TYPE D GRAVITATIONAL FIELDS

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William Morris Kinnersley

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ABSTRACT

The Newman-Penrose tetrad equations are set up for the principal tetrad of a Type D gravitational field in vacuum. With no further assumptions, the equations are integrated, yielding an exhaustive list of Type D vacuum metrics. The solutions all possess two commuting Killing vectors and depend on from one to four arbitrary constants. The Type D fields with expanding rays are six closely related versions of Kerr-NUT space, the Ehlers-Kundt "C" metric, and a new generalization of the "C" metric possessing rotation. For zero expansion we find the three Ehlers-Kundt "B" metrics, plus rotating generalizations of each.

The six Kerr-NUT metrics are interpreted as spinning particles with timelike, lightlike, or spacelike momentum and angular momentum vectors occurring in all possible combinations. The "C" metric is tentatively identified as a gravitational analog of the runaway solutions encountered in electrodynamics, i. e. , a point mass executing hyperbolic motion.

Next we consider Type D fields with electromagnetism present. We find that all of the above vacuum metrics can be readily "charged" by adding a non-null electromagnetic field whose principal null vectors coincide with the gravitational ones. We also discuss some interesting generalizations of the Schwarzschild and "C" metrics containing the geometrical optics limit of a null electromagnetic field which propagates along one principal null congruence. In the Schwarzschild case they generalize Vaidya's "shining star" metric to include the field of a particle traveling along an arbitrarily accelerated world-line.

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I. INTRODUCTION

Two of the oldest and most widely known solutions to Einstein's field equations are the Schwarzschild and Nordstrom metrics, which describe the gravitational fields of a point mass and a point charge respectively. The Schwarzschild vacuum solution can be "electrified" to yield the Nordstrom solution by the simple substitution $m \to m - e^2/r$ in the metric. (Here e and m are the charge and mass in suitable units, and r is the radial co-ordinate defined such that the area of a sphere centered at the origin is $4\pi r^2$.)

When the Petrov algebraic classification of gravitational fields was introduced,[1] it was noticed that both of these metrics are "Type D", which meant in this instance that there is only one independent component of the Riemann curvature tensor. Other Type D metrics have been discovered by Levi-Civita,[2] Newman et al.,[3,4] Kerr,[5] and Carter.[6] In each case the solutions exhibit a high degree of symmetry, and in some cases electrifications are also known.

Whether by design or coincidence, the Type D solutions have played key roles in bringing to light many interesting and unsuspected problems, thereby acting as a stimulus for further research. We mention in particular the occurrence of event horizons,[7] the analytic incompleteness of manifolds,[8] the conical behavior of NUT space,[9] and the hunt for a rotating source to match the Kerr metric.[10] Finding and understanding more Type D fields might therefore prove quite useful.

In this work we will use the Newman-Penrose null tetrad methods[11] to find all Type D vacuum fields and a wide class of Type D fields with electromagnetism present. Since the NP Formalism is unfamiliar to many people, we give a complete description of it here. For those who are interested in the results but not the methods, we give explicitly the metric coefficients for each solution encountered. Although the primary aim of this work is derivation rather than description, some first steps toward understanding the nature of the solutions are made. Many questions remain to be answered.

II. TETRAD METHODS IN RELATIVITY

A. Importance

A tetrad field consists of a set of four orthonormal vectors at each point of space-time. The differential geometry of such a structure immersed in a general Riemannian manifold was first considered by Ricci[12] in 1895. By the 1930's, tetrads were being used to advantage in connection with General Relativity. The most spectacular success of the tetrad method has come since 1961, when Sachs[13] and Newman and Penrose[14] began using null tetrads to study gravitational radiation. Today tetrads are used, explicitly or implicitly, in such a variety of ways with so many different notations that the essential unity of the subject is often overlooked. For example, the method of anholonomic co-ordinates of Schouten,[15] the differential forms used by Cartan[16] and recently Misner,[17] the curved space spinor calculus pursued by Baade and Jehle,[18] and also the Fock-Ivanenko coefficients[19] are all equivalent to the introduction of a tetrad field.

The interpretation and significance of the tetrad approach is a point not universally agreed upon. In the earlier geometrical view espoused by Einstein with its emphasis on the metric tensor, a tetrad was at best an auxiliary field, and often regarded as an unfortunate complication. More recently the relative importance of metric and tetrad has undergone a complete reversal.

A tetrad provides at each point a local Minkowski reference frame. For this reason, Estabrook regards it as a natural extension to General Relativity of the familiar "local observer" concept of the special theory. I prefer to think that only the relative orientation of the tetrads at different points is important. Given a tensor at one point we can define "the same tensor" at other points to be the one with the same tetrad components. In other words, each choice of the tetrad field defines a particular "transport".

Next we should consider why tetrads are important. The immediate answer, of course, is that Einstein's field equations turn out to be easier to solve in the tetrad formalism than they are in terms of tensors. However, there are several deeper reasons why this is so. First (as we shall show in Equation (II. 1)) the array of tetrad vector components forms a "square root of the metric" in a natural way. In this sense it is a more fundamental object than the metric.

Second, the tetrad method seems more effective at isolating the simple gravitational fields for study. In the traditional approach one requires the metric to take a certain form or to depend on only a few co-ordinates, and one is guided mainly by aesthetics. There are examples of recent research in which this procedure is still being used to advantage.[20] On the other hand, vector fields have many geometric properties which are both physically relevant and easy to visualize. For instance they can be chosen geodesic or hypersurface-orthogonal. The importance of such choices will be discussed further in Section III-C.

There is one more reason for using tetrads in General Relativity which should not go without mention. It is totally impossible to introduce spinors in a curved space-time without using a tetrad field.[21] This is because the group of general co-ordinate transformations does not possess spinor representations.

The group of tetrad rotations does, since it is just the direct product of Lorentz groups at every point.

B. General Formalism

To begin the formal development, let the vectors of the tetrad be denoted by $h_a{}^\mu$, where μ is a tensor index; and where a numbers the vectors from 1 to 4 and is called a tetrad index.

Inner products of the vectors,

$$\eta_{ab} \equiv h_a{}^\mu h_{b\mu} ,$$

are assumed to be constants and given a priori. For orthonormal tetrad fields, η_{ab} is just the Minkowski matrix, Diag (-1, -1, -1, 1). Denote the corresponding inverse matrices by

$$h^a{}_\mu \equiv (h_a{}^\mu)^{-1}$$

$$\eta^{ab} \equiv (\eta_{ab})^{-1} .$$

The completeness relation is

$$g_{\mu\nu} = h^a{}_\mu h^b{}_\nu \eta_{ab} . \qquad \text{(II. 1)}$$

Then the following interpretation is a natural one to make: $g_{\mu\nu}$, $h_a{}^\mu$, η_{ab}, and their inverses are all regarded as "fundamental tensors," i. e., tensors which are used to raise and lower indices. The quantities $h_a{}^\mu$ change an index from a tensor index to a tetrad index or vice versa, while η^{ab} raises and lowers tetrad indices. Thus

$$T_{\mu\nu}$$

$$T_{a\nu} = h_a{}^{\mu} T_{\mu\nu}$$

$$T_{ab} = h_a{}^{\mu} h_b{}^{\nu} T_{\mu\nu}$$

$$T_{\mu}{}^{b} = h_a{}^{\nu} \eta^{ab} T_{\mu\nu}$$

are all assumed to be various manifestations of the "same" tensor T. This is largely a matter of convention. Then $g_{\mu\nu}$, $h_a{}^{\mu}$, η_{ab} are all manifestations of the same metric tensor g. The processes of changing from tensor index to tetrad index and back have been called "strangulation" and "resurrection" respectively.[22]

Associated with a tetrad field are several different types of derivative. A semicolon will be used to denote the usual covariant derivative, which ignores tetrad indices and treats $h_a{}^{\mu}$ as a set of vectors. Define

$$A^{\mu}{}_{a\nu} \equiv h_a{}^{\mu}{}_{;\nu} \, .$$

This field is known as the "Ricci rotation coefficient"[23] and is central in what follows. The strangled components $A_{ba\nu}$ are antisymmetric in a and b, since

$$(h_a{}^{\mu} h_{b\mu})_{;\nu} = h_{b\mu} h_a{}^{\mu}{}_{;\nu} + h_{a\mu} h_b{}^{\mu}{}_{;\nu} = 0 \, .$$

Let us calculate the change in the tetrad under an infinitesimal displacement:

$$\delta h_a{}^\mu \equiv \Omega_{ab} h^{b\mu} = h_a{}^\mu{}_{;\nu} \delta x^\nu$$

$$\Rightarrow \Omega_{ab} = A_{ba\nu} \delta x^\nu . \qquad \text{(II. 2)}$$

Thus $A_{ba\nu}$ measures the rotation of the tetrad in the ab plane when we step in the ν direction.

The "intrinsic derivative"[24] of any quantity (say $T_a{}^\mu$) with respect to a given tetrad is obtained by strangling completely, taking the covariant derivative of the resulting scalar, and then resurrecting. For example,

$$T_a{}^\mu{}_{|\nu} \equiv (T_a{}^\sigma h^b{}_\sigma)_{;\nu} h_b{}^\mu . \qquad \text{(II. 3)}$$

According to this definition, intrinsic differentiation commutes with strangulation and resurrection:

$$h^b{}_\mu T_a{}^\mu{}_{|\nu} = T_a{}^b{}_{|\nu} ,$$

and also

$$h_a{}^\mu{}_{|\nu} = 0 .$$

With the help of the Ricci rotation coefficient we can rewrite the intrinsic derivative in a suggestive form, with a correction term appearing for each tensor index:

$$T_a{}^{\mu}{}_{|\nu} = T_a{}^{\mu}{}_{;\nu} + T_a{}^{\sigma} A_{\sigma}{}^{\mu}{}_{\nu} \ .$$

The intrinsic derivative is closely related to the concept of transport. A transport $O_{\mu}{}^{\nu'}(x, x')$ is a tensor used to convey vectors from the point x to the point x'. The transport associated with a tetrad $h_a{}^{\mu}$ is

$$O_{\mu}{}^{\nu'}(x, x') = h^a{}_{\mu}(x) h_a{}^{\nu'}(x') \ .$$

If we are given $T_a{}^{\mu}$ at one point x_0 and define it everywhere by transport,

$$T_a{}^{\mu}(x) \equiv T_a{}^{\sigma}(x_0) h^b{}_{\sigma}(x_0) h_b{}^{\mu}(x) \ ,$$

then clearly the tetrad components $T_a{}^b$ are constants and

$$T_a{}^{\mu}{}_{|\nu} = 0 \ .$$

One can easily calculate the commutator of successive intrinsic differentiations. Applied to a tetrad index the commutator* is

$$T_{a|[\nu\sigma]} = T_{a|\tau} A^{\tau}{}_{[\nu\sigma]} \ . \qquad \text{(II. 4)}$$

Applied to a tensor index it is

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*actually one-half the commutator. $T_{a|[\nu\sigma]} \ \frac{1}{2}(T_{a|\nu\sigma} - T_{a|\sigma\nu})$.

$$T_{\mu|[\nu\sigma]} = T_{\mu|\tau}A^{\tau}{}_{[\nu\sigma]} + T_{\mu;[\nu\sigma]}$$
$$+ T_{\tau}(A^{\tau}{}_{\mu[\nu;\sigma]} - A_{\rho\mu[\nu}A^{\tau\rho}{}_{\sigma]}) . \qquad \text{(II. 5)}$$

Recalling that

$$T_{\mu;[\nu\sigma]} = -\frac{1}{2}T_{\tau}R^{\tau}{}_{\mu\nu\sigma}$$

and that strangulation commutes with intrinsic differentiation, we see that the only way Equations (II. 4) and (II. 5) can be made to agree is if

$$R^{\tau}{}_{\mu\nu\sigma} = 2A^{\tau}{}_{\mu[\nu;\sigma]} - 2A_{\rho\mu[\nu}A^{\tau\rho}{}_{\sigma]} , \qquad \text{(II. 6)}$$

or equivalently,

$$R^{\tau}{}_{\mu\nu\sigma} = 2A^{\tau}{}_{\mu[\nu|\sigma]} - 2A_{\rho\mu[\nu}A^{\tau\rho}{}_{\sigma]}$$
$$+ 2A^{\tau}{}_{\mu\rho}A^{\rho}{}_{[\nu\sigma]} . \qquad \text{(II. 7)}$$

As a particular example, apply the commutator to the co-ordinates x^{α}, which are scalars:

$$x^{\alpha}{}_{|\mu} = x^{\alpha}{}_{,\mu} = \delta^{\alpha}{}_{\mu}$$
$$x^{\alpha}{}_{|[\mu\nu]} = \delta^{\alpha}{}_{\tau}A^{\tau}{}_{[\mu\nu]} = A^{\alpha}{}_{[\mu\nu]} . \qquad \text{(II. 8)}$$

There is one gap remaining to be bridged. Since $A_{\tau\mu\nu}$ involves a covariant derivative, one would think that the Christoffel symbols are still necessary to calculate them. However because of its index symmetry, $A_{\tau\mu\nu}$ may be expressed in terms of $A_{\tau[\mu\nu]}$ which contains only partial derivatives. Any tensor $A_{\mu\nu\sigma}$, anti-symmetric in μ and ν satisfies

$$A_{\mu\nu\sigma} = A_{\mu[\nu\sigma]} + A_{\nu[\sigma\mu]} + A_{\sigma[\nu\mu]} . \qquad \text{(II. 9)}$$

The full procedure for calculating $R_{\mu\nu\sigma\tau}$ via the tetrad formalism is as follows: obtain $A_{\tau[\mu\nu]}$ from Equation (II. 8), $A_{\tau\mu\nu}$ from Equation (II. 9), and then $R_{\mu\nu\sigma\tau}$ from Equation (II. 7).

There is a third important type of derivative in tetrad analysis, called "invariant differentiation."[25] Its definition is similar to the one for intrinsic differentiation, but this time we first resurrect all indices, take the covariant derivative of the resulting tensor, and then strangle. That is,

$$T_{a}{}^{\mu}{}_{\cdot\nu} \equiv (T_{b}{}^{\mu} h^{b}{}_{\sigma})_{;\nu} h_{a}{}^{\sigma} . \qquad \text{(II. 10)}$$

As before, invariant differentiation commutes with strangulation and

$$h_{a}{}^{\mu}{}_{\cdot\nu} = 0 .$$

When a tensor T is expressed in terms of its tensor components $T_{\mu\nu\ldots}$, its invariant derivative is identical to its covariant derivative. For this reason the invariant derivative is the natural generalization of the covariant derivative to tetrad analysis. The invariant derivative can be rewritten using the $A^{b}{}_{a\nu}$ as:

$$T_a{}^{\mu}{}_{\cdot\nu} = T_a{}^{\mu}{}_{;\nu} - T_b{}^{\mu} A^{b}{}_{a\nu} ,$$

with a similar correction term for each tetrad index.

As can be seen from the definition, Equation (II. 10), the commutator of two invariant differentiations takes a familiar form, bringing in the Riemann tensor once for each index:

$$T_a{}^{\mu}{}_{\cdot[\nu\sigma]} = -\frac{1}{2} T_b{}^{\mu} R^{b}{}_{a\nu\sigma} - \frac{1}{2} T_a{}^{\tau} R^{\mu}{}_{\tau\nu\sigma} \quad . \qquad \text{(II. 11)}$$

When Equation (II. 10) is substituted into Equation (II. 11) we get an expression for $R^{\mu}{}_{\nu\sigma\tau}$ which agrees with Equation (II. 6).

The Riemann tensor satisfies a set of 20 equations known as the Bianchi Identities:

$$R^{\mu\nu}{}_{[\sigma\tau\cdot\rho]} = 0 \ .$$

Put in terms of the intrinsic derivative they are

$$R^{\mu\nu}{}_{[\sigma\tau|\rho]} - A_{\lambda}{}^{\mu}{}_{[\rho} R^{\lambda\nu}{}_{\sigma\tau]} - A_{\lambda}{}^{\nu}{}_{[\rho} R^{\mu\lambda}{}_{\sigma\tau]}$$
$$- A^{\lambda}{}_{\sigma[\rho} R^{\mu\nu}{}_{\lambda\tau]} - A^{\lambda}{}_{\tau[\rho} R^{\mu\nu}{}_{\sigma\lambda]} = 0 \ . \qquad \text{(II. 12)}$$

Since we will be considering space-times containing electromagnetic fields in later sections, we also need to write Maxwell's Equations in a form suitable for tetrad analysis. When no sources are present, Maxwell's Equations are

$$F_{\mu\ \cdot\nu}^{\ \nu} = 0$$

$$F_{[\mu\nu\cdot\sigma]} = 0 .$$

In terms of intrinsic derivatives they become

$$\begin{aligned} F_{\mu}^{\ \nu}|_{\nu} - A^{\lambda}_{\ \mu\nu}F_{\lambda}^{\ \nu} - A_{\lambda\ \nu}^{\ \nu}F_{\mu}^{\ \lambda} &= 0 \\ F_{[\mu\nu|\sigma]} - A_{\lambda[\mu\sigma}F^{\lambda}_{\ \nu]} - A_{\lambda[\nu\sigma}F_{\mu]}^{\ \ \lambda} &= 0 . \end{aligned} \tag{II. 13}$$

III. NEWMAN-PENROSE FORMALISM

A. NP Equations

The Newman-Penrose tetrad formalism[26] is currently one of the most successful versions of the general tetrad formalism outlined in Section II.

Let $e_x{}^\mu$, $e_y{}^\mu$, $e_z{}^\mu$, $e_t{}^\mu$ be any orthonormal tetrad with $e_t{}^\mu$ timelike and future-pointing, and η_{ab} = Diag (-1, -1, -1, 1). The corresponding "quasi-orthonormal" tetrad is defined by

$$h_1{}^\mu \equiv \ell^\mu = \tfrac{1}{2}\sqrt{2}\ (e_t{}^\mu + e_z{}^\mu)$$

$$h_2{}^\mu \equiv n^\mu = \tfrac{1}{2}\sqrt{2}\ (e_t{}^\mu - e_z{}^\mu)$$

$$h_3{}^\mu \equiv m^\mu = \tfrac{1}{2}\sqrt{2}\ (e_x{}^\mu + ie_y{}^\mu)$$

$$h_4{}^\mu \equiv \bar{m}^\mu = \tfrac{1}{2}\sqrt{2}\ (e_x{}^\mu - ie_y{}^\mu)\ .$$

With respect to this basis,

$$\eta_{ab} = \begin{pmatrix} 0 & 1 & & \\ 1 & 0 & & \\ & & 0 & -1 \\ & & -1 & 0 \end{pmatrix}$$

$$g_{\mu\nu} = \ell_\mu n_\nu + n_\mu \ell_\nu - m_\mu \bar{m}_\nu - \bar{m}_\mu m_\nu\ .$$

Note that ℓ^μ and n^μ are null vectors and m^μ is complex. Complex basis vectors are used here as in quantum mechanics because of their simple behavior under rotations. Under a rotation about the z-axis m^μ just picks up a phase factor. Now of course any set of equations involving an even number of variables can be trivially cast into complex notation by transforming from x, y to $x \pm iy$, etc. It is normally considered advantageous to do this only when the resulting equations are free of conjugates; that is, when they involve only analytic functions or anti-analytic functions, not a mixture of both. We will not be so fortunate. However, the necessary number of equations will be cut in half, and this is important since we henceforth abandon the summation convention for tetrad indices.

The 24 independent tetrad components of A_{abc} relative to our orthonormal basis are expressed in terms of 12 complex functions as follows:

$$A_{131} = \ell^\mu m_{\mu;\nu} \ell^\nu = -m^\mu \ell_{\mu;\nu} \ell^\nu = -\kappa$$

$$A_{132} = \ell^\mu m_{\mu;\nu} n^\nu = -m^\mu \ell_{\mu;\nu} n^\nu = -\tau$$

$$A_{133} = \ell^\mu m_{\mu;\nu} m^\nu = -m^\mu \ell_{\mu;\nu} m^\nu = -\sigma$$

$$A_{134} = \ell^\mu m_{\mu;\nu} \bar{m}^\nu = -m^\mu \ell_{\mu;\nu} \bar{m}^\nu = -\rho$$

$$A_{231} = n^\mu m_{\mu;\nu} \ell^\nu = -m^\mu n_{\mu;\nu} \ell^\nu = \bar{\pi}$$

$$A_{232} = n^\mu m_{\mu;\nu} n^\nu = -m^\mu n_{\mu;\nu} n^\nu = \bar{\nu}$$

$$
\begin{aligned}
A_{233} &= n^{\mu} m_{\mu;\nu} m^{\nu} = -m^{\mu} n_{\mu;\nu} m^{\nu} = \bar{\lambda} \\
A_{234} &= n^{\mu} m_{\mu;\nu} \bar{m}^{\nu} = -m^{\mu} n_{\mu;\nu} \bar{m}^{\nu} = \bar{\mu} \qquad \text{(III. 1)} \\
A_{121} &= \ell^{\mu} n_{\mu;\nu} \ell^{\nu} = -n^{\mu} \ell_{\mu;\nu} \ell^{\nu} = -(\epsilon + \bar{\epsilon}) \\
A_{122} &= \ell^{\mu} n_{\mu;\nu} n^{\nu} = -n^{\mu} \ell_{\mu;\nu} n^{\nu} = -(\gamma + \bar{\gamma}) \\
A_{123} &= \ell^{\mu} n_{\mu;\nu} m^{\nu} = -n^{\mu} \ell_{\mu;\nu} m^{\nu} = -(\bar{\alpha} + \beta) \\
A_{341} &= m^{\mu} \bar{m}_{\mu;\nu} \ell^{\nu} = -\bar{m}^{\mu} m_{\mu;\nu} \ell^{\nu} = \epsilon - \bar{\epsilon} \\
A_{342} &= m^{\mu} \bar{m}_{\mu;\nu} n^{\nu} = -\bar{m}^{\mu} m_{\mu;\nu} n^{\nu} = \gamma - \bar{\gamma} \\
A_{343} &= m^{\mu} \bar{m}_{\mu;\nu} m^{\nu} = -\bar{m}^{\mu} m_{\mu;\nu} m^{\nu} = -\bar{\alpha} + \beta .
\end{aligned}
$$

The above formulae involve 14 of the 24 independent components. The remaining 10 are obtained by complex conjugation; e. g.,

$$A_{141} = \ell^{\mu} \bar{m}_{\mu;\nu} \ell^{\nu} = \bar{A}_{131} .$$

The 10 independent tetrad components of the Weyl tensor* and the 6 independent components of the Maxwell tensor are denoted by:

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*The Weyl tensor $C_{\mu\nu\sigma\tau}$ is the traceless part of the Riemann tensor and represents the gravitational field in General Relativity. In vacuum, $C_{\mu\nu\sigma\tau} = R_{\mu\nu\sigma\tau}$.

$$
\begin{aligned}
C_{1313} &= -\psi_0 \\
C_{1213} &= -\psi_1 \\
C_{4213} &= \tfrac{1}{2} C_{1212} - \tfrac{1}{2} C_{1234} \\
&= \tfrac{1}{2} C_{3434} - \tfrac{1}{2} C_{1234} \\
&= -\psi_2 \\
C_{2123} &= -\bar{\psi}_3 \\
C_{2323} &= -\bar{\psi}_4 \\
C_{1314} &= C_{2324} = C_{1324} = 0 \\
F_{13} &= \varphi_0 \\
\tfrac{1}{2} F_{12} + \tfrac{1}{2} F_{34} &= \varphi_1 \\
F_{23} &= -\bar{\varphi}_2 \, ,
\end{aligned}
\qquad \text{(III. 2)}
$$

plus the obvious complex conjugate equations.

Finally, the intrinsic derivatives along each tetrad vector are each given a unique operator symbol:

$$
\begin{aligned}
|1 &\rightarrow D \\
|2 &\rightarrow \Delta \\
|3 &\rightarrow \delta \\
|4 &\rightarrow \bar{\delta} \, .
\end{aligned}
$$

With all this formidable terminology the more important equations of Section II can now be written out in full, for the case in which the only stress-energy present is electromagnetic. Equation (II. 7) for the Riemann tensor in terms of A_{abc} becomes:

$$D\rho - \bar{\delta}\kappa = \rho(\rho + \epsilon + \bar{\epsilon}) + \sigma\bar{\sigma} - \bar{\kappa}\tau - \kappa(3\alpha + \bar{\beta} - \pi) + \varphi_0\varphi_0 \qquad \text{(III. 3a)}$$

$$D\sigma - \delta\kappa = \sigma(\rho + \bar{\rho} + 3\epsilon - \bar{\epsilon}) - \kappa(\tau - \bar{\pi} + \bar{\alpha} + 3\beta) + \psi_0 \qquad \text{(III. 3b)}$$

$$D\beta - \delta\epsilon = \beta(\bar{\rho} - \bar{\epsilon}) + \sigma(\alpha + \pi) - \kappa(\mu + \bar{\gamma}) - \epsilon(\bar{\alpha} - \bar{\pi}) + \psi_1 \qquad \text{(III. 3c)}$$

$$D\alpha - \bar{\delta}\epsilon = \alpha(\rho + \bar{\epsilon} - 2\epsilon) + \beta\bar{\sigma} - \bar{\beta}\epsilon - \kappa\lambda - \bar{\kappa}\gamma + \pi(\rho + \epsilon) + \varphi_0\varphi_1 \qquad \text{(III. 3d)}$$

$$D\tau - \Delta\kappa = \tau(\rho + \epsilon - \bar{\epsilon}) + \rho\bar{\pi} + \sigma(\bar{\tau} + \pi) - \kappa(3\gamma + \bar{\gamma}) + \psi_1 + \varphi_0\varphi_1 \qquad \text{(III. 3e)}$$

$$D\gamma - \Delta\epsilon = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) - \gamma(\epsilon + \bar{\epsilon}) - \epsilon(\gamma + \bar{\gamma}) + \tau\pi - \nu\kappa + \psi_2 + \varphi_1\varphi_1 \qquad \text{(III. 3f)}$$

$$D\lambda - \bar{\delta}\pi = \lambda(\rho - 3\epsilon + \bar{\epsilon}) + \pi(\pi + \alpha - \bar{\beta}) + \bar{\sigma}\mu - \nu\bar{\kappa} + \varphi_0\varphi_2 \qquad \text{(III. 3g)}$$

$$D\mu - \delta\pi = \mu(\bar{\rho} - \epsilon - \bar{\epsilon}) + \pi(\bar{\pi} - \bar{\alpha} + \beta) + \sigma\lambda - \nu\kappa + \psi_2 \qquad \text{(III. 3h)}$$

$$D\nu - \Delta\pi = \pi(\mu + \gamma - \bar{\gamma}) + \mu\bar{\tau} + \lambda(\tau + \bar{\pi}) - \nu(3\epsilon + \bar{\epsilon}) + \psi_3 + \varphi_1\varphi_2 \qquad \text{(III. 3i)}$$

$$\delta\rho - \bar{\delta}\sigma = \rho(\bar{\alpha} + \beta) - \sigma(3\alpha - \bar{\beta}) + \tau(\rho - \bar{\rho}) + \kappa(\mu - \bar{\mu}) - \psi_1 + \varphi_0\varphi_1 \qquad \text{(III. 4a)}$$

$$\delta\tau - \Delta\sigma = \tau(\tau - \bar{\alpha} + \beta) + \sigma(\mu - 3\gamma + \bar{\gamma}) - \kappa\bar{\nu} + \varphi_0\varphi_2 \qquad \text{(III. 4b)}$$

$$\bar{\delta}\tau - \Delta\rho = \tau(\bar{\tau} + \alpha - \bar{\beta}) + \rho(\bar{\mu} - \gamma + \bar{\gamma}) - \sigma\lambda + \kappa\nu - \psi_2 \qquad \text{(III. 4c)}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \mu\rho + \gamma(\rho - \bar{\rho}) - \lambda\sigma + \epsilon(\mu - \bar{\mu}) - \psi_2 + \varphi_1\varphi_1 \qquad \text{(III. 4d)}$$

$$\bar{\delta}\mu - \delta\lambda = -\mu(\pi + \alpha + \bar{\beta}) + \bar{\mu}\pi + \lambda(3\beta - \bar{\alpha}) - \nu(\rho - \bar{\rho}) + \psi_3 - \varphi_1\varphi_2 \tag{III. 4e}$$

$$\Delta\beta - \delta\gamma = \beta(\gamma - \bar{\gamma} - \mu) + \gamma(\bar{\alpha} + \beta - \tau) - \mu\tau + \sigma\nu + \epsilon\bar{\nu} - \bar{\lambda}\alpha - \varphi_1\varphi_2 \tag{III. 4f}$$

$$\Delta\alpha - \bar{\delta}\gamma = \alpha(\bar{\gamma} - \bar{\mu}) + \gamma(\bar{\beta} - \bar{\tau}) + \nu(\rho + \epsilon) - \lambda(\beta + \tau) - \psi_3 \tag{III. 4g}$$

$$\Delta\mu - \delta\nu = -\mu(\mu + \gamma - \bar{\gamma}) + \nu(\bar{\alpha} + 3\beta - \tau) + \bar{\nu}\pi - \lambda\bar{\lambda} + \varphi_2\varphi_2 \tag{III. 4h}$$

$$\Delta\lambda - \bar{\delta}\nu = -\lambda(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}) + \nu(3\alpha + \bar{\beta} + \pi - \bar{\tau}) - \psi_4 \tag{III. 4i}$$

These are usually referred to as the NP Equations. Equations (III. 3) which contain the operator D are the radial equations and Equations (III. 4) are the non-radial equations.

Maxwell's Equations, Equation (II. 13), take the form

$$D\varphi_1 - \bar{\delta}\varphi_0 = 2\rho\varphi_1 + (\pi - 2\alpha)\varphi_0 - \kappa\varphi_2 \tag{III. 5a}$$

$$D\varphi_2 - \bar{\delta}\varphi_1 = (\rho - 2\epsilon)\varphi_2 + 2\pi\varphi_1 - \lambda\varphi_0 \tag{III. 5b}$$

$$\Delta\varphi_0 - \delta\varphi_1 = (2\gamma - \mu)\varphi_0 + 2\tau\varphi_1 - \sigma\varphi_2 \qquad \text{(III. 5c)}$$

$$\Delta\varphi_1 - \delta\varphi_2 = 2\mu\varphi_1 + (\tau - 2\beta)\varphi_2 - \nu\varphi_0 \; ; \qquad \text{(III. 5d)}$$

and the Bianchi Identities, Equation (II. 12), take the form

$$\begin{aligned} D\psi_1 - \bar{\delta}\psi_0 &= (4\rho + 2\epsilon)\psi_1 + (\pi - 4\alpha)\psi_0 - 3\kappa\psi_2 \\ &+ \bar{\varphi}_1 D\varphi_0 - \bar{\varphi}_0 \delta\varphi_0 + 2(\beta\varphi_0\bar{\varphi}_0 \\ &- \epsilon\varphi_0\bar{\varphi}_1 - \sigma\bar{\varphi}_0\varphi_1 + \kappa\varphi_1\bar{\varphi}_1) \end{aligned} \qquad \text{(III. 6a)}$$

$$\begin{aligned} D\psi_2 - \bar{\delta}\psi_1 &= 3\rho\psi_2 + (2\pi + 2\alpha)\psi_1 - 2\kappa\psi_3 \\ &- \lambda\psi_0 + \bar{\varphi}_1\bar{\delta}\varphi_0 - \bar{\varphi}_0\Delta\varphi_0 \\ &+ 2(\gamma\varphi_0\bar{\varphi}_0 - \alpha\varphi_0\bar{\varphi}_1 - \tau\bar{\varphi}_0\varphi_1 \\ &+ \rho\varphi_1\bar{\varphi}_1) \end{aligned} \qquad \text{(III. 6b)}$$

$$\begin{aligned} D\psi_3 - \bar{\delta}\psi_2 &= (2\rho - 2\epsilon)\psi_3 + 3\pi\psi_2 - \kappa\psi_4 - 2\lambda\psi_1 \\ &+ \bar{\varphi}_1 D\varphi_2 - \bar{\varphi}_0\delta\varphi_2 + 2(\mu\bar{\varphi}_0\varphi_1 \\ &- \pi\bar{\varphi}_1\varphi_1 - \beta\bar{\varphi}_0\varphi_2 + \epsilon\bar{\varphi}_1\varphi_2) \end{aligned} \qquad \text{(III. 6c)}$$

$$\begin{aligned} D\psi_4 - \bar{\delta}\psi_3 &= (\rho - 4\epsilon)\psi_4 + (4\pi + 2\alpha)\psi_3 - 3\lambda\psi_2 \\ &+ \bar{\varphi}_1\bar{\delta}\varphi_2 - \bar{\varphi}_0\Delta\varphi_2 + 2(\nu\bar{\varphi}_0\Delta\varphi_1 \\ &- \lambda\varphi_1\bar{\varphi}_1 - \gamma\bar{\varphi}_0\varphi_2 + \alpha\bar{\varphi}_1\varphi_2) \end{aligned} \qquad \text{(III. 6d)}$$

$$
\begin{aligned}
\Delta\psi_0 - \delta\psi_1 = {} & (4\gamma - \mu)\psi_0 - (2\beta + 4\tau)\psi_1 + 3\sigma\psi_2 \\
& + \bar{\varphi}_1 \delta\varphi_0 - \bar{\varphi}_2 D\varphi_0 - 2(\beta\varphi_0\bar{\varphi}_1 \\
& - \epsilon\varphi_0\bar{\varphi}_2 - \sigma\varphi_1\bar{\varphi}_1 + \kappa\varphi_1\bar{\varphi}_2)
\end{aligned}
\qquad \text{(III. 6e)}
$$

$$
\begin{aligned}
\Delta\psi_1 - \delta\psi_2 = {} & (2\gamma - 2\mu)\psi_1 - 3\tau\psi_2 + 2\sigma\psi_3 + \nu\psi_0 \\
& + \bar{\varphi}_1 \Delta\varphi_0 - \bar{\varphi}_2 \bar{\delta}\varphi_0 - 2(\gamma\varphi_0\bar{\varphi}_1 \\
& - \alpha\varphi_0\bar{\varphi}_2 - \tau\varphi_1\bar{\varphi}_1 + \rho\varphi_1\bar{\varphi}_2)
\end{aligned}
\qquad \text{(III. 6f)}
$$

$$
\begin{aligned}
\Delta\psi_2 - \delta\psi_3 = {} & -3\mu\psi_2 + (2\beta - 2\tau)\psi_3 + \sigma\psi_4 + 2\nu\psi_1 \\
& + \bar{\varphi}_1 \delta\varphi_2 - \bar{\varphi}_2 D\varphi_2 - 2(\mu\varphi_1\bar{\varphi}_1 \\
& - \pi\varphi_1\bar{\varphi}_2 - \beta\bar{\varphi}_1\varphi_2 + \epsilon\varphi_2\bar{\varphi}_2)
\end{aligned}
\qquad \text{(III. 6g)}
$$

$$
\begin{aligned}
\Delta\psi_3 - \delta\psi_4 = {} & -(2\gamma + 4\mu)\psi_3 + (4\beta - \tau)\psi_4 \\
& + \bar{\varphi}_1 \Delta\varphi_2 - \bar{\varphi}_2 \bar{\delta}\varphi_2 - 2(\nu\varphi_1\bar{\varphi}_1 \\
& - \lambda\varphi_1\bar{\varphi}_2 - \gamma\bar{\varphi}_1\varphi_2 + \alpha\varphi_2\bar{\varphi}_2) .
\end{aligned}
\qquad \text{(III. 6h)}
$$

Finally there are the commutation relations, Equation (II. 4), which become

$$
\begin{aligned}
\Delta D - D\Delta &= (\gamma + \bar{\gamma})D + (\epsilon + \bar{\epsilon})\Delta - (\bar{\tau} + \pi)\delta - (\tau + \bar{\pi})\bar{\delta} \\
\delta D - D\delta &= (\bar{\alpha} + \beta - \pi)D + \kappa\Delta - (\bar{\rho} + \epsilon - \bar{\epsilon})\delta - \sigma\bar{\delta} \\
\delta\Delta - \Delta\delta &= -\bar{\nu}D + (\tau - \bar{\alpha} - \beta)\Delta + (\mu - \gamma + \bar{\gamma})\delta + \bar{\lambda}\bar{\delta} \\
\bar{\delta}\delta - \delta\bar{\delta} &= (\bar{\mu} - \mu)D + (\bar{\rho} - \rho)\Delta + (\alpha - \bar{\beta})\delta + (\beta - \bar{\alpha})\bar{\delta}
\end{aligned}
\qquad \text{(III. 7)}
$$

Although Equations (III. 3) - (III. 6) look very complicated they contain a great deal of symmetry. There is also a strong analogy between Equations (III. 5) and (III. 6).*

B. Tetrad Transformations

In any given space-time an infinity of different tetrad fields is possible. At each point there are six degrees of freedom corresponding to the six Lorentz transformations which rotate one tetrad into another. (This is why $h_a{}^\mu$ has 16 independent components and $g_{\mu\nu}$ only 10.) To investigate the behavior of the NP variables under a tetrad transformation it is convenient to split the Lorentz group into three Abelian 2-parameter subgroups, as follows:

1) "Null" rotations which leave ℓ^μ fixed. These are given by

$$\ell^\mu \to \ell^\mu$$

$$m^\mu \to m^\mu + a\ell^\mu \qquad \text{(III. 8)}$$

$$n^\mu \to n^\mu + a\bar{m}^\mu + \bar{a}m^\mu + a\bar{a}\ell^\mu ,$$

where a is an arbitrary complex function of position.

2) Null rotations which leave n^μ fixed:

*A massless particle of spin s can be described by a $(2s + 1)$-component complex field obeying $4s$ equations. Only the numerical coefficients depend on s. For $s = 1, 2$ we get Equations (III. 5), (III. 6) above.

$$n^\mu \to n^\mu$$

$$m^\mu \to m^\mu + b\, n^\mu \qquad \text{(III. 9)}$$

$$\ell^\mu \to \ell^\mu + b\,\bar{m}^\mu + \bar{b}\, m^\mu + b\bar{b}\, n^\mu .$$

3) Two rotations which leave the <u>directions</u> of both ℓ^μ and n^μ fixed:

$$\ell^\mu \to A^{-1}\ell^\mu, \quad n^\mu \to A n^\mu \qquad \text{(III. 10)}$$

$$m^\mu \to e^{i\theta} m^\mu , \qquad \text{(III. 11)}$$

where A and θ are real functions.

Actually the analysis will be more transparent if before considering rotations, we first consider the two tetrad reflections $m^\mu \leftrightarrow \bar{m}^\mu$ and $\ell^\mu \leftrightarrow n^\mu$.

Under $m^\mu \leftrightarrow \bar{m}^\mu$, all variables go into their complex conjugates. Under $\ell^\mu \leftrightarrow n^\mu$,

$$
\begin{array}{llll}
\Psi_0 \leftrightarrow \bar{\Psi}_4 & & \kappa \leftrightarrow -\bar{\nu} & \\
\Psi_1 \leftrightarrow \bar{\Psi}_3 & & \rho \leftrightarrow -\bar{\mu} & \\
\Psi_2 \leftrightarrow \bar{\Psi}_2 & & \sigma \leftrightarrow -\bar{\lambda} & \\
\varphi_0 \leftrightarrow -\bar{\varphi}_2 & & \epsilon \leftrightarrow -\bar{\gamma} & \text{(III. 12)} \\
\varphi_1 \leftrightarrow -\bar{\varphi}_1 & & \pi \leftrightarrow -\bar{\tau} & \\
D \leftrightarrow \Delta & & \alpha \leftrightarrow -\bar{\beta} & \\
\delta \leftrightarrow \delta . & & &
\end{array}
$$

Under the null rotation leaving ℓ^{μ} fixed, Equation (III. 8),

$$
\begin{aligned}
\psi_0 &\to \psi_0 \\
\psi_1 &\to \psi_1 + \bar{a}\,\psi_0 \\
\psi_2 &\to \psi_2 + 2\bar{a}\,\psi_1 + \bar{a}^2\psi_0 \\
\psi_3 &\to \psi_3 + 3\bar{a}\,\psi_2 + 3\bar{a}^2\psi_1 + \bar{a}^3\psi_0 \\
\psi_4 &\to \psi_4 + 4\bar{a}\,\psi_3 + 6\bar{a}^2\psi_2 + 4\bar{a}^3\psi_1 + \bar{a}^4\psi_0 \\
\varphi_0 &\to \varphi_0 \\
\varphi_1 &\to \varphi_1 + \bar{a}\,\varphi_0 \\
\varphi_2 &\to \varphi_2 + 2\bar{a}\,\varphi_1 + \bar{a}^2\varphi_0 \\
D &\to D \\
\delta &\to \delta + aD \\
\Delta &\to \Delta + \bar{a}\,\delta + a\,\bar{\delta} + a\bar{a}D \\
\kappa &\to \kappa \\
\rho &\to \rho + \bar{a}\,\kappa \\
\sigma &\to \sigma + a\,\kappa \qquad\qquad \text{(III. 13)} \\
\epsilon &\to \epsilon + \bar{a}\,\kappa \\
\tau &\to \tau + a\,\rho + \bar{a}\,\sigma + a\bar{a}\,\kappa \\
\alpha &\to \alpha + \bar{a}(\rho + \epsilon) + \bar{a}^2\kappa \\
\beta &\to \beta + a\,\bar{\epsilon} + \bar{a}\,\sigma + a\bar{a}\,\kappa \\
\pi &\to \pi + 2\bar{a}\,\epsilon + \bar{a}^2\kappa + D\bar{a} \\
\gamma &\to \gamma + a\,\alpha + \bar{a}(\beta + \tau) + a\bar{a}(\rho + \epsilon) + \bar{a}^2\sigma \\
&\quad + a\bar{a}^2\kappa
\end{aligned}
$$

$$\lambda \to \lambda + \bar{a}(\pi + \alpha - \bar{\beta} - \bar{\tau}) + \bar{a}^2(\rho + 2\epsilon) + \bar{a}^3\varkappa$$
$$+ \bar{\delta}\,\bar{a} + \bar{a}D\bar{a}$$

$$\mu \to \mu + a\pi + 2\bar{a}\beta + 2a\bar{a}\epsilon + \bar{a}^2\sigma + a\bar{a}^2\varkappa$$
$$+ \delta\bar{a} + aD\bar{a}$$

$$\nu \to \nu + a\lambda + \bar{a}(\mu + 2\gamma) + a\bar{a}(2\alpha + \pi) + \bar{a}^2(\tau + 2\beta)$$
$$+ a\bar{a}^2(\rho + 2\epsilon) + \bar{a}^3\sigma + a\bar{a}^3\varkappa + \Delta\bar{a} + a\bar{\delta}\bar{a}$$
$$+ \bar{a}\delta\bar{a} + a\bar{a}D\bar{a}\,.$$

It is now unnecessary to write out the behavior under the other null rotations, Equation (III. 9), since this follows immediately upon interchanging ℓ^μ and n^μ.

Under the rotations of Equations (III. 10) and (III. 11),

$$\psi_0 \to A^{-2}e^{2i\theta}\,\psi_0$$
$$\psi_1 \to A^{-1}e^{i\theta}\,\psi_1$$
$$\psi_2 \to \psi_2$$
$$\psi_3 \to A e^{-i\theta}\,\psi_3$$
$$\psi_4 \to A^2 e^{-2i\theta}\,\psi_4$$
$$\varphi_0 \to A^{-1}e^{i\theta}\,\varphi_0$$
$$\varphi_1 \to \varphi_1$$
$$\varphi_2 \to A e^{-i\theta}\,\varphi_2$$
$$D \to A^{-1}D$$
$$\Delta \to AD$$
$$\delta \to e^{i\theta}\,\delta$$

$$
\begin{aligned}
\kappa &\rightarrow A^{-2} e^{i\theta} \kappa \\
\rho &\rightarrow A^{-1} \rho \\
\sigma &\rightarrow A^{-1} e^{2i\theta} \sigma \\
\epsilon &\rightarrow A^{-1} \epsilon - \frac{1}{2} A^{-2} DA + \frac{1}{2} iA^{-1} D\theta \\
\tau &\rightarrow e^{i\theta} \tau \\
\alpha &\rightarrow e^{-i\theta} \alpha - \frac{1}{2} A^{-1} e^{-i\theta} \bar{\delta} A + \frac{1}{2} ie^{-i\theta} \bar{\delta} \theta \\
\beta &\rightarrow e^{i\theta} \beta - \frac{1}{2} A^{-1} e^{i\theta} \delta A + \frac{1}{2} ie^{i\theta} \delta \theta \\
\pi &\rightarrow e^{-i\theta} \pi \\
\gamma &\rightarrow A\gamma - \frac{1}{2} \Delta A + \frac{1}{2} iA\Delta\theta \\
\lambda &\rightarrow Ae^{-2i\theta} \lambda \\
\mu &\rightarrow A\mu \\
\nu &\rightarrow A^2 e^{-i\theta} \nu .
\end{aligned}
\qquad \text{(III. 14)}
$$

It can be checked that the NP Equations are covariant (i. e., go into linear combinations of themselves) under all these transformations.

C. Geometric Considerations

Next we would like to illustrate the geometric significance of some of the NP variables. As pointed out in Section II, Equation (II. 2), the rotation coefficient A_{abc} corresponds to a rotation in the ab plane when we step δx in the c direction. For example, traveling along the ℓ^μ congruence we can use the quantities A_{ab1} from Equation (III. 1) to find the antisymmetric rotation matrix

$$\frac{1}{2}\Omega_{\mu\nu} = (\varepsilon+\bar{\varepsilon})\,\ell_{[\mu}n_{\nu]} + (\bar{\varepsilon}-\varepsilon)\,m_{[\mu}\bar{m}_{\nu]}$$

$$+ \bar{\kappa}\,n_{[\mu}m_{\nu]} + \kappa\,n_{[\mu}\bar{m}_{\nu]} \qquad \text{(III. 15)}$$

$$- \pi\,\ell_{[\mu}m_{\nu]} - \bar{\pi}\,\ell_{[\mu}\bar{m}_{\nu]} \ .$$

Suppose n^μ, m^μ, $\bar{m}^\mu$ are given at one point on a trajectory of ℓ^μ. Is there now a particularly simple way of choosing n^μ, m^μ, $\bar{m}^\mu$ at all other points of the trajectory? The freedoms available are the rotations, Equations (III. 8), (III. 10), and (III. 11) listed above. According to the behavior of π listed in Equation (III. 13), we can, by a suitable null rotation leaving ℓ^μ fixed, set $\pi = 0$. Then from Equation (III. 14) we see that a particular choice of A and θ will make $\varepsilon = 0$ also. Defining the real spacelike vector

$$a_\mu = \kappa\,\bar{m}_\mu + \bar{\kappa}\,m_\mu \ , \qquad \text{(III. 16)}$$

we have reduced $\Omega_{\mu\nu}$ to a simple bivector,

$$\frac{1}{2}\Omega_{\mu\nu} = n_{[\mu}a_{\nu]} \ .$$

By analogy with the definition for spacelike congruences,[27] we will say a tetrad is "Fermi-Walker transported" along ℓ^μ if and only if $\pi = \varepsilon = 0$. Note that under these circumstances, n^μ is even parallel-transported.

Starting again with an arbitrary tetrad, assume only that $\kappa = 0$. Then from Equation (III. 1),

$$\ell^{\mu}{}_{;\nu}\ell^{\nu} = g^{\mu\sigma}\ell_{\sigma;\nu}\ell^{\nu}$$

$$= (\ell^{\mu}n^{\sigma} + n^{\mu}\ell^{\sigma} - m^{\mu}\bar{m}^{\sigma} - \bar{m}^{\mu}m^{\sigma})\ell_{\sigma;\nu}\ell^{\nu}$$

$$= (\epsilon + \bar{\epsilon})\ell^{\mu} ,$$

so ℓ^{μ} must now be tangent to a geodesic congruence, $x^{\mu} = x^{\mu}(s)$ and

$$\ell^{\mu} = \frac{dx^{\mu}}{ds} .$$

The choice of A in Equation (III. 10) which makes $\epsilon + \bar{\epsilon} = 0$ also makes s an affine parameter. For a geodesic congruence, $a_{\mu} = 0$ and $\Omega_{\mu\nu} = 0$, hence Fermi-Walker transport reduces to parallel transport.

Corresponding remarks can be made for the n^{μ} congruence and the NP variables ν, γ, τ according to Equation (III. 12).

If instead we step in a spacelike direction along the congruence m^{μ}, the rotation matrix is

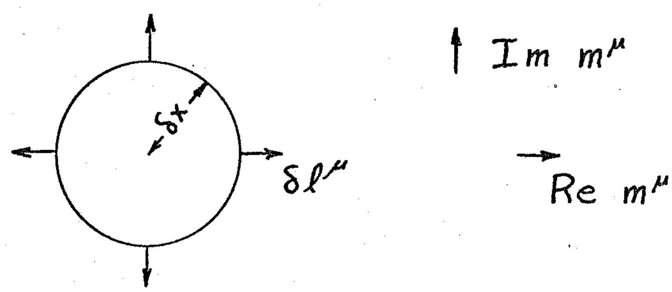
$$\frac{1}{2}\Omega_{\mu\nu} = -(\bar{\alpha} + \beta)\ell_{[\mu}n_{\nu]} + (\bar{\alpha} - \beta)m_{[\mu}\bar{m}_{\nu]}$$

$$+ \bar{\rho}\, n_{[\mu}m_{\nu]} + \sigma\, n_{[\mu}\bar{m}_{\nu]}$$

$$- \mu\, \ell_{[\mu}m_{\nu]} - \bar{\lambda}\, \ell_{[\mu}\bar{m}_{\nu]} .$$

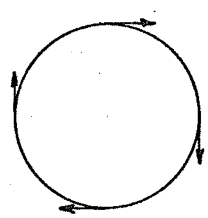
From the definition of $\Omega_{\mu\nu}$, Equation (II. 2), only the $\bar{\rho}$ and σ terms can change the direction of ℓ^{μ}. For instance $(\mathrm{Re}\rho)n_{[\mu}m_{\nu]}$

causes ℓ^μ to deviate toward m^μ, i.e., in the same direction we are stepping. This can be illustrated by looking down on the $m\bar{m}$ 2-flat at a bundle of trajectories of ℓ^μ coming out of the page. At the center of the diagram ℓ^μ points directly out of the page, but a distance δx away it has a small transverse component $\delta\ell^\mu$, represented by an arrow:

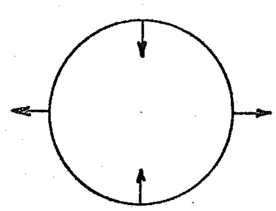


Reρ measures how much the ℓ^μ congruence is diverging as it passes through the $m\bar{m}$ 2-flat. Likewise Imρ, Reσ, and Imσ describe deviation of ℓ^μ in the $-im^\mu$, $\bar{m}^\mu$ and $i\bar{m}^\mu$ directions respectively. The diagrams are:

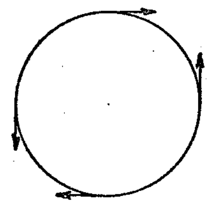
Im ρ:



Re σ:



$Im\ \sigma$:



Thus $\mathrm{Im}\rho$ measures the rotation of ℓ^μ, and $\mathrm{Re}\sigma$, $\mathrm{Im}\sigma$ measure two independent shearing motions.

Once more, by interchanging $\ell^\mu \leftrightarrow n^\mu$ (Equation (III. 12)) we see that μ, λ have the corresponding interpretations with respect to the n^μ congruence.

As a final geometric question we may ask under what circumstances the vector fields actually mesh to form finite 2-surfaces or 3-surfaces. According to the Frobenius Theorem,[28] the necessary and sufficient condition that an antisymmetric tensor $W_{\varkappa_1 \varkappa_2 \ldots \varkappa_m}$ in n space be orthogonal to an (n - m)-surface is

$$W_{[\varkappa_1 \varkappa_2 \ldots \varkappa_m, \lambda} W_{\nu_1] \nu_2 \ldots \nu_m} = 0 .$$

Since the calculations are straightforward only the results will be given.

1) The vector ℓ_μ is hypersurface-orthogonal iff

$$\varkappa = 0$$

$$\mathrm{Im}\,\rho = 0 . \qquad \text{(III. 17)}$$

2) The vector Re m_μ is hypersurface-orthogonal iff

$$\begin{aligned} \mathrm{Re}(\tau + \overline{\pi}) &= 0 \\ \mathrm{Im}(\lambda + \overline{\mu} + \gamma - \overline{\gamma}) &= 0 \\ \mathrm{Im}(\sigma + \overline{\rho} + \epsilon - \overline{\epsilon}) &= 0 \,. \end{aligned} \qquad \text{(III. 18)}$$

3) The bivector $\ell_{[\mu} n_{\nu]}$ is surface-orthogonal iff

$$\begin{aligned} \mathrm{Im}\,\rho &= 0 \\ \mathrm{Im}\,\mu &= 0 \,. \end{aligned} \qquad \text{(III. 19)}$$

4) The bivector $m_{[\mu} \overline{m}_{\nu]}$ is surface-orthogonal iff

$$\tau + \overline{\pi} = 0 \,. \qquad \text{(III. 20)}$$

5) The bivector Re $m_{[\mu} \ell_{\nu]}$ is surface-orthogonal iff

$$\begin{aligned} \mathrm{Im}\,\kappa &= 0 \\ \mathrm{Im}(\rho - \sigma - 2\epsilon) &= 0 \,. \end{aligned} \qquad \text{(III. 21)}$$

All other cases follow after suitable tetrad reflections.

D. Algebraically Special Fields

Now we turn our attention to the components of the electromagnetic field, namely φ_0, φ_1, and φ_2. Assume for simplicity that $\varphi_2 \neq 0$. (If this is not already the case, a small null rotation, Equation (III. 8), will make it so.) Try to find a null rotation, Equation (III. 9), to make φ_0 zero:

$$\begin{aligned} \varphi_0 &\to \varphi_0 + 2a\varphi_1 + a^2\varphi_2 = 0 \\ \varphi_1 &\to \varphi_1 + a\varphi_2 \\ \varphi_2 &\to \varphi_2 \ . \end{aligned} \tag{III. 22}$$

Equation (III. 22) is a complex quadratic equation and always has two roots. The corresponding directions of ℓ^{μ} are called the "principal null vectors" of the electromagnetic field. They coincide if and only if the discriminant is zero:

$$(2\varphi_1)^2 - 4\varphi_0\varphi_2 = 0 \ ,$$

or equivalently iff $a = -\varphi_1/\varphi_2$ is a root. In this case both φ_0, $\varphi_1 \to 0$ at once leaving only φ_2, and we say the field is a "null" electromagnetic field.* The other two rotations, Equations (III. 10) and (III. 11), preserve $\varphi_0 = \varphi_1 = 0$ but change φ_2 arbitrarily, so the field has no invariants in this case.

If the roots are distinct and $\varphi_1 \neq 0$, then a null rotation, Equation (III. 8), can be used to make $\varphi_0 = \varphi_2 = 0$, leaving φ_1 as the only component. Both ℓ^{μ} and n^{μ} are then the principal null vectors. In this case φ_1 is a genuine invariant because it is invariant under the remaining two rotations, Equations (III. 10) and (III. 11). The usual tensor invariants in terms of φ_1 are

---

*In the familiar 3-vector notation, a null field is one for which $\vec{E} \cdot \vec{B} = 0$ and $\vec{E}^2 = \vec{B}^2$.

$$-\frac{1}{8} F_{\mu\nu} F^{\mu\nu} = \mathrm{Re}\,(\varphi_1)^2$$

$$-\frac{1}{8} F_{\mu\nu} F^{\mu\nu *} = \mathrm{Im}\,(\varphi_1)^2 \ .$$

An analogous situation prevails for the components of the Weyl tensor, ψ_0, ψ_1, ψ_2, ψ_3, ψ_4. Assume ψ_4 has been made nonzero, and try to use a null rotation, Equation (III. 9), to set $\psi_0 = 0$:

$$\begin{aligned} \psi_0 &\rightarrow \psi_0 + 4a\psi_1 + 6a^2\psi_2 + 4a^3\psi_3 + a^4\psi_4 = 0 \\ \psi_1 &\rightarrow \psi_1 + 3a\psi_2 + 3a^2\psi_3 + a^3\psi_4 \ . \end{aligned} \tag{III. 23}$$

Equation (III. 23) always has four roots, and the corresponding directions of ℓ^μ are the principal null vectors of the gravitational field. If any of them happen to coincide, the field is called "algebraically special" and the various ways in which coincidence can occur lead to the Petrov classification.[29]

The condition for a polynomial to have a double root is that the derivative also have a root there, i. e., that

$$\psi_1 + 3a\psi_2 + 3a^2\psi_3 + a^3\psi^4 = 0.$$

This implies that there is a double root iff $\psi_0 = \psi_1 = 0$ for some choice of a. All possible cases will now be listed:

Petrov Type I: all roots distinct. By choice of tetrad we can make $\psi_0 = \psi_4 = 0$. The only invariants are then $\psi_1\psi_3 \neq 0$ and ψ_2.*

*We might also distinguish a subcase, Type I-A say, in which $\psi_2 = 0$.

Petrov Type II: one pair of roots coincide. We can make $\psi_0 = \psi_1 = \psi_4 = 0$. The only invariant is ψ_2.

Petrov Type D: two pairs of coincident roots. We can eliminate all components but ψ_2, which is then an invariant.

Petrov Type III: three roots coincide. Set $\psi_0 = \psi_1 = \psi_2 = \psi_4 = 0$, and there are no invariants in this case.

Petrov Type N: all four roots coincide. We can set everything zero except ψ_4, and there are no invariants in this case either.

In this paper we will be primarily concerned with Type D.

There are several theorems relating to the properties of algebraically special fields.

The Mariot-Robinson Theorem[30] for electromagnetism states that the principal null vector of a null field in a charge-free region is geodesic and shear-free. In the NP formalism the proof of this theorem is trivial: We choose the tetrad such that ℓ^μ is the principal null vector, thereby making $\varphi_0 = \varphi_1 = 0$ and $\varphi_2 \neq 0$. Then Maxwell's Equations, Equations (III. 5a) and (III. 5c) state that $\kappa = \sigma = 0$, thus ℓ^μ is geodesic and shear-free. This theorem reveals how very special the null fields are; even a beam of light emerging from a cylindrical lens does have shear, hence is not null.

The analogous result for gravitation is even stronger. The Goldberg-Sachs Theorem[31] says that the double principal null vector of any algebraically special vacuum field is geodesic and shear-free; and conversely, if there exists a geodesic shear-free null vector field in vacuum, then the Weyl tensor is algebraically special and has that vector as a double principal null vector. The first statement follows immediately from the Bianchi Identities, Equations (III. 6), when we choose a tetrad to make $\psi_0 = \psi_1 = 0$. The proof of the converse is complicated, and is given in reference 11. Again the inference to be drawn is that algebraically special fields are too special to represent realistic radiation. Even for the linearized radiating multipole solutions, the null congruences are clearly not geodesics (straight lines); hence those fields must be Petrov Type I. The same is therefore expected a fortiori for exact solutions.

The asymptotic behavior of an isolated radiating source has been studied in detail by Newman and Unti.[32] Their most important conclusion is the Peeling Theorem: that ψ_i dies off like r^{i-5}. The true field is Type I, but asymptotic approximations to it are algebraically special since $\psi_0 \approx 0$, and the farther we get from the source the more special the field becomes. Where $\psi_4 \sim r^{-1}$ is the only surviving component we have reached the "radiation zone" and Type N. This theorem provides a large part of the rationale for detailed studies of algebraically special fields.

IV. TYPE D VACUUM METRICS

In this section we will solve the NP Equations, assuming only that the gravitational field is a Type D vacuum field. Despite our use of tetrad methods throughout, we will give explicitly the metric for each solution.

Choose ℓ^μ and n^μ to lie along the double principal null vectors, thereby making $\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$. By the Goldberg-Sachs Theorem both ℓ^μ and n^μ are shear-free geodesics, or $\kappa = \sigma = \nu = \lambda = 0$. For convenience denote the co-ordinates by $x^1 = u$, $x^2 = r$, $x^3 = x$, $x^4 = y$. We will take them "comoving" along the ℓ^μ congruence; that is, r varies on a given trajectory and u, x, y label different trajectories.* If we choose r such that

$$\ell^\mu = \frac{dx^\mu}{dr} ,$$

the components of the tetrad may be written

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = (X^1,\ U,\ X^3,\ X^4)$$

$$m^\mu = (\xi^1,\ \omega,\ \xi^3,\ \xi^4) ,$$

where X^i and U are real and ξ^i, ω are complex. By means of the two tetrad rotations, Equations (III. 10), (III. 11) which leave the directions of ℓ^μ and n^μ fixed, we may set $\varepsilon = 0$. Note that the

---

*Any attempt to treat ℓ^μ and n^μ in a symmetrical manner would only lead to more complicated equations later on.

tetrad is still not completely determined, since a further rotation with $DA = D\theta = 0$ preserves $\epsilon = 0$.

Apply the commutators, Equation (III. 7), to each of the co-ordinates to get the NP version of Equations (II. 8):

$$DU = \omega(\bar{\tau} + \pi) + \bar{\omega}(\tau + \bar{\pi}) - (\gamma + \bar{\gamma}) \quad \text{(IV. 1a)}$$

$$DX^i = \xi^i(\bar{\tau} + \pi) + \bar{\xi}^i(\tau + \bar{\pi}) \quad \text{(IV. 1b)}$$

$$D\omega = \bar{\rho}\omega + (\bar{\pi} - \bar{\alpha} - \beta) \quad \text{(IV. 1c)}$$

$$D\xi^i = \bar{\rho}\xi^i \quad \text{(IV. 1d)}$$

$$\delta U - \Delta\omega = U(\tau - \bar{\alpha} - \beta) + \omega(\mu - \gamma + \bar{\gamma}) \quad \text{(IV. 2a)}$$

$$\delta X^i - \Delta\xi^i = X^i(\tau - \bar{\alpha} - \beta) + \xi^i(\mu - \gamma + \bar{\gamma}) \quad \text{(IV. 2b)}$$

$$\bar{\delta}\omega - \delta\bar{\omega} = \omega(\alpha - \bar{\beta}) + \bar{\omega}(\beta - \bar{\alpha}) + U(\rho - \bar{\rho}) + (\bar{\mu} - \mu) \quad \text{(IV. 2c)}$$

$$\bar{\delta}\xi^i - \delta\bar{\xi}^i = \xi^i(\alpha - \bar{\beta}) + \bar{\xi}^i(\beta - \bar{\alpha}) + X^i(\rho - \bar{\rho}), \quad \text{(IV. 2d)}$$

where i runs over the values 1, 3, 4.

The Bianchi Identities, Equation (III. 6), simplify considerably under the present assumptions. The only ones which survive are:

$$D\psi_2 = 3\rho\psi_2 \quad \text{(IV. 3a)}$$

$$\Delta\psi_2 = -3\mu\psi_2 \quad \text{(IV. 3b)}$$

$$\delta\psi_2 = 3\tau\psi_2 \quad \text{(IV. 3c)}$$

$$\bar{\delta}\psi_2 = -3\pi\psi_2 \,. \quad \text{(IV. 3d)}$$

In addition one can derive three important "integrability conditions" by applying the commutators to ψ_2:

$$
\begin{aligned}
(\Delta D - D\Delta)\psi_2 &= \Delta(3\rho\psi_2) - D(-3\mu\psi_2) \\
&= 3\psi_2(\Delta\rho + D\mu) \\
&= 3\psi_2[\rho(\gamma + \bar{\gamma}) - \tau(\bar{\tau} + \pi) + \pi(\tau + \bar{\pi})] \\
\Rightarrow \quad \Delta\rho + D\mu &= \rho(\gamma + \bar{\gamma}) + \pi\bar{\pi} - \tau\bar{\tau} \, .
\end{aligned}
\tag{IV. 4}
$$

Likewise from $(\bar{\delta}D - D\bar{\delta})\psi_2$ and $(\delta\Delta - \Delta\delta)\psi_2$ we get:

$$\bar{\delta}\rho + D\pi = \rho(\alpha + \bar{\beta}) \tag{IV. 5}$$

$$\delta\mu + \Delta\tau = -\mu(\bar{\alpha} + \beta) + \tau(\gamma - \bar{\gamma}) \, . \tag{IV. 6}$$

The other commutators give no new information.

For convenience here are the surviving radial NP Equations, Equation (III. 3), after all simplifications have been made:

$$D\rho = \rho^2 \tag{IV. 7a}$$

$$D\beta = \bar{\rho}\beta \tag{IV. 7b}$$

$$D\alpha = \rho(\alpha + \pi) \tag{IV. 7c}$$

$$D\tau = \rho(\tau + \bar{\pi}) \tag{IV. 7d}$$

$$D\gamma = \alpha(\tau + \bar{\pi}) + \beta(\bar{\tau} + \pi) + \tau\pi + \psi_2 \tag{IV. 7e}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \pi(\bar{\pi} - \bar{\alpha} + \beta) + \psi_2 \, . \tag{IV. 7f}$$

Since $D = \partial/\partial r$, the solution of Equation (IV. 7a) is easily seen to be

$$\rho = -(r + i\rho^{o})^{-1} . \qquad \text{(IV. 8)}$$

Here ρ^{o} is real, and the superscript "o" indicates it is independent of r. We will attempt to use ρ in preference to r as the independent variable. The solution $\rho = 0$, obtained by letting $\rho^{o} \to \infty$, must be considered separately later.

A. The Case $\rho \neq 0$

1. Radial Integration

An equation for $\bar{\delta}\rho$ may be derived from the commutator:

$$(\bar{\delta}D - D\bar{\delta})\rho = 2\rho\bar{\delta}\rho - D\bar{\delta}\rho$$
$$= \rho^{2}(\alpha + \bar{\beta} - \pi) - \rho\bar{\delta}\rho ,$$

giving:

$$D\bar{\delta}\rho - 3\rho\bar{\delta}\rho = \rho^{2}(\pi - \alpha - \bar{\beta}) . \qquad \text{(IV. 9)}$$

Using Equations (IV. 7a, b, c) the solution of this is

$$\bar{\delta}\rho = \rho(\alpha + \bar{\beta}) - 2\bar{\tau}^{o}\rho^{3} , \qquad \text{(IV. 10)}$$

where $\bar{\tau}^{o}$ is a "constant" of integration, i.e., independent of r. This result is substituted in Equation (IV. 5) to get

$$D\pi = 2\bar{\tau}^{o}\rho^{3}, \tag{IV. 11}$$

which has the solution

$$\pi = \pi^{o} + \bar{\tau}^{o}\rho^{2}. \tag{IV. 12}$$

Equations (IV. 1), (IV. 3a), (IV. 7b-e) can now be integrated easily one at a time, each one yielding a new complex integration constant. In this way the radial dependence of every tetrad variable except μ is determined; here are the solutions:

$$\beta = \bar{\rho}\,\beta^{o} \tag{IV. 13a}$$

$$\alpha = \rho\,\alpha^{o} - \pi^{o} + \rho^{2}\bar{\tau}^{o} \tag{IV. 13b}$$

$$\tau = \rho\,\eta^{o} + \rho\bar{\rho}\,\tau^{o} - \bar{\pi}^{o} \tag{IV. 13c}$$

$$\omega = \bar{\rho}\,\omega^{o} + \bar{\alpha}^{o} + \beta^{o} - \bar{\pi}^{o}/\bar{\rho} \tag{IV. 13d}$$

$$\xi^{i} = \bar{\rho}\,\xi^{oi} \tag{IV. 13e}$$

$$X^{i} = X^{oi} + \rho\eta^{o}\bar{\xi}^{oi} + \bar{\rho}\bar{\eta}^{o}\xi^{oi} + \rho\bar{\rho}(\bar{\tau}^{o}\xi^{oi} + \tau^{o}\bar{\xi}^{oi}) \tag{IV. 13f}$$

$$\psi_{2} = \rho^{3}\psi_{2}^{o} \tag{IV. 13g}$$

$$\begin{aligned}\gamma &= \gamma^{o} + \rho(\eta^{o}\alpha^{o} - \bar{\tau}^{o}\bar{\pi}^{o}) + \bar{\rho}(\bar{\eta}^{o}\beta^{o} - \tau^{o}\pi^{o}) \\ &\quad + \rho^{2}(\tfrac{1}{2}\psi_{2}^{o} + \bar{\tau}^{o}\eta^{o}) + \rho\bar{\rho}(\tau^{o}\alpha^{o} + \bar{\tau}^{o}\beta^{o}) \\ &\quad + \rho^{2}\bar{\rho}\,\tau^{o}\bar{\tau}^{o} + \pi^{o}\bar{\pi}^{o}/\rho\end{aligned} \tag{IV. 13h}$$

$$U = U^o + \rho[\bar{\tau}^o(\bar{\alpha}^o + \beta^o - \eta^o) + \eta^o\bar{\omega}^o - \frac{1}{2}\psi_2{}^o]$$
$$+ \bar{\rho}[\tau^o(\alpha^o + \bar{\beta}^o - \bar{\eta}^o) + \bar{\eta}^o\omega^o - \frac{1}{2}\bar{\psi}_2{}^o]$$
$$+ \rho\bar{\rho}(\bar{\tau}^o\omega^o + \tau^o\bar{\omega}^o - \tau^o\bar{\tau}^o) - (\rho/\bar{\rho})\bar{\tau}^o\bar{\pi}^o$$
$$- (\bar{\rho}/\rho)\tau^o\pi^o - r(\gamma^o + \bar{\gamma}^o + \eta^o\pi^o + \bar{\eta}^o\bar{\pi}^o)$$
$$+ \pi^o\bar{\pi}^o/\rho\bar{\rho} . \qquad \text{(IV. 13i)}$$

To get expressions for $\Delta\rho$ and μ we again need to use a commutator:

$$(\Delta D - D\Delta)\rho = 2\rho\Delta\rho - D\Delta\rho$$
$$= \rho^2(\gamma + \bar{\gamma}) - (\tau + \bar{\pi})\bar{\delta}\rho - (\bar{\tau} + \pi)\delta\rho .$$

The last member can be written out in full using Equations (III. 4a), (IV. 10), (IV. 12), and (IV. 13). The result is an equation for $\Delta\rho$ which can be integrated to give

$$\Delta\rho = -\rho^2 M^o + \rho^2\eta^o(\alpha^o + \bar{\beta}^o) + \rho(\gamma^o + \bar{\gamma}^o + \eta^o\pi^o)$$
$$+ \bar{\rho}\bar{\eta}^o\bar{\pi}^o + \rho\bar{\rho}[\bar{\eta}^o(\bar{\alpha}^o + \beta^o) - \tau^o\pi^o + \bar{\tau}^o\bar{\pi}^o$$
$$- \eta^o\bar{\eta}^o] - \rho^3(\frac{1}{2}\psi_2{}^o + \bar{\tau}^o\eta^o) - \rho^2\bar{\rho}[\frac{1}{2}\bar{\psi}_2{}^o \qquad \text{(IV. 14)}$$
$$- \tau^o(\alpha^o + \bar{\beta}^o) - \bar{\tau}^o(\bar{\alpha}^o + \beta^o) + \bar{\tau}^o\eta^o]$$
$$- \rho^3\bar{\rho}\tau^o\bar{\tau}^o + (\rho/\bar{\rho})\pi^o\bar{\pi}^o,$$

where M^o is the constant of integration. Finally we substitute this into Equation (IV. 4) and perform the radial integration for μ, getting

$$\mu = \mu^o + \rho(M^o - \bar{\tau}^o \bar{\pi}^o) + \bar{\rho}\tau^o \pi^o + \rho^2(\tfrac{1}{2}\psi_2{}^o + \bar{\tau}^o \eta^o) + \tfrac{1}{2}\rho\bar{\rho}\bar{\psi}_2{}^o + \rho^2\bar{\rho}\tau^o\bar{\tau}^o - \pi^o\bar{\pi}^o/\bar{\rho} . \tag{IV. 15}$$

Looking back for a moment, we see that the solution of Type D fields would present a far more difficult problem were it not for some very fortunate circumstances:

1) The Goldberg-Sachs Theorem makes many NP variables zero.
2) Three "bonus" integrability conditions arise.
3) The non-radial derivatives all disappear from Equations (III. 3).
4) The variables after radial integration are simple power series in ρ and $\bar{\rho}$.

2. Reduction of the Non-radial Equations

In the second stage of the solution we complete the elimination of r from the remaining NP Equations. We substitute the above results into the non-radial NP Equations and equate coefficients of like powers of ρ, $\bar{\rho}$. (Care must be taken since ρ and $\bar{\rho}$ are not independent variables; they are linked by $\rho - \bar{\rho} = 2i\rho^o \rho\bar{\rho}$.) In this manner we obtain differential equations involving ξ^{oi} and X^{oi}, and also some purely algebraic constraints among the integration constants.

To find the derivatives of ρ^o we differentiate Equation (IV. 8) for ρ :

$$\delta\rho = (r + i\rho^o)^{-2}(\delta r + i\delta\rho^o) = \rho^2(\omega + i\delta\rho^o) \quad \text{(IV. 16a)}$$

$$\bar{\delta}\rho = \rho^2(\bar{\omega} + i\bar{\delta}\rho^o) \quad \text{(IV. 16b)}$$

$$\Delta\rho = \rho^2(U + i\Delta\rho^o) . \quad \text{(IV. 16c)}$$

When these are expanded and compared with Equations (III. 4a), (IV. 10) and (IV. 14) they yield the following information:

$$\xi\cdot\rho^o = -\rho^o(\bar{\alpha}^o + \beta^o - \eta^o) - i\tau^o + 2i(\rho^o)^2\bar{\pi}^o \quad \text{(IV. 17a)}$$

$$X\cdot\rho^o = -\rho^o(\gamma^o + \bar{\gamma}^o + 2\bar{\eta}^o\bar{\pi}^o) + \frac{1}{2}i(M^o - \bar{M}^o) + i(\tau^o\pi^o - \bar{\tau}^o\bar{\pi}^o) \quad \text{(IV. 17b)}$$

$$\omega^o = -i\rho^o(\bar{\alpha}^o + \beta^o - \eta^o) - 2(\rho^o)^2\bar{\pi}^o \quad \text{(IV. 17c)}$$

$$U^o = -\frac{1}{2}(M^o + \bar{M}^o) + \eta^o(\alpha^o + \bar{\beta}^o) + \bar{\eta}^o(\bar{\alpha}^o + \beta^o) - \eta^o\bar{\eta}^o + \tau^o\pi^o + \bar{\tau}^o\bar{\pi}^o + i\rho^o(\eta^o\pi^o - \bar{\eta}^o\bar{\pi}^o), \quad \text{(IV. 17d)}$$

where the obvious shorthand $\xi\cdot\rho^o = \xi^{oi}\rho^o{}_{,i}$ and $X\cdot\rho^o = X^{oi}\rho^o{}_{,i}$ has been introduced.

The three non-radial Bianchi Identities, Equations (IV. 3b-d), provide the derivatives of $\psi_2{}^o$:

$$\xi\cdot\psi_2{}^o = -3\psi_2{}^o(\bar{\alpha}^o + \beta^o - \eta^o - 2i\rho^o\bar{\pi}^o) \quad \text{(IV. 18a)}$$

$$\bar{\xi}\cdot\psi_2{}^o = -3\psi_2{}^o(\alpha^o + \bar{\beta}^o) \quad \text{(IV. 18b)}$$

$$X\cdot\psi_2{}^o = -3\psi_2{}^o(\gamma^o + \bar{\gamma}^o + \mu^o + \eta^o\pi^o + \bar{\eta}^o\bar{\pi}^o) . \quad \text{(IV. 18c)}$$

Now for convenience we write out the remaining NP Equations:

$$\delta\tau = \tau(\tau - \bar{\alpha} + \beta) \quad \text{(IV. 19a)}$$

$$\Delta\rho - \bar{\delta}\tau = -\rho\bar{\mu} - \tau(\bar{\tau} + \alpha - \bar{\beta}) + \rho(\gamma + \bar{\gamma}) + \psi_2 \quad \text{(IV. 19b)}$$

$$D\mu - \delta\pi = \bar{\rho}\mu + \pi(\bar{\pi} - \bar{\alpha} + \beta) + \psi_2 \quad \text{(IV. 19c)}$$

$$\bar{\delta}\pi = -\pi(\pi + \alpha - \bar{\beta}) \quad \text{(IV. 19d)}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta + \rho\mu + \gamma(\rho - \bar{\rho}) - \psi_2 \quad \text{(IV. 19e)}$$

$$\bar{\delta}\mu = -\mu(\pi + \alpha + \bar{\beta}) + \bar{\mu}\pi \quad \text{(IV. 19f)}$$

$$\Delta\pi = -\pi(\mu + \gamma - \bar{\gamma}) - \mu\bar{\tau} \quad \text{(IV. 19g)}$$

$$\Delta\beta - \delta\gamma = -\beta(\mu - 2\gamma + \bar{\gamma}) + \gamma(\bar{\alpha} - \tau) - \mu\tau \quad \text{(IV. 19h)}$$

$$\Delta\alpha - \bar{\delta}\gamma = -\alpha(\bar{\mu} - \bar{\gamma}) + \gamma(\bar{\beta} - \bar{\tau}) \quad \text{(IV. 19i)}$$

$$\Delta\mu = -\mu(\mu + \gamma + \bar{\gamma}) . \quad \text{(IV. 19j)}$$

We now proceed to substitute Equations (IV. 13) into the above. Substitution in Equations (IV. 19a, b) gives the derivatives of π^o, τ^o, η^o:

$$\xi \cdot \pi^o = -\pi^o(\beta^o - \bar{\alpha}^o) - \mu^o \quad \text{(IV. 20a)}$$

$$\bar{\xi} \cdot \pi^o = -\pi^o(\alpha^o - \bar{\beta}^o) \quad \text{(IV. 20b)}$$

$$\xi \cdot \tau^o = -\tau^o(3\bar{\alpha}^o + \beta^o) \quad \text{(IV. 20c)}$$

$$\bar{\xi} \cdot \tau^o = -\tau^o(\alpha^o + 3\bar{\beta}^o - \bar{\eta}^o + 2i\,\rho^o\pi^o) - 2i\,\rho^o\bar{M}^o + \frac{1}{2}(\psi_2{}^o - \bar{\psi}_2{}^o) \quad \text{(IV. 20d)}$$

$$\xi \cdot \eta^o = -\eta^o(2\bar{\alpha}^o - \eta^o - 2i\,\rho^o\bar{\pi}^o) + 2\tau^o\bar{\pi}^o \quad \text{(IV. 20e)}$$

$$\bar{\xi} \cdot \eta^o = -2\eta^o\bar{\beta}^o - (M^o - \bar{M}^o) + 2\bar{\tau}^o\bar{\pi}^o . \quad \text{(IV. 20f)}$$

Equations (IV. 19c, d) merely confirm these results. Equation (IV. 19e) yields

$$\xi \cdot \alpha^o - \bar{\xi} \cdot \beta^o = 2\beta^o(\bar{\beta}^o - \alpha^o) + 2i\rho^o(\gamma^o + \mu^o + \alpha^o\bar{\pi}^o + \beta^o\pi^o) + M^o + 2(\rho^o)^2 \pi^o\bar{\pi}^o . \quad \text{(IV. 21)}$$

Equation (IV. 19f) yields the derivatives

$$\bar{\xi} \cdot \mu^o = -\mu^o(\alpha^o + \bar{\beta}^o) - 2i\rho^o\bar{\mu}^o\pi^o + (M^o + \bar{M}^o)\pi^o + 2i\rho^o\pi^o\bar{\pi}^o\bar{\eta}^o + 4(\rho^o\pi^o)^2\bar{\pi}^o \quad \text{(IV. 22a)}$$

$$\bar{\xi} \cdot M^o = -2M^o(\alpha^o + \bar{\beta}^o) + (\psi_2{}^o + 2\bar{\psi}_2{}^o)\pi^o - 2\mu^o\bar{\tau}^o + 2\bar{\tau}^o\pi^o\eta^o + 4i\rho^o\bar{\tau}^o\pi^o\bar{\pi}^o , \quad \text{(IV. 22b)}$$

and a very important relation,

$$\eta^o = 2i\rho^o\bar{\pi}^o . \quad \text{(IV. 23)}$$

When this last constraint is introduced, many of the above results simplify. If we differentiate it and compare with Equation (IV. 20f) we get another condition,

$$M^o - \bar{M}^o = 2i\rho^o\bar{\mu}^o + 4\bar{\tau}^o\bar{\pi}^o - 8(\rho^o)^2\pi^o\bar{\pi}^o ; \quad \text{(IV. 24)}$$

and since the real part of the right-hand side must vanish, we have also

$$2\tau^o\pi^o + 2\bar{\tau}^o\bar{\pi}^o = 8(\rho^o)^2\pi^o\bar{\pi}^o + i\rho^o(\mu^o - \bar{\mu}^o) . \quad \text{(IV. 25)}$$

Resuming substitution in the NP Equations, we get from Equations (IV. 6), (IV. 19g-j):

$$\xi \cdot \mu^{o} = -\mu^{o}(\bar{\alpha}^{o} + \beta^{o}) + 2i\rho^{o}\mu^{o}\bar{\pi}^{o} + (M^{o} + \bar{M}^{o})\bar{\pi}^{o} + 8(\rho^{o}\bar{\pi}^{o})^{2}\pi^{o} \tag{IV. 26a}$$

$$\xi \cdot M^{o} = -2M^{o}(\bar{\alpha}^{o} + \beta^{o}) + 3\bar{\psi}_{2}{}^{o}\bar{\pi}^{o} + 2i\rho^{o}\bar{\pi}^{o}(2M^{o} + 4\bar{M}^{o}) + 16i\rho^{o}\tau^{o}\pi^{o}\bar{\pi}^{o} - 16i\rho^{o}\bar{\tau}^{o}\bar{\pi}^{o}\bar{\pi}^{o} - 16(\rho^{o})^{2}\mu^{o}\bar{\pi}^{o} \tag{IV. 26b}$$

$$X \cdot \pi^{o} = -\pi^{o}(\gamma^{o} - \bar{\gamma}^{o}) \tag{IV. 26c}$$

$$X \cdot \tau^{o} = -\tau^{o}(\bar{\mu}^{o} + \gamma^{o} + 3\bar{\gamma}^{o}) + \frac{1}{2}(\psi_{2}{}^{o} - \bar{\psi}_{2}{}^{o})\bar{\pi}^{o} - 2i\rho^{o}\bar{M}^{o}\bar{\pi}^{o} + 4i\rho^{o}\tau^{o}\pi^{o}\pi^{o} \tag{IV. 26d}$$

$$X \cdot \alpha^{o} - \bar{\xi} \cdot \gamma^{o} = -\alpha^{o}(\bar{\mu}^{o} + \gamma^{o}) + \bar{\beta}^{o}\gamma^{o} + 2i\rho^{o}\pi^{o}(\gamma^{o} + \mu^{o} - \alpha^{o}\bar{\pi}^{o} + \beta^{o}\pi^{o}) + \bar{M}^{o}\pi^{o} + 2\bar{\tau}^{o}\pi^{o}\bar{\pi}^{o} + 4(\rho^{o}\pi^{o})^{2}\bar{\pi}^{o} \tag{IV. 26e}$$

$$X \cdot \beta^{o} - \xi \cdot \gamma^{o} = -\beta^{o}(\mu^{o} - \gamma^{o} + 2\bar{\gamma}^{o}) + \bar{\alpha}^{o}\gamma^{o} - 2i\rho^{o}\bar{\pi}^{o}(\gamma^{o} + \alpha^{o}\bar{\pi}^{o} - \beta^{o}\pi^{o}) + M^{o}\bar{\pi}^{o} + 2\tau^{o}\pi^{o}\bar{\pi}^{o} + 4(\rho^{o})^{2}\pi^{o}\bar{\pi}^{o}\bar{\pi}^{o} \tag{IV. 26f}$$

$$X \cdot \mu^{o} = -\mu^{o}(\mu^{o} + \gamma^{o} + \bar{\gamma}^{o}) + 2i\rho^{o}\pi^{o}\bar{\pi}^{o}(\mu^{o} - \bar{\mu}^{o}) \tag{IV. 26g}$$

$$X \cdot M^{o} = -2M^{o}(\mu^{o} + \gamma^{o} + \bar{\gamma}^{o}) - (\psi_{2}{}^{o} - \bar{\psi}_{2}{}^{o})\pi^{o}\bar{\pi}^{o} + 4i\rho^{o}\bar{M}^{o}\pi^{o}\bar{\pi}^{o} - 2\mu^{o}\tau^{o}\pi^{o} + 6(\rho^{o})^{2}\pi^{o}\bar{\pi}^{o}(\mu^{o} - \bar{\mu}^{o}) - 8i\rho^{o}\tau^{o}\pi^{o}\pi^{o}\bar{\pi}^{o} \tag{IV. 26h}$$

and an algebraic constraint,*

$$\mu^o = \bar{\mu}^o . \qquad \text{(IV. 26i)}$$

Differentiation of Equation (IV. 25) leads to

$$\bar{\pi}^o(\psi_2{}^o - \bar{\psi}_2{}^o) = -2\mu^o\tau^o + 4\,i\,\rho^o\bar{M}^o\bar{\pi}^o - 8\,i\,\rho^o\tau^o\pi^o\bar{\pi}^o . \qquad \text{(IV. 27)}$$

Finally, Equations (IV. 2) give only

$$\xi\cdot X^{oi} - X\cdot\xi^{oi} = (\mu^o + 2\bar{\gamma}^o - 2\,i\,\rho^o\pi^o\bar{\pi}^o)\xi^{oi} + 2\,i\,\rho^o\pi^o\bar{\pi}^o\bar{\xi}^{oi} + (2\,i\,\rho^o\bar{\pi}^o - \bar{\alpha}^o - \beta^o)X^{oi} \qquad \text{(IV. 28a)}$$

$$\bar{\xi}\cdot\xi^{oi} - \xi\cdot\bar{\xi}^{oi} = (-2\bar{\beta}^o - 2\,i\,\rho^o\pi^o)\xi^{oi} + (2\beta^o - 2\,i\,\rho^o\bar{\pi}^o)\bar{\xi}^{oi} - 2\,i\,\rho^o X^{oi} . \qquad \text{(IV. 28b)}$$

The lengthy algebra encountered in the derivation of these equations has been verified by a FORMAC computer program on the California Institute of Technology's IBM 7094.

To simplify the equations further we must use the tetrad freedom which remains. Under the rotations of Equations (III. 10), (III. 11) with $DA = D\theta = 0$, some of the recently introduced tetrad variables transform as follows:

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*It is interesting to note that this constraint is implied only by the very last equation to be used, Equation (IV. 19j).

$$\begin{aligned}
\pi^o &\rightarrow \pi^o \exp(-i\theta) \\
\tau^o &\rightarrow \tau^o A^2 \exp(i\theta) \\
\psi_2{}^o &\rightarrow \psi_2{}^o A^3 \\
U^o &\rightarrow U^o A^2 \; .
\end{aligned} \tag{IV. 29}$$

Select A such that $\psi_2{}^o \bar{\psi}_2{}^o$ becomes a constant. Then Equations (IV. 18) tell us that

$$\bar{\alpha}^o + \beta^o - 2i\rho^o \bar{\pi}^o = 0 \tag{IV. 30a}$$

$$\gamma^o + \bar{\gamma}^o + \mu^o = 0 \; . \tag{IV. 30b}$$

Select θ such that π^o is real and non-negative. It is actually advantageous at this point to abandon the complex notation and work with real and imaginary parts. To this end let us define four new functions b^o, t^o, V^o, and m^o by:

$$\beta^o = b^o + i\rho^o \pi^o \tag{IV. 31a}$$

$$\tau^o = 2(\rho^o)^2 \pi^o + it^o \tag{IV. 31b}$$

$$V^o = U^o + 4(\rho^o \pi^o)^2 \tag{IV. 31c}$$

$$\psi_2{}^o = m^o + i[-2\rho^o V^o + 4b^o t^o + 8(\rho^o)^3(\pi^o)^2] \tag{IV. 31d}$$

A comparison of Equations (IV. 20a, b), (IV. 30a) and the reality of π^o and μ^o show that b^o is real and

$$\mu^o = -4\pi^o b^o \; .$$

Then from Equations (IV. 25) and (IV. 27), t^o and m^o are also real.* From Equations (IV. 26c), (IV. 30b), (IV. 17d) and (IV. 24), we find

$$\gamma^o = 2 b^o \pi^o$$

$$M^o = -V^o - 2 i \pi^o t^o - 4 i \rho^o \pi^o b^o + 8(\rho^o \pi^o)^2 \ .$$

Every quantity has now been expressed in terms of real functions. Furthermore when we carry this change of variables into Equations (IV. 17), (IV. 20), (IV. 21), (IV. 22), (IV. 26), (IV. 28) we make the remarkable discovery that

$$\xi \cdot \rho^o = \bar{\xi} \cdot \rho^o$$

$$X \cdot \rho^o = 0 \ ,$$

and likewise for t^o, π^o, V^o and m^o! In other words they are all effectively functions of a single co-ordinate. The only equations remaining to be solved are

$$\xi \cdot \rho^o = t^o \qquad \text{(IV. 32a)}$$

$$\xi \cdot t^o = 2 b^o t^o - 8(\rho^o)^3 (\pi^o)^2 \qquad \text{(IV. 32b)}$$

$$\xi \cdot \pi^o = 2 b^o \pi^o \qquad \text{(IV. 32c)}$$

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*If $\pi^o = 0$, Equations (IV. 25), (IV. 27) are identities and give no information. However in that case, θ is not yet well-defined. We can use θ to make $t^o = -i\tau^o$ real and non-negative, and then the reality of m^o follows from Equations (IV. 20c, d).

$$\xi \cdot V^o = -3\, m^o \pi^o \tag{IV. 32d}$$

$$\xi \cdot m^o = 12(\rho^o)^2 \pi^o V^o - 24\, \rho^o b^o \pi^o t^o - 48(\rho^o)^4 (\pi^o)^3 , \tag{IV. 32e}$$

and

$$\xi \cdot b^o + \bar{\xi} \cdot b^o = V^o - 4(b^o)^2 - 12(\rho^o \pi^o)^2 \tag{IV. 33a}$$

$$\bar{\xi} \cdot \xi^{oi} - \xi \cdot \bar{\xi}^{oi} = -2b^o(\xi^{oi} - \bar{\xi}^{oi}) - 2\, i\, \rho^o X^{oi} \tag{IV. 33b}$$

$$\xi \cdot X^{oi} - X \cdot \xi^{oi} = -2\, i\, \rho^o (\pi^o)^2 (\xi^{oi} - \bar{\xi}^{oi}) \ . \tag{IV. 33c}$$

3. Solutions

At this point it is necessary to distinguish several cases.

Case I) $\pi^o = t^o = 0$. These are the NUT metrics. Since they were treated thoroughly in reference 3, a rigorous derivation will not be given here. However they will be recovered as a limiting case of our other solutions.

In all the remaining cases at least one of the variables ρ^o, V^o is non-constant and hence may be used to define the co-ordinate $x^3 = x$, e.g., $\rho^o = \rho^o(x)$. Then $X^{o3} = 0$ and ξ^{o3} is real. By co-ordinate transformations

$$\begin{aligned} u &\rightarrow u + f(x, y) \\ y &\rightarrow y + g(x, y) , \end{aligned} \tag{IV. 34}$$

make ξ^{o1}, ξ^{o4} imaginary.

Case II) $\pi^o = 0$, $t^o \neq 0$. In this case V^o and $\psi_2{}^o$ are both constants. Define the co-ordinate x by setting $\rho^o = ax + b$, where a, b are also constants. Then Equations (IV. 32a, b), (IV. 33a) become

$$a\xi^{o3} = t^o$$
$$\xi^{o3}{}_{,x} = 2b^o$$
$$(\xi^{o3})^2{}_{,xx} = 2V^o \quad ,$$

implying that ξ^{o3}, b^o also are functions of x alone. Integration gives

$$(\xi^{o3})^2 = V^o x^2 + Bx + C \, , \qquad \text{(IV. 35)}$$

where B, C are constants. Equation (IV. 33b) implies

$$(\xi^{o3}\xi^{o4})_{,x} = 0$$
$$(\xi^{03}\xi^{01})_{,x} = -i(ax + b) \, ,$$

with solutions

$$\xi^{o4} = ij(y)/\xi^{o3}$$
$$\xi^{o1} = [-\tfrac{1}{2}iax^2 - ibx + ik(y)]/\xi^{o3} \, . \qquad \text{(IV. 36)}$$

By further co-ordinate transformations of the form of Equation (IV. 34) we may eliminate k and set j = 1. The resulting tetrad is

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = \rho\bar{\rho}(r^2 + b^2,\ V^o r^2 + mr + [b\ell - a^2 C],\ 0,\ 2a) \qquad (IV.37)$$

$$m^\mu = \bar{\rho}([-\tfrac{1}{2} iax^2 - ibx]/\xi,\ 0,\ \xi,\ i/\xi),$$

where

$$\xi^2 = V^o x^2 + Bx + C$$

$$\rho = -(r + iax + ib)^{-1}$$

$$\psi_2{}^o = m + i\ell \qquad (IV.38)$$

$$\ell = aB - 2bV^o .$$

A detailed discussion of this solution and a presentation of the metric will be postponed until the remaining cases have been dealt with.

Case III) $t^o = 0$, $\pi^o \neq 0$. In this case $\rho^o = 0$ and $\psi_2{}^o = \bar{\psi}_2{}^o$ = const. Define the co-ordinate x by $V^o = wx + v$, where w, v are also constants. Equations (IV. 32d, c), (IV. 33a) become

$$w\xi^{o3} = -3m\pi^o$$

$$\xi^{o3}{}_{,x} = 2b^o \qquad (IV.39)$$

$$(\xi^{o3})^2{}_{,xx} = 2wx + 2v .$$

Their integration gives

$$(\xi^{o3})^2 = \frac{1}{3} wx^3 + vx^2 + Bx + C .$$

Equation (IV. 33b) implies

$$(\xi^{o3}\xi^{o4})_{,x} = 0$$

$$(\xi^{o3}\xi^{o1})_{,x} = 0 ,$$

with solutions

$$\xi^{o4} = i j(y)/\xi^{o3}$$

$$\xi^{o1} = i k(y)/\xi^{o3} . \qquad \text{(IV. 40)}$$

Once more, by the co-ordinate transformation of Equation (IV. 34) we can set $j = 1$, $k = 0$. The resulting tetrad is

$$\ell^{\mu} = (0, 1, 0, 0)$$

$$n^{\mu} = [1, w^2 r^2 \xi^2 (x + 3m/wr)/9m^2, 0, 0] \qquad \text{(IV. 41)}$$

$$m^{\mu} = [0, -wr\xi(x)/3m, -\xi(x)/r, -i/r\xi(x)] ,$$

where

$$\xi^2(x) = \frac{1}{3} wx^3 + vx^2 + Bx + C , \qquad \text{(IV. 42)}$$

and $\xi^2(x + 3m/wr)$ is the same cubic polynomial of its argument. Here again we have left the tetrad slightly more general than necessary, to facilitate the later discussion.

Case IV) $\pi^o \neq 0$, $t^o \neq 0$. Recall that ξ^{o3} and π^o are functions of a single co-ordinate x. Under $x \to f(x)$ we may set ξ^{o3} equal to any function of x. Fix x by demanding $\xi^{o3} = d\pi^o$, where d is a real scaling parameter to be chosen in a moment. Equations (IV. 32a, b) imply

$$\rho^o{}_{,xx} = -8(\rho^o)^3/d^2 \ . \qquad \text{(IV. 43)}$$

The general solution is

$$\rho^o = a\,\mathrm{cn}[(2a\sqrt{2}/d)(x - x_o)] \ ,$$

where cn is a Jacobian elliptic function* of modulus $k = \frac{1}{2}\sqrt{2}$, and a, x_o are integration constants. A simple redefinition of x permits us to set $d = -2a\sqrt{2}$, $x_o = 0$. Then

$$\begin{aligned} \rho^o &= a\,\mathrm{cn}\,x \\ t^o &= 2\sqrt{2}\,a^2\pi^o\,\mathrm{sn}\,x\,\mathrm{dn}\,x \ . \end{aligned} \qquad \text{(IV. 44)}$$

Equation (IV. 18a) is

*The elliptic functions of modulus $k = \sin 45^o$ are somewhat special, lying midway between the circular ($k = 0$) and the hyperbolic ($k = 1$) functions. For another vacuum metric which utilizes these functions, see reference 33.

$$\xi \cdot \psi_2{}^o = 6 i \rho^o \psi_2{}^o ,$$

which has the solution

$$\psi_2{}^o = (m + i\ell)(\mathrm{dn}\, x - \tfrac{1}{2} i \sqrt{2}\, \mathrm{sn}\, x)^3 , \qquad \text{(IV. 45)}$$

where m, ℓ are constants. Therefore

$$\begin{aligned} m^o = \mathrm{Re}\, \psi_2{}^o &= m(\mathrm{dn}^3 x - \tfrac{3}{2} \mathrm{dn}\, x\, \mathrm{sn}^2 x) \\ &+ \ell(\tfrac{3}{2}\sqrt{2}\, \mathrm{dn}^2 x\, \mathrm{sn}\, x - \tfrac{1}{4}\sqrt{2}\, \mathrm{sn}^3 x) . \end{aligned} \qquad \text{(IV. 46)}$$

We substitute this into Equation (IV. 32d) and get

$$\begin{aligned} V^o{}_{,x} = \tfrac{3}{4}\sqrt{2}\, a^{-1} [&m(\mathrm{dn}^3 x - \tfrac{3}{2} \mathrm{dn}\, x\, \mathrm{sn}^2 x) \\ &+ \ell(\tfrac{3}{2}\sqrt{2}\, \mathrm{dn}^2 x\, \mathrm{sn}\, x - \tfrac{1}{4}\sqrt{2}\, \mathrm{sn}^3 x)] , \end{aligned}$$

hence

$$V^o = b + \tfrac{3}{4}\sqrt{2}\, a^{-1} (m\, \mathrm{sn}\, x - \ell\sqrt{2}\, \mathrm{dn}\, x)\, \mathrm{cn}\, x , \qquad \text{(IV. 47)}$$

where b is a constant.

Now if we put these findings into Equation (IV. 32e) we get an equation for $(\pi^o)^2$:

$$\operatorname{sn} x \operatorname{dn} x\, [(\pi^o)^2]_{,x} - \operatorname{cn}^3 x\, (\pi^o)^2 = -\frac{1}{4} b a^{-2} \operatorname{cn} x + \frac{1}{16}\sqrt{2}\, m a^{-3} \operatorname{sn}^3 x + \frac{1}{4} \ell\, a^{-3} \operatorname{dn}^3 x \ ,$$

with the solution

$$(\pi^o)^2 = c \operatorname{sn} x \operatorname{dn} x + \frac{1}{4} b\, a^{-2} \operatorname{cn}^2 x - \frac{1}{8}\sqrt{2}\, a^{-3} \operatorname{cn} x\, (m \operatorname{sn} x + \ell\sqrt{2} \operatorname{dn} x)\ , \tag{IV. 48}$$

where c is a constant.

Finally, the solution of Equations (IV. 33b, c) for $i = 1, 4$ is

$$X^{oi} = D^i \operatorname{sn} x + E^i \operatorname{dn} x$$
$$\xi^{oi} = \frac{1}{2} i\sqrt{2}\, (\pi^o)^{-1} (-D^i \operatorname{dn} x + \frac{1}{2} E^i \operatorname{sn} x)\ . \tag{IV. 49}$$

By a co-ordinate transformation, Equation (IV. 34), we set $D^i = \delta_{i4}$ and $E^i = \delta_{i1}$, and the resulting vacuum solution is

$$\ell^\mu = (0, 1, 0, 0)$$
$$n^\mu = (X^1, U, 0, X^4) \tag{IV. 50}$$
$$m^\mu = -\frac{1}{2}\sqrt{2}\,\bar{\rho}\,[-\frac{1}{2} i(\pi^o)^{-1} \operatorname{sn} x,\ \sqrt{2}\,\pi^o(r^2 + 3a^2 \operatorname{cn}^2 x),\ 4a\pi^o,\ i(\pi^o)^{-1} \operatorname{dn} x]$$

where

$$\rho = -(r + ia\,\mathrm{cn}\,x)^{-1}$$

$$X^1 = \mathrm{dn}\,x + \sqrt{2}\,a\rho\bar{\rho}\,(r\,\mathrm{cn}\,x + a\sqrt{2}\,\mathrm{sn}\,x\,\mathrm{dn}\,x)\,\mathrm{sn}\,x$$

$$X^4 = \mathrm{sn}\,x - 2\sqrt{2}\,a\rho\bar{\rho}(r\,\mathrm{cn}\,x + a\sqrt{2}\,\mathrm{sn}\,x\,\mathrm{dn}\,x)\,\mathrm{dn}\,x$$

$$\begin{aligned} U = {} & V^o - 7a^2(\pi^o)^2\,\mathrm{cn}^2 x + 2ar\sqrt{2}\,(\pi^o)^2{}_{,x} + r^2(\pi^o)^2 \\ & + r\rho\bar{\rho}\,[m^o + 8a^3\sqrt{2}\,(\pi^o)^2\,\mathrm{sn}\,x\,\mathrm{dn}\,x] \\ & + \rho\bar{\rho}\,[a\ell^o\,\mathrm{cn}\,x + a^4(2\,\mathrm{cn}^4 x - 1)(\pi^o)^2] \end{aligned} \qquad \text{(IV. 51)}$$

$$\begin{aligned} \ell^o = \mathrm{Im}\psi_2{}^o = {} & -m\sqrt{2}\,(\tfrac{3}{2}\,\mathrm{dn}^2 x\,\mathrm{sn}\,x - \tfrac{1}{4}\,\mathrm{sn}^3 x) \\ & + \ell(\mathrm{dn}^3 x - \tfrac{3}{2}\,\mathrm{dn}\,x\,\mathrm{sn}^2 x)\,, \end{aligned}$$

m^o, V^o, $(\pi^o)^2$ are given by Equations (IV. 46) - (IV. 48), and a, b, c, ℓ, m are all constants.

The non-zero metric components are

$$\begin{aligned} g_{uu} &= -2U\,\mathrm{dn}^2 x - (X^4\pi^o)^2\,(r^2 + a^2\,\mathrm{cn}^2 x) \\ g_{ur} &= \mathrm{dn}\,x \\ g_{ux} &= -\tfrac{1}{4}\sqrt{2}\,a^{-1}\,(r^2 + 3a^2\,\mathrm{cn}^2 x)\,\mathrm{dn}\,x \\ g_{uy} &= -U\,\mathrm{sn}\,x\,\mathrm{dn}\,x + X^1X^4(\pi^o)^2\,(r^2 + a^2\,\mathrm{cn}^2 x) \\ g_{ry} &= \tfrac{1}{2}\,\mathrm{sn}\,x \\ g_{xx} &= -\tfrac{1}{16}\,(a\pi^o)^{-2}\,(r^2 + a^2\,\mathrm{cn}^2 x) \end{aligned} \qquad \text{(IV. 52)}$$

$$g_{xy} = -\frac{1}{8}\sqrt{2}\, a^{-1} (r^2 + 3a^2 \,\mathrm{cn}^2 x)\, \mathrm{sn}\, x$$

$$g_{yy} = -\frac{1}{2} U \,\mathrm{sn}^2 x - (X^1 \pi^o)^2 (r^2 + a^2 \,\mathrm{cn}^2 x) .$$

There is one degree of freedom left in the tetrad, namely Equation (III. 10) with A constant. It may be used to set $a = 1$, leaving four arbitrary parameters in this solution.

This concludes the list of solutions for $\rho \neq 0$.

4. Discussion

The most obvious fact about all of these solutions is that the tetrad components depend only on two co-ordinates, r and x. Hence they all possess at least two Killing vectors,* namely

$$V^\mu = (1, 0, 0, 0)$$

$$W^\mu = (0, 0, 0, 1) .$$

Since $\psi_2 \sim r^{-3}$ for large r, the solutions are all asymptotically flat at $r \to \infty$. Misner has emphasized[34] that this does not necessarily imply the existence of a global co-ordinate system which is asymptotically Minkowskian. However, in case one does exist, at $r \to \infty$ the Killing vectors must tend to some combinations of space-time translations

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*A vector V^μ is a Killing vector if $g_{\mu\nu}(x^\mu) = g_{\mu\nu}(x^\mu + V^\mu da)$ where da is an infinitesimal parameter. Thus V^μ is the generator of a group of mappings of the space onto itself which do not distort the geometry.

and rotations in a flat background metric.

Let us begin the discussion with Case II. In a search for spaces in which the geodesic equations are separable, Carter[35] found a metric (which he called the [A] metric) that is equivalent to our tetrad solution, Equation (IV. 37). In order to find a physical interpretation, define a new co-ordinate θ by

$$\theta = \int \xi^{-1}\, dx = \int (V^{o} x^{2} + Bx + C)^{-1/2}\, dx . \qquad \text{(IV. 53)}$$

Although this integral is expressible in terms of elementary functions, <u>six</u> cases must be distinguished depending upon the signs of V^{o} and $B^{2} - 4V^{o}C$. Further simplification can be achieved by using a tetrad rotation, Equation (III. 10), with A constant. (For instance we may set $V^{o} = \pm \frac{1}{2}, 0$.) The tetrads which result, along with the non-zero metric components, are now given:

Case II-A) $V^{o} < 0$.

$$\ell^{\mu} = (0, 1, 0, 0)$$

$$n^{\mu} = \rho\bar{\rho}\,[r^{2} + \ell^{2} + a^{2}, -\tfrac{1}{2}(r^{2} - 2mr - \ell^{2} + a^{2}), 0, a]$$

$$m^{\mu} = -\tfrac{1}{2}\sqrt{2}\,\bar{\rho}\,[ia\sin\theta + 2i\ell\cot\theta, 0, 1, i\csc\theta]$$

$$\rho = -(r + i\ell - ia\cos\theta)^{-1}$$

$$g_{uu} = \rho\bar{\rho}(r^{2} - 2mr - \ell^{2} + a^{2}\cos^{2}\theta)$$

$$g_{ur} = 1 \qquad \text{(IV. 54)}$$

$$g_{uy} = -2\ell\rho\bar{\rho}(r^2 - 2mr - \ell^2 + a^2)\cos\theta$$
$$+ 2a\rho\bar{\rho}(mr + \ell^2)\sin\theta$$

$$g_{ry} = -a\sin^2\theta - 2\ell\cos\theta$$

$$g_{\theta\theta} = -r^2 - (\ell - a\cos\theta)^2$$

$$g_{yy} = -\rho\bar{\rho}(r^2 + \ell^2 + a^2)^2\sin^2\theta + \rho\bar{\rho}(r^2 - 2mr$$
$$- \ell^2 + a^2)(a\sin^2\theta + 2\ell\cos\theta)^2$$

Case II-B) $V^o > 0,\ B^2 - 4V^oC > 0.$

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = \rho\bar{\rho}[r^2 + \ell^2 + a^2,\ \tfrac{1}{2}(r^2 + 2mr - \ell^2 + a^2),\ 0,\ a]$$

$$m^\mu = -\tfrac{1}{2}\sqrt{2}\,\bar{\rho}[-ia\sinh\theta + 2i\ell\coth\theta,\ 0,\ 1,\ i\operatorname{csch}\theta]$$

$$\rho = -(r - i\ell + ia\cosh\theta)^{-1}$$

$$g_{uu} = -\rho\bar{\rho}(r^2 + 2mr - \ell^2 + a^2\cosh^2\theta)$$

$$g_{ur} = 1 \qquad \text{(IV. 55)}$$

$$g_{uy} = 2\ell\rho\bar{\rho}(r^2 + 2mr - \ell^2 + a^2)\cosh\theta$$
$$-2a\rho\bar{\rho}(mr - \ell^2)\sinh^2\theta$$

$$g_{ry} = a\sinh^2\theta - 2\ell\cosh\theta$$

$$g_{\theta\theta} = -r^2 - (-\ell + a\cosh\theta)^2$$

$$g_{yy} = -\rho\bar{\rho}(r^2 + \ell^2 + a^2)^2 \sinh^2\theta - \rho\bar{\rho}(r^2 + 2mr - \ell^2 + a^2)(-a\sinh^2\theta + 2\ell\cosh\theta)^2$$

Case II-C) $V^o > 0,\ B^2 - 4V^oC = 0.$

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = \rho\bar{\rho}[r^2 + \ell^2,\ \tfrac{1}{2}(r^2 + 2mr - \ell^2),\ 0,\ a]$$

$$m^\mu = -\tfrac{1}{2}\sqrt{2}\,\bar{\rho}\,(-iae^\theta + 2i\ell,\ 0,\ 1,\ ie^{-\theta})$$

$$\rho = -(r - i\ell + iae^\theta)^{-1}$$

$$g_{uu} = -\rho\bar{\rho}(r^2 + 2mr - \ell^2 + a^2 e^{2\theta})$$

$$g_{ur} = 1 \qquad \text{(IV. 56)}$$

$$g_{uy} = 2\ell\rho\bar{\rho}(r^2 + 2mr - \ell^2)e^\theta - 2a\rho\bar{\rho}(mr - \ell^2)e^{2\theta}$$

$$g_{ry} = ae^{2\theta} - 2\ell e^\theta$$

$$g_{\theta\theta} = -r^2 - (-\ell + ae^\theta)^2$$

$$g_{yy} = -\rho\bar{\rho}(r^2 + \ell^2)^2 e^{2\theta} - \rho\bar{\rho}(r^2 + 2mr - \ell^2)(-ae^{2\theta} + 2\ell e^\theta)^2$$

Case II-D) $V^o > 0,\ B^2 - 4V^oC < 0.$

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = \rho\bar{\rho}[r^2 + \ell^2 - a^2,\ \tfrac{1}{2}(r^2 + 2mr - \ell^2 - a^2),\ 0,\ a]$$

$$m^\mu = -\frac{1}{2}\sqrt{2}\,\bar{\rho}(-ia\cosh\theta + 2i\ell\tanh\theta,\ 0,\ 1,\ i\,\mathrm{sech}\,\theta)$$

$$\rho = -(r - i\ell + ia\sinh\theta)^{-1}$$

$$g_{uu} = -\rho\bar{\rho}(r^2 + 2mr - \ell^2 + a^2\sinh^2\theta)$$

$$g_{ur} = 1 \qquad \text{(IV. 57)}$$

$$g_{uy} = 2\ell\rho\bar{\rho}(r^2 + 2mr - \ell^2 - a^2)\sinh\theta - 2a\rho\bar{\rho}(mr - \ell^2)\cosh^2\theta$$

$$g_{ry} = a\cosh^2\theta - 2\ell\sinh\theta$$

$$g_{\theta\theta} = -r^2 - (-\ell + a\cosh\theta)^2$$

$$g_{yy} = -\rho\bar{\rho}(r^2 + \ell^2 - a^2)^2\cosh^2\theta - \rho\bar{\rho}(r^2 + 2mr - \ell^2 - a^2)(-a\cosh^2\theta + 2\ell\sinh\theta)^2$$

Case II-E) $V^o = 0,\ B \neq 0.$

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = \rho\bar{\rho}(r^2 + b^2,\ mr + b\ell,\ 0,\ 1)$$

$$m^{\mu} = -\frac{1}{2}\sqrt{2}\,\bar{\rho}(-\frac{1}{4}i\ell\,\theta^{3} - ib\,\theta,\ 0,\ 1,\ i\,\theta^{-1})$$

$$\rho\ = -(r + ib + \frac{1}{2}\,i\ell\,\theta^{2})^{-1}$$

$$g_{uu} = -\rho\bar{\rho}(2mr + 2b\ell + \ell^{2}\theta^{2})$$

$$g_{ur} = 1 \qquad \text{(IV. 58)}$$

$$g_{uy} = \rho\bar{\rho}(\ell r^{2} - 2mbr - \ell b^{2})\theta^{2} - \frac{1}{2}\,\ell\,\rho\bar{\rho}(mr + b)\theta^{4}$$

$$g_{ry} = b\,\theta^{2} + \frac{1}{4}\,\ell\,\theta^{4}$$

$$g_{\theta\theta} = -r^{2} - (b + \frac{1}{2}\,\ell\,\theta^{2})^{2}$$

$$g_{yy} = -\rho\bar{\rho}(r^{2} + b^{2})^{2}\theta^{2} - 2\,\rho\bar{\rho}(mr + b\ell)(b\theta^{2} + \frac{1}{4}\ell\,\theta^{4})^{2}$$

Case II-F) $V^{o} = 0$, $B = 0$.

$$\ell^{\mu} = (0,\ 1,\ 0,\ 0)$$

$$n^{\mu} = \rho\bar{\rho}(r^{2},\ mr - \frac{1}{2}\,b^{2},\ 0,\ b)$$

$$m^{\mu} = -\frac{1}{2}\sqrt{2}\,\bar{\rho}(-i\,b\,\theta^{2},\ 0,\ 1,\ i)$$

$$\rho = -(r + i\,b\,\theta)^{-1}$$

$$g_{uu} = -2\,\rho\bar{\rho}\,mr$$

$$g_{ur} = 1 \qquad \text{(IV. 59)}$$

$$g_{uy} = -2\,b\,\rho\bar{\rho}(mr - \frac{1}{2}\,b^{2})\theta^{2} + \rho\bar{\rho}\,b\,r^{2}$$

$$g_{ry} = b\,\theta^2$$

$$g_{\theta\theta} = -r^2 - b^2\,\theta^2$$

$$g_{yy} = -\rho\bar{\rho}\,r^4 - 2b^2\,\rho\bar{\rho}(mr - \frac{1}{2}\,b^2)\theta^4\ .$$

Case II-A, first discovered by Demianski and Newman,[36] is known as Kerr-NUT space. For $\ell = 0$ it reduces to the Kerr metric,[37] widely hailed as the exterior gravitational field of a rotating mass. For $a = 0$, ρ^o becomes constant and all six metrics revert to Case I. As mentioned earlier, Case I was exhaustively treated by Newman, Tamburino, and Unti,[38] and found to have only three distinct solutions. In this limit, therefore, some of the six solutions given above must become equivalent. For $a = \ell = 0$ we get the three "A" metrics listed by Ehlers and Kundt,[39] of which one is the Schwarzschild metric.

Let us look at the norm of the ∂_u Killing vector,

$$V^\mu\, V^\nu\, g_{\mu\nu} = g_{uu}\ .$$

As $r \to \infty$ we see from Equations (IV. 54) - (IV. 59) that this norm goes to a constant: +1 for Case A; -1 for B, C, D; and 0 for E and F. Since the only flat-space Killing vectors with constant norm are the translations, we are justified in regarding V^μ as a timelike, spacelike, or null translation. Although V^μ may not be hypersurface-orthogonal everywhere, it is at $r \to \infty$, and there the hypersurface will be either a Euclidean, Lorentz, or null 3-flat. The "wavefronts" r = const., u = const. are two-dimensional manifolds in the 3-flat. The metrics induced on them tend in the limit $r \to \infty$ to:

Case A) $-r^2(d\theta^2 + \sin^2\theta\, dy^2)$

B) $-r^2(d\theta^2 + \sinh^2\theta\, dy^2)$

C) $-r^2(d\theta^2 + e^{2\theta}\, dy^2)$

D) $-r^2(d\theta^2 + \cosh^2\theta\, dy^2)$

E) $-r^2(d\theta^2 + \theta^2\, dy^2)$

F) $-r^2(d\theta^2 + dy^2)$.

Thus the wavefronts tend to spheres in A, pseudospheres in B, C, C, and planes in E and F. In all cases the co-ordinate system is geared to the ∂_y Killing vector. On a sphere any isometry is a rotation about some axis, leading to the usual spherical co-ordinates. On a plane there are two kinds of isometries, translations and rotations, and the corresponding co-ordinate systems are polar and Cartesian co-ordinates (Cases E and F). A pseudosphere, however, has three kinds of isometries, perhaps not as familiar. When it is embedded as the future hyperboloid in a Lorentz 3-space (as it is here), the three isometry classes are rotations about a timelike, spacelike, or null axis; Cases B, C, D respectively are the co-ordinate systems geared to these isometries. The explicit transformations relating the three co-ordinate systems are given in Appendix A.

For $a \neq 0$, ρ^o depends explicitly on θ and therefore the six Case II metrics are only axially symmetric and inequivalent. For $a = 0$ the metrics actually become spherically symmetric, acquiring two more Killing vectors. Cases B, C, D then differ only by a co-ordinate transformation, and likewise for E and F. The Killing vectors of these Case I metrics are described in reference 6.

A fortunate circumstance permits us to easily visualize the geometry of some of the Case II metrics at finite values of r. For $\ell = 0$ the curvature $\psi_2{}^o = m$ appears only once in the tetrad, in the quantity U. Defining

$$n'^{\mu} = n^{\mu} - mr\rho\bar{\rho}\,\ell^{\mu} ,$$

we see that ℓ^{μ}, n'^{μ}, m^{μ}, $\bar{m}^{\mu}$ comprise a flat-space tetrad and

$$g'^{\mu\nu} = g^{\mu\nu} - 2mr\,\rho\bar{\rho}\,\ell^{\mu}\ell^{\nu}$$

a flat-space metric. Vacuum metrics which differ from flat space by the square of a vector were studied by Kerr and Schild.[40] The flat-space limit of the ℓ^{μ} congruence for Case II-A has been discussed in detail,[41] and this is summarized in Appendix B.

We propose that all six metrics of Case II with $\ell = 0$ are the external fields of spinning particles, with a 4-momentum parallel to V^{μ}, and an angular momentum vector orthogonal to it and along the direction of axial symmetry. The six cases arise as the six different combinations in which spacelike, null, or timelike vectors may be orthogonal. Particles with spacelike momentum have been discussed in the framework of first quantization by Wigner.[42] He found relativistic wave equations corresponding to each of these cases. To firmly establish the spinning particle interpretation one would want to either (a) find interior solutions for these metrics, or (b) study the linearized fields of such particles. The widely-accepted interpretation of the Kerr metric thus far relies on approach (b). Since all six metrics arise from one metric, Equation (IV. 37) by continuously varying V^o, A, B, and since they have the expected symmetries, I believe this extended interpretation is justified.

Next we would like to discuss the solution of Case III, Equation (IV. 41). A metric equivalent to this tetrad was listed by Ehlers and Kundt[43] as the "C" metric. We have three degrees of freedom remaining: the constant tetrad rotation, and linear transformations of x, $x \to px + q$. These can be used to set $w = 3m$, $B = 0$ and $m = 1$, giving

$$
\begin{aligned}
\ell^\mu &= (0,\ 1,\ 0,\ 0) \\
n^\mu &= [1,\ r^2\xi^2(x + 1/r),\ 0,\ 0] \\
m^\mu &= [0,\ -r\xi(x),\ -\xi(x)/r,\ -i/r\xi(x)] \\
g_{uu} &= -r^2\,\xi^2(x + 1/r) \\
g_{ur} &= 1 \qquad\qquad\qquad\qquad\qquad \text{(IV. 60)} \\
g_{ux} &= -r^2 \\
g_{xx} &= -r^2/\xi^2(x)
\end{aligned}
$$

$$g_{yy} = -r^2 \xi^2(x)$$

where

$$\xi^2(x) = x^3 + vx^2 + C .$$

The "C" metric therefore has two free parameters.

If instead we take the singular limit $w \to 0$, the tetrad becomes

$$\ell^\mu = (0, 1, 0, 0)$$

$$n^\mu = (1, v + m/r, 0, 0)$$

$$m^\mu = (0, 0, -\xi/r, i/r\,\xi) ,$$

with

$$\xi^2 = vx^2 + Bx + C ,$$

which coincides with Equation (IV. 37) with $a = \ell = 0$, $V^o = v$. Hence in this limit we recover the three "A" metrics.

The wavefronts $r = \text{const.}$, $u = \text{const.}$ of the "C" metric are spacelike 2-surfaces whose intrinsic geometry may be studied by embedding them in a Euclidean or Lorentz 3-space. They are axially symmetric but may have cusps on the axis. They may also be open or closed, depending on the particular values of v and C. The only case in which the wavefronts are closed and free of cusps is the Schwarzschild limit $w \to 0$, $v > 0$.

Let us now examine the two Killing vectors. Since $\xi^2(x + 1/r)$ is a cubic polynomial, $V^\mu V^\nu g_{\mu\nu} = g_{uu} = 0$ may have as many as three roots. This implies that as many as three event horizons may exist where the ∂_u Killing vector becomes null. However in all cases as $r \to \infty$, $g_{uu} < 0$, $g_{yy} < 0$, $g_{uy} = 0$, and the two Killing vectors are spacelike and orthogonal with norms proportional to r. Therefore they do not tend to translation vectors, but rather to rotations or screw motions. The simplest assumption is that they are rotations (i. e., Lorentz transformations). Since they commute, the two isometries must then be a rotation about an axis and a boost along the same axis.

To gain further insight, consider the flat-space limit $w = 3m = 0$ of the tetrad of Equation (IV. 41). Now we have

$$\xi^2 = vx^2 + Bx + C .$$

Just as in Equation (IV. 53) in the discussion of Case II we define $\theta = \int \xi^{-1} dx$ and need to consider six different sub-cases. We will only write down the case $\theta = \cos^{-1} x$. The tetrad and metric become

$$\ell^\mu = (0, 1, 0, 0)$$

$$n^\mu = (1, -\tfrac{1}{2} + Ar\cos\theta + A^2 r^2 \sin^2\theta, 0, 0)$$

$$m^\mu = (0, Ar\sin^2\theta, -1/r\sqrt{2}, -i/r\sqrt{2}\sin\theta)$$

$$g_{uu} = 1 - 2Ar\cos\theta - A^2 r^2 \sin^2\theta$$

$$g_{ur} = 1 \tag{IV.61}$$

$$g_{u\theta} = -Ar^2 \sin\theta$$

$$g_{\theta\theta} = -r^2$$

$$g_{yy} = -r^2 \sin^2\theta ,$$

where A is a constant.

Flat-space metrics of this form have been considered by Newman and Unti[44] (cf. also Equation (V. 36)). From their results one can conclude that u, r, θ, y are null co-ordinates based on a timelike hyperbola (at $r = 0$) with radius $1/A$.

I would like to suggest, therefore, that the "C" metric is a "runaway" solution, similar to those encountered in electrodynamics. It may represent the field of a point mass executing hyperbolic motion. More work needs to be done to confirm this. For example, one would like to examine the linearized metrics in which m/r or m/A are small but non-zero.

Case IV is the one Type D vacuum metric which I believe to be new. We may say it is the "most general," in the sense that it contains all previous metrics as singular limits. For example, we can retrieve Case I from it via a co-ordinate transformation

$$x = px' ,$$

thereby introducing a redundant parameter p. Replace the elliptic functions by their Maclaurin series,

$$\mathrm{sn}\, x = x - \frac{1}{4} x^3 + \ldots$$

$$\mathrm{cn}\, x = 1 - \frac{1}{2} x^2 + \ldots$$

$$\mathrm{dn}\, x = 1 - \frac{1}{4} x^2 + \ldots$$

valid for small x. If we set

$$a = \ell$$

$$b = 1 + p^2 C$$

$$c = (m/2 + p\ell B)/8\ell^3$$

$$m = m$$

$$\ell = \ell$$

$$u = u$$

$$y = 2\ell p^{-1} y'$$

and let $p \to 0$, the resulting tetrad is Equation (IV. 37) with $\rho^0 = \ell$, i. e., NUT space.

To show the relation to Cases II and III, it is necessary to shift the origin of x by one-quarter wavelength of the cn function. (This distance is the complete elliptic integral K(k), listed in many tables. Some particular values are $K(0) = \pi/2$, $K(1) = \infty$, $K(\frac{1}{2}\sqrt{2}) = 1.3506$.) The relations we need are

$$\mathrm{sn}\,(x + K) = \mathrm{cn}\,x/\mathrm{dn}\,x$$

$$\mathrm{cn}\,(x + K) = -\frac{1}{2}\sqrt{2}\,\mathrm{sn}\,x/\mathrm{dn}\,x$$

$$\mathrm{dn}\,(x + K) = \frac{1}{2}\sqrt{2}/\mathrm{dn}\,x\,.$$

Now $\rho^{o} = -\frac{1}{2}\,a\sqrt{2}\,\mathrm{sn}\,x/\mathrm{dn}\,x$ has its zero at the origin. We again change the scale of x by $x = px'$ and use the Maclaurin expansions. If we set

$$a = -\frac{1}{4}\,p\sqrt{2}$$

$$b = v$$

$$c = C$$

$$m = -\frac{1}{2}\sqrt{2}$$

$$\ell = +\frac{1}{2}\sqrt{2}$$

$$u = \frac{1}{2}u'\sqrt{2} + \frac{1}{4}py'\sqrt{2}$$

$$y = u' - \frac{1}{2}\,py'$$

and take the limit $p \to 0$, the tetrad which results is Case III, and corresponds to the metric given in Equation (IV. 60).

If instead we set

$$a = -\sqrt{2}\,a'/p$$

$$b = V^{o}$$

$$c = C p^4 \sqrt{2}/16 a'^2$$

$$m = -\frac{1}{2} m' \sqrt{2}$$

$$\ell = +\frac{1}{2} m' \sqrt{2}$$

$$u = \frac{1}{2} u' \sqrt{2} + \sqrt{2}\, a' y'/p^2$$

$$y = u' - 2a' y'/p^2$$

the limit $p \to 0$ yields the tetrad of Case II, Equation (IV. 37) with $\rho^o = a'x$, i. e., the Kerr family of metrics. Kerr-NUT space can be obtained by exactly the same approach, but with somewhat more complicated substitutions.

B. The Case $\rho = 0$

1. Integration and Solutions

In the discussion of the case $\rho = 0$ which follows, we can also assume $\mu = 0$, because otherwise after interchange of ℓ^μ and n^μ the derivation of Section IV A would apply. According to Equation (III. 17), both ℓ^μ and n^μ are hypersurface orthogonal. We take co-ordinates comoving along ℓ^μ as before, but with the added condition that

$$\ell_\mu = u_{,\mu} = (1, 0, 0, 0) .$$

The orthonormality relations imply

$$\ell^{\mu} n^{\mu} = X^{1} = 1$$

$$\ell_{\mu} m^{\mu} = \xi^{1} = 0 \ .$$

Also, from Equation (III. 1), the fact that $\ell_{[\mu,\nu]} = 0$ implies

$$\tau = \bar{\alpha} + \beta \ .$$

Equations (IV. 4), (IV. 5), (IV. 7d) reduce to

$$\pi\bar{\pi} = \tau\bar{\tau}$$

$$D\pi = 0 \qquad \text{(IV. 62)}$$

$$D\tau = 0 \ ,$$

hence we may use the tetrad rotation $m^{\mu} \rightarrow m^{\mu} \exp(i\theta)$ with $D\theta = 0$ to set

$$\tau = -\pi \ . \qquad \text{(IV. 63)}$$

With all these simplifications, Equations (IV. 19), (IV. 6) now become

$$\Delta\tau = \tau(\gamma - \bar{\gamma}) \qquad \text{(IV. 64a)}$$

$$\delta\tau = 2\beta\tau \qquad \text{(IV. 64b)}$$

$$\bar{\delta}\tau = 2\alpha\tau + \psi_2 \qquad \text{(IV. 64c)}$$

$$\delta\tau = \tau(\alpha + \bar{\alpha} - \beta + \bar{\beta}) + \psi_2 \qquad \text{(IV. 64d)}$$

$$\bar{\delta}\tau = \tau(-\alpha + \bar{\alpha} + \beta + \bar{\beta}) \qquad \text{(IV. 64e)}$$

$$\delta\alpha - \bar{\delta}\beta = \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \psi_2 \qquad \text{(IV. 64f)}$$

$$\Delta\tau = \tau(\bar{\gamma} - \gamma) \qquad \text{(IV. 64g)}$$

$$\Delta\beta - \delta\gamma = 0 \qquad \text{(IV. 64h)}$$

$$\Delta\alpha - \bar{\delta}\gamma = 0 \;. \qquad \text{(IV. 64i)}$$

Equations (IV. 64a, g) are consistent only if

$$\gamma = \bar{\gamma} \qquad \text{(IV. 65a)}$$

$$\Delta\tau = 0 \;. \qquad \text{(IV. 65b)}$$

Equations (IV. 64h, i) then imply

$$\Delta\alpha = \Delta\beta = 0 \qquad \text{(IV. 66a)}$$

$$\delta\gamma = 0 \;. \qquad \text{(IV. 66b)}$$

From Equations (IV. 64b, d) we get a formula for ψ_2:

$$\psi_2 = \tau(-\alpha - \bar{\alpha} + 3\beta - \bar{\beta}) \,, \qquad \text{(IV. 67)}$$

while from Equations (IV. 64c, e):

$$\psi_2 = \tau(-3\alpha + \bar{\alpha} + \beta + \bar{\beta}) \,. \qquad \text{(IV. 68)}$$

The only way Equations (IV. 67), (IV. 68) can be consistent is if

$$\alpha + \beta = \bar{\alpha} + \bar{\beta} \qquad \text{(IV. 69a)}$$

$$\psi_2 = 2\tau(\beta - \alpha) \, . \qquad \text{(IV. 69b)}$$

Then Equations (IV. 64b, c) are

$$\delta\tau = \bar{\delta}\tau = 2\beta\tau \, . \qquad \text{(IV. 70)}$$

The Bianchi Identities, Equations (IV. 3), read

$$D\psi_2 = \Delta\psi_2 = 0 \qquad \text{(IV. 71a)}$$

$$\delta\psi_2 = \bar{\delta}\psi_2 = 3\tau\psi_2 \, . \qquad \text{(IV. 71b)}$$

The radial integration is trivial. All of the NP variables are independent of r except

$$\gamma = \gamma^o + r[(\tau + \bar{\tau})(\beta - \alpha) - \tau^2] \qquad \text{(IV. 72a)}$$

$$U = U^o - r[2\gamma^o + \omega^o(\tau - \bar{\tau})] - r^2[(\tau + \bar{\tau})(\beta - \alpha) - \tau^2] \qquad \text{(IV. 72b)}$$

$$\omega = \omega^o - r(\tau + \bar{\tau}) \qquad \text{(IV. 72c)}$$

$$X^i = X^{oi} - r(\tau - \bar{\tau})(\xi^i - \bar{\xi}^i) \, . \qquad \text{(IV. 72d)}$$

Choose co-ordinates such that $X^{oi} = \delta_{i1}$. When these expressions are substituted in Equations (IV. 66b), (IV. 2) we get the following:

$$\delta\gamma^o = -\omega^o[(\tau + \bar{\tau})(\beta - \alpha) - \tau^2] \qquad \text{(IV. 73a)}$$

$$\delta U^{o} - X \cdot \omega^{o} = \omega^{o}[2\gamma^{o} + (\tau - \bar{\tau})(\omega^{o} - \bar{\omega}^{o})] - U^{o}(\tau + \bar{\tau}) \quad \text{(IV. 73b)}$$

$$\bar{\delta}\omega^{o} - \delta\bar{\omega}^{o} = -(\omega^{o} - \bar{\omega}^{o})(\alpha + 2\beta + \bar{\beta}) \quad \text{(IV. 73c)}$$

$$\bar{\delta}\xi^{i} - \delta\bar{\xi}^{i} = -(\xi^{i} - \bar{\xi}^{i})(\alpha - \bar{\beta}) \quad \text{(IV. 73d)}$$

$$\delta X^{oi} - X \cdot \xi^{i} = \omega^{o}(\tau - \bar{\tau})(\xi^{i} - \bar{\xi}^{i}) . \quad \text{(IV. 73e)}$$

As a first step in simplifying these equations, we will show how ω^{o} may be eliminated. The co-ordinate transformation $r \rightarrow r + f(u, x, y)$ leaves previous conditions unchanged. Under this transformation

$$\omega^{o} \rightarrow \omega^{o} + \delta f + (\tau + \bar{\tau})f \ .$$

Thus a solution of

$$\delta f = -\omega^{o} - (\tau + \bar{\tau})f$$

will make $\omega^{o} \rightarrow 0$, provided such a solution exists. A partial differential equation like this always has certain integrability conditions which must be satisfied. Giving the first derivatives Df, Δf, δf legitimately defines f if and only if the commutators $D(\Delta f) - \Delta(Df)$, etc. agree with Equations (III. 7). Choose

$$Df = 0$$

$$X \cdot f = -U^{o} - 2\gamma^{o}f + f^{2}D\gamma$$

$$\delta f = -\omega^{o} - (\tau + \bar{\tau})f \ .$$

The only nontrivial commutators are $(\Delta\delta - \delta\Delta)f$ and $(\bar{\delta}\delta - \delta\bar{\delta})f$. These may be calculated using Equations (IV. 64f), (IV. 65b), (IV. 70) and (IV. 73a - c), and they do agree with Equations (III. 7). Hence we can find an f which sets $\omega^o = 0$.

Next we show how to eliminate U^o, γ^o by a combined co-ordinate transformation and tetrad rotation. The rotation is

$$(\ell')^{\mu} = A^{-1}(u)\,\ell^{\mu}$$
$$(n')^{\mu} = A(u)\, n^{\mu}, \qquad \text{(IV. 74)}$$

and the change of co-ordinates is

$$u' = \int_o^u A^{-1}(t)dt$$
$$r' = rA(u) + U^o R(u), \qquad \text{(IV. 75)}$$

which together preserve all previously imposed conditions, but send

$$\gamma^o \rightarrow A_{,u} - 2\gamma^o A + 2(U^o D\gamma)R$$
$$U^o \rightarrow (A^2 + [U^o D\gamma]R^2 + AR_{,u} + ARU^o{}_{,u}/U^o)U^o .$$

From Equation (IV. 73b) and $(\delta D - D\delta)\gamma$, we have

$$\delta(U^o D\gamma) = -U^o(\tau + \bar{\tau})D\gamma + U^o(\bar{\alpha} + \beta - \bar{\pi}^o)D\gamma = 0 .$$

Therefore $U^o D\gamma$ is a real function of u alone. We can prove the same thing for the quantity $U^o{}_{,u}/U^o$. The problem of eliminating γ^o, U^o thus reduces to the solution of two differential equations for A, R, given γ^o, $U^o D\gamma$, $U^o{}_{,u}/U^o$ as arbitrary functions of u. Under sufficient assumptions of continuity such equations always have solutions, which is all we need to know.

Now we are ready to solve the NP Equations. Since $\delta\psi_2 = \bar{\delta}\psi_2 \neq 0$, we can use $\psi_2\bar{\psi}_2$ to define a co-ordinate x via $\psi_2\bar{\psi}_2 = \psi_2\bar{\psi}_2(x)$. Then ξ^3 is real, and by a co-ordinate transformation $y \to y + f(x, y)$ we make ξ^4 imaginary. Equations (IV. 62), (IV. 65b), (IV. 66a), (IV. 70), (IV. 71a, b), (IV. 73d) show that ψ_2, τ, β, α, ξ^3 depend only on x. From Equations (IV. 61), (IV. 69a), (IV. 70),

$$\begin{aligned} \tau - \bar{\tau} &= \bar{\alpha} - \alpha + \beta - \bar{\beta} \\ &= 2(\beta - \bar{\beta}) \qquad \text{(IV. 76)} \\ \tau - \bar{\tau} &= \tau^{-1}\delta\tau - \bar{\tau}^{-1}\delta\bar{\tau} . \end{aligned}$$

Solve Equation (IV. 71b) for τ and substitute in this expression. The result may be written

$$\delta\left(\frac{\psi_2{}^{-4/3}\,\delta\psi_2}{\bar{\psi}_2{}^{-4/3}\,\delta\bar{\psi}_2}\right) = 0 \ , \qquad \text{(IV. 77)}$$

and integrated twice to give

$$E\psi_2{}^{-1/3} + \bar{E}\,\bar{\psi}_2{}^{-1/3} = 1 \qquad \text{(IV. 78)}$$

where E is an arbitrary complex constant. Actually it is convenient here, as in the metrics for $\rho \neq 0$, to introduce a redundant parameter by letting

$$E = -\frac{1}{2} i a^{-1} (m + i\ell)^{1/3} , \qquad \text{(IV. 79)}$$

where a, m, ℓ are real constants.

The co-ordinate freedom $x' = f(x)$ permits us to set $\psi_2 \bar{\psi}_2$ equal to any function of x we please. Let

$$\psi_2 \bar{\psi}_2 = \frac{m^2 + \ell^2}{(x^2 + a^2)^3} . \qquad \text{(IV. 80)}$$

Equations (IV. 78) - (IV. 80) imply

$$\psi_2 = \frac{m + i\ell}{(x + ia)^3} , \qquad \text{(IV. 81)}$$

and it is this simple form which the above definitions were contrived to achieve.

Next, Equations (IV. 81), (IV. 71b), (IV. 70), (IV. 61) are used to find expressions for τ, α, and β all in terms of ξ^3. These are substituted into Equation (IV. 69b), which then becomes

$$[(\xi^3)^2]_{,x} + \frac{2ia(\xi^3)^2}{x^2 + a^2} = -\frac{m + i\ell}{(x + ia)^2} . \qquad \text{(IV. 82)}$$

If $a \neq 0$ this has the real solution

$$(\xi^3)^2 = \frac{2amx + \ell(a^2 - x^2)}{2a(x^2 + a^2)} .$$

If $a = 0$, then ξ^3 can be real only if $\ell = 0$ also. The solution in this case is

$$(\xi^3)^2 = C + m/x ,$$

where C is an arbitrary constant.

Here are the tetrads and metrics which result.

Case V) $a \neq 0$.

$$\ell^\mu = (0, 1, 0, 0)$$

$$n^\mu = (1, -\frac{r^2 \ell}{2a(x^2 + a^2)}, 0, \frac{4ar}{x^2 + a^2})$$

$$m^\mu = (0, \frac{2rx\xi}{x^2 + a^2}, \xi, \frac{i}{\xi}) \qquad \text{(IV. 83)}$$

$$g_{uu} = [r^2 \ell / a(x^2 + a^2)] - [4r^2 a^2 \xi^2 / (x^2 + a^2)^2]$$

$$g_{ur} = 1$$

$$g_{ux} = -2rx/(x^2 + a^2)$$

$$g_{uy} = ar\xi^2/(x^2 + a^2)$$

$$g_{xx} = -1/2\xi^2$$

$$g_{yy} = -\xi^2/2 \ ,$$

where

$$\xi^2 = \frac{2amx + \ell(a^2 - x^2)}{2a(x^2 + a^2)} \ .$$

Case VI) $a = 0$.

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = (1,\ Cr^2/x^2,\ 0,\ 0)$$

$$m^\mu = (0,\ 2r\xi/x,\ \xi,\ i/\xi)$$

$$g_{uu} = -2Cr^2/x^2 \qquad \text{(IV. 84)}$$

$$g_{ur} = 1$$

$$g_{ux} = -2r/x$$

$$g_{xx} = -1/2\xi^2$$

$$g_{yy} = -\xi^2/2 \ ,$$

where

$$\xi^2 = C + m/x \ .$$

2. Discussion

Case V appears in the preprint by Carter,[45] where it is referred to as the [B(-)] metric. Case VI is equivalent (via a transformation of the form of Equations (IV. 74), (IV. 75)) to the trio of static "B" metrics listed by Ehlers and Kundt.[46]

Recall that in Case V the parameter a is redundant, so we may set $a = 1$ in the following discussion. The space is asymptotically flat at $x \to \infty$, however there is a restriction on the range of x which may actually exclude the flat region for certain values of m and ℓ. The permitted values of x must keep ξ^3 real, or equivalently,

$$-\ell x^2 + 2mx + \ell^2 > 0.$$

The discriminant is $4(m^2 + \ell^2) > 4m^2 > 0$ so $\xi^3 = 0$ always has two roots, say $x = x_\pm$, one positive and one negative.

Case V-A) $\ell < 0$. The permitted ranges of x are $x \geq x_+$ and $x \leq x_-$. At $|x| \to \infty$, $(\xi^3)^2 = -\ell$. Since $g_{uu} < 0$, $g_{yy} < 0$, the ∂_u and ∂_y Killing vectors are spacelike everywhere. At $|x| \to \infty$, $g_{uu} \to 0$, $g_{uy} \to 0$, $g_{yy} \to 2\ell$; hence the ∂_y vector remains spacelike while the ∂_u vector becomes a null vector orthogonal to it.

Case V-B) $\ell > 0$. The range of x is $x_- < x < x_+$. The ∂_y Killing vector is spacelike everywhere. At $x = x_\pm$, $g_{uu} = r^2\ell/(x_\pm{}^2 + 1) > 0$; but at $x = 0$, $g_{uu} = -3r^2\ell < 0$. Hence the ∂_u Killing vector is timelike near the endpoints, spacelike near $x = 0$, and has at least two "horizons" where it is a null vector.

Case V-C) $\ell = 0$. The allowed range of x is $mx \geq 0$. At $x = 0$ and at $|x| \to \infty$, $\xi^3 \to 0$. Both Killing vectors are spacelike for finite values of x, but at $|x| \to \infty$, they both become null and orthogonal, hence coincident. The only obvious statement one can make about the asymptotically flat region is that the co-ordinate system is badly behaved there.

The metric of Case VI may be obtained from that of Case V by taking the limit

$$\ell \to 0$$

$$a \to 0$$

$$\ell/a \to -2C \ .$$

The following transformation preserves the metric:

$$x \to px$$

$$y \to p^{-1} y$$

$$m \to p^3 m$$

$$C \to p^2 C \ .$$

In this way we cannot change the sign of C, but we can reduce it to one of the three values $+\frac{1}{2}$, $-\frac{1}{2}$, 0. These three metrics will be denoted VI-A, VI-B, and VI-C respectively, and are the limits as $a \to 0$ of V-A, V-B, V-C respectively.

V. ELECTRIFIED TYPE D METRICS

Although vacuum metrics are the most natural ones to study, because they presumably represent purely gravitational phenomena, it is sometimes interesting to consider the interaction of gravity with an explicit source of stress-energy. Introduction of a realistic source such as stellar material leads either to a very complicated equation of state, or else to a debatable idealized one. Simple stress-energy tensors can come from unquantized relativistic wave fields such as a Klein-Gordon or Dirac field. However, free scalar bosons do not exist, and macroscopically coherent fermion fields are inconsistent with quantization. The only source which is simple and unarguably realistic is electromagnetism.

In this section we will generalize the results of Section IV to the case in which an electromagnetic field is present. We still assume that the Weyl tensor is Type D. A priori there is no reason to expect the principal null vectors of $F_{\mu\nu}$ to have any particular relation to the gravitational null vectors. This most general situation has not yet been solved. We will treat only two cases: one in which the electromagnetic field is non-null with the same two principal vectors as the Weyl tensor, and the other in which the field is null and shares one of the gravitational principal null vectors.

A. Non-null Fields

The assumptions to be made for the non-null case are $\psi_0 = \psi_1 = \psi_3 = \psi_4 = \varphi_0 = \varphi_2 = \epsilon = 0$. The proof of the

Goldberg-Sachs Theorem ($\kappa = \nu = \sigma = \lambda = 0$) must be reexamined, since it was previously intended only for vacuum fields. The Maxwell Equations and Bianchi Identities become

$$D\varphi_1 = 2\rho\varphi_1 \tag{V. 1a}$$

$$\Delta\varphi_1 = -2\mu\varphi_1 \tag{V. 1b}$$

$$\delta\varphi_1 = 2\tau\varphi_1 \tag{V. 1c}$$

$$\bar{\delta}\varphi_1 = -2\pi\varphi_1 \tag{V. 1d}$$

$$D\psi_2 = 3\rho\psi_2 + 2\rho\varphi_1\bar{\varphi}_1 \tag{V. 2a}$$

$$\Delta\psi_2 = -3\mu\psi_2 - 2\mu\varphi_1\bar{\varphi}_1 \tag{V. 2b}$$

$$\delta\psi_2 = 3\tau\psi_2 - 2\tau\varphi_1\bar{\varphi}_1 \tag{V. 2c}$$

$$\bar{\delta}\psi_2 = -3\pi\psi_2 + 2\pi\varphi_1\bar{\varphi}_1 \tag{V. 2d}$$

$$3\kappa\psi_2 = 2\kappa\varphi_1\bar{\varphi}_1 \tag{V. 3a}$$

$$3\lambda\psi_2 = -2\lambda\varphi_1\bar{\varphi}_1 \tag{V. 3b}$$

$$3\sigma\psi_2 = -2\sigma\varphi_1\bar{\varphi}_1 \tag{V. 3c}$$

$$3\nu\psi_2 = 2\nu\varphi_1\bar{\varphi}_1\ . \tag{V. 3d}$$

Clearly, exceptions to the theorem can occur only if $\psi_2 = \pm\varphi_1\bar{\varphi}_1$. Putting this into Equations (V. 1), (V. 2) we get a contradiction unless $\rho = \mu = \tau = \pi = 0$.

Case 1) $\psi_2 = +\varphi_1\overline{\varphi}_1$, $\sigma = \lambda = 0$. The NP Equations (III. 4b, c) reduce to

$$\kappa\overline{\nu} = 0$$

$$\kappa\nu = \psi_2 ,$$

which is impossible except in flat space.

Case 2) $\psi_2 = -\varphi_1\overline{\varphi}_1$, $\kappa = \nu = 0$. The NP Equations (III. 3a), (III. 4h) reduce to

$$\sigma\overline{\sigma} = 0$$

$$\lambda\overline{\lambda} = 0 .$$

Hence this important theorem is still true, and from now on we assume $\kappa = \nu = \lambda = \sigma = 0$.

We get integrability conditions by applying commutators to both φ_1 and ψ_2. Since Equations (V. 1) are so similar to the vacuum Bianchi Identities, Equations (IV. 3), the commutators on φ_1 give exactly the same integrability conditions as obtained in Section IV (Equations (IV. 4) - (IV. 6)):

$$\Delta\rho + D\mu = \rho(\gamma + \overline{\gamma}) + \pi\overline{\pi} - \tau\overline{\tau} \qquad \text{(V. 4a)}$$

$$\overline{\delta}\rho + D\pi = \rho(\alpha + \overline{\beta}) \qquad \text{(V. 4b)}$$

$$\delta\mu + \Delta\tau = -\mu(\overline{\alpha} + \beta) + \tau(\gamma - \overline{\gamma}) . \qquad \text{(V. 4c)}$$

The commutators applied to ψ_2 give some different conditions:

$(\Delta D - D\Delta)\psi_2 \Rightarrow$

$$\pi\bar{\pi} - \tau\bar{\tau} + \bar{\rho}\mu - \rho\bar{\mu} = 0 ,$$

hence

$$\tau\bar{\tau} = \pi\bar{\pi} \tag{V. 5a}$$

$$\bar{\rho}\mu = \rho\bar{\mu} ; \tag{V. 5b}$$

$(\bar{\delta}D - D\bar{\delta})\psi_2 \Rightarrow$

$$\bar{\delta}\rho = \rho(\alpha + \bar{\beta} - 2\pi - \bar{\tau}) + \bar{\rho}\pi \tag{V. 6a}$$

$$D\pi = \rho(2\pi + \bar{\tau}) - \bar{\rho}\pi \tag{V. 6b}$$

$(\delta\Delta - \Delta\delta)\psi_2 \Rightarrow$

$$\Delta\tau = \tau(\gamma - \bar{\gamma}) - \mu(2\tau + \bar{\pi}) + \bar{\mu}\tau \tag{V. 7a}$$

$$\delta\mu = -\mu(\alpha + \bar{\beta} - 2\tau - \bar{\pi}) - \bar{\mu}\tau . \tag{V. 7b}$$

Apparently, the electrification of a Type D metric imposes severe restrictions above and beyond those present in the vacuum case.

1. The Case $\rho \neq 0$

The radial integration of Equations (V. 1a), (V. 2a) yields

$$\varphi_1 = \rho^2 \varphi_1{}^o \tag{V. 8a}$$

$$\psi_2 = \rho^3 \psi_2{}^o + 2\rho^3 \bar{\rho}\, \varphi_1{}^o \bar{\varphi}_1{}^o . \tag{V. 8b}$$

The other results of the radial integration may be taken over almost verbatim from the corresponding vacuum formulae, Equations (IV. 13) - (IV. 15). The only exception is that where $\tau^o \bar{\tau}^o$ occurs in γ, U, μ, it must be replaced by $\tau^o \bar{\tau}^o + \varphi_1{}^o \bar{\varphi}_1{}^o$. For instance μ is now given by

$$\mu = \mu^o + \rho(M^o - \bar{\tau}^o \bar{\pi}^o) + \bar{\rho}\tau^o \pi^o + \rho^2(\tfrac{1}{2}\psi_2{}^o + \bar{\tau}^o \eta^o)$$
$$+ \tfrac{1}{2}\rho\bar{\rho}\bar{\psi}_2{}^o + \rho^2\bar{\rho}(\tau^o \bar{\tau}^o + \varphi_1{}^o \bar{\varphi}_1{}^o) - \pi^o \bar{\pi}^o / \bar{\rho} \; .$$

Making substitution into the conditions, Equations (V. 5), (V. 6), we get

$$\mu^o = \bar{\mu}^o$$

$$M^o - \bar{M}^o = 2\bar{\tau}^o \bar{\pi}^o - 2\tau^o \pi^o + 2i\rho^o \mu^o$$

$$\eta^o = 2i\rho^o \bar{\pi}^o$$

$$\tau^o \pi^o + \bar{\tau}^o \bar{\pi}^o - 4(\rho^o)^2 \pi^o \bar{\pi}^o = 0 \; .$$

These are not new restrictions since they occur in the vacuum case (Equations (IV. 23) - (IV. 25), (IV. 26i)).

When we carry out the substitution into the non-radial NP Equations the results are identical to the vacuum results. The only changes that do occur are in the derivatives of $\varphi_1{}^o$, $\psi_2{}^o$ implied by Equations (V. 1), (V. 2):

$$\xi \cdot \varphi_1{}^o = -2\varphi_1{}^o(\bar{\alpha}^o + \beta^o - 4i\rho^o\bar{\pi}^o) \qquad \text{(V. 9a)}$$

$$\bar{\xi} \cdot \varphi_1{}^o = -2\varphi_1{}^o(\alpha^o + \bar{\beta}^o) \qquad \text{(V. 9b)}$$

$$X \cdot \varphi_1{}^o = -2\varphi_1{}^o(\mu^o + \gamma^o + \bar{\gamma}^o) \qquad \text{(V. 9c)}$$

$$\xi \cdot \psi_2{}^o = -3\psi_2{}^o(\bar{\alpha}^o + \beta^o - 4i\rho^o\bar{\pi}^o) + 4\varphi_1{}^o\bar{\varphi}_1{}^o\bar{\pi}^o \qquad \text{(V. 10a)}$$

$$\bar{\xi} \cdot \psi_2{}^o = -3\psi_2{}^o(\alpha^o + \bar{\beta}^o) + 4\varphi_1{}^o\bar{\varphi}_1{}^o\pi^o \qquad \text{(V. 10b)}$$

$$X \cdot \psi_2{}^o = -3\psi_2{}^o(\mu^o + \gamma^o + \bar{\gamma}^o). \qquad \text{(V. 10c)}$$

We use the tetrad freedom to make $\varphi_1{}^o\bar{\varphi}_1{}^o$ constant, obtaining the same conditions as Equation (IV. 30).

Cases I & II) $\pi^o = 0$. We get $\psi_2{}^o = m + i\ell = \text{const.}$ and $\varphi_1{}^o = \text{const.}$ Let $\varphi_1{}^o = e + id$, where e, d are real. The only spot where $\varphi_1{}^o$ occurs in the tetrad is in U:

$$U = U^o + \rho\bar{\rho}(mr + \ell\rho^o - e^2 - d^2) .$$

Therefore, all versions of NUT and Kerr-NUT Space may be electrified by the simple substitution $m \rightarrow m - (e^2 + d^2)/r$ in the tetrad or in the metric.

Case III) $\pi^o \neq 0$, $\rho^o = 0$. In this case $\varphi_1{}^o = e + id$ is constant but $\psi_2{}^o = \bar{\psi}_2{}^o = m^o$ is not. With $\xi^{o3} = \pi^o$ the equations to solve, Equations (IV. 39), (V. 10a) become

$$\pi^o{}_{,x} = 2b^o$$

$$2\pi^o b^o{}_{,x} + 4(b^o)^2 = V^o$$

$$V^o{}_{,x} = -3\psi_2{}^o$$

$$\psi_2{}^o{}_{,x} = 4(e^2 + d^2) \ .$$

Together these imply

$$(\pi^o)^2{}_{,xxxx} = -24(e^2 + d^2) \ ,$$

with the solution

$$(\pi^o)^2 = -(e^2 + d^2)x^4 - mx^3 + vx^2 + Bx + C \ , \qquad \text{(V. 11)}$$

where m, v, B, C are arbitrary constants. If we take the Case III vacuum solution as Equations (IV. 41), (IV. 42) with $w = -3m$, the Ehlers-Kundt "C" metric may be electrified by the substitution $m \rightarrow m + x(e^2 + d^2)$.

Case IV) $\pi^o \neq 0$, $\rho^o \neq 0$. As in the corresponding vacuum case we get

$$\rho^{o} = a \operatorname{cn} x \ .$$

Integration of Equations (V. 9a), (V. 10a) leads to

$$\begin{aligned} \varphi_1^{\,o} &= (e + id)(\operatorname{dn} x - \tfrac{1}{2} i\sqrt{2} \operatorname{sn} x)^2 \\ \psi_2^{\,o} &= (m + i\ell)(\operatorname{dn} x - \tfrac{1}{2} i\sqrt{2} \operatorname{sn} x)^3 \\ &\quad + 2ia^{-1}(e^2 + d^2)(\operatorname{dn} x - \tfrac{1}{2} i\sqrt{2} \operatorname{sn} x)^2 \operatorname{cn} x \ . \end{aligned} \tag{V. 12}$$

The remaining integrations parallel exactly the vacuum case. The tetrad is the same as Equation (IV. 50) with

$$\begin{aligned} V^{o} &= b + \tfrac{3}{4}\sqrt{2}\, a^{-1}(m \operatorname{sn} x - \ell\sqrt{2} \operatorname{dn} x) \operatorname{cn} x \\ &\quad - \tfrac{3}{2} a^{-2}(e^2 + d^2) \operatorname{cn}^2 x \\ (\pi^{o})^2 &= c \operatorname{sn} x \operatorname{dn} x + \tfrac{1}{4} ba^{-2} \operatorname{cn}^2 x \\ &\quad - \tfrac{1}{8}\sqrt{2}\, a^{-3}(m \operatorname{sn} x + \ell\sqrt{2} \operatorname{dn} x) \operatorname{cn} x \\ &\quad - \tfrac{1}{8} a^{-4}(e^2 + d^2) \operatorname{cn}^3 x \\ m^{o} &= m(\operatorname{dn}^3 x - \tfrac{3}{2} \operatorname{dn} x \operatorname{sn}^2 x) + \ell(\tfrac{3}{2}\sqrt{2} \operatorname{dn}^2 x \operatorname{sn} x \\ &\quad - \tfrac{1}{4}\sqrt{2} \operatorname{sn}^3 x) + 2\sqrt{2}\, a^{-1}(e^2 + d^2) \operatorname{sn} x \operatorname{cn} x \operatorname{dn} x \end{aligned} \tag{V. 13}$$

$$\ell^o = -m\sqrt{2}\,(\tfrac{3}{2}\,\mathrm{dn}^2 x\ \mathrm{sn}\ x - \tfrac{1}{4}\,\mathrm{sn}^3 x) + \ell(\mathrm{dn}^3 x$$

$$- \tfrac{3}{2}\,\mathrm{dn}\ x\ \mathrm{sn}^2 x) + 2a^{-1}(e^2 + d^2)\ \mathrm{cn}^3 x$$

$$U = V^o - 7a^2(\pi^o)^2 \mathrm{cn}^2 x + 2ar\sqrt{2}\,(\pi^o)^2{}_{,x} + r^2(\pi^o)^2$$

$$+ r\rho\bar{\rho}\,[m^o + 8a^3\sqrt{2}\,(\pi^o)^2\ \mathrm{sn}\ x\ \mathrm{dn}\ x]$$

$$+ \rho\bar{\rho}\,[a\ell^o\ \mathrm{cn}\ x + a^4(2\ \mathrm{cn}^4 x - 1)(\pi^o)^2 - (e^2 + d^2)]\ .$$

2. The Case $\rho = 0$

The electrification of the $\rho = 0$ vacuum metrics proceeds just as smoothly. The only NP Equations which change are the Maxwell Equations and the Bianchi Identities:

$$D\varphi_1 = \Delta\varphi_1 = 0 \tag{V. 14a}$$

$$\delta\varphi_1 = \bar{\delta}\varphi_1 = 2\tau\varphi_1 \tag{V. 14b}$$

$$D\psi_2 = \Delta\psi_2 = 0 \tag{V. 15a}$$

$$\delta\psi_2 = \bar{\delta}\psi_2 = 3\tau\psi_2 - 2\tau\varphi_1\bar{\varphi}_1\ . \tag{V. 15b}$$

The variables γ and U (cf. Equation (IV. 72)) acquire a term in φ_1:

$$\gamma = \gamma^o + r[(\tau + \bar{\tau})(\beta - \alpha) - \tau^2 + \varphi_1\bar{\varphi}_1]$$

$$U = U^o - r[2\gamma^o + \omega^o(\tau - \bar{\tau})] - r^2[(\tau + \bar{\tau})(\beta - \alpha) - \tau^2 + \varphi_1\bar{\varphi}_1].$$

The analysis follows the vacuum case down to Equation (IV. 76). This time we must solve Equation (V. 14b) for τ and substitute in Equation (IV. 7b). The resulting equation may be written

$$\delta \left(\frac{\varphi_1^{-3/2} \; \delta \varphi_1}{\bar{\varphi}_1^{\,-3/2} \; \delta \bar{\varphi}_1} \right) = 0 ,$$

and integrated twice to give

$$\bar{E} \varphi_1^{-1/2} + E \bar{\varphi}_1^{-1/2} = 1 .$$

We can choose x such that

$$\varphi_1 = \frac{e + id}{(x + ia)^2} , \qquad \text{(V. 16)}$$

where e, d, a are real constants. The rest of the derivation is a straightforward modification of Section IV and leads to these metrics:

Case V) $a \neq 0$

$$\ell^\mu = (0, 1, 0, 0)$$

$$n^\mu = (1, \frac{-r^2(e^2 + d^2 + a\ell)}{2a^2(x^2 + a^2)}, 0, \frac{4ar}{x^2 + a^2})$$

$$m^\mu = (0, \frac{2rx\xi}{x^2 + a^2}, \xi, \frac{i}{\xi})$$

$$g_{uu} = [r^2(e^2 + d^2 + a\ell)/a^2(x^2 + a^2)] - [4r^2a^2\xi^2/(x^2 + a^2)^2] \qquad (V.17)$$

$$g_{ur} = 1$$

$$g_{ux} = -2rx/(x^2 + a^2)$$

$$g_{uy} = ar\xi^2/(x^2 + a^2)$$

$$g_{xx} = -1/2\xi^2$$

$$g_{yy} = -\xi^2/2 \quad ,$$

where

$$\xi^2 = \frac{2a^2mx + a\ell(a^2 - x^2) - (e^2 + d^2)(x^2 + a^2)}{2a^2(x^2 + a^2)} \quad .$$

Case VI) $a = 0$.

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = (1,\ Cr^2/x^2,\ 0,\ 0)$$

$$m^\mu = (0,\ 2r\xi/x,\ \xi,\ i/\xi)$$

$$g_{uu} = -2Cr^2/x^2 \qquad (V.18)$$

$$g_{ur} = 1$$

$$g_{ux} = -2r/x$$

$$g_{xx} = -1/2\xi^2$$

$$g_{yy} = -\xi^2/2 \ ,$$

where

$$\xi^2 = C + \frac{m}{x} - \frac{e^2 + d^2}{x^2} \ .$$

As in the corresponding vacuum metrics we may set $a = +1$ and $C = \pm\frac{1}{2}, 0$.

B. Null Fields

1. Solutions

Next we would like to consider the situation which arises when the electromagnetic field is null with ℓ^{μ} as the principle null vector. Assume $\psi_0 = \psi_1 = \psi_3 = \psi_4 = \varphi_0 = \varphi_1 = \epsilon = 0$. The Mariot-Robinson Theorem immediately tells us that $\kappa = \sigma = 0$, i.e., the ℓ^{μ} congruence is geodesic and shear-free. However the Goldberg-Sachs Theorem is actually false in this case and, as we will see, the fact that n^{μ} need not be geodesic produces some very interesting results. The Bianchi Identities and Maxwell Equations under the current assumptions are

$$D\psi_2 = 3\rho\psi_2 \tag{V. 19a}$$

$$\Delta\psi_2 = -3\mu\psi_2 - \rho\varphi_2\bar{\varphi}_2 \tag{V. 19b}$$

$$\delta\psi_2 = 3\tau\psi_2 \tag{V. 19c}$$

$$\bar{\delta}\psi_2 = -3\pi\psi_2 \tag{V. 19d}$$

$$\lambda = 0 \tag{V. 20a}$$

$$3\nu\psi_2 = \bar{\varphi}_2\bar{\delta}\varphi_2 + 2\alpha\bar{\varphi}_2\varphi_2 \tag{V. 20b}$$

$$D\varphi_2 = \rho\varphi_2 \tag{V. 21a}$$

$$\delta\varphi_2 = (\tau - 2\beta)\varphi_2 \,. \tag{V. 21b}$$

Here are the only NP Equations in which ν or φ_2 make an appearance:

$$\Delta\pi - D\nu = \pi(\bar{\gamma} - \gamma) - \mu(\bar{\tau} + \pi) \tag{V. 22a}$$

$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu + \alpha(\bar{\gamma} - \bar{\mu}) - \gamma(\bar{\tau} - \beta) \tag{V. 22b}$$

$$\Delta\mu - \delta\nu = -\mu(\mu + \gamma + \bar{\gamma}) + \nu(\bar{\alpha} + 3\beta - \tau)$$

$$+ \bar{\nu}\pi - \varphi_2\bar{\varphi}_2 \tag{V. 22c}$$

$$\bar{\delta}\nu = \nu(\bar{\tau} - \pi - 3\alpha - \bar{\beta}) \tag{V. 22d}$$

$$\bar{\delta}\mu = -\mu(\pi + \alpha + \bar{\beta}) + \bar{\mu}\pi - \nu(\rho - \bar{\rho}) \,. \tag{V. 22e}$$

If the methods of Section V are followed, one is forced to conclude after a tedious argument that ρ must be real. For the sake of simplicity, we are willing to take $\rho = \overline{\rho}$ as an initial assumption. The solutions we seek will therefore be electromagnetic generalizations of the Schwarzschild and Ehlers-Kundt "C" metrics.

The radial integration is unchanged, except for the addition of Equation (V. 21a), which gives

$$\varphi_2 = \rho\varphi_2{}^o . \qquad \text{(V. 23)}$$

Equations (V. 19b-d) imply

$$\xi \cdot \psi_2{}^o = -3\psi_2{}^o(\overline{\alpha}^o + \beta^o) \qquad \text{(V. 24a)}$$

$$\overline{\xi} \cdot \psi_2{}^o = -3\psi_2{}^o(\alpha^o + \overline{\beta}^o) \qquad \text{(V. 24b)}$$

$$X \cdot \psi_2{}^o = -3\psi_2{}^o(\mu^o + \gamma^o + \overline{\gamma}^o) - \varphi_2{}^o\overline{\varphi}_2{}^o . \qquad \text{(V. 24c)}$$

Equations (V. 20b), (V. 21b) can be combined to give

$$3\nu\psi_2 = \overline{\delta}(\varphi_2\overline{\varphi}_2) + \varphi_2\overline{\varphi}_2(2\alpha + 2\overline{\beta} - \overline{\tau}) \ ; \qquad \text{(V. 25)}$$

substitution into this yields an expression for ν:

$$\nu = \nu^o + r\pi^o\varphi_2{}^o\overline{\varphi}_2{}^o/\psi_2{}^o \ , \qquad \text{(V. 26)}$$

and a derivative

$$\xi \cdot (\varphi_2{}^o\overline{\varphi}_2{}^o) = -4\varphi_2{}^o\overline{\varphi}_2{}^o(\alpha^o + \overline{\beta}^o) + 3\nu^o\psi_2{}^o . \qquad \text{(V. 27)}$$

The information obtainable from Equations (V. 22b-e) is

$$X \cdot \pi^o = -\pi^o(\gamma^o - \bar{\gamma}^o - \varphi_2{}^o\bar{\varphi}_2{}^o/\psi_2{}^o) \qquad \text{(V. 28a)}$$

$$X \cdot \alpha^o - \bar{\xi} \cdot \gamma^o = -\alpha^o(\mu^o + \gamma^o) + \bar{\beta}^o\gamma^o - M^o\pi^o + \nu^o \qquad \text{(V. 28b)}$$

$$X \cdot \mu^o = -\mu^o(\mu^o + \gamma^o + \bar{\gamma}^o - \varphi_2{}^o\bar{\varphi}_2{}^o/\psi_2{}^o) - 2\bar{\nu}^o\pi^o \qquad \text{(V. 28c)}$$

$$X \cdot M^o + \xi \cdot \nu^o = -2M^o(\mu^o + \gamma^o + \bar{\gamma}^o) - \nu^o(\bar{\alpha}^o + 3\beta^o) \qquad \text{(V. 28d)}$$

$$\bar{\xi} \cdot \nu^o = -\nu^o(3\alpha^o + \bar{\beta}^o) \qquad \text{(V. 28e)}$$

$$\pi^o \nu^o = 0 \,. \qquad \text{(V. 28f)}$$

We also need to recall Equations (IV. 21), (IV. 26f), (IV. 28), which for $\rho^o = 0$ simplify to

$$\xi \cdot \alpha^o - \bar{\xi} \cdot \beta^o = 2\beta^o(\bar{\beta}^o - \alpha^o) + M^o \qquad \text{(V. 29a)}$$

$$X \cdot \beta^o - \xi \cdot \gamma^o = -\beta^o(\mu^o + 2\bar{\gamma}^o) + \gamma^o(\bar{\alpha}^o + \beta^o) + M^o\bar{\pi}^o \qquad \text{(V. 29b)}$$

$$\xi \cdot X^{oi} - X \cdot \xi^{oi} = (\mu^o + 2\bar{\gamma}^o)\xi^{oi} - (\bar{\alpha}^o + \beta^o)X^{oi} \qquad \text{(V. 29c)}$$

$$\bar{\xi} \cdot \xi^{oi} - \xi \cdot \bar{\xi}^{oi} = -2\bar{\beta}^o\xi^{oi} + 2\beta^o\bar{\xi}^{oi} \,. \qquad \text{(V. 29d)}$$

Equation (V. 28f) leads to separate consideration of two cases.

Case 1) $\nu^o = 0$. We use the tetrad freedom to make π^o real and non-negative and $\psi_2{}^o = m$ constant. Then we have from Equations (V. 24), (V. 27),

$$\bar{\alpha}^o + \beta^o = 0$$

$$\varphi_2^o \bar{\varphi}_2^o = -3m(\mu^o + \gamma^o + \bar{\gamma}^o) \tag{V. 30}$$

$$\xi \cdot (\varphi_2^o \bar{\varphi}_2^o) = 0 \, .$$

As we can check from Equations (IV. 13), this makes $\tau = \bar{\alpha} + \beta = -\bar{\pi}^o$, which implies that ℓ^μ is equal to a gradient. Choosing co-ordinates such that $\ell_\mu = u_{,\mu}$, we have $X^1 = 1$ and $\xi^1 = 0$. Therefore, from Equation (V. 30), $\varphi_2^o \bar{\varphi}_2^o$ and $\mu^o + \gamma^o + \bar{\gamma}^o$ are functions of u alone. Under a recalibration of u,

$$u \to f(u)$$

$$\ell^\mu \to f'(u)\, \ell^\mu$$

$$\mu^o + \gamma^o + \bar{\gamma}^o \to [(\mu^o + \gamma^o + \bar{\gamma}^o)f' + f'']/(f')^2 ,$$

hence we can find an f which sets $\mu^o + \gamma^o + \bar{\gamma}^o = 0$. In a roundabout way we have now achieved exactly the same tetrad conditions as in Equation (IV. 30); however $m = m(u)$ is no longer a constant. Straightforward integration leads to a tetrad identical to that of Equation (IV. 41) except that $m(u)$ is an arbitrary function and

$$\varphi_2^o \bar{\varphi}_2^o = -m_{,u} \, . \tag{V. 31}$$

We have deliberately postponed using Equations (V. 20b) and (V. 21b) separately to determine the phase of $\varphi_2{}^o$. Let

$$\varphi_2{}^o = |\varphi_2{}^o| e^{i\chi} .$$

Then Equations (V. 20b), (V. 21b) imply

$$\xi \cdot \chi = 2 i \beta^o ,$$

or

$$\chi_{,x} = 0$$

$$\chi_{,y} = \frac{1}{2} w x^2 + v x + B .$$

Hence the electromagnetic field $\varphi_2{}^o$ can exist only if

$$w = v = 0$$

$$\chi = By + D ,$$

which correspond to the vacuum Cases II-E and II-F with $b = \ell = 0$. Then

$$\varphi_2{}^o = |\varphi_2{}^o| (u) e^{iy} \qquad \text{(V. 32)}$$

in Case II-E, and

$$\varphi_2{}^o = |\varphi_2{}^o| (u) e^{iD} = \varphi_2{}^o (u) \qquad \text{(V. 33)}$$

in Case II-F.

For Cases II-A-D and Case III a null electromagnetic field cannot be found. We postpone further comment on this situation to the discussion.

Case 2) $\nu^o \neq 0$. In this case $\pi^o = \mu^o = 0$. Choose the tetrad rotations to make ν^o real and positive and $U^o = V^o = -M^o$ = constant. Once again, $\bar{\alpha}^o + \beta^o = 0$, which implies $\tau = \bar{\alpha} + \beta$. Letting $\ell_\mu = u_{,\mu}$ we get $X^1 = 0$, $\xi^1 = 0$, and $\psi_2{}^o = m(u)$.

Since ν^o is real, Equations (V.28d, e) imply

$$2\beta^o\nu^o = -V^o(\gamma^o + \bar{\gamma}^o), \qquad \text{(V.34)}$$

and therefore β^o is also real.

From Equation (V.27), $\xi \cdot (\varphi_2{}^o\bar{\varphi}_2{}^o)$ is real, so we may choose co-ordinates such that ξ^{o3} is real and ξ^{o4} imaginary. By analogy with the vacuum metrics, choose

$$\xi^{o3} = (V^o x^2 + Bx + C)^{1/2}.$$

The solutions of Equations (V.29a, d), (V.28e) are

$$\beta^o = (2V^o x + B)/4\xi^{o3}$$

$$\xi^{o4} = i/\xi^{o3}$$

$$\nu^o = \xi^{o3} f(u).$$

Equations (V.28b), (V.29b) imply

$$\xi \cdot (\gamma^o + \bar{\gamma}^o) = -\nu^o ,$$

hence

$$\gamma^o + \bar{\gamma}^o = -x f(u) + g(u) .$$

Comparing with Equation (V. 34),

$$B f(u) = -2 V^o g(u) .$$

We take those linear combinations of Equation (V. 29c) and its conjugate which involve only $\gamma^o + \bar{\gamma}^o$, and treat X^{oi} as the unknowns. When these have been integrated for X^{oi}, we put the results back into the remaining linear combinations and solve for $\gamma^o - \bar{\gamma}^o$. The resulting tetrad is of the form

$$\begin{aligned} l^\mu &= (0,\ 1,\ 0,\ 0) \\ n^\mu &= (1,\ V^o + \frac{m}{r},\ X^3,\ X^4) \qquad \text{(V. 35)} \\ m^\mu &= (0,\ 0,\ -\xi/r,\ -i/r\xi) \end{aligned}$$

where

$$\xi^2 = V^o x^2 + Bx + C ,$$

$m = m(u)$, and X^3, X^4 depend on x, y, u. More explicit tetrads and metrics will be given in the discussion.

2. Discussion

In Special Relativity there is a theorem[47] which says that, given any shear-free geodesic null congruence ℓ^μ, it is possible to construct a null electromagnetic field with ℓ^μ as principal null vector. However in Special Relativity we do not have to satisfy the Bianchi Identities! Hence we cannot expect this theorem to extend to General Relativity. In particular it is invalid for Type D metrics because of the extra restriction imposed by the Bianchi Identity, Equation (V. 20b). Thus we found it possible to insert null electromagnetic fields only in the vacuum metrics II-E and F.

In the remaining cases, what interpretation can we give to the generalized tetrads we have just found? They have stress tensors proportional to $\ell_\mu \ell_\nu$, which suggests the presence of a stream of non-interacting particles with velocity vector ℓ_μ. Finding a phase for φ_2 is tantamount to finding a polarization vector

$$e_\mu = \varphi_2 \, m_\mu + \bar{\varphi}_2 \, \bar{m}_\mu \ ,$$

with

$$F_{\mu\nu} = 2 \, \ell_{[\mu} e_{\nu]} \ .$$

(This agrees with the definitions of φ_i in Equation (III. 2).) If a phase does not exist we may say we are merely dealing with unpolarized light. The metric which is the generalization of Case II-A with $\nu^o = 0$ has been previously discovered by

Vaidya,[48] who gave it this interpretation. The same stress tensor could as easily represent unpolarized neutrinos or even gravitons.[49] On the other hand we may forsake the attempt for a physical interpretation and regard these time-dependent generalizations as nothing more than a formal tool to use to study the static vacuum metrics.

As an example if we take the Vaidya metric with $\varphi_2^{\,o}\bar{\varphi}_2^{\,o}$ non-zero only in the interval $u_o \le u \le u_1$. The solution is initially and finally a static Schwarzschild field, but some stress energy escapes to the asymptotically flat region during the intervening period. The total loss of 4-momentum may be calculated by integrating $T^{\mu\nu}$ over a large sphere:

$$\Delta P^{\mu} = \int_{u_o}^{u_1} \oint T^{\mu\nu} d^3\Sigma_{\nu} \, ,$$

where

$$T^{\mu\nu} = \frac{1}{4\pi} \varphi_2 \bar{\varphi}_2 \ell^{\mu} \ell^{\nu}$$

$$d^3\Sigma_{\nu} = r_{,u} \, r^2 d^2\Omega \, du \, ,$$

and $d^2\Omega$ denotes solid angle. In the limit $r \to \infty$ the integral becomes a covariant quantity. Writing

$$\ell_{\mu} = -r_{,\mu} + V_{\mu}$$

where V_{μ} is the timelike Killing vector, and using Equations

(V. 23), (V. 31), we get

$$\Delta P^{\mu} = V^{\mu} \int_{u_o}^{u_1} (-m)\, du = (m(u_o) - m(u_1))V^{\mu} .$$

For $m(u_1) = 0$, this proves that the source of the Schwarzschild vacuum metric has 4-momentum $P^{\mu} = mV^{\mu}$.

The solutions we have found with $\nu^o \neq 0$ are more interesting. Let us take the tetrad Equation (V. 35) and put it in terms of the angular co-ordinate θ, used in Section IV and defined by

$$\theta = \int \xi^{-1}\, dx .$$

The various possibilities arising from the signs of V^o, $B^2 - 4V^oC$ will now be listed:

Case 2-A) $V^o < 0$.

$$\ell^{\mu} = (0, 1, 0, 0)$$

$$n^{\mu} = (1, -\tfrac{1}{2} + r\, a \cos\theta + m/r, X^3, X^4)$$

$$m^{\mu} = (0, 0, 1/r\sqrt{2}, i/r\sqrt{2} \sin\theta)$$

$$X^3 = -a \sin\theta + b \sin y + c \cos y$$

$$X^4 = b \cot\theta \cos y - c \cot\theta \sin y$$

$$g_{uu} = 1 - 2m/r - 2r\, a \cos\theta - r^2(X^3)^2 - r^2(X^4)^2 \sin^2\theta \qquad \text{(V. 36)}$$

$$g_{ur} = 1$$

$$g_{u\theta} = r^2(X^3)$$

$$g_{uy} = r^2(X^4)\sin^2\theta$$

$$g_{\theta\theta} = -r^2$$

$$g_{yy} = -r^2\sin^2\theta$$

$$\varphi_2^{\,o}\,\bar{\varphi}_2^{\,o} = -m_{,u} + 3\,m\,a\cos\theta\ .$$

Here, as in the succeeding solutions, m, a, b, c are all arbitrary functions of u.

Case 2-B) $V^o > 0,\ B^2 - 4V^oC > 0$.

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = (1,\ \tfrac{1}{2} - r\,a\cosh\theta + m/r,\ X^3,\ X^4)$$

$$m^\mu = (0,\ 0,\ 1/r\sqrt{2},\ i/r\sqrt{2}\sinh\theta)$$

$$X^3 = a\cosh\theta + b\cosh y + c\sinh y$$

$$X^4 = -b\tanh\theta\sinh y - c\tanh\theta\cosh y$$

$$g_{uu} = -1 - 2m/r + 2\,r\,a\cosh\theta - r^2(X^3)^2 - r^2(X^4)^2\sinh^2\theta$$

$$g_{ur} = 1 \qquad \text{(V. 37)}$$

$$g_{u\theta} = r^2(X^3)$$

$$g_{uy} = r^2(X^4)\sinh^2\theta$$

$$g_{\theta\theta} = -r^2$$

$$g_{yy} = -r^2 \sinh^2\theta$$

$$\varphi_2^{\,o} \bar{\varphi}_2^{\,o} = -m_{,u} - 3\,m\,a \cosh\theta \ .$$

Case 2-C) $V^o > 0, \ B^2 - 4V^oC = 0.$

$$\begin{aligned}
\ell^\mu &= (0,\ 1,\ 0,\ 0) \\
n^\mu &= (1,\ \tfrac{1}{2} - r\,a\,e^\theta + m/r,\ X^3,\ X^4) \\
m^\mu &= (0,\ 0,\ 1/r\sqrt{2},\ i\,e^{-\theta}/r\sqrt{2}) \\
X^3 &= a\,e^\theta + b \\
X^4 &= -\,by - c \\
g_{uu} &= -1 - 2m/r + 2\,r\,a\,e^\theta - r^2(X^3)^2 - r^2(X^4)^2\,e^{2\theta} \\
g_{ur} &= 1 \\
g_{u\theta} &= r^2(X^3) \\
g_{uy} &= r^2(X^4)e^{2\theta} \\
g_{\theta\theta} &= -r^2 \\
g_{yy} &= -r^2\,e^{2\theta} \\
\varphi_2^{\,o} \bar{\varphi}_2^{\,o} &= -m_{,u} - 3\,m\,a\,e^\theta .
\end{aligned} \qquad (V.\ 38)$$

Case 2-D) $V^o > 0, \ B^2 - 4V^oC < 0.$

$$\ell^\mu = (0,\ 1,\ 0,\ 0)$$

$$n^\mu = (1, \tfrac{1}{2} - ra\sinh\theta + m/r, X^3, X^4)$$

$$m^\mu = (0, 0, 1/r\sqrt{2}, i/r\sqrt{2}\cosh\theta)$$

$$X^3 = a\cosh\theta + b\cosh y + c\sinh y$$

$$X^4 = -b\tanh\theta\sinh y - c\tanh\theta\cosh y$$

$$g_{uu} = -1 - 2m/r + 2ra\sinh\theta$$

$$g_{ur} = 1 \qquad \text{(V. 39)}$$

$$g_{u\theta} = r^2(X^3)$$

$$g_{uy} = r^2(X^3)\cosh^2\theta$$

$$g_{\theta\theta} = -r^2$$

$$g_{yy} = -r^2\cosh^2\theta$$

$$\varphi_2{}^o\bar{\varphi}_2{}^o = -m_{,u} - 3ma\sinh\theta \ .$$

Case 2-E) $V^o = 0$, $B = 0$.

$$\ell^\mu = (0, 1, 0, 0)$$

$$n^\mu = (1, -rax - rb + m/r, X^3, X^4)$$

$$m^\mu = (0, 0, 1/r\sqrt{2}, i/r\sqrt{2})$$

$$X^3 = \tfrac{1}{2}a(x^2 - y^2) + bx + cy$$

$$X^4 = axy + by - cx$$

$$
\begin{aligned}
g_{uu} &= -2m/r + 2rax + 2rb - r^2(X^3)^2 - r^2(X^4)^2 \\
g_{ur} &= 1 \\
g_{u\theta} &= r^2(X^3) \\
g_{uy} &= r^2(X^4) \\
g_{\theta\theta} &= -r^2 \\
g_{yy} &= -r^2 \\
\varphi_2^{\,o}\,\bar{\varphi}_2^{\,o} &= -m_{,u} - 3max - 3mb\,.
\end{aligned} \tag{V. 40}
$$

In reference 44 the flat-space limit of these co-ordinate systems are examined and found to correspond to a family of null cones based on an accelerating world line. The co-ordinates r, θ, y are spherical co-ordinates on each cone, and u is the retarded time. The function $a(u)$ is the magnitude of the acceleration vector, while $b(u)$ and $c(u)$ measure the rate at which the direction of the acceleration vector is changing.

We can calculate ΔP^{μ} for these metrics just as we did for the Vaidya metric. If $\varphi_2^{\,o}\,\bar{\varphi}_2^{\,o}$ is non-zero for $u_o \leq u \leq u_1$ only, then solution 2-A is initially and finally a static Schwarzschild metric. The analysis is simplest for the case of a slowly accelerating particle. If we can pick a radius r such that $m \ll r \ll \frac{1}{a}$, then the metric is approximately Minkowskian, and

$$\frac{dP^\mu}{du} = \frac{1}{4\pi} \oint \varphi_2{}^o \bar{\varphi}_2{}^o \, \ell^\mu \, d^2\Omega$$

$$= - \dot{m} V^\mu + m a z^\mu$$

$$= - \frac{d}{du} (m V^\mu) \quad .$$

VI. CONCLUSION

We have found a total of fourteen distinct Type D vacuum metrics, Equations (IV. 54) - (IV. 59), (IV. 60), (IV. 52), (IV. 83), and (IV. 84). This marks the first time that all solutions of a given Petrov Type have been explicitly determined. All but one (Equation (IV. 52)) of the fourteen have been previously discovered. However, it is important to know that the list is now complete. The list is complete even in the sense that there remain no uncovered regions of these manifolds to be revealed by analytic continuation. Although the co-ordinate patches given can be and need to be continued analytically, the most we can expect is that several equivalent or inequivalent patches will be found to join together. For instance in the continuation of the Kerr metric, a family of patches with parameters m, a are analytically connected to a family with parameters -m, -a.[50]

What comes as a complete surprise is that the Type D vacuum metrics all have at least two Killing vectors. It is not known whether this could be called "accidental," or whether it is due to some profound connection between algebraically special fields and the existence of isometries.

We have extended the simple relation between the Schwarzschild and Nordstrom metrics by finding charged versions of each Type D vacuum solution. I believe this emphasizes nicely the close structural similarity between the gravitational and electromagnetic fields. In the search for Type D metrics containing null electromagnetic fields we were not as successful in a direct sense, but the metrics thereby uncovered (Equations (IV. 41) with $m = m(u)$ and Equations (V. 36) - (V. 40)) can help us better understand the vacuum solutions.

Further work needs to be done on most of these metrics to clarify their physical interpretations, as noted in the discussion. I hope also that now that Type D has proven possible to analyze, efforts will be made to obtain exhaustive treatments of the other special Petrov Types.

APPENDIX A. THREE CO-ORDINATE SYSTEMS ON PSEUDOSPHERES

Let us consider a Lorentz 3-space with signature (+, -, -). Let X, Y, T be Minkowski co-ordinates, so that

$$ds^2 = dT^2 - dX^2 - dY^2.$$

The pseudosphere is defined by $T^2 - X^2 - Y^2 = 1$, $T > 0$, the future hyperboloid. Construct three sets of intrinsic co-ordinates (x, y) on the pseudosphere as follows:

1) $$T = \cosh x$$
$$X = \sinh x \sin y$$
$$Y = \sinh x \cos y$$

2) $$T = \cosh x \cosh y$$
$$X = \cosh x \sinh y$$
$$Y = \sinh x$$

3) $$T = \frac{1}{2}(y^2 + 1)e^x + \frac{1}{2}e^{-x}$$
$$X = \frac{1}{2}(y^2 - 1)e^x + \frac{1}{2}e^{-x}$$
$$Y = y\,e^x.$$

We can check for all three of these that $T^2 - X^2 - Y^2 = 1$, $T > 0$. The metrics, $ds^2 = dT^2 - dX^2 - dY^2$, become

1) $-dx^2 - \sinh^2 x\, dy^2$

2) $-dx^2 - \cosh^2 x\, dy^2$

3) $-dx^2 - e^{2x}\, dy^2$,

which are the 2-metrics discussed in Section IV A. We can see that the ∂_y isometries are rotations in the (X, Y), (X, T), and (T + X, Y) planes respectively.

APPENDIX B. THE FLAT-SPACE LIMIT OF THE KERR NULL CONGRUENCE

The tetrad and metric of Case II-A for $m = \ell = 0$ is

$$\ell^{\mu} = (0,\ 1,\ 0,\ 0)$$

$$n^{\mu} = \rho\bar{\rho}\,(r^2 + a^2,\ -\tfrac{1}{2}(r^2 + a^2),\ 0,\ a)$$

$$m^{\mu} = -\tfrac{1}{2}\sqrt{2}\ \bar{\rho}$$

$$g_{uu} = 1$$

$$g_{ur} = 1$$

$$g_{ry} = -a \sin^2\theta$$

$$g_{\theta\theta} = -r^2 - a^2 \cos^2\theta$$

$$g_{yy} = -(r^2 + a^2)\sin^2\theta\ .$$

Define new co-ordinates t, R, Θ, φ by

$$\begin{aligned} t &= -u - r \\ R &= (r^2 + a^2 \sin^2\theta)^{1/2} \\ \Theta &= \tan^{-1}[r^{-1}(r^2 + a^2)^{1/2} \tan\theta] \\ \varphi &= y - \tan^{-1}(a/r)\ . \end{aligned} \tag{B.1}$$

When we calculate the new metric, we discover that t, R, Θ, φ are ordinary spherical co-ordinates:

$$g_{tt} = 1$$

$$g_{RR} = -1$$

$$g_{\Theta\Theta} = -R^2$$

$$g_{\varphi\varphi} = -R^2 \sin^2 \Theta \ .$$

The new components of ℓ^μ are

$$\ell^\mu = \left(-1, \frac{r}{R}, -\frac{a^2 \sin\theta \cos\theta}{R^2(r^2+a^2)^{1/2}}, \frac{a \sin\theta}{R(r^2+a^2)^{1/2} \sin\Theta}\right)$$

where r, θ are given implicitly by Equation (B. 1) or by

$$\frac{R^2 \sin^2 \Theta}{r^2 + a^2} + \frac{R^2 \cos^2 \Theta}{r^2} = 1$$

$$R^2 \sin^2 \Theta = (r^2 + a^2) \sin^2 \theta \ .$$

Projected into the 3-space (R, Θ, φ), the trajectories of ℓ^μ must be straight lines, with the affine parameter r measuring Euclidean distance along them. Hence they must all pass through the surface $r = 0$, which is a disk centered at the origin with radius a.

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