}{r}\right)^0 + \frac{\bar{P}_1}{2\sqrt{\bar{p}^2 - \bar{J}^2/r^2}} \left(\frac{Gm}{r}\right)^1 + O(G^2), \tag{5.22}$$

$$\begin{aligned} \frac{\partial\mathcal{H}}{\partial p^2} &= \overline{\frac{\partial\mathcal{H}}{\partial p^2}} \left(\frac{Gm}{r}\right)^0 + \overline{\frac{d\frac{\partial H}{\partial p^2}}{d(Gm/r)}} \left(\frac{Gm}{r}\right)^1 + O(G^2) \\ &= \overline{\frac{\partial\mathcal{H}}{\partial p^2}} \left(\frac{Gm}{r}\right)^0 + \left(\overline{\frac{\partial^2\mathcal{H}}{\partial p^2 \partial(Gm/r)}} + \overline{\frac{\partial p^2}{\partial(Gm/r)} \frac{\partial^2\mathcal{H}}{\partial(p^2)^2}}\right) \left(\frac{Gm}{r}\right)^1 + O(G^2) \\ &= \overline{\frac{\partial E}{\partial p^2}} \left(\frac{Gm}{r}\right)^0 + \left(\overline{\frac{\partial c_1}{\partial p^2}} + \bar{P}_1 \overline{\frac{\partial^2 E}{\partial(p^2)^2}}\right) \left(\frac{Gm}{r}\right)^1 + O(G^2), \end{aligned} \tag{5.23}$$

where c_1 is the G^1 coefficient of the potential in the conservative Hamiltonian (5.9). Solving the resulting equation first at G^2 order yields

$$\mathcal{F}_J^{(2)} = \frac{\nu \Delta J_{\text{hyp}}^{(2)}}{2\bar{J}\xi\sqrt{\sigma^2 - 1}}. \tag{5.24}$$

At G^3 order, we find

$$\begin{aligned} \mathcal{F}_J^{(3)} = \frac{2\nu\gamma}{\pi\bar{J}\xi(\sigma^2 - 1)} \Bigg[&\Delta J_{\text{hyp}}^{(3)} \\ &+ \frac{2\pi\nu^2}{8\gamma^8\xi^2\sqrt{\sigma^2 - 1}} \left(u_1(\sigma)\Delta J_{\text{hyp}}^{(2)} - 2(m^2\gamma^6\xi^2)u_2(\sigma)\frac{d}{dp^2}\Delta J_{\text{hyp}}^{(2)} \right) \Bigg], \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} u_1(\sigma) &= 1 + 2\sigma^2 - 4\sigma^4 - 4\nu(1 - 2\sigma + 2\sigma^2 - 4\sigma^4 + 3\sigma^5) \\ &\qquad - \nu^2(-1 + \sigma)^3(3 - 7\sigma - 6\sigma^2 + 6\sigma^3) \\ u_2(\sigma) &= 1 - 3\sigma^2 + 2\sigma^4. \end{aligned} \tag{5.26}$$

Inserting the expression for $\Delta J^{(2)}$ from [54] and the expression for $\Delta J^{(3)}$ derived in Chapter 4 into Eq. (5.24) and Eq. (5.25) (and weighting with $\bar{j}$ powers) yields

$$\mathcal{F}_J^{(2)} = A_J^{(2)} + B_J^{(2)} \log\left(\frac{1+\sigma}{2}\right) + C_J^{(2)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}, \tag{5.27}$$

where

$$A_J^{(2)} = \frac{-2\nu^2(-1+2\sigma^2)(-8+5\sigma^2)}{3\gamma^2\xi(-1+\sigma^2)}$$

$$B_J^{(2)} = 0$$

$$C_J^{(2)} = \frac{4\nu^2\sigma(-3+2\sigma^2)(-1+2\sigma^2)}{\gamma^2\xi(-1+\sigma^2)} \tag{5.28}$$

and

$$\mathcal{F}_J^{(3)} = A_J^{(3)} + B_J^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_J^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}, \tag{5.29}$$

where

$$
\begin{aligned}
A_J^{(3)} = \frac{\nu^2}{24\gamma^9\xi^3(-1+\sigma^2)^2} \Big[& 16\nu^4(-1+\sigma)^3(8-40\sigma-37\sigma^2+121\sigma^3+52\sigma^4 \\
& \quad -92\sigma^5-20\sigma^6+20\sigma^7) \\
& +64\nu^3(8-8\sigma-21\sigma^2+13\sigma^3+10\sigma^4+19\sigma^5-31\sigma^7+10\sigma^9) \\
& +2\gamma^5\xi^2\Big(-141+386\sigma-525\sigma^2-683\sigma^3+1377\sigma^4+240\sigma^5-711\sigma^6 \\
& \quad +105\sigma^7+16\gamma^3(1-2\sigma^2)^2(-8+5\sigma^2)\Big) \\
& +16\nu^2(-1+2\sigma^2)\Big(8-5\sigma^2+2\gamma^4\xi(8-25\sigma-4\sigma^2+37\sigma^3-6\sigma^4 \\
& \quad -18\sigma^5+8\sigma^6)\Big) \\
& +\gamma^4\nu\xi\Big(16\sigma(-25+87\sigma^2-92\sigma^4+36\sigma^6)+\gamma\xi(1715-5444\sigma+5641\sigma^2 \\
& \quad +3056\sigma^3-11049\sigma^4+4908\sigma^5+3675\sigma^6-2712\sigma^7+210\sigma^8)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
B_J^{(3)} = \frac{\nu^2}{4\gamma^4\xi(-1+\sigma^2)} \Big[& 2(-62+155\sigma+16\sigma^2-70\sigma^3-90\sigma^4+35\sigma^5) \\
& +\nu(253-944\sigma+701\sigma^2+360\sigma^3-105\sigma^4-440\sigma^5+175\sigma^6)\Big]
\end{aligned}
$$

$$
\begin{aligned}
C_J^{(3)} = \frac{-\nu^2}{4\gamma^9\xi^3(-1+\sigma^2)^2} \Big[& 32\gamma^8\xi^2\sigma(1-2\sigma^2)^2(-3+2\sigma^2) \\
& +16\nu^2\sigma(3-8\sigma^2+4\sigma^4)(-1+\nu^2(-1+\sigma)^3(1-5\sigma-2\sigma^2+2\sigma^3) \\
& \quad +4\nu(1-\sigma-\sigma^3+\sigma^5)) \\
& +\gamma^5\xi^2\sigma(-3+2\sigma^2)(-48+38\sigma+288\sigma^2-140\sigma^3-240\sigma^4+70\sigma^5 \\
& \quad +\nu(85-172\sigma-459\sigma^2+856\sigma^3+135\sigma^4-620\sigma^5+175\sigma^6)) \\
& +16\gamma^4\nu\xi(-1+2\sigma^2)(-3+21\sigma^2-28\sigma^4+12\sigma^6 \\
& \quad +2\nu(3-21\sigma^2+10\sigma^3+28\sigma^4-16\sigma^5-12\sigma^6+8\sigma^7))\Big]\,.
\end{aligned} \tag{5.30}
$$

As before, these expressions are available in the ancillary file.

5.2 Relative Radiation Reaction Force at 3PM Order

With both the energy and angular momentum fluxes in hand, we can finally compute the relative radiation reaction force to 3PM order. Eq. (5.5) for the force components, $\boldsymbol{F}_{\rm rr}$ can be written in polar coordinates as

$$
\boldsymbol{F}_{\rm rr} = F_r \mathbf{e}_r + \frac{F_\phi}{r}\mathbf{e}_\phi \tag{5.31}
$$

$$F_r = \frac{-\mathcal{F}_E + \dot{\phi}\mathcal{F}_J}{\dot{r}}, \qquad F_\phi = -\mathcal{F}_J \tag{5.32}$$

In this coordinate system, the Hamiltonian equations yield [56]

$$\dot{r} = 2p_r \frac{\partial \mathcal{H}(r,p^2)}{\partial p^2}, \qquad \dot{\phi} = \frac{2p_\phi}{r}\frac{\partial \mathcal{H}(r,p^2)}{\partial p^2}, \tag{5.33}$$

where $p_\phi = J/r = \bar{J}/r + O(G^2)$. We will only need $\dot{r}$ and $\dot{\phi}$, and therefore $\partial\mathcal{H}(r,p^2)/\partial p^2$, to G^1 order. For reference,

$$\frac{\partial \mathcal{H}(r,p^2)}{\partial p^2} = \frac{1}{2m\gamma\xi} - \frac{\nu^2}{2m\gamma^8\xi^3}\Big(1 + 2\sigma^2 + 2\nu^2(-1+\sigma)^2(1-4\sigma+2\sigma^2) \\ +\nu(-4+6\sigma-8\sigma^2+8\sigma^3)\Big)\left(\frac{Gm}{r}\right) + O(G^2). \tag{5.34}$$

The radial component of the radiation reaction force is given by

$$F_r = \left(\frac{Gm}{r}\right)^2\left(\frac{1}{rp_r}\right) \\ \times \left(\frac{J}{r}\right)^2\left(A_r^{(2)} + B_r^{(2)}\log\left(\frac{1+\sigma}{2}\right) + C_r^{(2)}\frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \\ + \left(\frac{Gm}{r}\right)^3\left(\frac{1}{rp_r}\right) \\ \times\Bigg[m^2\left(A_{r,1}^{(3)} + B_{r,1}^{(3)}\log\left(\frac{1+\sigma}{2}\right) + C_{r,1}^{(3)}\frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right) \\ + \left(\frac{J}{r}\right)^2\left(A_{r,2}^{(3)} + B_{r,2}^{(3)}\log\left(\frac{1+\sigma}{2}\right) + C_{r,2}^{(3)}\frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right)\Bigg], \tag{5.35}$$

where

$$A_r^{(2)} = \frac{-2\nu^2(-1+2\sigma^2)(-8+5\sigma^2)}{3\gamma^2\xi(-1+\sigma^2)}$$

$$B_r^{(2)} = 0$$

$$C_r^{(2)} = \frac{4\nu^2\sigma(-3+2\sigma^2)(-1+2\sigma^2)}{\gamma^2\xi(-1+\sigma^2)}$$

$$A_{r,1}^{(3)} = \frac{-\nu^3(1151-3336\sigma+3148\sigma^2-912\sigma^3+339\sigma^4-552\sigma^5+210\sigma^6)}{24\gamma^2(-1+\sigma^2)}$$

$$B_{r,1}^{(3)} = \frac{\nu^3(-5+76\sigma-150\sigma^2+60\sigma^3+35\sigma^4)}{4\gamma^2}$$

$$C_{r,1}^{(3)} = \frac{-\nu^3\sigma(-33+112\sigma^2-165\sigma^4+70\sigma^6)}{4\gamma^2(-1+\sigma^2)} \tag{5.36}$$

$$A^{(3)}_{r,2} = \frac{\nu^2}{24\gamma^9\xi^3(-1+\sigma^2)^2}\Big[16\nu^4(-1+\sigma)^3(8-40\sigma-37\sigma^2+121\sigma^3$$
$$+52\sigma^4-92\sigma^5-20\sigma^6+20\sigma^7)$$
$$+64\nu^3(8-8\sigma-21\sigma^2+13\sigma^3+10\sigma^4+19\sigma^5-31\sigma^7+10\sigma^9)$$
$$+2\gamma^5\xi^2\Big(-141+386\sigma-525\sigma^2-683\sigma^3+1377\sigma^4+240\sigma^5$$
$$-711\sigma^6+105\sigma^7+16\gamma^3(1-2\sigma^2)^2(-8+5\sigma^2)\Big)$$
$$+16\nu^2(-1+2\sigma^2)\Big(8-5\sigma^2+2\gamma^4\xi(8-25\sigma-4\sigma^2+37\sigma^3$$
$$-6\sigma^4-18\sigma^5+8\sigma^6)\Big)$$
$$+\gamma^4\nu\xi\Big(16\sigma(-25+87\sigma^2-92\sigma^4+36\sigma^6)$$
$$+\gamma\xi(1715-5444\sigma+5641\sigma^2+3056\sigma^3-11049\sigma^4$$
$$+4908\sigma^5+3675\sigma^6-2712\sigma^7+210\sigma^8)\Big)\Big]$$

$$B^{(3)}_{r,2} = \frac{\nu^2}{4\gamma^4\xi(-1+\sigma^2)}\Big[2(-62+155\sigma+16\sigma^2-70\sigma^3-90\sigma^4+35\sigma^5)$$
$$+\nu(253-944\sigma+701\sigma^2+360\sigma^3-105\sigma^4-440\sigma^5+175\sigma^6)\Big]$$

$$C^{(3)}_{r,2} = \frac{-\nu^2}{4\gamma^9\xi^3(-1+\sigma^2)^2}\Big[32\gamma^8\xi^2\sigma(1-2\sigma^2)^2(-3+2\sigma^2)$$
$$+16\nu^2\sigma(3-8\sigma^2+4\sigma^4)$$
$$\times\Big(-1+\nu^2(-1+\sigma)^3(1-5\sigma-2\sigma^2+2\sigma^3)+4\nu(1-\sigma-\sigma^3+\sigma^5)\Big)$$
$$+\gamma^5\xi^2\sigma(-3+2\sigma^2)\Big(-48+38\sigma+288\sigma^2-140\sigma^3-240\sigma^4+70\sigma^5$$
$$+\nu(85-172\sigma-459\sigma^2+856\sigma^3+135\sigma^4-620\sigma^5+175\sigma^6)\Big)$$
$$+16\gamma^4\nu\xi(-1+2\sigma^2)\Big(-3+21\sigma^2-28\sigma^4+12\sigma^6$$
$$+2\nu(3-21\sigma^2+10\sigma^3+28\sigma^4-16\sigma^5-12\sigma^6+8\sigma^7)\Big)\Big]. \qquad (5.37)$$

The angular component of the relative radiation reaction force is $(F_\phi/r)\boldsymbol{e}_\phi$, where

$$F_\phi = \left(\frac{Gm}{r}\right)^2 \left(\frac{J}{r}\right)\left(A_\phi^{(2)} + B_\phi^{(2)} \log\left(\frac{1+\sigma}{2}\right) + C_\phi^{(2)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right)$$

$$+ \left(\frac{Gm}{r}\right)^3 \left(\frac{J}{r}\right)\left(A_\phi^{(3)} + B_\phi^{(3)} \log\left(\frac{1+\sigma}{2}\right) + C_\phi^{(3)} \frac{\operatorname{arcsinh}\left(\sqrt{\frac{\sigma-1}{2}}\right)}{\sqrt{\sigma^2-1}}\right), \qquad (5.38)$$

with coefficients

$$A_\phi^{(2)} = \frac{2\nu^2(-1+2\sigma^2)(-8+5\sigma^2)}{3\gamma^2\xi(-1+\sigma^2)}$$

$$B_\phi^{(2)} = 0$$

$$C_\phi^{(2)} = \frac{-4\nu^2\sigma(-3+2\sigma^2)(-1+2\sigma^2)}{\gamma^2\xi(-1+\sigma^2)} \qquad (5.39)$$

$$
\begin{aligned}
A_\phi^{(3)} = \frac{-\nu^2}{24\gamma^9\xi^3(-1+\sigma^2)^2} \Big[& 16\nu^4\ (-1+\sigma)^3(8-40\sigma-37\sigma^2+121\sigma^3+52\sigma^4 \\
& \quad -92\sigma^5-20\sigma^6+20\sigma^7) \\
& +64\nu^3(8-8\sigma-21\sigma^2+13\sigma^3+10\sigma^4+19\sigma^5-31\sigma^7+10\sigma^9) \\
& +2\gamma^5\xi^2\Big(-141+386\sigma-525\sigma^2-683\sigma^3+1377\sigma^4+240\sigma^5-711\sigma^6 \\
& \quad +105\sigma^7+16\gamma^3(1-2\sigma^2)^2(-8+5\sigma^2)\Big) \\
& +16\nu^2(-1+2\sigma^2)\Big(8-5\sigma^2+2\gamma^4\xi(8-25\sigma-4\sigma^2+37\sigma^3-6\sigma^4 \\
& \quad -18\sigma^5+8\sigma^6)\Big) \\
& +\gamma^4\nu\xi\Big(16\sigma(-25+87\sigma^2-92\sigma^4+36\sigma^6)+\gamma\xi(1715-5444\sigma+5641\sigma^2 \\
& \quad +3056\sigma^3-11049\sigma^4+4908\sigma^5+3675\sigma^6-2712\sigma^7+210\sigma^8)\Big)\Big]
\end{aligned}
$$

$$
\begin{aligned}
B_\phi^{(3)} = \frac{-\nu^2}{4\gamma^4\xi(-1+\sigma^2)} \Big[& 2(-62+155\sigma+16\sigma^2-70\sigma^3-90\sigma^4+35\sigma^5) \\
& +\nu(253-944\sigma+701\sigma^2+360\sigma^3-105\sigma^4-440\sigma^5+175\sigma^6)\Big]
\end{aligned}
$$

$$
\begin{aligned}
C_\phi^{(3)} = \frac{\nu^2}{4\gamma^9\xi^3(-1+\sigma^2)^2} \Big[& 32\gamma^8\xi^2\sigma(1-2\sigma^2)^2(-3+2\sigma^2) \\
& +16\nu^2\sigma(3-8\sigma^2+4\sigma^4)\Big(-1+\nu^2(-1+\sigma)^3(1-5\sigma-2\sigma^2+2\sigma^3) \\
& \quad +4\nu(1-\sigma-\sigma^3+\sigma^5)\Big) \\
& +\gamma^5\xi^2\sigma(-3+2\sigma^2)\Big(-48+38\sigma+288\sigma^2-140\sigma^3-240\sigma^4+70\sigma^5 \\
& \quad +\nu(85-172\sigma-459\sigma^2+856\sigma^3+135\sigma^4-620\sigma^5+175\sigma^6)\Big) \\
& +16\gamma^4\nu\xi(-1+2\sigma^2)\Big(-3+21\sigma^2-28\sigma^4+12\sigma^6 \\
& \quad +2\nu(3-21\sigma^2+10\sigma^3+28\sigma^4-16\sigma^5-12\sigma^6+8\sigma^7)\Big)\Big]\,.
\end{aligned} \tag{5.40}
$$

All expressions and coefficients in this section are provided in the ancillary file. The 2PM contribution to the radiation reaction force was computed by Manohar et al. [54]. Our 2PM expression for $\boldsymbol{F}_{\rm rr}$ matches theirs up to a Schott term. To our knowledge, the 3PM correction to gravitational radiation reaction has never before been computed, and thus requires cross-checking. To this end, we have performed a systematic comparison of our 3PM fluxes $\mathcal{F}_E$ and $\mathcal{F}_J$ to PN expansions from literature.

Chapter 6

COMPARISON OF 3PM FLUXES TO PN LITERATURE

We saw in Section 3.1 that the Post-Newtonian, or PN, expansion is essentially a simultaneous expansion in weak gravitational field (PM) and non-relativistic velocity, $v \ll 1$, in units where $c = 1$. Consequently, given a PM-expanded quantity accurate to G^n order, one can then expand again in $v \ll 1$ yielding the terms of a PN expansion truncated at G^n order, but accurate to *all* orders in v. The resulting series can then be compared to pre-existing PN approximations of the same quantity, order-by-order in powers $(Gm/b)^x(v)^2y$. The combination $2(x+y)-2$ fixes the PN order, and the higher the PN order at which a PM expansion matches PN literature, the higher confidence one should have in its validity. For convenience, we will make use of a power-counting parameter η. To perform a PN expansion, we dress $(Gm/b) \to \eta^2(Gm/b)$ and $(v) \to \eta(v)$, expand in powers of η, and then set $\eta \to 1$. Half-integer y correspond to odd powers of v. Such terms are usually referred to by "half-PN" order, eg. 4.5PN.

Post-Newtonian approaches to the dynamics of gravitational inspiral have been developed by the classical gravity community for over half a century. The first expression for the gravitational-wave energy flux radiated from a binary system was computed at leading order in 1963 by Peters and Matthews [78, 79]. Since then, the state of the art has been extended to the third subleading, or *relative* 3PN order, both for the energy and angular momentum fluxes. The energy flux was computed to a breathtaking 6PN absolute order (3PN relative) in 2007 by Arun et al. [79], including terms up to G^7 order in the PM approximation. The 5PN absolute (3PN relative) expression for angular momentum flux followed shortly thereafter in 2009 [80].

The primary difficulty in comparing our 3PM flux expressions to these PN approximations is mismatch in gauges used to compute the two results. Unlike the total radiated charges $\Delta E, \Delta J$, the instantaneous fluxes are not gauge-invariant quantities. Further complicating the comparison is the absence of gravitational field degrees-of-freedom from either flux expression. We cannot simply perform a gauge transformation of the fields because they have been "integrated out" in passing to the effective theory. The gauge transformations of the full theory induce *residual gauge transformations* in the effective theory.

6.1 Induced Canonical Transformations

On one hand, gauge transformations induce residual active transformations of the binary coordinates. By assumption, the full gauge transformations do not vary the action of the complete system from Eq. (3.1). It follows that the effective action obtained by literally integrating out the gravitational field must be invariant under the residual coordinate transformations. Working in the Hamiltonian formalism, we conclude that the gauge transformations of the full theory induce *canonical transformations* of the binary phase-space $(\boldsymbol{r}, \boldsymbol{R}, \boldsymbol{p}, \boldsymbol{P})$.

Furthermore, the canonical transformations dictate the transformation of the effective Hamiltonian. Consequently, two candidate expressions for the effective Hamiltonian are equivalent up to gauge transformations if and only if the expressions are related by a canonical transformation. Indeed, Bern et al. demonstrate the gauge equivalence of their isotropic-gauge 3PM Hamiltonian to the harmonic-gauge PN Hamiltonian by constructing the associated canonical transformation in [56]. Conversely, if we are given two expressions for the Hamiltonian which we know are related by a gauge transformation, we can extract the residual canonical transformation from them. Because we must work with truncated expansions, such a pair of Hamiltonian expressions is not sufficient to uniquely fix a canonical transformation at all orders, but we will see that matching the G^3, $3PN$ Hamiltonians from literature is more than sufficient to fix the canonical transformation to the orders needed for our flux comparisons.

We are interested in proving that our energy and angular momentum fluxes, given in isotropic gauge, are equivalent to expressions from the literature, which are given in ADM gauge. As we will see, there is more to the residual gauge dependence of the fluxes than the induced canonical transformations. However, constraining these canonical transformations is the first step towards comparing the fluxes, so we will describe this process in detail. Recall that we computed each flux by constructing an ansatz and matching it, term-by-term, to the PM expansion of the corresponding radiative loss. Similarly, we will fix the residual coordinate transformations by constructing an ansatz for them and using it to transform an expression for the (PN-expanded) Hamiltonian in ADM gauge into isotropic gauge. We then equate the result, term-by-term, to the (PN expansion of) the isotropic-gauge Hamiltonian, and use the resulting system of equations to solve for the ansatz coefficients.

As a warm-up, consider the action of a canonical transformation on a *scalar* function $O(\chi)$ of the relative phase-space coordinate $\chi = (\boldsymbol{r}, \boldsymbol{p})$ in isotropic-gauge. Let

$\tilde{\chi} = \Phi(\chi)$ represent the canonical transformation mapping isotropic coordinates to another (target) coordinate system. In the target coordinates, the function is represented $\tilde{O}(\tilde{\chi})$. That these functions are equivalent up to the transformation Φ entails

$$\tilde{O}(\tilde{\chi}) = (O \circ \Phi^{-1})(\tilde{\chi}) = O(\chi). \tag{6.1}$$

Given an expression for a function $\tilde{O}(\tilde{\chi})$ found in the literature in, say, ADM gauge, we first replace $\tilde{\chi} \to \chi$, so that we are using only one "dummy variable" χ in both gauges. The coordinate transformation (now an active transformation) is encoded entirely in the functional form of $\tilde{O}(\chi) = (O \circ \Phi^{-1})(\chi)$.

We construct a transformation ansatz $\Phi[\vec{a}]$, where $\vec{a}$ are the undetermined coefficients, and use it to transform $\tilde{O}$,

$$\tilde{O}(\chi) \to (\tilde{O} \circ \Phi[\vec{a}])(\chi). \tag{6.2}$$

If the target function is really related to our isotropic-gauge function by a canonical transformation, then by Eq. (6.1),

$$(\tilde{O} \circ \Phi[\vec{a}])(\chi) = (O \circ \Phi^{-1} \circ \Phi[\vec{a}])(\chi). \tag{6.3}$$

Consequently, if we have constructed a large enough ansatz, there must exist coefficients $\vec{a}^\star$ such that

$$(\tilde{O} \circ \Phi[\vec{a}^\star])(\chi) = O(\chi), \tag{6.4}$$

where $O(\chi)$ is our isotropic-gauge function. In practice, we construct an ansatz for the *generator* $\mathcal{G}_C[\vec{a}]$. We then use it to construct the transformation $\Phi[\vec{a}]$ as a differential operator using a truncated exponential map,

$$\hat{\Phi}[\vec{a}](\bullet) = \exp\left(\epsilon\{\bullet, \mathcal{G}_C[\vec{a}]\}\right) = \hat{\Phi}_K[\vec{a}](\bullet) + O(\epsilon^{K+1}), \tag{6.5}$$

where

$$\hat{\Phi}_K[\vec{a}] = \sum_{k=0}^{K} \frac{\epsilon^k}{k!} \{\cdots\{\{\bullet, \underbrace{\mathcal{G}_C[\vec{a}]\}, \mathcal{G}_C[\vec{a}]\}, \cdots, \mathcal{G}_C[\vec{a}]\}}_{k}. \tag{6.6}$$

Composition of a phase-space function O with Φ is thus approximated to order ϵ^K by the action of this truncated operator,

$$O \cdot \Phi[\vec{a}] = \hat{\Phi}_K[\vec{a}]\, O + O(\epsilon^{K+1}). \tag{6.7}$$

To invert $\hat{\Phi}_K$, we simply flip the sign of ϵ in Eq. (6.6). For our purposes, ϵ is just a power-counting parameter. We will construct $\mathcal{G}_C$ such that the leading term in its PN expansion scales as a positive power $p \geq 2$ of η. Thus, the overall η power (and therefore PN order) of the nested Poisson brackets in Eq. (6.6) strictly increases with the power of ϵ. When performing the transformation, we simply determine the lowest K such that the $O(K+1)$ corrections to the truncated transformation will be higher order in η than the PN expansions we are comparing to. The exact value of K will depend on the leading PN order of the generator, but such a K always exists.

Ultimately, we will use this truncated transformation to compare Hamiltonians (and later fluxes) in different gauges. To simplify the exposition, we have so far assumed that our phase-space functions $O(\chi)$ transform as scalars under canonical transformations, $\tilde{O}(\chi) = O \circ \Phi^{-1}(\tilde{\chi})$. Hamiltonians, however, do not generally transform as scalars. Canonical transformations do not map the Hamiltonian in one coordinate system $\mathcal{H}(\chi)$ to the same function in the new coordinate system, $\tilde{H}(\tilde{\chi})$. Instead, the Hamiltonian is mapped to a new function with an unfortunate sobriquet: the "Kamiltonian" $\tilde{\mathcal{K}}(\tilde{\chi})$. Referring to Goldstein's Classical Mechanics [81], there is a subgroup of canonical transformations

$$\chi \to \Phi(\chi) \tag{6.8}$$

with no explicit time dependence under which the Hamiltonian *does* transform as a scalar. One may then compose this transformation with a *scaling* transformation

$$(\boldsymbol{r}, \boldsymbol{p}) \to \lambda(\boldsymbol{r}, \boldsymbol{p}) = (\mu \boldsymbol{r}, \nu \boldsymbol{p}), \tag{6.9}$$

where μ, ν are some real scaling parameters to produce a *extended canonical transformation*

$$\chi \to \tilde{\chi} = \lambda \circ \Phi(\chi) \tag{6.10}$$

In terms of the dummy χ, this corresponds to an active transformation

$$\mathcal{H}(\chi) \to \tilde{\mathcal{K}}(\chi) = (\mu\nu) \times \mathcal{H} \circ \Phi^{-1} \circ \lambda^{-1}(\chi). \tag{6.11}$$

Such transformations are called *extended canonical transformations*. We mention them because it is common in the PN literature to present *reduced* Hamiltonians in *reduced* coordinates. For example, in [7], this means that the Hamiltonians have undergone a canonical scale transformation. The transformation is straightforward, but if one forgets to include scaling transformations in their canonical generator ansatz, the matching equations will not admit any solutions.

We are now prepared to construct the generator ansatz and perform the PN matching. The ADM-gauge Hamiltonian $\mathcal{H}^{\text{ADM}}$ is given in Eq.(5.17) of [7] to 3PN order. We will compare it to the isotropic-gauge Hamiltonian $\mathcal{H}^{\text{iso}}$ by first expanding the PM expression from [56] (reproduced in Eqs. (3.37), (3.38)) in $\nu << 1$ to 3PN order. In ADM gauge, there are three independent degrees of freedom: the relative separation r, the radial momentum $p_r = \boldsymbol{p} \cdot \boldsymbol{r}/r$, and the momentum-squared $p^2 = |\boldsymbol{p}|^2$. In isotropic gauge, $p_r = 0$. At order G^3, we may ignore back-reaction from the radiation of E and J on the trajectory, and so $J = \bar{J} = r\sqrt{p^2 - p_r^2}$. We constructed ansätze for $\mathcal{G}_C$ as a power series in the ADM variables,

$$\mathcal{G}_C[\vec{a}] = \sum_{ijk} a_{ijk}\eta^{2i+2j+k}\,(r)\left(\frac{Gm}{r}\right)^i p^{2j} p_r^k. \tag{6.12}$$

We use this generator to construct $\hat{\Phi}[\vec{a}]$ according to Eq. (6.6), and then use Eq. (6.11) to construct the transformation from ADM to isotropic coordinates. Because we have defined Φ as the (nonscaling) map from isotropic to ADM coordinates, Φ, not Φ^{-1} will appear in the ADM to isotropic Hamiltonian transformation. The resulting matching equation reads

$$\mathcal{H}^{\text{iso}} = \mathcal{K}^{\text{iso}} \equiv \frac{1}{\mu\nu} \times \hat{\Phi}[\vec{a}]\left(\mathcal{H}^{\text{ADM}} \circ \lambda\right). \tag{6.13}$$

μ and ν are simply thrown on the pile of undetermined coefficients $\vec{a}$ to be determined.

As expected, these constraints admit solutions. Using the G^3, $3PN$ Hamiltonian approximations, these fix the generator to G^3,$2.5PN$ order, which is more than sufficient to match all fluxes at the orders considered in this work. Expressions for the Hamiltonians, canonical generator, and derived q, p transformations are all provided in the ancillary file.

6.2 Induced Gauge Transformations of the Instantaneous Fluxes

We have hinted that the gauge transformations of the fluxes are more complicated than those of scalars or Hamiltonians. In fact, the residual gauge transformations of the fluxes cannot be described with canonical transformations alone. To understand the transformation properties of the instantaneous fluxes $\mathcal{F}_{E/J}$, we need to be more precise about their definition in the full theory. In Chapter 3, we saw that an instantaneous flux may be defined with respect to a Noether current $\mathcal{J}_Q$ by specifying a foliation of the spacetime into Cauchy surfaces over which to integrate the current. The resulting fluxes are functions of some coordinate t parameterizing the timelike trajectory of the binary. A given time t corresponds to a particular point $x(t)$ along the binary trajectory, which is itself a codimension-three surface embedded in the spacetime. It follows that there is no unique extension of a point x along the trajectory with time coordinate t to a Cauchy surface defining a time-slice t for the entire spacetime. However, a particular choice of Cauchy surface at one time t may be extended to a foliation of the spacetime[1] by the flow of a timelike vector field, so it is sensible to speak of the instantaneous flux $\mathcal{F}_Q(t)$ as a function of all $t \in \mathbb{R}$. The point is that different foliations lead to different time dependencies for the fluxes associated to a single physical scattering process. The various Cauchy foliations are generally related by diffeomorphisms of the gravitational theory, and so the associated ambiguity in the instantaneous fluxes as functions of time may be understood as a manifestation of gravitational gauge freedom.

The fluxes defined in Chapter 5 as phase-space functions[2] of the effective theory thus carry two residual dependencies on the diffeomorphisms of the full theory. On one hand, a diffeomorphism acts directly on the spacetime as an active coordinate transformation. In the effective theory, such transformations induce canonical transformations of the binary phase-space. We will slightly abuse notation by using the symbol Φ to represent both the spacetime diffeomorphism in the full theory, $\Phi : x \mapsto \tilde{x} = \Phi(x)$, and the induced canonical transformation in the effective theory, $\Phi : \chi \mapsto \tilde{\chi} = \Phi(\chi)$, where the maps are distinguished by their arguments. The gauge transformation of the fluxes thus includes the active canonical transformation (in terms of χ) $\Phi : \mathcal{F}_Q(\chi) \mapsto \mathcal{F}_Q \circ \Phi^{-1}(\chi)$.

On the other hand, by deforming the Cauchy surface $\mathcal{S}_x$ through which the binary trajectory passes at spacetime point $x(t)$, the diffeomorphism has altered the value of the instantaneous flux *associated to the the original trajectory point* $x(t)$. Explicitly,

$$\Phi : \mathcal{F}_Q(x) \mapsto \tilde{\mathcal{W}}_Q(x) = \mathcal{W}_Q \circ \Phi^{-1}(x) \neq \mathcal{F}_Q \circ \Phi^{-1}(x). \tag{6.14}$$

Figures 6.2,6.2, and 6.2 depict example trajectories in a 1 + 1 spacetime, and will hopefully clarify this point. Let $\mathcal{F}_Q(x(t))$ denote the flux obtained by integrating the Noether current over the Cauchy surface at time t in Figure 6.2. The diffeomorphism relating the surfaces of Fig. 6.2 to those of Figure 6.2 is simply a global shift in the

[1]Provided the spacetime is globally hyperbolic, which we will always assume.

[2]We have been somewhat careless about specifying the *domain* of the flux functions. When we write $\mathcal{F}_Q(t)$, we are referring to the function obtained by substituting the phase-space trajectory $\chi(t)$ into our phase-space flux expressions, $\mathcal{F}_Q(t) = \mathcal{F}_Q(\chi) \circ \chi(t)$. If we apply the induced canonical transformation (not the full gauge transformation) to both the phase-space function $\mathcal{F}_Q(\chi)$ *and* the trajectory $\chi(t)$ and then compose them, $\mathcal{F}_Q(t)$ will, of course, be left invariant, provided the canonical transformation is not explicitly time-dependent (which we always assume). In this section, we will directly compare our phase-space expressions to corresponding expressions in the literature, without any need for trajectories. For this reason, we must account for canonical transformations.

spatial direction. Points on the trajectories of the constituents transform as $\Phi : x \mapsto y = x + a$, but the flux at any given time is unaffected because the Cauchy surfaces are stabilized by the shift. Therefore $\Phi : \mathcal{F}_Q(x) \mapsto \tilde{\mathcal{W}}_Q(x) = \mathcal{W}_Q \circ \Phi^{-1}(x) = \mathcal{F}_Q \circ \Phi^{-1}(x)$, so the flux transforms as a scalar. By contrast, the diffeomorphism relating Figure 6.2 to Figure 6.2 deforms the Cauchy surfaces, altering the associated fluxes as well as applying the usual active coordinate transformation. Therefore, $\Phi : \mathcal{F}_Q(x) \mapsto \tilde{\mathcal{W}}_Q(\tilde{x}) = \mathcal{W}_Q \circ \Phi^{-1}(x) \neq \mathcal{F}_Q \circ \Phi^{-1}(x)$.

In terms of the phase-space coordinates χ,

$$\Phi : \mathcal{F}_Q(\chi) \mapsto \tilde{\mathcal{W}}_Q(\chi) = \tilde{\mathcal{W}}_Q \circ \Phi^{-1}(\chi) \neq \mathcal{F}_Q \circ \Phi^{-1}(\chi) \tag{6.15}$$

in general. As in the case of the Hamiltonian/Kamiltonian transformation, both the function and the point at which it is evaluated have changed. The relationship between $\mathcal{F}$ and $\mathcal{W}$, however is generally much more complicated than the simple scaling relating $\mathcal{H}$ to $\mathcal{K}$.

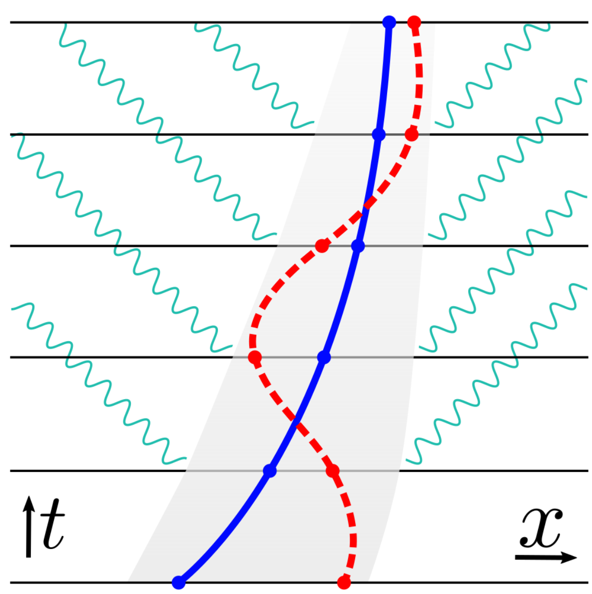


Figure 6.2.1: A binary system (red and blue) emitting gravitational radiation (green) through Cauchy surfaces (black). This diagram depicts only a finite spacetime patch, not the entire spacetime.

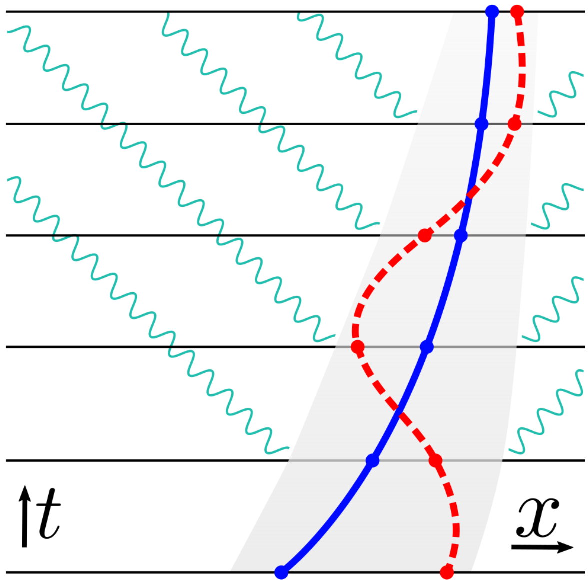


Figure 6.2.2: The same binary system shown in Figure 6.2, but with a different Cauchy foliation, related to the original by a global spatial shift.

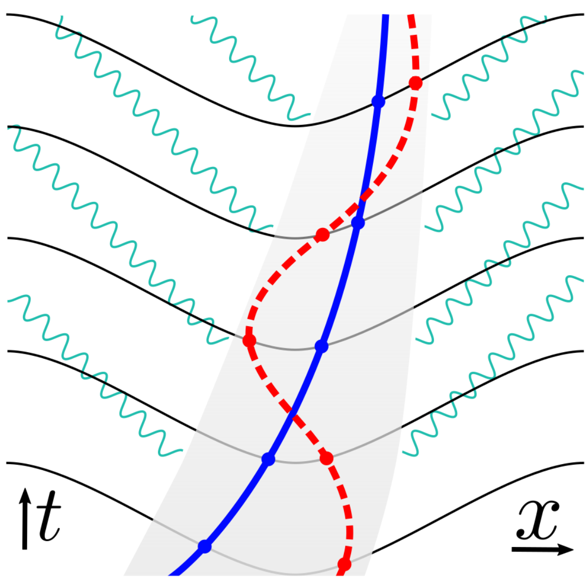


Figure 6.2.3: The same system shown in Figure 6.2, but now with a Cauchy foliation related to the original by a non-trivial small diffeomorphism.

6.3 Schott Terms, Flux Balance, and Small Diffeomorphisms

At first blush, the deformation-transformation $\mathcal{F}_Q \to \mathcal{W}_Q$ is a huge obstacle to our comparisons because it isn't at all obvious how to construct an ansatz for this map. If one can apply *any* diffeomorphism, then it is hard to imagine what constraints the induced transformation would need to satify. Fortunately, we do not need to consider the entire diffeomorphism group. We are only concerned with those diffeomorphisms which do not alter asymptotic fluxes, namely the *small gauge transformations*. The conservation of Noether charges in the full system implies

$$\int_{-\infty}^{\infty} dt\, \mathcal{F}_Q(t) = -\Delta Q_{\text{bin}} = \Delta Q_{\text{rad}}, \tag{6.16}$$

but we know that the total radiative loss, ΔQ_{rad}, can be computed by subtracting the flux of $\mathcal{J}_Q$ across $\mathcal{I}^-$ from the flux across $\mathcal{I}^+$. By assumption, the small diffeomorphisms do not alter these fluxes, so ΔQ_{rad} is guaranteed to be invariant under the small gauge transformations. Consulting Eq. 6.16, invariance of ΔQ_{rad} implies

$$\int_{-\infty}^{\infty} dt\, \left(\mathcal{W}_Q(t) - \mathcal{F}_Q(t)\right) = 0, \tag{6.17}$$

where, as before, $\mathcal{F}_Q(t) = \mathcal{F}_Q(\chi) \circ \chi(t)$ and $\mathcal{W}_Q(t) = \tilde{\mathcal{W}}_Q(\tilde{\chi}) \circ \tilde{\chi}(t)$, such that $\chi(t)$ and $\tilde{\chi}(t)$ are the trajectory coordinates in the two respective gauges.

Taken as a function of time along the trajectory, the residual gauge transformation of an instantaneous flux is given by

$$\mathcal{F}_Q(t) \to \mathcal{W}_Q(t) = \mathcal{F}_Q(t) + \mathcal{F}_Q^{\star}(t), \quad \text{where} \quad \int_{-\infty}^{\infty} dt\, \mathcal{F}_Q^{\star}(t) = 0, \tag{6.18}$$

and the term $\mathcal{F}_Q^{\star}(t)$ is called a *Schott term*. Taken as a function of phase-space, the flux also picks up the residual canonical coordinate transformation, so that the complete transformation becomes

$$\mathcal{F}_Q(\chi) \to \tilde{\mathcal{W}}_Q(\chi) = \tilde{\mathcal{F}}_Q(\chi) + \tilde{\mathcal{F}}_Q^{\star}(\chi). \tag{6.19}$$

This transformation can be described either as the addition of a Schott term $\mathcal{F}_Q^{\star}(\chi)$ followed by a canonical transformation Φ, or as a canonical transformation Φ followed by the addition of a Schott term $\tilde{\mathcal{F}}_Q^{\star}(\chi)$. Of course, the resulting flux $\tilde{\mathcal{W}}_Q(\chi)$ is the same either way, but the expressions of the Schott terms will differ,

$$\tilde{\mathcal{F}}_Q^{\star}(\chi) = \mathcal{F}_Q^{\star} \circ \Phi^{-1}(\chi). \tag{6.20}$$

For this reason, it will be important to keep track of which canonical coordinate system we use to formulate our ansatz for the Schott term. In Section 6.1, we matched the Hamiltonians by writing an ansatz for Φ, which maps isotropic coordinates to ADM coordinates, and then performing the inverse transformation from ADM to isotropic coordinates using this ansatz. Associating the $\tilde{\chi}$ coordinates with isotropic coordinates and using Eq. (6.19) to apply the ADM to isotropic flux transformation,

$$\mathcal{F}_Q^{\mathrm{ADM}} \rightarrow \mathcal{W}_Q^{\mathrm{iso}} = \mathcal{F}_Q^{\mathrm{ADM}} \circ \Phi + \mathcal{F}_Q^{\star iso}, \tag{6.21}$$

where the usual composition with Φ^{-1} has been replaced with Φ to invert the isotropic to ADM transformation. Our Schott terms are thus defined in *isotropic* coordinates.

A Schott term is a total time derivative. We will assume that it has no explicit time dependence and can be written as a function of phase-space. Thus, any Schott term can be written as the Poisson bracket of some phase-space function (which we will call the *Schott Generator* $\mathcal{G}_Q^\star$), and the isotropic-gauge Hamiltonian,

$$\mathcal{F}_Q^\star = \{\mathcal{G}_Q^\star, \mathcal{H}^{\mathrm{iso}}\}. \tag{6.22}$$

To guarantee that the time integral of the Schott term vanishes, it is sufficient to require the generator to fall off faster than r^{-1},

$$\lim_{r\to\infty} \mathcal{G}_Q^\star \times r^p = 0 \,\forall\, p \leq 1. \tag{6.23}$$

This constraint can be derived by analyzing the leading-order r scaling in the integrand of Eq. (5.8). The Schott generator ansätze are nearly identical to the canonical ansatz,

$$\mathcal{G}_E^\star[\vec{b}] = \sum_{ijk} b_{ijk} \eta^{2i+2j+k} \left(\frac{Gm}{r}\right)^i p^{2j} p_r^k. \tag{6.24}$$

$$\mathcal{G}_J^\star[\vec{c}_J] = J \sum_{ijk} c_{ijk} \eta^{2i+2j+k} \left(\frac{Gm}{r}\right)^i p^{2j} p_r^k. \tag{6.25}$$

Except that an overall factor of r has been added to the canonical generator, but not the Schott generator, because the fluxes have an extra factor of inverse (time~ distance) relative to the Hamiltonian. Additionally, the angular-momentum Schott generator, $\mathcal{G}_J^\star$ picks up an overall power of the initial scattering-plane angular momentum J which is also present in the fluxes $\mathcal{F}_J$. The Schott ansätze are then constructed from the generators and $\mathcal{H}^{\text{iso}}$ using Eq. (6.22),

$$\mathcal{F}_E^\star[\vec{b}] = \{\mathcal{G}_E^\star, \mathcal{H}^{\text{iso}}\} \tag{6.26}$$
$$\mathcal{F}_J^\star[\vec{c}] = \{\mathcal{G}_J^\star, \mathcal{H}^{\text{iso}}\}. \tag{6.27}$$

To construct the flux matching equations, we first PN expand our 3PM isotropic-gauge energy flux to 6PN absolute order and our 3PM isotropic-gauge angular momentum flux to 5PN absolute order. We will take the corresponding ADM-gauge fluxes at the same PN orders (including only terms up to G^3) from Eq. (5.1) of [79] ($\mathcal{F}_E$) and Eqs. (3.4),(3.5),(3.15) of [80] ($\mathcal{F}_J$). In the latter case, the ADM expression is spread out over three equations. The first gives the flux in harmonic coordinates, which only differs from the ADM expression at 2PN relative order. We gather the 0PN (rel.) and 1PN (rel.) J fluxes from Eq.(3.4). The ADM flux at 2PN (rel.) and 3PN (rel.) is given in Eq. (3.5). Finally, the 2.5PN[3] (rel.) correction to the ADM flux is given in Eq. (3.15).

Here we run into a slight problem with the ADM-gauge fluxes. In both papers, they are given as functions of the relative separation r and the relative *radial velocity* $\dot{r}$, whereas our fluxes are functions of separation and radial momentum p_r. At leading order, the relationship is simply $\mu\dot{r} = p_r$, but this picks up corrections at sub-leading orders in η and G which will affect our results. To compute the Legendre transformation at all relevant orders, we take the reduced ADM-gauge Lagrangian from Eq.(5.6) of [82] G^3/3PN (absolute) order, also given in $r, \dot{r}$ coordinates, and compute from it a PN expansion of p_r as a function of $r, \dot{r}$ valid to G^3/3PN (absolute) order. We then use this expression to convert the ADM gauge fluxes to phase-space coordinates. With these fluxes in hand, we can finally construct the matching equations,

$$\mathcal{F}_E^{\text{iso}} = \mathcal{W}_E^{\text{iso}} \equiv \mathcal{F}_E^\star[\vec{b}] + \hat{\Phi}[\vec{a}]\left(\mathcal{F}_E^{\text{ADM}} \circ \lambda\right) \tag{6.28}$$
$$\mathcal{F}_J^{\text{iso}} = \mathcal{W}_J^{\text{iso}} \equiv \mathcal{F}_J^\star[\vec{c}] + \hat{\Phi}[\vec{a}]\left(\mathcal{F}_J^{\text{ADM}} \circ \lambda\right), \tag{6.29}$$

where we must remember to include the scaling transformation by λ explicitly because the ADM fluxes, like the ADM Lagrangian, are given in *reduced* coordinates. Equations (6.28) and (6.29), as well as the Hamiltonian matching equation (6.13) are really shorthand for systems of equations. Each of these equations relates a PN

[3] This term, displayed in the G^3, 4.5PN (abs.) $\mathcal{F}_J/J$ cell in table 6.3, vanishes in isotropic gauge, but not in ADM gauge. That we are able to match it using a canonical transformation and a Schott term confirms that this term is pure gauge.

expansion on the left to a PN expansion on the right, and thus decomposes into a system of equations relating the coefficients of each monomial $G^n\eta^m$ in the PM/PN power counting parameters on the right and left hand sides.

Before solving any of Eqs. (6.13),(6.28), or (6.29), we preemptively zero out all coefficients of terms with η order zero or one. The zero-eta term simply vanishes in the Poisson bracket, and the η^1 term contributes pieces to the Schott terms and canonical transformations which have no dependence on G or η. This violates the assumption that the argument to the exponential map in Eq. (6.6) is proportional to a perturbative parameter (at least $O(\eta^1)$), upon which the exponential truncation scheme depends for its validity.

Odd powers of p_r and $|\boldsymbol{p}|$ can produce Schott and transformation terms with an overall half-PN *relative* scaling. Such terms do appear in the fluxes at order G^4 and higher, but they are absent at G^3 order, so we will not need them to complete the matching. We emphasize the parity of the relative PN order because the absolute PN orders of $\mathcal{H}$ and $\mathcal{F}_E$ are even up to order G^3 in both gauges, so we may preemptively zero out all coefficients of odd powers of p_r in $\Phi[\vec{a}]$ and $\mathcal{F}_E^\star[\vec{b}]$. Only the normalized ADM-gauge angular momentum flux $\mathcal{F}_J^{\mathrm{ADM}}/J$ acquires an odd-PN term, which appears at 4.5PN (absolute) order. We must therefore preserve odd powers of p_r in $\mathcal{F}_J^\star[\vec{c}]$. Finally, because the energy flux begins at G^3 order, we may zero out Schott generator coefficients b_{ijk} in $\mathcal{G}_E^\star$ with $i < 3$. The angular momentum is G^2 at leading PM order, so we also zero out coefficients c_{ijk} in $\mathcal{G}_J^\star$ with $i < 2$.

The dimensions of the resulting systems, which constrain coefficients $\vec{a}, \vec{b}, \vec{c}$, grows rapidly with PN order. It will therefore behoove us to eliminate as many of these coefficients as possible as early as possible. We chose to perform the Hamiltonian matching first because Eq. (6.13) depends only on the canonical parameters $\vec{a}$, and is considerably easier to solve than the flux matching equations.

Even with the canonical transformation fixed, equations (6.28) and (6.29) are prohibitively complicated at 3PN (relative) order, and we we were not able to solve them as simulataneous systems. Luckily, they can be solved iteratively by exploiting a familiar property of perturbative expansions. Substituting an expansion A in some parameter with leading order n into another expansion B in the same parameter will alter the coefficients of B at order n and higher, but *never* at lower orders. In our case, the generator ansätze are PN series, and so substituting them into the matching equations will cause nPN ansatz coefficients to appear in the coefficients of monomials of order nPN or higher, but never in lower order coefficients. Consequently, we may solve the system of equations iteratively in PN order. We begin by solving only the coefficient equations of the leading PN order terms appearing in the matching equations in terms of the leading PN order coefficients of the ansätze. This will fix some of these coefficients, eliminating degrees of freedom in the ansätze. We may then substitute these leading order solutions into the remaining equations, and repeat the process at the next-to-leading PN order (and so on). Because the coefficients at order n do not "contaminate" the equations at lower orders, the dimension of the system at any given step in this iteration is much less than the overall dimension of the complete system.

First, we perform the Hamiltonian matching using the unconstrained restricted canonical ansatz $\Phi[\vec{a}_0]$ and scaling ansatz λ_0. This stage fixes $\lambda_0 \to \lambda$, and $\Phi[\vec{a}_0] \to \Phi[\vec{a}]$. This canonical data is then passed into the energy flux matching equation, along with the unconstrained energy Schott ansatz $\mathcal{F}_E^\star[\vec{b}_0]$. Solving this equation fixes $\mathcal{F}_E^\star[\vec{b}_0] \to \mathcal{F}_E^\star[\vec{b}]$. Finally, we pass $\Phi[\vec{a}]$ and λ to the angular momentum flux matching equation, along with the unconstrained angular momentum Schott ansatz $\mathcal{F}_J^\star[\vec{c}_0]$. Solving this last matching equation fixes $\mathcal{F}_J^\star[\vec{c}_0] \to \mathcal{F}_J^\star[\vec{c}]$[4].

The orders of all the terms we have successfully matched are displayed in Table 6.3. In light of the successful matches, we can be quite confident that our 3PM, isotropic-gauge expressions for the instantaneous energy and angular momentum fluxes are correct. Because the radiation reaction force is so simply related to the instantaneous fluxes, these matches validate our G^3 expression for $\boldsymbol{F}_{\text{rr}}$ as well.

Abs. PN Order	2PN	3PN	4PN	4.5PN	5PN	5.5PN	6PN
Abs. η Order	η^6	η^8	η^{10}	η^{11}	η^{12}	η^{13}	η^{14}
$\mathcal{F}_E$	0	G^3v^2	G^3v^4	0	G^3v^6	0	G^3v^8
$\mathcal{F}_J/J$	G^2v^2 G^3	G^2v^4 G^3v^2	G^2v^6 G^3v^4	0 0*	G^2v^8 G^3v^6		

Table 6.3.1: PN Expansions of 3PM energy and angular momentum fluxes in isotropic gauge. Green cells indicate a match between our PM fluxes and the ADM-gauge fluxes from PN literature at the corresponding order. The G^3, 4.5PN(abs.) $\mathcal{F}_J/J$ entry is starred to remind us that this coefficient vanishes in isotropic gauge, but not in ADM gauge.

[4] All expressions required to perform this matching calculation, together with the resulting canonical transformation and Schott terms, are provided in the ancillary file.

APPENDIX

6.A Instantaneous vs. Asymptotic Variables

The purpose of these matching calculations is to extract instantaneous dynamics from asymptotic scattering data, such as the energy and angular momentum losses. It is crucially important that we do not confuse *instantaneous* data, which describe the phase space configuration of the black holes (and, in principle, the gravitational field) at an arbitrary instant in time, with *asymptotic* data, which describe only the initial and final phase space configurations (or the differences between them).

Initial (asymptotic) values of the canonical variables are typically distinguished with a subscripted ∞, or by a special symbol. For instance, in the CM frame,

$$p_{\text{initial}}^2 = p_\infty^2, \qquad \boldsymbol{r}_{\perp,\text{initial}} = b. \tag{6.30}$$

Unfortunately, there doesn't appear to be any standard notation for distinguishing other asymptotic variables from their instantaneous counterparts. In everything that follows, we will adopt a notation used occasionally by Bern et al. in [56]. We assume that any *bare* kinematic variable X denotes the *instantaneous* variable and interpret it as a function of time or another coordinate of the black hole trajectories (e.g. instantaneous relative distance r in the CM frame). We denote with an *overbar* all *initial*, *asymptotic* values $\bar{X}$ of kinematic variables X. That is,

$$\bar{X} = \lim_{t\to-\infty} X(t). \tag{6.31}$$

The only exceptions to this rule are the scattering observables, such as ΔE, ΔJ, and scattering angle χ, which are asymptotic quantities by definition. Taking another cue from [56], we collect all of our kinematic variables into a brief glossary 6.A.

m_a	mass of black hole $a \in \{1, 2\}$
$m = m_1 + m_2$	total mass
$\nu = \frac{m_1 m_2}{m^2}$	symmetric mass ratio
p_a	four-momentum of black hole a
$\sigma = \frac{p_1^\mu p_{2\mu}}{m_1 m_2}$	Lorentz factor
$\boldsymbol{p}_a$	three-momentum of black hole a
$\boldsymbol{p} = \boldsymbol{p}_{1,\mathrm{CM}} = -\boldsymbol{p}_{2,\mathrm{CM}}$	three-momentum in CM frame
$E_a = \sqrt{\boldsymbol{p}^2 + m_a^2}$	energy of black hole a in CM frame
$E = E_a + E_b$	total energy of black holes in CM frame
$\gamma = \frac{E}{m}$	energy-mass ratio
$\xi = \frac{E_1 E_2}{E^2}$	total energy of black holes in CM frame

6.B An Introduction to the Method of Regions

To illustrate the method of regions, we will adapt a simple (though somewhat artificial) example from[83]. Consider a one-dimensional integral over energies E involving a rational integrand with poles at scales q and m,

$$I = \int_0^\infty dE \frac{E}{(E^2 + q^2)(E^2 + m^2)}. \tag{6.32}$$

The integral can be evaluated directly, giving

$$I = \frac{-\ln \frac{q}{m}}{m^2 - q^2}, \tag{6.33}$$

which can then be expanded in the ratio $\alpha\beta = q/m$ for small α and β, taking advantage of the large separation of scales between the classical weak-field region (I) and the quantum strong-field region (III); and yielding

$$I = -\frac{\ln q/m}{m^2}\left(1 + (q/m)^2 + (q/m)^4 + \cdots\right). \tag{6.34}$$

We wish to reproduce this result by performing α, β expansions *before* evaluating the integral. Clearly, we cannot assume $E \ll m$ or $E \gg q$ generally in the integrand of Eq. (6.32), because these scale separations only hold in their respective regions, not globally in the E integration domain. Indeed, as pointed out in [83], assuming (for example) $E \gg q$ and expanding the integrand in $q/E \ll 1$ yields

$$I \to \int_0^\infty dE \frac{E}{E^2(E^2 + m^2)\left(1 - \frac{q^2}{m^2} + \frac{q^4}{m^4}\right)}, \tag{6.35}$$

which is a singular integral because it has a $1/E$ IR divergence owing to the E^{-2} factor appearing after expansion. This is clearly not correct, and the error was in performing an integrand expansion in a region $E \sim q$ where it did not apply. More precisely, the radius of convergence of the (Laurent) series of $E^2(1 + (q/E)^2)$ about $E \to \infty$ is determined by the distance to the nearest (complex) singularity at $E = \pm qi$. Therefore, the expansion is only valid for $|E| > q$.

To remedy this issue, we break the integral into two integrals over different regions:

$$I = \left(\int_0^{\Lambda_Q} + \int_{\Lambda_Q}^{\infty} \right) dE \frac{E}{(E^2 + q^2)(E^2 + m^2)}, \tag{6.36}$$

where the first integral belongs to the (classical) region $I \cup II$, and the second belongs to the (quantum) region III. We may now perform complementary expansions of the integrand in the two regions. In the classical region, we have $E < m$, and are therefore justified in expanding $(E^2 + m^2)^{-1}$ about $E = 0$. Conversely, in the quantum region, we have $E > q$, and are thus justified in expanding $(E^2 + q^2)^{-1}$ about $E \to \infty$. In particular,

$$\begin{aligned} I_{I \cup II} &= \int_0^{\Lambda_Q} dE \frac{E}{(E^2 + q^2) m^2 (1 + (E/m)^2)} \\ &= \int_0^{\Lambda_Q} dE \frac{E}{(E^2 + q^2) m^2} \left(1 + \frac{E^2}{m^2} + \frac{E^4}{m^4} + \cdots \right) \end{aligned} \tag{6.37}$$

One may then truncate the integrand series at order E^2/m^2 and perform the two integrals (with integrand numerators E and $-E^3$, respectively), yielding

$$I_{I \cup II} = -\frac{\Lambda_Q^2}{2m^4} - \frac{(q^2 + m^2) \log(q/\Lambda_Q) \log(1 + (q/\Lambda_Q)^2)}{m^4}. \tag{6.38}$$

Because $q < \Lambda_Q$ by definition, we may expand the result to lowest order in q/Λ_Q, where we find

$$I_{I \cup II} = -\frac{\Lambda_Q^2}{2m^4} - \frac{\log(q/\Lambda_Q)}{m^2} + \cdots . \tag{6.39}$$

An analogous expansion of the integrand of I_{III} under the assumption $E > q$ produces

$$I_{III} = +\frac{\Lambda_Q^2}{2m^4} - \frac{\log(\Lambda_Q/m)}{m^2} + \cdots . \tag{6.40}$$

Summing these two perturbative results gives

$$I_{I \cup II} + I_{III} = \frac{\log(q/m)}{m^2} + O((q/m)^2), \tag{6.41}$$

which matches the leading term of the direct integral result in Eq. (6.34).

Note that at no stage of the calculation did we actually use $q \ll m$, $q \ll \Lambda_Q$, or $\Lambda_Q \ll m$. We only used the inequalities $q < m$, $q < \Lambda_Q$, or $\Lambda_Q < m$ to justify our expansions. This is because, formally, the method of regions boils down to the observation of the fact that integration and Laurent expansion are interchangeable only when the domain of integration is a subset of the convergent disk associated to the Laurent series. the equivalence of expansion before and after integration is thus valid to all orders in the expansion, regardless of how close the scales may be, provided they satisfy the strict inequalities. If we are to justify truncating the series derived from the integrand expansion after a handful of terms, we must *further* assume that the scales (and the separator) are *widely* separated: $q \ll m$, $q \ll \Lambda_Q$, and $\Lambda_Q \ll m$.

The issue with this formulation of the method of regions is that we have introduced hard cutoffs into our momentum-space integrals by way of the separator Λ_Q. The example given above was simple enough that this cutoff didn't cause any trouble. In reality, we need to apply this expansion technique to multi-loop integrals over products of D-dimensional momentum spaces, where hard cutoffs are notoriously incompatible with symbolic loop integration techniques and with dimensional regularization. The true power of the method of regions becomes apparent when one realizes that it can be done without any explicit cutoffs in a manner completely compatible with (and in fact, reliant on) dimensional regularization. Let's revisit the erroneous attempt to expand the integrand in the quantum region without introducing a cutoff from Eq. (6.35). We saw that the resulting integral had an IR divergence. Similarly, expanding the full integrand in the classical region gives rise to a UV divergence. If we dimensionally regularize both integrals,

$$\begin{aligned}\int_0^\infty dE \frac{E^{1-\epsilon}}{(E^2+q^2)m^2}\left(1-\frac{k^2}{m^2}+\cdots\right) &= \frac{q^{-\epsilon}}{m^2}\left(\frac{1}{\epsilon}+O(\epsilon)\right)+\cdots \\ &= \frac{1}{m^2}\left(\frac{1}{\epsilon}-\ln q+O(\epsilon)\right)+\cdots \end{aligned} \tag{6.42}$$

$$\begin{aligned}\int_0^\infty dE \frac{E^{1-\epsilon}}{(E^2+m^2)E^2}\left(1-\frac{q^2}{E^2}+\cdots\right) &= \frac{m^{-\epsilon}}{m^2}\left(-\frac{1}{\epsilon}+O(\epsilon)\right)+\cdots \\ &= \frac{1}{m^2}\left(-\frac{1}{\epsilon}+\ln m+O(\epsilon)\right)+\cdots \end{aligned} \tag{6.43}$$

Adding these results produces $-(1/m^2)\ln(q/m) + O(\epsilon) + O((q/m)^2)$, and thus taking $\epsilon \to 0$ returns the correct leading-order result. On one hand, it shouldn't be surprising that dimensional regularization eliminated the need for a cutoff. After all, the *raison d'être* of dim-reg is the conversion of momentum-space divergences to ϵ singularities. The interesting thing about this dim-reg calculation is that the IR divergence from the quantum expansion of the integrand in the classical region, and the UV divergence from the classical expansion of the integrand in the quantum region exactly canceled. Such UV-IR cancellation is indicative of a scaleless integral vanishing in dim-reg. In fact, it is not hard to see that this is exactly what is going on in our example. For the sake of brevity, let

$$F(E) = \frac{E}{(E^2+q^2)(E^2+m^2)} \tag{6.44}$$

denote our full integrand. Additionally, let F_C denote the expansion of $F(E)$ in the classical region $E \ll m$,

$$F_C(E) = \frac{E}{(E^2+q^2)m^2}\left(1 - \frac{E^2}{m^2} + \frac{E^4}{m^4} + \cdots\right). \tag{6.45}$$

Conversely, let F_Q denote the expansion of $F(E)$ in the quantum region, $E \gg q$,

$$F_Q(E) = \frac{E}{(E^2+m^2)E^2}\left(1 - \frac{q^2}{E^2} + \frac{q^4}{E^4} + \cdots\right). \tag{6.46}$$

Finally, define $F_{CQ} = F_{QC}$ as the simultaneous expansion of both denominators, one in the classical, and the other in the quantum,

$$F_{CQ}(E) = \frac{E}{m^2E^2}\left(1 - \frac{E^2}{m^2} + \frac{E^4}{m^4} + \cdots\right)\left(1 - \frac{q^2}{E^2} + \frac{q^4}{E^4} + \cdots\right). \tag{6.47}$$

The crucial observation is that the terms of F_{CQ} are *monomials* in E. In other words, these terms are scaleless integrands, and therefore the integral

$$\int_0^\infty dE\, E^{-\epsilon} F_{CQ}(E) = 0 \tag{6.48}$$

in dim-reg. When we expanded under the full integral in Eq. (6.35), we produced an integral which, though not equivalent to the integral we wanted to compute, can nevertheless *itself* be broken into regions,

$$\begin{aligned} I \to \int_0^\infty dE\, F_Q(E) &= \int_0^{\Lambda_Q} dE\, F_Q(E) + \int_{\Lambda_Q}^\infty dE\, F_Q(E) \\ &= \int_0^{\Lambda_Q} dE\, F_{QC}(E) + \int_{\Lambda_Q}^\infty dE\, F_Q(E), \end{aligned} \tag{6.49}$$

where the expansion $F_Q \to F_{QC}$ in the classical region is justified because $E \ll m$ there. The same argument can be made for the other erroneous expansion,

$$I \to \int_0^\infty dE F_C(E) = \int_0^{\Lambda_Q} dE F_C(E) + \int_{\Lambda_Q}^\infty dE F_C(E)$$
$$= \int_0^{\Lambda_Q} dE F_C(E) + \int_{\Lambda_Q}^\infty dE F_{CQ}(E). \qquad (6.50)$$

Adding the two integrals and using $F_{QC} = F_{CQ}$, we find

$$\int_0^\infty dE F_C(E) + \int_0^\infty dE F_Q(E) = \left(\int_0^{\Lambda_Q} dE F_C(E) + \int_{\Lambda_Q}^\infty dE F_Q(E)\right) \qquad (6.51)$$
$$+ \left(\int_0^{\Lambda_Q} + \int_{\Lambda_Q}^\infty\right) dE F_{QC}(E)$$
$$= I.$$

Here, the right-hand-side of the first line in Eq. (6.51) is the correct regions expansion of the integral I, while the integrals on the second line combine into an integral of the doubly-expanded F_{CQ} over the *entire* domain of integration. In the regions parlance, such terms are called *overlap* integrals. Because F_{CQ} is scaleless, this overlap integral vanishes in dim-reg owing to UV-IR cancellation of $1/\epsilon$ poles. *Caveat emptor*: if there are more than two scales in an integral (for example, imagine the integrand of I also had a $(E^2 + m_s^2)^{-1}$ factor), but one only expands into two regions (e.g. I∪II and III), then the overlap integrand will *not be scaleless*! In our example, we would have expanded away the scale-ful factors $(E^2 + q^2)$ and $(E^2 + m^2)$ in the overlap integrand, but $(E^2 + m_s^2)$ would remain, and the overlap integral would not generically vanish. Even in this case, the overlap integral would depend on a single scale m_s, and may be simple enough to evaluate using other methods, at which point one may simply subtract the overlap integral from the full regions expansion. To guarantee that the overlap integral *vanishes*, however, one must generally decompose into as many regions as there are mass-scale poles in the denominator of the integrand.

To summarize, the method of regions allows us to expand an integrand in terms of a hierarchy of scales *as if* this expansion were valid everywhere in the integration domain, so long as we sum over the expanded integrands in all regions, ensure that we have associated regions to mass-scale poles on a one-to-one basis, and evaluate the integrals using dimensional regularization[5].

[5]In reality, not all region decompositions can be handled with dim-reg, and one must introduce

6.C δ-Regulated One-Loop Integrals

The soft limit is intricate and there are various contributions where terms that vanish sit on a pole and therefore can eventually contribute. In order to keep track of such terms it is natural to introduce a power counting variable δ such that all soft messengers have energies $\omega_i \sim \delta$. [6] We are only interested in the leading soft contributions, which are in general $O(\delta^0)$. In the language of the method of regions we are interested in regions where the some of loop momenta are

$$\ell \sim \delta \tag{6.52}$$

Soft factors are of order δ^{-1} the on-shell phase space for

$$\int \mathrm{d}^D \ell \theta(\ell^0)\delta(\ell^2) \sim \delta^2 \tag{6.53}$$

We wish to compute the following integral

$$I^0_{mn} = \mathrm{Im} \int d\Phi(k) \frac{m_m m_n}{(p_m \cdot k + i\epsilon)(p_n \cdot k - i\epsilon)} \tag{6.54}$$

This integral is scaleless in dimensional regularization, but by introducing a fictitious mass of order δ to the propagators, we encode the desired $O(\delta^0)$ terms in the $\ell \sim \delta$ region of the (no longer scaleless) integral. We then evaluate the integral using IBP relations and canonical differential equations.

As always, we introduce

$$y = u_1 \cdot u_2 = \frac{1+x^2}{2x} \tag{6.55}$$

The following integrals constitute a pure basis at one-loop

$$g_1 = \epsilon G_{2,0,1}\,, \quad g_2 = \epsilon^2 \sqrt{y^2-1} G_{1,1,1}\,. \tag{6.56}$$

The canonical DE is

$$\partial_x \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} = \epsilon \left[\frac{1}{x} \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} + \frac{1}{1-x} \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix} \right] \begin{pmatrix} g_1 \\ g_2 \end{pmatrix} \tag{6.57}$$

For the boundary conditions we use the fact that the integrals are regular at $x = 1$ (the eigenvalue at $x = 1$ has the wrong sign to correspond to a classical region). This fixes the following values at $x = 1$

$$g_1 = e^{\gamma_E} \epsilon \Gamma(1-\epsilon)\Gamma(2\epsilon)\,, \quad g_2 = 0\,. \tag{6.58}$$

additional regularization schemes to tame divergences (this is true, for example, in the case of collinear divergences). For our purposes, however, dim-reg will be sufficient.

[6] This δ can be interpreted as a cutoff separating the hard and soft modes.

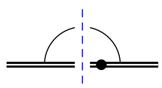
The system is so simple that we could, in principle, solve it to all orders in ϵ in terms of a hypergeometric function. For the present discussion it is sufficient to consider an expansion in ϵ

$$g_1 = \frac{1}{2} + \frac{4}{24}\pi^2\epsilon^2\,, \quad g_2 = \frac{1}{2}\log(x)\epsilon + \dots \tag{6.59}$$

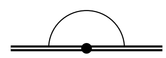
Therefore, the cusp integral is

$$G_{2,0,1} = \frac{1}{2\epsilon} + \dots\,, \quad G_{1,1,1} = \frac{\log(x)}{2\sqrt{y^2-1}\epsilon} + \dots \tag{6.60}$$

The cut integrals can be computed from the unitarity relation or through direct integration



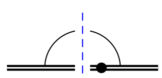
$$= 2\Im\Big[$$



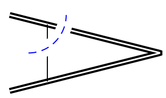
$$\Big] = 2\Im\left[(-\delta + \mathrm{i}\varepsilon)^{-\epsilon}\right] G_{2,0,1}$$

$$= -\delta^{-\epsilon}\sin(\pi\epsilon)e^{\gamma_{\mathrm{E}}\epsilon}\Gamma(1-\epsilon)\Gamma(2\epsilon) \tag{6.61}$$

For the triangle contribution, we exploit the fact that it vanishes at $x = 1$. Note that this integral is negative, while the undotted version (i.e. the phase-space integral) is positive. Up to the required order,



$$= -1 + \dots\,,$$



$$= \frac{\log(x)}{4\sqrt{y^2-1}} + \dots \tag{6.62}$$

where the triangle picks up an additional factor of $1/2$ because there are two (identical) cuts.

6.D Instantaneous Energy Flux at 4PM Order

Instantaneous Energy Flux with Asymptotic Coefficients

Our 4PM sanity check is conceptually quite simple, because Porto et al. have computed $\mathcal{F}_E$ at 3 and 4PM in terms of abstract 3 and 4PM radiative losses $\Delta E^{(3)}_{\text{hyp}}$ and $\Delta E^{(4)}_{\text{hyp}}$ in [75][7]. We can therefore compare our results without needing explicit expressions for these energy losses, which become very complicated at 4PM. Following Porto et al., we write a flux ansatz with coefficients depending only on asymptotic data,

$$\mathcal{F}_E = \sum_{n=3}^{\infty} \bar{\mathcal{F}}_E^{(n)} \left(\frac{m}{r}\right) \left(\frac{Gm}{r}\right)^n . \tag{6.63}$$

Consulting equation (5.8) and noting that the energy loss expansion begins at G^3 order, we see that we need p_r and $\partial H/\partial p^2$ to G^1 order to compute both $\bar{\mathcal{F}}_3$ and $\bar{\mathcal{F}}_4$. These expressions were given in Eqs. (5.22), (5.23), but are reproduced here for convenience,

$$p_r = \sqrt{p^2 - \bar{J}^2/r^2} = \sqrt{\bar{p}^2 - \bar{J}^2/r^2}\left(\frac{Gm}{r}\right)^0 + \frac{\bar{P}_1}{2\sqrt{\bar{p}^2 - \bar{J}^2/r^2}}\left(\frac{Gm}{r}\right)^1 + O(G^2), \tag{6.64}$$

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial p^2} &= \overline{\frac{\partial \mathcal{H}}{\partial p^2}}\left(\frac{Gm}{r}\right)^0 + \overline{\frac{d\frac{\partial H}{\partial p^2}}{d(Gm/r)}}\left(\frac{Gm}{r}\right)^1 + O(G^2) \\
&= \overline{\frac{\partial \mathcal{H}}{\partial p^2}}\left(\frac{Gm}{r}\right)^0 + \left(\overline{\frac{\partial^2 \mathcal{H}}{\partial p^2 \partial(Gm/r)}} + \overline{\frac{\partial p^2}{\partial(Gm/r)}\frac{\partial^2 \mathcal{H}}{\partial(p^2)^2}}\right)\left(\frac{Gm}{r}\right)^1 + O(G^2) \\
&= \overline{\frac{\partial E}{\partial p^2}}\left(\frac{Gm}{r}\right)^0 + \left(\overline{\frac{\partial c_1}{\partial p^2}} + \bar{P}_1 \overline{\frac{\partial^2 E}{\partial(p^2)^2}}\right)\left(\frac{Gm}{r}\right)^1 + O(G^2),
\end{aligned} \tag{6.65}$$

Inserting the expansions (6.63), (6.64), and (6.65) into the matching equation (5.8) and equating the G^4 terms yields the 4PM coefficient

$$\begin{aligned}
\bar{\mathcal{F}}_4 = \frac{3\pi\bar{\gamma}^2\nu\Delta E^{(4)}_{\text{hyp}}}{4\pi m\bar{\xi}(\bar{\sigma}^2-1)^{3/2}} - \frac{2\Delta E^{(3)}_{\text{hyp}}\nu^3}{m\pi\bar{\xi}^3\bar{\gamma}^6(\bar{\sigma}^2-1)^2}\Big[&(\bar{\sigma}-1)^3(10\bar{\sigma}^3 - 10\bar{\sigma}^2 - 9\bar{\sigma} + 5)\nu^2 \\
&+ 4(5\bar{\sigma}^5 - 8\bar{\sigma}^4 + \bar{\sigma}^3 + 4\bar{\sigma}^2 - 3\bar{\sigma} + 1)\nu + (8\bar{\sigma}^4 - 4\bar{\sigma}^2 - 1) \\
&- \bar{\gamma}^4(1 - 3\bar{\xi})(\bar{\sigma}^2 - 1)(2\bar{\sigma}^2 - 1)\Big].
\end{aligned} \tag{6.66}$$

Here, we disagree with Porto et al. (see [75] and [77]). Their result is missing the final term in square brackets of equation (6.66), highlighted in blue. Because

[7] By "abstract", we mean simply that they presented the fluxes in terms of named (but not evaluated) functions of initial kinematics: $\Delta E^{(n)}_{\text{hyp}}$.

this term is proportional to $\Delta E^{(3)}_{\rm hyp}$, it must result from one of the G^1 corrections to either p_r or $\partial\mathcal{H}/\partial p^2$ multiplying the G^3 flux term in the integrand of (5.8). A little calculation reveals that

$$\overline{\frac{\partial^2\mathcal{H}}{\partial(p^2)^2}} = \frac{(-1)(1-3\xi)}{4m^3\gamma^3\xi^3}, \tag{6.67}$$

suggesting that the $\partial\mathcal{H}/\partial p^2$ correction is the culprit. Indeed, the 4PM result of Porto et al. can be recovered from (5.8) by dropping the term

$$\overline{P_1}\,\overline{\frac{\partial^2 E}{\partial(p^2)^2}}\left(\frac{Gm}{r}\right)^1 \tag{6.68}$$

from the expansion (6.65) of $\partial\mathcal{H}/\partial p^2$ before integration. Furthermore, if one substitutes $p^2(r) \to \bar{p}^2$ everywhere in $\partial\mathcal{H}/\partial p^2$ *before* computing its PM expansion, one finds exactly Eq. (6.65) *minus the term* from Eq. (6.68). While we cannot say with certainty why this term is missing from the 4PM coefficient in [75], the authors do mention "a caveat regarding the flux as a function of $(r, \boldsymbol{p}^2)$ as in the tail Hamiltonian, and (r, ε)..." (see p.21 of [75]), where ε refers to the binding energy, which can be written as a function of asymptotic initial kinematics, and $\boldsymbol{p}^2$ is presumably the *instantaneous* three-momentum-squared in the CM frame.

Instantaneous Energy Flux with Instantaneous Coefficients

Computing the *instantaneous* 4PM energy flux coefficient requires only a slight modification relative to the asymptotic calculation performed in the previous section. Because $\mathcal{F}_E^{(4)}$ dresses the subleading flux contribution, the 4PM expansion of the instantaneous ansatz about the asymptotic kinematics gains a term arising from the expansion of $\mathcal{F}_E^{(4)}$ as a function of $p^2(r)$ about $\bar{p}^2$ to G^1 order. That is,

$$\mathcal{F}_E^{(4)} = \bar{\mathcal{F}}_E^{(4)} + \frac{\bar{P}_1}{m}\frac{d}{d\bar{p}^2}\bar{\mathcal{F}}_E^{(3)}. \tag{6.69}$$

It is thus straightforward to take the asymptotic expression for $\bar{\mathcal{F}}_E^{(4)}$ derived in the previous section, and the expression for $\bar{\mathcal{F}}_E^{(3)}$ from the expression for $\mathcal{F}_E^{(3)}$ derived in Chapter 5, to which it is equivalent. Inserting them into Eq. (6.69), we find

$$\begin{aligned}\mathcal{F}_E^{(4)} = \frac{3\nu\gamma^2}{4m\xi(\sigma^2-1)^{3/2}}\Bigg[\Delta E^{(4)}_{\rm hyp} \\ +\frac{8\nu^2}{3\pi\gamma^8\xi^2\sqrt{\sigma^2-1}}\left(g_1(\sigma)\Delta E^{(3)}_{\rm hyp} - 2(m^2\gamma^6\xi^2)g_2(\sigma)\frac{d}{dp^2}\Delta E^{(3)}_{\rm hyp}\right)\Bigg]\end{aligned} \tag{6.70}$$

where

$$g_1(\sigma) = 1 + 2\sigma^2 - 4\sigma^4 - 4\nu(1 - 2\sigma + 2\sigma^2 - 4\sigma^4 + \sigma^5) - \nu^2(-1 + \sigma)^3(3 - 7\sigma - 6\sigma^2 + 6\sigma^3) \tag{6.71}$$

$$g_2(\sigma) = 1 - 3\sigma^2 + 2\sigma^4. \tag{6.72}$$

We have chosen not to display this flux explicitly because $\Delta E^{(4)}_{\text{hyp}}$ is a complicated expression which can be found in [77] (where it is called $\Delta E^{(1)}_{\text{hyp}}$, owing to a *relative* PM indexing scheme). For convenience, we have collected all relevant expressions, including the explicit instantaneous 4PM energy flux, in the ancillary file.

Finally, we extended the gauge-matching procedure described in Chapter 6 to G^4 order and compared both our expression for the instantaneous 4PM energy flux (6.70) and the instantaneous-coefficient analog of the asymptotic 4PM energy flux given by Porto et al. [75, 77] (obtained by inserting their asymptotic expression for $\bar{\mathcal{F}}^{(4)}_E$ into Eq. (6.69)) to the PN-expanded energy flux from [79] up to 4PN order. The results have been collected in Table 6.D. At 4PN order, our expression agrees with the ADM-gauge flux, but the expression from Porto et al. does not.

Abs. PN Order	2PN	3PN	4PN
Abs. η Order	η^6	η^8	η^{10}
$\mathcal{F}_E$	0	G^4	$G^4\nu^2$
$\mathcal{F}_{E,\text{lit}}$	0	G^4	$G^4\nu^2$

Table 6.D.1: PN Expansion of instantaneous-coefficient energy flux in isotropic gauge at G^4 order. $\mathcal{F}_E$ denotes our expression, while $\mathcal{F}_{E,\text{lit}}$ denotes the expression from Porto et al. Green cells indicate a match with PN literature, while red cells indicate a mismatch.

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