THE EGOROFF PROPERTY AND RELATED PROPERTIES IN THE THEORY OF RIESZ SPACES

Thesis by

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ABSTRACT

A Riesz space L is said to be Egoroff if, whenever $u, u_{n,k} \in L$ and $(\forall n)(0 \leq u_{n,k} \uparrow_k u)$, there is a sequence u_m in L such that $0 \leq u_m \uparrow_m u$ and, for each n,m, there exists an index k(n,m) such that $u_m \leq u_{n,k(n,m)}$. This notion was introduced, in rather a different form, by Nakano. Banach function spaces are Egoroff, and Lorentz showed that, for any function seminorm ρ, the maximal seminorm ρ_1 among those which are dominated by ρ and which are σ-Fatou (a monotone seminorm ρ_2 is σ-Fatou if $[0 \leq u_n \uparrow u \Longrightarrow \rho_2(u_n) \uparrow \rho_2(u)]$) is precisely the "Lorentz seminorm" ρ_L, where $\rho_L(u) = \inf\{\lim_n \rho(u_n) : 0 \leq u_n \uparrow u\}$. In this thesis the extent to which $\rho_1 = \rho_L$ holds in general Riesz spaces is determined. In fact, $\rho_1 = \rho_L$ for every monotone seminorm ρ on a Riesz space L if, and only if, L is "almost-Egoroff". The almost-Egoroff property is closely related to the Egoroff property and, indeed, coincides with it in the case of Archimedean spaces. Analogous theorems for Boolean algebras are discussed. The almost-Egoroff property is shown to yield a number of results which ensure that, under certain conditions, a monotone seminorm is σ-Fatou when restricted to an appropriate super order dense ideal. Riesz spaces L possessing an integral Riesz norm ρ (i.e., a Riesz norm such that $[u_n \downarrow 0 \Longrightarrow \rho(u_n) \downarrow 0]$) are considered also, since in many cases these are known to be Egoroff. In particular if ρ is normal on L (i.e., [u_τ a directed system, $u_\tau \downarrow 0 \Longrightarrow \rho(u_\tau) \downarrow 0$]), then L is Egoroff. In this connection, a pathological space, possessing an integral Riesz norm which is nowhere normal, is constructed.

Introduction

The basic purpose of this thesis is to study the significance of the Egoroff property, and certain other properties derived from it, in various aspects of the theory of Riesz spaces. Upon occasion, we consider similar problems in the theory of Boolean algebras. The thesis is divided into nine sections. Section 1 introduces the Egoroff property along with related properties which prove to be important. Chief among these is the "almost-Egoroff" property, a property which coincides with the Egoroff property in the case of Archimedean spaces. In section 2 we discuss the rôle of the Egoroff property in the classical Riesz spaces derived from spaces of measurable functions, and in the associated measure algebras. The relationship of the Egoroff property to the well-known theorem of Egoroff is also pointed out in this section. Examples of Riesz spaces in which the Egoroff properties fail in various ways are presented in section 3. In section 4 we discuss, for the theory of Boolean algebras, analogues of results which take their most significant form in section 5. The latter section deals with the relationship between Egoroff properties of Riesz spaces and the behavior of the monotone seminorms which may be defined on them. In particular we show that the property which we have called the almost-Egoroff property represents a precise answer to the question: in which spaces can the maximal σ-Fatou seminorm dominated by a given (monotone) seminorm ρ be obtained in the form of the Lorentz seminorm ρ_L ? Sections 6 and 7 contain comments on some questions which stem naturally from the work of section 5. The almost-Egoroff property is applied in section 8 to prove a number of results of "Koshi type", i.e., results which ensure that certain monotone seminorms become σ-Fatou when restricted to the appropriate super order dense ideal. Section 9 deals with Riesz spaces on which there is defined an integral Riesz norm. These spaces are closely related to the Egoroff spaces. For example, it is known that if the Riesz space possesses a norm which is not only integral but also normal, then the space is Egoroff. Detailed study of a particular example reveals some hitherto unobserved pathology in this respect, namely, an integral Riesz norm which is nowhere normal.

1: Egoroff Conditions

In this thesis we shall be concerned, for the most part, with abstract Riesz spaces. For the general properties of these spaces we refer the reader to Bourbaki [1], chapter II, to Nakano [2], or to Luxemburg and Zaanen [3], Note VI and following. Unless stated explicitly, no assumptions are made concerning the order-completeness or Archimedean character of the spaces under consideration.

The basic Egoroff condition is an abstract formulation, in the Riesz space setting, of the property which lies behind Egoroff's classical result (see [5]) concerning spaces of measurable functions. Presently, in section 2, we shall discharge our historical obligation by demonstrating the relationship between Egoroff's theorem and the Egoroff condition we are about to enunciate.

It is convenient to introduce first a special notational device. If A is any system with a partial ordering " $\leqslant$ " , and $\{a_{n,k}\}$ is a double sequence of elements in A, we write $a \ll \{a_{n,k}\}$, where $a \in A$, to mean that

$$\forall n \exists k(n) \text{ such that } a \leqslant a_{n,k(n)} \quad .$$

Consider first a Boolean algebra $\mathcal{B}$. An element <u>$b \in \mathcal{B}$ is said to have the Egoroff property</u> if

$$[\forall n \ \ b_{n,k} \uparrow_k b] \Longrightarrow [\exists b_m \text{ such that } b_m \uparrow_m b \text{ and } \forall m \ b_m \ll \{b_{n,k}\}] \quad .$$

The Boolean algebra <u>$\mathcal{B}$ itself is said to be Egoroff</u> if every one of its elements has the Egoroff property.

In the case of a Riesz space L, we say that an element $u \in L^+$ has the Egoroff property if $[0 \leq u_{n,k} \uparrow_k u$, for each $n] \implies$

$$[\exists u_m \geq 0 \text{ such that } u_m \uparrow_m u \text{ and } \forall m \ u_m \ll \{u_{n,k}\}] .$$

More generally, we say that $f \in L$ has the Egoroff property if $|f|$ has that property, and we say that the space L is Egoroff if every one of its elements has the Egoroff property.

A related property which has been introduced is the weak Egoroff property. If $\mathcal{B}$ is a Boolean algebra, $b \in \mathcal{B}$ is said to have the weak Egoroff property if

$$[\forall n \ b_{n,k} \uparrow_k b] \implies [b = 0 \text{ or } \exists a \neq 0 \text{ such that } a \ll \{b_{n,k}\}] .$$

Following the pattern above, we say that a Boolean algebra $\mathcal{B}$ is weak Egoroff if each of its elements has the weak Egoroff property. In a Riesz space L, we say that an element $u \in L^+$ has the weak Egoroff property if

$$[0 \leq u_{n,k} \uparrow_k u, \text{ for each } n] \implies [u = 0 \text{ or } \exists v \gneq 0 \text{ such that } v \ll \{u_{n,k}\}].$$

More generally, we say $f \in L$ has the weak Egoroff property if $|f|$ has that property, and we say that the space L is weak Egoroff if every one of its elements has the weak Egoroff property.

It appears that Nakano was the first to isolate the Egoroff property for Riesz spaces, although he did so in rather a different form from that which we have given above. With certain reservations, our Egoroff property corresponds to his notion of "total continuity" (see [2], §14). Luxemburg introduced the notions of Egoroff and weak Egoroff Boolean algebras in [6], using the present terminology. The properties also occur in the present form in Luxemburg and Zaanen [3] (see, for example, Definition 20.5).

We wish to introduce here a third Egoroff-type property, which will play an important rôle in the sequel. We say an element $u \in L^+$, where L is a Riesz space, has the almost-Egoroff property if

$$[0 \leq u_{n,k} \uparrow_k u, \text{ for all } n] \Longrightarrow$$

$$[\forall \varepsilon > 0 \ \exists u_m^\varepsilon \geq 0 \text{ such that } u_m^\varepsilon \uparrow_m (1-\varepsilon)u \text{ and } \forall m \ u_m^\varepsilon \ll \{u_{n,k}\}].$$

An arbitrary element $f \in L$ is said to have the almost-Egoroff property if $|f|$ has that property, and the space L is called almost-Egoroff if every one of its elements has the almost-Egoroff property. There is no useful analogue of the almost-Egoroff property for the case of Boolean algebras.

Let us first make the simple observation that the Egoroff property is an "ideal property".

Theorem 1.1: (a) If f is an element in the Riesz space L which has the Egoroff property, then every element g in the ideal generated by f, i.e., in $\{g: \exists M \text{ such that } |g| \leq M|f|\}$, has also the Egoroff property.

(b) If an element b of the Boolean algebra $\mathcal{B}$ has the Egoroff property, then every element a such that $a \leq b$ has also the Egoroff property.

Proof: (a) Clearly we need only prove that if u has the Egoroff property and $0 \leq v \leq u$, then v has that property. But if $0 \leq v_{n,k} \uparrow_k v$, then $(v_{n,k} + (u - v)) \uparrow_k u$, and there exist u_m such that

$$0 \leq u_m \uparrow_m u \quad \text{and} \quad u_m \ll \{(v_{n,k} + (u - v))\} \ ; \ \text{but then}$$

$0 \leq (u_m - (u - v))^+ \ll \{v_{n,k}\}$ and $(u_m - (u-v))^+ \uparrow_m (u - (u - v))^+ = v$.

(b) If $a_{n,k} \uparrow_k a$, then

$$a_{n,k} \vee (b \wedge a') \uparrow_k a \vee (b \wedge a') = b.$$

Thus $\exists\, b_m \uparrow_m b$ such that

$$b_m \ll \{a_{n,k} \vee (b \wedge a')\},$$

so that $(b_m \wedge a) \uparrow_m a$ and

$$(b_m \wedge a) \ll \{(a_{n,k} \vee (b \wedge a')) \wedge a\} = \{a_{n,k}\} .$$

Similar theorems do not hold for the almost-Egoroff and weak Egoroff properties. In the case of the almost-Egoroff property, this will be made clear during the discussion of Example 3.3. It is much easier to see that we may have $0 \leqslant v \leqslant u$ in a Riesz space L where u has the weak Egoroff property while v does not have that property. Consider, for example, the direct sum $C[0, 1] \oplus R$, where R denotes the reals and $C[0, 1]$ the real continuous functions on the interval $[0, 1]$; it will be shown in Example 3.1 that no non-zero element of $C[0, 1]$ has the weak Egoroff property, so that if $f \in C[0, 1]$, $f \gneq 0$, then $v = (f, 0) \in C[0, 1] \oplus R$ has not the weak Egoroff property; on the other hand $u = (f, 1)$ has that property, and, in fact, it is clear that $0 \leq u_{n,k} \uparrow_k u$ implies that $(0, \alpha) \ll \{u_{n,k}\}$ for every $\alpha \lneq 1$.

There are certain simple techniques which are useful in dealing with Egoroff conditions and which will occur a number of times in the proofs of theorems in subsequent sections. To demonstrate these techniques in the most elementary setting let us derive an equivalent form of the Egoroff property for elements in a Boolean algebra.

Theorem 1.2: An element b in a Boolean algebra $\mathcal{B}$ has the Egoroff property if, and only if, whenever $b_{n,k} \uparrow_k b$ (for each n), there exists a sequence b_m such that $b_m \uparrow_m b$ and, for each m, there exists k(m) such that $b_m \leq b_{m,\, k(m)}$.

Proof: ($\Longrightarrow$) This is immediate.

($\Longleftarrow$) Suppose we have, for each n, $b_{n,k} \uparrow_k b$. Consider

$$a_{n,k} = b_{1,k} \wedge b_{2,k} \wedge \cdots \wedge b_{n,k} \quad .$$

Again we have, for each n, $a_{n,k} \uparrow_k b$ (because of the strong distributivity properties which hold in Boolean algebras). In view of our hypothesis, then, there exist b_m , $k(m)$ such that $b_m \uparrow_m b$ and $b_m \leqslant a_{m,k(m)}$. We need only show that, for each n,m, we have $k(n,m)$ with $b_m \leqslant b_{n,k(n,m)}$. But for $n \geqslant m$ we have

$$b_m \leqslant b_n \leqslant a_{n,k(n)} \leqslant b_{n,k(n)} \quad ,$$

and for $n \leqslant m$ we have

$$b_m \leqslant a_{m,k(m)} \leqslant b_{n,k(m)} \quad .$$

2: Classical Examples; Egoroff's Theorem

We wish to discuss here the significance of what we have called the Egoroff property for the well-known case of Riesz spaces of real measurable functions. We shall also show how the Egoroff property is involved in the classical theorem of Egoroff, although it is not the intention of this thesis to formulate generalizations of that theorem (cf. [2], Theorem 14.2).

Given a set X, a σ-complete Boolean algebra $\mathfrak{F}$ of subsets of X which is a subalgebra of the power set $p(X)$, and a countably additive measure μ on $\mathfrak{F}$, valued in $[0,\infty]$, let L denote the space of real-valued functions on X measurable with respect to $(X, \mathfrak{F}, \mu)$, with identification of almost everywhere (a.e.) equal functions. With the introduction of the usual linear operations and partial ordering, L becomes a Riesz space. Let $\mathcal{B}$ denote the measure algebra associated with $(X, \mathfrak{F}, \mu)$, i.e., $\mathcal{B}$ is the Boolean algebra composed of equivalence classes of elements from $\mathfrak{F}$, where $A, B \in \mathfrak{F}$ are identified whenever the symmetric difference $A \triangle B$ has measure zero.

We shall say the Egoroff theorem holds for $(X, \mathfrak{F}, \mu)$ whenever

$$[\, F_k \text{ measurable functions, and } F_k \rightarrow F \text{ a.e.} \,] \Longrightarrow$$

$$[\, \exists \text{ measurable subsets } X_m \uparrow \text{ such that } F_k \rightarrow F \text{ uniformly on each } X_m \text{, and } \mu(X - \bigcup_1^\infty X_m) = 0 \,].$$

This represents a convenient reformulation of the well-known result of Egoroff (cf. [5]) for the case of a finite real interval and Lebesgue measure.

The proofs which we shall give for the first three theorems presented in this section are somewhat abbreviated. We make without comment a number of steps which in fact require a small argument for their justification. These steps are all of a similar nature and arise in passing between statements concerning $\mathcal{B}$ or L and statements concerning the measurable sets and functions of the measure space $(X, \mathcal{F}, \mu)$ from which $\mathcal{B}$ and L were constructed. Naturally, the arguments depend upon properties of the null sets in $(X, \mathcal{F}, \mu)$ and, in particular, upon the fact that a countable union of null sets is again a null set. In order to give a clear idea of what is involved in these steps, without obscuring the proofs of Theorems 2.1, 2.2 and 2.3 with details, we propose simply to justify here a representative example of such a step. Suppose we have elements u_n, $u \in L$ such that $u_n \uparrow_n u$. We wish to be able to assert that, if we choose measurable functions F_n, F on X such that $F_n \in u_n$ and $F \in u$, then $F_n \uparrow_n F$ a.e. To see that this must be the case, let

$$G_n = \sup(\inf(F_1, F), \ \dots \ , \inf(F_n, F)) \quad (n = 1, 2, \dots) .$$

Now we have $G_n \in u_n$, and $G_n \uparrow_n \leq F$ everywhere; hence, if G is the measurable function which is the limit of the sequence G_n, we have $G_n \uparrow_n G \leq F$. If v is the element of L such that $G \in v$, then $u_n \leq v \leq u$, for all n. It follows that $v = u$, so that $G = F$ a.e. Clearly, then, we have $F_n \uparrow_n$, and, in fact, $F_n \uparrow_n G$ except possibly on the set

$$A = \{x: (\exists n)(F_n(x) \neq G_n(x)) \} .$$

For each n, however, the set $\{x: F_n(x) \neq G_n(x)\}$ is a null set, so that A, the union of a countable collection of null sets, is itself a null set. Thus $F_n \uparrow_n$ a.e. and $F_n \uparrow_n G$ a.e. ; finally, $F_n \uparrow_n F$ a.e., since, as we have remarked, $F = G$ a.e.

<u>Theorem 2.1</u>: If L and $\mathcal{B}$ are the Riesz space and Boolean algebra associated with a measure space $(X, \mathfrak{F}, \mu)$, using the notation developed above, then [$\mathcal{B}$ is Egoroff] $\Longleftrightarrow$ [L is Egoroff] .

<u>Proof</u>: ($\Longrightarrow$) Suppose we have $0 \leqslant u_{n,k} \uparrow_k u$ (for each n) in L. First of all, we may assume that $u_{n+1,k} \leqslant u_{n,k}$ for each n, k, since otherwise we replace $u_{n,k}$ by

$$\bar{u}_{n,k} = u_{1,k} \wedge u_{2,k} \wedge \dots \wedge u_{n,k} \; ;$$

we still have $\bar{u}_{n,k} \uparrow_k u$ (for each n), and

$$[v \ll \{\bar{u}_{n,k}\}] \Longrightarrow [v \ll \{u_{n,k}\}] \; .$$

Now choose measurable functions $F_{n,k}$, F on X such that $F_{n,k} \,\varepsilon\, u_{n,k}$ and $F \,\varepsilon\, u$. If we set

$$X_{n,k} = \{x: \; F_{n,k}(x) \geqslant F(x) - \tfrac{1}{n}\}$$

and let $b_{n,k}$ be the element of $\mathcal{B}$ such that $X_{n,k} \,\varepsilon\, b_{n,k}$, we certainly have $b_{n,k} \uparrow_k 1$ for each n. Assuming the Egoroff property in $\mathcal{B}$, we have a sequence $b_m \uparrow_m 1$ such that

$$\forall m \quad b_m \ll \{b_{n,k}\} \; .$$

If we now choose $X_m \in \mathfrak{F}$ such that $X_m \,\varepsilon\, b_m$ and let

$$G_m = (F - \tfrac{1}{m})^+ \cdot \chi_{X_m} \; ,$$

we have $0 \leqslant G_m \uparrow_m F$ a.e. Moreover, for every m, there exists k(m) such that $b_m \leqslant b_{m,k(m)}$, and hence, for a.e. x,

$$x \,\varepsilon\, X_m \Longrightarrow x \,\varepsilon\, X_{m,k(m)} \; , \text{ i.e., } \; F(x) - \tfrac{1}{m} \leqslant F_{m,k(m)}(x) \; .$$

If we recall that $F_{m,k(m)} \geqslant 0$ a.e., we see that

$$G_m = (F - \frac{1}{m})^+ \cdot \chi_{X_m} \leq F_{m,k(m)} \quad \text{a.e.}$$

Clearly, then, if we let u_m be the element of L such that $G_m \varepsilon u_m$, we have $0 \leq u_m \uparrow_m u$ and $\forall m \ u_m \leq u_{m,k(m)}$. Thus each $u_m << \{u_{n,k}\}$, since for $n \geqslant m$ we have $u_m \leq u_n \leq u_{n,k(n)}$ and for $n \leq m$ we have $u_m \leq u_{m,k(m)} \leq u_{n,k(m)}$.

($\Longleftarrow$) Recalling Theorem 1.1(b), we need only consider the situation where $b_{n,k} \uparrow_k 1$ (for each n) in $\mathfrak{B}$. If we now choose $X_{n,k} \varepsilon \mathfrak{F}$ such that $X_{n,k} \varepsilon b_{n,k}$ and let $u_{n,k}$ be the element in L such that $\chi_{X_{n,k}} \varepsilon u_{n,k}$, we have $0 \leq u_{n,k} \uparrow_k u$ (for each n), where u is the element of L containing the unit function on X. Assuming the Egoroff property in L, we have $0 \leq u_m \uparrow_m u$ such that $\forall m \ u_m << \{u_{n,k}\}$. Choosing measurable functions $F_m \varepsilon u_m$, we must have $F_m \uparrow 1$ a.e. so that, if we let $X_m = \{x: F_m(x) \gneq 0\}$, and let b_m be that element of $\mathfrak{B}$ which contains X_m, we certainly have $b_m \uparrow 1$. Moreover, if we fix m for a moment, for each n there exists k(n) such that $u_m \leq u_{n,k(n)}$; hence $F_m \leq \chi_{X_{n,k(n)}}$ a.e. so that, for a.e. x,

$$x \varepsilon X_m \Longrightarrow x \varepsilon X_{n,k(n)} .$$

It follows that $b_m \leq b_{n,k(n)}$, and hence that $\forall m \ b_m << \{b_{n,k}\}$.

<u>Theorem 2.2</u>: If $(X, \mathfrak{F}, \mu)$ is a measure space, then

[$\mathfrak{B}$ is Egoroff (equivalently, L is Egoroff)] $\Longrightarrow$

[the Egoroff theorem holds for $(X, \mathfrak{F}, \mu)$] .

<u>Proof</u>: Given measurable functions F_k on X, such that $F_k \longrightarrow F$ a.e., set

$$X_{n,k} = \{x: \forall k' \geqslant k \quad |F_{k'}(x) - F(x)| < \tfrac{1}{n}\} .$$

Clearly, for each n, $X_{n,k} \uparrow_k$ and $\mu(X - \bigcup_{k=1}^{\infty} X_{n,k}) = 0$.

Thus, if $b_{n,k}$ is the element of $\mathcal{B}$ such that $X_{n,k} \in b_{n,k}$, we have $b_{n,k} \uparrow_k 1$, for each n. Under the assumption that $\mathcal{B}$ is Egoroff, we have $b_m \uparrow_m 1$ with $b_m \ll \{b_{n,k}\}$, for each m. Choose k(m,n) such that $k(m,n) \uparrow_m$ for each fixed n and $b_m \leqslant b_{n,k(m,n)}$. If we choose $Y_m \in b_m$, we must have $\mu(X - \bigcup_1^{\infty} Y_m) = 0$ and, for each m, for a.e. x,

$$x \in Y_m \implies x \in \bigcap_{n=1}^{\infty} X_{n,k(m,n)} .$$

Thus if we let

$$X_m = \bigcap_{n=1}^{\infty} X_{n,k(m,n)} ,$$

we have $X_m \uparrow$, because of the choice of k(m,n), and $\mu(X - \bigcup_1^{\infty} X_m) = 0$.

Finally, $F_k \longrightarrow F$ uniformly on each X_m, since $k \geqslant k(m,n)$ and $x \in X_m$ imply that $|F_k(x) - F(x)| < \frac{1}{n}$.

<u>Theorem 2.3</u>: If the measure space $(X, \mathfrak{F}, \mu)$ is σ-finite, i.e., if there exist $X_m \in \mathfrak{F}$ such that $X_m \uparrow X$ and $\forall m \ \mu(X_m) < \infty$, then $\mathcal{B}$ is Egoroff (equivalently, L is Egoroff).

<u>Proof</u>: In the obvious way, μ induces on the σ-complete Boolean algebra $\mathcal{B}$ a measure, which we again denote by μ. This measure is valued in $[0, \infty]$ and is strictly positive; $c_n \uparrow c$ in $\mathcal{B} \implies \mu(c_n) \uparrow \mu(c)$ and $c_n \downarrow 0$ in $\mathcal{B} \implies \mu(c_n) \downarrow 0$ whenever $\mu(c_1) < \infty$.

Furthermore there exists a sequence $a_m \uparrow 1$ such that $\mu(a_m) < \infty$ for each m (for example, we may take a_m such that $X_m \in a_m$). Now consider a double sequence $b_{n,k} \uparrow_k 1$ (for each n). Clearly, for each m, n, $(a_m \wedge b'_{n,k}) \downarrow_k 0$ and, as

$$\mu(a_m \wedge b'_{n,1}) \leq \mu(a_m) < \infty ,$$

we can choose $k(m,n)$ such that

$$\mu(a_m \wedge b'_{n,k(m,n)}) < 2^{-(m+n)} .$$

Moreover, we may suppose that $k(m+1,n) \geqslant k(m,n)$ for all m,n. Now let

$$b_m = \bigwedge_{n=1}^{\infty} (a_m \wedge b_{n,k(m,n)}) .$$

Clearly $b_m \leq a_m$ and $b_m \uparrow$, due to the choice of $k(m,n)$. The construction of b_m ensures that $\forall m \;\; b_m \ll \{b_{n,k}\}$. Moreover,

$$a_m \wedge b'_m = \bigvee_{n=1}^{\infty} (a_m \wedge b'_{n,k(m,n)})$$

so that $\mu(a_m \wedge b'_m) < 2^{-(m+1)} + 2^{-(m+2)} + \ldots = 2^{-m}$.

Now, for all p,

$$\mu(a_m \wedge b'_p) \leq \mu(a_p \wedge b'_p) \leq 2^{-p}$$

so that $\mu(a_m \wedge b'_p) \longrightarrow 0$ as $p \longrightarrow \infty$. Hence $\bigvee_1^{\infty} b_p \geqslant a_m$ for each m, and since $a_m \uparrow 1$, we must have $b_m \uparrow 1$. We have shown that $1 \in \mathcal{B}$ has the Egoroff property, and this completes the proof of the theorem, in view of Theorem 1.1(b).

Theorems 2.3 and 2.1 above serve to show that there is a good supply of Egoroff Boolean algebras and Riesz spaces. Theorems 2.2 and 2.3 constitute a proof of the classical Egoroff theorem which makes evident the rôle played in that theorem by the "Egoroff property" introduced in section 1.

We should perhaps mention a general result which could be made the basis of a rather different proof of Theorem 2.3. We say a set A in a Boolean algebra $\mathcal{B}$ is <u>super order dense</u> in $\mathcal{B}$ if, for any $b \in \mathcal{B}$, there exists a sequence a_n in A such that $a_n \uparrow_n b$; the same terminology is used for subsets of Riesz spaces. We shall show presently that if A is super order dense in a Riesz space (Boolean algebra), and if each element of A has the Egoroff property, then the Riesz space (Boolean algebra) is Egoroff. Now in the proof of Theorem 2.3 one might first show that each of the elements a_m has the Egoroff property; since, for each m, $\mu(a_m) < \infty$, the argument could be patterned more directly after Egoroff's original proof, which dealt with an interval of finite measure. It would then follow, in view of Theorem 1.1(b), that every element in the ideal $I = \{b\colon (\exists m)\ (b \leq a_m)\}$ has the Egoroff property. Since I is clearly super order dense in $\mathcal{B}$, Theorem 2.3 would then be a consequence of the general result noted above, which we now present as a theorem.

<u>Theorem 2.4</u>: Let A be a subset of a Riesz space L (Boolean algebra $\mathcal{B}$) and suppose that A is super order dense in L (in $\mathcal{B}$). If each element of A has the Egoroff property, then L ($\mathcal{B}$) is Egoroff.

<u>Proof</u>: We shall give the proof in the case of a Riesz space L; the case of a Boolean algebra $\mathcal{B}$ may be treated in just the same way. Suppose u, $u_{n,k}$ are elements of L such that $0 \leq u_{n,k} \uparrow_k u$, for each n. Since A is super order dense in L, there is a sequence $a_p \in A$ such that

$a_p \uparrow_p u$. We have $|a_p| = v_p \uparrow_p u$, and each v_p has the Egoroff property, since every element of A has that property. Since, for each n,

$$(v_p \wedge u_{n,k}) \uparrow_k (v_p \wedge u) = v_p \quad ,$$

there exists a sequence u_m^p, for each p, such that

$$0 \leqslant u_m^p \uparrow_m v_p \quad \text{and} \quad u_m^p \ll \{v_p \wedge u_{n,k}\} \quad .$$

Let $u_m = u_m^1 \vee u_m^2 \vee \dots \vee u_m^m$. Now $u_m \uparrow_m \leqslant u$, since $v_p \leqslant u$ for each p; on the other hand, for each p, $u_m \geqslant u_m^p$ whenever $m \geqslant p$. Hence $u_m \uparrow u$, since $v_p \uparrow u$. Fixing m and n for the moment, note that, for each p, there exists k(p) such that $u_m^p \leqslant v_p \wedge u_{n,k(p)}$. Thus

$$u_m \leqslant u_{n,k(1)} \vee \dots \vee u_{n,k(m)} = u_{n, \max_{p \leqslant m} k(p)} \quad ,$$

so that $u_m \ll \{u_{n,k}\}$. Clearly, then, u has the Egoroff property.

3: Relationships Among the Several Egoroff Conditions; Examples

It is clear that if u is an element of a Riesz space L, then (i) u has the Egoroff property implies that u has the almost-Egoroff property, and (ii) u has the almost-Egoroff property implies that u has the weak Egoroff property; similarly, if an element b of a Boolean algebra $\mathcal{B}$ has the Egoroff property, then it certainly has also the weak Egoroff property. The examples below show that none of the reverse implications holds.

Example 3.1: Let $C[0, 1]$ denote the set of continuous, real-valued functions on the interval $[0, 1]$. $C[0, 1]$ becomes a Riesz space L when we define the linear operations in the usual way and introduce the ordinary pointwise partial ordering (i.e., $f \geqslant g$ if $f(x) \geqslant g(x)$ for all $x \in [0, 1]$). It is easy to show that no element different from 0 in this space has the weak Egoroff property. To see this, consider a function $u \geqslant 0$. Let $\{r_n\}$ be an enumeration of the rationals in $[0, 1]$, and let, for each n, $u_{n,k}$ be a sequence of functions in $C[0, 1]$ such that $0 \leqslant u_{n,k} \uparrow_k \leqslant u$, and for all $x \notin \{r_1, r_2, \dots, r_n\}$ $u_{n,k}(x) \uparrow_k u(x)$, while for $x \in \{r_1, r_2, \dots, r_n\}$ $u_{n,k}(x) = 0$. Clearly $u_{n,k} \uparrow_k u$ in $L = C[0, 1]$. On the other hand, if $0 \leqslant g \ll \{u_{n,k}\}$, then $g(r_n) = 0$ for all n, so that, since $\{r_n\}$ is dense and g is continuous, $g \equiv 0$.

Example 3.2: Consider the set $R^{(N^N)}$ of all real functions over the set N^N (N^N is the set of all functions from $N = \{1, 2, 3, \dots \}$ into N). With the introduction of the natural linear operations and the pointwise partial ordering, $L = R^{(N^N)}$ becomes a Riesz space. Consider the following subsets of $X = N^N$:

$$X_{n,k} = \{x: \ x \in N^N \text{ and } x(n) \leq k\} \ ;$$

identifying subsets of X with their characteristic functions, we clearly have, for each n, $X_{n,k} \uparrow_k X$ (in L). Nevertheless, this double sequence has the rather surprising property that for <u>no</u> choice of k(n) do we have $\sup_n X_{n,k(n)} = X$; in fact we can construct $x \in X$ such that $\forall n \ x \notin X_{n,k(n)}$ simply by setting $x(n) = k(n) + 1$ for all $n \in N$. It is now easy to show that X, the unit function in L, <u>has not the Egoroff property</u>, <u>nor even the almost-Egoroff property</u>. For suppose we have a sequence $0 \leq u_m << \{X_{n,k}\}$; then in particular we can choose k(m), for each m, such that $u_m \leq X_{m,k(m)}$. Hence

$$\sup_m u_m \leq \sup_m X_{m,k(m)} \neq X,$$

in view of the comments above, so that X cannot have the Egoroff property. More precisely, there exists $x \in X$ such that $\forall m \ X_{m,k(m)}(x) = 0$ so that $[\sup u_m](x) = 0$ and since $X(x) = 1$, it is clear that X cannot have the almost-Egoroff property.

Nevertheless, the whole space <u>$L = R^{(N^N)}$ is weak Egoroff</u>, for if $u \in L^+$, $u \neq 0$, then there is some x in X such that $u(x) \gneq 0$, and if $u_{n,k} \uparrow_k u$, then $u_{n,k}(x) \uparrow_k u(x)$ (for each n) so that

$$0 \lneq \tfrac{1}{2}u(x) \cdot \chi_{\{x\}} << \{u_{n,k}\} \ .$$

It is also quite evident from the above discussion that the power set $p(N^N)$ furnishes an example of a <u>Boolean algebra which is weak Egoroff but not Egoroff</u>.

<u>Example 3.3</u>: For this example it is convenient to have available a non-Archimedean extension *R of the real numbers R, i.e., we assume *R to be a totally ordered field containing R as a subfield and containing elements h, called infinitesimals, such that

$$r \in R \text{ and } r \gneqq 0 \Longrightarrow -r < h < r \quad .$$

A general construction for such *R is discussed in [7] ; however, much less sophisticated constructions would be sufficient for our present purposes. All we really need is a totally ordered non-Archimedean linear space over R.

Let L consist of the functions f on the set $X = N^N$ (see the previous example), with values in *R, and having the following form: $f(x) = r_f + h_f(x)$, where $r_f \in R$ and, for each $x \in X$, $h_f(x)$ is infinitesimal. Upon the introduction of the pointwise linear operations and the partial ordering induced by the pointwise order in *R, L becomes a Riesz space. Note that if $r_f \lneqq r_g$, then $f \leq g$ (and, in fact, $f \lneqq g$) regardless of the values of the functions h_f and h_g. Now consider the element u in L which has $r_u = 1$ and, for each x in X, $h_f(x) = 0$. We can easily show that u has the almost-Egoroff property. To see this, suppose that $0 \leq u_{n,k} \uparrow_k u$ (for each n); certainly $r_{u_{n,k}} \uparrow_k 1 \ (= r_u)$, for each n, so that, given $\varepsilon \gneqq 0$, we have some k(n), for each n, with

$$r_{u_{n,k(n)}} \gneqq 1 - \varepsilon = r_{(1-\varepsilon)u} \qquad ;$$

thus we have $(1 - \varepsilon)u \leq u_{n,k(n)}$. Hence, for each $\varepsilon \gneq 0$,

$$(1 - \varepsilon)u \ll \{u_{n,k}\} .$$

Consider now the element $h . X$ in L, where h is a fixed infinitesimal greater than zero. Using the notation of Example 3.2 we have $h . X_{n,k} \uparrow_k h . X$ (for each n), and, arguing just as we did in the discussion of Example 3.2, we see that $h . X$ has not the almost-Egoroff property. As $0 \leq h . X \leq u$, we have shown that <u>the almost-Egoroff property is not an ideal property</u>, as we promised to do at the end of section 1. Note also that u has not the Egoroff property, for in that case h would have that property as well, in view of Theorem 1.1(a). Thus we have exhibited an element which <u>has the almost-Egoroff property but not the Egoroff property</u>.

<u>Example 3.4</u>: We can demonstrate the existence of a Boolean algebra $\mathcal{B}$ not having the weak Egoroff property, at least if we assume the continuum hypothesis. This was pointed out by Professor Luxemburg. Consider the algebra of all subsets $p(X)$ of a set X having cardinality $c = 2^{\aleph_0}$. It was proved by Banach and Kuratowski, as the basic tool in their discussion of the non-measurability of the cardinal c (see [11]), that, if the continuum hypothesis holds, there exist subsets $X_{n,k}$ of X such that (i) for all n, $X_{n,k} \uparrow_k X$, and

(ii) for any choice of k(n), $\bigcap_{n=1}^{\infty} X_{n,k(n)}$ is countable.

This result is discussed (as Proposition C_{11}) in Sierpinski's well-known book [12] . If we now form the quotient algebra $p(X)/C$, where $\mathcal{C}$ is the ideal in $p(X)$ consisting of those subsets which are at most countable, we have our example $\mathcal{B}$. To see this, let h be the canonical map of $p(X)$ into $p(X)/_{\mathcal{C}} = \mathcal{B}$. Now, for each n,

$h(X_{n,k}) \uparrow_k 1 \in \mathcal{B}$, while, for any choice of $k(n)$, we have

$$\bigwedge_{n=1}^{\infty} h(X_{n,k(n)}) = 0.$$

Clearly, then, 1 has not the weak Egoroff property. Furthermore, if b is any element of $\mathcal{B}$ and we let $b_{n,k} = b \wedge h(X_{n,k})$, we have $b_{n,k} \uparrow_k (b \wedge 1) = b$, for all n, and, for any $k(n)$,

$$\bigwedge_{n=1}^{\infty} b_{n,k(n)} = (b \wedge 0) = 0 \;;$$

thus no non-zero element of $\mathcal{B}$ has the weak Egoroff property.

Thus we cannot hope to show that every Boolean algebra is weak Egoroff, since Gödel has shown (see [13]) that the continuum hypothesis is consistent with the other axioms of set theory (if those axioms are consistent).

We have now completed our list of examples designed to separate the various Egoroff conditions in Riesz spaces and Boolean algebras. It is important to note that the non-Archimedean character of the example (Example 3.3) of an element of a Riesz space having the almost-Egoroff property without the Egoroff property is essential, as the following result shows.

<u>Theorem 3.5</u>: If $u \in L^+$ is such that $\inf_n \frac{1}{n} u = 0$ (in particular, if the Riesz space is Archimedean), then

[u has the Egoroff property] $\Longleftrightarrow$ [u has the almost-Egoroff property].

<u>Proof:</u> We have already noted that the implication $(\Longrightarrow)$ always holds. For the other case, suppose that u has the almost-Egoroff property, and that $0 \leq u_{n,k} \uparrow_k u$ (for each n). Choose a sequence of real numbers

ε_p and elements u_m^p such that

$\varepsilon_p \uparrow 1$, $0 \leq u_m^p \uparrow_m \varepsilon_p u$, and $u_m^p \ll \{u_{n,k}\}$ (for all m, p).

$$\text{Now let} \quad u_m = u_m^1 \vee \dots \vee u_m^m \quad .$$

Clearly $0 \leq u_m \uparrow$ and $u_m \leq u$. Moreover, if v is such that $u_m \leq v$, for all m, then $u_m^p \leq v$, for all m, so that $v \geq \varepsilon_p u$; as this holds for each p, $v \geq \sup_p \varepsilon_p u$. But $\varepsilon_p \uparrow 1$ and we have assumed that $\inf_n \frac{1}{n} u = 0$, so that $\sup_p \varepsilon_p u = u$. Hence $u_m \uparrow u$. Also, $u_m \ll \{u_{n,k}\}$ since each $u_m^p \leq u_{n,k(p)}$, for some k(p) (depending also on m and n), so that

$$u_m \leq \sup_{p \leq m} u_{n,k(p)} = u_{n,(\max_{p \leq m} k(p))} \quad .$$

It follows that u has the Egoroff property.

4: A Characterization of those Boolean Algebras which are Egoroff

A function φ on a Boolean algebra $\mathcal{B}$ is called a finitely additive measure if φ is valued in $[0, \infty)$ and, for any $a, b \in \mathcal{B}$ such that $a \wedge b = 0$, $\varphi(a \vee b) = \varphi(a) + \varphi(b)$. It follows that φ is monotone, i.e., if $a \leq b$, then $\varphi(a) \leq \varphi(b)$.

Finitely additive measures on Boolean algebras have received considerable study, and, in particular, Yosida and Hewitt proved the following decomposition theorem (see [8]) : every finitely additive measure φ on a Boolean algebra $\mathcal{B}$ can be expressed uniquely as the sum $\varphi = \varphi_1 + \varphi_0$ of two finitely additive measures where φ_1 is σ-additive and φ_0 is purely finitely additive. A finitely additive measure ψ is called σ-additive if

$$[\, a, a_n \in \mathcal{B},\ \{a_n\} \text{ disjoint},\ a = \bigvee_{n=1}^{\infty} a_n \,] \Longrightarrow [\,\psi(a) = \sum_{n=1}^{\infty} \psi(a_n)\,],$$

and ψ is called purely finitely additive if the only σ-additive measure dominated by ψ is 0, i.e., if

$$[\,\psi \geq \psi_1 \text{ (pointwise on } \mathcal{B}\text{) and } \psi_1\ \sigma\text{-additive}\,] \Longrightarrow [\,\psi_1 \equiv 0\,].$$

For every finitely additive measure φ on a Boolean algebra $\mathcal{B}$ we can define a related function φ_L on $\mathcal{B}$ in the following way: for each $b \in \mathcal{B}$, set

$$\varphi_L(b) = \inf\{\lim_n \varphi(b_n) : b_n \uparrow b\}.$$

Obviously, $\varphi_L \leq \varphi$ and, if $\varphi \geq \psi$, then $\varphi_L \geq \psi_L$. Furthermore, it is easy to show that φ_L is always finitely additive.

A result which appears to go back to Woodbury [9] and Bauer [10] states that, for all finitely additive measures φ on a Boolean algebra $\mathcal{B}$, $\varphi_L = \varphi_1$, where φ_1 is the σ-additive part of φ in the decomposition of Yosida and Hewitt.

It is clear that, if φ is a finitely additive measure, then

$$[\varphi \text{ is } \sigma\text{-additive}] \Longleftrightarrow [\varphi_L = \varphi] \quad .$$

Hence, if φ is a given finitely additive measure on a Boolean algebra $\mathcal{B}$ and ψ is a σ-additive measure dominated by φ (i.e., $\varphi \geq \psi$), we have $\varphi_L \geq \psi_L = \psi$, so that $\varphi_1 = \varphi_L$ is simply the statement that φ_L is σ-additive. The fundamental result concerning finitely additive measures, then, may be expressed succinctly by saying that, for all finitely additive measures on a Boolean algebra $\mathcal{B}$, $\varphi_{LL} = \varphi_L$.

It is natural to ask in which cases the simple operation of passing from φ to φ_L remains idempotent when applied to a larger class of monotone functions on the Boolean algebra $\mathcal{B}$. Given a function $\rho : \mathcal{B} \longrightarrow [0, \infty]$ which is monotone (i.e., $a \leq b \Longrightarrow \rho(a) \leq \rho(b)$), we define ρ_L by

$$\rho_L(b) = \inf \{ \lim_n \rho(b_n) : b_n \uparrow b \} \quad .$$

We shall show that if the Boolean algebra $\mathcal{B}$ is Egoroff, then, for <u>all</u> monotone ρ, $\rho_{LL} = \rho_L$; what is more surprising is that if $\rho_{LL} = \rho_L$ holds, even for the well-known class of <u>outer measures</u> on $\mathcal{B}$, then $\mathcal{B}$ must be Egoroff. We say $\rho : \mathcal{B} \rightarrow [0, \infty]$ is an outer measure on the Boolean algebra $\mathcal{B}$ if ρ is monotone, $\rho(0) = 0$, and ρ is countably subadditive, i.e.,

$$a \leq \bigvee_1^\infty a_n \implies \rho(a) \leq \sum_1^\infty \rho(a_n) \quad .$$

We should point out that the statement $\rho_{LL} = \rho_L$ is equivalent to $[\forall a_n, a \in \mathcal{B}: \ a_n \uparrow a \implies \rho_L(a_n) \uparrow \rho_L(a)]$, i.e., to the statement that ρ_L is, to a certain extent, continuous with respect to order convergence in $\mathcal{B}$. This point of view plays the dominant rôle when we consider the analogous situation for Riesz spaces (where ρ is a monotone seminorm), as we shall do in section 5. In the case of a general outer measure ρ , the property we have been discussing is strictly weaker than the property $[\forall a_n \in \mathcal{B}: \ a_n \downarrow 0 \implies \rho_L(a_n) \downarrow 0]$. This is in contrast to the case of a finitely additive measure φ , where we clearly have $[\forall a_n, a \in \mathcal{B} : \ a_n \uparrow a \implies \varphi(a_n) \uparrow \varphi(a)]$
$\iff [\forall a_n \in \mathcal{B}: \ a_n \downarrow 0 \implies \varphi(a_n) \downarrow 0]$.

The following theorem represents, then, a characterization of those Boolean algebras which are Egoroff; it is also an introduction to the analogous, but considerably more complicated theorem concerning norms and seminorms on Riesz spaces, to be treated in section 5.

<u>Theorem 4.1</u>: A Boolean algebra $\mathcal{B}$ is Egoroff if, and only if, for every outer measure ρ on $\mathcal{B}$, we have $\rho_{LL} = \rho_L$. In fact :

(a) if $\mathcal{B}$ is Egoroff, then $\rho_{LL} = \rho_L$ for every monotone function $\rho : \mathcal{B} \longrightarrow [0, \infty]$; and

(b) if $\rho_{LL} = \rho_L$ for every finite-valued outer measure ρ (we may assume $\rho(1) = 1$) on $\mathcal{B}$, then $\mathcal{B}$ is Egoroff.

<u>Proof</u>: (a) Of course, $\rho_{LL} \leq \rho_L$. On the other hand, suppose

$\rho_{LL}(b) < \lambda$; in this case there must exist $b_n \uparrow_n b$ and, for each n, $b_{n,k} \uparrow_k b_n$ such that, for all n, k, $\rho(b_{n,k}) < \lambda$. Let

$$a_{n,k} = b_{n,k} \vee (b \wedge b'_n) \quad ;$$

for each n,

$$a_{n,k} \uparrow_k (b_n \vee (b \wedge b'_n)) = b.$$

Assuming the Egoroff property for b, we have $a_m \uparrow b$ such that, for all m, $a_m \ll \{a_{n,k}\}$. Now $a_m \wedge b_m \uparrow b$, and, for each m, there exists $k(m)$ such that

$$a_m \wedge b_m \leq a_{m,k(m)} \wedge b_m = (b_{m,k(m)} \vee (b \wedge b'_m)) \wedge b_m = b_{m,k(m)}.$$

Thus $\rho(a_m \wedge b_m) \leq \rho(b_{m,k(m)}) < \lambda$, so that $\rho_L(b) \leq \lambda$. Since this holds for any $\lambda > \rho_{LL}(b)$, we have $\rho_L(b) \leq \rho_{LL}(b)$.

(b) Suppose, for each n, $b_{n,k} \uparrow_k 1$. Let

$$\bar{b}_{n,k} = b_{1,k} \wedge b_{2,k} \wedge \dots \wedge b_{n,k} \quad ;$$

again we have, for each n, $\bar{b}_{n,k} \uparrow_k 1$. Note that unless there is some $\bar{n}$ such that $b_{\bar{n},k} \neq 1$ for every k, it is immediate that we can find $b_m \uparrow_m 1$ such that $b_m \ll \{b_{n,k}\}$; in fact, if there is no such $\bar{n}$ we have $1 \ll \{b_{n,k}\}$. Now let $a_n = b_{\bar{n},n}$, and let $a_{n,k} = a_n \wedge \bar{b}_{n,k}$. Define ρ on $\mathcal{B}$ by

$$\rho(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{2} & \text{if } \exists n,k \text{ such that } x \leq a_{n,k} \\ 1 & \text{otherwise.} \end{cases}$$

It is evident that ρ is an outer measure on $\mathcal{B}$. Moreover, as $a_{n,k} \uparrow_k a_n$ and $a_n \uparrow_n 1$, we have $\rho_{LL}(1) = \frac{1}{2}$. By our assumption, then, $\rho_L(1) = \frac{1}{2}$, and clearly this means that there exists a sequence

c_m and, for each m, there exist $n(m)$ and $k(m)$ such that $c_m \uparrow 1$ and $c_m \leq a_{n(m),k(m)}$. Now $n(m) \longrightarrow \infty$ as $m \longrightarrow \infty$. To see this observe that if, for some n, $n(m) = n$ for an infinite number of m, then for each c_m there exists $m' \geqslant m$ such that $n(m') = n$; but then, for each m,

$$c_m \leqslant c_{m'} \leqslant a_{n(m'),k(m')} \leqslant a_{n(m')} = a_n \neq 1 ,$$

so that $c_m \not\uparrow 1$, a contradiction. We can now show that each $c_m \ll \{b_{n,k}\}$, since, given fixed m and n, we can find $m' \geqslant m$ such that $n(m') \geqslant n$; but in that case

$$c_m \leqslant c_{m'} \leqslant a_{n(m'),k(m')} \leqslant \bar{b}_{n(m'),k(m')} \leqslant b_{n,\, k(m')} \quad .$$

Hence we have shown that $1 \in \mathcal{B}$ has the Egoroff property, and, in view of Theorem 1.1(b), this completes the proof of (b).

To every outer measure ρ on a Boolean algebra $\mathcal{B}$ there corresponds the subalgebra M_ρ of elements which are measurable with respect to ρ in the well-known sense of Carathéodory (i.e., $b \in M_\rho \iff [\forall a \in \mathcal{B} : \rho(a) = \rho(a \wedge b) + \rho(a \wedge b')]$). It is always true that M_ρ is a subalgebra of $\mathcal{B}$ and that ρ is countably additive on M_ρ ; indeed, if b_n is a disjoint sequence in M_ρ <u>and</u> $b = \bigvee_1^\infty b_n$ exists, then $b \in M_\rho$ and $\rho(b) = \sum_1^\infty \rho(b_n)$. In general it may happen that $M_\rho = \{0, 1\}$. The outer measure ρ is called <u>regular</u> if ρ is "generated" by (ρ, M_ρ), i.e., for each $a \in \mathcal{B}$,

$$\rho(a) = \inf\{\sum_1^\infty \rho(b_n) \; : \; a = \bigvee_{n=1}^\infty (a \wedge b_n) \text{ and } b_n \in M\} \quad .$$

If the Boolean algebra $\mathcal{B}$ is σ-complete, as is the case when $\mathcal{B}$ is the measure algebra associated with a measure space $(X, \mathfrak{F}, \mu)$ (cf. section 2), then M_ρ is itself a σ-complete Boolean algebra. In this situation it is not hard to show that every <u>regular</u> outer measure ρ on $\mathcal{B}$ has the property $\rho_L = \rho$, and, a fortiori, $\rho_{LL} = \rho_L$. Thus we may state the following theorem as a consequence of Theorem 4.1.

<u>Theorem 4.2</u>: If $\mathcal{B}$ is the measure algebra associated with a measure space $(X, \mathfrak{F}, \mu)$, then $\mathcal{B}$ is Egoroff if, and only if, $\rho_{LL} = \rho_L$ for every outer measure ρ on $\mathcal{B}$ which is not regular.

5: The Rôle of the Almost-Egoroff Property in Riesz Spaces

The Riesz spaces which occur naturally in classical analysis usually come equipped with norms, and almost invariably such a norm is compatible with the order structure of the space to the extent that the norm is monotone. Inevitably, then, a theory of monotone norms and seminorms has developed in the study of Banach function spaces and in the study of the more general Riesz spaces.

We shall call a function ρ on a Riesz space L a monotone seminorm if (i) $\rho : L \longrightarrow [0, \infty]$ (note that $+\infty$ is admitted as a value),

(ii) ρ is a seminorm in the usual sense, i.e.,
ρ is positive homogeneous ($\lambda \geqslant 0 \Longrightarrow \rho(\lambda f) = \lambda \rho(f)$)
and sublinear ($\rho(f+g) \leqslant \rho(f) + \rho(g)$) , and

(iii) ρ is monotone, i.e., $|g| \leqslant |f| \Longrightarrow \rho(g) \leqslant \rho(f)$
(note that this implies that $\rho(f) = \rho(|f|)$, or, as it is sometimes put, ρ is "absolute").

Such a ρ will be called a monotone norm if, in addition, $\rho(f) = 0 \Longrightarrow f = 0$. Following Luxemburg and Zaanen [3] , section 22, we reserve the term Riesz (semi)norm for the case where these objects satisfy the additional requirement that they assume only finite values in L.

A special but important class of monotone seminorms is the class of those seminorms which are σ-Fatou. We say that a monotone seminorm on a Riesz space L is σ-Fatou if, whenever $u_n, u \in L^+$ and $u_n \uparrow u$, we have $\rho(u_n) \uparrow \rho(u)$ (cf. [3] , section 5). For example, the norms in the classical L^p spaces $(1 \leqslant p \leqslant \infty)$ are σ-Fatou; moreover, the name " σ-Fatou" is inherited from these examples (L^p norms are σ-Fatou as a consequence of the famous "Fatou lemma").

We first wish to note that, although a given monotone seminorm ρ may not itself be σ-Fatou, there is always a largest element among those monotone seminorms dominated by ρ which do have that property. We formulate this comment as a theorem.

<u>Theorem 5.1</u>: If ρ is a monotone seminorm on a Riesz space L, then there exists a monotone seminorm $\rho_1 \leq \rho$ such that ρ_1 is σ-Fatou and, for any monotone seminorm $\rho_2 \leq \rho$ such that ρ_2 is σ-Fatou, $\rho_2 \leq \rho_1$ (ρ_1 is clearly unique).

<u>Proof</u>: Let A be the set of monotone seminorms on L which are dominated by ρ and which are σ-Fatou; the seminorm which is 0 throughout L is an element of A. We set, for f in L, $\rho_1(f) = \sup\{\rho_2(f) : \rho_2 \in A\}$. Clearly we need only show that ρ_1, which is obviously a monotone seminorm, is indeed σ-Fatou; but if $0 \leq u_n \uparrow u$ and $\lambda < \rho_1(u)$, then there is some $\rho_2 \in A$ such that $\lambda < \rho_2(u)$, and we have

$$\lim_n \rho_1(u_n) \geqslant \lim_n \rho_2(u_n) = \rho_2(u) > \lambda$$

so that $\rho_1(u_n) \uparrow \rho_1(u)$.

It was Lorentz (see [3] , section 7) who showed that, in the case where L is a (real) Banach function space, the seminorm ρ_1 of Theorem 5.1 can be calculated simply as ρ_L , the "Lorentz seminorm corresponding to ρ", where, for each $u \in L^+$,

$$\rho_L(u) = \inf\{\lim_n \rho(u_n) : 0 \leq u_n \uparrow u\} .$$

The real Banach function spaces are, however, rather special, and the question remained to be answered: to what extent does the simple relation " $\rho_1 = \rho_L$ " occur in the realm of general Riesz spaces, i.e., in which cases do we have an elementary construction for ρ_1 ? It turns out that a precise solution can be given to this problem in terms of the almost-Egoroff property introduced in section 1; in fact, $\rho_1 = \rho_L$ for every monotone seminorm on the Riesz space L just in case L is almost-Egoroff. Presently we shall prove this theorem (as a consequence of some sharper intermediate results), but first a few comments should be made.

It is easy to show that, for any monotone seminorm ρ , ρ_L is again a monotone seminorm; $\rho_L \leq \rho$ and, if ρ_2 is a monotone seminorm such that $\rho_2 \leq \rho$, then $\rho_{2L} \leq \rho_L$; furthermore, a monotone seminorm ρ is σ-Fatou if and only if $\rho_L = \rho$. Clearly, then, $\rho_L \geq \rho_{1L} = \rho_1$ and the statement $\rho_L = \rho_1$ can, in view of the characteristic property of ρ_1 , be given a convenient equivalent form as $\rho_{LL} = \rho_L$.

In the case where $\rho = \varphi$, a non-negative linear functional on L, as one would expect from our discussion in section 4 of the analogous situation for finitely additive measures on Boolean algebras, we have always $\varphi_{LL} = \varphi_L$ (on L^+) , regardless of the nature of the space L. This is simply saying that φ_L represents the projection of the non-negative linear functional φ onto the normal subspace of integrals in $L^\sim$, the order dual of L (see Luxemburg and Zaanen [3], section 20).

Let us first prove the theorem which generalizes the result of Lorentz concerning Banach function spaces.

<u>Theorem 5.2</u>: Let L be a Riesz space. If f is an element of L^+ and f has the almost-Egoroff property, then for every monotone seminorm ρ we have $\rho_{LL}(f) = \rho_L(f)$.

<u>Proof</u>: It is clear that $\rho_{LL}(f) \leq \rho_L(f)$. On the other hand, suppose $\lambda > \rho_{LL}(f)$; in this case there must exist $0 \leq f_n \uparrow_n f$ and, for each n, $0 \leq f_{n,k} \uparrow_k f_n$ such that $\rho(f_{n,k}) < \lambda$ for all n, k. Now if we let $g_{n,k} = f_{n,k} + (f - f_n)$, we have $g_{n,k} \uparrow_k f$, for each n. Since f has the almost-Egoroff property there exists, for each $\varepsilon > 0$, a sequence f_m^ε such that $0 \leq f_m^\varepsilon \uparrow_m (1 - \varepsilon)f$ and $f_m^\varepsilon \ll \{g_{n,k}\}$, for all m. Now if we set $h_m^\varepsilon = (f_m^\varepsilon + f_m - f)^+$, we have

$$h_m^\varepsilon \uparrow_m (1 - \varepsilon)f \quad \text{(we can assume } \varepsilon < 1\text{).}$$

Moreover, for each m there is some k(m) such that $f_m^\varepsilon \leq g_{m,k(m)}$ and

$$h_m^\varepsilon \leq g_{m,k(m)} + f_m - f = f_{m,k(m)} \quad .$$

We then have $\rho(h_m^\varepsilon) \leq \rho(f_{m,k(m)}) < \lambda$, so that $\rho_L((1 - \varepsilon)f) \leq \lambda$, i.e., $\rho_L(f) \leq (1 - \varepsilon)^{-1}.\lambda$. Since this is true for every $\varepsilon > 0$, we have $\rho_L(f) \leq \lambda$; hence $\rho_L(f) \leq \rho_{LL}(f)$.

<u>Corollary</u> (Lorentz) : If L is a Banach function space with function seminorm ρ , then $\rho_{LL} = \rho_L$.

<u>Proof</u>: The hypotheses say, essentially, that ρ is a monotone seminorm on the space L of measurable functions (with identification of functions equal a.e.) over a measure space $(X, \mathfrak{F}, \mu)$ which is σ-finite (cf. Luxemburg and Zaanen [3] , sections 2 and 3). Hence Theorem 2.3 (along with Theorem 2.1) tells us that L is Egoroff. A fortiori, ρ is

a monotone seminorm on an almost-Egoroff space, so that, by the theorem just proved, $\rho_{LL} = \rho_L$.

We now prove a strong converse to Theorem 5.2; the full strength of this result will be useful in our subsequent discussion.

Theorem 5.3: Let f be an element of L^+, where L is a Riesz space; if $\rho_{LL}(f) = \rho_L(f)$ for every monotone seminorm ρ on L such that $\rho(f) < \infty$, then f has the almost-Egoroff property.

Proof: Suppose that $0 \leq f_{n,k} \uparrow_k f$, for each n. Let

$$\bar{f}_{n,k} = f_{1,k} \wedge f_{2,k} \wedge \cdots \wedge f_{n,k} \quad ;$$

then $\bar{f}_{n,k} \uparrow_k f$, for each n, and $n \leq m \Longrightarrow \bar{f}_{n,k} \geqslant \bar{f}_{m,k}$. We shall show that, given $\varepsilon > 0$, there exists a sequence f_m^{ε} such that

$$0 \leqslant f_m^{\varepsilon} \uparrow_m (1 - \varepsilon)f \quad \text{and} \quad f_m^{\varepsilon} \ll \{\bar{f}_{n,k}\} ,$$

for each m; since, for all n, k, $\bar{f}_{n,k} \leqslant f_{n,k}$ we also have $f_m^{\varepsilon} \ll \{f_{n,k}\}$.

(a) First we show that, given any $\varepsilon_1 > 0$, there exists a sequence g_n such that $0 \leqslant g_n \uparrow f$ and

$$[\exists n: \ Mf \leqslant g_n] \Longrightarrow M \leqslant \varepsilon_1 \quad .$$

For each n, let

$$\beta_n = \sup \{\beta : (\exists k)(\beta f \leqslant \bar{f}_{n,k}) \} .$$

Clearly, for all n, $\beta_n \geqslant 0$; on the other hand, if $\beta_n \geqslant 1$ for all n, then we can construct f_m^{ε} immediately by setting $f_m^{\varepsilon} = (1 - \varepsilon)f$, since $\quad (1 - \varepsilon) < \beta_n \Longrightarrow \exists k(n) : \quad (1 - \varepsilon)f \leqslant \bar{f}_{n,k(n)}$,

i.e., we have $f_m^{\varepsilon} = (1 - \varepsilon)f \ll \{\bar{f}_{n,k}\}$.

We can assume, then, that there exists $\bar{n}$ such that $0 \leq \beta_{\bar{n}} \lneqq 1$. Consider, for $\beta \lneqq \beta_{\bar{n}}$, the sequence

$$h_k = \frac{f_{\bar{n},k} - \beta f}{(1-\beta)} \quad ;$$

clearly $h_k \uparrow_k f$ and the terms of the sequence are non-negative after a finite number, since $\beta \lneqq \beta_{\bar{n}}$. Denote the non-negative tail of the sequence $\{h_k\}$ by $g_1, g_2, \ldots$. If there exists k such that $Mf \leq g_k$, then there is a corresponding k' such that

$$Mf \leqslant \frac{f_{\bar{n},k'} - \beta f}{(1-\beta)} \quad , \text{ i.e., } ((1-\beta)M + \beta)f \leq f_{\bar{n},k'} \quad ,$$

so that $(1-\beta)M + \beta \leqslant \beta_{\bar{n}}$, or $M \leqslant \dfrac{\beta_{\bar{n}} - \beta}{(1-\beta)}$.

Thus for any $\varepsilon_1 > 0$ we can, by taking β sufficiently close to $\beta_{\bar{n}}$, ensure that $[\exists n\colon \; Mf \leq g_n] \Longrightarrow M \leq \varepsilon_1$.

(b) Next, for any $\varepsilon_2 > 0$, set

$$g_{n,k} = (f_{n,k} \wedge g_n) \vee \varepsilon_2 f \quad .$$

It is convenient to assume that $\varepsilon_2 < 1$; then, for each n,

$$g_{n,k} \uparrow_k (f \wedge g_n) \vee \varepsilon_2 f = g_n \vee \varepsilon_2 f \quad , \text{ and}$$

$$(g_n \vee \varepsilon_2 f) \uparrow_n (f \vee \varepsilon_2 f) = f \quad .$$

Now for $0 \leq x \in L$ set

$\rho(x) = \begin{cases} \inf\{ (\sum_n \alpha_n)\colon \; \alpha_n \geqslant 0 \text{ and } \sum_n \alpha_n g_{n,k(n)} \geqslant x \text{ for some choice of } k(n) \} \text{; the sums involved are understood to be finite, i.e., all but a finite number of the } \alpha_n \text{ are zero;} \\ +\infty \text{ if there is no such finite sum covering } x. \end{cases}$

It is not hard to see that ρ is a monotone seminorm on L: that ρ is monotone and positive homogeneous is clear; ρ is also subadditive, for if $\lambda > \rho(x) + \rho(y)$, then there exist α_n , $k(n)$, α'_n , $k'(n)$ such that $\lambda > \sum_n \alpha_n + \sum_n \alpha'_n$ and

$$\sum_n \alpha_n g_{n,k(n)} \geqslant x \quad ; \quad \sum_n \alpha'_n g_{n,k'(n)} \geqslant y \quad ,$$

so that we have

$$\sum_n (\alpha_n + \alpha'_n) g_{n,\max(k(n),\, k'(n))} \geqslant x + y$$

and $\rho(x + y) \quad \sum_n (\alpha_n + \alpha'_n) < \lambda$.

Moreover, $\rho(f) < \infty$; in fact, $g_{1,1} \geqslant \varepsilon_2 f$, so that $\rho(f) \leqslant (\varepsilon_2)^{-1}$.

Now $\rho(g_{n,k}) \leqslant 1$ so that, since $g_{n,k} \uparrow_k (g_n \vee \varepsilon_2 f)$, we have $\rho_L(g_n \vee \varepsilon_2 f) \leqslant 1$, for all n. But $(g_n \vee \varepsilon_2 f) \uparrow_n f$, so that $\rho_{LL}(f) \leqslant 1$. Hence, by our assumption, $\rho_L(f) \leqslant 1$. This means that, for any $\varepsilon_3 > 0$, there exists a sequence f_m such that $0 \leqslant f_m \uparrow_m f$ and, for each m, $\rho(f_m) < 1 + \varepsilon_3$.

(c) If we now set $f_m^{\varepsilon} = (f_m - \varepsilon f)^+$, we have $0 \leqslant f_m^{\varepsilon} \uparrow_m (f - \varepsilon f)^+ = (1 - \varepsilon) f$ (we may assume that $\varepsilon < 1$). The remainder of the proof consists in showing that, by appropriate choices of ε_1, ε_2, ε_3 (> 0), we can ensure that $f_m^{\varepsilon} \ll \{\bar{f}_{n,k}\}$.

Let us find, then, conditions on ε_1, ε_2, ε_3 which will ensure that, for a particular m_1, n_1 , there exists some k such that $f_{m_1}^{\varepsilon} \leqslant \bar{f}_{n_1,k}$. We shall find conditions which are independent of m_1, n_1, thereby establishing our result.

Now $\rho(f_m) < 1 + \varepsilon_3$ so that there exist $\alpha_n^m \geqslant 0$, $k^m(n)$ such that

$$\sum_n \alpha_n^m < 1 + \varepsilon_3 \quad \text{and} \quad \sum_n \alpha_n^m g_{n,k^m(n)} \geqslant f_m .$$

Suppose that γ is such that, for every $m \geqslant m_1$, $\sum_{n < n_1} \alpha_n^m > \gamma$.

$$\text{Now} \quad f_m - \sum_{n \geqslant n_1} \alpha_n^m g_{n,k^m(n)} \leqslant \sum_{n \leqslant n_1} \alpha_n^m g_{n,k^m(n)}$$

so that, for all $m \geqslant m_1$, we certainly have

$$f_m - (1 + \varepsilon_3 - \gamma)f \leqslant (1 + \varepsilon_3)(g_{n_1} \vee \varepsilon_2 f) \leqslant (1 + \varepsilon_3)(g_{n_1} + \varepsilon_2 f) .$$

But $f = \sup \{f_m : m \geqslant m_1\}$ so that we have

$$f - (1 + \varepsilon_3 - \gamma)f \leqslant (1 + \varepsilon_3)(g_{n_1} + \varepsilon_2 f) , \text{ i.e.,}$$

$$\left(\frac{\gamma - \varepsilon_3}{1 + \varepsilon_3} - \varepsilon_2 \right) f \leqslant g_{n_1} .$$

Hence, by the construction of part (a), $(\gamma - \varepsilon_3)(1 + \varepsilon_3)^{-1} - \varepsilon_2 \leqslant \varepsilon_1$, i.e., $\gamma \leqslant (1 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2) + \varepsilon_3$. Clearly, then, if we let $\varepsilon_4 = (1 + \varepsilon_3)(\varepsilon_1 + \varepsilon_2) + 2\varepsilon_3$, there must exist $m \geqslant m_1$ such that

$$\sum_{n < n_1} \alpha_n^m < \varepsilon_4 .$$

Now, for such an $m \geqslant m_1$, we have

$$\begin{aligned} f_m &\leqslant \varepsilon_4 f + \sum_{n \geqslant n_1} \alpha_n^m g_{n,k^m(n)} \\ &\leqslant \varepsilon_4 f + \sum_{n \geqslant n_1} \alpha_n^m (\bar{f}_{n,k^m(n)} + \varepsilon_2 f) . \end{aligned}$$

If we let $k = \max \{k^m(n) : \alpha_n^m \neq 0\}$ and recall that, for $n \geqslant n_1$,

$\bar{f}_{n,k} \leq \bar{f}_{n_1,k}$, we see that

$$f_m \leq \varepsilon_4 f + \Big(\sum_{n \geq n_1} \alpha_n^m \Big)(\bar{f}_{n_1,k} + \varepsilon_2 f) .$$

Now $m \geq m_1$ so that we have

$$f_{m_1} \leq f_m \leq \varepsilon_4 f + (1 + \varepsilon_3)\bar{f}_{n_1,k} + (1 + \varepsilon_3)\, \varepsilon_2 f$$

$$\leq (\varepsilon_4 + \varepsilon_3 + (1 + \varepsilon_3)\, \varepsilon_2) f + \bar{f}_{n_1,k}$$

$$\leq \varepsilon f + \bar{f}_{n_1,k}$$

provided we choose $\varepsilon_1, \varepsilon_2, \varepsilon_3$ (> 0) such that $((1 + \varepsilon_3)(\varepsilon_1 + 2\varepsilon_2) + 3\varepsilon_3) \leq \varepsilon$, which we clearly can do. In this case, $f_{m_1}^{\varepsilon} = (f_{m_1} - \varepsilon f)^+ \leq \bar{f}_{n_1,k}$, as required.

This completes the proof of the theorem.

The following result, mentioned earlier, is a direct consequence of Theorems 5.2 and 5.3.

<u>Theorem 5.4</u>: Let L be a Riesz space; then $\rho_{LL} = \rho_L$, for every monotone seminorm ρ on L, if, and only if, L is almost-Egoroff.

<u>Corollary</u>: Let L be an Archimedean Riesz space; then $\rho_{LL} = \rho_L$, for every monotone seminorm ρ on L, if, and only if, L is Egoroff.

<u>Proof</u>: Simply apply Theorem 3.5 to Theorem 5.4.

6: Riesz Seminorms

In Theorem 5.4 we showed that the almost-Egoroff property holds throughout a Riesz space L if, and only if, L has the following property:

$$P \equiv [\text{for every monotone seminorm } \rho \text{ on } L,\quad \rho_{LL} = \rho_L] \quad .$$

It is natural to ask whether the corresponding property for the more well-behaved Riesz seminorms (monotone seminorms which assume only finite values) has any such significance. We therefore consider, for a Riesz space L, the following property:

$$P' \equiv [\text{for every Riesz seminorm } \rho \text{ on } L,\quad \rho_{LL} = \rho_L] \quad .$$

The property P' is clearly weaker than property P and holds, therefore, in any Riesz space L which is almost-Egoroff. It is important to determine whether, in fact, P' may hold for every Riesz space. The following result, which depends upon the full strength of Theorem 5.3, shows that this is not the case.

Theorem 6.1: There exists a Riesz space L_0 in which property P' fails to hold; such an L_0 may even be weak Egoroff.

Proof: Let L be a Riesz space containing an element f which has not the almost-Egoroff property; to be more specific, we might choose L to be the space of Example 3.1 or that of Example 3.2. In view of Theorem 5.3, there must exist a monotone seminorm ρ on L such that $\rho_{LL}(f) \lneqq \rho_L(f)$ and $\rho(f) < \infty$. Now let $L_0 = \{g: g \in L \text{ and } \rho(g) < \infty\}$. L_0 is an ideal in L, so that L_0 is itself a Riesz space and, if $\bar{\rho}$ denotes the restriction of ρ to L_0 , we have $\bar{\rho}_L = \rho_L$, $\bar{\rho}_{LL} = \rho_{LL}$ on L_0 .

Thus $\bar{\rho}$, which is, of course, a Riesz seminorm on L_0 , violates property P', since $f \in L_0$. If L is weak Egoroff (in particular, if L is the space of Example 3.2), then L_0 is again weak Egoroff.

The theorem above makes it clear that property P' is not a consequence of the weak Egoroff property. On the other hand, we shall demonstrate that property P' is strictly weaker than the almost-Egoroff property. Consider the space $L = R^{(N^N)}$ of Example 3.2. It is not difficult to show (cf. [3], Example 20.8) that, whenever L has the form R^X, for some set X, every order bounded linear functional φ on L is an integral, i.e., $[u_n \downarrow 0 \Longrightarrow \varphi(u_n) \longrightarrow 0]$. To put it briefly, we are dealing with a space L such that $L^\sim = L^\sim_c$ (cf. [3], section 20). Let ρ be a Riesz seminorm on such a space L. For each element u in L^+, there exists, in view of the Hahn-Banach theorem, a linear functional φ_u on L such that $\varphi_u(u) = \rho(u)$ and, for all f in L, $\varphi_u(f) \leq \rho(f)$. Since ρ is monotone, φ_u is an order bounded linear functional, and hence φ_u is an integral. Now if $0 \leq u_n \uparrow u$ we have $\varphi_u(u_n) \rightarrow \varphi_u(u) = \rho(u)$, and, since $\varphi_u(u_n) \leq \rho(u_n)$, it is clear that $\rho(u_n) \uparrow \rho(u)$. We conclude that, for every Riesz seminorm ρ , $\rho_L = \rho$; a fortiori, L has property P'. We have exhibited, then, a space L ($= R^{(N^N)}$) which has property P' but which is not almost-Egoroff. It should be remarked that, by means of a more pertinent application of the Hahn-Banach theorem (in the form of Mazur's theorem), it can be shown (cf. [3], Note VII, Lemma 22.6) that in a space L such that $L^\sim = L^\sim_c$ every Riesz seminorm ρ is actually an "integral" seminorm, i.e.,

$$[u_n \downarrow 0 \Longrightarrow \rho(u_n) \downarrow 0] \;;$$

it is an immediate consequence of this fact that $\rho_L = \rho$.

Note that each of the Egoroff properties is "permanent" in a Riesz space, i.e., if a Riesz space L is Egoroff (almost-Egoroff, weak Egoroff) then every ideal of L is Egoroff (almost-Egoroff, weak Egoroff) when considered as a Riesz space in its own right. In this respect property P' proves itself quite different from any of the Egoroff conditions, for it is clearly not permanent. In fact, we have seen that the space $L = R^{(N^N)}$ has property P', while the proof of Theorem 6.1 shows that $R^{(N^N)}$ contains an ideal which does not have this property.

7: Lorentz Seminorms

In an almost-Egoroff space the Lorentz seminorms, i.e., those seminorms ρ such that $\rho = \bar{\rho}_L$, for some monotone seminorm $\bar{\rho}$, can be identified precisely as those monotone seminorms which are σ-Fatou. In any Riesz space, in fact, a monotone seminorm ρ which is σ-Fatou is certainly a Lorentz seminorm, since $\rho = \rho_L$. On the other hand, if the Riesz space in question is almost-Egoroff, Theorem 5.2 ensures that every Lorentz seminorm ρ_L is σ-Fatou. Moreover, Theorem 5.3 makes it clear that only in almost-Egoroff spaces is this simple characterization possible. The problem of characterizing the Lorentz seminorms in general Riesz spaces apparently remains unsolved. We wish to point out, however, that, regardless of the structure of the Riesz space on which they are defined, Lorentz seminorms are always "σ-Fatou null". We say ρ is a σ-Fatou null monotone seminorm if

$$[0 \leq u_n \uparrow u \text{ and } (\forall n)(\rho(u_n) = 0)] \Longrightarrow [\rho(u) = 0] .$$

This property is important in other connections and was introduced by Luxemburg and Zaanen (cf. [3], section 5).

Theorem 7.1: If ρ is a monotone seminorm on a Riesz space L, then ρ_L is σ-Fatou null.

Proof: Suppose $0 \leq u_n \uparrow u$, and, for each n, $\rho_L(u_n) = 0$. Given $\varepsilon > 0$, we can choose, for each n, $0 \leq u_{n,k} \uparrow_k u_n$ such that $\rho(u_{n,k}) < \varepsilon \cdot 2^{-n}$. Now let $v_n = u_{1,n} \vee \cdots \vee u_{n,n}$. Clearly $0 \leq v_n \uparrow u$. Moreover,

$$\rho(v_n) \leq \rho(u_{1,n}) + \cdots + \rho(u_{n,n}) \leq \varepsilon \cdot 2^{-1} + \varepsilon \cdot 2^{-2} + \cdots = \varepsilon .$$

Thus $\rho_L(u) \leq \varepsilon$, and since ε may be any positive number, we have $\rho_L(u) = 0$.

8: Descendants of a Theorem of S. Koshi

In his paper [14] of 1958 S. Koshi proved that, under certain conditions on a Boolean algebra $\mathcal{B}$, every finitely additive measure on $\mathcal{B}$ is countably additive on a suitable super order dense ideal. Subsequently Luxemburg showed that the Egoroff property affords a natural approach to this problem, by proving that in an Egoroff Boolean algebra every finitely additive measure is countably additive on a super order dense ideal (cf. [6] , Theorem 5.1); Koshi's theorem follows from this result. The corresponding result for non-negative linear functionals on a Riesz space appears in Luxemburg and Zaanen [3] (Note VI, Corollary 20.7). The following theorem includes an extension of the last result to almost-Egoroff spaces, where a rather different technique of proof seems to be necessary. We shall state this extension explicitly as a corollary.

Theorem 8.1: If ρ is a Riesz seminorm on a Riesz space L which is almost-Egoroff, and $[\rho - \rho_L]$ is also a monotone seminorm on L, then (a) ρ_L is "attained" throughout L, i.e., for each u in L^+, there exists a sequence u_n such that $0 \leq u_n \uparrow u$ and $\rho_L(u) = \lim_n \rho(u_n)$; and

(b) $I = \{f : \rho(f) = \rho_L(f)\}$ is a super order dense ideal in L.

Proof: Consider $u \in L^+$; for each n, there exists a sequence $u_{n,k}$ such that $0 \leq u_{n,k} \uparrow_k u$ and $\lim_k \rho(u_{n,k}) < \rho_L(u) + \frac{1}{n}$.

Since u has the almost-Egoroff property, we certainly have a sequence u_m such that $0 \leq u_m \uparrow \frac{1}{2}u$, and, for each m, $u_m << \{u_{n,k}\}$.

Now, for each n, $u_{n,k} \uparrow_k u$, so that, by Theorem 5.2,

$\rho_L(u_{n,k}) \uparrow_k \rho_L(u)$; hence there exists k(n) such that

$\rho_L(u_{n,k}) > \rho_L(u) - \frac{1}{n}$, whenever $k \geqslant k(n)$. Fixing m for the moment, and recalling that $u_m << \{u_{n,k}\}$, there exists $k'(n)$, for each n, such that $u_m \leqslant u_{n,k'(n)}$. Now for any $k \geqslant \max(k(n), k'(n))$ we have $u_m \leqslant u_{n,k}$ and

$$[\rho - \rho_L](u_{n,k}) \leqslant (\rho_L(u) + \frac{1}{n}) - (\rho_L(u) - \frac{1}{n}) = \frac{2}{n} ;$$

hence, as $[\rho - \rho_L]$ is monotone, $[\rho - \rho_L](u_m) \leqslant \frac{2}{n}$ for all n, so that $[\rho - \rho_L](u_m) = 0$. Finally, $0 \leqslant 2u_m \uparrow u$, and certainly $[\rho - \rho_L](2u_m) = 0$, so that $I = \{f : \rho(f) = \rho_L(f)\}$ is super order dense in L. Since $[\rho - \rho_L]$ is a monotone seminorm, I is certainly an ideal in L, so that we have proved part (b) of the theorem. Part (a) also follows immediately, since $2u_m \uparrow u$ and $\rho(2u_m) = \rho_L(2u_m) \leqslant \rho_L(u)$.

<u>Corollary</u>: If φ is a non-negative linear functional on an almost-Egoroff Riesz space L, then $I = \{ f : \varphi(|f|) = \varphi_L(|f|) \}$ is a super order dense ideal in L.

<u>Proof</u>: If we set $\rho(f) = \varphi(|f|)$, ρ is certainly a Riesz seminorm on L. Furthermore $[\rho - \rho_L](f) = \varphi_s(|f|)$ where φ_s is the (non-negative) singular linear functional associated with φ (cf. [3] , section 20), so that $[\rho - \rho_L]$ is certainly a monotone seminorm. Thus we can apply the previous theorem to get our result.

Note that if we modify the hypotheses of Theorem 8.1 by replacing "almost-Egoroff" with "weak Egoroff", we can prove (using the same technique) that $I = \{ f : \rho(f) = \rho_L(f) \}$ is a <u>dense</u> ideal in L in the sense that $[\, 0 \lneqq u \in L \Longrightarrow \exists v \in I$ such that $0 \lneqq v \leq u \,]$.

It should be pointed out that Riesz seminorms ρ satisfying the condition of Theorem 8.1 (i.e., $[\rho - \rho_L]$ is a monotone seminorm) are frequently encountered, quite apart from the case where ρ is generated by a non-negative linear functional (as in the Corollary above). In fact, the question was raised by Luxemburg and Zaanen (cf. [3] , Note IV, p. 262) whether $[\rho - \rho_L]$ is <u>always</u> a monotone seminorm, at least in the case where ρ is a function seminorm on a Banach function space. That this is not the case, even for function seminorms, is the significance of the following example.

<u>Example 8.2</u>: Let L be the space of all bounded real functions over the integers N and, for f in L, set

$$\rho(f) = \sup(\, |f(1)| \, , \, \limsup_{n \to \infty} |f(n)| \,) \, .$$

Now ρ is certainly a Riesz seminorm on L; indeed, ρ is a function seminorm on the function space L. However, we shall show that $[\rho - \rho_L]$ is not a monotone seminorm on L. In fact, $[\rho - \rho_L]$ <u>is not monotone</u>. To see this consider the elements u, v $\in$ L , where $u(n) = 1$ for all n, and $v(1) = 0$, $v(n) = 1$ for $n > 1$. We have $0 \leq v \leq u$, and clearly $\rho(u) = \rho(v) = 1$. Moreover, $\rho_L(u) = 1$ since

$$[\, 0 \leq u_n \uparrow_n u \Longrightarrow u_n(1) \uparrow u(1) = 1 \,] \;\; ;$$

on the other hand, $\rho_L(v) = 0$ since we have $0 \leq v_n \uparrow v$ where v_n is the characteristic function of the subset $\{2, 3, \dots , n\}$ of N so that $\rho(v_n) = 0$, for each n. Thus $[\rho - \rho_L](u) = 0$, $[\rho - \rho_L](v) = 1$, while $0 \leq v \leq u$. We should point out that our example L is an Egoroff space. One way to see this is to observe that L is an ideal in the Riesz space associated with the measure space $(N, p(N), \mu)$, where μ is the discrete measure, and to apply Theorem 2.3. Moreover ρ has the "Koshi property" , since $\rho = \rho_L$ on the super order dense ideal consisting of all functions in L having finite support.

In Theorem 8.1(a) we noted that under the conditions of that theorem, ρ_L is "attained" throughout L. In general this may not be the case. If we assume, however, that ρ is generated by a non-negative linear functional (as in the corollary to Theorem 8.1), we can show at least that the elements where ρ_L is attained form an ideal. In the absence of an Egoroff condition we cannot ensure that this ideal in any way exhausts the space.

<u>Theorem 8.3</u>: If φ is a non-negative linear functional on a Riesz space L, then $I = \{f : \varphi_L(|f|)$ is attained, i.e., $\exists u_n \uparrow |f|$ such that $\varphi(u_n) \uparrow \varphi_L(|f|) \}$

is an ideal in L.

<u>Proof</u>: Since it is clear that $[f \in I \iff |f| \in I]$ and $[f \in I , \alpha$ real $\implies \alpha f \in I]$, we need only show that

(i) $[u, v \in L^+, \; u, v \in I \implies u + v \in I]$, and

(ii) $[0 \leq v \leq u \in I \implies v \in I]$.

It is a simple matter to prove (i) since, if we have $0 \leq u_n \uparrow u$, $0 \leq v_n \uparrow v$ such that $\varphi(u_n) \uparrow \varphi_L(u)$, $\varphi(v_n) \uparrow \varphi_L(v)$, then $0 \leq (u_n + v_n) \uparrow (u + v)$ and $\varphi(u_n + v_n) = (\varphi(u_n) + \varphi(v_n))$; but this sequence increases to $(\varphi_L(u) + \varphi_L(v)) = \varphi_L(u + v)$.

To prove (ii), suppose that $0 \leq v \leq u$, and $0 \leq u_n \uparrow u$ such that

$\varphi(u_n) \uparrow \varphi_L(u)$. Now $0 \leq (u_n \wedge v) \uparrow_n v$ and

$0 \leq (u_n - (u_n \wedge v)) = ((u_n \vee v) - v) \uparrow_n (u - v)$.

Thus $\lim_n \varphi(u_n \wedge v) \geq \varphi_L(v)$ and $\lim_n \varphi(u_n - (u_n \wedge v)) \geq \varphi_L(u - v)$.

On the other hand, $\lim_n \varphi(u_n \wedge v) + \lim_n \varphi(u_n - (u_n \wedge v))$

$= \lim_n \varphi(u_n) = \varphi_L(u) = \varphi_L(v) + \varphi_L(u - v)$.

It follows that $\lim_n \varphi(u_n \wedge v) = \varphi_L(v)$ (and $\lim_n \varphi((u_n \vee v) - v)$

$= \varphi_L(u - v)$), so that $v \in I$.

There are other situations in which we can prove theorems of the "Koshi type", i.e., theorems which ensure that certain monotone seminorms are σ-Fatou when restricted to a super order dense ideal. It is a consequence of a theorem of Luxemburg (see [4], Note XIV, Theorem 44.2) that if L is a Riesz space possessing a Riesz norm ρ with respect to which L is separable, and if L is Egoroff, then $\rho = \rho_L$ on a super order dense ideal. The theorem of Luxemburg may itself be extended to almost-Egoroff spaces, but we restrict ourselves below to proving in the more general setting that part of his result which is of interest to us here.

Theorem 8.4: If ρ is a Riesz seminorm on an almost-Egoroff Riesz space L, and L is separable in the pseudometric induced by ρ, then $\rho = \rho_L$ on a super order dense ideal I.

Proof: Let f_n be a countable dense set in L. In virtue of the Hahn-Banach theorem there exist linear functionals φ_n such that $\varphi_n(|f_n|) = \rho(f_n)$

and, for all f in L, $|\varphi_n(f)| \leq \rho(f)$. Since ρ is monotone, each φ_n is an order bounded linear functional and we may consider the non-negative linear functionals φ_n^+ . In fact, we again have $|\varphi_n^+(f)| \leq \rho(f)$, for all f in L. Moreover, for each n,

$$\rho(f_n) \geq \varphi_n^+(|f_n|) \geq \varphi_n(|f_n|) = \rho(f_n) \text{ , i.e., } \varphi_n^+(|f_n|) = \rho(f_n) \text{ .}$$

Now, for every u in L^+, we have $\rho(u) = \sup_n \{\varphi_n^+(u)\}$. To see this, note first that, for all n, $\varphi_n^+(u) \leq \rho(u)$. On the other hand, given $\varepsilon > 0$, there exists f_n such that $\rho(u - f_n) < \frac{1}{2}\varepsilon$, since the f_n are dense in L. Thus $|\rho(u) - \rho(f_n)| < \frac{1}{2}\varepsilon$, i.e., $|\rho(u) - \varphi_n^+(|f_n|)| < \frac{1}{2}\varepsilon$. Moreover, $|u - |f_n|| \leq |u - f_n|$, since $u \geq 0$, so that $\rho(u - |f_n|) < \frac{1}{2}\varepsilon$, and $|\varphi_n^+(u - |f_n|)| < \frac{1}{2}\varepsilon$. Hence $|\rho(u) - \varphi_n^+(u)| < \varepsilon$ so that

$$\rho(u) - \varepsilon < \sup_n \{\varphi_n^+(u)\} \leq \rho(u) \text{ .}$$

Since ε may be arbitrarily small, we have our result, namely,

$$\rho(u) = \sup_n \{\varphi_n^+(u)\} \text{ .}$$

Now let I_n be a super order dense ideal in L such that

$$[f \in I_n \Longrightarrow \varphi_n^+(|f|) = \varphi_{nL}^+(|f|)] \text{ .}$$

The existence of such an ideal I_n for each n is guaranteed by the corollary to Theorem 8.1, since L is almost-Egoroff. Let $I = \bigcap_{n=1}^{\infty} I_n$. I is automatically an ideal and we shall show that, for f in I, $\rho(f) = \rho_L(f)$. There exists, for any $\varepsilon > 0$, some n such that $\varphi_n^+(|f|) > \rho(f) - \varepsilon$. Now $\rho_L(f) \geq \varphi_{nL}^+(|f|) = \varphi_n^+(|f|)$, since $|f| \in I \subset I_n$, so that, for all $\varepsilon > 0$, $\rho_L(f) > \rho(f) - \varepsilon$. Clearly, then, $\rho_L(f) = \rho(f)$.

It remains to prove that the ideal I is super order dense. Consider u in L^+. Since, for each n, I_n is super order dense, there exists a sequence $u_{n,k} \in I_n$ such that $0 \leq u_{n,k} \uparrow_k u$. Since L is almost-Egoroff, there exists a sequence u_m such that $0 \leq u_m \uparrow_m \frac{1}{2}u$ and $u_m \ll \{u_{n,k}\}$. The last property ensures that u_m is in each of the ideals I_n , so that the sequence u_m is in the ideal I. Finally, the sequence $2u_m$ is also in I, and $0 \leq 2u_m \uparrow_m u$.

Further results of "Koshi type" continue to appear (see, for example, Luxemburg [4] , Note XVI, Theorem 64.9), and it is not clear for which classes of monotone seminorms and Riesz spaces such theorems may be proved. Using the results of section 5, however, we can show that the almost-Egoroff Riesz spaces constitute perhaps the natural domain for such theorems.

<u>Theorem 8.5</u>: If L is a Riesz space such that for <u>every</u> monotone seminorm ρ on L there exists a super order dense ideal I_ρ in L such that $[f \in I_\rho \Longrightarrow \rho_L(f) = \rho(f)]$,
then L is almost-Egoroff.

<u>Proof</u>: Our result follows from Theorem 5.4 as soon as we have proved that, for every monotone seminorm ρ on L, $\rho_{LL} = \rho_L$.
To see that this is the case, consider u_n , $u \in L^+$ such that $0 \leq u_n \uparrow_n u$. There exists a sequence v_n such that $0 \leq v_n \uparrow_n u$

and each v_n is an element of the ideal I_ρ corresponding to ρ. Now, if $w_n = u_n \wedge v_n$, we have $0 \leq w_n \uparrow_n u$ with $w_n \in I_\rho$.
Now $\lim_n \rho(w_n) \geq \rho_L(u)$ so that

$$\lim_n \rho_L(u_n) \geq \lim_n \rho_L(w_n) = \lim_n \rho(w_n) \geq \rho_L(u) .$$

Hence, for any sequence u_n such that $0 \leq u_n \uparrow u$, we have

$$\rho_L(u_n) \uparrow \rho_L(u) , \text{ i.e., } \rho_{LL} = \rho_L .$$

9: Spaces Possessing an Integral Riesz Norm

Consider a Riesz space L with a Riesz norm ρ ; ρ is said to be integral if $[u_n \downarrow 0 \implies \rho(u_n) \downarrow 0]$ (this is the property called (A, i) by Luxemburg and Zaanen, in [3], section 33). Many (but not all) Egoroff Riesz spaces possess integral Riesz norms; on the other hand, it is an open question whether or not every integrally normed Riesz space is Egoroff. Several natural conditions, when added to the condition that a space L possess an integral Riesz norm, have been found to imply that L is Egoroff, and we shall mention these presently. First, however, we shall prove some simple results with the sole assumption that an integral norm exists, to give an indication of the connection between this assumption and the Egoroff property.

Theorem 9.1: If L is a Riesz space with an integral Riesz norm ρ , then (a) L is weak Egoroff, and

(b) if $u_{n,k}$, u are elements of L and, for each n, $0 \leq u_{n,k} \uparrow_k u$, then there exist indices $k(n)$ such that $\sup_n u_{n,k(n)} = u$.

Proof: Consider $0 \leq u_{n,k} \uparrow_k u$, as in (b). We can assume $u \neq 0$, so that $\rho(u) \gneq 0$. Since ρ is integral, $\rho(u - u_{n,k}) \downarrow_k 0$, for each n. Let k(n), for each n, be such that $\rho(u - u_{n,k(n)}) < \frac{1}{2}\rho(u).2^{-n}$. Now let $v_n = u_{1,k(1)} \wedge u_{2,k(2)} \wedge \cdots \wedge u_{n,k(n)}$; v_n is a decreasing sequence. Now

$$u - v_n = \bigvee_{i=1}^{n} (u - u_{i,k(i)}) \leq \sum_{i=1}^{n} (u - u_{i,k(i)}) ,$$

so that $\rho(u - v_n) \leq \frac{1}{2}\rho(u)(2^{-1} + 2^{-2} + \dots) = \frac{1}{2}\rho(u)$.

Hence $\rho(v_n) \geq \frac{1}{2}\rho(u)$ $(\gneq 0)$, for each n; it follows that $v_n \not\downarrow 0$, since ρ is integral. Thus there exists v such that $0 \lneq v \leq v_n$, for all n, so that $0 \lneq v \ll \{u_{n,k}\}$, and part (a) of the theorem is established. Part (b) also follows, for we have $\sup_n u_{n,k(n)} = u$. To see this observe that, since ρ is a norm, we need only show that $\inf_n \rho(u - u_{n,k(n)}) = 0$, which holds in view of the construction of k.

Theorem 9.1(a) is a result of Luxemburg (see [4], Note XVI, Theorem 65.3). Theorem 9.1(b) gives rather different information, for there are spaces which are weak Egoroff but which do not have the property (b) of the theorem (see Example 3.2).

Although we do not know whether or not every integrally normed Riesz space is Egoroff, the results of section 5 make possible an interesting reformulation of this "conjecture".

Theorem 9.2: The statement

A ≡ [every integrally normed Riesz space is Egoroff]

is logically equivalent to the statement

B ≡ [if a Riesz space L possesses an integral Riesz norm ρ_1, then, for every Riesz norm ρ on L, $\rho_{LL} = \rho_L$] .

Proof: That A implies B is an immediate consequence of Theorem 5.2. On the other hand, suppose A does not hold, i.e., we have a Riesz space L, with an integral Riesz norm ρ_2, which is not Egoroff. The existence of the Riesz norm ρ_2 implies at once that L is Archimedean, so that L is not almost-Egoroff, in view of Theorem 3.5. Now, using Theorem 5.3

just as we did in the proof of Theorem 6.1, we can show that there is a Riesz space L_0, which is an ideal in L, and a <u>Riesz</u> seminorm ρ on L_0 such that $\rho_{LL} \neq \rho_L$. Now let ρ_1 be the restriction of ρ_2 to L_0. Clearly ρ_1 is an integral Riesz norm on L_0. Moreover $(\rho_1 + \rho)$ is a Riesz <u>norm</u> on L_0 such that

$$(\rho_1 + \rho)_L = \rho_1 + \rho_L \neq \rho_1 + \rho_{LL} = (\rho_1 + \rho)_{LL}$$

($\rho_{1L} = \rho_1$, since ρ_1 is integral). This contradicts B.

Luxemburg has shown that if L possesses an integral Riesz norm ρ <u>and</u> L is locally ρ-complete, i.e., order bounded ρ-Cauchy sequences converge, then L is Egoroff (see [4], Note XVI, Theorem 65.2). Note that if the space in question is order complete, or even σ-Dedekind complete, then it is automatically locally ρ-complete (this may be seen as a consequence of a more general result in [4], Note XVI, Theorem 61.7).

If we assume that L possesses a Riesz norm which is not only integral but also <u>normal</u>, we obtain a different sort of condition under which it may be proved that L is Egoroff. This is a theorem of Luxemburg and Zaanen (see [3], Note XI, Theorem 35.1). A Riesz norm ρ on a Riesz space L is said to be <u>normal</u> if, for every directed system u_τ such that $u_\tau \downarrow_\tau 0$, we have $\rho(u_\tau) \downarrow 0$ (this is the property called (A, ii) by Luxemburg and Zaanen in [3], section 33). The case where L is order complete is included in this theorem as well, since an integral Riesz norm on a σ-Dedekind complete space is always normal; this result goes back, essentially, to Nakano (cf. [15], pp. 321-322).

In the light of the results just mentioned, it is important to investigate to what extent an integral Riesz norm "tends to be normal". Riesz spaces L were constructed fairly simply which possessed integral Riesz norms not normal on the whole space (cf. [3], the last space discussed in Example 29.11), but these norms were always normal when restricted to an appropriate <u>dense</u> ideal I in L (i.e., an ideal I with the property that, for each non-zero u in L^+, there exists v in I such that $0 \lneqq v \leq u$). We present below an example of a space L with an integral Riesz norm ρ which fails to be normal even to this extent; in fact the only ideal on which ρ is normal is the trivial ideal $\{0\}$. It should be remarked that, while the space of our example is not norm-complete, Luxemburg has shown (see [4], Note XVI, Theorem 65.5) that the existence of such an example implies the existence of a norm-complete example having the same pathological properties (and, in fact, we need only consider the ordinary norm-completion of the given space). We shall also show that our example is an Egoroff space; this indicates that the results known at present do not encompass all situations in which the existence of an integral norm on a Riesz space implies that the space is Egoroff, even if, indeed, this implication does not hold in general.

Consider then the following space, to be known as $L^{\#}$ throughout the remainder of this section. An element f in $L^{\#}$ is a sequence of functions: $f = (f^0, f^1, f^2, \ldots)$, where $f^n : R^n \longrightarrow R$, $n = 0, 1, 2, \ldots$. R denotes the reals and we make the convention that $R^0 = \{\phi\}$. We further require that each element of $L^{\#}$ be bounded, in the sense that, for a given f in $L^{\#}$, there is a constant M such that

$$\sup \{ |f^n(v)| : v \in R^n \} \leqslant M , \quad \text{for each } n.$$

The remaining condition on the elements of $L^{\#}$ is the essential one, and to describe it we need to introduce the operation c:

c is defined on any vector $v \in R^n$ $(n \geqslant 1)$, where $v = (v_1, v_2, \dots , v_n)$, by $c(v) = (v_1, v_2, \dots , v_{n-1})$.

Thus $c(R^n) = R^{n-1}$; note that, because of our convention, for any $t \in R$, $c(t) = \phi$. Now for each $f \in L^{\#}$ we define the "exceptional" sets $E_n(f)$ $(n \geqslant 1)$ corresponding to f by

$$E_n(f) = \{ v \in R^n : f^n(v) \neq f^{n-1}(c(v)) \} .$$

The essential requirement is that, for each $f \in L^{\#}$, $E_n(f)$ should be <u>finite</u> , for $n = 1, 2, 3, \dots$.

For the linear operations in $L^{\#}$ we make the obvious definitions: $(f + g)^n = f^n + g^n$; $(\lambda f)^n = \lambda . f^n$. It is easy to see that $L^{\#}$ is closed under these operations and becomes a linear space.
A partial ordering is introduced in $L^{\#}$ by the definition

$$[f \leqslant g \iff f^n \leqslant g^n , \ n = 0, 1, 2, \dots] .$$

Clearly $L^{\#}$ is a Riesz space with this ordering and

$$(f \vee g)^n = f^n \vee g^n , \quad (f \wedge g)^n = f^n \wedge g^n .$$

We define the monotone seminorm ρ on $L^{\#}$ by

$$\rho(f) = \sup_{n \geqslant 0} \{ (n + 1)^{-1} (\sup \{ |f^n(v)| : v \in R^n \}) \} ;$$

ρ is clearly a norm, and ρ is a <u>Riesz</u> norm because each f in $L^{\#}$ is bounded.

We denote the r-th iterate of the operation c by c^r , $r = 0, 1, 2, \ldots$ (c^0 is simply the identity mapping). For a given element f in $L^\#$, a vector v in R^n is called <u>f-regular</u> if, for each $r = 0, 1, 2, \ldots$,

$$[\, w \in R^{(n+r)} , \quad c^r(w) = v \implies f^{(n+r)}(w) = f^n(v) \,] \quad .$$

<u>Lemma 9.3</u>: If f is an element of $L^\#$, then the set of points (in $\bigcup_{n=0}^{\infty} R^n$) which are <u>not</u> f-regular is at most countable.

<u>Proof</u>: If v in R^n is not f-regular, there exists some integer r and w in $R^{(n+r)}$ such that $c^r(w) = v$ and $f^{(n+r)}(w) \neq f^n(v)$. Let r be the smallest integer with this property; then

$$f^{(n+r-1)}(c(w)) = f^n(v) \neq f^{(n+r)}(w)$$

so that $w \in E_{(n+r)}(f)$. Thus the set of points which are not f-regular is exactly the set $\{ c^r(w) : r \geq 1 \text{ and } w \in E_m(f), \text{ for some } m \}$, which is clearly countable, at most, since each $E_m(f)$ is finite.

<u>Lemma 9.4</u>: Suppose $f_m \downarrow_m 0$ in $L^\#$; then, for each fixed $v_0 \in R^n$ (for some $n \geq 0$), $f_m^n(v_0) \downarrow_m 0$.

<u>Proof</u>: Otherwise we have some $\varepsilon > 0$ such that, for all m, $f_m^n(v_0) \geq \varepsilon$. Consider the uncountable set $A(v_0) = \{ v \in R^{(n+1)} ; \ c(v) = v_0 \}$. By Lemma 9.3 the set of points which are not f_m-regular for some m is at most countable; moreover $\bigcup_{m=1}^{\infty} E_{(n+1)}(f_m)$ is countable at most, since each $E_{(n+1)}(f_m)$ is finite. Hence there exists some $w_0 \in A(v_0)$ such that w_0 is f_m-regular for each m, and, for each m, $w_0 \notin E_{(n+1)}(f_m)$, so that

$$f_m^{(n+1)}(w_0) = f_m^n(c(w_0)) = f_m^n(v_0) \geq \varepsilon \quad .$$

Thus if we define an element f_0 in $L^\#$ by setting $f_0^k \equiv 0$ for $k \leq n$,

$f_0^{(n+1)} = \varepsilon . \chi_{\{v_0\}}$, and stipulating that every point of $R^{(n+1)}$ be f_0-regular, we have $0 \lneq f_0$, and $f_0 \leq f_m$, for all m. It follows that $f_m \not\downarrow_m 0$, a contradiction.

<u>Lemma 9.5</u>: Suppose $f_m \downarrow_m 0$ in $L^{\#}$; then, for each $n \geq 0$, $f_m^n \downarrow_m 0$ <u>uniformly over R^n</u>.

<u>Proof</u>: Since R^0 contains just the single point ϕ, we have the result for $n = 0$ immediately by Lemma 9.4. We now proceed by induction on n. Assuming we have the result for n, and given $\varepsilon > 0$, let us find m such that $f_m^{(n+1)} \leq \varepsilon$ (throughout $R^{(n+1)}$). We do have p such that $f_p^n \leq \varepsilon$ so that $f_p^{(n+1)} \leq \varepsilon$ except on the set $E_{(n+1)}(f_p)$. By Lemma 9.4, since $E_{(n+1)}(f_p)$ is finite, there exists $m \geq p$ such that, for $v \in E_{(n+1)}(f_p)$, we have $f_m^{(n+1)}(v) \leq \varepsilon$. But for $v \in (R^{(n+1)} - E_{(n+1)}(f_p))$ we have $f_m^{(n+1)}(v) \leq f_p^{(n+1)}(v) \leq \varepsilon$.

<u>Theorem 9.6</u>: ρ is an integral norm on $L^{\#}$, i.e.,

$$f_m \downarrow 0 \text{ in } L^{\#} \implies \rho(f_m) \downarrow 0 .$$

<u>Proof</u>: Suppose $\varepsilon > 0$. We shall find $\bar{m}$ such that $\rho(f_{\bar{m}}) < \varepsilon$. Let M be a bound for f_1, i.e., for each n, $M \geq f_1^n$ $(\geq f_m^n)$. Clearly, for some p,

$$n > p \implies (n+1)^{-1}(\sup\{|f_m^n(v)| : v \in R^n\}) \leq (n+1)^{-1}M < \varepsilon$$

(for all m). On the other hand, in view of Lemma 9.5, we have some $\bar{m}$ such that $f_{\bar{m}}^n < \varepsilon$ for all $n \leq p$. Clearly, then,

$$\rho(f_{\bar{m}}) = \sup_{n \geq 0} \{(n+1)^{-1}(\sup\{|f_{\bar{m}}^n(v)| : v \in R^n\})\} < \varepsilon .$$

<u>Theorem 9.7</u>: ρ is nowhere normal on $L^{\#}$, i.e., given $f \gneq 0$, there is a directed system f_τ such that $f \geqslant f_\tau \downarrow_\tau 0$, but $\rho(f_\tau) \not\downarrow_\tau 0$.

<u>Proof</u>: There exist some n, and v in R^n, such that $f^n(v) = \varepsilon > 0$. Now let $A(v) = \{w \in R^{(n+1)} : c(w) = v\}$. For each <u>finite</u> subset τ of $A(v)$, define $g_\tau \in L^{\#}$ by setting $g_\tau^k \equiv 0$, for $k < n$,

$$g_\tau^n = \varepsilon \cdot \chi_{\{v\}}, \qquad g_\tau^{(n+1)} = \varepsilon \cdot \chi_{(A(v) - \tau)} \quad .$$

and stipulating that every point of $R^{(n+1)}$ be g_τ-regular. Clearly $\{g_\tau\}$ is directed downwards, and, in fact,

$$g_\tau \wedge g_{\tau'} = g_{(\tau \cup \tau')} \quad .$$

Moreover $\inf_\tau g_\tau = 0$. To see this, suppose $0 \leq g \leq g_\tau$, for all τ, and $g \in L^{\#}$. For $w \in R^{(n+r+1)}$, consider $\tau = \{c^r(w)\} \cap A(v)$; $g_\tau^{(n+1)}(c^r(w)) = 0$ and every point of $R^{(n+1)}$ is g_τ-regular ; hence $g_\tau^{(n+r+1)}(w) = 0$ so that $g^{(n+r+1)}(w) = 0$. Thus $g^m \equiv 0$ for any $m > n$. Furthermore, it is clear that, for any h in $L^{\#}$, if $h^m \equiv \alpha$ where α is a constant, then $h^{(m-1)} \equiv \alpha$. Hence $g = 0$.

It follows that, if we set $f_\tau = f \wedge g_\tau$, we have $f \geqslant f_\tau \downarrow_\tau 0$. However, for any w in $A(v) - (\tau \cup E_{(n+1)}(f))$ (which is certainly not empty, since both τ and $E_{(n+1)}(f)$ are finite), we have $(f^{(n+1)} \wedge g_\tau)(w) = \varepsilon$, so that $\rho(f_\tau) \geqslant \varepsilon(n+2)^{-1} > 0$. It follows that $\rho(f_\tau) \not\downarrow_\tau 0$.

<u>Theorem 9.8</u>: $L^{\#}$ is Egoroff.

<u>Proof</u>: Since every element of $L^{\#}$ is bounded, $L^{\#}$ is the ideal generated

by the element e determined by $e^n \equiv 1$, for all n. Thus, in view of Theorem 1.1(a), we need only show that the element e has the Egoroff property. Suppose we have $f_{m,k}$ in $L^{\#}$ such that, for each m, $0 \leq f_{m,k} \uparrow_k e$. For each natural number p, we shall show that $k(1), k(2), \ldots$ can be chosen so that f_p, defined by

$$f_p^n = (1 - \tfrac{1}{p}) \wedge f^n_{1,k(1)} \wedge f^n_{2,k(2)} \wedge \cdots \qquad (n = 0, 1, 2, \ldots) ,$$

is actually an element of $L^{\#}$, and $f_p^n \equiv (1 - \frac{1}{p})$ for $n \leq p$. To see this observe that for each m $f_{m,k} \uparrow_k e$ so that, in view of Lemma 9.5, we can find k(m) such that $f^n_{m,k(m)} \geq (1 - \frac{1}{p})$ for all $n \leq p + m$. That $f_p^n \equiv (1 - \frac{1}{p})$ for $n \leq p$ is now clear. To show that $f_p \in L^{\#}$ we must verify that, for each $n \geq 1$, $E_n(f_p)$ is finite. However, since for $m \geq n - p$ we have $f^n_{m,k(m)} \geq (1 - \frac{1}{p})$,

$$f_p^n = (1 - \tfrac{1}{p}) \wedge f^n_{1,k(1)} \wedge \cdots \wedge f^n_{(n-p),k(n-p)} \quad , \text{and}$$

$$f_p^{(n-1)} = (1 - \tfrac{1}{p}) \wedge f^{(n-1)}_{1,k(1)} \wedge \cdots \wedge f^{(n-1)}_{(n-p),k(n-p)} \quad .$$

It follows that

$$E_n(f_p) = E_n(\ (1 - \tfrac{1}{p})e \wedge f_{1,k(1)} \wedge \cdots \wedge f_{(n-p),k(n-p)}\ )$$
$= E_n(g)$, where $g \in L^{\#}$; hence $E_n(f_p)$ is finite.

Now it is clear that $\sup f_p = e$, since $f_p^n \equiv (1 - \frac{1}{p})$ for $n \leq p$. Moreover we can always choose the k(m) corresponding to $(p + 1)$ so that it is larger than that corresponding to p. In this case f_p is an increasing sequence, $0 \leq f_p \uparrow_p e$, and clearly, by construction, each $f_p \ll \{f_{m,k}\}$. We have proved, then, that e has the Egoroff property.

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