A SATELLITE THEORY AND ITS APPLICATIONS

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ABSTRACT

A theory of an Earth satellite has been developed by considering the Earth's bulge, atmospheric drag and the rotation of the atmosphere simultaneously. The equations of motion have been set up on a tilted equatorial plane coordinate system. All of the orbital variables have been expanded in a series in terms of a perturbative force parameter based on the Keplerian orbit. These equations have been linearized and then solved. By means of geometrical arguments, all of the above solutions have been expressed in the form of conventional orbital elements. In the limiting case, these solutions agree with the classical values. One previously neglected effect, the rotation of the line of apsides by drag, is identified and evaluated. The results have been used to show the correction due to the effect of the above-mentioned forces on the Earth's gravitational potential.

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LIST OF SYMBOLS

a	=	major axis
b	=	minor axis
c	=	angular momentum = $r^2\dot{\theta}$
C_D	=	drag coefficient
d	=	diameter of the satellite
D	=	drag force
E	=	orbital energy
f	=	flattening of the Earth, also the point at $\theta_0 = 2\pi$
$\underline{f}$	=	force (vector) acting on a unit mass
$\underline{F}_d$	=	force due to drag
$\underline{F}_p$	=	force due to the gravitational field
g_0	=	gravitational constant at sea level
G	=	universal gravitational constant
i	=	inclination angle of the orbital plane
$\underline{i}$	=	unit vector
$I_n(\zeta)$	=	modified Bessel functions with argument ζ
J_n	=	coefficient of spherical harmonics of gravitational potential
K	=	drag parameter = $(SC_D\rho_s)/2m$
m	=	mass of the satellite
M	=	mean anomaly
$M_\oplus$	=	mass of the Earth
n	=	mean motion
ON	=	line of nodes
OZ	=	polar axis of the Earth
OZ'	=	normal to the orbital plane

LIST OF SYMBOLS (Cont'd)

$O\Upsilon$	=	vernal equinox
p	=	semilatus rectum = $a(1 - \epsilon^2)$, also pressure
$P_n(Z)$	=	Legendre polynomials
r	=	radius measured from the center of Earth
R	=	radius of the Earth
$\mathcal{R}$	=	gas constant
S	=	characteristic area of the satellite
t	=	time
T	=	temperature
T_M	=	molecular-scale temperature
T_{sat}	=	period of the satellite
u	=	reciprocal of the radius = $1/r$
U	=	potential function of the Earth's gravitational field
V	=	velocity
$\underline{V}$	=	total velocity of the satellite
V_c	=	equivalent circular velocity
$\underline{V}_e$	=	relative velocity to the atmosphere
$\underline{V}_R$	=	resultant velocity of the satellite
Z	=	altitude measured from sea level
β	=	latitude angle
ϵ	=	eccentricity of the trajectory
ζ	=	parameter $= \nu \mu \epsilon / C_0^2$
θ	=	polar angle measured from apogee
$\tilde{\theta}$	=	polar angle measured on the π-plane from the ascending nodal line, $\tilde{\theta} = \theta + \theta_a$

LIST OF SYMBOLS (Cont'd)

θ_a = polar angle measured from the nodal line to apogee

λ = reciprocal of the scale height, also longitudinal angle

μ = gravitational parameter of the Earth = $GM_{\oplus}$

ν = parameter = $g_0R^2/\mathcal{R}T$

ρ = density of the air

$\sigma(r)$ = density ratio = ρ/ρ_s

ϕ = perturbation potential of the Earth's gravitational field, also longitude of the Earth

ψ = latitude angle measured from π-plane

ω = argument of apogee

$\underline{\omega}$ = angular velocity

ω_e = angular velocity of the Earth

Ω = argument of the ascending nodes

$(\ )_0$ = initial value, also Keplerian value

$(\ )_p$ = at perigee

$(\ )_s$ = reference point on the trajectory

$(\bar{\ })$ = values of the second ellipse, also on the new orbital plane

$(\ )|_f$ = perturbed quantity evaluated at $\theta_0 = 2\pi$

I. INTRODUCTION

To describe the motion of a celestial body in its orbit is a very old desire of man and has been subjected to an intensive study since the early ages. An impetus was given to this desire when the artificial satellite was launched. The understanding of the motion of these man-made celestial bodies has become essential.

In this paper, the Earth's bulge and aerodynamic resistance effects upon an artificial satellite within a single coordinate system, have been successfully analyzed. Various authors have solved the problem of bulge effect and drag effect separately. As perturbative forces are small, their solutions can be added to get the integrated effect. However, it is more desirable to obtain the complete solution of the combined effects directly. The equations of motion have been set up in such a way that the solutions would indicate the departure from the classical unperturbed Keplerian orbit. The drag treatment is similar to that of Chien (Ref. 1). The bulge effect has been analyzed only up to the second harmonic, but a generalization to include the higher harmonics is possible. The atmosphere has been assumed to be spherically symmetric and rotating with the Earth. The effect of perturbations is given by an indefinite integral, and the integration can be performed for any given interval in terms of the orbital angle. The secular perturbation in the rotation of line of apsides, given by this analysis, permits a correction to the value of the second harmonic, or the flattening of the Earth. This correction is small.

It is hoped that the present analysis will help in understanding further the motion of a satellite and also the shape of the Earth.

II. ANALYSIS

1. Choice of the Coordinate System

The problem of the motion of an artificial close Earth satellite is quite different from the traditional natural satellite. Applications to natural satellites did not require a solution, as specific and detailed, as was needed to represent the motions of artificial satellites. Equations of motion will change with different coordinate systems. The traditional classical approach to satellite motion was based on Lagrange's six planetary equations (Ref. 2). These equations have also been called the variation of parameters method. In order to determine the complete history of a moving body in a space, one needs six variables to determine six degrees of freedom. Each of the six equations is expressed in such a way as to represent the time derivative of a particular element, or a variable due to the corresponding change of its forcing components. Of course, these equations can be simplified or reduced in some cases, when a proper approximation procedure is used. Most of the earlier papers dealing with artificial satellites follow this traditional approach. The most complete, typical, and up-to-date paper, written in the traditional approach, is by Kozai (Ref. 3).

The orbit of a particle moving in a central attraction force field is an ellipse. For such an elliptical orbit, the total angular momentum and its energy are constants. The artificial satellite, moving about the Earth, has a nearly elliptical path. It is natural to look for a method of solving the equations of motion that makes use of the fact that the total angular momentum and its energy is nearly constant. The Hamiltonian formulation of the equations of motion provides such a method. It is clear that the Hamiltonian method may be suitably applied to a potential force field, but not to a dissipating force field. A typical paper using this approach was worked out by Brouwer (Ref. 4). Brouwer considered the separation of Hamiltonian in the Hamilton-Jacobi equation similar to that used in Delanny's Lunar theory.

The Hamiltonian method gives a clear insight into the dynamics of the motion of a satellite. However, the above method is not necessary in describing this motion. Integrating equations of motion in suitable coordinate systems is the most efficient way to approach the satellite problem. The first pioneer to adopt this method was King-Hele early in 1958 (Ref. 5).

Modification of King-Hele's approach has been proposed by Brenner and Latta (Ref. 6). The coordinate systems of King-Hele and Brenner and Latta are suitable for considering the gravitational effect, but not for the aerodynamic drag perturbations. To relate the drag perturbation forces to these coordinate systems makes the equations of motion not only highly coupled but also complicated. Further modification of the above approach is necessary to arrive at a suitable coordinate system. The present analysis uses spherical polar coordinates which are fixed in space. The reference plane is tilted from the equatorial plane, and only coincides with the osculating orbital plane at the initial instant. In this study, as in King-Hele, the reference plane is designated the "π-plane." The equations of motion have been set up as perturbations from this fixed reference plane.

Using the tilted spherical coordinates mentioned above, with proper manipulation, the equations of motion can be properly decoupled. The equations of motion can be directly integrated, resulting in the variation of radius vector. When dealing with the drag perturbation case, it is more convenient to assume that all of the values are perturbed from the Keplerian ones. The Keplerian trajectory is confined in a plane; hence the so-called "instantaneous orbital plane method" is the natural choice. On the other hand, it is better to use the fixed in space, geocentric latitude-longitude polar coordinates, as used by King-Hele, in dealing with the <u>secular</u> change. The transformation of equations of motion from the inertia frame to the language of the Keplerian variables through angle relationships is very complicated. In the present analysis, a combination of the above two methods is attempted. The spherical polar coordinates are now based on a π-plane, rather than the equatorial plane as is shown in Figure 1. The π-plane has been connected to the inertia frame, through definite angle relationships.

2. The Perturbation Forces

The gravity field of the Earth is nearly that of a point mass, or, in other words, the satellite is moving in a close to $-1/r$ potential field. Other forces acting on the satellite are very small in magnitude. Due to the above factors, the satellite orbit is very close to the classical Keplerian ellipse. However, other forces than the $-1/r$ potential field will cause the satellite motion to deviate from the Keplerian trajectory. The perturbation forces are the non-spherical components of the Earth's gravitational force,

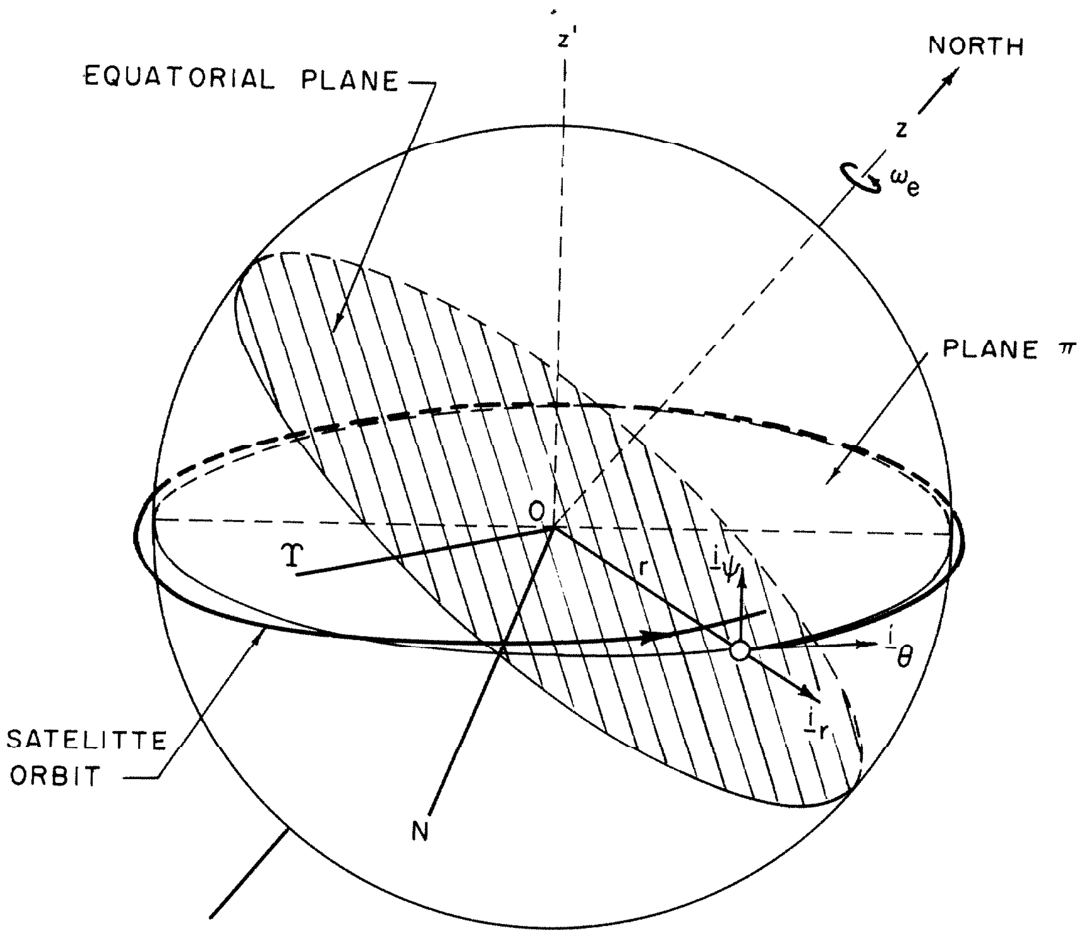

Figure 1. Schematic Diagram of Tilted Equatorial Plane Coordinates

the air resistance, the attractions of Sun, Moon, and other planets, the light pressure, the electromagnetic forces, the relativity effects, and so on (for treatment of these minor perturbation forces see References 7 to 11). Major perturbative forces will include the non-uniform gravitational force and the aerodynamic resistance, if the satellite orbit is close enough to the Earth. Other perturbing forces are minor in comparison to the two above-mentioned forces, for the ordinary Earth satellite. The influence of the Earth's oblateness and the atmospheric drag effect, have a coupled effect on the satellite orbit, since they act on the satellite simultaneously. Fortunately these two sources produce perturbations of different types, and, as a first approximation, it can be assumed that their effects are linearly separable. A discussion of the coupling effect on the forcing function of the equations of motion has been carried out in the second part of the applications of theory in this analysis.

Traditionally, it has been assumed that the external force field of the Earth can be expressed by the potential function U. If so, U must satisfy Laplace's equation, since

$$\underline{f} = -\nabla U \quad \text{then} \quad \nabla \cdot (-f) = \nabla^2 U = 0, \tag{2.1}$$

where $\underline{f}$ is the gravitational force acting on a unit mass outside of the Earth's surface.

Assuming that the Earth is symmetrical about its polar axis, there is no longitudinal dependence. Longitudinal dependence, or the sectorial harmonics, will not be considered in this analysis. The solution of Laplace's equation can be expressed in the Legendre polynomials, $P_n(\sin\beta)$, as

$$U(r,\beta) = \sum_{n=0}^{\infty} A_n r^{-n-1} P_n(\sin\beta), \qquad (2.2)$$

where r is a radial distance from the center of the Earth, β is the latitude and $P_n(Z)$, the Legendre polynomials, are defined as

$$P_n(Z) = \frac{1}{2^n n!} \frac{d^n}{dZ^n} (Z^2 - 1)^n .$$

From Newton's universal gravitational law, we know that for the central gravitational field of the Earth

$$U = -\frac{\mu}{r},$$

where $\mu = GM_{\oplus}$. G = universal gravitational constant and

$M_{\oplus}$ = Mass of the Earth.

So, when factoring this out from the above expression, we have

$$U = \frac{\mu}{r} [1 - \sum_{n=1}^{\infty} J_n (\frac{R}{r})^n P_n (\sin\beta)] \qquad (2.3)$$

where R is the mean radius of the Earth, J_n are numerical coefficients.

Equation 3 gives the Earth's gravitational potential which is symmetrical to its polar axis. This kind of potential has been called the zonal harmonics of the Earth. When one does not assume a symmetrical form of the Earth in solving equation 2.1, additional terms of the associated Legendre polynomials will appear in the solution 2.2. Longitudinal dependent quantities have been called tesseral and sectorial harmonics (for instance, cf. Ref. 12). These harmonics are very difficult to detect since they do not yield secular perturbations, or perturbations of a long time period. In the long range, longitudinal dependent perturbative effects are expected to be damped out by cancelling each other, and will not affect the final results. Only zonal harmonics have been considered in this analysis.

In representing the actual field of the Earth's axially symmetrical potential in the form of equation 2.3, one needs an infinite series to handle the problem. Due to difference in the order of magnitude of the coefficient of the zonal harmonics, a further simplification can be made by neglecting the small order terms. First of all $n = 1$, the J_1 case, has been omitted in the calculation. This is the consequence of the fact that the J_1 case will be zero if the origin of the geocentric coordinate system coincides with the Earth's center of gravity. It is believed that these two centers fail to coincide by not more than 200 meters (Ref. 13). Therefore the J_1 effect is very minor, and is usually not taken into account.

The physical properties of the other zonal harmonics were indicated by Jastrow in his Wright Brothers Lecture speech (Ref. 14). These harmonics represent the departure of geoid from its hydrostatic equilibrium status. An interesting exaggerated comparison of the J_n of the geoid has been quoted here in Figure 3. Among all of the J_n's, the largest in order of magnitude is J_2, which indicates the Earth's oblateness effect. The other J_n's are on the order of one thousandth of J_2. Therefore, in the first approximation only J_2 is retained in the equation 2.3.

The other major perturbative force taken into account is aerodynamic resistance, which is a dissipating force field and

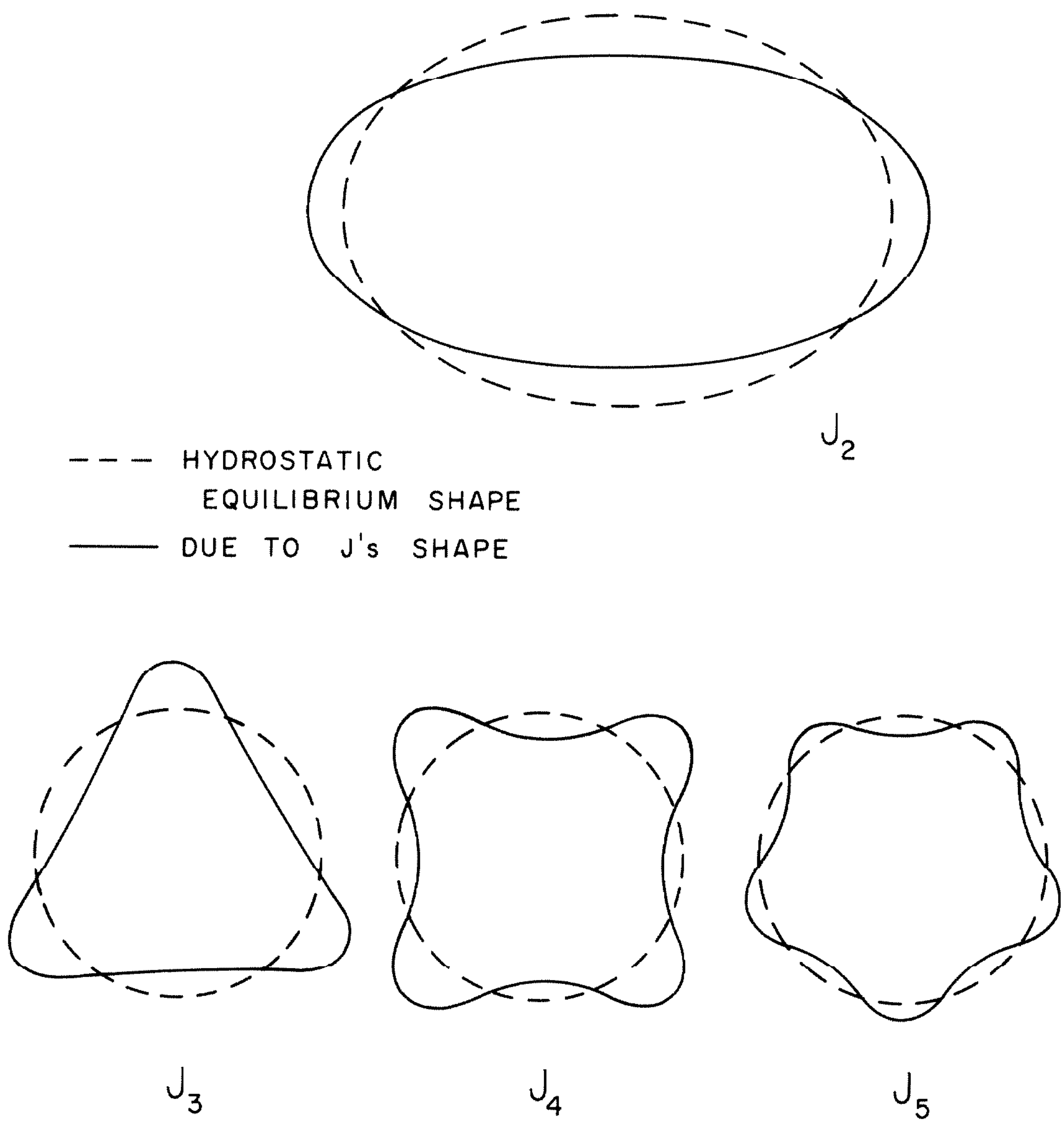

Figure 2. Harmonics of the Geoid (from Ref. 14) (Vertically Exaggerated)

fades away very swiftly with increases in altitude. Aerodynamic resistance is a strong function of the altitude. The order of magnitude of this force compared to the zonal harmonics is entirely dependent on the perigee height of the satellite. At really high altitude the other perturbative forces, namely, the attractions of the Moon, Sun, etc., might become more significant than aerodynamic resistance. In the ordinary range of the satellite, the drag force still appears to be a major perturbative force.

Drag force is produced by the relative motion of the satellite and the atmospheric media. The relative motion is composed of two parts, namely, the direct motion of the satellite, and the motion of the atmosphere, which is caused by the rotation of the Earth. Therefore, the drag of a satellite will differ, depending on whether it is moving in an easterly or westerly direction. The angular velocity of the atmosphere in this analysis has been assumed to be the same as that of the Earth. This may not be true at very high altitudes, due to the fact that the mean free paths of the atmospheric molecules are quite large. However, the lag in the angular velocity of the atmosphere and the Earth is very minute. This lag in angular velocity is a higher order effect. A correction can be made by adding an empirical factor upon the angular velocity in the analysis, when necessary.

The drag force, D, is generally defined in terms of a dimensionless coefficient, C_D, and can be written

$$D = \frac{1}{2} \rho V^2 S C_D$$

where S is some characteristic area of the body.

The drag coefficient C_D at high altitude, where the Newtonian theory is applicable, i.e., the body is small compared to the mean free path of the air molecules, is approximately a constant value close to 2. We may rewrite the above equation as the drag per unit mass and define ρ_s as the air density at some reference point. Then:

$$\frac{D}{m} = \frac{SC_D\rho_s}{2m} \left(\frac{\rho}{\rho_s}\right) V^2 = K\sigma(r)\, V^2 \qquad (2.4)$$

where K can be treated as a constant in the analysis and $\sigma(r)$ is the density ratio.

The atmosphere has been treated as spherically symmetric in shape. The atmosphere is obviously distorted by the Earth's bulge; however it is likely that this effect can be neglected for long lived satellites which pass through many cycles of the motion of the latitude of perigee. Besides this, if one properly chooses an average density, the change of density during the day and night at high altitude will not affect the ultimate result.

3. Derivations of the Equations of Motion

As stated in sections 1 and 2, we have used a fixed-in-space, tilted-equatorial-plane coordinate system to derive the equations of motion. We defined a π-plane, which is a Keplerian orbital plane defined at a convenient point on the trajectory, as our fundamental reference plane in spherical polar coordinates (see Figure 1). In this coordinate system, the Earth's equatorial is tilted. The nomenclature and sense of all the angles are indicated in Figure 3. The kinematic angle relationships will be discussed later. It is

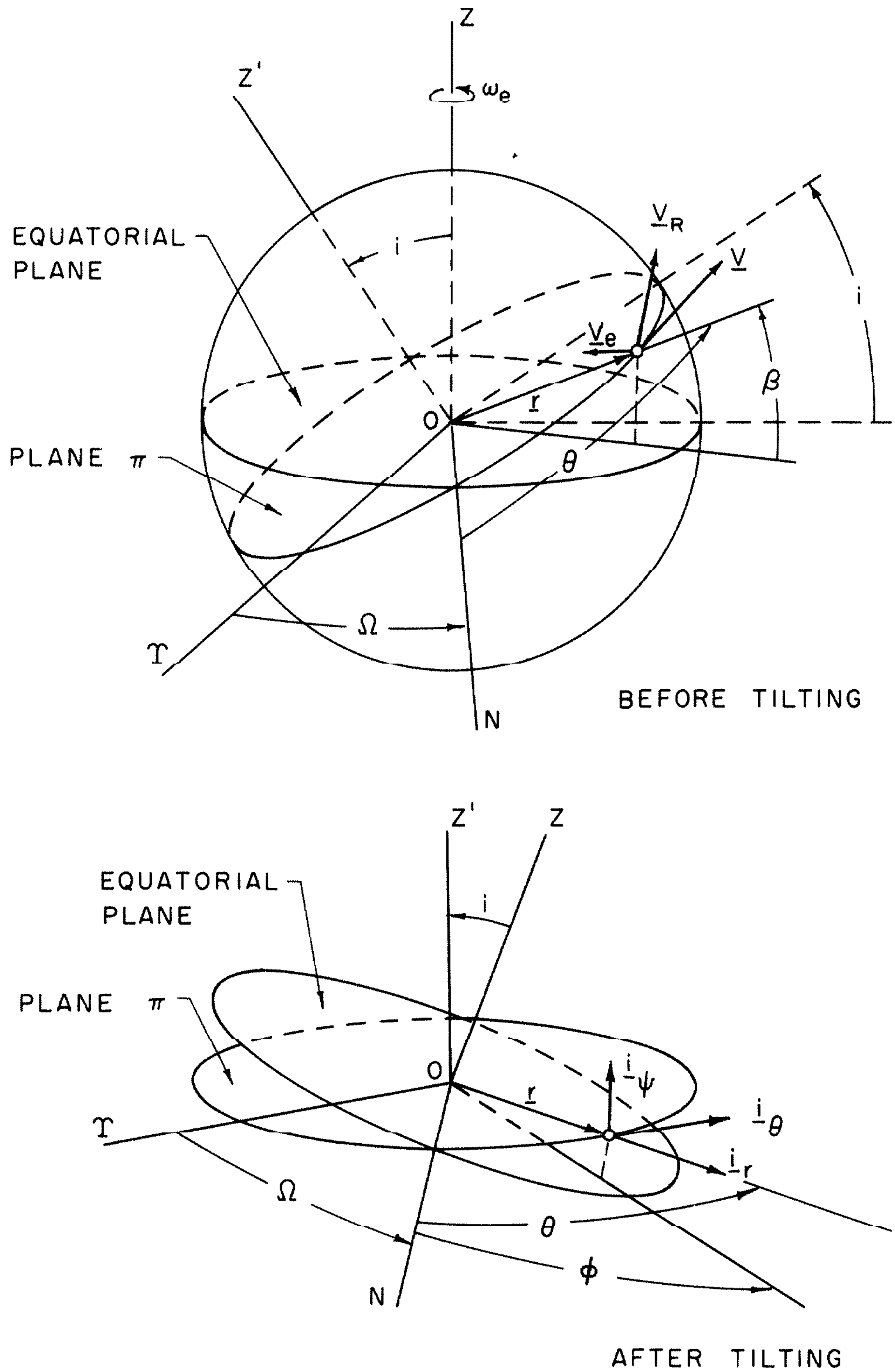

Figure 3. Nomenclature of the Coordinate System

worthwhile to point out here that O♈ is a reference line on the equatorial plane of the vernal equinox for some epoch, and Ω is a nodal angle measured along the equator. $\tilde{\theta}$ is a polar angle measured on the π-plane from the ascending nodal line. ϕ is also a polar angle but is measured on the equatorial plane, that is ϕ is a longitude of the Earth. All the $\underline{i}$'s are the unit vectors in the direction indicated on their subscript. The ψ angle will be a latitude-like angle measured away from the π-plane, and β is a latitude angle measured from the equatorial plane.

The linear velocity of the satellite moving in this tilted spherical polar coordinate is:

$$\underline{V} = \frac{d\underline{r}}{dt} = \dot{r}\,\underline{i}_r + r\cos\psi\,\dot{\theta}\,\underline{i}_\theta + r\,\dot{\psi}\,\underline{i}_\psi \tag{3.1}$$

and the angular velocity of the system is:

$$\underline{\omega} = \sin\psi\dot{\theta}\,\underline{i}_r - \dot{\psi}\,\underline{i}_\theta + \cos\psi\,\dot{\theta}\,\underline{i}_\psi \tag{3.2}$$

where the dots indicate differentiation with respect to time, and $\dot{\tilde{\theta}} = \dot{\theta}$ because $\tilde{\theta} = \theta_a + \theta$, and θ_a has been treated as a constant within one revolution.

The acceleration of a unit mass particle is therefore:

$$\begin{aligned}\frac{d\underline{V}}{dt} &= \frac{\partial\underline{V}}{\partial t} + \underline{\omega} \times \underline{V} \\ &= (\ddot{r} - r\cos\psi\,\dot{\theta}^2 - r\dot{\psi}^2)\,\underline{i}_r + \frac{1}{r\cos\psi}\frac{d}{dt}(r^2\cos^2\psi\dot{\theta})\,\underline{i}_\theta \\ &\quad + \frac{1}{r}\left[\frac{d}{dt}(r^2\dot{\psi}) + r^2\sin\psi\cos\psi\,\dot{\theta}^2\right]\,\underline{i}_\psi\,.\end{aligned} \tag{3.3}$$

In order to get the drag perturbation forces, we have to know the directional velocity components in $\underline{i}_r$, $\underline{i}_\theta$, and $\underline{i}_\psi$ respectively. The resultant velocity is composed of the orbiting velocity, and the relative velocity of the satellite to the rotating atmosphere of the Earth. When expressed mathematically, this velocity is

$$\underline{V}_R = \underline{V} + \underline{V}_e \qquad (3.4)$$

where $\underline{V}$ has been given in equation 3.1. $\underline{V}_e$ can be written in terms of the angular velocity, ω_e, of the rotation of the Earth, as follows:

$$\underline{V}_e = r\omega_e \cos\beta \, [(\sin\phi \cos\tilde{\theta} - \cos\phi \cos i \sin\tilde{\theta}) \, \underline{i}_r - (\cos\phi \cos i \cos\tilde{\theta} + \sin\phi \sin\tilde{\theta}) \, \underline{i}_\theta + \cos\phi \sin i \, \underline{i}_\psi] \qquad (3.5)$$

From the geometrical consideration of our specific coordinate system, we have the following angle relationships (see Figure 4):

$$\cos\tilde{\theta} \cos\psi = \cos\beta \cos\phi$$

$$\sin\tilde{\theta} \cos\psi = \sin\beta \sin i + \cos\beta \cos i \sin\phi$$

$$\sin\psi = \sin\beta \cos i - \cos\beta \sin i \sin\phi \; . \qquad (3.6)$$

From these equations, $\underline{V}_e$ can be rewritten in terms of our orbital variables. Equation 3.5 becomes:

$$\underline{V}_e = -r\omega_e \cos\tilde{\theta} \sin\psi \sin^2 i \, \underline{i}_r - r\omega_e (\cos i \cos\psi - \sin i \sin\tilde{\theta} \sin\psi) \, \underline{i}_\theta + r\omega_e \cos\tilde{\theta} \sin i \cos\psi \, \underline{i}_\psi \; . \qquad (3.7)$$

This equation applies only to a satellite moving on an easterly course, as is most usual. However, for the westerly moving satellite, the sign should be reversed. The resultant velocity, $\underline{V}_R$, is therefore a vector sum of equations 3.1 and 3.7. By using the above equations, we can write the drag force per unit mass as follows:

$$\frac{D}{m} = -\frac{1}{2}\rho V_R^2 \left(\frac{C_D S}{m}\right)\frac{\underline{V}_R}{V_R} = -K\sigma(r)\, V_R \underline{V}_R$$

$$= \underline{F}_d = F_{dr}\underline{i}_r + F_{d\theta}\,\underline{i}_\theta + F_{d\psi}\underline{i}_\psi$$

$$= -K\sigma(r)V_R\,[(\dot{r} - r\omega_e \cos\tilde{\theta}\sin^2 i \sin\psi)\,\underline{i}_r$$

$$+(\cos\psi\dot{\theta} - \omega_e \cos i \cos\psi + \omega_e \sin i \sin\tilde{\theta} \sin\psi)\,\underline{i}_\theta$$

$$+ (\dot{\psi} + \omega_e \cos\tilde{\theta} \sin i \cos\psi)\,\underline{i}_\psi] \quad . \qquad (3.8)$$

For the non-spherical perturbation potential forces of the Earth, which we have described in the previous section, we will consider only the first order perturbation. This means only the effects produced by J_2 will be considered. J_3, J_4 and so on are in the order of J_2^2, or a thousandth of J_2, hence can be regarded as the higher order terms, which are neglected in the first calculation. In other words, we consider only the oblateness effect of the Earth at the present time. Let ϕ be the perturbation potential. By using equations 2.3 and 3.6 which represent the geometrical angle relations, we obtain:

$$\phi = \frac{\mu}{2r} J_2 \left(\frac{R}{r}\right)^2 (1 - 3 \sin^2\beta)$$

$$= \frac{\mu}{2r} J_2 \left(\frac{R}{r}\right)^2 [1-3 (\sin\tilde{\theta} \sin i' \cos\psi + \cos i \sin\psi)^2] \qquad (3.9)$$

Therefore, the perturbation force due to oblateness J_2 alone can be derived as:

$$\underline{F}_p = F_{pr}\underline{i}_r + F_{p\theta}\underline{i}_\theta + F_{p\psi}\underline{i}_\psi$$

$$= \frac{\partial\phi}{\partial r}\underline{i}_r + \frac{1}{r\cos\psi}\frac{\partial\phi}{\partial\theta}\underline{i}_\theta + \frac{1}{r}\frac{\partial\phi}{\partial\psi}\underline{i}_\psi$$

$$= -\frac{3}{2} J_2 g_0 \left(\frac{R}{r}\right)^4 \Big\{[1-3(\sin\tilde{\theta} \sin i \cos\psi + \cos i \sin\psi)^2]\underline{i}_r$$

$$+ [\sin^2 i \sin 2\tilde{\theta} \cos\psi + \sin 2i \cos\tilde{\theta} \sin\psi]\underline{i}_\theta$$

$$+ [\sin\tilde{\theta} \sin 2i \cos 2\psi + \sin 2\psi (\cos^2 i - \sin^2\tilde{\theta} \sin^2 i)]\underline{i}_\psi\Big\}$$

$$(3.10)$$

The total perturbation force is therefore the sum of the forces of equations 3.8 and 3.9, namely,

$$\underline{F} = \underline{F}_p + \underline{F}_d$$

$$= F_r\underline{i}_r + F_\theta\underline{i}_\theta + F_\psi\underline{i}_\psi \ . \qquad (3.11)$$

The next step is to derive the inclination angle of the orbital plane, i, in terms of the other orbital elements (refer to Figure 4). It can be seen that the cross product of vectors $\underline{r}$ and $\underline{v}$ will be perpendicular to the orbital plane, and the angle between this cross product vector and the polar axis of the Earth, will be the actual instantaneous inclination angle, i.

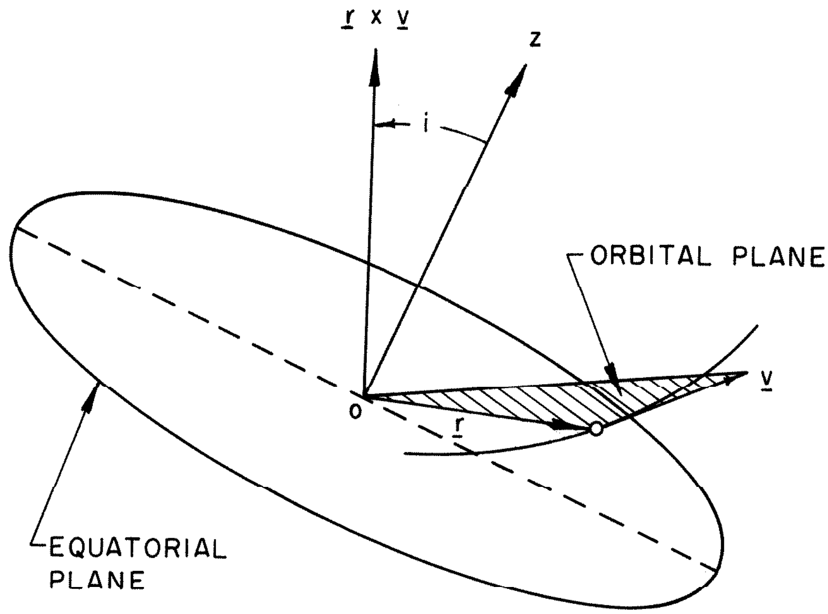

Figure 4. The Inclination Angle of the Orbital Plane with the Equatorial Plane

$$\underline{r} \times \underline{v} = \begin{vmatrix} \underline{i}_r & \underline{i}_\theta & \underline{i}_\psi \\ r & 0 & 0 \\ \dot{r} & r\cos\psi\dot{\theta} & r\dot{\psi} \end{vmatrix}$$

$$= -r^2\dot{\psi}\,\underline{i}_\theta + r^2 \cos\psi\,\dot{\theta}\,\underline{i}_\psi$$

By normalizing this vector we have:

$$\underline{n} = \frac{\underline{r} \times \underline{v}}{|\underline{r} \times \underline{v}|} = \frac{-\dot{\psi}\,\underline{i}_\theta + \cos\psi\,\dot{\theta}\,\underline{i}_\psi}{[\dot{\psi}^2 + \cos^2\psi\,\dot{\theta}^2]^{\frac{1}{2}}} \quad . \qquad (3.12)$$

Let the direction of the polar axis of the Earth be denoted by the unit vector $\underline{i}_z$. This vector can be decomposed into $\underline{i}_r$, $\underline{i}_\theta$, and $\underline{i}_\psi$ directions in the orbital plane as follows:

$$\underline{i}_z = (\cos i \sin\psi + \sin i \sin\tilde{\theta} \cos\psi)\,\underline{i}_r + \sin i \cos\tilde{\theta}\,\underline{i}_\theta$$

$$+ (\cos i \cos\psi - \sin i \sin\tilde{\theta} \sin\psi)\,\underline{i}_\psi \quad . \qquad (3.13)$$

The components of vector $\underline{i}_z$ will vary as the orbital variables change their values. However, the inclination angle of the orbital plane remains the same, since we are referring the polar axis to the same coordinate system. The angle between $\underline{n}$ and $\underline{i}_z$ will be the instantaneous inclination angle i. From analytical geometry we know that the angle between the two vectors can be expressed by the sum of its directional cosine products. We obtain:

$$\cos i = (\dot{\psi}^2 + \cos^2\psi\,\dot{\theta}^2)^{-\frac{1}{2}}[\dot{\theta}\cos\psi(\cos i_0 \cos\psi - \sin i_0 \sin\tilde{\theta} \sin\psi)$$

$$- \sin i_0 \cos\tilde{\theta}\,\dot{\psi}] \quad . \qquad (3.14)$$

This equation relates the inclination angle to the known orbital variables.

By the above derivations, we can write down the equations of motion of a satellite in the Earth's oblate gravitational potential field, with a rotating spherical atmosphere by combining equations 3.3, 3.4, 3.11, and 3.14 as follows:

$$\ddot{r} - r\cos^2\psi\,\dot{\theta}^2 - r\dot{\psi}^2 = -g_0\left(\frac{R}{r}\right)^2 + F_r \qquad (3.15)$$

$$2\cos\psi\dot{r}\dot{\theta} - 2r\sin\psi\,\dot{\theta}\dot{\psi} + r\cos\psi\,\ddot{\theta} = F_\theta \qquad (3.16)$$

$$r\ddot{\psi} + 2\dot{r}\dot{\psi} + r\sin\psi\cos\psi\,\dot{\theta}^2 = F_\psi \qquad (3.17)$$

$$V_R^2 = \dot{r}^2 + r^2\cos^2\psi\dot{\theta}^2 + r^2\dot{\psi}^2 + 2r^2\omega_e\cos\psi\,(\cos\tilde{\theta}\sin i\dot{\psi} - \cos i\cos\psi\,\dot{\theta}) + O(\omega_e\sin\psi,\ \omega_e^2) \qquad (3.18)$$

$$\cos i = (\dot{\psi}^2 + \cos^2\psi\dot{\theta}^2)^{-\frac{1}{2}}[\dot{\theta}\cos\psi(\cos i_0\cos\psi - \sin i_0\sin\tilde{\theta}\sin\psi) - \sin i_0\cos\tilde{\theta}\ \dot{\psi}], \qquad (3.19)$$

where F_r, F_θ and F_ψ are the perturbation force components in r, θ, and ψ directions respectively, they are given by equations 3.11, 3.8 and 3.10. The subscript 0 refers to the Keplerian values. The five equations listed above are for the five unknowns, namely r, θ, ψ, V_R and i.

These equations of motion should be able to reduce to the Keplerian situation as a limiting case if all the perturbation forces vanish. This can be easily examined. Considering the second equation, we have:

$$\frac{d}{dt}(r^2\dot{\theta}) = 0, \qquad \text{or } r^2\dot{\theta} = c = \text{constant},$$

this gives the constancy of the angular momentum. Similarly, the third equation becomes:

$$\frac{d}{dt}(r^2\dot{\psi}) = 0, \qquad \text{or } r^2\dot{\psi} = \text{constant}.$$

From the initial conditions $\psi = 0$, $\dot{\psi} = 0$, we obtain $\psi = 0$ as a result. This proves that the satellite motion is in a plane. Then, by putting this result into the fifth equation, we have

$$\cos i = \cos i_0, \quad \text{or} \quad i = i_0 \, .$$

This further proves that the plane is fixed in space. Now, the first equation can be reduced to

$$\ddot{r} - \frac{C^2}{r^3} + g_0 \left(\frac{R}{r}\right)^2 = 0 \, .$$

This is Kepler's equation of motion.

4. Rearrangement of the Equations

The equations as they stand are highly non-linear and coupled together. The perturbation forces in general are very small in magnitude, compared to the central gravitational attractions of the Earth. As a result, the variations in r, θ, ψ, and i are also very small. A linearization process is thus applicable. Since the Keplerian trajectory corresponds to the situation where all of the perturbation forces vanish, linearization is possible by expanding all of the variables around their Keplerian values.

A direct linearization will completely decouple equation 3.17 from 3.15 and 3.16. Equations 3.15 and 3.16 remain coupled together. The coupled equations can be solved directly by eliminating one variable from the other. However, a rearrangement has been tried for these two equations. By this rearrangement, we not only have simplified the manipulation, but also arrive at the expression for the orbital velocity V. It is useful to know V to understand the satellite motion. The above rearrangement was inspired through consideration of the plane trajectory case.

From equation 3.1 we know the orbital velocity, V, and:

$$V^2 = \dot{r}^2 + r^2 \cos\psi\dot{\theta} + r^2\dot{\psi}^2 \tag{4.1}$$

Differentiation of this gives:

$$V\dot{V} = \dot{r}\,\ddot{r} + r\,\dot{r}\cos^2\psi\dot{\theta}^2 - r^2 \sin\psi\cos\psi\,\dot{\psi}\,\dot{\theta}^2$$

$$+ r^2\cos^2\psi\,\dot{\theta}\,\ddot{\theta} + r\,\dot{r}\,\dot{\psi}^2 + r^2\,\dot{\psi}\,\ddot{\psi} \tag{4.2}$$

Rewrite equation 3.15:

$$\dot{r}\ddot{r} = F_r\dot{r} - g_0\left(\frac{R}{r}\right)^2 \dot{r} + r\cos^2\psi\,\dot{r}\,\dot{\theta}^2 + r\,\dot{r}\,\dot{\psi}^2$$

Substituting this into equation 4.2, we have:

$$V\dot{V} = -g_0\left(\frac{R}{r}\right)^2 \dot{r} + F_r\dot{r} + 2r\,\dot{r}\,\dot{\psi}^2 + 2r\cos^2\psi\,\dot{r}\,\dot{\theta}^2$$

$$-r^2\sin\psi\cos\psi\,\dot{\theta}^2\dot{\psi} + r^2\cos^2\psi\,\dot{\theta}\,\ddot{\theta} + r^2\dot{\psi}\,\ddot{\psi}$$

By substituting equations 3.16 and 3.17 into the above equation, we have:

$$V\dot{V} = -g_0 \left(\frac{R}{r}\right)^2 \dot{r} + F_r \dot{r} + r\cos\psi\, F_\theta \dot{\theta} + rF_\psi \dot{\psi} \; . \tag{4.3}$$

This is

$$\frac{d}{dt}\left[V^2 - \frac{2g_0 R^2}{r}\right] = 2F_r \dot{r} + 2r\cos\psi\, F_\theta \dot{\theta} + 2rF_\psi \dot{\psi} \; . \tag{4.4}$$

Similarly, equation 4.1 combined with equations 3.15 and 3.16 gives:

$$\begin{aligned} V^2 &= \dot{r}^2 + r^2 \cos^2\psi \dot{\theta}^2 + r^2 \dot{\psi}^2 \\ &= \frac{g_0 R^2}{r} - rF_r + \frac{1}{\cos\psi} F_\theta \frac{\dot{r}}{\dot{\theta}} + r^2 \dot{\theta} \frac{d}{dt}\left(\frac{\dot{r}}{r\dot{\theta}}\right) \\ &\quad + 2r\dot{r}\,\dot{\psi}\tan\psi \; . \end{aligned} \tag{4.5}$$

Equations 4.4 and 4.5 are equivalent differential equations of the plane orbital case, in which the force components have been taken along and perpendicular to the velocity vector.

By first changing the time derivative t to θ, or by changing the dynamic problem to an orbital problem, in equation 4.4 we have:

$$\because \; \frac{d}{dt} = \dot{\theta}\frac{d}{d\theta}$$

so

$$\frac{d}{d\theta}\left[V^2 - \frac{2g_0 R^2}{r}\right] = 2\frac{dr}{d\theta} F_r + 2r\cos\psi\, F_\theta + 2rF_\psi \frac{d\psi}{d\theta} \; . \tag{4.6}$$

Similarly, equation 4.5 becomes

$$\begin{aligned} \dot{\theta}^2 \frac{d^2}{d\theta^2}\left(\frac{1}{r}\right) + \frac{\cos\psi\dot{\theta}^2}{r} &= \frac{g_0 R^2}{r^4} - \frac{1}{r^2} F_r - \frac{1}{r\cos\psi} F_\theta \frac{dr}{d\theta} \\ &\quad + \frac{\dot{\theta}^2}{r}\left(\frac{d\psi}{d\theta}\right)^2 - \frac{2}{r^2}\tan\psi\, \dot{\theta}^2 \frac{dr}{d\theta}\frac{d\psi}{d\theta} \; . \end{aligned} \tag{4.7}$$

The equations 3.15 and 3.16 have been replaced by the equations 4.6

and 4.7.

5. Linearization

As we stated previously, the orbital trajectories of conventional Earth satellites are very close to that of Keplerian ones. It is reasonable to assume that all of the orbital parameters are slightly perturbed from the Keplerian values. Using subscript 0 to indicate the Keplerian parameters, and $\Delta(\ )$ to indicate the first order perturbed quantities, we can write:

$$
\begin{aligned}
r &= r_0 + \Delta r \\
\theta &= \theta_0 + \Delta\theta \\
V &= V_0 + \Delta V \\
i &= i_0 + \Delta i \\
\psi &= \psi_0 + \Delta\psi \\
&= \Delta\psi
\end{aligned}
\qquad (5.1)
$$

because

$$\psi_0 = 0 \,. \qquad (5.2)$$

Based on the above relations, the equations of motion derived in sections 3 and 4 can be linearized.

With the assumptions 5.1, we can linearize the derivative operators as:

$$\frac{d}{d\theta} = \frac{d\theta_0}{d\theta}\frac{d}{d\theta_0} = (1 - \frac{d\Delta\theta}{d\theta_0})\frac{d}{d\theta_0}$$

$$\frac{d}{dt} = \dot{\theta}\frac{d}{d\theta} = \dot{\theta}_0 (1 + \frac{d\Delta\theta}{d\theta_0})(1 - \frac{d\Delta\theta}{d\theta_0})\frac{d}{d\theta_0} = \dot{\theta}_0 \frac{d}{d\theta_0}$$

$$\frac{d^2}{dt^2} = \dot{\theta}_0 \frac{d}{d\theta_0}[\dot{\theta}_0 \frac{d}{d\theta_0}] = \dot{\theta}_0^2 \frac{d^2}{d\theta_0^2} + \dot{\theta}_0 \frac{d\dot{\theta}_0}{d\theta_0}\frac{d}{d\theta_0}$$

$$= \dot{\theta}_0^2 \frac{d^2}{d\theta_0^2} + \ddot{\theta}_0 \frac{d}{d\theta_0} . \qquad (5.3)$$

For the Keplerian case, we know that:

$$\frac{d}{d\theta_0}[V^2 - \frac{2g_0R^2}{r_0}] = 0 . \qquad (5.4)$$

Hence, by equations 5.1 and 5.3, the linearized equation 4.6 can be written as:

$$\frac{d}{d\theta_0}[2V_0\Delta V + \frac{2g_0R^2}{r_0^2}\Delta r] = 2\frac{dr_0}{d\theta_0}F_r + 2r_0F_0 , \qquad (5.5)$$

where the variables in F_r and F_θ should be properly linearized too. For convenience, the variable can be changed by letting:

$$u = \frac{1}{r} ,$$

then:

$$u = u_0 + \Delta u = \frac{1}{r_0+\Delta r} = \frac{1}{r_0}(1 - \frac{\Delta r}{r_0} + ---)$$

hence:

$$\Delta u = -\frac{\Delta r}{r_0^2} \qquad \text{and} \qquad \Delta r = -\frac{\Delta u}{u_0^2} \quad . \tag{5.6}$$

When equation 5.6 is rewritten in terms of u_0 and Δu, we have:

$$\frac{d}{d\theta_0}\left[2V_0\,\Delta V - 2g_0R^2\,\Delta u\right] = -\frac{2}{u_0^2}\,\frac{du_0}{d\theta_0}\,F_r + \frac{2}{u_0}\,F_\theta \quad . \tag{5.7}$$

Next, we will treat equation 4.7 similarly, since for the Keplerian case we know that:

$$\frac{d^2}{d\theta_0^2}\left(\frac{1}{r_0}\right) + \frac{1}{r_0} = \frac{g_0R^2}{r_0^4\dot{\theta}_0^2} \tag{5.8}$$

hence, subtracting this from the linearized equation 4.7, and also changing the variable to u, we have:

$$\frac{d^2(\Delta u)}{d\theta_0^2} + \left[1 - \frac{4}{u_0}\,\frac{g_0R^2}{C_0^2}\right]\Delta u + 2u_0\,\frac{d(\Delta\theta)}{d\theta_0} - \frac{du_0}{d\theta_0}\,\frac{d^2(\Delta\theta)}{d\theta_0^2}$$

$$= -\frac{F_r}{u_0^2\,C_0^2} - \frac{1}{u_0^3C_0^2}\,\frac{du_0}{d\theta_0}\,F_\theta \tag{5.9}$$

where $C_0 = r_0^2\dot{\theta}_0$ is the angular momentum of the Keplerian case.

In order to eliminate V from equation 5.5 at a later time, we must also linearize equation 4.1. Therefore:

$$2V_0\Delta V = \dot{\theta}_0^2\left[2\,\frac{dr_0}{d\theta_0}\,\frac{d(\Delta r)}{d\theta_0} + 2r_0\Delta r + 2r_0^2\,\frac{d(\Delta\theta)}{d\theta_0}\right]$$

By rewriting this equation in terms of variable u, we obtain:

$$2V_0\Delta V = C_0^2\left\{2\,\frac{du_0}{d\theta_0}\,\frac{d(\Delta u)}{d\theta_0} - \left[\frac{4}{u_0}\left(\frac{du_0}{d\theta_0}\right)^2 + 2u_0\right]\Delta u + 2u_0^2\,\frac{d(\Delta\theta)}{d\theta_0}\right\} \quad . \tag{5.10}$$

When solving ψ at a later time, we need to linearize the third equation of motion, namely, equation 3.17:

$$\frac{d^2(\Delta\psi)}{d\theta_0^2} + \left[\frac{1}{\dot{\theta}_0}\frac{d\dot{\theta}}{d\theta_0} + \frac{2}{r_0}\frac{dr_0}{d\theta_0}\right]\frac{d\Delta\psi}{d\theta_0} + \Delta\psi = \frac{F_\psi}{r_0\dot{\theta}_0^2}$$

Since, in the Keplerian case, the angular momentum is given by:

$$r_0^2\dot{\theta}_0 = C_0 = \text{constant}$$

therefore:

$$\frac{1}{\dot{\theta}_0}\frac{d\dot{\theta}_0}{d\theta_0} = -\frac{2}{r_0}\frac{dr_0}{d\theta_0}$$

The above equation is reduced to a forced pendulum equation:

$$\frac{d^2(\Delta\psi)}{d\theta_0^2} + \Delta\psi = \frac{F_\psi}{u_0^3C_0^2} \quad . \tag{5.11}$$

6. Solutions for Δr, and the Assumptions

Equations 5.7 and 5.9 may be solved by the help of equation 5.10. The equation for ψ, namely, equation 5.11 will be solved separately.

The merit of section 4, Rearrangement of the Equations, can be seen immediately, since with the known initial conditions, equation 5.7 can be immediately integrated. For convenience, the initial point of the satellite at its apogee position is chosen. Consequently:

$$\Delta u = \frac{d(\Delta u)}{d\theta_0} = \frac{d(\Delta\theta)}{d\theta_0} = \Delta\psi = \frac{d(\Delta\psi)}{d\theta_0} = 0 \quad \text{at } t = 0, \text{ or } \theta_0 = 0. \tag{6.1}$$

Substituting equation 5.10 into 5.7 and performing one integration, taking into account the initial conditions, we obtain:

$$2\frac{du_0}{d\theta_0}\frac{d(\Delta u)}{d\theta_0} - \left[\frac{4}{u_0}\left(\frac{du_0}{d\theta_0}\right)^2 + 2u_0 + 2\frac{g_0R^2}{C_0^2}\right]\Delta u + 2u_0^2\frac{d(\Delta\theta)}{d\theta_0}$$

$$= -\int_0^{\theta_0}\frac{2}{u_0^2C_0^2}\frac{du_0}{d\theta_0}F_r\,d\theta + \int_0^{\theta_0}\frac{2}{u_0C_0^2}F_\theta\,d\theta$$

which may be rewritten as:

$$\frac{d(\Delta\theta)}{d\theta_0} = -\frac{1}{u_0^2}\frac{du_0}{d\theta_0}\frac{d(\Delta u)}{d\theta_0} + \left[\frac{2}{u_0^3}\left(\frac{du_0}{d\theta_0}\right)^2 + \frac{1}{u_0} + \frac{g_0R^2}{u_0^2C_0^2}\right]\Delta u$$

$$-\frac{1}{u_0^2}\int_0^{\theta_0}\frac{1}{C_0^2u_0^2}\frac{du_0}{d\theta_0}F_r\,d\theta + \frac{1}{u_0^2}\int_0^{\theta_0}\frac{1}{C_0^2u_0}F_\theta\,d\theta \;. \qquad (6.2)$$

When differentiating once, this results in:

$$\frac{d^2(\Delta\theta)}{d\theta_0^2} = -\frac{1}{u_0^2}\frac{du_0}{d\theta_0}\frac{d^2(\Delta u)}{d\theta_0^2} + \left[\frac{4}{u_0^3}\left(\frac{du_0}{d\theta_0}\right)^2 - \frac{1}{u_0^2}\frac{d^2u_0}{d\theta_0^2} + \frac{1}{u_0} + \frac{g_0R^2}{C_0^2u_0^2}\right]\frac{d(\Delta u)}{d\theta_0}$$

$$+\left[-\frac{6}{u_0^4}\left(\frac{du_0}{d\theta_0}\right)^3 + \frac{4}{u_0^3}\left(\frac{du_0}{d\theta_0}\right)\frac{d^2u_0}{d\theta_0^2} - \frac{1}{u_0^2}\frac{du_0}{d\theta_0} - \frac{2}{u_0^3}\frac{du_0}{d\theta_0}\frac{g_0R^2}{C_0^2}\right]\Delta u$$

$$-\frac{1}{C_0^2u_0^4}\frac{du_0}{d\theta_0}F_r + \frac{2}{u_0^3}\frac{du_0}{d\theta_0}\int_0^{\theta_0}\frac{1}{C_0^2u_0^2}\frac{du_0}{d\theta_0}F_r\,d\theta$$

$$+\frac{1}{C_0^2u_0^3}F_\theta - \frac{2}{u_0^3}\frac{du_0}{d\theta_0}\int_0^{\theta_0}\frac{1}{C_0^2u_0}F_\theta\,d\theta \;. \qquad (6.3)$$

Substituting both equations 6.2 and 6.3 into equation 5.8 to eliminate $\Delta\theta$, we obtain:

$$[1 + \frac{1}{u_0^2} (\frac{du_0}{d\theta_0})^2] \frac{d^2(\Delta u)}{d\theta_0^2} - \frac{1}{u_0} (\frac{du_0}{d\theta_0}) [3 + \frac{1}{u_0} \frac{g_0 R^2}{C_0^2} - \frac{1}{u_0} \frac{d^2 u_0}{d\theta_0^2} + \frac{4}{u_0^2} (\frac{du_0}{d\theta_0})^2] \frac{d(\Delta u)}{d\theta_0}$$

$$+[3 - \frac{2 g_0 R^2}{u_0 C_0^2} + \frac{5}{u_0^2} (\frac{du_0}{d\theta_0})^2 + \frac{6}{u_0^4} (\frac{du_0}{d\theta_0})^4 - \frac{4}{u_0^3} (\frac{du_0}{d\theta_0})^2 \frac{d^2 u_0}{d\theta_0^2} + \frac{2}{u_0^3} (\frac{du_0}{d\theta_0})^2 \frac{g_0 R^2}{C_0^2}] \Delta u$$

$$= - \frac{1}{C_0^2 u_0^2} [1 + \frac{1}{u_0^2} (\frac{du_0}{d\theta_0})^2] F_r + \frac{2}{u_0} [1 + \frac{1}{u_0^2} (\frac{du_0}{d\theta_0})^2] \int_0^{\theta_0} \frac{1}{C_0^2 u_0^2} \frac{du_0}{d\theta_0} F_r \, d\theta$$

$$- \frac{2}{u_0} [1 + \frac{1}{u_0^2} (\frac{du_0}{d\theta_0})^2] \int_0^{\theta_0} \frac{1}{C_0^2 u_0} F_\theta \, d\theta \quad . \qquad (6.4)$$

A reduction is then possible by considering:

$$\frac{d^2 u_0}{d\theta_0^2} + u_0 = \frac{g_0 R^2}{C_0^2}$$

so that a factor of $[1 + \frac{1}{u_0^2} (\frac{du_0}{d\theta_0})^2]$ can be found in the coefficient of $\frac{d(\Delta u)}{d\theta_0}$ and Δu. As a result the above equation can be reduced to:

$$\frac{d^2(\Delta u)}{d\theta_0^2} - \frac{4}{u_0} \frac{du_0}{d\theta_0} \frac{d(\Delta u)}{d\theta_0} + [1 + \frac{6}{u_0^2} (\frac{du_0}{d\theta_0})^2 - \frac{2}{u_0} \frac{d^2 u_0}{d\theta_0^2}] \Delta u$$

$$= - \frac{1}{C_0^2 u_0^2} F_r + \frac{2}{u_0} \int_0^{\theta_0} \frac{1}{C_0^2 u_0^2} \frac{du_0}{d\theta_0} F_r d\theta - \frac{2}{u_0} \int_0^{\theta_0} \frac{1}{C_0^2 u_0} F_\theta \, d\theta \quad . \qquad (6.5)$$

Changing the variable from Δu back to Δr will eliminate the first order derivative and the coefficient of Δr becomes exactly the same as that of $\frac{d^2(\Delta r)}{d\theta_0^2}$. This results in:

$$\frac{d^2(\Delta r)}{d\theta_0^2} + \Delta r = \frac{1}{C_0^2 u_0^4} F_r - \frac{2}{C_0^2 u_0^3} \int_0^{\theta_0} [\frac{1}{u_0^2} \frac{du_0}{d\theta_0} F_r - \frac{1}{u_0} F_\theta] \, d\theta. \quad (6.6)$$

The complicated equations of 5.7 and 5.9 are combined, resulting in a simplified form of equation 6.6. The solution of this equation in the general form, with the given initial conditions as indicated in equation 6.1, may be immediately written down as:

$$\Delta r(\theta_0) = \int_0^{\theta_0} \sin(\theta_0 - \theta) \, \Xi(\theta) \, d\theta \quad (6.7)$$

where $\Xi(\theta)$ is the forcing function. In order to carry out the integration of equation 6.7, we must know the precise form of the forcing function, $\Xi(\theta)$. In other words, the perturbing forcing functions F_r, F_θ, and later F_ψ must be known explicitly as a function of θ. Refer to equations 3.8, 3.10, and 3.11, from which we may derive:

$$F_r = -\frac{3}{2} J_2 g_0 (\frac{R}{r_0})^4 [1 - 3(\sin\tilde{\theta}_0 \sin i_0 \cos\psi_0 + \cos i_0 \sin\psi_0)^2]$$

$$-K\sigma(r_0) V_{R_0} (\dot{r}_0 - r_0 \omega_e \cos\tilde{\theta}_0 \sin^2 i_0 \sin\psi_0)$$

$$F_\theta = -\frac{3}{2} J_2 g_0 (\frac{R}{r_0})^4 (\sin^2 i_0 \sin 2\tilde{\theta}_0 \cos\psi + \sin 2i_0 \cos\tilde{\theta}_0 \sin\psi_0)$$

$$-K\sigma(r_0) V_{R_0} r_0 (\cos\psi_0 \dot{\theta}_0 - \omega_e \cos i_0 \cos\psi_0 + \omega_e \sin i_0 \sin\tilde{\theta}_0 \sin\psi_0)$$

$$F_\psi = -\frac{3}{2} J_2 g_0 (\frac{R}{r_0})^4 [\sin\tilde{\theta}_0 \sin 2i_0 \cos 2\psi_0 + \sin 2\psi_0 (\cos^2 i_0 - \sin^2\tilde{\theta}_0 \sin^2 i_0)]$$

$$-K\sigma(r_0) V_{R_0} r_0 (\dot{\psi}_0 + \omega_e \cos\tilde{\theta}_0 \sin i_0 \cos\psi_0)$$

where $\sigma(r) = \frac{\rho}{\rho_s}$ is a density function as mentioned in equation 2.4 and

$$V_R = [\dot{r}^2 + r^2\cos^2\psi\dot{\theta}^2 + r^2\dot{\psi}^2 + 2r^2\omega_e\cos\psi(\cos\tilde{\theta}\sin i\dot{\psi} - \cos i\cos\psi\dot{\theta})$$

$$+ O(\omega_c \sin\psi, \omega_c^{\,2})]^{\frac{1}{2}} .$$

The above perturbing functions are very complex in their present form. However a simplification is possible, if the proper assumptions are made.

Let us focus our attention on the density function first. As mentioned in the introduction, assuming the atmosphere is spherical in shape, the pressure will vary with altitude. This variation of pressure is due to the weight of the atmosphere, and may be expressed as:

$$\frac{d}{dr}[4\pi\, r^2 p]\, dr = -4\pi\, r^2 \rho g dr, \qquad (6.8)$$

or

$$\frac{dp}{dr} + \frac{2}{r}p = -\rho g$$

where g is the local gravitational acceleration.

Usually it is true that:

$$\frac{2p}{r\rho g} << 1 \quad \text{or} \quad \frac{2\mathcal{R}T}{rg} << 1 .$$

If the atmosphere obeys the perfect gas law,

$$p = \rho\mathcal{R}T .$$

Equation 2.47 may then be reduced to:

$$\frac{dp}{dr} \doteq - \frac{p}{\mathcal{R}T} g = - \frac{p}{\mathcal{R}T} g_0 \left(\frac{R}{r}\right)^2 .$$

This can be integrated at once, if the temperature at satellite orbital height can be treated as a constant. This results in:

$$\frac{p}{p_s} = \frac{\rho}{\rho_s} = \sigma(r) = e^{\frac{g_0 R^2}{\mathcal{R}T}\left(\frac{1}{r} - \frac{1}{r_s}\right)} = e^{\nu\left(\frac{1}{r} - \frac{1}{r_s}\right)} \qquad (6.9)$$

where $\nu = \frac{g_0 R^2}{\mathcal{R}T}$ = a constant depends on the specific orbital altitude, and r_s is a radius of a reference point measured from the center of the Earth.

In current literature density ratio is often treated in the form of

$$\frac{\rho}{\rho_s} = \sigma(r) = e^{\lambda(r_s - r)} \qquad (6.10)$$

where λ is also a constant equal to the reciprocal of the scale height. Equation 6.10 is derived as a consequence of treating the gravitational acceleration g as a constant in addition to the above assumptions. Equation 6.10 is equivalent to the plane Earth's case. Therefore equation 6.9 is expressed more accurately than equation 6.10. In the limiting case, as the eccentricity of the satellite's orbit is vanishingly small, the expression of equations 6.10 and 6.9 becomes similar. The relation between the two constants is approximately as follows:

$$\lambda R^2 \doteq \nu .$$

In this analysis, equation 6.9 has been used for the density function $\sigma(r)$. For the Keplerian orbital case

$$\frac{1}{r_0} = \frac{1-\epsilon\cos\theta_0}{a(1-\epsilon^2)} = \frac{\mu}{C_0^2}(1-\epsilon\cos\theta_0) .$$

If the reference point is taken at perigee, i.e., $r_s = r_p$, then:

$$\frac{1}{r_p} = \frac{\mu}{C_0^2}(1+\epsilon) .$$

Therefore:

$$\sigma(r_0) = e^{\nu\left(\frac{1}{r_0}-\frac{1}{r_p}\right)} = e^{-\nu\frac{\mu\epsilon}{C_0^2}(1+\cos\theta_0)} = e^{-\frac{\nu\mu\epsilon}{C_0^2}}\sigma'[r(\theta_0)] . \qquad (6.11)$$

Furthermore, this can be expanded into a series of modified Bessel functions (for the details see Appendix). 6.11 may then be rewritten in the following form:

$$\sigma(r_0) = e^{-\zeta}[I_0(\zeta) - 2\cos\theta_0 I_1(\zeta) + 2\cos 2\theta_0 I_2(\zeta) + \ldots + 2(-1)^n \cos n\theta_0 I_n(\zeta) + \ldots] \qquad (6.12)$$

where $\zeta = \frac{\nu\mu\epsilon}{C_0^2}$, and $I_n(\zeta)$ are the modified Bessel functions with argument ζ.

Next, the resultant velocity V_R may be simplified as the following:

$$V_R = [V^2 + 2r^2\omega_e\cos\psi(\cos\tilde{\theta}\sin i\dot{\psi} - \cos i\cos\psi\dot{\theta}) + O(\omega_e\sin\psi, \omega_e^2)]^{\frac{1}{2}}$$

$$= V\left\{1 + \frac{1}{V^2}[2r^2\omega_e\cos\psi(\cos\tilde{\theta}\sin i\dot{\psi} - \cos i\cos\psi\dot{\theta}) + O(\omega_e^2)]\right\}^{\frac{1}{2}}$$

$$= V[1 + \frac{1}{V^2}r^2\omega_e\cos\psi(\cos\tilde{\theta}\sin i\dot{\psi} - \cos i\cos\psi\dot{\theta}) + \ldots] . \qquad (6.13)$$

Therefore, we obtain:

$$V_{R_0} = V_0 [1 - \frac{1}{V_0^2} r_0^2 \omega_e \cos i_0 \dot{\theta}_0 + \ldots]$$

$$\doteq V_0 [1 - \frac{C_0}{V_0^2} \omega_e \cos i_0] \quad . \tag{6.14}$$

From the results of equations 6.12 and 6.14 the perturbation forcing functions F_r, F_θ, and F_ψ may be rewritten in a simpler form as:

$$F_r = Ke^{-\zeta} C_0 \frac{du_0}{d\theta_0} V_0 [1 - \frac{C_0}{V_0^2} \omega_e \cos i_0] \, [I_0(\zeta) + \sum_{n=1}^{\infty} (-1)^n 2\cos n\theta_0 I_n(\zeta)]$$

$$- \frac{3}{2} J_2 g_0 R^4 u_0^4 [1 - 3 \sin^2(\theta_0 + \theta_a) \sin^2 i_0]$$

$$F_\theta = -Ke^{-\zeta} u_0 V_0 C_0 [1 - (\frac{C_0}{V_0^2} + \frac{1}{\dot{\theta}_0}) \omega_e \cos i_0] \cdot [I_0(\zeta)$$

$$+ \sum_{n=1}^{\infty} (-1)^n 2\cos n\theta_0 I_n(\zeta)] - \frac{3}{2} J_2 g_0 R^4 u_0^4 \sin^2 i_0 \sin 2(\theta_0 + \theta_a)$$

$$F_\psi = -Ke^{-\zeta} \frac{1}{u_0} V_0 \sin i_0 \omega_e \cos(\theta_0 + \theta_a) [I_0(\zeta) + \sum_{n=1}^{\infty} (-1)^n 2\cos n\theta_0 I_n(\zeta)]$$

$$- \frac{3}{2} J_2 g_0 R^4 u_0^4 \sin(\theta_0 + \theta_a) \sin 2i_0 \, . \tag{6.15}$$

A further simplification may be made, by assuming that the eccentricity, ϵ, of the orbit is minute and that terms higher than ϵ^2 may be neglected. This situation is true for most satellites. An extension to cover a higher order ϵ case can be made without any essential difficulties. This point will be discussed later.

In order to be correct to the ϵ order terms in the rotation of the line of apsides, we must accurately carry out calculations to the ϵ^2 order terms in the $\frac{d\Delta r}{d\theta_0}$ expression. This means that the $\Delta r(\theta_0)$ equation must be correct up to the ϵ^2 order terms.

Through the above assumption, binomial expansion may be used to simplify the expression u_0^n, V_0^2 in the forcing functions of 6.15. u_0 is a solution of Keplerian equation 5.8 which is:

$$u_0 = \frac{\mu}{C_0^2}(1 - \epsilon \cos\theta_0) .$$

By correcting up to the ϵ^2 order terms we obtain:

$$u_0^2 \doteq \frac{\mu^2}{C_0^4}(1 - 2\epsilon\cos\theta_0 + \epsilon^2\cos^2\theta_0)$$

$$u_0^{-2} \doteq \frac{C_0^4}{\mu^2}(1 + 2\epsilon\cos\theta_0 + 3\epsilon^2\cos^2\theta_0)$$

$$u_0^{-4} \doteq \frac{C_0^8}{\mu^4}(1 + 4\epsilon\cos\theta_0 + 10\epsilon^2\cos^2\theta_0)$$

$$V_0 = [\dot{r}_0^2 + r_0^2\,\dot{\theta}_0^2]^{\frac{1}{2}}$$

$$= C_0 u_0 \left[1 + \frac{1}{u_0^2}\left(\frac{du_0}{d\theta_0}\right)^2\right]^{\frac{1}{2}}$$

$$\doteq \frac{\mu}{C_0}(1 - \epsilon\cos\theta_0 + \tfrac{1}{2}\epsilon^2\sin^2\theta_0) \qquad \text{and so on.} \qquad (6.16)$$

From equations 6.15 and the above simplifications, the expression for $\Delta r(\theta)$, correct to the second order of ϵ, may be written as:

$$
\begin{aligned}
\Delta r(\theta_0) = & -\frac{3}{2}\,\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\int_0^{\theta_0}\sin(\theta_0-\theta)\left[1-3\,\sin^2\,(\theta+\theta_a)\sin^2 i_0\right]d\theta \\
& +\epsilon K e^{-\zeta}\frac{C_0^4}{\mu^2}\int_0^{\theta_0}\sin(\theta_0-\theta)\sin\theta\left[1-\frac{C_0^3}{\mu^2}\,(1+2\epsilon\cos\theta)\omega_e\cos i_0\right]\left[I_0+\sum_1^{\infty}(-1)^n 2\cos n\theta I_n\right]d\theta \\
& +3\epsilon^2 K e^{-\zeta}\frac{C_0^4}{\mu^2}\left[1-\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\right]\int_0^{\theta_0}\sin(\theta_0-\theta)\sin\theta\cos\theta\left[I_0+\sum_1^{\infty}(-1)^n 2\cos n\theta I_n\right]d\theta \\
& +3\epsilon\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\int_0^{\theta_0}(1+3\epsilon\cos\theta)\sin(\theta_0-\theta)\int_0^{\theta}\sin\theta'\left[1-3\sin^2(\theta'+\theta_a)\sin^2 i_0\right]d\theta' d\theta \\
& -6\epsilon^2\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\int_0^{\theta_0}\sin(\theta_0-\theta)\int_0^{\theta}\sin\theta'\cos\theta'\left[1-3\sin^2(\theta'+\theta_a)\sin^2 i_0\right]d\theta' d\theta \\
& -2\epsilon^2 K e^{-\zeta}\frac{C_0^4}{\mu^2}\left[1-\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\right]\int_0^{\theta_0}\sin(\theta_0-\theta)\int_0^{\theta}\sin^2\theta'\left[I_0+\sum_1^{\infty}(-1)^n 2\cos n\theta' I_n\right]d\theta' d\theta \\
& -3\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\,\sin^2 i_0\int_0^{\theta_0}\sin(\theta_0-\theta)\int_0^{\theta}(1-3\epsilon\cos\theta'+2\epsilon^2\cos^2\theta')\sin 2(\theta'+\theta_a)d\theta' d\theta \\
& -9\epsilon\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\,\sin^2 i_0\int_0^{\theta_0}\sin(\theta_0-\theta)\cos\theta\int_0^{\theta}(1-3\epsilon\cos\theta')\sin 2(\theta'+\theta_a)d\theta' d\theta \\
& +15\epsilon^2\frac{J_2}{g_0}\,\frac{\mu^2}{C_0^2}\,\sin^2\, i_0\int_0^{\theta_0}\sin(\theta_0-\theta)\cos^2\theta\int_0^{\theta}\sin 2(\theta'+\theta_a)d\theta' d\theta \\
& -2K e^{-\zeta}\frac{C_0^4}{\mu^2}\int_0^{\theta_0}\sin(\theta_0-\theta)(1+3\,\epsilon\cos\theta+5\epsilon^2\cos^2\theta)\ \times \\
& \left\{\int_0^{\theta}(1-\epsilon\cos\theta'+\tfrac{1}{2}\epsilon^2\sin^2\theta'-\epsilon^2\cos^2\theta')\left[1-\frac{C_0^3}{\mu^2}(2+4\epsilon\cos\theta'-\epsilon^2+7\epsilon^2\cos^2\theta')\,\omega_e\cos i_0\right]\right. \\
& \qquad \left.\times\left[I_0+\sum_1^{\infty}(-1)^n\,2\,\cos n\theta' I_n\right]\,d\theta'\right\}d\theta \qquad (6.17)
\end{aligned}
$$

$\Delta r(\theta_0)$ may be integrated explicitly for any arbitrary θ_0. We may then calculate the perturbation of the radius length to any point. The integration is straightforward, though lengthy.

In general, the change of orbital elements during the motion of the satellite may be classified into the following three categories: short periodic changes within one revolution, the long periodic changes according to the position of the argument of perigee or apogee, and the secular changes. The perturbing forcing functions are a function of θ_0, consequently the perturbing quantities will change from point to point on the trajectory. These short periodic changes are not of interest in this study. This study is concerned with the secular and the long periodic changes in the satellite trajectory. For this reason the integration limit is zero to 2π.

For the convenience of analysis at a later time, the symbol $(\ )\big|_f$ will be used to indicate the perturbed quantities, which are evaluated after one complete revolution of the satellite. The symbol of subscript f is used to denote the values at point $\theta_0 = 2\pi$. This may be expressed as:

$$A(\theta_0)\big|_f = A(\theta_0)\big|_{\theta_0=2\pi} \tag{6.18}$$

The result of integration of the perturbation of radial length Δr to the ϵ order, after one complete revolution, is as follows:

$$\Delta r\big|_f = -Ke^{-\zeta}\frac{C_0^4}{\mu^2}\pi\left\{4\left(1-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0\right)\left(I_0(\zeta)+I_1(\zeta)\right)\right.$$
$$\left.+\epsilon\left[2\left(3I_0(\zeta)+2I_1(\zeta)-I_2(\zeta)\right)+\left(I_0(\zeta)+16I_1(\zeta)+15I_2(\zeta)\right)\frac{C_0^3}{\mu^2}\omega_e\cos i_0\right]\right\}. \tag{6.19}$$

It is interesting to note that the oblateness effect of the Earth, which has been expressed in terms of J_2, produces no perturbations. There is no secular change in the radius due to the effect of J_2; that is, J_2 will not affect the life time of the satellite, since the bulge effect is part of a potential force field. It is surprising though to realize that there is no long periodic change to the first order perturbation assumptions in this study. The paper by Kozai (Ref. 3) also has proved that there is no long-periodic perturbation of the first order in the semi-major axis of the satellite trajectory.

The Earth's rotational effect, ω_e, has a considerable influence on the above expression Δr. The solution 6.19, as stated before, represents the trajectory of the easterly moving satellite. The sign of ω_e for the trajectory of the westerly moving satellite should be changed. Δr is subject to an order of 10 per cent, plus or minus, error in its calculation when the rotational effect of the Earth is ignored. Consequently, the estimation of the lifetime of the satellite, which is based on the shrinkage of the radius, will be subjected to an order of 10 per cent error if we do not take into account the Earth's rotational effect. For the above reason, the rotational atmospheric correction of Δr seems to be significant. The magnitude of the correction of Δr is about the same for most ordinary close Earth satellites, since the angular momentum of the satellite trajectory C_0 is only a weak function of the altitude.

7. The Other Solutions

The other solutions of the equations of motion will be discussed in this section. In order to arrive at the perturbations in the line of apsides, the line of nodes, and so on, we have to solve not only for Δr, $\Delta\theta$, $\Delta\psi$, but also their first derivatives. It is essential to carry out the first derivative of Δr to the ϵ^2 order. The perturbations in the line of nodes, the line of apsides, etc. will be discussed in the successive sections.

Since the expression for $\Delta r(\theta_0)$ is known, its derivative can be obtained by direct differentiation of equation 6.17. The perturbation of the derivative of Δr, correct to the ϵ^2 order, after one full revolution of the satellite, is obtained as follows:

$$\frac{d(\Delta r)}{d\theta_0}\Big|_f = -\frac{3}{2}\epsilon\frac{J_2}{g_0}\frac{\mu^2}{C_0^2}\pi(1-3\epsilon)\left[(2-3\sin^2 i_0) + 3\sin^2 i_0\cos 2\theta_a\right]$$
$$-6\epsilon K e^{-\zeta}\frac{C_0^4}{\mu^2}\pi^2\left[I_0(\zeta)+\epsilon I_1(\zeta)-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0(I_0(\zeta)-\epsilon I_1(\zeta))\right]$$

as a result the perturbation, $\frac{d(\Delta r)}{d\theta_0}\Big|_f$ includes both the secular and the long-periodic terms. Due to the rotation of the line of apsides, the argument of the apogee, θ_a, has a periodic value. Those terms containing θ_a will therefore have a long periodic change. The bulge and the drag effects of the above perturbation are of the order of ϵ and higher.

The next calculation derived from equation 6.1 solves $\frac{d(\Delta\theta)}{d\theta_0}$:

$$\frac{d(\Delta\theta)}{d\theta_0} = -\frac{1}{u_0^2}\frac{du_0}{d\theta_0}\frac{d(\Delta u)}{d\theta_0} + \left[\frac{2}{u_0^3}\left(\frac{du_0}{d\theta_0}\right)^2 + \frac{1}{u_0} + \frac{g_0 R^2}{u_0^2 C_0^2}\right]\Delta u$$

$$-\frac{1}{C_0^2 u_0^2}\int_0^{\theta_0}\left[\frac{1}{u_0^2}\frac{du_0}{d\theta_0}F_r - \frac{1}{u_0}F_\theta\right]d\theta$$

To simplify this equation, terms higher than the ϵ^2 order are neglected as in equation 6.16. By substituting the forcing function of 6.15, we obtain:

$$\begin{aligned}
\frac{d(\Delta\theta)}{d\theta_0} = {} & \epsilon\frac{\mu}{C_0^2}\sin\theta_0\frac{d(\Delta r)}{d\theta_0} - \frac{\mu}{C_0^2}(2-\epsilon\cos\theta_0)\Delta r + \frac{3}{2}\epsilon\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}(1-\cos\theta_0) \\
& -\frac{9}{2}\epsilon\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\sin^2 i_0\int_0^{\theta_0}\sin\theta\sin^2(\theta+\theta_a)d\theta \\
& -\frac{3}{2}\frac{J_2}{g_0}\sin^2 i_0\frac{\mu^3}{C_0^4}(1+2\epsilon\cos\theta_0)\int_0^{\theta_0}\sin 2(\theta+\theta_a)d\theta \\
& +\frac{9}{2}\epsilon\frac{J_2}{g_0}\sin^2 i_0\frac{\mu^3}{C_0^4}\int_0^{\theta_0}\cos\theta\sin 2(\theta+\theta_a)\,d\theta \qquad (7.2) \\
& -Ke^{-\zeta}\frac{C_0^2}{\mu}\left(1-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0\right)(1+2\epsilon\cos\theta_0)\int_0^{\theta_0}\left[I_0(\zeta)+\sum_1^{\infty}(-1)^n 2\cos n\theta I_n(\zeta)\right]d\theta \\
& +\epsilon Ke^{-\zeta}\frac{C_0^2}{\mu}\left(1+4\frac{C_0^3}{\mu^2}\omega_e\cos i_0\right)\int_0^{\theta_0}\cos\theta\left[I_0(\zeta)+\sum_1^{\infty}(-1)^n 2\cos n\theta I_n(\zeta)\right]d\theta .
\end{aligned}$$

Carrying out the indicated integration for $\theta_0 = 2\pi$, we arrive at the following:

$$\begin{aligned}
\left.\frac{d(\Delta\theta)}{d\theta_0}\right|_f = 2\pi Ke^{-\zeta}\frac{C_0^2}{\mu}\Big\{ & \left(1-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0\right)(3I_0(\zeta)+4I_1(\zeta)) \\
& +\epsilon\left[\!\left[2I_0(\zeta)+I_1(\zeta)-2I_2(\zeta)+[7I_0(\zeta)-16I_1(\zeta)-15I_2(\zeta)]\frac{C_0^2}{\mu}\omega_e\cos i_0\right]\!\right]\Big\} . \qquad (7.3)
\end{aligned}$$

This perturbation is due to the drag force of the atmosphere.

The perturbation of $\Delta\theta$ may be obtained by directly integrating equation 7.2. Substituting the expression for Δr, namely equation 6.17 into equation 7.2 and integrating, we obtain

$$\Delta\theta\Big|_f = \frac{3}{2}\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\pi[2(2-3\sin^2 i_0)-5\epsilon(2-3\sin^2 i_0)+(3+\epsilon)\sin^2 i_0\cos 2\theta_a]$$

$$+6Ke^{-\zeta}\frac{C_0^2}{\mu}\pi^2\left\{I_0(\zeta)-\epsilon[2I_0(\zeta)-I_1(\zeta)]+\frac{2}{3}\frac{C_0^3}{\mu^2}\omega_e\cos i_0[I_0(\zeta)-\epsilon I_1(\zeta)]\right\} \tag{7.4}$$

which is correct to the order of ϵ and contains both the secular and the long-periodic terms. The meaning and the significance of the above perturbation will be elaborated on in the latter sections of this study.

It has been shown in the previous section, after linearization, that the third equation of motion will be completely decoupled from the other equations. By rewriting equation 5.10 and considering equation 6.15, we may obtain

$$\frac{d^2(\Delta\psi)}{d\theta_0^2}+\Delta\psi = -\frac{1}{C_0^2u_0^3}\left\{\frac{3}{2}\frac{J_2}{g_0}\mu^2u_0^4\sin(\theta_0+\theta_a)\sin 2i_0\right.$$

$$\left.+Ke^{-\zeta}\frac{v_0}{u_0}\sin i_0\,\omega_e\cos(\theta_0+\theta_a)[I_0(\zeta)+\sum_1^\infty(-1)^n 2\cos n\theta_0 I_n(\zeta)]\right\}. \tag{7.5}$$

The solution of this equation may be immediately written down with the given initial conditions 6.1. If $\epsilon << 1$, and its higher than ϵ^2 terms are neglected, the resulting solution is:

$$\Delta\psi(\theta_0) = -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\sin 2i_0\int_0^{\theta_0}\sin(\theta_0-\theta)(1-\epsilon\cos\theta)\sin(\theta+\theta_a)d\theta$$

$$-Ke^{-\zeta}\frac{C_0^5}{\mu^3}\omega_e\sin i_0\int_0^{\theta_0}\sin(\theta_0-\theta)(1+3\epsilon\cos\theta)\cos(\theta+\theta_a)\left[I_0(\zeta)\right.$$

$$\left.+\sum_1^{\infty}(-1)^n\, 2\cos n\theta\, I_n(\zeta)\right]d\theta \;. \tag{7.6}$$

Therefore, the perturbation $\Delta\psi$, after a complete revolution of the satellite will be:

$$\begin{aligned}\Delta\psi\Big|_f &= \frac{3}{2}\pi\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\sin 2i_0\cos\theta_a \\ &-\pi Ke^{-\zeta}\frac{C_0^5}{\mu^3}\omega_e\sin i_0\left[I_0(\zeta)-\frac{3}{2}\epsilon I_1(\zeta)\right]\sin\theta_a\end{aligned} \tag{7.7}$$

$\Delta\psi\,|_f$ includes only the long-periodic change caused by the oblateness of the Earth, and the drag effects of the atmosphere.

By differentiating equation 7.6 for one satellite revolution we obtain:

$$\begin{aligned}\frac{d(\Delta\psi)}{d\theta_0}\Big|_f &= -\frac{3}{2}\pi\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\sin 2i_0\sin\theta_a \\ &-\pi Ke^{-\zeta}\frac{C_0^5}{\mu^3}\omega_e\sin i_0\left[I_0(\zeta)-\frac{9}{2}\epsilon I_1(\zeta)\right]\cos\theta_a\end{aligned} \tag{7.8}$$

which is also a long-periodic perturbation. The perturbation $\Delta\psi$ and its first derivative, may be used in describing the motion of the satellite's orbital plane.

The perturbation of the velocity, i.e., ΔV may be obtained by simplifying equation 5.9. By considering the relations expressed in 6.16, equation 5.9 may be written as:

$$\Delta V(\theta_0) = \frac{\mu^2}{C_0^3}(1-2\epsilon\cos\theta_0)\Delta r + \epsilon\frac{\mu^2}{C_0^3}\sin\theta_0\frac{d(\Delta r)}{d\theta_0} + \frac{\mu}{C_0}(1-\epsilon\cos\theta_0)\frac{d(\Delta\theta)}{d\theta_0} \tag{7.9}$$

The perturbation of the satellite's orbiting velocity, $\Delta V(\theta_0)$, can be calculated at any instant, by substituting $\Delta \mathbf{r}(\theta_0)$, $\frac{d\Delta r}{d\theta_0}(\theta_0)$, and $\frac{d\Delta\theta}{d\theta_0}(\theta_0)$ into the above equation. By letting $\theta_0 = 2\pi$, we obtain the following expression for one complete revolution of the satellite:

$$\Delta V\Big|_f = Ke^{-\zeta}\pi C_0\left\{(1-2\frac{C_0^3}{\mu^2}\omega_e \cos i_0)[2I_0(\zeta)+4I_1(\zeta)]\right.$$
$$\left.-\epsilon[10I_1(\zeta)+2I_2(\zeta)+(27I_0(\zeta)+54I_1(\zeta)+15I_2(\zeta))\frac{C_0^3}{\mu^2}\omega_e \cos i_0]\right\} . \quad (7.10)$$

This shows that only the drag force of the atmosphere causes the perturbation in the orbital velocity. It is interesting to note that the presence of air drag on the satellite results in an increase of velocity, when travelling in a near circular orbit. This paradox can be explained through considering the phenomenon of energy balance. The potential energy drop in this situation appears as an increase in kinetic energy, which is larger than the dissipating energy loss. The above equation demonstrates that the eccentricity of the satellite orbit will produce a negative perturbation in orbital velocity. Therefore, in a large eccentric orbit the perturbation result may be negative.

The last perturbation solvable directly from the equations of motion is the inclination angle i. By referring to equation 3.19 and applying the proper linearization, we obtain:

$$\cos i_0-(\Delta i)\sin i_0 = \cos i_0\left\{1-\tan i_0[\Delta\psi \sin\tilde{\theta}_0 + \frac{d\Delta\psi}{d\theta_0}\cos\tilde{\theta}_0\right.$$
$$\left.+ \text{higher order terms}]\right\} . \quad (7.11)$$

This equation, correct to the first order, can be written for any instant as:

$$\Delta i(\theta_0) = \Delta\psi(\theta_0)\sin(\theta_0+\theta_a) + \frac{d(\Delta\psi)}{d\theta_0}(\theta_0)\cos(\theta_0+\theta_a) \ . \qquad (7.12)$$

The perturbation, Δi, after one complete revolution of the satellite is:

$$\Delta i|_f = \sin\theta_a \, \Delta\psi|_f + \cos\theta_a \frac{d(\Delta\psi)}{d\theta_0}|_f$$

$$= -Ke^{-\zeta} \frac{C_0^5}{\mu^3} \omega_e \sin i_0 [I_0(\zeta) - 3\epsilon I_1(\zeta) - \frac{3}{2}\epsilon\cos 2\theta_a I_1(\zeta)] \qquad (7.13)$$

which proves that the inclination angle of the orbit has slow varying secular, and long periodic perturbations which are caused by the rotational effect of the Earth's atmosphere. This long periodic change is only proportional to the orbital eccentricity ϵ. Observation of the true satellite trajectory indicates that there is a very slow varying secular change in the orbital inclination angle. Other studies usually discount the atmospheric rotational effect, and are therefore unable to explain the observed secular change in the inclination angle of the orbital plane. This secular perturbation has a tendency to decrease the inclination angle of the easterly moving satellite's trajectory plane.

8. The Rotation of Line of Nodes

To determine the motion of the orbital plane of the satellite, we must know the motion of either of the ascending or descending nodes, and the orbital inclination angle as measured from the Earth's equator. To understand the motion of the satellite in the orbital plane, we must determine the amount of rotation of line of apsides, in addition to the previously discussed changes in radius and orbital velocity. The line of nodes locates the orbital plane of the trajectory. The line of apsides locates the orientation of the orbital ellipse. Both lines are vital in discussing any satellite theory. In this section we will derive the motion of line of nodes. In the next section we will treat the line of apsides.

The particular arrangement of coordinates in this study determines the specific angle relationships between various angles as indicated in equations 3.6. These angle relationships are purely geometrical. By combining the second and third equations of 3.6, we obtain:

$$\sin \beta = \sin i \sin \tilde{\theta} \cos \psi + \cos i \sin \psi \qquad (8.1)$$

This relation is true at any instant. β is the latitude angle, measured from the equator as we have previously indicated. We know that the argument of latitude $\tilde{\theta}$ will be perturbed from its initial Keplerian value at any moment. The same is true for the ψ angle. That is:

$$\tilde{\theta} = \tilde{\theta}_0 + \Delta\theta = (\theta_a + \theta_0) + \Delta\theta$$

By referring to the initial coordinate system, the latitude angle will change by the amount $\Delta\beta$, i.e.

$$\beta = \beta_0 + \Delta\beta$$

Since β is measured in the initial coordinate system, the inclination angle of the orbital plane remains the same. That is, the new latitude resulting from the change of $\tilde{\theta}$ and ψ can be arrived at by considering equation 8.1 as follows:

$$\sin(\beta_0 + \Delta\beta) = \sin i_0 \sin(\tilde{\theta}_0 + \Delta\theta) \cos\Delta\psi + \cos i_0 \sin\Delta\psi \ . \qquad (8.1')$$

At the final instant, $\theta_0 = 2\pi$, after the satellite makes one revolution, θ_0 to 2π, we may redefine the corresponding new Keplerian trajectory. This trajectory may be redefined by its osculating orbital values at that instant. The above concept will be elaborated on in the next section of the study. After one complete revolution, the satellite will be in a new orbital plane. In this new orbital plane $\psi = 0$. The perturbation of the line of nodes is defined by the angle difference between this new instantaneous orbital plane, and the initial plane, as measured on the equatorial plane extending from the center of Earth.

By using $(\overline{\ })$ notation to indicate the values on the new orbital plane, and referring to equation 8.1, we obtain:

$$\sin\bar{\beta} = \sin\bar{i} \sin\bar{\theta} \qquad (8.2)$$

which is due to the fact that, in the new orbital plane, $\bar{\psi} = 0$. Therefore:

$$\bar{\beta} = \beta_0 + \Delta\beta|_f \ ,$$

$$\sin\bar{\beta} = \sin(\beta_0 + \Delta\beta|_f) \ . \qquad (8.3)$$

At the final instant of $\theta_0 = 2\pi$, we can equate 8.2 and 8.1', thereby obtaining:

$$\sin\bar{i}\sin\bar{\theta} = \sin i_0 \sin(\tilde{\theta}_0 + \Delta\theta|_f)\cos\Delta\psi|_f + \cos i_0 \sin\Delta\psi|_f . (8.4)$$

The $\bar{i}$ in the new orbital plane is composed of:

$$\bar{i} = i_0 + \Delta i|_f$$

When considering $\bar{\theta}$ in the new orbital plane, we must take into account the shift of the nodal line of the plane, since $\bar{\theta}$ is measured from the new nodal line. When Ω is the angle of the ascending nodal line, the perturbation in Ω will be (referring to Figure 5):

$$\Omega = \Omega_0 + \Delta\Omega$$

then:

$$\bar{\theta} = [\theta_a + \Delta\theta|_f - \Delta\Omega|_f \cos(i_0 + \Delta i|_f)]$$

By substituting $\bar{i}$ and $\bar{\theta}$ into equation 8.4 and neglecting the higher order terms, since the perturbation angles are minute, we obtain

$$\Delta\Omega|_f = -\frac{1}{\sin i_0 \cos\theta_a} \Delta\psi|_f + \frac{\tan\theta_a}{\sin i_0} \Delta i|_f \,. \qquad (8.5)$$

Substituting $\Delta\psi|_f$, equation 7.7, and $\Delta i|_f$, equation 7.11, into the above equation, we obtain:

$$\Delta\Omega|_f = -3\pi \frac{J_2}{g_0} \frac{\mu^3}{C_0^4} \cos i_0 + \frac{3}{2}\epsilon K e^{-\zeta} \pi \frac{C_0^5}{\mu^3} \omega_e I_1(\zeta) \sin 2\theta_a \qquad (8.6)$$

which represents the perturbation of the line of nodes accurately to the first order, after the satellite makes one complete revolution.

The secular change of the line of nodes is a result of the bulge effect. The long periodic change is an effect of the atmospheric rotational effect of the Earth. The secular perturbation, or the regression of the line of nodes, in the above solution, is a well-known phenomenon. This means that the orbital plane of the satellite is rotating about the polar axis of the Earth. The minus

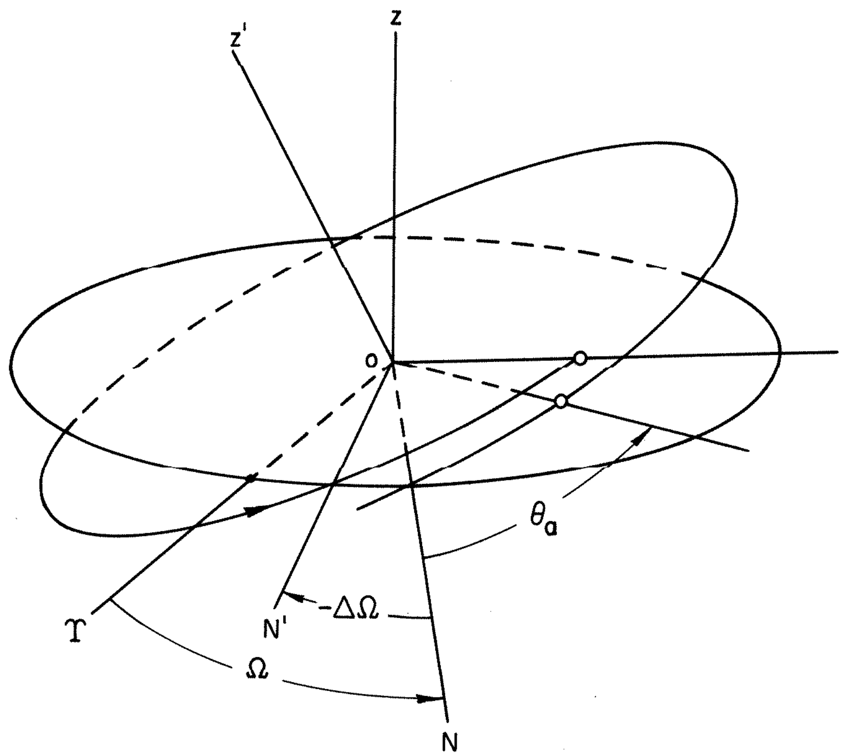

Figure 5. The Rotation of Line of Nodes

sign indicates that the nodal line regresses, that is, moves contrary to the satellite as measured along the equatorial plane.

9. The Rotation of Line of Apsides

At any point on a satellite trajectory, for a given set of orbital elements, a corresponding Keplerian orbit can always be found which has the same exact element values. Therefore, at every point on the orbital trajectory, a corresponding Keplerian ellipse may be formulated. That is, at any particular instant it may be assumed that, if the perturbation forces were suddenly nullified, the satellite trajectory would become a Keplerian ellipse. There is an analogy between this concept and the total pressure concept of fluid dynamics. In the total pressure concept, when fluid at any point is brought to rest isentropically, then the static pressure value of the fluid will reduce to the total pressure, that is, the corresponding stagnation values.

When considering the perturbation nullification concept, we become interested in two particular ellipses. These ellipses correspond to points $\theta_0 \neq 0$ and $\theta_0 = 2\pi$, since our integration limit is from $\theta_0 \neq 0$ to 2π. The initial ellipse, point $\theta_0 = 0$, and second ellipse, point $\theta_0 = 2\pi$, are indicated in Figure 6.

$(\bar{\ })$ notation is used to indicate the values of the second ellipse, and $(\ )|_f$ is used to indicate the values which have been evaluated at poin $\theta_0 = 2\pi$. Three plane orbital elements must be known in order to completely determine the values and orientation of the second ellipse. This process may be accomplished by matching the second ellipse elements with the known quantities at point f.

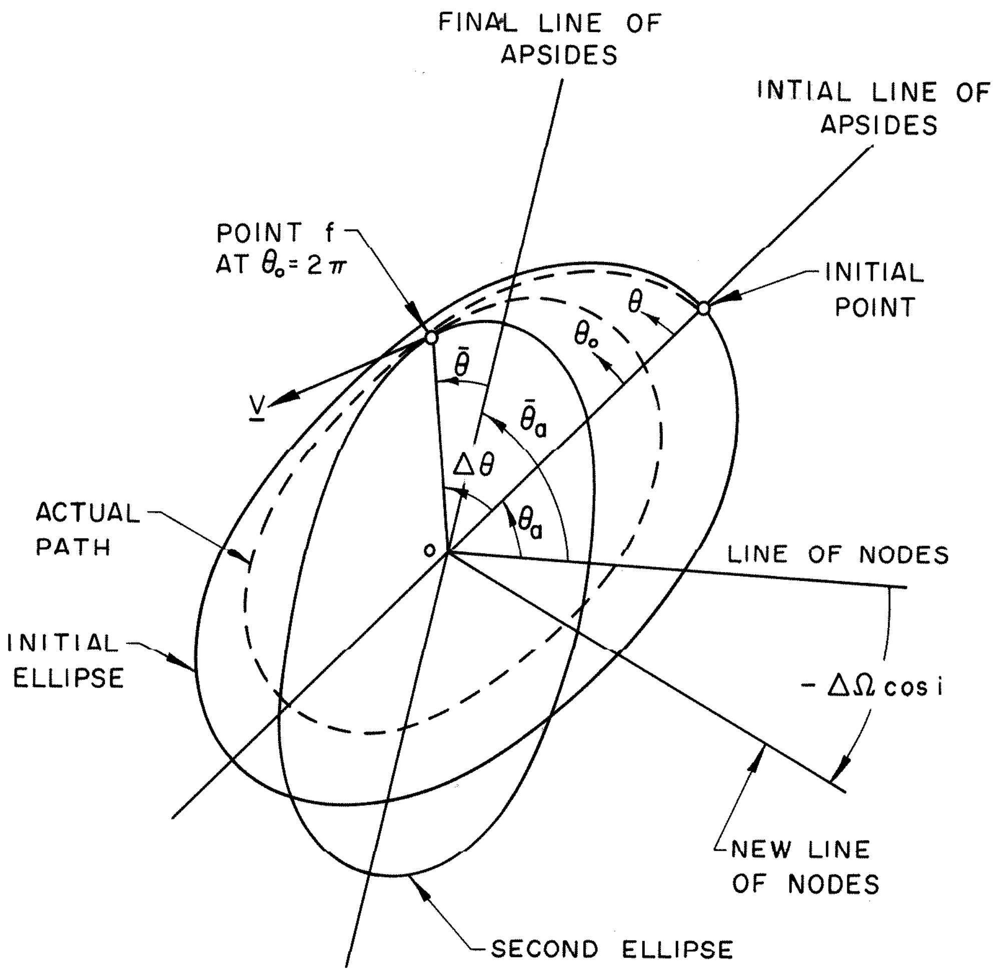

Figure 6. Schematic Diagram of Initial and Second Ellipse and Their Angle Relationships

It is known that at point f the radius length and its first derivative must be the same. The angular momentum of the second ellipse may be derived in the following manner:

$C = r^2\dot{\theta}$ is the definition for angular momentum where:

$r = r_0 + \Delta r$, and $\theta = \theta_0 + \Delta\theta$.

$$C = C_0 + \Delta C = (r_0 + \Delta r)^2 \, \dot{\theta}_0 \, (1 + \frac{d\Delta\theta}{d\theta_0}) \, .$$

If we neglect the higher order terms, we obtain

$$\overline{C}_0 = (1 + \frac{d\Delta\theta}{d\theta_0}\Big|_f) \, C_0 + 2[r_0\dot{\theta}_0 \, \Delta r]_f \, . \tag{9.1}$$

For the Keplerian trajectory, it is known that:

$$\frac{d^2 u_0}{d\theta_0{}^2} + u_0 = \frac{\mu}{C_0^2} \, .$$

Therefore, its solution is:

$$u_0 = \frac{\mu}{C_0^2} (1 - \epsilon\cos\theta_0) \, .$$

For the second Keplerian ellipse, it is known that:

$$\overline{u}_0 = \frac{\mu}{\overline{C}_0^2} (1 - \epsilon\cos\overline{\theta}_0) \, .$$

The above equation must equal the value of u at point f. That is:

$$\overline{u}_0\big|_f = u\big|_f \, , \tag{9.2}$$

which results in:

$$\frac{\mu}{\overline{C}_0^2} [1 - \overline{\epsilon}\cos\overline{\theta}_0]_f = \frac{1}{(r_0 + \Delta r)\big|_f} = \frac{1}{r_0\big|_f} (1 - \frac{\Delta r}{r_0}\Big|_f)$$

$$= u_0\big|_f \, [1 - u_0\big|_f \, (\Delta r)\big|_f]$$

$$= \frac{\mu}{C_0^2} (1 - \epsilon)[1 - \frac{\mu}{C_0^2}(1 - \epsilon) \, \Delta r\big|_f] \tag{9.3}$$

or,

$$1-\overline{\epsilon}\cos\overline{\theta}_0\Big|_f = \frac{\overline{C}_0^2}{C_0^2}(1-\epsilon)\left[1-\frac{\mu}{C_0^2}(1-\epsilon)\,\Delta r\Big|_f\right] \quad . \tag{9.4}$$

The derivatives of u must be matched at point f:

$$\frac{d\overline{u}_0}{d\theta_0}\Big|_f = \frac{du}{d\theta_0}\Big|_f = -u^2\Big|_f\frac{dr}{d\theta_0}\Big|_f = -u^2\Big|_f\frac{d\Delta r}{d\theta_0}\Big|_f \tag{9.5}$$

where we used the relation of

$$\frac{dr_0}{d\theta_0}\Big|_f = 0 \; .$$

By using equation 9.3 and carrying out the differentiation, we obtain:

$$\frac{\mu}{\overline{C}_0^2}\,\overline{\epsilon}\sin\overline{\theta}_0\Big|_f = -\frac{\mu^2}{C_0^4}(1-\epsilon)^2\left[1-\frac{\mu}{C_0^2}(1-\epsilon)\Delta r\Big|_f\right]^2\frac{d(\Delta r)}{d\theta_0}\Big|_f \, . \tag{9.6}$$

Equation 9.1 may be rewritten, since:

$$\dot{\theta}_0 = \frac{C}{r_0^2} = C_0u_0^2 \, ,$$

therefore:

$$[r_0\dot{\theta}_0]_f = C_0u_0\Big|_f = \frac{\mu}{C_0}(1-\epsilon),$$

and equation 9.1 becomes:

$$\overline{C}_0 = \left(1+\frac{d\Delta\theta}{d\theta_0}\Big|_f\right)C_0 + 2\frac{\mu}{C_0}(1-\epsilon)\,\Delta r\Big|_f \quad . \tag{9.7}$$

From these three equations, Namely equations 9.4, 9.6, and 9.7, with the known quantities of $\Delta r\big|_f$, $\frac{d(\Delta r)}{d\theta}\big|_f$, and $\frac{d(\Delta\theta)}{d\theta_0}\big|_f$ we may solve for the three new orbital elements, $\overline{\theta}_0\big|_f$, $\overline{\epsilon}$, and $\overline{C}_0$. Therefore the elements of the second ellipse may be completely determined. The above three equations are valid for any order of the ϵ.

From the geometrical considerations apparent in Figure 7, we find that the new argument of the latitude $\overline{\theta}_a$ may be expressed as:

$$\overline{\theta}_a = \theta_a + \Delta\theta|_f - \overline{\theta}|_f , \tag{9.8}$$

hence the perturbation of the true anomaly, for one complete revolution of the satellite is:

$$\overline{\theta}_a - \theta_a = \Delta\theta|_f - \overline{\theta}_0|_f . \tag{9.9}$$

If the orbital plane of the satellite remains in a fixed position, the above expression should be equal to the rotation of the line of apsides. However, the argument of the apogee is defined as measured from the line of nodes, which after one revolution regresses the amount, $\Delta\Omega$, as was proved in the previous section of the study. Therefore, the perturbation of line of apsides should take the change of Ω into account. Projecting the movement of the line of nodes into the orbital plane, we obtain:

$$\Delta\omega|_f = \Delta\theta|_f - \overline{\theta}_0|_f - \Delta\Omega|_f \cos (i_0+\Delta i|_f) . \tag{9.10}$$

The above equation is the final expression for the perturbation of the line of apsides.

In order to evaluate equation 9.10, we must first calculate the value of $\overline{\theta}_0|_f$. By studying the equations for the second ellipse, equations 9.4, 9.6, and 9.7, $\overline{\theta}_0|_f$ may be calculated. ϵ^2 and the higher terms will be dropped as in previous simplifications. Next we substitute $\frac{d(\Delta\theta)}{d\theta_0}|_f$ and $\Delta r|_f$ from equations 7.3 and 6.19 into the following equation:

$$\overline{C}_0 = [1 + \frac{d\Delta\theta}{d\theta_0}|_f] C_0 + 2\frac{\mu}{C_0}(1-\epsilon) \Delta r|_f . \tag{9.11}$$

The result is:

$$\overline{C}_0 = C_0 - \pi K e^{-\zeta} \frac{C_0^3}{\mu} \left\{ 2(1-2\frac{C_0^3}{\mu^2}\omega_e \cos i_0) I_0(\zeta) + \epsilon \left[2I_1(\zeta) - 2(20I_0(\zeta)+43I_1(\zeta)+15I_2(\zeta)) \frac{C_0^3}{\mu}\omega_e \cos i_0 \right] \right\} . \quad (9.12)$$

This equation indicates that only dissipating drag force contributes to the change in angular momentum of the satellite's orbital trajectory. Bulge effect plays no part in this change of angular momentum, since only the dissipating forces consume the orbital energy.

By combining equations 9.4 and 9.6 we obtain:

$$\tan\overline{\theta}_0\Big|_f = \frac{-\frac{\mu\overline{C}_0^2}{C_0^4}(1-\epsilon)^2\left[1-\frac{\mu}{C_0^2}(1-\epsilon)\Delta r\Big|_f\right]^2 \frac{d(\Delta r)}{d\theta_0}\Big|_f}{1-\frac{\overline{C}_0^2}{C_0^2}(1-\epsilon)\left[1-\frac{\mu}{C_0^2}(1-\epsilon)\Delta r\Big|_f\right]} . \quad (9.13)$$

By modifying equation 9.11 and keeping within the first order perturbation terms, we obtain:

$$\frac{\overline{C}_0^2}{C_0^2} = 1 + 2\frac{d\Delta\theta}{d\theta_0}\Big|_f + 4\frac{\mu}{C_0^2}\Delta r\Big|_f - 4\epsilon\frac{\mu}{C_0^2}\Delta r\Big|_f . \quad (9.14)$$

By substituting the above into equation 9.13, neglecting the higher order terms, and realizing that $\theta_0\big|_f$ is a very small quantity, we obtain:

$$\overline{\theta}_0\Big|_f \doteq \tan\overline{\theta}_0\Big|_f = \frac{-\frac{\mu}{C_0^2}(1-\epsilon)^2\left[1+2\frac{d\Delta\theta}{d\theta_0}\Big|_f + 2(1-\epsilon)\frac{\mu}{C_0^2}\Delta r\Big|_f\right]\frac{d(\Delta r)}{d\theta_0}\Big|_f}{\epsilon - 2(1-\epsilon)\frac{d\Delta\theta}{d\theta_0}\Big|_f - 3\frac{\mu}{C_0^2}(1-\epsilon)^2\Delta r\Big|_f} . \quad (9.15)$$

A further simplification of the equation is possible if:

$$\epsilon >> \frac{d\Delta\theta}{d\theta_0}\Big|_f \text{ and } \epsilon >> \frac{\mu}{C_0^2}\Delta r\Big|_f . \quad (9.16)$$

Usually the above statements are true, since the perturbations of $\frac{d\Delta\theta}{d\theta_0}\Big|_f$ and $\frac{\mu}{C_0^2}\Delta r\Big|_f$ are produced only by the drag effect, and their order of magnitude in one satellite period will be very minute. In general, equations 9.16 will hold true in most satellite cases. Assuming this, the denominator of the equation 9.15 may be expanded. Now we see why it is necessary to carry out $\frac{d\Delta r}{d\theta_0}\Big|_f$ correctly to the ϵ^2 order terms, since the denominator is an order of ϵ. The expression for $\theta_0\big|_f$ as indicated in equation 9.15 will not degenerate, due to the small denominator, since the numerator itself is proportional to $\frac{d\Delta r}{d\theta_0}\Big|_f$, which is on the order of ϵ.

After a simplification, $\overline{\theta}_0\big|_f$ becomes:

$$\overline{\theta}_0\Big|_f = \frac{3}{2}\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\pi\,(1-5\epsilon)\left[(2-3\sin^2 i_0)+3\sin^2 i_0\cos 2\theta_a\right]$$
$$+6Ke^{-\zeta}\frac{C_0^2}{\mu}\pi^2\left[I_0(\zeta)-2\epsilon I_0(\zeta)+\epsilon I_1(\zeta)-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0\left(I_0(\zeta)-2\epsilon I_0(\zeta)-\epsilon I_1(\zeta)\right)\right]. \tag{9.17}$$

Now, we have all of the elements needed to solve for the perturbation of the line of apsides. Substituting $\Delta\theta\big|_f$, $\overline{\theta}_0\big|_f$ and $\Delta\Omega\big|_f$ from equations 7.4, 9.17, and 8.6 into equation 9.10, we obtain the following result:

$$\Delta\omega\Big|_f = \frac{3}{2}\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\pi\left[(4-5\sin^2 i_0)+16\epsilon\sin^2 i_0\cos 2\theta_a\right]$$
$$+Ke^{-\zeta}\frac{C_0^5}{\mu^3}\pi\omega_e\cos i_0\left[8\pi(2-3\epsilon)I_0(\zeta)-16\epsilon\pi I_1(\zeta)\right.$$
$$\left.-\frac{3}{2}\epsilon I_1(\zeta)\sin 2\theta_a\right]. \tag{9.18}$$

This equation gives the rotation of line of apsides per satellite revolution. The first order secular term in the equation contains only J_2. Surprisingly, all of the secular parts of ϵJ_2, and the drag effects of the order of K and ϵK are cancelled out during the substitutions in equation 9.10. The secular part of the line of apsides, caused by the drag effect is due to the rotation of the Earth's atmosphere. However, the dominating major secular part is due to J_2, and is:

$$\Delta\omega_{maj.sec.} \sim \frac{3}{2}\frac{J_2}{g_0}\frac{\mu^3}{C_0^4}\pi\,(4-5\sin^2 i)$$

The rotation of the line of apsides, caused by the J_2 or flattening effect of the Earth is one of the most well-known features of satellite motion. When the inclination angle, $i = i_c$, and $i_c = 63.43^\circ$, that is $\sin^2 i_c = 4/5$, the rotation of the major axis of the satellite trajectory, due to the first order perturbation stops. The inclination angle i_c has usually been called the critical inclination angle. The importance of this critical angle is related to the fact that for $0 < i < i_c$, the motion of the argument of apogee is positive, or advanced, and for $i_c < i < \pi/2$, the motion of the argument of apogee is negative, or regressed. The rotation of the major axis advances about 15 degree/day for near equatorial orbits, and regresses about 4 degree/day for near polar orbits.

Besides this major perturbation, we found that the line of apsides will rotate due to the drag caused by atmospheric rotation of the Earth. Due to this drag, the line of apsides advances for an easterly moving satellite, and regresses for a westerly moving satellite. J_2 is the dominant effect in the secular part of the rotation of the major axis and drag is a very minor effect, nevertheless the drag effect is significant in precisely tracking the satellite, or in doing a geodetic study.

10. The Description of the Satellite Motion

In this section we are going to put together the results of the foregoing analysis, thereby giving a complete description of the motion of a satellite.

First, we will calculate the change in eccentricity of the orbit per revolution. Through considering the second and the initial Keplerian ellipses as introduced in the last section, we obtain:

$$\overline{\epsilon} \cos \overline{\theta}_0\Big|_f = 1 - \frac{\overline{C}_0^2}{C_0^2} (1-\epsilon) \left[1 - \frac{\mu}{C_0^2} (1-\epsilon) \Delta r\Big|_f \right] , \qquad (10.1)$$

and

$$\overline{C}_0 = C_0 \left(1 + \frac{d\Delta\theta}{d\theta_0}\Big|_f\right) + 2 \frac{\mu}{C_0} (1-\epsilon) \Delta r\Big|_f . \qquad (10.2)$$

It can be seen that $\overline{\theta}_0\big|_f$ is the first order perturbing quantity, $O(\eta)$, therefore the left hand side of equation 10.1 may be expanded. From equation 10.2 we obtain:

$$\overline{\epsilon} = 1 - \left[1-\epsilon + 2(1-\epsilon) \frac{d\Delta\theta}{d\theta_0}\Big|_f + 3(1-\epsilon)^2 \frac{\mu}{C_0^2} \Delta r\Big|_f + O(\eta^2)\right] .$$

Therefore:

$$\Delta\epsilon\Big|_f = \overline{\epsilon} - \epsilon = -2(1-\epsilon) \frac{d\Delta\theta}{d\theta_0}\Big|_f - 3(1-\epsilon)^2 \frac{\mu}{C_0^2} \Delta r\Big|_f + O(\eta^2) . \qquad (10.3)$$

The above equation is the perturbation of the eccentricity of the orbit, per satellite revolution. Equation 10.3 is valid for any value of ϵ. In order to be consistent with previous calculations, we will neglect in present calculations both the higher perturbing and eccentric terms. By substituting $\Delta r|_f$ and $\frac{d\Delta\theta}{d\theta_0}|_f$ from equations 6.19 and 7.3 into equation 10.3, we obtain:

$$\Delta\epsilon|_f = -Ke^{-\zeta}\pi\frac{C_0^2}{\mu}\Big\{4(1-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0)\,I_1(\zeta) + \epsilon[\![2[I_0(\zeta)-I_2(\zeta)] + 2[5I_0(\zeta)+8I_1(\zeta)+15I_2(\zeta)]\frac{C_0^3}{\mu^2}\omega_e\cos i_0]\!]\Big\}. \quad (10.4)$$

From the above result, we see that the perturbation in eccentricity is caused by the drag effect alone.

After the change in eccentricity of the orbit is found, it is relatively easy to calculate the change in major axis a, and the perigee distance r_p. From the geometrical relationship of the elliptical trajectory, we have:

$$r_a = a\,(1+\epsilon) \quad \text{and} \quad r_p = a\,(1-\epsilon). \quad (10.5)$$

Therefore:

$$\Delta a = \frac{1}{1+\epsilon}\,[\Delta r_a - a\Delta\epsilon]$$

and:

$$\Delta a|_f = \frac{1}{1+\epsilon}\,[\Delta r|_f - a\Delta\epsilon|_f]\quad. \quad (10.6)$$

By substituting $\Delta r|_f$ and $\Delta\epsilon|_f$ from equations 6.19 and 10.4, then neglecting $O(\epsilon^2)$ and higher terms, we obtain:

$$\Delta a|_f = -Ke^{-\zeta}\pi\frac{C_0^4}{\mu^2}\Big\{4(1-2\frac{C_0^3}{\mu^2}\omega_e\cos i_0)\,I_0(\zeta) + \epsilon[\![4I_1(\zeta)-[I_0(\zeta)+15I_2(\zeta)]\frac{C_0^3}{\mu^2}\omega_e\cos i_0]\!]\Big\}. \quad (10.7)$$

From equation 10.5, we obtain the perturbation of the perigee distance Δr_p as follows:

$$\Delta r_p\big|_f = (1-\epsilon)\,\Delta a\big|_f - a\Delta\epsilon\,\big|_f$$

$$= -Ke^{-\zeta}\,\pi\,\frac{C_0^4}{\mu^2}\Big\{4\Big(1-2\,\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\Big)\,[I_0(\zeta) - I_1(\zeta)]$$

$$+\epsilon\,\Big[\!\Big[\,6I_0(\zeta) - 4I_1(\zeta) - 2I_2(\zeta)$$

$$+[9I_0(\zeta) - 24I_1(\zeta) + 45I_2(\zeta)]\,\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\Big]\!\Big]\Big\} \qquad (10.8)$$

The loss of orbital energy per revolution can also be calculated, since the energy of the orbit is given by:

$$E = -\frac{\mu}{2a}\ .$$

Therefore:

$$\Delta E\big|_f = \frac{\mu}{2a^2}\,\Delta a\big|_f$$

$$= -Ke^{-\zeta}\,\pi\,\mu\Big\{2\Big(1-2\,\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\Big)\,I_0(\zeta)$$

$$+\epsilon\,\Big[\!\Big[2I_1(\zeta) - \frac{1}{2}\,[I_0(\zeta) + 15I_2(\zeta)]\,\frac{C_0^3}{\mu^2}\,\omega_e\cos i_0\Big]\!\Big]\Big\} \qquad (10.9)$$

The satellite motion can be completely determined from the preceding work, if the initial conditions are given. Our initial point in the satellite orbit has been chosen at apogee, and the values of the orbital elements at this instant have been designated by the subscript 0.

After one complete satellite revolution, the apogee distance measured from the center of Earth, will be decreased by the amount given in equation 6.19, which is correct to the first order as follows:

$$\Delta r_a = \Delta r\big|_f = -Ke^{-\zeta}\pi \frac{C_0^4}{\mu^2} \; 4(1-2\frac{C_0^3}{\mu^2}\omega_e \cos i_0)[I_0(\zeta)+I_1(\zeta)]$$

$$+\epsilon \,[\![\, 2[3I_0(\zeta) + 2I_1(\zeta)-I_2(\zeta)]$$

$$+ [I_0(\zeta)+16I_1(\zeta)+15I_2(\zeta)] \frac{C_0^3}{\mu^2}\omega_e \cos i_0] \qquad (6.19)$$

The perigee drop is given in equation 10.8.

The satellite orbit will spiral in, with a drop in eccentricity, $\Delta\epsilon\,\big|_f$, after one revolution, as is shown in equation 10.4. The orbital velocity of the satellite will be changed by the amount $\Delta V\big|_f$, as given in equation 7.10. The orbital plane will rotate about the polar axis in a direction contrary to the satellite. The amount of rotation, $\Delta\Omega\big|_f$, is given in equation 8.6. During the rotation of the orbital plane, the inclination angle will be changed by the amount $\Delta i\,\big|_f$, given in equation 7.12. The elliptical path of the trajectory is rotating in the direction of the moving satellite, as is evidenced by the rotation of the line of apsides. $\Delta\omega\big|_f$, the amount of rotation of the major axis, is obtained from equation 9.18.

11. Numerical Examples

Two numerical calculations will be carried out to estimate the perturbations of various orbital elements. The numerical data of the first example corresponds to the first United States manned orbiting satellite case, 1962 Gamma 1. The data of the second example corresponds to Vanguard I, 1958 Beta 2, which has been tracked carefully for many geodetic studies. The above-mentioned cases represent two extreme situations in satellite altitudes and eccentricities.

The data about the Earth used in this study is adopted as follows (p. 63, Ref. 15):

R = mean radius of the Earth = 6.367456×10^6 m

μ = $GM_{\oplus} = g_0 R^2 = 3.986016 \times 10^{14}\ m^3/sec^2$

J_2 = second coefficient of harmonics of the gravitational potential = 1.0825×10^{-3}

ω_e = angular velocity of the Earth = 0.7292×10^{-4} rad/sec

The density and the molecular-scale temperature data are obtained from "U. S. Standard Atmosphere, 1962" (p. 8 and p. 11, Ref. 16). The numerical values of modified Bessel functions $I_n(\zeta)$ are computed from their asymptotic form. Since the argument of these $I_n(\zeta)$'s in our cases are found to be great enough, their asymptotic form may be used. The asymptotic form is as follows (Reference 17):

$$I_n(\zeta) = \frac{e^{\zeta}}{\sqrt{2\pi\zeta}} \left\{ 1 - \frac{4n^2-1^2}{1!\,8\zeta} + \frac{(4n^2-1^2)(4n^2-3^2)}{2!\,(8\zeta)^2} \;---- \right\} \qquad (11.1)$$

In fact the above series converges very rapidly for small n.

Case 1.

Adopted data for 1962 Gamma 1:

Z_p = perigee altitude = 98 miles = 158 Km

Z_a = apogee altitude = 160 miles = 257 Km

S = frontal surface area = 28.2 ft^2

m = mass of the satellite = 2900 lbs = 90 slugs

i = inclination angle to equator = 32.5°

C_D = drag coefficient = 2

The eccentricity ϵ of this orbit can be calculated from:

$$\epsilon = \frac{r_a - r_p}{r_a + r_p} = .00759 , \tag{11.2}$$

where: r_a = geocentric radius of apogee = 4120 miles

r_p = geocentric radius of perigee = 4058 miles

The density and the molecular-scale temperature of the atmosphere at perigee height are:

$$\begin{aligned} \rho_s &= 1.265 \times 10^{-9} \text{ kg/m}^3 = 2.45 \times 10^{-12} \text{ slugs/ft}^3 \\ T_M &= 1080^\circ \text{ K} \end{aligned} \tag{11.3}$$

Using the gas constant at sea level:

$$\mathcal{R}_0 = 2.875 \times 10^6 \text{ cm}^2/\text{sec}^2 \, {}^\circ\text{K},$$

we can compute the non-dimensional parameter ζ as:

$$\zeta = \frac{\nu\mu\epsilon}{C_0^2} = \frac{\mu\epsilon}{\mathcal{R}_0 T_M r_p (1+\epsilon)} = 1.49 . \tag{11.4}$$

The drag parameter K can also be obtained as follows:

$$K = \frac{S C_D \rho_s}{2m} = 0.77 \times 10^{-12} \quad 1/\text{ft} \tag{11.5}$$

From the asymptotic form of equation (11.1), we obtain:

$$I_0(1.49) = .378 \times e^{1.49}$$

$$I_1(1.49) = .227 \times e^{1.49}$$

$$I_2(1.49) = .104 \times e^{1.49} \tag{11.6}$$

The semilatus rectum is:

$$p = r_p(1+\epsilon) = 6.56 \times 10^6 \text{ m} = 2.15 \times 10^7 \text{ ft.}, \tag{11.7}$$

and the angular momentum is:

$$C_0 = \sqrt{\mu p} = 5.11 \times 10^{10} \text{ m}^2/\text{sec} = 5.5 \times 10^{11} \text{ ft}^2/\text{sec} \qquad (11.8)$$

From these values we can obtain the following perturbation quantities per satellite revolution.

The perturbation in apogee radial distance is:

$$\Delta r_a = -K \frac{C_0^4}{\mu^2} e^{-\zeta} \pi \Big\{ 4\Big(1-2\frac{C_0^4}{\mu^2}\omega_e \cos i_0\Big)[I_0(\zeta) + I_1(\zeta)]$$

$$+\epsilon \Big[2[3I_0(\zeta)+2I_1(\zeta)-I_2(\zeta)] + [I_0(\zeta)+16I_1(\zeta)+15I_2(\zeta)] \frac{C_0^3}{\mu^2}\omega_e \cos i_0 \Big] \Big\}$$

$$= -2480 \text{ ft/cycle} \qquad (11.9)$$

It is surprising that the correction factor due to the rotation of atmosphere in the above equation is about 10% of the total value, since:

$$2\frac{C_0^3}{\mu^2}\omega_e \cos i_0 = .1035 \,, \qquad (11.10)$$

therefore the above correction is quite significant.

The secular part of the regression of the line of nodes is:

$$\Delta\Omega = -3\pi \frac{J_2}{g_0} \frac{\mu^3}{C_0^4} \cos i_0 = -.815 \times 10^{-2} \text{ rad./cycle} \qquad (11.11)$$

The secular part of the rotation of line of apsides is:

$$\Delta\omega = \frac{3}{2}\pi \frac{J_2}{g_0}\frac{\mu^3}{C_0^4}(4 - 5\sin^2 i_0)$$

$$+ 8Ke^{-\zeta}\frac{C_0^5}{\mu^3}\pi^2 \omega_e \cos i_0 [(2-3\epsilon)I_0(\zeta) - 2\epsilon I_1(\zeta)] \qquad (11.12)$$

$$= 1.235 \times 10^{-2} + .506 \times 10^{-4} = 1.2411 \times 10^{-2} \text{ rad./cycle}$$

In the above equation the drag effect is about a half-percent of that of the bulge effect. The secular part in the perturbation of the inclination angle is:

$$\Delta i = -Ke^{-\zeta} \pi \frac{C_0^5}{\mu^2} \omega_e \sin i_0 [I_0(\zeta) - 3\epsilon I_1(\zeta)]$$

$$= .644 \times 10^{-6} \text{ rad./cycle}, \quad (11.13)$$

which is quite small.

The perturbation in true anomaly is:

$$\overline{\theta}_a - \theta_a = .5531 \times 10^{-2} \text{ rad./cycle} . \quad (11.14)$$

The orbiting velocity of the satellite will increase by the amount ΔV at apogee, as follows:

$$\Delta V = Ke^{-\zeta} C_0 \pi \Big\{ (1 - 2 \frac{C_0^3}{\mu^2} \omega_e \cos i_0) [2I_0(\zeta) + 4I_1(\zeta)]$$

$$-\epsilon [\![10I_1(\zeta) + 2I_2(\zeta) + [27I_0(\zeta) + 54I_1(\zeta) + 15I_2(\zeta)] \frac{C_0^3}{\mu^2} \omega_e \cos i_0]\!] \Big\}$$

$$= 19.3 \text{ ft/sec, cycle} \quad (11.15)$$

As before, the correction of ΔV, due to the rotation of atmosphere, is of the order of 10%.

The perturbation in eccentricity of the orbit is:

$$\Delta \epsilon = -Ke^{-\zeta} \pi \frac{C_0^2}{\mu} \Big\{ (1 - 2 \frac{C_0^3}{\mu^2} \omega_e \cos i_0) \, 4I_1(\zeta)$$

$$+\epsilon [\![2I_0(\zeta) - 2I_2(\zeta) + 2 [5I_0(\zeta) + 8I_1(\zeta) + 15I_2(\zeta)] \frac{C_0^3}{\mu^2} \omega_e \cos i_0]\!] \Big\}$$

$$= .428 \times 10^{-4} \text{ 1/cycle}, \quad (11.16)$$

which is very small.

The perturbation of the major axis is:

$$\Delta a = \frac{1}{1+\epsilon}\,[\Delta r_a - a\Delta\epsilon\,] = -1.55 \times 10^3 \text{ ft/cycle,} \tag{11.17}$$

and the period of the satellite can be obtained as follows:

$$T_{sat.} = 2\pi\,(\frac{a^3}{\mu})^{\frac{1}{2}} = 5320 \text{ sec.} = 1.478 \text{ hr.} \tag{11.18}$$

Case 2.

Adopted data from 1958 Beta 2:

a = major axis = 1.3618 R

ϵ = eccentricity of the orbit = 0.1903

i = inclination angle to equator = 0.5977 rad.

d = diameter of the satellite = 6.4 in.

m = mass of the satellite = 3.25 lbs

C_D = drag coefficient = 2

The perigee radius r_p may be calculated as follows:

$$r_p = a(1-\epsilon) = 7.018 \times 10^6 \text{ m} = 4360 \text{ miles} .$$

At the above altitude, the density and the molecular-scale temperature of the atmosphere are:

$$\rho_s = 6 \times 10^{-13} \text{ Kg/m}^3 = 1.165 \times 10^{-15} \text{ slugs/ft}^3$$

$$T_M = 2650^\circ \text{ K} \tag{11.19}$$

The parameter ζ and K are:

$$\zeta = 11.92$$

$$K = .261 \times 10^{-14} \text{ 1/ft} \tag{11.20}$$

The modified Bessel functions are:

$$I_0(11.92) = .1215 \times e^{11.92}$$
$$I_1(11.92) = .112 \times e^{11.92}$$
$$I_2(11.92) = .098 \times e^{11.92} \qquad (11.21)$$

The semilatus rectum and the angular momentum are:

$$p = 8.37 \times 10^6 m = 2.75 \times 10^7 \text{ ft}$$
$$C_0 = 5.77 \times 10^{10} \text{ m}^2/\text{sec} = 6.2 \times 10^{11} \text{ ft}^2/\text{sec} \qquad (11.22)$$

The corrections in Δr, ΔV, and $\Delta\epsilon$, due to the effect of the rotation of atmosphere are as great as 14.6%, since:

$$2 \frac{C_0^3}{\mu^2} \omega_e \cos i_0 = 14.6 \times 10^{-2} \qquad (11.23)$$

The perturbation in the length of the radius of the apogee is very minute, as is shown by:

$$\Delta r_a = -6.35 \text{ ft/cycle} \qquad (11.24)$$

The rotations of line of nodes and the line of apsides are:

$$\Delta\Omega = -.498 \times 10^{-2} \text{ rad./cycle}$$

$$\Delta\omega = .73 \times 10^{-2} + .6 \times 10^{-7} \text{ rad./cycle} \qquad (11.25)$$

The drag portion of the rotation is about 10^{-5} of that of the bulge effect.

The perturbation of the inclination angle Δi and the eccentricity, $\Delta\epsilon$, of the satellite orbit are:

$$\Delta i = -.64 \times 10^{-9} \text{ rad/cycle}$$

$$\Delta\epsilon = -.91 \times 10^{-7} \text{ 1/cycle} \tag{11.26}$$

The perturbation of true anomaly is:

$$\overline{\theta}_a - \theta_a = .318 \times 10^{-2} \text{ rad./cycle} \tag{11.27}$$

The perturbation of the orbiting velocity, ΔV, and the major axis, Δa, are:

$$\Delta V = .173 \times 10^{-2} \text{ ft/sec, cycle} \tag{11.28}$$

$$\Delta a = -3.18 \text{ ft/cycle} \tag{11.29}$$

Finally, the period of the satellite can be computed as:

$$T_{sat.} = 8.05 \times 10^{3} \text{ sec} = 2.238 \text{ hr.} \tag{11.30}$$

12. The Possibility of Extension of Theory

A satellite motion theory has been developed which may be used in most artificial satellite cases. One of the merits of this analysis is the ability to handle the bulge and drag perturbations simultaneously, within one reference coordinate system. This permits us to further study the higher order perturbations, and their coupling relations. The set up of the equations of motion, 3.15 through 3.19, is rather general.

In order to simplify the calculations, the eccentricity ϵ of the orbit in this study has been kept to the first order accuracy. Exceptions to this are the perturbations of Δr and $\frac{d(\Delta r)}{d\theta_0}$, which have been evaluated to the second order accuracy. In principle, the present theory can be carried to the higher order of ϵ without much difficulty.

In this study the atmosphere has been assumed spherically symmetric. The effect of the ellipticity of the atmosphere is very small, but would be of interest in a further study.

Other minor perturbations caused by the celestial bodies, the electro-magnetic effect, the light pressure, and so on, can be included in the forcing functions if a further study is undertaken.

III. APPLICATIONS

1. Corrections to the Spherical Harmonics of the Earth

The major object of any satellite theory is the prediction, determination, modification, and selection of satellite orbits. Furthermore a satellite theory provides a method for geodetic study. Since an artificial satellite orbit is much affected by the external gravity field of the Earth, this field may be studied by observing changes in the satellite orbit.

Estimates of the shape of the Earth have been tremendously improved since the launching of satellites. Geodetic studies using satellite orbital data are superior to studies measuring the gravitational variations along the Earth's surface.

As stated in the second section of the previous chapter, the external potential of the Earth can be written as (Ref. 18):

$$U = \frac{\mu}{r}\left[1 + \sum_{n=1}^{\infty}\sum_{n=0}^{n}\left(\frac{R}{r}\right)^{n} P_n^m(\sin\beta)\left(C_{n,m}\cos m\lambda + S_{n,m}\sin m\lambda\right)\right] \tag{1.1}$$

and

$$C_{n,0} = -J_n = C_n, \quad S_{n,0} = 0$$

where P_n^m is the associated Legendre polynomial, λ the longitudinal angle, and $C_{n,m}$ and $S_{n,m}$ are the numerical coefficients. If the Earth is symmetrical about the polar axis, the above equation may be reduced to:

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n\left(\frac{R}{r}\right)^{n} P_n(\sin\beta)\right] . \tag{1.2}$$

The above form was used in this analysis. To determine the coefficients of spherical harmonics, or J_n, is equivalent to determining the shape of the Earth; since J_n's are proportional to the amount of departure from a shape in hydrostatic equilibrium.

It may be seen from the previous chapter in this study that the perturbations of the orbital elements will contain J_n's. The difference between adopted and actual J_n's yields the discrepancies in the observed data and theoretical calculations. From these discrepancies, a method for obtaining more accurate information about the actual J_n's may be established. There are many papers by various authors which estimate J_n values. An example of extensive study of J_n's is a paper by Lecar, Sorenson, and Eckels (Ref. 19). This study estimated the value of J_2 and J_4, without considering the effects of drag and Lunar and Solar attraction on the satellite. Later, O'Keefe, Eckels and Squares (Ref. 13) corrected the J_n values of Lecar et al., by including previously unaccounted for minor perturbations. The O'Keefe et al. study corrected not only the even order of J_n's proposed by Lecar et al., but also discovered and estimated the existence and values of the odd harmonics of the Earth. However, the present analysis has considered a factor not accounted for in the study by O'Keefe et al.; as a result of new considerations, a correction of the even order spherical harmonics has been carried out in this study.

O'Keefe et al. obtained the rate of movement of the lines of apsides and nodes due to J_n's alone. In a satellite tracking situation the observed data contains the effects of drag coupling, and Lunar and Solar attraction. Therefore one must decouple these effects from the actual observed data before such data can be used for estimating the Earth's spherical harmonics. O'Keefe et al. noticed that drag coupling contributes to the change of eccentricity and major axis length of the satellite orbit. Their paper also related the change in the length of the major axis to the change in the mean anomaly. The perturbation of the mean anomaly due to drag can be obtained from satellite tracking data. O'Keefe et al. estimated the drag coupling effect on the rate of rotations of the apsidal and nodal lines from observed data on the mean anomaly.

O'Keefe et al.'s approach is expressed in mathematical terms in the following presentation.

The secular part of the perturbation in the rate of change of the lines of apsides and nodes is:

$$\dot{\omega} = 3n \frac{J_2}{g_0} \frac{\mu^3}{C_0^4} \left(1 - \frac{5}{4} \sin^2 i\right) \tag{1.3}$$

$$\dot{\Omega} = -\frac{3}{2} n \frac{J_2}{g_0} \frac{\mu^3}{C_0^4} \cos i \tag{1.4}$$

The notation used above is slightly different from that used by O'Keefe et al. Where they used $A_{2.0} = \mu J_2 R^2$, we use J_2 explicitly. In the above equations, n = mean motion, which is defined as:

$$n = \frac{V_c}{a} = \mu \, a^{-\frac{3}{2}}, \tag{1.5}$$

where V_c is the equivalent circular velocity.

By making use of equation 1.5, $\dot{\omega}$ and $\dot{\Omega}$ can be expressed as:

$$\dot{\omega} = 3 \mu^{\frac{1}{2}} J_2 R^2 \frac{\left(1 - \frac{5}{4} \sin i\right)}{a^{\frac{7}{2}} (1 - \epsilon^2)^2} \tag{1.6}$$

$$\dot{\Omega} = \frac{3}{2}\mu^{\frac{1}{2}} J_2 R^2 \frac{\cos i}{a^{\frac{7}{2}}(1-\epsilon^2)^2} \tag{1.7}$$

The major secular drag effect on the satellite will decrease the length of major axis a, and the eccentricity ϵ of the orbit. Therefore the change of $\dot{\omega}$ and $\dot{\Omega}$, brought about by the above perturbations, will be:

$$\Delta\dot{\omega} = \frac{\partial\dot{\omega}}{\partial\epsilon}\Delta\epsilon + \frac{\partial\dot{\omega}}{\partial a}\Delta a \tag{1.8}$$

$$\Delta\dot{\Omega} = \frac{\partial\dot{\Omega}}{\partial\epsilon}\Delta\epsilon + \frac{\partial\dot{\Omega}}{\partial a}\Delta a \tag{1.9}$$

If it is assumed that the satellite's perigee radius will not be perturbed, $\Delta\epsilon$ can be written in terms of Δa, as follows:

$$\Delta\epsilon = \frac{1-\epsilon}{a}\Delta a \tag{1.10}$$

The integration of Δa may be related to the mean anomaly through the relation:

$$\int_{t_0}^{t} \Delta a\, dt \simeq -\frac{2}{3} a^{\frac{5}{2}} \Delta M , \tag{1.11}$$

By neglecting terms of higher order than ϵ^2, the expressions for $\Delta\omega$ and $\Delta\Omega$, from equations 1.6 and 1.7, may be written as:

$$\Delta\omega = \mu^{\frac{1}{2}} J_2 R^2 \frac{1}{p^2}\left(1 - \frac{5}{4}\sin^2 i\right)\left[\frac{7-\epsilon}{1+\epsilon}\right] J_2\, \Delta M \tag{1.12}$$

$$\Delta\Omega = -\frac{1}{2}\mu^{\frac{1}{2}} J_2 R^2 \frac{1}{p^2}\cos i \left[\frac{7-\epsilon}{1+\epsilon}\right] J_2\, \Delta M \tag{1.13}$$

where p = semilatus rectum = $a(1-\epsilon^2)$.

Equations 1.12 and 1.13 represent the perturbations of ω and

Ω, due to the coupling effect of drag with the oblateness of the Earth. Subtracting the values of $\Delta\omega$ and $\Delta\Omega$ from the observed data, O'Keefe et al. arrived at their so-called "corrected" values. In addition, they subtracted the effects due to the Moon and Sun, according to Kozai's calculations (Ref. 8).

In the present analysis, it was found that the line of apsides has a secular perturbation, due to the drag caused by atmospheric rotation. For the satellite Vanguard I case, which was used by O'Keefe et al. and various authors in their geodetic studies, the mentioned drag perturbation is very minute at very high altitudes. O'Keefe et al. neglected this minute drag perturbation in their study. As a result the error introduced is equivalent to almost one half of the solar effect, or one fifth of the lunar effect.

There exists a perturbation of the line of apsides caused by drag, therefore the correct expression for the rate of rotation of this line should be the sum of all of the perturbation effects. If the total rate of rotation of the apsidal line is $\dot{\omega}_{tot.}$, then:

$$\dot{\omega}_{tot.} = \dot{\omega}_J + \dot{\omega}_{Jd} + \dot{\omega}_d + \dot{\omega}_{dJ} + \dot{\omega}_{cb} , \qquad (1.14)$$

where:

$\dot{\omega}_J$ = rate of rotation of apsidal line due to the bulge of Earth

$\dot{\omega}_{Jd}$ = rate of rotation of apsidal line due to the effect of coupling of drag with $\dot{\omega}_J$

$\dot{\omega}_d$ = rate of rotation of apsidal line due to drag

$\dot{\omega}_{dJ}$ = rate of rotation of apsidal line due to the effect of coupling of bulge to $\dot{\omega}_d$

$\dot{\omega}_{cb}$ = rate of rotation of apsidal line due to the other celestial bodies

Usually $\dot{\omega}_{cb}$ considers only the Lunar and Solar effects, namely:

$$\dot{\omega}_{cb} = \dot{\omega}_{moon} + \dot{\omega}_{sun} \qquad (1.15)$$

O'Keefe et al. considered only a portion of equation 1.14, that is:

$$\dot{\omega}_{tot.} = \dot{\omega}_{J_2} + \dot{\omega}_{J_2 d} + \dot{\omega}_{cb} \qquad (1.16)$$

Therefore their correction was not complete.

In the Vanguard I case, it was found that ω_d and ω_{J_2} rotate in the same direction. Therefore $\dot{\omega}_d$ should be subtracted from the observed data. The $\dot{\omega}_{dJ}$ term, representing the drag caused by the non-spherical atmosphere, should be negligible, since $\dot{\omega}_d$ itself is very minute. Therefore it is reasonable to neglect the $\dot{\omega}_{dJ}$ term completely. The paper of O'Keefe et al. did not take into consideration effects of the order of $\dot{\omega}_{J_4 d}$ or higher. $\dot{\omega}_{J_4}$ will be on the order of a thousandth of $\dot{\omega}_{J_2}$. In the correction of the observed data, $\dot{\omega}_d$ and $\dot{\omega}_{J_4 d}$ do not cancel each other out, since the direction of these rates is similar. In this study, the effect due to drag is considered, thereby obtaining a closer approximation of the J_n values.

To compute $\dot{\omega}_d$ for the Vanguard I case, we refer to equation 9.18 of II. From this equation, the secular portion of the rotation of line of apsides may be stated as:

$$\omega_d = 8Ke^{-\zeta} \frac{C_0^5}{\mu^3} \pi^2 \omega_e \cos i_0 \left[(2-3\epsilon)I_0(\zeta) - 2\epsilon I_1(\zeta)\right] \qquad (1.17)$$

The numerical value of ω_d may be computed from section 11 of II as follows:

$$\omega_d = 0.586 \times 10^{-7} \text{ rad./cycle} . \qquad (1.18)$$

The period of the satellite in the Vanguard I case is:

$$T_{sat.} = 2\pi(\frac{a^3}{\mu})^{\frac{1}{2}} = 8.05 \times 10^3 \text{ sec,}$$

therefore, the rate of rotation of the line of apsides in this case is:

$$\dot{\omega}_d = 0.63 \times 10^{-6} \text{ rad./ day} . \qquad (1.19)$$

$\dot{\omega}_d$ is quite small indeed; in fact, it is in the order of one fifth of the Solar, and one tenth of the Lunar effects, respectively. However, in the paper by O'Keefe et al., there is a difference between the observed data and the calculated value of $\dot{\omega}$. The difference between the real observed data and the calculated value of $\dot{\omega}$ in O'Keefe et al. is accumulated. The error introduced by neglecting $\dot{\omega}_d$ is almost of the order of one half of the Solar and one fifth of the Lunar effects, respectively. This small error quite possibly could produce a slight modification in the numerical value of J_2, or a correction to the value representing the shape of the Earth.

Discrepancies between the observed data and the calculated values for $\dot{\omega}$ and $\dot{\Omega}$ may be adjusted through using the following error equations:

$$\frac{\partial\dot{\omega}}{\partial J_2}\,\Delta J_2 + \frac{\partial\dot{\omega}}{\partial J_4}\,\Delta J_4 = \Delta\dot{\omega}$$

$$\frac{\partial\dot{\Omega}}{\partial J_2}\,\Delta J_2 + \frac{\partial\dot{\Omega}}{\partial J_4}\,\Delta J_4 = \Delta\dot{\Omega} \qquad (1.20)$$

By referring to the paper of Lecar et al., the coefficients of the above equations can be calculated as follows:

$$\frac{\partial\dot{\omega}}{\partial J_2} = \frac{3\pi R^2}{p^2}\left(2 - \frac{5}{2}\sin^2 i\right)$$

$$\frac{\partial\dot{\omega}}{\partial J_4} = -\frac{30\pi R^4}{4p^2}\left[\left(1+\frac{3}{4}\epsilon^2\right)\left(1-5\sin^2 i+\frac{35}{8}\sin^4 i\right)+\cos^2 i\left(1+\frac{3}{2}\epsilon^2\right)\left(1-\frac{7}{4}\sin^2 i\right)\right]$$

$$\frac{\partial\dot{\Omega}}{\partial J_2} = -\frac{3\pi R^2}{p^2}\cos i$$

$$\frac{\partial\dot{\Omega}}{\partial J_4} = \frac{30\pi R^4}{4p^2}\cos i\left(1+\frac{3}{2}\epsilon^2\right)\left(1-\frac{7}{4}\sin^2 i\right) \qquad (1.21)$$

By using the above equations, the numerical values for the Vanguard I case can be computed as follows:

$$\frac{\partial\dot{\omega}}{\partial J_2} = 6.6105 \qquad \frac{\partial\dot{\omega}}{\partial J_4} = -1.3744$$

$$\frac{\partial\dot{\Omega}}{\partial J_4} = -4.5225 \qquad \frac{\partial\dot{\Omega}}{\partial J_4} = 3.0848 \qquad (1.22)$$

As a result, we may carry out the correction of the Vanguard I case data by using equations 1.20. The modifications of the data will be based on the numerical values which were computed by Lecar et al. O'Keefe et al., also used these numerical values, which are:

$$J_2 = 1082.1 \times 10^{-6}$$

$$J_4 = -2.359 \times 10^{-6} \qquad (1.23)$$

O'Keefe et al., using the above values, calculated the rate of rotation of the apsidal and the nodal lines of the Vanguard I case, as follows:

$$\dot{\omega}_{cal} = +.0767684 \frac{\text{rad.}}{\text{day}}$$

$$\dot{\Omega}_{cal} = -.0525610 \frac{\text{rad.}}{\text{day}} \quad . \qquad (1.24)$$

The actual observed values in the Vanguard I case are:

$$\dot{\omega}_{ob} = +.0767920 \frac{\text{rad.}}{\text{day}}$$

$$\dot{\Omega}_{cal} = -.0525622 \frac{\text{rad.}}{\text{day}} \quad . \qquad (1.25)$$

The Solar and Lunar effects in this case are:

	$\dot{\omega}$ (rad./day)	$\dot{\Omega}$ (rad./day)
Solar Portion	$+.31 \times 10^{-5}$	$-.23 \times 10^{-5}$
Lunar Portion	$+.68 \times 10^{-5}$	$-.49 \times 10^{-5}$.

(1.26)

Therefore, by referring to the value of $\dot{\omega}_d$ in equation 1.23, we obtain the discrepancies between the calculated and observed values, resulting from the bulge effects. The discrepancies are:

$$\Delta\dot{\omega} = .131 \times 10^{-4} \text{ rad./day} = .1222 \times 10^{-5} \text{ rad./cycle}$$

$$\Delta\dot{\Omega} = .6 \times 10^{-4} \text{ rad./day} = .5595 \times 10^{-6} \text{ rad./cycle} \quad . \qquad (1.27)$$

By substituting the above values of $\Delta\dot{\omega}$ and $\Delta\dot{\Omega}$ and the coefficients in equations 1.22 into equations 1.20, we may solve for ΔJ_2 and ΔJ_4, as follows:

$\Delta J_2 = .32 \times 10^{-6}$

$\Delta J_4 = .651 \times 10^{-6}$

The above values for ΔJ_2 and ΔJ_4 may be used in modifying the values of J_2 and J_4 in (1.23), as follows:

$J_{2_{corr.}} = 1082.42 \times 10^{-6}$

$J_{4_{corr.}} = -1.71 \times 10^{-6}$

A comparison table of values arrived at in three studies follows:

	Lecar et al.	O'Keefe et al.	Our values
J_2	1082.1×10^{-6}	1082.53×10^{-6}	1082.42×10^{-6}
J_4	-2.359×10^{-6}	-1.7×10^{-6}	-1.71×10^{-6}
f	$\frac{1}{298.32}$	$\frac{1}{298.24}$	$\frac{1}{298.27}$

f is the flattening of the Earth defined by

$$f = \frac{a-b}{b}$$

a = semi-major axis, b = semi-minor axis.

It is interesting to observe from the following table that men have been working to improve their knowledge of this planet Earth for the past hundred years.

FLATTENING OF THE EARTH*

Name of Authors	Year	1/f
Everest	1830	300.80
Bessel	1841	299.15
Clarke	1866	294.98
Clarke	1880	293.47
International Adopted	before 1957	297.00
O'Keefe	1958	298.38
Jacchia	1958	298.28
Lecar, Sorenson, Eckels	1959	298.32
O'Keefe, Eckels, Squares	1959	298.24
King-Hele	1960	298.24
Kozai	1960	298.30
Kaula	1961	298.24
de Vaucouleurs	1961	298.20

* Reproduced from References 15 and 20.

2. Coupling Effect in the Forcing Function

In satellite motion, orbital elements are constantly changing their values. Strictly speaking, all of these changes should be considered in our calculations. Changes in orbital elements appear on both sides of the equations of motion; yet changes in the forcing function more significantly affect our solutions.

The values of orbital elements are affected by both the oblateness of the Earth and drag effects. As a result, we expect to arrive at terms such as $O(J_2^2)$, $O(J_2K)$ and $O(K^2)$. The term K, drag parameter, has been defined as: $K = (SC_D\rho_s)/2m$. The above expressions are the self and cross coupling terms. In order of magnitude, these terms may be significant, when compared to higher order harmonics like J_3, J_4, J_5, and so on. Many authors consider the higher order harmonics in their calculations, without investigating the possible coupling terms as illustrated above.

From prior analysis, we know that the forcing functions which contain J_2 and K, in their $\underline{i}_r$, $\underline{i}_\theta$ and $\underline{i}_\psi$ direction components, are as follows:

$$F_r = -\frac{3}{2} J_2 g_0 \left(\frac{R}{r}\right)^4 \left\{1-3\left[\sin(\theta+\theta_a)\sin i \cos\psi + \cos i \sin\psi\right]^2\right\}$$

$$-K\sigma(r)V_R\left[\dot{r} - r\omega_e \cos(\theta+\theta_a)\sin^2 i \sin\psi\right]$$

$$F_\theta = -\frac{3}{2} J_2 g_0 \left(\frac{R}{r}\right)^4 \left[\sin^2 i \sin 2(\theta+\theta_a)\cos\psi + \sin 2i \cos(\theta+\theta_a)\sin\psi\right]$$

$$-K\sigma(r)V_R r\left[\cos\psi\dot{\theta} - \omega_e \cos i \cos\psi + \omega_e \sin i \sin(\theta+\theta_a)\sin\psi\right]$$

$$F_\psi = -\frac{3}{2} J_2 g_0 \left(\frac{R}{r}\right)^4 \left\{\sin(\theta+\theta_a)\sin 2i \cos 2\psi\right.$$

$$\left. + \sin 2\psi\left[\cos^2 i - \sin^2 i \sin(\theta+\theta_a)\right]\right\}$$

$$- K\sigma(r)V_R r\left[\dot{\psi} + \omega_e \cos(\theta+\theta_a)\sin i \cos\psi\right] . \tag{2.1}$$

Next we shall calculate the perturbations of the above components.

Taking F_r as an example, we know that after perturbing it will become:

$$F_r = F_{r_0} + \Delta F_r\,(J_2, K)$$

$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{(r_0+\Delta r)^4}\Big\{1-3[\sin^2(\theta_0+\theta_a+\Delta\theta)\sin^2(i_0+\Delta i)\cos^2\Delta\psi$$

$$+\frac{1}{2}\sin(\theta_0+\theta_a+\Delta\theta)\sin 2(i_0+\Delta i)\sin 2\Delta\psi + \cos^2(i_0+\Delta i)\sin^2\Delta\psi]\Big\}$$

$$- K\sigma(r)(V_0+\Delta V)[(\dot r_0+\Delta\dot r)-(r_0+\Delta r)\omega_e\cos(\theta_0+\theta_a+\Delta\theta)$$

$$\times \sin^2(i_0+\Delta i)\sin^2\Delta\psi] \quad .$$

Expansion and simplification of the above results in the following two parts. The main part is:

$$F_{r_0} = -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}(1-3\sin^2\tilde\theta_0\sin^2 i_0)-K\sigma(r_0)\dot r_0 V_0(1-\frac{C_0}{V_0^2}\omega_e\cos i_0) \; ,$$

and the perturbing part is:

$$\Delta F_r(J_2, K) = -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}[4\frac{\Delta r}{r_0}(3\sin^2\tilde\theta_0\sin^2 i_0-1)-3\Delta\theta\sin 2\tilde\theta_0\sin^2 i_0$$

$$- 3\Delta i\sin 2i_0\sin^2\tilde\theta_0 - 3\Delta\psi\sin\tilde\theta_0\sin 2i_0]$$

$$+K\sigma(r_0)[\Delta\psi r_0 V_0(1-\frac{C_0}{V_0^2}\omega_e\cos i_0)\omega_e\cos\tilde\theta_0\sin^2 i_0-\Delta V\dot r_0] \; . \qquad (2.2)$$

A similar treatment of F_θ and F_ψ results in:

$$F_\theta = F_{\theta_0} + \Delta F_\theta (J_2, K)$$

$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\sin^2 i_0 \sin 2\tilde{\theta}_0 - K\sigma(r_0)V_0 r_0 \dot{\theta}_0 [1-(\frac{C_0}{V_0^2}+\frac{1}{\dot{\theta}_0})\omega_e \cos i_0]$$

$$-\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}[2\Delta\theta \sin^2 i_0 \cos 2\tilde{\theta}_0 - 4\frac{\Delta r}{r_0}\sin^2 i_0 \sin 2\tilde{\theta}_0$$

$$+ \Delta i \sin 2i_0 \sin 2\tilde{\theta}_0 + \Delta\psi \sin 2i_0 \cos \tilde{\theta}_0]$$

$$-K\sigma(r_0)r_0 V_0(1-\frac{C_0}{V_0^2}\omega_e \cos i_0)\left\{\dot{\theta}_0 \frac{d\Delta\theta}{d\theta_0} + (\dot{\theta}_0 - \omega_e \cos i_0)\left[\frac{\Delta r}{r_0}\right.\right.$$

$$\left.\left. + \frac{\Delta V}{V_0(1-\frac{C_0}{V_0^2}\omega_e \cos i_0)}\right] + (\Delta i + \Delta\psi \sin\tilde{\theta}_0)\omega_e \sin i_0\right\} \quad (2.3)$$

$$F_\psi = F_{\psi_0} + \Delta F_\psi (J_2, K)$$

$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\sin\tilde{\theta}_0 \sin 2i_0 - K\sigma(r_0)V_0 r_0(1-\frac{C_0}{V_0^2}\omega_e \cos i_0)\omega_e \cos\tilde{\theta}_0 \sin i_0$$

$$-\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}[\Delta\theta \cos\tilde{\theta}_0 \sin 2i_0 - 4\frac{\Delta r}{r_0}\sin\tilde{\theta}_0 \sin 2i_0 + 2\Delta i \cos 2i_0 \sin\tilde{\theta}_0$$

$$+ \Delta\psi(\cos^2 i_0 - \sin^2\tilde{\theta}_0 \sin^2 i_0)]$$

$$-K\sigma(r_0)V_0 r_0(1-\frac{C_0}{V_0^2}\omega_e \cos i_0)[\dot{\theta}_0 \frac{d\Delta\psi}{d\theta_0} + \frac{\Delta r}{r_0}\omega_e \cos\tilde{\theta}_0 \sin i_0$$

$$-\Delta\theta\omega_e \sin\tilde{\theta}_0 \sin i_0 + \Delta i \omega_e \cos\tilde{\theta}_0 \cos i_0]. \quad (2.4)$$

Since $\Delta\theta$ is proportional to J_2 and K, Δr is proportional to K and so on, we see that the coupling terms $O(J_2^2$, JK and $K^2)$ exist. In order to compare the order of magnitude of the above coupling terms with the higher order harmonics* J_3, J_4, ..., we must examine the perturbing gravitational potential, ϕ, of the Earth, since:

$$U = \frac{\mu}{r}\left[1 - \sum_{n=1}^{\infty} J_n \left(\frac{R}{r}\right)^n P_n(\sin\beta)\right],$$

where

$$\sin\beta = \sin\tilde{\theta}\ \sin i\ \cos\psi + \cos i\ \sin\psi\ .$$

Therefore:

$$\phi = \frac{1}{2}\frac{\mu}{r}\left(\frac{R}{r}\right)^2 J_2(1-3\sin^2\beta) - \frac{1}{2}\frac{\mu}{r}\left(\frac{R}{r}\right)^3 J_3\,(5\sin^3\beta - 3\sin\beta)$$
$$- \frac{1}{8}\ \frac{\mu}{r}\left(\frac{R}{r}\right)^4 J_4\,(3 - 30\sin^2\beta + 35\sin^4\beta) + \dots \qquad (2.5)$$

As a result of the above, the force components due to gravitational potential are:

$$F_{pr} = \frac{\partial\phi}{\partial r}$$
$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r^4}(1-3\sin^2\beta) + 2\frac{\mu^2 R}{g_0}\frac{J_3}{r^5}(5\sin^3\beta - 3\sin\beta)$$
$$+ \frac{5}{8}\frac{J_4}{g_0}\frac{\mu^3}{r^6}(3 - 30\sin^2\beta + 35\sin^4\beta) \qquad (2.6)$$

$$F_{p\theta} = \frac{1}{r\cos\psi}\ \frac{\partial\phi}{\partial\theta}$$
$$= -\frac{1}{2}\frac{\mu^2}{r^4}\cos\beta\ \cos\tilde{\theta}\ \sin i\left[6\frac{J_2}{g_0}\sin\beta + 3\frac{R}{r}\frac{J_3}{g_0}(5\sin^2\beta - 1)\right.$$
$$\left. + 5\left(\frac{R}{r}\right)^2\frac{J_4}{g_0}\sin\beta\ (7\sin^2\beta - 3)\right] \qquad (2.7)$$

---

*Others usually have considered the even harmonics in their calculations since the odd harmonics give the long-periodic type change only. Nevertheless, J_3 has been included here for generality.

$$F_{p\psi} = \frac{1}{r}\frac{\partial\phi}{\partial\psi}$$

$$= \frac{1}{2}\frac{\mu^2}{r^4}(\sin\tilde{\theta}\ \sin i\ \sin\psi - \cos i\ \cos\psi)\cos\beta\ [6\frac{J_2}{g_0}\sin\beta$$

$$+ 3\frac{R}{r}\frac{J_3}{g_0}(5\ \sin^2\beta - 1) + 5(\frac{R}{r})^2\frac{J_4}{g_0}\sin\beta\ (7\sin^2\beta - 3)]\ . \qquad (2.8)$$

In this study the order of magnitude of various terms will be compared. For easier calculation, several assumptions are going to be made, in order to simplify our problem. In a more accurate calculation, the following simplifications need not be made.

Assumptions:

1. The satellite trajectory is so close to a circular orbit that terms for eccentricity higher than the ϵ order may be neglected.
2. The modified Bessel functions $I_n(\zeta)$ for n = 0, 1 and the large argument ζ cases can be approximated by retaining only the first order term in the following series:

   $$I_n(\zeta) = \frac{e^\zeta}{\sqrt{2\pi\zeta}}\left\{1 - \frac{4n^2 - 1^2}{1!\ 8\zeta!} + ---\right\}$$

   Hence:

   $$I_0(\zeta) \doteq I_1(\zeta) \sim \frac{e^\zeta}{\sqrt{2\pi\zeta}}$$

3. The effect due to the rotation of atmosphere may be neglected in this particular calculation.
4. If any term in the forcing function oscillates, we consider the oscillating term's amplitude only.

From these simplifications and prior calculations, we obtain

$$\Delta r \sim -8K\pi \frac{C_0^4}{\mu^2\sqrt{2\pi\zeta}}$$

$$\Delta\theta \sim 6\pi \frac{J_2}{g_0} \frac{\mu^3}{C_0^4} + 6K\pi^2 \frac{C_0^2}{\mu\sqrt{2\pi\zeta}}$$

$$\Delta\psi \sim \frac{3}{2}\pi \frac{J_2}{g_0} \frac{\mu^3}{C_0^4}$$

$$\Delta V \sim 3KC_0\sqrt{\frac{2\pi}{\zeta}}$$

$$\frac{d(\Delta\theta)}{d\theta_0} \sim 14K\pi \frac{C_0^2}{\mu\sqrt{2\pi\zeta}}$$

$$\frac{d(\Delta\psi)}{d\theta_0} \sim -\frac{3}{2}\pi \frac{J_2}{g_0} \frac{\mu^3}{C_0^4}$$

$$\Delta i \sim 0, \quad \frac{d(\Delta r)}{d\theta_0} \sim 0$$

Also,

$$r_0 \sim \frac{C_0^2}{\mu}, \quad V_0 \sim \frac{\mu}{C_0} \quad . \tag{2.9}$$

The forcing function components, including the coupling terms and the higher order harmonics up to J_4, in $\underline{i}_r$, $\underline{i}_\theta$ and $\underline{i}_\psi$ directions, may be expressed in terms of the above equations, as follows:

$$F_r = F_{r_0}(J_2, K) + \Delta F_r (J_2, K) + \Delta F_{pr}(J_3, J_4)$$

$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}(1-3\sin^2\beta) - K\sigma(r_0)\dot{r}_0 V_0 (1 - \frac{C_0}{V_0^2}\omega_e \cos i_0)$$

$$\boxed{\frac{3}{2}\frac{\mu^6}{g_0 C_0^8} J_2} \qquad\qquad \boxed{0}$$

$$-\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}[4\frac{\Delta r}{r_0}(3\sin^2\beta - 1) - 3\Delta\theta \sin 2\tilde{\theta}_0 \sin^2 i_0$$

$$\boxed{\frac{48\pi}{\sqrt{2\pi\zeta}}\frac{\mu^5}{g_0 C_0^6} J_2 K} \qquad \boxed{27\pi[\frac{\mu^9}{g_0^2 C_0^{12}} J_2^2 + \frac{\pi}{\sqrt{2\pi\zeta}}\frac{\mu^5}{g_0 C_0^6} J_2 K]}$$

$$-3\Delta i \sin 2i_0 \sin^2\tilde{\theta}_0 - 3\Delta\psi \sin\tilde{\theta}_0 \sin 2i_0]$$

$$\boxed{0} \qquad\qquad \boxed{\frac{9}{2}\frac{\pi\mu^9}{g_0^2 C_0^{12}} J_2^2}$$

$$+ K\sigma(r_0)[\Delta\psi\, r_0 V_0 (1 - \frac{C_0}{V_0^2}\omega_e \cos i_0)\omega_e \cos\tilde{\theta}_0 \sin^2 i_0 - \Delta V \dot{r}_0]$$

$$\boxed{0} \qquad\qquad \boxed{0}$$

$$+ 2\frac{\mu^2 R}{g_0}\frac{J_3}{r^5}(5\sin^3\beta_0 - 3\sin\beta_0) + \frac{5}{8}\frac{J_4}{g_0^2}\frac{\mu^3}{r_0^6}(3 - 30\sin^2\beta + 35\sin^4\beta)$$

$$\boxed{6(\frac{R}{r_0})\frac{\mu^6}{g_0 C_0^8} J_3} \qquad\qquad \boxed{\frac{15}{8}\frac{\mu^9}{g_0^2 C_0^{12}} J_4}$$

(2.10)

$$F_\theta = F_{\theta_0}(J_2, K) + \Delta F_\theta(J_2, K) + \Delta F_{p\theta}(J_3, J_4)$$

$$= -\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\sin^2 i_0 \sin 2\tilde{\theta}_0 - K\sigma(r_0)V_0\frac{C_0}{r_0}\left[1-\left(\frac{C_0}{V_0^2}+\frac{r_0^2}{C_0}\right)\omega_e \cos i_0\right]$$

$$\boxed{3\frac{\mu^6}{g_0 C_0^8} J_2} \qquad \boxed{\frac{\mu^2}{C_0^2} K}$$

$$-\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\left[2\Delta\theta \sin^2 i_0 \cos 2\tilde{\theta}_0 - 4\frac{\Delta r}{r_0}\sin^2 i_0 \sin 2\tilde{\theta}_0\right.$$

$$\boxed{18\pi\frac{\mu^2}{g_0}\left[\frac{\mu^7}{g_0 C_0^{12}} J_2^2 + \frac{\pi}{\sqrt{2\pi\zeta}} JK\right]} \qquad \boxed{\frac{48\pi}{\sqrt{2\pi\zeta}}\frac{\mu^5}{g_0 C_0^6} J_2 K}$$

$$\left. + \Delta i \sin 2i_0 \sin 2\tilde{\theta}_0 + \Delta\psi \sin 2i_0 \cos\tilde{\theta}_0\right]$$

$$\boxed{0} \qquad \boxed{\frac{9}{4}\pi\frac{\mu^9}{g_0^2 C_0^{12}} J_2^2}$$

$$-K\sigma(r_0)r_0V_0\left(1-\frac{C_0}{V_0^2}\omega_e \cos i_0\right)\left\{\frac{C_0}{r_0^2}\frac{d\Delta\theta}{d\theta_0} + (\dot{\theta}_0 - \omega_e \cos i_0)\left[\frac{\Delta r}{r_0}\right.\right.$$

$$\boxed{\frac{14\pi\mu}{\sqrt{2\pi\zeta}} K^2} \qquad \boxed{\frac{8\pi\mu}{\sqrt{2\pi\zeta}} K^2}$$

$$\left.\left. + \frac{\Delta V}{V_0\left(1-\frac{C_0}{V_0^2}\omega_e \cos i_0\right)}\right] + (\Delta i + \Delta\psi \sin\tilde{\theta}_0)\,\omega_e \sin i_0\right\}$$

$$\boxed{\frac{6\pi\mu}{\sqrt{2\pi\zeta}} K^2} \qquad \boxed{0} \qquad \boxed{0}$$

$$-\frac{1}{2}\frac{\mu^2}{r_0^4}\cos\beta_0 \cos\tilde{\theta}_0 \sin i_0 \left[3\frac{R}{r}\frac{J_3}{g_0}(5\sin^2\beta_0 - 1) + 5\left(\frac{R}{r_0}\right)^2 \frac{J_4}{g_0}\sin\beta_0(7\sin^2\beta_0 - 3)\right]$$

$$\boxed{\frac{3}{2}\frac{R}{r_0}\frac{\mu^6}{g_0 C_0^8} J_3} \qquad \boxed{\frac{15}{2}\left(\frac{R}{r_0}\right)^2 \frac{\mu^6}{g_0 C_0^8} J_4} \qquad (2.11)$$

$$F_{\psi} = F_{\psi_0}(J_2, K) + \Delta F_{\psi}(J_2, K) + \Delta F_{p\psi}(J_3, J_4)$$

$$= -\underbrace{\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\sin\tilde{\theta}_0 \sin 2i_0}_{3\frac{\mu^6}{g_0 C_0^8}J_2} - \underbrace{K\sigma(r_0)V_0 r_0\left(1-\frac{C_0}{V_0^2}\omega_c \cos i_0\right)\omega_e \cos\tilde{\theta}_0 \sin i_0}_{0}$$

$$-\frac{3}{2}\frac{J_2}{g_0}\frac{\mu^2}{r_0^4}\Big[\underbrace{\Delta\theta\cos\tilde{\theta}_0 \sin 2i_0}_{9\pi\left[\frac{\mu^9}{g_0^2 C_0^{12}}J_2^2 + \frac{\mu^5\pi}{g_0 C_0^6\sqrt{2\pi\zeta}}J_2 K\right]} - \underbrace{4\frac{\Delta r}{r_0}\sin\tilde{\theta}_0 \sin 2i_0}_{\frac{48\pi}{\sqrt{2\pi\zeta}}\frac{\mu^5}{g_0 C_0^6}J_2 K}$$

$$+\underbrace{2\Delta i \cos 2i_0 \sin\tilde{\theta}_0}_{0} + \underbrace{\Delta\psi\left(\cos^2 i_0 - \sin^2\tilde{\theta}_0 \sin^2 i_0\right)}_{\frac{9}{4}\pi\frac{\mu^9}{g_0^2 C_0^{12}}J_2^2}\Big]$$

$$-K\sigma(r_0)V_0 r_0\left(1-\frac{C_0}{V_0^2}\omega_e \cos i_0\right)\Big[\underbrace{\dot{\theta}_0\frac{d\Delta\psi}{d\theta_0}}_{\frac{3}{2}\pi\frac{\mu^5}{g_0 C_0^6}J_2 K} + \underbrace{\frac{\Delta r}{r_0}\omega_e \cos\tilde{\theta}_0 \sin i_0}_{0}$$

$$-\underbrace{\Delta\theta\omega_e \sin\tilde{\theta}_0 \sin i_0}_{0} + \underbrace{\Delta i\,\omega_e \cos\tilde{\theta}_0 \cos i_0}_{0}\Big]$$

$$-\frac{1}{2}\frac{\mu^2}{r_0^4}\cos\beta_0 \cos i_0\Big[\underbrace{3\frac{R}{r}\frac{J_3}{g_0}\left(5\sin^2\beta_0 - 1\right)}_{\frac{3}{2}\frac{R}{r_0}\frac{\mu^6}{g_0 C_0^8}J_3} + \underbrace{5\left(\frac{R}{r}\right)^2\frac{J_4}{g_0}\sin\beta_0\left(7\sin^2\beta - 3\right)}_{\frac{15}{2}\left(\frac{R}{r}\right)^2\frac{\mu^6}{g_0 C_0^8}J_4}\Big]. \tag{2.12}$$

In order to calculate the numerical values of each term appearing in equations 2.10, 2.11, and 2.12, we must calculate the drag parameter K and the non-dimensional parameter ζ. The drag parameter K is proportional to the parameter $(SC_D)/m$.

$$\frac{SC_D}{m} \sim \frac{1}{\rho_{sat.} d} \; , \quad [\frac{ft^2}{slug}]$$

where $\rho_{sat.}$ is the average density of the satellite, and d is the characteristic length of the satellite. For ordinary satellites the value for the above is slightly greater than 1. For the balloon type satellite, such as Echo I, the value for the above is nearly 3000. For extremely small particles like meteorites, this value is very large and may be somewhere between 200 - 1000. The drag parameter K vs. the altitude of conventional satellites has been plotted in Figure 7. The non-dimensional parameter ζ, vs. the satellite's altitude, for small eccentricities of the orbit, is shown in Figure 8.

From Figures 7 and 8, the numerical values of the order of magnitude of various terms in equations 2.10, 2.11 and 2.12, for conventional satellites, have been calculated. The results of these calculations are shown in Figure 9. It is interesting to see in Figure 9 that terms such as $[J_2^2]$ are of the same order as $[J_3]$ and $[J_4]$, whose values are almost independent of the satellite's altitude. The cross coupling terms in Figure 9, such as $[J_2K]$ are strong functions of the satellite's altitude. Around 100 miles above sea level, the indicated cross coupling terms

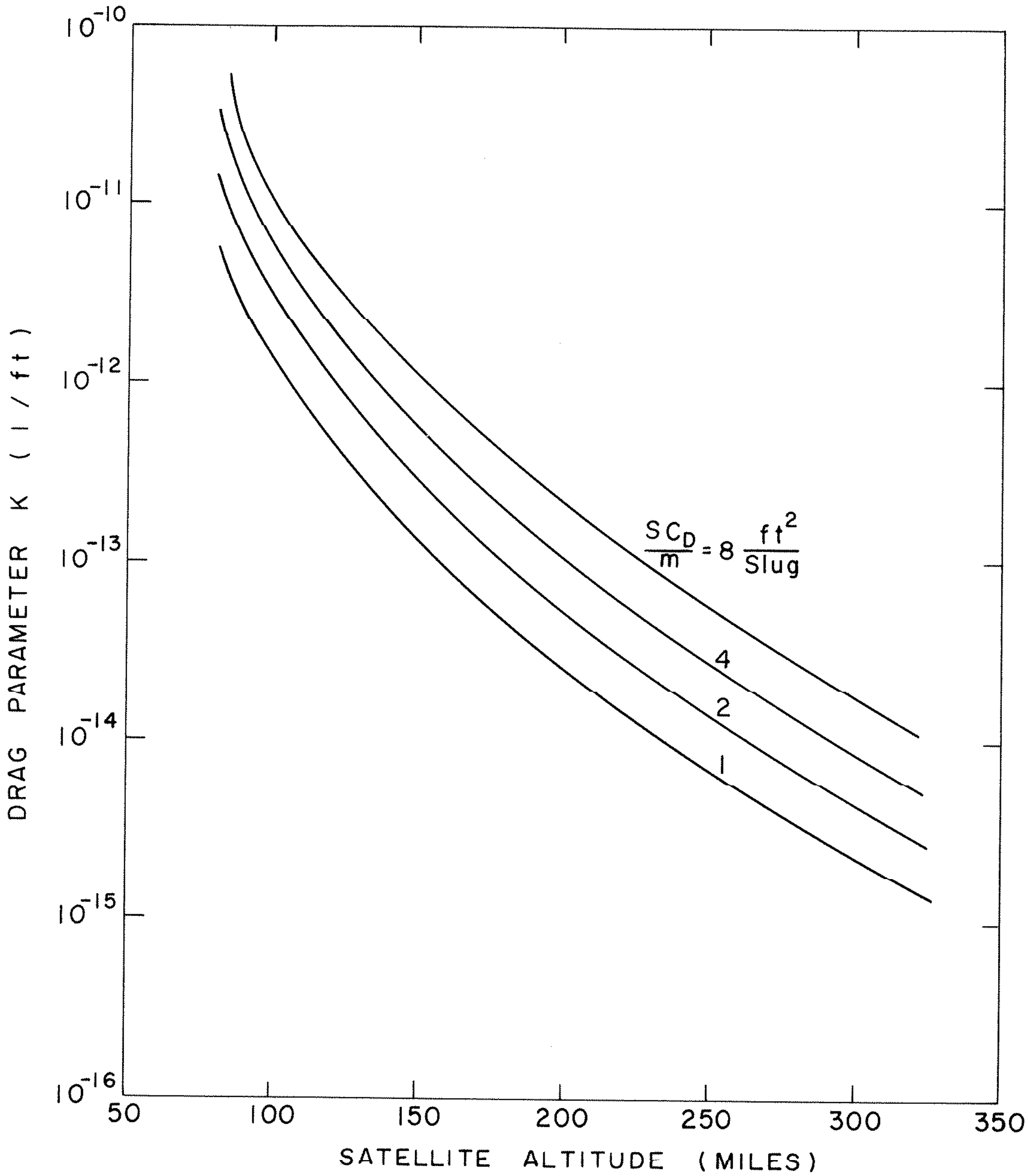

Figure 7. Drag Parameter K vs. Satellite Altitude

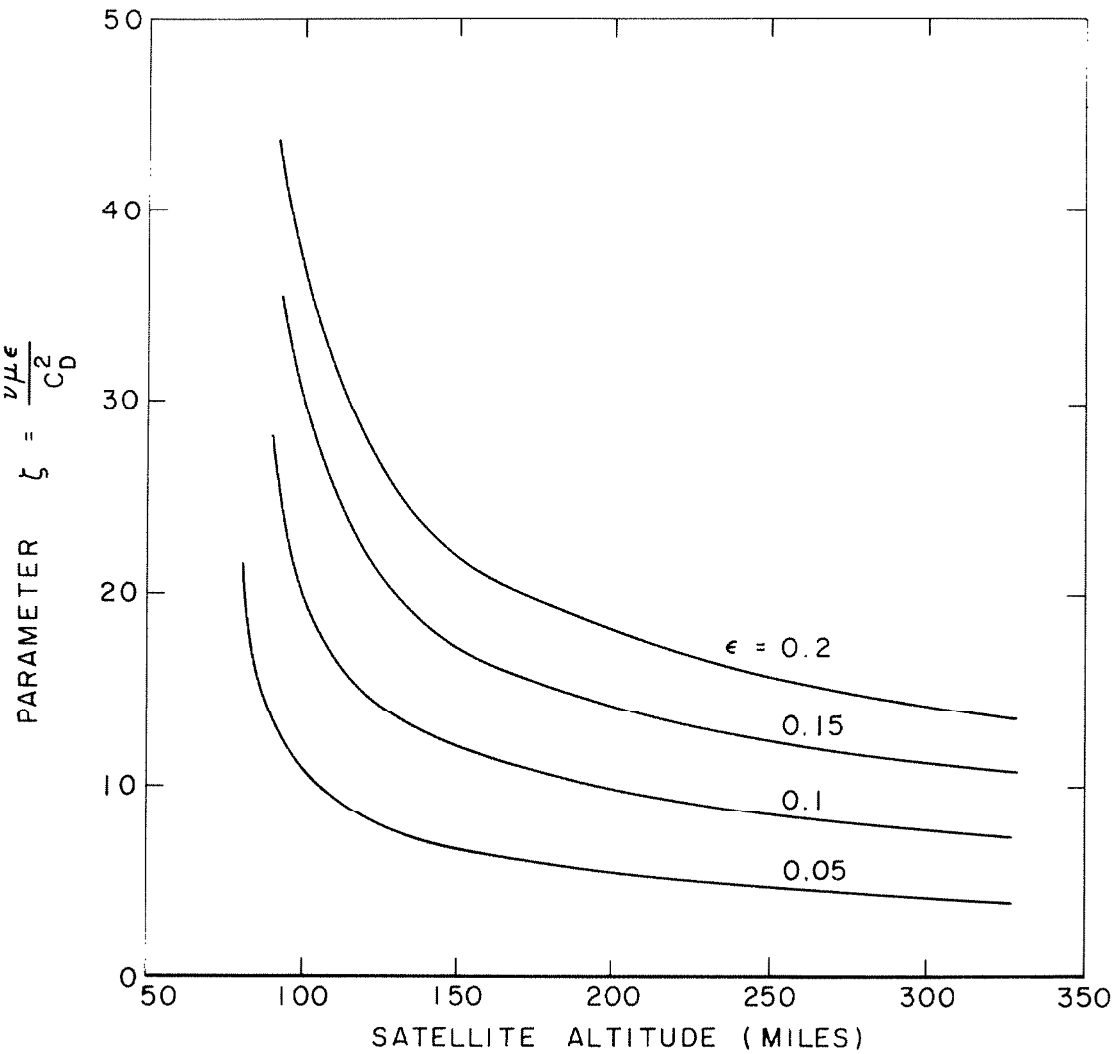

Figure 8. Non-dimensional Parameter ζ vs. Satellite Altitude

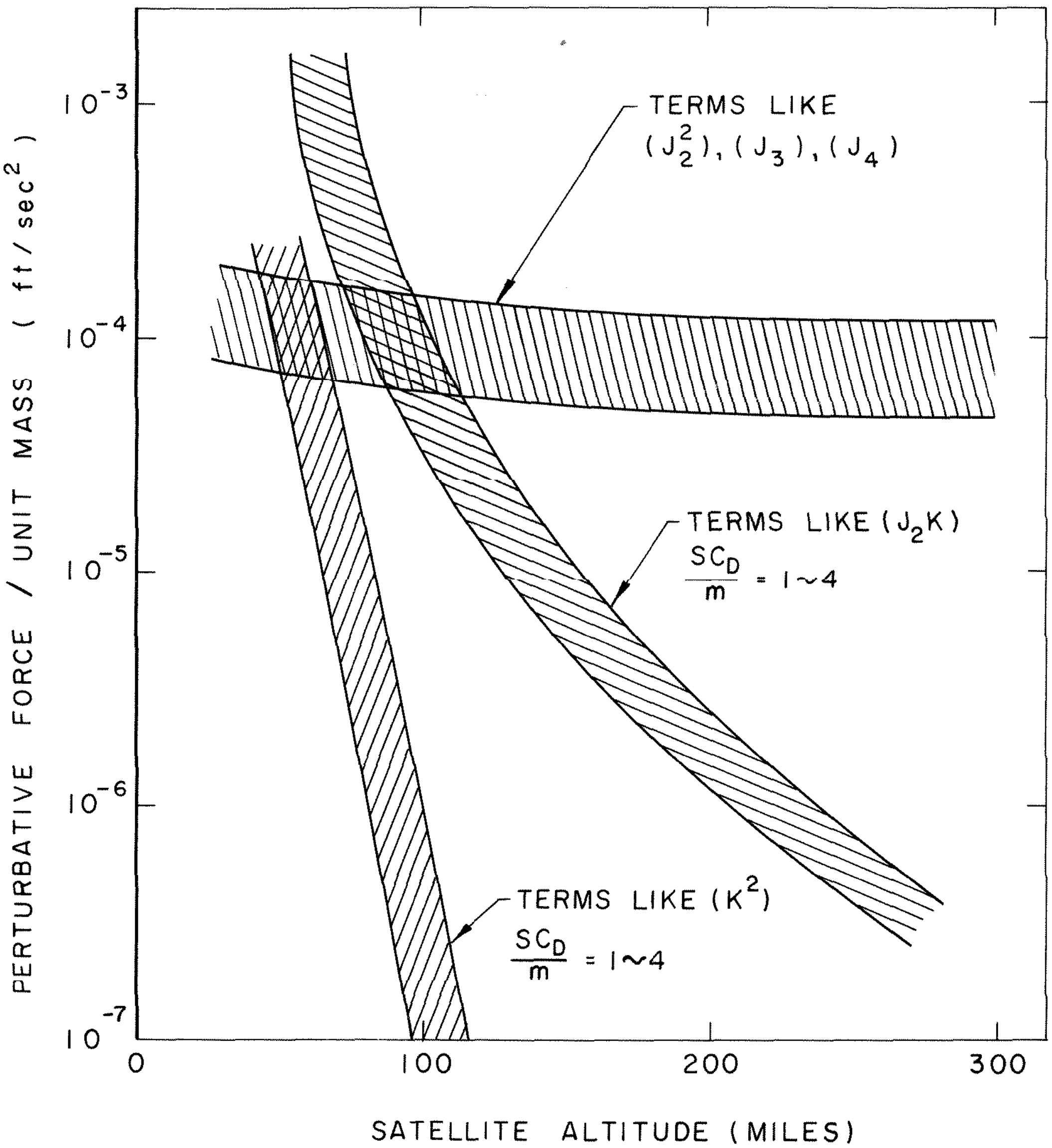

Figure 9. Comparison of the Order of Magnitude of Coupling Terms to the Higher Order Harmonics Terms

are as significant as the $[J_3]$ or $[J_4]$ terms. At an altitude of 200 miles, the values of the cross coupling terms quickly drops to only a few percent of the $[J_3]$ or $[J_4]$ terms. Terms such as $[K^2]$ are very much smaller than $[J_3]$ and $[J_4]$ at an altitude of 100 miles or more, therefore in practice they may be neglected.

IV. CONCLUSION

In this study, in order to include the bulge and the drag perturbative forces in a single fixed reference frame, the conventional spherical polar coordinates have been modified. This modification results in a geocentric coordinate system, with a tilted equatorial plane.

Equations of motion based on the above-mentioned system were found to decouple properly after linearization. The third equation of motion is completely decoupled from the other equations. The first and the second equations may be combined to produce a forced pendulum type equation, for the variable Δr. The above is a very desirable form to work with, since the solution of the equation may always be written as an integral, in terms of the forcing function. The decoupled third equation, like that for Δr, is also a forced pendulum type equation for the variable $\Delta\psi$. All of the other solutions arrived at in this study may be derived from these two solutions.

Since the coordinate system being used is based on the Keplerian trajectory, the solutions indicate the extent of departure from the perturbation free orbit. The equations of motion have been decoupled in such a way that their solutions clearly indicate in-plane and out-plane motions. Perturbations in Δr and $\Delta\theta$ determine the new position in the orbital plane. Perturbations in $\Delta\psi$ and $\frac{d(\Delta\psi)}{d\theta_0}$ determine the extent of shifting and twisting of the orbital plane. The above perturbations are not easily seen in other studies.

By considering the geometrical situations, the solutions for all perturbations derived have been expressed in the language of conventional orbital elements. Geometrical considerations lead to the idea of the so-called second Keplerian ellipse, based on the values at point $\theta_0 = 2\pi$. The differences between the second and initial ellipse determines the amount of perturbation, since these differences are caused by the perturbative forces.

The amount of correction arrived at by considering the effects of atmospheric rotation is extensive and significant. Effects of atmospheric rotation appear in both the secular and long-periodic terms. The perturbation of the inclination angle is caused solely by the above effect.

The perturbation of true anomaly is represented by $\theta_a - \theta_a$ in this analysis. It is interesting to observe that the bulge effect in the secular part of the above perturbation is caused by the forcing function of the equation in the r-direction. The other forcing function components have no effect on the above perturbation, when just the first order of accuracy is considered.

It is found that the line of apsides rotates, due to the effect of the Earth's atmospheric rotation. The magnitude of apsidal rotation is minute; nevertheless we obtain a basis for reexamining the existing harmonic values of J_2 and J_4. In this study, a completely accurate estimate of the above value is not claimed. In precisely calculating the above harmonics, one must consider drag effect on the apsidal line, a factor neglected by most authors. The extent of correction to the zonal harmonics has been shown.

We have shown the order of magnitude of the coupling terms in the forcing functions of the equations of motion. These coupling terms for the bulge and the drag effects are significant in calculations of higher order accuracy.

Treatment in this analysis has been quite general. This study may be extended to include terms of higher order of eccentricity ϵ. Principally, all of the other minor perturbative forces, such as non-spherical atmosphere, Solar, Lunar or the electromagnetic force effects can be included in the analysis if desired.

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APPENDIX

Expansion of the Density Function σ in a Series of Bessel Functions

The density ratio function, $\sigma(r)$, derived as an exponential form in equation 6.11 in II, is:

$$\sigma[r(\theta)] = e^{-\nu\frac{\mu\epsilon}{C_0^2}(1+\cos\theta)} = e^{-\zeta(1+\cos\theta)} . \qquad (1)$$

The exponential type function of $e^{\frac{x}{2}(z-\frac{1}{z})}$ can be expanded into a series of Bessel functions (see, for example, p. 101 and p. 357 of Reference 21), as follows:

$$e^{\frac{x}{2}(z-\frac{1}{z})} = J_0(x) + zJ_1(x) + z^2J_2(x) + --- + z^nJ_n(x) + ---$$

$$- \frac{1}{z}J_1(x) + \frac{1}{z^2}J_2(x) + --- + \frac{(-1)^n}{z^n}J_n(x) + --- . \qquad (2)$$

Now let $z = ie^{i\phi}$, $x = iy$, and by the definition of modified Bessel function $I_n(y) = i^{-n}J_n(iy)$, the above series becomes:

$$e^{\frac{iy}{2}(ie^{i\phi}+ie^{-i\phi})} = e^{-y\cos\phi}$$

$$= J_0(x)+zJ_1(x)+ --- \frac{1}{z}J_1(x) + \frac{1}{z^2}J_2(x) + ---$$

$$= \sum_0^\infty z^nJ_n(x) + \sum_1^\infty \frac{(-1)^n}{z^n}J_n(x)$$

$$= \sum_0^\infty (ie^{i\phi})^n J_n(iy) + \sum_1^\infty (-1)^n(ie^{i\phi})^{-n} J_n(iy)$$

$$= \sum_0^\infty i^{2n} e^{in\phi} i^{-n}J_n(iy) + \sum_1^\infty (-1)^n e^{-in\phi} i^{-n}J_n(iy)$$

$$= \sum_0^\infty (-1)^n e^{in\phi} I_n(y) + \sum_1^\infty (-1)^n e^{-in\phi} I_n(y)$$

$$= I_0(y) + \sum_1^\infty (-1)^n 2\cos n\phi \, I_n(y) . \qquad (3)$$

Therefore, equation 1 can be expanded into a series. From equation 3 wc obtain:

$$\sigma[r(\theta)] = e^{-\zeta} [I_0(\zeta) - 2\cos\theta \, I_1(\zeta) + 2\cos 2\theta \, I_2(\zeta) + --- + 2\,(-1)^n \cos n\theta \, I_n(\zeta) + ---] \quad . \tag{4}$$

The above is equation 6.12 in II.