RG-Flows, AdS/CFT Correspondence and Stability of Non-Dilatonic Branes

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Abstract

The possibility of having multiple renormalization group (RG) flows (one of which is supersymmetric) between two fixed points is investigated in the context of anti de Sitter / conformal field theory (AdS/CFT) correspondence. An analysis of a toy-model potential suggests that such flows are likely to exist. Superpotential methods are used in the context of finite temperature AdS/CFT to derive a black brane solution which approximates various finite temperature RG-flows in AdS/CFT near the horizon. This solution is also used in formulating a notion of univerality classes of instabilities of black branes. Instabilities of D3, M2 and M5-branes are investigated numerically, and the results confirm the predictions of the proposal of universality classes.

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Contents

Chapter 1

Introduction and summary

1.1 General introduction

Much of this dissertation is related to string theory, so it might be appropriate to do a quick heuristic introduction to the subject before delving into the details of the work.

The success of quantum mechanics of particles in calculating things like the spectrum of the Hydorgen atom has led people to consider applying the same approach, i.e., "quantization," to fields. As a simplistic example of a field, consider a string (a "usual string," not the abstract one, yet!), streched between two fixed points. In classical mechanics, one can obtain the (linearized) equations of motion from the action

$$S = \int_{t_1}^{t_2} \int_0^L \left[(\frac{\partial f}{\partial t})^2 - (\frac{\partial f}{\partial z})^2 \right] dz dt \,, \tag{1.1}$$

where $f(z, t)$ is the displacement of the string, which can be considered as a field that lives on the closed interval $[0, L]$. One can get the motion of the string by applying appropriate boundary/initial conditions to $f(z, t)$.

Even though the string has a continuous infinity of degrees of freedom, we can choose a discrete set of basis functions $f_n(x)$, and represent the time evolution of the string as the time evolution of the coefficients of these basis functions, as in

$$f(x,t) = \sum_n f_n(x) c_n(t) \,. \tag{1.2}$$

If we use as basis an appropriate set of Fourier modes, the time evolution of the coefficients is in fact decoupled;

$$\frac{d^2 c_n(t)}{dt^2} = -\omega_n^2 c_n(t) \,, \tag{1.3}$$

which means the string behaves as a collection of independent harmonic oscillators.

The quantum wavefunction of a system of particles whose position in classical mechanics is given by $(x_1(t), x_2(t), ..., x_n(t))$, is denoted by $\psi(x_1, ..., x_n)$. This wavefunction gives the probability amplitude for the positions of particles. In applying the philosophy of quantum mechanics to a field such as f, we can consider a wave*functional*

$$\Psi[f(z)] \,, \tag{1.4}$$

since $f(z)$ denotes the displacement of the particle labeled by z. We can imagine having a (perhaps formal) functional Schrödinger equation, which gives the time evolution of this wave functional in terms of the hamiltonian of the system. One can also think about the Heisenberg picture, or evaluate amplitudes by path integrals. In the path integral approach, one integrates e^{iS} over all time evolutions with given inital and final conditions to get the probability amplitude of ending up with a given final state, given the initial state (or uses the formalism to calculate correlation functions, scattering amplitudes, etc.). See [1] for an elaboration of these ways of looking at the quantum theory of fields.

Even though the infinity of the number of degrees of freedom brings in some confusion, the fact that the classical system is equivalent to an infinite collection of independent harmonic oscillators helps us guess that the quantum version can also be described by such a collection. As long as the action is quadratic in the field and its first derivatives (i.e., the field theory is "free"), one can perform a decomposition of the quantum system into harmonic oscillators, and the infinities that come up (such as the energy of the ground state) can be removed by simple procedures. However, once non-quadratic terms go in the action, things are not so simple.

Let us focus on the path integral formulation of quantum mechanics. Writing the action as a sum of a quadratic part and a non-quadratic "interaction" part, and expanding the integrand e^{iS} of the path integral to a Taylor series in the interaction part, one gets the path integral of the interacting theory as a series of "correlation functions" of the *free theory*. Terms in the series can be represented by diagrams called Feynman diagrams, and it turns out that terms which correspond to diagrams with loops are in general divergent. For some theories it is possible to get rid of these divergences by absorbing them into the parameters in the action, and get sensible, finite results for correlation functions, or scattering cross sections of "particle-type" excitations of the fields, etc. Theories for which this procedure works are called renormalizable theories. The work of K. Wilson showed that such theories have the property that they are robust to modifications of the theory at very small distance scales.

Electromagnetism, the weak force, and the strong force have all been put in this framework of *quantum field theory*, which consists of starting with a classical field theory and quantizing. In fact what one used to call particles (e.g. electrons, photons) before the developement of quantum field theory are now considered as excitations of fields. This approach had spectacular success for a wide range of problems.

Gravity has a distinct place among the interactions we observe. Einstein's theory of general relativity treats gravity not as "yet another field," but gives it the distinct position of describing the geometry, or the "setting" all the other fields live in. The theory tells us how various fields affect the geometry, and how they are affected by it through Einstein's equations. There are arguments that show the universe can't be "part classical and part quantum," so one is forced to consider a quantum version of gravity. However, if one is to take the geometrical point of view seriously, then deep problems block the way to a quantization of the gravitational field. If we are to let the geometry be a quantum object, then the spacetime intervals between events do not have well defined values. In flat Minkowski space, Heisenberg picture operators of quantum fields at two events with spacelike separation are supposed to commute, and it is not clear what to do when the metric is a quantum object—one can't tell if two events are separated by a spacelike or a timelike interval. A metric also gives information about the topology of the spacetime, and making the metric a quantum object, one also opens the way for having spacetime which is in a superposition of different topologies. It indeed becomes difficult to talk about correlation functions of the other fields in such a shaky background!

If one is to make some progress in all this fog, one can perhaps take a more pragmatic approach, and postpone all these philosophical issues for a while (hopefully not till eternity!). In fact, it is possible to ask practical questions related to the expected quantum nature of gravity which might be answered without having a full-fledged quantum gravity, just as it was possible to ask and answer in part [2] the question brought by the Lamb shift experiment, before the advent of QED. [1]

The most straightforward and pragmatic approach that comes to mind is the one mentioned above: treating gravity as "yet another field." In fact, if we fix the background to be the flat Minkowski spacetime, and expand the Einstein-Hilbert action in terms of the perturbation of the metric, and give the resulting formula to somebody who hasn't heard of general relativity (but knows QFT!), this will seem to her as nothing but a complicated field theory, waiting to be quantized. She will see no problems about fuzzy geometries, quantum topologies, etc. She will perhaps take the quadratic part of this action, use a decomposition into harmonic oscillators, then include the nonlinear terms as interactions of these, and hopefully get a sensible perturbation expansion for correlation functions. These then can be used to calculate physical quantities such as scattering amplitudes. Perhaps after we get the results from our relativity-ignorant, field theorist friend, we can use them as food for thought about our previous concerns related to the seemingly distinctive nature of the gravitational interaction.

It turned out that this approach doesn't work. The field theory obtained by treating gravity as yet another field turned out to be non-renormalizable, thus one cannot

[1] One could perhaps make this analogy even more precise; we might want to calculate the first-order shift due to quantum gravitational effects in the energy levels of an "atom" bound by the gravitational force [3]!

squeeze meaningful numbers out of the theory.[2] As mentioned above, renormalizability is related to the independence of a given theory to short distances. One may say that the non-renormalizibility of gravity forces us to consider the actual short distance physics. There have been various proposals for a path to a meaningful quantization of gravity, I will only mention one of them here: string theory.[3] I will next try to give a heuristic description of perturbative string theory.

It is usually said that in contrast to conventional quantum field theory, string theory treats elementary particles as one dimensional objects, rather than point particles. The idea here is to generalize the notion of Feynman diagrams; in field theory, the diagrams are "graphs," and in string theory they are "generalized" to two dimensional surfaces. In order to calculate a scattering cross section in field theory, one sums over Feynman diagrams with given states for external legs. In string theory, one sums –vaguely speaking– over all two dimensional surfaces with given states on the external legs. These legs can be thought of cylinders (for "closed" strings), or strips (for "open" strings) that shoot off from a two dimensional surface. String theory tells us how to calculate the amplitudes for each diagram from a given embedding of a "string Feynman diagram" in spacetime.

In order to make this vague discussion slightly more accurate, we should mention that string theory in its current form is a supersymmetric theory, which means it has symmetries that relate bosons and fermions. The most basic implication of this is that the surfaces that describe the Feynman diagrams for strings have fermionic fields that live on them; the maps that embed a surface in spacetime can be thought of as bosonic fields that live on the surface, and they will have fermionic partners. All this is pretty strange, admittedly, however one encounters happy surprizes once one starts

[2]In fact there were other difficulties besides non-renormalizability. The Einstein-Hilbert action is invariant under diffeomorphisms (which might seem to our field theorist friend as some curious gauge invariance). The gauge fixing is done by using ghosts, which have been invented by Feynman during his explorations in quantum gravity, and have been put on a more systematic basis by Fadeev and Popov, as well as de Witt.

[3]In fact, string theory was not born out of such a proposal originally, but was invented during explorations of strong interactions. For a quick review of the history of the subject, see [4].

to calculate things with this formalism.

One of the most striking facts about string theory is that there aren't different types of strings for all of the "fundamental" particles, but all the different kinds of particles are thought to be excitations of fundamental strings. The external states that appear in the legs of string Feynmans diagram can be chosen to be normal modes of the string, and each normal mode represents a particle. In fact, the first thing to calculate with string theory is the spectrum of these normal modes. Such a calculation tells that the spectrum includes a massless spin-2 particle. There are arguments [3] which suggest that a massless spin-2 particle has to be "the" graviton. Thus, string theory in its spectrum of particles contains the graviton, hence we should in principle be able to calculate Feynman diagrams that involve gravitons [5, 6]. The happy surprize is that, the diagrams calculated in this way give finite answers![4] Thus, string theory might be the solution to the problems about the non-renormalizability of gravity.

Defining a theory through its Feynman diagrams may seem somewhat weird; one more commonly starts with a classical field theory, and quantizes the theory and gets the Feynman rules for its perturbation expansion. String theory is unique in this respect, its Feynman rules were found before a nonperturbative formulation of the theory existed. In fact it is fair to say that we still don't have a nonperturbative formulation of string theory.[5] Given the amplitudes for Feynman diagrams, one can go backwards and ask which classical field theory would give those amplitudes when quantized. The first approximation to the answer to this question, for tree-level diagrams that involve only the massless particles, is given by various supergravity theories.

In perturbative string theory, a crucial role is played by conformal invariance, more precisely the conformal invariance of the theory that describes the fields that live on the string (such as the embedding maps and their supersymmetric partners). In fact,

[4] I am not aware of a general proof of the claim that string theory gives finite amplitudes to *all* orders in perturbation theory.

[5] "String field theory" (which describes string interactions by assuming one spacetime field for each normal mode of the string) may be considered to be one step in this direction.

conformal invariance is the source of another method for deriving the equations of motion for the aforementioned supergravity theories; treating strings as moving in "fixed background fields" and imposing conformal invariance, one gets, for example, Einsteins equations for the background metric.

One problem with this approach is that, it requires the classical background spacetime to be ten-dimensional. Since we only observe a four dimensional spacetime, people have devised possible scenarios where a ten-dimensional spacetime can appear to be four-dimensional. Some of the proposals involve 6 of the 10 dimensions being curled into tiny manifolds, the excitations in these directions being energetically very expensive. Other scenarios involve more modern developments involving higher-dimensional objects in string theory (which I will mention below). Basically one thinks of the observed universe as being confined to a four-dimensional subspace (a "membrane") in the ten-dimensional spacetime.

It should be mentioned that if the graviton is just another one of the whole family of different modes of the string, then the foregoing discussion of embedding a string Feynman diagram in a spacetime is somewhat strange. The background spacetime that the string moves in is also supposed to be "made of" strings, so treating it as a fixed background is at best an approximation, perhaps similar in spirit to an investigation of the propagation (or scattering) of photons in a *classical* background electromagnetic field[6]. The graviton case is more subtle than the photon case, however,[7] due to the arguments mentioned above in the introduction to quantum gravity in general. The conceptual problems mentioned there continue to exist in the string theory approach as well: what is one supposed to do when the background spacetime itself becomes a quantum object? Again, one is led to taking a pragmatic approach and exploring the theory, leaving such deep questions to later.

The last decade has seen an enormous amount of developement in string theory,

[6] As opposed to treating the electromagnetic field as a "single" quantum field with its wavefunctional describing a photonic excitation on top of a "coherent" background with a given expectation value.

[7] That is, if we are to assume that gravity does describe the geometry, and not take $g_{\mu\nu}$ to be "yet another field."

much of which is related to the discovery of certain extended objects called "branes." When one is exploring the spectrum of open strings, one has to impose some boundary conditions at the endpoints. The traditional choice was to impose Neumann (free end) boundary conditions. However, one can also impose Dirichlet (fixed end) boundary conditions in some directions, which would restrict the endpoints of the open string to a submanifold of the spacetime manifold. It turns out such a submanifold itself is a dynamical object; a so-called D-brane. D-branes can interact with the closed strings that are not confined to them, and they have spectra of open strings that can live on them. For a D-brane in a flat spacetime, the massless spectrum of open strings is that of a supersymmetric Yang-Mills theory: for the case of a p-dimensional brane, it is the theory found by dimensionally reducing the $D = 10$, $\mathcal{N} = 1$ super Yang-Mills theory to $p + 1$ dimensions. If one considers a number N of D-branes on top of each other, one gets a nontrivial gauge group for the Yang-Mills theory, such as $U(N)$. It turns out that in the low energy limit, the interactions of the open strings that fill this spectrum are also described by the aforementioned super Yang-Mills theories.

The investigation of the interactions of the closed strings that live in the bulk of the spacetime with the open strings that are confined to the brane has been the source of an enormous amount of information and conjectures about supersymmetric Yang-Mills theories. In particular, one can find regimes in which the bulk theory and the "worldvolume" theory, i.e. the supersymmetric Yang-Mills theory that lives on the worldvolume of the brane, decouple. In order to explain the basic idea behind the dualities that originate from this line of developement let us mention another route to the branes of string theory.

As mentioned above, the low energy limit of string theories are given by various supergravity theories. One sector of the string spectrum (the so-called Ramond-Ramond sector) includes particles that can be thought of as elementary excitations of antisymmetric tensor fields (i.e., differential forms) of various ranks. The action for a point particle coupled to a one-form is given by

$$S = \int_{\Sigma_1} A \,,$$

where $A = A_\mu dx^\mu$ is the one-form potential (just as the four-potential of electrodynamics), and Σ_1 is the worldline of the particle. Analogously, one can write the action for a p-dimensional object coupled to a $(p+1)$-form as

$$S = \int_{\Sigma_{p+1}} C$$

where C is the $(p+1)$-form potential, and Σ_{p+1} is the worldvolume of the p-brane.

Long before the discovery of D-branes, the existence of the $(p+1)$-form fields (the so-called Ramond-Ramond or RR fields) in the supergravity theories led people to construct solutions to the classical equations of motion where these fields are excited. The simplest such solutions are generalizations of the Reissner-Nordström solution, which describes a black hole with electric charge. Just as the Reissner-Nordström solution has the symmetries of a static point particle (i.e., the group $SO(3)$ of three-dimensional rotations, and time translations), the simplest so-called p-brane solutions that are charged under the $(p+1)$-form potentials have the symmetries of static, flat, p-dimensional objects, i.e., $SO(10-(p+1))$ (rotations in the transverse directions), time translation, and the Euclidean group $ISO(p)$ in the spatial worldvolume directions.[8] These solutions have event horizons, and are called black branes.

One of the most important discoveries in string theory in the past decade was that of the fact that D-branes are charged under the Ramond-Ramond fields [7]. This can be seen by calculating scattering amplitudes of the strings that describe elementary excitations of the RR fields from D-branes. Thus, one has two descriptions of objects charged under these fields: as perturbative string theory objects on which strings can end, or as supergravity solutions. The inspection of the regimes of validity of these descriptions lead to various "duality" conjectures, which will be elaborated in more detail in the following sections. The most celebrated result in this direction is the so-called AdS/CFT conjecture by Maldacena [8]: $\mathcal{N} = 4$ super Yang-Mills theory with gauge group $SU(N)$ in $(3+1)$-dimensions (a conformal field theory) in the large N, strong 't Hooft coupling limit is dual to classical type IIB supergravity

[8]This hints at the fact that as opposed to "everyday" objects, p-branes do not have longitudinal degrees of freedom.

on a background of five-dimensional anti-de Sitter space times a five-sphere, with N units of Ramond-Ramond five-form flux through the sphere. This duality allowed us to learn a great deal about the strong coupling behavior of the super Yang-Mills theory by using results from classical supergravity.

The explorations of black branes have also helped with our understanding of the thermodynamics of black holes, whose foundations lay in an analysis of quantum fields in curved (classical) spacetimes. The rationale of this is as follows. Just as there is a regime in which the Shcrödinger equation with a $1/r$ potential describes the physics with reasonable accuracy without the need for quantum electrodynamics, one can imagine a regime in which a quantum theory of fields coupled to gravity by treating the latter as a classical background describes the physics reasonably accurately, without the need for quantum gravity. Hawking's groundbreaking work in this direction discovered what is now called "Hawking radiation" of a black hole. One of the most fundamental results in black hole thermodynamics is the identification of the entropy of a black hole with its area in Planck units. This is based on both classical area theorems which basically state that the area of a black hole can't decrease, and more firmly on Hawking's result. However, these results are in thermodynamics, not statistical mechanics. This means they don't give an interpretation of black hole entropy as the logarithm of the number of microstates of the black hole, the latter would presumably be related to quantum gravity.

D-branes are described in terms of the fields that live on them, and their interactions with the "bulk" of the spacetime. If the curvatures are much smaller than the Planck scale, a description in terms of classical black brane solutions to supergravity is expected to be accurate. The dualities between these descriptions mentioned above have also given insights about the thermodynamics of black branes. Calculating the entropy of a black hole in string theory "directly" might be still out of reach, but by using dualities at certain regimes in which the bulk of the spacetime is decoupled from the fields that live on the brane, people have identified the thermodynamics of certain black holes/branes with those of the dual field theories that live on branes. The investigation of the thermodynamics of such "worldvolume" theories has confirmed the area formula for the entropy of black holes from a "microscopic" standpoint. (See [9] for a pioneering work in this direction, or [10] for a review.)

Much of this dissertation will be concerned with one of the most celebrated of the dualities between various descriptions of the brane dynamics: the so-called "AdS/CFT correspondence" of Maldacena mentioned above. Chapter 2 is concerned directly with the correspondence. We first investigate the possibility of having multiple renormalization flows between field theories that are dual to supergravity theories with asymptotically anti-de Sitter backgrounds, using some toy models. Then we investigate an exact solution to gravity coupled to a scalar, which confirms previous conjectures [11] about the equations of state of various finite-temperature field theories, in the context of AdS/CFT correspondence. Chapter 3 is about the instabilities and thermodynamics of black branes in the context of supergravity. We explore the stability properties of some of the black branes of string theory, namely D3, M2, and M5-branes, and confirm the expectations of [12, 13], where it was conjectured that black branes are dynamically stable if and only if they have the property of being "locally thermodynamically stable", which is a generalization of the notion of having positive specific heat. Part of the motivation for this conjecture also comes from the AdS/CFT conjecture. We also make some observations about the transition from stability to instability, and propose a notion of "universality classes" of the instabilities of black branes. This work utilizes the aforementioned exact solution as a unifying theme.

The exploration of the string theory landscape still continues, and I have barely mentioned a couple of the lines of developement. There have been many mathematical developements that originate in string theory, such as mirror symmetry. The various string theories that have been thought to be distinct have been related to each other through various dualities. The general consensus seems to be that there is a unique underlying theory, and the previously discovered string theories have been various points in its "moduli space." An 11-dimensional theory (whose low energy limit is 11-dimensional supergravity) also seems to exist and is thought to be linked to the others through a "web of dualities."

As mentioned above, there have also been many attempts in embedding the standard model of particle physics in string theory, by curling up the 6 extra dimensions in various ways, or considering theories that live on branes, on orbifolds, etc. Perhaps the richness of the theory has also been a liability in that one cannot yet see a way that the observed universe would sit in string theory in a *natural* way.

A bug that spends all her life on a huge magnetic rock might try to develop a theory that describes her observed "universe". Perhaps she can come up one day with the idea of atoms as small magnets, and their interactions—a theory of ferromagnetism. Her explorations of this theory might give some testable predictions, and give satisfactory results, nevertheless, she may still be disappointed that her theory doesn't explain the reason for the direction of the magnetic field she observes; her theory will be consistent with the magnetic field pointing in any direction.

Just as the bug should not expect her theory of ferromagnetism to explain the direction of the magnetic field she observes, perhaps we, too should not expect our "fundamental" theories to explain all our observations, but leave some facts (why do we observe the given gauge groups, etc.) to be explained only by "cosmic historical coincidences", not the fundamental dynamics of our theories. Of course there *might* actually be fundamental reasons for the observed properties of particles; one can't make conclusions without exploring the possibilities. However, even if it eventually turns out that some candidate "theory of everything" such as string theory is "correct," but is incapable of naturally predicting some things such as the observed gauge groups, etc., this doesn't necessarily mean that the theory has no predictive power. If one can figure out the particular "vacuum" of the theory that describes the universe we live in (or at least some properties of it), one might be able to use the fundamemental theory to make testable predictions about the corrections to the standard model, say.

Whether string theory describes the universe is of course the most important question in this subject. However, explorations of the theory have been very fruitful in helping us develop a better understanding of theories that unrefutedly have something to do with the universe, i.e. quantum field theories. Starting with the next section, I will be talking about some subset of this line of developement.

1.2 AdS/CFT correspondence

It became clear in the mid-90's that string theory contains not only strings, but also higher dimensional objects that carry Ramond-Ramond charges. The existence of such objects has been predicted in the context of string dualities [14, 15, 16], and in a groundbreaking paper [7], Polchinski demonstrated that these objects are embedded in string theory in a very simple way: as Dirichlet-branes, or D-branes. A Dp-brane is a p-dimensional object, a sort of topological defect on which strings are allowed to end. When a closed string touches a D-brane, it can open up and turn into open strings, whose endpoints are confined to the Dp-brane. The perturbative string description of D-branes is expected to be valid when the string coupling g_s is small. The fact that such defects in fact carry Ramond-Ramond charges was demonstrated in [7].

We will mostly be interested in the case $p = 3$. The spectrum of massless open strings that live on a D3-brane is that of $\mathcal{N} = 4$, $D = 4$ supersymmetric $U(1)$ gauge theory. If we consider not a single D-brane, but N of them sitting on top of each other, then each of the two endpoints of an open string has N D-branes to choose from, and the spectrum of massless open string states turns out to be that of $\mathcal{N} = 4$ $D = 4$, $U(N)$ gauge theory [17], which is a superconformal field theory. It turns out that in the low energy limit, the interactions of these massless states are also given by that theory,[9] the coupling constant g_{YM} being related to the string coupling constant by $g_{YM}^2 = 4\pi g_s$. As mentioned above, the open strings that live on the brane can interact with the closed strings which are free to move in the bulk of the spacetime. A first approximation to these interactions can be obtained by "covariantizing" the Dirac-Born-Infeld action [18].

Low energy behavior of string theory is given by supergravity. Solutions to type II supergravity theories that represent objects with Ramond-Ramond charges have been

[9] A more accurate description is given in terms of the Dirac-Born-Infeld action, see [18] for a review.

found long before Polchinski's calculation [19]—much in the same way that one would start with Maxwell's equations in vacuum, and by imposing spherical symmetry end up with a (singular) solution that represents a point charge. In the case of p-branes, one looks for solutions with p-dimensional euclidean symmetry and $(8-p)$-dimensional spherical symmetry, as well as symmetry under time translations.[10] The metric for the extremal D3-brane of type IIB supergravity is given by

$$ds^2 = f^{-1/2}(r)\left[-dt^2 + d\vec{x}^2\right] + f^{1/2}(r)\left[dr^2 + r^2 d\Omega_{8-p}^2\right], \tag{1.5}$$

where

$$f(r) = 1 + \frac{L^4}{r^4}, \tag{1.6}$$

and $d\vec{x}^2 = \sum_{i=1}^{3} dx_i^2$. Note that this metric is asymptotically flat—as $r \to \infty$, it asymptotes to 10-dimensional Minkowski metric.

The supergravity solution is thought to be an approximation to the full string theory solution that becomes accurate at low energies and small curvatures, where the effects of massive string states can be ignored. The length scale for strings is given by $l_s = \sqrt{\alpha'}$, thus the metric (1.5) is expected to be an accurate approximation when $L \gg \sqrt{\alpha'}$.

One can get the tension and Ramond-Ramond charge of a D-brane by calculating a closed string exchange amplitude between two branes [18]. These quantities are used in the supergravity side to get the correct L in (1.5). For the case of N D3-branes sitting on top of each other, one gets

$$L^4 = 4\pi g_s N \alpha'^2 . \tag{1.7}$$

This tells that the regime of validity for the supergravity solution $L \gg l_s$ corresponds to $g_s N \gg 1$ (note that we should also restrict the energies of the physical processes that probe the geometry to $E \ll 1/l_s$). If in addition the coupling g_s is small, one can ignore loop corrections, and arrive at the approximation of classical supergravity. In this limit one can in principle calculate scattering amplitudes for particles sent from

[10]It is possible to generalize this to cases with less symmetry, such as rotating branes, but we will not consider such cases here.

the asymptotically flat region by using classical supergravity. Such calculations have been done in [20, 21, 22].

Another approach to calculating scattering amplitudes is by using the aforementioned couplings of the closed strings to the worldvolume super Yang-Mills (SYM) theory. In order for a perturbative calculation on the worldvolume theory at large N to be accurate, the 't Hooft coupling $\lambda = g_{YM}^2 N = g_s N$ needs to be small. Thus, we see that the regimes of validity for the perturbative worldvolume SYM theory calculation and the supergravity calculation are mutually exclusive. However, a calculation of the dilaton absorption amplitude [20] showed that the results obtained from the two sides agree perfectly. In fact this is to be taken as a prediction of string theory for the strong coupling behavior of SYM theory.

The asymptotically flat geometry (1.5) approaches that of $AdS_5 \times \mathbf{S}^5$ as $r \to 0$ [23]. One can see this by ignoring the 1 in (1.6), and defining $z = L^2/r$, which gives

$$ds^2 = \frac{L^2}{z^2}(-dt^2 + d\vec{x}^2 + dz^2) + L^2 d\Omega_5^2 \,. \tag{1.8}$$

The first term is the metric of AdS space, and the second term is the standard metric on $\mathbf{S}^5$. This suggests that the scattering of a particle sent from the asymptotically flat region is actually a scattering from this $AdS_5 \times \mathbf{S}^5$ "throat"—in the classical supergravity calculation, some part of the wave is reflected, and some of it is translated into the throat. In the limit of low energies for the incoming particle, the throat and the asymptotically flat region in fact decouple.

A similar decoupling also occurs in the perturbative worldvolume calculation—the worldvolume $\mathcal{N} = 4$ super Yang-Mills theory decouples from the closed strings in the bulk. These facts seem to suggest that the worldvolume SYM theory with gauge group $U(N)$ at large N in the strong 't Hooft coupling limit, and classical IIB supergravity on the near-brane background $AdS_5 \times \mathbf{S}^5$ with N units of 5-form flux on the 5-sphere[11] actually describe the same system.

This is the so-called Maldacena conjecture. In fact Maldacena formulated a much stronger form of this. According to [8], the full string theory on the background

[11]This near-brane metric indeed turns out to be a solution to type IIB supergravity, together with the N units of 5-form flux through $\mathbf{S}^5$.

spacetime $AdS_5 \times \mathbf{S}^5$ with N units of Ramond-Ramond (RR) 5-form flux through the $\mathbf{S}^5$ [12] is dual to the conformal $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$. One of the fundamental checks of the conjecture is the matching of the symmetries of the two sides: AdS_5 has the isometry group $SO(2,3)$, which is also the conformal group in 4 dimensions. The supersymmetric extension of the isometry group is $SU(2,2|4)$, and this is realized on the SYM side as the superconformal group. For a more detailed discussion of the matching of the symmetries, see [24].

If for a given radius of curvature $L = (4\pi g_s N \alpha'^2)^{1/4}$ one wants to work with string theory in the weak coupling limit $g_s \ll 1$, then one needs to take the large N limit with $\lambda = g_{YM}^2 N = 4\pi g_s N$ kept fixed. Thus, the 't Hooft limit [25] ($N \to \infty$ with λ constant) of $D = 4$, $\mathcal{N} = 4$ SYM theory with gauge group $U(N)$ is dual to tree level type IIB string theory on the $AdS_5 \times \mathbf{S}^5$ background. If the further limit of large λ is considered, then, L becomes much larger than l_s, and the corrections from massive string states can be ignored, so the dual theory on $AdS_5 \times \mathbf{S}^5$ becomes classical type IIB *supergravity.*

This last form of Maldacena's conjecture (the "weak version") is the one I will be using in the following chapters. Actually, once one obtains the precise relations between the two sides of the duality as we will see in section 2.1.3, it is possible to start with any gravity theory in AdS, and define a dual conformal field theory (CFT). Thus, the AdS/CFT correspondence has a much wider range of application than the initial motivation from D3-branes might suggest. In the following, I will occasionally consider the correspondence in this more general context.

For a detailed review of the correspondence with a more extensive list of references, as well as the case of M-branes, see [24].

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[12] Unfortunately string theory on this background with RR flux is not within the reach of current calculational techniques.

1.3 RG-flows in AdS/CFT

AdS/CFT correspondence [8, 26, 27] is a duality between a $(d+1)$-dimensional gravity theory on an asymptotically AdS spacetime (the bulk theory), and a d dimensional gauge theory without gravity (the boundary theory). The gauge theory can be thought of as living on the boundary of the spacetime that the gravity theory lives in. A precise statement of how the two sides of the correspondence are related is given in section 2.1.3. The correspondence relates small distances in the bulk theory to large distances in the boundary theory in a sense that will be elaborated in section 2.2.1. In the so-called weak version of the correspondence, it is believed that renormalization flows of the boundary theory on $\mathbf{R}^{3,1}$ are represented by classical solutions to the bulk theory (a supergravity theory) which have d-dimensional Poincaré symmetry. The dependence of the fields on a "radial coordinate" is related to the dependence of the coupling constants of the boundary theory to the energy (or length) scale of the RG-flow. There are many such "radial flow" solutions in the literature, together with comparisons with the expectations from the gauge theory side (which might not always be made in detail since the supergravity solutions are generally singular, so the supergravity approximation breaks down, at least at first sight (more on this later)). Many of the solutions explored preserve some supersymmetry, which allows one to find them by using first-order equations, thus reducing the amount of effort necessary. This reduction of order works even in cases without supersymmetry, as reviewed in section 2.2.3. In section 2.3.3, it is shown that this method can also be used for finding some black hole solutions.

Some of the supersymmetric radial flow solutions interpolate between two supersymmetric "fixed points," i.e., they asymptote to two supersymmetric solutions to supergravity at "large radii" and "small radii." At these two endpoints, the scalars of the supergravity theory asymptote to extrema of the scalar potential. On the boundary theory side, such an interpolating flow solution represents a renormalization flow between an IR fixed point and a UV fixed point, both of which correspond to supersymmetric gauge theories.

The existence of other, not necessarily supersymmetric, renormalization flows between two supersymmetric fixed points is an interesting issue that can be attacked more easily on the gravity side of the correspondence than on the gauge theory side. Given a supersymmetric flow solution to supergravity that asymptotes to two supersymmetric fixed points at the two ends, it shouldn't be too hard to see if we can also find a non-supersymmetric solution that interpolates between the same fixed points. If such a solution can be found, the presription of AdS/CFT correspondence could in principle be used to read off the relevant deformations of the boundary theory which would result in such a non-supersymmeric flow between supersymmetric fixed points. This would give an interesting example of a symmetry of the UV fixed point broken on an RG-flow, but recovered at the IR fixed point.

In section 2.2.2, it will be shown by using some toy models that it is quite likely that such multiple radial flows between the fixed points exist, and some criteria are proposed for checking whether given extrema of the scalar potential that appears in the gravity theory are good candidates for being endpoints of such multiple flows (supersymmetric or not).

Another aspect of AdS/CFT correspondence is related to nonzero temperatures. One can consider turning on a nonzero temperature in a gauge theory, and investigate the resulting thermodynamics. As explained in section 2.3, nonzero temperatures in the gauge theory side of AdS/CFT are closely related to black-brane backgrounds of the gravity theory. The thermodynamics of the gauge theory is thus related to the thermodynamics of the relevant black-brane solutions.

Gubser has investigated the thermodynamics of various deformations of the original AdS/CFT correspondence, and has conjectured various equations of state by giving an approximation scheme to black brane generalizations of radial flow solutions. The solutions involve various scalars that are coupled to gravity, but in some cases there is a regime in which a single scalar dominates the potential, which can in turn be approximated by a simple exponential term (i.e., $V(\phi) = V_0 e^{2\gamma\phi}$). Thus, exact black brane solutions to a gravity theory coupled to a single scalar with such an exponential potential would be useful for investigations of thermodynamics of various deformations of the AdS/CFT correspondence. Such an exact solution is given in section 2.3.3 by using a "superpotential" method, and thus the approximations in [11] are put on a firmer footing.

Another theme of [11] is related to the question of just when a nakedly singular radial flow solution to gravity should be considered unphysical. It is proposed that if a singular flow solution cannot be a limit of black brane solutions, i.e., if it has no "near-extremal" generalizations, then it shouldn't be considered physical. A necessary condition for the existence of near extremal generalizations is shown to be the negativity of the scalar potential at the singularity. A somewhat generalized version of the proof of [11] is given in section 2.3.2.

It turns out that the black brane solutions to gravity plus a scalar with an exponential potential mentioned above are also useful for the exploration of the stability properties of a class of black brane solutions.

1.4 Instabilities of black branes

1.4.1 Stability of black holes

Many exact solutions to Einstein's equations (and the equations of motion of other fields that may be present) are known. One criterion in determining the relevance of a given solution to astrophysics or cosmology is the stability of the solution: will a small perturbation of the solution die out in time, oscillate, or grow? If the solution is not stable, one can't expect it to be relevant to complicated astrophysical situations. On the other hand, there exist solutions that are not expected to be directly relevant to astrophysics to begin with, but their stability is nevertheless an interesting problem. For example, there are many black brane solutions in string theory whose dynamics in various limits are described by quantum field theories/gauge theories, as in the case of AdS/CFT correspondence. The stability properties of such solutions can give insights about the behavior of the dual field theories, among other things.

The simplest approach to the stability problem is linear stability analysis. Such an analysis is performed by parametrizing the perturbations by some "small constant" ϵ, and ignoring terms of order ϵ^2 and higher. Even so, the resulting equations are in general complicated, and one resorts to numerical methods.

The analysis of the best-known black hole solution, namely, the Schwarzschild solution has been initiated by Regge and Wheeler [28]. The approach is to perturb the Schwarzschild metric, and study the evolution of the perturbations using the linearized version of *vacuum* Einstein equations. If one can find "small perturbations" that grow in time, then the solution is unstable. The notion of "small perturbations" is somewhat subtle, however. The components of the metric are not coordinate invariant quantities, and a given perturbation to a background can in principle have small components in one coordinate system and large components in another,[13] particularly when the two coordinate systems are related by a singular transformation.

The Schwarzschild solution

$$ds^2 = -(1 - \frac{r_0}{r})dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1.9}$$

where $r_0 = 2m$ is asymptotically flat, and the coordinate system used above becomes the standard one of Minkowski spacetime as $r \to \infty$. One treats perturbations that are large (i.e., those that grow without bound) in the standard flat Minkowski coordinates as "physically large" perturbations. Since we want to investigate the time evolution of physically small perturbations, we should impose the boundary condition that the perturbations be well behaved as $r \to \infty$.

The boundary condition at the other end is a bit more subtle. First off, one has to decide which space one is going to perturb. Do we want to investigate the stability of the maximal analytic extension of the Schwarzschild solution? The maximal analytic extension of Schwarzschild solution is a "wormhole," and one does not expect black holes formed by the collapse of stars in the universe to result in the wormhole spacetime; the initial conditions aren't right. In this context, we might want to perturb the solution in the region $r > 2m$, or we may want to include the region that has the future singularity, together with the region $r > 2m$.

[13] In particular, one needs to check that the perturbations being worked on are not pure gauge, perhaps by using some gauge invariant quantities, see [29]

The traditional approach is to perturb the solution outside the horizon, and impose the boundary condition that the perturbations are well behaved at the horizon (as well as at $r \to \infty$). This still doesn't nail down the boundary conditions precisely. Since the Schwarzschild coordinates are not well behaved at the horizon, well behavedness of the perturbations at the horizon in Schwarzschild coordinates is not a satisfactory criterion. An alternative approach is to go to coordinates well behaved at the horizon [30], and impose well behaved boundary conditions in such coordinates. Vishveshwara [31] has done such an analysis, and concluded that the exterior of the horizon of the Schwarzschild solution is stable to small perturbations: no perturbation well behaved both at the horizon and at infinity grows in time.[14]

1.4.2 Stability of black branes

Black branes are solutions to equations of motion of gravity coupled to various fields, with event horizons that have some translational symmetry. Perhaps the simplest example is the five-dimensional vacuum black string solution, which is just the Schwarzschild metric multiplied by the real line:

$$ds^2 = -(1 - \frac{r_0}{r})dt^2 + \frac{dr^2}{1 - \frac{r_0}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dw^2 \,. \tag{1.10}$$

The spatial part of this metric has the "cylinderical" isometry group $SO(3) \times \mathbf{R}$ where $\mathbf{R}$ denotes the group of translations along the w direction.

Initially it was thought that this black string solution is stable [32], but it turned out [33, 29] that some of the unstable modes dismissed as singular in [32] were in fact well behaved when analyzed in Kruskal-Szekeres coordinates. In exploring the stability of (1.10), one deals with the dependence of the perturbations on w and t by

[14] One might also note that the demonstration of the nonexistence of a single well-behaved unstable mode might not be considered to be a proof of stability if the set of well behaved modes does not form a complete basis. In principle it might be possible to combine singular unstable modes in a way that cancels their singularities. A rigorous proof of stability is much harder than demonstrating instability.

using a Fourier decomposition

$$\delta g_{\mu\nu}(t,r,\theta,\phi,w) = e^{i(\omega t+kw)} H_{\mu\nu}(r,\theta,\phi)\,, \tag{1.11}$$

and looks for $H_{\mu\nu}$ that is well behaved in the sense explained above. If such solutions exist for complex ω, then we can say that the solution is unstable.

The Fourier decomposition in the w direction reduces the problem to a four-dimensional one. The w-dependence induces a "mass" term for the four-dimensional "graviton" field, which changes the nature of the problem from that of the stability of the four-dimensional Schwarzschild solution, and in fact brings in an instability. Focusing attention to spherically symmetric perturbations, Gregory and Laflamme have shown that for wavelengths in the w direction bigger than a certain threshold related to the Schwarzschild radius r_0, there are unstable modes, and have numerically obtained a dispersion relation for the instability, relating its wavelength to its exponential growth coefficient.

1.4.3 Thermodynamics and stability

Gregory and Laflamme mention a thermodynamical argument that perhaps makes the existence of an instability seem not that surprizing. Black hole area theorems state that the total area of black hole horizons in an asymptotically flat spacetime never decreases "in time," which is one of the main hints in the identification of the black hole horizon area with entropy [34, 35]. The statistical nature of this entropy has been investigated by candidate theories of quantum gravity. For reviews of the string theory point of view, see, e.g., [10, 36, 37, 38, 39, 40].

In the black string solution above, the area and the mass are infinite, but one can consider these quantities "per unit length in the w direction," or make them finite by considering a compactification by making w a periodic coordinate. One gets $M \sim Lr_0$ and $S \sim r_0^2 L$, where L denotes the length in the w direction, so the entropy for a given length L is related to the mass for the same length by $S \sim M^2/L$.

Next consider an array of 5-dimensional black holes. A single 5-dimensional black hole is described by the 5-dimensional version of the Schwarzschild solution:

$$ds^2 = -(1 - (\frac{r_0^{(5)}}{r})^2)dt^2 + \frac{dr^2}{1 - (\frac{r_0^{(5)}}{r})^2} + r^2 d\Omega_3^2\,, \tag{1.12}$$

where $d\Omega_3^2$ denotes the standard metric on the 3-sphere, and $r_0^{(5)}$ is the 5-dimensional Schwarzschild radius. An array of such black holes will not be a static solution since the black holes will attract each other. If, however, the initial separation between the black holes is much larger than the sizes of the black holes, then one expects the initial mass and entropy of a portion of the array that has N black holes to be given reasonably accurately by the mass and entropy of a single black hole (obtained from an "approximate Schwarzschild radius" for the members of the array), multiplied by N.

The mass of a single 5-dimensional black hole is given by (see, for a example, [10]) $M \sim r_0^2$, and its entropy is given by $S \sim r_0^3$. Thus, such a black hole has an equation of state of the form $S \sim M^{3/2}$, which will also be valid for an array of such black holes, assuming they are widely separated. This shows that for L large enough, a black string has less entropy per unit length than that of an array of widely separated black holes with roughly the same mass/length. This is the argument used in [33, 29] to suggest that the instability could have been expected on thermodynamical grounds. However, even though the fact that there is another solution with the same mass but higher entropy might suggest that the black string is meta-stable, it can't be called strong evidence for a *dynamical* instability.

The thermodynamic approach has been put under further scrutiny in [41, 42, 43, 44, 12, 13], and a relation between "local thermodynamical stability" of black brane solutions and their "dynamical" stability has been conjectured in [12, 13]. Local thermodynamic stability is a generalization of the requirement of positive specific heat. If a material has negative specific heat and is in an isothermal configuration, one half of it can *increase* its temperature by giving heat to the other half, which would drive further heat flow between the two halves, and hence an instability.[15]

[15]Note that this intuitive picture applies to a black hole/brane less straightforwardly than it does

A black brane is called "locally thermodynamically stable" if the Hessian (the matrix of second derivatives) of its entropy with respect to the conserved charges (e.g., mass, p-form charges, angular momenta) is a negative semi-definite matrix. This is a natural generalization of the requirement of non-negative specific heat. The Gubser-Mitra conjecture formulated in [12, 13] states that a black brane is dynamically unstable to small perturbations if and only if it is locally thermodynamically unstable. The authors of [12, 13] give numerical support to the conjecture by examining the stability of charged black hole solutions to $\mathcal{N} = 8, D = 4$ SUGRA that are asymptotically AdS. In the large black hole limit, the black holes become dynamically unstable exactly when they become locally thermodynamically unstable, except for a little discrepancy which is probably due to numerical error.

It makes sense to talk about the size of a quantity only when there is some other quantity that we can compare it to. In the case of a black hole in AdS (i.e., an asymptotically AdS black hole), we compare the radius of the black hole to the radius of curvature of AdS. For a Schwarzschild black hole in an asymptotically flat spacetime, however, there isn't a reference radius that can be compared to the Schwarzschild radius, so the fact that the Schwarzschild black hole is stable, independent of its radius, even though it is thermodynamically unstable does not contradict the intuition of large black holes in AdS approximating black branes.

Reall outlined an approach to the proof of the Gubser-Mitra conjecture in [45]. His approach is based on the relation between threshold unstable modes (i.e., marginally stable modes) of a black brane and negative modes of the Lichnerowicz operator on a euclidean black hole background, which is found by compactifying the worldvolume directions of the black hole and doing a Wick rotation. If a black brane is locally thermodynamically unstable, one can find a negative mode of the Lichnerowicz operator on the euclidean black hole. One can then expect to be able to "oxidize" this negative mode, and after Wick rotating back, get a threshold unstable mode of the black brane solution. This would show that thermodynamical instability implies dynamical instability (there are subtleties in this approach, see [45] for details). In order

to a material.

to prove the other direction of the conjecture, one needs to start with an unstable mode in the black brane solution, and prove that the brane is thermodynamically unstable. Reall argues that given an unstable mode, one can get a negative mode of the euclidean black hole related to the black brane. What is left now is to show that the existence of such negative modes implies thermodynamical instability. This may seem plausible, though a rigorous proof is lacking.

The works by Gubser and Mitra [45] and Reall [12, 13] have given insights to just when a black brane becomes unstable, however, the endpoint of "time evolution" of dynamical instabilities is still an unresolved issue. Initally it was thought that the horizon of a black string would eventually "pinch off," but by using the Raychaudhuri equation, Horowitz and Maeda [46] have shown that this can't happen for finite affine parameter on a geodesic on the horizon, and conjectured that it can't happen for infinite affine parameter, either. See Chapter 3 for further references.

Another problem that needs further exploration is the transition from the unstable regime to the stable one. One can for instance ask questions about the behavior of the wavelengths of the threshold instabilities as one approaches the limit of instability. This last question, together with a simplified approach to the stability of the non-dilatonic branes (D3, M2 and M5-branes) of string theory, is explored in Chapter 3. The stability of the solution to gravity coupled to a scalar given in section 2.3.3 plays a unifying role in this analysis. In section 3.4, a simplified perturbation ansatz and a dimensional reduction reduces the problem to one in three dimensions with two scalars (the sizes of the compactification sphere and torus) that have a potential in the form of a linear combination of exponential terms. There are regimes (small and large non-extremalities) in which different scalars dominate the potential, and by figuring out which scalars dominate, one can approximate the problem with that of a *single* scalar with an exponential potential $V = V_0 e^{2\gamma\phi}$ (with appropriate γs for each regime). This is the basis of the notion of "universality classes" of horizon instabilities introduced in Chapter 3: if it is possible to approximate the potential by an exponential term in some regime, then the equation of state of the brane in that regime will be given by that of the brane solution for the exponential potential, i.e., by a relation of the form

$$S/V \propto T^{\alpha}\,, \tag{1.13}$$

where α is given in terms of γ, and the instabilities that may be present will be in the "universality class" identified by γ. The expectations from the universality arguments and the Gubser-Mitra conjecture are confirmed by a numerical analysis of the stability of D3, M2 and M5-branes.

Chapter 2

AdS/CFT correspondence, RG-flows and finite temperature

2.1 AdS/CFT correspondence

The AdS/CFT correspondence relates a gravity theory in anti-de Sitter spacetime (or more generally an asymptotically AdS spacetime) to a field theory on the boundary of the latter. In order to give meaning to this statement, one first needs to make sense of the notion of the boundary of an infinite space like AdS, and then needs to make the relation between the two sides of the correspondence precise, giving a prescription that relates various quantities appearing on both sides. Before I give the precise form of the correspondence, I will review some facts about AdS and its boundary, as well as some properties of scalar fields in AdS, which might at first sight seem unusual since they are a bit different from those of scalars that live in Minkowski spacetime.

2.1.1 Anti-de Sitter space and its boundary

Penrose has introduced a "conformal compactification" procedure that helps define a notion of boundary for an "infinite" spacetime [1]. The main idea behind this is the

[1]Geroch, Kronheimer and Penrose have introduced an alternative approach to the definition of the boundary of a spacetime, which considers equivalence classes of timelike curves as points at

fact that local rescalings of a metric of a spacetime do not change its causal structure. i.e., if ds^2 is a metric with Minkowski signature, then the sets of spacelike, timelike and null curves as defined by ds^2 will be identical to those defined by

$$\tilde{ds}^2 = e^{2A} ds^2 \,, \tag{2.1}$$

where A is an arbitrary function that is well behaved throughout the spacetime. Given an "infinite" spacetime, one tries to find an A that diverges in a suitable way as one goes to "infinite distances," and perhaps a coordinate system for which the infinity is at finite parameter distance. If this is done appropriately, $\tilde{ds}^2$ will be well defined at infinity, allowing one to utilize it to give a causal structure to the boundary of the spacetime. One can draw diagrams called Penrose-Carter diagrams in order to describe this causal structure pictorially.

To make things concrete, let's start with the d-dimensional AdS metric given in a particular coordinate system as

$$ds^2 = R^2(-\cosh^2\rho\, d\tau^2 + d\rho^2 + \sinh^2\rho\, d\Omega_{d-2}^2) \,, \tag{2.2}$$

where R denotes the radius of curvature of AdS, $d\Omega_{d-2}^2$ denotes the standard metric on the $(d-2)$-sphere, ρ is a "radial" coordinate with $0 \leq \rho < \infty$, and τ is a time coordinate ($\frac{\partial}{\partial\tau}$ is a global timelike Killing vector) with $-\infty < \tau < \infty$. Defining a coordinate θ by $\tan\theta = \sinh\rho$ (so that AdS is confined to the region $0 \leq \theta < \pi/2$), one gets

$$ds^2 = \frac{R^2}{\cos^2\theta}(-d\tau^2 + d\theta^2 + \sin^2\theta\, d\Omega_{d-2}^2) \,, \tag{2.3}$$

which suggests that in (2.1), we use $e^{2A} = (\frac{R^2}{\cos^2\theta})^{-1}$, i.e.,

$$\tilde{ds}^2 = (-d\tau^2 + d\theta^2 + \sin^2\theta d\Omega_{d-2}^2) \,. \tag{2.4}$$

The metric $\tilde{ds}^2$ gives the region $0 \leq \theta < \pi/2$ the same causal structure as that of AdS, but is also well defined at $\theta = \pi/2$, which was at infinite distance in AdS. We can extend the parameter range to $0 \leq \theta \leq \pi$, in which case the metric (2.4) describes

infinity. For a review, see [47], and see [48] for an alternative definition and further references.

the d-dimensional version of the so-called Einstein static universe, which is basically a spatial three-sphere times the time axis.

$\frac{\partial}{\partial\tau}$ is a global timelike Killing vector for $\tilde{ds}^2$ as well, and at each "instant" of constant τ, the space described by the region $0 \leq \theta \leq \pi/2$ is a half $(d-1)$-sphere. Thus, the boundary of AdS_d can be thought of as the boundary of this $(d-1)$-sphere times the real line, i.e., $\mathbf{S^{d-2}} \times \mathbf{R}$.[2] Note that one cannot assign a canonical metric to the boundary, since one can always factor out a function well-behaved at $\theta = \pi/2$ from the term in paranthesis in (2.3), and this will change the precise form of the metric (2.4). Thus, one can only define an equivalence class of metrics under rescalings, i.e., a conformal structure at the boundary, which in turn gives the boundary a causal structure.

It is not hard to see that null geodesics in AdS originating on the worldline of an observer moving on a timelike geodesic can reach the boundary and come back to the observer in finite proper time of the observer. Thus, determining the initial conditions at a given τ of fields that live in an AdS background will not in general be sufficient to determine the time evolution, since there will be data flowing in from the boundary of AdS. One can specify boundary conditions at infinity in order to circumvent this problem. I'll return to this issue in section 2.1.2.

It will also be useful to mention a euclidean version of the "Penrose procedure" explained above. This might seem a bit confusing, since one cannot talk about a causal structure in the euclidean case. The notion of conformal compactification still makes sense, however, and consists of performing a local rescaling of the metric in order to bring infinity to finite distance, and giving a conformal structure to it.[3] The euclidean version of anti-de Sitter space is called the hyperbolic space, and it has constant negative sectional curvature for all planes at all points. One can use a conformal compactification similar to the one mentioned above to include its boundary, which turns out to be a $(d-1)$-dimensional sphere for the case of d-dimensional

[2]The "timelike infinity" in (2.4) is still "at infinity," but this will not bother us since in the context of AdS/CFT, we will treat the time coordinate on the boundary as a time coordinate of the dual field theory.

[3]This conformal compactification also commutes with "Wick rotation," if done properly.

AdS (incidentally, this sphere can be thought of as the conformal compactification of d-dimensional *flat* euclidean space, which is a fact relevant to the AdS/CFT correspondence). Let us perform the conformal compactification of euclidean AdS in Poincaré coordinates, for which the metric takes the form

$$ds^2 = \frac{R^2}{(x^0)^2} \sum_{\mu=0}^{d-1} (dx^\mu)^2 \,, \tag{2.5}$$

where $x^0 > 0$[4]. This is already in a form suitable for identifying the boundary: We see that $x^0 = 0$ is "at infinite distance" from any given point, and forms the boundary of euclidean AdS, together with the point at $x^0 = \infty$. The latter is indeed a single point since the distance between any two points $(x_1^0, \vec{x}_1)$ and $(x_2^0, \vec{x}_2)$ vanishes as the maximum of x_1^0 and x_2^0 goes to zero keeping the other coordinates constant. These two pieces together make up the boundary of euclidean AdS into a $(d-1)$-dimensional sphere.[5]

2.1.2 A quick review of scalars in AdS

The action for a massive scalar field in a fixed background spacetime is

$$-\int d^d x \sqrt{g}\, \frac{1}{2} (\partial_\mu \phi\, \partial_\nu \phi\, g^{\mu\nu} + m^2 \phi^2) \,. \tag{2.6}$$

If we assume the background metric is flat Minkowski metric, a scalar with a negative m^2 will not have a stable static configuration. This is not surprizing intuitively, since the equation of motion that follows from the action is the Klein-Gordon equation, which is the long wavelength limit of the equations of motion of coupled pendula

[4]The notation is not meant to imply x^0 is a time coordinate—it is not, not even after Wick rotating back to Minkowski signature.

[5]It is possible to give the conformal compactification in a coordinate system for which the spherical nature of the boundary is more transparent. The coordinate transformation (or the isometry, depending on the point of view) that relates the two forms of the metric will induce a conformal transformation on the boundary, which will be "singular" in that it has to send one of the points of the boundary sphere to infinite distance.

doing small oscillations, and negative m^2 means that the pendula are upside down. It turns out that in AdS, a certain range of negative m^2s are possible while still maintaining stability [49, 50, 51]. (The pendulum argument becomes invalid perhaps because AdS has sectional curvature not only in "spatial" planes, but also in mixed (space/time) planes.) The so-called Breitenlohner-Freedman bound states that the theory on anti-de Sitter space defined by (2.6) has a stable vacuum if and only if

$$m^2 R^2 \geq -\frac{d^2}{4}\,, \tag{2.7}$$

where R is the radius of curvature of AdS. Thus strong negative curvature allows the scalar to have a large negative mass while still preserving stability. Breitenlohner and Freedman arrive at this result by examining the energies[6] of perturbations of the scalar. With the appropriate boundary conditions, the normal modes of the scalar in AdS have finite, positive conserved energies when the bound (2.7) is satisfied, thus guaranteeing stability.

It turns out that for

$$m^2 R^2 > -\frac{d^2}{4} + 1\,, \tag{2.8}$$

there is a single set of normal modes with finite energy, and one can use them to quantize the scalar field in AdS by introducing a Fock space via creation and annihilation operators. However, for

$$-\frac{d^2}{4} < m^2 R^2 \leq -\frac{d^2}{4} + 1\,, \tag{2.9}$$

there are two such sets, corresponding to two different boundary conditions at infinity which, if we think of AdS as living inside Einstein static universe as in (2.3), represent "Neumann" and "Dirichlet" boundary conditions [50, 51]. In this case one can use either set for constructing the Fock space and quantization, i.e., there are two possible quantizations.

[6]AdS has a global timelike vector, which allows us to define a notion of energy.

2.1.3 A precise statement of the correspondence

I will first review the correspondence for a euclidean signature metric, then mention how things change in Minkowski signature.

In the commonly accepted proposal for the precise statement of Maldacena's AdS/CFT correspondence [27, 26], it is assumed that given a field in the bulk theory, there will be a "corresponding" field (or operator) in the boundary theory. This relation is naturally given in the original context of the correspondence motivated by extremal D3-branes, since fields corresponding to open string states that live on the D3-brane will have definite coupling terms to the bulk fields given in string theory, and it is proposed that from these couplings one can read off which bulk field corresponds to which boundary field.[7]

Given the boundary operator $\mathcal{O}$ that corresponds to the bulk field ϕ, the statement of the AdS/CFT correspondence is [8]

$$\left\langle \exp\left(\int_{boundary} \phi_0 \mathcal{O}\right)\right\rangle_{CFT} = Z_S(\phi_0)\,. \tag{2.10}$$

Here, $Z_S(\phi_0)$ denotes the partition function of the bulk theory (which may be string theory or supergravity depending on the version of the AdS/CFT conjecture being used) with the boundary condition that $\phi \to \phi_0$ as one approaches the boundary, the precise meaning of which will be given below. The left-hand side of (2.10) is an expectation value taken in the boundary theory. This shows that the boundary value of the bulk field ϕ behaves as a source for $\mathcal{O}$.

One should note that (2.10) may not make sense in general, since the quantities on the two sides might be divergent. However, one can still use the equation as a formal aid in relating the correlation functions of the two sides of the correspondence.

Let ϕ be a free scalar in the bulk theory (such free fields may not exist in realistic versions of the correspondence, at least due to a coupling to gravity, but this discussion

[7] One might also use arguments relating to the representations of the (super)conformal group (which is the (super)isometry group on the AdS side) in order to identify the bulk field that corresponds to a given boundary field.

[8] I will only consider the case of bulk scalars. For other fields including gauge fields, see e.g., [26, 24].

helps illustrate the main points of the correspondence). The action for the scalar is given by (2.6). For a given configuration ϕ_0 on the boundary, we would like to calculate the bulk partition function with the boundary condition that $\phi \to \phi_0$ as one approaches the boundary. In the classical SUGRA approximation, all we need to do is to calculate the action of the classical solution that satisfies the appropriate boundary conditions, i.e.,

$$Z_S(\phi_0) = \exp(-I_S(\phi_0)) , \tag{2.11}$$

where $I_S(\phi_0)$ is the action of the classical SUGRA solution satisfying the given boundary condition.

Let us now be a bit more precise about the relation between ϕ_0 and ϕ. If we use the representation (2.5), then we might try to identify the boundary value of the field ϕ by

$$\phi_0(\vec{x}) = \lim_{x_0 \to 0} \phi(x^0, \vec{x}) . \tag{2.12}$$

Note that a relation of this form between ϕ and ϕ_0 seems to depend on the choice of the coordinate system. In fact, we could just as well use another representation of the AdS metric, and let the limit on the right-hand side be replaced by $\lim_{r \to r_{boundary}}$ for some other radial variable r for which the AdS boundary is at $r = r_{boundary}$. Taking the "active" transformation point of view, two such representations of the metric will be related by an isometry of AdS, and this will induce a conformal transformation on the boundary, which will in turn relate the two identifications of ϕ_0. We will see below that due to these considerations, the nature of ϕ_0 is a bit nontrivial for the massive case, however, it turns out that the simple minded definition above works nicely for massless scalars in that given ϕ_0, one can find a unique ϕ that satisfies (2.12) (uniqueness can be proven by an integration by parts argument [26]). Let us thus first focus on the massless case, with the boundary asymptotics $\phi \to \phi_0$ meaning (2.12).

Given a $\phi_0(\vec{x}')$, we can obtain a solution $\phi(x_0, \vec{x})$ to the bulk equations with the required asymptotics by using a propagator. The relevant propagator is given in

Poincaré coordinates by

$$G(x_0, \vec{x}; \vec{x}') = c\frac{x_0^d}{(x_0^2 + |\vec{x} - \vec{x}'|^2)^d}, \tag{2.13}$$

where c is a constant. This has the required δ-function behavior at the boundary, as shown in [26]. Using $G(\vec{x}, \vec{x'})$, we can calculate the action $I_S(\phi_0)$ of a solution with given boundary asymptotics, and thus using functional derivatives of (2.10) we can get correlators of the boundary field $\mathcal{O}$.[9]

We see that the "region of influence" at a fixed x_0 of a boundary point $\vec{x}'$ gets larger as we go farther from the boundary (i.e., to bigger x_0). Likewise, a point $(x_0, \vec{x})$ is affected by a region in the boundary whose size grows as x_0 gets bigger. These facts plays an important role in the identification of large distances in the bulk theory with small distances in the boundary theory, which will be reviewed in more detail in section 2.2.

As stated above, things work out a little less smoothly for a massive scalar field. We will be interested in scalars with $m^2 \leq 0$, since these will be the ones that are well behaved near the boundary (which is related to their being dual to relevant perturbations of the boundary theory). For a massive scalar, solutions well behaved in the interior of AdS do not approach nonzero and finite values as $x_0 \to 0$, thus making the simple identification (2.12) impossible. Namely, the boundary behavior of ϕ is [52]

$$\phi(x_0, \vec{x}) \sim x_0^{d-\Delta}\left(\phi_0(\vec{x}) + O(x_0^2)\right) + x_0^{\Delta}\left(A(\vec{x}) + O(x_0^2)\right), \tag{2.14}$$

where Δ and $d - \Delta$ are the two roots of $\Delta(\Delta - d) = m^2R^2$, i.e.,

$$\Delta_\pm = \frac{d}{2} \pm \sqrt{\frac{d^2}{4} + m^2R^2}. \tag{2.15}$$

For AdS/CFT applications, ϕ_0 will be the source function, and A will be the "physical fluctuation" in the boundary theory, once the proper choice for Δ has been made [52].

[9] In practice the correlation functions have not been calculated directly in this way, but –just as in field theory– the rules for the so-called "Witten diagrams" (i.e., Feynman diagrams with external legs at the boundary of AdS) have been obtained by this method. I will return to this below.

Negative m^2 makes it impossible to find a solution to the scalar equations of motion with the boundary condition $\phi \to \phi_0$ if ϕ_0 is nonzero, as both solutions to (2.15) are positive, so ϕ vanishes at the boundary due to (2.14). Ignoring A for the moment, a way to define nonvanishing boundary values for ϕ is as follows. In order to isolate the zero of ϕ near $x_0 = 0$, we choose a function f that vanishes near the boundary as $f \sim x_0$, and then we factor out $f^{d-\Delta}$ from ϕ. The remaining part will be nonzero in general, and will be the boundary source coupling to the field dual to ϕ.

Factoring out the zero in order to identify the source ϕ_0 at the boundary is a bit like factoring out an infinity in the metric as one approaches the boundary, in order to define a conformal structure at the boundary. In the case of ϕ, the boundary field obtained (ϕ_0) is a "conformal density" that transforms under conformal transformations[10] with a particular "conformal weight." We can read off from (2.14) that ϕ_0 will transform under $f \to \tilde{f} = e^w f$ as $\phi_0 \to \tilde{\phi}_0 = e^{(d-\Delta)w}\phi_0$, hence the conformal weight of ϕ_0 is $(d - \Delta)$. The boundary operator dual to ϕ which couples to ϕ_0 as in (2.10) therefore has conformal dimension Δ.

As in the discussion of stability and quantization of scalars in AdS, the cases $-\frac{d^2}{4} < m^2R^2 \leq -\frac{d^2}{4} + 1$ and $-\frac{d^2}{4} + 1 < m^2R^2 < 0$ have different characteristics [53, 52]. For $-\frac{d^2}{4} + 1 < m^2R^2 < 0$, one needs to choose the larger root in (2.15), and thus ϕ_0 will give the leading behavior of ϕ near the boundary. In order to obtain ϕ for a given ϕ_0, one constructs the massive version of the boundary to bulk propagator (2.13).

When $-\frac{d^2}{4} < m^2R^2 \leq -\frac{d^2}{4} + 1$, either of the roots in (2.15) can be used in (2.14), i.e., there are two possible dimensions for the boundary field $\mathcal{O}$ that corresponds to ϕ, a fact closely related to the existence of two inequivalent quantizations in AdS for this range of masses. A and ϕ_0 exchange roles when we change the choice of Δ. Which root of (2.15) to use will depend on the case at hand, see [53, 52] for a discussion.

Once one has the boundary to bulk propagators, it is possible to use them in (2.10)

[10] Using a different f in identifying ϕ_0 through a transformation like $f \to \tilde{f} = e^w f$ is analogous to a conformal transformation $\tilde{ds}^2 = e^{2w} ds^2$

at least formally and get expectation values and correlation functions by functionally differentiating with respect to the source and setting the latter to zero afterwards. As mentioned in a previous footnote, this is not the way such correlation functions were calculated in general—one derives the Feynman rules for Witten diagrams from the generating function, and calculates the correlation functions perturbatively, using tree-level diagrams in the case of classical supergravity. This turned out to be a long ordeal with subtleties. The general program was began for scalars and gauge fields in [54] (see also [55]), where it was also pointed out that extra care is needed when calculating the correlation functions of two-point functions of scalars, in order to get the numerical factor right to make the Ward identities satisfied.[11]. In [56], an investigation of four-point functions of operators dual to the axian/dilaton sector was begun. A series of papers derived relevant fluctuation equations and exchange amplitudes: gauge boson exchange between scalars in [57], scalar exchange in [58], graviton and gauge boson propagator in AdS [59], the graviton exchange amplitudes and the final result for the four-point amplitudes of the chiral primaries dual to the dilaton/axion sector in [60].

In a hypothetical calculation of correlation functions by taking the functional derivatives of (2.10), one can, in principle leave the source function nonzero rather than setting it to zero. This would correpsond to a deformation of the boundary theory by a source term, and if ϕ_0 is a nonzero *constant*, it would be a coupling constant for the perturbing operator $\mathcal{O}$. Such a deformation would in general break the conformal invariance of the boundary theory, and generate a renormalization group flow. Thus, the correlation functions obtained in a hypothetical calculation like this will be those of a non-conformal field theory, with a nontrivial RG-flow. Real calculations of correlation functions in RG-flows in AdS/CFT are not as straightforward as this simplistic summary might seem to suggest, as will be mentioned in section 2.2.

Note that a nonzero source term could also give a nonzero vacuum expectation value to $\mathcal{O}$, the boundary field dual to ϕ. This expectation value turns out to be related to A [52]. In fact, it is also possible to investitage states with nonzero ex-

[11] An alternative derivation of the relevant factor is given in [52].

pectation values of operators of pure $\mathcal{N} = 4$ SYM theory from the SUGRA side. A nonzero expectation value will also break the conformal invariance of the theory, and will be represented by a SUGRA solution that has a radial dependence. See section 2.2.1 for further details and references.

Finally, let us mention the case of Minkowski signature. The Minkowski version of (2.10) is

$$Z_S(\phi_0) = \langle T e^{i \int_{boundary} \phi_0 \mathcal{O}} \rangle \,, \tag{2.16}$$

where T denotes time ordering. It turns out that in AdS proper (i.e., with Minkowski signature rather than euclidean), fixing the leading boundary behavior of ϕ doesn't uniquely fix the bulk field—there is a "non-normalizable" part of ϕ fixed by ϕ_0, but there is also a set (or two sets, depending on the mass range) of "normalizable" modes that we are free to turn on, without affecting the leading boundary behavior [53]. These normalizable modes are precisely the ones used in [49] for quantizing the scalar field on AdS, and they represent the freedom of specifying the *state* of the boundary theory. Thus, we can use the non-normalizable modes to turn on perturbations of the bulk theory, and use the normalizable ones to specify the state, both of which will in turn contribute to the expectation value of the corresponding boundary operators. These expectation values can be calculated by the methods explained above.

2.2 RG-Flows in AdS/CFT

2.2.1 A review of RG-flows in AdS/CFT

The existence of a relation between long radial distances in the gravity theory and short distances in the Yang-Mills dual has been known since the earliest works on the AdS/CFT correspondence. For example, Maldacena mentions in [8] that the "radial" position of a string in AdS will be related to the "mass" of the corresponding configuration in the boundary theory, and that a radial motion in AdS corresponds to a motion in energy scales of the boundary theory, UV regime of the boundary theory corresponding to the large AdS radii,[12] and the IR regime correponding to small radii. While giving his proposal for the precise specification of the correspondence in [26], Witten notes that the divergences due to the infinite volume of AdS that occur in the integrals of classical SUGRA theory will be the bulk versions of the ultraviolet divergences of the boundary theory, and that these can be removed by the addition of local counterterms to the gravity action, mimicking the renormalization process of the gauge theory. This suggests that integrating in the radial direction while evaluating the classical action in (2.11) roughly corresponds to integrating over momenta in the corresponding gauge theory path integral.

Witten and Susskind have further elaborated the relation between the length scales of the bulk and the boundary theories in [61], and demonstrated that the short distance cutoff in calculating a two-point correlation function of a scalar field corresponds to a large distance cutoff in calculating the same quantity in the AdS side. They also demonstrated that the divergence of the energy of an infinite string that asymptotes to two points on the boundary is related to the divergence of the self energies of the point charges at the endpoints of the string—the point charges that the string corresponds to in the boundary theory. The distinction between these two types of "UV/IR" relations, one related to the cutoffs for calculations of correlation functions and the other related to the characteristic energies of probes has been more explicitly demonstrated in [62, 63].

Various groups have considered deformations of the original correspondence between $D = 4, \mathcal{N} = 4$ SYM and IIB SUGRA on $AdS_5 \times \mathbf{S}^5$. By using the identification of superconformal symmetries of both sides and representations of the superconformal group, one can guess which fields in the SUGRA correspond to which operators in the SYM theory. As mentioned in section 2.1.3, by turning on appropriate bulk fields one can investigate the SUGRA duals of deformed versions of $\mathcal{N} = 4$ $D = 4$ SYM theory, or its states with expectation values for the fields. Such deformations/expectation values in general will not have conformal invariance and will have nontrivial RG-flows.

[12]actually, one should say "radii near the boundary," which of course might be "small radii," depending on the coordinate system used, see, e.g., (2.5).

The relation between the length scales of the bulk and boundary theories mentioned above suggests that these flows are represented in the SUGRA theory by backgrounds in which the SUGRA fields have nontrivial radial dependences.

In practice, many of the explorations of RG-flows in AdS/CFT have not been made by deforming the IIB SUGRA background on $AdS_5 \times \mathbf{S}^5$, but by using a certain reduction of the ten-dimensional theory to 5 dimensions, the main reason being the technical difficulties in working with the full spectrum of IIB SUGRA on the 5-sphere. The five-dimensional theory in question is $\mathcal{N} = 8$, $D = 5$ gauged SUGRA. This theory does not include the full Kaluza-Klein tower of IIB SUGRA compactified on $\mathbf{S}^5$, but is believed to be a so-called "consistent truncation" of IIB SUGRA on $\mathbf{S}^5$. This means that even though an arbitrary solution of the parent theory may not be represented as a solution of the dimensionally reduced theory, one can in principle *lift* any solution of the five-dimensional theory to a solution of the 10-dimensional theory. Note that this is more than a simple Kaluza-Klein truncation of harmonics, the lifted solutions are *exact* solutions of the parent theory, not just approximate ones. [13] Such lifts have in fact been performed for various RG-flows, and have helped clarify the interpretations of some of the five-dimensional solutions given in the literature.

$\mathcal{N} = 8$, $D = 5$ gauged SUGRA has 42 scalars, parametrizing the coset space $E_{6(6)}/USp(8)$[14]. The potential of these scalars is invariant under the gauged subalgebra $SO(6)$ of $E_{6(6)}$. Typically in the AdS/CFT RG-flow literature, attention is restricted to a submanifold of the scalar manifold in a manner consistent with the equations of motion. The relevant part of the SUGRA action is given by

$$S = \int \sqrt{g} \left[R - \frac{1}{2} \partial_\mu \phi^i \partial_\nu \phi^j \mathcal{G}_{ij}(\vec{\phi}) - V(\phi) \right] d^5 x \,, \tag{2.17}$$

where $\mathcal{G}_{ij}$ is the metric on the scalar manifold. In many cases one ends up with a potential that is a linear combination of exponential terms, as in

$$V(\vec{\phi}) = \sum_n \alpha_n e^{\vec{\varsigma_n} \cdot \vec{\phi}} \tag{2.18}$$

[13]See [64] for an example of consistent truncation for the case of $D = 11$ SUGRA on $\mathbf{S}^7$.

[14]Or more precisely, the space of scalars of the theory is the 42-dimensional coset manifold $E_{6(6)}/USp(8)$.

where α_n is a constant, $\vec{\zeta_n}$ is a constant vector, and $\vec{\phi}$ denotes the scalars collectively. The potentials that come up in the literature are sometimes given in terms of "superpotentials" through relations of the form

$$V \sim \left|\partial_{\vec{\phi}} W\right|^2 - W^2(\vec{\phi}) \,, \tag{2.19}$$

where I suppressed constant factors. Relations of this form have been obtained directly from supergravity, giving specific Ws. However, relations of the form (2.19) have been useful in other contexts also, some unrelated to supersymmetry. We will return to this issue in section 2.2.3.

The RG-flows of field theories on four-dimensional Minkowski spacetime are represented by SUGRA solutions with four-dimensional Poincare symmetry and a "radial" dependence. Namely, they take the following form in a suitable choice of coordinates:

$$\begin{aligned} ds^2 &= e^{2A(r)} dx^\mu dx^\nu \eta_{\mu\nu} + dr^2 \\ \phi^i &= \phi^i(r) \,. \end{aligned} \tag{2.20}$$

Assuming for simplicity that $\mathcal{G}_{ij} = \delta_{ij}$ in (2.17), the equations of motion for the ansatz (2.20) are given by

$$\begin{aligned} 6\,(A')^2 &= -\frac{1}{2} V(\vec{\phi}) + \frac{1}{4} |\vec{\phi}'|^2 \\ \phi_i'' + 4A'\,\phi_i' &= \frac{\partial V}{\partial \phi_i} \,. \end{aligned} \tag{2.21}$$

The first equation is the Einstein equation $G^r_r = T^r_r$, and the second one is the generic scalar equation of motion. All the remaining nontrivial Einstein equations are identical, and can be obtained by differentiating the $G^r_r = T^r_r$ equation and combining with the scalar equations. For future reference, I'll also quote the equation $G^t_t - G^r_r = T^t_t - T^r_r$:

$$3A'' = -\frac{1}{2} |\vec{\phi}'|^2 \,. \tag{2.22}$$

This equation tells us that $A(r)$ is concave down, i.e., $A'' \leq 0$. This is the basis of the holographic version of the so-called "c-theorem" [65, 66, 67], which states that a certain quantity (in this case A') is monotonic throughout the RG-flow. It is possible to generalize this a bit; $A'' < 0$ holds whenever the energy-momentum tensor of the scalars satisfies a certain energy condition called the "weaker energy condition". A' is related to the conformal anomaly of the boundary theory [67], see [66].

Holographic RG-flows are induced by either deforming the boundary theory (see, e.g., [65, 68, 69, 70, 66, 71, 72, 73, 74, 75, 76, 77, 78, 79]) or by giving expectation values to boundary operators (e.g., [80, 81, 82, 83, 52, 84, 85, 86, 86, 87, 88, 89]). One can in principle calculate correlation functions for a given RG-flow by considering fluctuations of the background fields, and evaluating Witten diagrams. The actual calculations, however, are riddled with technical difficulties and subtleties. Some of the papers cited above have such calculations, and others have approached the issue in a more general setting. [90] investigates the fluctuations of scalars and the metric around flow-type solutions, but encounters problems with correlation functions calculated from these fluctuations. These problems are resolved in [91, 92, 93, 94, 95] by using the formalism of holographic renormalization [96, 97]. For other work on correlation functions in holographic RG-flows, see e.g., [98, 99, 100].

The fact that the SUGRA description can be trusted at strong 't Hooft coupling and perturbative field theory calculations are expected to give good approximations only for weak coupling has made it difficult at times to compare the two sides of the duality for the flow solutions, but in some cases it has been possible to compare some robust expectations such as the behavior of Wilson loops calculated using the formalism in [101, 102], or those protected by supersymmetry.

Even though the explicit flows found and the analysis of their fluctuations and correlation functions have given considerable amount of evidence that the radial flow in the SUGRA does indeed correspond to the RG-flow of the dual SYM theory, they leave something to be desired. How exactly does the map between the radial flow and the RG-flow work? Is it possible to derive the equations of the RG-flow (e.g., Callan-Symanzik eqns) from the SUGRA equations or vice versa? Can we really "see" that the integration of the classical SUGRA action up to larger and larger radii corresponds to integrating over larger and larger momenta in the SYM path integral?

These questions, detailed answers to which might give deep insights to the nature of the holographic hypothesis [103, 104] have been attacked by various methods in various papers, giving different insights.

In order to gain some understanding of the nature of RG-flows in AdS/CFT, Porrati and Starinets [105] have focused on the class of 4-D Poincaré invariant SUGRA solutions that represent flows between a UV fixed point and an IR fixed point. The metric of such solutions asymptotes to that of AdS space at both ends (the UV end is a true AdS boundary), possibly with different radii of curvature. The authors of [105] have shown that for such solutions, in the UV(IR) limit, correlation functions depend only on the UV(IR) region of the metric. They also mention that it might be possible to relate the radial equations of motion of the SUGRA theory to the RG-flow equations for the correlation functions (even though the former equations are second order), and they show an example of such a relation for the two point function.

A more satisfactory result would be to obtain first-order differential equations at the SUGRA side which give the radial dependence of general correlation functions of the SYM theory, i.e., the SUGRA duals of Callan-Symanzik equations of the boundary SYM theory. In [106], Balasubramanian and Kraus have derived equations that resemble the Callan-Symanzik equations by utilizing propagators in AdS. Their approach can be summarized as follows. In order to calculate correlation functions for a deformed theory, one needs to utilize the formula (2.10) by setting the value ϕ_0 of the bulk field ϕ at the boundary. If we do this boundary not to be at infinity, but "close" to it, i.e., at some large radius, this would be analogous to to setting the coupling constants at some energy scale in the boundary theory. Now, if we have a propagator that relates the values of the fields between two different radii, we can use it to translate the values of the bulk fields given at some interior radius to those at some exterior radius where we might cut off the radial integration. This is similar in spirit to the specification of the coupling constants of a quantum field theory at a given energy scale, and relating these values to the corresponding ones at the UV cutoff scale (except the coupling constants are now promoted to fields, which can nevertheless be set to constant values). One can thus represent the correlation functions found from the AdS side in terms of the values of the fields either at the inner radius or the outer radius, the relation between the two representations being given via the Green's functions. Equating these two versions and collecting all the dependence on the inner radius to one side of the equation, the other side (and hence both sides) must be independent of the latter. Again, this is very much in the spirit of the derivation of Callan-Symanzik equations. Sending the cutoff (the "large," or exterior radius) to the boundary of AdS, one gets the holographic Callan-Symanzik equations. The authors of [106] also remark how cool it would be if we could start with the RG-flow of a given QFT and get a "natural" metric on a space formed by the whole flow, hence obtaining a spacetime from any given RG-flow in a natural way. See [96] for related work.

Another derivation of "holographic Callan-Symanzik equations" has been given by de Boer, and Verlinde^2 [107], who use the hamiltonian formulation of general relativity (with the radial coordinate playing the role of time) to derive a radial evolution equation for the correlation functions.

Their argument goes roughly as follows. The SUGRA action for a solution with the values of the fields specified at a given radius ρ_0 will be a nonlocal functional of the fields at ρ_0—which they should, in order to give nontrivial holographic correlation functions when one takes the functional derivatives of (2.10). The action will nevertheless have some local part, which will contain terms similar in form to the lagrangian. Substituting this separation to local and nonlocal parts into the "Hamilton-Jacobi constraint" (which is a functional differential equation for the action, derived from the hamiltonian constraint of gravity), one gets an equation satisfied by the nonlocal terms. Functional differentiation turns this into an equation satisfied by the correlation functions. After getting rid of the divergences that come up as one sends the cutoff of the radial integration to the boundary, one gets the holographic version of Callan-Symanzik equations as proposed by Verlinde^2 and de Boer. The cancellation of the divergences of the local terms in the Hamilton-Jacobi constraint also allows the authors of [107] to derive a relation between the potential of a scalar in the gravity theory, and a sort of "superpotential," which might in fact be unrelated to supersymmetry. I will come back to the superpotential in section 2.2.3. Finally, for the relation between holographic RG-flows and Polchinski's RG equation, see [108].

2.2.2 Multiple flows between fixed points

The equations of motion (2.21) imply that the fixed points of RG-flows of gauge theories (where the coupling constants are independent of the energy scale) correspond to extrema of V in the SUGRA side. As mentioned in section 2.2.1, solutions have been given in the literature for which the scalars interpolate between two such extrema, representing RG-flows of field theories between UV and IR fixed points. When it is known that an RG-flow preserves some supersymmetry, supersymmetry conditions on the SUGRA side can be used to get first-order equations. For example the flow given in [66] is between the maximally supersymmetric point (the "central maximum" of the potential) of five-dimensional SUGRA which represents the $\mathcal{N} = 4$ SYM theory, and another extremum which is thought to represent another supersymmetric CFT. It is known from the field theory side that some supersymmetry is preserved throughout the RG-flow, which gives rise to first-order gradient flow equations for the scalars.

Given two fixed points, one might wonder whether it is possible to have two distinct flows between them. In particular, given a *supersymmetric* flow between supersymmetric fixed points as in [66], one can explore the possibility of finding other flows between the fixed points; flows which break all the supersymmetry, but recover it asymptotically in the UV and the IR. If found, such flows would be dual to RG-flows with the same property. This would be an interesting example of a symmetry of the UV fixed point of a field theory broken during the RG-flow, and recovered in the IR. In the remainder of this section, I will explore the possibility of such multiple flows between the extrema of some toy-model potentials. These toy models will give insights about what might happen for more realistic potentials.

We can read off from (2.21) that for a flow with $A' \geq 0$[15], the equation of motion

[15] Since reversing the sign of r changes the sign of A', the case we are ignoring by assuming $A' \geq 0$ is one where A' has opposite signs at the two fixed points of the flow. Since $A'' \leq 0$ (c.f. the c-theorem mentioned at the end of section 2.2), this means $A' > 0$ in the IR and $A' < 0$ in the UV.

of the scalars is that of a particle under the influence of the inverted version of the potential V, and a "time-dependent" (role of time being played by the radius) friction coefficient namely A'. Using this analogy, we can make statements about the behavior of the scalars. Let us first focus on the case of a single scalar.

Single scalar

It should be possible to find solutions that asymptote to the maxima of V in the UV, since a maximum of V corresponds to a minimum of the potential in the particle analogy, and UV corresponds to the "far future" according to the conventions I will be using. For very sharp maxima, the convergence will perhaps be oscillatory, and for "smoother" maxima, the friction term will be dominant and the convergence to the maximum will be exponential (these statements will be justified below). This is just a restatement of the analysis of Breitenlohner and Freedman [50, 51], with the regimes of oscillatory and exponential behavior reversed due to the fact that we are considering a "motion" in radius, not in time.

Note that if the initial velocity of the particle is fine-tuned, it might also be possible to shoot it to the top of a hill, which means one might also apporach *minima* of V in the UV. As for the IR fixed point, the fact that a particle can originate near a maximum of a potential in the distant past and move away from it tells us that minima of V can be IR fixed points to which the RG-flow solution asymptotes in the far IR.[16]

Thus, if we have a single scalar with a potential V that has two quadratic extrema, we can find an RG-flow type solution that asymptotes between these extrema if one extremum is a maximum (the UV point) and the other is a minimum (the IR point).[17]

However, for such a solution we do not have an AdS boundary ($A^{UV\prime} > 0$ is needed for the metric to have an AdS boundary at large r).

[16]and that this cannot be in an oscillatory manner, since if a particle starts at an extremum in the infinite past with infinitesimal speed, the dissipation of energy will prevent the particle from passing the same extremum with finite speed.

[17]It might also be possible to find a flow between two minima, but this would require a fine tuning of the potential, just as it could be possible to send a particle from one hilltop in the infinite past

A linearized stability analysis near a quadratic extremum of V confirms these expectations. Near an extremum p ($\phi = \phi^{(p)}$),

$$V(\phi) \approx V_0^{(p)} + \frac{m^{2(p)}}{2}(\delta\phi)^2\,, \tag{2.23}$$

and the linearized equations of motion are

$$\begin{aligned} 6(A'^{(p)})^2 &= -\frac{1}{2}V_0^{(p)} \\ \delta\phi'' + 4A'^{(p)}\,\delta\phi' &= m^{2(p)}\delta\phi\,, \end{aligned} \tag{2.24}$$

where $\delta\phi = \phi - \phi^{(p)}$, $A'^{(p)}$ is the value of A' at the fixed point p, $V_0^{(p)} = V(\phi^{(p)})$, and $m^{2(p)} = \frac{\partial^2 V}{\partial \phi^2}\Big|_{\phi=\phi^{(p)}}$. The value of V has to be negative at the two endpoints of the flow so that the metric asymptotes to that of AdS (with different radii) at the two ends. Also, I will assume $A' > 0$ throughout the flow (see footnote on the previous page).

The exact solution to the linearized equations is

$$\begin{aligned} \delta\phi =& C_1 \exp\Big[\frac{1}{2}\Big(- 4A'^{(p)} - \sqrt{(4A'^{(p)})^2 + 4m^{2(p)}}\Big)r\Big] \\ &+ C_2 \exp\Big[\frac{1}{2}\Big(- 4A'^{(p)} + \sqrt{(4A'^{(p)})^2 + 4m^{2(p)}}\Big)r\Big]\,. \end{aligned} \tag{2.25}$$

The Breitenlohner-Freedman bound is satisfied if and only if the term in the square-roots is positive, in which case the solutions will be purely exponential rather than oscillatory. $A'^{(p)} > 0$ ensures that there is at least a one parameter family of solutions (those with $C_2 = 0$) that converge to the fixed point p as $r \to \infty$. These are not physically distinct solutions however, since the system is symmetric under shifts in r, and for $C_2 = 0$, $r \to r + r_0$ corresponds to $C_1 \to C_1 e^{\lambda_- r_0}$, i.e., the members of the one-parameter family of solutions can be obtained from one another by translations in r. If p is a *maximum* of the potential, i.e., if $m^{2(p)} < 0$, there will be a *two*

to another in the infinite future, if the heights of the hills are adjusted properly. I will not consider such fine-tuned cases here.

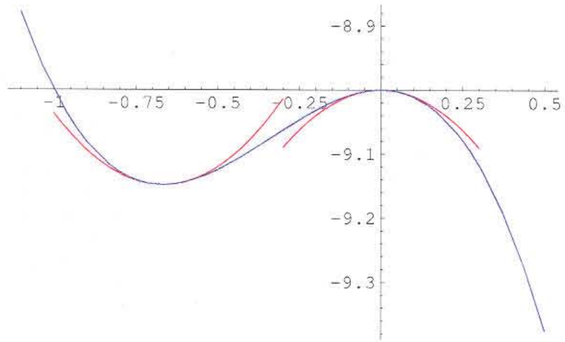

Figure 2.1: $V(\phi)$ vs. ϕ for a V that is a third-order polynomial, superimposed with the quadratic approximations to V near its extrema. The coefficients of the polynomial have been chosen in a way that makes sure the Breitenlohner-Freedman bound is satisfied at the maximum.

parameter family of solutions that converge to p as $r \to \infty$, as expected from the particle analogy (given a parabolic potential, the particle ends up at the minimum, independent of the initial conditions).

If we want to have a solution that converges to p in the IR, i.e., as $r \to -\infty$, then for $A'^{(p)} > 0$ we can only find a one parameter family of such solutions, and these exist when the quadratic extremum is a minimum, i.e., $m^{2(p)} > 0$. Matching this family of IR solutions to the UV, we can hope to get a one parameter family of flows interpolating between the two extrema, but these flows will be related by r-translations.

A potential that allows a flow between its extrema is shown in figure 2.1. A numerical attempt in finding the flow between the extrema is shown in figure 2.2. Numerical integration has been started at the IR end, since the numerics is stable to small errors in the initial conditions there. The IR end of a bunch of flows with inital conditions that are close to those used for the solution given in figure 2.2 are shown in figure 2.3.

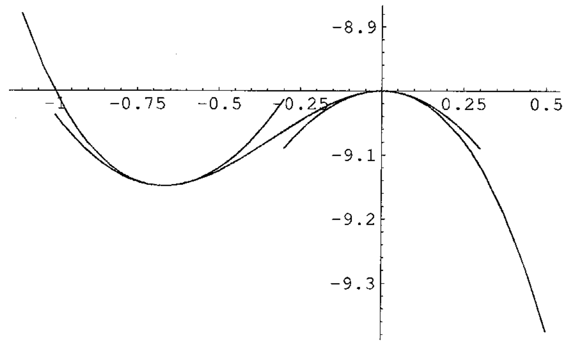

Figure 2.1: $V(\phi)$ vs. ϕ for a V that is a third-order polynomial, superimposed with the quadratic approximations to V near its extrema. The coefficients of the polynomial have been chosen in a way that makes sure the Breitenlohner-Freedman bound is satisfied at the maximum.

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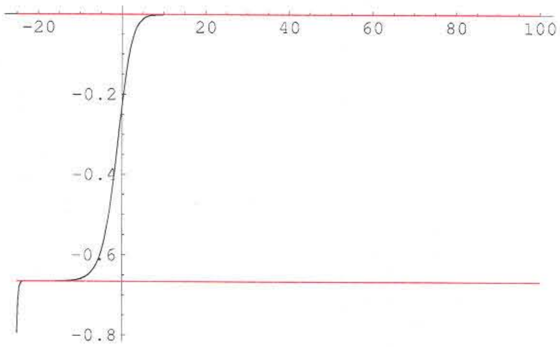


Figure 2.2: A numerical attempt in finding the solution between the minimum in the IR and maximum in the UV. The imperfect numerics means we start to pick up the exploding exponentials in the IR. The IR asymptotics have been used as initial conditions at $r = -20$. The horizontal lines denote the extrema of the potential.

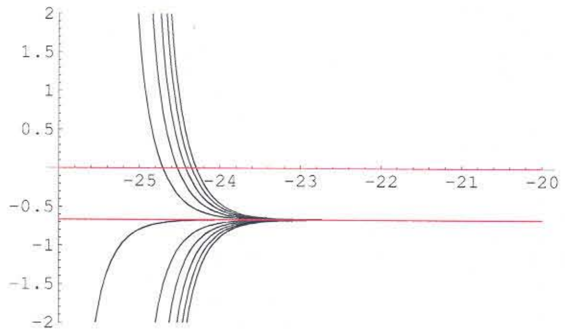
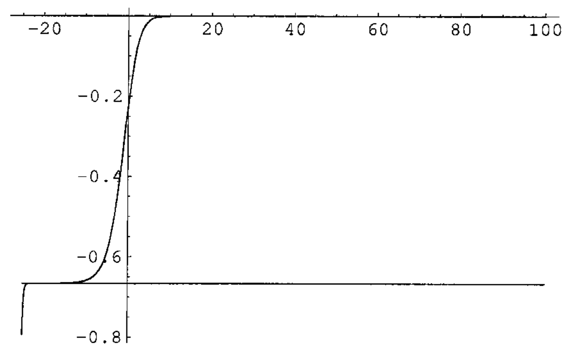


Figure 2.3: The IR end of a bunch of flows whose initial conditions are close to the ones used for the solution in figure 2.2 (namely, $\phi(-20)$ was fixed, and $\phi'(-20)$ was varied). The horizontal lines denote the extrema of the potential. These plots seem to suggest that the exact initial conditions required for a flow between the fixed points is in the vicinity of the initial conditions used here.



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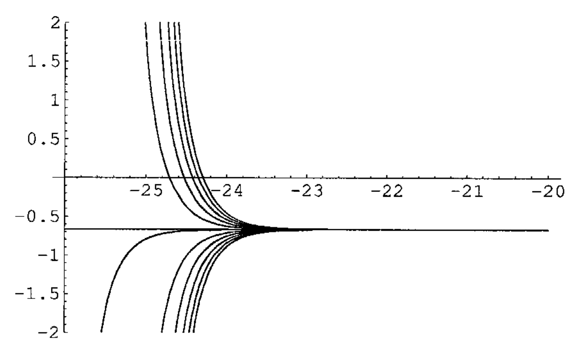

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Two scalars

The preceding analysis strongly suggests that for the case of a single scalar, there is essentially a single flow between a minimum of V in the IR and a maximum in the UV. It also suggests how one might find more than one flow between the critical points of a potential of *two* scalars. One class of potentials that could give such multiple flows consists of those that have a local maximum and a local minimum near each other. Perhaps there would be a range of directions at the minimum in which the scalars could start in the IR and end up in the nearby minimum in the UV. An example of such a potential is

$$V(\phi_1, \phi_2) = -b - \alpha\phi_1 + (\beta\phi_1^2 + \phi_2^2 - c)^2 . \tag{2.26}$$

(I will assume that all the constants b, c, α and β and are positive.) This is basically a tilted "sombrero" (Mexican hat) potential: the term $-\alpha\phi_1$ will be used to give a tilt to the hat, so that it has one absolute minimum and a local maximum, to which RG-flow solutions can asymptote in the UV and the IR, respectively (a third extremum will turn out to be a saddle point for the particular choice of parameters that will be used below). β will be used to stretch the hat a bit so that we can set the eigenvalues of the mass matrix at the minimum to be equal (more on this later). b will be chosen so as to make sure that the Breitenlohner-Freedman bound is satisfied at the max. Finally, c determines the "size" of the hat. A plot of V for the choice of parameters used below for numerical calculations is given in Figure 2.4.

As mentioned above, we hope to find flows that originate at the minimum of the potential in the far IR and end at the maximum in the far UV. As in the single-scalar case, we will perform a numerical search for such solutions by getting the initial conditions from the IR asymptotics, and integrating numerically from the far IR to the far UV—once again, the numerics will be more stable with respect to small changes in the initial conditions if we start integrating at the IR end. The IR asymptotics of the scalars will be described by the equations

$$\delta\phi_i'' + 4A'^{IR}\,\delta\phi_i' = (m_i^{IR})^2\delta\phi_i\,, \tag{2.27}$$

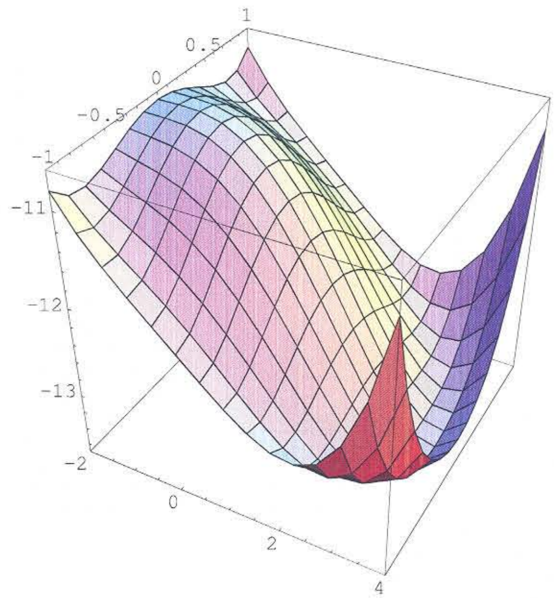
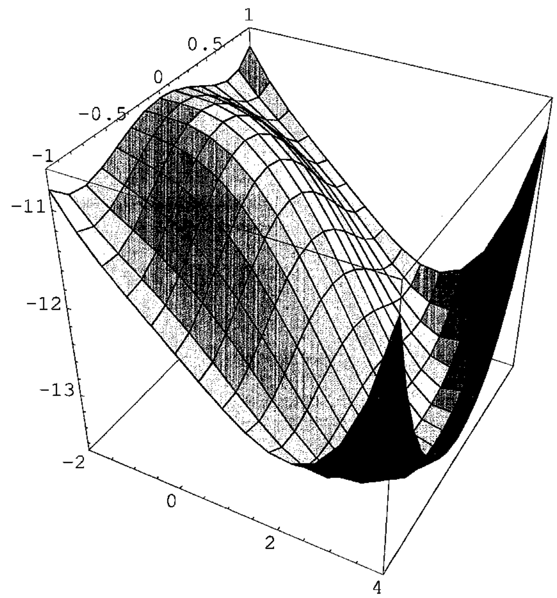


Figure 2.4: A plot of the "tilted sombrero" potential. ϕ_1 runs from -2 to 4 and ϕ_2 runs from -1 to 1.

where $(m_i^2)^{IR} = \left(\frac{\partial^2 V}{\partial \phi_i^2}\right)^{IR}$ with $i = 1, 2$, and A'^{IR} is the value of A' at the IR fixed point, obtained from the linearized Einstein equation

$$6(A'^{(IR)})^2 = -\frac{1}{2} V_0^{(IR)} \,. \tag{2.28}$$

We would like to choose the parameters of the potential in a way that gives a two parameter family of solutions that converge to the IR fixed point as $r \to -\infty$. While we should probably be able to do this with $(m_1^{IR})^2 \neq (m_2^{IR})^2$, the asymptotic analysis might be a bit more intricate in that case: if the sizes of the two scalars are not of the same order near the fixed point, then one might need to include orders higher than linear in order to perform a consistent asymptotic analysis (e.g., if $\delta\phi_1 \sim e^r$ and $\delta\phi_2 \sim e^{2r}$ as $r \to -\infty$, $\delta\phi_2$ and $(\delta\phi_1)^2$ are of the same order). In order to avoid such a complication, I will choose the parameters in V in a way that ensures $(m_1^{IR})^2 = (m_2^{IR})^2$. The choice of parameters I used for numerics is



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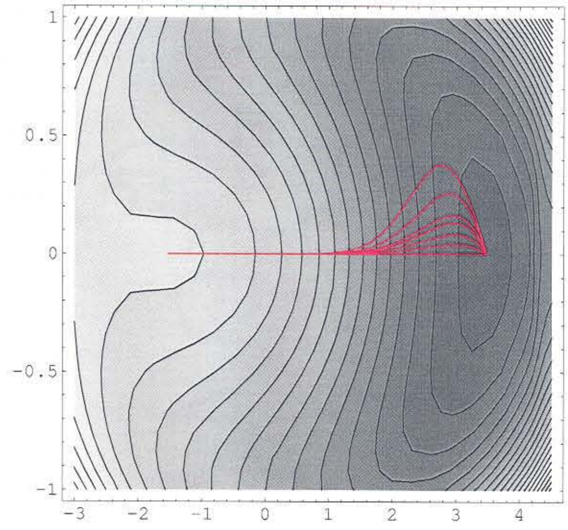


Figure 2.5: Various flows between the critical points of the potential. The horizontal axis is ϕ_1, the vertical axis is ϕ_2.

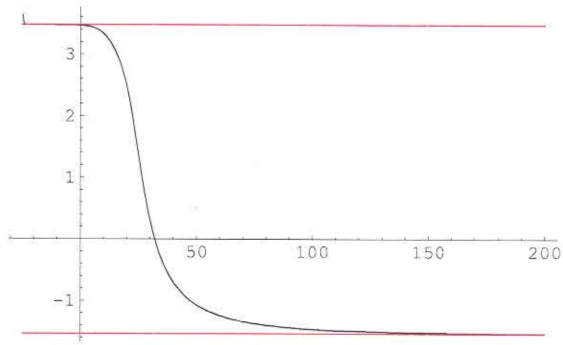


Figure 2.6: Typical $\phi_1(r)$ vs. r.

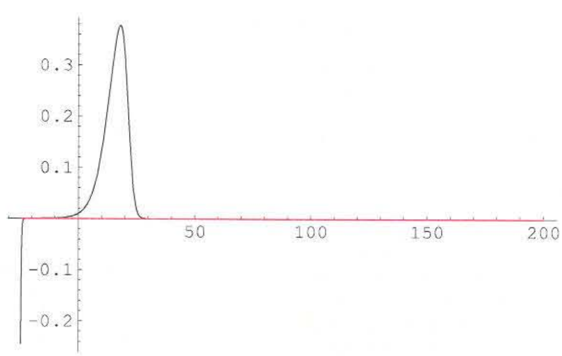
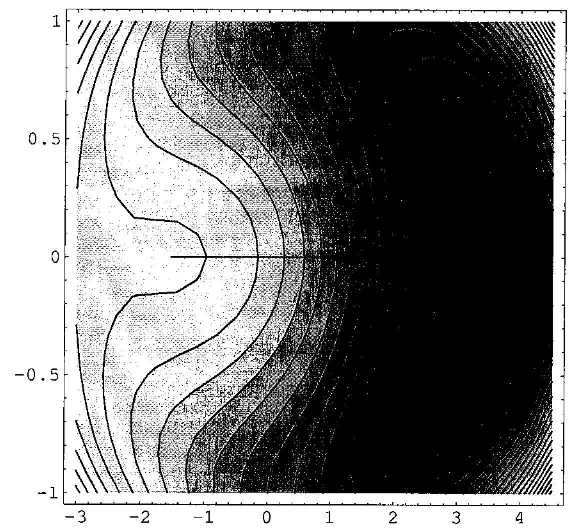


Figure 2.7: Typical $\phi_2(r)$ vs. r.



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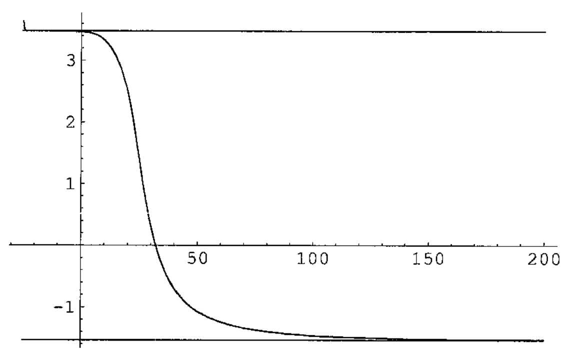

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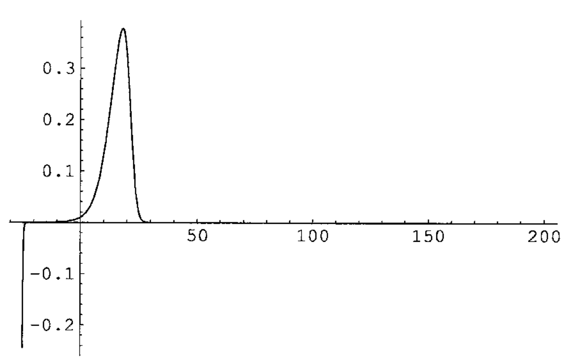

Figure 2.7: Typical $\phi_2(r)$ vs. r.

$$
\begin{aligned}
b &= 12 \\
c &= 1 \\
\alpha &= 1/2 \\
\beta &\approx 0.1098 \,.
\end{aligned} \tag{2.29}
$$

With this choice of parameters, the Breitenlohner-Freedman bound is satisfied at the max of the potential, and the potential has 3 extrema; a maximum, a minimum, and a saddle point at

$$
\begin{aligned}
\vec{\phi}^{UV} &\approx (-1.538, 0) \\
\vec{\phi}^{IR} &\approx (+3.477, 0) \\
\vec{\phi}^{saddle} &\approx (-1.939, 0) \,,
\end{aligned} \tag{2.30}
$$

respectively. The mass matrix (i.e., the Hessian of V) is diagonal at both UV and IR fixed points, and its eigenvalues are given by

$$(m_1^{IR})^2 \approx 1.31 \qquad (m_2^{IR})^2 \approx 1.31 \,, \tag{2.31}$$

and

$$(m_1^{UV})^2 \approx -0.097 \qquad (m_2^{UV})^2 \approx -2.96 \,. \tag{2.32}$$

Plugging the value of A'^{IR} in the linearized equations of motion (2.27), we get the solution

$$\delta\phi_i^{IR} = a_i e^{\eta_1 r} + b_i e^{\eta_2 r} \,, \tag{2.33}$$

where $\eta_1 \approx -4.55$ and $\eta_2 \approx 0.288$. Setting $a_i = 0$ for $i = 1, 2$ gives a two parameter family of solutions that converge to the IR as $r \longrightarrow -\infty$. Getting the initial conditions from this IR asymptotics at a finite r and integrating the full equations of motion numerically to find solutions that converge to the UV fixed point as $r \longrightarrow \infty$, I found the samples given in Figure 2.5. Figures 2.6 and 2.7 show sample plots of ϕ_1 and ϕ_2. The imperfectness of the numerics shows up at the IR end of these plots as the solutions begin to pick up the exploding exponentials.

This example of multiple flows involved a maximum and a minimum that are neighbors. This suggests that multiple flows between fixed points could perhaps be a somewhat generic phenomenon, at least if suitable neighboring maxima and minima occur generically in the scalar potentials that come up in AdS/CFT.

The particle analogy helps us guess what other types of extrema could be UV and IR fixed points of a family of RG-flows. For example, a fine-tuning of the potential should be necessary to find multiple flows between two minima. Another possibility is non-unique flows between extrema that are inflection points in one or both directions. For example, a max in the UV and a "double inflection point" (i.e., a point around which the potential resembles the function $f(x,y) = x^3 + y^3$) in the IR, or a max in the UV and a point that is an inflection point in one direction and a min in the other direction in the IR should be good candidates for giving multiple flows, without the need for fine tuning.

Looking for neighboring supersymmetric minima and maxima of the scalar potential of $\mathcal{N} = 8$, $D = 5$ gauged supergravity that would allow multiple flows in the manner demonstrated above could be an interesting project which could give insights about the dual quantum field theories.

2.2.3 "Superpotential" without supersymmetry: a first-order formalism

As mentioned in section 2.2, in many of the holographic RG-flow geometries that appear in the literature, the potential for the "active scalars" was naturally given in terms of a "superpotential" by a relation of the form $V \sim W'^2 - W^2$. More precisely, for the D-dimensional action ($D = 5$ for many of the AdS-CFT applications)

$$S = \int d^D x \sqrt{g} \left[R - \frac{1}{2}(\partial\vec{\phi})^2 - V(\vec{\phi}) \right] , \tag{2.34}$$

the potential is given by

$$V = \frac{1}{2} \left| \partial_{\vec{\phi}} W \right|^2 - \frac{1}{4} \frac{D-1}{D-2} W^2 . \tag{2.35}$$

The "superpotential" W comes up rather naturally in the context of supergravity. For example, in [66], attention is restricted to a submanifold of the scalar manifold of $\mathcal{N} = 8$, $D = 5$ gauged supergravity theory, and it turns out that the role of W is played by eigenvalues of a tensor of $USp(8)$, an invariance group of the supergravity theory.

In cases where it is known that the RG-flow solution in question is known to preserve some supersymmetry, it has been possible to obtain first-order equations from the supersymmetry conditions. Namely, for some solutions of the form

$$\begin{aligned} ds^2 &= e^{2A(r)} \left(dt^2 + d\vec{x}^2\right) + dr^2 \\ \vec{\phi} &= \vec{\phi}(r) \, , \end{aligned} \tag{2.36}$$

the supersymmetry conditions turn out to be equivalent to

$$\vec{\phi}' = W_{\vec{\phi}} \, , \qquad A' = -\frac{1}{2(D-2)} W \, . \tag{2.37}$$

It can be easily checked that for a W that is related to V as in (2.35), these equations guarantee that the equations of motion (2.21) are satisfied. This might suggest the following general method for generating flow-type solutions. Suppose for a given V, rather than using the W obtained from supergravity, we treat (2.35) as a *differential equation* for W (an ODE if there is just a single scalar, a PDE otherwise), and use Ws obtained from this equation in (2.37) to generate flow solutions. As in the supersymmetric case, the solutions to the first-order equations (2.37) will automatically be solutions to the scalar equations of motion and the Einstein equations.

This method and some of its generalizations were utilized in [109] in the context of brane world scenarios. A parameter counting suggests that it might be possible to obtain all flow type solutions in this manner [11]. In [88], the same method was used in the exploration of flow (or "kink") type solutions to supergravity, and it is related to a sort of Bogomolnyi method by completing squares. A similar approach is taken in [110], which also mentions old work on the stability of vacua of gauged supergravity theories. For the case with a single scalar, [110] says that (2.35) is a necessary and sufficient condition for the stability of an extremum of the potential.

In [107], the potential/superpotential relation is derived in a different and rather interesting way during the derivation of holographic Callan-Symanzik equations. The requirement of cancellation of the divergences of the SUGRA action in its original form and a form separated into "local" and "nonlocal" terms gives a relation of the form (2.35). The first-order equations (2.37) show up in this work as Hamilton's equations of scalars and gravity, the role of time being played by the radial coordinate.

The usefulness of the "superpotential" method stems from the fact that it allows us to work with first-order equations rather than second order ones. However, solving for W might be impractical for the case with multiple scalars and/or complicated V's. As mentioned above, the method has been used in the literature for finding D-dimensional solutions with $(D-1)$-dimensional Poincaré symmetry, such as the RG flow solutions as mentioned above. It would be interesting to find extensions of the method to cases with less symmetry, for example those that have black hole horizons. I will next give a somewhat trivial example of this, a less trivial example will be given in section 2.3.3.

The example I will give consists of a derivation of the the "AdS-Schwarzschild brane" solution by using a superpotential method somewhat different from the one mentioned above. The AdS-Schwarzschild solution is well-known, and the main aim here is to demonstrate that the method can in fact general non-supersymmetric solutions, and to suggest that similar methods might be useful for generating less trivial solutions.

For the action (2.34), let's take the following ansatz for the metric (we will work with $D=5$):

$$ds^2 = -dt^2 e^{2B(r)} + d\vec{x}^2 e^{2A(r)} + dr^2 e^{2(A(r)-B(r))}\,, \tag{2.38}$$

where $d\vec{x}^2 = \sum_{i=1}^{3} dx_i^2$. This metric can be put in the possibly more familiar form

$$ds^2 = e^{2A(r)}\left(-h(r)dt^2 + d\vec{x}^2\right) + \frac{dr^2}{h(r)}\,, \tag{2.39}$$

if one makes the identification $e^{2B(r)} = h(r)e^{2A(r)}$. A radius r_0 at which $h(r)$ changes sign from positive to negative as one goes inward from infinity will be a horizon.[18]

[18]One really needs to check the global causal structure of the spacetime in order to identify the

Now let's plug this ansatz together with $\phi = \phi(r)$ into the action (2.34), and formally divide out by the infinite integral $\int d\vec{x}^3$ over the three transverse directions. The result is

$$S = \int dr \left(6e^{2(A+B)}(A'^2 + A'B') - \frac{1}{2}e^{2A+2B}\phi'^2 - e^{4A}V(\phi) \right) . \qquad (2.40)$$

Note that in order to get this result, we have freely integrated by parts to get rid of terms with second derivatives. If one wants to include boundaries, a bit more care is needed at this step.

The action (2.40) is of the form

$$S = \int dr (\mathcal{T} - \mathcal{V}) , \qquad (2.41)$$

where $\mathcal{T}$ is a "kinetic energy" term quadratic in the first derivatives of the functions A, B and ϕ, and $\mathcal{V} = e^{2A+2B}V(\phi)$ is a potential for these functions. We have reduced the problem to a one-dimensional one.

Now, in [111] (see section 4), for a mathematically similar problem the following method is mentioned: first express $\mathcal{T}$ as

$$\mathcal{T} = \frac{1}{2}\mathcal{G}_{ij}\frac{\partial \alpha^i}{\partial r}\frac{\partial \alpha^j}{\partial r} , \qquad (2.42)$$

where α^i denotes any one of A, B or ϕ, and $\mathcal{G}_{ij}$ is a matrix with entries that are functions of the α^is. It can be shown that if one can find a "superpotential" $W(A, B, \phi)$ such that

$$\mathcal{V} = -\frac{1}{2}\mathcal{G}^{ij}\frac{\partial W}{\partial \alpha^i}\frac{\partial W}{\partial \alpha^j} , \qquad (2.43)$$

then the first-order equations

$$\frac{\partial \alpha^i}{\partial r} = \mathcal{G}^{ij}\frac{\partial W}{\partial \alpha^j} \qquad (2.44)$$

guarantee that the equations of motion that follow from the action S are satisfied. Here, $\mathcal{G}^{ij}$ denotes the inverse of the matrix $\mathcal{G}_{ij}$.

We'll next assume that $V(\phi)$ is a constant, and ignore ϕ, in order to derive the "AdS-Schwarzschild brane" solution. The ansatz

$$W(A, B, \phi) = \mu e^{4A} + \lambda e^{2(A+B)} . \qquad (2.45)$$

nature of the horizon.

gives

$$\mathcal{V} = e^{4A}\left[-\frac{1}{2}\left(\frac{8\lambda}{3}\mu\right)\right], \tag{2.46}$$

i.e., the scalar potential is just a cosmological constant term: $V(\phi) = -\frac{1}{2}\left(\frac{8\lambda}{3}\mu\right)$. The first-order equations (2.37) read

$$\begin{aligned} \frac{dA}{dr} &= \frac{\lambda}{3} \\ \frac{dB}{dr} &= \frac{2}{3}\mu e^{2(A-B)} - \frac{\lambda}{3}, \end{aligned} \tag{2.47}$$

whose solution is given by

$$\begin{aligned} A(r) &= \frac{\lambda(r-r_0)}{3} \\ B(r) &= \frac{1}{2}\log\left(\frac{\mu}{\lambda} + Ce^{-4\lambda r/3}\right) + \frac{\lambda(r-r_0)}{3}. \end{aligned} \tag{2.48}$$

Here r_0 and C are integration constants. In order to compare this with the form (2.39), we use $h(r) = e^{2(B(r)-A(r))}$, which gives

$$h(r) = \frac{\mu}{\lambda} + Ce^{-\frac{4\lambda r}{3}}. \tag{2.49}$$

The combined rescaling $r = \tilde{r}\beta$, $t = \tilde{t}/\beta$ preserves the form of the metric given by (2.39), but the metric functions $h(r)$ and $A(r)$ are replaced with

$$\begin{aligned} \tilde{A}(\tilde{r}) &= A(\beta\tilde{r}) \\ \tilde{h}(\tilde{r}) &= h(\beta\tilde{r})/\beta^2. \end{aligned} \tag{2.50}$$

Choosing $\beta = \sqrt{\frac{\mu}{\lambda}}$ brings the solution into the form

$$\begin{aligned} \tilde{A}(\tilde{r}) &= \frac{\tilde{r}-\tilde{r}_0}{l} \\ \tilde{h}(\tilde{r}) &= 1 - be^{-\frac{4\tilde{r}}{l}}. \end{aligned} \tag{2.51}$$

where $\frac{1}{l} = \frac{\sqrt{\mu\lambda}}{3}$, and C and r_0 have been replaced by the related constants b and $\tilde{r}_0$. This is perhaps a more familiar form of the AdS-Schwarzschild solution. One can obtain the version given in [112] (see, eq. 2.3) by making the identifications $\frac{U}{(4\pi g N)^{1/4}} = e^{r/l}$ and $(4\pi g N)^{1/4} = l$ (and ignoring the 5-sphere in [112]).

This shows that black hole solutions can be found by using a first-order formalism. However, this was a somewhat trivial example since we set the scalar potential to a constant, and ignored the scalar. Extensions of the method to cases with a non-constant scalar potential could be very useful in generating black brane solutions with scalars. Such solutions are useful for finite temperature AdS/CFT, brane world models, and other cosmological applications. One example of a black brane solution with a scalar will be given in section 2.3.3.

2.3 Finite temperature and black holes in AdS/CFT

2.3.1 General considerations

Finite temperature field theory calculations can be done by taking the path integral in periodic euclidean time with period $\frac{1}{T}$ (fermionic fields being antiperiodic in the euclidean time direction—depending on the spin structure). In the strong form of the AdS/CFT correspondence, the partition function in the boundary theory is identified with the partition function of string theory inside the asymptotically AdS space with certain boundary conditions. In the case of finite temperature theory, this amounts to taking the path integral over asymptotically (euclidean) AdS spaces, with a periodic euclidean time direction [8, 26, 113]. In order to calculate the partition function in the classical SUGRA limit (large N, large 't Hooft coupling in the SYM side) one finds solutions to the classical euclidean SUGRA equations which have the prescribed boundary behavior, and then calculates the corresponding classical euclidean actions S_i. The contribution of the ith solution to the gauge theory partition function is then given by $e^{-S_i/T}$. Witten argues in [26] that in the large N limit, only the solution with the smallest euclidean action will be relevant to the partition function.

For the case of undeformed $\mathcal{N} = 4$ SYM theory on $\mathbf{S^3}$, there are two solutions known with the appropriate asymptotics [114]. One is the euclidean AdS-Schwarzschild black hole, the other is the quotient of AdS by a discrete group. There is a phase transition between these solutions found by Hawking and Page, in that the AdS-Schwarzschild contribution to free energy dominates for large temperatures, and the contribution of "thermal AdS" dominates for small temperatures. Interpreting this phase transition as one of $\mathcal{N} = 4$, $D = 4$ SYM theory, Witten argued in [113] that if one wants to consider finite temperature SYM on $\mathbf{R}^3$ rather than $\mathbf{S}^3$, one needs to work in the high temperature limit (as in that case the radius of $\mathbf{S}^3$ gets much bigger than that of $\mathbf{S^1}$, and hence the three-sphere approximates $\mathbf{R}^3$). In that limit, the AdS-Schwarzschild solution –or rather a certain scaling limit of it– gives the dominating contribution to the partition function. The horizon, which is a 3-sphere in AdS-Schwarzschild, becomes an $\mathbf{R}^3$ due to the scaling limit, hence the solution describes a black 3-brane in AdS_5 which is none other than the one we derived in section 2.2.3 by a superpotential method. The thermodynamics of such black branes is then related to the thermodynamics of $\mathcal{N} = 4$ SYM theory on $\mathbf{R^3}$.

Similarly, we can expect that for the singular RG-flow solutions given in the literature, there exist black-brane solutions that have three-dimensional euclidean symmetry, as in

$$\begin{aligned} ds^2 &= e^{2A(r)}(-h(r)dt^2 + d\vec{x}^2) + \frac{dr^2}{h(r)} \\ \vec{\phi} &= \vec{\phi}(r)\,. \end{aligned} \tag{2.52}$$

These solutions should include the original RG-flow solutions as the "extremal limits," just as the AdS solution is the extremal limit of AdS-Schwarzschild solution (found by setting $b = 0$ in (2.51)). We can expect the thermodynamics of the deformed SYM theory to be related to the thermodynamics of such black brane solutions. Since the equations of motion of the SUGRA theories are complicated, it is difficult to find such solutions exactly in the absence of supersymmetry conditions that give first-order equations.

One can, however, find approximations to such solutions that are reasonably accurate under certain circumstances. For example, for the case of "small non-extremality," one can expect the black-brane solution to be only slightly different from the extremal solution outside the horizon. Another scheme for approximation stems from the fact that many of the scalar potentials that come up in RG-flows in AdS/CFT are linear combinations of exponentials as in (2.18). It can be shown for many extremal flows that one of the exponential terms dominates in the far IR, and the scalars run in an asymptotic direction in the scalar manifold, which is determined by the dominant exponent.

Defining the linear combination of scalars that appears in the dominant term as a new scalar, one sees that it is possible to approximate the IR behavior of extremal flows by solutions for a scalar potential that consists of a single exponential term. One then expects that near-extremal generalizations of RG-flow solutions near their horizons be approximated reasonably accurately by black brane solutions to gravity plus a scalar with an exponential potential.

Using this approximation scheme, Gubser calculated equations of state for various deformations and states of $\mathcal{N} = 4$ $D = 4$ SYM theory. These will be put on a firmer footing in section 2.3.3 when we give an exact black brane solution. However, let us first review a criterion for the existence of near extremal generalizations of singular RG-flow type solutions in general.

2.3.2 A criterion for singularities

Most of the solutions to the flow equations of motion (2.21) for the potentials that come up in the literature blow up in the infrared, sending scalars to infinity and resulting in a nakedly singular geometry. Only carefully tuned initial conditions at the UV end are capable of generating flows between the UV fixed point and an IR fixed point. Some of the singular solutions signal the inadequacy of the SUGRA approximation, suggesting one should consider stringy corrections which would hopefully "resolve" the singularity ([78, 79]), and others are considered "unphysical". However, it turns out that not all singular solutions signal problems; for instance, one can get a singular solution to the five-dimensional theory by dimensionally reducing a perfectly well-behaved solution of ten-dimensional SUGRA equations. A criterion for distinguishing the acceptable singularities from the "bad" ones has been given in [11], which I will review below. Let me note in passing that a solution with a "bad" singularity that needs a stringy resolution might still be of use for investigating physical processes that probe only the weakly curved region.

The criterion used in [11] was based on the requirement of existence of "near extremal generalizations," i.e., given a singular solution of the form (2.20), we ask if there is a one parameter family of solutions of the form (2.52), for which (2.20) is the "extremal" limit. We call the singularity a "good" one if the answer is affirmative, and a "bad" one otherwise. It turns out that this criterion effectively rules out a bunch of "unphysical singularities," whose problems are seemingly unrelated to issues of finite temperature, at least at first sight, see [11] for details (other criteria have been explored in [115, 116]).

If this criterion is to have practical applications, one needs to develop a procedure that would tell whether a given extremal solution allows non-extremal generalizations, by looking at the solution. A necessary condition was presented in [11], and I will give a slightly generalized version of it next.

Let us once again start with the action

$$S = \int \sqrt{g}\left[R - \frac{1}{2}\partial_\mu \phi^i \partial_\nu \phi^j \mathcal{G}_{ij}(\vec{\phi}) - V(\phi)\right] d^D x\,, \tag{2.53}$$

in D-dimensions, and leave $\mathcal{G}$ to be an arbitrary positive definite metric on the scalar manifold. We will be working on solutions of the form (2.52). h will turn out to be monotonically increasing, and the radius at which it changes sign will be the position of the horizon, whose nature (cosmological or black hole) can be determined by looking at the global causal structure of the metric. We use the following combinations of Einstein's equations: $G_t^t - G_x^x = T_t^t - T_x^x$, $G_t^t - G_r^r = T_t^t - T_r^r$ and $G_r^r = T_r^r$. These give, respectively,

$$
\begin{aligned}
h'' + (D-1)A'h' &= 0 \\
(D-2)A'' &= -\frac{1}{2}\mathcal{G}_{ij}\phi^{i'}\phi^{j'} \\
\frac{(D-2)}{2}A'(h' + (D-1)A'h) &= \frac{1}{4}\mathcal{G}_{ij}\phi^{i'}\phi^{j'}h - \frac{V}{2}\,.
\end{aligned}
\tag{2.54}
$$

The basic idea behind the criterion is the following. If we can show that

$$
T^r_r = \frac{1}{4}\mathcal{G}_{ij}\phi^{i'}\phi^{j'}h - \frac{V}{2} \tag{2.55}
$$

is monotonically decreasing in r, then using the fact that h vanishes at the horizon and $\phi^{i'}$ vanishes at infinity,[19] we can see that the value of V at the horizon must be less than or equal to its value at infinity. This property pertaining to black hole solutions of the form (2.52) can be translated to a property of an extremal solution which is a limit of such black hole solutions in a suitable norm. This will give the criterion we need: the value of the potential at the singularity has to be less than its value at infinity in order for a nakedly singular solution to have a near-extremal generalization.

In order to show that T^r_r is monotonic in r, let us take the derivative of the Einstein equation $G^r_r = T^r_r$:

$$
\begin{aligned}
\frac{dT^r_r}{dr} &= \frac{dG^r_r}{dr} \\
&= \frac{(D-2)}{2}(A''(h' + 2(D-1)hA') + A'(h'' + (D-1)A'h')) \\
&= \frac{(D-2)}{2}A''(h' + 2(D-1)hA') \\
&= -\frac{1}{4}\mathcal{G}_{ij}\phi^{i'}\phi^{j'}(h' + 2(D-1)hA')\,,
\end{aligned}
\tag{2.56}
$$

[19]Note that we are interested in asymptotically AdS solutions, for which the scalars asymptote to an extremum of the potential near the boundary.

where we used $G^t_t - G^x_x = T^t_t - T^x_x$ in the third line, and $G^t_t - G^r_r = T^t_t - T^r_r$ in the last line. We can see the negativity of $\frac{dT^r_r}{dr}$ as follows. As mentioned above, we assume the kinetic matrix $\mathcal{G}_{ij}$ to be positive definite. h' is positive outside the horizon, since $h' > 0$ at the horizon and the $G^t_t - G^x_x = T^t_t - T^x_x$ equation tells us that $h'e^{(D-1)A}$ is a constant. It also follows from this that $h > 0$ outside the horizon. Finally, $A' > 0$ since $A'' < 0$ (see the $G^t_t - G^r_r = T^t_t - T^r_r$ equation above), and $A' > 0$ asymptotically for the solutions we are interested in.

We have thus given a semi-rigorous argument which effectively says that for a singular RG-flow type solution of the form (2.20) to be a "limit" of non-extremal black brane solutions, the value of the potential near the singularity has to be less than its value at $r = \infty$, i.e., the UV. In AdS/CFT applications, $V(\phi) < 0$ at the singularity is immediately implied, since an asymptotically AdS space needs a negative cosmological constant asymptotically.

2.3.3 An exact solution for gravity plus a scalar

As mentioned at the end of section 2.3.1, exact black hole solutions for gravity plus a scalar with an exponential potential would be of use in explorations of thermodynamics of RG-flows in AdS/CFT. For example, in many of the examples mentioned in [11], only one of the exponential terms in the scalar potential dominates in the IR, so near the horizon, one can approximate the near-extremal version of a flow solution with a black brane solution for a single scalar with a pure exponential potential.

I will next give an exact black brane solution for such a gravity plus scalar system. The thermodynamics and dynamical stability of this solution will be investigated, and used as a tool for in developing a notion of universality classes of black brane instabilities in Chapter 3.

We will work with the action (2.53) for a single scalar coupled to gravity., with $\mathcal{G}_{\phi\phi} = 1$. The Einstein equations have already been given in (2.54), and the scalar equation of motion is

$$h(\phi'' + (D-1)A'\phi') + h'\phi' = \frac{\partial V}{\partial \phi}\,. \tag{2.57}$$

For small non-extremalities (i.e., for solutions for which $h(r)$ is close to 1 for most of the region outside the horizon), one can expect $A(r)$ and $\phi(r)$ to be close to their values for the extremal ($h(r) = 1$) case. One can thus try to find approximate solutions by plugging the extremal versions of $A(r)$ and $\phi(r)$ in one of the equations of motion, and integrating to find $h(r)$. It turns out that this "approximate" method is actually exact for an exponential potential. In order to get A and ϕ, we use the first-order formalism of (2.35) and (2.37), with $W = ce^{\gamma\phi}$, which when plugged in (2.35) gives

$$V(\phi) = c^2 \left(\frac{1}{2}\gamma^2 - \frac{1}{4}\frac{D-1}{D-2}\right) e^{2\gamma\phi}. \tag{2.58}$$

We will restrict attention to negative V, since this is the case with an event horizon.[20] Solving for A and ϕ from (2.37) and substituting in the equations of motion to get h, we find

$$\begin{aligned} e^{2A} &= r^{\frac{1}{\gamma^2(D-2)}} \\ h &= 1 - Br^{1-\frac{D-1}{2\gamma^2(D-2)}} \\ \phi &= -\frac{1}{\gamma}\log(\gamma^2 cr), \end{aligned} \tag{2.59}$$

where we have made certain gauge choices (e.g., in setting the origin of r and choosing the constant term in h). Here, B is a non-extremality parameter. Note that $A(r)$ and $\phi(r)$ are independent of B. One can easily plug this in the equations of motion (2.54) and (2.57), and chek that they are satisfied.

In fact, these solutions have been previously found by Chamblin and Reall in the context of dilatonic domain walls [117]. These authors found that the solution above (as well as some other related solutions that are not black holes) admit domain walls that do not "backreact on the geometry" if their coupling to the scalar is tuned properly.

[20]The derivation of the criterion given in section 2.3.2 doesn't apply here as I gave it, since ϕ' doesn't go to zero at infinity, however, it turns out that black brane solutions exist only when $V < 0$ in this case as well, see [117].

We will use this solution and its stability properties in Chapter 3 for formulating a notion of universality classes of the instabilities of black branes.

Chapter 3

Universality classes for horizon instabilities

[1]

3.1 Introduction

The Gregory-Laflamme instability [33, 29] has been conjectured to arise precisely when local thermodynamic instabilities exist, for horizons with infinite extent and a translational symmetry [12, 13]. The idea is that, barring finite volume effects or some unexpected consequences of a curved horizon, the horizon will tend to become lumpy through real-time dynamics precisely if it can gain entropy by doing so. Based on this line of thinking, one might poetically ascribe the dynamical stability of the Schwarzschild black hole in four dimensions merely to the fact that the horizon is smaller in total size than the shortest wavelength instability.

A fairly thorough analysis in [45], exploiting a connection between time-independent, finite-wavelength perturbations in Lorentzian signature and negative modes in euclidean signature, gives considerable confidence that the conjecture of [12, 13] is correct, as well as making clearer the reasons why thermodynamic and dynamical stability should be connected. The analysis of [45] does leave several points open,

[1] This chapter is more or less a copy-paste of [118]

including the following:

1. For the black brane solutions familiar from string theory, just when is stability lost?

2. What happens near the threshold of stability?

3. Once an instability develops, what does the system evolve into?

Question 3) has been the subject of recent scrutiny for the case of uncharged black branes (see for instance [46, 119, 120, 121, 122]). Question 1) is straightforward if one grants the analysis of [45], since one needs only compute the specific heat as a function of non-extremality. We will address the question in section 3.2 from this thermodynamic point of view, and in section 3.4 by directly searching for time-independent, finite-wavelength perturbations, thus providing an independent check on the validity of the analysis of [45]. We will also make a start on question 2), based on our numerics. In section 3.3 we will introduce a notion of universality classes for black brane horizons, based on solutions described in [117]. For the questions we will address, a single characteristic exponent defines a universality class: this exponent describes how horizon entropy scales with temperature.

Our notion of universality classes is of some intuitive use in understanding the stability properties of various charged brane solutions. In particular, the well-known non-extremal Dp-branes of type II string theory interpolate between one universality class far from extremality (where the entropy scales as a negative power of temperature, and Gregory-Laflamme instabilities exist) to a different universality class near extremality (where entropy scales as a positive power of temperature, and Gregory-Laflamme instabilities probably do not exist).

Other recent work on the Gregory-Laflamme instability includes [123, 124, 125]. In particular, [123] explores the stability of charged brane solutions, using methods somewhat different from ours, but with conclusions that overlap and agree with those in section 3.4.

3.2 Thermodynamic considerations

The Dp-brane solutions to type II string theory have the following string frame metric and dilaton:

$$\begin{aligned} ds^2 &= G_{tt}dt^2 + G_{rr}dr^2 + G_{\theta\theta}d\Omega_{n+1}^2 + G_{yy}dy^i dy^i \\ &= -\frac{h}{\sqrt{f}}dt^2 + \frac{\sqrt{f}}{h}dr^2 + \sqrt{f}r^2 d\Omega_{n+1}^2 + \frac{1}{\sqrt{f}}dy^i dy^i \qquad (3.1) \\ e^{2\phi} &= f^{(n-4)/2}, \end{aligned}$$

where

$$h = 1 - \frac{r_0^n}{r^n} \qquad f = 1 + \frac{r_0^n \sinh^2\alpha}{r^n}. \qquad (3.2)$$

The string metric $G_{\mu\nu}$ is related to the Einstein metric $g_{\mu\nu}$ by $g_{\mu\nu} = e^{-\phi/2}G_{\mu\nu}$. In equations (3.1) and (3.2), $n = 7-p$. We will assume throughout that the metric (3.1) is being used to describe a large number N_p of coincident Dp-branes, so that the supergravity approximation is reliable (except perhaps near the horizon at extremality, where some of the solutions have a null singularity).

The thermodynamic properties of Dp-branes are well-known, so we will present here only a brief summary. The original literature, including [19, 126, 127], can be consulted for further details. The entropy of the Dp-brane solution (3.1) is

$$S = \frac{A_{\text{Einstein}}}{4G_N} = \frac{A_{\text{string}}}{4G_N e^{2\phi}} = \frac{4\pi\Omega_{n+1}}{2\kappa^2}Vr_0^{n+1}\cosh\alpha\,, \qquad (3.3)$$

where A_{Einstein} is the horizon area measured using the Einstein frame metric, A_{string} is the horizon area in the string frame metric, V is the coordinate volume in the y^i directions, and Ω_{n+1} is the volume of the sphere S^{n+1}:

$$\Omega_m = \text{Vol}\, S^m = 2\frac{\pi^{\frac{m+1}{2}}}{\Gamma\left(\frac{m+1}{2}\right)}. \qquad (3.4)$$

If one compactified the y^i on a torus whose volume at infinity was V, then (3.3) would be the total entropy of the resulting horizon. The ADM mass (most easily calculated using the Einstein frame metric) is

$$M = \frac{\Omega_{n+1} V}{2\kappa^2} r_0^n \left(\frac{n+2}{2} + \frac{n}{2} \cosh 2\alpha \right) . \tag{3.5}$$

The temperature can be extracted from the surface gravity:

$$T = \frac{\kappa}{2\pi} = \frac{1}{2\pi} \frac{\partial g_{tt} / \partial r}{2\sqrt{g_{tt} g_{rr}}} \bigg|_{r=r_0} = \frac{n}{4\pi r_0 \cosh \alpha} . \tag{3.6}$$

One can substitute G_{tt} and G_{rr} for g_{tt} and g_{rr} in (3.6), provided the dilaton is non-singular at the horizon (which would just mean that the horizon is a null singularity, and that only happens for extremal Dp-branes).

It is straightforward to check that $T = \partial M / \partial S$.

The Dp-brane charge is quantized, and can be expressed in terms of the following constraint:

$$\tilde{R}^n \equiv \tfrac{1}{2} r_0^n \sinh 2\alpha = \frac{2\kappa^2}{n\Omega_{n+1}} N_p \tau_p = \frac{N_p \sqrt{2\kappa^2 (2\pi)^{7-2p} (\alpha')^{3-p}}}{n\Omega_{n+1}} = \frac{(2\pi)^n}{n\Omega_{n+1}} \sqrt{\alpha'}^n g N_p \, , \tag{3.7}$$

where N_p is an integer and τ_p is the tension of a single extremal Dp-brane, and we have used the relation

$$16\pi G_N = 2\kappa^2 = (2\pi)^7 g^2 (\alpha')^4 \, , \tag{3.8}$$

valid in uncompactified type II string theory. The formula (3.7), taken from [126], can be checked against the standard formula for D-brane tension:

$$2\kappa^2 \tau_p^2 = (2\pi)^{7-2p} (\alpha')^{3-p} . \tag{3.9}$$

Holding N_p fixed, the behavior of the temperature in different limits is well-known. In all cases, $T \to 0$ in the small α limit, since this is the large mass limit where the charge is negligible. For $p = 1, 2, 3,$ and 4, $T \to 0$ also in the extremal limit, whereas for $p = 5$, T approaches a constant, and for $p = 6$, T diverges. There are no other remarkable qualitative features of the dependence of T on mass other than a consequence of what we have already said: for $p < 5$, there is a special mass M_* where T reaches a global maximum. This is the border of thermodynamic stability:

for masses less than M_*, the specific heat is positive, and for masses greater than M_* it is negative. Holding N_p fixed, one finds that (3.6) is extremized when

$$\sinh^2\alpha = \sinh^2\alpha_* = \frac{1}{n-2}\,, \tag{3.10}$$

or alternatively when

$$\frac{M}{N_p\tau_p V} = \frac{M_*}{N_p\tau_p V} = \frac{n^2-2}{n\sqrt{n-1}} \tag{3.11}$$

(note that $N_p\tau_p V$ is the extremal mass). Another characterization of the same condition is

$$r_0 = r_0^* = \left(\frac{n-2}{\sqrt{n-1}}\right)^{1/n}\tilde{R}\,, \tag{3.12}$$

where $\tilde{R}$ is the length scale introduced in (3.7).

The M2 and M5-brane metrics are given by

$$\begin{aligned} ds_{M2}^2 &= f_{M2}^{-2/3}\left(-h_{M2}dt^2 + d\vec{x}_2^{\,2}\right) + f_{M2}^{1/3}\left(\frac{dr^2}{h_{M2}} + r^2 d\Omega_7^2\right) \\ ds_{M5}^2 &= f_{M5}^{-1/3}\left(-h_{M5}dt^2 + d\vec{x}_5^{\,2}\right) + f_{M5}^{2/3}\left(\frac{dr^2}{h_{M5}} + r^2 d\Omega_4^2\right)\,, \end{aligned} \tag{3.13}$$

where f and h are given by (3.2) with $n = 6$ for the M2-brane and $n = 3$ for the M5-brane. Indeed, the thermodynamics for the M2-brane and the D1-brane are identical. This is not accidental: wrapping an M2-brane around a circle gives a fundamental string of type IIA, which is equivalent through some further dualities to the D1-brane (and, more to the point, has the same Einstein metric up to a slightly different identification of parameters). For similar reasons, the M5-brane and D4-brane have the same thermodynamics. In particular, the condition for the maximum Hawking temperatures for the M2-brane and the M5-brane are given precisely by (3.10), (3.11), and (3.12), with $n = 6$ and 3, respectively.

3.3 Universality classes and the general problem

A fairly broad statement of the question of p-brane stability in the supergravity approximation can be summed up as follows: starting with some p-brane solution in supergravity, one wishes to consider perturbations that break some part of the translational invariance and determine whether any of them grow with time. Such considerations can often be reduced to a problem in $2+1$ dimensions, and in the following paragraphs we shall give more particulars about how this comes about.

One starts with a two-derivative action in $\hat{D} > 2$ spacetime dimensions, containing Einstein gravity and perhaps also form fields and scalars. The brane in question is some solution with a horizon and a weakly curved region far from it (typically asymptotically flat space), where the whole solution admits as symmetries the euclidean group of translations and rotations in p dimensions.[2] There should also be some Killing vector which is timelike in the weakly curved region and null at the horizon, which can be used to construct a notion of energy; and there may be assorted spacelike Killing vectors as well, indicating rotational symmetries. The brane is of spatial co-dimension $\hat{D} - 1 - p$, and far from the brane one generally expects an approximate $SO(\hat{D} - 1 - p)$ rotational symmetry group. The brane may have various conserved charges and angular momenta (the latter corresponding to generators of $SO(\hat{D}-1-p)$). In this rather general context, we would like to ascertain the stability properties of the solution against small perturbations (determined by thermodynamic stability, according to [12, 13, 45]), perhaps also the dispersion relation for such perturbations, and finally the end state of evolution along an unstable direction—though this last is surely a much harder problem in general than the other two.

Although we believe that the notion of universality classes we introduce below will have some applicability to the general case (including for example smeared brane solutions), let us focus on the case where the $SO(\hat{D}-1-p)$ symmetry is exact. Then

[2] Smaller symmetry groups can arise. In "smeared" brane solutions, for instance, composed of a density of p-branes distributed in several orthogonal directions, the symmetry group would be translations in all dimensions plus a product of two rotation groups for the directions parallel to and perpendicular to the individual p-branes.

perturbations can be labeled uniquely by their wave-number $\vec{k}$ along the brane and their angular momentum quantum numbers under $SO(\hat{D}-1-p)$. It seems reasonable to assume that the s-wave perturbations become unstable first, provided they exist as dynamical perturbations (as opposed to being perturbations which are pure gauge). The perturbations in question thus preserve a large subgroup of the symmetry of the original solution: $SO(\hat{D}-1-p)$ times that part of the euclidean group that preserves $\vec{k}$. This makes it particularly natural to consider a Kaluza-Klein "reduction" where one integrates the action over an $S^{\hat{D}-2-p}$ orbit of $SO(\hat{D}-1-p)$,[3] and drops the integration over an $\mathbf{R}^{p-1}$ perpendicular to $\vec{k}$. The result is a 2+1-D action, which in a fairly broad set of circumstances can be brought into the form

$$S = \int d^3x \sqrt{g} \left(R - \frac{1}{2} G_{ij}(\vec{\phi}) \partial_\mu \phi^i \partial^\mu \phi^j - V(\vec{\phi}) \right) . \tag{3.14}$$

Einstein gravity, form fields, and scalars in $\hat{D}$ dimensions reduce down to the type of action written in (3.14), plus perhaps abelian and non-abelian gauge fields. The abelian gauge fields can be dualized to scalars, unless there are Chern-Simons terms in the action. Non-abelian gauge fields in general cannot be dualized, but examples in the literature often have fields excited corresponding to an abelian subgroup. Thus (3.14), while obviously not completely general, does cover a broad range of cases. Furthermore, it has been observed [11] that $V(\vec{\phi})$ is commonly a sum of exponentials of canonically normalized scalars: for example, the integrated curvature of $S^{\hat{D}-2-p}$ scales as a power of its radius, but the canonically normalized scalar measuring the size of $S^{\hat{D}-2-p}$ is some multiple of the logarithm of the radius.

In section 3.3.1 we will consider properties of solutions to (3.14), in part recapitulating arguments of [11]. In section 3.3.2 we briefly remark on the reason to search for static perturbations. Then in section 3.3.3 we will make an explicit numerical study

[3] An interesting aside is that in circumstances involving angular momentum, where obviously $SO(\hat{D}-1-p)$ is not preserved even by the original solution, Kaluza-Klein reduction is still often useful, with the somewhat mysterious "consistent truncation" ansatze providing an exact lower-dimensional account of both the solution and some low partial wave perturbations [12, 13]. We suspect that the coincidences arising in consistent truncation that matter for analyses of horizon perturbations arise from bosonic symmetries rather than supersymmetry.

of stability of a special case of the solutions considered in section 3.3.1.

3.3.1 Solutions to gravity plus scalars

Let us now study the slightly more general situation where

$$S = \int d^D x \sqrt{g} \left(R - \frac{1}{2} G_{ij}(\vec{\phi}) \partial_\mu \phi^i \partial^\mu \phi^j - V(\vec{\phi}) \right) , \tag{3.15}$$

where $D = d + 1$ is unrelated to $\hat{D}$. Motivated by the previous discussion, we will be most interested in the case $D = 3$. Solutions to the equations of motion of this action were studied in [11], of the form

$$ds^2 = a(r)^2(-dt^2 + d\vec{x}^2) + dr^2 , \qquad \vec{\phi} = \vec{\phi}(r) , \tag{3.16}$$

with $a(r)$ a monotonically increasing function of r, running from 0 to ∞. The large a region in these solutions is weakly curved (in fact, the main interest in [11] was asymptotically anti-de Sitter space), and the small a region is in most cases singular. These solutions have no horizons (or at best degenerate or singular horizons), and their symmetry under boosts amounts to the statement that they are extremal. All such solutions can be derived using a first-order formalism (see, for example, [110, 66, 109]), which is a direct analog of the Hamilton-Jacobi method of generating FRW cosmologies [128]: if one starts with an action of the same form as (3.14), but in D dimensions, and finds $W(\vec{\phi})$ such that

$$V(\phi) = \frac{1}{2} G^{ij} \frac{\partial W}{\partial \phi^i} \frac{\partial W}{\partial \phi^j} - \frac{1}{4} \frac{D-1}{D-2} W^2 , \tag{3.17}$$

then a solution to

$$\frac{\partial \phi^i}{\partial r} = -G^{ij} \frac{\partial W}{\partial \phi^j} , \qquad \frac{\partial a / \partial r}{a} = \frac{1}{2(D-2)} W(\vec{\phi}) \tag{3.18}$$

will also solve the equations of motion.

Non-extremal generalizations of (3.16) have the form

$$ds^2 = a(r)^2(-h(r)dt^2 + d\vec{x}^2) + \frac{dr^2}{h(r)} , \qquad \vec{\phi} = \vec{\phi}(r) . \tag{3.19}$$

We assume in general that $h(r) \to 1$ in the region where $a(r) \to \infty$, and $h(r) < 1$ everywhere. If $h(r_H) = 0$ and r_H is the largest value of r for which this is true, then

$r = r_H$ is an event horizon with respect to the region where $a(r) \to \infty$ (which is also at large r). Provided $h'(r_H) \neq 0$, this is a non-degenerate horizon with finite Hawking temperature.

Solutions of the form (3.19) involve a number of constants of integration. The counting of them has been explained in [11], and we will recapitulate briefly here. The relevant equations of motion are all ordinary differential equations in r once we assume the form (3.19). They comprise the second-order equations for the scalars, of which we suppose there are n; the G_{rr} component of Einstein's equation, which is a first-order equation; and one further Einstein equation (for instance, the combination $G^t{}_t - G^x{}_x$ of Einstein's equations), which is second order. There are $2n+3$ integration constants altogether. One amounts to an additive shift in the radial variable r, which is just a coordinate transformation. Another is fixed by the requirement that $h(r) \to 1$ in the region where $a(r)$ is large. The presence of a non-degenerate horizon fixes n more, as explained in [11] (one way to understand the presence of these n horizon constraints is that in the Wick-rotated euclidean solution, where the horizon becomes a point, the scalars must be smooth everywhere, so their radial derivatives must vanish at this point). This leaves $n + 1$ parameters that specify the solution. One of these is the temperature of the horizon, and the other n pertain to the asymptotics of the scalars in the large a region. Of these n parameters, some but not all may be fixed by demanding regularity in the large a region. In an AdS_{d+1}/CFT_d context, where the large a region is asymptotically anti de-Sitter, the n parameters correspond to the coefficients of $e^{-(d-\Delta_i)r/L}$ in the expansion of the scalars around their constant limiting values as $r \to \infty$. These coefficients amount on the CFT side to the mass parameters of gauge singlet operators added to the lagrangian. If some of the operators in question are irrelevant, then the corresponding scalars have positive m^2, and the larger of the two solutions to the linearized scalar equations of motion blows up at large r: this is a case where parameters are fixed to zero by demanding regularity in the large a region. In the case where the dual operators are relevant, and the corresponding scalars have negative m^2, then the coefficient of $e^{-(d-\Delta_i)r/L}$ is truly a variable parameter. In the case of asymptotically anti-de Sitter solutions, an additional scaling symmetry can be used to eliminate one additional parameter: this amounts to applying a scale transformation on the CFT side to set, say, the temperature equal to 1.

Let us now consider the situation where the asymptotics of the scalars is entirely fixed in the large a region, leaving us with one free parameter. One can show (see [11] for the case where $d = 4$) that

$$h = 1 - B \int_r^\infty \frac{dr_1}{a(r_1)^d}, \tag{3.20}$$

and then B is the free parameter. (The identity (3.20) holds even though $a(r)$ may in general change when B changes. It is a convenient parametrization because the influence of B on $a(r)$ is small in the large a region). In an AdS_{d+1}/CFT_d context, this would correspond to specifying a deformation of the gauge theory lagrangian by relevant operators and then varying either the temperature or the energy density. Evidently, in the limit $B \to 0$, one recovers a solution of the form (3.16). Suppose that in this limiting solution, $V(\vec{\phi}(r))$ decreases monotonically as $a(r)$ decreases, going to ∞ as $a \to 0$.[4] It can then be argued (though not wholly rigorously) [11] that

- Solutions with regular horizons exist for all $B > 0$.

- If $V(\vec{\phi})$ is the sum of exponentials of canonically normalized scalars, then in the $B = 0$ solution, the scalars have an asymptotic direction as $a(r) \to 0$: that is, $\vec{\phi}(r) \sim \vec{\phi}_0 + \vec{v}\varphi(r)$ for some constants $\vec{\phi}_0$ and $\vec{v}$, which we may normalize so that φ is itself canonically normalized. Then $V(\vec{\phi}(r)) \approx e^{2\gamma\varphi}$ in the region of small a.

- In near-extremal solutions, with sufficiently small B, the solution at small a is well-approximated by a Chamblin-Reall solution for the exponent γ, which we explain below in section 3.3.3. The whole solution can be well-approximated by the original extremal solution patched onto a Chamblin-Reall solution. These approximations become progressively better as $B \to 0$.

[4] It is possible that this condition could be relaxed somewhat: we must certainly have $V(\vec{\phi}(r))$ less than its limiting value at large a, and probably the monotonicity property is also needed for a less than a certain upper bound.

- The entropy and temperature of near-extremal solutions are related by a power law: $S \propto VT^{\alpha}$, where

$$\alpha = \frac{D-2}{1-2\gamma^2(D-2)} \,. \tag{3.21}$$

These features suggest the notion of universality class that we are going to use. Suppose that an extremal solution to an action with canonically normalized scalars has a "scaling region" where $\vec{\phi}(r) \approx \vec{\phi}_0 + \vec{v}\varphi(r)$ and $V \approx e^{2\gamma\varphi}$ for many decades of variation of $a(r)$. Then non-extremal solutions with the horizon well within the scaling region should again be well-approximated by the original extremal solution patched onto a Chamblin-Reall solution near the horizon. The approximation should become good in the limit where the scaling region in the extremal solution extends far above and below the value of a where the horizon is located in the non-extremal solution. The same power law, $S \propto VT^{\alpha}$, should pertain.

With the idea of universality classes in hand, we can address stability of the horizon in a simple way. In section 3.3.3, we will study explicitly perturbations around Chamblin-Reall solutions, involving the metric and the "active" scalar φ. We will find perturbations which are normalizable both at the horizon and far from it. As the full solution is well approximated by matching the extremal solution onto a Chamblin-Reall solution, perturbations should be well approximated by similarly matched solutions—only, because the perturbations we are considering are normalizable within the Chamblin-Reall region, they approach zero near the matching region, and the perturbations of fields in the extremal region are very small. It does not seem plausible that there are normalizable perturbations which are large in the extremal region and small as one enters the Chamblin-Reall region near the horizon. To sum up, we believe the stability properties of the whole solution are determined by the stability properties of the near-horizon Chamblin-Reall solution, which also determines the scaling of entropy with temperature.

It may seem that we are focusing rather narrowly on a rather special class of perturbations: not only invariant under the $SO(\hat{D}-1-p)$ that we started with in carrying out the Kaluza-Klein reduction, but also involving only the metric and the active scalar near the horizon. This amounts to focusing on a type of fluctuations which we might describe as adiabatic, since the fields are varying locally in the same proportions that they would do globally if we simply changed the temperature. But this is precisely the mode of instability that the thermodynamic arguments of [12, 13, 45] suggest. Instabilities in other modes are conceivable, and one could even study them by considering scalar fluctuations "transverse" to the solution. But we would find it very surprising if such fluctuations gave rise to normalizable instabilities when the fluctuations we consider do not. To put it another way, for that to happen, the logic of [45] must fail. Our rather detailed predications about the nature of the instability amount in a sense to an elaboration of the connection between thermodynamic and dynamical instabilities.

3.3.2 Static perturbations

In the rest of this paper, we will investigate Gregory-Laflamme type instabilities of various black branes. For all the unstable black branes Gregory and Laflamme analyzed [33, 29], they found at linear order in perturbation theory that there is a static perturbation, and that instabilities occur at longer wavelengths than this static perturbation.

As remarked in the introduction, Reall [45] in his approach to the proof of the conjecture of [12, 13] investigated the role of such time-independent perturbations, and argued that they are of central importance to the relation between thermodynamic and dynamical instability. In particular he related such perturbations of a black-brane solution to the negative modes of the Lichnerowicz operator on the euclidean black hole background found by compactifying the black p-brane on a p-torus and doing a Wick rotation. The existence of such negative modes is then related to thermodynamic stability. (Negative modes of euclidean black holes have also been analyzed in the context of finite temperature stability of flat spacetime by Gross, Perry and Yaffe [129].)

Thus when looking for instabilities of black branes, instead of considering instabilities with arbitrary growth rates, we will restrict our attention to static threshold perturbations. With the relation between thermodynamic and dynamical stabilities in mind, we expect the dispersion curve of a Gregory-Laflamme type instability to end at such a threshold point at the small wavelength limit.

3.3.3 The Chamblin-Reall solutions

As argued in section 3.3.1, close to the horizon, black brane solutions [117] for gravity coupled to a scalar ϕ with a potential of the form $V = V_0 e^{2\gamma\phi}$ provide good approximations to a broad range of supergravity solutions, including ones that are non-extremal versions of those that represent RG-flows in AdS/CFT.

The action for the gravity plus scalar system is

$$\begin{aligned} S &= \int d^D x \sqrt{g} \left(R - \frac{1}{2}(\partial\phi)^2 - V(\phi) \right) \\ V(\phi) &= V_0 e^{2\gamma\phi} . \end{aligned} \tag{3.22}$$

We will restrict our attention to negative V_0, since that is the case that admits black hole solutions. The black brane solutions are given by

$$ds_0^2 = a_0^2(r)(-h_0(r)dt^2 + d\vec{x}_{D-2}^2) + \frac{dr^2}{h_0(r)} , \tag{3.23}$$

where

$$\begin{aligned} h_0(r) &= 1 - \left(\frac{r_0}{r}\right)^{\frac{D-1}{2\gamma^2(D-2)} - 1} \\ a_0(r) &= r^{\frac{1}{2\gamma^2(D-2)}} \\ \phi_0(r) &= -\frac{1}{\gamma} \log(\gamma^2 c r) , \end{aligned} \tag{3.24}$$

and c is given by the positive root of the equation

$$V_0 = c^2 \left(\frac{1}{2}\gamma^2 - \frac{1}{4}\frac{D-1}{D-2} \right) . \tag{3.25}$$

In the notation of [117] (differing slightly from ours), these solutions correspond to the "type II" case, with $b^2 < D - 1$, $M > 0$, and $V_0 < 0$. Note that in the gauge presented above, the metric is independent of V_0: it only depends on γ, whereas the scalar profile depends on both γ and V_0.

With respect to the general discussion of non-extremal horizons in section 3.3.1, the solution (3.24) is rather special, in that the functions $a(r)$ and $\phi(r)$ are entirely independent of the parameter r_0 that determines the location of the horizon. (The equivalent parameter in section 3.3.1 is B). In other words, one can get at the solution (3.24) by first obtaining the extremal solution and then using (3.20) without changing $a(r)$. In general, $h \neq 1$ "back-reacts" on the solution, changing $a(r)$ and $\phi(r)$. This property of (3.24) also hints at the origin of the relation (3.25): One plugs $W = ce^{\gamma\phi}$ in (3.17) to obtain V, just as one would do when looking for extremal solutions.

As noted in [117], for a discrete set of γ's it is possible to arrive at the solutions (3.24), (3.25) by a dimensional reduction of spacetimes of the form $Schwarzschild_{q+2} \times R^{D-2}$ on the q-sphere of $Schwarzschild_{q+2}$. For such cases, V_0 and γ are given in terms of q as

$$V_0 = -q(q-1)$$
$$\gamma = \sqrt{\frac{D+q-2}{2q(D-2)}}. \tag{3.26}$$

At least for this discrete set of solutions, we certainly expect to find dynamical instabilities: they correspond to the instabilities of an uncharged black string in $q+3$ spacetime dimensions. The γ's that can be obtained by a dimensional reduction on a q-sphere, where q runs from 1 to ∞, fall in the range

$$\sqrt{\frac{1}{2(D-2)}} < \gamma < \sqrt{\frac{D-1}{2(D-2)}}. \tag{3.27}$$

(The case $q = 1$, which gives the upper limit on γ, is degenerate: $V_0 = 0$ for that case.) It turns out that (3.27) is precisely the range for which the D-dimensional black brane has a negative specific heat. The conjecture of [12, 13] then implies that one should have dynamical instabilities for the γ's precisely in this range, including the ones that *can't* be obtained by dimensional reduction. In the next section, we will give numerical evidence in favor of this for the case $D = 3$. (The same conclusion for other D then follows from straightforward Kaluza-Klein reduction).

3.3.4 Numerical study of the stability of Chamblin-Reall solutions

Choosing $D = 3$, we searched numerically for threshold dynamical instabilities of the Chamblin-Reall solutions by restricting attention to perturbations of the form

$$\begin{aligned} ds^2 &= -e^{2A(r,x)}\left(a_0^2(r)h_0(r)dt^2\right) + e^{2B(r,x)}\left(a_0^2(r)dx^2 + \frac{dr^2}{h_0(r)}\right) \\ \phi(r) &= \phi_0(r) + \lambda\cos kx\phi_1(r)\,, \end{aligned} \tag{3.28}$$

where

$$\begin{aligned} A(r,x) &= \lambda\cos kxA_1(r) \\ B(r,x) &= \lambda\cos kxB_1(r)\,, \end{aligned} \tag{3.29}$$

and working to linear order in λ. It is possible to show that this ansatz for the perturbations is consistent with the equations of motion.

The Einstein equations in $D = 3$ read

$$R^\mu_\nu = T^\mu_\nu - g^\mu_\nu T^\alpha_\alpha\,. \tag{3.30}$$

Four of these equations are nontrivial, namely, the ones involving R^t_t, R^r_r, R^x_x and R^r_x. Using the R^r_x equation, one can solve algebraically for B_1 in terms of A_1, A_1', and ϕ_1. The R^t_t equation and the scalar equation of motion involve B_1, but not its derivatives. Plugging in B_1 as found from the R^r_x equation into the R^t_t equation and the scalar equation of motion, we end up with the following two equations involving A_1 and ϕ_1:

$$0 = \left(-\left(\gamma^6\, k^2\, r^2\right) + r^{\gamma^{-2}} - 2\,\gamma^2\, r^{\gamma^{-2}} + \gamma^4\, r^{\gamma^{-2}}\,\left(1 + k^2\, r\right)\right)\, A_1(r) + \gamma\left(-1 + \gamma^2\right)^2 r^{\gamma^{-2}}\,\phi_1(r)$$
$$+ \frac{1}{2}\gamma^2\left(-3\,\gamma^4\, r^2 + r^{\gamma^{-2}}\left(-4\, r + r^{\gamma^{-2}}\right) - 2\,\gamma^2\, r^{\gamma^{-2}}\left(-5\, r + 2\, r^{\gamma^{-2}}\right)\right)\, A_1'(r)$$
$$- \gamma^4\, r\left(r - r^{\gamma^{-2}}\right)\left(\gamma^2\, r - r^{\gamma^{-2}}\right)\, A_1''(r)$$
$$0 = 2\left(-1 + \gamma^2\right)^2 r^{\gamma^{-2}}\, A_1(r) + \gamma\left(2\, r^{\gamma^{-2}} - \gamma^2\, r^{\gamma^{-2}}\left(4 + k^2\, r\right) + \gamma^4\left(k^2\, r^2 + 2\, r^{\gamma^{-2}}\right)\right)\,\phi_1(r)$$
$$- \gamma^2\left(-r + r^{\gamma^{-2}}\right)\left(-3\, r^{\gamma^{-2}} + \gamma^2\left(-r + 4\, r^{\gamma^{-2}}\right)\right)\, A_1'(r) + \gamma\left(-\left(\gamma^2\, r\right) + r^{\gamma^{-2}}\right)^2 \phi_1'(r)$$
$$+ \gamma^3\, r\left(-r + r^{\gamma^{-2}}\right)\left(-\left(\gamma^2\, r\right) + r^{\gamma^{-2}}\right)\,\phi_1''(r)\,. \tag{3.31}$$

(It is possible to do a consistency check by arriving at the same equations using the R^x_x and R^r_r equations).

We used the linearity of the equations to fix the value of ϕ_1 at the horizon. Then, using a a shooting algorithm, we looked for values of k and $A_1(r_0)$ that would give solutions that are well-behaved at the horizon and infinity, for a given γ. The results for k as a function of γ are shown in Figure 3.1. As can be seen from the figure, the wavelengths of the threshold instabilities diverge as one approaches the thermodynamic stability limit given by $\gamma = \gamma^* \equiv \frac{1}{\sqrt{2}}$ (the lower limit in (3.27)). $\gamma = 1$, which corresponds to the upper limit in (3.27), is where the causal structure of the Chamblin-Reall solutions changes: in the notation of Chamblin and Reall [117], going from $\gamma < 1$ to $\gamma > 1$ corresponds to going from the $b^2 < D - 1$ case to the $b^2 > D - 1$ case of the type II solutions.

For $\gamma < 1/\sqrt{2}$, we found no static perturbation. This plus the thermodynamic stability of the solutions strongly suggests that they are stable.

In order to address the question of what happens near the threshold of instability, we attempted to fit the behavior of k as a function of γ to a power law near $\gamma = \gamma^*$. The fit was considerably better if the threshold of stability was shifted to $\gamma^*_{num} \approx 1.003\gamma^*$: then we found $k \sim \sqrt{\gamma - \gamma^*_{num}}$. The fit to a power law was still less than

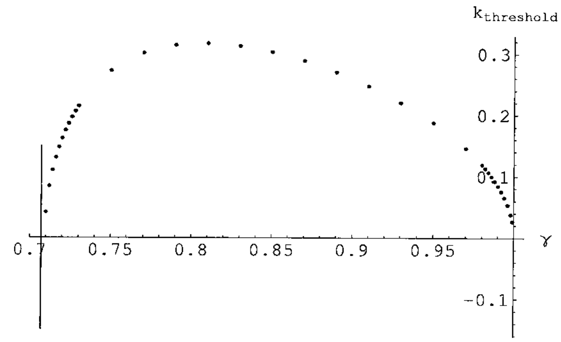

Figure 3.1: Stability of Chamblin-Reall solutions. The dots represent threshold instabilities of the Chamblin-Reall solutions. The vertical line at $\gamma^* = 1/\sqrt{2}$ indicates the thermodynamic stability limit.

spectacular. We suspect that numerical error contributed both to this and to the discrepancy between γ^* and γ^*_{num}.

3.4 Crossing the threshold of stability: non-dilatonic branes

Next, we would like to investigate the dynamical stability of D3, M2 and M5-branes. Following the philosophy introduced in the beginning of section 3.3, we will do a Kaluza-Klein reduction of the metrics (3.1) and (3.13) on all but one of the spatial worldvolume directions and S^{n+1}, to end up with a three-dimensional background. We will be working with s-wave perturbations of this background, keeping only the two scalars that describe the sizes of S^{n+1} and the torus, ignoring all the higher harmonics. There will be two contributions to the potential of these scalars: one due to the curvature of the S^{n+1} that we compactify on, another due to the kinetic term of the 4-form (3-form) potential in IIB (eleven-dimensional) supergravity. Both of these terms will turn out to be exponentials of a linear combination of the two scalars, hence the notion of universality classes introduced in section 3.3 will be relevant in this context.

3.4.1 Kaluza-Klein reduction

We want to perform the dimensional reduction in a way that would give an Einstein-Hilbert action for gravity, without a scalar-dependent coefficient in front of the three-dimensional Ricci scalar. For this purpose, let's start with a general metric

$$ds_D^2 = d\bar{s}_r^2 + e^{2K} ds_p^2 + e^{2L} ds_q^2 \,, \tag{3.32}$$

where $D = p + q + r$, $d\bar{s}_r^2$ is a metric on the Lorentzian r-manifold X_r, ds_p^2 and ds_q^2 are metrics on compact Riemannian manifolds Y_p and Z_q, respectively, and K and L are functions that depend only on the coordinates of X_r. For example, for the case of the D3-brane, $p = 2$, $Y_p = T^2$ (two worldvolume directions are compactified), $q = 5$, $Z_q = S^5$, $r = 3$. Defining

$$d\bar{s}_r^2 = e^{2N} ds_r^2 \,, \tag{3.33}$$

with

$$N = -\frac{pK + qL}{r-2} \,, \tag{3.34}$$

one gets

$$\sqrt{-g_D} R(ds_D^2) = \sqrt{-g_r}\sqrt{g_p}\sqrt{g_q}\Big[R(ds_r^2) + e^{2(N-K)} R(ds_p^2) + e^{2(N-L)} R(ds_q^2) - \frac{1}{r-2}(p(r+p-2)\partial_\mu K \partial_\nu K g_r^{\mu\nu} + q(r+q-2)\partial_\mu L \partial_\nu L g_r^{\mu\nu} + 2pq\partial_\mu K \partial_\nu L g_r^{\mu\nu})\Big] \,, \tag{3.35}$$

where $R(ds^2)$ denotes the Ricci scalar for the metric ds^2. The first term gives the Einstein-Hilbert lagrangian for ds_r^2 (without a scalar-dependent factor—as we wanted), and the rest are the kinetic and potential terms for the scalars K and L. In our applications, $R(ds_p^2)$ and $R(ds_q^2)$ will be the Ricci scalars of T^p and S^q, respectively, i.e., 0 and $q(q-1)$.

Using these results, we get the dimensionally reduced form of the D3-brane metric and the scalars as

$$ds_0^2 = -r^{10} f h dt^2 + r^{10}\frac{f^2}{h}dr^2 + r^{10} f dx^2$$

$$K_0(r) = -\frac{1}{4}\log f \tag{3.36}$$

$$L_0(r) = \log r + \frac{1}{4}\log f$$

where f and h are given by (3.2) with $n = 4$. The three-dimensional action is given by

$$S = \int d^3x\sqrt{g_3}\left(R - \frac{1}{2}\partial_\mu\phi^i\partial_\nu\phi^j G_{ij} g_3^{\mu\nu} - V(\vec{\phi})\right) \tag{3.37}$$

where

$$G_{ij} = \begin{pmatrix} 12 & 20 \\ 20 & 60 \end{pmatrix} \qquad \vec{\phi} = \begin{pmatrix} K \\ L \end{pmatrix} \tag{3.38}$$

$$V(\vec{\phi}) = -20e^{-4K-12L} + 8\tilde{R}_{D3}^8 e^{-4K-20L}, \tag{3.39}$$

and $\tilde{R}_{D3}$ is the length scale defined in (3.7), with $n = 4$. The first term in V comes from (3.35), and the second one comes from the dimensional reduction of the F_5^2 term in the IIB SUGRA action. We have omitted overall factors from (3.37), and we have introduced notation that somewhat obscures dimensional checks: for instance, e^L has the dimensions of length. In practice, we will simply choose an arbitrary numerical value for $\tilde{R}_{D3}$ for the numerics in section 3.4.2. Another way to think of this is that we choose an arbitrary value for the Planck length, which is permissible because our analysis is entirely classical.

Since the second term in the potential is due to the charge of the D3-brane, we expect this term to be of little significance in the large non-extremality limit. By performing a linear transformation of the scalars K and L to canonically normalized ones (i.e., those with $G_{ij} = \delta_{ij}$ in equation (3.15)) one of which is aligned with the exponent in the first term of the potential, one finds that this term corresponds to a

Chamblin-Reall coefficient $\gamma = \sqrt{\frac{3}{5}}$. That this value is in the unstable range (3.27) is consistent with expectations, since the brane is expected to be unstable when the effects of the charge are negligible.

Another consistency check can be done by calculating the relevant exponent α in the Chamblin-Reall equation of state by using (3.21). The equation of state of an uncharged brane can be read off from (3.3) and (3.6) to be $S \propto VT^{-(n+1)}$. The exponent obtained from (3.21) by setting $\gamma = \sqrt{\frac{3}{5}}$ agrees precisely with this: one gets $S \propto VT^{-5}$ in both cases.

When the brane is near-extremal, $R^4 \equiv r_0^4 \sinh^2\alpha \gg r_0^4$, so for radii close to the horizon, $R \gg r$. Using this in (3.36), we see that L_0 is approximately constant near the horizon, i.e., for a near-extremal D3-brane the "active" scalar close to the horizon is K. Thus, in order to read off the relevant Chamblin-Reall exponent γ in the near-extremal regime, one sets $L = const$ in the action, and rescales K to make it canonically normalized: $\tilde{K} = \sqrt{12}K$. Then, both terms in the potential become a multiple of $e^{-\frac{2\tilde{K}}{\sqrt{3}}}$ which shows that $\gamma = \frac{1}{\sqrt{3}}$.[5] This value of γ is in the thermodynamically stable range, as expected for a near extremal D3-brane. Equation (3.21) gives the equation of state of the near-extremal D3-brane to be $S \propto VT^3$, which is the equation of state of a 3+1-dimensional CFT, in line with the AdS/CFT correspondence.

The expected transition from the stability for near-extremal branes to instability at large non-extremalities is also confirmed by the numerical analysis of dynamical instabilities, which we present in the next section.

The analysis for black M2 and M5-branes is similar. M2-brane versions of (3.36),

[5]The sign of γ is arbitrary since the field redefinition $\phi \to -\phi$ is equivalent to switching the sign of γ in (3.22), (3.24) and (3.25).

(3.38), and (3.39) are

$$
\begin{aligned}
ds_0^2 &= -r^{14} f h dt^2 + r^{14}\frac{f^2}{h}dr^2 + r^{14} f dx^2 \\
K_0(r) &= -\frac{1}{3}\log f \\
L_0(r) &= \log r + \frac{1}{6}\log f \\
G_{ij} &= \begin{pmatrix} 4 & 14 \\ 14 & 112 \end{pmatrix} \\
V(\vec{\phi}) &= -42 e^{-2K-16L} + 18\tilde{R}_{M2}^{12} e^{-2K-28L} ,
\end{aligned}
\tag{3.40}
$$

and M5-brane versions are

$$
\begin{aligned}
ds_0^2 &= -r^{8} f h dt^2 + r^{8}\frac{f^2}{h}dr^2 + r^{8} f dx^2 \\
K_0(r) &= -\frac{1}{6}\log f \\
L_0(r) &= \log r + \frac{1}{3}\log f \\
G_{ij} &= \begin{pmatrix} 40 & 32 \\ 32 & 40 \end{pmatrix} \\
V(\vec{\phi}) &= -12 e^{-8K-10L} + \frac{9}{2}\tilde{R}_{M5}^{6} e^{-8K-16L} ,
\end{aligned}
\tag{3.41}
$$

where h and f are given by (3.2) and $\tilde{R}_{M2}$ and $\tilde{R}_{M5}$ are given by (3.7) with $n = 6$ for the M2-brane and $n = 3$ for the M2-brane. Once again for large non-extremality the potential terms due to the charges can be neglected, and by using canonically normalized scalars one can calculate the γ's that are relevant for the terms due to the curvatures of S^{n+1}. One gets $\gamma = \frac{2}{\sqrt{7}}$ for the M2-brane and $\gamma = \frac{\sqrt{5}}{2\sqrt{2}}$ for the M5-brane. As in the D3-brane case, these are consistent with the equations of state of uncharged branes: $S \propto VT^{-7}$ for the M2-brane and $S \propto VT^{-4}$ for the M5-brane. In the near-extremal limits, the "active" scalar is K for the M2 and M5-branes as well, and one gets the corresponding Chamblin-Reall coefficients as $\gamma = \frac{1}{2}$ for the M2-brane and $\sqrt{\frac{2}{5}}$ for the M5-brane. The equations of state one gets from these are consistent with expectations from AdS/CFT: $S \propto VT^2$ for the M2-brane and $S \propto VT^5$ for the M5-brane.

3.4.2 Numerical stability analysis for non-dilatonic branes

We will describe the stability analysis for the D3-brane in some detail, and just show results for the M2-brane and the M5-brane.

In order to investigate the dynamical stability of the three-dimensional background (3.36), obtained by Kaluza-Klein reduction from the D3-brane, we introduce the perturbations

$$ds^2 = -e^{2A(r,x)}\left(r^{10} f h dt^2\right) + e^{2B(r,x)}\left(r^{10}\frac{f^2}{h}dr^2 + r^{10} f dx^2\right)$$

$$\begin{aligned} A(r,x) &= \lambda \cos kx A_1(r) \\ B(r,x) &= \lambda \cos kx B_1(r) \\ K(r,x) &= K_0(r) + \lambda \cos kx K_1(r) \\ L(r,x) &= L_0(r) + \lambda \cos kx L_1(r)\,, \end{aligned} \tag{3.42}$$

and work to linear order in λ. K and L are the scalars introduced in (3.37).

There are four nontrivial Einstein equations (those involving R^t_t, R^r_r, R^x_x and R^r_x), and two scalar equations of motion. Using the R^r_x equation, one can solve for $B_1 = B_1(A_1, A_1', K_1, L_1)$. The R^t_t equation and both of the scalar equations of motion involve B_1, but not its derivatives. Plugging in B_1 as found from the R^r_x equation into the R_t^t equation and the scalar equations of motion, we end up with three equations involving A_1, K_1 and L_1 (it is possible to do a consistency check by arriving at the same equations using the R_x^x and R_r^r equations). Defining $R^4 = r_0^4 \sinh^2 \alpha$, the equations read

$$
\begin{aligned}
0 =& r \Big(16\, r^2\, {r_0}^4 \left(5\, r^8 + 10\, r^4\, R^4 + 3\, R^8 - 2\, R^4\, {r_0}^4\right) \\
&\quad + k^2 \left(r^4 + R^4\right)^2 \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4 \left(R^4 - {r_0}^4\right)\right) \Big) A_1(r) \\
&- 32\, r^3\, {r_0}^4 \left(5\, r^8 + 10\, r^4\, R^4 + 3\, R^8 - 2\, R^4\, {r_0}^4\right) K_1(r) \\
&- 160\, r^3 \left(2\, r^4\, R^4\, {r_0}^4 + R^8\, {r_0}^4 + r^8 \left(2\, R^4 + 3\, {r_0}^4\right)\right) L_1(r) \\
&- 2 \Big(5\, r^{16} + R^8\, {r_0}^8 - 5\, r^{12} \left(R^4 + {r_0}^4\right) + r^8 \left(20\, R^4\, {r_0}^4 + 6\, {r_0}^8\right) \\
&\quad + r^4 \left(5\, R^8\, {r_0}^4 - 3\, R^4\, {r_0}^8\right) \Big) A_1'(r) \\
&- r \left(r^4 + R^4\right) \left(r^4 - {r_0}^4\right) \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4 \left(R^4 - {r_0}^4\right)\right) A_1''(r)
\end{aligned}
\tag{3.43}
$$

$$
\begin{aligned}
0 =& 8\, r^3\, {r_0}^4\, \left(10\left(r^4+R^4\right)^2 - 4\, R^4\, \left(R^4+{r_0}^4\right)\right)\, A_1(r) \\
&+ \Bigg(-32\, r^3\, {r_0}^4\, \left(5\, r^8 + 10\, r^4\, R^4 + 3\, R^8 - 2\, R^4\, {r_0}^4\right) \\
&\quad + 3\, k^2\, r\left(r^4+R^4\right)^2\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right)\Bigg)\, K_1(r) \\
&+ 5\Bigg(k^2\, r\left(r^4+R^4\right)^2\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right) \\
&\quad - 32\, r^3\, \left(2\, r^4\, R^4\, {r_0}^4 + R^8\, {r_0}^4 + r^8\, \left(2\, R^4 + 3\, {r_0}^4\right)\right)\Bigg)\, L_1(r) \\
&+ \left(r^4 - {r_0}^4\right)\, \left(15\, r^{12} + 3\, R^8\, {r_0}^4 + 5\, r^8\, \left(10\, R^4 + 3\, {r_0}^4\right) + r^4\, \left(15\, R^8 - 2\, R^4\, {r_0}^4\right)\right)\, A_1'(r) \\
&- 3\, \left(r^4+R^4\right)\, \left(5\, r^4 - {r_0}^4\right)\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right)\, K_1'(r) \\
&- 5\, \left(r^4+R^4\right)\, \left(5\, r^4 - {r_0}^4\right)\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right)\, L_1'(r) \\
&- 3\, \left(r^4+R^4\right)\, \left(r^5 - r\, {r_0}^4\right)\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right)\, K_1''(r) \\
&- 5\, \left(r^4+R^4\right)\, \left(r^5 - r\, {r_0}^4\right)\, \left(5\, r^8 - R^4\, {r_0}^4 + 3\, r^4\, \left(R^4 - {r_0}^4\right)\right)\, L_1''(r)
\end{aligned}
\tag{3.44}
$$

$$
\begin{aligned}
0 =& 8\,r^3\,{r_0}^4\,\left(6\,\left(r^4+R^4\right)^2-4\,R^4\,\left(R^4+{r_0}^4\right)\right)\,A_1(r) \\
&+\Bigg(-32\,r^3\,{r_0}^4\,\left(3\,r^8+6\,r^4\,R^4+R^8-2\,R^4\,{r_0}^4\right) \\
&\quad +k^2\,r\,\left(r^4+R^4\right)^2\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right)\Bigg)\,K_1(r) \\
&+\Bigg(3\,k^2\,r\,\left(r^4+R^4\right)^2\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right) \\
&\quad -32\,r^3\,\left(3\,R^8\,{r_0}^4+r^8\,\left(6\,R^4+9\,{r_0}^4\right)+r^4\,\left(-4\,R^8+2\,R^4\,{r_0}^4\right)\right)\Bigg)\,L_1(r) \\
&+\left(r^4-{r_0}^4\right)\,\left(9\,r^{12}+R^8\,{r_0}^4+r^8\,\left(34\,R^4+9\,{r_0}^4\right)+5\,r^4\,\left(R^8-2\,R^4\,{r_0}^4\right)\right)\,A_1'(r) \\
&-\left(r^4+R^4\right)\,\left(5\,r^4-{r_0}^4\right)\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right)\,K_1'(r) \\
&-3\,\left(r^4+R^4\right)\,\left(5\,r^4-{r_0}^4\right)\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right)\,L_1'(r) \\
&-\left(r^4+R^4\right)\,\left(r^5-r\,{r_0}^4\right)\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right)\,K_1''(r) \\
&-3\,\left(r^4+R^4\right)\,\left(r^5-r\,{r_0}^4\right)\,\left(5\,r^8-R^4\,{r_0}^4+3\,r^4\,\left(R^4-{r_0}^4\right)\right)\,L_1''(r)\,.
\end{aligned}
\tag{3.45}
$$

Solving these linear ODE's numerically, one extracts using a shooting algorithm the value of k where a static perturbation exists, for any specified non-extremal mass. The results of this numerical investigation are summarized in Figure 3.2. For large non-extremality, the threshold wavenumbers fit nicely onto the curve that gives those of an uncharged 3-brane. (This curve was obtained by reading off the wavenumber of the threshold instability of an uncharged 3-brane from the plots in [33, 29], and doing an appropriate scaling to get the values for an arbitrary r_0.) As one approaches the thermodynamic stability limit, the wavenumbers approach to zero. In fact, by zooming into this region, we were able to show that the wavenumbers go to zero approximately with a power-law behavior, with the critical exponent being close to 1/2:

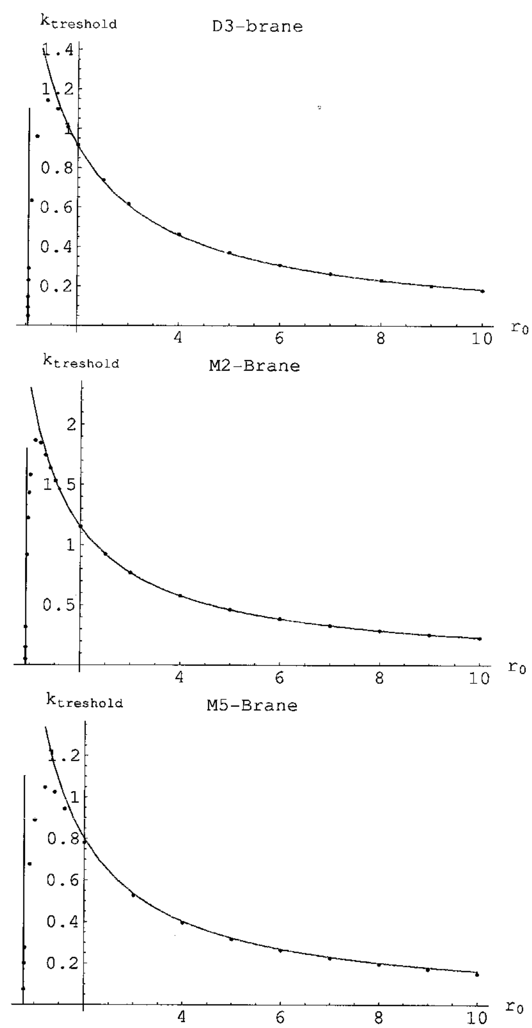

Figure 3.2: Instabilities of black D3, M2, and M5-branes. The dots represent threshold instabilities of the black branes we found. The vertical lines indicate the thermodynamic stability limits, which were found by (3.12), with $\tilde{R} = 1$ for the D3-brane, $\tilde{R} = (1/3)^{1/6}$ for the M2-brane, and $\tilde{R} = (2/3)^{1/3}$ for the M5-brane. The curves indicate wavenumbers of threshold instabilities of *uncharged* black branes, with r_0 being the Schwarzschild radius.

that is, $k \sim \sqrt{r_0 - r_0^*}$ where r_0^* is the critical value where the instability disappears. Because M is a non-singular function of r_0, one could also write $k \sim \sqrt{M - M_*}$. A naive way of understanding this is to think of some effective theory describing the perturbations in terms of a bosonic field with $m^2 < 0$. Then the criterion for the perturbations to be static is $\omega^2 = k^2 + m^2 = 0$, where k is the wave-number of a static perturbation. The resulting equation, $k = \sqrt{-m^2}$, predicts the observed scaling, provided $m^2 \sim r_0^* - r_0$. Such analytic behavior is a standard assumption: our argument basically amounts to a naive application of Landau theory.[6]

The numerical stability analyses for the M2-brane and M5-brane are similar to the one for the D3-brane. The results are summarized in Figure 3.2. Once again, near the thermodynamic stability limit, the wavenumbers go approximately as $\sqrt{r_0 - r_0^*}$.

3.5 Conclusions

The stability of charged p-brane solutions was first attacked in [29]. Although the methods presented there are in principle practicable for any solution, they require considerable computational fortitude to apply. We have argued that a simple notion of universality classes is fairly broadly applicable to the question of brane stability: one commonly finds a thermodynamic relation $S \propto VT^\alpha$ for charged p-brane solutions in some limiting regime of parameters, and when one does, the stability properties depend only on α—stability pertaining when $\alpha > 0$. Given our treatment of the problem via a Kaluza-Klein reduction / truncation to a 2+1-dimensional action involving scalars and gravity, the dispersion relation of unstable modes would also appear to be the same for any solution in the universality class labeled by a given α.

We have further shown, in section 3.4, that when one approaches a boundary of stability, the critical wavelength separating stable from unstable modes diverges, and that it does so for the cases studied with a critical exponent that could be guessed

[6]It is actually a little too naive, because it predicts an incorrect dispersion relation. In [33, 29], it was found that $\omega \to 0$ as $k \to 0$, whereas using $\omega^2 = k^2 + m^2$ would suggest finite ω at $k = 0$. Possibly an improved understanding could be based on hydrodynamic considerations.

on the grounds of a naive effective field theory argument. It would be interesting to see if other exponents arise from non-generic situations, for instance a case where the temperature T depends on mass as $T_{\max} - T \sim |M - M_*|^\beta$ for some $\beta \neq 2$. It is possible that spinning brane solutions provide a venue for more intricate thermodynamics [43, 44], but one would encounter there the complication of chemical potentials for the various angular momenta.

For the p-branes of type II string theory and M-theory, away from extremality but without angular momenta, thermodynamic stability pertains up to an upper mass limit given in equation (3.11). For the D3-brane, the M2-brane, and the M5-brane, our numerical analysis supports the claim [12, 13, 45] that dynamical and thermodynamic stability coincide. Extending the analysis to other p-branes should not be too difficult.

In conclusion, it seems that the notion of universality classes that we introduced in section 3.3, together with the behavior at a boundary of stability explored in section 3.4, represent a fairly comprehensive description of the qualitative features of the Gregory-Laflamme instability in linearized perturbation theory. Of course, it would be desirable to go beyond classical perturbation theory in understanding universality classes of behaviors for non-uniform horizons. Some of the methods of this paper may prove useful in that broader context as well.

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