Quantum States: With a View Toward Homological Algebra

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To my dear mother who does not read English,

致妈妈：

谁言寸草心，报得三春晖。

ABSTRACT

The thesis comprises three papers covering different topics in quantum many-body physics. The first paper examines translationally invariant Pauli stabilizer codes, introducing invariants called charge modules and discussing their properties. The second paper explores invertible (G-invariant) states of 1D bosonic quantum lattice systems (or spin chains), demonstrating a full classification using group cohomology. The third paper analyzes the relation between ordinary correlators and Kubo's canonical correlators for thermal states of systems with short-range interactions. Overall, the thesis highlights the power of mathematics, especially homological methods, in understanding quantum states.

PUBLISHED CONTENT AND CONTRIBUTIONS

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TABLE OF CONTENTS

Chapter 1

OVERVIEW

The thesis is about **states** of quantum spin systems. We focus on identifying structures underlying entire classes of states that can be analyzed using mathematical techniques, particularly homological algebra. In this overview, we first define a quantum spin system and its states using the language of C^*-algebra. Then, we introduce three classes of quantum states relevant to our discussion.

1.1 Quantum spin system and its states

Quantum spin systems are widely used for studying the behavior of many-body quantum systems. They have applications in a variety of fields including condensed matter physics, quantum information theory, and quantum statistical mechanics. In recent years, they have gained attention and popularity among mathematicians due to their intriguing connections to several mathematical disciplines, including representation theory, topology, and operator algebra. These connections have spurred new developments in both mathematics and physics, with quantum spin systems serving as a bridge between the two fields.

A *quantum spin system* consists of an infinite discrete space (Γ, d), often called a lattice[1], as well as an algebra of observables $\mathscr{A}$. Let $P_0(\Gamma)$ be the set of finite subsets of Γ. The algebra of *local observables* is

$$\mathscr{A}_{\text{loc}} = \varinjlim_{X \in \mathcal{P}_0(\Gamma)} \bigotimes_{x \in X} M_{d_x}, \tag{1.1}$$

where M_{d_x} is the algebra of $d_x \times d_x$ matrices with $\sup_{x \in X} d_x < \infty$. We also denote $\bigotimes_{x \in X} M_{d_x}$ by $\mathscr{A}_X$ for all $X \in \mathcal{P}_0(\Gamma)$. The C^*-algebra $\mathscr{A}$ of *quasi-local observables* is the norm completion of $\mathscr{A}_{\text{loc}}$. Later we impose additional assumptions on (Γ, d):

- in Chapter 2, $\Gamma = \mathbb{Z}^D$ for any positive integer D;
- in Chapter 3, $\Gamma = \mathbb{Z}$. These models are also known as *spin chains*;
- in Chapter 4, (Γ, d) satisfies a growth bound. For a finite subset $X \in \mathcal{P}_0(\Gamma)$, we define $B_r(X) = \{y \in \Gamma : d(y, X) < r\}$. We assume there is a constant

[1]The use of the term "lattice" does not necessarily indicate that all systems possess an abelian group as their lattice, although it is often the case.

$D > 0$, such that for all $X \in \mathcal{P}_0(\Gamma)$ there exists $C > 0$ with

$$|B_r(X)| \leq C|X|(1+r)^D.$$

Remark 1. When concerning a system on an infinite lattice, it is necessary to use C^*-algebra of observables instead of Hilbert space of states. The latter is more commonly seen in the context of quantum information where the Hilbert space is a tensor product of $\mathbb{C}^d$ called qudits. These two approaches are equivalent for a system on a finite lattice. However, the latter breaks down on infinite lattices since infinite tensor product of Hilbert spaces is ill-defined [1].

A *state* is a positive linear function $\psi : \mathscr{A} \to \mathbb{C}$ such that $\psi(\mathrm{Id}) = 1$. Here $\mathrm{Id} \in \mathscr{A}$ is the identity operator. The set $\mathfrak{S}$ of all states is clearly convex. A state is called *pure* if it is extremal[2] and *mixed* otherwise. Sometimes, we want to "stack" two systems $\mathscr{A}_1$ and $\mathscr{A}_2$ to produce a new system $\mathscr{A}_1 \otimes \mathscr{A}_2$. For two respective states ψ_1 and ψ_2, this produces a new state $\psi_1 \otimes \psi_2$ on the stacked system.

The rest of this chapter provides a summary of three distinct classes of states of physical and mathematical interests. We defer precise definitions and detailed references to later chapters.

1.2 Stabilizer states

Let $\mathscr{U}(\mathscr{A}_{\mathrm{loc}})$ denote the group of unitary elements in $\mathscr{A}_{\mathrm{loc}}$. For any $\mathcal{U} \in \mathscr{U}(\mathscr{A}_{\mathrm{loc}})$ and $\psi \in \mathfrak{S}$, a new state $\psi_{\mathcal{U}}$ is defined by

$$\psi_{\mathcal{U}}(\mathcal{A}) = \psi(\mathcal{U}^*\mathcal{A}\mathcal{U}),$$

where $\mathcal{A} \in \mathscr{A}$. Thus it defines an action of $\mathscr{U}(\mathscr{A}_{\mathrm{loc}})$ on $\mathfrak{S}$. There is a special subgroup $\mathscr{P}$ of $\mathscr{U}(\mathscr{A}_{\mathrm{loc}})$ called the Pauli operators. A *stabilizer state* is a pure state invariant under action of certain *abelian* subgroup of $\mathscr{P}$. This abelian subgroup is sometimes referred to as the stabilizers (of the state). In order to understand stabilizer states, it is sufficient to study their respective groups of stabilizers. Surprisingly, this leads us to the world of commutative algebra.

Any two Pauli operators $\mathcal{A}, \mathcal{B}$ satisfies the commutation relation

$$\mathcal{A}\mathcal{B} = \xi\mathcal{B}\mathcal{A},$$

where ξ is always a root of unity. Moreover, there exists a mapping from $\mathscr{P}$ to certain R-module[3] P, equipped with a pairing $\omega : P \times P \to R$, such that ξ is

[2] An extremal point of a convex set is a point within the set that cannot be expressed as a convex combination of any two distinct points within the set.

[3] Here R is always a Gorenstein ring dependent on details of the lattice.

computable from ω. Assuming states are translation-invariant, then their groups of stabilizers correspond precisely to submodules L of P such that $L^{\omega} := \{x \in P \mid \omega(x,l) = 0 \text{ for all } l \in L\} = L$. We call them *Lagrangian submodules* of P due to their resemblance to Lagrangian subspaces of a symplectic vector space. Finally, homological methods from commutative algebra are brought in to study Lagrangian submodules. Previously discovered invariants of stabilizer states are found to correspond to derived functors such as $\mathrm{Ext}^1_R(P/L, R)$. More importantly, additional homological invariants, such as higher Ext and local cohomologies, are used to further our understanding of stabilizer states.

1.3 Invertible states

Motivated by physical notions of quantum phases, an equivalence relation on states is defined called LGA-equivalence. A goal is to completely classify states modulo LGA-equivalence. This goal is attained on spin chains for a class of states that are *invertible*. Below, we sketch the definition of invertibility.

A state ψ is called factorized if $\psi(\mathcal{A}\mathcal{B}) = \psi(\mathcal{A})\psi(\mathcal{B})$ whenever $\mathcal{A}$ and $\mathcal{B}$ are local observables supported on two different sites. While all factorized pure states of $\mathscr{A}$ are LGA-equivalent, the converse is far from being true. In fact, a pure state that is LGA-equivalent to a factorized pure state is known as short-range entangled (SRE). A pure state is called *invertible* if it stacks with another pure state and produces an SRE state.

1.4 KMS states

KMS states are the only mixed states this thesis studies. They satisfy the so-called Kubo-Martin-Schwinger condition requiring a certain "time-periodicity". This condition relates the expectation values of observables in the system at different times, and it is a condition for a state to be in thermal equilibrium.

For a quantum spin system, time evolution $\tau : \mathbb{R} \to \mathrm{Aut}(\mathscr{A})$ is a strongly continuous one-parameter group of C^*-algebra automorphisms. Given τ and a number $\beta \in [0, \infty)$ called inverse temperature, a state ϕ is called β-KMS[4] if for all $A, B \in \mathscr{A}_{loc}$ there exists a function $F_{A,B}(t)$ which is holomorphic on the open strip $S_\beta = \{z \in \mathbb{C} : 0 < \mathrm{Im} z < \beta\}$, and bounded and continuous on $\overline{S}_\beta$, such that

$$F_{A,B}(t) = \phi(A\tau_t(B)) \tag{1.2}$$

[4]KMS stands for Kubo-Martin-Schwinger.

and

$$F_{A,B}(t + i\beta) = \phi(\tau_t(B)A), \tag{1.3}$$

for all $t \in \mathbb{R}$.

The holomorphic nature of $F_{A,B}$ allows us to apply the residue theorem from complex analysis, which enables us to study the decay rates of functions that measure the static linear response of the state.

Chapter 2

TRANSLATION-INVARIANT PAULI STABILIZER CODE

2.1 Introduction

Pauli stabilizer codes are spin systems whose ground state (and excitations) are described by eigenequations for a set of mutually commuting operators, each of which is a tensor product of finitely many Pauli matrices, or generalizations thereof called clock and shift matrices. Initially these models were studied as a class of quantum error-correcting codes [2, 3]. Due to their mathematical tractability and nontrivial properties, they have also become popular as exactly solvable models of exotic phases of quantum matter. Qubits (or qudits) are typically placed on sites of a D-dimensional square lattice. Perhaps the most famous example is the toric code [4].

One may ask which quantum phases can be realized as Pauli stabilizer codes. It has been shown [5, 6] that for codes on $\mathbb{Z}^2$ lattice with prime-dimensional qudits, stacks of toric codes are the only nontrivial phases with a unique ground state in infinite volume. The story is richer for qudits of composite dimension. Namely, it was shown [7] that every abelian anyon model which admits a gapped boundary [8] may be represented by a Pauli stabilizer code. It was conjectured that the list of models constructed therein is exhaustive (up to finite depth quantum circuits and stabilization). This proposal depends on several assumptions, one of which is that all local excitations are mobile and hence can be created at endpoints of string operators. In this chapter we prove this, extending earlier results for prime-dimensional qudits. The situation is even more complicated for $D > 2$ [9] due to existence of so-called fractons. These results show that mathematical study of stabilizer codes is an interesting and nontrivial problem. It is also closely related to classification of Clifford Quantum Cellular Automata [10, 11].

Let us recall how similar classification problems were handled in algebraic topology. Historically, researchers first discovered some basic invariants, such as Euler characteristic or fundamental group. Later they developed more systematic methods, including (co)homology and homotopy theory. In our situation, the module of topological point excitations [12] and (for the case $D = 2$) topological spin and braiding [6] are the known invariants. It is natural to look for machinery that produces their generalizations in order to make progress in the classification problem.

In this chapter, we develop such tools for translationally invariant Pauli stabilizer codes with qudits of arbitrary (perhap not even uniform) dimension placed on a lattice described by a finitely generated abelian group Λ. This setup incorporates infinitely extended as well as finite spatial directions. Of course physics crucially depends on $D = \mathrm{rk}(\Lambda)$ (the number of independent infinite directions). We describe stabilizer codes by symplectic modules over a group ring R of Λ and their Lagrangian (or more generally, isotropic) submodules. This is closely related to the approach developed in [12]. In contrast to treatment therein, the emphasis is on modules with direct physical interpretation, rather than their presentations with maps from free modules[1].

We propose a definition of modules Q^p of charges of p-dimensional excitations (anyons, fractons, strings, etc.) for every non-negative integer p. The construction of Q^p uses standard homological invariants of modules. In the case of local excitations ($p = 0$), our definition agrees with the known one. For general p, the physical interpretation of mathematically defined Q^p is most justified under the assumption that all charge modules have zero Krull dimension (which we interpret as the requirement that the excitations are mobile). In this case, we define for every element of Q^p an operator with $(p+1)$-dimensional support which creates an excitation on its boundary. This excitation is uniquely defined modulo excitations which can be created by p-dimensional operators. Its mobility (moving around with p-dimensional operators) is established. We show also that every element of Q^p gives rise to a $(D-p-1)$-form symmetry [13]. Furthermore, a braiding pairing between Q^p and Q^q (with $p+q = D-2$) is defined and its basic properties (such as symmetry) are established.

It is natural to expect that for codes with only mobile excitations, the underlying abelian groups of Q^p and pairings between them described above are (a part of) data of some Topological Quantum Field Theory (TQFT), e.g., an abelian higher gauge theory[2]. Such correspondence exists in every example known to authors. Modules Q^p have more structure, which does not seem to be captured by a TQFT: they are acted upon by the group of translations. In some cases this allows us to distinguish models with the same topological order which are distinct as Symmetry Enriched Topological (SET) phases with translational symmetry.

Section 2 details the mathematical set-up of translationally invariant stabilizer codes

[1] The latter approach is very useful in concrete computations. We prefer ours in general considerations.

[2] Say, with action $\frac{1}{4\pi}\sum_{p=1}^{D-1}\sum_{i,j} K_p^{ij}\int A_p^i \mathrm{d}A_{D-p}^j$, where A_p^i are p-form $\mathrm{U}(1)$ gauge fields and K matrices are non-degenerate and satisfy $K_p^{ij} = (-1)^{p+1}K_{D-p}^{ji}$.

in terms of commutative algebra. Rudiments of symplectic geometry over group rings of Λ are laid out here. Section 3 makes the connection between topological excitations and the functor Ext. Section 4 discusses operations on stabilizer codes, e.g., coarse-graining and stacking. In particular we prove that charge modules are invariant to coarse-graining and that they provide obstructions to obtaining a system from a lower dimensional one by stacking. Section 5 ventures a definition of mobility for excitations in any dimension. We also include a proof for the conjecture that in any 2D code with unique ground state, all excitations are mobile and can be created with string operators. In Section 6, we specialize to codes with only mobile excitations. It is shown that in this case, charges may be described by cohomology classes of a certain Čech complex. We show how to obtain interesting operators and physical excitations from Čech cocycles. Moreover, we define braiding in terms of a cup product in the Čech complex and show that our proposal reduces to what is expected for $D = 2$. Several examples are worked out in Section 7. Some known mathematical definitions and facts used in the main text are reviewed in appendices: Gorenstein rings in Appendix A, local cohomology in Appendix B, and Čech cohomology in Appendix C.

Let us mention some problems which are left unsolved in this work. Firstly, results of Section 6 are restricted to so-called Lagrangian stabilizer codes such that charge modules have Krull dimension zero. We would like to remove some of these assumptions in the future, for example to treat models with spontaneous symmetry breaking or fractons. Secondly, we did not prove that braiding is non-degenerate. We expect that this can be done by relating braiding to Grothendieck's local duality, in which we were so far unsuccessful. We expect also that the middle-dimensional braiding admits a distinguished quadratic refinement for $D = 4k+2$ (which is already known to be true for $D = 2$ from previous treatments) and that it is alternating (rather than merely skew-symmetric) for $D = 4k$. Thirdly, it is not known in general to what extent invariants we defined determine a stabilizer code, presumably up to symplectic transformations (corresponding to Clifford Quantum Cellular Automata), coarse graining, and stabilization. We hope that in the future a one-to-one correspondence between equivalence classes of stabilizer codes with only mobile excitations and some (abelian) TQFTs will be established.

2.2 Stabilizer codes and symplectic modules

In order to obtain homological invariants of a stabilizer code, we need to translate it to the language of modules. In this section we generalize [12] to codes with arbitrary (prime or composite) qudit dimensions. Multiple qudits are placed on each lattice site. A d-dimensional qubit is acted upon by shift and clock matrices X, Z, which satisfy

$$XZ = \mathrm{e}^{\frac{2\pi \mathrm{i}}{d}} ZX, \qquad X^d = Z^d = 1. \tag{2.1}$$

For brevity, products of Z and X (possibly acting on finitely many different qudits) and phase factors will be called Pauli operators. Unlike [12], our framework does not require qudits in a model to have a uniform dimension. Instead, an array of qudits with various dimensions populates each lattice site. We let n be a common multiple of dimensions of all qudits in a model.

All rings are commutative with unity and $\mathbb{Z}_n$ is the ring $\mathbb{Z}/n\mathbb{Z}$.

Definition 1. Let n be a positive integer and Λ a finitely generated abelian group. $\mathbb{Z}_n[\Lambda]$ is the group ring of Λ over $\mathbb{Z}_n$. When n and Λ are clear from the context, we denote $R = \mathbb{Z}_n[\Lambda]$. For $\lambda \in \Lambda$, we denote the corresponding element of R by x^λ. If $r = \sum_{\lambda \in \Lambda} r_\lambda x^\lambda$ (with all but finitely many $r_\lambda \in \mathbb{Z}_n$ equal to zero), we call r_0 the scalar part of r. Moreover, we let $\overline{r} = \sum_{\lambda \in} r_\lambda x^{-\lambda}$. Operation $r \mapsto \overline{r}$ is called the antipode.

Example 2. Suppose that $\Lambda = \mathbb{Z}^D$. Then R is the ring of Laurent polynomials in D variables $x_1, \ldots, x_D$, corresponding to D elements of a basis of $\mathbb{Z}^D$. A general element of R is a sum of finitely many monomials $x_1^{\lambda_1} \cdots x_D^{\lambda_D}$ with $\mathbb{Z}_n$ coefficients; exponents λ_i are in $\mathbb{Z}$. Here we use the more economical notation in which such a monomial is simply denoted x^λ. One may think of λ as a multi-index.

For a lattice Λ with the same array of qudits on each site, ring $R = \mathbb{Z}_n[\Lambda]$ describes certain basic operations on Pauli operators. An element x^λ translates a Pauli operator on the lattice by $\lambda \in \Lambda$, while a scalar $m \in \mathbb{Z}_n$ raises a Pauli operator to m-th power. As n is a common multiple of qudit dimensions, taking the n-th power of any Pauli operator gives a scalar. This action endows the collection P of all local Pauli operators modulo overall phases with an R-module structure. We will sometimes call elements of P operators for conciseness. Addition in P corresponds to composition of operators, which is commutative because we are disregarding phases. Specifically, if qudits on each site have respective dimensions $n_1, \ldots, n_q$, then P is isomorphic to the module $\bigoplus_{j=1}^q \mathbb{Z}_{n_j}[\Lambda]^{\oplus 2}$. It is not a free module unless $n = n_1 = n_2 = \cdots = n_q$. We will see that it nevertheless shares some homological properties of free modules, which is important in the study of invariants. In most cases, understanding of proofs is not necessary to read the remainder of the chapter.

We will also define an antipode-sesquilinear symplectic form $\omega : P \times P \to R$ on P, which captures commutation relations satisfied by Pauli operators. More precisely, if T, T' are Pauli operators corresponding to elements $p, p' \in P$, then

$$TT' = \exp\left(\frac{2\pi \mathrm{i}}{n}\omega(p,p')_0\right) T'T. \tag{2.2}$$

Thus it is the scalar part of ω which has most direct physical interpretation, whereas $\omega(p, p')$ encodes also commutation rules of all translates of T, T'. Algebraically ω is much more convenient to work with, essentially because the scalar part map $R \to \mathbb{Z}_n$ is not a homomorphism of R-modules. Sesquilinearity of ω implies that $\omega(x^\lambda p, x^\lambda p') = \omega(p, p')$, which is the statement that commutation relations of Pauli operators are translationally invariant.

Stabilizer code is a collection of eigenequations for a state[3] Ψ of the form

$$T\Psi = \Psi, \tag{2.3}$$

where T are Pauli operators (with phase factors chosen so that 1 is in the spectrum of T). If such equations are imposed for two operators T, T', then existence of solutions requires that $p, p' \in P$ satisfy $\omega(p, p')_0 = 0$. In a translationally invariant code, the same condition has to be satisfied for all translates of T, T', i.e., $\omega(p, p') = 0$. It follows that the images in P of operators defining the code generate a submodule L with $\omega|_L \equiv 0$. Such submodules of (P, ω) are called isotropic. The stabilizer code determines a unique state if L is Lagrangian, i.e., it is isotropic and every $p \in P$ such that $\omega(p, p') = 0$ for every $p' \in L$ is in L. Throughout the chapter, we refer to codes with this property as *Lagrangian codes*.

In quantum computation, one wishes to use spaces of states satisfying (2.3) to store and protect information. When error occurs, there are violations of eigenequations called syndromes. On the other hand, one may also think of solutions of (2.3) as ground states of a certain Hamiltonian. Then syndromes are also regarded as energetic excitations. Excited states are described by

$$T\Psi = \mathrm{e}^{\frac{2\pi \mathrm{i}}{n}\varphi(p)}\Psi, \tag{2.4}$$

where $p \in L$ corresponds to T and φ is a $\mathbb{Z}_n$-linear functional. The excitation is local (supported in a finite region) if $\varphi(x^\lambda p)$ vanishes for all but finitely many $\lambda \in \Lambda$.

[3] Here we regard Ψ as a vector in some Hilbert space on which Pauli operators act. This Hilbert space is not specified a priori. However, one can reinterpret the eigenequations as $\Psi^{\mathrm{pre}}(T) = 1$, where Ψ^{pre} is a state on the algebra of local operators. The Hilbert space and Ψ may be then constructed from Ψ^{pre} using the GNS construction.

The discussion above establishes a correspondence between a translationally invariant Pauli stabilizer code and an isotropic submodule of (P, ω). This correspondence will allow us to tap into the power of homological algebra. For the rest of the section, we develop the right-hand side of the correspondence with additional generality.

Definition 3. For a $\mathbb{Z}_n$-module M, let $M^\# = \mathrm{Hom}_{\mathbb{Z}_n}(M, \mathbb{Z}_n)$. If M is an R-module, then $M^\#$ is made an R-module as follows: $r\varphi(m) = \varphi(rm)$ for $r \in R$, $\varphi \in M^\#$ and $m \in M$. Moreover, we can define

$$M^\#_\Lambda = \{\varphi \in M^\# \mid \forall m \in M \;\; \varphi(x^\lambda m) \neq 0 \text{ for finitely many } \lambda \in \Lambda\}. \tag{2.5}$$

Definition 4. Let M be an R-module. We define $\overline{M}$ to be the R-module which coincides with M as an abelian group, but with antipode R-action. In other words, if $m \in M$, we denote the corresponding element of $\overline{M}$ by $\overline{m}$ and put $x^\lambda \overline{m} = \overline{x^{-\lambda} m}$. Furthermore, we let $M^* = \mathrm{Hom}_R(\overline{M}, R)$. M^* is identified with the module of $\mathbb{Z}_n$-linear maps $f : M \to R$ such that $f(rm) = \overline{r} f(m)$ for $r \in R$ and $m \in M$.

The following Lemma provides a useful description of M^*.

Lemma 5. Let M be an R-module. The map taking $\varphi \in M^*$ to its scalar part $\varphi_0 \in \overline{M}^\#_\Lambda$ (i.e., $\varphi_0(m) = \varphi(m)_0$ for $m \in M$) is an R-module isomorphism with inverse given by the formula

$$\varphi(m) = \sum_{\lambda \in \Lambda} \varphi_0(x^\lambda m) x^\lambda. \tag{2.6}$$

Definition 6. We denote the total ring of fractions of R by K.

Please see Appendix A.1 for some definitions referred to below.

Lemma 7. R is a Gorenstein ring of dimension $\mathrm{rk}(\Lambda)$, the free rank of Λ. Its total ring of fractions K is a QF ring.

Proof. $\mathbb{Z}_n$ is a QF ring by Baer's test. Thus $(-)^\#$ is an exact functor and $R^\#$ is an injective R-module, as $\mathrm{Hom}_R(-, R^\#) = (-)^\#$. Now suppose that $\mathrm{rk}(\Lambda) = 0$. Then R is finite, so $\dim(R) = 0$. We have a bilinear form

$$R \times R \ni (r, r') \mapsto (rr')_0 \in \mathbb{Z}_n, \tag{2.7}$$

which yields an isomorphism $R \cong R^\#$. Hence R is a QF ring.

Next, let Λ be arbitrary. We can split $\Lambda = \Lambda_1 \oplus \Lambda_2$, where Λ_1 is finite and Λ_2 free. We have $R = \mathbb{Z}_n[\Lambda_1][\Lambda_2]$, which is a Laurent polynomial ring in $D = \mathrm{rk}(\Lambda)$ variables over the QF ring $\mathbb{Z}_n[\Lambda_1]$. By Lemmas 61, 63, R is a Gorenstein ring. Standard dimension theory shows that $\dim(R) = D$.

Invoking 61, K is also Goreinstein. It remains to show that $\dim(K) = 0$. As $\mathbb{Z}_n[\Lambda_1]$ is Artinian, it is the product $\prod_{i=1}^{s} A_i$ of some Artin local rings A_i. Thus $R = \prod_{i=1}^{s} A_i[\Lambda_2]$. An element of R is a zero-divisor if and only if its component in some $A_i[\Lambda_2]$ is a zero divisor, so $K = \prod_{i=1}^{s} K_i$, where K_i is the total ring of fractions of $A_i[\Lambda_2]$. We will show that each K_i is Artinian.

Let $\mathfrak{m}_i$ be the maximal ideal of A_i. Then $\mathfrak{m}_i$ is nilpotent and every element of $A_i \setminus \mathfrak{m}_i$ is a unit. Clearly $\mathfrak{m}_i[\Lambda_2]$ is a prime ideal in $A_i[\Lambda_2]$. We claim that it is the unique minimal prime. Indeed, if $\mathfrak{q} \subset A_i[\Lambda_2]$ is a prime ideal, then $\mathfrak{q} \cap A_i$ is prime in A_i, hence equal to $\mathfrak{m}_i$. Thus $\mathfrak{m}_i[\Lambda] \subset \mathfrak{q}$ and the claim is established. Next, McCoy theorem [14] and nilpotence of $\mathfrak{m}_i$ imply that $\mathfrak{m}_i[\Lambda_2]$ is the set of zero divisors of $A_i[\Lambda_2]$, so every non-minimal prime ideal of $A_i[\Lambda_2]$ is killed in K_i. □

Recall that an element of R is said to be regular if it is not a zero divisor. The torsion submodule of an R-module M is the set of all elements of M annihilated by a regular element of R. Equivalently, it is the kernel of the natural map $M \to M \otimes_R K$. If M coincides with its torsion submodule, it is called a torsion module. If the torsion submodule of M is 0, then M is said to be torsion-free. Quotient of any module by its torsion submodule is torsion-free.

Lemma 8. Let M be an R-module.

1. M is torsion if and only if $M^* = 0$.
2. If M is finitely generated, then M is torsion-free if and only if it can be embedded in some free module R^t.

Proof. 1. $\Longleftarrow$: Let $M^* = 0$. Then $\mathrm{Hom}_K(\overline{M} \otimes_R K, K) = M^* \otimes_R K = 0$ (since Hom commutes with localization), so $\overline{M} \otimes_R K = 0$ by Lemmas 7, 64. Thus $\overline{M}$, and hence M, is a torsion module.

2. $\Longrightarrow$: As M is torsion-free, it embeds in $M \otimes_R K$, which in turn embeds in K^t by Lemmas 7 and 64. Let $e_1, \ldots, e_t$ be a basis of K^t. Since M is finitely generated, there exists a regular element $d \in R$ such that the image of M in K^t is contained in the R-linear span of $d^{-1}e_1, \ldots, d^{-1}e_t$, which is R-free. □

Definition 9. Quasi-symplectic module is a finitely generated R-module M equipped with a $\mathbb{Z}_n$-bilinear pairing $\omega : M \times M \to R$ satisfying:

1. $\omega(m', rm) = r\omega(m', m) = \omega(\bar{r}m', m)$ for $r \in R$ and $m, m' \in M$,
2. $\omega(m, m)_0 = 0$ for every $m \in M$,
3. the map $\flat : M \ni m \mapsto \omega(\cdot, m) \in M^*$ is injective.

We write M^*/M for the quotient of M^* by the image of $\flat$. If $M^*/M = 0$, i.e., $\flat$ is an isomorphism, (M, ω) is called a symplectic module. If N is another quasi-symplectic module, an isomorphism $f : M \to N$ is said to be symplectic if $\omega(f(m), f(m')) = \omega(m, m')$ for every $m, m' \in M$.

Proposition 10. Let M be a quasi-symplectic module. Then

1. M is torsion-free.
2. For every $m, m' \in M$ we have $\omega(m, m') = -\overline{\omega(m', m)}$.
3. M^*/M is a torsion module. More generally, if $N \subset M$ is a submodule, the cokernel of $M \ni m \mapsto \omega(\cdot, m)|_N \in N^*$ is a torsion module.

Proof. 1. If $m \in M$ is a torsion element, then $m \in \ker(\flat) = 0$.

2. For $r \in R$, let r_λ be the coefficient of $x^\lambda \in R$. One has $r_\lambda = (rx^{-\lambda})_0$. Plugging into $\omega(m, m)_0 = 0$ an element $m = m' + m''$ gives

$$\omega(m', m'')_0 = -\omega(m'', m')_0. \tag{2.8}$$

Taking $m'' = x^\lambda m$ yields $\omega(m', m)_{-\lambda} = -\omega(m, m')_\lambda$, establishing the claim.

3. For this part, we denote the functor $\mathrm{Hom}_K(-, K)$ by $(-)^\vee$. We have a short exact sequence

$$0 \to M \otimes_R K \xrightarrow{\flat'} M^* \otimes_R K \to (M^*/M) \otimes_R K \to 0, \tag{2.9}$$

where $\flat' = \flat \otimes_R \mathrm{id}_K$. We may identify $M^* \otimes_R K$ with $(\overline{M} \otimes_R K)^\vee$, since Hom commutes with localization. As $\flat'$ is injective, the homomorphism

$$\flat'' = \mathrm{Hom}_K(\flat', K) : (\overline{M} \otimes_R K)^{\vee\vee} \to (M \otimes_R K)^\vee \tag{2.10}$$

is surjective by Lemma 7. We identify $(\overline{M} \otimes_R K)^{\vee\vee} = \overline{M} \otimes_R K$, by Lemma 64. Using 2. we find that for any $m, m' \in M$ and $k, k' \in K$:

$$\flat''(\overline{m} \otimes k)(m' \otimes k') = -\overline{\flat'(m \otimes k)(\overline{m'} \otimes k')}. \tag{2.11}$$

It follows at once that also $\flat'$ is surjective. Thus the short exact sequence (2.9) yields $(M^*/M) \otimes_R K = 0$, i.e., M^*/M is a torsion module.

Now let $N \subset M$ be a submodule. We have a short exact sequence

$$0 \to N \to M \to M/N \to 0. \tag{2.12}$$

Applying $*$ gives

$$0 \to (M/N)^* \to M^* \to N^* \to \mathrm{Ext}^1_R(\overline{M/N}, R). \tag{2.13}$$

We have $\mathrm{Ext}^1_R(\overline{M/N}, R) \otimes_R K = \mathrm{Ext}^1_K(\overline{M/N} \otimes_R K, K) = 0$, since Ext commutes with localization. Hence $\mathrm{Ext}^1_R(\overline{M/N}, R)$ is a torsion module. As both homomorphisms $M \to M^*$ and $M^* \to N^*$ have torsion cokernel, so does their composition. □

Corollary 11. Suppose that Λ is finite and let M be a quasi-symplectic R-module. Then M is symplectic. More generally, if $N \subset M$ is a submodule, then the map $M \ni m \mapsto \omega(\cdot, m)|_N \in N^*$ is surjective.

Proof. The assumption guarantees that R is a finite ring, so every element is either a zero-divisor or invertible. Hence torsion modules vanish. □

Definition 12. Let M be a quasi-symplectic module and $N \subset M$ a submodule. Set $N^\omega = \{m \in M \mid \omega(\cdot, m)|_N = 0\}$. N is called isotropic (resp. Lagrangian) if $N \subset N^\omega$ (resp. $N = N^\omega$).

Recall that the saturation $\mathrm{sat}_M(N)$ of a submodule $N \subset M$ is defined to be the module of all $m \in M$ such that $rm \in N$ for some regular element $r \in R$. If $N = \mathrm{sat}_M(N)$, then N is said to be saturated (in M). This is equivalent to M/N being torsion-free.

Proposition 13. Let N be a submodule of a quasi-symplectic module M.

1. If $L \subset N$, then $N^\omega \subset L^\omega$.

2. $N^\omega = N^{\omega\omega\omega}$.

3. If N is isotropic, $N \subset N^{\omega\omega} \subset N^\omega$, with equalities if N is Lagrangian.

4. $N^{\omega\omega} = \mathrm{sat}_M(N)$.

5. $N^{\omega\omega}/N$ is a torsion module.

Proof. Points 1–3 are established with simple manipulations.

4. Clearly $\mathrm{sat}_M(N) \subset N^{\omega\omega}$. For the reverse inclusion, it is sufficient to check that if N is saturated then $N^{\omega\omega} \subset N$. Let $m \in M \setminus N$. We will construct $z \in N^{\omega}$ such that $\omega(m, z) \neq 0$, showing that $m \notin N^{\omega\omega}$.

Put $L = N + Rm$. As N is saturated, L/N is torsion-free. Hence by Lemma 8 we have $(L/N)^* \neq 0$. Choose a nonzero element $\varphi \in (L/N)^*$. Composing with the quotient map $L \to L/N$ we obtain $\varphi' \in L^*$ which annihilates N and $\varphi'(m) \neq 0$. By Proposition 10 there exists a regular element $r \in R$ and $z \in M$ such that $r\varphi' = \omega(\cdot, z)|_L$. The element z is as desired.

5. Follows immediately from 4 and the definition of $\mathrm{sat}_M(N)$. □

Corollary 14. Suppose that Λ is finite and let M be a quasi-symplectic R-module. Then for every submodule $N \subset M$ we have $N^{\omega\omega} = N$.

Proof. As in Corollary 11. □

Proposition 15. Let M be a quasi-symplectic module and $N \subset M$ an isotropic submodule.

1. $N^{\omega\omega}/N$ is the torsion module of N^{ω}/N.

2. There exists an induced quasi-symplectic module structure on $N^{\omega}/N^{\omega\omega}$.

3. There exists a canonical embedding $M/N^{\omega} \to N^*$ with torsion cokernel.

4. There exists a canonical embedding $N^{\omega} \to (M/N)^*$ with torsion cokernel. If M is symplectic, this embedding is an isomorphism.

Proof. 1 follows from Proposition 13. The bilinear form ω on M restricted to N^{ω} has kernel $N^{\omega\omega}$, which establishes 2. By Proposition 10, we have a map $M \to N^*$ with torsion cokernel. Its kernel is clearly N^{ω}, proving 3.

4. Dualizing the short exact sequence $0 \to N \to M \to M/N \to 0$ gives

$$0 \to (M/N)^* \to M^* \to N^*, \tag{2.14}$$

so $(M/N)^*$ may be identified with the set of $\varphi \in M^*$ with trivial restriction to N. Next we note that $\flat(N^\omega) = \flat(M) \cap (M/N)^*$, so

$$(M/N)^*/\flat(N^\omega) = (M/N)^*/(\flat(M) \cap (M/N)^*) \subset M^*/M. \tag{2.15}$$

□

Definition 16. Let M be an R-module. We say that M is quasi-free if there exists a $\mathbb{Z}_n$-module M_0 such that $M \cong M_0 \otimes_{\mathbb{Z}_n} R$. We will also interpret elements of $M_0 \otimes_{\mathbb{Z}_n} R$ as polynomials in x^λ with coefficients in M_0, thus writing $M_0 \otimes_{\mathbb{Z}_n} R = M_0[\Lambda]$.

Remark 17. Let M, M_0 be as in Definition 16. Then M_0 is determined by M up to isomorphism. M is finitely generated over R if and only if M_0 is finitely generated over $\mathbb{Z}_n$. Moreover, an R-module M is free if and only if it is quasi-free and free as a $\mathbb{Z}_n$-module.

Proposition 18. Let P_0 be a finitely generated $\mathbb{Z}_n$-module equipped with a bilinear form $\omega_0 : P_0 \times P_0 \to \mathbb{Z}_n$ which is

- alternating: $\omega_0(p_0, p_0) = 0$ for every $p_0 \in P_0$,
- nondegenerate: $\omega_0(\cdot, p_0) = 0$ implies $p_0 = 0$.

Let $P = P_0[\Lambda]$ and define a $\mathbb{Z}_n$-bilinear form $\omega : P \times P \to R$ by

$$\omega(p_0 x^\lambda, p_0' x^\mu) = \omega_0(p_0, p_0') x^{\mu-\lambda}, \qquad \text{for } p_0, p_0' \in P_0,\ \lambda, \mu \in \Lambda. \tag{2.16}$$

Then (P, ω) is a symplectic module.

Proof. First note that P_0 and $P_0^\#$ have the same number of elements. Thus the map $P_0 \ni p_0 \mapsto \omega_0(\cdot, p_0) \in P_0^\#$, being injective by definition, is bijective.

A short calculation shows that conditions 1 and 2 in the Definition 9 are satisfied. Using the description of P^* in Lemma 5, it is easy to see that $\flat$ is an isomorphism. □

Physically, P_0 is the group generated by clock and shift matrices acting on qubits on a single lattice site, considered modulo phases.

Definition 19. A stabilizer code is a tuple $\mathfrak{C} = (n, \Lambda, L, P)$, where P is symplectic module over $R = \mathbb{Z}_n[\Lambda]$ as constructed in Proposition 18 and $L \subset P$ is an isotropic submodule. We will also abbreviate $\mathfrak{C} = (\Lambda, L, P)$ or (L, P) when there is no danger of confusion. To $\mathfrak{C}$ we associate

- integer dimension $D = \mathrm{rk}(\Lambda)$, the free rank of Λ,
- quasi-symplectic module $S(\mathfrak{C}) = L^{\omega}/L^{\omega\omega}$,
- torsion module $Z(\mathfrak{C}) = L^{\omega\omega}/L$,
- torsion module $Q(\mathfrak{C}) = L^{*}/(P/L^{\omega})$.

We say that $\mathfrak{C}$ is saturated if $L \subset P$ is saturated ($Z(\mathfrak{C}) = 0$) and Lagrangian if $L \subset P$ is Lagrangian ($Z(\mathfrak{C}) = S(\mathfrak{C}) = 0$). An isomorphism of stabilizer codes $(L, P) \to (L', P')$ is a symplectic isomorphism $P \to P'$ taking L to L'.

Let us interpret physically objects defined above. Let $\mathcal{H}$ be a Hilbert space on which local Pauli operators act irreducibly and let $\mathcal{H}_0 \subset \mathcal{H}$ be the space of solutions of (2.3) in $\mathcal{H}$. We assume that $\mathcal{H}_0 \neq 0$. One can show that operators in L^{ω} act irreducibly in $\mathcal{H}_0$. Since they commute with operators in $L^{\omega\omega}$, the latter act in $\mathcal{H}_0$ as scalars. This is a trivial statement for operators in L, but for operators in $L^{\omega\omega} \setminus L$ the conclusion relies on the irreducibility of $\mathcal{H}$, through Schur's lemma. Values of the latter operators may be changed by acting on a state with a suitable automorphism of the local operator algebra (more precisely, a non-local Pauli operator) which preserves all operators in L. This gives a state which is not representable by an element of $\mathcal{H}$ (belongs to a different superselection sector). Hence we have the following interpretations:

- $Z(\mathfrak{C})$ labels order parameters for spontaneously broken symmetries. If $L^{\omega\omega}$ is Lagrangian, isomorphism classes of representations $\mathcal{H}$ with $\mathcal{H}_0 \neq 0$ are in bijection with $Z(\mathfrak{C})^{\#}$ (and hence also with $Z(\mathfrak{C})$ if $Z(\mathfrak{C})$ is finite).
- Elements of $S(\mathfrak{C})$ are Pauli operators acting in $\mathcal{H}_0$ (sometimes called logical operators) modulo operators which act in $\mathcal{H}_0$ as scalars. Hence $\dim(\mathcal{H}_0)$ is the square root of the number of elements[4] of $S(\mathfrak{C})$.

By the discussion around (2.4) and Lemma 5, module L^{*} parametrizes local excitations. Therefore $Q(\mathfrak{C}) = L^{*}/(P/L^{\omega})$ is the module of local excitations modulo excitations which can be created by acting with local operators.

[4] $S(\mathfrak{C})$ is at most countably infinite. It is is not finite, $\dim(\mathcal{H}_0)$ in this statement has to be interpreted as the Hilbert dimension, not the algebraic dimension.

2.3 Topological charges

In this section we define a series of homological invariants Q^i, with Q^0 isomorphic to Q in Definition 19. Moreover, we show that Q^0 is isomorphic to the module of topological point excitations defined in [12] and derive some general properties of Q^i. Firstly, we show that $Q^i(\mathfrak{C}) = 0$ for $i > D-1$ (and also for $i = D-1$ for saturated codes). Secondly, we obtain bounds on Krull dimensions of $Q^i(\mathfrak{C})$. We expect Q^i to describe i-dimensional excitations (or defects). This is shown in Section 6 for Lagrangian codes such that all Q^i have Krull dimension zero. Computations of Q^i for certain specific codes are presented in Section 7.

We remark that it follows immediately from our results that for saturated codes $\mathfrak{C}$ with $D = 2$, the module $Q(\mathfrak{C})$ either vanishes or has Krull dimension zero. Together with the discussion in Section 5 it implies that all point excitations are mobile, i.e., they can be transported around by suitable string operators. This result has previously been shown only for codes with qudits of prime dimension [12]. Methods adapted therein do not generalize to the case of composite qudit dimension due to the failure of Hilbert's syzygy theorem, a crucial ingredient of the proof.

Lemma 20. If M is a quasi-free module and N is free over $\mathbb{Z}_n$, then for $i > 0$

$$\mathrm{Ext}^i_R(M,N) = 0, \qquad \mathrm{Tor}_i^R(M,N) = 0. \tag{2.17}$$

Proof. Every $\mathbb{Z}_n$-module is a direct sum of cyclic modules, so without loss of generality $M = \mathbb{Z}_k[\Lambda]$ with $k|n$. Let $l = \frac{n}{k}$. We have a free resolution

$$\dots \to R \xrightarrow{k} R \xrightarrow{l} R \xrightarrow{k} R \xrightarrow{\mathrm{mod}\, k} M \to 0. \tag{2.18}$$

Erasing M and applying $\mathrm{Hom}_R(-,N)$ we obtain the sequence

$$0 \to N \xrightarrow{k} N \xrightarrow{l} N \to \dots \tag{2.19}$$

which is exact in every degree $i > 0$. This establishes the claim for Ext. The argument for Tor is analogous. □

Proposition 21. Let $\mathfrak{C} = (L,P)$ be a stabilizer code. We have

$$Q(\mathfrak{C}) \cong \mathrm{Ext}^1_R(\overline{P/L}, R). \tag{2.20}$$

Proof. Consider the short exact sequence

$$0 \to L \to P \to P/L \to 0. \tag{2.21}$$

We apply $*$, use Lemma 20, and identify $(P/L)^* = L^\omega$, $P^* = P$ to get

$$0 \to L^\omega \to P \to L^* \to \operatorname{Ext}^1_R(\overline{P/L}, R) \to 0, \tag{2.22}$$

so $\operatorname{Ext}^1_R(\overline{P/L}, R) \cong L^*/(P/L^\omega) = Q(\mathfrak{C})$. □

Proposition 21 motivates the definition of generalized charge modules.

Definition 22. Generalized charge modules of a stabilizer code $\mathfrak{C} = (L, P)$ are defined as

$$Q^i(\mathfrak{C}) = \operatorname{Ext}^{i+1}_R(\overline{P/L}, R), \qquad i \geq 0. \tag{2.23}$$

Proposition 23. For $i > 0$ we have a canonical isomorphism

$$Q^i(\mathfrak{C}) \cong \operatorname{Ext}^i_R(\overline{L}, R). \tag{2.24}$$

Proof. Inspect the long exact sequence obtained by applying $(-)^*$ to (2.21). □

The next proposition shows that our definition of $Q(\mathfrak{C})$ agrees with topological point excitations in [12].

Proposition 24. Let (L, P) be a stabilizer code and let $\sigma : F \to P$ be a homomorphism with F quasi-free and $\operatorname{im}(\sigma) = L$. Let T be the torsion submodule of the cokernel of $\sigma^* : P^* \to F^*$. Then $T \cong Q^0(\mathfrak{C})$.

Proof. Choose a quasi-free module F' and a homomorphism $\iota : F' \to F$ with image $\ker(\sigma)$. One may extend it to a quasi-free resolution of P/L:

$$\cdots \to F' \xrightarrow{\iota} F \xrightarrow{\sigma} P \to P/L \to 0. \tag{2.25}$$

By Lemma 20 this resolution may be used to compute $\operatorname{Ext}^\bullet(\overline{P/L}, R)$. Thus we erase P/L and apply $(-)^*$, yielding the complex

$$0 \to P^* \xrightarrow{\sigma^*} F^* \xrightarrow{\iota^*} F'^* \to \dots \tag{2.26}$$

whose homology $\ker(\iota^*)/\operatorname{im}(\sigma^*)$ in degree 1 is $Q^0(\mathfrak{C})$. This exhibits $Q^0(\mathfrak{C})$ as a submodule of $\operatorname{coker}(\sigma^*)$. It is contained in T because $Q^0(\mathfrak{C})$ is torsion. It only remains to show that every $\varphi \in F^*$ representing an element of T is in $\ker(\iota^*)$. Indeed, let $r\varphi = \sigma^*(\psi)$ for some $r \in R$ not a zero-divisor and $\psi \in P^*$. Then $r\,\iota^*(\varphi) = 0$, so $\iota^*(\varphi) = 0$ since F'^* is torsion-free. □

Recall that the dimension $\dim(M)$ of an R-module M is defined as the Krull dimension of the quotient ring $R/\mathrm{Ann}(M)$, where $\mathrm{Ann}(M)$ is the annihilator of M. A nonzero module has a nonnegative Krull dimensions. By convention, the zero module has Krull dimension $-\infty$.

Proposition 25. Let $\mathfrak{C}$ be a stabilizer code.

1. $Q^i(\mathfrak{C}) = 0$ for $i \geq D$.
2. $Q^{D-1}(\mathfrak{C}) \cong \mathrm{Ext}^D_R(\overline{Z(\mathfrak{C})}, R)$. In particular $Q^{D-1}(\mathfrak{C}) = 0$ if $\mathfrak{C}$ is saturated.
3. $\dim(Q^i(\mathfrak{C})) \leq D - 1 - i$. In particular $Q^i(\mathfrak{C})$ is a torsion module.
4. If $\mathfrak{C}$ is saturated, then $\dim(Q^i(\mathfrak{C})) \leq D - 2 - i$.

In particular, saturated 1D codes have no topological charge.

Proof. 1 follows from the definition of a Gorenstein ring. 3 follows from Lemma 66. Now suppose that $\mathfrak{C}$ is saturated. Then P/L is torsion-free, so by Lemma 8 there exists a short exact sequence

$$0 \to P/L \to F \to M \to 0 \tag{2.27}$$

with F finite free. Applying $(-)^*$ gives a long exact sequence from which

$$\mathrm{Ext}^{i+1}_R(\overline{P/L}, R) \cong \mathrm{Ext}^{i+2}_R(\overline{M}, R), \qquad i \geq 0. \tag{2.28}$$

In particular $\mathrm{Ext}^D_R(\overline{P/L}, R) = 0$. Invoking Lemma 66 establishes 4.

2. We have a short exact sequence

$$0 \to L^{\omega\omega}/L \to P/L \to P/L^{\omega\omega} \to 0. \tag{2.29}$$

Apply $*$ and use $\mathrm{Ext}^D_R(\overline{P/L^{\omega\omega}}, R) = 0$, established in the proof of 2. □

2.4 Operations on Pauli stabilizer codes

One Pauli stabilizer code may give rise to various other codes. For example, one may "compatify" some (even all) spatial directions, i.e., replace Λ by a quotient group. Another possibility is stacking of infinitely many copies of a certain code to create a code with higher dimension. Finally, one has coarse-graining, which does not change the code, but forgets about some of its translation symmetry. In this section we discuss stacking and coarse-graining (in particular how they affect invariants of a code), but compactifications are postponed to future work. Moreover, we explain that the choice of n (which has to be a common multiple of qubit dimensions) does not matter and that the whole theory reduces to the case when n is a prime power.

Definition 26. Let $\mathfrak{C} = (n, \Lambda, L, P)$ be a stabilizer code and let k be a positive integer divisible by n. Then we may regard L and P as $\mathbb{Z}_k[\Lambda]$-modules, yielding a stabilizer code $\mathfrak{C}' = (k, \Lambda, L, P)$. We will not distinguish between $\mathfrak{C}$ and $\mathfrak{C}'$. The proposition below shows that this does not affect charge codes. Given data (Λ, L, P) we choose n (needed to define the ring R) as the smallest positive integer annihilating the abelian group P.

Proposition 27. Let $\mathfrak{C}, \mathfrak{C}'$ be as above. Then $S(\mathfrak{C}')$ coincides with $S(\mathfrak{C})$ regarded as a $\mathbb{Z}_k[\Lambda]$-module. Similarly, $Z(\mathfrak{C}) = Z(\mathfrak{C}')$ and $Q^i(\mathfrak{C}') = Q^i(\mathfrak{C})$.

Proof. If $M \subset P$ is a submodule, M^ω is the same over $\mathbb{Z}_n[\Lambda]$ and $\mathbb{Z}_k[\Lambda]$. This establishes the first two equalities. For the last one, note that $\mathrm{Ext}^\bullet_R(-, R)$ may be computed using quasi-free resolutions by Lemma 20, a quasi-free $\mathbb{Z}_n[\Lambda]$-module is also quasi-free over $\mathbb{Z}_k[\Lambda]$, and for any $\mathbb{Z}_n[\Lambda]$-module M we have

$$\mathrm{Hom}_{\mathbb{Z}_k[\Lambda]}(M, \mathbb{Z}_k[\Lambda]) \cong \mathrm{Hom}_{\mathbb{Z}_n[\Lambda]}(M, \mathbb{Z}_n[\Lambda]). \tag{2.30}$$

□

Definition 28. Direct sum of stabilizer codes is defined by

$$(n, \Lambda, L, P) \oplus (m, \Lambda, L', P') = (\gcd(n, m), \Lambda, L, P). \tag{2.31}$$

Clearly $S(\mathfrak{C}), Z(\mathfrak{C})$ and $Q^i(\mathfrak{C})$ are additive.

Proposition 29. Let $\mathfrak{C} = (n, \Lambda, L, P)$ be a stabilizer code and let $n = \prod_{i=1}^{r} p_i^{n_i}$ be the prime decomposition of n. Then

$$\mathfrak{C} = \bigoplus_{i=1}^{r} (p_i^{n_i}, \Lambda, L_i, P_i), \tag{2.32}$$

where $P_i = \{m \in P \mid p_i^{n_i} m = 0\}$, $L_i = L \cap P_i$.

Proof. Chinese remainder theorem. □

Note that Proposition 29 implies that the study of stabilizer code with general n reduces to the case when n is a prime power.

Definition 30. Let Γ be a finitely generated abelian group and let $\iota : \Lambda \to \Gamma$ be a homomorphic embedding. For any $\mathbb{Z}_n[\Lambda]$-module M let $\iota_* M = M \otimes_{\mathbb{Z}_n[\Lambda]} \mathbb{Z}_n[\Gamma]$. Then ι_* is an exact functor because $\mathbb{Z}_n[\Gamma]$ is free over $\mathbb{Z}_n[\Lambda]$. In particular for a stabilizer code (Λ, L, P) we have $\iota_* L \subset \iota_* P$, allowing us to define

$$\iota_*(\Lambda, L, P) = (\Gamma, \iota_* L, \iota_* P). \tag{2.33}$$

The operation introduced in Definition 30 may be thought of as stacking of Γ/Λ layers of the system described by (Λ, L, P). Let us note that

$$S(\iota_*\mathfrak{C}) = \iota_* S(\mathfrak{C}), \qquad Z(\iota_*\mathfrak{C}) = \iota_* Z(\mathfrak{C}), \qquad Q^i(\iota_*\mathfrak{C}) = \iota_* Q^i(\mathfrak{C}). \tag{2.34}$$

Due to these simple formulas, the structure of charge modules may be used to show that a certain system can not be obtained from a lower dimensional system by stacking. Here we note only a simple criterion based on whether charge modules vanish.

Proposition 31. Suppose that $\mathfrak{C}$ is a stabilizer code with $Q^i(\mathfrak{C}) \neq 0$. Then $\mathfrak{C}$ is not isomorphic to any $\iota_*(\Lambda, L, P)$ with $\mathrm{rk}(\Lambda) < i + 1$. If $\mathfrak{C}$ is saturated, $\mathrm{rk}(\Lambda) = i + 1$ is also excluded.

Proof. Formula (2.34) and Proposition 25. □

Proposition 32. Suppose that $\mathfrak{C}$ is a stabilizer code which is not saturated. Then $\mathfrak{C}$ is not isomorphic to any $\iota_*(\Lambda, L, P)$ with $\mathrm{rk}(\Lambda) = 0$.

Proof. Zero-dimensional systems have $Z(\mathfrak{C}) = 0$ by Corollary 14. The claim follows from (2.34). □

Definition 33. Let $\iota : \Gamma \to \Lambda$ be a finite index embedding. If M is a $\mathbb{Z}_n[\Lambda]$ module, we let $\iota^* M$ be M treated as $\mathbb{Z}_n[\Gamma]$-module. We define

$$\iota^*(\Lambda, L, P) = (\Lambda, \iota^* L, \iota^* P). \tag{2.35}$$

This operation is called coarse graining.

Proposition 34. Coarse graining satisfies

$$S(\iota^*\mathfrak{C}) = \iota^* S(\mathfrak{C}), \qquad Z(\iota^*\mathfrak{C}) = \iota^* Z(\mathfrak{C}), \qquad Q^i(\iota^*\mathfrak{C}) = \iota^* Q^i(\mathfrak{C}). \tag{2.36}$$

Proof. Let $M \subset P$ be a submodule and $p \in P$. Then $p \in M^\omega$ if and only if the scalar part of $\omega(m, p)$ vanishes. The scalar part is unchanged by coarse graining, so $(\iota^* M)^\omega = \iota^*(M^\omega)$. This establishes first two equalities in (2.36). For the last one, ι^* is an exact functor which takes free modules to free modules and commutes with $(-)^*$, as one verifies using Lemma 5. □

2.5 Mobility theorem

A local excitation is said to be mobile if there exist local Pauli operators which "move" it in all non-compact directions of the lattice. By "move", we mean destroying the excitation and creating its displaced copy, without creating additional excitations.

Recall that $Q = L^*/(P/L^\omega)$ describes all local excitations modulo those creatable by local Pauli operators. According to the previous paragraph, an excitation $e \in L^*$ can be displaced by an element $\gamma \in \Lambda$ if and only if $(x^\gamma - 1)e \in P/L^\omega$. In conclusion, mobility of all local excitations is equivalent to the existence of a subgroup $\Gamma \subset \Lambda$ of finite index such that $x^\gamma - 1$ annihilates Q for each $\gamma \in \Gamma$. We now show that this condition is also equivalent to the vanishing of the Krull dimension of R-module Q.

Lemma 35. If n, r are positive integers, let $L_n(r)$ be the largest integer such that $(x-1)^{L_n(r)}$ divides $x^{n^r} - 1$ in $\mathbb{Z}_n[x]$. For example, $L_p(r) = p^r$ for any prime number p. One has $\limsup_{r\to\infty} L_n(r) = \infty$.

Proof. Factorization $x^n - 1 = -(x-1)^2 \sum_{j=0}^{n-1} (j+1)x^j$ implies that $L_n(1) \geq 2$. We will show that $L_n(2r) \geq L_n(r)^2$. Write $x^{n^r} - 1 = (x-1)^{n^r} f(x)$. Then

$$\begin{aligned} x^{n^{2r}} - 1 = (x^{n^r})^{n^r} - 1 &= (x^{n^r} - 1)^{L_n(r)} f(x^{n^r}) \\ &= (x-1)^{L_n(r)^2} f(x)^{L_n(r)} f(x^{n^r}). \end{aligned} \tag{2.37}$$

□

Proposition 36. If $\mathfrak{a} \subset R$ is an ideal, then $\dim(R/\mathfrak{a}) = 0$ if and only if there exists a subgroup $\Gamma \subset \Lambda$ of finite index such that $x^\gamma - 1 \in \mathfrak{a}$ for every $\gamma \in \Gamma$.

Proof. $\Longleftarrow$: $R/\mathfrak{a}$ is a finite ring, so $\dim(R/\mathfrak{a}) = 0$.

$\Longrightarrow$: choose $\lambda_1, \ldots, \lambda_D \in \Lambda$ which generate a subgroup of finite index and put $x_i = x^{\lambda_i}$. If $\mathfrak{m} \subset R$ is a maximal ideal, then $R/\mathfrak{m}$ is a finite field, so there exists a positive integer such that $x_i^r - 1 \in \mathfrak{m}$. As $\dim(R/\mathfrak{a}) = 0$, there exist finitely many maximal ideals $\mathfrak{m} \subset R$ containing $\mathfrak{a}$. Thus it is possible to choose r such that

$$(x_1^r - 1, \dots, x_D^r - 1) \subset \bigcap_{\mathfrak{m} \supset \mathfrak{a}} \mathfrak{m} = \sqrt{\mathfrak{a}}. \tag{2.38}$$

Since R is Noetherian, $\sqrt{\mathfrak{a}}^N \subset \mathfrak{a}$ for large enough N. Lemma 35 implies that there exist N, L such that

$$(x_1^L - 1, \dots, x_D^L - 1) \subset ((x_1^r - 1)^N, \dots, (x_D^r - 1)^N) \subset \mathfrak{a}. \tag{2.39}$$

We may take Γ to be the span of $L\lambda_1, \dots, L\lambda_D$. □

Corollary 37. All local excitations of a stabilizer code $\mathfrak{C}$ are mobile if and only if $\dim(Q(\mathfrak{C})) = 0$. In particular this is true if $D = 1$ or $\mathfrak{C}$ is saturated and $D = 2$.

Proof. The second part of the statement follows from dimension bounds in Proposition 25. □

Though logically equivalent, the condition $\dim(R/\mathfrak{a}) = 0$ avoids mentioning a finite index subgroup of Λ. It is also the easier condition to establish in a proof, due to the large number of results in dimension theory. An example is given by Corollary 37 above.

A direct characterization of mobility for i-dimensional topological charges in Q^i, $i > 0$ may be possible, given an interpretation of charges in terms of extended excitations. We leave this to future efforts. Instead we make the conjectural definition that mobility for Q^i is still equivalent to $\dim(Q^i) = 0$. We sometimes call a code $\mathfrak{C}$ mobile if $\dim(Q^i(\mathfrak{C})) = 0$ for all i. In the next section we will see that under this assumption elements of $Q^i(\mathfrak{C})$ may indeed be interpreted as excitations, which are mobile in a suitable sense.

2.6 Codes with only mobile excitations

This section is devoted to analysis of topological charges for mobile codes. Mobility allows us to describe topological charges in terms of Čech cocycles. Cup product for Čech cohomology fits a physical process commonly known as braiding. It furnishes an algebraic description of exchange relations for mobile excitations. A direct physical interpretation of Čech cocycles is also given.

Mathematical preliminaries

If A is a ring and M an A-module, let $\mathrm{E}_A(M)$ be the injective envelope of M. We refer to Appendix A.2 for other definitions and facts used below.

Proposition 38. Let $\mathfrak{a} \subset R$ be an ideal such that $\dim(R/\mathfrak{a}) = 0$. Then

$$\Gamma_{\mathfrak{a}}(R^{\#}) \cong \bigoplus_{\mathfrak{m}} \mathrm{E}_R(R/\mathfrak{m}), \tag{2.40}$$

the sum being taken over maximal ideals of R containing $\mathfrak{a}$.

Proof. Lemma 69 allows us to reduce to the case of $\mathfrak{a}$ being itself a maximal ideal $\mathfrak{m}$. We put $k = R/\mathfrak{m}$. R-module $R^{\#}$ represents the exact cofunctor $(-)^{\#}$ on the category of R-modules, so it is injective. By [15, Proposition 3.88], $\Gamma_{\mathfrak{m}}(R^{\#})$ is also injective. It is easy to see that $k^{\#} \cong \{\varphi \in R^{\#} \mid \mathfrak{m}\varphi = 0\}$ is an essential submodule of $\Gamma_{\mathfrak{m}}(R^{\#})$, so $\Gamma_{\mathfrak{m}}(R^{\#}) = \mathrm{E}_R(k^{\#})$. The proof will be completed by showing that $k^{\#} \cong k$ as an R-module. As $k^{\#}$ is annihilated by $\mathfrak{m}$, it is a k-vector space. We have to argue that its dimension over k is 1. Let p be the characteristic of k. Every element of $k^{\#}$ factors through $\mathbb{Z}_p$, so

$$\dim_{\mathbb{Z}_p}(k^{\#}) = \dim_{\mathbb{Z}_p}(\mathrm{Hom}_{\mathbb{Z}_p}(k, \mathbb{Z}_p)) = \dim_{\mathbb{Z}_p}(k), \tag{2.41}$$

and hence $\dim_k(k^{\#}) = \frac{\dim_{\mathbb{Z}_p}(k^{\#})}{\dim_{\mathbb{Z}_p}(k)} = 1$. □

Lemma 39. Every maximal ideal of R has height D.

Proof. R is a product of rings $\mathbb{Z}_{p^t}[\Lambda]$ where p is prime and $t \in \mathbb{N}$, so we may assume that $n = p^t$ with no loss of generality. Then R is an extension of $S = \mathbb{Z}_p[\Lambda]$ by a nilpotent ideal, so its poset of prime ideals is isomorphic to that of S. The result for S is standard, see e.g., [16, Corollary 13.4]. □

Proposition 40. Let $\mathfrak{a} \subset R$ be an ideal such that $\dim(R/\mathfrak{a}) = 0$ and let M be a quasi-free module. Then $\mathrm{H}^j_{\mathfrak{a}}(M) = 0$ for $j \neq D$ and

$$\mathrm{H}^D_{\mathfrak{a}}(M) \cong \left(\bigoplus_{\mathfrak{m}} \mathrm{E}_R(R/\mathfrak{m})\right) \otimes M, \tag{2.42}$$

the sum being taken over maximal ideals of R containing $\mathfrak{a}$.

Proof. Lemma 69 allows us to reduce to the case of $\mathfrak{a}$ being a maximal ideal $\mathfrak{m}$. First consider the case $M = R$. By maximality of $\mathfrak{m}$ and $\mathrm{H}^j_{\mathfrak{m}}(R)$ being $\mathfrak{m}$-torsion, every element of $R \setminus \mathfrak{m}$ acts as an invertible endomorphism of $\mathrm{H}^j_{\mathfrak{m}}(R)$. Thus we have R-module isomorphisms $\mathrm{H}^j_{\mathfrak{m}}(R) \cong \mathrm{H}^j_{\mathfrak{m}}(R)_{\mathfrak{m}} \cong \mathrm{H}^j_{\mathfrak{m}'}(R_{\mathfrak{m}})$, where $\mathfrak{m}'$ is the extension of $\mathfrak{m}$ in $R_{\mathfrak{m}}$. The second isomorphism follows from Lemma 72. By Lemma 39, $R_{\mathfrak{m}}$ is a Gorenstein ring of dimension D, so Lemma 70 gives $\mathrm{H}^j_{\mathfrak{m}}(R) = 0$ for $j \neq D$ and $\mathrm{H}^D_{\mathfrak{m}}(R) \cong \mathrm{E}_{R_{\mathfrak{m}}}(R_{\mathfrak{m}}/\mathfrak{m}') \cong \mathrm{E}_R(R/\mathfrak{m})$.

Local cohomology can be computed using the Čech complex, so the result for $M = R$ shows that $\check{\mathrm{C}}^\bullet(\mathbf{t}, R)$ is a flat resolution of $\mathrm{E} := \mathrm{E}_R(R/\mathfrak{m})$, up to a degree shift. Since $\check{\mathrm{C}}^\bullet(\mathbf{t}, M) \cong \check{\mathrm{C}}^\bullet(\mathbf{t}, R) \otimes_R M$, this implies that for any module M we have $\check{\mathrm{H}}^p(\mathbf{t}, M) \cong \mathrm{Tor}^R_{D-p}(E, M)$. Now specialize to the case of M being quasi-free and invoke Lemma 20. □

Let $\Gamma \subset \Lambda$ be a subgroup such that Λ/Γ is finite and let $\gamma_1, \ldots, \gamma_D$ be a basis of Γ. We put

$$x_i = x^{\gamma_i}, \qquad t_i = 1 - x^{\gamma_i}, \qquad \mathfrak{a} = (t_1, \ldots, t_D), \tag{2.43}$$

and consider the Čech complex $\check{\mathrm{C}}^\bullet(\mathbf{t}, R)$ (see Appendix A.3). Lemma 71 and Propositions 38, 40 show that its only nonzero cohomology module $\check{\mathrm{H}}^D(\mathbf{t}, R)$ is isomorphic to $\Gamma_{\mathfrak{a}}(R^\#)$. Our next goal is to construct an explicit isomorphism.

Definition 41. Let $\mathbb{Z}_n[[\Lambda]]$ be the set of formal sums $\sum_{\lambda\in\Lambda} r_\lambda x^\lambda$. This is an abelian group, but in general not a ring: the product

$$\left(\sum_{\lambda\in\Lambda} r_\lambda x^\lambda\right)\left(\sum_{\mu\in\Lambda} r'_\mu x^\mu\right) = \sum_{\lambda\in\Lambda}\left(\sum_{\mu\in\Lambda} r_{\lambda-\mu} r'_\mu\right) x^\lambda \tag{2.44}$$

is well-defined only if for every $\lambda \in \Lambda$ there are only finitely many $\mu \in \Lambda$ such that both $r_{\lambda-\mu}$ and r'_μ is nonzero. This condition is always satisfied if one of the two factors is in R, so $\mathbb{Z}_n[[\Lambda]]$ is an R-module. Using the pairing

$$\mathbb{Z}_n[\Lambda] \times \mathbb{Z}_n[[\Lambda]] \ni (r, r') \mapsto (rr')_0 \in \mathbb{Z}_n, \tag{2.45}$$

we identify $\mathbb{Z}_n[[\Lambda]]$ with $R^\#$.

Recall that $\check{\mathrm{C}}^D(\mathbf{t}, R) = R_{t_1\ldots t_D}$ and that $\check{\mathrm{H}}^D(\mathbf{t}, R)$ is the quotient of $R_{t_1\ldots t_D}$ by the sum of images of $R_{t_1\ldots t_{j-1}t_{j+1}\ldots t_D}$ (module of coboundaries).

Definition 42. We consider formal Laurent expansions of $\frac{1}{t_i}$ (regarded as elements of $R^\#$) into positive and negative powers of x_i:

$$\left(\frac{1}{t_i}\right)_+ = \sum_{j=0}^\infty x^j, \qquad \left(\frac{1}{t_i}\right)_- = -\sum_{j=1}^\infty x^{-j}. \tag{2.46}$$

The residue homomorphism Res : $R_{t_1 \dots t_D} \to R^\#$ is defined by

$$\frac{r}{t_1^{k_1} \cdots t_D^{k_D}} \mapsto r \prod_{i=1}^{D} \left[\left(\frac{1}{t_i}\right)_+^{k_i} - \left(\frac{1}{t_i}\right)_-^{k_i} \right]. \tag{2.47}$$

This is well-defined because $t_i \left(\frac{1}{t_i}\right)_\pm = 1$.

Proposition 43. ker(Res) is the module of coboundaries and the image of Res is $\Gamma_{\mathfrak{a}}(R^\#)$. Therefore Res induces an isomorphism $\check{\mathrm{H}}^D(\mathbf{t}, R) \to \Gamma_{\mathfrak{a}}(R^\#)$.

Proof. A Čech coboundary is a sum of elements as on the left-hand side of (2.47) with at least one k_i equal to zero, each of which is annihilated by Res. Moreover, the right-hand side of (2.47) is annihilated by $t_i^{k_i}$, so it belongs to $\Gamma_{\mathfrak{a}}(R^\#)$. We have obtained an induced homomorphism $\check{\mathrm{H}}^D(\mathbf{t}, R) \to \Gamma_{\mathfrak{a}}(R^\#)$. From now on the symbol Res refers to this induced homomorphism. Let z be the cohomology class of $\frac{1}{t_1 \dots t_D}$. Clearly $\mathfrak{a} \subset \mathrm{Ann}(z)$. We evaluate

$$\mathrm{Res}(z) = \sum_{\gamma \in \Gamma} x^\gamma. \tag{2.48}$$

One checks that the annihilator of the right-hand side is $\mathfrak{a}$, so $\mathrm{Ann}(z) \subset \mathfrak{a}$. We deduce that the submodule M of $\check{\mathrm{H}}^D(\mathbf{t}, R)$ generated by z intersects ker(Res) trivially. Clearly M is an essential submodule of $\check{\mathrm{H}}^D(\mathbf{t}, R)$, so Res is injective. Propositions 38, 40 imply that it is an isomorphism. □

Physical interpretations of charges

For the rest of this section we assume that $\mathfrak{C} = (\Lambda, L, P)$ is a Lagrangian stabilizer code such that $\dim(Q^i(\mathfrak{C})) = 0$ for every i. Proposition 36 allows us to choose a subgroup $\Gamma \subset \Lambda$ of finite index such that $x^\gamma - 1$ annihilates all $Q^i(\mathfrak{C})$. With this Γ, we consider the Čech complex as discussed around (2.43).

Charges as Čech cocyles

Proposition 44. We have $Q^i(\mathfrak{C}) \cong \check{\mathrm{H}}^{i+1}(\mathbf{t}, P/L)$ for $0 \le i \le D-2$.

Proof. We can continue the quotient map $P \to P/L$ to a quasi-free resolution $P_\bullet \to P/L$ with $P_0 = P$. Applying $(-)^*$ yields a complex

$$0 \to L \to P \to P_1^* \to P_2^* \to \dots, \tag{2.49}$$

where we used isomorphisms $(P/L)^* \cong L$ and $P^* \cong P$. From this we have also a cochain complex $K^\bullet$ with $K^0 = P/L$, $K^i = P_i^*$ for $i > 0$:

$$K^\bullet \ : \ 0 \to P/L \to P_1^* \to P_2^* \to \dots \tag{2.50}$$

Its cohomology is trivial in degree zero and $\mathrm{Ext}_R^\bullet(\overline{P/L}, R)$ elsewhere. Next, we form a double complex $\check{\mathrm{C}}^\bullet(\mathbf{t}, K^\bullet)$, with the following properties:

- $\check{\mathrm{C}}^0(\mathbf{t}, K^\bullet) \cong K^\bullet$ has cohomology described above. If $p > 0$, the complex $\check{\mathrm{C}}^p(\mathbf{t}, K^\bullet)$ is exact because $\check{\mathrm{C}}^p(\mathbf{t}, -) = \check{\mathrm{C}}^p(\mathbf{t}, R) \otimes_R -$ is an exact functor annihilating the cohomology of $K^\bullet$.
- $\check{\mathrm{C}}^\bullet(\mathbf{t}, K^0)$ has cohomology $\check{\mathrm{H}}^\bullet(\mathbf{t}, P/L)$. If $q > 0$, the complex $\check{\mathrm{C}}^\bullet(\mathbf{t}, K^q)$ has nonzero cohomology only in degree D, by Proposition 40.

The isomorphism is established either by a diagram chase or using the double complex spectral sequence. For a reader not familiar with these techniques, we sketch the more elementary approach below.

We let d be the differential induced from $K^\bullet$ and δ the Čech differential. Let $i \in \{1, \dots D-1\}$ and consider $q \in \mathrm{Ext}_R^i(\overline{P/L}, R)$ represented by an element $q^{(0)} \in K^i$ annihilated by d. Then also $\delta q^{(0)}$ is annihilated by d, so by exactness of $\check{\mathrm{C}}^1(\mathbf{t}, K^\bullet)$ there exists $q^{(1)} \in \check{\mathrm{C}}^1(\mathbf{t}, K^{i-1})$ such that $dq^{(1)} = \delta q^{(0)}$. Hence $\delta q^{(1)}$ is annihilated by d. If $i = 1$, this implies that $\delta q^{(1)} = 0$ because $d : \check{\mathrm{C}}^2(\mathbf{t}, K^0) \to \check{\mathrm{C}}^2(\mathbf{t}, K^1)$ is injective. If $i > 1$, we conclude that there exists $q^{(2)} \in \check{\mathrm{C}}^2(\mathbf{t}, K^{i-2})$ such that $dq^{(2)} = \delta q^{(1)}$. Continuing like this inductively we obtain a sequence of elements $q^{(j)} \in \check{\mathrm{C}}^j(\mathbf{t}, K^{i-j})$, $0 \le j \le i$, such that

$$dq^{(0)} = 0, \qquad dq^{(j)} = \delta q^{(j-1)} \quad \text{for } j \neq 0, \qquad \delta q^{(i)} = 0. \tag{2.51}$$

The Čech cohomology class of $q^{(i)}$ is declared to be the image of q in $\check{\mathrm{H}}^i(\mathbf{t}, P/L)$. With similar reasoning one checks that this cohomology class does not depend on arbitrary choices in the construction of $q^{(i)}$. Thus a well-defined homomorphism $h : \mathrm{Ext}_R^i(\overline{P/L}, R) \to \check{\mathrm{H}}^i(\mathbf{t}, P/L)$ is obtained. Performing the same steps reversed yields a homomorphism in the opposite direction, easily seen to be an inverse of h. □

Remark 45. If we assume that $x^\gamma - 1$ annihilates $Q^i(\mathfrak{C})$ for every $i \le d$ for some $0 \le d \le D-2$, we may still obtain $Q^i(\mathfrak{C}) \cong \check{\mathrm{H}}^{i+1}(\mathbf{t}, P/L)$ for $0 \le i \le d$. The proof of Proposition 44 goes through with essentially no modifications. Moreover, even with no restrictions on $\dim(Q^i(\mathfrak{C}))$ we may construct a homomorphism $\check{\mathrm{H}}^i(\mathbf{t}, P/L) \to \mathrm{Ext}^i_R(\overline{P/L}, R)$ for $1 \leq i \leq D-1$. If $\dim(\mathrm{Ext}^i_R(\overline{P/L}, R)) \neq 0$, this homomorphism can not be surjective.

Charges as topological excitations

Next we provide a concrete interpretation of our charge modules $Q^i(\mathfrak{C})$ (reinterpreted as Čech cocycles by Proposition 44) in terms of operators and physical excitations.

Definition 46. We define $\widehat{P} = P \otimes_R R^{\#}$. Recall that $P \cong P_0[\Lambda]$ for some finite abelian group Λ, so $\widehat{P} \cong P_0[[\Lambda]]$. We will sometimes multiply elements of $R^{\#}$ and $\widehat{P}$. Such product is well-defined under a condition analogous to the one discussed in Definition 41. Symplectic form on P extends to a pairing between P and $\widehat{P}$ valued in $R^{\#}$. Under suitable conditions one may also pair two elements of $\widehat{P}$.

Elements of $\widehat{P}$ describe products of Pauli operators (up to phase) with possibly infinite spatial support. Such expressions do not necessarily define bona fide operators on a Hilbert space, but they make sense as automorphisms of the algebra of local operators. Hence they may be applied to states, in general yielding a state in a different superselection sector. The extended symplectic forms captures their "commutation rules" with local Pauli operators.

Definition 47. Let $s = (s_1, \ldots, s_D)$ be a tuple of elements of the multiplicative group $\{\pm\}$. We think of s as a label of an orthant in $\Gamma \cong \mathbb{Z}^D$. For every s we define an embedding of $P_{t_1 \ldots t_D}$ (and hence also of every $P_{t_{i_0 \ldots i_p}}$ for a sequence $1 \leq i_0 < \cdots < i_p \leq D$, since P is torsion-free) in $\widehat{P}$ as follows:

$$\frac{p}{t_1^{k_1} \cdots t_D^{k_D}} \mapsto p \prod_{i=1}^{D} \left(\frac{1}{t_i}\right)_{s_i}^{k_i}. \tag{2.52}$$

If π is an element of $P_{t_1 \ldots t_D}$, we denote the element of $\widehat{P}$ obtained this way by π^s, to emphasize dependence on s.

Consider a cocycle $\varphi \in \check{\mathrm{C}}^p(\mathbf{t}, P/L)$. We lift φ to a cochain $\widetilde{\varphi} \in \check{\mathrm{C}}^p(\mathbf{t}, P)$. Then $\sigma = \delta\widetilde{\varphi} \in \check{\mathrm{C}}^{p+1}(\mathbf{t}, L)$ is a cocycle. Note that the map taking the cohomology class of φ to the cohomology class of σ is the connecting homomorphism in the long exact sequence of Čech cohomology. Consider images in $\widehat{P}$ of components of $\widetilde{\varphi}$ and σ. Two observations are in order. Firstly, $\widetilde{\varphi}^s_{i_1 \ldots i_p}$ describes an infinite Pauli operator whose support is extended only in directions $i_1 \ldots i_p$, and moreover is contained in a shifted orthant specified by s. Secondly, each $\sigma^s_{i_0\dots i_p}$ is ω-orthogonal to L. Hence we have an identity

$$\sum_{j=0}^{p}(-1)^j\omega\left(\cdot,\widetilde{\varphi}^s_{i_0\dots i_{j-1}i_{j+1}\dots i_p}\right)\Big|_L = 0 \qquad \text{in } \overline{L}^{\#}. \tag{2.53}$$

Let us rewrite this as

$$\omega\left(\cdot,\widetilde{\varphi}^s_{i_1\dots i_p}\right)\Big|_L = \sum_{j=1}^{p}(-1)^j\omega\left(\cdot,\widetilde{\varphi}^s_{i_0\dots i_{j-1}i_{j+1}\dots i_p}\right)\Big|_L. \tag{2.54}$$

By comparing supports of the two sides of this equation we can see that action of $\widetilde{\varphi}^s_{i_1\dots i_p}$ creates an excitation (violation of the stabilizer condition) which is supported on a thickened boundary of the support of $\widetilde{\varphi}^s_{i_1\dots i_p}$. Hence $\widetilde{\varphi}^s_{i_1\dots i_p}$ represents a p-dimensional extended operator which creates an excitation on the $(p-1)$-dimensional boundary of its support. This excitation does not depend on the lift of the cocycle φ to $\widetilde{\varphi}$.

Next, let us suppose that φ represents the trivial cohomology class. That is, we have $\varphi = \delta\psi$ for some $\psi \in \check{\mathrm{C}}^{p-1}(\mathbf{t}, P/L)$. We lift ψ to a cochain $\widetilde{\psi}$ valued in P and choose $\widetilde{\varphi} = \delta\widetilde{\psi}$. Then

$$\widetilde{\varphi}^s_{i_1\dots i_p} = \sum_{j=1}^{p}(-1)^{j-1}\widetilde{\psi}^s_{i_1\dots i_{j-1}i_{j+1}\dots i_p}, \tag{2.55}$$

which shows that the $(p-1)$-dimensional excitation created by $\widetilde{\varphi}^s_{i_1\dots i_p}$ can be created by operators $\widetilde{\psi}^s_{i_1\dots i_{j-1}i_{j+1}\dots i_p}$, each of which is extended in only $p-1$ (rather than p) directions.

Note that even though an excitation corresponding to a p-cocycle φ is created by an operator with p-dimensional support, it can be shifted by an element of Γ by the action of a $(p-1)$-dimensional operator. Indeed, $x^\gamma - 1$ annihilates cohomology, so $(x^\gamma - 1)\varphi$ is a coboundary. The result follows from the discussion of the previous paragraph.

Summarizing, an element of $Q^p(\mathfrak{C}) \cong \check{\mathrm{H}}^{p+1}(\mathbf{t}, P/L)$ gives rise to an excitation extended in p dimensions, determined modulo excitations created by p-dimensional operators.

Charges as higher form symmetries

Now let $\varphi \in \check{\mathrm{C}}^p(\mathbf{t}, P/L)$ be a cocycle. We consider the expression

$$\widetilde{\varphi}^{\mathrm{Res}}_{i_1\dots i_p} = \sum_{s_{i_1},\dots,s_{i_p}\in\{\pm\}} s_{i_1}\cdots s_{i_p}\,\widetilde{\varphi}^s_{i_1\dots i_p}. \tag{2.56}$$

This makes sense because $\widetilde{\varphi}^s_{i_1 \dots i_p}$ does not depend on s_j for $j \notin \{i_1, \dots, i_p\}$. $\widetilde{\varphi}^{\mathrm{Res}}_{i_1 \dots i_p}$ is a p-dimensional extended operator. By the earlier discussion, the excitation it creates is supported in the union of a finite collection of subsets infinitely extended in at most $p-1$ directions. On the other hand, there exists some k such that each $t^k_{i_j}$ annihilates it. One checks that a nonzero element with such property must be infinitely extended in all p directions. We obtain the conclusion that $\omega(\cdot, \widetilde{\varphi}^{\mathrm{Res}}_{i_1 \dots i_p})\Big|_L = 0$, i.e., $\widetilde{\varphi}^{\mathrm{Res}}_{i_1 \dots i_p}$ preserves the state defined by the stabilizer condition.

Since the cochain $\widetilde{\varphi}$ allows to construct a symmetry $\widetilde{\varphi}^{\mathrm{Res}}_{i_1 \dots i_p}$ of the ground state for every coordinate p-plane (labeled by $i_1 < \cdots < i_p$), it defines a $(D-p)$-form symmetry of $\mathfrak{C}$. Let us now investigate to what extent this $(D-p)$-form symmetry is uniquely determined by the cohomology class of φ.

Firstly, let us fix the cocycle φ and ask for the dependence on the choice of the lift $\widetilde{\varphi}$. For two different lifts $\widetilde{\varphi}, \widetilde{\varphi}'$, the difference $\widetilde{\varphi}'^{\mathrm{Res}}_{i_1 \dots i_p} - \widetilde{\varphi}^{\mathrm{Res}}_{i_1 \dots i_p}$ is an infinite sum of elements of L, i.e., it represents a product of local operators separately preserving the ground state. A p-dimensional ($p \geq 1$) operator of this form should be regarded as a trivial $(D-p)$-form symmetry.

To understand the dependence on the cocycle φ representing a given cohomology class, let us suppose that $\varphi = \delta\psi$. We lift ψ and choose $\widetilde{\varphi} = \delta\widetilde{\psi}$. With this choice, expression (2.56) vanishes on the nose.

Summarizing, we have argued that the definition (2.56) defines a $(D-p)$-form symmetry of $\mathfrak{C}$, which depends only on the cohomology class of φ. This means that we have an alternative interpretation of $Q^p(\mathfrak{C})$ as a group of $(D-p-1)$-form symmetries of $\mathfrak{C}$ (possibly nontrivially acted upon by Λ).

Braiding

Definition 48. Let $\varphi \in \check{\mathrm{H}}^p(\mathbf{t}, P/L)$, $\psi \in \check{\mathrm{H}}^q(\mathbf{t}, P/L)$. The cup product defined in the Appendix A.3 yields an element

$$\overline{\varphi} \smile \delta\psi \in \check{\mathrm{H}}^{p+q}(\mathbf{t}, \overline{P/L} \otimes_R L), \tag{2.57}$$

where δ is the connecting homomorphism $\check{\mathrm{H}}^q(\mathbf{t}, P/L) \to \check{\mathrm{H}}^{q+1}(\mathbf{t}, L)$ in a long exact sequence. Using the map (with a slight abuse of notation) in Čech cohomology induced by the symplectic pairing $\omega : \overline{P/L} \otimes_R L \to R$ we obtain a class

$$\omega(\overline{\varphi} \smile \delta\psi) \in \check{\mathrm{H}}^{p+q}(\mathbf{t}, R). \tag{2.58}$$

This class is trivial if $p+q \neq D$, by Proposition 40. Let us suppose that $p+q=D$. Then we may define

$$\Omega(\varphi,\psi) = \mathrm{Res}\,(\omega(\overline{\varphi} \smile \delta\psi)) \in \Gamma_{\mathfrak{a}}(R^{\#}). \tag{2.59}$$

Proposition 49. Let $\varphi \in \check{\mathrm{H}}^p(\mathbf{t},P/L)$, $\psi \in \check{\mathrm{H}}^{D-p}(\mathbf{t},P/L)$. We have:

1. Graded skew-symmetry: $\Omega(\varphi,\psi) = -(-1)^{p(D-p)}\overline{\Omega(\psi,\varphi)}$.
2. Translation covariance: $\Omega(\varphi, r\psi) = \Omega(\overline{r}\varphi,\psi) = r\Omega(\varphi,\psi)$.
3. Commutation rule of operators introduced in Subsection 2.6:

$$\Omega(\varphi,\psi) = \mathrm{Res}(\omega(\widetilde{\varphi}_{1\dots p}, \widetilde{\psi}_{p+1\dots D})) = \omega(\widetilde{\varphi}^{\mathrm{Res}}_{1\dots p}, \widetilde{\psi}^{\mathrm{Res}}_{p+1\dots D}). \tag{2.60}$$

Proof. 1 follows from the graded commutativity and graded Leibniz rule of the cup product and antipode skew-symmetry of ω. 2 is obvious.

3. From the relevant definitions we have

$$\Omega(\varphi,\psi) = \mathrm{Res}\sum_{j=0}^{D-p}(-1)^j\omega(\widetilde{\varphi}_{1\dots p}, \widetilde{\psi}_{p\dots p+j-1,p+j+1\dots D}). \tag{2.61}$$

Let $0 < j \leq D-p$. The j-th term on the right-hand side of (2.61) is the residue of an element of $R_{t_1\dots t_{p-j-1}t_{p-j+1}\dots D}$, so it vanishes. The 0-th term is equal to the right-hand side of (2.60). □

We propose to interpret the scalar part of Ω as a higher dimensional version of braiding. Thus $\Omega(\varphi,\psi)$ encodes braiding of excitations described by φ,ψ as well as their translates. We will see later that for $D=2$ our proposal reduces to known expressions, providing evidence for our interpretation.

Recall that we have a decomposition $\check{\mathrm{H}}^{p+1}(\mathbf{t},P/L) = \bigoplus_{\mathfrak{m}}\Gamma_{\mathfrak{m}}\check{\mathrm{H}}^{p+1}(\mathbf{t},P/L)$, where $\mathfrak{m}$ are maximal ideals of R containing $\mathfrak{a}$. Its summands are charges characterized by specific behavior under translations, so we interpret $\mathfrak{m}$ as momentum "quantum numbers". Note that for every $\mathfrak{m}$, the ideal $\overline{\mathfrak{m}}$ obtained by acting with the antipode also contains $\mathfrak{a}$, as $\overline{\mathfrak{a}} = \mathfrak{a}$. We think of $\overline{\mathfrak{m}}$ as momentum opposite to $\mathfrak{m}$. The following proposition shows that two charges with fixed momentum may braid nontrivially only if their momenta are opposite.

Proposition 50. Suppose that $\varphi \in \Gamma_{\mathfrak{m}}\check{\mathrm{H}}^{p+1}(\mathbf{t},P/L), \psi \in \Gamma_{\mathfrak{m}'}\check{\mathrm{H}}^{D-p-1}(\mathbf{t},P/L)$. If $\overline{\mathfrak{m}} \neq \mathfrak{m}'$, then $\Omega(\varphi,\psi)=0$.

Proof. For some j, φ is annihilated by $\mathfrak{m}^j$ and ψ by $\mathfrak{m}'^j$. Therefore $\Omega(\varphi, \psi)$ is annihilated by $\overline{\mathfrak{m}}^j + \mathfrak{m}'^j$. If $\overline{\mathfrak{m}} \neq \mathfrak{m}'$, this sum is R. □

Decomposition of $\check{\mathrm{H}}^{p+1}(\mathbf{t}, P/L)$ into $\mathfrak{m}$-torsion parts is not invariant to coarse-graining. In fact, after sufficient coarse-graining we can assure that $\mathfrak{a}$ contains all $x^\lambda - 1$. Then, for n being a prime power, $\mathfrak{a}$ is contained in only one maximal ideal. Decomposition into $\mathfrak{m}$-torsion parts (and more generally, the module structure on $Q^i(\mathfrak{C})$) is an invariant protected by the translation symmetry and hence in principle can be used to distinguish SET phases with the same topological order.

Braiding and spin in 2D

We will now specialize to 2D Lagrangian codes. The assumption $\dim(Q) = 0$ is automatically satisfied, as stated in Corollary 37. Hence we have well-defined braiding. Expression (2.60) agrees with the standard braiding formula as a commutator of two orthogonal string operators. Let us explain this in more detail.

Consider a Lagrangian $\mathfrak{C} = (\mathbb{Z}^2, P, L)$. We have $R = \mathbb{Z}_n[x_1^\pm, x_2^\pm]$. There exists some $l > 0$ such that $t_i = 1 - x_i^l \in \mathrm{Ann}(Q(\mathfrak{C}))$. Therefore we have the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P/L & \xrightarrow{\iota_0} & L^* & \longrightarrow & Q(\mathfrak{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\delta} & & & & \\ 0 & \longrightarrow & \check{\mathrm{C}}^1(\mathbf{t}, P/L) & \xrightarrow{\iota_1} & \check{\mathrm{C}}^1(\mathbf{t}, L^*) & \longrightarrow & 0. & & \end{array}$$

For any $e \in L^*$, we have $\delta e = (e, e) = \iota_1(\frac{p_1}{x_1^l - 1}, \frac{p_2}{x_2^l - 1})$ with $p_i = (x_i^l - 1)e \in P/L$. One may check that $e \longmapsto (\frac{p_1}{x_1^l - 1}, \frac{p_2}{x_2^l - 1})$ defines an isomorphism between L^* and 1-cocycles, with elements of P/L mapped onto coboundaries. In particular this map induces an isomorphism $L^*/(P/L) \to \check{\mathrm{H}}^1(\mathbf{t}, P/L)$. A lift of p_i to P represents a Pauli operator which moves the excitation e by l units in the i-th direction. For this reason, p_i is sometimes called an i-mover.

Let $e_1, e_2 \in L^*$ be two excitations and let $p_i(e_j)$ be their movers. We can then form arbitrarily long string operators

$$(x_i^{-cl} + \cdots + 1 + \cdots x_i^{cl}) p_i(e_j), \tag{2.62}$$

which transport (displaced) excitations described by e_j by $(2c + 1)l$ units of length. Braiding may be related [6, 17] to the commutator phase

$$\omega((x_1^{-cl} + \cdots + x_1^{cl}) p_1(e_1), (x_2^{-cl} + \cdots + x_2^{cl}) p_2(e_2))_0 \tag{2.63}$$

with sufficiently large c. This expression is asymptotically independent of c because the two strings operators cross at most along a finite set. Taking c to infinity, this expression matches the scalar part of (2.60) with

$$\varphi = \left(\frac{p_1(e_1)}{x_1^l - 1}, \frac{p_2(e_1)}{x_2^l - 1} \right), \qquad \psi = \left(\frac{p_1(e_2)}{x_1^l - 1}, \frac{p_2(e_2)}{x_2^l - 1} \right). \tag{2.64}$$

We remark that it is also equal to the evaluation of the Laurent polynomial $\omega(p_1(e_1), p_2(e_2))$ at $x_1 = x_2 = 1$.

One can also define the topological spin function

$$\begin{aligned} \theta(e) =& \omega((x_1^{-cl} + \cdots + x_1^{-l})p_1(e), (x_2^{-cl} + \cdots + x_2^{-l})p_2(e))_0 \\ & -\omega((x_2^{-cl} + \cdots + x_2^{-l})p_2(e), (1 + x_1^l + \cdots + x_1^{cl})p_1(e))_0 \\ & -\omega((1 + x_1^l + \cdots + x_1^{cl})p_1(e), (x_1^{-cl} + \cdots + x_1^{-l})p_1(e))_0 \end{aligned} \tag{2.65}$$

with sufficiently large c (the right-hand side, as a function of c, is eventually constant). It is a quadratic refinement of the braiding pairing. Formula (2.65) appeared first in [6], where the case of prime-dimensional qudits was studied.

2.7 Examples

In this section we discuss examples with concrete codes. They serve several purposes. Firstly, they show that invariants we proposed are nontrivial, calculable, and yield what is expected on physical grounds in models which are already well understood. Secondly, they support our physical interpretation of mathematical objects and the conjecture that braiding is non-degenerate. Finally, the last example illustrates certain technical complications that do not arise for codes with prime-dimensional qudits.

In examples presented below we take P to be a free module R^{2t} with the symplectic form

$$\omega\left(\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} a' \\ b' \end{pmatrix} \right) = \begin{pmatrix} a^\dagger & b^\dagger \end{pmatrix} \underbrace{\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}_{\text{denote } \lambda} \begin{pmatrix} a' \\ b' \end{pmatrix}, \tag{2.66}$$

where $a, a', b, b' \in R^t$ and $\dagger$ denotes transposition composed with antipode. Following [12], we represent L as the image of a homomorphism $\sigma : R^s \to R^{2t}$, described by a $2t \times s$ matrix with entries in R.

We will also work with cocycles in $\check{C}^\bullet(\mathbf{t}, P/L)$. In calculations it is convenient to identify them with cochains in $\check{C}^\bullet(\mathbf{t}, P)$ which are closed modulo $\check{C}^\bullet(\mathbf{t}, L)$, with two cochains identified if they differ by a cochain in $\check{C}^\bullet(\mathbf{t}, L)$.

3D $\mathbb{Z}_n$-toric code

We take $\Lambda = \mathbb{Z}^3$ and denote generators of R corresponding to three basis vectors by x, y, z, so that R is a Laurent polynomial ring in three variables x, y, z. 3D toric code is defined by $P = R^6$, $L = \mathrm{im}(\sigma)$ with

$$\sigma = \begin{pmatrix} 1-\bar{x} & 0 & 0 & 0 \\ 1-\bar{y} & 0 & 0 & 0 \\ 1-\bar{z} & 0 & 0 & 0 \\ 0 & 0 & z-1 & y-1 \\ 0 & z-1 & 0 & 1-x \\ 0 & 1-y & 1-x & 0 \end{pmatrix}. \tag{2.67}$$

We have the following free resolution of P/L

$$0 \longrightarrow R \xrightarrow{\tau} R^4 \xrightarrow{\sigma} R^6 \longrightarrow P/L \longrightarrow 0, \qquad \tau = \begin{pmatrix} 0 \\ x-1 \\ 1-y \\ z-1 \end{pmatrix}. \tag{2.68}$$

Erasing P/L and applying $(-)^*$ we obtain

$$0 \longrightarrow P \xrightarrow{\sigma^\dagger \lambda} R^4 \xrightarrow{\tau^\dagger} R \longrightarrow 0. \tag{2.69}$$

Here matrix $\epsilon = \sigma^\dagger \lambda$ (rather than $\sigma^\dagger$) is present because the canonical isomorphism $P \to P^*$ is given by λ if both P and P^* are identified with R^6. From this resolution we easily get

$$\begin{aligned} &\mathrm{Ext}^1_R(\overline{P/L}, R) \cong \mathbb{Z}_n, && \text{generated by the class of } \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \in R^4, \\ &\mathrm{Ext}^2_R(\overline{P/L}, R) \cong \mathbb{Z}_n, && \text{generated by the class of } 1 \in R. \end{aligned} \tag{2.70}$$

Both Ext modules are annihilated by $x-1, y-1, z-1$.

Let us show how Čech cochains can be obtained from classes found above. In the construction of the Čech complex we may take $(x_1, x_2, x_3) = (x, y, z)$. Recall that we defined $t_i = 1 - x_i$. Now consider $\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}^{\mathrm{T}} \in R^4$. Applying the Čech differential gives

$$R^4_{t_1} \oplus R^4_{t_2} \oplus R^4_{t_3} \ni \left(\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}1\\0\\0\\0\end{pmatrix}\right) \tag{2.71}$$
$$= \left(\epsilon\begin{pmatrix}0\\0\\0\\-t_1^{-1}\\0\\0\end{pmatrix}, \epsilon\begin{pmatrix}0\\0\\0\\0\\-t_2^{-1}\\0\end{pmatrix}, \epsilon\begin{pmatrix}0\\0\\0\\0\\0\\-t_3^{-1}\end{pmatrix}\right).$$

The final expression is the image through ϵ of a certain element of $\check{\mathrm{C}}^1(\mathbf{t}, P)$. Let us call this cochain φ. By construction, it is closed modulo L. Let us show how this can be checked by an explicit computation:

$$\check{\mathrm{C}}^2(\mathbf{t}, R^6) = R^6_{t_2 t_3} \oplus R^6_{t_1 t_3} \oplus R^6_{t_1 t_2} \ni \delta\varphi$$
$$= \left(\begin{pmatrix}0\\0\\0\\0\\t_2^{-1}\\-t_3^{-1}\end{pmatrix}, \begin{pmatrix}0\\0\\0\\t_1^{-1}\\0\\-t_3^{-1}\end{pmatrix}, \begin{pmatrix}0\\0\\0\\t_1^{-1}\\-t_2^{-1}\\0\end{pmatrix}\right) \tag{2.72}$$
$$= \left(\sigma\begin{pmatrix}0\\-t_2^{-1}t_3^{-1}\\0\\0\end{pmatrix}, \sigma\begin{pmatrix}0\\0\\-t_1^{-1}t_3^{-1}\\0\end{pmatrix}, \sigma\begin{pmatrix}0\\0\\0\\-t_1^{-1}t_2^{-1}\end{pmatrix}\right).$$

One can go through a similar procedure with the element generating Ext^2. Let us record the final result:

$$\check{\mathrm{C}}^2(\mathbf{t}, R^6) \ni \psi = \left(\begin{pmatrix}\bar{t}_2^{-1}\bar{t}_3^{-1}\\0\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-\bar{t}_1^{-1}\bar{t}_3^{-1}\\0\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\\bar{t}_1^{-1}\bar{t}_2^{-1}\\0\\0\\0\end{pmatrix}\right). \tag{2.73}$$

Having these formulas in hand we evaluate

$$\Omega(\varphi, \psi)_0 = 1 \in \mathbb{Z}_n. \tag{2.74}$$

Hence braiding is a non-degenerate pairing in this example.

It is well known that toric code is closely related to $\mathbb{Z}_n$ gauge theory. With this interpretation, line operators corresponding to φ are Wilson lines. They create electric excitations at their endpoints. Cocycle ψ corresponds to electric flux (surface) operators, which create a magnetic field on the boundary. Braiding between the two excitations is an Aharonov-Bohm type phase. We remark also that the relation between generators of L, described by the map τ, corresponds to Bianchi identity.

4D $\mathbb{Z}_n$-toric code

In a 4D version of the $\mathbb{Z}_n$ toric code we have $P = R^8$. We let $x_1, \dots, x_4$ be four variables corresponding to generators of $\mathbb{Z}^4$ and denote basis vectors of P by $e_1, \dots, e_4, a_1, \dots, a_4$. Consider the free module R^7 with basis $\{g\} \cup \{f_{ij}\}_{1 \le i < j \le 4}$. We define $L = \mathrm{im}(\sigma)$, where $\sigma : R^7 \to P$ is given by

$$\sigma(g) = \sum_{i=1}^{4} (1 - \overline{x}_i) e_i, \qquad \sigma(f_{ij}) = -(x_i - 1) a_j + (x_j - 1) a_i. \tag{2.75}$$

Elements $\sigma(g), \sigma(f_{ij})$ generate L. To continue σ to a resolution of P/L, we need to describe relations between generators. Consider the free module R^4 with basis $\{b_{ijk}\}_{1 \le i < j < k \le 4}$. Define $\tau_1 : R^4 \to R^7$ by

$$\tau_1(b_{ijk}) = (x_i - 1) f_{jk} - (x_j - 1) f_{ik} + (x_k - 1) f_{ij}. \tag{2.76}$$

Then $\mathrm{im}(\tau_1) = \ker(\sigma)$, but we still have to take care of relations between relations. Let $\tau_2 : R \to R^4$ be given by

$$\tau_2(1) = (x_1 - 1) b_{234} - (x_2 - 1) b_{134} + (x_3 - 1) b_{124} - (x_4 - 1) b_{123}. \tag{2.77}$$

We have constructed a free resolution

$$0 \to R \xrightarrow{\tau_2} R^4 \xrightarrow{\tau_1} R^7 \xrightarrow{\sigma} R^8 \to P/L \to 0. \tag{2.78}$$

Proceeding as in the 3D case we found

$$Q^0 \cong \mathbb{Z}_n, \qquad Q^1 = 0, \qquad Q^2 \cong \mathbb{Z}_n, \tag{2.79}$$

all annihilated by $x_i - 1$. After some tedious calculations we found also the Čech cochains $\varphi \in \check{C}^1(\mathbf{t}, P)$ and $\psi \in \check{C}^3(\mathbf{t}, P)$ corresponding to generators of Q^0 and Q^2:

$$\varphi_i = \frac{a_i}{x_i - 1}, \qquad \psi_{i^c} = \frac{(-1)^i e_i}{\prod_{j \neq i}(\overline{x}_j - 1)}, \tag{2.80}$$

where i^c denotes the triple of indices complementary to i. Given these expressions it is easy to check that

$$\Omega(\varphi, \psi)_0 = 1. \tag{2.81}$$

Again, braiding is non-degenerate.

4D $\mathbb{Z}_n$ 2-form toric code

By a 2-form version of the toric code we mean a code in which degrees of freedom are assigned to lattice plaquettes. Starting from dimension 4, such a code is neither trivial nor equivalent to the standard ("1-form") toric code. Module $P \cong R^{12}$ has basis $\{e_{ij}, a_{ij}\}_{1 \leq i < j \leq 4}$, with nontrivial symplectic pairings of the form $\omega(e_{ij}, a_{ij}) = 1$. Consider the free module R^8 with basis $\{g_i, f_{i^c}\}_{i=1}^4$. We define $L = \mathrm{im}(\sigma)$, where $\sigma : R^8 \to P$ is given by

$$\begin{aligned} \sigma(g_i) &= -\sum_{j<i}(1 - \overline{x}_j)e_{ji} + \sum_{j>i}(1 - \overline{x}_j)e_{ij}, \\ \sigma(f_{ijk}) &= -(x_i - 1)a_{jk} + (x_j - 1)a_{ik} - (x_k - 1)a_{ij}. \end{aligned} \tag{2.82}$$

$\ker(\sigma)$ coincides with the image of $\tau : R^2 \to R^8$ such that

$$\tau\begin{pmatrix}1\\0\end{pmatrix} = \sum_i (1 - \overline{x}_i)g_i, \qquad \tau\begin{pmatrix}0\\1\end{pmatrix} = \sum_i (-1)^i(x_i - 1)f_{i^c}. \tag{2.83}$$

This defines a free resolution

$$0 \to R^2 \xrightarrow{\tau} R^8 \xrightarrow{\sigma} R^{12} \to P/L \to 0, \tag{2.84}$$

from which we derive

$$Q^0 = 0, \qquad Q^1 \cong \mathbb{Z}_n \oplus \mathbb{Z}_n, \qquad Q^2 = 0, \tag{2.85}$$

with Q^1 annihilated by all $x_i - 1$. Two Čech cochains corresponding to generators of Q^1 take the form

$$\varphi_{ij} = \frac{a_{ij}}{(x_i - 1)(x_j - 1)}, \qquad \psi_{ij} = \frac{(-1)^{i+j} e_{ij^c}}{(\overline{x_i} - 1)(\overline{x_j} - 1)}, \tag{2.86}$$

where ij^c is the pair of indices complementary to ij. We find

$$\Omega(\varphi, \psi)_0 = 1, \tag{2.87}$$

so braiding is non-degenerate.

$\mathbb{Z}_n$ Ising model

For the Ising model in zero magnetic field we have $P = R^2$ and

$$\sigma = \begin{pmatrix} x_1 - 1 & \cdots & x_D - 1 \\ 0 & \cdots & 0 \end{pmatrix}, \tag{2.88}$$

where $D \geq 1$ is arbitrary. We see that $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in L^{\omega\omega} \setminus L$ and $L^{\omega\omega}/L \cong \mathbb{Z}_n$, in accord with the interpretation of $L^{\omega\omega}/L$ in terms of order parameters for spontaneously broken symmetries. Next, we note that $P/L \cong R \oplus R/\mathfrak{a}$, where $\mathfrak{a} = (x_1 - 1, \cdots, x_D - 1)$. Hence for every $i > 0$ we have

$$\operatorname{Ext}^i_R(\overline{P/L}, R) \cong \operatorname{Ext}^i_R(R/\mathfrak{a}, R). \tag{2.89}$$

As elements $x_i - 1$ form a regular sequence in R, this Ext vanishes for $i \neq D$ and $\operatorname{Ext}^D_R(\overline{P/L}, R) \cong R/\mathfrak{a}$. Therefore the only nonzero Q^i is $Q^{D-1} \cong \mathbb{Z}_n$. This is consistent with the interpretation of Q^i in terms of i-dimensional excitations: the Ising model features domain walls, which are objects of spatial codimension 1. However, our formalism does not provide a systematic construction of this domain wall (Ising model is not a Lagrangian code). Let us also remark that we expect that there exists a generalization of braiding that allows us to pair Q^{D-1} with $L^{\omega\omega}/L$. Physically such pairing should describe how the value of order parameter changes as the domain wall is crossed.

$\mathbb{Z}_n$ toric code on a cylinder

Consider the 2D cylinder geometry $\Lambda = \mathbb{Z}_L \times \mathbb{Z}$. Thus $R = \mathbb{Z}_n[x, y^{\pm}]/(x^L - 1)$. We let $P = R^4$ and $L = \operatorname{im}(\sigma)$, where

$$\sigma = \begin{pmatrix} 1 - \overline{x} & 0 \\ 1 - \overline{y} & 0 \\ 0 & y - 1 \\ 0 & x - 1 \end{pmatrix}. \tag{2.90}$$

Let us put $W_x = \sum_{j=0}^{L-1} x^{j-1} \in R$. Note that $(x-1)W_x = 0$, so

$$(y-1)\begin{pmatrix} 0 \\ 0 \\ W_x \\ 0 \end{pmatrix} = W_x \begin{pmatrix} 0 \\ 0 \\ y - 1 \\ x - 1 \end{pmatrix} \in L. \tag{2.91}$$

Since $y-1$ is a regular element, it follows that $\begin{pmatrix} 0 & 0 & W_x & 0 \end{pmatrix}^{\mathrm{T}} \in L^{\omega\omega}$. A similar calculation shows that $\begin{pmatrix} 0 & W_x & 0 & 0 \end{pmatrix}^{\mathrm{T}} \in L^{\omega\omega}$. Classes of these two elements generate $L^{\omega\omega}/L \cong \mathbb{Z}_n \times \mathbb{Z}_n$. One may check also that $L^{\omega} = L^{\omega\omega}$. Hence there exist n^2 superselection sectors containing a ground state, and in each of these sectors the ground state is unique. This is different than for the toric code on a torus, for which there is only one superselection sector containing an n^2-dimensional space of ground states. This illustrates the difference in physical interpretations of modules $Z(\mathfrak{C})$ and $S(\mathfrak{C})$.

Let us also mention that in the present example $Q(\mathfrak{C}) \cong \mathbb{Z}_n \times \mathbb{Z}_n$, as on a plane (but not on a torus). Even though the code is effectively one-dimensional (one direction being finite), this does not contradict Proposition 25 because $Z(\mathfrak{C}) \neq 0$.

$\mathbb{Z}_{p^t}$ plaquette model

Let $n = p^t$, where p is a prime number and t a positive integer. We consider a $\mathbb{Z}_{p^t}$ version of Wen's plaquette model [18] on a plane. Thus we take $P = R^2$ and let L be the span of $s = \begin{pmatrix} 1 - xy & x - y \end{pmatrix}^{\mathrm{T}}$. L is freely generated by s, so there exists an element $\varphi \in L^*$ such that $\varphi(s) = 1$. Clearly $(x-y)\varphi$ and $(xy-1)\varphi$ are representable by elements of P and we have

$$Q \cong R/(x-y, xy-1). \tag{2.92}$$

There exists an abelian group isomorphism $Q \cong \mathbb{Z}_n \times \mathbb{Z}_n$ (as for the toric code), but in contrast to the case of toric code Q is acted upon nontrivially by translations. Hence this model is in a different SET phase (with translational symmetry) than the toric code. On the other hand, these models are well-known to be equivalent if translational symmetry is ignored.

For a subgroup of Λ acting trivially on Q, we can take the subgroup of index 4 generated by x^2, y^2. With this choice, we found the following Čech cocycle φ representing the generator of Q (corresponding via the isomorphism (2.92) to the class of 1):

$$\varphi_1 = \frac{\begin{pmatrix} x & x^2 \end{pmatrix}^{\mathrm{T}}}{1-x^2}, \qquad \varphi_2 = \frac{\begin{pmatrix} -y & y^2 \end{pmatrix}^{\mathrm{T}}}{1-y^2}. \tag{2.93}$$

Classes of cocycles φ and $x\varphi$ form a $\mathbb{Z}_n$ basis of Čech cohomology.

We will find the decomposition of Q into $\mathfrak{m}$-torsion parts. If $p \neq 2$, maximal ideals of R containing the annihilator of Q are of the form

$$\mathfrak{m}^{\pm} = (p, x \mp 1, y \mp 1). \tag{2.94}$$

The case $p = 2$ is special because then $\mathfrak{m}^{+} = \mathfrak{m}^{-}$. We assume that $p \neq 2$ from now on. $\mathfrak{m}^{\pm}$-torsion submodules of Q correspond to cocycles $\varphi^{\pm} = (1 \pm x)\varphi$. They also form a $\mathbb{Z}_n$ basis of Čech cohomology. By Proposition 50, φ^{+} is Ω-orthogonal to φ^{-}. Indeed, a calculation gives

$$\Omega(\varphi, \varphi) = (x + y) \sum_{k,l \in \mathbb{Z}} x^{2k} y^{2l}, \tag{2.95}$$

and therefore

$$\begin{aligned} \Omega(\varphi^{+}, \varphi^{-}) &= (1 + \bar{x})(1 - x)\Omega(\varphi, \varphi) = 0, \\ \Omega(\varphi^{\pm}, \varphi^{\pm}) &= (1 \pm \bar{x})(1 \pm x)\Omega(\varphi, \varphi) = \pm 2 \sum_{k,l \in \mathbb{Z}} (\pm x)^{k} (\pm y)^{l}. \end{aligned} \tag{2.96}$$

Remark 51. Redefining s to $\begin{pmatrix} 1 + xy & x + y \end{pmatrix}$ gives a second code, which is related to the one above by a local unitary transformation (which is y^2-invariant but not y-invariant). Simple calculation gives $Q \cong R/(x + y, xy + 1)$, so this code is in a different SET phase than the previous one.

Haah's code and X-cube model

Haah's code and X-cube model (over $\mathbb{Z}_2$) are defined by

$$\sigma_{\text{Haah}} = \begin{pmatrix} 1 + xy + yz + zx & 0 \\ 1 + x + y + z & 0 \\ 0 & 1 + \bar{x} + \bar{y} + \bar{z} \\ 0 & 1 + \bar{x}\bar{y} + \bar{y}\bar{z} + \bar{z}\bar{x} \end{pmatrix}, \tag{2.97}$$

$$\sigma_{X\text{-cube}} = \begin{pmatrix} 1 + \bar{x} + \bar{y} + \bar{x}\bar{y} & 0 & 0 \\ 1 + \bar{y} + \bar{z} + \bar{y}\bar{z} & 0 & 0 \\ 1 + \bar{x} + \bar{z} + \bar{x}\bar{z} & 0 & 0 \\ 0 & 1 + z & 0 \\ 0 & 1 + x & 1 + x \\ 0 & 0 & 1 + y \end{pmatrix}.$$

In both cases σ is injective, so L is free. This implies that $Q^i = 0$ for $i > 0$, so our approach confirms that corresponding phases of matter do not admit nontrivial spatially extended excitations. Computation of Q^0 of course agrees with what is known.

$\mathbb{Z}_4$ toric code with condensation

We consider a code with composite-dimensional qudits. Let $R = \mathbb{Z}_4[x^{\pm}, y^{\pm}]$, $P = R^4$ and $L = \mathrm{im}(\sigma)$, where

$$\sigma = \begin{pmatrix} 2+2\bar{x} & 0 & 0 & 0 \\ 2+2\bar{y} & 0 & 0 & 0 \\ 0 & 1-y & 2 & 0 \\ 0 & 1-x & 0 & 2 \end{pmatrix}. \tag{2.98}$$

Let us define a matrix

$$\tau = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 1-y & 2 & 0 \\ 0 & 1-x & 0 & 2 \end{pmatrix}. \tag{2.99}$$

We have an infinite free resolution

$$\cdots \to R^4 \xrightarrow{\tau} R^4 \xrightarrow{\tau} R^4 \xrightarrow{\tau} R^4 \xrightarrow{\sigma} P \to P/L \to 0 \tag{2.100}$$

from which one obtains

$$Q^0 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2, \qquad Q^i = 0 \text{ for } i > 0, \tag{2.101}$$

as in the ordinary toric code. In spite of vanishing of higher Q^i, there exists no finite free resolution–see characterization in Proposition 52 below.

We remark also that $\mathrm{Ext}^i_R(L, R) = 0$ for $i > 0$. If n was prime, we would be able to deduce from this that L is a free module. In the present example, L is not even quasi-free. Indeed, if L was quasi-free, $L/2L$ would be a free module over $S = \mathbb{Z}_2[x^{\pm}, y^{\pm}]$. On the other hand, it is not difficult to check that $\mathrm{Ext}^1_S(L/2L, S) \neq 0$. This motivates the following result.

Proposition 52. Let $n = p^t$ for a prime number p. An R-module has finite projective dimension if and only if it is free over $\mathbb{Z}_n$. If this condition is satisfied, there exists a free resolution of length not exceeding D.

Proof. $\implies$: A projective R-module P is a summand of a free R-module, which is clearly free over $\mathbb{Z}_n$. Thus P is also projective over $\mathbb{Z}_n$. Projective modules over $\mathbb{Z}_n$ are free.

Now let $P_\bullet \to M$ be a finite projective resolution of a module M. By the paragraph above, this is also a free resolution of M considered as a $\mathbb{Z}_n$ module. Thus M has finite projective dimension over $\mathbb{Z}_n$. Such $\mathbb{Z}_n$-modules are free.

$\Longleftarrow$: Let M be free over $\mathbb{Z}_n$. We choose a $\mathbb{Z}_p[\Lambda]$-free resolution

$$0 \to P_D \to \cdots \to P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} M/pM \to 0 \tag{2.102}$$

of length D. This is possible by Hilbert's syzygy theorem. We will lift the resolution of M/pM to a resolution of M of the same length. Let $K_i = \ker(\partial_i)$.

For the purpose of this proof it will be convenient to denote reduction of an element mod p by an overline. We have

$$\partial_0 \begin{bmatrix} \overline{r}_1 \\ \vdots \\ \overline{r}_n \end{bmatrix} = \sum_{i=1}^{n} \overline{r_i m_i} \tag{2.103}$$

for some $m_1, \ldots, m_n \in M$ such that $\overline{m}_i$ generate M/pM. Then by Nakayama, m_i generate M. Define $\widehat{P}_0 = R^n$ and $\widehat{\partial}_0 : R^n \to M$ by

$$\widehat{\partial}_0 \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} = \sum_{i=1}^{n} r_i m_i. \tag{2.104}$$

By construction, $\widehat{\partial}_0$ is surjective. Let $\widehat{K}_0 = \ker(\widehat{\partial}_0)$. Reducing the short sequence $0 \to \widehat{K}_0 \to R^n \xrightarrow{\widehat{\partial}_0} M \to 0$ mod p yields

$$0 \to \widehat{K}_0/p\widehat{K}_0 \to P_0 \xrightarrow{\partial_0} M/pM \to 0, \qquad \mathrm{Tor}_1^R(\widehat{K}_0, R/(p)) = 0. \tag{2.105}$$

Here we used the simple fact that an R-module N is free over $\mathbb{Z}_n$ if and only if $\mathrm{Tor}_1^R(N, R/(p)) = 0$, which can be verified using the resolution

$$\cdots \to R \xrightarrow{p} R \xrightarrow{p^{t-1}} R \xrightarrow{p} R \to R/(p) \to 0. \tag{2.106}$$

Results in (2.105) imply that $\widehat{K}_0/p\widehat{K}_0$ may be identified with K_0, and $\widehat{K}_0$ is free over $\mathbb{Z}_n$.

Now replace M by K_0 and P_0 by P_1 and repeat. Proceeding like this inductively we find short exact sequences

$$0 \to \widehat{K}_D \to \widehat{P}_D \to \widehat{K}_{D-1} \to 0,$$

$$\ldots,$$

$$0 \to \widehat{K}_1 \to \widehat{P}_0 \to M \to 0,$$

such that each P_i is free and $\widehat{K}_i/p\widehat{K}_i \cong K_i$. In particular $\widehat{K}_D = 0$ by Nakayama. Short sequences compose into a free resolution of M of length D:

$$0 \to \widehat{P}_D \to \cdots \to \widehat{P}_0 \to M \to 0. \tag{2.107}$$

□

[1] Blazej Ruba and Bowen Yang. "Homological invariants of pauli stabilizer codes". In: *arXiv preprint arXiv:2204.06023* (2022). DOI: 10.48550/arXiv.2204.06023. URL: https://doi.org/10.48550/arXiv.2204.06023.

Chapter 3

SYMMETRY PROTECTED TOPOLOGICAL SPIN CHAIN

3.1 Introduction

Gapped phases of lattice systems in 1d are well understood by now [19, 20]. Their classification depends on whether one considers bosonic or fermionic systems, as well as whether other symmetries are present. For example, without imposing any symmetry there is only one bosonic phase (the trivial one). Fermionic phases without symmetries beyond the fermion parity $\mathbb{Z}_2^F$ are classified by $\mathbb{Z}_2$, where the non-trivial element can be realized by the Kitaev chain. Bosonic phases with a unitary on-site symmetry G are classified by the abelian group $H^2(G, U(1))$, while fermionic phases with a symmetry $G \times \mathbb{Z}_2^F$ are classified by $\mathbb{Z}_2 \times \mathcal{F}$ where $\mathcal{F}$ is an extension of $H^1(G, \mathbb{Z}_2)$ by $H^2(G, U(1))$. An element of an abelian group describing the phase will be called its index. Most of these classification results have been obtained by assuming that every phase of a 1d lattice system can be described by an injective Matrix Product State (MPS).

Although ground states of generic gapped Hamiltonians are not MPS, for finite 1d systems one can find an MPS approximation of any such state. There is an efficient numerical procedure (DMRG) for doing this. Thus from a practical standpoint the situation in 1d is very satisfactory: given a gapped Hamiltonian, there is an algorithmic procedure for identifying its phase. But from a theoretical standpoint the MPS approach leaves much to be desired. For example, it is not obvious that the index determined through a highly non-unique MPS approximation depends only on the original state. Neither is it obvious that it is an invariant of the phase (i.e., that it depends only on an appropriately defined equivalence class of states).

Recently, Y. Ogata and collaborators [21, 22, 23] (see also [24]) developed an approach to the classification of phases of 1d systems which does not rely on using an injective MPS. Instead they work with arbitrary 1d states satisfying the split property. The split property is the statement that the weak closure of the algebra of observables on any half-line is a Type I von Neumann algebra. In the bosonic case, this is equivalent to saying that the Hilbert space of the system can be written as a tensor product of Hilbert spaces $\check{\mathrm{H}}_- \otimes \check{\mathrm{H}}_+$, so that observables localized on the positive (resp. negative) half-line act non-trivially only on $\check{\mathrm{H}}_+$ (resp. $\check{\mathrm{H}}_-$). The split property, while seemingly obvious, typically fails for gapless systems in infinite volume, because von Neumann algebras more exotic than algebras of bounded operators in a Hilbert space show up. Nevertheless, it was shown by T. Matsui [25] that states of 1d lattice systems satisfying the area law (and in particular all gapped ground states of local Hamiltonians [26]) satisfy the split property. It is possible in this approach to define an index of a state and show that it is invariant with respect to a natural notion of equivalence (automorphic equivalence). This property ensures that the index is unchanged under continuous deformations of the Hamiltonian which do not close the gap [27, 28].

In this chapter we use a variant of this approach to classify bosonic phases of matter with a certain additional property ("invertibility", see the next paragraph). Our definition of a quantum phase of matter is along the lines of Refs. [29, 30] and is based on the notion that a quantum phase of matter is a pattern of entanglement in a many-body wave-function, a point of view expounded in the monograph Ref. [30]. In Ref. [30] two states were said to be in the same phase if, after stacking with some unentangled state of an ancillary system, they could be connected by a finite-depth quantum circuit. For the case of systems with a unitary symmetry, one additionally requires the quantum circuit to preserve the symmetry. Our definition is the same, except we replace quantum circuits with their fuzzy analogs, locally-generated automorphisms (LGAs) [31]. This notion of a phase does not make reference to Hamiltonians and applies to arbitrary pure states of lattice systems.

To focus on ground states of gapped Hamiltonians, Refs. [21, 22, 23] impose the split property. In this chapter we focus on states which are “invertible” in the sense of A. Kitaev [32]. Loosely speaking, an invertible state is a state which lacks long-range order. More precisely, a system is in an invertible state if, when combined with some ancillary system, it can be completely disentangled by applying an LGA. The ancillary degrees of freedom may be entangled between themselves but not with the original system. We show that for states of 1d systems invertibility implies the split property, but unlike the latter, it has a clear physical meaning and can be generalized to higher dimensions. Our definition of a quantum phase of matter applies both to bosonic and fermionic systems, but in this chapter we focus on bosonic systems, since the fermionic case is more technically involved.

Our main results are as follows. If no symmetry is imposed, then we show that every invertible bosonic 1d state is in the trivial phase. In the case when a finite unitary on-site symmetry G is present, we define an index valued in an abelian group for arbitrary G-invariant invertible 1d systems and prove that it is a complete invariant of such phases. That is, two G-invariant invertible 1d states are in the same phase if and only if their indices coincide. Our definition of the index is equivalent to that in [21, 22] but is formulated in terms of properties of domain walls for symmetries. We also show that a non-trivial index is an obstruction for a system to have an edge which preserves both the symmetry G and invertibility.

The organization of the chapter is as follows. In Section 2 we formulate our definition of phases and invertible phases as suitable equivalence classes of pure states of lattice systems. In Section 3 we define an index for invertible 1d states with unitary symmetries and show that entangled pair states realize all possible values of the index. A reader who is mostly interested in the properties of systems without symmetries can skip this section and proceed to Section 4. In Sections 4.1 and 4.2 we study invertible states of 1d systems without symmetries. We show that any such state is in a trivial phase. In Section 4.3 we classify phases of invertible 1d states with a finite unitary symmetry. In Appendix A we show that G-invariant pure factorized states of 1d systems are all in the trivial phase according to our definition. In Appendix B we show that the index of a state can be determined provided one has access to a sufficiently large but finite piece of the system. This implies that a non-trivial index is an obstruction for a system to have an edge which preserves both the symmetry G and invertibility.

3.2 States and phases of 1d lattice systems

A 1d system is defined by its algebra of observables $\mathscr{A}$. In the bosonic case, this is a C^*-algebra defined as a completion of the $*$-algebra of the form

$$\mathscr{A}_\ell = \varinjlim_{M} \bigotimes_{j\in[-M,M]} \mathscr{A}_j, \tag{3.1}$$

where $\mathscr{A}_j = \mathrm{End}(\mathscr{V}_j)$ for some finite-dimensional Hilbert space $\mathscr{V}_j$. The numbers $d_j = \mathrm{d}im\mathscr{V}_j$ are not assumed to be bounded, but we assume that $\log d_j$ grows at most polynomially with $|j|$. Elements of the algebra $\mathscr{A}$ are called (quasi-local) observables. For any finite subset $\Gamma \subset \mathbb{Z}$ we define a sub-algebra $\mathscr{A}_\Gamma = \otimes_{j\in\Gamma}\mathscr{A}_j$. Its elements are called local observables localized on Γ. The union of all $\mathscr{A}_\Gamma$ is the algebra of local observables $\mathscr{A}_\ell$.

There is a distinguished dense $*$-sub-algebra $\mathscr{A}_{a\ell}$ of $\mathscr{A}$ which we call the algebra of almost local observables. $\mathcal{A} \in \mathscr{A}$ is an almost local observable if its commutators with observables in $\mathscr{A}_j$ decay faster than any power of $|j|$. More precisely, there exists $k \in \mathbb{Z}$ and a monotonically-decreasing positive (MDP) function $h : \mathbb{R} \to \mathbb{R}$ which decays faster than any power such that for any $\mathcal{B} \in \mathscr{A}_j$ one has $\|[\mathcal{A}, \mathcal{B}]\| \leq \|\mathcal{A}\| \cdot \|\mathcal{B}\| h(|j-k|)$. We will say that $\mathcal{A}$ is h-localized on k.

Almost local observables are the building blocks for Hamiltonians with good locality properties. Such a Hamiltonian is a formal linear combination

$$F = \sum_{j \in \mathbb{Z}} F_j, \tag{3.2}$$

where F_j is a self-adjoint almost local observable which is h localized on j (with the same function h for all j), and such that the sequence $\|F_j\|$, $j \in \mathbb{Z}$, is bounded. One may call a Hamiltonian of this kind an almost local Hamiltonian. Note that F is not a well-defined element of $\mathscr{A}$ unless the sum is convergent. To emphasize this, we prefer to call such a formal linear combination a 0-chain rather than a Hamiltonian. Note also that $\mathrm{ad}_F(\mathcal{A}) = [F, \mathcal{A}] = \sum_j [F_j, \mathcal{A}]$ is a well-defined almost local observable for any $\mathcal{A} \in \mathscr{A}_{a\ell}$. It is easy to see that ad_F is an derivation of $\mathscr{A}_{a\ell}$.

The main use of 0-chains is to define a distinguished class of automorphisms of $\mathscr{A}$ which preserve the sub-algebra $\mathscr{A}_{a\ell}$. Consider a self-adjoint 0-chain $F(s)$ which depends continuously[1] on a parameter $s \in [0, 1]$. Then we define a one-parameter family of automorphisms $\alpha_F(s)$, $s \in [0, 1]$, as a solution of the differential equation

$$\frac{d}{ds} \alpha_F(s)(\mathcal{A}) = \alpha_F(s)(i[F(s), \mathcal{A}]), \tag{3.3}$$

with the initial condition $\alpha_F(0) = \mathrm{Id}$. One can show that this differential equation has a unique solution for any 0-chain $F(s)$ [33]. It follows from Lieb-Robinson bounds that $\alpha_F(s)$ maps $\mathscr{A}_{a\ell}$ to itself (see Lemma A.2 of [31]). We will use a short-hand $\alpha_F(1) = \alpha_F$.

Automorphisms of the form α_F for some $F(s)$ will be called locally-generated automorphisms (or LGAs). They are "fuzzy" analogs of finite-depth unitary quantum circuits. Note that locally-generated automorphisms form a group. Indeed, the composition of α_F and α_G can be generated by the 0-chain

$$G(s) + \alpha_G(s)^{-1}(F(s)). \tag{3.4}$$

The inverse of α_F is an automorphism generated by

$$-\alpha_F(s)(F(s)). \tag{3.5}$$

[1] We say that a 0-chain $F(s)$ depends continuously on s if $F_j(s)$ depends continuously on s for all $j \in \mathbb{Z}$.

A state on $\mathscr{A}$ is a positive linear function $\psi : \mathscr{A} \to \mathbb{C}$ such that $\psi(\mathrm{Id}) = 1$. Given a state ψ and a locally-generated automorphism α_F, we can define another state by $\alpha_F(\psi) := \psi \circ \alpha_F(\mathcal{A}) = \psi(\alpha_F(\mathcal{A}))$. We will say that $\alpha_F(\psi)$ is related by an LGA to ψ, or LGA-equivalent to ψ. We denote by $\Phi(\mathscr{A})$ the set of LGA-equivalence classes of pure states on $\mathscr{A}$.

The notion of LGA-equivalence makes sense for arbitrary pure states of lattice systems. If the states in question are unique ground states of gapped almost local Hamiltonians, then they are LGA-equivalent if and only if the corresponding Hamiltonians can be continuously deformed into each other without closing the gap [27, 28]. It is also easy to see that if a pure state ψ is a unique ground state of a gapped almost local Hamiltonian, then so is every state which is LGA-equivalent to ψ. Thus for gapped ground states, LGA-equivalence is the same as homotopy equivalence. More generally, it has been proposed to define a zero-temperature phase as an equivalence class of a (not necessarily gapped) pure state under the action of finite-depth quantum circuits [30]. One obvious defect of such a definition is that quantum circuits cannot change the range of correlations by more than a finite amount. Thus according to this definition, ideal atomic insulators which have a finite correlation range are not in the same phase as any state with exponentially decaying correlations. Replacing finite-depth quantum circuits with LGAs fixes this defect.

A state ψ is called factorized if $\psi(\mathcal{A}\mathcal{B}) = \psi(\mathcal{A})\psi(\mathcal{B})$ whenever $\mathcal{A}$ and $\mathcal{B}$ are local observables supported on two different sites. It is easy to see that all factorized pure states of $\mathscr{A}$ are LGA-equivalent.[2] Thus factorized pure states on $\mathscr{A}$ define a distinguished element $\tau(\mathscr{A}) \in \Phi(\mathscr{A})$. Following [30], we will say that a state on $\mathscr{A}$ is Short-Range Entangled (SRE) if it is LGA-equivalent to a pure factorized state on $\mathscr{A}$.

To compare states on different 1d lattice systems, we define the notion of stable equivalence following A. Kitaev [32]. We say that a pure state ψ_1 on a system $\mathscr{A}_1$ is stably equivalent to a pure state ψ_2 on a system $\mathscr{A}_2$ if there exist systems $\mathscr{A}_1'$ and $\mathscr{A}_2'$ and factorized pure states ψ_1' and ψ_2' on them such that the pairs $(\mathscr{A}_1 \otimes \mathscr{A}_1', \psi_1 \otimes \psi_1')$ and $(\mathscr{A}_2 \otimes \mathscr{A}_2', \psi_2 \otimes \psi_2')$ are LGA-equivalent.

A 1d phase is defined to be a stable equivalence class of a pure state ψ on a 1d system $\mathscr{A}$. We will denote the set of (1d) phases by Φ. Any two factorized pure states on any two 1d systems are stably equivalent. Thus it makes sense to say that the stable

[2] It is important here that we are dealing with bosonic systems. The statement is not true in the fermionic case.

equivalence class of factorized pure states is the trivial phase $\tau \in \Phi$. It is also easy to see that the tensor product of systems and their states descends to a commutative associative binary operation $\otimes$ on Φ, and that τ is the neutral element with respect to this operation. In other words, $(\Phi, \otimes, \tau)$ is a commutative monoid.

Next we define what we mean by an invertible phase. Following A. Kitaev [32], we will say that a pure state ψ on $\mathscr{A}$ is in an invertible phase if there is a 1d system $\mathscr{A}'$ and a pure state ψ' on it such that $\psi \otimes \psi'$ is in a trivial phase. In other words, $x \in \Phi$ is an invertible phase if it has an inverse. Therefore invertible phases form an abelian group which we denote Φ^*.

Note that we defined a phase without any reference to a Hamiltonian. In general, given a pure state, it is not clear whether it is a ground state of any almost-local Hamiltonian.

It will be shown below that all invertible 1d systems are in a trivial phase[3], i.e., Φ^* consists of a single element τ. To get a richer problem, we will study systems and phases with on-site symmetries. Recall that we assumed that each $\mathscr{A}_j$ is a matrix algebra, i.e., it has the form $\mathscr{A}_j = \mathrm{End}(\mathscr{V}_j)$ for some finite-dimensional vector space $\mathscr{V}_j$ (the "on-site Hilbert space"). One says that a group G acts on a system $\mathscr{A}$ by unitary on-site symmetries if one is given a sequence of homomorphisms $R_j : G \to U(\mathscr{V}_j)$, $j \in \mathbb{Z}$, where $U(\mathscr{V}_j)$ is the unitary group of $\mathscr{V}_j$. Since each $\mathscr{V}_j$ is finite-dimensional, it decomposes as a sum of a finite number of irreducible representations of G.

Homomorphisms R_j give rise to a G-action on $\mathscr{A}$, $\mathscr{A}_\ell$ and $\mathscr{A}_{a\ell}$: $(g, \mathcal{A}) \mapsto \mathrm{Ad}_{R(g)}(\mathcal{A}) = R(g)\mathcal{A}R(g)^{-1}$, where

$$R(g) = \prod_{j \in \mathbb{Z}} R_j(g), \quad g \in G. \tag{3.6}$$

The latter is a formal product of unitary local observables $R_j(g) \in \mathscr{A}_j$ such that the automorphism $\mathrm{Ad}_{R(g)}$ is well-defined. A state ψ on a system $\mathscr{A}$ with an on-site action of G is said to be G-invariant if it is invariant under $\mathrm{Ad}_{R(g)}$ for all $g \in G$: $\psi \circ \mathrm{Ad}_{R(g)} = \psi$ for all $g \in G$.

When defining LGA-equivalence of G-invariant states on $\mathscr{A}$, one needs to restrict to those LGAs which are generated by G-invariant self-adjoint 0-chains, i.e., self-adjoint 0-chains $F(s) = \sum_j F_j(s)$ such that all observables $F_j(s)$ are G-invariant. As a

[3] This is not true for fermionic systems, the Kitaev chain is a counter-example.

result, the automorphism it generates is called G-equivariant as it commutes with the action of the group. We denote by $\Phi_G(\mathscr{A})$ the set of G-equivariant LGA-equivalence classes of G-invariant pure states of $\mathscr{A}$ (with some particular G-action which is not indicated explicitly).

There is a distinguished element $\tau_G \in \Phi_G(\mathscr{A})$, but its definition is not completely obvious. The most general G-invariant factorized pure state on $\mathscr{A}$ is uniquely defined by the condition that when restricted to $\mathscr{A}_j$ it takes the form

$$\psi(\mathcal{A}_j) = \langle v_j|\mathcal{A}_j|v_j\rangle, \quad \mathcal{A}_j \in \mathscr{A}_j, \tag{3.7}$$

where $v_j \in \mathscr{V}_j$ is a unit vector which transforms in a one-dimensional representation of G. Not every $\mathscr{A}$ admits such states. If such states on $\mathscr{A}$ exist, they need not all correspond to the same element in $\Phi_G(\mathscr{A})$. For example, consider a system where $\mathscr{V}_j$ is the trivial one-dimensional representation of $G = \mathbb{Z}_2$ for all j except $j = 0$, while $\mathscr{V}_0$ is a sum of the trivial and the non-trivial one-dimensional representations with normalized basis vectors e_+ and e_-. There are two different $\mathbb{Z}_2$-invariant factorized pure states corresponding to $v_0 = e_+$ and $v_0 = e_-$, and they are clearly not related by a G-equivariant LGA. We will call a state of the form (3.7) where all v_j are G-invariant a special G-invariant factorized pure state. We will define $\tau_G \in \Phi_G(\mathscr{A})$ to be the G-equivariant LGA-equivalence class of a special G-invariant factorized pure state. This definition makes sense because for any particular $\mathscr{A}$ all special G-invariant factorized pure states are in the same G-equivariant LGA-equivalence class.

Next we define the notion of a G-invariant phase. We say that a G-invariant pure state ψ_1 of a system $\mathscr{A}_1$ is G-stably equivalent to a pure state ψ_2 on a system $\mathscr{A}_2$ if there exist systems $\mathscr{A}_1'$ and $\mathscr{A}_2'$ and special G-invariant factorized pure states ψ_1' and ψ_2' on them such that the pairs $(\mathscr{A}_1 \otimes \mathscr{A}_1', \psi_1 \otimes \psi_1')$ and $(\mathscr{A}_2 \otimes \mathscr{A}_2', \psi_2 \otimes \psi_2')$ are related by a G-equivariant LGA. A G-invariant phase is defined to be a G-stable equivalence class of a G-invariant pure state ψ of a 1d system $\mathscr{A}$. We will denote the set of G-invariant phases by Φ_G. Φ_G is a commutative monoid, with the trivial G-invariant phase as the neutral element τ_G. Invertible elements in this monoid form an abelian group which we denote Φ_G^*. Elements of this group will be called G-invertible phases.

It is shown in Appendix A that any G-invariant factorized pure state (that is, a state of the form (3.7) where all $v_j \in \mathscr{V}_j$ transform in one-dimensional representations of G) belongs to the trivial G-invariant phase. This provides a sanity check on our definition of a phase: from a physical viewpoint, a nontrivial state must exhibit some entanglement and cannot be factorized.

In this chapter we will be studying G-invariant phases which are also invertible, that is, G-invariant phases which are mapped to Φ^* by the forgetful homomorphism $\Phi_G \to \Phi$. It is easy to see that every G-invertible phase is invertible. Less obviously, we will show that every G-invariant phase which maps to Φ^* is G-invertible. We will define an index taking values in an abelian group which classifies such phases. This abelian group is nothing but $\Phi^*_G \subset \Phi_G$.

3.3 An index for invertible 1d systems

Domain wall states

Consider a G-invariant pure state ψ of $\mathscr{A}$. For any $g \in G$ we let

$$\rho^g_{>j} = \prod_{k>j} \mathrm{Ad}_{R_k(g)}, \tag{3.8}$$

which is a restriction of a locally generated automorphism $\mathrm{Ad}_{R(g)}$ to a half-line $j > 0$. Clearly, $\rho^g_{>j} \circ \rho^h_{>j} = \rho^{gh}_{>j}$. We define the domain wall state $\psi^g_{>j}$ by

$$\psi^g_{>j}(\mathcal{A}) = \psi(\rho^g_{>j}(\mathcal{A})). \tag{3.9}$$

From now on we suppose that ψ defines a G-invariant invertible phase. Thus there exists a system $\mathscr{A}'$, a factorized pure state ψ' on $\mathscr{A}'$, and a locally-generated automorphism α of $\mathscr{A} \otimes \mathscr{A}'$ that transforms the state $\Psi = \psi \otimes \psi'$ to a factorized pure state Ω on $\mathscr{A} \otimes \mathscr{A}'$, i.e., $\Psi = \Omega \circ \alpha$. We can extend the action of ρ^g on $\mathscr{A}$ to an action on $\mathscr{A} \otimes \mathscr{A}'$ by making it trivial on $\mathscr{A}'$. Then the automorphism

$$\tilde{\rho}^g = \prod_{k \in \mathbb{Z}} \mathrm{Ad}_{\alpha(R_k(g))} \,. \tag{3.10}$$

preserves the state Ω, and its restriction to a half-line $k > j$ is given by $\tilde{\rho}^g_{>j} = \alpha \circ \rho^g_{>j} \circ \alpha^{-1}$.

We will need the following result from Appendix C of [31] whose proof we reproduce here for convenience.

Proposition 53. Let Ω be a factorized pure state on $\mathscr{A}$. Let α be an automorphism of $\mathscr{A}$ which asymptotically preserves Ω, in the sense that there exists a monotonically decreasing positive function $h(r) = O(r^{-\infty})$ on $[0, \infty)$ such that for any $\mathcal{A}$ localized on $[r, \infty)$ or $(-\infty, -r]$ one has $|\Omega(\alpha(\mathcal{A})) - \Omega(\mathcal{A})| \leq h(r)\|\mathcal{A}\|$. Then the state $\Omega^\alpha : \mathcal{A} \mapsto \Omega(\alpha(\mathcal{A}))$ is unitarily equivalent to Ω.

Proof. Let $\mathscr{A}_+$ and $\mathscr{A}_-$ be the C^*-subalgebras of $\mathscr{A}$ corresponding to $j > 0$ and $j \leq 0$, respectively. Let $\Omega_\pm$ be the restriction of Ω to $\mathscr{A}_\pm$, and $\Omega^\alpha_\pm$ be the restriction of Ω^α to $\mathscr{A}_\pm$. By Cor. 2.6.11 in [34], the state Ω^α_+ is quasi-equivalent to Ω_+ and the state Ω^α_- is quasi-equivalent to Ω_-. Therefore $\Omega = \Omega_+ \otimes \Omega_-$ is quasi-equivalent to $\Omega^\alpha_+ \otimes \Omega^\alpha_-$. Since both Ω and Ω^α are pure states, it remains to show that Ω^α is quasi-equivalent to $\Omega^\alpha_+ \otimes \Omega^\alpha_-$.

Let $\rho_j, \rho^\alpha_j \in \mathscr{A}_j$ be the density matrices for the restriction of Ω and Ω^α to $\mathscr{A}_j$. The density matrix ρ_j is pure, while ρ^α_j is mixed, in general. But since for any $\mathcal{A} \in \mathscr{A}_j$ we have $|\mathrm{T}r(\rho^\alpha_j - \rho_j)\mathcal{A}| \leq h(|j|)\|\mathcal{A}\|$, we have $\|\rho_j - \rho^\alpha_j\|_1 \leq h(|j|)$, where $\|\cdot\|_1$ is the trace norm. Thus for large $|j|$ the entropy S^α_j of ρ^α_j rapidly approaches zero. Specifically, by Fannes' inequality, for all sufficiently large $|j|$ we have $S^\alpha_j \leq h(|j|)\log(d_j/h(|j|))$. Here $d_j^2 = \dim \mathscr{A}_j$. Since we assumed that $\log d_j$ grows at most polynomially with $|j|$, $S^\alpha = \sum_j S^\alpha_j < \infty$. Therefore the entropy of the restriction of Ω^α to any finite region of $\mathbb{Z}$ is upper-bounded by S^α. By Proposition 2.2 of [35] (where the proof of equivalence of (i) and (ii) does not use translation invariance) and Theorem 1.5 of [25], the state Ω^α is quasi-equivalent to $\Omega^\alpha_+ \otimes \Omega^\alpha_-$. □

The automorphism $\rho^g_{>j}$ does not preserve ψ, but it asymptotically preserves it, by Lemma A.4 of [31]. Therefore $\tilde{\rho}^g_{>g}$ asymptotically preserves Ω. Then the above proposition implies that the state $\Omega^g_{>j}$ defined by $\Omega^g_{>j}(\mathcal{A}) = \Omega(\tilde{\rho}^g_{>j}(\mathcal{A}))$ is unitarily equivalent to Ω. Since $\Omega \circ \alpha = \Psi$ and $\rho^g_{>j}$ acts non-trivially only on $\mathscr{A}$, the states ψ and $\psi^g_{>j}$ are also unitarily equivalent.

Let $(\Pi, \check{\mathrm{H}}, |0\rangle)$ be the GNS data of the original state ψ constructed in the usual way. We can identify $\Pi(\rho^g_{>j}(\mathcal{A}))$ with the GNS representation of $\psi^g_{>j}$. Specifically, let $(\Pi^g_{>j}, \check{\mathrm{H}}^g_{>j}, |0\rangle^g_{>j})$ be the GNS data of $\psi^g_{>j}$. In particular, this coincides with $(\Pi, \check{\mathrm{H}}, |0\rangle)$ when g is the identity. $\check{\mathrm{H}}^g_{>j}$ is the completion of $\mathscr{A}/I^g_{>j}$ where the left ideal

$$I^g_{>j} := \{\mathcal{A} | \psi(\rho^g_{>j}(\mathcal{A}^*\mathcal{A})) = 0\}. \tag{3.11}$$

Since $\rho^g_{>j}(I^h_{>j}) = I^{hg^{-1}}_{>j}$, $\rho^g_{>j}$ induces a linear isometry of Hilbert spaces

$$\iota^g_{>j} : \check{\mathrm{H}}^h_{>j} \to \check{\mathrm{H}}^{hg^{-1}}_{>j},$$

such that $\iota^g_{>j} \circ \iota^h_{>j} = \iota^{gh}_{>j}$ for all $g, h \in G$. Note ι does not have a fixed source or target. It is a (categorified) group action over all Hilbert spaces $\check{\mathrm{H}}^g_{>j}$. It then follows that

$$\iota^g_{>j}\Pi^g_{>j}(\mathcal{A})(\iota^g_{>j})^{-1} = \Pi(\rho^g_{>j}(\mathcal{A})). \tag{3.12}$$

This, together with the unitary equivalence proven above, implies that for any $g \in G$ and any $j \in \mathbb{Z}$ there exists a unitary operator $U^g_{>j}$ such that

$$\Pi(\rho^g_{>j}(\mathcal{A})) = U^g_{>j}\Pi(\mathcal{A})\left(U^g_{>j}\right)^{-1}. \tag{3.13}$$

Since the commutant of $\Pi(\mathscr{A})$ consists of scalars, this equation defines $U^g_{>j}$ up to a multiple of a complex number with absolute value 1.

Finally, we are ready to define the index. From Equation (3.13) and $\rho^g_{>j} \circ \rho^h_{>j} = \rho^{gh}_{>j}$, we infer that

$$U^g_{>j}U^h_{>j}\Pi(\mathcal{A})(U^g_{>j}U^h_{>j})^{-1} = U^{gh}_{>j}\Pi(\mathcal{A})(U^{gh}_{>j})^{-1}. \tag{3.14}$$

Since the commutant of $\Pi(\mathscr{A})$ consists of scalars, it follows that

$$U^g_{>j}U^h_{>j} = \nu_{>j}(g,h)U^{gh}_{>j}, \tag{3.15}$$

where $\nu_{>j}(g,h)$ is a complex number with absolute value 1. Associativity of operator product implies that $\nu_{>j}(g,h)$ is a 2-cocycle of the group G. Its cohomology class is independent of the different choices of $\{U^g_{>j} : g \in G\}$. We claim this cohomology class is an index for bosonic systems. Firstly we show that it is independent of j. Then we demonstrate that any G-equivariant locally generated automorphism preserves the index. Lastly, we verify that the index of a state in the trivial phase is trivial.

Given $i < j$, we let

$$R^g_{(i,j]} = \prod_{i<k\leq j} R_k(g). \tag{3.16}$$

By (3.13), $\Pi(R^h_{(i,j]}) = \Pi(\rho^g_{>j}(R^h_{(i,j]})) = U^g_{>j}\Pi(R^h_{(i,j]})(U^g_{>j})^{-1}$, i.e., $\Pi(R^h_{(i,j]})$ and $U^g_{>j}$ commute. Also it is easy to see that

$$U^g_{>i} = \mu(g)\Pi(R^g_{(i,j]})U^g_{>j}, \tag{3.17}$$

where $\mu(g)$ is a complex number with absolute value 1. Therefore,

$$\begin{aligned} &\nu_{>i}(g,h)\mu(gh)\Pi(R^{gh}_{(i,j]})U^{gh}_{>j} \\ =&\nu_{>i}(g,h)U^{gh}_{>i} = U^g_{>i}U^h_{>i} \\ =&\mu(g)\Pi(R^g_{(i,j]})U^g_{>j}\mu(h)\Pi(R^h_{(i,j]})U^h_{>j} \\ =&\mu(g)\mu(h)\Pi(R^{gh}_{(i,j]})U^g_{>j}U^h_{>j} \\ =&\mu(g)\mu(h)\nu_{>j}(g,h)\Pi(R^{gh}_{(i,j]})U^{gh}_{>j}. \end{aligned} \tag{3.18}$$

It is important here that $R_j(g)$ is an ordinary (non-projective) representation of G. From now on, we fix a domain wall position j as it does not affect the index. We also omit this choice from all notations, for example $\rho^g_{>j}$ becomes simply $\rho^g_>$.

Next, we prove that the index is invariant under an automorphism $\beta := \beta(1)$ generated by a G-equivariant self-adjoint 0-chain $F(t) = \sum_j F_j(t)$. Note the site j here is not the domain wall position which has been fixed. In general $\rho^g_> \circ \beta \neq \beta \circ \rho^g_>$, however their commutator $\beta \circ \rho^g_> \circ \beta^{-1} \circ \rho^{g^{-1}}_> = \mathrm{Ad}_{\mathcal{B}_g}$ for an almost local observable $\mathcal{B}_g$. Furthermore,

$$\mathcal{B}_{gh} = \mathcal{B}_g \rho^g_>(\mathcal{B}_h). \tag{3.19}$$

Before giving the proof, we notice that these facts indeed lead us to the desired invariance of cohomology.

Define a new state

$$\psi^\beta(\mathcal{A}) := \psi(\beta(\mathcal{A})). \tag{3.20}$$

Repeating the argument above for state ψ^β, we get

$$\Pi(\beta(\rho^g_>(\mathcal{A}))) = W^g_> \Pi(\beta(\mathcal{A}))(W^g_>)^{-1}. \tag{3.21}$$

However,

$$\begin{aligned} &\Pi(\beta(\rho^g_>(\mathcal{A}))) \\ =&\Pi(\beta \circ \rho^g_> \circ \beta^{-1} \circ \rho^{g^{-1}}_>(\rho^g_>(\beta(\mathcal{A})))) \\ =&\Pi(\mathcal{B}_g) U^g_> \Pi(\beta(\mathcal{A}))(U^g_>)^{-1} \Pi(\mathcal{B}_g)^{-1}. \end{aligned} \tag{3.22}$$

Therefore,

$$W^g_> = \mu(g)\Pi(\mathcal{B}_g) U^g_>, \tag{3.23}$$

where $\mu(g)$ is a complex number with norm 1. And finally,

$$\begin{aligned} W^g_> W^h_> =&\mu(g)\mu(h)\Pi(\mathcal{B}_g)U^g_>\Pi(\mathcal{B}_h)U^h_> \\ =&\mu(g)\mu(h)\Pi(\mathcal{B}_g\rho^g_>(\mathcal{B}_h))U^g_>U^h_> \\ =&\mu(g)\mu(h)\mu(gh)^{-1}\nu(g,h)W^{gh}. \end{aligned} \tag{3.24}$$

To prove that such $\mathcal{B}_g$ exists, notice that the automorphism $\beta(t) \circ \rho^g_> \circ \beta^{-1}(t) \circ \rho^{g^{-1}}_>$ is generated by the 0-chain $F_g(t) := \rho^g_> \circ \beta(t)(\rho^{g^{-1}}_>(F(t)) - F(t))$, which is almost local. For any almost local self-adjoint $\mathcal{A}(t)$ depending continuously on t, we can define an almost local unitary observable $E_{\mathcal{A}}(t)$ satisfying the equation

$$-i\frac{d}{dt}E_{\mathcal{A}}(t) = E_{\mathcal{A}}(t)\mathcal{A}(t). \tag{3.25}$$

When viewed as a 0-chain, the automorphism generated by $\mathcal{A}(t)$ is simply $\mathrm{Ad}_{E_{\mathcal{A}(t)}}$. We let $E_g(t) := E_{F_g}(t)$, then it follows that $\mathcal{B}_g = E_g(1)$. To establish identity (3.19), we need two properties of the observables $E_{\mathcal{A}}(t)$:

$$\alpha(E_{\mathcal{A}}(t)) = E_{\alpha(\mathcal{A})}(t), \tag{3.26}$$

and

$$E_{\mathcal{A}}(t)E_{\mathcal{B}}(t) = E_C(t), \tag{3.27}$$

where $C(t) = \alpha_{\mathcal{B}}^{-1}(t)(\mathcal{A}(t)) + \mathcal{B}(t)$. Both are easily checked from Equation (3.25).

Then

$$E_g(t) \cdot \rho_>^g(E_h(t)) = \rho_>^g(E_G(t) \cdot E_h(t)) = \rho_>^g(E_X(t)). \tag{3.28}$$

By the first property, $G(t) = \beta(t)(\rho_>^{g^{-1}}(F(t)) - F(t))$. By the second property,

$$X(t) = \alpha_{F_h}^{-1}(t)(G(t)) + F_h(t) \tag{3.29}$$

$$= \rho_>^h \circ \beta(t) \circ \rho_>^{h^{-1}} \circ \beta^{-1}(t) \circ \beta(t)\big(\rho_>^{g^{-1}}(F(t)) - F(t)\big) \tag{3.30}$$

$$+ \rho_>^h \circ \beta(t)(\rho_>^{h^{-1}}(F(t)) - F(t)) \tag{3.31}$$

$$= \rho_>^h \circ \beta(t)\big(\rho_>^{(gh)^{-1}}(F(t)) - F(t)\big). \tag{3.32}$$

Therefore, $\rho_>^g(X(t)) = F_{gh}(t)$, which is what we need.

At last, we verify that the index of the trivial phase is trivial. By the previous result, it suffices to compute the index of a factorized state. Since $\psi(\rho_>^g(\mathcal{A})) = \psi(\mathcal{A})$ for any factorized G-invariant pure state ψ, the GNS representation $\Pi = \Pi^g$. Equations (3.12) and (3.13) imply

$$\iota_>^g \Pi(\mathcal{A})(\iota_>^g)^{-1} = \iota_>^g \Pi_>^g(\mathcal{A})(\iota_>^g)^{-1} = \Pi(\rho_>^g(\mathcal{A})) = U_>^g \Pi(\mathcal{A})(U_>^g)^{-1}. \tag{3.33}$$

As $\iota_>^g \iota_>^h = \iota_>^{gh}$, the index is trivial.

Taken together, these results imply that the index of an invertible G-invariant pure state on $\mathscr{A}$ depends only on its G-invariant phase.

Remark 2. Let $\mathscr{A}_{>j} = \otimes_{k>j}\mathscr{A}_k$ and $\mathscr{A}_{\leq j} = \otimes_{k\leq j}\mathscr{A}_k$. We can define commuting von Neumann algebras acting in $\check{\mathrm{H}}_\psi$ by letting $\mathcal{M}_{>j} = \Pi_\psi(\mathscr{A}_{>j})''$ and $\mathcal{M}_{\leq j} = \Pi_\psi(\mathscr{A}_{\leq j})''$. These two algebras are each other commutants and generate the whole $B(\check{\mathrm{H}}_\psi)$. Then the definition of $U_{>j}^g$ implies that $U_{>j}^g \in \mathcal{M}_{>j}$. Similarly, replacing in (3.13) the automorphism $\rho_{>j}^g$ with $\rho_{\leq j}^g = \rho^g \left(\rho_{>j}^g\right)^{-1}$, we can define a unitary $U_{\leq j}^g \in \mathcal{M}_{\leq j}$. Then it is easy to see that the index we defined above is the same as the index defined in [21].

The stacking law agrees with the group structure of $H^2(G, U(1))$. This is seen by taking the domain wall unitary of the stacked system to be the tensor product of the domain wall unitary operators of the respective subsystems. According to (3.15), the stacked 2-cocycle is the product of the 2-cocycles of the subsystems.

An important property of the index is that it is locally computable: it can be evaluated approximately given the restriction of the state to any sufficiently large segment of the lattice. This is shown in Appendix B. Local computability of the index implies there can be no G-invariant invertible interpolation between a state with a non-trivial index and a G-invariant factorized pure state. Put more concisely, an invertible G-invariant system with a non-trivial index cannot have a G-invariant non-degenerate edge.

Examples

In this section we compute the index of some standard examples of non-trivial invertible states.

In the bosonic case, Matrix Product States furnish examples of invertible states invariant under a (finite) symmetry group G. To construct such an example, we pick a projective unitary representation Q of G on a finite-dimensional Hilbert space $\mathscr{W}$. Thus we are given unitary operators $Q(g) \in U(\mathscr{W})$, $g \in G$, satisfying

$$Q(g)Q(h) = \nu(g,h)Q(gh), \tag{3.34}$$

where $\nu(g,h)$ is a 2-cocycle with values in $U(1)$. We take the local Hilbert space $\mathscr{V}_j$ to be $\mathscr{W}_j \otimes \mathscr{W}_j^*$, where $\mathscr{W}_j$ is isomorphic to $\mathscr{W}$ for all $j \in \mathbb{Z}$. Then G acts on $\mathscr{V}_j$ via $R(g) = Q(g) \otimes Q(g)^*$. This is an ordinary (non-projective) action.

To define a G-invariant state on $\mathscr{A}$, we first specify it on $\mathscr{A}_l$ and then extend by continuity. We note first that

$$\mathscr{A}_\ell = \otimes_{k\in\mathbb{Z}} \operatorname{End}(\mathscr{W}_k \otimes \mathscr{W}_k^*) = \otimes_{k\in\mathbb{Z}} \operatorname{End}(\mathscr{W}_k^* \otimes \mathscr{W}_{k+1}). \tag{3.35}$$

It is understood here that in both infinite tensor product all but a finite number of elements are identity elements. Thus we can get a G-invariant pure state on $\mathscr{A}_\ell$ by picking a G-invariant vector state on $\operatorname{End}(\mathscr{W}_k^* \otimes \mathscr{W}_{k+1})$ for all $k \in \mathbb{Z}$. If the representation Q is irreducible, there is a unique choice of such a state: the one corresponding to the vector $\frac{1}{\sqrt{d}} 1_{\mathscr{W}} \in \mathscr{W}^* \otimes \mathscr{W}$, where d is the dimension of $\mathscr{W}$. In the physics literature such a state on $\mathscr{A}_\ell$ is known as an entangled-pair state. Let us denote this state ψ.

Note that ψ is in an invertible phase. Indeed, any vector state on $\mathrm{End}(\mathscr{W}_k^* \otimes \mathscr{W}_k)$ can be mapped to a factorized vector state by a unitary transformation. The product of these unitary transformations for all k gives us a locally-generated automorphism connecting ψ with a factorized pure state on $\mathscr{A}$. In fact, it is easy to see that the G-invariant phase of ψ is G-invertible. The inverse system is obtained by replacing $\mathscr{W}$ with $\mathscr{W}^*$ and Q with Q^*.

To describe the GNS Hilbert space corresponding to ψ, let us pick an orthonormal basis $|n\rangle$, $n = 1, \ldots, d$, in $\mathscr{W}$ and denote by $|j, n\rangle$, $\langle j, n|$ the corresponding orthonormal basis vectors for $\mathscr{W}_j$ and $\mathscr{W}_j^*$. Let us also denote by $|1'_j\rangle \in \mathscr{W}_j^* \otimes \mathscr{W}_{j+1}$ the vector $\frac{1}{\sqrt{d}} \sum_{n=1}^{d} \langle j, n| \otimes |j+1, n\rangle$. Then the GNS Hilbert space is the completion of the span of vectors of the form

$$\left(\bigotimes_{k<J} |1'_k\rangle\right) \otimes \langle J, n_J| \otimes |J+1, m_J\rangle \otimes \ldots \otimes \langle J', n_{J'}| \otimes |J'+1, m_{J'}\rangle \otimes \left(\bigotimes_{l>J'} |1'_l\rangle\right), \quad (3.36)$$

where J, J' ($J \leq J'$) are integers.

It is easy to check that the operator $U^g_{>j}$ is given (up to a scalar multiple) by

$$U^g_{>j} = \bigotimes_{k \leq j} 1_k \otimes Q_{j+1}(g) \otimes \bigotimes_{l>j} R'_l(g), \quad (3.37)$$

where $Q_{j+1}(g) \in \mathrm{End}(\mathscr{W}_{j+1})$ is given by $a \mapsto Q(g)a$, $R'_l(g) \in End\left(\mathscr{W}_l^* \otimes \mathscr{W}_{l+1}\right)$ is given by $a \otimes b \mapsto Q(g)^* a \otimes Q(g) b$, and $1_k \in \mathrm{End}(\mathscr{W}_k \otimes \mathscr{W}_{k+1}^*)$ is the identity operator. This is a well-defined operator because $|1'_l\rangle$ is invariant under $R'_l(g)$. Now, since the operators $R'_l(g)$ define an ordinary (non-projective) representation of G, we see that the index of the state ψ is given precisely by the 2-cocycle $\nu(g, h)$.

3.4 A classification of invertible phases of bosonic 1d systems

Preliminaries

In the following, for a state ψ we denote by $\check{\mathrm{H}}_\psi$ and Π_ψ the corresponding GNS Hilbert space and GNS representation. For a region A of the lattice, we denote by $\psi|_A$ the restriction of a state ψ to $\mathscr{A}_A$.

An LGA α generated by F which is f-local can be represented by an ordered conjugation with $\overrightarrow{\prod}_{j \in \Lambda} e^{iG_j}$ for almost local observables G_j which are g-localized for some $g(r)$ that depends on $f(r)$ only. Indeed, suppose we have an f-localized 0-chain. The automorphism

$$\alpha_{F_{(-\infty,k+1]}} \circ \left(\alpha_{F_{(-\infty,k]}}\right)^{-1} \quad (3.38)$$

is a conjugation by an almost local unitary e^{iG_j} which is g-localized for some $g(r)$ that depends on $f(r)$ only.

We define a restriction $G = F|_A$ of a 0-chain F to a region A by

$$G_j = \int \prod_{k \in \bar{A}} dV_k \, \mathrm{Ad}_{\prod_{k\in\bar{A}} V_k}(F_j), \tag{3.39}$$

where the integration is over all on-site unitaries $V_k \in \mathscr{A}_k$ with Haar measure. Note that $[G, \mathcal{A}] = 0$ for any $\mathcal{A} \in \mathscr{A}_{\bar{A}}$, i.e., $G = F|_A$ is localized on A. It is also easy to see that $F - F|_A$ is almost localized on $\bar{A}$.

These properties of restriction have several immediate consequences. Let A and B be a left and a right half-chain, correspondingly. We can represent F as

$$F = F|_A + \mathcal{F}_0 + F|_B \tag{3.40}$$

for some almost local observable $\mathcal{F}_0$. By Lemma A.4 from [31] we can represent α_F as a product

$$\alpha_F = \alpha_{F|_A} \circ \alpha_{F|_B} \circ \mathrm{Ad}_{\mathcal{U}_0} \tag{3.41}$$

for some almost local unitary observable $\mathcal{U}_0$. Similarly, let A, B, and C be three regions $(-\infty, j)$, $[j, k]$, and (k, ∞), correspondingly, for some sites j and k. Then Lemma A.4 from [31] implies

$$\alpha_F = \alpha_{F|_A} \circ \alpha_{F|_C} \circ \alpha_{F|_B} \circ \mathrm{Ad}_{\mathcal{U}_j} \circ \mathrm{Ad}_{\mathcal{U}_k} \tag{3.42}$$

for some almost local unitaries $\mathcal{U}_j$ and $\mathcal{U}_k$ g-localized at sites j and k, correspondingly. Here $g(r)$ does not depend on j and k.

Here and below we will denote by $\Gamma_r(j)$ the interval $[j - r, j + r]$. If $j = 0$, we use a shorthand Γ_r. Let $f : \mathbb{Z}_+ \to \mathbb{R}_+$ be an MDP function such that $f(r) = O(r^{-\infty})$. Let A be a subset of $\mathbb{Z}$ and $j \in \mathbb{Z}$. We will say that two states ψ_1 and ψ_2 on $\mathscr{A}$ are f-close on A far from j if for any observable $\mathcal{A}$ localized on $A \cap \bar{\Gamma}_r(j)$ we have $|\psi_1(\mathcal{A}) - \psi_2(\mathcal{A})| \leq \|\mathcal{A}\| f(r)$. Note that since all our algebras of observables are simple, all representations of $\mathscr{A}$ are faithful, and thus $\|\mathcal{A}\|$ can be replaced with $\|\Pi(\mathcal{A})\|$ in any representation Π of $\mathscr{A}$.

Lemma 3.4.1. Let ψ be a pure state on $\mathscr{A}$ which is f-close on $\mathbb{Z}$ far from $j = 0$ to a pure factorized state ψ_0 for some MDP function $f(r) = O(r^{-\infty})$. Then ψ and ψ_0 are unitarily equivalent and one can be produced from the other by a conjugation with $e^{i\mathcal{G}}$, where $\mathcal{G}$ is an almost local self-adjoint observable g-localized at $j = 0$ and bounded $\|\mathcal{G}\| \leq C$ for some $g(r)$ and C which only depend on $f(r)$.

Proof. Unitary equivalence of states ψ and ψ_0 follows from Corollary 2.6.11 of [34]. Let $(\Pi_0, \check{\mathrm{H}}_0, |0\rangle)$ be the GNS data for ψ_0. The state ψ is a vector state corresponding to $|\psi\rangle \in \check{\mathrm{H}}_0$. Let $\mathcal{V}_n$ be a subspace of $\check{\mathrm{H}}_0$ spanned by vectors which can be produced from $|0\rangle$ by an observable localized on Γ_n. Note that $\mathcal{V}_1 \subset \mathcal{V}_2 \subset \mathcal{V}_3 \subset \ldots$.

Let n_0 be such that $f(n_0) < 1/2$. Let us temporarily fix $n \geq n_0$ and not indicate it explicitly. Let us estimate the angle between the vector $|\psi\rangle$ and the subspace $\mathcal{V} = \mathcal{V}_n$. The Hilbert space $\check{\mathrm{H}}_0$ is isomorphic to $\check{\mathrm{H}}_\Gamma \otimes \check{\mathrm{H}}_{\bar{\Gamma}}$, where the Hilbert spaces $\check{\mathrm{H}}_\Gamma$ and $\check{\mathrm{H}}_{\bar{\Gamma}}$ carry representations of $\mathscr{A}_\Gamma$ and $\mathscr{A}_{\bar{\Gamma}}$, respectively. The restrictions of vector states ψ and ψ_0 to $\mathscr{A}_{\bar{\Gamma}}$ can be described by density matrices ρ and ρ_0 on $\check{\mathrm{H}}_{\bar{\Gamma}}$. The density matrix ρ_0 is pure, but ρ is mixed, in general. We have

$$\|\rho - \rho_0\|_1 \leq \varepsilon, \tag{3.43}$$

where $\varepsilon = f(n) \in (0, 1)$. Fuchs–van de Graaf inequality implies that for fidelity we have

$$F(\rho, \rho_0) := \|(\rho)^{1/2}(\rho_0)^{1/2}\|_1 \geq 1 - \frac{\varepsilon}{2}. \tag{3.44}$$

Let

$$|\psi\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |\eta_i\rangle \otimes\, |\xi_i\rangle. \tag{3.45}$$

be the Schmidt decomposition of $|\psi\rangle$. Here $N \leq \mathrm{d}im\check{\mathrm{H}}_\Gamma$, $|\eta_i\rangle$, $i = 1, \ldots, N$, are orthonormal vectors in $\check{\mathrm{H}}_\Gamma$, $|\xi_i\rangle$, $i = 1, \ldots, N$, are orthonormal vectors in $\check{\mathrm{H}}_{\bar{\Gamma}}$, and λ_i, $i = 1, \ldots, N$, are positive numbers satisfying $\sum_i \lambda_i = 1$. Since $|0\rangle$ is factorized, its Schmidt decomposition contains only a single term:

$$|0\rangle = |0_\Gamma\rangle \otimes |0_{\bar{\Gamma}}\rangle. \tag{3.46}$$

Let $a_i = \langle \xi_i | 0_{\bar{\Gamma}}\rangle$. The fidelity of ρ and ρ_0 can be expressed in terms of λ_i and a_i:

$$F(\rho, \rho_0) = \left(\sum_i \lambda_i |a_i|^2\right)^{1/2}. \tag{3.47}$$

We define $|v\rangle = \sum_i \sqrt{\lambda_i} a_i^* |\eta_i\rangle$ and let

$$|\chi\rangle = \left(\sum_k \lambda_k |a_k|^2\right)^{-1/2} |v\rangle \otimes |0_{\bar{\Gamma}}\rangle. \tag{3.48}$$

Then it is easy to see that

$$|\langle\psi|\chi\rangle| = \left(\sum_i \lambda_i |a_i|^2\right)^{1/2} = F(\rho, \rho_0) \geq 1 - \frac{\varepsilon}{2}. \tag{3.49}$$

Since $\varepsilon < 1/2$, $|\psi\rangle$ is not orthogonal to the subspace $\mathcal{V}$.

For any $n \geq n_0$ let $|\chi_n\rangle \in \mathcal{V}_n$ be as above (geometrically, it is the normalized projection of $|\psi\rangle$ to $\mathcal{V}_n$). The estimate (3.49) implies

$$|\langle\chi_n|\chi_{n+1}\rangle| \geq 1 - 2\varepsilon_n, \tag{3.50}$$

where $\varepsilon_n = f(n)$. Let $\mathcal{U}_{n_0} = e^{i\mathcal{G}_{n_0}}$ be a unitary localized on Γ_{n_0} that implements a rotation of $|0\rangle$ to $|\chi_{n_0}\rangle$ with $\|\mathcal{G}_{n_0}\| \leq \pi$. We can also choose unitary observables $\mathcal{U}_n$ for $n \geq n_0$ localized on Γ_{n+1} and satisfying $\|1 - \mathcal{U}_n\| \leq (4\varepsilon_n)^{1/2}$ which implement rotations of $|\chi_n\rangle$ to $|\chi_{n+1}\rangle$, and which therefore can be written as $\mathcal{U}_n = e^{i\mathcal{G}_n}$ for an observable $\mathcal{G}_n$ local on Γ_{n+1} with $\|\mathcal{G}_n\| \leq 2(2\varepsilon_n)^{1/2}$. The ordered product of all such unitaries over $n \geq n_0$ can be written as $\mathcal{U} = e^{i\mathcal{G}}$. By construction, this unitary maps $|0\rangle$ to $|\psi\rangle$. Moreover, since $\|\mathcal{G}_n\| \leq 2(2f(n))^{1/2}$ for $n \geq n_0$, $\mathcal{G}$ is g-localized for some MDP function $g(r) = O(r^{-\infty})$ that only depends on $f(r)$, and $\|\mathcal{G}\| \leq \sum_{n=n_0}^{\infty} 2(2f(n))^{1/2} + \pi$, a quantity that also depends only on $f(r)$.

□

Corollary 3.4.1.1. Let ψ_0 and ψ be distinct vector states on $\mathscr{A}$ which satisfy the conditions of the above lemma. Let ψ_s, $s \in [0,1]$, be a path of vector states corresponding to a normalization of the path of vectors $s|0\rangle + (1-s)|\psi\rangle$. Then there exists a continuous path of self-adjoint almost local observables $\mathcal{G}(s)$ h-localized at $j = 0$ such that $\psi_s = \alpha_{\mathcal{G}}(s)(\psi_0)$ and $\|\mathcal{G}(s)\| \leq C$, for some MDP function $h(r) = O(r^{-\infty})$ and $C > 0$ which only depend on f.

Proof. By the above lemma $|\psi\rangle = \Pi_0(\mathcal{U})|0\rangle$ for some g-localized $\mathcal{U}$. Therefore the states ψ_s are all g-close to the state ψ_0. The vectors $|\chi_{n,s}\rangle$ depend continuously on s, therefore we can choose the unitaries $\mathcal{U}_{n,s}$ and the observables $\mathcal{G}_{n,s}$ so that they are continuous functions of s. Let $\tilde{\mathcal{U}}_{n,s}$ be a product $\mathcal{U}_{1,s}\cdots\mathcal{U}_{n,s}$ generated by an almost local observable $\tilde{\mathcal{G}}_{n,s}$. It follows from Equation (3.4) that $\|\tilde{\mathcal{G}}_{n+1,s} - \tilde{\mathcal{G}}_{n,s}\| \leq 2(2\varepsilon_n)^{1/2}$. Therefore, the limit $\mathcal{G}(s) = \lim_{n\to\infty} \tilde{\mathcal{G}}_{n,s}$ is a continuous function of s. □

Corollary 3.4.1.2. Lemma 3.4.1 and Corollary 3.4.1.1 hold if we replace ψ_0 by an SRE state ϕ and ψ by a state $\tilde{\phi}$ which is f-close to ϕ.

Proof. Let α_F be an LGA, such that $\phi \circ \alpha_F = \psi_0$. The state $\tilde{\phi} \circ \alpha_F$ is g-close to ψ_0 for some $g(r) = O(r^{-\infty})$ that depends on $f(r)$ only. Both Lemma 3.4.1 and Corollary 3.4.1.1 hold for $\phi \circ \alpha_F$ and $\tilde{\phi} \circ \alpha_F$, and therefore they both hold for ϕ and $\tilde{\phi}$. □

A classification of invertible phases without symmetries

Short-Range Entangled states are invertible by definition. In this section we show that the converse is also true,[4] and thus all invertible phases of bosonic 1d systems without symmetries are trivial.

We will say that a pure 1d state has bounded entanglement entropy if the entanglement entropies of all intervals $[j, k] \subset \mathbb{Z}$ are uniformly bounded.

Remark 3. It was shown by Matsui [25] that if ψ has bounded entanglement entropy then it has the split property: the von Neumann algebras $\mathcal{M}_A = \Pi_\psi(\mathscr{A}_A)''$ and $\mathcal{M}_{\bar{A}} = \Pi_\psi(\mathscr{A}_{\bar{A}})''$ for a half-line A are Type I von Neumann algebras. Since they are each other's commutants and generate $B(H_\psi)$, they must be Type I factors. Thus $\mathcal{M}_A \simeq B(\check{\mathrm{H}}_A)$ for some Hilbert space $\check{\mathrm{H}}_A$, and the restriction $\psi|_A$ to a half-line A can be described by a density matrix ρ_A on $\check{\mathrm{H}}_A$. In fact, [25] shows that $\check{\mathrm{H}}_A$ can be identified with the GNS Hilbert space of one of the Schmidt vector states of ψ, which are all unitarily equivalent.

Lemma 3.4.2. Both SRE 1d states and invertible 1d states have bounded entanglement entropy.

Proof. Suppose we have a state ψ obtained from a factorized pure state ψ_0 by conjugation with almost local unitaries $\mathcal{U}_j$ and $\mathcal{U}_k$ which are g-localized at sites j and k, correspondingly. Since conjugation by $\mathcal{U}_{j,k}$ is an automorphism of $\mathscr{A}$ which is almost localized on j, k, for any $\mathcal{A}_l \in \mathscr{A}_l$ we have

$$|\psi_0(\mathcal{U}_j\mathcal{U}_k\mathcal{A}_l\mathcal{U}_k^*\mathcal{U}_j^*) - \psi_0(\mathcal{A}_l)| \leq (g(|j-l|) + g(|k-l|))\,\|\mathcal{A}_l\|. \tag{3.51}$$

By Fannes' inequality [36], the entropy of the site l in the state ψ is bounded by $h(|j-l|) + h(|k-l|)$ for some MDP function $h(r) = O(r^{-\infty})$ that depends only on $g(r)$ and the asymptotics of d_j for $j \to \pm\infty$. Therefore such a state has a uniform bound on the entanglement entropy of any interval $[j, k]$. The decomposition Equation (3.42) then implies that the same is true for any SRE or invertible state. □

Let ψ be a possibly mixed state on a half-line A which is f-close to a pure factorized state ψ_0 on A. By Corollary 2.6.11 of [34], ψ is normal in the GNS representation of ψ_0 and can be described by a density matrix.

[4] This is not true in the case of fermionic systems: Kitaev chain provides a counter-example.

Lemma 3.4.3. Let ψ be a pure 1d state on $\mathscr{A}$. Suppose there is an $R > 0$ such that ψ is f-close far from $j = 0$ to a pure factorized state ω^+ on $(R, +\infty)$ and is f-close far from $j = 0$ to a pure factorized state ω^- on $(-\infty, -R)$. Then it is g-close on $\mathbb{Z}$ far from $j = 0$ to $\omega^+ \otimes \omega^-$ for some MDP function $g(r) = O(r^{-\infty})$ which depends only on f and the asymptotics of d_j for $j \to \pm\infty$.

Proof. Since the states are split, we can describe them using density matrices on appropriate Hilbert spaces. The decomposition of $\mathbb{Z}$ into the union $(-\infty, -n) \sqcup \Gamma_n \sqcup (n, +\infty)$ gives rise to a tensor product decomposition $\check{\mathrm{H}} = \check{\mathrm{H}}_n^- \otimes \check{\mathrm{H}}_{\Gamma_n} \otimes \check{\mathrm{H}}_n^+$. Let ψ_n^+ and ω_n^+ be restrictions of ψ and ω^+ to $(n, +\infty)$, and ψ_n^- and ω_n^- be restrictions of ψ and ω^- to $(-\infty, -n)$. Let $\rho_n^\pm$ and $\sigma_n^\pm$ be the corresponding density matrices. Let ψ_n be a restriction of ψ to $(-\infty, -n) \cup (n, +\infty)$ with the corresponding density matrix ρ_n. For $n > R$ we have

$$\|\rho_n^\pm - \sigma_n^\pm\|_1 \leq f(n). \tag{3.52}$$

Since trace norm is multiplicative under tensor product, we have

$$\|(\rho_n^- \otimes \rho_n^+) - (\sigma_n^- \otimes \sigma_n^+)\|_1 \leq \|\rho_n^- - \sigma_n^-\|_1 + \|\rho_n^+ - \sigma_n^+\|_1 \leq 2f(n). \tag{3.53}$$

On the other hand, Fannes' inequality implies that for sufficiently large n the entropy of $\rho_n^\pm$ is upper-bounded by MDP function $h(n) = O(n^{-\infty})$, where $h(n)$ depends only on $f(n)$ and the asymptotics of d_j for $j \to +\infty$. Therefore mutual informations $I(\rho_n^- : \rho_n^+)$ are also upper-bounded by $h(n)$, and the quantum Pinsker inequality implies

$$\|\rho_n - (\rho_n^- \otimes \rho_n^+)\|_1 \leq 2\sqrt{h(n)}. \tag{3.54}$$

Combining this with Equation (3.53), we get

$$\|\rho_n - (\sigma_n^- \otimes \sigma_n^+)\|_1 \leq 2f(n) + 2\sqrt{h(n)}. \tag{3.55}$$

□

We say that a set of (ordered) eigenvalues $\{\lambda_j\}$ has $g(r)$-decay if $\varepsilon(k) \leq g(\log(k))$ for some MDP function $g(r) = O(r^{-\infty})$, where $\varepsilon(k) = \sum_{j=k+1}^{\infty} \lambda_j$.

Lemma 3.4.4. Let ψ be a state on a half-line A which is f-close far from the origin of A to a pure factorized state ψ_0. Then its density matrix (in the GNS Hilbert space of this factorized state) has eigenvalues with $g(r)$-decay for some $g(r) = O(r^{-\infty})$ that depends only on $f(r)$ and the asymptotic behavior of $d_j = \mathrm{d}im\, \mathcal{V}_j$ for $j \to \infty$. Conversely, for any density matrix on a half-line A (in the GNS Hilbert space of a pure factorized state) whose eigenvalues have $g(r)$-decay there is a state on that half-line which has the same eigenvalues and is f-close far from the origin of A to this pure factorized state. Furthermore, one can choose $f(r)$ so that it depends only on $g(r)$ and the asymptotic behavior of $d_j = \mathrm{d}im\,\mathcal{V}_j$ for $j \to \infty$.

Proof. Suppose ψ is f-close to a pure factorized state far from the origin. It can be purified on the whole line (e.g., in a system consisting of the given system on a half-line and its reflected copy on the other half-line). Moreover, by Lemma 3.4.3 we can choose this pure state to be f'-close far from the origin to a pure factorized state on the whole line for some MDP function $f'(r) = O(r^{-\infty})$ that depends only on f. By Lemma 3.4.1, it can be produced from a pure factorized state on the whole line by a unitary observable $\mathcal{U}$ which is h-localized for some h which depends only on f'. Let $|\psi_0\rangle$ be a GNS vector for the corresponding factorized state ψ_0. By Lemma A.1 of [31], there is an MDP function $h'(r) = O(r^{-\infty})$ such that for any $r > 0$ there is a unitary observable $\mathcal{U}^{(r)}$ localized on a disk Γ_r of radius r such that

$$||\Pi_{\psi_0}(\mathcal{U})|\psi_0\rangle - \Pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle|| \leq h'(r). \tag{3.56}$$

On the other hand we have

$$||\Pi_{\psi_0}(\mathcal{U})|\psi_0\rangle - \Pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle|| \geq \|\rho - \rho^{(r)}\|_1, \tag{3.57}$$

where ρ is the density matrix for ψ and $\rho^{(r)}$ is the density matrix for $\Pi_{\psi_0}(\mathcal{U}^{(r)})|\psi_0\rangle$ on A. The tracial distance between any two density matrices ρ and ρ' can be bounded from below in terms of their eigenvalues [37]:

$$\|\rho - \rho'\|_1 \geq \sum_{j=1}^{\infty} |\lambda_j(\rho) - \lambda_j(\rho')|, \tag{3.58}$$

where the eigenvalues λ_i are ordered in decreasing order. Applying this to ρ and $\rho^{(r)}$ and noting that $\rho^{(r)}$ has rank at most $\dim \check{\mathrm{H}}_{A\cap\Gamma_r}$, we get

$$\|\rho - \rho^{(r)}\|_1 \geq \varepsilon(\dim \check{\mathrm{H}}_{A\cap\Gamma_r}). \tag{3.59}$$

Combining (3.56), (3.57), and (3.59) we get

$$\varepsilon(\dim \check{\mathrm{H}}_{A\cap\Gamma_r}) \leq h'(r). \tag{3.60}$$

Since $\dim \check{\mathrm{H}}_{A\cap\Gamma_r}$ is upper-bounded by $\exp(cr^{\alpha})$ for some positive constants c and α, we have $\varepsilon(k) \leq g(\log(k))$ for some $g(r) = O(r^{-\infty})$ which depends only on $f(r)$, c and α.

Conversely, suppose we are given a density matrix on a half-line A with eigenvalues $\lambda_1 \geq \lambda_2 \geq ....$ We may assume that the dimension of $\mathscr{A}_A$ is infinite, since otherwise the statement is obviously true. Pick any pure factorized state ψ_0 on A and choose a basis in each on-site Hilbert space $\mathcal{V}_j$, $j \in A$, such that for all j the first basis vector gives the state $\psi_0|_{\mathscr{A}_j}$. This gives a lexicographic basis $|n\rangle$, $n \in \mathbb{N}$, in the GNS Hilbert space of ψ_0. By our assumption on the growth of dimensions of $d_j = dim\, \mathcal{V}_j$, there are positive constants c and α such that for any r and any $n < e^{cr^\alpha}$ the vector state $|n\rangle\langle n|$ coincides with ψ_0 outside of Γ_r. Therefore the state $\sum_{n=1}^{\infty} \lambda_n |n\rangle\langle n|$ is f-close to ψ_0, where $f(r) = g(cr^\alpha) = O(r^{-\infty})$. □

By Lemma 3.4.4 any restriction $\psi|_A$ of a state ψ with $g(r)$-decay of Schmidt coefficients to a half-line A can be purified by a state on $\bar{A}$, which is f-close far from the origin of A to a pure factorized state for some $f(r)$ that depends on $g(r)$ only. We call such state a truncation of ψ to A. Clearly, if ω is a truncation of ψ to A, then $\omega|_A = \psi|_A$. The following lemma shows that truncations exist for all invertible states.

Lemma 3.4.5. Let ψ be an invertible state with an inverse ψ', such that that $\Psi = \psi \otimes \psi'$ can be produced by an f-local LGA $(\alpha_F)^{-1}$ from a pure factorized state $\Psi_0 = \psi_0 \otimes \psi_0'$. Then ψ has $g(r)$-decay of Schmidt coefficients for some $g(r) = O(r^{-\infty})$ that depends only on $f(r)$.

Proof. By Lemma 3.4.2 and the results of [25], Ψ, ψ, and ψ' have the split property. Therefore for any half-line $A \subset \mathbb{Z}$ the corresponding GNS Hilbert spaces factorize into Hilbert spaces for A and Hilbert spaces for $\bar{A}$, and the restrictions $\psi|_A$, $\psi'|_A$, $\Psi|_A$ can be described by density matrices ρ_A, ρ'_A and P_A in the Hilbert spaces for A [25]. The restriction of $\Psi \circ \alpha_{F|_{\bar{A}}}$ on $\bar{A}$ is f-close far from the origin of A to a pure factorized state, and therefore by Lemma 3.4.4 the density matrix P_A has $g(r)$-decay of Schmidt coefficients for some MDP function $g(r) = O(r^{-\infty})$ that depends on f. Since $P_A = \rho_A \otimes \rho'_A$, the same is true for ρ_A and ρ'_A.

□

Lemma 3.4.6. Any truncation of an invertible state ψ to any half-line is in a trivial phase.

Proof. Let ψ be an invertible state on $\mathscr{A}$ with an inverse ψ' such that that $\Psi = \psi \otimes \psi'$ can be produced by an f-local LGA $(\alpha_F)^{-1}$ from a pure factorized state $\Psi_0 = \psi_0 \otimes \psi_0'$. Here $f(r) = O(r^{-\infty})$ is an MDP function.

Let ϕ_k be a truncation of ψ to $[k, \infty)$, and let ϕ'_k be a truncation of ψ' to $[k, \infty)$. Since $(\phi_k \otimes \phi'_k) \circ \alpha_{F|_{[k,\infty)}}$ is g-close to $\psi_0 \otimes \psi'_0$ for some $g(r) = O(r^{-\infty})$ that depends on $f(r)$ only, Lemma 3.4.1 implies that ϕ_k is invertible with the inverse ϕ'_k. Let $\tilde{\phi}_k$ be a pure state on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$, where $\mathscr{A}^{(1)}$ and $\mathscr{A}^{(2)}$ are two copies of $\mathscr{A}$, with the following two properties: (1) its restriction to $(-\infty, k)$ coincides with the factorized pure state $(\psi_0 \otimes \psi_0)|_{(-\infty,k)}$; (2) its restriction to $A = [k, +\infty)$ is a purification of $\psi|_A$ on $\mathscr{A}^{(1)}_A$ by some state on $\mathscr{A}^{(2)}_A$ which is f'-close to a factorized state for some $f'(r) = O(r^{-\infty})$ that depends on f only. The existence of such a state follows from Lemma 3.4.5. Similarly, we can define a state $\tilde{\phi}'_k$ on $\mathscr{A}'^{(1)} \otimes \mathscr{A}'^{(2)}$ which is an inverse of $\tilde{\phi}_k$. We let $\alpha_{\tilde{G}}$ be an LGA that maps $\tilde{\phi}_k \otimes \tilde{\phi}'_k$ to $\psi_0 \otimes \psi_0 \otimes \psi'_0 \otimes \psi'_0$. This LGA can be chosen to be the identity on $(-\infty, k)$ and g'-local with $g'(r) = O(r^{-\infty})$ depending on $g(r)$ only.

Let us first show that the state $\psi_0 \otimes \phi_k \otimes \psi'_0 \otimes \psi_0$ can be transformed into $\tilde{\phi}_k \otimes \psi'_0 \otimes \psi_0$ by applying a certain LGA β, then an LGA h_2-localized at k (for some MDP function $h_2(r) = O(r^{-\infty})$), and finally the inverse of β. The sequence of steps is shown schematically in Figure 3.1, where it is also indicated that β is a composition of two LGAs described in more detail below.

Equivalently, we can apply β to both states and then show that the resulting states are related by an LGA h_2-localized at k. β is a composition of two LGAs. The first one has the form $\mathrm{Id} \otimes \mathrm{Id} \otimes \alpha_G$, where α_G maps $\psi'_0 \otimes \psi_0$ to $\phi'_k \otimes \phi_k$, see Figure 3.1. The second one has the form $\mathrm{Id} \otimes \alpha_{F|_{[k+1,\infty)}} \otimes \mathrm{Id}$. The product of these two LGAs maps the two states of interest to the states $\Xi_k \otimes \phi_k$ and $\tilde{\Xi}_k \otimes \phi_k$, where both Ξ_k and $\tilde{\Xi}$ are h_1-close on $\mathbb{Z}$ far from k to the same pure factorized state on $\mathscr{A} \otimes \mathscr{A} \otimes \mathscr{A}'$. Here $h_1(r) = O(r^{-\infty})$ is an MDP function which depends only on $f(r)$. By Lemma 3.4.1 Ξ_k and $\tilde{\Xi}_k$ are related by a conjugation with a unitary observable which is h_2-localized at k for some $h_2(r) = O(r^{-\infty})$ depending only on $f(r)$.

Let us fix $L \in \mathbb{N}$. Since the state $\tilde{\phi}'_k$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$ is invertible, by Lemma 3.4.5 its restriction to $(-\infty, k+L)$ has $\tilde{g}(r)$-decay of Schmidt coefficients with $\tilde{g}(r)$ depending only on $f(r)$. Also, since the restriction of the state $\tilde{\phi}'_k$ to $(-\infty, k)$ is factorized, the nonzero Schmidt coeffients are the same as for the state $\tilde{\phi}_k|_{[k,k+L)}$. In particular, the number of nonzero Schmidt coefficients does not exceed the dimension of $\mathscr{A}_{[k,k+L)} \otimes \mathscr{A}_{[k,k+L)}$.

Let us tensor $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)}$ with $\mathscr{A}^{(3)} \otimes \mathscr{A}^{(4)}$. By the proof of Lemma 3.4.4 one can find a state χ_k on $\mathscr{A}^{(1)}_{[k,k+L)} \otimes \mathscr{A}^{(2)}_{[k,k+L)} \otimes \mathscr{A}^{(3)}_{[k,k+L)} \otimes \mathscr{A}^{(4)}_{[k,k+L)}$ which is a purification of $\tilde{\phi}_k|_{[k,k+L)}$ on $\mathscr{A}^{(1)}_{[k,k+L)} \otimes \mathscr{A}^{(2)}_{[k,k+L)}$ and such that its restriction to $\mathscr{A}^{(3)}_{[k,k+L)} \otimes \mathscr{A}^{(4)}_{[k,k+L)}$

is h-close to a factorized state far from $k+L$ for some $h(r)=O(r^{-\infty})$ that depends on $f(r)$ only. The state χ_k can be produced from $(\psi_0\otimes\psi_0\otimes\psi_0\otimes\psi_0)|_{[k,k+L)}$ by a conjugation with a unitary $\mathcal{V}^{(0)}\in\mathscr{A}^{(1)}_{[k,k+L)}\otimes\mathscr{A}^{(2)}_{[k,k+L)}\otimes\mathscr{A}^{(3)}_{[k,k+L)}\otimes\mathscr{A}^{(4)}_{[k,k+L)}$.

Consider the states $\psi_0|_{[k,\infty)}\otimes\psi_0|_{[k,\infty)}\otimes\tilde{\phi}_k|_{[k,\infty)}$ and $\chi_k\otimes(\psi_0|_{[k+L,\infty)}\otimes\psi_0|_{[k+L,\infty)}\otimes\tilde{\phi}_{k+L})$ on the algebra $\mathscr{A}^{(3)}_{[k,\infty)}\otimes\mathscr{A}^{(4)}_{[k,\infty)}\otimes\mathscr{A}^{(1)}_{[k,\infty)}\otimes\mathscr{A}^{(2)}_{[k,\infty)}$ (see Figure 3.2). They are stably related by an almost local unitary which is h'-localized at $k+L$, with $h'(r)=O(r^{-\infty})$ which depends only on $f(r)$. Indeed, we can first tensor both states with $\psi'_0\otimes\psi'_0\otimes\psi_0\otimes\psi_0$ restricted to $[k,+\infty)$, then produce on these ancillas the state $\tilde{\phi}'_k\otimes\tilde{\phi}_k$ with a g'-local LGA, and apply $\alpha_{\tilde{G}|_{(-\infty,k+L)}}\circ\alpha_{\tilde{G}|_{[k+L,\infty)}}$ acting on the tensor product of the original states and $\tilde{\phi}'_k$ in an obvious way. In the same way as in the previous paragraph one can argue that these states are related by an almost local at $k+L$ unitary. This implies that the original states are also related by almost local at $k+L$ unitary $\mathcal{U}^{(0)}$.

We have shown that the states $\tilde{\phi}_k$ and $\tilde{\phi}_{k+L}$ are related (after tensoring with a total of six copies of factorized states ψ_0 and ψ'_0) by a conjugation with an almost local unitary $\mathcal{U}^{(0)}$ followed by a conjugation with a strictly local unitary $\mathcal{V}^{(0)}$. Similarly, we can construct such unitaries $\mathcal{U}^{(n)}$, $\mathcal{V}^{(n)}$ relating stabilizations of $\tilde{\phi}_{k+nL}$ and $\tilde{\phi}_{k+(n+1)L}$. By Lemma B.3.2 we can choose L such that an ordered product of conjugations with $\prod_{n=0}^{\infty}\mathcal{U}^{(n)}$ is an LGA. Since $\mathcal{V}^{(n)}$ commute with $\mathcal{U}^{(n')}$ for $n<n'$, an ordered product $\prod_{n=0}^{\infty}\mathcal{V}^{(n)}\mathcal{U}^{(n)}$ is equal to $\prod_{m=0}^{\infty}\mathcal{V}^{(m)}\prod_{n=0}^{\infty}\mathcal{U}^{(n)}$ and therefore is also an LGA. By construction it relates $\tilde{\phi}_k$ to a factorized state, and therefore ϕ_k is in the trivial phase.

□

Theorem 54. *Any invertible bosonic 1d state ψ is in a trivial phase.*

Proof. Let ψ' be an inverse state for ψ, and let $(\alpha_F)^{-1}$ be an f-local LGA that produces $(\psi\otimes\psi')$ from a pure factorized state $(\psi_0\otimes\psi'_0)$. It enough to show that the state $\psi_0\otimes\psi\otimes\psi'_0\otimes\psi_0$ on $\mathscr{A}^{(1)}\otimes\mathscr{A}^{(2)}\otimes\mathscr{A}'^{(3)}\otimes\mathscr{A}^{(4)}$ is stably SRE.

Let A be a half-line $[0,\infty)$. Since $\mathscr{A}^{(1)}_A$ is identical to $\mathscr{A}^{(2)}_A$ there is a pure state on $\mathscr{A}^{(1)}_{\bar{A}}\otimes\mathscr{A}^{(2)}_A$ identical to ψ. Let ω_- its truncation to $\bar{A}$. Similarly, let ω_+ be a truncation to A of a pure state on $\mathscr{A}^{(2)}_{\bar{A}}\otimes\mathscr{A}^{(1)}_A$ identical to ψ. By Lemma 3.4.6, the state $(\omega_-\otimes\omega_+)\otimes\psi'_0\otimes\psi_0$ is stably SRE. Therefore it is enough to show that $(\omega_-\otimes\omega_+)\otimes\psi'_0\otimes\psi_0$ and $\psi_0\otimes\psi\otimes\psi'_0\otimes\psi_0$ are related by an LGA. In fact, as we show below, such an LGA can be generated by an almost local observable.

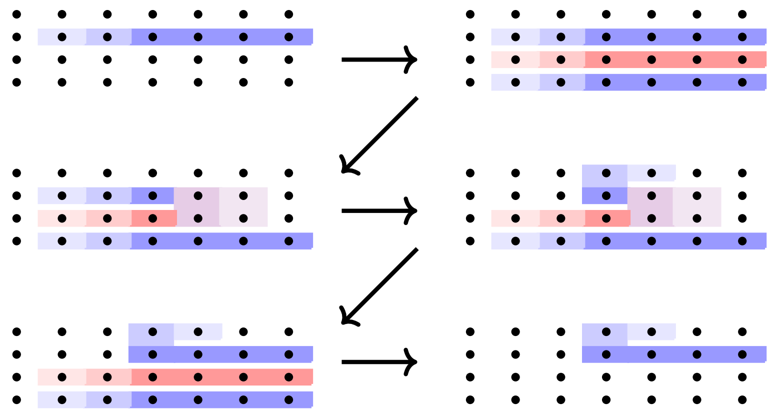


Figure 3.1: The sequence of steps that transforms $\psi_0 \otimes \phi_k \otimes \psi'_0 \otimes \psi_0$ into $\tilde{\phi}_k \otimes \psi'_0 \otimes \psi_0$. The regions shaded in blue denote the entangled parts of ψ, while the regions shaded in red denote the entangled parts of ψ'. The regions where the state is close to a factorized state are schematically indicated by a faded shading.

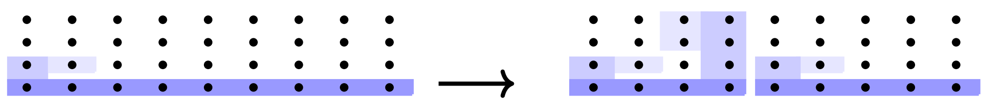


Figure 3.2: States $\tilde{\phi}_k$ and $\chi_k \otimes (\psi_0|_{k+L,\infty} \otimes \psi_0|_{k+L,\infty} \otimes \tilde{\phi}_{k+L})$.

First we apply to both states an LGA which acts only on the last two factors and produces $\psi' \otimes \psi$ out of $\psi'_0 \otimes \psi_0$. This gives us $\psi_0 \otimes \psi \otimes \psi' \otimes \psi$ and $(\omega_- \otimes \omega_+) \otimes \psi' \otimes \psi$.

Second, we apply a composition of $\alpha_{F|_{\bar{A}}}$ and $\alpha_{F|_A}$ on $\mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$ to states $(\omega_- \otimes \omega_+) \otimes \psi'$ and $\psi_0 \otimes \psi \otimes \psi'$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$. This transformation is shown schematically in Figure 3.3. By Lemma 3.4.3 this gives two states which are both g-close far from 0 to a pure factorized state $\psi_0 \otimes \psi_0 \otimes \psi'_0$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(3)}$, for some MDP function $g(r) = O(r^{-\infty})$. By Lemma 3.4.1 these states are related by a conjugation with an almost local unitary. Applying the first two steps backwards, we conclude that the two original states are related by an LGA generated by an almost local observable.

□

A classification of invertible phases with symmetries

In this section we show that the index defined in Section 3 completely classifies invertible phases of 1d bosonic lattice systems with unitary symmetries.

Theorem 55. *A G-invariant invertible 1d state ψ is in the trivial G-invariant phase if and only if it has a trivial index.*

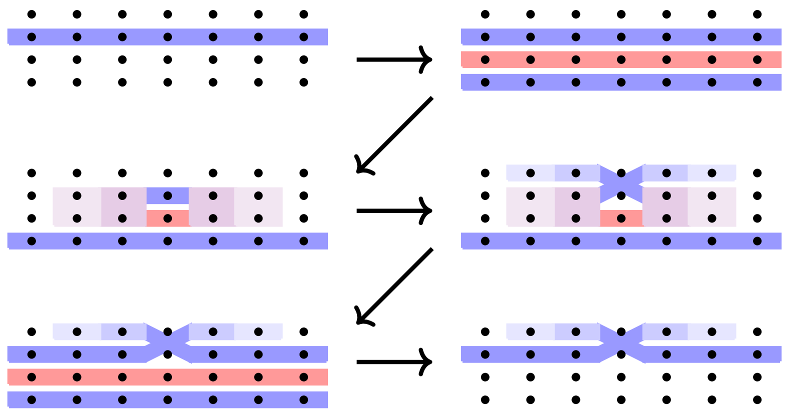


Figure 3.3: The sequence of steps that transforms $\psi_0 \otimes \psi \otimes \psi'_0 \otimes \psi_0$ into $(\omega_- \otimes \omega_+) \otimes \psi'_0 \otimes \psi_0$. The coloring in the same as in Figure 3.1.

In order to prove the theorem, we need the following result from representation theory. Its proof can be found in various sources such as Theorem 4.4 in [38].

Lemma 3.4.7. Let V be a finite dimensional faithful representation of a finite group G. For any irreducible representation W of G, $d(L) = \dim \mathrm{Hom}_G(W, V^{\otimes L}) > 0$ for large enough L. Moreover, $d(L)$ grows exponentially with L.

Proof of theorem. Let V be a finite dimensional representation of G whose subrepresentations contain every irreducible representation of G (including the trivial one). Such a V always exists, with $\mathbb{C}[G]$ being one example. Clearly V is faithful. For all $j \in \mathbb{Z}$, let $\mathcal{V}'_j = V$, $\mathscr{A}'_j = \mathrm{E}nd(\mathcal{V}'_j)$, and let $\mathscr{A}'$ be the norm-completion of $\otimes_j \mathscr{A}'_j$. Let ψ' be a special G-invariant factorized pure state on $\mathscr{A}'$ (it exists because V contains the trivial representation).

Further, for any $j \in \Lambda$ we pick a one-dimensional representation W_j of G (to be fixed later) and let $\mathscr{A}''_j$ be the norm-completion of $\otimes_j \mathrm{E}nd(W_j \oplus T_j)$, where $T_j \simeq \mathbb{C}$ is the trivial representation of G. We also choose normalized basis vectors $w_j \in W_j$ and $t_j \in T_j$. If we ignore the G-action, then $\mathscr{A}''$ corresponds to an infinite chain of qubits. We denote by ψ'' a G-invariant factorized pure state ψ'' on $\mathscr{A}''$ whose restriction to $\mathscr{A}''_j$ is given by $\langle w_j| \cdot |w_j\rangle$. Note that ψ'' is a special G-invariant factorized pure state if and only if all W_j are trivial representations.

We will show that with appropriate choice of representations W_j the state $\Psi = \psi \otimes \psi' \otimes \psi''$ on $\mathscr{A} \otimes \mathscr{A}' \otimes \mathscr{A}''$ tensored with a finite number of copies of factorized pure states can be disentangled by a G-equivariant LGA (that is, can be mapped by a G-equivariant LGA to a G-invariant factorized pure state). Since, as shown in

Appendix A, any G-invariant factorized pure state is G-stably equivalent to a special G-invariant factorized pure state, this implies that ψ is in a trivial G-invariant phase.

By Lemma 3.4.2 and [25], the state ψ has the split property. Thus the von Neumann algebra $\mathcal{M}_{>k} = \Pi_\psi(\mathscr{A}_{>k})''$ on a half-line (k,∞) is a Type I factor. In other words, $\mathcal{M}_{>k}$ is isomorphic to the algebra of bounded operators on a Hilbert space $\mathcal{W}_{>k}$. By Remark 2 and triviality of the index, the operators $U^g_{>k}$ define a unitary representation of G on $\mathcal{W}_{>k}$. The split property also implies that ψ restricted to (k,∞) is a normal state with a density matrix ρ_k (a positive operator on $\mathcal{W}_{>k}$ with unit trace). As the restriction of ψ to $\mathscr{A}_{>k}$ is G-invariant, each eigenspace of ρ_k is a representation of G. Therefore, each summand in the direct sum decomposition of $\mathcal{W}_{>k}$ into irreducible (necessarily finite dimensional) representations of G is spanned by eigenvectors of ρ_k with equal eigenvalues.

By Lemma 3.4.7 each of these irreducible summands is contained in $V^{\otimes L}$ for large enough L with exponentially growing multiplicity (with respect to L). Therefore in the same way as in Lemma 3.4.4 we can construct a G-invariant state on $\mathscr{A}_{>k} \otimes \mathscr{A}'_{>k}$ which is f-close to a G-invariant factorized pure state, has the same eigenvalues as ρ_k, and the eigenspace for each eigenvalue transforms in the same representation of G as the corresponding eigenspace of ρ_k. Here $f(r) = O(r^{-\infty})$ is an MDP function which depends only on the localization of the LGA which produces $\psi \otimes \psi'$ from a pure factorized state and in particular is independent of k. Therefore there is a G-invariant pure state $\Xi^{(k)}$ on $\mathscr{A} \otimes \mathscr{A}'$ that coincides with $\psi \otimes \psi'$ on $(-\infty, k]$, while on $(k, +\infty)$ it is f-close far from k to a G-invariant factorized pure state. In other words, $\Xi^{(k)}$ is a truncation of the invertible state $\psi \otimes \psi'$ to $(-\infty, k]$.

Similarly to the proof of Lemma 3.4.6 we can also define G-invariant states $\tilde{\Xi}^{(k)}$ on $\mathscr{A}^{(1)} \otimes \mathscr{A}'^{(1)} \otimes \mathscr{A}^{(2)} \otimes \mathscr{A}'^{(2)}$ which have the following properties: (1) $\tilde{\Xi}^{(k)}$ is pure factorized when restricted to (k,∞), and (2) its restriction to $(-\infty, k]$ is a purification of $(\psi \otimes \psi')|_{(-\infty,k]}$ by a state on $\mathscr{A}^{(2)}_{(-\infty,k]} \otimes \mathscr{A}'^{(2)}_{(-\infty,k]}$ which is g-close to a pure factorized state for some $g(r) = O(r^{-\infty})$ which depends only on f. In the following we assume that all states are tensored with a finite number of copies of factorized states, so that all stable equivalences correspond to equivalences without stabilization.

Let us fix $L \in \mathbb{N}$. By the proof of Lemma 3.4.6 the state $\tilde{\Xi}^{(k-L)}$ can be stably produced from $\tilde{\Xi}^{(k)}$ by first conjugation with an almost local at $k+L$ unitary, producing a G-invariant state $\tilde{\Theta}^{(k-L)}$, followed by a conjugation with a unitary strictly local on $[k, k+L)$. Let $|\tilde{\Xi}^{(k)}\rangle$ and $|\tilde{\Theta}^{(k-L)}\rangle$ be vectors representing the states $\tilde{\Xi}^{(k)}$ and $\tilde{\Theta}^{(k-L)}$

in the GNS Hilbert space $\check{\mathrm{H}}_{\tilde{\Xi}^{(k)}}$. The vector $\tilde{\Xi}^{(k)}$ is the vacuum vector and thus is G-invariant. The vector $|\tilde{\Theta}^{(k-L)}\rangle$, in general, transforms in a one-dimensional representation of G. We choose W_k to be the dual of this representation, while W_l for $k-L<l<k$ to be the trivial representation.

For each $k \in \mathbb{Z}$ let $\Upsilon^{(k)}$ be a pure factorized state on $\mathscr{A}''$ which coincides with ψ'' on $(-\infty, k]$ and whose restriction to $\mathscr{A}''_j$, $j \in (k, +\infty)$, is given by $\langle t_j| \cdot |t_j\rangle$. This state is G-invariant and restricts to pure factorized G-invariant states both on $(-\infty, k]$ and on $(k, +\infty)$. Thus $\Upsilon^{(k-L)}$ can be obtained from $\Upsilon^{(k)}$ by a conjugation with a strictly local at k unitary. Therefore the vector $|\Upsilon^{(k-L)}\rangle$ in the GNS Hilbert space of the state $\Upsilon^{(k)}$ transforms in the representation W_k. Consequently, the vector $|\tilde{\Theta}^{(k-L)} \otimes \Upsilon^{(k-L)}\rangle \in \check{\mathrm{H}}_{\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)}}$ is G-invariant, just like the vacuum vector $|\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)}\rangle$. Therefore their arbitrary linear combinations are also G-invariant.

Let Ω_s, $s \in [0,1]$, be the path of of vector states corresponding to the normalization of the path of vectors $s|\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)}\rangle + (1-s)|\tilde{\Theta}^{(k-L)} \otimes \Upsilon^{(k-L)}\rangle$. By Corollary 3.4.1.2 there is an LGA α_P generated by an almost local observable $P(s)$, such that $\Omega_s = \alpha_P(s)(\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)})$. Let $P^G(s)$ be the observable obtained from $P(s)$ by averaging over the group action. Then $P^G(s)$ generates an automorphism α_{P^G} which is G-equivariant, and since the state Ω_s is G-invariant for all s, we still have $\Omega_s = \alpha_{P^G}(s)(\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)})$. The automorphism α_{P^G} is a conjugation by some G-invariant almost local unitary $e^{i\mathscr{G}_k}$ which is h'-localized at $k-L$ for some $h'(r) = O(r^{-\infty})$ which depends only on $f(r)$. The state $\tilde{\Theta}^{(k-L)} \otimes \Upsilon^{(k-L)}$ can be obtained from $\tilde{\Xi}^{(k)} \otimes \Upsilon^{(k)}$ by conjugation with this observable.

A similar averaging argument shows that $\tilde{\Xi}^{(k-L)} \otimes \Upsilon^{(k-L)}$ can be obtained from $\tilde{\Theta}^{(k-L)} \otimes \Upsilon^{(k-L)}$ by a conjugation with a strictly local on $(k-L, k]$ G-invariant observable. Therefore, in the same way as in the proof of Lemma 3.4.6 and Theorem 54, by taking L large enough we can construct a G-invariant LGA that disentangles Ψ tensored with several factorized states. □

Corollary 55.1. The index defines a group isomorphism between Φ^*_G and $H^2(G, U(1))$.

Proof. In Section 3, we have shown that the index defines a group homomorphism from Φ^*_G to $H^2(G, U(1))$. The MPS construction in Section 3 implies that the homomorphism is surjective. Moreover, the theorem shows that this homomorphism has trivial kernel. Therefore, it defines a group isomorphism. □

Corollary 55.2. Every G-invariant invertible system is G-invertible.

Proof. Let $(\mathscr{A}, \psi)$ be a G-invariant invertible system. Construct another G-invariant system $(\mathscr{A}', \psi')$ whose index is the group inverse to that of $(\mathscr{A}, \psi)$. This can be done using to the Entangled Pair State construction in Section 3. The stacked system $(\mathscr{A} \otimes \mathscr{A}', \psi \otimes \psi')$ has trivial index according to the stacking rule shown in Section 3. The theorem implies that this stacked system is in the trivial G-invariant phase. Hence, $(\mathscr{A}, \psi)$ is G-invertible. □

[1] Anton Kapustin, Nikita Sopenko, and Bowen Yang. "A classification of invertible phases of bosonic quantum lattice systems in one dimension". In: *Journal of Mathematical Physics* 62.8 (2021), p. 081901. DOI: 10.1063/5.0055996. URL: https://doi.org/10.1063/5.0055996.

Chapter 4

STATIC LINEAR RESPONSE FOR QUANTUM SPIN SYSTEMS

4.1 Introduction

Condensed matter experiments often measure the linear response to a small local perturbation of a system in thermodynamic equilibrium in order to gain information about the unperturbed system. One of the simplest quantities to measure is the static response [39] to an infinitesimal perturbation $H \to H + \lambda B$ on the Hamiltonian by an observable B. The change in the equilibrium expectation value of an observable A is determined by a function $\langle\langle A, B \rangle\rangle$ called Kubo's canonical correlation function. It is natural to measure the rate at which this function decays with respect to the spatial distance between A and B. In the theoretical study of quantum many body systems, knowledge of this decay rate is important. For example, the assignment of a local temperature to a subsystem relies crucially on the exponential decay of canonical correlators [40]. Another example is the proof of the energy Bloch's theorem [41] which assumes a fast enough decay rate as well.

A study of decay rates of canonical correlators in general quantum systems is, therefore, warranted. While there has been some progress for finite quantum spin system at high temperature (see [40]), our understanding of the topic is far from complete. On the other hand, a lot is known about ordinary correlation functions and their decay in thermal states. In 1969, Araki [42] proved the exponential and uniform clustering of ordinary correlators for one dimensional quantum spin chains at positive temperature. The clustering of ordinary correlators has been proved for certain higher dimensional spin systems, both classical [43] and quantum [44]. It would be interesting to prove analogous results for canonical correlators. There could be two applications if such results exist. Firstly, it would justify the theoretical assumptions about rapid decay of canonical correlators away from phase transition. Secondly, near a continuous phase transition, ordinary correlators have power-like decay. The exponents of the decay rates are believed to be universal. A natural question is whether this translates into power-like decay with the same exponents for canonical correlators. The answer is affirmative for classical systems where ordinary and canonical correlators are exactly equal. It would be interesting to verify if this result holds also for quantum systems at positive temperature.

In this chapter we prove such results in the framework of infinite-volume quantum spin systems at positive temperature. We believe our methods can be adapted to many other discrete or continuous models.

4.2 Setup and statement of the main result

Setup and notation

We consider a quantum spin system defined on an infinite discrete metric space (Γ, d). Let $P_0(\Gamma)$ be the set of finite subsets of Γ. For a finite subset $X \in \mathcal{P}_0(\Gamma)$, we define $B_r(X) = \{y \in \Gamma : d(y, X) < r\}$. We also assume there is a constant $D > 0$, such that for all $X \in \mathcal{P}_0(\Gamma)$ there exists $C > 0$ with

$$|B_r(X)| \leq C|X|(1 + r)^D. \tag{4.1}$$

The algebra of local observables is

$$\mathscr{A}_{\text{loc}} = \varinjlim_{X \in \mathcal{P}_0(\Gamma)} \bigotimes_{x \in X} M_{d_x}, \tag{4.2}$$

where M_{d_x} is the algebra of $d_x \times d_x$ matrices with $\sup_{x \in X} d_x < \infty$. We also denote $\bigotimes_{x \in X} M_{d_x}$ by $\mathscr{A}_X$ for all $X \in \mathcal{P}_0(\Gamma)$. The C^*-algebra $\mathscr{A}$ of quasi-local observables is the norm completion of $\mathscr{A}_{\text{loc}}$. For $A \in \mathscr{A}_{\text{loc}}$, its *support* supp A is the smallest $X \in \mathcal{P}_0(\Gamma)$ such that $A \in \mathscr{A}_X$.

In a system of finite volume, time evolution is generated by a self-adjoint operator called the Hamiltonian. This is replaced by an interaction in an infinite-volume system. An interaction Φ is a function from $\mathcal{P}_0(\Gamma)$ to $\mathscr{A}$ such that $\Phi(X) = \Phi(X)^* \in \mathscr{A}_X$. In order to determine a time evolution on the infinite system, an interaction has to be reasonably local. Mathematically, this translates into a decay condition on $\|\Phi(X)\|$ as X grows in size. Various viable conditions have been explored by others (see [45, 44, 46, 47]). For clarity we shall follow a single convention and assume for all $x \in \Gamma$ the following holds for some positive constants μ and v:

$$\sum_{Z \ni x} \|\Phi(Z)\| \, |Z| \exp(\mu \text{diam}(Z)) \leq v/2 < \infty, \tag{4.3}$$

where $\text{diam}(Z) = \max_{y,z \in Z} d(y, z)$ for $Z \in \mathcal{P}_0(\Gamma)$. Under this condition, we are able to determine an infinite-volume time evolution $\tau : \mathbb{R} \to \text{Aut}(\mathscr{A})$ as a strongly continuous one-parameter group of C^*-algebra automorphisms. Specifically, it is shown (see [48, 49, 50, 51]) that for all A in a dense subset of $\mathscr{A}$, the following limit is well-defined

$$\delta(A) = \lim_{|\Lambda| \to \infty} \sum_{X \subset \Lambda} [\Phi(X), A], \tag{4.4}$$

where $\Lambda \in \mathcal{P}_0(\Gamma)$ grows to eventually include any point in Γ. Then τ_t is obtained on a dense subset by the exponentiation of δ through a Taylor series. The convergence of both δ and τ are shown in the references above. In particular, $\tau_t(A)$ is differentiable with respect to $t \in \mathbb{R}$ for all $A \in \mathscr{A}_{\text{loc}}$ (see Lemma 3.3.5 in [51]). We will use this fact later.

When working with the algebra of observables (instead of the Hilbert space of pure states), a state $\phi : \mathscr{A} \longrightarrow \mathbb{C}$ is a positive linear functional of norm 1, or equivalently ϕ is bounded with $\|\phi\| = \phi(1) = 1$. This immediately implies that for any observable $A \in \mathscr{A}$, we have a bound $|\phi(A)| \leq \|A\|$.

Given a time evolution τ and an inverse temperature $\beta \in [0, \infty]$, a state ϕ is called β-KMS [1] if for all $A, B \in \mathscr{A}_{loc}$ there exists a function $F_{A,B}(t)$ which is holomorphic on the open strip $S_\beta = \{z \in \mathbb{C} : 0 < \text{Im} z < \beta\}$, and bounded and continuous on $\overline{S}_\beta$, such that

$$F_{A,B}(t) = \phi(A\tau_t(B)) \tag{4.5}$$

and

$$F_{A,B}(t + i\beta) = \phi(\tau_t(B)A), \tag{4.6}$$

for all $t \in \mathbb{R}$. For $\beta > 0$ the KMS condition implies that ϕ is invariant under the automorphisms τ [49]. The KMS states are well studied as an infinite-volume thermal state at temperature $T = 1/\beta$. In particular, it has been shown [52] that such a state always exists in a quantum spin system for any β. More detailed exposition on the topic can be found in [49, 50] . For a fixed β-KMS state the ordinary and Kubo's canonical correlators are respectively

$$\langle A, B\rangle_\phi = \phi(AB) - \phi(A)\phi(B) \tag{4.7}$$

and

$$\langle\langle A, B\rangle\rangle_\phi = \frac{1}{\beta}\int_0^\beta F_{A,B}(ib)db - \phi(A)\phi(B). \tag{4.8}$$

Main result

Given a KMS state at positive temperature, we establish spatial decay for its canonical correlators assuming spatial decay for the ordinary correlators. Only two other ingredients are needed: approximate locality (see the next section) and the KMS condition. The large number of existing results on correlators decays often come in two forms: exponential clustering or power-like decay (see [44, 46, 47, 42, 43, 40,

[1] KMS stands for Kubo-Martin-Schwinger.

53]). We choose to use a general monotone decreasing function $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as the decay rate instead. While this has the benefit of generality, it may appear opaque for applications. Thus, we also discuss our result for $g(l) = e^{-l/\xi}$ and $g(l) = l^{-n}$ with some $n, \xi > 0$.

Theorem 1. In a quantum spin system as defined above, let ϕ be a β-KMS state with $0 < \beta < \infty$. Given $X, Y \in \mathcal{P}_0(\Gamma)$ with $l = d(X, Y) > 0$, if for all $A \in \mathscr{A}_X$, $B \in \mathscr{A}_Y$,

$$|\langle A, B\rangle_\phi| \leq c\,\|A\|\,\|B\|\min\{|X|, |Y|\}g(l), \tag{4.9}$$

where c is a constant, then

$$|\langle\langle A, B\rangle\rangle_\phi| \leq c'\,\|A\|\,\|B\|\min\{|X|^2, |Y|^2\}g'(l), \tag{4.10}$$

where c' is another constant and $g'(l)$ is $g(l/4)$ or $e^{-\mu l/2}$ whichever one has slower decay. In particular, if $g(l) = l^{-n}$ then $g'(l) = l^{-n}$. If $g(l) = e^{-l/\xi}$, then $g'(l) = e^{-l/\xi'}$ where ξ' is another constant independent of A, B, and l.

4.3 Approximate locality

Locality in a physical system broadly refers to a limit on the propagation speed of information. In relativistic systems, no information is allowed to travel faster than the speed of light. However, non-relativistic systems often do not possess such a sharp bound. Nevertheless, an approximate version of locality can emerge in a multitude of many body systems. In this section, we first define approximate locality for a quantum spin system. Then we state the Lieb-Robinson bounds which guarantees approximate locality for the spin systems considered in this chapter.

Definition 56. In a quantum spin system with time evolution τ, we say the system is *approximately local* if for any $A \in \mathscr{A}_X$ and $r > 0$, there exists a family of operators $A^r(t) \in \mathscr{A}_{B_r(X)}$ differentiable with respect to $t \in \mathbb{R}$, such that $A^r(0) = A$ and

$$\|\tau_t(A) - A^r(t)\| \leq c\,\|A\|\,|X|e^{-2\mu r}(e^{v|t|} - 1), \tag{4.11}$$

where c, μ, v are positive absolute constants.

Remark 57. The condition can be interpreted as a bound on information leaked out of an effective light-cone. In relativistic field theory, the condition above holds without any leakage.

It turns out many non-relativistic quantum spin systems possess *approximate locality*. This fact was originally observed by Lieb and Robinson in [54] and became known as the Lieb-Robinson bounds. There exists many improved versions of this result for example in [55, 45, 47, 56] just to name a few. Below, we state a version discussed in [55].

Theorem 2. In a quantum spin system on (Γ, d), assume the interaction Φ is such that for all $x \in \Gamma$, the following holds:

$$\sum_{Z \ni x} \|\Phi(Z)\| \, |Z| \exp(\mu \mathrm{diam}(Z)) \leq v/2 < \infty, \tag{4.12}$$

for some positive constants c, μ, and v. Let A, B be operators supported on sets X, Y, respectively. Then, if $l = d(X, Y) > 0$,

$$\|[B, \tau_t(A)]\| \leq c \, \|A\| \, \|B\| \min(|X|, |Y|) e^{-\mu l}(e^{v|t|} - 1). \tag{4.13}$$

Proposition 58. A spin system satisfying the Lieb-Robinson bounds possesses approximate locality and vice versa.

Proof. ($\Rightarrow$): Given quasi-local $A \in \mathscr{A}$ and $X \in \mathcal{P}_0(\Gamma)$, we define for an integer $n > 0$

$$\mathcal{T}_X^n(A) = \int_{B_n(X) \setminus X} dU U A U^\dagger, \tag{4.14}$$

where the integral is over all unitary matrices of the form $U = \bigotimes_{x \in B_n(X) \setminus X} U_x$ with $U_x \in \mathscr{A}_{\{x\}}$ using the Haar measure. We also assume the Haar measure in the integral is normalized so that

$$\|\mathcal{T}_X^n(A)\| \leq \int_{B_n(X) \setminus X} dU \|UAU^\dagger\| = \|A\| \int_{B_n(X) \setminus X} dU = \|A\|. \tag{4.15}$$

Moreover, $\{\mathcal{T}_X^n(A)\}_n$ is a Cauchy sequence: given $\varepsilon > 0$, there exists $T \in \mathscr{A}_{\mathrm{loc}}$ such that $\|T - A\| < \varepsilon/2$. For any large m, n such that $\mathrm{supp}\, T \subset B_n(X)$ and $m > n$,

$$\begin{aligned}
&\|\mathcal{T}_X^m(A) - \mathcal{T}_X^n(A)\| \\
\leq &\int_{B_n(X) \setminus X} dU \|U(\mathcal{T}_{B_n(X)}^{m-n}(A) - A)U^\dagger\| \\
= &\left\| \int_{B_m(X) \setminus B_n(X)} dW W A W^\dagger - T + T - A \right\| \\
\leq &\left\| \int_{B_m(X) \setminus B_n(X)} dW W(A - T)W^\dagger \right\| + \|T - A\| < \varepsilon,
\end{aligned} \tag{4.16}$$

where the last step relies on the fact $[T, W] = 0$ for all $W \in \mathscr{A}_{B_m(X) \setminus B_n(X)}$. Let $\mathcal{T}_X(A) = \lim_{n \to \infty} \mathcal{T}_X^n(A)$. It is easy to see that

$$\|\mathcal{T}_X(A)\| \leq \|A\|. \tag{4.17}$$

Thus $\mathcal{T}_X$ is a bounded linear operator on $\mathscr{A}$.

We are now ready to show approximate locality. Given $A \in \mathscr{A}_X$ and $r > 0$, we define

$$A^r(t) = \mathcal{T}_{B_r(X)}(\tau_t(A)), \tag{4.18}$$

and for $n > r$,

$$A^r_n(t) = \mathcal{T}^{n-r}_{B_r(X)}(\tau_t(A)), \tag{4.19}$$

Clearly $A^r(0) = A$. From (4.17),

$$\|A^r(t)\| \leq \|A\|. \tag{4.20}$$

Since $U\tau_t(A)U^\dagger = \tau_t(A) + U[\tau_t(A), U^\dagger]$, Theorem 2 implies

$$\left\|\tau_t(A) - A^r_n(t)\right\| \leq \int_{B_n(X)\backslash B_r(X)} dU \left\|[U^\dagger, \tau_t(A)]\right\| \leq c\|A\|e^{-\mu r}(e^{v|t|} - 1). \tag{4.21}$$

Let $n \to \infty$ on both sides, we have

$$\|\tau_t(A) - A^r(t)\| \leq c\|A\|e^{-\mu r}(e^{v|t|} - 1). \tag{4.22}$$

As $\tau_t(A)$ is differentiable with respect to t for local $A \in \mathscr{A}_X$ and $\mathcal{T}_{B_r(X)}$ is bounded and linear, $A^r(t) = \mathcal{T}_{B_r(X)}(\tau_t(A))$ is also differentiable.

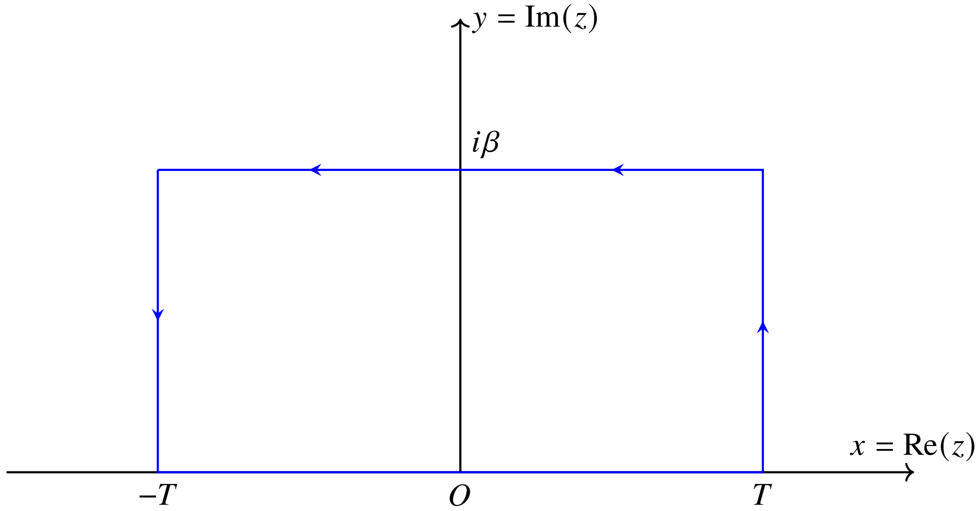
($\Leftarrow$): Given $A \in \mathscr{A}_X, B \in \mathscr{A}_Y$ with $l = d(X,Y) > 0$, we can take $r = l/2$ and deduce from approximate locality the following:

$$\|[B, \tau_t(A)]\| \leq \|[B, \tau_t(A) - A^r(t)]\| \leq c\,\|A\|\,\|B\|\,|Y|e^{-\mu l}(e^{v|t|} - 1), \tag{4.23}$$

where we assume $\min(|X|, |Y|) = |Y|$ without loss of generality. In the first step, we used the fact $[B, A^r(t)] = 0$ for $r = l/2$. □

4.4 Proof of main result

By the KMS condition, $f(z) := F_{A,B}(z)$ is bounded and analytic on S_β. At positive temperature β is finite, so the interval $[0, \beta]$ is compact. It is, therefore, sufficient to establish the desired decay for the integrand $f(ib) - \phi(A)\phi(B)$. As long as the bound is uniform in $b \in [0, \beta]$, it survives after the integration. This approach would not apply to systems at zero temperature where $\beta = \infty$.



Contour Integration

Fix $b \in (0, \beta)$ and let $w(z) = e^{-z^2-b^2}$. The function w is bounded and analytic on the strip S_β. Its exponent is conveniently chosen so that $w(ib) = 1$. We integrate the function $\frac{f(z)w(z)}{z-ib}$ along the contour Γ_T shown on the figure above. By an extended version of the residue theorem [57],

$$\begin{aligned} 2\pi i f(ib)w(ib) &= \int_{\Gamma_T} \frac{f(z)w(z)}{(z-ib)} dz \\ &= \int_{-T}^{T} \frac{f(t)w(t)}{t-ib} dx + i\int_0^\beta \frac{f(T+iy)w(T+iy)}{T+iy-ib} dy \\ &- \int_{-T}^{T} \frac{f(t+i\beta)w(t+i\beta)}{t+i\beta-ib} dt - i\int_0^\beta \frac{f(-T+iy)w(-T+iy)}{-T+iy-ib} dy. \end{aligned} \tag{4.24}$$

$f(z)$ and $w(z)$ are bounded on S_β, therefore, the 2nd and 4th terms vanish as $T \to \infty$.

$$\begin{aligned} f(ib) =& \frac{1}{2\pi i}\left(\int_{-\infty}^{\infty} \frac{f(t)w(t)}{t-ib} dt - \int_{-\infty}^{\infty} \frac{f(t+i\beta)w(t+i\beta)}{t+i\beta-ib} dt\right) \\ =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)f(t)(t+i\beta-ib) - w(t+i\beta)f(t+i\beta)(t-ib)}{(t-ib)(t+i\beta-ib)} dt \\ =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t+i\beta)(f(t)-f(t+i\beta))}{t+i\beta-ib} dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)f(t)i\beta + (w(t)-w(t+i\beta))f(t)(t-ib)}{(t-ib)(t+i\beta-ib)} dt \\ =& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib} dt \\ &+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)f(t)i\beta + (w(t)-w(t+i\beta))f(t)(t-ib)}{(t-ib)(t+i\beta-ib)} dt, \end{aligned} \tag{4.25}$$

where in the last equality we used the fact $f(t+i\beta) = \phi(\tau_t(B)A)$. If we substitute $A = I$, the above identity reduces to

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)\phi(B)i\beta + (w(t) - w(t+i\beta))\phi(B)(t-ib)}{(t-ib)(t+i\beta-ib)}dt = \phi(B), \tag{4.26}$$

or

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)i\beta + (w(t) - w(t+i\beta))(t-ib)}{(t-ib)(t+i\beta-ib)}dt = 1. \tag{4.27}$$

This identity can be verified more directly using the residue theorem bearing in mind that $w(ib) = 1$. With this identity we write

$$\begin{aligned}
& f(ib) - \phi(A)\phi(B) \\
=& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib}dt \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t)(f(t) - \phi(A)\phi(B))i\beta + (w(t) - w(t+i\beta))(f(t) - \phi(A)\phi(B))(t-ib)}{(t-ib)(t+i\beta-ib)}dt \\
=& \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib}dt \\
&+ \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{\langle A,\tau_t(B)\rangle_\phi (w(t)i\beta + (w(t) - w(t+i\beta))(t-ib))}{(t-ib)(t+i\beta-ib)}dt.
\end{aligned} \tag{4.28}$$

Therefore,

$$\begin{aligned}
& 2\pi|\phi(\tau_{-ib}(A)B) - \phi(A)\phi(B)| \\
\leq & \left| \int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib}dt \right| \\
&+ \left| \int_{-\infty}^{\infty} \frac{\langle A,\tau_t(B)\rangle_\phi w(t)}{(t-ib)}dt \right| \\
&+ \left| \int_{-\infty}^{\infty} \frac{\langle A,\tau_t(B)\rangle_\phi w(t+i\beta)}{(t+i\beta-ib)}dt \right|.
\end{aligned} \tag{4.29}$$

We proceed to bound each term separately. Throughout the next sections, we absorb all constant coefficients into a symbol c. Thus c may change from step to step.

First Term

The decay for this term relies completely on approximate locality:

$$
\begin{aligned}
&\left| \int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib} dt \right| \\
\leq &\int_{-\mu l/2v}^{\mu l/2v} \frac{|w(t+i\beta)| \, \|[A,\tau_t(B)]\|}{|t+i\beta-ib|} dt \\
&+ \int_{|t|>\mu l/2v} \frac{|w(t+i\beta)| \, \|[A,\tau_t(B)]\|}{|t+i\beta-ib|} dt \\
\leq &\int_{-\mu l/2v}^{\mu l/2v} \frac{|w(t+i\beta)| \, \|[A,\tau_t(B)]\|}{|t|} dt \\
&+ 2\, \|A\| \, \|B\| \int_{|t|>\mu l/2v} \frac{|w(t+i\beta)|}{|t|} dt. \qquad (4.30)
\end{aligned}
$$

Finally we substitute $|w(t+i\beta)| = e^{-t^2+\beta^2-b^2}$ and then use approximate locality (or the Lieb-Robinson bounds) to show exponential decay:

$$
\begin{aligned}
&e^{\beta^2-b^2} \int_{-\mu l/2v}^{\mu l/2v} \frac{\|[A,\tau_t(B)]\|}{|t|} e^{-t^2} dt \\
\leq &c\, \|A\| \, \|B\| \min(|X|,|Y|) e^{\beta^2-\mu l} \int_0^{\mu l/2v} \frac{e^{vt}-1}{t} e^{-t^2} dt \\
\leq &c\, \|A\| \, \|B\| \min(|X|,|Y|) e^{\beta^2-\mu l} \left(\int_0^1 \frac{e^{vt}-1}{t} e^{-t^2} dt + \int_1^{\mu l/2v} e^{vt} dt \right) \\
= &c\, \|A\| \, \|B\| \min(|X|,|Y|) e^{\beta^2-\mu l} \left(\int_0^1 \frac{e^{vt}-1}{t} e^{-t^2} dt + \frac{e^{\mu l/2}}{v} + \frac{e^{v}}{v} \right) \\
\leq &c\, \|A\| \, \|B\| \min(|X|,|Y|) e^{-\mu l/2}. \qquad (4.31)
\end{aligned}
$$

The second integral becomes

$$
\begin{aligned}
&4\, \|A\| \, \|B\| \int_{t>\mu l/2v} \frac{e^{\beta^2-b^2-t^2}}{t} dt \\
\leq &c\, \|A\| \, \|B\| \int_{t>\mu l/2v} \frac{e^{-t^2}}{t} dt. \qquad (4.32)
\end{aligned}
$$

This decays exponentially in l^2 which is faster than the previous integral, so overall

$$
\left| \int_{-\infty}^{\infty} \frac{w(t+i\beta)\phi([A,\tau_t(B)])}{t+i\beta-ib} dt \right| \leq c\, \|A\| \, \|B\| \min(|X|,|Y|) e^{-\mu l/2}. \qquad (4.33)
$$

Note this term decays exponentially with l regardless of the rate of decay assumed for the ordinary correlator. The bound obtained is manifestly independent of b.

Second and Third Terms

Bounding the second and third terms similar, so we only provide the detail of the former:

$$\begin{aligned}
&\left|\int_{-\infty}^{\infty}\frac{\langle A,\tau_t(B)\rangle_\phi w(t)}{t-ib}dt\right| \\
\leq&\left|\int_{-\mu l/2v}^{\mu l/2v}\frac{\langle A,\tau_t(B)\rangle_\phi w(t)}{t-ib}dt\right| \\
&+2\|A\|\,\|B\|\int_{|t|>\mu l/2v}\frac{|w(t)|}{|t|}dt.
\end{aligned} \tag{4.34}$$

We have used the simple bound $|\langle A,\tau_t(B)\rangle_\phi| = |\phi(A\tau_t(B)) - \phi(A)\phi(B)| \leq \|A\tau_t(B)\| + \|A\|\,\|B\| \leq 2\|A\|\,\|B\|$ for the last step.

Since $w(t) = e^{-t^2-b^2}$, the second integral clearly decays exponentially with respect to l^2.

By approximate locality, let $B^{3l/4}(t)$ be the local approximation to $\tau_t(B)$. Then

$$\begin{aligned}
&\left|\int_{-\mu l/2v}^{\mu l/2v}\frac{w(t)}{t-ib}\langle A,\tau_t(B)\rangle_\phi dt\right| \\
=&\left|\int_{-\mu l/2v}^{\mu l/2v}\frac{w(t)}{t-ib}\big(\langle A,B^{3l/4}(t)\rangle_\phi + \phi(A(\tau_t(B)-B^{3l/4}(t))) - \phi(A)\phi((\tau_t(B)-B^{3l/4}(t)))\big)dt\right| \\
\leq&\left|\int_{-\mu l/2v}^{\mu l/2v}\frac{w(t)}{t-ib}\langle A,B^{3l/4}(t)\rangle_\phi dt\right| + 2\|A\|\int_{-\mu l/2v}^{\mu l/2v}\frac{|w(t)|}{|t-ib|}\left\|\tau_t(B)-B^{3l/4}(t)\right\|dt.
\end{aligned} \tag{4.35}$$

By approximate locality, the second integral is bounded as follows,

$$\begin{aligned}
&2\|A\|\int_{-\mu l/2v}^{\mu l/2v}\frac{|w(t)|}{|t-ib|}\left\|\tau_t(B)-B^{3l/4}(t)\right\|dt \\
\leq&2\|A\|\int_{-\mu l/2v}^{\mu l/2v}\frac{e^{-t^2}}{|t|}\left\|\tau_t(B)-B^{3l/4}(t)\right\|dt \\
\leq&c\|A\|\,\|B\|\min(|X|,|Y|)e^{-3\mu l/2}\int_0^{\mu l/2v}\frac{e^{vt}-1}{t}e^{-t^2}dt \\
\leq&c\|A\|\,\|B\|\min(|X|,|Y|)e^{-3\mu l/2}\left(\int_0^1\frac{e^{vt}-1}{t}e^{-t^2}dt + \frac{e^{\mu l/2}}{v} + \frac{e^v}{v}\right) \\
\leq&c\|A\|\,\|B\|\min(|X|,|Y|)e^{-\mu l}.
\end{aligned} \tag{4.36}$$

We are now left with

$$\left|\int_{-\mu l/2v}^{\mu l/2v}\frac{e^{-t^2-b^2}}{t-ib}\langle A,B^{3l/4}(t)\rangle_\phi dt\right|, \tag{4.37}$$

whose decay rate depends on g the decay rates of ordinary correlators.

By (4.9), we have

$$|\langle A, B^{3l/4}(t)\rangle_\phi| \leq c \min(|X|, |Y|)\, \|A\|\, \|B\|\, g(l/4). \tag{4.38}$$

Remark 59. We have $\min(|X|, |Y|)$ above instead of $|X|$ because we could have switched the roles of A and B in the entire proof.

Now, we fix an $\epsilon > 0$ and estimate

$$\begin{aligned}
&\left|\int_{-\mu l/2v}^{\mu l/2v} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi dt\right| \\
=&\left|\int_0^{\mu l/2v} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi - \frac{e^{-t^2-b^2}}{t+ib}\langle A, B^{3l/4}(-t)\rangle_\phi dt\right| \\
\leq&\left|\int_0^{\epsilon} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi - \frac{e^{-t^2-b^2}}{t+ib}\langle A, B^{3l/4}(-t)\rangle_\phi dt\right| \\
&+\left|\int_\epsilon^{\mu l/2v} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi - \frac{e^{-t^2-b^2}}{t+ib}\langle A, B^{3l/4}(-t)\rangle_\phi dt\right| \\
\leq&\left|\int_0^{\epsilon} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi - \frac{e^{-t^2-b^2}}{t+ib}\langle A, B^{3l/4}(-t)\rangle_\phi dt\right| \\
&+\int_\epsilon^{\infty} \frac{e^{-t^2}}{t}|\langle A, B^{3l/4}(t)\rangle_\phi| + \frac{e^{-t^2}}{t}|\langle A, B^{3l/4}(-t)\rangle_\phi| dt.
\end{aligned} \tag{4.39}$$

To get the desired decay of the last term in Equation (4.39), we apply (4.38) to $|\langle A, B^{3l/4}(t)\rangle_\phi|$ as well as to $|\langle A, B^{3l/4}(-t)\rangle_\phi|$. Note $\int_\epsilon^\infty \frac{e^{-t^2}}{t} dt$ is a finite constant for a fixed ϵ, so

$$\int_\epsilon^{\infty} \frac{e^{-t^2}}{t}|\langle A, B^{3l/4}(t)\rangle_\phi| + \frac{e^{-t^2}}{t}|\langle A, B^{3l/4}(-t)\rangle_\phi| dt \leq c \min(|X|, |Y|)\, \|A\|\, \|B\|\, g(l/4). \tag{4.40}$$

Finally, we are left with the terms

$$\begin{aligned}
&\left|\int_0^{\epsilon} \frac{e^{-t^2-b^2}}{t-ib}\langle A, B^{3l/4}(t)\rangle_\phi - \frac{e^{-t^2-b^2}}{t+ib}\langle A, B^{3l/4}(-t)\rangle_\phi dt\right| \\
\leq&\int_0^{\epsilon} e^{-t^2-b^2} \frac{|\langle A, B^{3l/4}(t)\rangle_\phi (t+ib) - \langle A, B^{3l/4}(-t)\rangle_\phi (t-ib)|}{t^2+b^2} dt \\
\leq&\int_0^{\epsilon} \frac{|\langle A, B^{3l/4}(t)\rangle_\phi| b + |\langle A, B^{3l/4}(-t)\rangle_\phi| b}{t^2+b^2} dt \\
&+\int_0^{\epsilon} \frac{t|\langle A, B^{3l/4}(t)\rangle_\phi - \langle A, B^{3l/4}(-t)\rangle_\phi|}{t^2+b^2} dt.
\end{aligned} \tag{4.41}$$

To the first integral, apply (4.38) again while noting that the remaining integral $\int_0^\epsilon \frac{b}{t^2+b^2}dt$ gives $\arctan(\epsilon/b)$ which is bounded for any b. Therefore,

$$\int_0^\epsilon \frac{|\langle A, B^{3l/4}(t)\rangle_\phi|b + |\langle A, B^{3l/4}(-t)\rangle_\phi|b}{t^2+b^2}dt \leq c\min(|X|,|Y|)\,\|A\|\,\|B\|\,g(l/4). \tag{4.42}$$

To estimate the final integral, recall that $B^{3l/4}(t)$ is differentiable with respect to t. By the mean value theorem,

$$\begin{aligned} &\int_0^\epsilon \frac{t|\langle A, B^{3l/4}(t)\rangle_\phi - \langle A, B^{3l/4}(-t)\rangle_\phi|}{t^2+b^2}dt \\ \leq &\int_0^\epsilon \frac{|\langle A, B^{3l/4}(t)\rangle_\phi - \langle A, B^{3l/4}(-t)\rangle_\phi|}{t}dt \qquad (4.43)\\ \leq &4\epsilon \sup_{-\epsilon<t<\epsilon}\left|\frac{d}{dt'}\Big|_{t'=t}\langle A, B^{3l/4}(t')\rangle_\phi\right| = 4\epsilon \sup_{-\epsilon<t<\epsilon}\left|\langle A, \frac{dB^{3l/4}(t')}{dt'}\Big|_{t'=t}\rangle_\phi\right|. \qquad (4.44) \end{aligned}$$

Since $B^{3l/4}(t')$ is supported on $B^{3l/4}(Y)$ for all t', the support of $\frac{dB^{3l/4}(t')}{dt'}\Big|_{t'=t}$ is also contained in $B^{3l/4}(Y)$. Next we bound its norm,

$$\begin{aligned} &\left\|\frac{dB^{3l/4}(t')}{dt'}\Big|_{t'=t}\right\| \leq \left\|\frac{d}{dt'}\Big|_{t'=0}\tau_{t+t'}(B)\right\| = \left\|\lim_{|\Lambda|\to\infty}\sum_{Z\subset\Lambda}[\Phi(Z),\tau_t(B)]\right\| \\ \leq &\lim_{|\Lambda|\to\infty}\left\|\sum_{Z\subset\Lambda:Z\cap Y\neq\emptyset}[\Phi(Z),\tau_t(B)]\right\| + \lim_{|\Lambda|\to\infty}\left\|\sum_{Z\subset\Lambda:Z\cap Y=\emptyset}[\Phi(Z),\tau_t(B)]\right\|. \qquad (4.45) \end{aligned}$$

We bound the former term using (4.3),

$$\lim_{|\Lambda|\to\infty}\left\|\sum_{Z\subset\Lambda:Z\cap Y\neq\emptyset}[\Phi(Z),\tau_t(B)]\right\| \leq 2\,\|B\|\sum_{y\in Y}\sum_{Z\ni y}\|\Phi(Z)\| \leq v|Y|\,\|B\|\,. \tag{4.46}$$

As for the latter, apply (4.13) and then (4.3)

$$\begin{aligned} &\lim_{|\Lambda|\to\infty}\left\|\sum_{Z\subset\Lambda:Z\cap Y=\emptyset}[\Phi(Z),\tau_t(B)]\right\| \\ \leq &\sum_{Z\cap Y=\emptyset}\|[\Phi(Z),\tau_t(B)]\| \\ \leq &\sum_{Z\cap Y=\emptyset} 2\,\|\Phi(Z)\|\,\|B\|\,|Z|e^{-\mu d(Z,Y)}(e^{v|t|}-1) \\ \leq &2\,\|B\|\,(e^{v\epsilon}-1)\sum_{Z\cap Y=\emptyset}\|\Phi(Z)\|\,|Z|e^{-\mu d(Z,Y)} \\ \leq &2\,\|B\|\,(e^{v\epsilon}-1)\sum_{r=1}^{\infty}e^{-\mu r}\sum_{d(z,Y)\in(r-1,r]}\sum_{Z\ni z}\|\Phi(Z)\|\,|Z| \\ \leq &v\,\|B\|\,(e^{v\epsilon}-1)\sum_{r=1}^{\infty}e^{-\mu r}D(r), \qquad (4.47) \end{aligned}$$

where $|t| < \epsilon$ and $D(r) := |\{z \in \Gamma : d(z, Y) \in (r-1, r]\}|$. Clearly, $D(r)$ grows polynomially with respect to r, so $\sum_{r=1}^{\infty} e^{-\mu r} D(r)$ is bounded by a constant linearly dependent on $D(1)$. This is dominated by $|Y|$ due to (4.1). Overall,

$$\left\| \frac{dB^{3l/4}(t')}{dt'}\Big|_{t'=t} \right\| < C(\mu, v, \epsilon) \|B\| \, |Y|. \tag{4.48}$$

Then we apply (4.9) to $\left|\langle A, \frac{dB^{3l/4}(t')}{dt'}\big|_{t'=t}\rangle_\phi\right|$ and obtain

$$\int_0^\epsilon \frac{t|\langle A, B^{3l/4}(t)\rangle_\phi - \langle A, B^{3l/4}(-t)\rangle_\phi|}{t^2+b^2} dt \leq c \|A\| \, \|B\| \min(|X|, |Y|)|Y| g(l/4). \tag{4.49}$$

Furthermore, the canonical correlator is symmetric, so if we swap A and B from the very beginning, we get the same overall bound except an additional factor of $|X|$ instead of $|Y|$. Thus, we actually have

$$\int_0^\epsilon \frac{t|\langle A, B^{3l/4}(t)\rangle_\phi - \langle A, B^{3l/4}(-t)\rangle_\phi|}{t^2+b^2} dt \leq c \|A\| \, \|B\| \min(|X|^2, |Y|^2) g(l/4). \tag{4.50}$$

The procedure for bounding the third term is entirely similar.

Summary

In summary,

$$|\langle\langle A, B\rangle\rangle_\phi| \leq c \|A\| \, \|B\| \min\{|X|^2, |Y|^2\} g'(l), \tag{4.51}$$

where $g'(l)$ is the slower decaying one between $g(l/4)$ and $e^{-\mu l/2}$.

4.5 Discussion

In the present chapter, we focus on quantum spin systems with short-range interactions. In particular, interactions are assumed to satisfy a concrete condition (4.3). However, our analysis can be adapted readily if we substitute (4.3) with many other conditions for short-range interactions such as those analyzed in [44, 45]. Lieb-Robinson bounds have also been proven for certain spin systems with long-range interactions (see [44, 45, 58]). However, these bounds may not lead to approximate locality defined here. For example, [58] demonstrates that systems with long-range interactions could have unbounded information propagation speeds. Therefore, it is interesting to investigate decay rates of canonical correlators in systems with long-range interactions.

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Appendix A

NOTIONS IN COMMUTATIVE ALGEBRA

A.1 Gorenstein rings

Definition 60. Noetherian ring A is called a Gorenstein ring if its injective dimension (as a module over itself) is finite. If it is zero, i.e., A is an injective A-module, then A is called a QF[1] ring.

Lemma 61. Let A be a Gorenstein ring. Every localization of A is a Gorenstein ring. The injective dimension of A is equal to the Krull dimension.

Proof. Corollaries 1.3 and 5.6 in [59]. □

Remark 62. It is popular to define the Gorenstein property first for Noetherian local rings and then declare a general Noetherian ring to be Gorenstein if its localization on any prime ideal is Gorenstein. Such rings do not necessarily have finite dimension. This situation is not encountered in this thesis, so it is more convenient to stick to the more restrictive Definition 60.

Lemma 63. If a ring A is Gorenstein, so is the polynomial ring $A[x]$.

Proof. Follows immediately from [Stacks, Tag 0A6J]. □

Lemma 64. Let A be a QF ring. Every A-module M embeds in a free module (of finite rank if M is finitely generated). The natural module map $M \to \mathrm{Hom}_A(\mathrm{Hom}_A(M, A), A)$ is injective (an isomorphism if M is finitely generated). In particular $M = 0$ if and only if $\mathrm{Hom}_A(M, A) = 0$.

Proof. See [15, Theorem 15.11]. □

Lemma 65. Let A be a commutative ring, M an A-module and $r \geq 0$ an integer. If $\dim(M) \leq r$, then the localization $M_{\mathfrak{p}}$ vanishes for all prime ideals $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) > r$. If M is finitely generated, the converse is true.

[1] QF stands for quasi-Frobenius.

Proof. Observe that $\dim(M) \leq r$ if and only if $\mathrm{Ann}(M)$ is not contained in any prime ideal $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) > r$. Suppose that this condition is satisfied and let $\mathfrak{p}$ be such that $\dim(R/\mathfrak{p}) > r$. Then $R \setminus \mathfrak{p}$ contains an element of $\mathrm{Ann}(M)$, so $M_{\mathfrak{p}} = 0$. Next, let M be finitely generated. Then $S^{-1}M = 0$ for a multiplicative set $S \subset R$ if and only if $S \cap \mathrm{Ann}(M) \neq \emptyset$. Thus $M_{\mathfrak{p}} = 0$ for a prime ideal $\mathfrak{p}$ if and only if $\mathrm{Ann}(M)$ is not contained in $\mathfrak{p}$. □

Lemma 66. Let A be a Gorenstein ring of dimension D and let M be a finitely generated A-module. Then $\dim(\mathrm{Ext}^i_A(M, A)) \leq D - i$.

Proof. Let $\mathfrak{p} \subset A$ be a prime ideal with $\dim(A/\mathfrak{p}) \geq D - i$. Then $A_{\mathfrak{p}}$ is a Gorenstein ring with $\dim A_{\mathfrak{p}} \leq D - i$, so $\mathrm{Ext}^{i+1}_A(M, A)_{\mathfrak{p}} \cong \mathrm{Ext}^{i+1}_{A_{\mathfrak{p}}}(M_{\mathfrak{p}}, A_{\mathfrak{p}}) = 0$. Now invoke Lemma 65. □

A.2 Local cohomology

Definition 67. Let A be a Noetherian commutative ring and $\mathfrak{a} \subset A$ an ideal. If M is an A-module, $\Gamma_{\mathfrak{a}}(M) = \{m \in M \mid \exists j \in \mathbb{N}\ \mathfrak{a}^j m = 0\}$ is called $\mathfrak{a}$-torsion submodule of M. Modules M such that $M = \Gamma_{\mathfrak{a}}(M)$ are said to be $\mathfrak{a}$-torsion. $\Gamma_{\mathfrak{a}}$ is a left exact functor. Its right derived functors $\mathrm{H}^j_{\mathfrak{a}}$ are called local cohomology functors. More explicitly, $\mathrm{H}^j_{\mathfrak{a}}(M)$ is defined as the j-th degree cohomology of the complex $\Gamma_a(I^\bullet)$, where $M \to I^\bullet$ is an injective resolution.

Note that by construction, every $\mathrm{H}^j_{\mathfrak{a}}(M)$ is a subquotient of an $\mathfrak{a}$-torsion module and hence is $\mathfrak{a}$-torsion. Moreover, $\mathrm{H}^0_{\mathfrak{a}}(M) \cong \Gamma_{\mathfrak{a}}(M)$.

Lemma 68. Let M be an A-module.

1. $\mathrm{H}^j_{\mathfrak{a}}(M) \cong \mathrm{H}^j_{\sqrt{\mathfrak{a}}}(M)$, where $\sqrt{\mathfrak{a}} = \{a \in A \mid \exists j \in \mathbb{N}\ a^j \in \mathfrak{a}\}$.

2. If $\mathfrak{a}_1, \ldots, \mathfrak{a}_t$ are coprime, then $\mathrm{H}^j_{\mathfrak{a}_1 \ldots \mathfrak{a}_t}(M) \cong \bigoplus_{i=1}^{t} \mathrm{H}^j_{\mathfrak{a}_i}(M)$.

Proof. 1. As A is Noetherian, $(\sqrt{\mathfrak{a}})^N \subset \mathfrak{a}$ for some N, so $\Gamma_a = \Gamma_{\sqrt{a}}$.

2. By induction, for any $i \neq j$ and $k \in \mathbb{N}$ ideals $\mathfrak{a}_i^k, \mathfrak{a}_j^k$ are coprime. Letting $K(I) = \{m \in M \mid Im = 0\}$ for an ideal I, Chinese remainder theorem gives $K(\mathfrak{a}_1^k \ldots \mathfrak{a}_t^k) = \bigoplus_{i=1}^t K(\mathfrak{a}_i^k)$. Next use $\Gamma_{\mathfrak{a}_1 \ldots \mathfrak{a}_t}(M) = \bigcup_{k=0}^\infty K(\mathfrak{a}_1^k \ldots \mathfrak{a}_t^k)$. □

Lemma 69. Let $\mathfrak{a}$ be an ideal such that $\dim(A/\mathfrak{a}) = 0$. Then for any A-module M and any j we have a natural isomorphism

$$\mathrm{H}^j_{\mathfrak{a}}(M) \cong \bigoplus_{\mathfrak{m}} \mathrm{H}^j_{\mathfrak{m}}(M), \tag{A.1}$$

where the sum is over maximal ideals $\mathfrak{m} \subset A$ containing $\mathfrak{a}$.

Proof. We have $\sqrt{\mathfrak{a}} = \bigcap_{\mathfrak{m}} \mathfrak{m}$. Moreover, there exists finitely many maximal ideals containing $\mathfrak{a}$ and they are pairwise coprime. In particular their intersection coincides with the product. We invoke Lemma 68. □

Lemma 70. Let A be a local Gorenstein ring with maximal ideal $\mathfrak{m}$ and residue field k and let D be the dimension of A. Then $\mathrm{H}^j_{\mathfrak{m}}(A) = 0$ for $j \neq D$ and $\mathrm{H}^D_{\mathfrak{m}}(A)$ is an injective envelope of k.

Proof. See [60, Theorem 11.26]. □

A.3 Čech complex

Now let $\mathbf{t} = (t_1, \dots, t_r)$ be a sequence of elements of A and let M be an A-module. We define in terms of its localizations

$$\check{\mathrm{C}}^0(\mathbf{t}, M) = M, \qquad \check{\mathrm{C}}^p(\mathbf{t}, M) = \bigoplus_{1 \le i_1 < \cdots < i_p \le r} M_{t_{i_1} \dots t_{i_p}}. \tag{A.2}$$

If $\varphi \in \check{\mathrm{C}}^p(\mathbf{t}, M)$, we let $\varphi_{i_1 \dots i_p}$ be its component in $M_{t_{i_1} \dots t_{i_p}}$ for every sequence $1 \le i_1 < \cdots < i_p \le r$. A differential $\delta : \check{\mathrm{C}}^p(\mathbf{t}, M) \to \check{\mathrm{C}}^{p+1}(\mathbf{t}, M)$ is defined by

$$(\delta\varphi)_{i_0 \dots i_p} = \sum_{j=0}^{p} (-1)^j \varphi_{i_0 \dots i_{j-1} i_{j+1} \dots i_p}, \tag{A.3}$$

in which $\varphi_{i_0 \dots i_{j-1} i_{j+1} \dots i_p}$ is implicitly mapped from $M_{t_{i_0} \dots t_{i_{j-1}} t_{i_{j+1}} \dots t_{i_p}}$ to $M_{t_{i_0} \dots t_{i_p}}$ by the localization homomorphism. This makes $\check{\mathrm{C}}^\bullet(\mathbf{t}, M)$ a cochain complex. Its cohomology is denoted by $\check{\mathrm{H}}^\bullet(\mathbf{t}, M)$ and called Čech cohomology.

Lemma 71. One has $\check{\mathrm{H}}^\bullet(\mathbf{t}, M) \cong \mathrm{H}^\bullet_{\mathfrak{a}}(M)$, where $\mathfrak{a} = (t_1, \dots, t_r)$.

Proof. See [60, Theorem 7.13]. □

Lemma 72. Let U be a multiplicatively closed subset of A, $A' = U^{-1}A$ and let $\mathfrak{a}'$ be the extension of $\mathfrak{a}$ in A'. Then for any A-module M we have $\mathrm{H}^j_{\mathfrak{a}'}(U^{-1}M) \cong U^{-1}\mathrm{H}^j_{\mathfrak{a}}(M)$.

Proof. Follows from Lemma 71 because the corresponding property of Čech cohomology is easy to verify. □

Clearly we have $\check{C}^\bullet(\mathbf{t}, M) = \check{C}^\bullet(\mathbf{t}, A) \otimes_A M$. Since modules $\check{C}^p(\mathbf{t}, A)$ are flat, this implies that $\check{C}^\bullet(\mathbf{t}, -)$ takes short exact sequences of modules to short exact sequences of complexes. Hence every short exact sequence of modules induces a long exact sequence in Čech cohomology.

Let $\varphi \in \check{C}^{p+1}(\mathbf{t}, M)$, $\psi \in \check{C}^{q+1}(\mathbf{t}, N)$ with $p, q \geq 0$. We define the cup product $\varphi \smile \psi \in \check{C}^{p+q+1}(\mathbf{t}, M \otimes_A N)$ by

$$(\varphi \smile \psi)_{i_0 \dots i_{p+q}} = \varphi_{i_0 \dots i_p} \otimes \psi_{i_p \dots i_{p+q}}. \tag{A.4}$$

It is associative and satisfies the graded Leibniz rule

$$\delta(\varphi \smile \psi) = \delta\varphi \smile \psi + (-1)^p \varphi \smile \delta\psi, \tag{A.5}$$

hence induces a product $\check{H}^{p+1}(\mathbf{t}, M) \otimes_A \check{H}^{q+1}(\mathbf{t}, N) \to \check{H}^{p+q+1}(\mathbf{t}, M \otimes_A N)$.

Let $\tau : N \otimes_A M \to M \otimes_A N$ be the standard isomorphism. For brevity we denote induced maps of Čech complexes and in Čech cohomology with the same symbol. Mimicking formulas in [61] we define products

$$\smile_1 : \check{C}^{p+1}(\mathbf{t}, M) \otimes_A \check{C}^{q+1}(\mathbf{t}, N) \to \check{C}^{p+q}(\mathbf{t}, M \otimes_A N), \tag{A.6}$$

$$(\varphi \smile_1 \psi)_{i_0 \dots i_{p+q-1}} = \sum_{j=0}^{p-1} (-1)^{(p-j)(q+1)} \varphi_{i_0 \dots i_j i_{j+q} \dots i_{p+q-1}} \otimes \psi_{i_j \dots i_{j+q}}.$$

They satisfy the following identity:

$$\begin{aligned} &\varphi \smile \psi - (-1)^{pq} \tau(\psi \smile \varphi) \\ =&(-1)^{p+q+1} \left[\delta(\varphi \smile_1 \psi) - \delta\varphi \smile_1 \psi - (-1)^p \varphi \smile_1 \delta\psi\right]. \end{aligned} \tag{A.7}$$

If φ, ψ are cocycles and $[\varphi], [\psi]$ are their cohomology classes, this gives

$$[\varphi] \smile [\psi] = (-1)^{pq} \tau\left([\psi] \smile [\varphi]\right). \tag{A.8}$$

In this sense the cup product is graded commutative.

Remark 73. The Čech complex and the cup product depend on the ordering of elements t_i however, cohomologies (and the cup products) do not. We refer for example to [Stacks, Tag 01FG], and discussion in [61].

Appendix B

TECHNICAL RESULTS ON SPIN CHAIN

B.1 Stable equivalence of G-invariant factorized pure states

In this section we study G-invariant factorized pure states defined by (3.7) where the vectors v_j may transform in non-trivial one-dimensional representations of G. We will show that all such states are in the trivial G-invariant phase.

Consider first a 1d system where for $j \neq 0$ $\mathscr{V}_j = \mathbb{C}$ is the trivial representation of G, while $\mathscr{V}_0 = \mathscr{W}$ is a finite-dimensional representation containing a unit vector w transforming in a non-trivial one-dimensional representation of G. Consider a G-invariant factorized pure state where $v_0 = w$ (all other v_j are unique up to a scalar multiple). Physically, this corresponds to a non-trivial G-invariant ground state of a 0d system regarded as a G-invariant state of a 1d system. We are going to show that this 1d state is in a trivial G-invariant stable phase.

Without loss of generality we may assume that $\mathscr{W}$ contains a G-invariant vector w'. Indeed, if this is not the case, we can tensor the above system with a similar system where $\mathscr{W}$ is replaced with $\mathscr{W}^* \oplus \mathbb{C} \cdot e$, where G acts on the second summand by the trivial representation, and $v_0 = e$. The auxiliary system is in the trivial G-invariant phase, so this does not affect the G-invariant phase of the system we are interested in. Then the composite system has $\mathscr{V}_0 = \mathscr{W} \otimes \mathscr{W}^* \oplus \mathscr{W} \otimes \mathbb{C} \cdot e$, with a G-invariant factorized pure state corresponding to v_0 in the second summand. The first summand now contains a G-invariant vector $w \otimes w^*$.

Consider now an auxiliary system $\mathscr{A}'$ where $\mathscr{V}'_j = \mathbb{C}$ is the trivial representation for $j \leq 0$ and $\mathscr{V}'_j = \mathscr{W}^* \otimes \mathscr{W}$ for $j > 0$. The algebra of local observables $\mathscr{A} \otimes \mathscr{A}'$ has the form

$$\operatorname{End}(\mathscr{W}) \otimes \operatorname{End}(\mathscr{W}^* \otimes \mathscr{W}) \otimes \operatorname{End}(\mathscr{W}^* \otimes \mathscr{W}) \otimes \dots \tag{B.1}$$

We pick a G-invariant factorized pure state ψ' on $\mathscr{A}'$ defined by the condition that for any $A \in \mathscr{A}'_j = \operatorname{End}(\mathscr{V}'_j)$, $j > 0$, it is a vector state corresponding to $w^* \otimes w$ which is G-invariant. Consider now the state $\psi \otimes \psi'$ on $\mathscr{A} \otimes \mathscr{A}'$. As G acts trivially on the vector state in $\mathscr{A}'_j$ for all j, the composite state belongs to the same G-invariant phase as $(\psi, \mathscr{A})$.

After re-writing the algebra of observables as

$$\mathrm{End}(\mathscr{W} \otimes \mathscr{W}^*) \otimes \mathrm{End}(\mathscr{W} \otimes \mathscr{W}^*) \otimes \dots \tag{B.2}$$

it is easy to see that the state $\psi \otimes \psi'$ is related by a G-equivariant LGA to a special factorized pure state. Indeed, since $\mathscr{W}$ contains a G-invariant vector w', $\mathscr{W} \otimes \mathscr{W}^*$ contains a G-invariant 2-plane spanned by vectors $w \otimes w^*$ and $w' \otimes w'^*$. Let U be a unitary operator on $\mathscr{W} \otimes \mathscr{W}^*$ which acts by identity on the orthogonal complement of this plane and rotates $w \otimes w^*$ into $w' \otimes w'^*$. Consider a local unitary circuit on $\mathscr{A} \otimes \mathscr{A}'$ which acts on $\mathscr{A} \otimes \mathscr{A}'$ by conjugation with $U \otimes U \otimes U \otimes \dots$. It maps $\psi \otimes \psi'$ to a factorized pure state on $\mathscr{A} \otimes \mathscr{A}'$ with $v_0 = w'$ and $v_j = w'^* \otimes w'$ for $j > 0$. All these vectors are G-invariant. Thus ψ is a special G-invariant factorized pure state.[1]

In general, we can write the algebra $\mathscr{A}$ as a tensor product of sub-algebras $\mathscr{A}_{\geq 0}$ and $\mathscr{A}_{<0}$ corresponding to $j \geq 0$ and $j < 0$. Since the state ψ on $\mathscr{A}$ is assumed factorized, it is sufficient to consider the restriction of ψ to one of these sub-algebras, say $\mathscr{A}_{\geq 0}$. Then we apply the argument of the above paragraph to each of the factors $\mathscr{A}_j$ separately. Note that the auxiliary system $\mathscr{A}'$ in this case has $\log d'_j$ growing with j even if the dimension d_j of $\mathscr{A}_j$ is bounded. However, it is easy to see that if $\log d_j$ grows at most as a power of j, then so does $\log d'_j$. Thus it is still true that ψ is in the trivial G-invariant phase.

B.2 Local computability of the index

An important property of the index is its local computability, i.e., that one can compute $\nu(g,h)$ up to $O(L^{-\infty})$ accuracy while having access only to a disk of radius L. Let us fix a disk $\Gamma_L = [-L, L]$, and let R^g_L be a unitary $\prod_{j\in\Gamma_L} R_j(g)$. By Theorem 54 for any invertible state ψ there is a pure factorized state ψ'_0 on $\mathscr{A}'$ (with a trivial G action) such that $\psi \otimes \psi'_0$ can be produced from a pure factorized state Ω on $\mathscr{A} \otimes \mathscr{A}'$ by some LGA α_F for an f-local F for some MDP function $f(r) = O(r^{-\infty})$. In what follows we redefine ψ to be the SRE state $\psi \otimes \psi'_0$.

First, note that by Lemma 3.4.3 and Corollary 3.4.1.2 we have

$$U^g_{>j}|\psi\rangle = \Pi(\mathcal{V}^g_{>j})|\psi\rangle, \tag{B.3}$$

for some observable $\mathcal{V}^g_{>j}$ which is g-localized at j for some function g that depends on f only. We can find an observable $\mathcal{V}^g_L$ local on Γ_L such that $\|\mathcal{V}^g_{>j} - \mathcal{V}^g_L\| = O(L^{-\infty})$.

[1] This argument is a version of the Eilenberg swindle.

The index can be computed as

$$\nu(g,h) = \langle\psi|(U^{gh}_{>j})^{-1}U^{g}_{>j}U^{h}_{>j}|\psi\rangle = \langle\psi|\Pi(\mathcal{V}^{gh}_{>j})^{-1}U^{g}_{>j}\Pi(\mathcal{V}^{h}_{>j})|\psi\rangle$$
$$= \langle\psi|\Pi(\mathcal{V}^{gh}_{>j})^{-1}\Pi(\rho^{g}_{>j}(\mathcal{V}^{h}_{>j}))\Pi(\mathcal{V}^{g}_{>j})|\psi\rangle = \psi\left((\mathcal{V}^{gh}_{>j})^{-1}\rho^{g}_{>j}(\mathcal{V}^{h}_{>j})\mathcal{V}^{g}_{>j}\right). \quad \text{(B.4)}$$

Therefore

$$\nu(g,h) = \psi\left((\mathcal{V}^{gh}_{L})^{-1}R^{g}_{L}\mathcal{V}^{h}_{L}(R^{g}_{L})^{-1}\mathcal{V}^{g}_{L}\right) + O(L^{-\infty}). \quad \text{(B.5)}$$

Note that the $O(L^{-\infty})$ term depends on $f(r)$ only, and by taking L large enough we can compute the index with any given accuracy. Therefore if we have an interpolating G-invariant invertible state ψ which is f-close to a G-invariant invertible state ψ_1 on the left half-chain and f-close to a G-invariant invertible state ψ_2 on the right half-chain, then the indices of ψ_1 and ψ_2 must be the same. In particular, a non-trivial index for ψ_1 is an obstruction for the existence of such an interpolation between ψ_1 and a pure factorized state ψ_2.

B.3 Multiplicative Lieb-Robinson bound

Lemma B.3.1. Let $\mathcal{A}_i, i = 0, 1, 2, \ldots$ be quasi-local observables such that $\sum_i \|\mathcal{A}_i - 1\|$ converges. Then the sequence of observables $C_n = \prod_{i=1}^{n} \mathcal{A}_i$ is norm-convergent.

Proof. Let $C_n = \prod_{i=0}^{n} \mathcal{A}_i$ and $\mathcal{B}_i = \mathcal{A}_{i+1} - 1$. Then

$$C_n = \sum_{i=0}^{n-1} C_i\mathcal{B}_i + C_0. \quad \text{(B.6)}$$

This implies $\|C_n\| \leq \sum_{i=0}^{n-1} \|\mathcal{B}_i\|\|C_i\| + \|C_0\|$, which by the discrete Gronwall inequality [62] implies

$$\|C_n\| \leq \|C_0\| \exp\left(\sum_{i=0}^{n-1} \|\mathcal{B}_i\|\right) \leq \|C_0\| \exp\left(\sum_{i=0}^{\infty} \|\mathcal{B}_i\|\right) < \infty.$$

Thus the right-hand side of Equation (B.6) converges in norm as $n \to \infty$. □

Corollary B.3.1.1. Let $\mathcal{V}_k$, $k = 0, 1, 2, \ldots$ be local unitaries localized on $[(-k - 1/2)L, (k+1/2)L]$ and satisfying $\|\mathcal{V}_k - 1\| \leq h((k+1/2)L)$ for some MDP function $h(r) = O(r^{-\infty})$. Then product $\mathcal{V}_0\mathcal{V}_1\mathcal{V}_2\ldots$ exists. Furthermore, it is f-local at 0 for some MDP function $f(r) = O(r^{-\infty})$ determined by h.

Proof. As $h(r) = O(r^{-\infty})$ and $\|\mathcal{V}_k - 1\| \leq h((k+1/2)L)$, $\sum_k \|\mathcal{V}_k - 1\| \leq \sum_k h((k+1/2)L)$ converges. Therefore, the product $C_\infty = \mathcal{V}_0\mathcal{V}_1\mathcal{V}_2\ldots$ exists by the lemma above. Let $C_n = \mathcal{V}_0\mathcal{V}_1\mathcal{V}_2...\mathcal{V}_n$ and $\mathcal{B}_i = \mathcal{V}_{i+1} - I$. Then

$$C_n = \sum_{i=0}^{n-1} C_i\mathcal{B}_i + C_0. \tag{B.7}$$

For any $\mathcal{A} \in \mathscr{A}_j$,

$$\begin{aligned} \|[C_\infty, \mathcal{A}]\| = \|[\sum_{i=0}^{\infty} C_i\mathcal{B}_i + C_0, \mathcal{A}]\| &\leq \sum_{(i+\frac{1}{2})L>j} \|[C_i\mathcal{B}_i, \mathcal{A}]\| \\ &\leq 2\|\mathcal{A}\| \sum_{(i+\frac{1}{2})L>j} \|\mathcal{B}_i\| \leq 2\|\mathcal{A}\| \sum_{(i+\frac{1}{2})L>j} h((i+1/2)L). \end{aligned} \tag{B.8}$$

Thus we may let $f(r) = \sum_{s>r} h(s) = O(r^{-\infty})$. □

In what follows for any sequence of automorphisms α_n, $n \in \mathbb{Z}$, we let $\overrightarrow{\prod_n}\alpha_n$ be the formal expression $\ldots \circ \alpha_{-1} \circ \alpha_0 \circ \alpha_1 \circ \ldots$. Similarly, we denote by $\overleftarrow{\prod_n}\alpha_n$ the formal expression $\ldots \circ \alpha_1 \circ \alpha_0 \circ \alpha_{-1} \circ \ldots$. These expressions are well-defined automorphisms if all but a finite number of α_n are identities. The following lemma describes a class of situations when the formal expressions make sense even if an infinite number of α_n are nontrivial.

Lemma B.3.2. Let $\Lambda = \mathbb{Z} \subset \mathbb{R}$. For any MDP function $f(r) = O(r^{-\infty})$ there is $L \in \mathbb{N}$ such that any ordered composition $\overrightarrow{\prod_{n=-\infty}^{\infty}}\alpha_{\mathcal{B}^{(n)}}$ of LGAs generated by f-local at $j = nL$ observables $\mathcal{B}^{(n)}$ for $n \in \mathbb{Z}$ is an LGA.

Proof. First, note that it is enough to show this for $\overrightarrow{\prod_{n=0}^{\infty}}\alpha_{\mathcal{B}^{(n)}}$. Second, to prove the latter it is enough to show that with an appropriate choice of L for any f-local at 0 observable $\mathcal{A}$ the observable $(\overleftarrow{\prod_{n=1}^{N}}\alpha_{\mathcal{B}^{(n)}})(\mathcal{A})$ is almost local at 0 with localization depending on f only (in particular, independent of N), and that as $N \to \infty$ it converges in norm to some element of $\mathscr{A}$.

Let $\mathcal{U}^{(n)} := e^{i\mathcal{B}^{(n)}}$ be a unitary that corresponds to $\alpha_{\mathcal{B}^{(n)}}$. It can be represented as a product $(\mathcal{V}_0^{(n)}\mathcal{V}_1^{(n)}\mathcal{V}_2^{(n)}...)$ of strictly local unitaries $\mathcal{V}_k^{(n)}$ on $B_n((k+\frac{1}{2})L) := [(n-k-\frac{1}{2})L, (n+k+\frac{1}{2})L]$, so that $\|\mathcal{V}_k^{(n)} - 1\| \leq h((k+\frac{1}{2})L)$ for some MDP function $h(r) = O(r^{-\infty})$ that depends on $f(r)$ only. This is achieved by letting $(\mathcal{V}_0^{(n)}\mathcal{V}_1^{(n)}...\mathcal{V}_k^{(n)}) = e^{i\mathcal{B}|_{B_n((k+\frac{1}{2})L)}}$.

Since conjugation of a strictly local observable $\mathcal{A}$ with a unitary $\mathcal{U}$ strictly local in the localization set of $\mathcal{A}$ does not change the property $\|\mathcal{A} - 1\| < \varepsilon$ and preserves the localization set, we can rearrange unitaries $\mathcal{V}_k^{(n)}$ in the product

$$(\mathcal{V}_0^{(1)}\mathcal{V}_1^{(1)}\mathcal{V}_2^{(1)}...)(\mathcal{V}_0^{(2)}\mathcal{V}_1^{(2)}\mathcal{V}_2^{(2)}...)...(\mathcal{V}_0^{(N)}\mathcal{V}_1^{(N)}\mathcal{V}_2^{(N)}...) \tag{B.9}$$

in the following order:

$$(\tilde{\mathcal{V}}_0^{(1)})(\tilde{\mathcal{V}}_0^{(2)}\tilde{\mathcal{V}}_1^{(1)})(\tilde{\mathcal{V}}_0^{(3)}\tilde{\mathcal{V}}_1^{(2)}\tilde{\mathcal{V}}_2^{(1)})...(\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)})..., \tag{B.10}$$

where $\tilde{\mathcal{V}}_k^{(n)}$ is obtained from $\mathcal{V}_k^{(n)}$ by conjugation with $\mathcal{V}_l^{(m)}$ with m, l satisfying $n + 1 \le m \le n + k$ and $0 \le l \le n + k - m$. Importantly, $\tilde{\mathcal{V}}_k^{(n)}$ is strictly local on the same interval as $\mathcal{V}_k^{(n)}$ and still satisfies $\|\tilde{\mathcal{V}}_k^{(n)} - 1\| \le h((k + \frac{1}{2})L)$. The infinite product Equation (B.10) is a well-defined almost local observable by Corollary B.3.1.1. Indeed, $\|\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)} - 1\| \le \sum_{i=1}^n \|\tilde{\mathcal{V}}_{n-i}^{(i)} - 1\|$ by repeatedly applying the inequality $\|\mathcal{A}\mathcal{B} - 1 - \mathcal{A} + \mathcal{A}\| \le \|\mathcal{A}\|\|\mathcal{B} - 1\| + \|\mathcal{A} - 1\|$. Since for any fixed N we have $\mathcal{V}_k^{(n)} = 1$ for $n > N$, $\|\tilde{\mathcal{V}}_0^{(n)}\tilde{\mathcal{V}}_1^{(n-1)}...\tilde{\mathcal{V}}_{n-1}^{(1)} - 1\| \le \sum_{i=1}^N \|\tilde{\mathcal{V}}_{n-i}^{(i)} - 1\| \le \sum_{i=1}^N h((n - i + \frac{1}{2})L)$ satisfies the assumption of Corollary B.3.1.1. After this rearrangement the infinite product Equation (B.10) still converges to the same unitary observable as Equation (B.9).

Let $\tilde{\mathcal{U}}^{(n)} = \tilde{\mathcal{V}}_0^{(n)}...\tilde{\mathcal{V}}_{n-1}^{(1)}$. We can represent $\mathcal{A} = \sum_{p=0}^{\infty} \mathcal{A}_p$ with $\sum_{p=0}^n \mathcal{A}_p = \mathcal{A}|_{B_0(n+1/2)}$. Let $\mathcal{A}_p^{(0)} := \mathcal{A}_p$, $\mathcal{A}_p^{(n)} := \sum_{k=0}^{n-1} \tilde{\mathcal{U}}^{(n)*}[\mathcal{A}_p^{(k)}, \tilde{\mathcal{U}}^{(n)}]$. Note that $\mathcal{A}_p^{(n)} \in \mathscr{A}|_{B_0(p+\frac{1}{2})}$ for $n \le p$, and $\mathcal{A}_p^{(n)} \in \mathscr{A}|_{B_0(n+\frac{1}{2})}$ for $n > p$. Therefore we have

$$\begin{aligned}\|\mathcal{A}_0^{(n)}\| = \sum_{k=0}^{n-1} \|[\mathcal{A}_0^{(k)}, \tilde{\mathcal{U}}^{(n)}]\| &\le \sum_{k=0}^{n-1} \sum_{l > \frac{n-k-1}{2}}^{n-1} \|[\mathcal{A}_0^{(k)}, \tilde{\mathcal{V}}_l^{(n-l)}]\| \le \\ &\le \sum_{k=0}^{n-1} \sum_{l > \frac{n-k-1}{2}}^{\infty} 2\|\mathcal{A}_0^{(k)}\| h((l + \frac{1}{2})L) \le 2 \sum_{k=0}^{n-1} \|\mathcal{A}_0^{(k)}\| g_{n-k}, \end{aligned} \tag{B.11}$$

where $g_n := g(nL/2)$ for $g(n) := \sum_{l \ge n}^{\infty} h(l)$.

Any MDP function $g(r) = O(r^{-\infty})$ can be upper-bounded by a reproducing MDP function $\tilde{g} = O(r^{-\infty})$ for lattice $\Lambda \subset \mathbb{R}$ [63], i.e., an $O(r^{-\infty})$ MDP function satisfying

$$\sup_{j,k \in \Lambda} \sum_{l \in \Lambda} \frac{\tilde{g}(|j - l|)\tilde{g}(|l - k|)}{\tilde{g}(|j - k|)} < \infty. \tag{B.12}$$

We can further upper-bound $\tilde{g}(r)$ by a reproducing $O(r^{-\infty})$ MDP function $g'(r) = A\tilde{g}(r)^{\alpha}/r^{\nu} = O(r^{-\infty})$ for some constants A, $0 < \alpha < 1$ and $\nu > d$. Since $1/r^{\nu}$ is also reproducing, we have

$$A^2 \sum_{k=1}^{n-1} \frac{\tilde{g}(kL/2)^{\alpha} \tilde{g}((n-k)L/2)^{\alpha}}{(kL/2)^{\nu}((n-k)L/2)^{\nu}} < \frac{C}{L^{\nu}} A \frac{\tilde{g}(nL/2)^{\alpha}}{(nL/2)^{\nu}}, \tag{B.13}$$

and therefore for $g'_n := g'(nL/2) \geq g_n$ we have $\sum_{k=1}^{n-1} g'_k g'_{n-k} < (C/L^{\nu}) g'_n$ for some constant C. For L sufficiently large, we have $(C/L^{\nu}) < 1/2$. Therefore, for such L after setting $a_n = 2ng'_n$ we get

$$\begin{aligned} &2(g_n \cdot 1 + g_{n-1}a_1 + g_{n-2}a_2 + \ldots + g_1 a_{n-1}) \leq \\ &\leq 2(g'_n \cdot 1 + 2g'_{n-1}g'_1 + 4g'_{n-2}g'_2 + \ldots + 2(n-1)g'_1 g'_{n-1}) \leq 2ng'_n = a_n. \end{aligned} \tag{B.14}$$

Together with Equation (B.11) this implies that $\|\mathcal{A}_0^{(n)}\|/\|\mathcal{A}_0\|$ can be upper-bounded by $a_n = O(n^{-\infty})$, and the sequence $\sum_{k=0}^{n} \mathcal{A}_0^{(k)}$ converges in norm to some almost local observable. By construction, it is h-localized at 0 for some MDP function $h(r) = O(r^{-\infty})$ which depends on f only.

In the same way one can estimate the norms of $\mathcal{A}_p^{(n+p)}$ for $n, p > 0$ and bound $\|\mathcal{A}_p^{(n+p)}\|/\|\mathcal{A}_p\|$ by a sequence $a_n = O(n^{-\infty})$. Together with $\|\mathcal{A}_p\| = \|\sum_{q=0}^{p} \mathcal{A}_p^{(q)}\|$ that ensures convergence of $\sum_{p=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{A}_p^{(n)}$ to some almost local at 0 observable whose localization depends on f only.

□