Amenability, countable equivalence relations, and their full groups

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Abstract

This thesis consists of an introduction and four independent chapters.

In Chapter 2, we study homeomorphism groups of metrizable compactifications of the natural numbers. Those groups can be represented as almost zero-dimensional Polishable subgroups of the group S_∞. We show that all Polish groups are continuous homomorphic images of almost zero-dimensional Polishable subgroups of S_∞. We also find a sufficient condition for these groups to be one dimensional.

In Chapter 3, we study the connections between properties of the action of a countable group Γ on a countable set X and the ergodic theoretic properties of the corresponding shift action of $\Gamma \curvearrowright M^X$, where M is a measure space. In particular, we show that the action $\Gamma \curvearrowright X$ is amenable iff the shift $\Gamma \curvearrowright M^X$ has almost invariant sets. This is joint work with Alexander Kechris.

In Chapter 4, we prove that if the Koopman representation associated to a measure-preserving action of a countable group on a standard non-atomic probability space is non-amenable, then there does not exist a countable-to-one Borel homomorphism from its orbit equivalence relation to the orbit equivalence relation of any modular action (i.e., an action on the boundary of a countably splitting tree), generalizing previous results of Hjorth and Kechris. As an application, for certain groups, we connect antimodularity to mixing conditions. This is joint work with Inessa Epstein.

In Chapter 5, we study full groups of countable, measure-preserving equivalence relations. Our main results include that they are all homeomorphic to the separable Hilbert space and that every homomorphism from an ergodic full group to a separable group is continuous. We also find bounds for the minimal number of generators of a dense subgroup of full groups allowing us to distinguish full groups of equivalence relations generated by free, ergodic actions of the free groups $\mathbf{F}_n$ and $\mathbf{F}_m$ if m and n are sufficiently far apart. We also show that an ergodic equivalence relation is generated by an action of a finitely generated group iff its full group has a finitely generated dense subgroup. This is joint work with John Kittrell.

Contents

Chapter 1

Introduction

1.1 Definable equivalence relations

The study of definable equivalence relations on standard Borel spaces in connection with different mathematical classification problems has been extensively pursued during the last two decades. The main notion that allows us to classify equivalence relations is the partial (quasi-)ordering of Borel reducibility which is defined as follows. If E and F are two equivalence relations on standard Borel spaces X and Y, respectively, we say that E is *Borel reducible* to F and write $E \leq_B F$ if there exists a Borel map $f\colon X \to Y$ such that $x \mathrel{E} y \iff f(x) \mathrel{F} f(y)$. If the equivalence relations E and F are associated with classification problems (for example, isomorphism of certain type of mathematical structures), the notion of Borel reducibility can be interpreted in the sense that the problem defined by E is not harder than the one defined by F, or, equivalently, we can assign in a Borel way F-classes as complete invariants for E-classes. If $E \leq_B F$ and $F \leq_B E$ we say that E and F are *Borel bireducible* and regard them as being of equal complexity. For example, the spectral theorem (a typical instance of a classification theorem in mathematics) implies that unitary equivalence of normal operators on a separable Hilbert space is Borel bireducible with measure equivalence of Borel measures on an uncountable Polish space (that is, we can classify normal operators using measures as invariants and vice versa). See Kechris [39] for further motivation and discussion.

An important class of equivalence relations is the one given by Borel actions of Polish (completely metrizable, separable) groups on standard Borel spaces. This motivates the general study of Polish groups and their actions. For example, it is a well-known and very useful fact that there exist *universal* Polish groups (i.e., ones that embed every Polish group as a closed subgroup) (Uspenskiĭ [73, 74]). However, the dual problem is open: it is not known whether there exists a Polish group such that every Polish group is a continuous homomorphic image of it. Even the seemingly simpler question of whether every Polish group is a homomorphic image of a zero-dimensional Polish group is also open. A step towards better understanding this problem is the following theorem.

Theorem 1.1.1. *Every Polish group is the continuous homomorphic image of a Polish, almost zero-dimensional (hence, at most one-dimensional) group which can be embedded as a* $\mathbf{\Pi}^0_3$ *subgroup of* S_∞*, the group of permutations of the natural numbers. In particular, all Polish groups are quotients of Polishable subgroups of* S_∞*.*

Results similar to Theorem 1.1.1 were later independently obtained by Ding and Gao [11] who used a completely different method.

1.2 Countable Borel equivalence relations and orbit equivalence

A special class of equivalence relations generated by group actions is the one consisting of the *countable* ones, i.e., the Borel equivalence relations whose equivalence classes are countable. By a classical result of Feldman and Moore [18], every countable Borel equivalence relation is generated by a Borel action of a countable group. It turns out that in the same way that topological dynamics is relevant for the study of general Polish group actions, ergodic theory provides a useful framework for studying the actions of countable groups. If the space on which the equivalence relation lives is endowed with an invariant (or, more generally, quasi-invariant) probability measure, the setting becomes analogous to the one studied in ergodic theory and functional analysis for many years. A serious limitation of this approach, however, is that then results only hold "up to measure 0," while often many interesting features of Borel equivalence relations are concentrated on null sets.

Nonetheless, *orbit equivalence*, the theory of countable equivalence relations in the presence of a measure (usually with an emphasis on the group actions generating them), has become a common focus for researchers in ergodic theory, operator algebras, and descriptive set theory, and the exchange of ideas between the fields has led to many exciting developments. An important research objective is, given a free, measure-preserving action of a countable group Γ on a standard probability space (X, μ), to understand how much information the orbit equivalence relation "remembers" about the group and the action. On the one extreme, if Γ is amenable (e.g., solvable), the orbit equivalence relation remembers nothing but the fact that the group is amenable; more precisely, a combination of classical results of Dye and Ornstein–Weiss shows that all ergodic actions of amenable groups are orbit equivalent. On the other, Zimmer [77] proved that in certain situations, the orbit equivalence relation remembers a significant amount of information about the group and the action. More recently, a variety of even more striking rigidity phenomena have been discovered: for example, Furman [20] showed that the orbit equivalence relation of the usual action $\mathrm{SL}(n, \mathbf{Z}) \curvearrowright \mathbf{T}^n$ for $n \geq 3$ (essentially) remembers both the group and the action, and Popa [58] proved that the same holds for Bernoulli actions $\Gamma \curvearrowright 2^\Gamma$ of all groups Γ with property (T). Those superrigidity theorems can be used to obtain results in the purely Borel setting as well, particularly about the richness of the ordering $\leq_B$, and, in many cases, those are the only known methods (see Adams–Kechris [1] and Thomas [70]).

The connection of orbit equivalence with operator algebras comes from the *group measure space* construction of Murray and von Neumann. With every measure-preserving action $\Gamma \curvearrowright (X, \mu)$, one can associate a type II_1 von Neumann factor $L^\infty(X) \rtimes \Gamma$ together with its Cartan subalgebra $L^\infty(X)$. Feldman and Moore [18] proved that two measure-preserving actions $\Gamma \curvearrowright (X, \mu)$ and $\Delta \curvearrowright (Y, \nu)$ are orbit equivalent iff there is an isomorphism between the associated II_1 factors sending $L^\infty(X)$ to $L^\infty(Y)$. This motivates a lot of the current research in the area.

An object intimately connected with a measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is the associated Koopman representation κ of Γ on $L^2(X, \mu)$. Many properties of the action (usually referred to as *spectral properties*: e.g., ergodicity, weak mixing, mixing, etc.) can be conveniently expressed in terms of this representation. Since the constant functions are invariant under κ, one usually considers the restriction κ_0 of κ to the orthogonal complement of the constants. Studying spectral properties of $\mathbf{Z}$-actions comprises a large portion of classical ergodic theory.

Invariants for orbit equivalence are scarce and usually difficult to compute. Perhaps the oldest one is E_0-ergodicity (sometimes referred to as strong ergodicity) which was first considered by Schmidt [63] and is defined as follows: First recall that E_0 is the equivalence relation of eventual equality on $2^\mathbf{N}$. A measure-preserving equivalence relation E on (X, μ) (or the action generating it) is said to be E_0-*ergodic* if every measurable homomorphism from E to E_0 (a map $f \colon X \to 2^\mathbf{N}$ satisfying $x_1 \mathrel{E} x_2 \implies f(x_1) \mathrel{E_0} f(x_2)$) is constant on a co-null set. Connes and Weiss [9] constructed non-E_0-ergodic Gaussian actions for every non-property (T) group, but it was interesting to find simpler examples. Natural actions to look at are the so-called *generalized Bernoulli shifts*: every time a group Γ acts on a countable set A, it also acts on 2^A (by shift) where $2 = \{0, 1\}$ is equipped with (say) the $(1/2, 1/2)$ measure and 2^A is given the product measure. The action $\Gamma \curvearrowright A$ is called *amenable* if $\ell^\infty(A)$ admits a Γ-invariant mean. Interestingly, there are non-amenable groups which admit amenable actions with infinite orbits; those include free groups, various other free products (Glasner–Monod [28]), and inner amenable groups. For those groups, the following theorem, proved together with Alexander Kechris, produces non-E_0-ergodic shifts:

Theorem 1.2.1. *Let an infinite, countable group* Γ *act on a countable set* A*. Then the following are equivalent:*

(i) the action of Γ *on* A *is amenable;*

(ii) the action of Γ *on* 2^A *is not* E_0*-ergodic;*

(iii) the Koopman representation κ_0 *has almost invariant vectors.*

A form of incompatibility between equivalence relations stronger than being non-isomorphic is the non-existence of countable-to-one homomorphisms from one of them to the other. Since Borel reductions between countable equivalence relations are, in particular, countable-to-one homomorphisms, this is also relevant for the Borel theory. Hjorth [31] showed that there are no countable-to-one homomorphisms from (restrictions to co-null sets of) the equivalence relation of the shift of the free group $\mathbf{F}_2 \curvearrowright 2^{\mathbf{F}_2}$ to the orbit equivalence relation of a modular action of any group. (A *modular action* is an action on the space of ends of a countable rooted tree induced by an action by automorphisms on the tree; a special case of those (when the tree is finitely splitting) are the profinite actions.) Thus he proved that there are intermediate treeable equivalence relations, solving an important open problem. Later, Kechris [40], adapting his method, proved that there is a representation-theoretic obstruction (again, for groups containing $\mathbf{F}_2$) for the existence of such homomorphisms. The following theorem, which is joint work with Inessa Epstein, provides a spectral property of the action that serves as an obstruction for the existence of such homomorphisms and is not connected with the free group but only with amenability.

Theorem 1.2.2. *Let a countable group* Γ *act by measure-preserving transformations on the standard probability space* (X, μ)*. Then, if the Koopman representation* κ_0 *associated with the action is not amenable, there does not exist a countable-to-one homomorphism from (a restriction to a co-null set of) its orbit equivalence relation to the orbit equivalence relation of any modular action.*

Theorem 1.2.2 has various interesting corollaries: it implies the previous results of [31] and [40]; it also shows that such homomorphisms do not exist for non-amenable shifts, weakly mixing actions of property (T) groups, or mixing actions of groups with the Haagerup approximation property, thus resolving questions raised in [40].

1.3 Full groups

Orbit equivalence and topological groups come together in the study of full groups of equivalence relations. If E is a measure-preserving equivalence relation on a standard probability space (X, μ), its *full group*, denoted by $[E]$, is the group of all automorphisms T of (X, μ) such that for a.e. $x \in X$, $Tx \mathrel{E} x$. Full groups were first considered by Dye [15], and their importance stems from the fact that they are complete invariants for the equivalence relations up to isomorphism. More precisely, two ergodic equivalence relations are isomorphic iff their full groups are (algebraically) isomorphic and furthermore, every isomorphism between full groups comes from a conjugacy [15]. A natural Polish topology on the full groups, which allows one to use methods and ideas from the descriptive set theory of Polish groups to study them, is induced by the uniform metric $d(T, S) = \mu(\{x : Tx \neq Sx\})$.

Dye's theorem suggests that the algebraic structure of full groups is rich enough to "remember" the topology since every algebraic automorphism of an ergodic full group is automatically a homeomorphism. Confirming this intuition, John Kittrell and I proved the following (all results in this section are from our joint work [46]):

Theorem 1.3.1. *Let E be an ergodic, measure-preserving, countable equivalence relation. Then every homomorphism $f : [E] \to G$, where G is a separable topological group, is automatically continuous. In particular, the uniform topology is the finest separable group topology on $[E]$ and hence, the unique Polish topology.*

Hence, the structure of $[E]$ as an abstract group alone is sufficient to recover the topology, and any statement about $[E]$ as a topological group can, at least in principle, be translated into a statement referring only to its algebraic structure. Automatic continuity for group homomorphisms is a phenomenon which appeared recently in the work of Kechris–Rosendal [43], Rosendal–Solecki [60], and Rosendal [59]. One way to think about it is that if the source group is sufficiently complicated, the axiom of choice is unable to produce pathological homomorphisms to separable groups. See the papers [43, 59, 60] for discussion and examples.

In order to prove that two equivalence relations are non-isomorphic, it suffices to find a (topological group) property of their full groups which differentiates them. One could perhaps even hope to distinguish the full groups as topological spaces alone (forgetting the group structure). This, however, turns out to be impossible as they are all homeomorphic.

Theorem 1.3.2. *Let E be a countable, measure-preserving equivalence relation on the standard probability space (X, μ) which is not equality a.e. Then the full group $[E]$ with the uniform topology is* *homeomorphic to the Hilbert space ℓ^2.*

It was previously known that full groups are contractible (using the argument of Keane [36]).

Another possible invariant one could look at is the number of *topological generators* of $[E]$, i.e., the minimal number of generators of a dense subgroup of $[E]$, which we denote by $t([E])$. Since the group $[E]$ is separable, we always have $t([E]) \leq \aleph_0$. Gaboriau's theory of cost [21] allows one to conclude that if E is generated by a free action of $\mathbf{F}_n$, then $t([E]) \geq n$ (in fact, B. Miller observed that in this case, one actually has $t([E]) \geq n + 1$). This shows that if $t([E])$ is finite for some E, then it is a non-trivial invariant. The question of whether $t([E])$ can be finite was raised by Kechris [37]. Combining an example of Matui [52] from Cantor dynamics with an inductive procedure, we were able to prove the following.

Theorem 1.3.3. *Let E be an ergodic equivalence relation on (X, μ). Then the following are equivalent:*

(i) E can be generated by an action of a finitely generated group;

(ii) $[E]$ is topologically finitely generated.

For an equivalence relation E generated by a free action of $\mathbf{F}_n$, we have the following explicit bounds: $n + 1 \leq t([E]) \leq 3n + 3$. In particular, if m and n are far apart, this gives the first concrete distinction between full groups of equivalence relations generated by free actions of $\mathbf{F}_m$ and $\mathbf{F}_n$.

1.4 Organization of the thesis

This thesis consists of four separate papers. Chapter 2, which appeared in [72], discusses Polishable subgroups of the infinite symmetric group. Theorem 1.1.1 is proved there. Chapter 3, which is joint work with Alexander Kechris, appeared in [44] and concerns the relationships between group actions on countable sets and the corresponding measure-preserving shifts. Its main result is Theorem 1.2.1. Chapter 4, which will appear in [16], is joint work with Inessa Epstein; Theorem 1.2.2 and various generalizations and applications of it are proved there. Finally, Chapter 5 is joint work with John Kittrell; it discusses properties of full groups of equivalence relations, and Theorems 1.3.1, 1.3.2, and 1.3.3 are proved there. Parts of it are also included in Kittrell's dissertation [45].

Each chapter has its own introduction motivating the work and discussing relevant background.

Chapter 2

Compactifications of N and Polishable Subgroups of S_∞

2.1 Introduction

It is well known that every compact metrizable topological space X can be realized in a unique way as the remainder $\widetilde{X} \setminus \mathbf{N}$ of a metrizable compactification $\widetilde{X}$ of the countable discrete space of the natural numbers $\mathbf{N}$ (see Propositions 2.2.1 and 2.2.3). This allows us to associate with each compact metrizable X the homeomorphism group $\mathrm{Homeo}(\widetilde{X})$ and a certain subgroup of it, called the structure group of X (see Definition 2.2.5 below). These groups were first studied by Lorch [49], who proved the following interesting result:

Theorem 2.1.1 (Lorch)**.** *Two compact metrizable spaces are homeomorphic if and only if their structure groups are isomorphic.*

Both the group $\mathrm{Homeo}(\widetilde{X})$ and the structure group of X can be viewed as Polishable subgroups of S_∞, the group of all permutations of $\mathbf{N}$ (see Proposition 2.2.4 below). We study the topological dimension of the Polish topologies of those groups as well as their descriptive complexity. In particular, we prove the following (see Theorem 2.3.1, Corollary 2.4.7, and Theorem 2.4.8 below):

Theorem 2.1.2. *The group* $\mathrm{Homeo}(\widetilde{X})$ *is almost zero-dimensional (and thus at most one-dimensional). It is one-dimensional if the group* $\mathrm{Homeo}(X)$ *contains a path of finite length (in the natural complete metric of the group). Both* $\mathrm{Homeo}(\widetilde{X})$ *and the structure group of X are $\mathbf{\Pi}^0_3$ subgroups of S_∞ and they are $\mathbf{\Pi}^0_3$-complete iff X is infinite.*

As an interesting corollary of the construction, we show the following (see Corollary 2.4.11):

Theorem 2.1.3. *Every Polish group is a continuous homomorphic image of an almost zero-dimensional Polishable subgroup of S_∞.*

This is related to the open problem of whether every Polish group is a factor of a zero-dimensional Polish group.

In the last section of the paper we study Polishable ideals on **N** and certain almost zero-dimensional Polishable subgroups of S_∞ associated with them.

Recall that a topological space is called *Polish* if it is separable and completely metrizable; a topological group is Polish if its topology is Polish. A Borel subgroup H of a Polish group G is called *Polishable* if there exists a Polish group topology on H which has the same Borel structure as the one inherited from G. By [38, 9.10], the Polish topology of a Polishable H is always finer than the inherited topology. Two examples of Polish groups are the homeomorphism groups of compact metrizable spaces with the compact-open topology, which coincides with the uniform convergence topology, and the group of permutations of the natural numbers S_∞ with the pointwise convergence topology. A complete metric on S_∞ is given by

$$d(f,g) = 2^{-\min\{f \neq g\}} + 2^{-\min\{f^{-1} \neq g^{-1}\}}. \tag{2.1.1}$$

The support of a permutation $f \in S_\infty$, denoted by $\operatorname{supp} f$, is the set of points moved by f. For a detailed treatment of Polish spaces, and Polish and Polishable groups, the reader is referred to [38].

In any metric space we will denote by $B_r(x)$ the open ball with center x and radius r. Since we will often work with different topologies on the same space, to avoid confusion, we will sometimes explicitly mention the topology: e.g., (X,τ) is the space X with the topology τ. Throughout this paper, I denotes the unit interval $[0,1]$ and $Q = I^{\mathbf{N}}$ is the Hilbert cube.

2.2 Compact spaces as remainders of compactifications of N

The following fact is well known; we include a simple proof, due to H. Toruńczyk, and note the effectiveness of the construction.

Proposition 2.2.1. *For every compact metrizable space X, there exists a metrizable compactification $\widetilde{X}$ of* **N** *(taken with the discrete topology) such that the remainder $\widetilde{X} \setminus \mathbf{N}$ is homeomorphic to X.*

Proof. Fix a countable dense set $D = \{a_k\}$ in X. Set $\widetilde{X} = X \times \{0\} \cup A$, where $A = \bigcup \{a_1, \dots, a_n\} \times \{1/n\}$. Then A is countable, discrete, and dense in $\widetilde{X}$. $\widetilde{X}$ is compact as a closed subspace of $X \times [0,1]$. □

We will think of the space $\widetilde{X}$ as the union $X \cup \mathbf{N}$ and we will also fix a compatible metric d on $\widetilde{X}$. Consider the homeomorphism group $\mathrm{Homeo}(\widetilde{X})$. With the topology induced by the metric

$$\partial'(f,g) = \sup_{x \in \widetilde{X}} d(f(x), g(x)),$$

it is a Polish group. The metric ∂' is not complete but it is equivalent to the complete metric ∂ defined by

$$\partial(f,g) = \partial'(f,g) + \partial'(f^{-1}, g^{-1}).$$

Lemma 2.2.2. *Let $f\colon X \to X$ be a homeomorphism and $\tilde{f}$ a homeomorphism of $\widetilde{X}$ such that f and $\tilde{f}$ agree on X. If $g\colon X \to X$ is another homeomorphism and $\partial(f,g) < r$, then there exists a homeomorphism $\tilde{g}$ of $\widetilde{X}$ such that $\tilde{g}$ extends g and $\partial(\tilde{f}, \tilde{g}) < r$.*

Proof. Set $\epsilon = (r - \partial(f,g))/6$. Since $\tilde{f}, g, \tilde{f}^{-1}, g^{-1}$ are all uniformly continuous, we can find a $\delta < \epsilon$ so small that

$$\begin{aligned} &\forall x,y \in \widetilde{X} \quad d(x,y) < \delta \implies d(\tilde{f}(x), \tilde{f}(y)) < \epsilon \text{ and } d(\tilde{f}^{-1}(x), \tilde{f}^{-1}(y)) < \epsilon, \\ &\forall x,y \in X \quad d(x,y) < \delta \implies d(g(x), g(y)) < \epsilon \text{ and } d(g^{-1}(x), g^{-1}(y)) < \epsilon. \end{aligned}$$

Using a standard back-and-forth argument, we will define a permutation $h\colon \mathbf{N} \to \mathbf{N}$ and then will set $\tilde{g} = g \cup h$. First find a number N so big that $\forall n > N \; d(n, X) < \delta$. Find points $x_n \in X$, such that $d(n, x_n) = d(n, X)$ for each n and note that the set $\{x_n : n \in \mathbf{N}\}$ is dense in X. Set $h_0 = \tilde{f}|_{[0,N] \cup \tilde{f}^{-1}([0,N])}$. Now suppose we are at a forward step of the construction, say number $2k-1$, and let $n = \min\{\mathbf{N} \setminus \operatorname{dom} h_{2k-2}\}$. Find an m, such that $d(g(x_n), x_m) < 2^{-k}\delta$ and $m \notin \operatorname{ran} h_{2k-2}$. Define h_{2k-1} to agree with h_{2k-2} on $\operatorname{dom} h_{2k-2}$ and set $h_{2k-1}(n) = m$. Now prove that this extension does not move us too far from $\tilde{f}$. We have the following estimates:

$$\begin{aligned} d(h_{2k-1}(n), \tilde{f}(n)) &\le d(m, x_m) + d(x_m, g(x_n)) \\ &\qquad + d(g(x_n), f(x_n)) + d(f(x_n), \tilde{f}(n)) \\ &\le d(g(x_n), f(x_n)) + 3\epsilon; \\ d(h_{2k-1}^{-1}(m), \tilde{f}^{-1}(m)) &\le d(\tilde{f}^{-1}(m), f^{-1}(x_m)) + d(f^{-1}(x_m), g^{-1}(x_m)) \\ &\qquad + d(g^{-1}(x_m), x_n) + d(x_n, n) \\ &\le d(f^{-1}(x_m), g^{-1}(x_m)) + 3\epsilon. \end{aligned} \tag{2.2.1}$$

At a backward step $2k$ proceed similarly, let $m = \min\{\mathbf{N} \setminus \operatorname{ran} h_{2k-1}\}$. Find n such that $d(g(x_n), x_m) < 2^{-k}\delta$ and $n \notin \operatorname{dom} h_{2k-1}$. Define h_{2k} to agree with h_{2k-1} on $\operatorname{dom} h_{2k-1}$ and set $h_{2k}(n) = m$. (2.2.1) will again hold with h_{2k} replacing h_{2k-1}.

Set $h = \bigcup_{k=0}^{\infty} h_k$ and $\tilde{g} = g \cup h$. We will first prove that $\tilde{g}$, so defined, is a homeomorphism of $\widetilde{X}$. It is enough to show that for any sequence $\{n_k\}$ $n_k \to z \iff h(n_k) \to g(z)$ for $z \in X$. This is easily seen, in fact

$$\begin{aligned} n_k \to z \iff x_{n_k} \to z &\iff g(x_{n_k}) \to g(z) \\ &\iff x_{h(n_k)} \to g(z) \iff h(n_k) \to g(z). \end{aligned}$$

Now, using (2.2.1), we also check that $\partial(\tilde{f}, \tilde{g}) < r$:

$$\begin{aligned} \partial(\tilde{f}, \tilde{g}) &= \sup_{x \in \widetilde{X}} d(\tilde{f}(x), \tilde{g}(x)) + \sup_{x \in \widetilde{X}} d(\tilde{f}^{-1}(x), \tilde{g}^{-1}(x)) \\ &\leq \partial'(f, g) + 3\epsilon + \partial'(f^{-1}, g^{-1}) + 3\epsilon = \partial(f, g) + 6\epsilon < r. \end{aligned}$$

□

The next proposition is essentially contained in [49]. It also follows from the proof of Lemma 2.2.2 above.

Proposition 2.2.3. *If X, Y are compact metrizable, $f: X \to Y$ a homeomorphism, $\widetilde{X}$ and $\widetilde{Y}$ are compactifications of* **N** *as above and $X \hookrightarrow \widetilde{X}$ and $Y \hookrightarrow \widetilde{Y}$ given embeddings onto the remainders, then there exists a homeomorphism $\tilde{f}: \widetilde{X} \to \widetilde{Y}$, such that the diagram*

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{\;\tilde{f}\;} & \widetilde{Y} \\ \uparrow & & \uparrow \\ X & \xrightarrow[\;f\;]{} & Y \end{array}$$

commutes.

In particular, Propositions 2.2.1 and 2.2.3 show that with every compact metrizable space we can associate a unique metrizable compactification $\widetilde{X}$ of $\mathbf{N}$, such that $X \cong \widetilde{X} \setminus \mathbf{N}$.

Proposition 2.2.4. *The Polish group* $\operatorname{Homeo}(\widetilde{X})$ *can be identified with a Polishable subgroup of S_∞.*

Proof. Since $\mathbf{N}$ is the set of all isolated points in $\widetilde{X}$, for every homeomorphism $f \in \operatorname{Homeo}(\widetilde{X})$ we must have $f(\mathbf{N}) = \mathbf{N}$ and $f(X) = X$. Therefore the restriction map $R: \operatorname{Homeo}(\widetilde{X}) \to S_\infty$, $R(f) =$

$f|_{\mathbf{N}}$ is a well defined group homomorphism. It is injective because $\mathbf{N}$ is dense in $\widetilde{X}$ and hence a homeomorphism is entirely determined by its action on $\mathbf{N}$. R is also continuous as the composition of the identity map from $\mathrm{Homeo}(\widetilde{X})$ to the same space, equipped with the pointwise convergence topology, and the restriction from the latter to S_∞ (which also carries the pointwise convergence topology). Therefore we can identify $\mathrm{Homeo}(\widetilde{X})$ with a Borel subgroup of S_∞ which is Polishable because $\mathrm{Homeo}(\widetilde{X})$ itself is Polish. □

Now, following Lorch, we consider the pointwise stabilizer of X in $\mathrm{Homeo}(\widetilde{X})$.

Definition 2.2.5 (Lorch)**.** The closed subgroup $H(X) = \{f \in \mathrm{Homeo}(\widetilde{X}) : \forall x \in X f(x) = x\}$ of $\mathrm{Homeo}(\widetilde{X})$ is called the *structure group* of X.

We will write H instead of $H(X)$ if there is no danger of confusion. The restriction map

$$q\colon \mathrm{Homeo}(\widetilde{X}) \to \mathrm{Homeo}(X), \quad q(f) = f|_X$$

is a continuous group homomorphism, has kernel H, and, by Proposition 2.2.3, is onto $\mathrm{Homeo}(X)$. Therefore H is a closed normal subgroup of $\mathrm{Homeo}(\widetilde{X})$ and $\mathrm{Homeo}(X) \cong \mathrm{Homeo}(\widetilde{X})/H$ as topological groups.

The restriction of the metric d to $\mathbf{N}$ is totally bounded and induces the discrete topology on $\mathbf{N}$. Let $\mathrm{UHomeo}(\mathbf{N}, d)$ denote the group of all uniform homeomorphisms of $\mathbf{N}$ with respect to the metric d (i.e., all uniformly continuous permutations $\mathbf{N} \to \mathbf{N}$ with uniformly continuous inverses). $\mathrm{UHomeo}(\mathbf{N}, d)$ becomes a topological group with the uniform convergence topology. It is clear that any $f \in \mathrm{UHomeo}(\mathbf{N}, d)$ extends to a homeomorphism $\tilde{f}$ of $\widetilde{X}$ and conversely, any homeomorphism of $\widetilde{X}$ restricts to a uniform homeomorphism of $\mathbf{N}$. It is easy to check that this correspondence is a topological group isomorphism between $\mathrm{Homeo}(\widetilde{X})$ and $\mathrm{UHomeo}(\mathbf{N}, d)$, so from now on we can identify these two groups.

As was pointed out by A.S. Kechris, this viewpoint may also be relevant to the problem of characterizing the complexity of homeomorphism of compact metrizable spaces. More precisely, in view of the universality of the Hilbert cube Q (cf. [38, 4.14]), we can think of the hyperspace $K(Q)$ of all compact subsets of Q (equipped with the Vietoris topology) as the space of all compact metrizable spaces and define the equivalence relation E_h on $K(Q)$ by

$$K \, E_h \, L \iff K \text{ is homeomorphic to } L.$$

Similarly, we can consider the Borel set $D \subseteq I^{\mathbf{N}\times\mathbf{N}}$ consisting of all totally bounded, discrete metrics on $\mathbf{N}$ of diameter not greater than 1 and define the equivalence relation E_u on D by

$$d_1 \, E_u \, d_2 \iff (\mathbf{N}, d_1) \text{ and } (\mathbf{N}, d_2) \text{ are uniformly homeomorphic.}$$

For two equivalence relations E and F defined on the standard Borel spaces X and Y, respectively, we write $E \leq_B F$ if there exists a Borel map $f \colon X \to Y$ satisfying

$$x \, E \, y \iff f(x) \, F \, f(y).$$

If $E \leq_B F$ and $F \leq_B E$, we say that E and F are *Borel bireducible*. See [39] and the references therein for general background on the theory of equivalence relations and [24, Chapter 10] for more details on different (open) classification problems.

We have the following corollary from Propositions 2.2.1 and 2.2.3:

Corollary 2.2.6. *E_h and E_u are Borel bireducible.*

Proof. First construct a map $K(Q) \to D$ which reduces E_h to E_u. Given $K \in K(Q)$, by [38, 12.13], we can find in a Borel way a dense countable subset of K and then use the construction from the proof of Proposition 2.2.1 to define a metric on $\mathbf{N}$. Proposition 2.2.3 shows that this map is indeed a reduction.

Conversely, to reduce E_u to E_h, consider first the inclusion map $i \colon D \to Q^{\mathbf{N}} = I^{\mathbf{N}\times\mathbf{N}}$. By the proof of [38, 4.14] and using the total boundedness of the elements of D, if we consider $i(d)$, $d \in D$ as a countable subset of Q, then the closure of $i(d)$ in Q is homeomorphic to the completion of $(\mathbf{N}, d)$. The closure map $c \colon Q^{\mathbf{N}} \to K(Q)$ defined by $c((a_n)) = \overline{\{a_n : n \in \mathbf{N}\}}$ is Borel and the composition $c \circ i$ is the desired reduction. □

2.3 Descriptive complexity of $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$

Both $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$ are Borel subgroups of S_∞, so we can ask where they fit in the Borel hierarchy.

Theorem 2.3.1. *If X is a one point space, then $\mathrm{Homeo}(\widetilde{X}) = H(X) = S_\infty$. If X has more than one but finitely many points, both $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$ are $\mathbf{\Sigma}^0_2$-complete. Finally, if X is infinite, both $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$ are $\mathbf{\Pi}^0_3$-complete.*

Proof. Put $G = \mathrm{Homeo}(\widetilde{X})$ and $H = H(X)$. The first statement of the theorem is obvious. Let now $X = \{x_i\}_{i=0}^k$ be finite and, without loss of generality, assume that $d(x_i, x_j) = 1$ for $i \neq j$. Then $\mathbf{\Sigma}_2^0$ descriptions of G and H are given by:

$$f \in G \iff \exists \sigma \in S_{k+1}\, \exists \delta\, \forall m \in \mathbf{N} \quad d(m, x_i) < \delta \implies d(f(m), x_{\sigma(i)}) < 1/2,$$
$$f \in H \iff \forall i \leq k\, \exists \delta\, \forall m \in \mathbf{N} \quad d(m, x_i) < \delta \implies d(f(m), x_i) < 1/2.$$

Both G and H cannot be G_δ because they contain the permutations with finite support, which are dense in S_∞ (see Exercise 9.11 in [38]).

Let finally X be infinite and $\{a_k\}_{k=0}^\infty$ be a countable dense set. First of all, the following are $\mathbf{\Pi}_3^0$ descriptions of G and H:

$$f \in G \iff \forall \epsilon\, \exists \delta \forall m, n \in \mathbf{N} \quad d(m, n) < \delta \implies d(f(m), f(n)) < \epsilon,$$
$$f \in H \iff f \in G \text{ and } (\forall \epsilon \exists \delta \forall k\, \forall m \in \mathbf{N} \quad d(m, a_k) < \delta \implies d(f(m), a_k) < \epsilon).$$

Now consider the $\mathbf{\Pi}_3^0$-complete set $C \subseteq 2^{\mathbf{N}\times\mathbf{N}}$ defined by

$$A \in C \iff \forall n\, \{k : (n, k) \in A\} \text{ is finite.}$$

(We look at the elements of $2^{\mathbf{N}\times\mathbf{N}}$ as subsets of $\mathbf{N} \times \mathbf{N}$. For more information on $\mathbf{\Pi}_3^0$-complete sets see [38, 23.A].) We will construct a continuous map $\Phi\colon 2^{\mathbf{N}\times\mathbf{N}} \to S_\infty$, such that

$$(A \in C \implies \Phi(A) \in H) \quad \text{and} \quad (A \notin C \implies \Phi(A) \notin G), \tag{2.3.1}$$

thus Φ is a reduction of C to both G and H. Fix a convergent sequence $\{x_k\}_{k=0}^\infty$ of distinct elements of X, $x_k \to y$, and let $\{b_{k,j}\}$ be a 2-indexed sequence of distinct elements of $\mathbf{N}$, satisfying the following conditions:

(i) $\forall k, j\; d(b_{k,j}, x_k) < 2^{-(k+j)}$,

(ii) $\mathbf{N} \setminus \{b_{k,j} : k, j \in \mathbf{N}\}$ is dense in X.

Note that a sequence $\{b_{k_n, j_n}\}_{n=0}^\infty$ converges to a point of X iff either $k_n \to \infty$ (in which case $b_{k_n,j_n} \to$

y), or k_n is eventually constant and $j_n \to \infty$ (then $b_{k_n,j_n} \to x_{\lim k_n}$). Now define

$$\Phi(A) = \prod_{(k,j)\in A} (b_{2k,j}\ b_{2k+1,j}),$$

where $(m\ n)$ denotes the transposition in S_∞ which switches m and n. If $A \in C$, then the only limit point of $\operatorname{supp}\Phi(A)$ is y and it is easy to see that $\Phi(A) \cup \mathrm{id}_X$ is a homeomorphism of $\widetilde{X}$. If, on the other hand, $A \notin C$, then any continuous extension of $\Phi(A)$ to X must switch x_{2k} and x_{2k+1} for some k, which is impossible because of (ii). Hence $\Phi(A) \notin G$ and (2.3.1) is verified. □

2.4 Topological properties of the groups $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$

On the groups $\mathrm{Homeo}(\widetilde{X})$ and $H(X)$ we have two natural topologies, the Polish topology τ and the topology σ inherited from S_∞, i.e., the topology of pointwise convergence on $\mathbf{N}$. Clearly $\sigma \subseteq \tau$. We have the following easy fact.

Proposition 2.4.1. *(H,τ) is zero-dimensional.*

Proof. Note first that if $f, g \in H$ then there exists $a \in \mathbf{N}$, such that $\partial'(f,g) = d(f(a), g(a))$. We will now show that every open ball $B_r(1_H)$ in H is also closed. Indeed, let $\{g_n\}$ be a sequence in $B_r(1)$, such that $g_n \to g \in H$. There exists $a \in \mathbf{N}$ for which $\partial'(1_H, g) = d(a, g(a))$ but for some n, $g_n(a) = g(a)$ (because convergence in the topology of H implies convergence in the coarser topology of S_∞). Therefore, for this n,

$$\partial'(1_H, g) = d(a, g_n(a)) \le \partial'(1_H, g_n) < r.$$

Hence $g \in B_r(1_H)$ and the proof is complete. □

To continue our analysis, we need the notion of almost zero-dimensionality, first introduced in Oversteegen–Tymchatyn [56]. Recall that a basis for a topological space X is a collection $\mathcal{B}$ of (not necessarily open) subsets of X such that for every open $U \subseteq X$ and every $x \in U$ there exists $B \in \mathcal{B}$ with $B \subseteq U$ and x contained in the interior of B. Similarly, we say that $\mathcal{B}$ is a neighborhood basis at the point x if for every open U containing x, there exists $B \in \mathcal{B}$ with $B \subseteq U$ and x contained in the interior of B. An *open basis* is a basis consisting of open sets.

Definition 2.4.2 ([10, 56])**.** A separable metrizable space is *almost zero-dimensional* if there exists a basis for its topology consisting of intersections of clopen sets.

Note that almost zero-dimensionality is a hereditary property. An important fact about almost zero-dimensional spaces is the following:

Theorem 2.4.3 (Oversteegen–Tymchatyn [56], cf. Levin–Pol [48])**.** *Every almost zero-dimensional space is at most one-dimensional.*

The original definition of almost zero-dimensionality Oversteegen and Tymchatyn used to prove their theorem is somewhat different from Definition 2.4.2 (which we borrowed from Dijkstra–van Mill–Steprāns [10]) but the equivalence of the two definitions is proved in [10]. Almost zero-dimensional topologies are intimately related to certain coarser zero-dimensional topologies on the same space. This was noticed by van Mill and Dijkstra who suggested the following:

Definition 2.4.4. Let $(X, \mathcal{T})$ be a separable metrizable space. We say that a separable metrizable zero-dimensional topology $\mathcal{W}$ on X *witnesses the almost zero-dimensionality* of $(X, \mathcal{T})$ if $\mathcal{W} \subseteq \mathcal{T}$ and $(X, \mathcal{T})$ has a basis consisting of closed sets of $\mathcal{W}$.

As S. Solecki pointed out, using a result of his, we can exactly determine when the topology of a zero-dimensional Polish group witnesses the almost zero-dimensionality of a Polishable subgroup. To do this, we shall need some of the machinery developed in Solecki [68].

Let (H, τ) be a Polishable subgroup of a Polish group (G, σ) and $\{V_n : n \in \mathbf{N}\}$ be an *open* neighborhood basis at 1 for (H, τ), satisfying the conditions

$$V_n = V_n^{-1} \quad \text{and} \quad V_{n+1}^3 \subseteq V_n. \tag{2.4.1}$$

Let $F_n = \overline{V_n}^\sigma$ and for $x, y \in G$, define

$$\delta_l(x, y) = \inf\{2^{-k} : x^{-1}y \in F_k\},$$
$$\delta_r(x, y) = \inf\{2^{-k} : xy^{-1} \in F_k\},$$

and

$$d_l(x, y) = \inf\{\sum_{i=0}^{n-1} \delta_l(x_i, x_{i+1}) : x_0 = x, x_n = y, x_i \in G\}$$

and similarly d_r. Then

$$\begin{aligned} \widetilde{H} = \{g \in G : \forall V((1 \in V \text{ and } V \text{ is } \tau\text{-open}) \\ \implies \exists h_1, h_2 \in H \; g \in \overline{h_1 V}^\sigma \cap \overline{V h_2}^\sigma)\} \end{aligned} \tag{2.4.2}$$

is a $\mathbf{\Pi}^0_3$ Polishable subgroup of G with a Polish topology $\tilde{\tau}$ defined by the metric $\rho = d_l + d_r$ restricted to $\widetilde{H}$. Furthermore, the inequalities

$$\delta_l \geq d_l \geq \frac{1}{2}\delta_l \quad \text{and} \quad \delta_r \geq d_r \geq \frac{1}{2}\delta_r \tag{2.4.3}$$

hold. H is a dense subgroup of $(\widetilde{H}, \tilde{\tau})$ and for any $\mathbf{\Pi}^0_3$ set $A \subseteq G$ with $H \subseteq A$, $A \cap \widetilde{H}$ is comeager in $(\widetilde{H}, \tilde{\tau})$. For all of the above, see [68].

Lemma 2.4.5. *Let* $\{B_n\}$ *be an* arbitrary *basis at* 1 *for* (H, τ). *Then* $\widetilde{B}_n = \overline{B_n}^{\sigma} \cap \widetilde{H}$ *defines a basis at* 1 *for* $\widetilde{H}$.

Proof. Let for each $k \in \mathbf{N}$, $\widetilde{U}_k \subseteq \widetilde{H}$ be the open ball (in the metric ρ) with center 1 and radius 2^{-k}. Fix k and find n such that $B_n \subseteq V_{k+2}$. Then $\widetilde{B}_n \subseteq F_{k+2}$ and for any $x \in \widetilde{B}_n$,

$$d_l(1, x) \leq \delta_l(1, x) \leq 2^{-(k+2)} < 2^{-(k+1)}$$

and similarly $d_r(1, x) < 2^{-(k+1)}$. Hence $\rho(1, x) < 2^{-k}$ and $\widetilde{B}_n \subseteq \widetilde{U}_k$.

Conversely, for a fixed n, find k such that $V_k \subseteq B_n$. Then for any $x \in \widetilde{U}_{k+1}$,

$$\delta_l(1, x) \leq 2d_l(1, x) \leq 2\rho(1, x) < 2^{-k}.$$

Hence $x \in F_k \cap \widetilde{H} \subseteq \widetilde{B}_n$, $\widetilde{U}_{k+1} \subseteq \widetilde{B}_n$ and we are done. □

Proposition 2.4.6. *Let* (G, σ) *be a zero-dimensional Polish group and* (H, τ) *a Polishable subgroup. Then the following are equivalent:*

(i) H *is* $\mathbf{\Pi}^0_3$ *in* G*;*

(ii) $\sigma|_H$ *witnesses the almost zero-dimensionality of* (H, τ)*;*

(iii) *every open set in* (H, τ) *is* $\mathbf{\Sigma}^0_2$ *in* $(H, \sigma|_H)$.

Proof. (i) ⇒ (ii). Let $(\widetilde{H}, \tilde{\tau})$ be defined as in (2.4.2). Since H is $\mathbf{\Pi}^0_3$, by [68], H is comeager in $\widetilde{H}$, so we must have $\widetilde{H} = H$ (see [38, Exercise 9.11]). Then the basis $\{\widetilde{B}_n\}$ of closed sets of $\sigma|_H$, defined in Lemma 2.4.5 (starting with an arbitrary basis $\{B_n\}$ of H), shows that (ii) is true.

(ii) ⇒ (iii). Let $\mathcal{B}$ be a basis for τ consisting of closed sets in $\sigma|_H$. Since τ is Lindelöf, every open set is a countable union of elements of $\mathcal{B}$ and thus $\mathbf{\Sigma}^0_2(\sigma|_H)$.

(iii) ⇒ (i). This follows easily from a result in Farah–Solecki [17]. For $A \subseteq G$ and a τ-open $V \subseteq H$, we define the *Vaught transform* $A^{\triangle V}$ as

$$A^{\triangle V} = \big\{g \in G : \{h \in H : hg \in A\} \text{ is non-meager in } (V, \tau)\big\}.$$

We will use a claim from the proof of [17, Theorem 3.1].

Claim. For $A \subseteq G$, $A \in \mathbf{\Sigma}^0_2(\sigma)$ and any τ-open $U \subseteq H$, $A^{\triangle U} \cap \widetilde{H}$ is $\tilde{\tau}$-open.

Let V be any open τ-neighborhood of 1 in H. Since $V \in \mathbf{\Sigma}^0_2(\sigma|_H)$, there exists $A \subseteq G$, $A \in \mathbf{\Sigma}^0_2(\sigma)$, $A \cap H = V$. Then $1 \in A^{\triangle V}$ and by the Claim, $A^{\triangle V} \cap \widetilde{H}$ is $\tilde{\tau}$-open. Therefore $A^{\triangle V} \cap H$ is $\tilde{\tau}|_H$-open and it is not hard to check that $A^{\triangle V} \cap H \subseteq V^{-1}V$. For any τ-open neighborhood U of 1, we can find V as above with $V^{-1}V \subseteq U$. Furthermore, (H, τ) is a Polishable subgroup of $(\widetilde{H}, \tilde{\tau})$ and by [38, 9.10], $\tilde{\tau}|_H \subseteq \tau$. Thus the set $\{A^{\triangle V} \cap H : V \text{ is a } \tau\text{-neighborhood of } 1\}$ (where A depends on V) is a basis at 1 for τ consisting of $\tilde{\tau}|_H$-open sets and hence $\tilde{\tau}|_H = \tau$. Therefore $(H, \tilde{\tau}|_H)$ is a Polish subgroup of $\widetilde{H}$. Since H is dense in $\widetilde{H}$, we must have $H = \widetilde{H}$ (see [38, Exercise 9.11]). □

Now, going back to the group $\mathrm{Homeo}(\widetilde{X})$, by Theorem 2.3.1, it is $\mathbf{\Pi}^0_3$ in S_∞, hence Proposition 2.4.6 applies and we have the following corollary.

Corollary 2.4.7. $(\mathrm{Homeo}(\widetilde{X}), \tau)$ *is almost zero-dimensional.*

$\mathrm{Homeo}(\widetilde{X})$ can be zero-dimensional, e.g., if X is a one point space, then $\mathrm{Homeo}(\widetilde{X}) \cong S_\infty$. Below we give a sufficient condition for $(\mathrm{Homeo}(\widetilde{X}), \tau)$ not to be zero-dimensional. Recall that the length of a path $f\colon [a, b] \to Y$ in a metric space (Y, d) is defined as

$$\mathrm{len}(f) = \sup\{\sum_{i=0}^{n-1} d(f(x_i), f(x_{i+1})) : a = x_0 < x_1 < \cdots < x_n = b\}.$$

If $x, y \in [a, b]$ write $\mathrm{len}(x, y)$ for the length of the path $f|_{[x,y]}$.

Theorem 2.4.8. *If the group* $\mathrm{Homeo}(X)$ *has the property that there exists a homeomorphism* $g \neq \mathrm{id}_X$ *which can be connected to* id_X *via a path of finite length (in the complete metric* ∂*), then* $\mathrm{Homeo}(\widetilde{X})$ *is not zero-dimensional.*

Proof. Put $G = \mathrm{Homeo}(\widetilde{X})$, $K = \mathrm{Homeo}(X)$ and let f be a path of finite length defined on the unit interval $[0, 1]$ with $f(0) = 1_K$ and $f(1) = g \neq 1_K$. Set $r = \partial(1_K, g)$. The quotient map $q\colon G \to K$ is Lipschitz and by Lemma 2.2.2 it sends open balls to open balls of the same radius. Suppose that G is zero-dimensional; then there exists a clopen set $U \subseteq B_r(1_G)$. Towards a contradiction, define inductively transfinite sequences $\{t_\alpha\}$, $\{h_\alpha\}$ and $\{\tilde{h}_\alpha\}$, $\alpha < \omega_1$ of elements of $[0,1]$, K and G, respectively, satisfying the following conditions:

- $f(t_\alpha) = h_\alpha = q(\tilde{h}_\alpha)$; $\tilde{h}_\alpha \in U$;
- $\alpha < \beta \Longrightarrow t_\alpha < t_\beta$;
- $\partial(\tilde{h}_\alpha, \tilde{h}_\beta) \le 2\operatorname{len}(t_\alpha, t_\beta)$.

Set $t_0 = 0$, $h_0 = 1_K$, $\tilde{h}_0 = 1_G$. Suppose that the sequences have been defined for $\alpha < \beta$. If $\beta = \gamma + 1$ is a successor find an ϵ, $0 < \epsilon < \partial(h_\gamma, g)$ such that $B_\epsilon(\tilde{h}_\gamma) \subseteq U$. Set $t_\beta = \sup\{t \in [t_\gamma, 1] : \partial(h_\gamma, f(t)) = \epsilon/2\}$, $h_\beta = f(t_\beta)$. Using Lemma 2.2.2, find an $\tilde{h}_\beta \in G$, satisfying $q(\tilde{h}_\beta) = h_\beta$, $\partial(\tilde{h}_\gamma, \tilde{h}_\beta) < 3\epsilon/4$ and hence, $\tilde{h}_\beta \in U$. Finally, to verify the third condition, notice that for any $\alpha < \beta$

$$\begin{aligned}\partial(\tilde{h}_\alpha, \tilde{h}_\beta) &\le \partial(\tilde{h}_\alpha, \tilde{h}_\gamma) + \partial(\tilde{h}_\gamma, \tilde{h}_\beta) \le 2\operatorname{len}(t_\alpha, t_\gamma) + 3\epsilon/4 \\ &< 2\operatorname{len}(t_\alpha, t_\gamma) + 2\partial(h_\gamma, h_\beta) \le 2\operatorname{len}(t_\alpha, t_\beta).\end{aligned}$$

Now consider the case when $\beta < \omega_1$ is a limit ordinal. Since β is countable, there exists an increasing sequence $\{\gamma_n\}$ with $\lim \gamma_n = \beta$. By compactness of $[0,1]$, t_{γ_n} converges. By the inductive hypothesis, $\sum_n \partial(\tilde{h}_{\gamma_n}, \tilde{h}_{\gamma_{n+1}}) \le 2\sum_n \operatorname{len}(t_{\gamma_n}, t_{\gamma_{n+1}}) < \infty$, so $\{\tilde{h}_{\gamma_n}\}$ is Cauchy and therefore converges. Set $\tilde{h}_\beta = \lim \tilde{h}_{\gamma_n}$, $h_\beta = q(\tilde{h}_\beta)$, $t_\beta = \lim t_{\gamma_n}$. By continuity, $f(t_\beta) = h_\beta$ and $\tilde{h}_\beta \in U$ because U is closed. Now fix $\alpha < \beta$ and verify the last condition:

$$\partial(\tilde{h}_\alpha, \tilde{h}_\beta) = \lim_{n\to\infty} \partial(\tilde{h}_\alpha, \tilde{h}_{\gamma_n}) \le \sup_n 2\operatorname{len}(t_\alpha, t_{\gamma_n}) \le 2\operatorname{len}(t_\alpha, t_\beta).$$

As a result of the construction, we obtain an order preserving embedding $\omega_1 \to [0,1]$, which is clearly impossible. □

Proposition 2.4.9. *There exists a path* $f\colon \left[\frac{1}{2}, \frac{3}{4}\right] \to \operatorname{Homeo}(I)$ *with* $\operatorname{id}_I = f\left(\frac{1}{2}\right) \neq f\left(\frac{3}{4}\right)$ *and of finite length (in the complete metric* ∂*).*

Proof. For each $t \in [\frac{1}{2}, \frac{3}{4}]$ consider the homeomorphism $f(t)\colon I \to I$ which maps linearly $[0, \frac{1}{2}]$ onto $[0, t]$ and $[\frac{1}{2}, 1]$ onto $[t, 1]$. It is easy to see that $\partial(f(t), f(s)) \le 3|t - s|$, so we have our path. □

Endow the Hilbert cube $Q = I^{\mathbf{N}}$ with its standard metric

$$d((x_0, x_1, \dots), (y_0, y_1, \dots)) = \sum_{n=0}^{\infty} 2^{-n}|x_n - y_n|.$$

Then we have the following

Corollary 2.4.10. *There is a path of finite length* $f\colon [\frac{1}{2}, \frac{3}{4}] \to \mathrm{Homeo}(Q)$ *with* $\mathrm{id}_Q = f(\frac{1}{2}) \neq f(\frac{3}{4})$, *and hence* $\mathrm{Homeo}(\widetilde{Q})$ *is one-dimensional.*

Proof. The map $i\colon \mathrm{Homeo}(I) \to \mathrm{Homeo}(Q)$, defined by $i(h)(x_0, x_1, \dots) = (h(x_0), x_1, \dots)$, is an isometric embedding. □

It is an open problem whether every Polish group is a homomorphic image of a zero-dimensional Polish group. However, we have the following interesting corollary, again pointed out by Kechris:

Corollary 2.4.11. *Every Polish group is a factor of an almost zero-dimensional Polishable subgroup of* S_∞.

Proof. Let K be a Polish group. It is well known that $\mathrm{Homeo}(Q)$ is a universal Polish group (see Uspenskiĭ [73]), hence there exists an embedding $i\colon K \to \mathrm{Homeo}(Q)$ onto a closed subgroup of $\mathrm{Homeo}(Q)$. Let $q\colon \mathrm{Homeo}(\widetilde{Q}) \to \mathrm{Homeo}(Q)$ be the quotient map. Then $q^{-1}(i(K))$ is a closed subgroup of $\mathrm{Homeo}(\widetilde{Q})$ and $q^{-1}(i(K))/H \cong K$. □

Remark. Corollary 2.4.11 is false if we restrict ourselves to closed subgroups of S_∞. In fact, using the characterization that the closed subgroups of S_∞ are exactly the Polish groups which admit a basis at the identity consisting of open subgroups (see Becker–Kechris [2, Theorem 1.5.1]), it is not hard to show that any factor of a closed subgroup of S_∞ is isomorphic to a closed subgroup of S_∞.

2.5 Polishable ideals on N

Recall that an ideal on **N** is a collection of subsets of **N** closed under finite unions and taking subsets. To avoid trivialities, we will also assume that every ideal contains the ideal of finite sets Fin. An ideal is called Polishable if it is a Polishable subgroup of the Cantor group $2^{\mathbf{N}}$ (with symmetric difference as the group operation). A *lower semi-continuous (or lsc) submeasure* on **N** is a function $\phi\colon \mathcal{P}(\mathbf{N}) \to [0, \infty]$, satisfying

- $\phi(\emptyset) = 0$;

- $a \subseteq b \implies \phi(a) \leq \phi(b)$ for any $a, b \subseteq \mathbf{N}$;
- $\phi(a \cup b) \leq \phi(a) + \phi(b)$; $\phi(\{n\}) < \infty$ for $n \in \mathbf{N}$;
- $\phi(\bigcup_k a_k) = \lim_k \phi(a_k)$, whenever $a_0 \subseteq a_1 \subseteq \cdots$.

With every lsc submeasure we associate the following two ideals:

$$\mathrm{Exh}(\phi) = \{a \subseteq \mathbf{N} : \lim_n \phi(a \setminus n) = 0\} \text{ and } \mathrm{Fin}(\phi) = \{a \subseteq \mathbf{N} : \phi(a) < \infty\}.$$

(As is customary, we identify the natural number n with the set of its predecessors.) It is easy to see that $\mathrm{Exh}(\phi) \subseteq \mathrm{Fin}(\phi)$ and $\mathrm{Fin}(\phi)$ is $\mathbf{\Sigma}^0_2$, while $\mathrm{Exh}(\phi)$ is $\mathbf{\Pi}^0_3$ in $2^{\mathbf{N}}$. Since the ideals $\mathrm{Exh}(\phi)$ and $\mathrm{Fin}(\phi)$ do not change if we replace ϕ with the submeasure ϕ', $\phi'(a) = \phi(a) + \sum_{n \in a} 2^{-n}$, we can restrict our considerations to submeasures ϕ satisfying $\phi(\{n\}) > 0$ for all n. An ideal I is called a *P-ideal* if for every sequence $\{a_n\}$ of elements of I there exists $a \in I$, such that $a_n \setminus a$ is finite for all n. The following is a summary of the results of Solecki [66, 67] which we shall need.

Theorem 2.5.1 (Solecki)**.** *An ideal I is an analytic P-ideal iff it is Polishable iff there exists a finite, lsc submeasure ϕ with $I = \mathrm{Exh}(\phi)$. I is $\mathbf{\Sigma}^0_2$ Polishable iff there exists a lsc ϕ with $I = \mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$.*

If $I = \mathrm{Exh}(\phi)$, then the Polish topology on I is induced by the metric $d(a, b) = \phi(a \bigtriangleup b)$, where $\bigtriangleup$ denotes the operation of symmetric difference.

We say that two ideals I and J are *isomorphic* if there exists a permutation $f : \mathbf{N} \to \mathbf{N}$, such that $a \in I \iff f(a) \in J$. We denote the trivial ideal $\mathcal{P}(\mathbf{N})$ simply by $\mathbf{N}$. If I and J are ideals on $\mathbf{N}$ then $I \oplus J$ is the ideal on $\mathbf{N} \times 2$ defined by

$$I \oplus J = \{a \times \{0\} \cup b \times \{1\} : a \in I \text{ and } b \in J\}.$$

An ideal is a *trivial modification of* Fin if it is of the form $\{a : a \cap b \text{ is finite}\}$ for some $b \subseteq \mathbf{N}$. If an ideal I is Polishable, we will denote the topological space I with its Polish topology by I^τ. Since every Polishable ideal is $\mathbf{\Pi}^0_3$, Proposition 2.4.6 implies that I^τ is almost zero-dimensional, as witnessed by the topology inherited from the compact group $2^{\mathbf{N}}$. It is also easy to check that $(I \oplus J)^\tau$ is homeomorphic to $I^\tau \times J^\tau$.

We associate with each Polishable ideal I the subgroup $S_I \leq S_\infty$ defined by

$$S_I = \{f \in S_\infty : \operatorname{supp} f \in I\}.$$

Below we will make use of a lemma which can be proved in the same way as the fact that all automorphisms of S_∞ are inner. A very detailed exposition can be found in Lorch [50].

Lemma 2.5.2. *If G_1 and G_2 are isomorphic subgroups of S_∞, both containing all permutations with finite support, then they are conjugate, i.e., there exists $f \in S_\infty$, such that $G_2 = f^{-1}G_1 f$.*

Theorem 2.5.3. *With the above definition, S_I is a Polishable subgroup of S_∞, which in its Polish topology is almost zero-dimensional. It is zero-dimensional iff I^τ is zero-dimensional. The Borel complexity of S_I in S_∞ and I in $2^{\mathbf{N}}$ is the same. Furthermore, the groups S_I and S_J are isomorphic (algebraically) iff I and J are isomorphic ideals.*

Proof. Let ϕ be a lsc submeasure, such that $I = \mathrm{Exh}(\phi)$. S_I acts on I in a natural way: $g \cdot a = \{g(n) : n \in a\}$. The first thing we will check is that this action is continuous in the second variable, i.e.,

$$\forall g \in S_I\, \forall \epsilon\, \exists \delta\, \forall a \in I \quad \phi(a) < \delta \implies \phi(g \cdot a) < \epsilon. \tag{2.5.1}$$

(Continuity at $\varnothing$ is sufficient because $g\cdot(a \triangle b) = (g\cdot a) \triangle (g\cdot b)$.) Fix $g \in S_I$ and $\epsilon > 0$. Find $N \in \mathbf{N}$, such that $\phi(\operatorname{supp} g \cap [N, \infty)) < \epsilon/2$ and $\delta < \epsilon/2$ so small that $\phi(a) < \delta \implies a \cap g^{-1} \cdot [0, N) = \varnothing$. Now for any $a \in I$ with $\phi(a) < \delta$, we have

$$\phi(g \cdot a) \le \phi(a \cup (\operatorname{supp} g \cap [N, \infty))) \le \phi(a) + \phi(\operatorname{supp} g \cap [N, \infty)) < \epsilon.$$

Define the left invariant metric ∂' on S_I by $\partial'(f, g) = \phi(\{f \neq g\})$. It is clear that every open ball in this metric is Borel in S_∞. We next check that multiplication is continuous. Fix $f_0, g_0 \in S_I$ and $\epsilon > 0$. Using (2.5.1), find $\delta < \epsilon/2$ so small that $\phi(a) < \delta \implies \phi(g_0^{-1} \cdot a) < \epsilon/2$. Now for any $f, g \in S_I$ with $\max(\partial'(g, g_0), \partial'(f, f_0)) < \delta$, we have

$$\begin{aligned}\partial'(fg, f_0 g_0) &= \phi(\{fg \neq f_0 g_0\}) \le \phi(\{g \neq g_0\}) + \phi(\{f g_0 \neq f_0 g_0\}) \\ &= \partial'(g, g_0) + \phi(g_0^{-1} \cdot \{f \neq f_0\}) \le \epsilon/2 + \epsilon/2 = \epsilon.\end{aligned}$$

The map $f \mapsto f^{-1}$ is continuous because the metric is left invariant and multiplication is continuous. The next thing we show is that the metric $\partial(f, g) = \partial'(f, g) + \partial'(f^{-1}, g^{-1})$ is complete. Let $\{f_n\}$ be a Cauchy sequence in this metric. Without loss of generality, we can assume that $\phi(\{n\}) \ge 2^{-n}$, so ∂ dominates the standard complete metric on S_∞ (2.1.1). Therefore the pointwise limit $g = \lim_n f_n$ exists. We check that $g \in S_I$. Fix $\epsilon > 0$ and $N \in \mathbf{N}$, such that $\forall m, n > N\ \phi(\{f_m \neq f_n\}) < \epsilon/4$. Fix

$m > N$. Let $M \in \mathbf{N}$ be so big that $\phi(\operatorname{supp} f_m \setminus M) < \epsilon/4$. Suppose, towards a contradiction, that $\phi(\operatorname{supp} g \setminus M) > \epsilon$. Then there is $M_1 > M$, such that $\phi(\operatorname{supp} g \cap [M, M_1]) > \epsilon/2$. Find $k > N$, such that f_k agrees with g on $[0, M_1]$. Then $\phi(\operatorname{supp} f_k \setminus M) > \epsilon/2$, which contradicts the choice of N and M. Therefore $\operatorname{supp} g \in \operatorname{Exh}(\phi) = I$. Now it remains to check that $\phi(\{f_n \neq g\}) \to 0$. Again fix an ϵ and find N, such that $\forall m, n > N \ \phi(\{f_m \neq f_n\}) < \epsilon/2$. Fix $m > N$. Let M be such that $\phi(\{f_m \neq g\} \setminus M) < \epsilon/2$. Find $n > m$, such that f_n and g agree on M. Then

$$\phi(\{f_m \neq g\}) \leq \epsilon/2 + \phi(\{f_m \neq g\} \cap M) = \epsilon/2 + \phi(\{f_m \neq f_n\} \cap M) < \epsilon.$$

Finally, the topology defined by ∂ is separable because the group of permutations with finite support is dense in S_I (since $\forall f \in S_I \ \phi(\operatorname{supp} f \setminus n) \to 0$ and thus permutations in S_I can be approximated in the metric ∂' by permutations of finite support). This completes the proof that S_I is Polishable.

If $I = \mathrm{Fin}$ or $I = \mathbf{N}$, the remaining statements are clear. Suppose now that this is not the case and let $b \notin I$ be an infinite set, such that $\mathbf{N} \setminus b$ is infinite and in I. Fix a bijection h between b and $\mathbf{N} \setminus b$. Let $I' = I|_b = \{a \cap b : a \in I\} = \operatorname{Exh}(\phi|_b)$. Then I' is Polishable and $I \cong I' \oplus \mathbf{N}$. Let $p\colon 2^{\mathbf{N}} = 2^b \times 2^{\mathbf{N} \setminus b} \to 2^b$ be the projection and consider the continuous maps $\Phi\colon S_\infty \to 2^{\mathbf{N}}$ and $\Psi\colon 2^b \to S_\infty$ defined by

$$\Phi(f) = \operatorname{supp} f \quad \text{and} \quad \Psi(a) = \prod_{n \in a} (n \ h(n)).$$

By the definition of S_I, $f \in S_I \iff \Phi(f) \in I$. Furthermore, for $a \in 2^b$, $\operatorname{supp} \Psi(a) = a \cup h(a)$ and hence

$$a \in I \iff p(a) \in I' \iff \Psi(p(a)) \in S_I.$$

Those reductions prove the statement about the Borel complexity of I and S_I. The fact that S_I is $\mathbf{\Pi}^0_3$, together with Proposition 2.4.6, imply that the Polish topology of S_I is almost zero-dimensional.

Let now I^τ be zero-dimensional and $\{U_k\}$ be a clopen basis at $\varnothing$. Then $\{\Phi^{-1}(U_k)\}$ is a clopen basis for S_I at 1. Conversely, if S_I is zero-dimensional, notice that $\Psi(I') = \Psi(2^b) \cap S_I$ is a closed subgroup of S_I and hence the group homomorphism $\Psi|_{I'}\colon I' \to S_I$ is a homeomorphic embedding $I'^\tau \hookrightarrow S_I$. Therefore I'^τ is zero-dimensional and since $I^\tau \cong I'^\tau \times 2^{\mathbf{N}}$, I^τ is also zero-dimensional.

The last statement is a direct consequence of Lemma 2.5.2. □

Finally, we use our methods from the proof of Theorem 2.4.8 to sketch an alternative proof of the following fact, due to Solecki:

Proposition 2.5.4 (Solecki [69]). *For a $\mathbf{\Sigma}^0_2$ Polishable ideal I, the following are equivalent:*

(i) I^τ *is zero-dimensional;*

(ii) I *is a trivial modification of* Fin.

Proof. (ii) ⇒ (i). Let $I = \{a : a \cap b \text{ is finite}\}$ for some $b \subseteq \mathbf{N}$. If b is finite, then $I = \mathbf{N}$. If b is co-finite, then $I = \mathrm{Fin}$. Finally, if b is infinite and co-infinite, $I = \mathbf{N} \oplus \mathrm{Fin}$ and $I^\tau \cong \mathrm{Fin}^\tau \times \mathbf{N}^\tau \cong \mathbf{N} \times 2^{\mathbf{N}}$ is zero-dimensional.

(i) ⇒ (ii). Use Theorem 2.5.1 to find a lsc submeasure ϕ, such that $I = \mathrm{Exh}(\phi) = \mathrm{Fin}(\phi)$. Suppose, towards a contradiction, that I^τ is zero-dimensional but (ii) is not satisfied. Then it is not hard to see that

$$\forall \epsilon > 0 \quad \{n : \phi(\{n\}) < \epsilon\} \notin I. \tag{2.5.2}$$

Indeed, if not, find $\epsilon > 0$ with $\{n : \phi(\{n\}) < \epsilon\} \in I$ and set $b = \{n : \phi(\{n\}) \geq \epsilon\}$. Then $I = \{a : a \cap b \text{ is finite}\}$, a contradiction. Let $U \subseteq \{a : \phi(a) < 1\}$ be clopen. We will construct inductively a transfinite sequence $\{a_\alpha\}_{\alpha<\omega_1}$ of elements of U, satisfying $\alpha < \beta \implies a_\alpha \subsetneq a_\beta$, thus obtaining the desired contradiction. Start with $a_0 = \emptyset$. At successor steps, given a_β, use the openness of U and (2.5.2) to find $n \notin a_\beta$, such that $a_\beta \cup \{n\} \in U$ and set $a_{\beta+1} = a_\beta \cup \{n\}$. At a limit α set $a_\alpha = \bigcup_{\beta<\alpha} a_\beta$ and $\lim_n a_{\gamma_n} = a_\alpha$ in the Polish topology of I for any sequence $\{\gamma_n\}$ cofinal in α (use $I = \mathrm{Exh}(\phi)$ here). Hence $a_\alpha \in U$. □

Chapter 3

Amenable Actions and Almost Invariant Sets

3.1 Introduction

Let X be a countable set and Γ a countable, infinite group acting on X. Let M be a standard Borel space and ν an arbitrary Borel probability measure on M which does not concentrate on a single point. Consider the measure space (M^X, ν^X) where ν^X stands for the product measure (which we will also denote by μ). The action of Γ on X gives rise to an action on M^X (called a *generalized Bernoulli shift*) by measure-preserving transformations:

$$(\gamma \cdot c)(x) = c(\gamma^{-1} \cdot x), \quad \text{for } c \in M^X.$$

The classical Bernoulli shifts are obtained by letting $\mathbf{Z}$ act on itself by translation.

There are natural connections between many properties of the action of Γ on X and ergodic theoretic properties of the corresponding Bernoulli shift. We summarize some of those in Section 3.2. In studying generalized Bernoulli shifts, it is often useful to consider the unitary representations of Γ arising from the actions, namely the representation λ_X on $\ell^2(X)$ given by

$$(\lambda_X(\gamma) \cdot f)(x) = f(\gamma^{-1} \cdot x), \quad \text{for } f \in \ell^2(X),$$

and the *Koopman representation* κ on $L^2(M^X, \mu)$ given by

$$(\kappa(\gamma) \cdot f)(c) = f(\gamma^{-1} \cdot c), \quad \text{for } f \in L^2(M^X, \mu).$$

Since the representation κ trivially fixes the constants, we will often also consider its restriction κ_0 to $L^2_0(M^X, \mu) = \{f \in L^2 : \int f = 0\}$. We recall some basic definitions about unitary representations. Let π, σ be representations of a countable group Γ. If σ is isomorphic to a subrepresentation of π, we write $\sigma \leq \pi$. In particular, if $\sigma = 1_\Gamma$, the trivial (one-dimensional) representation of Γ, and $\sigma \leq \pi$, we say that π *has invariant vectors*. If $Q \subseteq \Gamma$ is finite, and $\epsilon > 0$, we say that a *unit* vector $v \in H$ is (Q, ϵ, π)-*invariant* if

$$\forall \gamma \in Q \quad \|\pi(\gamma) \cdot v - v\| < \epsilon.$$

If for all pairs (Q, ϵ), there exists a (Q, ϵ, π)-invariant vector, we say that π *has almost invariant vectors* and write $1_\Gamma \prec \pi$.

Recall that the action of Γ on X is called *amenable* if there exists a Γ-invariant mean on $\ell^\infty(X)$. The action is said to satisfy the *Følner condition* if for all finite $Q \subseteq \Gamma$ and all $\epsilon > 0$, there exists a finite $F \subseteq X$ such that

$$\forall \gamma \in Q \quad |F \triangle \gamma \cdot F| < \epsilon |F|. \tag{3.1.1}$$

The following equivalences are well known and can be proved in exactly the same way as the corresponding ones for amenability of groups (see, for example, Bekka–de la Harpe–Valette [4]).

Theorem 3.1.1. *The following are equivalent for an action of* Γ *on X:*

(i) the action is amenable;

(ii) the action satisfies the Følner condition;

(iii) $1_\Gamma \prec \lambda_X$.

Clearly, all actions of amenable groups are amenable and if an action has a finite orbit, it is automatically amenable. There are also non-amenable groups which admit amenable actions with infinite orbits. Important examples are the non-amenable, inner amenable groups with infinite conjugacy classes (consider the action of Γ on $\Gamma \setminus \{1\}$ by conjugation; see Bédos–de la Harpe [3] for definitions and examples). Interestingly, free groups also admit transitive, faithful, amenable actions (van Douwen [75]). Y. Glasner and N. Monod in a recent paper [28] study the class of groups which admit transitive, faithful, amenable actions and give some history, references, and further examples. Grigorchuk–Nekrashevych [29] describe yet another example of faithful, transitive, amenable actions of free groups. On the other hand, every amenable action of a group with Kazhdan's property (T) has a finite orbit.

An action of a countable group Γ on a measure space (Y, μ) by measure preserving transformations *has almost invariant sets* if there is a sequence $\{A_n\}$ of measurable sets with measures bounded away from 0 and 1 such that for all $\gamma \in \Gamma$,

$$\mu(\gamma \cdot A_n \vartriangle A_n) \to 0 \text{ as } n \to \infty.$$

It is easy to see that the existence of almost invariant sets implies the existence of almost invariant vectors for the Koopman representation κ_0 (look at the characteristic functions) but the converse may fail, as was first proved by Schmidt [63] (for another example, see Hjorth–Kechris [32, Theorem A3.2]). In fact, the existence of almost invariant sets depends only on the orbit equivalence relation which, in the ergodic case, is equivalent to non E_0-ergodicity (Jones–Schmidt [35]), while the existence of almost invariant vectors depends on the group action (see [32] again). Recall that E_0 is the equivalence relation on $2^{\mathbf{N}}$ defined by

$$(x_n)\, E_0\, (y_n) \iff \exists m \forall n > m\ x_n = y_n.$$

An equivalence relation E on a measure space (Y, μ) is E_0*-ergodic* if for every Borel map $f\colon Y \to 2^{\mathbf{N}}$ which satisfies

$$x\, E\, y \implies f(x)\, E_0\, f(y),$$

there is a single E_0 equivalence class whose preimage is μ-conull. For a discussion on E_0-ergodicity and the related concepts of almost invariant vectors and sets, see [32, Appendix A].

Now we can state the main theorem of this paper which connects the amenability of the action of Γ on X and the existence of almost invariant sets for the corresponding Bernoulli shift and almost invariant vectors for the Koopman representation:

Theorem 3.1.2. *Let an infinite, countable group Γ act on a countable set X. The following are equivalent:*

(i) the action of Γ on X is amenable;

(ii) the action of Γ on M^X has almost invariant sets;

(iii) the Koopman representation κ_0 has almost invariant vectors.

This result has an implication concerning orbit equivalence. Schmidt [63] showed that every non-amenable group Γ that does not have property (T) has at least two non-orbit equivalent, ergodic actions (this was extended later by Hjorth [31] to all non-amenable groups). The preceding result shows that if Γ is non-amenable but admits an action on X which is amenable and has infinite orbits (this class of groups is a subclass of non-property (T) groups), then one in fact has two ergodic, free a.e. generalized shifts which are not orbit equivalent: the generalized shift on 2^X and the usual shift on 2^Γ (ergodicity follows from Proposition 3.2.1 below and freeness can easily be achieved by adding an additional orbit to X, see Proposition 3.2.4). For example, for non-amenable, inner amenable groups Γ, the usual shift on 2^Γ and the conjugacy shift on $2^{\Gamma \setminus \{1\}}$ are not orbit equivalent. Also any non-abelian free group admits two non-orbit equivalent free, ergodic generalized shifts.

Since in most cases the existence of almost invariant vectors is easier to check than the existence of almost invariant sets, it will be interesting to know whether there are other cases in which the two concepts coincide. A relatively broad class of examples of measure-preserving actions, studied by several authors (see the monograph Schmidt [64] for discussion and references and also Kechris [40]), consists of the actions by automorphisms on compact Polish groups (equipped with the Haar measure). The generalized Bernoulli shifts with a homogeneous base space M also fall into that class.

Question 3.1.3. Let Γ act on a compact Polish group G by automorphisms (which necessarily preserve the Haar measure). Is it true that the action has almost invariant sets iff the corresponding Koopman representation κ_0 has almost invariant vectors?

The rest of the paper is organized as follows: in Section 3.2, we recall some necessary and sufficient conditions for a Bernoulli shift to be ergodic, mixing, etc.; in Section 3.3, we carry out a detailed spectral analysis of the Koopman representation of generalized Bernoulli shifts and prove a few preliminary lemmas; and finally, in Section 3.4, we give a proof of Theorem 3.1.2.

Below Γ and G will always be countable, infinite groups and Q will denote a *finite* subset of the group.

3.2 Group actions and generalized shifts

In this section, we record several known facts which characterize when a generalized Bernoulli shift is ergodic, weakly mixing, mixing, or free a.e.

Proposition 3.2.1. *The following are equivalent:*

(i) the action of Γ *on* M^X *is ergodic;*

(ii) the action of Γ *on* M^X *is weakly mixing;*

(iii) the action of Γ *on* X *has infinite orbits;*

(iv) $1_\Gamma \not\leq \lambda_X$.

Proof. We shall need the following standard lemma from group theory (for a proof, see, e.g., [40, Lemma 4.4]):

Lemma 3.2.2 (Neumann)**.** *Let* Γ *be a group acting on a set* X*. Then the following are equivalent:*

(a) all orbits are infinite;

(b) for all finite $F_1, F_2 \subseteq X$*, there exists* $\gamma \in \Gamma$ *such that* $\gamma \cdot F_1 \cap F_2 = \varnothing$*.*

(i) $\Rightarrow$ (iii) Suppose that there is a finite orbit $F \subseteq X$. Let $A \subseteq M$, $0 < \nu(A) < 1$. Then the set $\{c \in M^X : c(F) \subseteq A\}$ is non-trivial and invariant under the action.

(iii) $\Rightarrow$ (ii) It suffices to show that the diagonal action of Γ on $M^X \times M^X$ is ergodic. This action is the same as the Bernoulli shift corresponding to the disjoint sum of the action of Γ on X with itself. The latter action has infinite orbits by (iii). Suppose $A \subseteq M^{X \sqcup X}$ is invariant and $0 < \mu(A) < 1$. Then we can find $A' \subseteq M^{X \sqcup X}$ depending only on a finite set of coordinates $F \subseteq X \sqcup X$ such that $\mu(A' \bigtriangleup A) < \epsilon/3$ and $\mu(A') - \mu(A')^2 > \epsilon$ for some $\epsilon > 0$. By Lemma 3.2.2, there is $\gamma \in \Gamma$ such that $\gamma \cdot F \cap F = \varnothing$. By independence, $\mu(A' \cap \gamma \cdot A') = \mu(A')^2$. On the other hand,

$$\mu(A' \cap \gamma \cdot A') \geq \mu(A') - 3\mu(A \bigtriangleup A') > \mu(A') - \epsilon,$$

a contradiction.

(ii) $\Rightarrow$ (i) and (iii) $\Leftrightarrow$ (iv) are obvious. ☐

Recall that π is called a c_0*-representation* if for all $v \in H_\pi$, $\lim_{\gamma \to \infty} \langle \pi(\gamma) \cdot v, v \rangle = 0$.

Proposition 3.2.3. *The following are equivalent:*

(i) the action of Γ *on* M^X *is mixing;*

(ii) κ_0 *is a* c_0*-representation;*

(iii) λ_X *is a* c_0*-representation;*

(iv) the stabilizers $\Gamma_x = \{\gamma \in \Gamma : \gamma \cdot x = x\}$ *for* $x \in X$ *are finite.*

Proof. (ii) ⇒ (iv) Let $A \subseteq M$, $0 < \nu(A) < 1$. Suppose Γ_x is infinite for some x and consider the set $B = \{c \in M^X : c(x) \in A\}$. Then $0 < \mu(B) < 1$ and $\gamma \cdot B = B$ for infinitely many γ so the shift is not mixing.

(iv) ⇒ (ii) It suffices to show that the mixing condition is satisfied for sets $A, B \subseteq M^X$ depending only on finitely many coordinates. Let $F_1, F_2 \subseteq X$ be finite, A depend on F_1, and B depend on F_2. By (iv), there are only finitely $\gamma \in \Gamma$ for which $\gamma \cdot F_1 \cap F_2 \neq \varnothing$, hence

$$\lim_{\gamma \to \infty} \mu(\gamma \cdot A \cap B) = \mu(A)\mu(B)$$

and we are done.

Finally, the equivalences (i) ⇔ (ii) and (iii) ⇔ (iv) are easy to prove. □

Proposition 3.2.4. *If the measure ν has atoms, the following are equivalent:*

(i) the action of Γ on M^X is free a.e.;

(ii) for each $\gamma \in \Gamma \setminus \{1\}$, the set $\{x \in X : \gamma \cdot x \neq x\}$ is infinite.

If ν is non-atomic, (i) *is equivalent to*

(iii) the action of Γ on X is faithful.

Proof. Suppose first that ν has an atom $a \in M$. If for some $\gamma \neq 1$ the set $H_\gamma = \{x : \gamma \cdot x \neq x\}$ is finite, then

$$\mu(\gamma \cdot c = c) \geq \mu(\forall x \in H_\gamma \ c(x) = a) = \nu(\{a\})^{|H_\gamma|} > 0,$$

so the action of Γ on M^X is not free a.e.

Conversely, if H_γ is infinite for all $\gamma \neq 1$, find infinite sets $Y_\gamma \subseteq X$ such that $\gamma \cdot Y_\gamma \cap Y_\gamma = \varnothing$. Then

$$\begin{aligned} \mu(\gamma \cdot c = c) &\leq \mu\big(\forall x \in Y_\gamma \ c(x) = c(\gamma^{-1} \cdot x)\big) \\ &= \prod_{x \in Y_\gamma} \mu\big(c(x) = c(\gamma^{-1} \cdot x)\big) = 0. \end{aligned}$$

If the action of Γ on X is not faithful, then the action on M^X is not faithful either, so in particular it is not free. Conversely, if ν is non-atomic and $\gamma \cdot x \neq x$ for some $x \in X$,

$$\mu(\gamma \cdot c = c) \leq \mu\big(c(x) = c(\gamma^{-1} \cdot x)\big) = 0.$$

□

3.3 Spectral analysis of the Koopman representation

For each subgroup $\Delta \le \Gamma$, we have the *quasi-regular representation* $\lambda_{\Gamma/\Delta}$ on $\ell^2(\Gamma/\Delta)$ given by

$$(\lambda_{\Gamma/\Delta}(\gamma) \cdot f)(\delta\Delta) = f(\gamma^{-1}\delta\Delta).$$

Notice that if S is a transversal for the action of Γ on X (i.e., $S \subseteq X$ and S intersects each orbit in exactly one point), then

$$\lambda_X \cong \bigoplus\nolimits_{x \in S} \lambda_{\Gamma/\Gamma_x}, \tag{3.3.1}$$

where Γ_x denotes the stabilizer of the point x. The first aim of this section is to verify that κ is also equivalent to a sum of quasi-regular representations. This is well-known but the authors were unable to find a specific reference.

Let $\{f_i : i \in I\}$ be a (finite or countably infinite) orthonormal basis for $L^2(M, \nu)$ such that $f_{i_0} \equiv 1$ for some $i_0 \in I$. Set $I_0 = I \setminus \{i_0\}$ and notice that since ν does not concentrate on a single point, $I_0 \neq \varnothing$. For a function $q\colon X \to I$, write

$$\operatorname{supp} q = q^{-1}(I_0)$$

and let $\mathcal{A} = \{q : |\operatorname{supp} q| < \infty\}$. For $q \in \mathcal{A}$, define $h_q \in L^2(M^X, \mu)$ by

$$h_q(c) = \prod_{x \in X} f_{q(x)}(c(x)).$$

Lemma 3.3.1. *The collection $\{h_q : q \in \mathcal{A}\}$ forms an orthonormal basis for $L^2(M^X)$.*

Proof. First we check that $\|h_q\| = 1$. Indeed,

$$\begin{aligned}
\|h_q\|^2 = \langle h_q, h_q \rangle &= \int |h_q|^2 \, \mathrm{d}\mu \\
&= \prod_{x \in X} \int |f_{q(x)}(z)|^2 \, \mathrm{d}\nu(z) \\
&= \prod_{x \in X} \|f_{q(x)}\|^2 \\
&= 1.
\end{aligned}$$

Now suppose $q_1 \neq q_2$. Then

$$\begin{aligned}\langle h_{q_1}, h_{q_2} \rangle &= \int h_{q_1} \overline{h_{q_2}} \, \mathrm{d}\mu \\ &= \prod_{x \in X} \int f_{q_1(x)}(z) \overline{f_{q_2(x)}(z)} \, \mathrm{d}\nu(z) \\ &= 0\end{aligned}$$

because $\int f_{q_1(x_0)}(z)\overline{f_{q_2(x_0)}(z)}\,\mathrm{d}\nu(z) = \langle f_{q_1(x_0)}, f_{q_2(x_0)} \rangle = 0$ for some x_0 for which $q_1(x_0) \neq q_2(x_0)$.

Finally, we verify that the h_qs are total in $L^2(M^X)$. Let $\mathcal{F}$ be the measure algebra of M^X. Fix an exhausting sequence $F_1 \subseteq F_2 \subseteq \cdots$ of finite subsets of X and denote by $\mathcal{F}_n$ the σ-subalgebra of $\mathcal{F}$ generated by the projections $\{p_x : x \in F_n\}$. Notice that $L^2(M^X, \mathcal{F}_n)$ is canonically isomorphic to $L^2(M^{F_n})$ which, in turn, is canonically isomorphic to $\bigotimes_{x \in F_n} L^2(M)$. Under this isomorphism, a function h_q with $\operatorname{supp} q \subseteq F_n$ corresponds to the tensor $\bigotimes_{x \in F_n} f_{q(x)}$. Hence $\{h_q : \operatorname{supp} q \subseteq F_n\}$ is total in $L^2(M^X, \mathcal{F}_n)$. But $\bigcup_n \mathcal{F}_n$ generates $\mathcal{F}$, so $\bigcup_n L^2(M^X, \mathcal{F}_n)$ is dense in $L^2(M^X)$ and we are done. □

Notice that Γ acts on $\mathcal{A}$ in a natural way:

$$(\gamma \cdot q)(x) = q(\gamma^{-1} \cdot x).$$

This action induces a representation on $L^2(M^X)$ (by permuting the basis $\{h_q : q \in \mathcal{A}\}$) and clearly this representation is equal to κ. Let now T be a transversal for the action of Γ on $\mathcal{A}$ (i.e., $T \subseteq \mathcal{A}$ and T intersects each orbit in exactly one point). Let for each $q \in \mathcal{A}$, Γ_q denote the stabilizer of q. The preceding discussion implies that

$$\kappa \cong \bigoplus_{q \in T} \lambda_{\Gamma/\Gamma_q}.$$

Notice that the constant function $q_0 \equiv i_0$ is an orbit of the action of Γ on $\mathcal{A}$ consisting of a single element, so $q_0 \in T$. Let $T_0 = T \setminus \{q_0\}$. We have just proved

Proposition 3.3.2.

$$\kappa_0 \cong \bigoplus_{q \in T_0} \lambda_{\Gamma/\Gamma_q}. \tag{3.3.2}$$

We also record a few facts about quasi-regular representations which will be used later.

Lemma 3.3.3. *Let G be a countable group and $K \leq H \leq G$ with $[H : K] < \infty$. Then $\lambda_{G/H} \leq \lambda_{G/K}$.*

Proof. Let $n = [H : K]$ and let $p\colon G/K \to G/H$ be the natural projection. Define the map $\Phi\colon \ell^2(G/H) \to \ell^2(G/K)$ by

$$\Phi(f) = \frac{1}{\sqrt{n}} f \circ p.$$

It is easy to check that Φ is an isometric embedding which intertwines $\lambda_{G/H}$ and $\lambda_{G/K}$. □

Lemma 3.3.4. *Let G be a countable group and $K \leq H \leq G$. Let $Q \subseteq G$, $\epsilon > 0$ and assume there is a $(Q, \epsilon, \lambda_{G/K})$-invariant vector. Then there exists a $(Q, \epsilon, \lambda_{G/H})$-invariant vector.*

Proof. Let $v \in \ell^2(G/K)$ be $(Q, \epsilon, \lambda_{G/K})$-invariant. By considering $|v|$ instead of v, we can assume that $v \geq 0$ ($|v|$ is $(Q, \epsilon, \lambda_{G/K})$-invariant by the triangle inequality). Define $w \in \ell^2(G/H)$ by

$$w(D) = \sqrt{\sum_{C \subseteq D} v^2(C)}, \quad D \in G/H$$

where C runs over elements of G/K. We have

$$\|w\|^2 = \sum_{D \in G/H} \sum_{C \subseteq D} v^2(C) = \sum_{C \in G/K} v^2(C) = \|v\|^2 = 1.$$

Furthermore, for each $\gamma \in Q$,

$$\begin{aligned}
\langle \gamma \cdot w, w \rangle &= \sum_{D \in G/H} w(\gamma^{-1} D) w(D) \\
&= \sum_{D \in G/H} \sqrt{\sum_{C \subseteq \gamma^{-1} D} v^2(C)} \sqrt{\sum_{C \subseteq D} v^2(C)} \\
&\geq \sum_{D \in G/H} \sum_{C \subseteq D} v(C) v(\gamma^{-1} C), \quad \text{by Cauchy-Schwartz,} \\
&= \sum_{C \in G/K} v(C) v(\gamma^{-1} C) \\
&= \langle \gamma \cdot v, v \rangle .
\end{aligned}$$

Hence,

$$\begin{aligned}
\|\gamma \cdot w - w\|^2 &= 2\|w\|^2 - 2\langle \gamma \cdot w, w \rangle \\
&\leq 2\|v\|^2 - 2\langle \gamma \cdot v, v \rangle = \|\gamma \cdot v - v\|^2 < \epsilon^2
\end{aligned}$$

and w is $(Q, \epsilon, \lambda_{G/H})$-invariant. □

Lemma 3.3.5. *Let π_i, $i = 1, 2, \ldots$ be unitary representations of a countable group G on the Hilbert spaces H_i. Suppose that $1_G \prec \bigoplus_{i=1}^{\infty} \pi_i$. Then for each $Q \subseteq G$ and $\epsilon > 0$, there exists n and $v_n \in H_n$ which is (Q, ϵ, π_n)-invariant.*

Proof. Fix Q and ϵ. Let $H = \bigoplus_i H_i$. There exists $v \in H$, $v = \bigoplus_i v_i$, such that $\|\pi(\gamma) \cdot v - v\| < \epsilon/\sqrt{m}\,\|v\|$ where $m = |Q|$. We have

$$\begin{aligned} &\left\|\pi(\gamma) \cdot \textstyle\bigoplus_i v_i - \bigoplus_i v_i\right\|^2 < \epsilon^2/m \left\|\textstyle\bigoplus_i v_i\right\|^2 \quad \text{for all } \gamma \in Q, \\ &\sum_{\gamma \in Q} \sum_i \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \sum_i \|v_i\|^2 \\ &\sum_i \sum_{\gamma \in Q} \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \sum_i \epsilon^2 \|v_i\|^2 . \end{aligned}$$

Hence, for some i,

$$\sum_{\gamma \in Q} \|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \|v_i\|^2 ,$$

and in particular, for each $\gamma \in Q$,

$$\|\pi_i(\gamma) \cdot v_i - v_i\|^2 < \epsilon^2 \|v_i\|^2 .$$

□

3.4 Proof of Theorem 3.1.2

We start with the implication (i) ⇒ (ii). We shall need to use the Central Limit Theorem for random variables several times and we find it convenient to employ probabilistic notation. For all necessary background in probability theory, a good reference is Durrett [13]. In this section, we will use P instead of μ to denote the measure on M^X. Recall that a sequence ξ_k of random variables *converges in distribution* to ξ (written as $\xi_k \Rightarrow \xi$) if the distribution measures of ξ_k converge to the distribution measure of ξ in the weak* topology. For this, it is necessary and sufficient that $P(\xi_k \in A) \to P(\xi \in A)$ for every Borel set A for which $P(\xi \in \partial A) = 0$ (∂A denotes the topological boundary of A). The Central Limit Theorem states that if $\{\xi_k\}$ is a sequence of independent, identically distributed random variables with finite mean m and variance σ^2, then

$$\frac{\sum_{i=1}^{k} \xi_i - km}{\sigma\sqrt{k}}$$

converges in distribution to a standard normal random variable (see [13, Theorem 2.4.1]). Recall also that a distribution is *continuous* if the measure associated to it is non-atomic. Finally, a sequence ξ_k *converges in probability* to ξ if for all $\epsilon > 0$, $P(|\xi_k - \xi| > \epsilon) \to 0$ as $k \to \infty$. We need the following two lemmas.

Lemma 3.4.1. *Let $\xi_k, \eta_k, \zeta_k, k = 1, 2, \dots$ be random variables such that $\xi_k \Rightarrow \xi$, where ξ is a random variable with continuous distribution, and η_k, ζ_k converge in probability to* 0. *Then $P(\eta_k \leq \xi_k \leq \zeta_k) \to 0$ as $k \to \infty$.*

Proof. Fix $\epsilon > 0$ and find δ such that $P(|\xi| \leq \delta) < \epsilon$. Find N so big that for $k > N$, $\left|P(|\xi_k| \leq \delta) - P(|\xi| \leq \delta)\right| < \epsilon$, $P(\eta_k < -\delta) < \epsilon$, and $P(\zeta_k > \delta) < \epsilon$. Then, for all $k > N$,

$$P(\eta_k < \xi_k \leq \zeta_k) \leq P(|\xi_k| \leq \delta) + P(\eta_k < -\delta) + P(\zeta_k > \delta) \leq 4\epsilon.$$

□

Lemma 3.4.2. *Let $\xi_k \Rightarrow \xi$, $\alpha_k \in \mathbf{R}$, $\alpha_k \geq 0$, $\alpha_k \to 0$. Then $\alpha_k \xi_k \to 0$ in probability.*

Proof. It suffices to show that for all $\delta > 0$, $P(\alpha_k|\xi_k| > \delta) \to 0$. Fix $\epsilon > 0$. Find a such that $P(|\xi| > a) < \epsilon/2$ and $P(|\xi| = a) = 0$. For all large enough k, we will have $\left|P(|\xi_k| > a) - P(|\xi| > a)\right| < \epsilon/2$ and $\delta/\alpha_k > a$. For all those k (assuming also $\alpha_k > 0$),

$$P(|\xi_k| > \delta/\alpha_k) \leq P(|\xi_k| > a) < P(|\xi| > a) + \epsilon/2 < \epsilon.$$

□

Suppose now that the action of Γ on X is amenable. Without loss of generality, take $M = I = [-1, 1]$ and assume that the measure ν is centered at 0 (i.e., $\int_I x \, \mathrm{d}\nu(x) = 0$). We will find a sequence $\{A_k\}$ of subsets of I^X with measures bounded away from 0 and 1, satisfying for all $\gamma \in \Gamma$,

$$P(\gamma \cdot A_k \bigtriangleup A_k) \to 0 \text{ as } k \to \infty. \tag{3.4.1}$$

Enumerate $\Gamma = \{\gamma_n\}$. By (3.1.1), there exists a sequence $\{F_k\}$ of finite subsets of X satisfying

$$\forall i \leq k \quad \frac{|F_k \bigtriangleup \gamma_i \cdot F_k|}{|F_k|} < 1/k. \tag{3.4.2}$$

For each $x \in X$, let $p_x\colon I^X \to I$ be the corresponding projection function. We view the p_xs as independent, identically distributed, real random variables with distribution given by the measure ν. Note that all of their moments are finite because they are bounded. By our assumptions, the mean $\mathbf{E}\, p_x = 0$. Set $\sigma^2 = \operatorname{Var} p_x = \mathbf{E}\, p_x^2 > 0$. Let $r_k = |F_k|$ and set

$$A_k = \{\sum_{x \in F_k} p_x > 0\}.$$

First suppose that the sequence $\{r_k\}$ is bounded by a number K. Notice that $P(A_k)$ only depends on the number r_k and not on the actual set F_k. Therefore, in this case, we have only finitely many possibilities for $P(A_k)$, so $P(A_k)$ are bounded away from 0 and 1. Also, by (3.4.2), for $k > K$ and $i \le k$, $\gamma_i \cdot F_k = F_k$, hence $\gamma_i \cdot A_k = A_k$ and the sequence $\{A_k\}$ is almost invariant.

Now consider the case when $\{r_k\}$ is unbounded. By taking a subsequence, we can assume that $r_k \to \infty$. We first show that the measures of A_k are bounded away from 0 and 1. Indeed, by the Central Limit Theorem,

$$P(A_k) = P\left(\frac{\sum_{x \in F_k} p_x}{\sqrt{r_k}\sigma} > 0\right) \to P(\chi > 0) = 1/2,$$

where χ denotes a standard normal variable. Next we prove that (a subsequence of) A_k is almost invariant. By taking subsequences, we can assume that for each $\gamma \in \Gamma$, either $\{|\gamma \cdot F_k \mathbin{\triangle} F_k|\}_k$ is bounded, or $|\gamma \cdot F_k \mathbin{\triangle} F_k| \to \infty$. Fix $\gamma \in \Gamma$ and set $n_k = |\gamma \cdot F_k \setminus F_k| = |F_k \setminus \gamma \cdot F_k|$, $N_k = |\gamma \cdot F_k \cap F_k|$. Let

$$\xi_k = \sum_{x \in F_k \cap \gamma \cdot F_k} p_x,$$

$$\eta_k = \sum_{x \in F_k \setminus \gamma \cdot F_k} p_x,$$

$$\zeta_k = \sum_{x \in \gamma \cdot F_k \setminus F_k} p_x.$$

ξ_k, η_k, ζ_k are independent,

$$\mathbf{E}\, \xi_k = \mathbf{E}\, \eta_k = \mathbf{E}\, \zeta_k = 0, \quad \operatorname{Var} \xi_k = N_k \sigma^2, \quad \operatorname{Var} \eta_k = \operatorname{Var} \zeta_k = n_k \sigma^2,$$

and

$$A_k = \{\xi_k + \eta_k > 0\}, \quad \gamma \cdot A_k = \{\xi_k + \zeta_k > 0\}.$$

Suppose first that $\{n_k\}$ is bounded and let K be an upper bound for n_k. Notice that $|\eta_k|, |\zeta_k| \leq K$. We have

$$\begin{aligned} P(A_k \setminus \gamma \cdot A_k) &= P(\xi_k + \eta_k > 0 \text{ and } \xi_k + \zeta_k \leq 0) \\ &\leq P(-K < \xi_k \leq K) \\ &= P(-K/(\sigma\sqrt{N_k}) < \xi_k/(\sigma\sqrt{N_k}) \leq K/(\sigma\sqrt{N_k})). \end{aligned} \tag{3.4.3}$$

By the Central Limit Theorem, $\xi_k/(\sigma\sqrt{N_k}) \Rightarrow \chi$ and clearly $K/(\sigma\sqrt{N_k}) \to 0$. By Lemma 3.4.1, the expression (3.4.3) converges to 0.

Now suppose $n_k \to \infty$. Let $\xi'_k = \xi_k/(\sigma\sqrt{N_k})$, $\eta'_k = \eta_k/(\sigma\sqrt{n_k})$, $\zeta'_k = \zeta_k/(\sigma\sqrt{n_k})$. By the Central Limit Theorem, $\xi'_k \Rightarrow \chi$, $\eta'_k \Rightarrow \chi$, $\zeta'_k \Rightarrow \chi$. We have

$$\begin{aligned} P(A_k \setminus \gamma \cdot A_k) &= P(\xi_k + \eta_k > 0 \text{ and } \xi_k + \zeta_k \leq 0) \\ &= P(\zeta_k \leq -\xi_k < \eta_k) \\ &= P(\sqrt{n_k}\zeta'_k \leq -\sqrt{N_k}\xi'_k < \sqrt{n_k}\eta'_k) \\ &= P\left(\sqrt{\frac{n_k}{N_k}}\zeta'_k \leq -\xi'_k < \sqrt{\frac{n_k}{N_k}}\eta'_k\right). \end{aligned} \tag{3.4.4}$$

By (3.4.2), $\sqrt{n_k/N_k} \to 0$. By Lemma 3.4.2, $\sqrt{n_k/N_k}\zeta'_k, \sqrt{n_k/N_k}\eta'_k \to 0$ in probability. Finally, by Lemma 3.4.1, (3.4.4) converges to 0.

The implication (ii) $\Rightarrow$ (iii) is clear so we proceed to show (iii) $\Rightarrow$ (i). By Theorem 3.1.1, it suffices to show that $1_\Gamma \prec \lambda_X$. Fix $Q \subseteq \Gamma$ and $\epsilon > 0$. We will find a (Q, ϵ, λ_X) invariant vector in $\ell^2(X)$. By (iii), (3.3.2), and Lemma 3.3.5, there exists $q \in \mathcal{A}$ and $v_1 \in \ell^2(\Gamma/\Gamma_q)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_q})$ invariant. Let $F = \operatorname{supp} q$ and notice that since $q \neq q_0$, $F \neq \varnothing$. Denote by Γ_F and $\Gamma_{(F)}$ the setwise and pointwise stabilizers of F, respectively. Since $\Gamma_q \leq \Gamma_F \leq \Gamma$, by Lemma 3.3.4, there exists $v_2 \in \ell^2(\Gamma/\Gamma_F)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_F})$ invariant. Since $\Gamma_{(F)} \leq \Gamma_F \leq \Gamma$ and $[\Gamma_F : \Gamma_{(F)}] < \infty$, by Lemma 3.3.3, there exists $v_3 \in \ell^2(\Gamma/\Gamma_{(F)})$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_{(F)}})$ invariant. Fix $x \in F$. Since $\Gamma_{(F)} \leq \Gamma_x \leq \Gamma$, by Lemma 3.3.4, there exists $v_4 \in \ell^2(\Gamma/\Gamma_x)$ which is $(Q, \epsilon, \lambda_{\Gamma/\Gamma_x})$ invariant. Since by (3.3.1), $\lambda_{\Gamma/\Gamma_x} \leq \lambda_X$, we are done.

Chapter 4

Modular Actions and Amenable Representations

4.1 Introduction

Let Γ be a countable, infinite group which acts in a Borel way on a standard Borel space X. The action gives rise to a Borel orbit equivalence relation E_Γ^X with countable classes. Conversely, every countable Borel equivalence relation is given by a group action (Feldman–Moore [18]). It is of interest to compare equivalence relations arising from different groups and different actions of the same group.

Let E, F be equivalence relations on the spaces X, Y, respectively. A *homomorphism from E to F* is a map $f\colon X \to Y$ such that

$$x\, E\, y \implies f(x)\, F\, f(y)$$

for all $x, y \in X$. A map is *countable-to-one* if the preimage of every point is countable. Countable-to-one homomorphisms occur in different contexts: examples arise from orbit equivalences and stable orbit equivalences as well as Borel reductions between countable equivalence relations. A countable-to-one homomorphism is, in fact, a combination of an inclusion and a Borel reduction (see Thomas [70, Section 4]). In the Borel setting, one is interested in Borel homomorphisms, while in the presence of a measure, one usually considers measurable homomorphisms which are defined only almost everywhere.

Following Hjorth [31], call a Borel group action on a standard Borel space X *modular* if there exists a sequence of countable Borel partitions $\mathcal{A}_1 > \mathcal{A}_2 > \cdots$ of X, each one refining the previous, which separate points in X and are invariant under the action. Note that if there is a Γ-invariant, ergodic measure on X, then, possibly excluding a null set, all partitions are finite and the action on each partition is transitive. It is shown in [41] that, on an invariant set of full measure, every measure-preserving, ergodic, modular action is isomorphic to an action on the boundary of a rooted, locally finite tree induced by an action by automorphisms on the tree or, which is the same, an inverse limit of actions on finite sets. Also, it is not hard to see that a group admits a free, ergodic, modular action iff it is *residually finite* (i.e., the intersection of all of its normal subgroups of finite index is trivial); see [41] again.

Say that a countable Borel equivalence relation is of *modular type* if it is induced by a modular action. Hjorth considered equivalence relations of modular type in order to show that there exist more than two treeable equivalence relations (up to Borel bireducibility), which was an important problem in the theory of Borel equivalence relations. (An equivalence relation is *treeable* if to each equivalence class can be assigned in a Borel way the structure of a tree.) More precisely, he proved the following result:

Theorem 4.1.1 (Hjorth [31])**.** *Let $\Gamma \curvearrowright X$ be a modular action and E be the orbit equivalence relation. Let $M \subseteq 2^{\mathbf{F}_2}$ be a set of full measure (where $\mathbf{F}_2$ is the free group with 2 generators and $2^{\mathbf{F}_2}$ is equipped with the standard Bernoulli measure and the shift action of $\mathbf{F}_2$). Then there does not exist a countable-to-one Borel homomorphism from $E_{\mathbf{F}_2}^{2^{\mathbf{F}_2}}|_M$ to E.*

The above theorem implies that any free, measure-preserving, modular action of $\mathbf{F}_2$ gives rise to an intermediate treeable equivalence relation. (For more on the theory of countable Borel equivalence relations, and in particular the treeable ones, see Jackson–Kechris–Louveau [34].) It is interesting to try to generalize Theorem 4.1.1 to include actions other than $\mathbf{F}_2 \curvearrowright 2^{\mathbf{F}_2}$. Kechris [41] defined the notion of an *antimodular* action (one whose orbit equivalence relation does not admit a countable-to-one homomorphism to an equivalence relation of modular type) and, in the presence of an invariant measure, isolated a representation-theoretic property which implies antimodularity. Since in most of our considerations below, we will have a measure present, we find it convenient to introduce a notion of a.e. antimodularity: we say that a measure-preserving action $\Gamma \curvearrowright (X, \mu)$ is *μ-antimodular* if its restriction to any invariant conull subset of X is antimodular (or, equivalently, for any (not necessarily invariant) conull $A \subseteq X$, the restricted equivalence relation $E_\Gamma^X|_A$ does not admit a countable-to-one homomorphism to an equivalence relation of modular type).

Recall that if Γ acts on X preserving a measure μ, the *Koopman representation* κ of Γ on the Hilbert space $L^2(X, \mu)$ is the unitary representation given by

$$(\kappa(\gamma)f)(x) = f(\gamma^{-1} \cdot x).$$

We will usually consider the restriction κ_0 of κ to the orthogonal complement of the constant functions $L^2_0(X) = \{f \in L^2(X) \mid \int f = 0\}$ and, by abuse of terminology, call it also the Koopman representation. If σ and π are unitary representations of the same group, we write $\sigma \leq \pi$ if σ is contained in π (i.e., is isomorphic to a subrepresentation of π) and $\sigma \prec \pi$ if σ is weakly contained in π. For all necessary background on unitary representations, an excellent reference is Bekka–de la Harpe–Valette [4]. The action of Γ on X is called *tempered* if $\kappa_0 \prec \lambda_\Gamma$, where λ_Γ is the left-regular representation of Γ. Kechris [41] adapted Hjorth's method from [31] to show that if $\mathbf{F}_2 \leq \Gamma$, then every tempered action of Γ is antimodular. He also asked whether the hypotheses of this theorem can be weakened, for example, whether "$\mathbf{F}_2 \leq \Gamma$" can be replaced by "Γ is non-amenable" and "the action is tempered" by "κ_0 does not weakly contain a finite-dimensional representation of Γ." (If Γ is amenable, then by well-known results of Dye and Ornstein–Weiss (see [42]), the orbit equivalence relation is hyperfinite, and therefore induced by a modular action of $\mathbf{Z}$, on a set of measure 1.) In the present paper, we answer those two questions, the first one in the affirmative and the second in the negative (cf. Corollary 4.1.4 and Proposition 4.5.2).

It turns out that another representation-theoretic property of measure-preserving actions is relevant in this situation, namely the property of κ_0 being amenable in the sense of Bekka [5]. We recall the definition and a few basic facts from [5]. A unitary representation π of Γ on a Hilbert space $\mathcal{H}$ is *amenable* if there exists a Γ-invariant state on the C*-algebra $B(\mathcal{H})$ of bounded linear operators on $\mathcal{H}$, i.e., a bounded linear functional M on $B(\mathcal{H})$ satisfying $M \geq 0$, $M(I) = 1$, and

$$M(\pi(\gamma)S\pi(\gamma^{-1})) = M(S)$$

for all $\gamma \in \Gamma$ and $S \in B(\mathcal{H})$. The notion of an amenable representation captures many known instances of amenability in a single framework. For example, a group is amenable iff all of its representations are amenable, an action of a countable group on a countable set I is amenable iff the corresponding representation on $\ell^2(I)$ is amenable (cf. Lemma 4.3.1 and the remark after it), etc. A useful characterization of amenability is the following:

$$\pi \text{ is amenable} \iff 1_\Gamma \prec \pi \otimes \overline{\pi} \tag{4.1.1}$$

[6, Theorem 5.1]. The latter condition is sometimes referred to as the absence of *stable spectral gap*. For more examples and further discussion, see [5]. (In [5], the theory is developed for locally compact groups but we only need the discrete case here.)

Let (X, μ) be a standard Lebesgue space (i.e., a standard Borel space equipped with a non-atomic, Borel, probability measure μ) and Γ a countable group acting by measure-preserving transformations on it. Now consider another (arbitrary) probability space (Y, ν) and a measurable cocycle $\alpha\colon X \times \Gamma \to \mathrm{Aut}(Y)$, where $\mathrm{Aut}(Y)$ denotes the group of measure-preserving automorphisms of Y. α gives rise to a measure-preserving action $\Gamma \curvearrowright^{\alpha} X \times Y$ as follows:

$$\gamma \cdot (x, y) = (\gamma \cdot x, \alpha(x, \gamma) \cdot y).$$

Conversely, by a well-known theorem of Rokhlin, every ergodic extension of the action $\Gamma \curvearrowright X$ arises in this fashion (see [27, 3.3]).

Now we can state the main theorem of this paper.

Theorem 4.1.2. *Let Γ act by measure-preserving transformations on the standard Lebesgue space (X, μ). Let (Y, ν) be an arbitrary probability space and $\alpha\colon X \times \Gamma \to \mathrm{Aut}(Y)$ a measurable cocycle whose image is contained in a countable subgroup of $\mathrm{Aut}(Y)$. Then, if the Koopman representation κ_0 associated with the action $\Gamma \curvearrowright X$ is not amenable, the action $\Gamma \curvearrowright^{\alpha} X \times Y$ is $\mu \times \nu$-antimodular.*

Remark. We do not know whether the condition that the image of α is countable is necessary.

Note the following immediate corollary which is obtained in the case when Y consists of a single point.

Corollary 4.1.3. *Suppose that $\Gamma \curvearrowright (X, \mu)$ is measure-preserving. Then if κ_0 is non-amenable, the equivalence relation E_{Γ}^{X} is μ-antimodular.*

We can apply that to a variety of situations where we know that the Koopman representation is non-amenable and produce examples of antimodular actions. For example, if Γ is non-amenable, then its left-regular representation is not amenable [5, Theorem 2.2] and

$$\rho \text{ is non-amenable and } \pi \prec \rho \implies \pi \text{ is non-amenable} \tag{4.1.2}$$

[5, Corollary 5.3], so we have:

Corollary 4.1.4. *Every tempered action of a non-amenable group is antimodular.*

Recall that an action $\Gamma \curvearrowright (X, \mu)$ is called *weakly mixing* if the Koopman representation κ_0 does not contain finite-dimensional subrepresentations and *mixing* if κ_0 is a c_0*-representation*, i.e.,

$$\langle \kappa_0(\gamma) f, f \rangle \to 0 \quad \text{as} \quad \gamma \to \infty$$

for all $f \in L_0^2(X)$. It is clear that a weakly mixing action cannot be modular and Kechris [41] asked whether weak mixing (or even mixing) always implies antimodularity. One has to exclude amenable groups from consideration, however, since any ergodic action of an amenable group is orbit equivalent to a modular action of $\mathbf{Z}$. We will see in Section 4.5 that in general weak mixing (and even the stronger condition that κ_0 does not *weakly* contain a finite-dimensional representation) does not imply antimodularity. However, such an implication does exist for certain groups and for special actions of arbitrary non-amenable groups as we see below.

If a group has property (T), then all of its amenable representations contain a finite-dimensional subrepresentation (the converse is also true; cf. Bekka–Valette [6]) and hence:

Corollary 4.1.5. *Let* Γ *have property (T). Then every weakly mixing* $\Gamma \curvearrowright (X, \mu)$ *is* μ*-antimodular.*

Recall that a group Γ has the *Haagerup approximation property (HAP)* if it has a c_0-representation π such that $1_\Gamma \prec \pi$. (For more on groups with HAP, see Cherix et al. [8].) Since for any representation π, if π is a c_0-representation, then $\pi \otimes \overline{\pi}$ is also a c_0-representation, using (4.1.1), we obtain:

Corollary 4.1.6. *If* Γ *does not have HAP, every mixing action* $\Gamma \curvearrowright (X, \mu)$ *is* μ*-antimodular.*

A class of actions for which mixing implies antimodularity for arbitrary non-amenable groups is given by the generalized Bernoulli shifts (cf. Corollary 4.3.4). We do not know an example of a mixing action of a non-amenable group which is not antimodular.

Our final application is to the theory of orbit equivalence and Borel reducibility. We use Theorem 4.1.2 to show that every residually finite, non-amenable group admits at least three non-orbit equivalent actions as well as two non-Borel bireducible ones. For general non-amenable groups, it is only known that they admit at least two non-orbit equivalent actions (Schmidt [62], Connes–Weiss [9], Hjorth [31]). For definitions and further discussion, see Section 4.4.

The organization of the paper is as follows. In Section 4.2, we prove Theorem 4.1.2; in Section 4.3, generalized Bernoulli shifts and actions on compact Polish groups by automorphisms are considered; in Section 4.4, we discuss the applications to orbit equivalence and Borel reducibility; and finally, in Section 4.5, we give an example which shows that the hypothesis in Corollary 4.1.3 cannot be replaced by the weaker “κ_0 does not weakly contain a finite-dimensional representation,” answering the previously mentioned question of Kechris.

Below Γ will always be a countable, infinite group and $Q \subseteq \Gamma$ a finite set. All vector spaces will be complex and all representations unitary.

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4.2 Proof of Theorem 4.1.2

We argue towards a contradiction. Suppose that the action $\Gamma \curvearrowright X \times Y$ is not $\mu \times \nu$-antimodular. Suppose also that the image of α is contained in the countable subgroup Λ of $\mathrm{Aut}(Y)$. Let $\mathrm{MALG}(X)$ denote the measure algebra of (X, μ). Let $Z = \mathbf{N} \times Y$ and let σ be the measure on Z which is the product of the counting measure on $\mathbf{N}$ and ν.

The following technical proposition extracts from the combinatorial information given by the existence of a homomorphism to an equivalence relation of modular type the data we need to construct a κ_0-invariant state on $B(L^2_0(X))$.

Proposition 4.2.1. *For every finite $Q \subseteq \Gamma$ and $\epsilon > 0$, there exists a Borel map $\Phi\colon Z \to \mathrm{MALG}(X)$ such that the following are satisfied:*

(i) for almost all $x \in X$,

$$\int_Z \chi_{\{x \in \Phi(z)\}}(z)\, \mathrm{d}\sigma(z) = 1;$$

(ii)

$$\int_{\{\mu(\Phi(z)) > \epsilon\}} \mu(\Phi(z))\, \mathrm{d}\sigma(z) < \epsilon;$$

(iii) for all $\tau \in Q$, there exists $T \in \mathrm{Aut}(Z, \sigma)$ such that

$$\int_Z \mu\big(\Phi(Tz) \ \triangle\ \tau \cdot \Phi(z)\big)\, \mathrm{d}\sigma(z) < \epsilon.$$

To visualize what the proposition claims, it helps to consider the case when Y is a single point. Then condition (i) says that Φ defines a partition of X into countably many pieces, condition (ii) says that all pieces have measure smaller than ϵ, and, finally, condition (iii) says that the partition is “almost invariant” with respect to the pair (Q, ϵ).

Proof. Let Δ be a countable group which acts modularly on a standard Borel space W. We suppose that there is a conull $C \subseteq X \times Y$ and a countable-to-one Borel homomorphism $\theta\colon C \to W$ from E_Γ^C to E_Δ^W. By [41, 1.2], we can assume that θ is injective. Fix a symmetric $Q \subseteq \Gamma$ containing 1 and $1/2 > \epsilon > 0$. Denote by $\mathcal{B}$ the Borel σ-algebra of W and let

$$\mathcal{B}_0 \subseteq \mathcal{B}_1 \subseteq \cdots \subseteq \mathcal{B}_k \subseteq \mathcal{B}_{k+1} \subseteq \cdots \subseteq \mathcal{B}$$

be atomic (finite or countable) Boolean algebras which witness that the action $\Delta \curvearrowright W$ is modular, i.e., each $\mathcal{B}_k$ is invariant under Δ and $\bigcup_k \mathcal{B}_k$ generates $\mathcal{B}$. If $B \in \mathcal{B}$ and $A \subseteq X \times Y$, denote

$$\hat{B} = \theta^{-1}(B) \quad \text{and} \quad A^y = \{x \in X \mid (x,y) \in A\}.$$

Since θ is a Borel homomorphism, there exists a Borel map $g\colon C \times \Gamma \to \Delta$ such that

$$g((x,y),\tau) \cdot \theta(x,y) = \theta(\tau \cdot (x,y)).$$

Lemma 4.2.2. *For any $\eta > 0$, there is a Borel set $M \subseteq C$ and $k \in \mathbf{N}$ such that the following hold:*

(1) $\mu \times \nu(M) > 1 - \eta$;

(2) for any $\tau \in Q$, the functions $(x,y) \mapsto g((x,y),\tau)$ and $(x,y) \mapsto \alpha(x,\tau)$ are constant on $\hat{B} \cap M$ for each atom $B \in \mathcal{B}_k$;

(3) if $(x,y) \in \hat{B} \cap M$ for some atom $B \in \mathcal{B}_k$, then $\mu(\hat{B}^y) < \eta$;

(4) the set $\{B \in \mathcal{B}_k : M \cap \hat{B} \neq \varnothing\}$ is finite.

The lemma and proof are similar to [31, Claim I]. However, we additionally require that the cocycle α be constant on the atoms and M only intersect atoms with vertical sections of sufficiently small measure.

Proof. It suffices to find the required pair (M,k) for a single element $\tau \in Q$. Indeed, since Q is finite, in the end, we can take the intersection of the Ms and the maximum of the ks.

Let $\Delta_0 \subseteq \Delta$, $\Lambda_0 \subseteq \Lambda$ be finite sets such that off a set of $\mu \times \nu$-measure less than $\eta/2$, we have $g((x,y),\tau) \in \Delta_0$ and $\alpha(x,\tau) \in \Lambda_0$. Partition X into $A_1,\ldots,A_m \subseteq X$ such that $\mu(A_j) < \eta/2$. Then for $\delta \in \Delta_0$, $\lambda \in \Lambda_0$, and $j \le m$, let

$$M(\delta,\lambda,j) = \{(x,y) \in C \mid g((x,y),\tau) = \delta \text{ and } \alpha(x,\tau) = \lambda \text{ and } x \in A_j\}.$$

We have

$$\bigcup_{\substack{\delta\in\Delta_0,\lambda\in\Lambda_0,\\ j\le m}} M(\delta,\lambda,j) \ge 1 - \eta/2. \tag{4.2.1}$$

By the assumption that θ is injective, the set of θ-preimages of $\bigcup_k \mathcal{B}_k$ is dense in the measure algebra of $X \times Y$. Thus, for any $\delta \in \Delta_0$, $\lambda \in \Lambda_0$, and $j \le m$, there are $k(\delta,\lambda,j) \in \mathbf{N}$ and $B(\delta,\lambda,j) \in \mathcal{B}_{k(\delta,\lambda,j)}$ such that

$$\mu \times \nu\big(\hat{B}(\delta,\lambda,j) \;\triangle\; M(\delta,\lambda,j)\big) < \frac{\eta}{6m|\Delta_0||\Lambda_0|}. \tag{4.2.2}$$

Also, since $M(\delta,\lambda,j) \subseteq A_j$ and $\mu(A_j) \le \eta/2$,

$$\begin{aligned}\int_{\{y\in Y\mid \mu(\hat{B}(\delta,\lambda,j)^y)>\eta\}} \mu(\hat{B}(\delta,\lambda,j)^y)\,\mathrm{d}\nu(y) &\le 2\int_Y \mu(\hat{B}(\delta,\lambda,j)^y \;\triangle\; M(\delta,\lambda,j)^y)\,\mathrm{d}\nu(y)\\ &= 2\mu\times\nu\big(\hat{B}(\delta,\lambda,j) \;\triangle\; M(\delta,\lambda,j)\big)\\ &< \frac{\eta}{3m|\Delta_0||\Lambda_0|}\end{aligned}$$

and hence,

$$\mu\Big(\bigcup_{\substack{\delta\in\Delta_0,\lambda\in\Lambda_0,\\ j\le m}} \big\{(x,y) \in \hat{B}(\delta,\lambda,j) \mid \mu(\hat{B}(\delta,\lambda,j)^y) > \eta\big\}\Big) < \eta/3. \tag{4.2.3}$$

Finally, let $k = \max\{k(\delta,\lambda,j)\}$ and

$$\begin{aligned}M = &\bigcup_{\substack{\delta\in\Delta_0,\lambda\in\Lambda_0,\\ j\le m}} \hat{B}(\delta,\lambda,j) \cap M(\delta,\lambda,j)\\ &\smallsetminus \bigcup_{\substack{\delta\in\Delta_0,\lambda\in\Lambda_0,\\ j\le m}} \big\{(x,y) \in \hat{B}(\delta,\lambda,j) \mid \mu(\hat{B}(\delta,\lambda,j)^y) > \eta\big\}.\end{aligned}$$

Then (2), (3), and (4) are satisfied by definition, and by (4.2.1), (4.2.2), and (4.2.3),

$$\mu \times \nu(M) > 1 - (\eta/2 + \eta/6 + \eta/3) = 1 - \eta,$$

which verifies (1). □

Apply Lemma 4.2.2 with $\eta = \epsilon^2/4$ to obtain M and k and fix them from now on. Let $\{B_i \mid i \in \mathbf{N}\}$ enumerate the atoms of $\mathcal{B}_k$ (if $\mathcal{B}_k$ contains only finitely many atoms, add empty sets to the enumeration). The functions $(x, y) \mapsto g((x, y), \tau)$ and $(x, y) \mapsto \alpha(x, \tau)$ are constant on each $\hat{B}_i \cap M$ and, abusing notation, we will write $g(i, \tau)$ and $\alpha(i, \tau)$ (for those i for which $\hat{B}_i \cap M \neq \varnothing$). Now define the map $\Phi: Z \to \mathrm{MALG}(X)$ by $\Phi(i, y) = \hat{B}_i^y$. Note that $\{\hat{B}_i\}$ is a partition of $X \times Y$ and hence, for almost every $y \in Y$, $\{\Phi(i, y) \mid i \in \mathbf{N}\}$ is a partition of X.

That condition (i) is satisfied follows from the fact that for each x, the collection $\big\{\{y \mid x \in \Phi(i, y)\} \mid i \in \mathbf{N}\big\}$ forms a partition of Y. We proceed to check (ii). Using Lemma 4.2.2 (1) and (3), we have:

$$\begin{aligned}\int_{\{\mu(\Phi(z))>\epsilon\}} \mu(\Phi(z))\,\mathrm{d}\sigma(z) &= \int_{X\times Y\times\mathbf{N}} \chi_{\{(x,y)\in\hat{B}_i \text{ and } \mu(\hat{B}_i^y)>\epsilon\}}\,\mathrm{d}(x,y,i)\\ &\leq \int_{((X\times Y)\setminus M)\times\mathbf{N}} \chi_{\{(x,y)\in\hat{B}_i\}}\,\mathrm{d}(x,y,i)\\ &= \mu((X\times Y)\setminus M) < \eta < \epsilon.\end{aligned}$$

We are left with verifying (iii). Fix $\tau \in Q$. We will construct $T \in \mathrm{Aut}(Z)$ such that

$$\int_Z \mu\big(\Phi(Tz) \,\triangle\, \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) \leq 7\epsilon.$$

Set

$$G = \{(i, y) \in Z \mid \frac{\mu(\Phi(i,y)\setminus M^y)}{\mu(\Phi(i,y))} < \epsilon\}.$$

We have

$$\begin{aligned}\int_{Z\setminus G} \mu(\Phi(i,y))\,\mathrm{d}\sigma(i,y) &\leq \frac{1}{\epsilon}\int_{Z\setminus G} \mu(\Phi(i,y)\setminus M^y)\,\mathrm{d}\sigma(i,y)\\ &\leq \frac{1}{\epsilon}\mu\times\nu(X\times Y\setminus M) \leq \epsilon/4.\end{aligned} \tag{4.2.4}$$

Define

$$\begin{gathered}N_0 = \big\{(x,y)\in M \mid \exists i \quad (i,y)\in G \text{ and } x\in\Phi(i,y)\big\},\\ N = N_0 \cap \tau^{-1}\cdot N_0.\end{gathered}$$

By (4.2.4),

$$\mu\times\nu(N_0) \geq 1 - \mu\times\nu(C\setminus M) - \epsilon/4 \geq 1 - \epsilon/2$$

and hence,

$$\mu \times \nu(X \times Y \setminus N) \leq 2(\epsilon/2) = \epsilon.$$

Let $Z_0 = \{(i, y) \in Z \mid \Phi(i, y) \cap N^y \neq \varnothing\}$. Notice that

$$\begin{aligned}\int_{Z \setminus Z_0} \mu(\Phi(z))\, \mathrm{d}\sigma(z) &= \int_Y \sum_{\{i \mid \Phi(i,y) \cap N^y = \varnothing\}} \mu(\Phi(i, y))\, \mathrm{d}\nu(y) \\ &\leq \int_Y \mu(X \setminus N^y)\, \mathrm{d}\nu(y) \\ &\leq \mu \times \nu(X \times Y \setminus N) \leq \epsilon.\end{aligned} \tag{4.2.5}$$

Define the partial automorphism $T_0 \colon Z_0 \to Z$ by

$$T_0(i, y) = (j, \alpha(i, \tau) \cdot y) \iff g(i, \tau) \cdot B_i = B_j.$$

Lemma 4.2.3. *Given $(i, y) = z \in Z_0$, the following hold:*

(1) $\tau \cdot (\Phi(i, y) \cap M^y) \subseteq \Phi(T_0(i, y))$;

(2) $\tau^{-1} \cdot \left(\Phi(T_0(i, y)) \cap M^{\alpha(i,\tau)\cdot y}\right) \subseteq \Phi(i, y)$;

(3) $\frac{\mu(\Phi(z))}{\mu(\Phi(T_0 z))} \in \left[1 - \epsilon, \frac{1}{1-\epsilon}\right] \subseteq (1 - 2\epsilon, 1 + 2\epsilon)$;

(4) $\mu\left(\tau \cdot \Phi(z) \,\triangle\, \Phi(T_0 z)\right) \leq 3\epsilon \cdot \mu\left(\Phi(T_0 z)\right)$;

(5) *T_0 is injective and measure-preserving.*

Proof. (1). Take $x \in \Phi(i, y) \cap M^y$. Then

$$\theta(\tau \cdot (x, y)) = g(i, \tau) \cdot \theta(x, y) \in g(i, \tau) \cdot B_i = B_j$$

for some j. So $\tau \cdot (x, y) \in \hat{B}_j$. Also, $\tau \cdot (x, y) = (\tau \cdot x, \alpha(i, \tau) \cdot y)$. By our definition of T_0, $\tau \cdot (x, y) \in \Phi(T_0(i, y))$.

(2). Let $x \in \Phi(T_0(i, y)) \cap M^{\alpha(i,\tau)\cdot y}$ and let j be such that $T_0(i, y) = (j, \alpha(i, \tau) \cdot y)$. Then $(x, \alpha(i, \tau) \cdot y) \in \hat{B}_j$. Let $x_1 \in \Phi(i, y) \cap N^y$. From (1),

$$\tau \cdot x_1 \in \Phi(T_0(i, y)) \cap M^{\alpha(i,\tau)\cdot y}.$$

Note that

$$\tau^{-1} \cdot (\tau \cdot (x_1, y)) = (x_1, y) \in \hat{B}_i.$$

This implies that $g(j, \tau^{-1}) \cdot B_j = B_i$. So then

$$(\tau^{-1} \cdot x, \alpha(x, \tau^{-1})\alpha(i, \tau) \cdot y) = \tau^{-1} \cdot (x, \alpha(i, \tau) \cdot y) \in \hat{B}_i. \tag{4.2.6}$$

Also, $\alpha(i, \tau) = \alpha(x_1, \tau)$ and $\alpha(j, \tau^{-1}) = \alpha(\tau \cdot x_1, \tau^{-1})$. By the cocycle identity,

$$\alpha(x, \tau^{-1})\alpha(i, \tau) = \alpha(j, \tau^{-1})\alpha(i, \tau) = \alpha(\tau \cdot x_1, \tau^{-1})\alpha(x_1, \tau) = 1,$$

and hence, combining with (4.2.6), $\tau^{-1} \cdot x \in \Phi(i, y)$.

(3). From (1), we have that $\mu(M^y \cap \Phi(i, y)) \le \mu\big(\Phi(T_0(i, y))\big)$. Since $\Phi(i, y) \cap N^y \neq \varnothing$, $(i, y) \in G$ and we obtain

$$\mu\big(\Phi(T_0(i, y))\big) \ge \mu(M^y \cap \Phi(i, y)) \ge (1 - \epsilon)\mu(\Phi(i, y))$$

which then allows us to conclude that

$$\frac{\mu(\Phi(i, y))}{\mu\big(\Phi(T_0(i, y))\big)} \le \frac{1}{1 - \epsilon}.$$

Similarly, by (2) and the fact that $T_0(i, y) \in G$,

$$\mu(\Phi(i, y)) \ge \mu\big(\Phi(T_0(i, y)) \cap M^{\alpha(i,\tau)\cdot y}\big) \ge (1 - \epsilon)\mu\big(\Phi(T_0(i, y))\big)$$

which then leads to

$$\frac{\mu(\Phi(i, y))}{\mu\big(\Phi(T_0(i, y))\big)} \ge 1 - \epsilon.$$

(4). Using the fact that the action of Γ on X is measure-preserving and (1) and (3), we have:

$$\begin{aligned}
\mu\big(\tau \cdot \Phi(i, y) \setminus \Phi(T_0(i, y))\big) &= \mu\big(\Phi(i, y) \setminus \tau^{-1} \cdot \Phi(T_0(i, y))\big) \\
&\le \mu(\Phi(i, y) \setminus M^y) \\
&< \epsilon \cdot \mu(\Phi(i, y)) \\
&< \epsilon(1 + 2\epsilon)\mu\big(\Phi(T_0(i, y))\big) \\
&< 2\epsilon \cdot \mu\big(\Phi(T_0(i, y))\big).
\end{aligned}$$

Similarly, using (2), $\mu\big(\tau^{-1}\cdot\Phi(T_0(i,y))\smallsetminus\Phi(i,y)\big)<\epsilon\cdot\mu\big(\Phi(T_0(i,y))\big)$.

(5). Suppose that $T_0(i_1,y_1)=T_0(i_2,y_2)$ for some $(i_1,y_1),(i_2,y_2)\in Z_0$. Take $x_1\in\Phi(i_1,y_1)\cap N^{y_1}$, $x_2\in\Phi(i_2,y_2)\cap N^{y_2}$. Let

$$B_j = g(i_1,\tau)\cdot B_{i_1} = g(i_2,\tau)\cdot B_{i_2}.$$

$\tau\cdot(x_1,y_1),\tau\cdot(x_2,y_2)\in\hat{B}_j\cap M$, so

$$B_{i_1} = g(j,\tau^{-1})\cdot B_j = B_{i_2}.$$

Hence $i_1 = i_2$ and

$$y_1 = \alpha(i_1,\tau)^{-1}(\alpha(i_1,\tau)\cdot y_1) = \alpha(i_2,\tau)^{-1}(\alpha(i_2,\tau)\cdot y_2) = y_2.$$

Let now $A\subseteq Z_0$. We claim that $\sigma(T_0(A))=\sigma(A)$. Indeed, we have

$$T_0(A) = \bigcup_{i=1}^{\infty}\{(j,\alpha(i,\tau)\cdot y)\mid (i,y)\in A \text{ and } g(i,\tau)\cdot B_i = B_j\}.$$

Since the map $B_i\mapsto g(i,\tau)\cdot B_i$ is injective and $\alpha(i,\tau)$ is measure-preserving for all i, we have:

$$\begin{aligned}\sigma(T_0(A)) &= \sum_{i=1}^{\infty}\nu\big(\{\alpha(i,\tau)\cdot y\mid (i,y)\in A\}\big)\\ &= \sum_{i=1}^{\infty}\nu\big(\{y\mid (i,y)\in A\}\big)\\ &= \sigma(A).\end{aligned}$$

□

Note that by Lemma 4.2.2 (4), $Z_0\subseteq\{0,1,\ldots,n\}\times Y$ for some $n\in\mathbf{N}$ and, in particular, $\sigma(Z_0)<\infty$. This also implies that T_0 can be extended to a full measure-preserving automorphism T of Z. Use Lemma 4.2.3 (4) to obtain

$$\begin{aligned}\int_{Z_0}\mu\big(\Phi(Tz)\ \triangle\ \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) &\le \int_{Z_0}3\epsilon\cdot\mu(\Phi(Tz))\,\mathrm{d}\sigma(z)\\ &\le 3\epsilon\int_{Z}\mu(\Phi(z))\,\mathrm{d}\sigma(z)\le 3\epsilon.\end{aligned} \tag{4.2.7}$$

Also, by Lemma 4.2.3 (3) and (4.2.5),

$$\begin{aligned}
&\int_{Z\smallsetminus Z_0} \mu\big(\Phi(Tz) \vartriangle \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) \\
&\quad\le \int_{Z\smallsetminus Z_0} \mu(\Phi(Tz))\,\mathrm{d}\sigma(z) + \int_{Z\smallsetminus Z_0} \mu\big(\tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) \\
&\quad= \int_Z \mu(\Phi(Tz))\,\mathrm{d}\sigma(z) - \int_{Z_0} \mu(\Phi(T_0 z))\,\mathrm{d}\sigma(z) + \int_{Z\smallsetminus Z_0} \mu(\Phi(z))\,\mathrm{d}\sigma(z) \\
&\quad\le 1 - (1-2\epsilon)\int_{Z_0} \mu(\Phi(z))\,\mathrm{d}\sigma(z) + \epsilon \\
&\quad\le 1 - (1-2\epsilon)(1-\epsilon) + \epsilon \le 4\epsilon.
\end{aligned} \tag{4.2.8}$$

Finally, combine (4.2.7) and (4.2.8) to obtain

$$\begin{aligned}
&\int_Z \mu\big(\Phi(Tz) \vartriangle \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) \\
&\quad= \int_{Z_0} \mu\big(\Phi(Tz) \vartriangle \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) + \int_{Z\smallsetminus Z_0} \mu\big(\Phi(Tz) \vartriangle \tau\cdot\Phi(z)\big)\,\mathrm{d}\sigma(z) \\
&\quad\le 7\epsilon.
\end{aligned}$$

□

Now we proceed to construct an invariant state on $B(L_0^2(X))$. For $A \in \mathrm{MALG}(X)$, let

$$\eta_A = \chi_A - \mu(A),$$

where χ_A denotes the characteristic function of the set A. Then $\eta_A \in L_0^2(X)$ and $\|\eta_A\|^2 = \mu(A) - \mu(A)^2$. Also, for any $S \in B(L_0^2(X))$,

$$\begin{aligned}
|\langle S\eta_A, \eta_A\rangle - \langle S\eta_B, \eta_B\rangle| &\le \|S\|\,(\|\eta_A\| + \|\eta_B\|)\,\|\eta_A - \eta_B\| \\
&\le \|S\|\,\big(\sqrt{\mu(A)} + \sqrt{\mu(B)}\big)\sqrt{\mu(A \vartriangle B)},
\end{aligned} \tag{4.2.9}$$

as is verified by direct computation.

Enumerate $\Gamma = \{\gamma_n\}$ and set $Q_n = \{\gamma_1, \gamma_2, \ldots, \gamma_n\}$. Let for each n, $\Phi_n\colon Z \to \mathrm{MALG}(X)$ be a $(Q_n, 1/n)$-invariant map as given by Proposition 4.2.1. Let $M_n \in B(L_0^2(X))^*$ be the positive linear functional defined by

$$M_n(S) = \int_Z \big\langle S\eta_{\Phi_n(z)}, \eta_{\Phi_n(z)}\big\rangle\,\mathrm{d}\sigma(z).$$

Note that by (i) and an application of Fubini,

$$\int_Z \mu(\Phi_n(z))\, \mathrm{d}\sigma(z) = 1.$$

Hence,

$$\begin{aligned} |M_n(S)| &\leq \int_Z |\langle S\eta_{\Phi_n(z)}, \eta_{\Phi_n(z)}\rangle|\, \mathrm{d}\sigma(z) \\ &\leq \int_Z \|S\| \left\|\eta_{\Phi_n(z)}\right\|^2 \mathrm{d}\sigma(z) \\ &\leq \|S\| \int_Z \mu(\Phi_n(z))\, \mathrm{d}\sigma(z) = \|S\|\,. \end{aligned}$$

Therefore $\|M_n\| \leq 1$. Let now M be any weak* limit point of the set $\{M_n\}$. We will show that M is a κ_0-invariant state on $B(L^2_0(X))$ which will complete the proof of the theorem. M is clearly positive. Let I denote the identity operator on $L^2_0(X)$. We have

$$\begin{aligned} M_n(I) &= \int_Z \langle \eta_{\Phi_n(z)}, \eta_{\Phi_n(z)}\rangle\, \mathrm{d}\sigma(z) \\ &= \int_Z \mu(\Phi_n(z)) - \mu(\Phi_n(z))^2\, \mathrm{d}\sigma(z) \\ &= 1 - \int_Z \mu(\Phi_n(z))^2\, \mathrm{d}\sigma(z) \to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Indeed, by (ii),

$$\begin{aligned} \int_Z \mu(\Phi_n(z))^2\, \mathrm{d}\sigma(z) &= \int_{\{\mu(\Phi_n(z))>1/n\}} \mu(\Phi_n(z))^2\, \mathrm{d}\sigma(z) \\ &\quad + \int_{\{\mu(\Phi_n(z))\leq 1/n\}} \mu(\Phi_n(z))^2\, \mathrm{d}\sigma(z) \\ &\leq \frac{1}{n} + \frac{1}{n}\int_Z \mu(\Phi_n(z))\, \mathrm{d}\sigma(z) = \frac{2}{n}. \end{aligned}$$

Hence, $M(I) = 1$.

To show that M is invariant, it suffices to check that for all $\tau \in \Gamma$ and $S \in B(L^2_0(X))$,

$$M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau)) - M_n(S) \to 0.$$

Indeed, since M is a weak* limit point of the M_ns, for every $\epsilon > 0$, there exist infinitely many n such that

$$|M(S) - M_n(S)| < \epsilon \quad \text{and} \quad |M(\kappa_0(\tau^{-1})S\kappa_0(\tau)) - M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau))| < \epsilon.$$

Then

$$
\begin{aligned}
|M(S) - M(\kappa_0(\tau^{-1})S\kappa_0(\tau))| &\leq |M(S) - M_n(S)| + |M_n(S) - M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau))| \\
&\quad + |M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau)) - M(\kappa_0(\tau^{-1})S\kappa_0(\tau))| \\
&\leq |M_n(S) - M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau))| + 2\epsilon
\end{aligned}
$$

which shows that $M(S) = M(\kappa_0(\tau^{-1})S\kappa_0(\tau))$.

Fix $\tau \in \Gamma$ and $S \in B(L_0^2(X))$. For all n big enough that $\tau \in Q_n$, apply Proposition 4.2.1 to obtain $T_n \in \mathrm{Aut}(Z)$ satisfying (iii). Using (4.2.9) and Cauchy–Schwartz, we have:

$$
\begin{aligned}
&|M_n(\kappa_0(\tau^{-1})S\kappa_0(\tau)) - M_n(S)| = \\
&\quad = \left| \int_Z \langle \kappa_0(\tau^{-1})S\kappa_0(\tau)\eta_{\Phi_n(z)}, \eta_{\Phi_n(z)} \rangle \, \mathrm{d}\sigma(z) - \int_Z \langle S\eta_{\Phi_n(z)}, \eta_{\Phi_n(z)} \rangle \, \mathrm{d}\sigma(z) \right| \\
&\quad = \left| \int_Z \langle S\eta_{\tau\cdot\Phi_n(z)}, \eta_{\tau\cdot\Phi_n(z)} \rangle \, \mathrm{d}\sigma(z) - \int_Z \langle S\eta_{\Phi_n(T_n z)}, \eta_{\Phi_n(T_n z)} \rangle \, \mathrm{d}\sigma(z) \right| \\
&\quad \leq \int_Z \left| \langle S\eta_{\tau\cdot\Phi_n(z)}, \eta_{\tau\cdot\Phi_n(z)} \rangle - \langle S\eta_{\Phi_n(T_n z)}, \eta_{\Phi_n(T_n z)} \rangle \right| \mathrm{d}\sigma(z) \\
&\quad \leq 2\,\|S\| \int_Z \left(\mu(\Phi_n(z))^{\frac{1}{2}} + \mu(\Phi_n(T_n z))^{\frac{1}{2}} \right) \mu\big(\tau\cdot\Phi_n(z) \bigtriangleup \Phi_n(T_n z)\big)^{\frac{1}{2}} \, \mathrm{d}\sigma(z) \\
&\quad \leq 2\,\|S\| \left(\left(\int_Z \mu(\Phi_n(z)) \, \mathrm{d}\sigma(z) \right)^{\frac{1}{2}} + \left(\int_Z \mu(\Phi_n(T_n z)) \, \mathrm{d}\sigma(z) \right)^{\frac{1}{2}} \right) \cdot \\
&\qquad\qquad \cdot \left(\int_Z \mu\big(\tau\cdot\Phi_n(z) \bigtriangleup \Phi_n(T_n z)\big) \, \mathrm{d}\sigma(z) \right)^{\frac{1}{2}} \\
&\quad \leq 2\,\|S\| \frac{2}{\sqrt{n}} \to 0 \quad \text{as} \quad n \to \infty.
\end{aligned}
$$

This completes the proof of the theorem.

4.3 Amenable Koopman representations and almost invariant vectors

In this section, we describe two situations in which the amenability of the Koopman representation is equivalent to the existence of almost invariant vectors ($1_\Gamma \prec \kappa_0$). Note that by (4.1.2), $1_\Gamma \prec \pi$ implies that π is amenable for any representation π but the converse is not true in general, even for Koopman representations (consider, for example, a modular, ergodic action of a property (T) group). A special situation when it is true is given by the lemma below.

Let I be a countable set and let Γ act on I. Recall that the action is called *amenable* if there is a Γ-invariant mean on $\ell^\infty(I)$.

Lemma 4.3.1. *Let Γ be a countable group, π be a unitary representation of Γ on a separable Hilbert space $\mathcal{H}$ and let $\{\xi_i\}_{i\in\mathbf{N}}$ be an orthonormal basis for $\mathcal{H}$ invariant under π. Then*

$$\pi \text{ is amenable} \implies 1_\Gamma \prec \pi.$$

Proof. Set $I = \{\xi_i : i \in \mathbf{N}\}$. We can identify $\mathcal{H}$ with $\ell^2(I)$ and the representation π with the representation of Γ on $\ell^2(I)$ induced by the action of Γ on I. Let $B(\mathcal{H})$ denote the space of bounded operators on $\mathcal{H}$. π amenable implies that there is a state M on $B(\mathcal{H})$ invariant under π, i.e.,

$$M(\pi(\gamma)S\pi(\gamma)^{-1}) = M(S), \quad \text{for all } \gamma \in \Gamma, S \in B(\mathcal{H}).$$

For each $\phi \in \ell^\infty(I)$ consider the multiplication operator $T_\phi \in B(\mathcal{H})$ defined by

$$T_\phi f = \phi f$$

and notice that

$$\pi(\gamma)T_\phi\pi(\gamma)^{-1} = T_{\gamma\cdot\phi},$$

where $(\gamma\cdot\phi)(\xi) = \phi(\pi(\gamma^{-1})\cdot\xi)$. Hence $\phi \mapsto M(T_\phi)$ defines a Γ-invariant mean on $\ell^\infty(I)$ and the action of Γ on I is amenable. But this implies that $1_\Gamma \prec \pi$. □

Remark. The proof of Lemma 4.3.1 also shows that if Γ acts on a countable set I, the corresponding representation is amenable iff the action is amenable.

Let now (X_0, μ_0) be a probability space. If I is countable and $\Gamma \curvearrowright I$, we have a measure-preserving action $\Gamma \curvearrowright X_0^I$ by permuting the coordinates, which is called a *generalized Bernoulli shift*.

Proposition 4.3.2. *Let $\Gamma \curvearrowright X_0^I$ be a generalized Bernoulli shift. Let κ_0 be the corresponding Koopman representation of Γ. Then the following are equivalent:*

(i) the action $\Gamma \curvearrowright I$ is amenable;

(ii) κ_0 is amenable;

(iii) $1_\Gamma \prec \kappa_0$.

Proof. The equivalence of (i) and (iii) follows from [44, Theorem 1.2]. From the analysis of the Koopman representation of generalized Bernoulli shifts carried out in [44, Section 3], it follows that there is a basis of $L^2_0(X_0^I)$ invariant under κ_0. Hence Lemma 4.3.1 applies and we have (ii) ⇒ (iii). Lastly, the implication (iii) ⇒ (ii) follows from (4.1.2). □

Corollary 4.3.3. *Let $\Gamma \curvearrowright (X_0, \mu_0)^I$ be a generalized Bernoulli shift. If the action $\Gamma \curvearrowright I$ is non-amenable, then the action $\Gamma \curvearrowright X_0^I$ is μ_0^I-antimodular.*

Corollary 4.3.4. *Let $\Gamma \curvearrowright (X_0, \mu_0)^I$ be a mixing generalized Bernoulli shift. Then if Γ is non-amenable, the action $\Gamma \curvearrowright X_0^I$ is μ_0^I-antimodular.*

Proof. $\Gamma \curvearrowright X_0^I$ mixing implies that for each $i \in I$, the stabilizer Γ_i is finite (see, e.g., [44, Proposition 2.3]). Let λ_I be the representation of Γ on $\ell^2(I)$, λ_Γ denote the left-regular representation of Γ, and for $H \leq \Gamma$, let $\lambda_{\Gamma/H}$ be the quasi-regular representation on $\ell^2(\Gamma/H)$. Let $A \subseteq I$ be a transversal for the action $\Gamma \curvearrowright I$. Then

$$\lambda_I = \bigoplus_{i \in A} \lambda_{\Gamma/\Gamma_i}.$$

It is not hard to see that $\lambda_{\Gamma/\Gamma_i} \leq \lambda_\Gamma$ (cf. [44, Lemma 3.3]) and hence, $\lambda_I \prec \lambda_\Gamma$. By Theorem 4.1.2 and Proposition 4.3.2, if the action $\Gamma \curvearrowright X_0^I$ is not antimodular, λ_I is amenable, hence by (4.1.2), λ_Γ is amenable, contradicting the non-amenability of Γ. □

Hjorth's Theorem 4.1.1 can now be obtained as a special case of either Corollary 4.3.3 or Corollary 4.3.4 if we put $\Gamma = I = \mathbf{F}_2$, $X_0 = 2$, and let Γ act on I by left translation.

Now consider the case of an action on a compact Polish group (equipped with its normalized Haar measure) by (topological group) automorphisms.

Proposition 4.3.5. *Let Γ act on the compact Polish group G by automorphisms and let κ_0 be the corresponding Koopman representation of Γ on $L^2_0(G)$. Then*

$$\kappa_0 \text{ is amenable} \implies 1_\Gamma \prec \kappa_0.$$

Proof. Fix an invariant state M on $B(L^2_0(G))$. We adopt the notation from Folland [19, Chapter 5] (see also Kechris [41]). Let $\hat{G}_0$ denote the set of (equivalence classes of) *nontrivial* irreducible representations of G. Recall that

$$\{\pi_{ij} : i, j \leq d_\pi; \pi \in \hat{G}_0\}$$

is an orthogonal basis for $L^2_0(G)$, where $d_\pi = \dim \pi$ and the π_{ij}s are the matrix coefficients of π. For

$\phi \in \ell^\infty(\hat{G}_0)$, define the operator $T_\phi \in B(L^2_0(G))$ by

$$T_\phi \pi_{ij} = \phi(\pi)\pi_{ij}.$$

Notice that

$$T_{\gamma\cdot\phi} = \kappa_0(\gamma) T_\phi \kappa_0(\gamma)^{-1}$$

and hence $\phi \mapsto M(T_\phi)$ defines a Γ-invariant mean on $\ell^\infty(\hat{G}_0)$.

Let $I = \{\chi_\pi : \pi \in \hat{G}_0\}$ (where χ_π denotes the character corresponding to π) and notice that I is invariant under κ_0. We can also identify $\ell^\infty(I)$ and $\ell^\infty(\hat{G}_0)$ and conclude that the action of Γ on I is amenable. Denote $ZL^2_0(G) = \operatorname{span}\{\chi_\pi : \pi \in \hat{G}_0\}$ and $\sigma = \kappa_0|_{ZL^2_0(G)}$. We can identify $ZL^2_0(G)$ with $\ell^2(I)$ and by the amenability of the action of Γ on I, $1_\Gamma \prec \sigma$. But $\sigma \leq \kappa_0$ and we are done. □

Remark. Note that generalized Bernoulli shifts with a homogeneous base space (i.e., X_0 non-atomic or purely atomic with atoms of the same measure) are a special case of actions on abelian compact groups by automorphisms. However, for arbitrary X_0, this is not the case. It is shown in Kechris–Tsankov [44] that for generalized Bernoulli shifts, $1_\Gamma \prec \kappa_0$ implies the existence of almost invariant sets, while it is open whether the same holds for actions on compact Polish groups by automorphisms.

4.4 Applications to orbit equivalence and Borel reducibility

Recall that two measure-preserving actions $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$ are called *orbit equivalent* if there exist conull, invariant sets $A \subseteq X$ and $B \subseteq Y$ and a measurable bijection $f: A \to B$ such that

$$\forall x, y \in X \quad x \, E^X_\Gamma \, y \iff f(x) \, E^Y_\Delta \, f(y).$$

In this section, we use modular actions and Theorem 4.1.2 to show that residually finite groups have at least three non-orbit equivalent, free, ergodic actions.

Dye started the theory of orbit equivalence by showing that all ergodic actions of $\mathbf{Z}$ are orbit equivalent. Later, Ornstein and Weiss showed that, in fact, all ergodic actions of amenable groups are orbit equivalent. For all of this, see [42]. In the other direction, Schmidt [62] and Connes–Weiss [9] showed that non-property (T), non-amenable groups have at least two non-orbit equivalent, free, ergodic actions. The invariant they used was E_0-ergodicity (called strong ergodicity by them) which we proceed to define. E_0 is the equivalence relation on $2^{\mathbf{N}}$ given by

$$(x_n)\,E_0\,(y_n) \iff \exists n \forall m > n \quad x_m = y_m.$$

A measure-preserving group action $\Gamma \curvearrowright X$ (or the orbit equivalence relation it defines) is said to be E_0-*ergodic* if for every homomorphism from E_Γ^X to E_0, there exists a single E_0 equivalence class whose preimage is conull. For a measure-preserving, ergodic action $\Gamma \curvearrowright X$, *not* being E_0-ergodic is equivalent to possessing *almost invariant sets*, i.e., a sequence of measurable sets $\{A_n\}$ with measures bounded away from 0 and 1 satisfying for each $\gamma \in \Gamma$,

$$\mu(\gamma \cdot A_n \bigtriangleup A_n) \to 0 \quad \text{as} \quad n \to \infty$$

(Jones–Schmidt [35]). The definition shows that E_0-ergodicity is an invariant of orbit equivalence, while the existence of almost invariant sets is usually easier to verify in particular cases. For more information on the topic, see Hjorth–Kechris [32, Appendix A].

Many non-amenable groups are now known to have a continuum of non-orbit equivalent actions: for example, property (T) groups (Hjorth [31]) and non-abelian free groups (Gaboriau–Popa [23]). The latter result was recently extended by Ioana [33] to include all countable groups containing a copy of $\mathbf{F}_2$. It is not known whether all non-amenable groups admit a continuum of non-orbit equivalent actions. For more information on orbit equivalence and related topics, see the surveys Gaboriau [22] and Shalom [65], as well as the book Kechris–Miller [42].

First we note the following simple corollary of Theorem 4.1.2.

Corollary 4.4.1. *Let $\Gamma \curvearrowright (X,\mu)$ and $\Gamma \curvearrowright (Y,\nu)$ be two measure-preserving actions where the measure μ is non-atomic. Then if the Koopman representation κ_0 corresponding to the action $\Gamma \curvearrowright X$ is non-amenable, the product action $\Gamma \curvearrowright X \times Y$ is $\mu \times \nu$-antimodular.*

Proof. The action $\Gamma \curvearrowright Y$ defines a homomorphism $\phi\colon \Gamma \to \mathrm{Aut}(Y)$ and we can take $\alpha(x,\gamma) = \phi(\gamma)$ in Theorem 4.1.2. □

Theorem 4.4.2. *Suppose that Γ is a countable, non-amenable, residually finite group. Then Γ has at least three non-orbit equivalent, free, measure-preserving, ergodic actions.*

Proof. It is already known that countable groups with property (T) admit continuum many such actions (Hjorth [31]). We thus assume that Γ does not have property (T). First we construct two antimodular actions of Γ that are not orbit equivalent to each other. Since Γ does not have property (T), there is a measure-preserving, ergodic action $\Gamma \curvearrowright (Y, \nu)$ that is not E_0-ergodic (see Connes–Weiss [9]). Consider the shift action $\Gamma \curvearrowright 2^\Gamma$. It is ergodic, free a.e., and E_0-ergodic (Jones–Schmidt [35]; cf. Proposition 4.3.2). By Corollary 4.3.3, it is also antimodular. Now consider the diagonal action $\Gamma \curvearrowright 2^\Gamma \times Y$. It is free a.e. and ergodic. Corollary 4.4.1 implies that it is also antimodular. Since $\Gamma \curvearrowright Y$ is not E_0-ergodic, $\Gamma \curvearrowright 2^\Gamma \times Y$ is not E_0-ergodic either (almost invariant sets in Y lift to the product). Finally, since Γ is residually finite, there exists a free, modular, ergodic action $\Gamma \curvearrowright Z$.

Now our three actions are: $\Gamma \curvearrowright 2^\Gamma$, $\Gamma \curvearrowright 2^\Gamma \times Y$, and $\Gamma \curvearrowright Z$. E_0-ergodicity distinguishes the first two, and by antimodularity, they are not orbit equivalent to the third. □

An equivalence relation E on a standard Borel space X is *Borel reducible* to an equivalence relation F on Y (written as $E \leq_B F$) if there exists a Borel homomorphism π from E to F such that

$$x \mathrel{E} y \iff \pi(x) \mathrel{F} \pi(y) \quad \forall x, y \in X.$$

$E \leq_B F$ expresses that, in some sense, the equivalence relation F is more complicated than E. We say that E and F are *Borel bireducible* if $E \leq_B F$ and $F \leq_B E$. For more on the subject of Borel reducibility of countable Borel equivalence relations, we refer the reader to Jackson–Kechris–Louveau [34], Hjorth–Kechris [32], and, for motivation and more general background, to Kechris [39].

We have the following application of Theorem 4.1.2.

Theorem 4.4.3. *Suppose that Γ is a countable, non-amenable, residually finite group. Then Γ has two free, measure-preserving actions whose orbit equivalence relations are not Borel bireducible.*

Proof. In the terminology of the proof of Theorem 4.4.2, we just consider the equivalence relations $E_\Gamma^{2^\Gamma}$ and E_Γ^Z. □

4.5 A counterexample

In this section, we construct an example of a group action orbit equivalent to a modular action (of another group) such that its Koopman representation does not weakly contain any finite-dimensional representation (and, in particular, is weakly mixing).

Let Γ be a residually finite group, $\{H_i\}_{i \in I}$ a countable family of normal subgroups of Γ of finite index such that $\bigcap_{i \in I} H_i = \{1\}$ and $X = \varprojlim \Gamma / H_i$ be the profinite completion of Γ with respect to this family. Let μ be the (normalized) Haar measure on X. Γ embeds as a dense subgroup of X and the left translation action of Γ on X is free, ergodic, and modular (see [41, 5G]). Denote by λ the left-regular representation of X on $L^2(X)$ and notice that by the density of Γ in X, a subspace of $L^2(X)$ is invariant under λ iff it is invariant under $\lambda|_\Gamma$. Hence, by the Peter–Weyl theorem, any irreducible subrepresentation of $\lambda|_\Gamma$ is finite-dimensional and has finite multiplicity in $\lambda|_\Gamma$.

We need the following preliminary lemma.

Lemma 4.5.1. *Let π be a finite-dimensional, irreducible representation of a countable group Γ with property (T). Then for every normalized positive definite function ϕ on Γ associated to π, the following holds: whenever $0 \le a_k \le 1$, ψ_k and θ_k are normalized positive definite functions on Γ, the cyclic representations corresponding to the θ_ks do not contain π, and $(1 - a_k)\psi_k + a_k\theta_k \to \phi$ pointwise, it is always the case that $a_k \to 0$.*

The following proof, simpler than our original one, was suggested by the referee.

Proof. Let $\mathcal{P}_1$ be the set of normalized positive definite functions on Γ considered as a subset of $\ell^\infty(\Gamma)$ equipped with the weak* topology which coincides on $\mathcal{P}_1$ with the pointwise convergence topology. Since Γ is discrete, $\mathcal{P}_1$ is compact. For a positive definite function β on Γ, denote by ρ_β the cyclic representation associated to β. We have $\pi = \rho_\phi$.

Suppose that a_k does not converge to 0. Then, by the compactness of $\mathcal{P}_1$, we can find $\psi, \theta \in \mathcal{P}_1$ and $a > 0$ such that $\psi_{k_n} \to \psi$, $\theta_{k_n} \to \theta$ pointwise, and $a_{k_n} \to a$ for some subsequence $\{k_n\} \subseteq \mathbf{N}$. This implies that $(1 - a)\psi + a\theta = \phi$. Since ρ_ϕ is irreducible and $a > 0$, we conclude that $\rho_\theta = \rho_\phi = \pi$ (see [4, C.5.1]). Now, since $\theta_{k_n} \to \theta$ pointwise, it follows that $\rho_{\theta_{k_n}} \to \rho_\theta = \pi$ in the Fell topology; hence $\pi \prec \bigoplus_n \rho_{\theta_{k_n}}$ and by property (T) and the irreducibility of π, we conclude that $\pi \le \rho_{\theta_{k_n}}$ for some n, a contradiction. □

Proposition 4.5.2. *Let Γ and X be as in the beginning of the section. If moreover Γ has property (T), there exists a group Δ and an action $\Delta \curvearrowright X$ by measure-preserving transformations generating E_Γ^X such that the Koopman representation of Δ on $L_0^2(X)$ does not weakly contain any finite-dimensional representation of Δ (and, in particular, is weakly mixing).*

Proof. Recall that for a measure-preserving equivalence relation E, *the full group of E* (denoted by $[E]$) is the group of all measure-preserving transformations T preserving E, i.e., satisfying $T(x) \mathrel{E} x$ for almost all $x \in X$. If T is a measure-preserving transformation, $[T]$ denotes the full group of the equivalence relation generated by T.

Let E be the equivalence relation induced by the action of Γ. Since E is ergodic, there is an ergodic $T \in [E]$ (see [37, 3.5]). Hence, $[T] \leq [E]$. Let Λ be a non-trivial, countable, amenable group which does not have non-trivial finite-dimensional representations (for example, $\mathrm{SL}(2; \overline{F}_2)$ where $\overline{F}_2$ is the algebraic closure of the field with two elements; cf. Dye [14]). By Dye's theorem and Ornstein–Weiss (see [42, Theorem 10.7]), we can embed Λ in $[T]$ (and therefore in $[E]$) so that the resulting action $\Lambda \curvearrowright X$ is ergodic. Let Δ be the subgroup of $[E]$ generated by Γ and Λ. Δ inherits a natural action on X from $[E]$. Denote by κ_0 the Koopman representation of Δ on $L^2_0(X)$.

Suppose, towards a contradiction, that $\pi \prec \kappa_0$ for some finite-dimensional representation π of Δ on a Hilbert space $\mathcal{H}_\pi$. Without loss of generality, we can assume that π is irreducible. By the properties of Λ, $\pi|_\Lambda$ is trivial and hence $\pi|_\Gamma$ is irreducible. We have $\pi|_\Gamma \prec \kappa_0|_\Gamma$ and since Γ has property (T), $\pi|_\Gamma \leq \kappa_0|_\Gamma$. Let $\mathcal{K}_\pi$ be the sum of all subspaces of $L^2_0(X)$ invariant under $\kappa_0|_\Gamma$ on which $\kappa_0|_\Gamma$ is equivalent to $\pi|_\Gamma$. Then, by the above observations, $\dim \mathcal{K}_\pi < \infty$.

Fix a unit vector $\xi \in \mathcal{H}_\pi$. There exists a sequence $\{\eta_k\}$ of unit vectors in $L^2_0(X)$ such that

$$\langle \kappa_0(\delta) \cdot \eta_k, \eta_k \rangle \to \langle \pi(\delta) \cdot \xi, \xi \rangle \qquad \text{for all } \delta \in \Delta.$$

Set $\phi(g) = \langle \pi(g) \cdot \xi, \xi \rangle$ for $g \in \Gamma$. Write $\eta_k = \eta^1_k + \eta^2_k$ where $\eta^1_k \in \mathcal{K}_\pi$ and $\eta^2_k \in \mathcal{K}^\perp_\pi$, $\left\|\eta^1_k\right\|^2 + \left\|\eta^2_k\right\|^2 = 1$. Now we have, for all $g \in \Gamma$,

$$\left\|\eta^1_k\right\|^2 \left\langle \kappa_0(g) \cdot \frac{\eta^1_k}{\left\|\eta^1_k\right\|}, \frac{\eta^1_k}{\left\|\eta^1_k\right\|} \right\rangle + \left\|\eta^2_k\right\|^2 \left\langle \kappa_0(g) \cdot \frac{\eta^2_k}{\left\|\eta^2_k\right\|}, \frac{\eta^2_k}{\left\|\eta^2_k\right\|} \right\rangle \to \phi(g)$$

and Lemma 4.5.1 allows us to conclude that $\eta^2_k \to 0$.

On the other hand, $\kappa_0|_\Lambda$ does not have invariant vectors and hence

$$\forall 0 \neq \eta \in \mathcal{K}_\pi \; \exists g \in \Lambda \quad |\langle \kappa_0(g) \cdot \eta, \eta \rangle| < \|\eta\|^2$$

($|\langle \kappa_0(g) \cdot \eta, \eta \rangle| = \|\eta\|^2$ for all g implies that $\kappa_0|_\Lambda$ restricted to $\mathbf{C}\eta$ is a one-dimensional representation of Λ, hence trivial, hence η is an invariant vector). By compactness (of the unit sphere in $\mathcal{K}_\pi$), there exists a finite $Q \subseteq \Lambda$ and $\epsilon > 0$ such that

$$\forall 0 \neq \eta \in \mathcal{K}_\pi \; \exists g \in Q \quad |\langle \kappa_0(g) \cdot \eta, \eta \rangle| < (1 - \epsilon) \|\eta\|^2 . \tag{4.5.1}$$

Let k be so big that $\left\|\eta_k^2\right\| < \epsilon/4$. Then we calculate, for any $g \in \Lambda$,

$$
\begin{aligned}
|\langle\kappa_0(g)\cdot\eta_k, \eta_k\rangle| &= \left|\left\langle\kappa_0(g)\cdot(\eta_k^1+\eta_k^2), \eta_k^1+\eta_k^2\right\rangle\right| \\
&\leq \left|\left\langle\kappa_0(g)\cdot\eta_k^1, \eta_k^1\right\rangle\right| + \left|\left\langle\kappa_0(g)\cdot\eta_k^2, \eta_k^2\right\rangle\right| + 2\left\|\eta_k^1\right\|\left\|\eta_k^2\right\| \\
&\leq \left|\left\langle\kappa_0(g)\cdot\eta_k^1, \eta_k^1\right\rangle\right| + \left\|\eta_k^2\right\|^2 + 2\left\|\eta_k^1\right\|\left\|\eta_k^2\right\| \\
&\leq \left|\left\langle\kappa_0(g)\cdot\eta_k^1, \eta_k^1\right\rangle\right| + 3\epsilon/4.
\end{aligned}
$$

But by (4.5.1), for each η_k^1, there exists $g \in Q$ such that $\left|\left\langle\kappa_0(g)\cdot\eta_k^1, \eta_k^1\right\rangle\right| < 1-\epsilon$. Therefore there exists $g_0 \in Q$ such that for infinitely many ks,

$$|\langle\kappa_0(g_0)\cdot\eta_k, \eta_k\rangle| < 1-\epsilon/4$$

and in particular,

$$\langle\kappa_0(g_0)\cdot\eta_k, \eta_k\rangle \nrightarrow 1 = \langle\pi(g_0)\cdot\xi, \xi\rangle,$$

a contradiction. □

Chapter 5

Topological Properties of Full Groups

5.1 Introduction

The study of measure-preserving actions of countable groups on standard probability spaces up to orbit equivalence was initiated by Dye in the 1950s and since then the subject has become an important meeting point of ergodic theory, operator algebras, and Borel equivalence relations. This paper concentrates on the study of one invariant of orbit equivalence, namely, the full group of the orbit equivalence relation.

Let X be a standard Borel space and μ a non-atomic, Borel, probability measure on it. Denote by $\mathrm{Aut}(X,\mu)$ the group of all measure-preserving automorphisms of (X,μ) (modulo null sets). An equivalence relation E on X is called *countable* if all of its equivalence classes are countable, *finite* if all of its equivalence classes are finite, and *aperiodic* if all of its equivalence classes are infinite. Say that E is *measure-preserving* if every Borel automorphism T of X which preserves E is measure-preserving, i.e.,

$$(\forall x \in X \quad Tx\, E\, x) \implies T \text{ preserves } \mu.$$

All equivalence relations below are assumed countable, Borel, and measure-preserving. The *full group* of E, denoted by $[E]$, is defined by:

$$[E] = \{T \in \mathrm{Aut}(X,\mu) : Tx\, E\, x \quad \text{for a.e. } x \in X\}.$$

By a classical result of Feldman–Moore [18], every equivalence relation E is the orbit equivalence relation of some measure-preserving group action $\Gamma \curvearrowright X$, where Γ is a countable group. If $\Phi \subseteq \mathrm{Aut}(X,\mu)$ is a countable set of automorphisms, write E_Φ^X for the equivalence relation generated by Φ and $[\Phi]$ for the full group $[E_\Phi^X]$. If $\Phi = \{T\}$ is a singleton, write E_T^X instead of $E_{\{T\}}^X$ and $[T]$ instead of $[\{T\}]$.

E is called *ergodic* if every measurable E-invariant set is either null or co-null. Two equivalence relations E and F (on measure spaces (X,μ) and (Y,ν), respectively) are *isomorphic* if there exists a measure-preserving isomorphism $f\colon X \to Y$ such that

$$x_1 \, E \, x_2 \iff f(x_1) \, F \, f(x_2) \quad \text{for a.e. } x_1, x_2 \in X.$$

Two measure-preserving countable group actions $\Gamma \curvearrowright X$ and $\Delta \curvearrowright Y$ are *orbit equivalent* if their orbit equivalence relations are isomorphic. An equivalence relation is called *hyperfinite* if can be written as an increasing union of finite equivalence relations or, equivalently, is generated by a single automorphism. Dye proved that all ergodic, hyperfinite equivalence relations are isomorphic and Ornstein–Weiss showed that all equivalence relations generated by amenable groups are hyperfinite; in particular, all ergodic actions of amenable groups are orbit equivalent (see [42] for proofs of these facts). For non-amenable groups, however, there are many examples of non-orbit equivalent actions (both actions of different groups and different actions of the same group). Finding invariants for orbit equivalence has proved to be difficult and there are many results which indicate that satisfactory complete invariants in fact do not exist. (It follows from the work of Gaboriau–Popa [23], Törnquist [71], and Kechris [37] that actions of the free group $\mathbf{F}_2$, up to orbit equivalence, are not classifiable by countable structures. Recently this was extended to groups containing $\mathbf{F}_2$ by Ioana [33] and Kechris [37].)

The presence of a measure allows one to define a topology on the full groups which greatly facilitates their study. There are two group topologies on $\mathrm{Aut}(X,\mu)$, introduced by Halmos, which are relevant for us. Recall that the *measure algebra* of (X,μ), denoted by MALG_μ, is the collection of all measurable subsets of X modulo null sets. It becomes an abelian group under the operation of symmetric difference and the metric defined by

$$\rho(A,B) = \mu(A \vartriangle B)$$

is invariant under the group operation. The group $\mathrm{Aut}(X,\mu)$ acts faithfully by topological group automorphisms on MALG_μ. Both of the topologies on $\mathrm{Aut}(X,\mu)$ we consider are given by this action: the *weak topology* is the topology of pointwise convergence on MALG_μ and the *uniform topology* is the topology of uniform convergence. Both topologies are completely metrizable; the first one is separable and the second one is not. A metric compatible with the uniform topology, which is moreover invariant under group multiplication from both sides, is given by

$$d(T,S) = \mu(\{T \neq S\}). \tag{5.1.1}$$

The full groups are closed subgroups in $\mathrm{Aut}(X,\mu)$ in the uniform topology and they turn out to be separable, hence Polish. Unless otherwise mentioned, we assume them to be equipped with this Polish topology. For all of this, see [37].

Full groups were first considered by Dye and our main motivation to study them comes from the following theorem (see [37] for a recent exposition of the proof).

Theorem 5.1.1 (Dye [15])**.** *Let E and F be two countable, measure-preserving, ergodic equivalence relations on the standard probability space (X,μ). Then the following are equivalent:*

(i) E and F are isomorphic;

(ii) $[E]$ and $[F]$ are isomorphic (algebraically);

(iii) there exists $f \in \mathrm{Aut}(X,\mu)$ such that $f[E]f^{-1} = [F]$.

Moreover, every algebraic isomorphism between $[E]$ and $[F]$ is realized by a conjugacy.

Theorem 5.1.1 suggests that the algebraic structure of full groups is rich enough to "remember" the topology since, by the "moreover" assertion, every algebraic automorphism of an ergodic full group is automatically a homeomorphism. We pursue this point further in Section 5.3 where we prove one of our main results (cf. Theorem 5.3.1).

Theorem 5.1.2. *Let E be an ergodic, measure-preserving, countable equivalence relation. Then every homomorphism $f\colon [E] \to G$, where G is a separable topological group, is automatically continuous. In particular, the uniform topology is the finest separable group topology on $[E]$ and hence, the unique Polish topology.*

Hence, the structure of $[E]$ as an abstract group alone is sufficient to recover the topology and any statement about $[E]$ as a topological group can, at least in principle, be translated into a statement referring only to its algebraic structure. Automatic continuity is a phenomenon which appeared recently in the work of Kechris–Rosendal [43], Rosendal–Solecki [60], and Rosendal [59] (see Section 5.3 for more examples and further discussion). Automatic continuity has also implications for group actions on various spaces; for example, any action of a group with this property by homeomorphisms on a compact metrizable space or by linear isometries on a separable Banach space is automatically continuous.

Dye's theorem shows that full groups are complete invariants for orbit equivalence; in order to prove that two equivalence relations are non-isomorphic, it suffices to find a (topological group) property of their full groups which differentiates them. The only known (to the authors) result of that flavor to date is the following.

Theorem 5.1.3 (Giordano–Pestov [25])**.** *Let E be a countable, measure-preserving, ergodic equivalence relation. Then $[E]$ is hyperfinite iff $[E]$ is extremely amenable.*

Recall that a topological group is called *extremely amenable* if every time it acts continuously on a compact space, the action has a fixed point. The property of extreme amenability is enjoying a growing popularity; see Pestov [57] for discussion and references.

Combining Theorems 5.1.2 and 5.1.3 yields the following.

Corollary 5.1.4. *Let E be ergodic, hyperfinite. Then* any *action of $[E]$ by homeomorphisms on a compact, metrizable space has a fixed point.*

One should contrast that with the fact that any discrete group admits a free action on a compact space (see [57]).

Going back to orbit equivalence, one could perhaps hope to distinguish the full groups as topological spaces alone (forgetting the group structure). This, however, turns out to be impossible as they are all homeomorphic (cf. Corollary 5.2.5).

Theorem 5.1.5. *Let E be a countable, measure-preserving equivalence relation on the standard probability space (X, μ) which is not equality a.e. Then the full group $[E]$ with the uniform topology is homeomorphic to the Hilbert space ℓ^2.*

It was previously known that full groups are contractible (this follows from the argument of Keane [36]).

Another possible invariant (suggested by Kechris [37]) one could look at is the number of *topological generators* of $[E]$, denoted by $t([E])$ (i.e., the minimal number of generators of a dense subgroup of $[E]$). Since the group $[E]$ is separable, we always have $t([E]) \leq \aleph_0$. It is also easily seen that $t([E]) \geq 2$ if E is not trivial. Indeed, if E has a positive set of classes of size greater than 2, then $[E]$ is non-abelian and if E is generated by a non-trivial involution, $[E]$ is isomorphic to $(\mathrm{MALG}_\mu, \triangle)$

which is also not monothetic. The next simple observation is that if $\Gamma \leq [E]$ is countable and dense, then Γ must generate E (since $[\Gamma]$ is closed and by density, $[\Gamma] = [E]$). Now Gaboriau's theory of cost [21] (see also Kechris–Miller [42]) allows one to conclude that if E is generated by a free action of $\mathbf{F}_n$, then $t([E]) \geq n$. (In this case, it is not hard to show the slightly better bound $t([E]) \geq n+1$ (Miller); cf. Corollary 5.4.12.) So, to show that $t([E])$ is a non-trivial invariant, it suffices to find equivalence relations E for which $t([E])$ is finite. Our first result in this direction is an upper bound for $t([E])$ for ergodic, hyperfinite E (cf. Theorem 5.4.2).

Theorem 5.1.6. *Let E be ergodic, hyperfinite. Then $t([E]) \leq 3$.*

Using this and an inductive procedure, we further show the following (cf. Corollary 5.4.11).

Theorem 5.1.7. *Let E be an ergodic equivalence relation on (X, μ). Then the following are equivalent:*

(i) E can be generated by an action of a finitely generated group;

(ii) E has finite cost;

(iii) $[E]$ is topologically finitely generated.

Specific calculations for $t([E])$, for example, for E generated by free, ergodic actions of free groups, would allow to distinguish those equivalence relations. It is known that free actions of free groups with different number of generators are orbit inequivalent but the only known way to show that is using cost (see Gaboriau [21]). Here we provide estimates for $t([E])$ which distinguish free, ergodic actions of $\mathbf{F}_m$ and $\mathbf{F}_n$ when m and n are sufficiently far apart. The proof of the lower bound, however, still depends on Gaboriau's results on cost (cf. Corollary 5.4.12).

Theorem 5.1.8. *Let the equivalence relation E be generated by a free, ergodic action of the free group $\mathbf{F}_n$, $n \geq 1$. Then*

$$n + 1 \leq t([E]) \leq 3n + 3.$$

It will be interesting to sharpen those estimates and try to find a proof for the lower bound independent of cost.

The organization of this paper is as follows. In Section 5.2, we prove Theorem 5.1.5; in Section 5.3, we prove the automatic continuity results, and, finally, in Section 5.4, we discuss topological generators.

Below, (X, μ) is a standard probability space and all equivalence relations which we consider are countable and measure-preserving. We habitually ignore null sets if there is no danger of confusion.

If $A \subseteq X$, $A^{\complement}$ denotes the complement of A. If $T \in \mathrm{Aut}(X,\mu)$, $\operatorname{supp} T$ denotes the *support* of T, i.e., the set $\{x \in X : Tx \neq x\}$.

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5.2 Full groups are homeomorphic to ℓ^2

Identifying the topological type of big symmetry groups has been an ongoing enterprise for the last few decades. During that time, infinite-dimensional topology has developed many tools which allow that. Recall that a topological space Y is called an *absolute (neighborhood) retract* (abbreviated A(N)R) if every time it embeds as a closed subspace of a normal space Z, the image of the embedding is a (neighborhood) retract of Z. The prototypical examples of absolute retracts are the convex subsets of normed linear spaces. A basic fact is that an ANR is an AR iff it is contractible (see [76, Theorem 5.2.15]). We recommend the book van Mill [76] as a basic reference for infinite-dimensional topology.

Many groups of interest were proved to be ANRs (see, for example, Luke–Mason [51]). Dobrowolski–Toruńczyk [12] achieved a major breakthrough by showing that every non-locally compact Polish group whose underlying topological space is an ANR is homeomorphic to an ℓ^2-manifold. Moreover, every contractible ℓ^2-manifold is in fact homeomorphic to ℓ^2 (see [7, Chapter IX, Theorem 7.3]). As a result, now many big Polish groups are known to be homeomorphic to ℓ^2. Examples include $\mathrm{Aut}(X,\mu)$ with the weak topology (Nhu [55]), the group of orientation-preserving homeomorphisms of the unit interval (Anderson), the isometry group of the Urysohn space (Melleray [53]), and many others.

Regarding full groups, we have the following.

Theorem 5.2.1. *Let E be a countable, measure-preserving equivalence relation on (X,μ). Then $[E]$ is an absolute retract.*

Proof. Keane [36] showed that $\mathrm{Aut}(X,\mu)$ is contractible in both the weak and the uniform topologies using induced transformations and the same argument shows that $[E]$ is contractible. (We discuss induced transformations below and outline his method.) In order to verify that $[E]$ is an ANR, it suffices to produce a basis $\mathcal{B}$ for its topology with the following property:

$$\text{every nonempty finite intersection of elements of } \mathcal{B} \text{ is homotopically trivial} \qquad (*)$$

(see [76, Theorem 5.2.12]). We will prove that the basis consisting of the open balls of the uniform metric d defined by (5.1.1) has this property.

Let

$$\mathrm{PER} = \{T \in \mathrm{Aut}(X,\mu) : \exists n \; T^n = 1\}.$$

The Rokhlin lemma implies that $\mathrm{PER} \cap [E]$ is uniformly dense in $[E]$. If ξ is a finite partition of X, denote by $\hat{\xi}$ the finite Boolean algebra generated by ξ. We say that a set $B \in \mathrm{MALG}_\mu$ is *independent* of ξ if

$$\forall A \in \xi \quad \mu(A \cap B) = \mu(A)\mu(B).$$

Below, $I = [0,1]$ will be the closed unit interval and S^n will denote the n-dimensional sphere. We split the proof into a sequence of lemmas.

Lemma 5.2.2. *For all $T \in \mathrm{PER}$ and finite partitions ξ of X, there exists a continuous map $B\colon I \to \mathrm{MALG}_\mu$ satisfying the conditions:*

(i) $\mu(B(\lambda)) = \lambda$;

(ii) $\lambda \le \lambda' \implies B(\lambda) \subseteq B(\lambda')$;

(iii) for all $\lambda \in I$, the set $B(\lambda)$ is T-invariant;

(iv) for all $\lambda \in I$, $B(\lambda)$ is independent of ξ.

Proof. Let $T^n = 1$. By splitting X into pieces (and refining ξ appropriately), we can assume that all $x \in X$ have T-orbits of length exactly n. Let

$$\begin{aligned} \xi' &= \xi \vee T\xi \vee \cdots \vee T^{n-1}\xi \\ &= \{A_0 \cap T(A_1) \cap \cdots \cap T^{n-1}(A_{n-1}) : A_0, A_1, \ldots, A_{n-1} \in \xi\}. \end{aligned}$$

The partition ξ' is clearly T-invariant, so we have an action of T on it. Let $\{C_1, C_2, \ldots, C_k\} \subseteq \xi'$ be a transversal for the T-orbits in ξ' and $Y = C_1 \cup C_2 \cup \cdots \cup C_k$. By refining ξ if necessary, we can assume that all orbits have length n. There exists a map $h\colon I \to \mathrm{MALG}_\mu(Y)$ such that

- $\mu(h(\lambda)) = \lambda \cdot \mu(Y)$;
- $\lambda \leq \lambda' \implies h(\lambda) \subseteq h(\lambda')$;
- $\mu(h(\lambda) \cap C_i) = \mu(h(\lambda))\mu(C_i)$, $i = 1, \ldots, k$.

Finally, let

$$B(\lambda) = \bigcup_{j=0}^{n-1} T^j(h(\lambda)).$$

It is clear that this B satisfies the requirements. Continuity follows from (i) and (ii). □

Let ρ be any compatible metric on S^n.

Lemma 5.2.3. *For all $\delta > 0$, there exists $m \in \mathbf{N}$, points $z_1, \ldots, z_m \in S^n$, and a continuous map $g \colon S^n \to \mathbf{R}^m$ such that for all $z \in S^n$, the following conditions are satisfied:*

(i) for all i, $g_i(z) \geq 0$ and $\max_{i \leq m} g_i(z) = 1$;

(ii) at most $n+1$ of the numbers $g_1(z), g_2(z), \ldots, g_m(z)$ are non-zero;

(iii) $\forall i \leq m \quad g_i(z) > 0 \implies \rho(z, z_i) < \delta$,

where g_i denotes the i-th coordinate of g.

Proof. Consider S^n as a simplicial complex such that all of its simplices have ρ-diameter smaller than δ. Let $\{z_1, \ldots, z_m\}$ be the 0-skeleton of the complex. Then for each $z \in S^n$ there is a minimal set $\beta(z) \subseteq \{1, \ldots, m\}$ with $|\beta(z)| \leq n+1$ such that z belongs to the simplex $\{z_i : i \in \beta(z)\}$. Each z can be written uniquely as a convex combination $\sum_{i \in \beta(z)} a_i z_i$. Set

$$g_i(z) = \begin{cases} \frac{a_i}{\max_i a_i} & \text{if } i \in \beta(z), \\ 0 & \text{if } i \notin \beta(z). \end{cases}$$

It is easy to verify that g satisfies the requirements. □

Since the basis of open balls is translation invariant, the following lemma suffices to verify $(*)$.

Lemma 5.2.4. *Let $\mathsf{B}_{r_1}(Q_1), \ldots, \mathsf{B}_{r_k}(Q_k)$ be open balls in $[E]$ ($\mathsf{B}_{r_j}(Q_j)$ denotes the ball with center Q_j and radius r_j). Let $U = \bigcap_{j=1}^k \mathsf{B}_{r_j}(Q_j)$ and suppose that $1 \in U$. Then any continuous map $f \colon (S^n, s_0) \to (U, 1)$, where $s_0 \in S^n$, is nullhomotopic in U.*

Proof. We will build a continuous map $F\colon S^n \times I \to U$ satisfying $F(z,0) = f(z)$, $F(z,1) = 1$, and $F(s_0, \lambda) = 1$ for all $z \in S^n$, $\lambda \in I$.

By the compactness of S^n, there exists $\epsilon > 0$ such that

$$f(S^n) \subseteq \bigcap_i \mathsf{B}_{r_i - 6\epsilon}(Q_i).$$

Let $\delta < \epsilon/(n+1)$ be such that $1/\delta$ is an integer. Since f is uniformly continuous, there exists $\delta' > 0$ such that

$$\forall z, w \in S^n \quad \rho(z,w) < \delta' \implies d(f(z), f(w)) < \delta. \tag{5.2.1}$$

Apply Lemma 5.2.3 with $\delta = \delta'$ to obtain points $z_1, \ldots, z_m \in S^n$ and a map $g\colon S^n \to \mathbf{R}^m$ with the properties described in the lemma. Note that by properties (i) and (iii),

$$\forall z \in S^n \exists i \leq m \quad d(f(z), f(z_i)) < \delta < \epsilon.$$

Let $T_1, T_2, \ldots, T_m \in \mathrm{PER} \cap [E]$ be such that $d(f(z_i), T_i) < \delta$ for all i. Let ξ_0 be the partition of X generated by the collection

$$\big\{\{f(z_i) = Q_j\}, \{Q_j = 1\} : i = 1, \ldots, m; j = 1, \ldots, k\big\}.$$

Apply Lemma 5.2.2 to T_1 and ξ_0 to obtain a map $B_1\colon I \to \mathrm{MALG}_\mu$. Inductively, assuming that B_i and ξ_{i-1} have been built, let ξ_i be generated by ξ_{i-1} and the collection

$$\{B_i(\delta), B_i(2\delta), \ldots, B_i(1)\}$$

and apply Lemma 5.2.2 to T_{i+1} and ξ_i to obtain B_{i+1}. Our construction and Lemma 5.2.2 (iv) ensures that $B_i(\lambda)$ is independent of $B_j(q\delta)$ for all integers $q \leq 1/\delta$ and $j < i$ and all of them are independent of ξ_0. Hence, for any $A \in \hat{\xi}_0$ and any tuple $(q_1, q_2, \ldots, q_m)$ of integers with $q_i \leq 1/\delta$,

$$\begin{aligned}\mu\Big(A \cap \bigcap_{i=1}^{m} B_i(q_i\delta)\Big) &= \mu\Big(A \cap \bigcap_{i=1}^{m-1} B_i(q_i\delta)\Big)\mu(B_m(q_m\delta)) \\ &= \cdots \\ &= \mu(A) \prod_i \mu(B_i(q_i\delta)).\end{aligned} \tag{5.2.2}$$

Now define a map $h\colon S^n \times I \to \mathrm{MALG}_\mu$ by

$$h(z,\lambda) = \bigcap_{i=1}^{m} B_i(1-\lambda g_i(z)).$$

The idea is that $h(z,\lambda)$ is "almost invariant" under $f(z)$ and "almost independent" of the partition ξ_0. This will allow us to show that the induced transformation $f(z)_{h(z,\lambda)}$ is in U and $F(z,\lambda) = f(z)_{h(z,\lambda)}$ will furnish the desired homotopy. The following claim summarizes the properties of h we need.

Claim. The following statements hold for h:

(i) h is continuous;

(ii) $h(z,0) = X$, $h(z,1) = \varnothing$;

(iii) $\mu\big(f(z)(h(z,\lambda)) \triangle h(z,\lambda)\big) < 2\epsilon$;

(iv) for all $A \in \hat{\xi}_0$, $|\mu(h(z,\lambda) \cap A) - \mu(h(z,\lambda))\mu(A)| < 2\epsilon$.

Proof. (i) and (ii) are clear from the definition and the properties of g. We proceed to verify (iii) and (iv). Fix z and λ and set $T = f(z)$. Let $\alpha = \{i : g_i(z) > 0\}$. By Lemma 5.2.3 (ii), $|\alpha| \leq n+1$. Set $D_i = B_i(1-\lambda g_i(z))$ and let

$$C = h(z,\lambda) = \bigcap_{i\in\alpha} D_i.$$

By (5.2.1) and Lemma 5.2.3 (iii),

$$\forall i \in \alpha \quad d(T,T_i) < 2\delta,$$

so, in particular, $\mu(T(A) \triangle T_i(A)) < 2\delta$ for any $A \in \mathrm{MALG}_\mu$, $i \in \alpha$. Using Lemma 5.2.2 (iii), we have:

$$\begin{aligned}
\mu(T(C) \triangle C) &= \mu\Big(T\Big(\bigcap_{i\in\alpha} D_i\Big) \triangle \bigcap_{i\in\alpha} D_i\Big) \\
&\leq \sum_{i\in\alpha} \mu(T(D_i) \triangle D_i) \\
&\leq \sum_{i\in\alpha} \mu(T(D_i) \triangle T_i(D_i)) + \sum_{i\in\alpha} \mu(T_i(D_i) \triangle D_i) \\
&\leq 2|\alpha|\delta + 0 < 2\epsilon
\end{aligned}$$

which verifies (iii).

Now fix $A \in \hat{\xi}_0$. Let for each $i \in \alpha$, $q_i \leq 1/\delta$ be such that

$$|\mu(D_i \triangle B_i(q_i\delta))| \leq \delta.$$

Then

$$\begin{aligned} \mu\Big(C \triangle \bigcap_{i\in\alpha} B_i(q_i\delta)\Big) &= \mu\Big(\bigcap_{i\in\alpha} D_i \triangle \bigcap_{i\in\alpha} B_i(q_i\delta)\Big) \\ &\leq \mu\Big(\bigcup_{i\in\alpha} D_i \triangle B_i(q_i\delta)\Big) \\ &\leq |\alpha|\delta < \epsilon. \end{aligned} \tag{5.2.3}$$

By (5.2.2),

$$\mu\Big(\bigcap_{i\in\alpha} B_i(q_i\delta)\Big) = \prod_{i\in\alpha} \mu(B_i(q_i\delta))$$

and

$$\mu\Big(A \cap \bigcap_{i\in\alpha} B_i(q_i\delta)\Big) = \mu(A) \prod_{i\in\alpha} \mu(B_i(q_i\delta)),$$

which together with (5.2.3), allow us to calculate:

$$\begin{aligned} |\mu(A \cap C) - \mu(A)\mu(C)| &< \Big|\mu\Big(A \cap \bigcap_{i\in\alpha} B_i(q_i\delta)\Big) - \mu(A)\mu\Big(\bigcap_{i\in\alpha} B_i(q_i\delta)\Big)\Big| + 2\epsilon \\ &= 2\epsilon, \end{aligned}$$

verifying (iv). □

For $T \in [E]$ and $A \in \mathrm{MALG}_\mu$, let $T_A \in [E]$ denote the *induced transformation*, i.e.,

$$T_A(x) = \begin{cases} x, & \text{if } x \notin A, \\ T^{s(x)}(x), & \text{if } x \in A, \end{cases}$$

where $s(x)$ is the least $s > 0$ for which $T^s(x) \in A$. T_A is well defined (almost everywhere) by Poincaré's recurrence lemma and as follows from Keane [36], the map $(A, T) \mapsto T_A$ is continuous $\mathrm{MALG}_\mu \times [E] \to [E]$. (From here it is not hard to see that $[E]$ is contractible. Indeed, identifying (X, μ) with (I, Lebesgue measure) and defining $C\colon I \to \mathrm{MALG}_\mu$ by $C(\lambda) = [0, \lambda]$, it is immediate that the map $I \times [E] \to [E]$ given by $(\lambda, T) \mapsto T_{C(\lambda)}$ is a contraction.)

Now set

$$F(z,\lambda) = f(z)_{h(z,\lambda)}.$$

We only need to verify that $F(S^n, I) \subseteq U$. Fix $z \in S^n$, $\lambda \in I$, and $j \in \{1,\dots,k\}$. Set $T = f(z)$, $C = h(z,\lambda)$. Find z_i such that $d(T, f(z_i)) < \epsilon$. Note also that by (iii) of the Claim,

$$\mu(C \cap \{T \neq T_C\}) = \mu(C \setminus T^{-1}(C)) = \mu(T(C) \setminus C) < 2\epsilon. \tag{5.2.4}$$

Using (iv) of the Claim, (5.2.4), and the choice of ϵ, for all $j = 1,\dots,k$, we have:

$$\begin{aligned}
d(Q_j, T_C) &= \mu(\{Q_j \neq T_C\}) \\
&= \mu(C \cap \{Q_j \neq T_C\}) + \mu(C^{\mathsf{C}} \cap \{Q_j \neq T_C\}) \\
&\leq \mu(C \cap \{Q_j \neq f(z_i)\}) + \mu(C \cap \{f(z_i) \neq T\}) \\
&\qquad + \mu(C \cap \{T \neq T_C\}) + \mu(C^{\mathsf{C}} \cap \{Q_j \neq 1\}) \\
&\leq (\mu(C)\mu(\{Q_j \neq f(z_i)\}) + 2\epsilon) + d(T, f(z_i)) \\
&\qquad + 2\epsilon + (\mu(C^{\mathsf{C}})\mu(\{Q_j \neq 1\}) + \epsilon) \\
&\leq \mu(C)d(Q_j, f(z_i)) + \mu(C^{\mathsf{C}})d(Q_j, 1) + 6\epsilon \\
&< (r_j - 6\epsilon) + 6\epsilon = r_j.
\end{aligned}$$

Hence, $T_C \in \mathrm{B}_{r_j}(Q_j)$ for all j and we are done. □

This completes the proof of the theorem. □

Corollary 5.2.5. *Let E be a countable, measure-preserving equivalence relation on (X,μ) which is not equality a.e. Then $[E]$ is homeomorphic to ℓ^2.*

Proof. By the theorem of Dobrowolski–Toruńczyk cited above it suffices to check that $[E]$ is not locally compact. This follows from the simple observation that every non-trivial full group contains an involution and the full group of any involution is isomorphic (as a topological group) to $(\mathrm{MALG}_\mu, \triangle)$ which is easily verified to not be locally compact. □

Remark. In fact, Bessaga–Pełczyński [7] have shown that MALG_μ is homeomorphic to ℓ^2. Corollary 5.2.5 can be considered a generalization of this result since if T is a non-trivial involution, the full group $[T]$ is isomorphic to $(\mathrm{MALG}_\mu, \triangle)$.

5.3 Automatic continuity

The phenomenon of automatic continuity of group homomorphisms has recently enjoyed a lot of attention. Classical results state that Baire measurable homomorphisms between Polish groups (Pettis) and measurable homomorphisms between locally compact groups (Kleppner [47]) are necessarily continuous but recently such results have been obtained (for particular source groups) without any restrictions on the homomorphisms except that their target be separable. One way of thinking of this strong automatic continuity property is that the algebraic structure of the groups possessing it is so rigid that the axiom of choice is unable to produce pathological (non-continuous in the natural topology) homomorphisms. Kechris–Rosendal [43] showed that every homomorphism from a group with ample generics (including the group of permutations of the integers, the group of measure-preserving homeomorphisms of the Cantor space and the Lipschitz homeomorphisms of the Baire space) to a separable group is continuous; Rosendal–Solecki [60] proved the same result for the homeomorphism group of the Cantor space, the group of order-preserving automorphisms of $\mathbf{Q}$, and the orientation-preserving homeomorphisms of the real line and the circle; and Rosendal [59] proved automatic continuity for the homeomorphism groups of compact 2-manifolds. For details and further discussion, see those three papers.

Rosendal–Solecki [60] introduced a property implying automatic continuity which we proceed to describe. Let G be any topological group. A set $W \subseteq G$ is called *countably syndetic* if countably many left translates of W cover G. We say that G is *Steinhaus* if there exists a number n such that for any symmetric, countably syndetic set W, W^n contains an open neighborhood of the identity. It is proved in [60] that if G is Steinhaus, then any homomorphism $f \colon G \to H$, where H is an arbitrary separable topological group, is continuous.

Theorem 5.3.1. *Let E be an ergodic, measure-preserving, countable equivalence relation on the standard probability space (X, μ). Then the full group $[E]$ is Steinhaus. In particular, every homomorphism from $[E]$ to a separable group is continuous.*

Proof. We borrow ideas and methods from [60]. Fix $W \subseteq [E]$ a symmetric, countably syndetic set, i.e., let there exist $k_1, k_2, \ldots \in [E]$ such that

$$\bigcup_n k_n W = [E]. \tag{5.3.1}$$

For $B \in \mathrm{MALG}_\mu$, denote by H_B the subgroup of $[E]$ consisting of the transformations whose support is contained in B. Say that a set $U \subseteq [E]$ is *full* for B if

$$\forall T \in H_B \exists S \in U \quad T|_B = S|_B.$$

We will use repeatedly and without mentioning the simple fact that because of the ergodicity of E, for all pairs of sets $A, B \in \mathrm{MALG}_\mu$ of the same measure, there exists an involution $T \in [E]$ with $T(A) = B$ and $\operatorname{supp} T \subseteq A \cup B$ (see [42, Lemma 7.10]). The following lemma and its proof are similar to Claim 1 in the proof of [60, Theorem 12].

Lemma 5.3.2. *Let $\{B_1, B_2, \dots\}$ be a collection of pairwise disjoint subsets of X. Then there exists $n \in \mathbf{N}$ such that W^2 is full for B_n.*

Proof. It suffices to show that $k_n W$ is full for B_n for some n since then

$$W^2 = (k_n W)^{-1}(k_n W)$$

is also full for B_n. Suppose this is not the case. Then for each n, there exists $T_n \in H_{B_n}$ such that for all $S \in k_n W$, $S|_{B_n} \neq T_n|_{B_n}$. Define $T \in [E]$ by

$$Tx = \begin{cases} T_n x & \text{if } x \in B_n, \\ x & \text{if } x \notin \bigcup_n B_n. \end{cases}$$

Then $T \notin k_n W$ for all n, contradicting (5.3.1). □

Lemma 5.3.3. *There exists a non-empty $B \in \mathrm{MALG}_\mu$ such that $H_B \subseteq W^{36}$.*

Proof. Let $\{B_1, B_2, \dots\}$ be any collection of pairwise disjoint non-empty subsets of X. By Lemma 5.3.2, there is n_0 such that W^2 is full for B_{n_0}. Set $B' = B_{n_0}$. We will show that W^2 contains a non-trivial involution whose support is contained in B'. Indeed, let $T \in [E]$ be any involution with $\operatorname{supp} T = B'$. The group $[T] < [E]$ is uncountable and separable, hence there exists n and $S_1, S_2 \in [T]$ such that $0 < d(S_1, S_2) < \mu(B')/2$ and $S_1, S_2 \in k_n W$. Set $S = S_1 S_2$. Then

$$S = S_1 S_2 = S_1^{-1} S_2 \in (k_n W)^{-1}(k_n W) = W^2$$

and

$$\mu(\operatorname{supp} S) = d(1, S) = d(S_1, S_2) < \mu(B')/2.$$

Note that every involution $U \in H_{B'}$ with $\mu(\operatorname{supp} U) = \mu(\operatorname{supp} S)$ is conjugate to S in $H_{B'}$. Indeed, let $C \subseteq \operatorname{supp} S$ and $D \subseteq \operatorname{supp} U$ be such that $C \cup S(C) = \operatorname{supp} S$, $C \cap S(C) = \varnothing$, $D \cup S(D) = \operatorname{supp} U$, $D \cap U(D) = \varnothing$. Find $V_1 \in H_{B'}$ such that $V_1(C) = D$. Define $V_2 \in H_{B'}$ by

$$V_2 x = \begin{cases} V_1 x & \text{if } x \in C, \\ U V_1 S x & \text{if } x \in S(C) \end{cases}$$

on $\operatorname{supp} S$ and extend it arbitrarily to an element of $H_{B'}$. Then it is easy to check that $V_2 S V_2^{-1} = U$. By the fullness of W^2 for B', there exists $V_3 \in W^2$ such that $V_3|_{B'} = V_2$. Then $U = V_3 S V_3^{-1}$ and thus, $U \in W^6$.

Let now $B \subseteq B'$ be such that $\operatorname{supp} S \subseteq B$ and $\mu(B) = 2\mu(\operatorname{supp} S)$ and note that every involution in H_B can be written as a product $U_1 U_2$ where $U_1, U_2 \in H_{B'}$ are involutions with $\mu(\operatorname{supp} U_1) = \mu(\operatorname{supp} U_2) = \mu(\operatorname{supp} S)$. Therefore all involutions in H_B are contained in W^{12}. Finally, by the argument in Ryzhikov [61], every element of H_B is the product of three involutions, so $H_B \subseteq W^{36}$. □

Now we are ready to complete the proof of the theorem. Fix B as given by Lemma 5.3.3. Let $T_1, T_2, \dots$ be a sequence of elements of $[E]$ with $\lim_{n\to\infty} T_n = 1$. We will show that some T_m is in W^{38}, thus proving that W^{38} contains an open neighborhood of 1. Set $C_n = \operatorname{supp} T_n$. By passing to a subsequence, we can assume that $\sum_n \mu(C_n) < \mu(B)$. Let

$$D = \bigcup_n k_n(C_n).$$

Fix $A \subseteq B$ with $\mu(A) = \mu(D)$. There exists $S' \in [E]$ such that $S'(A) = D$. By (5.3.1), there is m such that $S' \in k_m W$. If we set $S = k_m^{-1} S'$, we have $S \in W$ and

$$S(A) = k_m^{-1}(D) \supseteq C_m.$$

Hence, $S^{-1} T_m S \in H_B \subseteq W^{36}$ and

$$T_m = S(S^{-1} T_m S) S^{-1} \in W W^{36} W = W^{38},$$

proving that $[E]$ is Steinhaus with exponent 38. □

Remark. Note that the condition of ergodicity cannot be omitted. If T is a non-trivial involution, the group $[T]$ is isomorphic to $(\mathrm{MALG}_\mu, \triangle)$ and the latter admits many different topologies. Indeed, MALG_μ is a vector space over $\mathbf{Z}/2\mathbf{Z}$, hence has a Hamel basis, hence is isomorphic to any abelian group of exponent 2 and cardinality continuum (for example, $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$).

5.4 Topological generators

5.4.1 The hyperfinite case

In order to distinguish equivalence relations by the number of topological generators of their full groups, the first thing one has to show is that this number is not always infinite. We start with the simplest case, the equivalence relation generated by a single ergodic automorphism. In this subsection, E will always be ergodic and hyperfinite. By Dye's theorem, up to isomorphism, there exists only one hyperfinite, ergodic equivalence relation, so we have the flexibility to consider any measure-preserving, ergodic automorphism as the generator of E. In order to produce a finitely generated dense subgroup of $[E]$, we will use a specific minimal homeomorphism of the Cantor space as a topological model and the theory of topological full groups of minimal homeomorphisms, as developed by Giordano–Putnam–Skau [26]. In fact, our dense subgroup comes directly from an example of Matui [52].

We recall some concepts and definitions from [26]. Let ϕ be an *aperiodic* (with all of its orbits infinite) homeomorphism of the Cantor space X. For every homeomorphism γ of X preserving the orbits of ϕ, define its associated cocycle $n_\gamma \colon X \to \mathbf{Z}$, where $\mathbf{Z}$ denotes the discrete group of the integers, by

$$n_\gamma(x) = n \iff \gamma(x) = \phi^n(x).$$

Define the *topological full group* of ϕ by

$$[[\phi]] = \{\gamma \in \mathrm{Homeo}(X) : \forall x \; \exists n \in \mathbf{Z} \; \gamma(x) = \phi^n(x) \text{ and } n_\gamma \text{ is continuous}\}. \tag{5.4.1}$$

Since there are only countably many continuous functions $X \to \mathbf{Z}$, the group $[[\phi]]$ is always countable. It is a remarkable fact, discovered by Matui, that those groups are sometimes finitely generated. Below we prove that topological full groups of minimal homeomorphisms are dense in the measure-theoretic full group and thus provide a rich supply of finitely-generated dense subgroups of $[E]$.

Let ϕ be a minimal homeomorphism of the Cantor space X and μ be any ϕ-invariant, Borel, probability measure on X. Define the *index map* $I\colon [[\phi]] \to \mathbf{R}$ by:

$$I(\gamma) = \int n_\gamma \, \mathrm{d}\mu.$$

It turns out to be a homomorphism $[[\phi]] \to \mathbf{Z}$ which does not depend on the choice of μ. Its kernel is denoted by $[[\phi]]_0$. For all of this, see [26].

Proposition 5.4.1. *Let ϕ be a minimal homeomorphism of the Cantor space X and μ be an invariant measure. Let E be the equivalence relation induced by ϕ. Then the countable group $[[\phi]]_0$ is dense in $[E]$.*

Proof. By [37, Proposition 3.7], it is sufficient to show that given pairwise disjoint clopen subsets $A_1, \dots, A_k \subseteq X$, integers $n_1, \dots, n_k$ such that the sets $\phi^{n_1}(A_1), \dots, \phi^{n_k}(A_k)$ are pairwise disjoint, and $\epsilon > 0$, we can produce $\psi \in [[\phi]]_0$ with the property

$$\mu(\{\psi \neq \phi^{n_i}\} \cap A_i) < \epsilon \quad \text{for all } i = 1, \dots, k.$$

Let N_0 be an integer so big that $|n_i|/N_0 < \epsilon$ for all i. Let $D \subseteq X$ be clopen with

$$D, \phi(D), \dots, \phi^{N_0 - 1}(D) \quad \text{disjoint.} \tag{5.4.2}$$

Build a Kakutani–Rokhlin stack with base D compatible with the sets $A_i, \phi^{n_i}(A_i)$, $i = 1, \dots, k$, i.e., find numbers $m \in \mathbf{N}$ (the number of towers in the stack) and $J(1), \dots, J(m) \in \mathbf{N}$ (the heights of the towers), and a clopen partition

$$\{Z(l, j) : l = 1, \dots, m; j = 0, \dots, J(l) - 1\}$$

of X satisfying the conditions:

- $\phi(Z(l, j)) = Z(l, j+1)$, $j = 0, \dots, J(l) - 2$;
- $\phi(\bigcup_{l=1}^m Z(l, J(l) - 1)) = \bigcup_{l=1}^m Z(l, 0)$;
- $\bigcup_{l=1}^m Z(l, 0) = D$;
- each one of the sets $A_i, \phi^{n_i}(A_i)$ $(i = 0, \dots, k-1)$ is the union of some elements of the partition.

For details on how to achieve this, see Herman–Putnam–Skau [30, Lemma 4.1]. Because of (5.4.2), the height of each tower in the stack is at least N_0. Let σ'_l be the partial function $J(l) \to J(l)$ (as customary, we identify $J(l)$ with the set $\{0, 1, \dots, J(l) - 1\}$) defined by

$$\sigma'_l(a) = b \iff \exists i \; Z(l, a) \subseteq A_i \text{ and } b = a + n_i.$$

Since the collections $\{A_i\}_{i<k}$ and $\{\phi^{n_i}(A_i)\}_{i<k}$ are each pairwise disjoint, σ'_l is injective, so it extends to a permutation $\sigma_l \in S_{J(l)}$. Finally, define

$$\psi(x) = \phi^n(x) \iff x \in Z(l, j) \text{ and } n = \sigma_l(j) - j.$$

For the value of the index homomorphism, we have the following simple computation (using that μ is invariant under ϕ and hence, $\mu(Z(l, j_1)) = \mu(Z(l, j_2))$ for $j_1, j_2 < J(l)$):

$$I(\psi) = \int n_\psi \, \mathrm{d}\mu = \sum_{l=1}^{m} \mu(Z(l, 0)) \sum_{j=0}^{J(l)-1} (\sigma_l(j) - j) = 0.$$

On each of the A_is, ψ agrees with ϕ^{n_i} everywhere except possibly on the set

$$U_i = \bigcup_l \bigcup_{j \in C_{l,i}} Z(l, j)$$

where $C_{l,i} = \{0, \dots, 1 - n_i\}$ if $n_i < 0$ and $C_{l,i} = \{J(l) - n_i, \dots, J(l) - 1\}$ if $n_i \geq 0$. But

$$\mu(U_i) = \sum_{l=1}^{m} \frac{|n_i|}{J(l)} \mu\Big(\bigcup_{j<J(l)} Z(l, j) \Big) \leq \frac{|n_i|}{N_0} < \epsilon,$$

so we are done. □

Matui [52] has recently characterized the minimal homeomorphisms ϕ for which the group $[[\phi]]$ (or, equivalently, the group $[[\phi_0]]$) is finitely generated. He proved that $[[\phi]]$ is finitely generated iff ϕ is conjugate to a minimal subshift and satisfies an additional technical condition. He also calculated in a few examples specific generators for those groups. Currently, there are examples of minimal homeomorphisms whose topological full groups have 3 generators [52, Examples 6.1 and 6.2]. It is not known whether such topological full groups with 2 generators exist.

Thus, by Proposition 5.4.1, we have the following theorem which answers a question of Kechris.

Theorem 5.4.2. *Let E be an ergodic, hyperfinite equivalence relation. Then $t([E]) \leq 3$.*

Since $[E]$ is not abelian, we must have $t([E]) \geq 2$. However, we do not know what the exact value of $t([E])$ is.

Question 5.4.3. Let E be ergodic, hyperfinite. Is $t([E])$ equal to 2 or 3?

Remark. The same method for producing dense subgroups of $[E]$ (using a Cantor topological model) works for non-hyperfinite equivalence relations as well. Miller [54] has shown that if X is a Cantor space, Γ is a countable group acting on X by homeomorphisms, and μ is a Γ-invariant probability measure, then the topological full group $[[\Gamma]]$ (defined by a formula analogous to (5.4.1)) is dense in $[E_\Gamma^X]$. Unfortunately, little is known about topological full groups which arise from actions of groups other than $\mathbf{Z}$ (especially in the non-amenable case).

It is also interesting to try to find an elementary construction of a dense subgroup of $[E]$ (for a hyperfinite E) with few generators for some concrete realization of E. (The examples of Matui are concrete enough but his computations of the generators rely heavily on the C^*-algebraic machinery of Giordano–Putnam–Skau.) In this direction, Kittrell [45], using E_0 (the equivalence relation on $(\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$ generated by the action of the subgroup $(\mathbf{Z}/2\mathbf{Z})^{<\mathbf{N}}$ by translation) as a model and purely combinatorial techniques, has found 18 generators for a dense subgroup of $[E]$.

5.4.2 The general case

Theorem 5.4.2 provides an example of a situation where $[E]$ is topologically finitely generated. Below, we develop techniques to characterize exactly when this happens. The following proposition is the main tool we shall use.

If G is a group and $A \subseteq G$ a subset, denote by $\langle A \rangle$ the subgroup of G generated by A.

Proposition 5.4.4. *Let (X, μ) be a standard probability space and $T_0, T_1, \ldots \in \mathrm{Aut}(X, \mu)$ be involutions. Let E be the equivalence relation generated by them. Then $\langle \bigcup_{j \in \mathbf{N}} [T_j] \rangle$ is dense in $[E]$.*

Proof. Since $[E]$ is generated by involutions, it suffices to approximate involutions. Fix $I \in [E]$, $I^2 = 1$. For $T \in \mathrm{Aut}(X, \mu)$ and $A \subseteq X$ such that $A \cap T(A) = \emptyset$, define $T^A \in [T]$ by

$$T^A x = \begin{cases} Tx & \text{if } x \in A, \\ T^{-1}x & \text{if } x \in T(A), \\ x & \text{otherwise.} \end{cases}$$

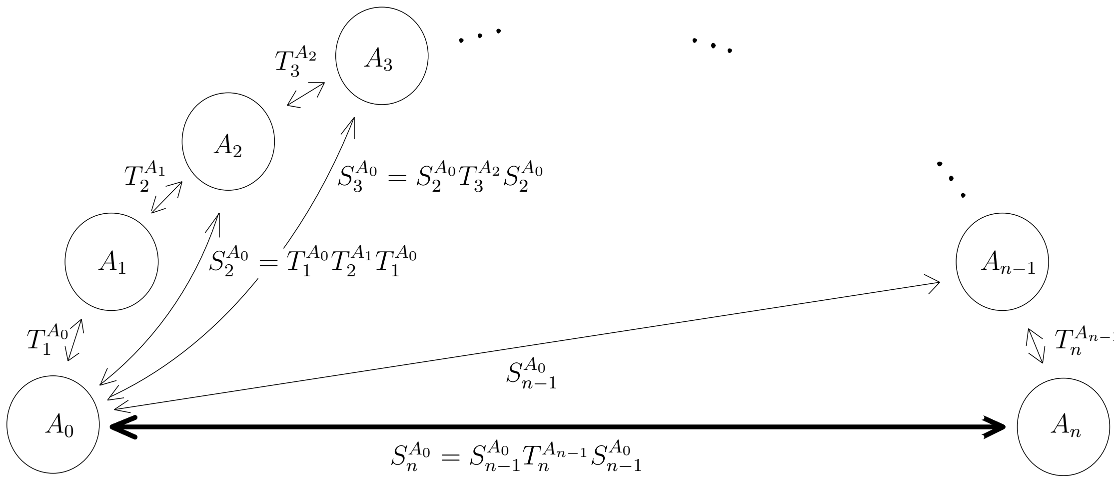

Figure 5.1: Building $S_n^{A_0}$ out of $T_1^{A_0}, T_2^{A_1}, \ldots, T_n^{A_{n-1}}$

The proof proceeds by first showing that we can approximate I^A for certain well chosen sets A and then gluing those together using Lemma 5.4.6 in order to approximate I.

Lemma 5.4.5. *Let $T_1, \ldots, T_n \in \mathrm{Aut}(X, \mu)$ and set $S_k = T_k T_{k-1} \cdots T_1$ for $k = 1, \ldots, n$. Let $A_0 \subseteq X$ be Borel and define $A_k = S_k(A_0)$ for $k = 1, \ldots, n$. Suppose that $A_0, A_1, \ldots, A_n$ are pairwise disjoint. Then $S_n^{A_0} \in \langle T_1^{A_0}, T_2^{A_1}, \ldots, T_n^{A_{n-1}} \rangle$.*

Proof. We have $S_1^{A_0} = I_1^{A_0}$ by definition. It will then be enough to show that

$$S_{k+1}^{A_0} = S_k^{A_0} T_{k+1}^{A_k} S_k^{A_0} \text{ for } k = 1, \ldots, n-1.$$

This can best be seen from the diagram on Figure 5.1. □

Lemma 5.4.6. *Let $T \in \mathrm{Aut}(X, \mu)$ be an involution and $\{X_n\}_{n \in \mathbf{N}}$ be a Borel partition of X. Let*

$$\Phi = \{T^A : A \cap T(A) = \varnothing \textit{ and } A \subseteq X_n \textit{ for some } n\}.$$

Then $T \in \overline{\langle \Phi \rangle}$.

Proof. By refining the partition if necessary, we can assume that it is T-invariant. Let for each n, A_n be a Borel transversal for $T|_{X_n}$, i.e., $A_n \subseteq X_n$, $A_n \cap T(A_n) = \varnothing$, $A_n \cup T(A_n) = X_n$. Then

$$T = \prod_{n \in \mathbf{N}} T^{A_n} \tag{5.4.3}$$

and since for each N, $\prod_{n<N} T^{A_n} \in \langle\Phi\rangle$, we are done (all of the terms in the product (5.4.3) have disjoint supports, hence commute, and the product converges). □

Now return to the proof of Proposition 5.4.4. Let $\mathcal{G} = \bigcup_{j\in\mathbf{N}} \operatorname{graph} T_j$. $\mathcal{G}$ is a Borel graph on X whose connected components are the E-equivalence classes. Label the edge (x, y) of $\mathcal{G}$ with j if $T_j x = y$ (some edges may have more than one label). For $s \in \mathbf{N}^{<\mathbf{N}}$, denote by $|s|$ the length of s. For each $x \in X$, let $s_x \in \mathbf{N}^{<\mathbf{N}}$ be the sequence of labels of the lexicographically least among the shortest $\mathcal{G}$-paths from x to $I(x)$ so that

$$T_{s_x(|s_x|-1)} T_{s_x(|s_x|-2)} \cdots T_{s_x(0)}(x) = I(x).$$

For $x \in X$ and $k \le |s_x|$, set

$$J_k(x) = T_{s_x(k-1)} T_{s_x(k-2)} \cdots T_{s_x(0)}(x).$$

(Thus $\mathrm{id} = J_0, J_1, J_2, \ldots$ are partial automorphisms of E.) By the choice of s_x, the points

$$x, J_1(x), \ldots, J_{|s_x|}(x) \text{ are distinct.} \tag{5.4.4}$$

The mapping $x \mapsto s_x$ is clearly Borel. For each $s \in \mathbf{N}^{<\mathbf{N}}$, let

$$X_s = \{x \in X : s_x = s\}.$$

By Lemma 5.4.6, in order to approximate I, it suffices to approximate I^A for sets A for which $A \cap I(A) = \varnothing$ and $A \subseteq X_s$ for some s. Fix such A and s. Let $\mathcal{B}$ be a countable dense subalgebra of MALG_μ. By (5.4.4), for each $x \in A$, there exist pairwise disjoint $U_0, \ldots, U_{|s|} \in \mathcal{B}$ such that $J_i(x) \in U_i$ for all $i \le |s|$. Let $\mathcal{A}$ be the countable set of all sequences $\alpha = (U_0, U_1, \ldots, U_{|s|})$ of pairwise disjoint elements of $\mathcal{B}$ of length $|s| + 1$. For $\alpha \in \mathcal{A}$, let

$$A_\alpha = \{x \in A : \forall i \le |s| \; J_i(x) \in \alpha(i)\}.$$

Thus $\bigcup_{\alpha\in\mathcal{A}} A_\alpha = A$. Let $\{\alpha_n\}_{n\in\mathbf{N}}$ be an enumeration of $\mathcal{A}$ and inductively define

$$B_n = A_{\alpha_n} \setminus \bigcup_{j<n} B_j.$$

Then $\{B_0, B_1, \ldots\}$ is a partition of A and for each n, the sets $B_n, J_1(B_n), \ldots, J_{|s|}(B_n)$ are pairwise disjoint (since $J_i(A_\alpha) \subseteq \alpha(i)$ by the definition of A_α). By Lemma 5.4.5, $I^{B_n} \in \langle \bigcup_{j\in\mathbf{N}}[T_j] \rangle$ for all n and applying Lemma 5.4.6 again shows that $I^A \in \overline{\langle \bigcup_{j\in\mathbf{N}}[T_j] \rangle}$. □

Recall that if $\{E_1, E_2, \dots\}$ is a countable collection of equivalence relations, their *join*, denoted by $E_1 \vee E_2 \vee \cdots$, is the smallest equivalence relation which contains all of them. Since every equivalence relation is generated by involutions, Proposition 5.4.4 generalizes to the following.

Theorem 5.4.7. *Let $E_1, E_2, \dots$ be countable, measure-preserving equivalence relations on (X, μ) and E be their join. Then $\langle \bigcup_{n\in\mathbf{N}}[E_n] \rangle$ is dense in $[E]$.*

In order to continue our analysis, we will need the notion of cost of an equivalence relation introduced by Levitt and further developed by Gaboriau. We briefly recall the definition and refer the reader to [42] for more details. If E is an equivalence relation, we denote by $[[E]]$ the set of all partial automorphisms of E, i.e., all partial Borel bijections of X whose graphs are contained in E. Since E is measure-preserving, for all $\psi \in [[E]]$, $\mu(\operatorname{dom}\psi) = \mu(\operatorname{rng}\psi)$. An *L-graphing* of an equivalence relation E is a countable subset $\Psi \subseteq [[E]]$ such that E is the smallest equivalence relation containing the graphs of all elements of Ψ. The *cost* of Ψ is defined as

$$\operatorname{cost}\Psi = \sum_{\psi\in\Psi} \mu(\operatorname{dom}\psi)$$

and the cost of E is given by

$$\operatorname{cost} E = \inf\{\operatorname{cost}\Psi : \Psi \text{ is an L-graphing of } E\}.$$

The cost can be finite or infinite and if E is ergodic, $\operatorname{cost} E \geq 1$. One of the main results of Gaboriau's theory [21] is that if E is generated by a free, ergodic action of $\mathbf{F}_n$, $\operatorname{cost} E = n$, i.e., the L-graphing given by the group generators is optimal in this case.

The following lemma was proved by Ben Miller.

Lemma 5.4.8. *Let E be an ergodic equivalence relation of cost less than n. Then there exist finite equivalence relations $F_1, \dots, F_n$ such that $F_1 \vee \cdots \vee F_n = E$.*

Proof. Since E is ergodic, $\operatorname{cost} E \geq 1$ and hence, $n \geq 2$. By [42, Lemma 27.7] and its proof, there exist $\phi_1, \dots, \phi_{n-1} \in [E]$ and $\psi \in [[E]]$ with $\mu(\operatorname{dom}\psi) < 1$ such that $\phi_1, \dots, \phi_{n-1}, \psi$ generate E and, moreover, ϕ_1 is ergodic. Let $E_1, \dots, E_{n-1}$ denote the orbit equivalence relations of $\phi_1, \dots, \phi_{n-1}$, respectively. We will build $F_1, \dots, F_{n-1}$ as finite approximations of $E_1, \dots, E_{n-1}$ and use F_n to glue the pieces together.

Without loss of generality, we can assume that $\phi_2, \dots, \phi_{n-1}$ are aperiodic. (If, say, ϕ_2 is periodic on the positive set D, we can set F_2 to be equal to $E_{\phi_2}^D$ on D and proceed with the aperiodic part exactly as below.) Set $B = \operatorname{dom}\psi$ and let $\epsilon < (1-\mu(B))/2n$. Let $A_1, \dots, A_{n-1}$ be complete sections for $E_1, \dots E_{n-1}$ such that $\mu(A_i) < \epsilon$ and $A_i \cap \phi_i(A_i) = \varnothing$ for $i = 1, \dots, n-1$. Since $\phi_1, \dots, \phi_{n-1}$ are aperiodic, A_i is ϕ_i-birecurrent. For each $i = 1, \dots, n-1$, define the finite equivalence relation F_i by:

$$x \, F_i \, y \iff \exists n \in \mathbf{Z} \; \phi_i^n(x) = y \text{ and } \forall k \in (0, n] \cup (n, 0] \; \phi_i^k(x) \notin A_i, \tag{5.4.5}$$

i.e., F_i is given by splitting the orbits of ϕ_i into finite pieces using the complete section A_i. Define $\xi_i \in [[E]]$ to be the involution $\phi_i|_{A_i} \cup \phi_i^{-1}|_{\phi_i(A_i)}$. Note that the equivalence relation generated by F_i and ξ_i is E_i. Set $B_1 = \operatorname{dom}\xi_1$ and let $B_2, B_3, \dots, B_{n-1}$ be disjoint subsets of $X \setminus (B \cup B_1)$ such that $\mu(B_i) = \mu(\operatorname{dom}\xi_i) = 2\mu(A_i)$. Let $\theta' \in [E_1]$ be such that $\theta'(\operatorname{rng}\psi) = B$ and let $\theta_i \in [E_1]$ be such that $\theta_i(\operatorname{dom}\xi_i) = B_i$ for $i = 2, \dots, n-1$. Define $\psi' = \theta'\psi$ and $\eta_i = \theta_i \xi_i \theta_i^{-1}$ for $i = 2, \dots, n-1$. Note that ψ' is an automorphism of B. Again, without loss of generality, we can assume that ψ' is aperiodic. Let C be a complete section for $E_{\psi'}^B$ such that $\mu(C) < \epsilon$ and $C \cap \psi'(C) = \varnothing$. Let the finite equivalence relation F_n' on B be the splitting of the orbits of ψ' into finite pieces using the complete section C (defined by a formula similar to (5.4.5)). Define $\xi_0 \in [[E]]$ to be the involution $\psi'|_C \cup \psi'^{-1}|_{\psi'(C)}$. Let B_0 be a set of measure $\mu(\operatorname{dom}\xi_0)$ disjoint from $B \cup B_1 \cup B_2 \cup \dots \cup B_{n-1}$. Let $\theta_0 \in [E_1]$ be such that $\theta_0(\operatorname{dom}\xi_0) = B_0$ and define $\eta_0 = \theta_0 \xi_0 \theta_0^{-1}$. Finally, define F_n by

$$F_n = F_n' \cup E_{\eta_0}^{B_0} \cup E_{\eta_1}^{B_1} \cup \dots \cup E_{\eta_{n-1}}^{B_{n-1}} \cup \operatorname{id}|_{X \setminus (B \cup B_0 \cup B_1 \cup \dots \cup B_{n-1})}.$$

Now it is easy to see that $E_i \subseteq E_1 \vee F_i \subseteq F_1 \vee F_n \vee F_i$ for $i = 1, \dots, n-1$ and $E_\psi^X \subseteq E_1 \vee F_n \subseteq F_1 \vee F_n$, showing that $F_1 \vee \dots \vee F_n = E$. □

Lemma 5.4.9. *Let $F \subseteq E$ be equivalence relations on (X, μ) where F is finite and E is ergodic. Then there exists an ergodic, hyperfinite equivalence relation E' such that $F \subseteq E' \subseteq E$.*

Proof. Since F is finite, the space $Y = X/F$ is standard Borel. Let $\pi \colon X \to Y$ be the canonical projection. Set $\nu = \pi_* \mu$ and define the equivalence relation E/F on Y by

$$y_1 \, E/F \, y_2 \iff \exists x_1, x_2 \in X \quad x_1 \, E \, x_2 \text{ and } \pi(x_1) = y_1 \text{ and } \pi(x_2) = y_2.$$

Since μ is non-atomic, ν is non-atomic and since E is μ-ergodic, E/F is ν-ergodic. Pick any ergodic $T \in [E/F]$ (such a T exists by [37, Theorem 3.5]) and let F' be the equivalence relation on Y generated by T. Finally, let $E' = \pi^{-1}(F')$. We will check that this E' works.

The inclusions $F \subseteq E' \subseteq E$ are obvious. Next, the ergodicity of E' follows from the ergodicity of F'. Finally, write $F' = \bigcup_n F'_n$ as the increasing union of finite equivalence relations on Y. Let for each n, $E'_n = \pi^{-1}(F'_n)$. Since F is finite, all the E'_ns are finite. Also, $E' = \bigcup_n E'_n$ and the union is clearly increasing, so E' is hyperfinite. □

Theorem 5.4.10. *Let E be an ergodic equivalence relation with $\operatorname{cost} E < n$ for some $n \in \mathbf{N}$. Then $t([E]) \leq 3n$.*

Proof. By Lemma 5.4.8, there exist finite equivalence relations $F_1, \ldots, F_n$ such that $\bigvee_{i=1}^n F_i = E$. Use Lemma 5.4.9 to find, for each $i \leq n$, an ergodic, hyperfinite equivalence relation E_i such that $F_i \subseteq E_i \subseteq E$. Then, $E = \bigvee_i E_i$ and applying Theorem 5.4.2 and Theorem 5.4.7, we obtain the desired upper bound. □

Corollary 5.4.11. *Let E be an ergodic equivalence relation on (X, μ). Then the following are equivalent:*

(i) E can be generated by an action of a finitely generated group;

(ii) E has finite cost;

(iii) $[E]$ is topologically finitely generated.

Proof. (i) ⇒ (ii) is obvious and (ii) ⇒ (i) was proved by Hjorth–Kechris (see [42, 27.7]). (iii) ⇒ (i) is also clear (as every group dense in $[E]$ generates E) and finally, (ii) ⇒ (iii) follows from Theorem 5.4.10. □

For free actions of free groups, the theory of cost allows us to obtain a lower bound for $t([E])$ as well.

Corollary 5.4.12. *Let E be generated by a free, ergodic action of $\mathbf{F}_n$. Then*

$$n + 1 \leq t([E]) \leq 3(n + 1).$$

Proof. To prove the lower bound, suppose, towards a contradiction, that there is a set of automorphisms $\Phi \subseteq [E]$, $|\Phi| = n$ with $\langle \Phi \rangle$ dense in $[E]$. Since $E = E^X_\Phi$ has cost n, Φ must act freely. Indeed,

Φ, considered as an L-graphing, realizes the cost of E, hence, by [42, 19.1], it is a treeing, hence the action is free. Therefore

$$\langle\Phi\rangle \subseteq \{T \in [E] : d(1,T) = 1\} \cup \{1\}$$

which is a closed, nowhere dense set in $[E]$, a contradiction.

The upper bound follows from Theorem 5.4.10. □

This corollary provides the first topological group distinction between $[E^X_{\mathbf{F}_m}]$ and $[E^X_{\mathbf{F}_n}]$, at least when m and n are sufficiently far apart.

It will be interesting to try to improve those bounds. For example, if an action $\mathbf{F}_n \curvearrowright X$ is mixing and $\mathbf{F}_n = \langle\gamma_1, \dots, \gamma_n\rangle$, then every $E^X_{\gamma_i}$ is hyperfinite and *ergodic*, so applying Theorems 5.4.2 and 5.4.7 yields the upper bound $t([E^X_{\mathbf{F}_n}]) \leq 3n$.

Question 5.4.13. Let E be generated by a free, ergodic action of $\mathbf{F}_n$. Is the number $t([E])$ independent of the action? If yes, what is it?

On another note, Proposition 5.4.4 allows us to associate with each uniformly closed, separable group $G \leq \mathrm{Aut}(X,\mu)$ a largest equivalence relation F_G such that $[F_G] \leq G$.

Proposition 5.4.14. *Suppose $G \leq \mathrm{Aut}(X,\mu)$ is uniformly closed and separable. Then there is a largest countable equivalence relation F_G such that its full group is contained in G. Moreover, $[F_G]$ is normal in G.*

Proof. Let $\mathcal{F}$ be a maximal family of involutions whose full groups are contained in G and which are almost everywhere different on their supports, i.e.,

$$\forall T,S \in \mathcal{F} \quad \mu(\{x : Tx = Sx \neq x\}) > 0 \implies T = S.$$

and $\mathcal{F} \subseteq G$ is maximal with this property. Since G is separable, $\mathcal{F}$ must be countable. Indeed, if $\mathcal{F}$ is uncountable, there exist an uncountable $\mathcal{A} \subseteq \mathcal{F}$ and $\epsilon > 0$ such that for all $T \in \mathcal{A}$, $\mu(\operatorname{supp} T) > \epsilon$. Then for all $T,S \in \mathcal{A}$, $d(T,S) > 2\epsilon$, contradicting the separability of G. Now let $F_G = E^X_{\mathcal{F}}$. Proposition 5.4.4 and the fact that G is closed imply that $[F_G] \leq G$.

Suppose now that $[E] \leq G$ for some equivalence relation E but $[E] \nleq [F_G]$. Then there exists an involution $T \in [E]$ such that $T \notin [F_G]$. Thus there is a non-null T-invariant set $A \subseteq \operatorname{supp} T$ such that

$$\forall x \in A \forall S \in \mathcal{F} \quad Tx \neq Sx.$$

Now set $T' = T|_A \cup \mathrm{id}|_{A^C}$. It is clear that $[T'] \leq [E] \leq G$ and T' is everywhere different on its support from the elements of $\mathcal{F}$, contradicting the maximality of $\mathcal{F}$.

Finally, normality is clear since the property of being a full group is preserved under conjugation.

□

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