I. THE ROTATION OF A GRAVITATING SPHERE IN A MONATOMIC GAS

II. THE DRAG OF A BODY MOVING TRANSVERSELY IN A CONFINED STRATIFIED FLUID

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Michael Ralph Foster

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ABSTRACT

I. The Rotation of a Gravitating Sphere in a Monatomic Gas

The flow resulting from the steady rotation of a gravitating sphere in a monatomic gas at rest is studied in a variety of special cases. The low speed rotation problem involves the solution of non-uniform Stokes equations and exhibits the interesting property that, if the field is large enough to make a "scale height" very small compared to the sphere radius, the motion is very weak and occurs primarily in a thin boundary layer on the sphere. The asymptotic theory for the gravitational field strength very large with arbitrary rotation speed shows essentially the same boundary layer, regardless of Reynolds number; the perturbation theory presents some interesting mathematical problems as well. The high speed rotation case is finally considered, and solutions have been obtained only for a gas with small Prandtl number. Even then, the flow structure is very complex. Depending on the relative sizes of the Prandtl number and inverse Reynolds number, there are six possibilities. In every case, there is a thin Prandtl boundary layer on the surface of the sphere and an essentially incompressible jet in the equatorial plane. In some cases, a thermal layer outside the Prandtl boundary layer is required to adjust the temperature, and in every case but one, it is necessary to infer the existence of still another layer, which is inviscid but rotational, that adjusts the uniform flow into the layer required by the strong hydrostatic constraints on the outer flow to that necessary for Prandtl boundary layer entrainment. In some cases these layers are unstable to small disturbances if the

temperature on the sphere is sufficiently large.

II. The Drag of a Body Moving Transversely in a Confined Stratified Fluid

The motion of a body through a stratified fluid bounded by vertical plates is studied in the case when the motion of the body is sufficiently slow to make the inertia of the fluid negligible. The case studied is for a very small coefficient of diffusion (for salt in water, for example). The density changes are quite large, and the drag is quite easily computed without appeal to the structure of any boundary layers or shear layers, depending only on changes of potential energy of the fluid. The solution exhibits regions where the fluid is unstably stratified, and hence mixes. Depending upon how complete the mixing process is, the body might experience a thrust!

The equations for boundary layers are given, but details of their solution are not dealt with here, because of their quasi-linear nature. The horizontal shear layers consist of a simple density adjustment layer surrounded by a thicker and quite complicated non-linear dynamical layer. The more conventional Stewartson layers do not appear here; these layers, because of the non-linearities, are quite complex, and details of their structure have not yet been fully worked out.

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LIST OF SYMBOLS

A = part of pressure field in Section 2

a = sphere radius; inner expansion of pressure in 2

B = $\Omega a/\sqrt{RT_o}$

GM = gravitational field strength per unit mass per unit area

R = gas constant; $\tilde{h} - T_1/1-T_1$ in Section 4

Re = Reynolds number

t = time

u, v, w = velocities

$\underline{\underline{e}}$ = deformation tensor

$\underline{r}$ = radius vector, $r = |r|$

T = temperature

p = pressure, Laplace variable

$\underline{k}$ = unit vector in the direction of $\underline{\Omega}$

$\underline{u}$ = velocity vector

U, V = r dependent functions in Section 2

Q = part of temperature field in Section 2

$\overline{U}$ = U/δ

F = outer expansion of $\overline{U}$ in Section 2

G = outer expansion of V in Section 2

f = inner expansion of U_1 in Section 2; temperature gradient

g = inner expansion of V in Section 2

q = inner expansion of Q in Section 2

LIST OF SYMBOLS (Cont'd)

y	=	boundary layer coordinate
P	=	pressure
z	=	boundary layer variable
E	=	boundary layer entrainment
h	=	static enthalpy
F	=	related to velocity field in Section 4
T_1	=	$T^{(o)}(a)/T_o$
W	=	angular momentum/unit mass
α	=	GM/aRT_o, also $\partial T/\partial y$ (a)
β	=	$\frac{\partial W}{\partial y}(a)\frac{1}{\sin\theta}$
γ	=	a function of θ in the pressure, Section 3
ν	=	kinematic viscosity
κ	=	thermometric conductivity
Ω	=	angular velocity
σ	=	Prandtl number
ρ	=	density
$\underline{\underline{\tau}}$	=	shear stress tensor
Φ_v	=	viscous dissipation
θ	=	polar angle
μ	=	viscosity
Γ	=	$\exp[-\frac{\alpha(r-1)}{r}]$
λ	=	$(B/Re)^2$
δ	=	α^{-1}
ψ	=	stream function

LIST OF SYMBOLS (Cont'd)

ω = temp. dependence of transport coeff.

χ = $RT_o \delta^3/\Omega_o$

Σ = div $\underline{\underline{\tau}}$

ξ = boundary layer variable; dummy variable

η = boundary layer cond.

ε = thickness of inviscid rotational layer

ϕ

$(\tilde{\ })$ = boundary layer variable in Section 3; thermal layer variable in Section 4

$(\ )^*$ = boundary layer variables in Section 4

$[x]_c$ = denotes a jump in x across a line c

$(\bar{\ })$ = denotes a boundary layer variable in Section 4

$(\underline{\ })$ = denotes a vector

$(\underline{\underline{\ }})$ = denotes a tensor of second rank

1. Introduction

The rotation of a massive, and hence gravitating sphere in a compressible monatomic gas presents a very interesting and perhaps geophysically relevant problem to attack. As will be pointed out in detail later, the solution to the incompressible version of the problem, where the field can be absorbed into the pressure gradient, is now well understood in both small and large Reynolds number regimes. Jeffrey,[1] Bickley,[2] and Collins[3] have quite completely accounted for the small Reynolds number flow; Howarth[4] and Squire[5] have together correctly accounted for the large Reynolds number solution, with Stewartson[6] resolving a difficulty due to Nigam.[7]

The compressible problem without gravity is clearly no difficult matter, and the solutions do not change substantially from their incompressible counterparts. The addition of gravity as well as compressibility, however, does bring a host of complications. The gravitational field introduces a new length into the problem, physically corresponding to an atmospheric scale height, so a new dimensionless number may be formed, $\alpha \equiv GM/aRT_o = \frac{\text{sphere radius}}{\text{scale height}}$ where GM/a^2 is the field strength per unit mass on the sphere of radius a, R is the particular gas constant, and T_o is the temperature on the surface of the sphere. In addition, there are the more conventional dimensionless combinations

$$Re = \frac{\Omega a^2}{\nu_o}, \quad B^2 = \frac{(\Omega a)^2}{RT_o}, \quad \frac{\nu_o}{\kappa_o} = \sigma$$

where $(\ )_o$ will always denote quantities evaluated on the sphere. Compressibility also means that T_∞/T_o will be a parameter of importance, though it will be O(1) throughout.

Considering that the solution in general could be represented as a point in an $(\alpha, \mathrm{Re}, B, \sigma)$ space, it is clear that the lines of approach are endless. The techniques of singular perturbation theory are very useful in understanding certain restricted regions of this space.

Section 2 deals with the simplest of the asymptotic problems, viz., the structure of the solutions near the Re = B = 0 plane. This is essentially a Stokes problem which degenerates to Bickley's solution as $\alpha \downarrow 0$. Because of the algebraic complexities of the general solution, the $\alpha \to \infty$ special cases are considered. The $\alpha \to \infty$ limit is singular, being characterized by the collapse of most of the fluid into a thin layer of width a/α onto the spherical surface. Analysis of the unsteady development of the primary flow, as well as a quite general solution for weak meridional temperature variations on the sphere in the $\alpha \to \infty$ limit complete this part of the work.

Because of the interesting nature of the $\alpha \to \infty$ Stokes solution, Section 3 contains the general $\alpha \to \infty$ problem with the pressure on the surface held fixed. Some difficulties encountered in the perturbation procedure indicate why, if $\underline{u}_p(\underline{r};\mathrm{Re}, B, \alpha, \sigma)$ denotes the solution, that

$$\lim_{\alpha \to \infty} \lim_{\substack{\mathrm{Re} \to 0 \\ B \to 0}} \underline{u}_p \neq \lim_{\substack{\mathrm{Re} \to 0 \\ B \to 0}} \lim_{\alpha \to \infty} \underline{u}_p, \quad r \text{ fixed,}$$

except in the boundary layer. Holding Re fixed as $\alpha \to \infty$ with p_o fixed essentially means that $p_\infty \to 0$ as $\alpha \to \infty$, so the Reynolds number based on ρ_∞ goes to zero with $1/\alpha$. Viewed in this light, the Stokes-like nature of the solution is no surprise.

Finally, Section 4 deals with the solutions for $Re \to \infty$. This is the most difficult of the asymptotic problems dealt with here; it has been possible to make progress only for $\sigma \ll 1$, for a steady solution. In some T_∞/T_o ranges, for certain values of σ, the solution exhibits a certain non-uniqueness which is believed to be associated with the fact that, for those same values of the parameters, Rayleigh-type instability becomes possible in the fluid. A variety of boundary layers occur as well as an equatorial-plane jet.

2. Asymptotic Solutions for $Re \to 0$, $B \to 0$

The equations of motion for a viscous, heat-conducting monatomic gas in a central gravitational field are just

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \underline{u} = 0 \tag{1}$$

$$B^2 \rho \frac{\partial \underline{u}}{\partial t} + B^2 \rho (\underline{u} \cdot \nabla) \underline{u} + \nabla p = \frac{B^2}{Re} \nabla \cdot \underline{\underline{\tau}} - \rho \alpha \underline{r}/r^3 \tag{2}$$

$$\frac{\nabla^2 T^{\omega+1}}{\omega+1} + \frac{2}{5} B^2 \Phi_v = \rho Re \frac{\partial T}{\partial t} + Re\, \rho\, \underline{u} \cdot \nabla T - \frac{2}{5} Re\, \underline{u} \cdot \nabla p \tag{3}$$

$$p = \rho T \tag{4}$$

$$\underline{\underline{\tau}} = T^{\omega}[\underline{\underline{e}} - \frac{2}{3} \underline{\underline{I}} \nabla \cdot \underline{u}] \tag{5}$$

where the variables are similar to the usual Stokes variables[8], with ρ_o, Ωa, T_o, $\rho_o R T_o$, $\Omega \mu_o$, Ω^{-1}, the dimensional density, velocity, temperature, pressure, stress, and time respectively. Here $\underline{\underline{e}}$ is the usual deformation tensor, $e_{ij} = u_{i,j} + u_{j,i}$, and Φ_v is the viscous dissipation. The only variance from the usual Stokes scaling is the pressure, which scaling comes from the static atmosphere. We now seek asymptotic solutions of (1) - (5) as Re, B $\to$ 0 subject to

$$\left.\begin{aligned} \underline{u} &= \underline{r} \times \underline{k} \\ \rho &= 1 \\ T &= T(\theta) \end{aligned}\right\} \quad \text{On } r = 1 \tag{6}$$

where θ is the polar angle, and $|\underline{u}| \to 0$, $T = T_\infty$ as $r \to \infty$.

A. Steady Solution for $T(1) = T(\infty) = 1$

In this section, the solution is given for a steady state solution of (1) - (5) with $f(\theta) = 1$ in boundary condition (6) and $T_\infty = 1$ in (7). For convenience, call $B^2/Re^2 = \lambda$, a parameter proportional to the Knudsen number, and hence ord $\lambda \leqslant 1$ in order to assure the validity of a continuum formulation. Also, because of its frequent occurrence, let $\Gamma \equiv \exp - \alpha(\frac{r-1}{r})$.

The perturbation scheme proceeds as

$$\begin{aligned} p &= \Gamma[1 + \frac{B^2}{Re} p_1 + B^2 p_2 + \ldots] \\ \rho &= \Gamma[1 + \frac{B^2}{Re} \rho_1 + B^2 \rho_2 + \ldots] \\ T &= 1 + \frac{B^2}{Re} T_1 + B^2 T_2 + \ldots \\ \underline{u} &= \underline{u}_1 + Re\, \underline{u}_2 + \ldots \quad \text{et al.} \end{aligned} \tag{8}$$

If λ, σ, and α are all order one, the insertion of (8) into (1.7) gives, to leading order, the Stokes equations

$$\nabla \cdot \underline{u}_1 = \alpha \frac{\underline{r} \cdot \underline{u}_1}{r^3}$$

$$\Gamma \nabla p_1 - \Gamma T_1 \frac{\alpha \underline{r}}{r^3} = \nabla \cdot \underline{\underline{\tau}}_1 \qquad (9)$$

$$\frac{1}{\sigma} \nabla^2 T_1 = \frac{2}{5\lambda} \alpha \frac{\underline{r} \cdot \underline{u}_1}{r^3} \Gamma$$

$$p_1 = \rho_1 + T_1$$

subject to $T_1(1) = \rho_1(1, 0) = 0$, $\underline{u}_1(1) = \underline{r} \times \underline{k}$; $T_o = \underline{u}_o = 0$ at ∞. G. B. Jeffrey, in 1915,[1] first carried out the solution to (9) with $\alpha \equiv 0$. Because all of the thermodynamic boundary conditions are homogeneous, Jeffrey's solution is correct for any α, viz.

$$\left. \begin{aligned} \underline{u}_1 &= \frac{\underline{r} \times \underline{k}}{r^3} \\ p_1 &= \rho_1 = T_1 = 0 \end{aligned} \right\} \qquad (10)$$

which can be verified by direct substitution. The next step is to calculate the secondary flow by substitution of (8) into (1)-(7) again and, using (10), performing the same limit to give just

$$\nabla \cdot \underline{u}_2 = \frac{\alpha \underline{r} \cdot \underline{u}_2}{r^3} \qquad (11)$$

$$\Gamma (u_1 \cdot \nabla) \underline{u}_1 + \Gamma \nabla p_2 - \Gamma T_2 \frac{\alpha \underline{r}}{r^3} = \nabla^2 \underline{u}_2 + \frac{1}{3} \nabla (\frac{\alpha \underline{r}}{r^3} \cdot \underline{u}_2) \qquad (12)$$

$$\frac{1}{\sigma} \nabla^2 T_2 + \frac{18}{5} \frac{\sin^2\theta}{r^6} = \frac{2}{5\lambda} \frac{\alpha(\underline{r} \cdot \underline{u}_2)}{r^3} \Gamma \qquad (13)$$

$$p_2 = \rho_2 + T_2 \qquad (14)$$

$$\left. \begin{aligned} &\text{subject to } \rho_2(0) = T_2 = 0, \ \underline{u}_2 = 0 \text{ on } r = 1 \\ &\qquad |\underline{u}| \to 0, \ \rho_2, \ T_2 \to 0 \text{ as } r \to \infty \end{aligned} \right\} \qquad (15)$$

The solutions to (11)-(15) were found (actually the incompressible equivalents) by Bickley,[2] in 1938 for $\alpha \equiv 0$, and the entire perturbation procedure was investigated (for $\alpha \equiv 0$) quite completely by Collins.[3] It is Bickley's solution that, surprisingly perhaps, lends itself to suitable generalization in this context. To proceed, one notes that, if (u_2, v_2, w_2) are the velocity components in the radial, polar, and azimuthal directions,

$$u_2 = U(r)\,(3\cos^2\theta - 1)$$
$$v_2 = V(r)\sin\theta\cos\theta \qquad (16)$$
$$w_2 = 0$$

One can split the solution to (13) into two particular integrals and a homogeneous solution. The result of that calculation is

$$T_2 = \frac{2}{5}\sigma(3\cos^2\theta - 1)\left[Q(r) - \frac{Q(1)}{r^3} + \frac{1}{2}\left(\frac{1}{r^4} - \frac{1}{r^3}\right)\right] + \frac{1}{5}\sigma\left(\frac{1}{r} - \frac{1}{r^4}\right) \qquad (17)$$

where Q is a particular integral of

$$r^2\frac{d^2Q}{dr^2} + 2r\frac{dQ}{dr} - 6Q = \frac{\alpha}{\lambda}\Gamma U(r) \qquad (18a)$$

i. e., $$Q = r^2\int^r \frac{1}{\eta^6}\int^\eta \frac{\alpha}{\lambda}\xi^2\Gamma(\xi)U(\xi)\,d\xi\,d\eta \qquad (18b)$$

Substitution of (16) into (11) gives

$$V + r\frac{dU}{dr} + 2U = \frac{\alpha}{r}U \qquad (19)$$

Then, substitution of (16) and (17) into (12) yields two further equations; cross-differentiation to eliminate the pressure gives

$$L_1(U) = 6\Gamma\left(\frac{2}{5}\alpha\sigma\left(Q(r) - \frac{Q(1)}{r^3}\right) - \frac{1}{r^3}\right) \qquad (20a)$$

where

$$L_1 \equiv r^4 \frac{d^4}{dr^4} + 8r^3 \frac{d^3}{dr^3} - \left(\frac{\alpha^2}{r^2} - \frac{6\alpha}{r}\right) r^2 \frac{d^2}{dr^2} - 6\left(4 - \frac{\alpha}{r}\right) r \frac{d}{dr} + 8\left(3 + \frac{\alpha^2}{r^2}\right) \qquad (20b)$$

In addition,

$$p_2 = A(r)(3\cos^2 - 1) - \frac{1}{6}\left(\frac{1}{r^4} - 1\right) + \frac{\alpha\sigma}{25}\left(\frac{3}{2} + \frac{1}{r^5} - \frac{5}{2r^2}\right) + \frac{1}{3}\left(1 + (\alpha - 6)U_{rr}(1) - U_{rrr}(1)\right) \qquad (21a)$$

where

$$A = -\frac{1}{6r^4} + \frac{1}{6\Gamma}\left[r^2 \frac{d^3U}{dr^3} + (6r - \alpha)\frac{d^2U}{dr^2} + \frac{8\alpha}{r^2} U\right] \qquad (21b)$$

The simultaneous solution of (18) and (20) gives the complete solution, using the boundary conditions

$$U(1) = \frac{dU}{dr}(1) = 0, \quad U, \frac{dU}{dr} \to 0, \quad r \to \infty \qquad (22)$$

Then, p_2 and T_2 are completely determined by (17) and (21). The exact solution for any α can in principle be carried out by a Frobenius method, but the result would be unwieldy and of little use for understanding the nature of the solution. Hence, what follows is an examination of the solution to (17)-(22) for two extreme values of α.

(i) Weak Field Solution; $\alpha \ll 1$

If $\alpha \equiv 0$ in the equations (17)-(22), then (20) is especially simple, being a 4th order equation of equidimensional type, having homogeneous solutions $1/r^2$ and $1/r^4$. Together with a particular solution, these solutions may be used with (22) to give Bickley's 1938[2] solution,

$$
\begin{aligned}
U_o &= -\frac{1}{8}\left(\frac{r-1}{r^2}\right)^2, \quad V_o = \frac{1}{4}\,\frac{r-1}{r^4} \\
Q_o &= 0 \\
A_o &= \frac{1}{3r^4} - \frac{1}{4r^3} \\
p_{20} &= A_o(3\cos^2 \ -1) - \frac{1}{6r^4}
\end{aligned}
\tag{23}
$$

where the subscript denotes the fact that this is the $\alpha \equiv 0$ solution. A procedure like

$$
\begin{aligned}
U &= U_o + \alpha U_1 + \ldots \\
V &= V_o + \alpha V_1 + \ldots
\end{aligned}
$$

et al.

will yield the next order solution; the operators are still equidimensional and so the solution is straightforward though somewhat involved. Without giving the details here, the result is simply

$$
\begin{aligned}
U_1 &= \frac{5r+2}{6r} U_o + \frac{3}{140}\left(\frac{13}{2} - \frac{5}{2}\sigma\right)\left(\frac{2}{r^4}\log r + \frac{1}{r^4} - \frac{1}{r^2}\right) \\
V_1 &= -\frac{1}{2}\left(\frac{2r-1}{r}\right)V_o + \frac{3}{35}\left(\frac{13}{2} - \frac{5}{2}\sigma\right)\frac{1}{r^4}\log r \\
Q_1 &= -\frac{1}{8\lambda}\left[\frac{2}{5}\,\frac{1}{r^3}\log r + \frac{1}{6r^4} - \frac{1}{4r^2}\right]
\end{aligned}
\tag{24}
$$

and similarly for A_1. It is clear that this is a regular perturbation expansion which might proceed indefinitely. One significant thing about this result is that (18) and (20) were not coupled, whereas for general α, (20) cannot be solved first as was done here. The reason for the decoupling is that the order one pressure field is only slowly-varying, and hence the work it does on the moving fluid is quite small compared with the rate of energy dissipation of the primary flow. The streamlines for $\alpha \equiv 0$ are shown in Figure 1.

(ii) Strong Field Solution; $\alpha >> 1$

The asymptotic theory for $\alpha \to \infty$ is considerably more complex than the theory for $\alpha \to 0$. It is immediately clear from (20) that $\alpha \to \infty$ presents a singular perturbation problem. It is convenient in this context to let $\delta \equiv \alpha^{-1}$, and to put $U = \delta\overline{U}$. Rewriting (18), (19), and (20) in this notation gives

$$r^2 \frac{d^2Q}{dr^2} + 2r \frac{dQ}{dr} - 6Q = \frac{\Gamma}{\lambda} \overline{U} \tag{25}$$

$$V + r\delta \frac{d\overline{U}}{dr} + 2\delta\overline{U} = \frac{1}{r} \overline{U} \tag{26}$$

$$\overline{L}(\overline{U}) = -\frac{6\Gamma\delta}{r^3} + \frac{12}{5}\sigma\Gamma\left[Q(r) - \frac{Q(1)}{r^3} + \frac{1}{2}(r^{-4} - r^{-3})\right] \tag{27}$$

where

$$\overline{L} \equiv \delta^2 r^4 \frac{d^4}{dr^4} + 8\delta^2 r^3 \frac{d^3}{dr^3} - \left(\frac{1}{r^2} - \frac{6\delta}{r}\right) r \frac{d^2}{dr^2} - 6\left(4\delta^2 - \frac{\delta}{r}\right) r \frac{d}{dr} + 8\left(3\delta^2 + \frac{1}{r^2}\right) \tag{28}$$

subject to

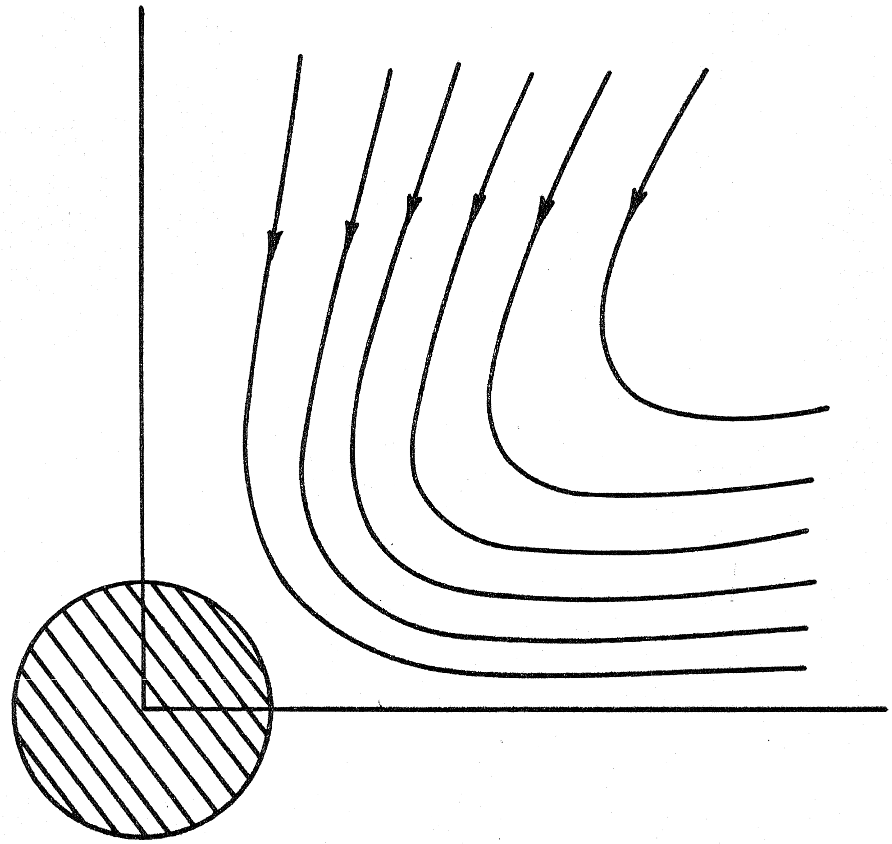


FIG. I STRUCTURE OF THE Re → O SOLUTION FOR $\alpha \equiv 0$ [from Bickley, [2]]

$$\left.\begin{aligned} &\overline{U}(1) = \frac{dU}{dr}(1) = 0 \quad \text{on} \quad r = 1 \\ &\overline{U},\ \overline{U}_r \to 0,\ r \to \infty \end{aligned}\right\} \tag{29}$$

Outer Expansion

We proceed in the conventional way by hypothesizing an outer expansion

$$\overline{U} = F_0 + \delta F_1 + \delta^2 F_2 + \ldots \tag{30}$$

$$V = G_o + \delta G_1 + \ldots$$

$$Q = Q_o + \delta Q_1 + \ldots \qquad \text{et al.,}$$

and inserting it into (25)-(28). Doing the limit process $\delta \to 0$, r fixed, it is important to notice that now $\Gamma = \exp(-\frac{r-1}{\delta r})$ and hence, for r fixed,

$$\lim_{\substack{\delta \to 0 \\ r \text{ fixed}}} \delta^k \Gamma(r, \delta) = 0 \quad \text{for any } k$$

This means that, in particular, $Q_n \equiv 0$, $n = 0, 1, 2, \ldots$, and also that (27) is a homogeneous equation for $\overline{U}$. Suppose F_k is the first non-zero term in (30); then,

$$\left.\begin{aligned} &r^2 \frac{d^2 F_k}{dr^2} - 8F_k = 0 \\ &F_k \to 0,\ r \to \infty \end{aligned}\right. \tag{31}$$

where the condition given on the solution is the only one we can as yet pose. Clearly,

$$F_k = C\, r^{-m},\quad m = \tfrac{1}{2}(\sqrt{33} - 1) \approx 2.4 \tag{32}$$

is the required solution of (31), but C is not known, since one cannot satisfy (29) with any choice but the trivial one. The general form of the flow is shown in Figure 2.

Inner Expansion

Examination of (25)-(28) indicates that there is a singular layer of size δ on the spherical surface. Let

$$\begin{aligned} \overline{U} &= f_o(y) + \delta f_1(y) + \dots \\ \overline{V} &= g_o(y) + \delta g_1(y) + \dots \\ Q &= q_o(y) + \delta q_1(y) + \dots \end{aligned} \tag{33}$$

where $y = (r-1)/\delta$. Then, $\Gamma = e^{-y}$, and insertion of these expansions into (27) and (28) gives, on letting $\delta \to 0$, y fixed,

$$\frac{d^4 f_\ell}{dy^4} - \frac{d^2 f_\ell}{dy^2} = 0 \tag{34}$$

if f_ℓ is the first non-zero f_k and $\ell < 3$. It can be easily verified that the solution to (34) is not sufficiently general to match with (32) and satisfy (29). Hence, the conclusion is $\ell \equiv 3$ in which case (34) is modified to become

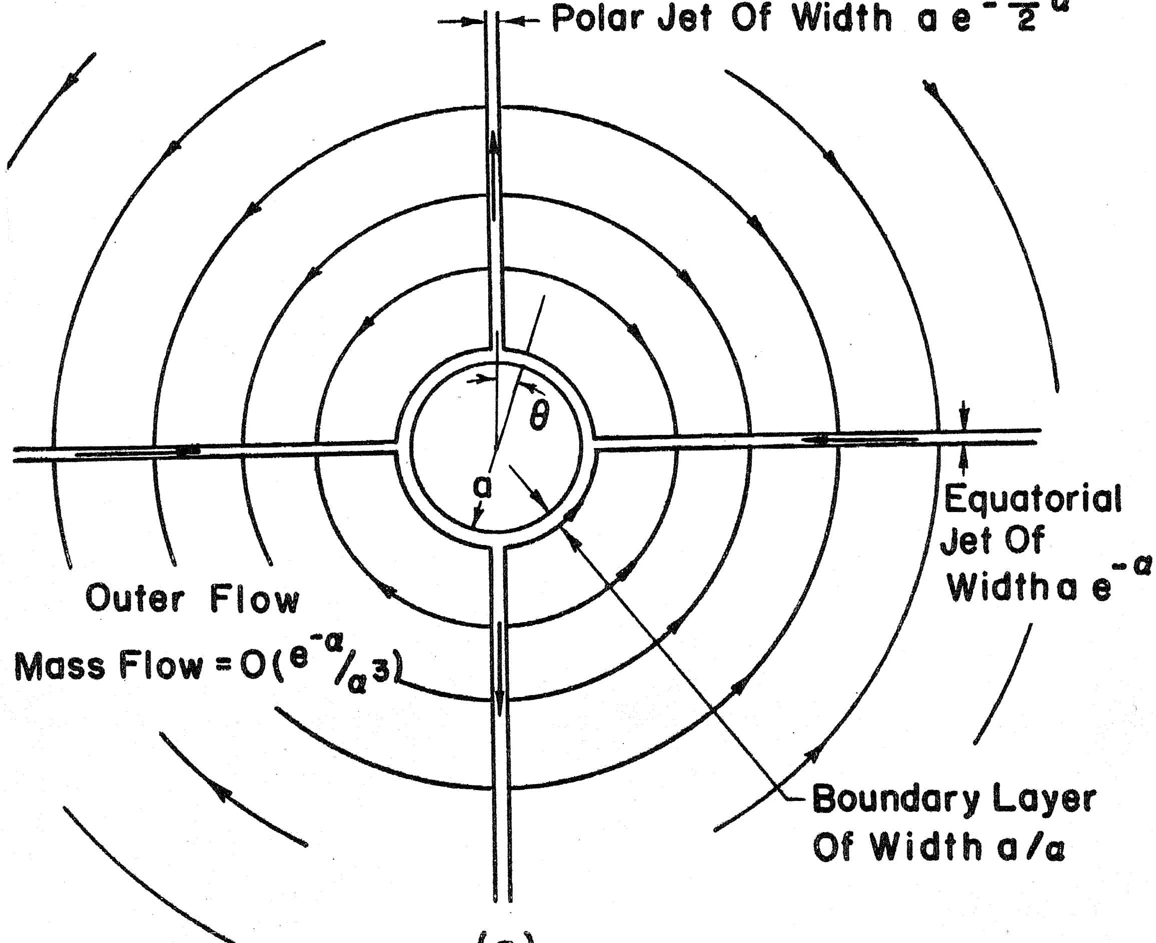
$$\frac{d^4 f_3}{dy^4} - \frac{d^2 f_3}{dy^2} = -6e^{-y} - \frac{6\sigma}{5} y\, e^{-y} \tag{35}$$

which is to be solved together with

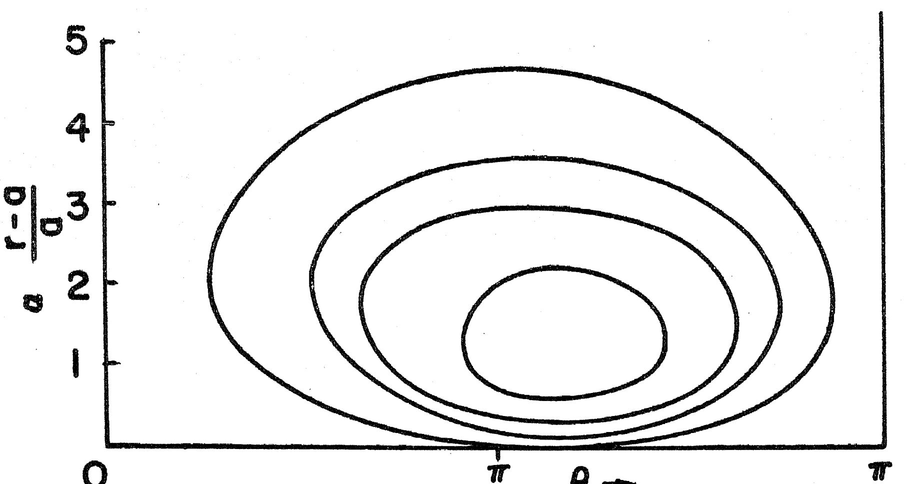
$$\frac{d^2 q_5}{dy^2} = \frac{e^{-y}}{\lambda} f_3 \tag{36}$$

$$g_3 = f_3 - \frac{df_3}{dy} \tag{37}$$

with $f_o = f_1 = f_2 = g_o = g_1 = g_2 = q_o = q_1 = \dots q_4 = 0$



(a)



(b) Boundary Layer Streamlines ($\sigma = 1$)

Mass Flow = $O(\alpha^{-4})$

FIG. 2 THE STRUCTURE OF THE $Re \to 0, \alpha \to \infty$ SOLUTION IN THE MERIDIONAL PLANE subject to the boundary conditions

$$f_3(0) = \frac{df_3}{dy}(0) = 0 \tag{38}$$

The solution to (35)-(38) is

$$f_3 = 3((y+1)e^{-y}-1) + \frac{3\sigma}{10}[(y^2+5y+5)e^{-y}-5] \tag{39}$$

$$g_3 = 3((2y+1)e^{-y}-1) + \frac{3\sigma}{10}[(2y^2+8y+5)e^{-y}-5] \tag{40}$$

$$q_5 - q_5(1) = \frac{3}{8\lambda}\left[\frac{1}{5}\sigma y^2 + \left(2 + \frac{6\sigma}{5}\right)y\right] e^{-2y}$$

$$+ \frac{3}{\lambda}\left(1 + \frac{1}{2}\sigma\right)(1-e^{-y}) - \frac{3}{8\lambda}\left(1 + \frac{7\sigma}{10}\right)(1-e^{-2y}) \tag{41}$$

Matching

Put $r-1 = \eta\, y_\eta$ where $\text{ord}\,\delta < \text{ord}\,\eta < 1$ into (39) and into (32) and let $\eta \to 0$, holding y_η fixed. Then, $\delta^k F_k(\eta, y_\eta) - \delta^3 f_3(\eta, y_\eta) \to \delta^k C + (3 + \frac{3}{2}\sigma)\delta^3$ so, to make this quantity zero, take $C = -3 - \frac{3}{2}\sigma$, and $k = 3$. Notice that putting (30) into (26) gives

$$G_o = G_1 = G_2 = 0, \quad G_3 = \frac{1}{r} F_3,$$

and since $g_3 \to -3 - \frac{3}{2}\sigma$ as $\eta \to 0$, y_η fixed,

$$\lim_{\substack{\eta \to 0 \\ y_\eta \text{ fixed}}} (\delta^3 G_3 - \delta^3 g_3) = 0$$

also. Hence, the velocity field is now uniformly valid to $O(\delta^3)$. The pressure is given in (21) and insertion of these expansions into that expression gives

$$A = -\frac{1}{6} + O(\delta)$$

$$p_2 = \frac{1}{2}\sin^2\theta + O(\delta) \tag{42}$$

in the boundary layer and the outer flow pressure vanishing at infinity is

$$p_2 = \frac{1}{2}(1 + \frac{1}{2}\sigma)m(m-1)(m+4)\frac{\delta^4}{\Gamma}(3\cos^2\theta - 1)r^{-m-1} + O(\delta^5\Gamma^{-1})$$

$$+ \frac{\sigma}{25\delta}[\frac{1}{r^5} - \frac{5}{2r^2}] + O(1) \tag{43}$$

Asymptotically, as $\delta \to 0$, it is clear that (43) gives

$$\Gamma p_2 = \frac{1}{2}(1 + \frac{1}{2}\sigma)m(m-1)(m+4)\frac{\delta^4(3\cos^2\theta - 1)}{r^{m+1}} + \ldots$$

and insertion into (8) gives the outer expansion in δ,

$$p = \Gamma + \frac{1}{2}(1 + \frac{1}{2}\sigma)m(m-1)(m+4)\delta^4 B^2 (\frac{3\cos^2\theta - 1}{r^{m+1}}) \tag{44}$$

Regarded as an asymptotic expansion in the sequence $1, B^2/Re, Re^2, ReB^2, \ldots$, (44) is a Poincaré expansion.[9] Further, use of (42) allows the formation of the composite expansion

$$p \sim e^{-\frac{r-1}{\delta r}} + B^2\left[\frac{1}{2}e^{-y}\sin^2\theta + \frac{1}{2}\delta^4(1 + \frac{1}{2}\sigma)m(m-1)(m+4)\frac{3\cos^2\theta - 1}{r^{m+1}}\right] \tag{45}$$

Again, this is a Poincaré expansion as $B \to 0$, δ small and fixed. However (43) represents a generalized asymptotic expansion[10] for p_2 only if the second term is uniformly smaller than the first, i.e.,

$$\delta^5 \gg e^{-1/\delta}\sigma \tag{46a}$$

which is certainly true as $\delta \rightarrow 0$, but from a computational point of view places a serious restriction on δ. Though (45) is acceptable under $B \rightarrow 0$, $\delta \rightarrow 0$ in that order, practically, it means more than $B \ll \delta$. Rather, from (44), one needs

$$B^2 \ll e^{-1/\delta}/\delta^4 \tag{46b}$$

Numerically, (46) represents a very serious restriction on B (or Re) and mathematically it means that one is confined to the region of B-δ space where

$$\sigma\delta^{-5}e^{-1/\delta} = O(1) \qquad B = 0(e^{-1/2\delta}\delta^{-2}) \tag{47}$$

It seems clear, in fact, that the first of these restrictions arises because (8) is too restrictive and that in fact p is the sum of two Poincare expansions in B and δ combined, viz.,

$$p = \Gamma[1 + \frac{B^2}{Re} p_1 + B^2 p_2 + \dots]$$
$$+ \delta^4 B^2 \left\{[p_1^* + \delta p_2^* + \delta^2 p_3^* + \dots] + O(Re)\right\}$$

So, though it seems at first strange, replace (45) by

$$p \sim e^{-\frac{r-1}{\delta r}}\left[1 + \frac{B^2\sigma}{25\delta}(\frac{1}{r^5} - \frac{5}{2r^2})\right]$$
$$+B^2\left[\frac{1}{2}e^{-y}\sin^2\theta + \frac{1}{2}\delta^4(1+\frac{1}{2}\sigma)m(m-1)(m+4)(\frac{3\cos^2\theta - 1}{r^{m+1}})\right] \tag{48}$$

Now, (48) is the final composite expansion for the pressure, and apparently requires, to be valid, a somewhat weaker condition,

$$B^2 \ll \underset{\delta}{MIN}(\delta/\sigma,\ \delta^{-4}e^{-1/\delta}) \tag{49}$$

or mathematically, $B = O(\delta^{1/2})$. In summary, the composite expansions for the velocity field are

$$V \sim -3(1+\frac{1}{2}\sigma)\frac{\delta^3}{r^{m+1}} + 3\delta^4(1+y)e^{-y} + \frac{3\sigma}{10}(2y^2+y+5)e^{-y} \tag{50a}$$

$$U \sim -3(1+\frac{1}{2}\sigma)\frac{\delta^4}{r^m} + 3\delta^4(1+y)e^{-y} + \frac{3\sigma}{10}(y^2+5y+5)e^{-y} \tag{50b}$$

and the temperature,

$$T_2 \sim \frac{1}{5}\sigma\left(\frac{1}{r}-\frac{1}{r^4}\right) + \frac{2}{5}\sigma(3\cos^2 -1)\left[\frac{1}{2}\left(\frac{1}{r^4}-\frac{1}{r^3}\right)+\delta^5(q_5(y)-q_5(0))\right] \tag{50c}$$

The streamlines in the outer flow are circular arcs since $V >> U$ everywhere except in a region of width $re^{-1/2\delta r}$ on the polar axis and $O(re^{-1/\delta r})$ in the equatorial plane where the streamlines turn inward. The boundary layer streamlines are sketched in Figure 2b; the streamfunction is

$$\psi(\theta, y) = f_3 e^{-y}\sin^2\theta\cos\theta$$

and the vorticity distribution is

$$\omega = -3\delta^2 \sin\theta\cos\theta\left[2+(1+\frac{2}{5}\sigma)y\right]e^{-y}$$

which has a minimum at $y = \left(\left(\frac{5}{\sigma}\right)^2 + \frac{15}{\sigma} + \frac{7}{2}\right)^{1/2} - \frac{5}{\sigma}$. One can show by integration that this flow carries no net mass; it is purely recirculation.

B. <u>Unsteady Development of the Primary Flow</u>

The azimuthal momentum equation, from (2), is

$$\rho\frac{\partial w}{\partial t} = \nabla^2 w - \frac{w}{r^2\sin^2\theta} \tag{51}$$

in the limit Re, B $\rightarrow$ 0. Further, in the same limit, (1) gives ρ = const so replace ρ in (51) by Γ. Then (51) is

$$\nabla^2 w - \frac{w}{r^2 \sin^2\theta} = e^{-\alpha(\frac{r-1}{r})} \frac{\partial w}{\partial t} \tag{52a}$$

which is to be solved for

$$\begin{aligned} w(1,\theta) &= H(t) \sin\theta \\ w \rightarrow 0, &\quad r \rightarrow \infty \end{aligned} \tag{52b}$$

where H(t) is the Heaviside stepfunction.

Put w = $\sin\theta$ s(r, t) and then let $S(p, r) \equiv \int_0^\infty e^{-pt} s(r,t)dt$.

The result is an ordinary differential equation

$$\begin{aligned} &\frac{d}{dr} r^2 \frac{dS}{dr} - (2 + r^2 p\Gamma) S = 0 \\ &S(1,p) = 1/p \\ &S \rightarrow 0, \quad r \rightarrow \infty \end{aligned} \tag{53}$$

which can be reduced to a form perhaps somewhat more useful by $S = r^{-1/2}\phi$ which gives

$$r^2 \frac{d^2\phi}{dr^2} + r \frac{d\phi}{dr} - \left[(\tfrac{3}{2})^2 + r^2 p\Gamma(r,\alpha)\right] \phi = 0 \tag{54}$$

The solution to this problem for $\alpha \equiv 0$ was originally given by Ghildyal in 1961.(11) If $\alpha \equiv 0$, the relevant solution to (54) is $K_{3/2}(p^{1/2}r)$. Denoting the solution with a superscript (o),

$$w^{(o)}(r,t) = \frac{1}{r^2} \left\{ (r-1) \left[1 - \mathrm{erf}(\frac{r-1}{\sqrt{2t}} + \sqrt{t})\right] e^{r+t-1} + 1 - \mathrm{erf}(\frac{r-1}{\sqrt{2t}}) \right\} \tag{55}$$

(i) Asymptotic Solution for Any α, $r \to \infty$

Clearly, if $r >> 1$, one can replace Γ by $e^{-\alpha}$. The (54) is again a Bessel equation and the inversion is the same as where $\alpha \equiv 0$. Hence, as $r \to \infty$

$$w(r,t) \sim \text{const. } w^{(o)}\left(re^{-\alpha/2}, t\right)$$

Hence, it would appear from this that the unsteadiness fills more and more of the fluid as $\alpha \to \infty$, provided the constant is O(1).

As $r \to 1$, $\Gamma \to 1$, so sufficiently close to the sphere, the solution is precisely that given by (55). In this context, that means $r - 1 << 1/\alpha$.

(ii) Solution as $\alpha \to 0$

It is clear that (54) is quite difficult to solve for general α. However, if $\alpha << 1$, then a regular perturbation expansion in α is quite straightforward. Let

$$w(r,t) = w^{(o)}(r,t) + \alpha w^{(1)}(r,t) + \ldots \tag{56}$$

Then, if $S = s^{(o)} + \alpha s^{(1)} + \ldots$, $s^{(1)} = e^{-\sqrt{p}(r-1)}\, g(r,p)$, then (53) gives

$$r^2 \frac{d^2g}{dr^2} + 2r\frac{dg}{dr}(1-r\sqrt{p}) - 2(1+r\sqrt{p})g$$
$$= -\left(1-\frac{1}{r}\right)\frac{\sqrt{p}\,r+1}{\sqrt{p}+1} \tag{57}$$

The general solution of (57) is quite involved. However, the transient character of the motion may be examined from (57) by letting p be large. In that case, the solution is easy, and

$$g = \frac{1}{2\sqrt{p}} \left(- \frac{1}{r} + 1 - \frac{\log r}{r}\right)$$

and therefore, for t sufficiently small,

$$w^{(1)} \sim \frac{1}{2\sqrt{\pi t}} \left(1 - \frac{1}{r} - \frac{\log r}{r}\right) e^{-\frac{(r-1)^2}{4t}} \tag{58}$$

In addition to this result, if $t \to \infty$, the solution to (57) is approximately $g \sim \frac{1}{2} p(1 - \frac{1}{r})$. Therefore, this is easily invertible,[12] the result being

$$w^{(1)} \sim \frac{(r-1)^2}{8tr} \left[\frac{(r-1)^2}{2t} - 3\right] e^{-\frac{(r-1)^2}{4t}} \tag{59}$$

as $t \to \infty$.

(iii) Solution as $\alpha \to \infty$

If one defines $S = \Phi/r^2$, then (53) becomes

$$r^2 \frac{d^2\Phi}{dr^2} - 2r \frac{d\Phi}{dr} = pr^2 e^{-\frac{r-1}{\delta r}} \Phi \tag{60}$$

Outer Solution

If we let $\delta \to 0$, s, r fixed in (60), the result is

$$r^2 \frac{d^2\Phi_o}{dr^2} - 2r \frac{d\Phi_o}{dr} = 0$$

the only bounded solution of which is Φ_o = constant.

Inner Solution

If y $\frac{r-1}{\delta}$, then (60) is

$$\frac{d^2\Phi_i}{dy^2} = p\delta^2 e^{-y} \Phi_i$$

and after some manipulation, Φ_i can be shown to be a solution of

$$z^2 \frac{d^2\Phi_i}{dz^2} + z\frac{d\Phi_i}{dz} - z^2\Phi_i = 0, \quad z \equiv 2\delta p^{1/2} e^{-1/2y}$$

which is Bessel's equation with solution $I_o(z)$.

Thus, $\Phi_i = \frac{1}{p} I_o(2p^{1/2}\delta e^{-y/2})/I_o(2p^{1/2}\delta)$ which is quite easily inverted by the residue theorem, and if ϕ_i is the inverse, then

$$\phi_i = 1 - 2\sum_{i=1}^{\infty} \frac{e^{-\nu_i^2 t/4\delta^2}}{\nu_i} \frac{J_o(\nu_i e^{-y/2})}{J_1(\nu_i)}$$

where ν_i are the roots of $J_o(\nu) = 0$. Now, Φ_o = constant means that ϕ_o is a function of time only and letting $y \to \infty$ in this result implies that

$$\phi_o = 1 - 2\sum_{i=1}^{\infty} \frac{e^{-\nu_i^2 t/4\delta^2}}{\nu_i J_1(\nu_i)}$$

and hence the composite expansion gives just

$$w \sim \frac{1}{r^2}\left(1 - 2\sum_{i=1}^{\infty} \frac{e^{-\nu_i^2 t/4\delta^2}}{\nu_i}\right) - 2\sum_{i=1}^{\infty} \frac{e^{-\nu_i^2 t/4\delta^2}}{\nu_i} \frac{J_o(\nu_i e^{-y/2}) - 1}{J_1(\nu_i)} \qquad (61)$$

or, what is equivalent

$$w \sim \frac{1}{r^2}\left(1 - 2\sum_{i=1}^{\infty} \frac{e^{-\nu_i^2 t/4\delta^2}}{\nu_i} \frac{J_o(\nu_i e^{-y/2})}{J_1(\nu_i)}\right) \qquad (62)$$

C. Steady Solution for a Slightly Non-uniform Temperature Distribution on a Sphere with a Strong Field

In this section, we seek solutions of (1)-(5) under the same expansion procedure (8), but for

$$T(1) = 1 + \frac{B^2}{Re} f(\theta) .$$

(i) The Stokes Problem

We seek a solution of (9) as $\alpha \to \infty$ ($\delta \to 0$) subject to

$$T_1(1) = f(\theta)$$
$$\rho_1(1, 0) = 0 \tag{63}$$
$$\underline{u}_1(1) = \underline{r} \times \underline{k}$$

It is quite clear that the azimuthal momentum equation in (9) is unchanged, and as in (10),

$$w_1 = \sin\theta/r^2 \tag{64}$$

The difference here is that there is now a meridional flow to this order, driven by pressure gradients and the non-uniform field created by the non-uniform temperature.

Inner Expansion

A careful ordering investigation very similar to that described in (2. A. ii) shows that $u_1 = O(\delta^3)$, $v_1 = O(\delta^2)$ so that the inner expansion begins

$$u_1 = \delta^3 f_1 + \ldots$$
$$v_1 = \delta^2 g_1 + \ldots$$
$$p_1 = a_1 + \ldots$$

$$T_1 = q_1 + \ldots$$

and then, with $r-1 = \delta y$, the boundary layer equations from (9) are

$$\frac{\partial f_1}{\partial y} + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta} g_1 \sin\theta = f_1 \tag{65a}$$

$$e^{-y}\frac{\partial a_1}{\partial\theta} = \frac{\partial^2 g_1}{\partial y^2} \tag{65b}$$

$$\frac{\partial a_1}{\partial y} = q_1 \tag{65c}$$

$$\frac{\partial^2 q_1}{\partial y^2} = 0 \tag{65d}$$

Equation (65d) clearly implies that $q_1 = f(\theta)$ so $a_1(y,\theta) = yf(\theta) + b(\theta)$. Then, (65b) is just

$$e^{y}\frac{\partial^2 g_1}{\partial y^2} = f'(\theta)y + b'(\theta)$$

Together with (65a) and $f_1(0,\theta) = g_1(0,\theta) = 0$ gives easily

$$\begin{aligned} f_1 &= \frac{1}{2}\nu(\nu+1)f(\theta)\left[1-(1+y)e^{-y}\right] \\ g_1 &= -\frac{1}{2}f'(\theta)\left[1-(1+2y)e^{-y}\right] \end{aligned} \tag{66}$$

where $f(\theta)$ has been taken to be the Legendre Polynomial of order ν without loss of generality. In addition, $b(\theta) = \frac{3}{2}(f(0)-f(\theta)) + f(0)$ using the final condition in (63).

Outer Expansion

Guided by the form of (66) as $y \to \infty$, let the outer expansions begin

$$
\begin{aligned}
u_1 &= \delta^3 \frac{1}{2} \nu(\nu+1) f(\theta) F(r) + \ldots \\
v_1 &= -\delta^2 \frac{1}{2} f'(\theta) G(r) + \ldots \\
p_1 &= \delta^3 \Gamma^{-1} P(r,\theta) + \ldots \\
T_1 &= Q(r,\theta) + \ldots
\end{aligned}
\tag{67}
$$

and insertion into (9) and then letting $\delta \to 0$ holding r fixed gives

$$F(r) = rG(r) \tag{68}$$

$$r^2 \frac{d^2G}{dr^2} + 2r \frac{dG}{dr} - \frac{4}{3} \nu(\nu+1) G(r) = 0 \tag{69}$$

$$\nabla^2 Q = 0 \tag{70}$$

$$P = f(\theta)\nu(\nu+1) \left[\frac{1}{6} F'(r) - \frac{4}{3r} F(r)\right] \tag{71}$$

The solution of (70) is quite easy, being

$$Q = \frac{f(\theta)}{r^{\nu+1}} \tag{72}$$

and (69) also yields a simple result, viz.,

$$G(r) = C/r^{\ell_\nu} \tag{73}$$

where $\ell_\nu = \frac{1}{2}\left(1 + \sqrt{(1+4\nu)(1+\frac{4}{3}\nu)}\right)$. The matching on T_1 has tacitly been done in (72), since (65d) really implies that T_1 has no boundary layer, so the solution to (70) should be uniformly valid. To match (73) to (66) requires $C \equiv 1$, and hence,

$$G(r) = 1/r^{\ell_\nu} \tag{74a}$$

$$F(r) = 1/r^{\ell_\nu - 1} \tag{74b}$$

$$Q = f(\theta)/r^{\nu+1} \tag{74c}$$

and (71) gives just

$$P = -\frac{1}{6}\left(\frac{1}{3} + \ell_\nu\right)\nu(\nu+1)\frac{f(\theta)}{r^{\ell_\nu}} \tag{74d}$$

The General Solution

This problem is entirely linear; hence, let A_ν be such that

$$A_\nu = \frac{1}{2}(2\nu+1)\int_{-1}^{1} f(\cos^{-1}x)P_\nu(x)dx \tag{75}$$

Then, the general solution to (70) is

$$T_1 = \sum_{\nu=0}^{\infty} \frac{A_\nu}{r^{\nu+1}} P_\nu(\cos\theta) + O(\delta)$$

and combination of each result for (67), (74), and (66) into a composite expansion gives, in general,

$$\begin{aligned} u_1 &\sim \delta^3 \sum_{\nu=0}^{\infty} \frac{1}{2}\nu(\nu+1)[r^{1-\ell_\nu} - (y+1)e^{-y}] A_\nu P_\nu(\cos\theta) \\ v_1 &\sim \frac{1}{2}\delta^2 \sin\theta \sum_{\nu=0}^{\infty} (r^{-\ell_\nu} - (2y+1)e^{-y}) A_\nu P'_\nu(\cos\theta) \end{aligned} \tag{76}$$

One can quite easily form a streamfunction for the boundary layer solution, $x \equiv \cos\theta$,

$$\psi = e^{-y}[1-(1+y)e^{-y}](1-x^2)\sum_{\nu=0}^{\infty} A_\nu \frac{dP_\nu}{dx} \tag{77}$$

It is very clear that (77) will exhibit $\psi = 0$ lines at places other than $x = \pm 1$; $\psi = 0$ also where $\frac{dP_\nu}{dx} = 0$ for every ν; such a thing will happen at $x = 0$ provided all the A_ν's vanish for odd ν. There will also be zeros where $\sum_{\nu=0}^{\infty} A_\nu \frac{dP_\nu}{dx} = 0$; this is a high order polynomial and hence has many roots. As an example, suppose A_2 and A_4 are the only non-zero A's for some $f(\theta)$. Then, $\psi = 0$ on $x = \pm(3/7)^{1/2}[1+\frac{6}{5}(A_2/A_4)]^{\frac{1}{2}}$

which has values $|x| < 1$ only for $-\frac{5}{6} \leqslant A_2/A_4 \leqslant 10/9$. However, for A_2/A_4 in this range, the $\psi = 0$ lines are: $y = 0$, $y = \infty$, $x = \pm 1$, $x = 0$, and $x = x_c$, i.e., a four-cell flow.

The possibilities of (77) are endless, but the flow structure is qualitatively like that shown previously in Figure 2b.

<u>The Constraint</u>

The pressure in the boundary layer is just

$$a_1(y,\theta) = \sum_{\nu=0}^{\infty} A_\nu \left[\frac{1}{2} P_\nu(\cos\theta)(2y-3) + \frac{5}{2}\right] \tag{78}$$

and in the outer flow,

$$P = -\frac{1}{6}\sum_{\nu=1}^{\infty} \nu(\nu+1)\left(\frac{1}{3}+\ell_\nu\right)P_\nu(\cos\theta)r^{-\ell_\nu} \tag{79}$$

and then (79) together with (67) and (8) gives the constraint on the solution

$$Re^{-1}\, B^2 << e^{-1/\delta}/\delta^3$$

which again represents a severe restriction on O(Re), in terms of which this may be rewritten as

$$Re = 0(\lambda^{-1}\, \delta^{-3}\, e^{-1/\delta}) \tag{80}$$

(ii) <u>Meridional Flow if $A_\nu \equiv 0$ for $\nu \neq 0$</u>

If $A_\nu \equiv 0$ for every ν but $\nu = 0$, then (76), (78) and (79) give $u_1 = v_1 = 0$, but

$$a_1 = \frac{1}{2} A_o\,[(2y-3) + 5]$$

$$P \equiv 0$$

Then, the next order equations in the Re expansion are to be considered, and they have a form nearly identical to (11)-(14), viz.,

$$\delta\nabla\cdot \underline{u}_2 = \frac{\underline{r}\cdot \underline{u}_2}{r^3} \tag{81a}$$

$$\Gamma(\underline{u}_1\cdot\nabla)\underline{u}_1+\Gamma\nabla p_2-\Gamma(T_2+\lambda\rho_1 T_1)\frac{\underline{r}}{\delta r^3} = \nabla^2\underline{u}_2+\tfrac{1}{3}\nabla[\nabla\cdot\underline{u}_1] + \omega\,\underline{\underline{e}}_1\cdot\nabla T_1 \tag{81b}$$

$$\frac{1}{\sigma}\nabla^2 T_2 + \frac{18}{5}\frac{\sin^2\theta}{r^6} = \frac{\Gamma}{\delta\lambda}\left(\frac{\underline{r}\cdot\underline{u}_2}{r^3}\right) \tag{81c}$$

$$p_2 = \rho_2 + T_2 + \lambda\rho_1 T_1 \tag{81d}$$

It is easy to show that $\underline{\underline{e}}_1\cdot\nabla T_1 = \frac{3A_o}{r^6}(\underline{k}\times\underline{r})$, i. e., it has a component in the azimuthal direction only. Now, $T_1 = A_o/r$ here by (74c), and a substitution

$$\begin{aligned} p_2 &= p_2^* + \frac{\lambda}{\delta}\int_c^r \frac{\rho_1 T_1}{r^2}\,dr \\ \rho_2 &= \rho_2^* + \frac{\lambda}{\delta}\int_c^r \frac{\rho_1 T_1}{r^2}\,dr - \lambda\rho_1 T_1 \end{aligned} \tag{82}$$

into (81) gives a modified (81b) and (81d). The new equations are

$$\Gamma(\underline{u}_1\cdot\nabla)\,\underline{u}_1 + \Gamma\nabla p_2^* - \Gamma\frac{\underline{r}}{\delta r^3}T_2 = \nabla^2\underline{v}_2 + \tfrac{1}{3}\nabla(\nabla\cdot\underline{u}_2) + \underset{*}{\frac{3A_o}{r^6}(\underline{k}\times\underline{r})} \tag{81b*}$$

$$p_2^* = \rho_2^* + T_2 \tag{81d*}$$

Equations (81*) are now identical in form with (11)-(14) apart from the term labeled with an asterisk. However, that term has a component in the azimuthal direction only, and hence, under (82), the problem is reduced to the problem solved in (2. A. ii). However, (*) in (81b*) does introduce an additional swirl velocity of O(Re), and (81b*) in the azimuthal direction is

$$\nabla^2 w_2 - \frac{w_2}{r^2 \sin^2\theta} = -\frac{3\omega A_o \sin\theta}{r^5}$$

which is easily reduced to an ordinary differential equation with the substitution $w_2 = A_o W(r)\sin\theta$ which gives

$$\frac{d}{dr}\left(r^2 \frac{dW}{dr}\right) - 2W = -\frac{3\omega}{r^3}$$

and then, easily,

$$w_2 = \frac{3\omega A_o}{8} \sin\theta\left(\frac{1}{r^3} - \frac{1}{r^2}\right) \tag{84}$$

If $8\pi\Omega\mu_o a^3$ is the torque on the sphere when $A_o \equiv 0$, one can easily get that

$$\frac{\text{Torque}}{8\pi\Omega\mu_o a^3} = 1 + \omega A_o \mathrm{Re}(1+\lambda/8) \tag{85}$$

3. Asymptotic Theory for a Strong Field and Fixed Viscosity

In this section, we develop an asymptotic theory for $\alpha \to \infty$, Re, B fixed. The only restriction on what follows appears to be

$$\lim_{\frac{GM}{aRT_o} \to \infty} \frac{\Omega^{\frac{1}{2}} a^2 RT_o}{GM\nu_o^{\frac{1}{2}}} = 0 \tag{86}$$

which means physically that the isothermal scale height, a^2RT_o/GM is thin compared to a Prandtl boundary layer thickness $(\nu_o/\Omega)^{\frac{1}{2}}$. The boundary conditions to be satisfied are (6).

A. The Boundary Layer

An examination of (1)-(5) as $\delta \to 0$ in a thin layer on the sphere, under condition (86), will readily convince the reader that any distinguished limit must be associated with a layer of width δ. Hence, write $r - 1 = \delta y$ as before, but, using the scaling in (1)-(5), put

$$u = \delta\chi(\delta)\,\tilde{u}$$

$$v = \chi(\delta)\,\tilde{v}$$

where $\chi(\delta)$ is a function of δ to be determined. Denoting all other variables in the boundary layer by $(\tilde{\ })$, the equations become

$$\frac{\partial\tilde{\rho}\tilde{u}}{\partial y} + \frac{1}{\sin\theta}\,\frac{\partial\tilde{\rho}\tilde{v}\sin\theta}{\partial\theta} = 0 \tag{87a}$$

$$\frac{\tilde{\rho}}{\delta} + \frac{1}{\delta}\frac{\partial\tilde{p}}{\partial y} - \tilde{\rho}\,B^2\,\tilde{w}^2 + B^2\chi^2\delta\tilde{\rho}[\tilde{u}\,\frac{\partial\tilde{u}}{\partial y} + \tilde{v}\,\frac{\partial\tilde{u}}{\partial\theta}] = O(\chi B^2/\delta) \tag{87b}$$

$$\frac{\partial\tilde{p}}{\partial\theta} + \tilde{\rho}B^2\chi^2[\tilde{u}\,\frac{\partial\tilde{v}}{\partial y} + \tilde{v}\,\frac{\partial\tilde{v}}{\partial\theta}] - \tilde{\rho}B^2\tilde{w}^2\cot\theta = \frac{B^2\chi}{\delta^2}\,\frac{\nu_o}{\Omega a^2}\,\frac{\partial}{\partial y}\,\tilde{T}^\omega\,\frac{\partial\tilde{v}}{\partial y} \tag{87c}$$

$$\frac{\partial}{\partial y}\,\tilde{T}^\omega\,\frac{\partial\tilde{w}}{\partial y} = O(\frac{\Omega a^2}{\nu_o}\,\chi\delta^2) \tag{87d}$$

$$\frac{\partial}{\partial y}\,\tilde{T}^\omega\,\frac{\partial\tilde{T}}{\partial y} + \sigma B^2\tilde{T}^\omega\,(\frac{\partial\tilde{w}}{\partial y})^2 + O(\sigma\chi^2) = O(\chi\delta^2) \tag{87e}$$

$$\tilde{p} = \tilde{\rho}\,\tilde{T} \tag{87f}$$

Provided that $\lim_{\delta\to 0}\chi < \infty$, it is clear that χ cannot be even $O(1)$. If it were, (87c)-(87f) indicate $\tilde{w}$, $\tilde{v}$ and hence $\tilde{T}$ are unchanged across the layer, and that means that (87a) and (87b) also have trivial solutions. So, proceeding on the fact $\chi = O(\delta)$, one notes that $\chi = O(\delta^2)$ is one distinguished limit of (87). However, carrying out the general solution shows that one cannot satisfy the boundary conditions on $y = 0$ and still have solutions that are not transcendentally large as $y \to \infty$. Thus, try $\chi = O(\delta^3)$. This turns out to give quite general solutions with enough arbitrariness. Let the inner expansions begin,

$$\tilde{p} = \tilde{p}_o + \delta\tilde{p}_1 + \ldots$$

$$\tilde{\rho} = \tilde{\rho}_o + \delta\tilde{\rho}_1 + \ldots$$

$$\tilde{T} = \tilde{T}_o + \delta\tilde{T}_1 + \ldots$$

$$\tilde{w} = \tilde{w}_o + \delta\tilde{w}_1 + \ldots$$

$$\tilde{u} = \tilde{u}_o + \delta\tilde{u}_1 + \ldots$$

$$\tilde{v} = \tilde{v}_o + \delta\tilde{v}_1 + \ldots$$

with $\chi(\delta) = \dfrac{RT_o}{\Omega\nu_o}\,\delta^3$. Putting these into (87) gives the O(1) equations, dropping the $(\tilde{\ })$ notation for the moment,

$$\frac{\partial\rho_o u_o}{\partial y} + \frac{1}{\sin\theta}\,\frac{\partial\rho_o v_o \sin\theta}{\partial\theta} = 0 \tag{88a}$$

$$\frac{\partial p_o}{\partial y} + \rho_o = 0 \tag{88b}$$

$$\frac{\partial p_o}{\partial\theta} - \rho_o w_o^2 B^2 \cot\theta = 0 \tag{88c}$$

$$\frac{\partial}{\partial y}\, T_o^{\omega}\,\frac{\partial w_o}{\partial y} = 0 \tag{88d}$$

$$\frac{\partial}{\partial y} T_o^{\omega} \frac{\partial T_o}{\partial y} = 0 \tag{88e}$$

$$p_o = \rho_o T_o \tag{88f}$$

If the temperature is uniform on the sphere, the use of (6) with (88d), (88e) gives quite easily $T_o = 1$, $w_o = \sin\theta$. Further, (88b) and (88c) with (88f) are easily solved to give

$$p_o = \rho_o = \exp[\tfrac{1}{2}B^2\sin^2\theta - y] \tag{89}$$

using (6) again. Repeating the limit process yields the O(δ) equations,

$$\frac{\partial^2 T_1}{\partial y^2} = \frac{\partial^2 w_1}{\partial y^2} = 0$$

first of all. A little thought shows that all this and (88d, e) are saying is that T_1 and w_1 change very slowly across this layer, so we replace T and w by their Taylor series expansions near r = 1. Regarding these as known, let $T_1 = y\,\alpha(\theta)$ and $w_1 = y\beta(\theta)\sin\theta$. Now, one can proceed with the remainder of the O(δ) equations,

$$p_1 = \rho_1 + \alpha(\theta) y\, p_o \tag{90a}$$

$$\frac{\partial p_1}{\partial y} + p_1 = p_o[(\alpha+2)y + B^2 \sin^2\theta] \tag{90b}$$

$$\frac{\partial p_1}{\partial \theta} = \frac{\partial^2 v_o}{\partial y^2} + B^2 \cos\theta\sin\theta[2p_o\,\beta(\theta)y + \rho_1] \tag{90c}$$

after a little manipulation has been done. The first two of these equations may be solved simultaneously and result in

$$p_1 = p_o\,[(\tfrac{1}{2}\alpha + 1)y^2 + B^2 y \sin\theta + \gamma(\theta)] \tag{91}$$

where $\gamma(\theta)$ is as yet arbitrary. Substituting this into (90c) results in

$$\frac{\partial^2 v_o}{\partial y^2} = p_o[(2+\alpha-2\beta)\, B^2 \sin\theta\cos\theta y + \frac{1}{2}\alpha'(\theta)y + \gamma'(\theta)]$$

which may be integrated to give, with $v_o = 0$ on $y = 0$,

$$v_o/p_o = \frac{1}{2}\alpha'(y^2+4y) + (2-2\beta+\alpha)B^2\sin\theta\cos\theta\; y$$
$$+(1-e^y)\{3\alpha'+2(2-2\beta+\alpha)B^2\sin\theta\cos\theta + \gamma'\} \tag{92}$$

Equations (89) and (92) inserted into (88a) give a first order equation for u_o which is easily solved and application of $u_o(0) = 0$ then evaluates $\gamma'(\theta)$; integrating once for $\gamma(\theta)$,

$$\gamma(\theta) = -\frac{3}{2}B^2\sin^2\theta - \frac{3}{2}\int_0^\theta B^2[\alpha(\phi)-2p(\phi)]\sin\phi\cos\phi\, d\phi$$
$$+\frac{7}{4}[\alpha(0) - \alpha(\theta)] \tag{93}$$

and insertion of this into (91) gives p_1 complete and into (92) and the expression for u_o gives eventually,

$$\tilde{u}_o = e^{\frac{1}{2}B^2\sin^2\theta}[\frac{1}{4}\langle\alpha'\rangle(y^2+5y+5/2)e^{-y} + \frac{1}{4}\langle(2-2\beta+\alpha)\sin\theta\cos\theta\rangle$$
$$B^2(2y+1)e^{-y} - (1-\frac{1}{2}e^{-y})(\frac{5}{4}\langle\alpha'\rangle + \frac{1}{2}B^2\langle(2-2\beta+\alpha)\sin\theta\cos\theta\rangle)] \tag{94}$$

$$\tilde{v}_o = e^{\frac{1}{2}B^2\sin^2\theta}[\frac{1}{2}\alpha'(y^2+4y)e^{-y}+y(2-2\beta+\alpha)B^2\sin\theta\cos\theta\; e^{-y}$$
$$+(e^{-y}-1)(\frac{5}{4}\alpha' + \frac{1}{2}B^2(2-2\beta+\alpha)\sin\theta\cos\theta)] \tag{95}$$

where (94) contains an operator denoted by $\langle\;\rangle$, defined as

$$\langle\psi\rangle \equiv \frac{1}{\sin\theta}\, e^{-B^2\sin^2\theta} \frac{d}{d\theta}\left(\sin\theta\, e^{B^2\sin^2\theta}\, \psi\right) \tag{96}$$

One can easily calculate the mass flux out of the layer

$$M = 4\pi\nu_o^{-1}\,\bar{p}_o\, a^3 \int_o^{\pi} \rho_o u_o \sin\theta d\theta$$

at any value of y. Multiplication of ρ_o and u_o gives function of y times a θ derivative over $\sin\theta$, so the integral is trivial and it is identically zero for all y. This solution is interestingly enough just a generalization of the solution given in (2. A. ii) and one may formally recover (39) and (40) by letting $B \to 0$ in (94) and (95). There is a net circulation in each meridional plane,

$$2\pi a^2 \left\{ -\frac{5}{4}\int_o^{\pi} \sin\theta\, e^{\frac{1}{2}B^2\sin^2\theta}\, \alpha(\theta)d\theta \right.$$
$$+ \frac{3}{4}B^2 \int_o^{\pi} e^{\frac{1}{2}B^2\sin^2\theta}\, \alpha(\theta)\sin^2\theta\cos\theta d\theta$$
$$\left. + B^2 \int_o^{\pi} e^{\frac{1}{2}B^2\sin^2\theta}\, \sin^2\theta\, \cos\theta\beta(\theta)d\theta \right\}$$

Solutions (94) and (95) are complete without any matching to the outer flow since it is this boundary layer flow which drives the flow in the bulk of the fluid. The details of the solution do, of course, depend upon the details of the temperature and swirl velocity solutions near the sphere but the form of the solution does not.

B. The Outer Flow; The Temperature and Swirl Velocity

Notice that, as $y \to \infty$ in (94) and (95), $\tilde{u}_o$ is O(1) and $\tilde{v}_o$ is O(1). Hence, u and v are respectively $O(\delta^4)$ and $O(\delta^3)$ in the outer flow. In that case, it is quite clear that the azimuthal component of (2) and (3) have inertial terms of negligible size. Hence, if the outer expansions begin

$$T = T_o + \delta^3 T_3 + \dots$$
$$w = w_o + \delta^6 w_6 + \dots$$

the correct outer limit gives

$$\nabla^2 w_o - \frac{w_o}{r^2 \sin^2\theta} + \omega r \frac{\partial}{\partial r}\left(\frac{w_o}{r}\right) \frac{\partial \log T_o}{\partial r}$$

$$+ \omega \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{w_o}{\sin\theta}\right) \frac{\partial \log T_o}{\partial\theta} = 0 \qquad (97)$$

$$\nabla \cdot T_o^\omega \nabla T_o + \sigma B^2 T_o^\omega r^2 \left[\frac{\partial}{\partial r} \frac{w_o}{r}\right]^2 = 0 \qquad (98)$$

where $w_o = \sin\theta$, $T_o = 1$ on $r = 1$

(99)

and $w_o \to 0$, $T \to T_\infty$ as $r \to \infty$.

This problem constitutes a well-posed elliptic system, and the solution is no difficulty in principle, but because of the non-linearity and coupling, is difficult in practice. If $\sigma B^2 \to 0$ then the solution to (98) is just

$$T_o = \left[T_\infty^{\omega+1} - \frac{T_\infty^{\omega+1} - 1}{r}\right]^{\frac{1}{\omega+1}} + O(\sigma B^2) \qquad (100)$$

and then $w_o = F(r)\sin\theta + O(\sigma B^2)$ into (97) gives

$$r^2 \frac{d^2F}{dr^2} + r\frac{dF}{dr}\left\{2 - \frac{\omega}{(\omega+1)r} \frac{1 - T_\infty^{\omega+1}}{1 + \frac{1 - T_\infty^{\omega+1}}{r}}\right\}$$

(101)

$$-F\left\{2 - \frac{\omega}{(\omega+1)\,r}\;\frac{1 - T_{\infty}^{\omega+1}}{1 + \frac{1-T_{\infty}^{\omega+1}}{r}}\right\} = 0 \tag{101 Cont'd}$$

$$F(1) = 1, \;\; F(\infty) = 0$$

When $T_{\infty} \equiv 1$, $F = 1/r^2$ which is also the solution for all T_{∞} for r sufficiently large. The other tractable case is when $\omega = 0$. This is not especially meaningful physically, since $\frac{1}{2} \leqslant \omega \leqslant 1$ for a gas, but it does give some information as to the structure of the solution. Clearly (97) gives

$$w_o = \frac{\sin\theta}{r^2} + O(\omega) \tag{102}$$

Substitution into (98) gives

$$\nabla^2 T_o = -9\,\sigma B^2\,\frac{\sin^2\theta}{r^6}$$

the solution of which is easily done by Legendre polynomials,

$$T_o = T_{\infty} + (1-T_{\infty})\frac{1}{r} + \frac{1}{2}\sigma B^2\left(\frac{1}{r} - \frac{1}{r^4}\right) + \frac{\sigma B^2}{2}(3\cos^2\theta - 1)\left(\frac{1}{r^4} - \frac{1}{r^3}\right) + O(\omega\sigma B^2) \tag{103}$$

C. <u>The Outer Flow; The Meridional Plane</u>

So, the outer flow splits into two parts, one just treated involving the solution of two coupled equations for the temperature and swirl velocity fields, and the other the flow in the meridional planes involving also the pressure field. This part will be dealt with here.

The momentum equations may be written symbolically as

$$\nabla p + \frac{\underline{r}}{\delta r^3 T} p - p\underline{C} = \delta^3 \underline{\Sigma} \tag{104}$$

where these vectors are two-dimensional vectors in the meridional plane, $\underline{\Sigma} = \nabla \cdot \underline{\underline{\tau}} / \delta^3$ and $\underline{C} = \frac{B^2 w_o^{\;2}}{r}(1, \cot\theta)$ is the centrifugal force term. The reason for the δ^3 scale on $\nabla \cdot \underline{\underline{\tau}}$ is the order of $(\tilde{u}, \tilde{v})$ as $y \to \infty$, viz., $\tilde{v}$ is $O(\delta^3)$. Consider the radial component of (104),

$$\frac{\partial p}{\partial r} + \frac{1}{\delta r^2 T} p - pC_r = \delta^3 \Sigma_r \tag{104'}$$

Having solved for w and T in B, we know that C_r and T are uniformly $O(\delta^o)$. The outer region is by definition one where gradients are O(1), hence, let $\delta \to 0$ holding r fixed in (104') and the result, if $p = \delta^4 P_4 + \ldots$, is

$$P_4 = r^2 T \Sigma_r \tag{105}$$

Having noted this, insertion of (105) into the θ-component of (104) and doing the limit gives

$$\Sigma_\theta = O(\delta)$$

Further, the form of (105) implies from mass conservation that u and v must be of the same order in this region. Let

$$u = \delta^3 (RT_o/\Omega\nu_o) U_o + \ldots$$

$$v = \delta^3 (RT_o/\Omega\nu_o) V_o + \ldots$$

Then $\underline{\Sigma} = \underline{\Sigma}_o + \ldots,\quad \underline{\underline{\tau}} = \delta^3 (RT_o/\Omega\nu_o)\, \underline{\underline{\tau}}_o + \ldots$

and the equations to be solved are just

$$\Sigma_{o_\theta} = 0$$
$$\nabla \cdot [r\, \underline{U}_o\, \underline{r} \cdot (\nabla \cdot \underline{\underline{\tau}}_o)] = 0 \tag{106}$$

Doing the matching with (94), (95) by any technique, like an intermediate limit, immediately gives the boundary conditions on (106), viz.,

$$U_o(1) = 0$$
$$V_o(1) = -e^{\frac{1}{2}B^2 \sin^2\theta} \left\{ \frac{5}{4}\alpha' + \frac{1}{2}B^2(2-2\beta+\alpha)\sin\theta\cos\theta \right\} \tag{107}$$

The density field is, of course, $P_4/T = r^2 \Sigma_{or}$. For completeness, we put here the expressions for $\underline{\Sigma}_o$,

$$\Sigma_{o_r} = T_o^\omega \Bigg[\nabla^2 U_o - \frac{2U_o}{r^2} - \frac{2}{r^2 \sin\theta} \frac{\partial V_o \sin\theta}{\partial\theta} + \frac{1}{3}\frac{\partial}{\partial r}(\nabla \cdot \underline{U}_o)$$
$$+ \omega \frac{\partial \log T_o}{\partial r} [2 \frac{\partial U_o}{\partial r} - \frac{2}{3} \nabla \cdot \underline{U}_o] + \frac{\omega}{r} \frac{\partial \log T_o}{\partial\theta} [\frac{1}{r} \frac{\partial U_o}{\partial\theta} + r \frac{\partial}{\partial r}(\frac{V_o}{r})] \Bigg] \tag{108a}$$

$$\Sigma_{o_\theta} = T_o^\omega \Bigg[\nabla^2 V_o - \frac{V_o}{r^2 \sin^2\theta} + \frac{2}{r^2} \frac{\partial U_o}{\partial\theta} + \frac{1}{3r} \frac{\partial}{\partial\theta} \nabla \cdot \underline{U}_o$$
$$+ \omega \frac{1}{r} \frac{\partial \log T_o}{\partial\theta} \left[\frac{2}{r} \frac{\partial V_o}{\partial\theta} + \frac{2U_o}{r} - \frac{2}{3r} \nabla \cdot \underline{U}_o \right]$$
$$+ \omega \frac{\partial \log T_o}{\partial r} \left[r \frac{\partial}{\partial r}(\frac{V_o}{r}) + \frac{1}{r} \frac{\partial U_o}{\partial\theta} \right] \Bigg] \tag{108b}$$

It is quite clear that the hope of solving (106)-(108) is minimal indeed. A rough solution by a Galerkin method is given in Appendix A. The pressure (105) is algebraically determined and there is no arbitrariness to be removed by matching. The composite expansion for the pressure, is symbolically, then,

$$p \sim e^{\frac{1}{2}B^2 \sin^2\theta - y} + \delta^4 r^2 T_o \Sigma_{or} \tag{109}$$

Notice that (109), and worse than that, (106)-(108) are not the equations solved under the limit sequence Re, B $\rightarrow$ 0, $\delta \rightarrow 0$, though the boundary layers <u>are</u> similar. The reasons for this are discussed by means of a more careful treatment of (104) in Appendix B. If Re and B are O(1), that work shows the restriction on the solution to be

$$\delta^4 >> e^{-1/\delta} \tag{110}$$

which is indeed true as $\delta \rightarrow 0$, for δ sufficiently small. A pictorial representation of the solution is shown in Figure 3.

(i) <u>The $B \equiv 0$, $T_\infty = 1$ Problem</u>

In this case, Section B showed $T_o \equiv 1$, $w_o = \sin\theta/r^2$. Then $\alpha \equiv 1$ and $\beta = -2$. Hence, (106), (107) become

$$\frac{1}{T_o} \Sigma_{r_o} = \nabla^2 U_o - \frac{2U_o}{r^2} - \frac{2}{r^2 \sin^2\theta} \frac{\partial V_o \sin\theta}{\partial\theta} + \frac{1}{3}\frac{\partial}{\partial r} \nabla \cdot \underline{U}_o \tag{111a}$$

$$\nabla \cdot [r^2 \underline{U}_o \Sigma_{or}] = 0$$

plus

$$\nabla^2 V_o - \frac{V_o}{r^2 \sin^2\theta} + \frac{2}{r^2}\frac{\partial U_o}{\partial\theta} + \frac{1}{3r}\frac{\partial}{\partial\theta} \nabla \cdot \underline{U}_o = 0 \tag{111b}$$

Equation (111b) is the equation solved in (2. A. ii) but not together with (111a), which was replaced there by a linear equation. The essential difference, as pointed out in Appendix B, is that the outer flow in the (2. A. ii) problem is in hydrostatic equilibrium; here, it is <u>not</u>.

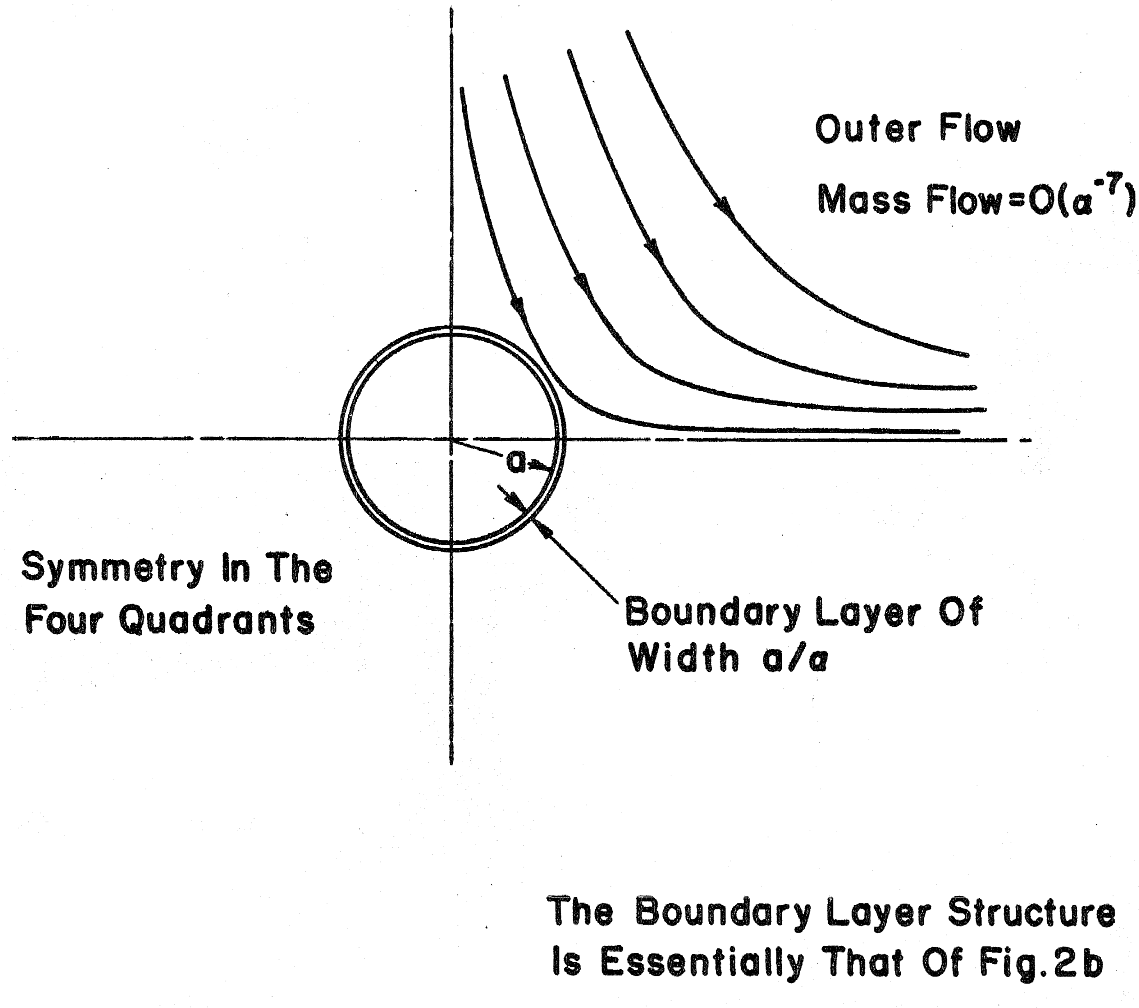


FIG. 3 THE STRUCTURE OF THE $\alpha \to \infty$ SOLUTION IN THE MERIDIONAL PLANE

D. The Far Field

The arguments presented in C, and as elucidated in Appendix B, on equation (104') are invalid for r sufficiently large, because then the second term is less important than the first. Sufficiently far from the sphere, the sphere itself may be replaced by a point torque at $r = 0$,

$$-\frac{16}{3}\pi\Omega a^3 \mu_o \underline{k} \times \nabla \delta(\underline{r})$$

This vector has a component only in the $\underline{r} \times \underline{k}$ direction, so clearly, if $r \to \infty$, the right hand sides of (104) vanish very rapidly, since the meridional velocity components vanish at ∞, but $p \to$ const which may be small in δ but always dominates the right hand sides for r sufficiently large. Thus, (104) becomes

$$\frac{\partial p}{\partial r} + \frac{p}{\delta r^2 T_\infty} = 0$$
$$\frac{\partial p}{\partial \theta} = 0 \qquad (112)$$

since the centrifugal forces vanish more rapidly than $1/r^2$. Hence, $p = p_\infty e^{+\frac{1}{\delta r T_\infty}}$ and now the swirl equation, if $\underline{u} = \underline{k} \times \nabla\phi + \ldots$ is reducible to

$$\nabla^2\phi = 4\pi\delta(\underline{r})\frac{\mu_o}{\mu_\infty}$$

The solution of which is easily written down,

$$\phi = -\frac{\mu_o}{\mu_\infty}\frac{1}{r}$$

so

$$\nabla\phi = \frac{\mu_o}{\mu_\infty}\ \frac{\underline{r}}{r^3}$$

and so

$$\underline{u} \sim \frac{\mu_o}{\mu_\infty}\ \frac{\underline{k} \times \underline{r}}{r^3}$$

$$p \sim p_\infty(\delta)\ e^{1/\delta r T_\infty} \qquad (113)$$

Though $\underline{u}$ is precisely determined, $p_\infty(\delta)$ is not known and must presumably be evaluated by a matching procedure to the pressures from Part C. This process appears to be enormously difficult, and has not as yet been accomplished, partly because solutions of (106), (107) are so difficult in themselves. So, $O(p_\infty)$ as $\delta \to 0$ is not known, though it seems certain that $\lim_{\delta \to 0} p_\infty = 0$.

4. Asymptotic Theory for Small Viscosity and Small Prandtl Number

Perhaps the most natural attack on the general problem is to seek solutions in the limit $\Omega a^2/\nu_o \to \infty$, holding the field strength fixed. In that case, one expects that, if such a layer exists at all, the Prandtl boundary layer will be much thinner than an isothermal scale height, and its dynamics become the mechanism driving the flow. This Reynolds number regime is then characterized by the constraint

$$\lim_{\nu_o \to 0} \frac{GM\nu_o^{\frac{1}{2}}}{\Omega^{\frac{1}{2}} a^2 R T_o} = 0 \qquad (114)$$

which is just the reciprocal of (86). Under (114), it has been possible to complete the solution in a consistent way only if

$$\sigma \ll 1 \; . \tag{115}$$

The extension for arbitrary Prandtl number is fraught with difficulties and has not yielded to solution as yet.

As before, it is the motion induced by centrifugal forces near the sphere that controls the dynamics of the entire gas, so we proceed by analyzing the boundary layer first.

A. The Viscous Boundary Layer

Let $z \equiv \frac{r-a}{a}$. The boundary layer equations on the surface of the rotating sphere are, as given in 1951 by Howarth,[4]

$$\rho u \frac{\partial v}{\partial z} + \rho v \frac{\partial v}{\partial \theta} - \rho w^2 \cot\theta = \frac{\partial}{\partial z} \frac{\mu}{a} \frac{\partial v}{\partial z} \tag{116}$$

$$\rho u \frac{\partial w}{\partial z} + \rho v \frac{\partial w}{\partial \theta} + \rho v w \cot\theta = \frac{\partial}{\partial z} \frac{\mu}{a} \frac{\partial w}{\partial z} \tag{117}$$

$$\frac{\partial \rho u}{\partial z} + \frac{1}{\sin\theta} \frac{\partial \rho v \sin\theta}{\partial \theta} = 0 \tag{118}$$

$$\left(\rho u \frac{\partial}{\partial z} + \rho v \frac{\partial}{\partial \theta}\right) \left\{ h + \frac{v^2+w^2}{2} + \frac{GM}{a} z \right\} = \left(\frac{1}{\sigma} - 1\right) \frac{\partial}{\partial z} \frac{\mu}{a} \frac{\partial h}{\partial z} + \frac{\partial}{\partial z} \frac{\mu}{a} \frac{\partial}{\partial z} \left(h + \frac{v^2+w^2}{2}\right) \tag{119}$$

$$\frac{\partial p}{\partial z} = 0 \tag{120}$$

$$p = \frac{2}{5} \rho h \tag{121}$$

where the variables are here dimensional, h being the enthalpy. Actually, for $\mu \alpha h$ and $\sigma = 1$, these equations may be treated quite generally, a Howarth-Doronitzyn transformation[13] being all that is required to make these into incompressible equations. Solutions may be done quite easily if h = const or $\frac{\partial h}{\partial z} = 0$ on $z = 0$, being characterized by slight vertical scale distortion of $O(B^2)$.

However, serious constraints on the outer flow require $\sigma \ll 1$ to make this all consistent. In that case, (119) implies

$$\frac{\partial^2 h}{\partial z^2} = 0$$

so h is unchanged across the layer, and, by (120), so are p and, hence, ρ. Non-dimensionalizing by p_o, h_o, ρ_o, μ_o, the values on the sphere, Ωa for the (v, w) velocities and $\sqrt{\nu_o \Omega}$ for u gives

$$u^* \frac{\partial v^*}{\partial y} + v^* \frac{\partial v^*}{\partial \theta} - w^{*2} \cot\theta = \frac{\partial^2 v^*}{\partial y^2} \tag{122a}$$

$$u^* \frac{\partial w^*}{\partial y} + v^* \frac{\partial w^*}{\partial \theta} + v^* w^* \cot\theta = \frac{\partial^2 w^*}{\partial y^2} \tag{122b}$$

$$\frac{\partial u^*}{\partial y} + \frac{1}{\sin\theta} \frac{\partial v^* \sin\theta}{\partial \theta} = 0 \tag{122c}$$

where $z = \left(\frac{\nu_o}{\Omega a^2}\right)^{\frac{1}{2}} y$.

These equations are to be solved subject to the boundary conditions

$$\begin{aligned} &u^*(0, \theta) = v^*(0, \theta) = 0, \;\; w^*(0, \theta) = \sin\theta \\ &u^*(0, y) = v^*(0, y) = 0 \end{aligned} \tag{123}$$

and $\quad v^*(\infty, \theta) = w^*(\infty, \theta) = 0 \qquad (124)$

which, together with (122) constitute a well-posed problem for (u^*, v^*, w^*), a parabolic system. Despite a controversy after Howarth first discussed the solution to (122)-(124) in connection with the rotation of a sphere in an unbounded incompressible fluid, [4], started by Nigam, [5], in 1954 and apparently successfully resolved by

Stewartson in 1958,[6] it is very clear now that the solution to the equations is as described by Howarth in 1951. The boundary conditions at the edge given in (124) are certainly not obviously correct in this problem. They are valid of course if $\alpha \equiv 0$, so one might expect, as indeed is true, that for certain ranges of the parameters α and σ, they are also correct. There will be some discussion of this point later; the true justification for (124) is a posteriori, i.e., that their use allows the completion of the solution in a self-consistent way. An integral of (122c) shows that the entrainment is

$$u^*(\infty, \theta) = -\frac{1}{\sin\theta} \frac{d}{d\theta} \int_0^\infty v^* \sin\theta \, dy \tag{125}$$

B. The Outer Solution

The next step in the solution of the problem is to compute the flow in the bulk of the fluid, i.e., away from the boundary. Now, certainly the outer flow must supply the boundary layer with the amount of fluid required by (125). Denoting $u^*(\infty, \theta)$ by $-E(\theta)$, we have that

$$u(a, \theta) = -T_1 (\nu_o \Omega)^{\frac{1}{2}} E(\theta) \tag{126a}$$

as a boundary condition on the outer flow, where $T_1 \equiv h(0, \theta)/h_o$. The thermodynamic conditions

$$p(a, \theta) = p_o, \;\; \rho(a, \theta) = \rho_o/T_1, \; h(0, \theta) = h_o T_1 \tag{126b}$$

must be satisfied as well, and also

$$h(\infty, \theta) = h_\infty \tag{126c}$$

Now (126a) and the equation of continuity imply that the velocities are $O(Re^{-\frac{1}{2}})$ which $\downarrow O$ as $Re \uparrow \infty$ so the pressure-temperature field is hydrostatic to leading order, and in fact, the outer expansion begins

$$p = p^{(o)} + \ldots$$

$$\rho = \rho^{(o)} + \ldots$$

$$h = h^{(o)} + \ldots$$

$$u = Re^{-\frac{1}{2}} u^{(o)} + \ldots$$

$$v = Re^{-\frac{1}{2}} v^{(o)} + \ldots$$

so the relevant equations are

$$\nabla p^{(o)} = \rho^{(o)} \nabla \frac{GM}{r} \tag{127}$$

$$\nabla \cdot (\rho^{(o)} \underline{u}^{(o)}) = 0 \tag{128}$$

$$p^{(o)} = \frac{2}{5} \rho^{(o)} h^{(o)} \tag{129}$$

$$\rho^{(o)} \underline{u}^{(o)} \cdot \nabla h^{(o)} - \underline{u}^{(o)} \cdot \nabla p^{(o)} = \frac{1}{\sigma} Re^{+\frac{1}{2}} \nabla \cdot [\mu^{(o)} \nabla h^{(o)}] \tag{130}$$

to be solved subject to

$$\left.\begin{aligned} &u^{(o)}(a, \theta) = -\Omega a E(\theta) \\ &h^{(o)}(\infty, \theta) = h_\infty, \; h^{(o)}(a, \theta) = h_o T_1, \; \rho^{(o)}(a, \theta) = \rho_o T_1^{-1}, \; p^{(o)}(a, \theta) = p_o \end{aligned}\right\} \tag{131}$$

The solutions to (127)-(131) depend crucially upon $O(\sigma)$ and $O(Re)$; the several possibilities are explored below.

CASE 1, $1 >> \sigma >> Re^{-\frac{1}{2}}$

If this constraint is used, then (130) reduces, in the limit $Re \rightarrow \infty$, to an adiabatic thermal field,

$$\left.\begin{aligned} &\rho^{(o)} u^{(o)} \frac{d}{dr} \left(h^{(o)} - \frac{GM}{r}\right) = 0 \\ &\frac{dp^{(o)}}{dr} = \rho^{(o)} \frac{d}{dr} \frac{GM}{r} \end{aligned}\right\} \qquad (132)$$

The solution of (132), with (131), is

$$h^{(o)} = h_{\infty} + \frac{GM}{r}$$

$$\frac{p^{(o)}}{p_o} = \left[\frac{1 + \frac{2}{5}\alpha a/r}{1 + \frac{2}{5}\alpha}\right]^{2/5} \qquad (133)$$

where we can evaluate T_1 for this case,

$$T_1 = T_{\infty} + \frac{2}{5}\alpha \qquad (134)$$

An ordering investigation of (127)-(130) indicates the outer expansion proceeds in the following way

$$p = p^{(o)} + (Re^{-\frac{1}{2}}/\sigma)\, p^{(1)} + \ldots$$

$$h = h^{(o)} + (Re^{-\frac{1}{2}}/\sigma)\, h^{(1)} + \ldots$$

$$\rho = \rho^{(o)} + (Re^{-\frac{1}{2}}/\sigma)\, \rho^{(1)} + \ldots$$

Putting these into (127)-(131) gives the following equations,

$$\frac{dp^{(1)}}{dr} = \rho^{(1)} \frac{d}{dr} \left(\frac{GM}{r}\right) \qquad (135a)$$

$$p^{(1)} = \frac{2}{5}[\rho^{(o)}h^{(1)} + \rho^{(1)}h^{(o)}] \tag{135b}$$

$$\rho^{(o)}u^{(o)}\frac{dh^{(1)}}{dr} = \rho_o\frac{\Omega a^2}{\omega+1}\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\frac{h^{(o)\omega+1}}{h_o^{\omega}} \tag{135c}$$

which are to be solved with the boundary conditions

$$p^{(1)}(0,\theta) = 0,\ h^{(1)}(\infty,\theta) = 0,\ u^{(o)}(a,\theta) = -\Omega a E(\theta). \tag{135d}$$

Notice that (135c) requires that $u^{(o)}$ be a function of r only. In that case, (128) is exceedingly simple,

$$\frac{1}{r}\frac{dr^2u^{(o)}}{dr} + \frac{1}{\sin\theta}\frac{\partial v^{(o)}\sin\theta}{\partial\theta} + ru^{(o)}\frac{d}{dr}\log(\rho^{(o)}/\rho_o) = 0$$

Putting $u^{(o)} = (a/r)^2\frac{\rho_o}{\rho^{(o)}}F(r)$ and solving the above equation for $v^{(o)}$, requiring it to be regular on $\theta = 0$ and $\theta = \pi$, gives

$$\begin{aligned} v_{\pm}^{(o)} &= \frac{\cos\theta \mp 1}{\sin\theta}\frac{a^2}{r}\frac{\rho_o}{\rho^{(o)}}\frac{dF}{dr} \\ u^{(o)} &= (\frac{a}{r})^2\frac{\rho_o}{\rho^{(o)}}F(r) \end{aligned} \tag{136}$$

where ± denotes the solution for $\theta \lessgtr \pi/2$.

From (136), it follows that $v^{(o)}$ has a jump across $\pi/2$,

$$[v^{(o)}(r,\frac{\pi}{2})] = \frac{2a^2}{r}\frac{\rho_o}{\rho^{(o)}}\frac{dF}{dr} \tag{136'}$$

The value of $[v^{(o)}(r,\pi/2)]$ can be found from the jet solution given in Section E, so F is determined to within a constant. Hence, regarding $[v^{(o)}]$ as known,

$$F(r) = F(a) + \frac{1}{2}\int_a^r \frac{\rho^{(o)}(\xi)}{\rho_o}[v^{(o)}(\xi)]\frac{\xi d\xi}{a^2} \tag{137}$$

and so

$$u^{(o)} = \left(\frac{a}{r}\right)^2 \frac{\rho_o}{\rho^{(o)}} F(a) + \frac{1}{2}\int_a^r \frac{\rho^{(o)}(\xi)}{\rho^{(o)}(r)} [v^{(o)}(\xi)] \frac{\xi d\xi}{r^2} \tag{138}$$

If we knew $u^{(o)}(a)$, then

$$F(a) = T_1^{-1} u^{(o)}(a,\theta) \tag{139}$$

The boundary condition (135d) is inconsistent with (139), hence between the viscous boundary layer of A and this outer flow, there must be an adjustment in the radial velocity as well as the temperature. Hence, one might expect an inviscid layer to adjust u and a thermal boundary layer for the temperature discontinuity; how in fact these adjustments are made can be found in C and D. Since F(a) is just a number, it is clear that, since there is no place else for the mass flux to go,

$$F(a) = -\Omega a \int_o^{\pi/2} E(\theta)\sin\theta d\theta \tag{140}$$

Having found F(r) from $[v^{(o)}]$ and F(a) from (140), the velocity field is complete and all that is left to solve is just

$$\frac{\rho^{(o)}F(r)}{\rho_o \Omega a^2} \frac{dh^{(1)}/h_o}{dr} = \frac{\omega}{r^4}\left(\frac{2\alpha}{5T_1}\right)^2 \left[\frac{T_1}{T_\infty + \frac{2}{5}\frac{\alpha a}{r}}\right]^{1-\omega}$$

and

$$\frac{2}{5}\frac{d}{dr}\left\{\rho^{(o)}h^{(1)} + h^{(o)}\rho^{(1)}\right\} = \rho^{(1)}\frac{d}{dr}\left(\frac{GM}{r}\right)$$

subject to

$$h^{(1)}(\infty,\theta) = 0, \quad p^{(1)}(a) = 0.$$

One can easily show that the streamlines are curves on which $F(r)[1 \mp \cos\theta] = \text{const}$, for $\theta \lessgtr \pi/2$. A sketch of these lines is given in Figure 2; they will be qualitatively unchanged so long as F(r) is a decreasing function of r on $r > a$.

CASE 11, $\sigma = O(Re^{-\frac{1}{2}})$

Presumably, if $\sigma = O(Re^{-\frac{1}{2}})$, convection and diffusion of heat are of equal importance, so (130) gives

$$\frac{\rho^{(o)}}{\rho_o}\frac{u^{(o)}}{a^2}\left[\frac{dh^{(o)}}{dr} + \frac{GM}{r^2}\right] = \left(\frac{Re^{-\frac{1}{2}}}{\sigma}\right)\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\frac{h^{(o)\omega+1}/h_o^{\omega}}{\omega+1} \tag{141}$$

The same arguments given in CASE 1 apply here as well to the velocity field, so (138), (140) are to be used to calculate $u^{(o)}$ and $v^{(o)}$. The difficulty is that the kernel of (138) is not known; equations (127) and (129) give

$$p^{(o)} = p_o \exp\left\{-\int_a^r \frac{5GM}{2\lambda^2 h^{(o)}(\lambda)}d\lambda\right\} \tag{142a}$$

so the kernel is easily found to be such that

$$u^{(o)} = \left(\frac{a}{r}\right)^2 F(a)\,\frac{h^{(o)}(r)}{h_o}\,e^{-\int_a^r \frac{5GMd\lambda}{2\lambda^2 h^{(o)}(\lambda)}}$$
$$+ \frac{1}{2}\int_a^r [v^{(o)}(\xi)]\,\frac{h^{(o)}(r)}{h^{(o)}(\xi)}\,e^{-\int_r^{\xi} \frac{5GMd\lambda}{2\lambda^2 h^{(o)}(\lambda)}}\,\frac{\xi d\xi}{r^2} \tag{142b}$$

Substitution of (142b) into (141) gives a hopelessly complex integro-differential equation for $h^{(o)}$. Clearly, the equation is of sufficiently high order to impose ($T_1 \equiv 1$)

$$h^{(o)}(a,\theta) = h_o,\; h^{(o)}(\infty,\theta) = T_\infty h_o$$

Hence, the only thing that prevents this solution from being uniformly valid into the edge of the velocity boundary layer is that, again, we require an adjustment in $u^{(o)}$. [See Section E.]

CASE 111, $Re^{-\frac{1}{2}} >> \sigma >> Re^{-3/2}$

A look at (130) shows that diffusion of heat is now dominant, hence, just as given in another context in (100),

$$\frac{h^{(o)}}{h_o} = [1 + (1-\frac{a}{r})s]^{\frac{1}{\omega+1}}, \quad s \equiv T_\infty^{\omega+1} - 1$$

$$\frac{p^{(o)}}{p_o} = \exp\left\{-\alpha\,\frac{\omega+1}{\omega}\left[\left\{1 + (1-\frac{a}{r})s\right\}^{\frac{\omega}{\omega+1}} - 1\right]\right\} \tag{143}$$

To complete the solution to the next order, one notes that temperature perturbations are $O(\sigma Re^{\frac{1}{2}})$ from (130) and if $\sigma >> Re^{-3/2}$, then (135a) and (135b) apply to the $(\;)^{(1)}$ variables with

$$p = p^{(o)} + \sigma Re^{\frac{1}{2}}\, p^{(1)} + \ldots$$

$$\rho = \rho^{(o)} + \sigma Re^{\frac{1}{2}}\, \rho^{(1)} + \ldots$$

$$h = h^{(o)} + \sigma Re^{\frac{1}{2}}\, h^{(1)} + \ldots$$

and gives the energy equation

$$\frac{\rho^{(o)}u^{(o)}}{\rho_o \Omega a^2}\left[\frac{dh^{(o)}}{dr} - \frac{d}{dr}\,\frac{GM}{r}\right] = \frac{1}{r^2}\,\frac{d}{dr}\,r^2\,\frac{d}{dr}\left(\frac{h^{(o)}}{h_o}\right)^{\omega} h^{(1)} \tag{144}$$

$$h^{(1)}(a,\theta) = h^{(1)}(\infty,\theta) = 0$$

so

$$s\Omega(\omega+1)\left(\frac{h^{(o)}}{h_o}\right)^{\omega} h^{(1)}/h_o = \frac{1}{a}\int_{1+s}^{1+s(1-a/r)} K(\lambda)(\lambda-1)d\lambda - \frac{1}{r}\int_{1+s}^{1} K(\lambda)(\lambda-1)d\lambda$$

(145a)

where $K(\lambda) = \left\{\frac{2}{5}\alpha \frac{\omega+1}{s} + \lambda^{-\omega/\omega+1}\right\} F\left(\frac{as}{1+s-\lambda}\right)$ (145b)

The pressure field is a solution of

$$\frac{r^2 h^{(0)}}{GM}\frac{dp^{(1)}}{dr} + p^{(1)} = \rho^{(o)} h^{(1)}$$
$$p^{(1)}(a,\theta) = 0 \qquad (145c)$$

Of course, (136)-(138) still apply in this situation, and again, a thermal layer is obviously unnecessary, but a $u^{(o)}$ adjustment layer remains important to determine a uniformly valid solution.

The $\sigma \gg Re^{-3/2}$ constraint is what is required for the thermodynamic quantities to be $O(Re^{\frac{1}{2}}\sigma)$. If σ drops below $Re^{-3/2}$, the $(\ )^{(1)}$ quantities are $O(Re^{-1})$ and are <u>not</u> hydrostatic, but dynamical. Hence, we are led to the next case.

CASE IV, $\sigma \leqslant Re^{-3/2}$

For reasons just delineated, we begin the outer expansions

$$p = p^{(o)} + Re^{-1}\, p^{(1)} + \ldots$$
$$\rho = \rho^{(o)} + Re^{-1}\, \rho^{(1)} + \ldots$$
$$h = h^{(o)} + \sigma Re^{\frac{1}{2}}\, h^{(1)} + \ldots$$

Of course, (143) remains valid here, but the perturbation equations are quite different, viz.,

$$\rho^{(o)}(\underline{u}^{(o)}\cdot\nabla)\underline{u}^{(o)} + \nabla p^{(1)} = \rho^{(1)}\nabla\left(\frac{GM}{r}\right) \qquad (146a)$$

$$\rho^{(o)} u^{(o)} \frac{d}{dr}\left[h^{(o)} - \frac{GM}{r}\right] = \rho_o \Omega a^2 \nabla^2\left(\frac{h^{(o)}}{h_o}\right)^{\omega} h^{(1)} \qquad (146b)$$

$$p^{(1)} = \frac{2}{5} h^{(o)} \rho^{(1)} + \frac{2}{5}(\sigma Re^{3/2})\, \rho^{(o)} h^{(1)} \qquad (146c)$$

The appropriate boundary conditions are

$$\begin{aligned} &p^{(1)}(a,\theta) = h^{(1)}(a,\theta) = h^{(1)}(\infty,\theta) = 0 \\ &v^{(o)}(0,r) = v^{(o)}(\pi,r) = 0 \\ &[v^{(o)}(\tfrac{\pi}{2},r)] \text{ given from jet solution} \\ &u^{(o)}(a,\theta) = -\Omega a E(\theta) \end{aligned} \qquad (146d)$$

A subcase of this case is when $\sigma = o\,(Re^{-3/2})$; in that eventuality (146b) is irrelevant and (146a) becomes just

$$(\underline{u}^{(o)}\cdot \nabla)\,\underline{u}^{(o)} + \frac{2}{5}\,h^{(o)}\,\nabla(p^{(1)}/p^{(o)}) = 0 \qquad (147a)$$

with (128),

$$h^{(o)}\nabla\cdot\underline{u}^{(o)} + u^{(o)}\,\frac{d}{dr}\left\{\frac{5}{2}\,\frac{GM}{r} - h^{(o)}\right\} = 0 \qquad (147b)$$

If r is sufficiently large, it is trivial to show that (147) goes to the incompressible form

$$\begin{aligned} &(\underline{u}^{(o)}\cdot \nabla)\,\underline{u}^{(o)} + \nabla[\tfrac{2}{5}\,h_o T_\infty p^{(1)}/p^{(o)}] = 0 \\ &\nabla\cdot\underline{u}^{(o)} = 0 \end{aligned} \qquad (148)$$

One can learn a lot from the curl of (147a)

$$\begin{aligned} &(\underline{u}^{(o)}\cdot \nabla)\underline{\omega}^{(o)} + u^{(o)}\left[\frac{2}{5}\,\frac{GM}{h^{(o)}r^2} + \frac{d}{dr}\log h^{(o)}\right]\underline{\omega}^{(o)} \\ &= \frac{2}{5}\,\frac{1}{r}\,\frac{dh^{(o)}}{dr}\;\underline{r}\times\nabla(p^{(1)}/p^{(o)}),\quad \underline{\omega}^{(o)} \equiv \text{curl}\,\underline{u}^{(o)} \end{aligned}$$

namely, that the flow is highly rotational, vorticity being generated by $\partial(p^{(1)}/p^{(o)})/\partial\theta \neq 0$ and $dh^{(o)}/dr \neq 0$.

This result is perhaps more what one might expect in this problem, but this expected sort of $\sigma \to 0$, $Re \to \infty$ result will occur only for $\sigma \ll Re^{-3/2}$.

C. The Thermal Layer

A study of the outer flow showed that one needs a thin boundary layer on the sphere, external to the velocity boundary layer, which removes what is found to be a jump in the temperature of the fluid; such a layer, as is easily seen from B, is necessary only when σ is in the range covered by CASE 1. For purposes of studying this layer, let $u = \phi(Re,\sigma)\Omega a\tilde{u}$, $v = \frac{\phi}{\delta}\chi(Re,\sigma)\Omega a\tilde{v}$ where δ is the layer thickness, $(r-a) = a\delta\eta$, and $\rho = \tilde{\rho}\rho_o$, $h = \tilde{h}h_o$. Then ϕ, δ, χ are functions of Re and σ such that ord $\chi \leq 1$, $Re^{-\frac{1}{2}} <$ ord $\delta < 1$, and $1 >$ ord$\phi \geq Re^{-\frac{1}{2}}$. The equations that result are

$$(\tilde{\rho}\,\tilde{u}\frac{\partial}{\partial\eta} + \chi\tilde{\rho}\,\tilde{v}\frac{\partial}{\partial\theta})(\tilde{h} - \frac{2}{5}\frac{GM}{RaT_o}) = \frac{Re^{-1}}{\sigma\phi\delta}\frac{\partial}{\partial\eta}\tilde{h}\frac{\partial\tilde{h}}{\partial\eta} \tag{149a}$$

$$\frac{\partial\tilde{\rho}\tilde{u}}{\partial\eta} + \chi\frac{\partial\tilde{\rho}\tilde{v}\sin\theta}{\sin\theta\,\partial\theta} = 0 \tag{149b}$$

$$\frac{\partial}{\partial\eta}\left[\tilde{\rho}\,\tilde{u}\frac{\partial\tilde{v}}{\partial\eta} + \tilde{\rho}\,\tilde{v}\chi\frac{\partial\tilde{v}}{\partial\theta}\right] = \frac{GM}{\Omega^2 a^3}\left(\frac{\delta^3}{\phi^2\chi}\right)\frac{\partial\tilde{\rho}}{\partial\theta} \tag{149c}$$

$$\tilde{\rho}\,\tilde{h} = 1 \tag{149d}$$

We now proceed to an investigation of these equations in what turns out to be three regimes in σ for σ in $(1, Re^{-\frac{1}{2}})$.

(i) $1 \gg \sigma \gg Re^{-1/6}$

This structure is delineated with $\phi = Re^{-\frac{1}{2}}$, which is an obvious choice but not necessarily correct a priori. Then (149a) is

$\delta = Re^{-\frac{1}{2}}/\sigma \ll 1$. We know that the rotational layer is external to this thermal layer. Velocity matching clearly requires

$$\chi \frac{Re^{-\frac{1}{2}}}{\delta} = \frac{Re^{-\frac{1}{2}}}{\varepsilon} \quad \text{or} \quad \chi = \frac{\delta}{\varepsilon} \ll 1.$$

With this choice,

$$\frac{\delta^3}{\phi^2 \chi} = \frac{\varepsilon \delta^2}{Re^{-1}} = \frac{\varepsilon}{\sigma} \ll 1$$

since it will be shown in Section E(i) that $\varepsilon = Re^{-1/8}\sigma^{1/4}$. Hence, the leading order equations are

$$-E(\theta)\frac{\partial \tilde{h}}{\partial \eta} = \frac{\partial}{\partial \eta}\, \tilde{h}^{\omega} \frac{\partial \tilde{h}}{\partial \eta} \tag{150a}$$

$$\tilde{\rho}\, \tilde{u} = E(\theta) \tag{150b}$$

$$\frac{\partial^2 \tilde{v}}{\partial \eta^2} = 0 \tag{150c}$$

$$\tilde{\rho}\, \tilde{h} = 1 \tag{150d}$$

The general solution is seen to be, for $R \equiv \frac{\tilde{h} - T_1}{1 - T_1}$,

$$\frac{E(\theta)\eta}{T_1^{\omega}} = \int_R^1 [1 + (\frac{1}{T_1} - 1)\lambda]^{\omega} \frac{d\lambda}{\lambda} \tag{151a}$$

which has a simple representation for $\omega = 1$,

$$\frac{1-T_1}{T_1}(1-R) - \log R = \frac{E(\theta)\eta}{T_1}$$

Also, the solution of (150c) that matches to the rotational layer is

$$\tilde{v} = \alpha(\theta) = \bar{v}(0, \theta) .$$

From (149b), there is a $\tilde{u}$ velocity of $O(\chi)$, as well as $\tilde{\rho}$ and $\tilde{h}$

perturbation of $O(\chi)$. So, denoting these $O(\chi)$ perturbations by $(\;)'$,

$$\frac{\partial \rho' \tilde{u} + u' \tilde{\rho}}{\partial \eta} + \frac{1}{\sin\theta} \frac{\partial \tilde{\rho} \sin\theta}{\partial \theta} = 0$$

So, integrating,

$$\rho' \tilde{u} + u' \tilde{\rho} = - \frac{1}{\sin\theta} \int_c^{\eta} \frac{\partial}{\partial \theta} \tilde{\rho} \tilde{v} \sin\theta \, d\eta$$

and, as $\eta \to \infty$, $\rho' \to 0$, as $u' \sim - \frac{1}{\sin\theta} \eta \frac{d}{d\theta} \tilde{v} \sin\theta$, which matches with $\xi \frac{\partial \bar{u}}{\partial \xi}(0, \theta)$ in the rotational layer, so things go through easily.

(ii) <u>$\sigma = O(Re^{-\frac{1}{2}})$</u>

This is a distinguished value of σ. Here, $\chi = O(1)$, and the two layers are merged, so

$$\tilde{\rho}\, \tilde{u} \frac{\partial \tilde{h}}{\partial \eta} + \chi\, \tilde{\rho}\, \tilde{v} \frac{\partial \tilde{h}}{\partial \theta} = \frac{\partial}{\partial y} \tilde{h}^{\omega} \frac{\partial \tilde{h}}{\partial \eta} \qquad (152a)$$

$$\frac{\partial \tilde{\rho} \tilde{u}}{\partial \eta} + \frac{\chi}{\sin\theta} \frac{\partial \tilde{\rho}\, \tilde{v} \sin\theta}{\partial \theta} = 0 \qquad (152b)$$

$$\frac{\partial}{\partial \eta} (\tilde{\rho}\, \tilde{u} \frac{\partial \tilde{v}}{\partial \eta} + \tilde{\rho}\, \tilde{v}\, \chi \frac{\partial \tilde{v}}{\partial \theta}) = \frac{\partial \tilde{p}}{\partial \theta} \qquad (152c)$$

$$\tilde{\rho}\, \tilde{h} = 1 \qquad (152d)$$

In this particular case, the thermal layer and the rotational layer (discussed in Section E) are merged into a complicated inviscid, rotational, heat-conducting flow. Hence, one can expect to be able to satisfy the boundary conditions

$$\tilde{u}(\infty,\theta) = u^{(o)}(a)/\Omega a$$

$$\tilde{u}(0,\theta) = -E(\theta)$$

$$\tilde{\rho}(0,\theta) = \tilde{h}(0,\theta) = 1 \tag{153}$$

$$\tilde{h}(\infty,\theta) = T_1$$

$$\tilde{v}(\infty,\theta) = \tilde{v}(\eta, 0) = \tilde{v}(\eta,\tfrac{\pi}{2}) = \tilde{v}(\eta,\pi) = 0$$

Some comments on the structure of the solutions to (152), (153) and the well-posed nature of the problem may be found in Appendix D, including the as asymptotic solution as $\eta \rightarrow \infty$, viz.,

$$\tilde{h} \sim T_1 + G(\theta)\, e^{\frac{u^{(o)}(a)\eta}{T_1^{\omega+1}\Omega a}}$$

$$\tilde{v} \sim -\frac{T_1^{2\omega+1}\Omega^3 a^3}{[u^{(o)}(a)]^3}\frac{dG}{d\theta}\, e^{\frac{u^{(o)}(a)\eta}{T_1^{\omega+1}\Omega a}} \tag{154}$$

$$\tilde{u} - u^{(o)}(a) \sim \frac{T_1^{3\omega+2}\Omega^4 a^4}{[u^{(o)}(a)]^4}\ \frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dG}{d\theta}\, e^{\frac{u^{(o)}(a)\eta}{T_1^{\omega+1}\Omega a}}$$

where $G(\theta)$ is arbitrary in the asymptotic result, but clearly related to a single function $E(\theta)$, as is $u^{(o)}(a) < 0$.

(iii) $\underline{Re^{-1/6} >> \sigma >> Re^{-\frac{1}{2}}}$

This final regime may be seen by putting $O(\chi) \leq 1$ in (149) and doing the appropriate limits for $\phi = Re^{-\frac{1}{2}}$. Hence, $\delta = Re^{-\frac{1}{2}}/\sigma$ again. Equation (149c) is

$$\frac{\partial\tilde{\rho}}{\partial\theta} = 0 \tag{155}$$

so (149d) gives $\tilde{h} = \tilde{h}(\eta)$, and hence (149a) is

$$\tilde{\rho}\,\tilde{u}\frac{\partial}{\partial\eta}(\tilde{h} - \frac{2}{5}\,GM/aRT_o) = \frac{\partial}{\partial\eta}\,\tilde{h}^{\omega}\,\frac{\partial\tilde{h}}{\partial\eta} \tag{156}$$

so clearly $\tilde{\rho}\,\tilde{u}$ depends only on η as well, so (149b) gives

$$\chi\tilde{\rho}\,\tilde{v} = \frac{\cos\theta \mp 1}{\sin\theta}\;\frac{d\tilde{\rho}\tilde{u}}{d\eta} \quad \text{for} \quad \theta \lesseqgtr \pi/2 \tag{157}$$

However, we would like $\tilde{v}$ to vanish on $\pi/2$ as well; this is clearly impossible from (157), hence (157) is to be replaced by

$$\frac{d\tilde{\rho}\tilde{u}}{d\eta} = 0 \tag{158}$$

which means $O(\chi) < 1$ and (156) is simply

$$\frac{\partial}{\partial\eta}\,\tilde{h}^{\omega}\,\frac{\partial\tilde{h}}{\partial\eta} - \frac{F(a)}{\Omega a}\;\frac{\partial\tilde{h} - GM/aRT_o}{\partial\eta} \tag{159}$$

the solution of which is just that given in (i), but with $E(\theta)$ replaced by $-F(a)$, viz.,

$$\frac{F(a)}{\Omega a T_1^{\;\omega}}\,\eta = \int_1^R [1 + (\frac{1}{T_1} - 1)\lambda]^{\omega}\,\frac{d\lambda}{\lambda} \tag{160}$$

Just what $O(\chi)$ is will be determined from higher approximations and matching as $\eta \to 0$ and $\eta \to \infty$, it is unnecessary to determine it to carry out the leading order solution.

In contrast to what happened in (i), where the rotational layer is external to the thermal layer, here, the implication of (158) is that the rotational layer is interior to this thermal layer but outside the velocity boundary layer.

For future use (in Section E), the Taylor series expansion for $\tilde{h}$ near $\eta = 0$ is, from (160),

$$R = 1 + \frac{F(a)\eta}{\Omega a} + \frac{1}{2!}\frac{F^2(a)}{(\Omega a)^2}[\omega - 1 - \omega T_1]\eta^2 + \ldots \quad (161)$$

D. The Equatorial Jet

The fact that regularity of the solutions for the outer flow in Cases 1-111 required a jump in v across $\theta = \pi/2$ is really a secondary indication of the fact that a swirling laminar jet exists in the $\theta = \frac{\pi}{2}$ plane to carry mass entrained by the boundary layer and angular momentum to infinity. This is in marked contrast to the slow rotation solutions which had angular momentum diffusing to infinity in the entire fluid; here the angular momentum is convected away in a thin sheet. In 1955, Squire[(5)] computed the structure of such a jet in a uniform environment, without swirl; the solution presented here is a logical but by no means trivial extension of Squire's work.

One might expect that buoyancy forces in the environment of the jet to limit its flow to some finite height. Thus far, the reason for the $\sigma << 1$ restriction has not been clear; the boundary layer and the outer flow can be done for all σ. However, if the jet extends only to a finite height, it represents a ring source of warm swirling gas to the outer flow; one supposes that the streamlines close in that case, but this can easily be shown to violate entropy and angular momentum conservation for the fluid. Thus, unless $\sigma << 1$, one cannot piece together a self-consistent solution which probably means (124) is no longer correct and perhaps even (116)-(121) are the wrong equations in that case.

If $\sigma << 1$, by the arguments of A, heat diffusion is dominant in such a layer of thickness $Re^{-\frac{1}{2}}$. So,

$$\frac{\partial p}{\partial \theta} = \frac{\partial h}{\partial \theta} = 0 \tag{162}$$

Thus, the thermal conductivity is large enough to assume that the pressure and temperature and hence density are at every point in the jet their respective values just outside. The reason why the jet has thickness $O(Re^{-\frac{1}{2}})$ is the following: The mass flux in the boundary layer on the sphere is $O(\rho_o \Omega a^2 \sqrt{\nu_o \Omega})$, and the angular momentum flux is $O(\rho_o \sqrt{\nu_o \Omega^3}\, a^4)$. Further, if one is to have a jet, the inertial terms balance the viscous terms, so, if u is the velocity along the jet and δ is the width, $u\delta \propto \nu_o^{\frac{1}{2}}$ by the mass flux order, and to balance diffusion and inertia requires $u \propto \nu_o/\delta^2$ which together give $u \alpha 1$ and $\delta \propto \nu_o^{\frac{1}{2}}$. Similarly, w is O(1), so the equations of motion are just

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \theta} - \frac{w^2}{r} = \frac{\nu^{(o)}}{r^2}\frac{\partial^2 u}{\partial \theta^2} \tag{163a}$$

$$u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \theta} + \frac{uw}{r} = \frac{\nu^{(o)}}{r^2}\frac{\partial^2 w}{\partial \theta^2} \tag{163b}$$

$$\frac{\partial \rho^{(o)} r^2 u}{\partial r} + \frac{\partial r \rho^{(o)} v}{\partial \theta} = 0 \tag{163c}$$

where $(\ )^{(o)}$ denotes the value outside the boundary layer. These equations contain no streamwise pressure gradient because the outer flow is hydrostatic, i. e.,

$$\nabla p^{(o)} = \rho^{(o)} \nabla\left(\frac{GM}{r}\right)$$

and hence the pressure gradient exactly balances the gravitational force in the jet as well as in the outer flow. Equations (163) may be transformed to nearly incompressible form by a simple Howarth-Doronitzyn transformation,

$$\eta = \frac{\rho^{(o)}}{\rho_o}\theta, \quad r = r$$
$$\hat{v} = \frac{\rho^{(o)}}{\rho_o}\left[v + \theta u r \frac{d \log \rho^{(o)}}{dr} \right] \tag{164}$$

which modifies the equations to

$$u \frac{\partial u}{\partial r} + \frac{\hat{v}}{r} \frac{\partial u}{\partial \eta} - \frac{w^2}{r} = \frac{\nu_o}{r^2} Q(r) \frac{\partial^2 u}{\partial \eta^2} \tag{165a}$$

$$u \frac{\partial w}{\partial r} + \frac{\hat{v}}{r} \frac{\partial w}{\partial \eta} + \frac{uw}{r} = \frac{\nu_o}{r^2} Q \frac{\partial^2 w}{\partial \eta^2} \tag{165b}$$

$$\frac{1}{r} \frac{\partial r^2 u}{\partial r} + \frac{\partial \hat{v}}{\partial \eta} = 0 \tag{165c}$$

where $Q(r) = \frac{p^{(o)}}{p_o} (h^{(o)}/h_o)^{1-\omega}$. Defining $x = r/a$ and using a stream-function ψ,

$$u = \frac{1}{x^2 a^2} \psi_\eta, \quad \hat{v} = -\frac{1}{xa^2} \psi_x \tag{166}$$

(165) reduces to

$$\psi_\eta \psi_{x\eta} - \frac{2}{x} \psi_\eta^2 - \psi_x \psi_{\eta\eta} - a^2 x W^2 = a \nu_o Q \psi_{\eta\eta\eta}$$
$$\psi_\eta W_x - \psi_x W_\eta = a \nu_o Q W_{\eta\eta} \tag{167}$$

where $W = axw$. The details of the solution to (167) by a similarity method are given in Appendix C. Only the results are given here for brevity.

If one writes

$$\psi = a^2\sqrt{\nu_o\Omega}\; A(x)\; F(\xi)$$
$$W = \Omega a^2\; B(x) F'(\xi) \qquad (168)$$

then the solution is found to be

$$F(\xi) = \tanh\xi, \quad \xi \equiv \eta/\delta(x) \qquad (169a)$$

$$B(x) = \frac{3H}{4A(x)}, \quad \delta(x) = \left(\frac{\nu_o}{\Omega a^2}\right)^{\frac{1}{2}} \frac{2Q(x)}{A'(x)} \qquad (169b, c)$$

where the function A(x) is found to be,

$$A(x) = \tfrac{1}{2} m \left\{ 1 + \frac{36}{m^3} \int_1^x \lambda^2 Q(\lambda) \sqrt{M^2 + H^2(1-\lambda^{-2})}\; d\lambda \right\}^{1/3} \qquad (169d)$$

The solution as given in (169) is dependent on three parameters which involve the initial mass flux, momentum flux, and angular momentum flux; the numbers come from the details of the boundary layer solution and in particular on structure of the solutions in the turning region $r = O(a)$, $\theta = O(\pi/2)$, discussed by Stewartson,[6] and in E. Define

$$m = \int_{-\infty}^{\infty} \frac{u(a,\theta)}{\sqrt{\nu_o\Omega}}\, d\theta \qquad (170a)$$

$$M = \int_{-\infty}^{\infty} \frac{u^2(a,\theta)}{\sqrt{\nu_o\Omega^3 a^2}}\, d\theta \qquad (170b)$$

and $$H = \int_{-\infty}^{\infty} \frac{u(a,\theta) w(a,\theta)}{\sqrt{\nu_o\Omega^3 a^2}}\, d\theta \qquad (170c)$$

The asymptotic structure is shown to be of the form

$$A(x) \sim \left(\frac{3}{2}\right)^{1/3} [Q(\infty)]^{1/3} (M^2+H^2)^{1/6} \; x$$

$$\delta(x) \sim \left(\frac{\nu_o}{\Omega a^2}\right)^{\frac{1}{2}} \left(\frac{16}{3}\right)^{1/3} \frac{[Q(\infty)]^{2/3}}{(M^2 + H^2)^{1/6}} \tag{171}$$

The actual physical thickness is asymptotic to

$$\left(\frac{\nu_o}{\Omega}\right)^{\frac{1}{2}} \times \left(\frac{16}{3}\right)^{1/3} \frac{\rho_o}{\rho^{(o)}(\infty)} \frac{[Q(\infty)]^{2/3}}{(M^2 + H^2)^{1/6}} \tag{172}$$

In Cases 1, 11, and 111 respectively, the quantity $[Q(\infty)]^{2/3}\rho_o/\rho^{(o)}(\infty)$ is

1. $T_\infty^{\frac{5-2\omega}{3}} \left[1 + \frac{2}{5}\frac{GM}{aRT_o}\right]^{-5/6}$

11. $T_\infty^{\frac{5-2\omega}{3}} \exp\left\{\frac{5}{6}\int_o^\infty \frac{GM}{\lambda^2} h^{(o)-1}(\lambda)d\lambda\right\}$

111. $T_\infty^{\frac{5-2\omega}{3}} \exp\left\{\frac{GM}{3aRT_o}\frac{\omega+1}{\omega}(T_\infty^\omega - 1)\right\}$

So, since boundary layer theory applies only when the layer is thin, (172) and these give respectively what are apparently in some cases more stringent conditions on the solutions than (114), i. e.,

1. $Re^{\frac{1}{2}} \gg T_\infty^{\frac{5-2\omega}{3}} \left[1 + \frac{2}{5}\frac{GM}{aRT_o}\right]^{-5/6}$

11. $Re^{\frac{1}{2}} \gg T_\infty^{\frac{5-2\omega}{3}} \exp\left\{\frac{5}{6}\int_o^\infty \frac{GM}{\lambda^2}\frac{1}{h^{(o)}(\lambda)}d\lambda\right\}$ (173)

111. $Re^{\frac{1}{2}} \gg T_\infty^{\frac{5-2\omega}{3}} \exp\left\{\frac{GM(\omega+1)}{3aRT_o\omega}[T_\infty^\omega - 1]\right\}$

The quantity of importance in determining the structure of the outer flow is $v(\infty, r) - v(-\infty, r) \equiv [v]$. With (143) one has

$$F(r) - F(a) = -\Omega a \int_1^x \frac{\rho^{(o)}(x)}{\rho_o} \frac{dA}{dx} dx$$

If x is sufficiently large, $\rho^{(o)}/\rho_o \rightarrow$ const, so

$$F(r) \sim -\Omega a \frac{\rho^{(o)}(\infty)}{\rho_o} A(x)$$

$$\sim -\Omega r \left(\frac{3}{2}\right)^{1/3} \frac{\rho^{(o)}(\infty)}{\rho_o} [Q(\infty)]^{1/3} (M^2 + H^2)^{1/6}$$

which gives the asymptotic structure of the flow for CASES 1-111, the streamfunction being just $r(1 \mp \cos\theta) = $ const in $\theta \lessgtr \pi/2$.

E. The Inviscid Rotational Layers

Recall that, in Section B, we found it impossible to match the boundary layer entrainment to the outer flow in CASES 1-111, hence the hypothesis of another thin layer to adjust the mass flow to that distribution required by the boundary layer. In addition, we found that CASE 1 breaks into three subcases, (ii) having a merged thermal layer and rotational layer. Thus, we have four distinct structures to study, viz. CASE 1, (i) and (iii), CASE 11, and CASE 111. It is convenient to leave the equations in physical variables. Certainly the layer is hydrostatic,

$$\frac{\partial p}{\partial z} = -\frac{GM}{a} \rho \qquad (174a)$$

Also, since the temperature is uniform to leading order, the mass conservation has the form

$$\frac{\partial u}{\partial z} + \frac{1}{\sin\theta} \frac{\partial v \sin\theta}{\partial \theta} = 0 \tag{174b}$$

There must be a pressure gradient along the sphere, so the horizontal inertia balances $\partial p/\partial\theta$, and cross differentiation with (174a) gives just

$$\rho_R \frac{\partial}{\partial z} \left(u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial \theta}\right) = \frac{GM}{a} \frac{\partial \rho'}{\partial \theta} \tag{174c}$$

If (174c) is satisfied, it is quite easy to show, from the equation of state, that

$$\rho_R \frac{\partial h'}{\partial \theta} + h_R \frac{\partial \rho'}{\partial \theta} = 0 \tag{174d}$$

where ρ_R and h_R are reference values to be determined in the particular case studied, and ( )' means a perturbation quantity. Eliminating (174d),

$$h_R \frac{\partial}{\partial z} \left(u \frac{\partial v}{\partial z} + v \frac{\partial v}{\partial \theta}\right) + \frac{GM}{a} \frac{\partial h'}{\partial \theta} = 0 \tag{174e}$$

As it happens, it is simplest to write

$$h = \overline{h}(r) + \frac{Re^{-1}}{\varepsilon^3} \hat{h} (\xi, \theta) \tag{175}$$

where $z = \varepsilon \xi$. This is essentially a two-variable expansion, i. e.,

$$\frac{\partial h}{\partial z} = a \frac{d\overline{h}}{dr} + \frac{Re^{-1}}{\varepsilon^4} \frac{\partial \hat{h}}{\partial \xi}$$

The energy equation is crucial, and if we use (175), the equation is

$$\begin{aligned} \rho(r) \left(u \frac{\partial}{\partial z} + v \frac{\partial}{\partial \theta}\right)\left[\overline{h} - \frac{GM}{r} + \frac{Re^{-1}}{\varepsilon^3} \hat{h}\right] \\ = \frac{1}{a\sigma} \frac{\partial}{\partial z} \mu \frac{\partial}{\partial z} \left[\overline{h}(r) + \frac{Re^{-1}}{\varepsilon^3} \hat{h}\right] \end{aligned} \tag{176}$$

Clearly, the terms are of the following order

$$O(Re^{-\frac{1}{2}}) + O(Re^{-3/2}/\varepsilon^4) = O(Re^{-1}/\sigma) + O(Re^{-2}/\sigma\varepsilon^5)$$

There are constraints on $O(\varepsilon)$ in the general case, namely $1 > O(\varepsilon) > O(Re^{-\frac{1}{2}})$. We now consider the various cases in sequence.

(i) CASE 1, [i]

Here, the thermal layer lies beneath this layer, so $\varepsilon \gg Re^{-\frac{1}{2}}/\sigma$ necessarily. These constraints imply that the second and third terms of (176) are largest, so $\varepsilon = O(\sigma Re^{-\frac{1}{2}})^{1/4}$ and the energy equation is

$$\rho_o u \left(\frac{\partial h}{\partial z} - \frac{GM}{a^2}\right)_{r=a} + \rho_o\left(u\frac{\partial}{\partial \xi} + \varepsilon v \frac{\partial}{\theta}\right)\frac{Re^{-1}}{\varepsilon^4}\hat{h} = \frac{1}{a\sigma}\frac{\partial}{\partial z}\mu\frac{\partial \bar{h}}{\partial z} \tag{177}$$

All this can be simplified by the use of the results of C, CASE 1, especially (135c) to give, with $\hat{h} = h_o\bar{h}$, $u = Re^{-\frac{1}{2}}\Omega a\bar{u}$, $v = Re^{-3/8}\sigma^{-\frac{1}{4}}\Omega a\bar{v}$, the set of equations

$$\bar{u}\frac{\partial \bar{h}}{\partial \xi} + \bar{v}\frac{\partial \bar{h}}{\partial \theta} = -\left(\frac{2\alpha}{5T_1}\right)^2 \frac{\Omega a}{F(a)}\left[\bar{u} - F(a)/\Omega a\right] \tag{178a}$$

plus

$$\frac{GM}{\Omega^2 a^3}\frac{\partial \bar{h}}{\partial \theta} + \frac{\partial}{\partial \xi}\left(\bar{u}\frac{\partial \bar{v}}{\partial \xi} + \bar{v}\frac{\partial \bar{v}}{\partial \theta}\right) = 0 \tag{178b}$$

$$\frac{\partial \bar{u}}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\bar{v}\sin\theta) = 0 \tag{178c}$$

to be solved with

$$\begin{aligned} &\bar{u}(0,\theta) = -E(\theta) \\ &\bar{u}(\infty,\theta) = F(0)/\Omega a \\ &\bar{v}(0,\theta) = \bar{v}(\xi,0) = \bar{v}(\xi,\tfrac{\pi}{2}) = \bar{v}(\xi,\pi) = 0 \\ &\bar{h}(\infty,\theta) = 0 \end{aligned} \tag{178d}$$

Now, it is shown in Appendix D that (178) is not a well-posed problem. The reason, as carried out there mathematically, is that this layer is unstable to small disturbances, so the (178) solution contains an undetermined function for good reason. That this flow is unstable is intuitively obvious, since (178a) shows that the mean temperature gradient in the layer is negative, which leads to the well-known Rayleigh instability.

(ii) CASE 1, [iii]

Here, the rotational layer is sandwiched between the velocity boundary layer and the thermal layer. In (175), $\overline{h}(r)$ is actually $\overline{h}(\eta) = \overline{h}((r-a)/(aRe^{-\frac{1}{2}}/\sigma))$, so actually, that term is $O(\sigma)$, as is term three. A little investigation of the limits indicates that the second term is irrelevant, and the final one balances the $O(\sigma)$ term. Therefore, $\varepsilon = (Re^{-1}/\sigma)^{2/5} << Re^{-\frac{1}{2}}/\sigma$ if $\sigma << Re^{-1/6}$. A little manipulation of that equation, and using (159) and (161) will yield, with $h = h_o\overline{h}$, $u = \Omega a Re^{-\frac{1}{2}}\,\overline{u}$, $v = \Omega a \sigma^{2/5} Re^{-1/10}\,\overline{v}$,

$$\frac{\partial \overline{u}}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial \overline{v}\sin\theta}{\partial \theta} = 0 \tag{180a}$$

$$\frac{\partial^2 \overline{h}}{\partial \xi^2} = \frac{F(a)}{\Omega a}\left[\,\overline{u} - \frac{F(a)}{\Omega a}\right](1-T_1) \tag{180b}$$

$$\frac{GM}{\Omega^2 a^3}\frac{\partial \overline{h}}{\partial \theta} + \frac{\partial}{\partial \xi}\left(\overline{u}\,\frac{\partial \overline{v}}{\partial \xi} + \overline{v}\,\frac{\partial \overline{v}}{\partial \theta}\right) = 0 \tag{180c}$$

which is to be solved for

$$\overline{u}(0,\theta) = -E(\theta)$$

$$\overline{u}(\infty,\theta) = F(a)/\Omega a \tag{180d}$$

$$\bar{v}(\infty,\theta) = \bar{v}(\xi, 0) = \bar{v}(\xi, \pi/2) = \bar{v}(\xi,\pi) = 0$$

(180d) Cont'd

$$\bar{h}(0, \theta) = \bar{h}(\infty, \theta) = 0$$

Appendix D exhibits the nature of the solution to the problem which is well-posed only if $T_1 > 1$, for the same reasons described in (i) and developed mathematically in Appendix D.

(iii) CASE 11

Here, $\bar{h}$ in (176) is $h^{(o)}(r)$, so, from (141), the first and third terms of (176) are $O(Re^{-1}/\sigma)$. If the layer is thin, then the second term is irrelevant, so $\varepsilon = O(Re^{-1/5})$ and hence, with $u = \Omega a Re^{-\frac{1}{2}}\bar{u}$, $v = \Omega a Re^{-4/5}\,\bar{v}$, and $\varepsilon = Re^{-1/5}$, $\hat{h} = h_o\bar{h}$ the result is just

$$\frac{a}{h_o}\,\frac{d}{dr}\left(h^{(o)} - \frac{GM}{r}\right)\Big|_{r=a}\left(\frac{Re^{-\frac{1}{2}}}{\sigma}\right)\left\{\bar{u} - \frac{F(a)}{\Omega a}\right\} = \frac{\partial^2 \bar{h}}{\partial \xi^2} \tag{182a}$$

$$\frac{\partial \bar{u}}{\partial \xi} + \frac{1}{\sin\theta}\,\frac{\partial \bar{v}\sin\theta}{\partial \theta} = 0 \tag{182b}$$

$$\frac{GM}{\Omega^2 a^3}\,\frac{\partial \bar{h}}{\partial \theta} + \frac{\partial}{\partial \xi}\left(\bar{u}\,\frac{\partial \bar{v}}{\partial \xi} + \bar{v}\,\frac{\partial \bar{v}}{\partial \theta}\right) = 0 \tag{182c}$$

which are to be solved with

$$\bar{h}(\infty, \theta) = \bar{h}(0, \theta) = 0$$

$$\bar{u}(0, \theta) = -E(\theta), \quad \bar{u}(\infty, \theta) = F(a)/\Omega a \tag{182d}$$

$$\bar{v}(\infty, \theta) = \bar{v}(\xi, 0) = \bar{v}(\xi, \tfrac{\pi}{2}) = \bar{v}(\xi, \pi) = 0$$

Again, this problem is well-posed provided

$$\frac{d}{dr}\left(h^{(o)} - \frac{GM}{r}\right)\Big|_{r=a} > 0$$

See Appendix D for details.

(iv) CASE 111

The scaling arguments are essentially those in (iii). Now, the first term is identically of the same order as the third. Balancing the fourth and first terms then gives $\varepsilon = Re^{-3/10}/\sigma^{1/5}$ and the resulting equations are the same as (182) except that it is possible to evaluate easily $\frac{d}{dr}(h^{(o)} - GM/r)\Big|_{r=a}$. Hence, with $u = \Omega a Re^{-\frac{1}{2}}\bar{u}$, $v = \Omega a Re^{-1/5}\sigma^{1/5}\bar{v}$, $\hat{h} = h_o\bar{h}$, we get

$$\left\{\frac{s}{\omega+1} + \frac{2}{5}\alpha\right\}\left[\bar{u} - \frac{F(a)}{\Omega a}\right] = \frac{\partial^2\bar{h}}{\partial\xi^2} \tag{183a}$$

$$\frac{\partial\bar{u}}{\partial\xi} + \frac{1}{\sin\theta}\frac{\partial\bar{v}\sin\theta}{\partial\theta} = 0 \tag{183b}$$

$$\frac{GM}{\Omega^2 a^3}\frac{\partial\bar{h}}{\partial\theta} + \frac{\partial}{\partial\xi}\left(\bar{u}\frac{\partial\bar{v}}{\partial\xi} + \bar{v}\frac{\partial\bar{v}}{\partial\theta}\right) = 0 \tag{183c}$$

to be solved with (182d) again. This problem is likewise discussed in Appendix D, and the layer is stable to small disturbances only for

$$T_\infty^{\omega+1} + \frac{2}{5}(\omega+1)\alpha > 1$$

Also, (183) with (182d) is well-posed only with this condition.

F. Summary

Because of the complexities of the solution for $\sigma \to 0$, the prime features are summarized here. All of the $\sigma \to 0$ solutions, CASES 1-1V, exhibit a Prandtl boundary layer of width $(\nu_o/\Omega)^{\frac{1}{2}}$, plus an equatorial jet of the same thickness. In the table below, the first phrase in the column denoted "Outer Flow" denotes the character of the thermal field off the boundaries; the second phrase refers to the character of the pressure perturbations and/or the velocity field. Under "Rotational Layer," the nature of the flow is given as well as the layer thickness, and similarly under "Thermal Layer." In these two columns, the layer marked (*) is the thinner.

Case	Regime	Outer Flow	Rotational Layer	Thermal Layer
1	$(Re^{-1/6}, 1)$	adiabatic, hydrostatic	$\varepsilon = Re^{-1/8}\sigma^{1/4}$ thermal convection	$\delta = Re^{-\frac{1}{2}}/\sigma E(\theta)$*
	$Re^{-1/6}$		$\delta = \varepsilon = Re^{-\frac{1}{2}}/\sigma$; the two layers are merged here.	
	$(Re^{-\frac{1}{2}}, Re^{-\frac{1}{6}})$		$\varepsilon = (Re^{-1}/\sigma)^{2/5}$, thermal* diffusion and convection	$\delta = Re^{-\frac{1}{2}}/\sigma$
11	$Re^{-\frac{1}{2}}$	diffusion and convection, hydrostatic	$\varepsilon = Re^{-1/5}$, thermal diffusion and convection	No Thermal Layer in this Range
111	$(Re^{-\frac{3}{2}}, Re^{-\frac{1}{2}})$	diffusion, hydrostatic	$\varepsilon = Re^{-3/10}/\sigma^{1/5}$ diffusion and convection	
1V	$(O, Re^{-3/2})$	diffusion, dynamical and rotational	No Rotational Layer	

Here (a, b) is understood to mean $a \ll \sigma \ll b$ and an entry like $Re^{-\frac{1}{2}}$ means $\sigma = O(Re^{-\frac{1}{2}})$. Notice that, as $\sigma \to 0$, that is, looking vertically downward through the table, the thermal layer, thinner than the rotational layer in (1)(i), thickens to the same order, then becomes thicker than the rotational layer, and finally heat diffusion occurs everywhere in the fluid at CASE 11. The rotational layer fattens as well, though not so rapidly as the thermal layer, and the flow ceases to be hydrostatic only when CASE 1V is reached.

Properly speaking, the first entry in the table is open to doubt since the rotational layer solution is non-unique. It may be that this particular steady solution is not a solution if the initial-boundary value problem for the Navier Stokes equations st $\rightarrow \infty$.

In the context of a conventional monatomic gas, this work is meaningless, since the Chapman-Enskog theory that gives $\mu \sim T^{\omega}$, $k \sim T^{\omega}$ also gives $\sigma = 2/3$.[14] Hence, the proper way to interpret this is perhaps that the thermal conduction term involves radiation heat transfer in an optically thick atmosphere, in which case what has been done goes through correctly if $\omega = 3$ in the thermal conduction coefficient only.[15]

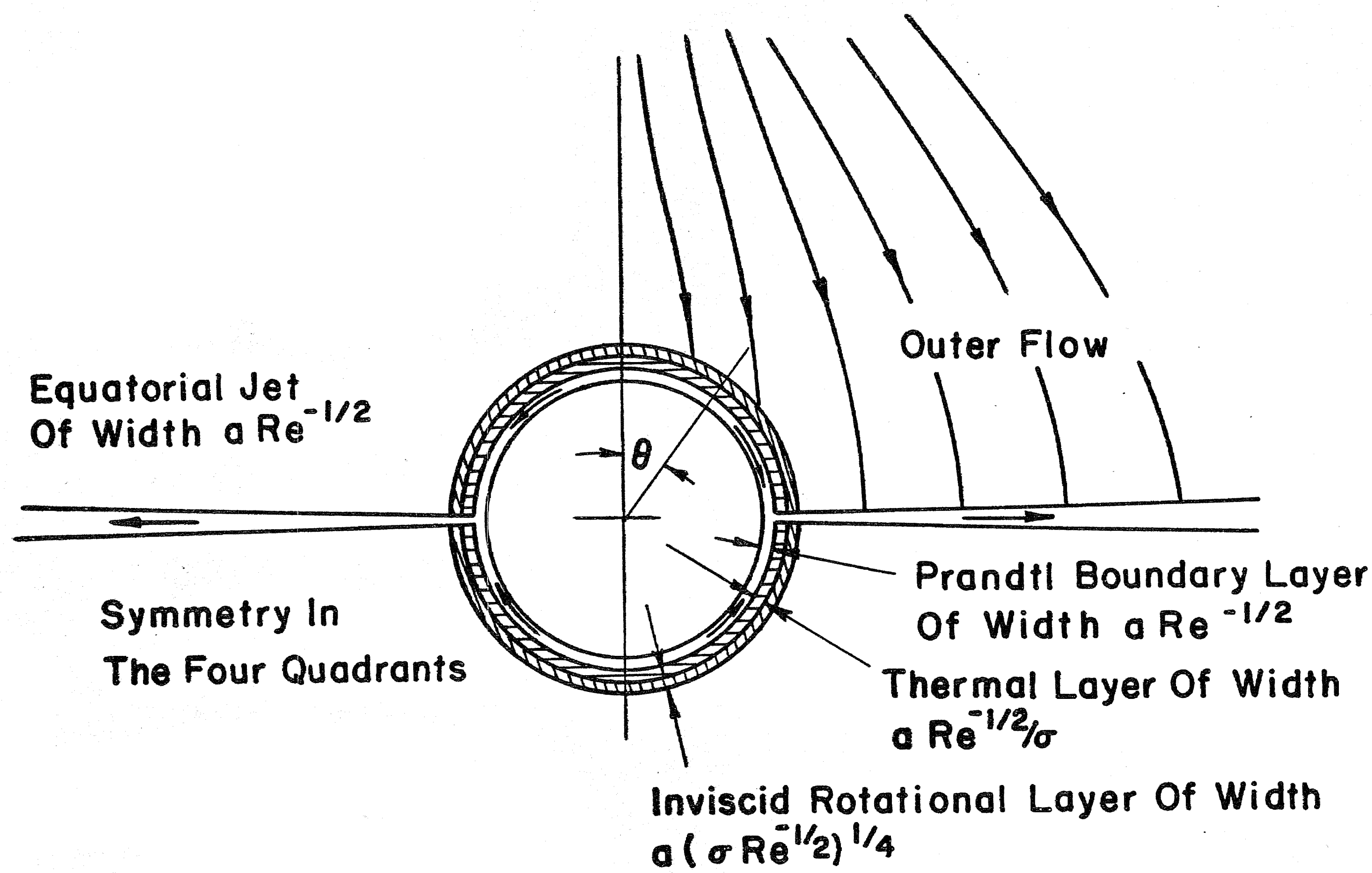
5. Conclusion

The work presented here does not exhaust the possibilities of the problem; the really interesting case remaining is $Re \rightarrow \infty$, $\sigma = O(1)$ which is certainly the one of most geophysical relevance. For the earth α is 10^3-10^5, but Re is enormous (even with an "eddy" viscosity). Intuitively, the $\sigma = O(1)$ flow appears to differ substantially from that given for $\sigma \rightarrow 0$ in Section 4, because (124) is probably incorrect; simple buoyancy force arguments seem to suggest angular momentum should be distributed throughout the entire fluid.

Of course, a rigid rotation cannot be a solution since the pressure field would be given by (for the barotropic case)

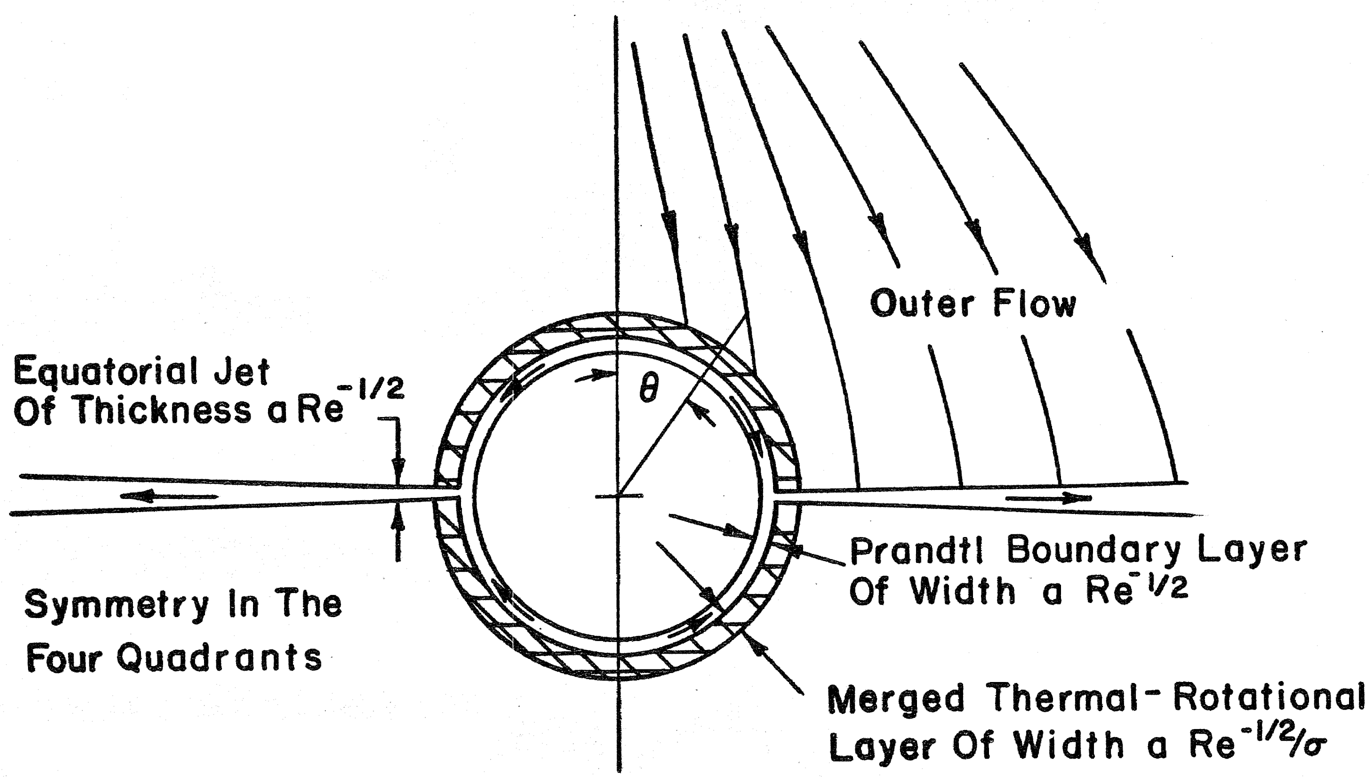
$$\nabla \int \frac{dp}{\rho} = \nabla \left(\frac{GM}{r} + \frac{1}{2} \Omega^2 r^2 \sin^2\theta\right)$$

and



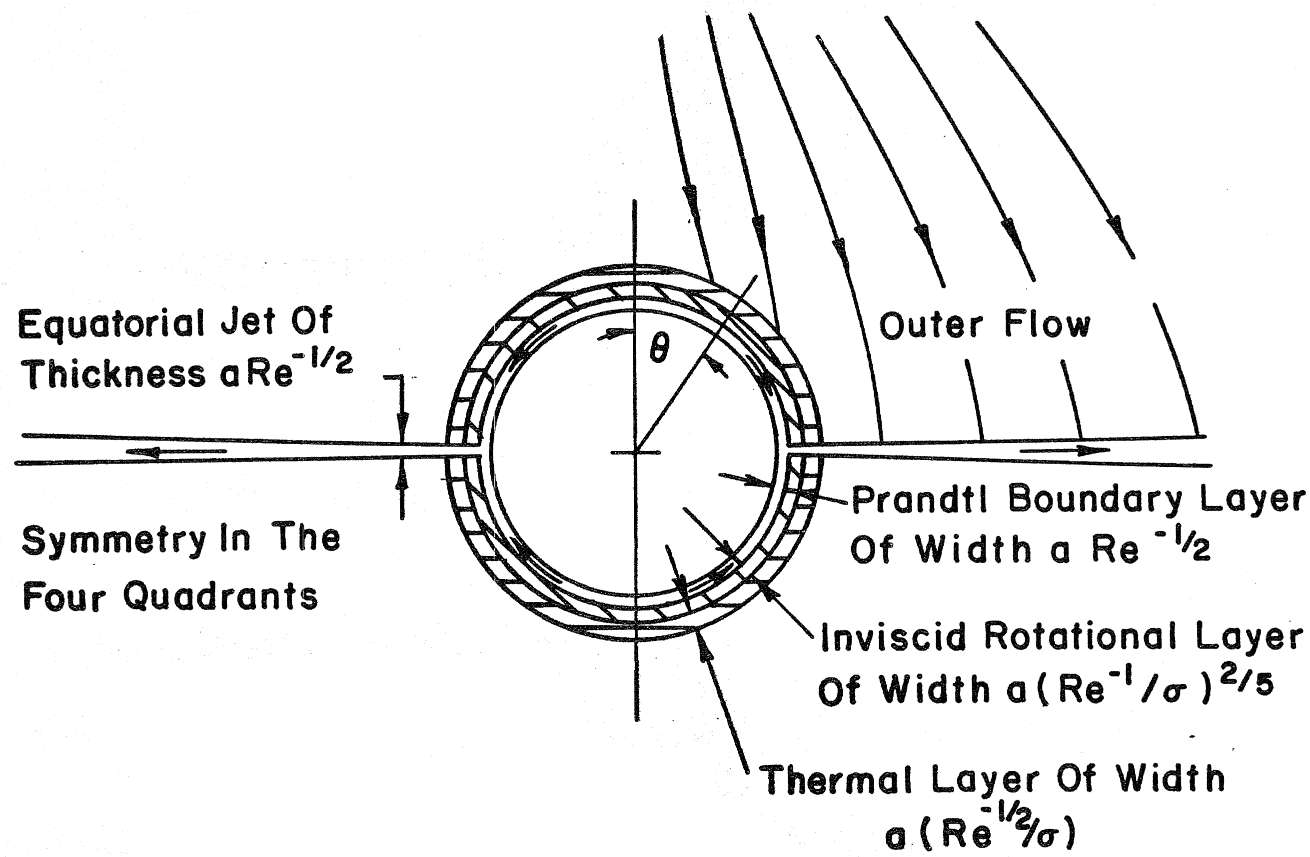
Radial Velocity Is Independent Of θ
The Outer Flow Is Adiabatic

FIG. 4 STRUCTURE OF THE $\mathrm{Re} \to \infty$, $\sigma \to 0$ FLOW
CASE I, [i] $1 \gg \sigma \gg \mathrm{Re}^{-1/6}$



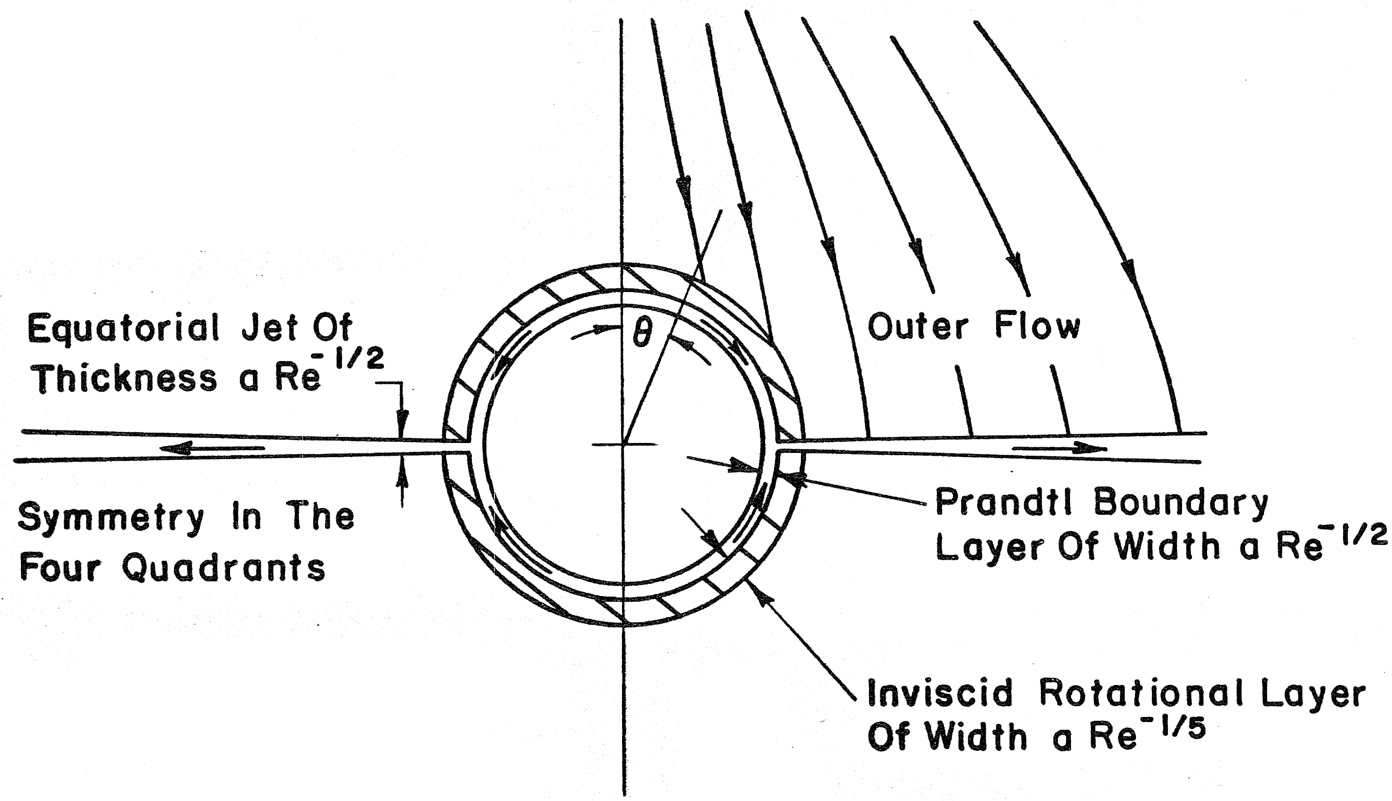
Radial Velocity Is Independent Of θ
The Outer Flow Is Adiabatic

FIG. 5 STRUCTURE OF THE $Re \to \infty$, $\sigma \to 0$ FLOW
CASE I, [ii] $\sigma = O(Re^{-1/6})$



Radial Velocity Is Independent Of θ
The Outer Flow Is Adiabatic

FIG. 6 STRUCTURE OF THE $Re \to \infty$, $\sigma \to 0$ FLOW
CASE I, [iii] $Re^{-1/6} \gg \sigma \gg Re^{-1/2}$



Radial Velocity Is Independent Of θ
Outer Flow Is Thermally Convective And Diffusive

FIG. 7 STRUCTURE OF THE $Re \rightarrow \infty, \sigma \rightarrow 0$ FLOW
CASE II $\sigma = O(Re^{-1/2})$

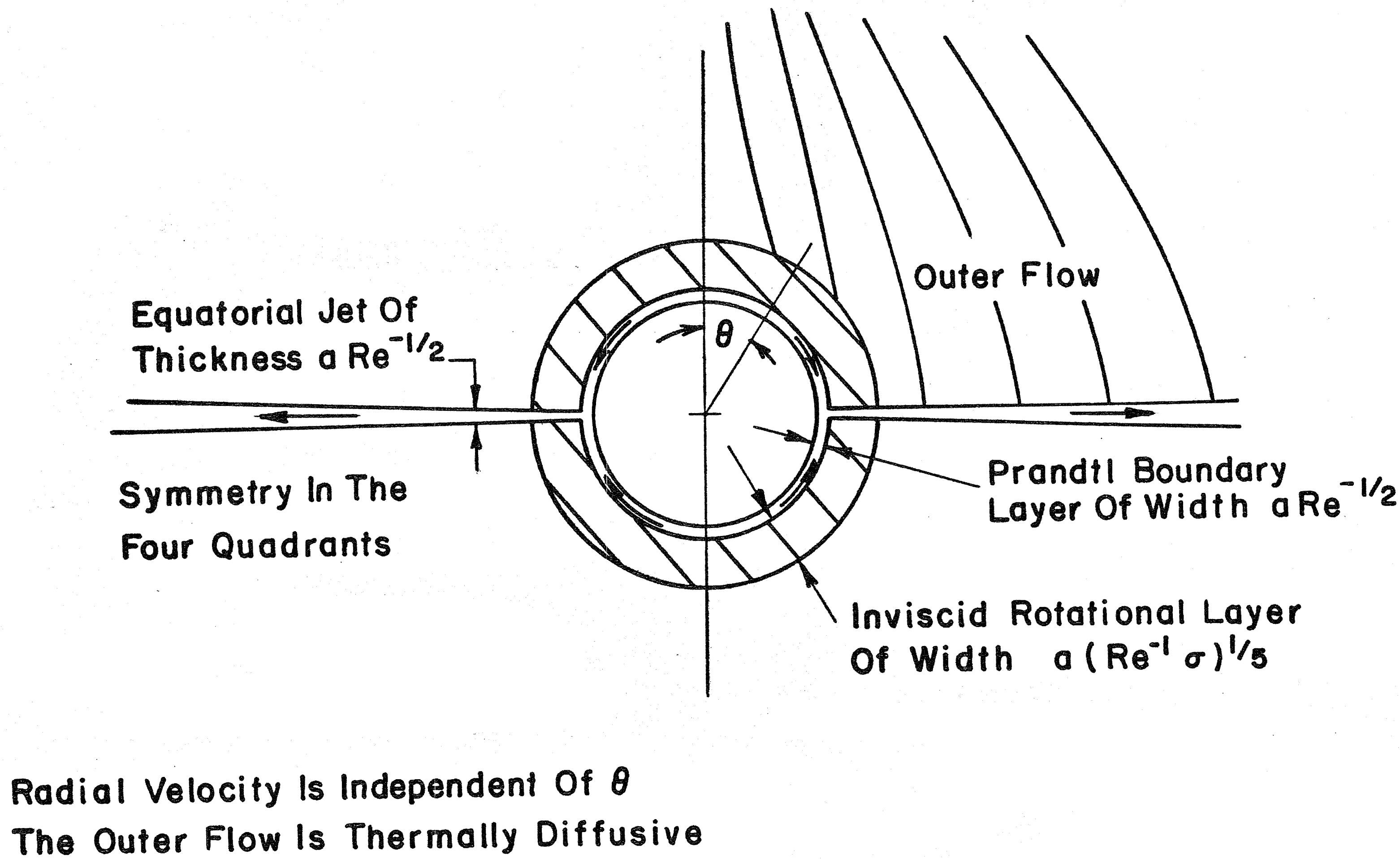


FIG. 8 STRUCTURE OF THE $Re \to \infty$, $\sigma \to 0$ FLOW
CASE III $Re^{-1/2} \gg \sigma \gg Re^{-3/2}$

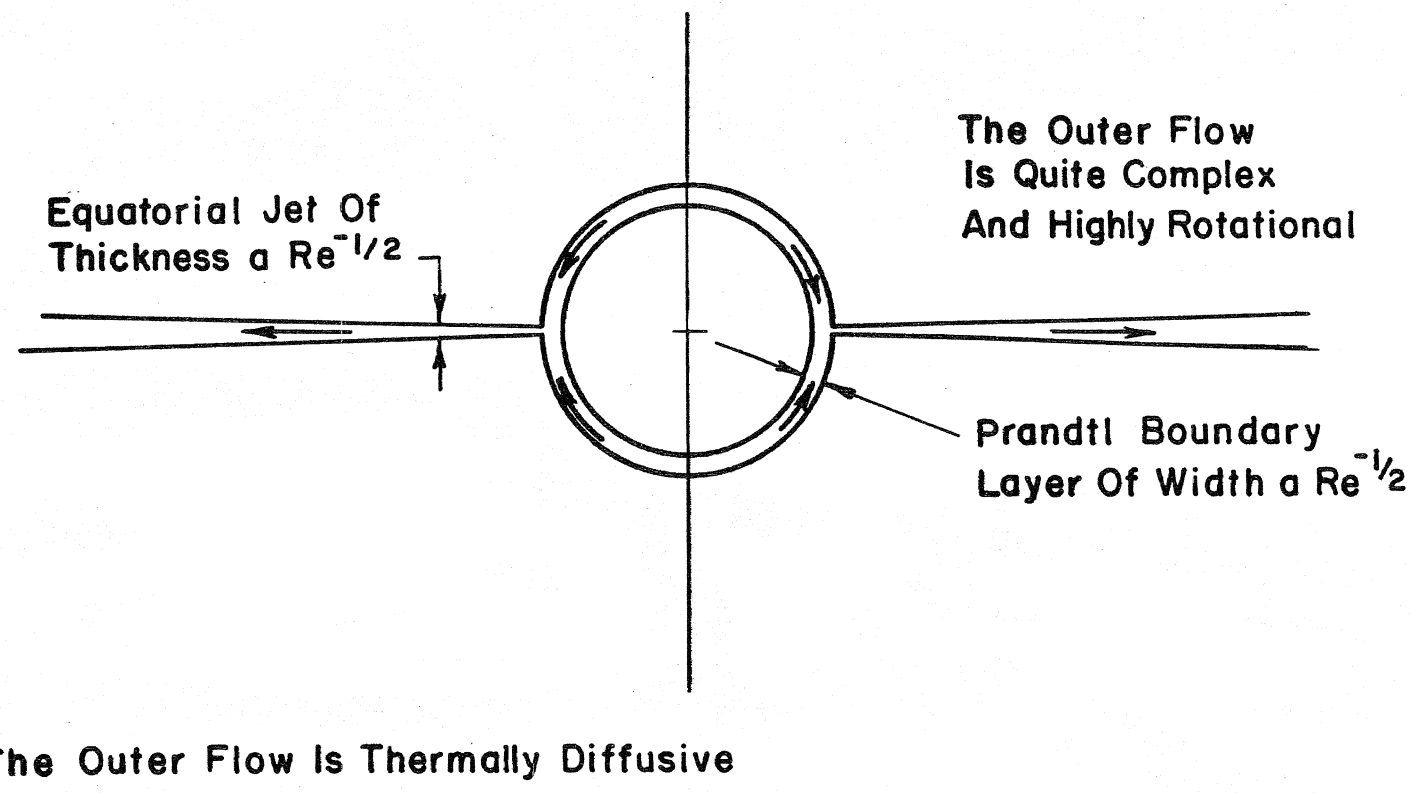


FIG. 9 STRUCTURE OF THE $Re \rightarrow \infty, \sigma \rightarrow 0$ FLOW
CASE IV $\sigma = O(Re^{-3/2})$

$$\frac{\gamma}{\gamma-1}\left[\left(\frac{p}{p(a)}\right)^{\frac{\gamma-1}{\gamma}} - 1\right] = \alpha\left(\frac{a}{r} - 1\right) + \frac{1}{2}B^2\left(\left(\frac{r}{a}\right)^2 - 1\right)\sin^2\theta$$

for the isentropic case. Clearly $p \to \infty$ with (r/a) for any B and α. One can make p a decreasing function of (r/a) provided we cut off this solution at a radius r_* such that

$$r_*^2 << GM/\Omega^2 a$$

However, this sort of solution is unacceptable from the point of view of a continuum formulation like that used here.

The terrestrial applications of this work are minimal anyway because solar heating and various other significant effects have not been included. However, it is conceivable that planets whose surfaces are sheltered by heavy clouds or are sufficiently far from the sun might show some phenomena predicted here. Certainly, there are planets that rotate slowly enough to make the results of Section 2 or 3 relevant.

In addition, astrophysical applications are possible, especially for the work in Section 4, which represents a radiating gas. The solar atmosphere in particular might be the most natural application of this latter work.

References

1. Jeffrey, G. B., "On the Steady Rotation of a Solid of Revolution in a Viscous Fluid," Proc. of the London Mathematical Society, Series 2, 14, (1914, 1915), 327.
2. Bickley, W. G., "Secondary Flow due to a Sphere Rotating in a Viscous Fluid," Philosophical Magazine, 25, (1938), 746.
3. Collins, W. D., "Steady Rotation of a Sphere in a Viscous Liquid," Mathematika, 2, (1955), 42.
4. Howarth, L., "Note on the Boundary Layer on a Rotating Sphere," Philosophical Magazine, 42, (1951), 1308.
5. Squire, H. B., "Radial Jets," 50 Jahre Grenzschictforschung. eds. H. Görtler and W. Tollmien, Vieweg (1955), 47.
6. Stewartson, K., "On rotating laminar boundary layers," Grenzshictforschung, IUTAM Symposium, Freiburg (1958), 57.
7. Nigam, S. D., "Note on the Boundary Layer on a Rotating Sphere," Zeitschrift für angewandte Mathmatik and Physik, 5 (1954), 151.
8. Lagerstrom, P., "Laminar Flow Theory" in Theory of Laminar Flows, ed. F. K. Moore, Princeton University Press, Princeton, New Jersey, (1964), 20.
9. Erdelyi, A., Asymptotic Expansions, Dover Publications, New York (1956).
10. Cole, J. D., Perturbation Methods in Applied Mathematics, Blaisdell Publishing Company, Waltham, (1968).
11. Ghildyal, "Unsteady motion of an infinite liquid due to the uniform rotation of a sphere," Quarterly of Applied Mathematics, 18, (1961), 369.

References (Cont'd)

12. Abrahamowitz, M. and Stegun, I., Handbook of Mathematical Functions, Dover Publications, New York (1965).

13. Stewartson, K., Theory of Laminar Boundary Layers in Compressible Fluids, Oxford University Press, London (1964).

14. Chapman, S. and Cowling, T. G., The Mathematical Theory of Non-Uniform Gases, Cambridge University Press, London, (1952), 168.

15. Vincenti, W. and Kruger, C., Introduction to Physical Gasdynamics, John Wiley and Sons, New York, (1965), 465.

16. Kaplun, S., Fluid Mechanics and Singular Perturbations, eds. Lagerstrom, P., Howard, L., and liu, C-S., Academic Press, New York, (1967), 9.

APPENDIX A

Galerkin Method Solution of the Large Field Outer Flow

We consider here solutions to (106), (107) by an approximate technique; to a large extent, the results depend on the approximation chosen, and no great claims are made for the usefulness or accuracy of such a solution. The attempt here is just to demonstrate the plausibility of such a solution to the quasi-linear, elliptic problem posed in (106). We will, since this is crude anyway, consider the mathematically simplest case, $\omega \equiv 0$, and thus we need to solve, first of all

$$\nabla^2 V_o - \frac{V_o}{r^2 \sin^2\theta} + \frac{2}{r^2}\frac{\partial U_o}{\partial\theta} + \frac{1}{3r}\frac{\partial}{\partial\theta}\nabla\cdot \underline{U}_o = 0$$

If we try, for V_o, $V_o = A(\theta)\,[r^{-n}]$, then it is possible to show that this equation is satisfied identically provided

$$U_o = B(\theta)\,[r^{-n} - r^{-8}]$$

where

$$B(\theta) = C + \frac{4}{n-8}\,\frac{1}{\sin\theta}\,\frac{d}{d\theta}\,[A(\theta)\sin\theta]$$

and an arbitrary function of θ was chosen to make $U_o = 0$ on $r = 1$ as required by (107). Further, one can compute from (108a) the value of $P_4/T \equiv \rho_4$ and

$$\rho_4 = r^{-n}\left[\frac{1}{\sin\theta}\,\frac{d}{d\theta}\sin\theta\,\frac{dB}{d\theta} + \frac{4}{3}(n+1)(n-2)B - \frac{n+7}{3}\,\frac{1}{\sin\theta}\frac{d}{d\theta}A\sin\theta\right]$$

$$-r^{-8}\left[\frac{1}{\sin\theta}\,\frac{d}{d\theta}\sin\theta\,\frac{dB}{d\theta} + 40B\right]$$

Consider a region $1 \leq r \leq R$ surrounding the sphere, say D. Then, the integral form of mass conservation is

$$\oint_{\partial D} \rho_4 \underline{U}_o \cdot \underline{n} dS = 0$$

However, $\underline{U}_o \cdot \underline{n} \equiv 0$ on $r = 1$ as well as on $\theta = 0$ and $\theta = \pi$. Therefore, all that remains is

$$\int_0^\pi \rho_4(R,\theta) U_o(R,\theta) \sin\theta d\theta = 0$$

This would certainly be true for an exact solution. The typical sort of Galerkin approach is to require

$$I \equiv \int_0^\pi \rho_4 U_o \sin\theta d\theta = \text{minimum } (n, C)$$

Putting ρ_4 and U_o into this gives

$$I = r^{-n}[r^{-n} - r^{-8}] \left\{ \frac{4}{3}(n-2)(n+1)a_1 - a_2 - \frac{n+7}{3} a_3 \right\}$$

$$-r^{-8}[r^{-n} - r^{-8}] \left\{ 40a_1 - a_2 \right\}$$

where $a_1 = \int_0^\pi B^2(\theta)\sin\theta d\theta$, $a_2 = \int_0^\pi (\frac{dB}{d\theta})^2 \sin\theta d\theta$, $a_3 = \int_0^\pi B \frac{dA}{d\theta} \sin\theta d\theta$

To make this a good approximation for all R, we get the two auxiliary conditions

$$\int_0^\pi [40B^2 - (\frac{dB}{d\theta})^2] \sin\theta d\theta = \text{minimum}$$

$$4(n-2)(n+1) - 120 - (n+7)a_3/a_1 = \text{minimum}$$

from making the two curly brackets vanish independently, subject to the conditions that C and n are real and $n > 0$.

Example

One simple case is for $\sigma B^2 \rightarrow 0$. Then,

$$A(\theta) = -K \sin\theta\cos\theta, \quad K = \frac{1}{2}B^2 [\frac{15}{2} \sigma + (T_1+5)]$$

so

$$B = C - \frac{4K}{n-8} (3\cos^2\theta - 1)$$

Doing the integrals,

$$a_1 = C^2 + \frac{2}{5} (\frac{8K}{n-8})^2$$

$$a_2 = \frac{12}{5} (\frac{8K}{n-8})^2$$

$$a_3 = \frac{2}{3} KC - \frac{32K^2}{n-8}$$

Thus, $40a_1 - a_2 = \min$ gives

$$40C^2 + \frac{68}{5} (\frac{8K}{n-\theta})^2 = \min \text{ if } C \equiv 0.$$

The second condition then reduces to

$$(\frac{7n}{4})^2 - \frac{9}{4} (\frac{7n}{4}) - \frac{199}{2} = \min$$

and it is identically zero for $n \approx 45/7 \approx 6.43$.

So

$$V_o \approx -K\sin\theta\cos\theta \; r^{-6.4}$$

$$U_o \approx 2.5K(3\cos^2\theta - 1)[r^{-6.4} - r^{-8}]$$

Hence, the streamlines are solutions of

$$\frac{dr}{rd\theta} = -2.5 \; (\frac{3\cos^2\theta - 1}{\cos\theta\sin\theta}) [1 - r^{-1.6}]$$

which, on integration, gives

$$r = \left[1 + \frac{C}{(\sin^2\theta\cos\theta)^{2.5}}\right]^{1/1.6}$$

which is very similar to Bickley's[(2)] solution. These streamlines are plotted in Figure 3.

APPENDIX B

The Structure of the $\alpha \to \infty$ Pressure Field

The purpose here is to deal with the problem posed in (104),

$$\nabla p + \frac{\underline{r}}{\delta r^3 T} p - p\underline{C} = \delta^3 \underline{\Sigma} \tag{104}$$

$$\mathrm{div} \rho \underline{u} = 0$$

Suppose we integrate (104') directly assuming that Σ_{o_r} is known. The result is

$$p = g(\theta)\, e^{-I/\delta} e^{J} + \delta^3 \int_1^r \Sigma_{o_r}(\bar{r}, \theta) e^{J(r)-J(\bar{r})} e^{\frac{\bar{I}-I}{\delta}} d\bar{r}$$

where $I \equiv \int_1^r \frac{d\bar{r}}{\bar{r}^2 T(\bar{r}, \theta)}$, $J \equiv B^2 \int_1^r \frac{w^2(\bar{r}, \theta)}{\bar{r} T(\bar{r}, \theta)} d\bar{r}$

In this expression, we have both hydrostatic pressure and that arising from Σ_{o_r}. It is quite easy to show by integrating by parts that

$$\int_1^r \Sigma_{o_r}(\bar{r}, \theta)\, e^{J(r)-J(\bar{r})} e^{-\frac{1}{\delta}(I(r)-I(\bar{r}))} d\bar{r}$$

$$= \delta r^2 T(r, \theta) \Sigma_{o_r} + O(\delta^2) \quad \text{as} \quad \delta \to 0 .$$

It appears that $g(\theta)$ must be determined by putting this expansion for pressure,

$$p = g(\theta) e^{-I/\delta}\, e^J + \delta^4 r^2 T \Sigma_{o_r} + O(\delta^5) \tag{B1}$$

into the θ-component of (104). Keeping leading order terms only, the result is

$$g(\theta)\, e^{-I/\delta} e^{J} \left\{ \frac{B^2 \sin\theta\cos\theta}{r} + \frac{1}{r}\frac{\partial J}{\partial\theta} - \frac{1}{r\delta}\frac{\partial I}{\partial\theta} - C_\theta \right\}$$

$$+ \delta^4 \left[\frac{1}{r} \frac{\partial r^2 T \Sigma_{o_r}}{\partial\theta} - r^2 T C_\theta \Sigma_{o_r} \right] = \delta^3 \Sigma_{o_\theta} \qquad \text{(B2)}$$

to be solved together with conservation of mass,

$$\nabla \cdot \left[\left(\frac{g}{T} e^{-I/\delta} e^{J} + \delta^4 r^2 \Sigma_{o_r} \right) \underline{u}_o \right] = 0 \qquad \text{(B3)}$$

It is not difficult to convince oneself that the limit process $\delta \to 0$, r fixed will give, for (B2)

$$\Sigma_{0_\theta} = 0 \qquad (106)$$

A little further work shows that indeed, $\delta \to 0$ in (B2) with $1 \geq \text{ord}(r-1) > \text{ord}\,\delta$ will give (106) also, so there is no difficulty matching the velocity field in an overlap domain. However, since Σ_{o_r} is $O(B^2)$, at least for $B \to 0$, the crucial quantity in completing the solution is

$$O\left(\frac{1}{B^2}\, \delta^{-4} e^{-1/\delta}\right) .$$

If this quantity is large ($B \to 0$, δ fixed), then the hydrostatic pressure is dominant and the continuity equation is

$$\nabla \cdot \left[\frac{g(\theta)}{T}\, e^{-I/\delta}\, e^{J}\, \underline{u}_o \right] = 0 \qquad \text{(B4)}$$

It is this that leads to the structure described in Section 2, with $u_o = O(v_o \delta)$. On the other hand, if $\delta \to 0$, B fixed, then we get

$$\nabla \cdot (r^2 \Sigma_{o_r} \underline{u}_o) = 0 \qquad \text{(B5)}$$

which is the (non-linear) equation used in Section 3. These equations are used as $B^2 << \delta^{-4}e^{-1/\delta}$, $>> \delta^{-4}e^{-1/\delta}$ respectively.

In the general case, clearly there are two kinds of pressure in the outer flow, one hydrostatic, the other, Stokes-like. It is this complication that makes the $\delta \to 0$ and $B \to 0$ problem non-analytic in the sense that the solution in the vicinity of $\delta = B = 0$ depends on the path $\delta = f(B)$ that one follows to the origin.

For $B = O(1)$, one can show that it is necessary to retain both terms in the conservation of mass equation in a thin layer of width ηa where $1 > \mathrm{ord}\eta > \delta$. In fact, for the special $\eta = \delta\log(\delta^{-4})$, both terms are precisely $O(\delta^4)$. Therefore, if $\delta\log\delta^{-4} > \mathrm{ord}\eta > \delta$, the correct mass conservation equation is (B4), <u>not</u> (B5). Hence, the solution of (B4) with (106) gives a velocity field which may be matched nicely through an overlap domain $1 > \mathrm{ord}\,\eta > \delta$ to the boundary layer solution. However, to talk of matching the pressure field which is a solution of (B1) and (B5) to the pressure on the sphere from a boundary layer solution is clearly meaningless since such an overlap domain does not, in fact, exist. One can think of a region of width ηa, $\delta\log^{-4} > \mathrm{ord}\eta > \delta$ where the pressure is hydrostatic, but the velocity field is that of the outer flow from (105) and (B5). It is not a distinguished region and the velocities are unchanged across it, but it is none the less there, its primary consequence being that one <u>should</u> not expect to match the pressures of the outer flow to the boundary layer solutions in the $\delta \to 0$ theory. One could alternatively argue to the same conclusion from quite general arguments given by Lagerstrom in Reference 16.

APPENDIX C

The Equatorial Jet Solution

In this section, we derive the solution to (167)

$$\psi_\eta \psi_{\eta x} - \frac{2}{x}\psi_\eta^2 - \psi_x \psi_{\eta\eta} - a^2 x W^2 = a\nu_o Q(r)\psi_{\eta\eta\eta}$$

$$\psi_\eta W_x - \psi_x W_\eta = a\nu_o Q W_{\eta\eta}$$

subject to

$$\psi(1,\infty) = \frac{1}{2}\sqrt{\nu_o \Omega}\, a^2 m$$

$$\int_{-\infty}^{\infty} [\psi_\eta(1,\eta)]^2 d\eta = \sqrt{\nu_o \Omega^3}\, a^5 M$$

$$\int_{-\infty}^{\infty} \psi_\eta W(1,\eta)\, d\eta = \sqrt{\nu_o \Omega^3}\, a^6 H$$

which when stated in a slightly variant form in (170). A substitution $\psi = a^2\sqrt{\nu_o \Omega}\, A(x)F(\xi)$, $W = \Omega a^2 B(x)G(\xi)$ where $\xi \equiv (\frac{\Omega a^2}{\nu_o})^{\frac{1}{2}}\eta/\delta(x)$ gives the equations

$$[\frac{\delta}{A}(\frac{A}{\delta})' - \frac{2}{x}]\,(F')^2 - \frac{A'}{A}FF'' - (\frac{xB^2\delta^2}{A^2})\,G^2 = \frac{Q}{\delta A}F'''$$

$$\frac{B'}{B}F'G - \frac{A'}{A}FG' = \frac{Q}{\delta A}G''$$

The integral constraints then reduce to

$$A(1)F(\infty) = \tfrac{1}{2}m$$

$$A^2(1)\int_{-\infty}^{\infty}[F'(\xi)]^2 d\xi = M\delta(1)$$

$$A(1)B(1)\int_{-\infty}^{\infty} F'(\xi)G(\xi)d\xi = H$$

One can integrate the (167) equations across the layer or integrate the ordinary differential equations directly to obtain

$$\left(\frac{B'}{B} + \frac{A'}{A}\right) \int_{-\infty}^{\infty} F'G \, d\xi = 0$$

and $$\left(\frac{(A/\delta)'}{(A/\delta)} - \frac{2}{x} + \frac{A'}{A}\right) \int_{-\infty}^{\infty} (F')^2 d\xi = \frac{xB^2\delta^2}{A^2} \int_{-\infty}^{\infty} G^2 d\xi$$

The first of these equations expresses conservation of flux of angular momentum, and since the integral cannot vanish since it has a given value, we get the ordinary differential equation

$$\frac{B'}{B} + \frac{A'}{A} = 0$$

which immediately gives $B = \text{const}/A$. The second equation is conservation of jet momentum. If there were no swirl, $H \equiv 0$ which means $B \equiv 0$ so this would mean linear momentum is a constant down the jet. However, in fact the jet gains momentum through the action of centrifugal forces, that is, the right hand side of the equation when $B \neq 0$. Now, normalizing in an appropriate way, we have

$$\frac{(A/\delta)'}{A/\delta} - \frac{2}{x} + \frac{A'}{A} = \frac{xB^2\delta^2}{A^2}$$

Finally it is clear that the diffusion term must have the same x-dependence as another term, hence $A' = kQ/\delta$. Putting this information into the original equation gives

$$k^{-1}F''' + FF'' + (F')^2 = \frac{xB^2\delta^2}{AA'} [G^2 - (F')^2]$$

$$k^{-1}G'' + FG' + F'G = 0$$

and, in summary, the ordinary differential equations in x are

$$A' = kQ(x)/\delta$$

$$\frac{(A/\delta)'}{(A/\delta)} - \frac{2}{x} + \frac{A'}{A} = \frac{xB^2\delta^2}{A^2}$$

$$\frac{B'}{B} + \frac{A'}{A} = 0$$

These three equations for A, B, δ can be solved as functions of the given function Q(x). That being the case, one can easily verify that the solution does not in general have $xB^2\delta^2/AA'$ = constant. Therefore, the only possibility of solution to the equations for $F(\xi)$ and $G(\xi)$ is for the right hand side of the first equation to vanish identically. In that case, the equations are identical, so the solution is complete. A little algebra and applications of the integral conditions for A(1), $\delta(1)$, B(1) then give the solutions

$$F(\xi) = \tan \xi$$

and

$$A(x) = \frac{1}{2}m\left[1 + \frac{36}{m^3}\int_1^x \lambda^3 Q(\lambda)\sqrt{M^2+H^2(1-1/\lambda^2)}\, d\lambda\right]^{1/3}$$

$$B(x)A(x) = \frac{3H}{4}$$

$$\delta(x) = 2Q(x)/A'(x)$$

Asymptotic form as $x \to \infty$

Consider the integral

$$I(x) = \int_1^x \lambda^2 Q(\lambda)\,(a-b/\lambda^2)^{\frac{1}{2}}\, d\lambda, \quad a > b$$

Now $Q(\infty) \neq 0$, and suppose we choose an $N < \infty$ such that $|Q(\infty)-Q(N)|$ as small as we like, say $|Q(\infty)-9(N)| < \delta$. For simplicity, in fact, suppose that

$$|Q(\infty)-Q(\lambda)| < \frac{a}{\lambda^p} \quad \text{for all } \lambda > N$$

and some $p > 0$. Then,

$$I(x) = Q(\infty) \int_1^x \lambda^2 [a - \frac{b}{\lambda^2}]^{\frac{1}{2}} d\lambda + \int_1^x [Q-Q(\infty)] \lambda^2 [a - \frac{b}{\lambda^2}]^{\frac{1}{2}} d\lambda$$

So $$I(x) \sim Q(\infty) \frac{1}{3} a^{\frac{1}{2}} x^3 + \varepsilon_1 + \varepsilon_2$$

and one can easily show that

$$|\varepsilon_1| \leqslant N^3 a^{\frac{1}{2}} Q_{max}$$

$$|\varepsilon_2| \leqslant \frac{a^{\frac{1}{2}} x^{3-p}}{3-p}$$

So, $p > 0$, the leading order asymptotic expansion is

$$I(x) \sim Q(\infty) \frac{1}{3} a^{\frac{1}{2}} x^3$$

Thus,

$$A(x) \sim [\frac{3}{2} Q(\infty)]^{\frac{1}{3}} \sqrt[6]{M^2 + H^2}\, x, \quad \delta \sim [\frac{16}{3} Q^2(\infty)]^{\frac{1}{3}} (M^2+H^2)^{-1/6}$$

APPENDIX D

Analysis of the Thin Rotational Layers

All of the rotational, inviscid layers of Section 4 are non-linear (or quasi-linear actually), and hence are not easily solved. The real purpose of this Appendix is just to demonstrate the existence and uniqueness of the solutions in a constructive, non-rigorous fashion. There are essentially three different such layers, which will be studied in sequence.

1. The Thermal-Rotational Layer, $\sigma = O(Re^{-1/6})$

Recall equations (152)

$$\left(\rho u \frac{\partial}{\partial \eta} + \rho v \frac{\partial}{\partial \theta}\right) h = \frac{\partial}{\partial \eta} h^{\omega} \frac{\partial h}{\partial \eta} \tag{D1}$$

$$\frac{\partial \rho u}{\partial \eta} + \frac{1}{\sin\theta} \frac{\partial \rho v \sin\theta}{\partial \theta} = 0 \tag{D2}$$

$$\frac{\partial}{\partial \eta}\left(\rho u \frac{\partial v}{\partial \eta} + \rho v \frac{\partial v}{\partial \theta}\right) = \frac{\partial \rho}{\partial \theta} \tag{D3}$$

$$\rho h = 1 \tag{D4}$$

where the $(\tilde{\ })$ has been dropped, and $\chi \tilde{v}$ is replaced by v. The boundary conditions are

$$\begin{aligned} & u(\infty, \theta) = F(a)/\Omega a \\ & u(0, \theta) = -E(\theta) \\ & \rho(0, \theta) = h(0. \theta) = 1, \; h(\infty, \theta) = T_1 \\ & v(\infty, \theta) = v(\eta, 0) = v(\eta, \pi/2) = v(\eta, \pi) = 0 \end{aligned} \tag{D5}$$

The asymptotic solution is easy, since $|v| \to 0$ and $u \to F(a)/\Omega a$, (D1) linearizes to

$$-\frac{F(a)}{T_1^{\omega+1}\Omega a}\frac{\partial h}{\partial \eta}+\frac{\partial^2 h}{\partial \eta^2}=0$$

so $$h \sim T_1 + G(\theta)\, e^{\frac{F(a)\eta}{\Omega a T_1^{\omega+1}}}, \text{ as } \eta \to \infty$$

and (D4) is also quite simply, giving

$$v \sim -\frac{T_1^{2\omega+1}}{[F(a)]^3}\,\Omega^3 a^3 \frac{dG}{d\theta}\, e^{\frac{F(a)\eta}{T_1^{\omega+1}\Omega a}}$$

and finally

$$u - F(a)/\Omega a \sim \frac{T_1^{3\omega+2}\Omega^4 a^4}{[F(a)]^4}\,\frac{1}{\sin\theta}\frac{d}{d\theta}\sin\theta\frac{dG}{d\theta}\, e^{\frac{F(a)\eta}{T_1^{\omega+1}\Omega a}}$$

Since there is one arbitrary function in the asymptotic expansion, $G(\theta)$, it must be simply related to $E(\theta)$ in a one-to-one way.

2. The Convective Rotational Layer

This layer occurs only in CASE 1, (i), and is given by equations (178). It is a simple matter to rescale these equations and write them in the simpler form,

$$\begin{aligned} & u\frac{\partial h}{\partial \xi} + v\frac{\partial h}{\partial \theta} = u+1 \\ & \frac{\partial h}{\partial \theta} + \frac{\partial}{\partial \xi}\left(u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \theta}\right) = 0 \qquad \text{(D6)} \\ & \frac{\partial v}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial v \sin\theta}{\partial \theta} = 0 \end{aligned}$$

These equations are difficult to solve. Consider the following device for studying them. Suppose that

$$u(0, \theta) = -1 + \varepsilon f(\theta)$$

where $f = O(1)$ and $\varepsilon \downarrow 0$. Putting this into (D6) gives what is essentially an Oseen approximation to the equations,

$$\frac{\partial h'}{\partial \xi} + u' = 0$$

$$\frac{\partial h'}{\partial \theta} - \frac{\partial^2 v'}{\partial \xi^2} = 0 \qquad \text{(D7)}$$

$$\frac{\partial u'}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial v \sin\theta}{\partial \theta} = 0$$

$$u'(0, \theta) = f(\theta)$$

where $\varepsilon\phi'$ denotes the perturbation quantity. One can now write a single equation for u'

$$\frac{\partial^4 u'}{\partial \xi^4} - \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial u'}{\partial\theta} = 0 \qquad \text{(D8)}$$

$$u'(\ \theta) = f(\theta)$$

Let $f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta)$ where $P_n(x)$ is a Legendre Polynomial. Then, (D8) is

$$\frac{\partial^4 b_n}{\partial \xi^4} + (n+1)nb_n = 0$$

where $u' = \sum_{o}^{\infty} b_n(\xi)P_n(\cos\theta)$. The solutions bounded at ∞ are two, so the solution is

$$u' = \sum_{n=0}^{\infty} a_n \cos\mu_n \xi\, e^{-\mu_n \xi} P_n(\cos\theta) + \sum_{o}^{\infty} u_n^* P_n(\cos\theta)$$

where $\mu_n = \left(\frac{n(n+1)}{4}\right)^{1/4}$ and u_n^* is an eigenfunction

$$u_n^* = C_n \sin\mu_n \xi \, e^{-\mu_n \xi}$$

which is a solution of (D7) with $f(\theta) \equiv 0$. Hence, the solution is non-unique.

<u>Stability</u>

The unsteady version of (D8) is just

$$\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial \xi}\right)^2 \frac{\partial^2 u'}{\partial \xi^2} - \frac{a}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial u'}{\partial\theta} = 0 \qquad \text{(D9)}$$

where a = 1 for the case here and a = -1 if the temperature gradient were positive. If we look for a solution

$$u' = P_n(\cos\theta)\, e^{i(\sigma t - \lambda y)}$$

the algebraic equation for σ is

$$(\sigma - \lambda) \quad = \pm \sqrt{-an(n+1)}$$

If a = -1, then there are wave-like solutions of (D9)

$$e^{i\left[\left(\lambda \pm \frac{\sqrt{n(n+1)}}{\lambda}\right)t - \lambda y\right]}$$

which are, more or less, internal waves. In <u>our</u> case, a = 1, so the solutions are

$$e^{i\lambda(t-y)} \quad e^{\mp \frac{\sqrt{n(n+1)}}{\lambda} t}$$

which grow in time. Therefore, the non-uniqueness of the solution of (D7), and presumably of (D6), is because the layer structure is unstable to small disturbances.

3. The Diffusive Rotational Layer

The other rotational layers, namely those given by (180), (182), and (183), all have the same form, which can be scaled to give just

$$\frac{\partial^2 h}{\partial \xi^2} = \alpha(u+1)$$

$$\frac{\partial}{\partial \xi}\left[u\frac{\partial v}{\partial \xi} + v\frac{\partial v}{\partial \theta}\right] + \frac{\partial h}{\partial \theta} = 0$$

$$\frac{\partial u}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial v \sin\theta}{\partial \theta} = 0$$

For purposes of study, consider the Oseen equations as described in (D7),

$$\frac{\partial^2 h'}{\partial \xi^2} = \alpha u'$$

$$\frac{\partial^2 v'}{\partial \xi^2} = \frac{\partial h'}{\partial \theta}$$

$$\frac{\partial u'}{\partial \xi} + \frac{1}{\sin\theta}\frac{\partial v' \sin\theta}{\partial \theta} = 0 \qquad \text{(D10)}$$

$$v'(\infty)=0, \quad u(0,\theta) = f(\theta), \quad h'(0) = h'(\infty) = 0$$

These equations are easily solved, and

$$u' = \sum_{n=0}^{\infty} a_n \left\{\sin(\sin\tfrac{\pi}{5}\, \nu_n z) - \tan\tfrac{2\pi}{5}\cos(\sin\tfrac{\pi}{5}\nu_n z)\right\} e^{-\nu_n \cos\frac{\pi}{5} z}\, P_n(\cos\theta)$$

where a_n is defined as before, and $\nu_n = [\alpha n(n+1)]^{1/5}$, provided $\alpha > 0$. If, however, $\alpha < 0$, there are three linearly independent solutions, so once again a non-uniqueness characterizes the solution of (D10).

Stability

The stability question then involves a solution of the time-dependent version of (D10) with homogeneous boundary conditions. The time-dependent equation to be solved is

$$\left(\frac{\partial^2}{\partial\xi^2} - \frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial\xi}\right)\frac{\partial^2 u'}{\partial\xi^2} - \frac{\alpha}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial u'}{\partial\theta} = 0 \tag{D11}$$

If one puts in a plane wave $e^{i(\sigma t-\lambda y)}P_n(\cos\theta)$, the dispersion relation is

$$\sigma = \frac{\lambda+i\lambda^2}{2} \pm \sqrt{\frac{1}{4}(\lambda-i\lambda^2)^2 + \frac{\alpha n(n+1)}{\lambda^2}} \tag{D12}$$

If λ is large, the solutions are $\sigma = \lambda$, $\sigma = i\lambda^2$, both of which are stable perturbations. However, the possibility of instability is at small λ. For sufficiently small λ for any $\alpha n \neq 0$, the approximate form of the dispersion relation is

$$\sigma \sim \frac{\lambda+i\lambda^2}{2} \pm \frac{1}{\lambda}\sqrt{\alpha n(n+1)}$$

which has a negative imaginary part if and only if $\alpha < 0$ for $n > 0$. The neutrally stable disturbance is a solution of

$$(i\lambda)^5 - \frac{1}{2}(i\lambda)^4 - 2\alpha n(n+1) = 0$$

which is easily shown to have a real root λ only if $\alpha < 0$. Hence, the stability requirement $\alpha > 0$ is exactly the condition for (D10), and also clearly (180), (182), (183) to be well posed.

LIST OF FIGURES

LIST OF SYMBOLS

U = forward velocity of the body

$\underline{u}$ = velocity vector

u, w } horizontal and vertical velocities

g = gravitational force per unit mass

t = time

p = pressure

L = horizontal length scale, $L_1 + L_2$

L_1 = distance between top of body and right wall at $t = 0$

L_2 = distance between top of body and left wall at $t = 0$

x = horizontal coordinate

z = vertical coordinate

T = time scale

θ = dimensionless time scale

Pe = Peclet number

R = $U\nu/gL^2$

h = half-height of body

S = initial density profile

$\underline{n}$ = unit outward normal to a surface

$F(z)$ = representation of the body

z_o = notation for a particular characteristic in (-)

D = drag

V = potential energy

n, s } normal-direction and tangential-direction coordinates

LIST OF SYMBOLS (Cont'd)

$\overline{U}$, $\overline{W}$ } dimensionless boundary layer variables in Section 5

y = boundary layer coordinate

ρ = density

κ = thermometric conductivity or coefficient of diffusion

μ = coefficient of viscosity

ρ_o = reference density

ϵ = Froude number, $u/\sqrt{gL}$, or slenderness ratio in Section 4

ν = kinematic viscosity

β = $(\partial\rho/\partial z)_o/\rho_o$, related to an initial density profile

τ = Ut, a time variable

ζ = location of the w = 0 line, shear layer coordinate in Section 6

ξ = dummy variable, dimensionless coordinate in Section 5

δ = dimensionless width of mixing fluid in Section 4

η = boundary layer coordinate in Section 5

ω = vorticity

α = related to boundary layer solution in Section 5

$(\ )^*$ = refers to dimensionless quantity, shear layer variables, Section 6

$\pm$ = refers to in front or behind body

$(\hat{\ })$ = refers to quantity in unstably stratified region

$(\bar{\ })$ = refers to boundary layer variable

$(\tilde{\ })$ = refers to variables in density adjustment layer

THE DRAG OF A BODY MOVING TRANSVERSELY IN A CONFINED STRATIFIED FLUID

1. Introduction

In recent years great amounts of effort have been expended on the dynamics of rotating and/or stratified fluids in confined regions. Part of the reason for doing the work is the hope of application to important geophysical problems like ocean circulation and meteorology. Recently Saffman and Moore,[1,2] have studied the vertical motion of a body through a rotating fluid bounded by horizontal plates. There is a general belief that rotating fluids and stratified fluids behave analogously; that belief was made precise by Veronis,[3] for a stratified fluid under Boussinesq approximation in a steady flow with $\sigma = O(1)$. The term Prandtl number and symbol, σ, will be used throughout this part, even though it may be in fact a number involving diffusion of salt through water, sometimes called a Schmidt number.

In the work that follows, we study the horizontal motion of a symmetric two-dimensional body through a stratified fluid bounded by vertical plates, for the particular case when the Peclet number is very large, and the Froude number, $U^2/\sqrt{gL}$, very small. Boundary and shear layers are discussed in Sections 5 and 6 for one particular range of large Peclet number, though that work is highly incomplete at this stage. However, the drag computed in Section 4 is independent of the structure of the boundary layers or shear layers, provided only that such layers <u>do</u> exist.

2. Formulation of the Problem

The Boussinesq equations for a viscous fluid in the presence of a uniform gravitational field are,[4]

$$\nabla \cdot \underline{u} = 0 \tag{1}$$

$$\frac{D\rho}{Dt} = \kappa \nabla^2 \rho \tag{2}$$

$$\rho_o \frac{D\underline{u}}{Dt} + \nabla p = -\rho g \underline{k} + \mu \nabla^2 \underline{u} \tag{3}$$

Here, variations in ρ can be due to temperature, salt, et al., and κ is the corresponding diffusion coefficient.

We will consider here the motion of a body of vertical dimension 2h at constant speed U through a stratified fluid bound by vertical walls a distance $L = L_1 + L_2$ apart. The body moves to the right and at $t = 0$, it is at a distance L_1 from the right-hand wall.

Throughout the work, we will suppose the motion is slow in the sense that

$$\varepsilon \equiv \frac{U}{\sqrt{gL}} << 1 \tag{4}$$

(This ε corresponds to a Rossby number in rotating fluids.) We now non-dimensionalize (1)-(3) with

$$\underline{u} = \varepsilon\sqrt{gL}\, \underline{u}^*, \;\; \rho = \rho_o \rho^*, \;\; p = \rho_o g L p^*, \;\; \nabla = \frac{1}{L} \nabla^*, \;\; t = Tt^*$$

which gives just

$$\nabla^* \cdot \underline{u}^* = 0 \tag{5}$$

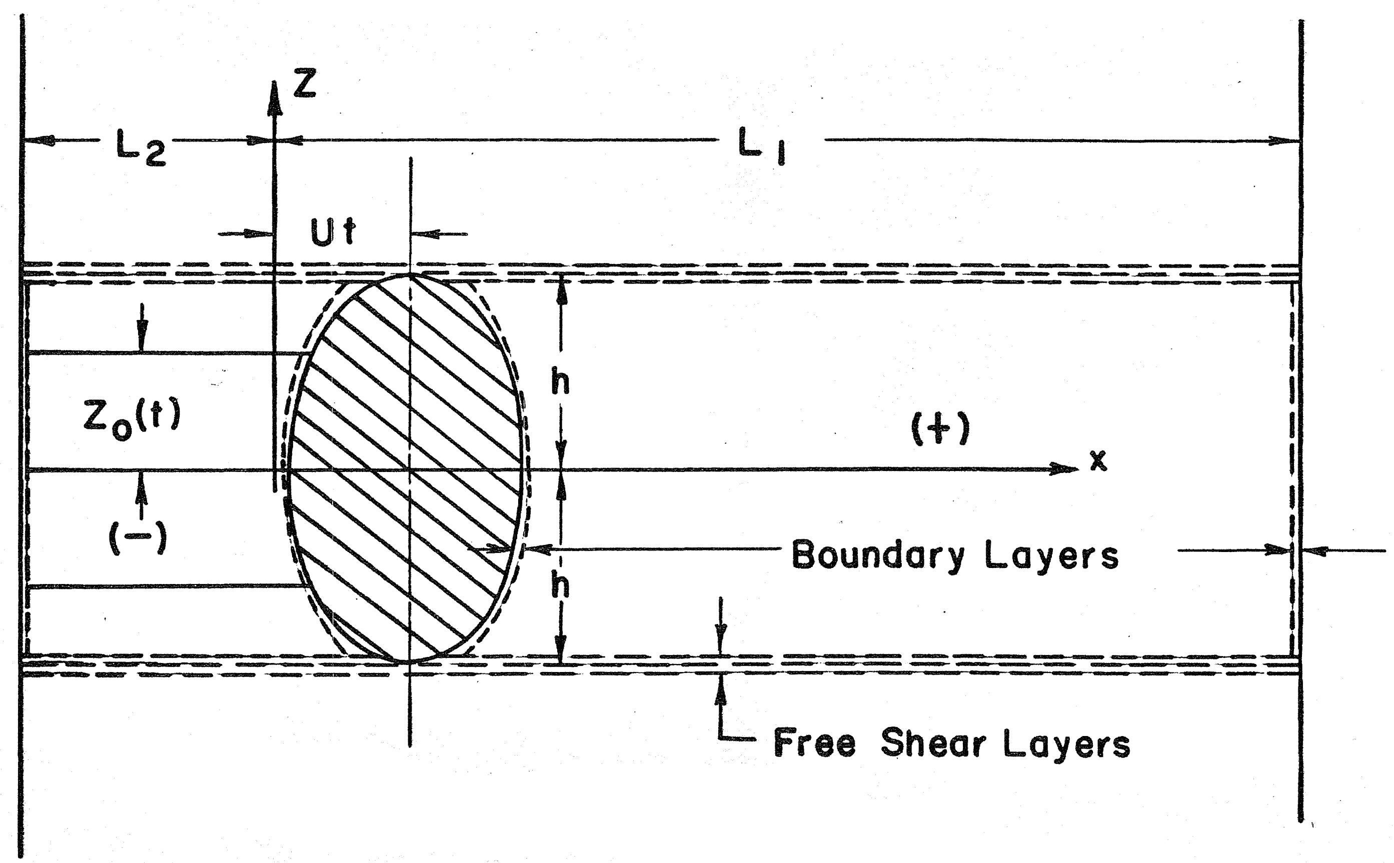


FIG. 1 FLOW STRUCTURE AND COORDINATE SYSTEM FOR BODY MOVING TO THE RIGHT AT SPEED U

$$\theta \frac{\partial \rho^*}{\partial t^*} + (\underline{u}^* \cdot \nabla^*) \rho^* = Pe^{-1} \nabla^{*2} \rho^* \quad (6)$$

$$\varepsilon^2 \theta \frac{\partial \underline{u}^*}{\partial t^*} + \varepsilon^2 (\underline{u}^* \cdot \nabla^*) \underline{u}^* + \nabla^* p^* = -\rho^* \underline{k} + R \nabla^{*2} \underline{u}^* \quad (7)$$

where the following dimensionless combinations arise

$$\varepsilon = U/\sqrt{gL}$$

$$Pe = \frac{UL}{\kappa} = \varepsilon \frac{\sqrt{gL^3}}{\kappa} \quad (8)$$

$$R = \frac{U\nu}{gL^2} = \varepsilon \frac{\nu}{\sqrt{gL^3}}$$

plus the time scale parameter

$$\theta = \left(\frac{L}{g\varepsilon^2 T^2}\right)^{\frac{1}{2}}$$

There is of course an additional parameter, h/L, which will be $O(1)$ in this treatment. Where appropriate, comments will be made as to the restrictions on $O(h/L)$.

There are three time scales associated with this problem, viz., those for momentum and density diffusion, L^2/ν and L^2/κ, respectively and the kinematic scale L/U related to the change in geometry (cf. Figure 1). Notice in particular that

$$Pe = \frac{L^2/\kappa}{L/U} = \frac{\text{diffusion time}}{\text{flow time}}$$

Now, if $Pe \gg 1$ then the density diffusion can be neglected over the total elapsed time for the flow. This point will be discussed further in Section 4 and then in Section 6 in connection with a "mixing time" U/g. On the kinematic time scale, i.e., $T = L/U$, (5)-(7) become

$$\nabla^* \cdot \underline{u}^* = 0 \tag{9}$$

$$Pe \frac{D^*\rho^*}{Dt^*} = \nabla^{*2} \rho^* \tag{10}$$

$$\varepsilon^2 \frac{D^*\underline{u}^*}{DT^*} + \nabla^* p^* + \rho^* \underline{k} = R\nabla^{*2} \underline{u}^* \tag{11}$$

The problem to be solved will be for the particular range of parameters

$$Pe >> 1, \ \varepsilon << 1, \ R << 1$$

in which case the following problem is posed

$$\frac{D\rho}{Dt} = 0 \tag{12}$$

$$\nabla p + \rho g \underline{k} = 0 \tag{13}$$

$$\nabla \cdot \underline{u} = 0 \tag{14}$$

subject to the initial condition

$$\rho(x, z, 0) = S(z), \ S'(z) \leq 0 \tag{15}$$

and the boundary conditions

$$\underline{u} \cdot \underline{n} = \underline{u}_B \cdot \underline{n} \tag{16}$$

on solid boundaries with normal $\underline{n}$, where $\underline{u}_B$ is the velocity of the surface. There will, of course, be boundary and shear layers since one cannot expect solutions to (12)-(14) to be uniformly valid in space. Further, at this stage we restrict ourselves to convex bodies symmetric about $z = 0$.

Physically, this problem would be valid for salt and water solution; there are other interesting limits which will be discussed briefly later.

3. The Interior Solution

In this section, we solve the problem (12)-(16) formulated in the previous section. Clearly, (13) means ρ is independent of x, so (12) implies w is independent of x and so (14) is immediately integrable

$$u = f(z,t) - x \frac{\partial w}{\partial z}(z,t) \tag{17}$$

We will now solve the problem for a particular case.

A. The Flat Plate

Clearly, (16) and (17) give, using the condition on $x = \pm L_{\frac{1}{2}}$,

$$u_{\pm} = (\pm L_{\frac{1}{2}} - x)\frac{\partial w}{\partial z} \tag{18}$$

where $\pm$ denote quantities in front or behind the plate. Using the boundary condition on the body, then,

$$\frac{\partial w_{\pm}}{\partial z} = \frac{U}{\pm L_{\frac{1}{2}} - Ut}$$

By symmetry, in the particular case $S(z) = \rho_o(1-\beta z)$, this integrates to

$$w_{\pm} = \frac{Uz}{\pm L_{\frac{1}{2}} - Ut} \tag{19}$$

Then, we solve (13) which is now

$$\frac{\partial \rho}{\partial t} + \frac{Uz}{\pm L_{\frac{1}{2}} - Ut}\frac{\partial \rho}{\partial z} = 0 \tag{20}$$

which is solvable by method of characteristics for first order equations. The characteristic directions are solutions of

$$\frac{dz}{dt} = + \frac{Uz}{\pm L_{\substack{1\\2}} - Ut} \quad ,$$

viz., $z(L_{\substack{1\\2}} \mp Ut) = \text{const.}$ In particular, then

$$\rho_{\pm} = G[z(L_{\substack{1\\2}} \mp Ut)] \tag{21}$$

Now, on $t = 0$, $\rho_{\pm} = \rho_o(1-\beta z)$, so then (21) gives the solution for all time as

$$\rho_{\pm} = \rho_o(1-\beta z\,[1 \mp Ut/L_{\substack{1\\2}}\,]) \tag{22}$$

However, notice that the characteristic that leaves $t = 0$ at $z = h$ (see Figure 2) has the equation, in the (-) region,

$$z(1 + Ut/L_2) = h$$

so that the points behind the plate satisfying

$$\frac{h}{1+Ut/L_2} < z < h$$

are not accessible along characteristics originating on $t = 0$ for $|z| < h$. In fact, (22) should be rewritten as

$$\rho_+ = \rho_o\,[1-\beta z(1-Ut/L_1)]\,, \quad |z| < h \tag{23a}$$

$$\rho_- = \rho_o\,[1-\beta z(1+Ut/L_2)]\,, \quad |z| < h[1+Ut/L_2]^{-1} \tag{23b}$$

Now, the volume flux that leaves the region in front of the body enters the region behind the plate along the lines $z = \pm h$. From (19),

$$(L_1 - Ut)\, w_+(\pm h) + (L_2 + Ut)\, w_-(\pm h) = 0 \tag{24a}$$

so the volume flux condition is satisfied. If mass flux is <u>also</u> to be conserved, as it must be,

$$\hat{\rho}_-(\pm h) = \rho_+(\pm h) \tag{24b}$$

where $(\hat{\ })_-$ denotes variables in the (-) region where $h(1+Ut/L_2)^{-1} < |z| < h$. All of this assumes, of course, that there exist shear layers on $z = \pm h$ capable of delivering this mass flux to the rear of the body. This is a boundary condition to be applied to (21) to give just

$$\hat{\rho}_- = \rho_o\left[1 - \beta h \frac{L_1 + L_2}{L_1} + \beta z \frac{L_2 + Ut}{L_1}\right], \quad \frac{h}{1+Ut/L_2} < z < h \tag{25a}$$

and

$$\hat{\rho}_- = \rho_o\left[1 + \beta h \frac{L_1 + L_2}{L_1} + \beta z \frac{L_2 + Ut}{L_1}\right], \quad -\frac{h}{1+Ut/L_2} > z > -h \tag{25b}$$

Of course, for all $|z| > h$, $u = w \equiv 0$, and hence

$$\rho(z,t) = \rho_o(1 - \beta z), \quad |z| > h \tag{25c}$$

So, the solution is complete. It remains to calculate the drag in Section 4.

B. <u>The General Case</u>

Here, the formulation of A will be extended to include a body represented by

$$\begin{aligned} x &= F_+(z) \quad \text{in } (+) \\ x &= -F_-(z) \quad \text{in } (-) \end{aligned} \tag{26a}$$

where $F_{\pm}(\pm h) = 0$ and further

$$F_{\pm}(-z) = F_{\pm}(z) \tag{26b}$$

Now, (18) is still correct. Then, using (16) on the body,

$$u_{\pm} \mp F'_{\pm}(z)\, w_{\pm} = U \text{ on } x = Ut \pm F_{\pm}(z) \tag{27}$$

Using this result and (18) gives the ordinary differential equation

$$\left\{\pm L_{\frac{1}{2}} - Ut \mp F_{\pm}(z)\right\} \frac{\partial w_{\pm}}{\partial z} \mp F'_{\pm}(z)\, w_{\pm} = U \tag{28}$$

The solution is easily found by the usual methods,

$$w_{\pm}(z,t) = U \frac{z - \zeta_{\pm}(t)}{\pm L_{\frac{1}{2}} - Ut \mp F_{\pm}(z)} \tag{29}$$

where $z = \zeta_{\pm}(t)$ is the $w \equiv 0$ line. Requiring (24a) to be true immediately gives $\zeta_{+} \equiv \zeta_{-}$. Just what ζ is, will be determined at a later stage by

$$p_{+}(\pm h) = p_{-}(\pm h) \tag{30}$$

The density will be constant on lines

$$\frac{dz}{dt} = w_{\pm}(z,t)$$

which is, with $\tau \equiv Ut$,

$$\frac{d\tau}{dz} = \frac{\pm(L_{\frac{1}{2}} - F_{\pm}(z)) - \tau}{z - \zeta(t)} \tag{31}$$

Sketches of these curves are given in Figure 2 in the very special case $\zeta \equiv 0$. It is quite clear from the geometry that

$$0 < \tau < L_1 - F_{+}(0), \quad \zeta < h$$

and the lines passing through z = +h have slope

$$\frac{d\tau}{dz} = \frac{\pm L_{\frac{1}{2}} - \tau}{h-\zeta} \gtrless 0$$

so there is the same difficulty as in A for the general case, viz., there will be a region behind the body adjacent to the z = ±h lines which cannot be reached along characteristics from $\tau = 0$, $|z| < h$. Hence, the general problem involves the solution of an initial-boundary value problem behind the body. We now specialize to the particular case

$$S(z) = \rho_o(1-\beta z) \tag{32}$$

One can carry $\zeta(t)$ through to the end and determine it by (30), or notice that in this special case, symmetry considerations imply $\zeta \equiv 0$. So, (31) is just

$$\frac{d\tau}{dz} = \frac{\pm L_{\frac{1}{2}} \mp F_{\pm}(z) - \tau}{z} \tag{33}$$

which is integrable, and gives solutions

$$\tau = \pm L_{\frac{1}{2}} \frac{z-z'}{z} \mp \frac{1}{z} \int_{z'}^{z} F_{\pm}(\lambda) d\lambda \tag{34}$$

where z' is the z-intercept. If z' = +h for example, in (-)

$$\tau z = L_2(h-z) - \frac{1}{z} \int_{z}^{h} F_{-}(\lambda) d\lambda$$

However, we have

$$\int_{z}^{h} F_{-} \, d\lambda \leqslant (h-z) F(0)$$

so

$$\tau \geq [L_2 - F(0)] \frac{h-z}{z} \geq 0, \quad 0 < z \leq h$$

and hence, on this characteristic $z < h$ for all $\tau > 0$, so it does indeed bound a region inaccessible from $\tau = 0$. Now, ρ depends only on the value of z', hence

$$\rho_{\pm} = G\left([\tau \mp L_2] z \pm \int_0^z F_{\pm}(\lambda) d\lambda\right) \tag{35}$$

In front of the body, we have simply an initial value problem (cf. Figure 2), so (32) gives

$$G\left[-L_1 z + \int_0^z F_+(\lambda) d\lambda\right] = \rho_o (1 - \beta z)$$

so

$$\rho_+ = \rho_o(1 - \beta\eta_+ \{[\tau - L_1] z + \int_0^z F_+(\lambda) d\lambda\}) \tag{36a}$$

where $z = \eta_+(\xi)$ is the solution of

$$-L_1 z + \int_0^z F_+ \, d\lambda = \xi \tag{36b}$$

So long as one is inside the lines $z = z_o(t)$ [cf. Figure 2], exactly the same calculation can be made behind the body, the result being

$$\rho_- = \rho_o(1 - \beta\eta_- \{(\tau + L_2) z - \int_0^z F_-(\lambda) d\lambda\}) \tag{37a}$$
$$|z| < z_o(\tau)$$

where $\eta_-(\xi)$ is the solution $z = \eta_-(\xi)$ of

$$L_2 z - \int_0^z F_- d\lambda = \xi \tag{37b}$$

Clearly, $z_o(t)$ is the solution of

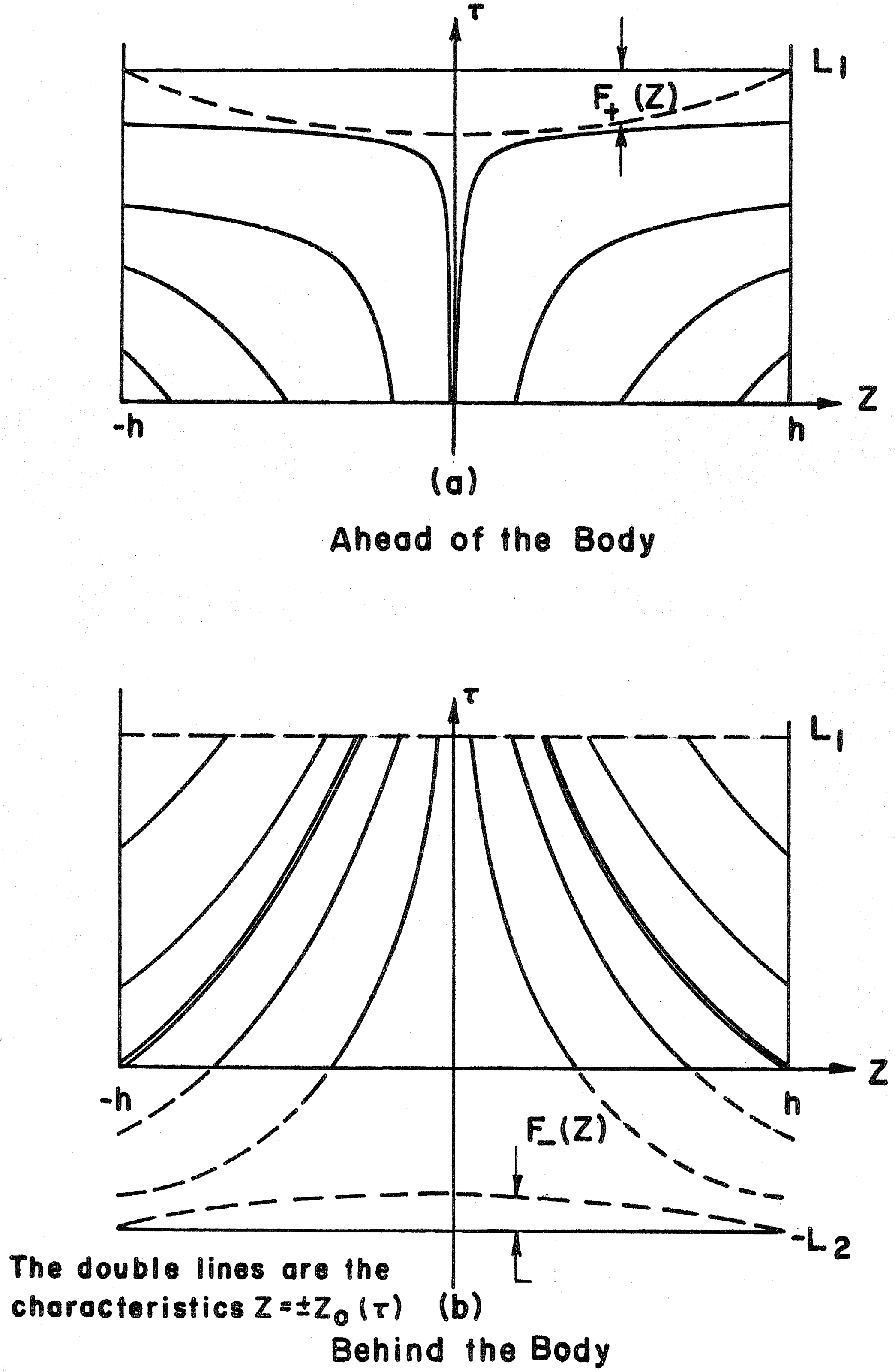


FIG. 2 CHARACTERISTIC DIRECTIONS FOR THE DENSITY FOR THE CASE $\zeta_{\pm}(\tau) \equiv 0$

$$\tau z_o(\tau) = L_2(h-z_o(\tau)) - \int_{z_o(\tau)}^{h} F_-(\lambda)d\lambda \tag{38}$$

Clearly, as before in (24b),

$$\rho_+(\pm h) = \rho_-(\pm h)$$

which then gives, from (35), for $z_o < z < h$,

$$G([\tau+L_2]\, h-\int_o^h F_- d\lambda) = \rho_o(1-\beta\eta_+\{(\tau-L_1)h + \int_o^h F_+(\lambda)d\lambda\})$$

A little manipulation then gives

$$\hat{\rho}_- = \rho_o(1-\beta\eta_+\{\tau z-L_1 h - L_2(h-z)+\int_o^h F_+ d\lambda+ \int_z^h F\text{-}d\lambda\}) \tag{39}$$

$$z_o < z < h$$

and similarly for the other strip,

$$\hat{\rho}_- = \rho_o(1-\beta\eta_+\{\tau z+L_1 h+L_2(h+z) - \int_o^h F_+ d\lambda + \int_z^h F_- d\lambda\}) \tag{40}$$

$$-h < z < -z_o$$

Recall that in $z_o < |z| < h$, $\rho_z > 0$ in Part A. Here, as well,

$$\frac{\partial\hat{\rho}_-}{\partial z} = \frac{\rho_o p_- \tau + L_2 - F-(z)}{L_1 - F_+ \left(\frac{1-\hat{\rho}_-(z,t)/\rho_o}{\beta}\right)} > 0 \tag{41}$$

so what was true in A is also true in general.

Note that one can substitute the solution into (2) to obtain a more careful restriction on the solution than Pe>> 1, viz.,

$$S'(h)\,\frac{hU}{L} >> S''\kappa$$

or $\quad (\frac{h}{L}) P_e >> LS''(h)/S'(h)$

which shows clearly that (h/L) cannot be too small or the solution fails.

4. The Drag

The drag of the body moving through the fluid is obviously

$$D = \int_{-h}^{h} \frac{p_+(z)}{\sqrt{1+F'^2_+}} dz - \int_{h}^{h} \frac{p_-(z)}{\sqrt{1+F'^2_-}} dz \qquad (42)$$

which, by one integration by parts and use of the symmetry gives

$$D = 2g \int_0^h [\rho_+(z)f_+(z) \; - \rho_-(z)f_-(z)]\, dz \qquad (43a)$$

where

$$f_{\pm}(z) = \int_0^z \frac{d\lambda}{\sqrt{1+\{F'(\lambda)\}^2}} \qquad (43b)$$

A. The Flat Plate

Recall from 3. A. that the solution is

$$\rho_+ = \rho_o[1-\beta(1-\tau/L_1)] \;, \quad |z|<h$$

$$\rho_- = \rho_o[1-\beta(1+\tau/L_2)] \;, \quad |z|<z_o \qquad (44)$$

$$\rho_- = \rho_o\, [1+\frac{\beta}{L_1}(z[L_2+\tau] - Lh)] \;, \qquad z_o< z< h$$

with

$$z_o = \frac{h}{1+\tau/L_2}$$

Then one can easily compute the drag from (43) and (44) and the result is

$$D_1 = \frac{2}{3}\rho_o\beta g h^3 \frac{L_1+L_2}{2L_1}\left[1 - \frac{1}{\left(1+\frac{Ut}{L_2}\right)^2}\right] \tag{45a}$$

However, as was pointed out, in general the fluid in $\hat{\rho}_-$ is unstable, so suppose it mixes to a uniform density. In that case, (69c) is replaced by

$$\hat{\rho}_- = \rho_o\left(1 - \frac{1}{2}\beta h\left[2 - \frac{Ut}{L_1}\right] \operatorname{sgn}(z)\right)$$

in which case the drag becomes

$$D_2 = D_1 + \frac{1}{6}\rho_o\beta g h^3 \frac{L_2}{L_1}\frac{(Ut/L_2)^3}{(1+Ut/L_2)^2} \tag{45b}$$

However, as shown in Figure 3(b), the new density profile is still statically unstable. Suppose then that the mixing process extends below $z = z_o$ an amount δz_o to a level that makes the density continuous. Then, one finds that

$$\hat{\rho}_- = \rho_o(1-\beta h(1-\delta))$$

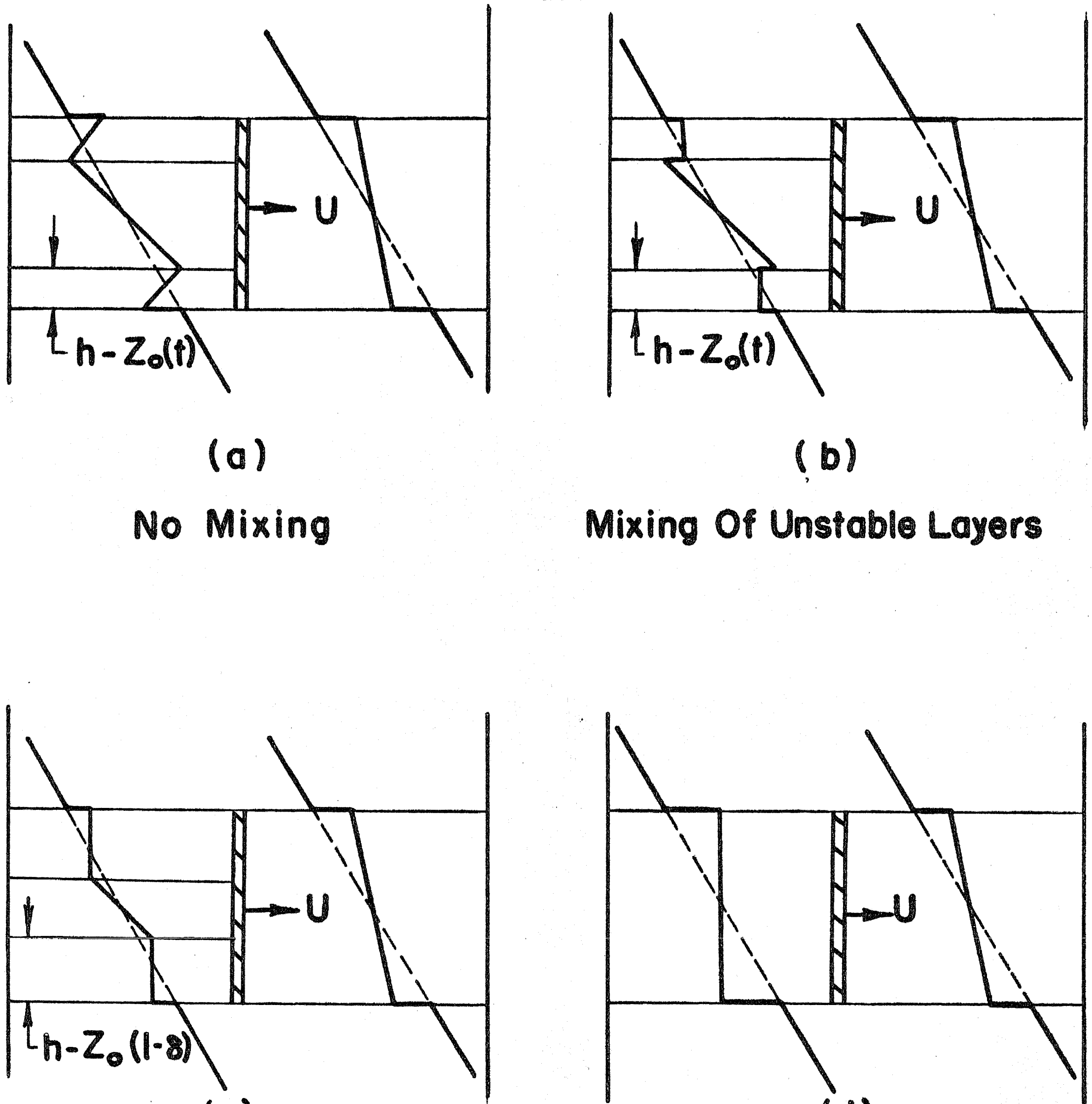
where δ is found to be

$$\delta = \frac{Ut}{L_2}\left[\sqrt{1+L_2/L_1} - 1\right]$$

and the drag is found to be

$$D_3 = D_2 + \frac{1}{3}(D_2 - D_1)\left[\frac{2L_1}{L_2}\left(\sqrt{1+\frac{L_2}{L_1}} - 1\right) - 1\right] \tag{45c}$$

If we suppose <u>all</u> of the fluid mixes, then $\rho_- \equiv \rho_o$ and hence



(a)

No Mixing

(b)

Mixing Of Unstable Layers

(c)

Mixing Penetration Into Unstable Region Far Enough To Make Profile Stable

(d)

Mixing Of All Fluid Behind The Plate

FIG. 3 POSSIBLE DENSITY PROFILES UNDER VARIOUS ASSUMPTIONS ABOUT THE MIXING PROCESS FOR THE FLAT PLATE

$$D_4 = -\frac{2}{3}\rho_o p\, gh^3 < 0 \tag{45d}$$

This curious result will be discussed later.

B. <u>A Thin Body</u>

Suppose that the body is very slender, i. e.,

$$\frac{F_\pm(0)}{h} << 1$$

In that case, write $F_\pm = \varepsilon g_\pm$, $\varepsilon << 1$. All of the formalism of Section 3. B. may now be exceedingly simplified. If $(\ )^{(p)}$ refers to the solution for the plate, then it is quite easy to get

$$\rho_+ = \rho_+^{(p)} + \frac{\rho_o \varepsilon \beta}{L_1}\left\{ \int_o^z g_+(\lambda)d\lambda - \int_o^{z(1-\tau/L_1)} g_-(\lambda)d\lambda \right\} + O(\varepsilon^2)$$

$$\rho_- = \rho_-^{(p)} + \frac{\rho_o \varepsilon \beta}{L_2}\left\{ \int_o^z g_-(\lambda)d\lambda + \int_o^{z(1+\tau/L_2)} g_+(\lambda)d\lambda \right\} + O(\varepsilon^2)$$

$$\rho_- = \rho_-^{(p)} + \frac{\rho_o \varepsilon \beta}{L_1}\left\{ \int_z^h g_- d\lambda + \int_{\frac{L}{L_1}h - \frac{L_{2+\tau}}{L_1}z}^{h} g_+(\lambda)d\lambda \right\} + O(\varepsilon^2) \tag{46}$$

and

$$z_o(\tau) = \frac{h}{1+\tau/L_2}\left[1 - \frac{\varepsilon}{hL_2} \int_{\frac{h}{1+\tau/L_2}}^{h} g_-(\lambda)d\lambda \right]$$

Any other than slender body shapes are of course consistent with the general solution in Section 3, but in practice are very difficult because the inversion of (36b) and (37b) to give an explicit representation is very difficult; for a cylinder, for example, one finds that (36b) becomes

$$\xi = -L_1 z + \frac{1}{2} z \sqrt{h^2 - z^2} + h^2 \sin^{-1}(z/h)$$

which is obviously analytically impossible to invert. Using (46) with $g_{\pm}(z) = h \cos(\frac{z}{2h})$ will then give

$$\rho_+ = \rho_+^{(p)} + \frac{2\varepsilon \rho_o \beta h^2}{\pi L_1} \left[\sin \frac{\pi z}{2h} - \sin \frac{\pi z}{2h} (1 - \tau / L_1) \right] \tag{47a}$$

$$\rho_- = \rho_-^{(p)} + \frac{2\varepsilon \rho_o \beta h^2}{\pi L_2} \left[\sin \frac{\pi z}{2h} + \sin \frac{\pi z}{2h} (1 + \tau / L_2) \right] \tag{47b}$$

$$\hat{\rho}_- = \hat{\rho}_-^{(p)} + \frac{2\varepsilon \rho_o \beta h^2}{\pi L_1} \left[2 - \sin(\frac{\pi z}{2h}) - \sin \frac{\pi}{2} [\frac{L}{L_1} - \frac{L_2 + \tau}{L_1} \frac{z}{h}] \right] \tag{47c}$$

and

$$z_o = \frac{h}{1 + \frac{\tau}{L_2}} \left[1 - \frac{2\,h}{L_2 \pi} (1 - \sin \frac{\pi}{2(1 + \tau / L_2)}) \right] \tag{47d}$$

From (47), the drag with the (43) formulation and $g_+ \equiv g_-$ is

$$D = 2g \int_0^h z [\rho_+(z) - \rho_-(z)]\, dz + O(\varepsilon^2) \tag{48}$$

since the cosine factor in the integral is $O(\varepsilon^2)$. Here, again, $(\ )^{(p)}$ denotes the flat plate solution. If $\varepsilon(\ )_1$ denotes the corrections to the $(\ )_o$ quantities, then the drag is

$$D = D_o + 2g\varepsilon \left[-\int_{\frac{h}{1+\tau/L_2}}^{h} \hat{\rho}_{-_1} z dz - \int_0^{\frac{h}{1+\tau/L_2}} \rho_{-_1} z dz + \int_0^h \rho_{+_1} z dz \right]$$

$$+ O(\varepsilon^2) \tag{49}$$

For the special case given in (47), the result of this computation for $L_1 \equiv L_2$ is

$$D_1 = D_{10} + \varepsilon \frac{16\rho_o \beta h^4 g}{\pi^3 L_1} \left[\frac{\pi(1-(\tau/L_1)^2)-4\tau/L_1}{(1-(\tau/L_1)^2)^2} \cos\frac{\pi\tau}{2L_1} + (2-(\frac{\pi}{2})^2) \left[1 - \frac{1}{(1+Ut/L_1)^2}\right]\right]$$

$$= \left\{1 - \frac{24\varepsilon h}{\pi^3 L_1} [(\frac{\pi}{2})^2 - 2]\right\} D_{10} + \frac{16\rho_o \beta h^4 g}{\pi^3 L_1} \varepsilon \frac{\pi(1-\tau^2/L_1{}^2)-4\tau_1/L_1}{[1-(\tau_1/L_1)^2]^2} \cos\frac{\pi\tau}{2L_1} \qquad (50a)$$

and

$$D_2 = D_{20} + \varepsilon \frac{16\rho_o \beta h^4 g}{\pi^3 L_1} \left[\frac{\pi}{2(1-\tau/L_1)} \sin\frac{\pi\tau}{2L_1} - \frac{1}{(1-\tau/L_1)^2} \cos\frac{\pi\tau}{2L_1} - \sin\frac{\pi}{2(1+\tau/L_1)} + \cos\frac{\pi}{2(1+\tau/L_1)} \left[\frac{\pi+1+2\tau/L_1}{2[1+\tau/L_1]}\right]\right] \qquad (50b)$$

and similarly for D_3. Also,

$$D_4 = D_{40} + \varepsilon \frac{16\rho_o \beta h^4 g}{\pi^3 L_1} \left[1 + \frac{\pi}{2(1-\tau/L_1)} \cos\frac{\pi}{2}(1-\frac{\tau}{L_1}) - \frac{\sin\frac{\pi}{2}(1-\tau/L_1)}{(1-\tau/L_1)^2}\right] \qquad (50c)$$

C. Remarks on the Drag

We noticed in A that, though the drag is positive with slight mixing, when the region behind the plate is completely mixed, then the plate experiences a thrust. One can alternatively compute the drag by considerations of potential energy directly. Recall that Figure 3 shows the successive possible density configurations, depending on the assumption involved in the mixing process. If the plate experiences a drag, that means that the plate is doing work on the fluid, so the potential energy must increase in that case. If we let the potential energy reference be zero on $z = 0$, then, in the case when the body is a plate, $V = -\ell \int_0^{z_1} g\rho(z,t) z\,dz$. Then, in front of the plate,

$$V_f(h) = (1-\tau_1)[V(0) - \frac{1}{3}\tau \rho_o \beta h^3] \leq V(0)$$

and behind the plate, a similar expression. The central region behind the plate, $|z| < z_o$, similarly has V increasing with time. The two regions $z_o < |z| < h$ have potential energy decreasing. The sum decreases, so there is drag on the plate. However, the resultant density structure is unstable. If it mixes as in Figure 3(b), the two unstable strips have their potential energy further increased so the drag is higher yet, and it increases still if the profile is as pictured in Figure 3(c). This apparently is the maximum drag configuration, because any further penetration of the mixing zone will, just from the geometry, decrease the potential energy, and hence lower the drag as well.

On the other hand, suppose in the mixing process that all of the fluid behind the plate gets mixed to a uniform density ρ_o. This clearly is a state of <u>much</u> lower potential energy, so, since the potential energy in front of the plate decreases as well, the plate actually experiences a thrust. So, the unstable stratification behind the plate triggers mixing that not only stabilizes the fluid, but mixes enough to destroy the density gradient altogether, thus greatly decreasing the potential energy and pushing the plate against the far wall.

5. The Boundary Layers on Non-horizontal Solid Surfaces

If n is a coordinate normal to the boundary and s is a measure of distance along it, then clearly the equations of motion in the boundary layer are

$$\frac{\partial p}{\partial n} = 0 \tag{51}$$

$$\frac{dp}{ds} + \frac{\rho g}{\sqrt{1+F'^2}} = \mu \frac{\partial^2 \overline{w}}{\partial n^2} \tag{52}$$

$$\frac{\partial \rho}{\partial t} + \overline{u} \frac{\partial \rho}{\partial n} + \overline{w} \frac{\partial \rho}{\partial s} = 0 \tag{53}$$

$$\frac{\partial \overline{u}}{\partial n} + \frac{\partial \overline{w}}{\partial s} = 0 \tag{54}$$

where $\overline{u}$ and $\overline{w}$ are velocities in the normal and tangential directions respectively. If δ is the thickness of the layer, we are here considering the case for which

$$\frac{\kappa}{\delta^2} << U/h \tag{55a}$$

and also neglect the inertia terms in the momentum equations, which means that

$$\varepsilon << R^{\frac{1}{2}} \tag{55b}$$

from (10). We have neglected unsteady terms in (52) because they are not important to the layer structure on the kinematic scale L/U. There do exist other limits of the equations apart from the one described here that involve these unsteady terms. None the less, these limits seem to describe the initial stages of growth and formation of this quasi-steady layer. Since $\overline{w}$ is O(U), however, and s is O(h), one must keep the time rate of change of the density in (53). Clearly the only balance possible is for

$$\delta = (\nu U/g)^{\frac{1}{2}}$$

so formally, if one puts $n = LR^{\frac{1}{2}}\eta$, $s = L\xi$, $\rho = \rho_o\overline{\rho}$, $\overline{u} = UR^{\frac{1}{2}}\overline{U}$, $\overline{w} = U\overline{W}$, then the non-dimensional equations are

$$\frac{\partial p}{\partial \eta} = 0$$

$$\frac{\overline{\rho}(\xi,\eta) - \overline{\rho}(\xi,\infty)}{\sqrt{1+F'(\xi)^2}} = \frac{\partial^2 \overline{W}}{\partial \eta^2} \quad (56)$$

$$\frac{\partial \overline{\rho}}{\partial t^*} + \overline{U}\frac{\partial \overline{\rho}}{\partial \eta} + \overline{W}\frac{\partial \overline{\rho}}{\partial \xi} = 0 \quad (57)$$

$$\frac{\partial \overline{U}}{\partial \eta} + \frac{\partial \overline{W}}{\partial \xi} = 0 \quad (58)$$

Then, (55a) becomes just

$$\kappa << \nu U^2/gh$$

or $$\left(\frac{L}{h}\right) \mathrm{Pe}\, R^{-1} >> 1 \quad (59)$$

which imposes an additional constraint on the solution. Further, this layer ceases to be thin near places when $F' \to \infty$, i. e.,

$$\frac{dF}{dz} << \left(\frac{h}{L}\right)^2 R^{-1} \tag{60}$$

This layer is nothing like the Ekman-layer (or actually "buoyancy layer" in Veronis, [(3)]) because, primarily, of (59) and the original Pe >> 1 restriction. One can easily combine (56) and (57) to make

$$\left\{\frac{\partial}{\partial t^*} + \overline{U}\frac{\partial}{\partial \eta} + \overline{W}\frac{\partial}{\partial \xi}\right\}\sqrt{1+F'^2}\ \frac{\partial^2 \overline{W}}{\partial \eta^2} + [W(\xi,\eta) - W(\xi,\infty)]\frac{\partial \overline{\rho}(\xi,\infty)}{d\xi} = 0 \tag{61}$$

subject to the following boundary conditions

$$\begin{aligned} \overline{W}(\xi, 0) &= 0 \\ \overline{U}(\xi, 0) &= 0 \end{aligned} \tag{62}$$

where we have written

$$\overline{W}(\xi,\infty) = \pm \left.\frac{w + F'(z)u}{U\sqrt{1+F'^2}}\right|_{\text{on the surface}} \tag{63}$$

and $\quad \overline{\rho}(\xi,\infty) = \rho/\rho_o \Big|_{\text{on the surface}}$

We have defined η so that it is always positive in the layer, and the sign in (63) is chosen + or - as the fluid is to the right or left of the boundary.

An asymptotic solution to (61) may be obtained for the case $\rho_{o_\xi} < 0$, and is just the solution for the flat plate,

$$\left\{\frac{\partial}{\partial t} + \overline{W}(\infty,\xi)\frac{\partial}{\partial \xi}\right\}\frac{\partial^2 \overline{W} - \overline{W}(\infty)}{\partial \eta^2} + \overline{\rho}_\xi(\xi,\infty)[\overline{W} - \overline{W}(\infty)] = 0$$

and by Laplace transform a solution of this may be obtained, for the (+) region,

$$\overline{W} - \overline{W}(\infty) = f(\xi(1-t^*)) \int_0^\infty \frac{\sin p\pi}{\pi} x^p e^{+\rho_{o_\xi} t^*/x} e^{-x\eta} dx$$

$$-1 < p < +1, \quad p \neq 0 \tag{64a}$$

which has the asymptotic expansion as $\eta \to +\infty$,

$$\overline{W} - \overline{W}(\infty) \sim \left(\frac{-\overline{\rho}_{o_\xi} t^*}{\eta} \right)^{p/2} \frac{\sin p\pi}{\left(-\overline{\rho}_{o_\xi}^{\,3} \eta^5 \pi^2\right)^{1/4}} f e^{-\sqrt{-\overline{\rho}_{o_\xi} t^* \eta}} \tag{64b}$$

Notice the single arbitrary function in (64b) that would be determined by (62). In general, the structure of the solutions to (61)-(63) is not simple. If $\omega(\eta) \equiv -\frac{\partial W}{\partial \eta}(\xi, \eta, t^*)$, then an integral of (61) gives

$$\frac{\partial \omega(0)}{\partial t^*} + \frac{1}{\sqrt{1+F'(\xi)^2}} \frac{\partial}{\partial \xi} \sqrt{1+F'(\xi)^2} \int_0^\infty \omega^2 d\eta$$

$$+ \overline{\rho}_{o_\xi} \int_0^\infty \eta \omega \, d\eta = 0$$

This layer does not of course satisfy boundary conditions on ρ; an interior layer is required for that purpose. From arguments similar to those given one can show the equation for ρ to be

$$\frac{\partial^2 \rho}{\partial y^2} + y^2 \alpha(\xi, t) \frac{\partial \rho}{\partial y} = 0 \tag{65a}$$

where y is in the n-direction, scaled with the layer thickness $(R/Pe)^{\frac{1}{3}}$. Here, we have used the asymptotic expansion for $\overline{U}$ from the outer layer solution, i. e.,

$$\overline{U}(\xi, \eta, t) \sim -\eta^2 \alpha(\xi, t^*) + \ldots \tag{65b}$$

If the boundary condition on ρ is

$$\rho(0, \xi, t^*) = \rho_o \, f(\xi, t^*)$$

then the solution is

$$\frac{\rho}{\rho_o} = f(\xi, t^*) + [\overline{\rho}(0, \xi^*, t^*) - f(\xi, t^*)] \frac{\int_o^y e^{-\frac{1}{3}\alpha(\xi)\lambda^3} d\lambda}{\int_o^\infty e^{-\frac{1}{3}\alpha\lambda^3} d\lambda} \tag{66}$$

If the surface is insulating (or, if ρ variations are due to variations in salt concentration) then

$$\frac{\partial \rho}{\partial y}(0, \xi, t^*) = 0$$

Now, (65a) must be replaced by

$$\frac{\partial^2 \rho}{\partial y^2} + y^2 \alpha \frac{\partial \rho}{\partial y} = y^2 \alpha \frac{\partial \rho}{\partial y}(\infty) \tag{67a}$$

Then, we get

$$\rho_y = \rho_y(\infty) \left\{ 1 - e^{-\frac{1}{3}\alpha(\xi, t^*) y^3} \right\} \tag{67b}$$

6. <u>The Horizontal Shear Layers</u>

We found in Section 3 that in general there will be a jump in (u, w, ρ) across $z = \pm h$ (see Figure 1). The jump in w produces the largest mass flux in a shear layer, so it is the discontinuity to be dealt with first. The approach, as in Section 5, will be to study the shear layers for $Pe \equiv \infty$. Having done that, if $Pe >> 1$, or some more restrictive requirement, one can insert an interior diffusive layer. Making the usual boundary layer approximations, and again supposing κ so small that

$$\kappa << \delta U \tag{68}$$

where δ is the layer thickness, the leading order terms are

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0 \tag{69}$$

$$\frac{\partial p}{\partial z} + \rho g = 0 \tag{70}$$

$$\frac{\partial p}{\partial x} = \mu \frac{\partial^2 u}{\partial z^2} \tag{71}$$

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + w \frac{\partial \rho}{\partial z} = 0 \tag{72}$$

One can explore the possible limits in a variety of ways. Doing it intuitively, it is clear that there must be vorticity produced by the gravitational field as the flow turns. So, (70) and (71) give

$$g \frac{\partial \rho}{\partial x} + \mu \frac{\partial^3 u}{\partial z^3} = 0 \tag{73}$$

If one puts $\rho = \rho_o(z,t) + \delta\hat{\rho}\,(z/\delta, x, t)$ into (72), the result is

$$\frac{\partial \rho_o}{\partial t} + w \frac{\partial \rho_o}{\partial z} + \delta u \frac{\partial \hat{\rho}}{\partial x} + w \frac{\partial \hat{\rho}}{\partial z'} = 0 \tag{74}$$

where $z' \equiv z/\delta$. From (69), $u = \frac{1}{\delta}\hat{u}$ and (73) then gives clearly the thickness

$$\delta = (\nu UL/gh^3)^{1/5} = O(R^{1/5})$$

and (68) becomes

$$\mathrm{PeR}^{-1/5}\left(\frac{h}{L}\right)^{2/5} >> 1 \tag{75}$$

which is less restrictive than (59). If we scale everything with the obvious quantities, then (69), (73), (74), with $\zeta \equiv (z \pm h)/LR^{1/5}$,

$$\frac{\partial^3 u^*}{\partial \zeta^3} + \frac{\partial \rho^*}{\partial x^*} = 0 \tag{76}$$

$$\frac{\partial u^*}{\partial x^*} + \frac{\partial w^*}{\partial \zeta} = 0 \tag{77}$$

$$u^* \frac{\partial \rho^*}{\partial x^*} + w^* \frac{\partial \rho^*}{\partial \zeta} + \frac{\rho_{o_z} L}{\rho_o}\bigg|_{\pm h} (w^* - w^*(\pm\infty)) = 0 \tag{78}$$

where the $w^*(\pm\infty)$ term enters in the layers $\zeta < 0$, but is identically zero for the layers adjacent to the undisturbed fluid in $|z| > h$. If we denote by $\pm$ the layers in front and behind the body on $|z| = h-$, and by $(\;)'$, the single layer on $|z| = h+$, then there are a few obvious boundary conditions,

$$\begin{aligned} \rho^*(\pm\infty, x, t) &= 0 \\ u^*(\pm\infty, x, t) &= 0 \end{aligned} \tag{79a}$$

and by matching,

$$\begin{aligned} w^*_{\pm}(-\infty, x, t) &= w_{\pm}(h, t) \text{ on } z = h \\ w^*_{\pm}(\infty, x, t) &= w_{\pm}(-h, t) \text{ on } z = -h \end{aligned} \tag{79b}$$

plus $\quad w^{*\prime}(\pm\infty, x, t) = 0 \text{ on } z = \pm h \qquad (79c)$

The interior density layer that lies between the layers to smooth the density discontinuity seems to require the interface to be a streamline.

Hence, if

$$\zeta = \eta_{\pm}(x) \tag{79d}$$

denotes the interface between these back-to-back layers, we must clearly have

$$\begin{aligned} w^*_{\pm}(\eta(x), x, t) &= \eta'_{\pm}(x) u^*_{\pm}(\eta, x, t) \\ w^{*\prime}(\eta(x), x, t) &= \eta'_{\pm}(x) u^{*\prime}(\eta(x), x, t) \end{aligned} \tag{79e}$$

Quite clearly the tangential velocities must be continuous, so

$$w^*_{\pm}(\eta(x), x, t)\eta'_{\pm}(x) + u^*_{\pm}(x, t) = w^{*\prime}(\eta, x, t)\eta'_{\pm} + u^{*\prime}(x, t)$$

$$\text{for } x \gtrless Ut \tag{79f}$$

The reason why $\eta_{\pm}(x)$ is non-zero is just that it must be chosen such that the pressure perturbation from an integral of (71) is continuous across the interface. It is easy to show that this is equivalent to requiring that

$$\int_{-\infty}^{\infty} \rho^*_{\pm}(\zeta, x, t) d\zeta = 0 \tag{79g}$$

By analogy from the rotating flow (Stewartson)layers, [2] the correct condition on u_ζ will be

$$\frac{\partial u^{*\prime}}{\partial n}(\eta, x, t) = \frac{L_1 - Ut}{L_1 + L_2} \frac{\partial u^*_+}{\partial n}(\eta, x, t) + \frac{L_2 + Ut}{L_1 + L_2} \frac{\partial u^*_-}{\partial n}(\eta, x, t) \tag{79h}$$

where $\frac{\partial}{\partial n}$ is symbolic of

$$\frac{\partial}{\partial n} = \frac{\partial}{\partial \zeta} - \eta'(x) \frac{\partial}{\partial x}$$

A. Asymptotic Solution in ±

For $|\zeta| \to \infty$ and $w^*(\pm\infty) \neq 0$, that is, for solutions denoted by + or -, an asymptotic expansion of the solution to (76)-(79) is possible. We write

$$w^*_{\pm} - w^*_{edge} = w_o + w_1 + w_2 + \ldots$$

$$u^*_{\pm} = u_o + u_1 + u_2 + \ldots$$

$$\rho^*_{\pm} = \rho_o + \rho_1 + \rho_2 + \ldots$$

where this will in fact be an asymptotic expansion if $\phi_{n+1} = o(\phi_n)$, $|\zeta| \to \infty$, $w^*_{edge} \neq 0$. The hierarchy is

$$\frac{\rho_{o_z}{}^{L}}{\rho_o} w_{n+1} + w^*(\pm\infty) \frac{\partial \rho_{n+1}}{\partial \zeta} = -u_n \frac{\partial \rho_n}{\partial x^*} - w_n \frac{\partial \rho_n}{\partial \zeta}$$

$$\frac{\partial u_{n+1}}{\partial x^*} + \frac{\partial w_{n+1}}{\partial \zeta} = 0 \tag{80}$$

$$\frac{\partial^3 u_{n+1}}{\partial \zeta^3} + \frac{\partial \rho_{n+1}}{\partial x^*} = 0, \qquad n = -1, 0, 1, \ldots$$

and $u_{-1} = w_{-1} = \rho_{-1} \equiv 0$. To leading order, (80) gives

$$\frac{\rho_{o_z}{}^{L}}{\rho_o} (\pm h)\, w_o + \frac{w_{\pm}(\pm h)}{U} \frac{\partial \rho_o}{\partial \zeta} = 0$$

$$\frac{\partial u_o}{\partial x^*} + \frac{\partial w_o}{\partial \zeta} = 0 \tag{81}$$

$$\frac{\partial^3 u_o}{\partial \zeta^3} + \frac{\partial \rho_o}{\partial x^*} = 0$$

Now, (81) can be reduced to

$$\frac{\partial^5 u_o}{\partial \zeta^5} + \frac{\rho_{o_z} LU}{\rho_o(h) w_\pm(h,t)} \frac{\partial^2 u_o}{\partial x^{*2}} = 0 \tag{82}$$

Now, it is always true, from Section 3, that

$$\frac{\rho_{o_z} LU}{\rho_o(h) w_\pm(h,t)} < 0$$

so, if $\mu \equiv [-\pi^2 \rho_{o_z} LU/\rho_o(h) w_\pm(h,t)]^{1/5}$, then an asymptotic expansion for the layers on z = h- begins with

$$u_o = \sin\pi x \; e^{\mu\zeta\cos\pi/5}[a\sin(\mu \sin\tfrac{\pi}{5}\zeta) + b\cos(\mu\sin\tfrac{\pi}{5}\zeta)]$$

$$w_o = -\frac{\pi\cos\pi x}{\mu} \; e^{\mu\zeta\cos\pi/5}\,[(a\cos\tfrac{\pi}{5} + b\sin\tfrac{\pi}{5})\sin(\mu\sin\tfrac{\pi}{5}\zeta)$$

$$+ (b\cos\tfrac{\pi}{5} - a\sin\tfrac{\pi}{5})\cos(\mu\sin\tfrac{\pi}{5}\zeta)] \;, \quad \zeta \to -\infty \tag{83}$$

One can show, with considerable labor, that w_1, ρ_1, u_1 all decay like $(e^{\mu\zeta\cos\zeta/5})^2$, $\zeta \to -\infty$, so the series is indeed asymptotic.

Approximate (Galerkin) solutions to (76)-(79) are not at all easy and have not as yet been found, since they involve solution of the simultaneous, non-linear ordinary differential equations.

B. The Density Layer

Clearly the equation to be solved to remove the density jump is

$$u\frac{\partial\rho}{\partial\tilde{x}} + w\frac{\partial\rho}{\partial\tilde{z}} = \kappa\frac{\partial^2\rho}{\partial\tilde{z}^2} \tag{84}$$

In this layer, $\frac{\partial u}{\partial\tilde{z}} = 0$ so that $u = UR^{-1/5}\, c(\tilde{x})$ from the layer described previously, and so

$$w = -\tilde{z}\, UR^{-1/5}\frac{dc}{d\tilde{x}}$$

so that (84) becomes

$$c(\tilde{x})\frac{\partial\rho}{\partial\tilde{x}} - \tilde{z}\frac{dc}{d\tilde{x}}\frac{\partial\rho}{\partial\tilde{z}} = \frac{\kappa}{UR^{-1/5}}\frac{\partial^2\rho}{\partial\tilde{z}^2} \tag{85}$$

The z and x used here are actually, in general, not the previous z and x, but a new orthogonal system with z in the direction $\underline{k} - \eta(x)$. If we now put

$$s \equiv \int_1^{\tilde{x}} c(\tilde{x})d\tilde{x}$$

then $\rho = \rho(s, yc(\tilde{x}))$, where y is a coordinate scaled with the thickness $L(R^{1/5}/Pe)$ of this layer, and (85) is

$$\frac{\partial\rho}{\partial s} = \frac{\partial^2\rho}{\partial(yc)^2}$$

and the solution is

$$\rho = \frac{1}{2}\left[\rho(-\infty)+\rho(+\infty)\right] + \frac{1}{2}\left[\rho(+\infty)-\rho(-\infty)\right]\operatorname{erf}\left\{\frac{y\,|c(\tilde{x})|}{2\sqrt{s(\tilde{x})}}\right\} \tag{86}$$

7. <u>Conclusion</u>

Although many details are left undone in the sense that solutions for the boundary layers and shear layers have not been constructed, though the equations and boundary conditions have been set down, we seem to have a self-consistent asymptotic theory for the slow motion of a body through a fluid of small thermal conductivity. The drag calculation required only

$$\varepsilon \ll 1, \; R \ll 1, \; Pe \gg L/h$$

However, the shear layers and boundary layers were fitted under further restrictions, the most severe of which, for $h = O(L)$, were, from the boundary layer structure,

$$R^{-1}\, Pe >> 1, \quad \varepsilon << R^{\frac{1}{2}}$$

The second of these conditions is reminiscent of Rossby No. << (Ekman No, $)^{\frac{1}{2}}$ in certain rotating flow problems. We also found that (h/L) must not be too small, viz.,

$$\frac{h}{L} >> \frac{1}{Pe}$$

$$\frac{h}{L} >> R^{\frac{1}{2}}/Pe^{5/2}$$

It appears that this flow will never be analogous to a rotating flow. In the work on the analogy between rotating fluids and stratified fluids, Veronis,[3] uses a velocity scale determined by the buoyancy forces. If the buoyancy forces are small, so are the velocities, and only the Rayleigh number and a number (equivalent to Rossby number) measuring the scale of the density variations compared to a mean density are required to characterize the problem. However, this problem has a velocity scale determined by the motion of the boundaries, not by the buoyancy forces. Further, slow motion (small Froude number) does not necessarily imply small changes in the density field. Hence, it should be no surprise that this flow bears no similarity to an analogous rotating flow. It appears to be the case that stratified fluids behave analogously to rotating fluids only if the boundary conditions are not kinematic but rather on ρ or $\frac{\partial \rho}{\partial n}$ (as for salts). In the rise of a body through a rotating fluid,[1,2] the fluid is squeezed out in Ekman layers. Here, the fluid is squeezed out through the entire slug of fluid ahead of the body which is precisely why the drag is $O(\varepsilon^o)$ here, and $O(\varepsilon)$ for the rotating fluid.

The small Peclet number problem is not especially trivial, and though it appears that the drag is O(1) in that case, the details of the flow have not been worked out as yet. This problem remains an item for further research.

An experimental check of these predictions seems quite feasible; glycerine has a large Prandtl number (~ 7000), and certainly the Schmidt number for salt is very large as well, so it seems possible to arrange the problem as shown in Figure 1. Of particular interest would be the mixing process behind the body.

REFERENCES

1. Saffman and Moore, "The Rise of a Body Through a Rotating Fluid," Journal of Fluid Mechanics, 31 (1968), 635.

2. Moore, D. W. and Saffman, P. G., "The Structure of the Free Vertical Shear Layers in a Rotating Fluid and the Motion Produced by a Slowly Rising Body," Philosophical Transactions, A., of the Royal Society, to appear.

3. Veronis, G., "Analogous Behavior of Homogeneous Rotating Fluids and Stratified, Non-rotating Fluids," Tellus, 19 (1967), 326.

4. Boussinesq, J., Theorie analytique de la Chaleur, Gaither-Villars, Paris, 2 (1903).