On the Categorical Approach to the Frobenius Trace

Thesis by
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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2023
Defended May 19, 2023

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ACKNOWLEDGEMENTS

I feel very fortunate and grateful to be in this position. I owe my PhD work, time, and process to many people.

First, I would like to thank my advisor, Xinwen Zhu. For both his guidance and collaboration. He has taught me how to identify important content from noise, how to tackle unknown problems, and how to think about mathematics. The seminars and special topics classes organized by Prof. Zhu helped me find inspiration for research questions and help start our joint project on the unipotent categorical Langlands correspondence.

I want to thank my collaborators Timo Richarz and Jakob Scholbach. For their time, intellectual generosity, inspiration, and companionship.

I would also like to thank all my colleagues in the math department and the PMA division who made these years so much more enjoyable.

Finally, I want to thank my family and friends who made it all possible. To my parents, whose love and support I can't thank enough, and to my partner, Naomi, who was a guiding light during difficult times and whose presence makes every good moment infinitely better.

ABSTRACT

Motivated by the study of the local and global Langlands correspondence from a geometric prespective, we establish two results of a general nature regarding categories of sheaves in algebraic geometry. The first result, motivated by the work of Drinfeld and Lafforgue on the Langlsnds correspondence over function fields, establishes a categorical enhancement of the Künneth formula for categories of Weil sheaves, generalizing a famous result of Drinfeld. In the second part, motivated by the geometric approach to the study of representations of reductive groups over local fields, we develop a method to calculate the categorical trace of monoidal categories arising from convolution pattern in algebraic geometry.

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Part I

Introduction

Chapter 1

OVERVIEW

This thesis can be divided into two parts which are mathematically independent but philosophically related. In this part, we give an informal introduction to the motivations and results. The guiding question of this thesis comes from the Langlands program and specifically the geometric approach to it.

Let F be a global field, such as a number field or the field of rational functions on a projective curve over a finite field. The Langlands program (Langlands, 1970) consists of many results and conjectures establishing a deep connection between representation theory of analytic nature with arithmetic properties of the field F. Let's recall the main protagonist of this program[1], the space of *automorphic functions*.

For every place v of F we denote by F_v the v-adic completion of F at v. We will denote by $\mathcal{O}_F$ the ring of integers of F and for every non-archimedean place v of F, we will denote by $\mathcal{O}_v$ the ring of integers of F_v. For every finite set S of places of F which contains all the archimedean places, we denote

$$\mathbb{A}_{F,S} = \prod_{v \in S} F_v \times \prod_{v \notin S} \mathcal{O}_v$$

considered as a topological ring with the product topology. We denote by $\mathbb{A}_F$ the *adele ring* of F. Recall that it can be described, as a topological ring, as the direct limit

$$\mathbb{A}_F = \varinjlim_S \mathbb{A}_{F,S}.$$

This is sometimes referred to as the "restricted product" of the fields F_v.

Let G be a connected reductive group over F, i.e. SL_n, GL_n, Sp_{2n}. We can now define the space of auytomnorphic functions:

$$\mathcal{A}_G^2 = L^2(G(F)\backslash G(\mathbb{A}_F)), \mathbb{C})$$

namely, it is the space of functions that are square-integrable with respect to some invariant measure on the coset space $G(F)\backslash G(\mathbb{A}_F)$. This space of functions carries an action of $G(\mathbb{A}_F)$ by the left action on the coset space. It is sometimes easier to

[1]From the point of view of representation theorists.

focus on specific subspaces of the automorphic space. For any compact subgroup $K \subseteq G(\mathbb{A}_F)$ we can consider the space of automorphic functions of level K:

$$\mathcal{A}^2_{G,K} = L^2(G(F)\backslash G(\mathbb{A}_F)/K), \mathbb{C})$$

Example 1.0.1. *Take $F = \mathbb{Q}$ and $G = \mathrm{SL}_2$ and $K = \prod_p \mathrm{SL}_2(\mathbb{Z}_p)$. We can relate the space of automorphic functions in the adelic description to a more familiar object. By the strong approximation theorem (Kneser, 1966), for element $g \in \mathrm{SL}_2(\mathbb{A}_\mathbb{Q})$ represented as elements $(g_v)_v$ where $g_p \in \mathrm{SL}_2(\mathbb{Q}_p)$ for prime p and $g_\infty \in \mathrm{SL}_2(\mathbb{R})$, there exists a matrix $\gamma \in \mathrm{SL}_2(\mathbb{Q})$ such that for γg we have $(\gamma g)_p \in \mathrm{SL}_2(\mathbb{Z}_p)$ for all prime p. This means that we can identify the double coset spaces*

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})/K \cong (\mathrm{SL}_2(\mathbb{Q}) \cap K)\backslash\mathrm{SL}_2(\mathbb{R})$$

Finally, the intersection $\mathrm{SL}_2(\mathbb{Q}) \cap K$ consists of matrices whose entries are rational numbers that are integral for any prime p, which is the same as being integers. This gives an identification:

$$\mathrm{SL}_2(\mathbb{Q})\backslash\mathrm{SL}_2(\mathbb{A}_\mathbb{Q})/K \cong \mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$$

relating the automorphic functions in the adelic description to the more familiar notion.

An important way to study representations in general is to find symmetries of them in the form of commuting linear operators. An important family of such operators are the Hecke operators. For every non-archimidean place v of F we consider the spherical Hecke algebra:

$$\mathrm{H}_v = C_c(G(\mathcal{O}_v)\backslash G(F_v)/G(\mathcal{O}_v), \mathbb{C}).$$

This algebra is commutative and acts on the space of automorphic functions by convolution:

$$h * f(g) = \int_{G(F_v)} f(gs^{-1})h(s)\,\mathrm{d}s$$

In the revolutionary seminal work (V. Lafforgue, 2018), Vincent Lafforgue showed how to expand this algebra of commuting operators in the function field case by introducing the *excursion operators*. In the next section, we will try to give a sense of the structures involved and the motivations for the first part in the next chapter. Finally, we will mainly focus on results that come from the algebraic-geometric approach to stufying the space of automorphic functions. In that case, we focus our attention on the subset of continuous compactly supported automorphic functions:

$$\mathcal{A}_G = C_c(G(F)\backslash G(\mathbb{A}_F)), \mathbb{C})$$

The second part of the thesis concerns the local side of the Langlands program. The main question considered is understanding representations of $G(F_v)$ for all the places v. These are significant for the study of local fields but can also help understand the global picture. For example, Every irreducible continuous representation π of the adele group $G(\mathbb{A}_F)$ is equivalent to a "restricted product"

$$\pi = \bigotimes_v{}' \pi_v,$$

but we will not go into this more deeply here. In the second part of this overview, we will review the geometric and categorical structures surrounding Hecke, and Iwahori-Hecke algebras, and the categorical trace, which motivates the second part of this thesis.

Chapter 2

DRINFELD'S LEMMA AND THE GLOBAL FUNCTION FIELD LANLGANDS CORRESPONDENCE

2.1 The Langlands Correspondence over function fields

Let X be a projective curve over a finite field $\mathbb{F}_q$ with function field F. Let G be a split reductive group defined over F For simplicity, we will assume in this section that the center of G is trivial and we will focus our attention to the *unramified* part of the space of automorphic functions. Namely, we consider the subspace of functions

$$C_c(G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}), \overline{\mathbb{Q}_\ell}) \subseteq C_c(G(F)\backslash G(\mathbb{A}_F, \overline{\mathbb{Q}_\ell})$$

where $\mathcal{O} = \prod_{x \in X} \mathcal{O}_{X,x}$. These are the functions which invariant under the action of a maximal compact subgroup corresponding to the integral points on the function field. This double coset space is a discrete groupoid.

The study of the function field version of the Langlands correspondence enjoys a rich connection to algebraic geometry thanks to the existance of an algebraic-geometric description of the double cosets $G(F)\backslash G(\mathbb{A}_F/G(\mathcal{O})$. Namely, it can be identified with the set of rational point of an algebraic stack, the stack of G-bundles on the curve. We will describe geometric objects such as schemes and stacks via their functor of points (Stacks, Tag 01J5). For convenience, for every scheme Y over $\mathbb{F}_q$ and an bF_q algebra R, we will denote by Y_R the fiber product $Y \times_{\operatorname{Spec} \mathbb{F}_q} \operatorname{Spec} R$.

Let $\operatorname{Bun}_G(X)$ denote the moduli stack of G-bundles on the curve X. This is the stack described by the following moduli problem: for a commutative $\mathbb{F}_q$-algebra R, the groupoid of R-points of $\operatorname{Bun}_G(X)$ is given by the groupoid

$$\operatorname{Bun}_G(X)(R) = \{\mathcal{E} \text{ a } G\text{-torsor on } X_R\}.$$

The connection to the automorphic space comes from the existance of a natural isomorphism of groupoids

$$\operatorname{Bun}_G(X)(\mathbb{F}_q) \simeq G(F)\backslash G(\mathbb{A}_F)/G(\mathcal{O}) \tag{2.1}$$

this is known as *adelic uniformization*. This equivalence is based on the fact that any G-torsor can be trivialized outside a finite set of points. For example, let's fix a set S of points $x_1, \ldots, x_n in|X|$. By a fundamental result of Beauville and

Laszlo(Beauville and Laszlo, 1995), to describe a G-bundle which is trivial on $X \setminus S$, it is enough to specify the "gluing data" consisting of modifications of the trivial bundle around each of the points $x_1, \ldots, x_n$. More explicitly, for every point x_i we can consider the "disc" and "punctured disc"

$$\mathrm{D}_i = \operatorname{Spec} \mathcal{O}_{x_i}, \quad \mathrm{D}_i^* = \operatorname{Spec} F_{x_i}$$

A modification of the trivial bundle $\mathcal{E}_0$ around a point x_i is an automorphism of the restriction $\mathcal{E}_0|_{D_i^*}$. These are exactly given by elements $g_i \in G(F_{x_i})$, so any given choice of elements $g_1, \ldots, g_n$ specifies such a bundle. However, some sets of these choices correspond to isomorphic bundles. First, two modifications that differ by an automatism of the restriction $\mathcal{E}_0|_{D_i}$ to the disc give rise the same bundles. So, accounting for these means a choice of an element in the product of coset spaces:

$$\prod_i G(F_{x_i})/G(\mathcal{O}_{X,x_i})$$

Finally, two such choices can give isomorphic bundles if they differ by an automorphism of the trivial bundle restricted to $X \setminus S$. Such automorphisms are parameterized by $G(\mathcal{O}_X[x_1^{-1}, \ldots, x_n^{-1}])$. Thus, the groupoid of G-bundles which are trivial on $X \setminus S$ can be identified with

$$G(\mathcal{O}_X[x_1^{-1}, \ldots, x_n^{-1}]) \backslash \Big(\prod_i G(F_{x_i})/G(\mathcal{O}_{X,x_i}) \Big)$$

taking the direct limit over all finite subsets of $|X|$ results in the description (2.1).

Such a description for the space of unramified automorphic functions comes to use when combined with the function-sheaf dictionary. Namely, this is a map:

$$\mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell}) \to C_c(\mathrm{Bun}_G(X)(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$$

taking a constructible complex $\mathcal{F} \in \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell})$ to the function

$$f_{\mathcal{F}} \colon \mathrm{Bun}_G(X)(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}$$

defined by sending a point $x \in |X|$ to

$$f_{\mathcal{F}}(x) = \sum_i (-1)^i \cdot \mathrm{Tr}(\mathrm{Frob}, \mathrm{H}^i(\mathcal{F}_x, \overline{\mathbb{Q}_\ell})),$$

where $\mathcal{F}_x$ is the stalk of $\mathcal{F}$ at x, considered as a chain complex of $\overline{\mathbb{Q}_\ell}$ vector spaces. Thus, studying sheaves on $\mathrm{Bun}_G(X)$ could help us understand the space of automorphic functions.

As a first attempt, let us try to follow the classical way automorphic function spaces are studied and try to port that into the geometric setting. An important symmetry of the space of unramified automorphic functions is the action of Hecke algebras. These can be understood in a geometric way. For every $x \in |X|$ we can parameterize the modifications of a bundle at the point x. Namely, consider the following moduli problem

$$\mathrm{Hecke}_{G,x}(R) = \left\{(\mathcal{E}_0, \mathcal{E}_1, \alpha) | \mathcal{E}_0, \mathcal{E}_1 \in \mathrm{Bun}_G(X)(R),\ \alpha\colon \mathcal{E}_0|_{X_R \setminus \{x_R\}} \simeq \mathcal{E}_1|_{X_R \setminus \{x_R\}}\right\}$$

where x_R is the subset $\operatorname{Spec} R \xrightarrow{x} X_R$. The functor $\mathrm{Hecke}_{G,x}$ is ind-representable and is related to the *affine Grassmannian*, which parameterized modifications of the trivial bundle, see (Zhu, 2016a). Such a structure gives rise to an integral transform map. Namely, we have a correspondence:

$$\begin{tikzcd} & \mathrm{Hecke}_{G,x} \arrow[dl, "h_l"'] \arrow[dr, "h_r"] & \\ \mathrm{Bun}_G(X) & & \mathrm{Bun}_G(X) \end{tikzcd}$$

given by $h_l(\mathcal{E}_0, \mathcal{E}_1, \alpha) = \mathcal{E}_0$ and $h_r((\mathcal{E}_0, \mathcal{E}_1, \alpha) = \mathcal{E}_1$. We can use this correspondence to get an action of $\mathrm{D}_{\mathrm{ctf}}(\mathrm{Hecke}_{G,x})$ on the category $\mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell})$. Namely, for every $\mathcal{K} \in \mathrm{D}_{\mathrm{ctf}}(\mathrm{Hecke}_{G,x})$ we have a functor:

$$S_{\mathcal{K}}\colon \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell}) \to \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell})$$

given by the "integral transform":

$$S_{\mathcal{K}}(\mathcal{F}) = (h_r)_!(h_l^*(\mathcal{F}) \otimes \mathcal{K}), \quad \mathcal{F} \in \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell})$$

Under the function-sheaf dictionary, $S_{\mathcal{K}}(\mathcal{F})$ is identified with the usual Hecke operator. This presentation connects the Hecke operator at the point (or place) x with a geometric version of it. One may ask how to make sense of the variation of the point x? how can we endode the fact that the Hecke operators depend on the choice of a point?

We can consider a version of the Hecke stack which lets the point vary. Namely, consider the moduli problem:

$$\mathrm{Hecke}_G(R) = \left\{(x, \mathcal{E}_0, \mathcal{E}_1, \alpha) | x \in X_R,\ \alpha\colon \mathcal{E}_0|_{X_R \setminus \{x_R\}} \simeq \mathcal{E}_1|_{X_R \setminus \{x_R\}}\right\}$$

which gives rise to the correspondence:

$$\begin{tikzcd} & \mathrm{Hecke}_G \arrow[dl, "h_l"'] \arrow[dr, "h_r"] & \\ X \times \mathrm{Bun}_G(X) & & \mathrm{Bun}_G(X) \end{tikzcd}$$

Such a structure gives us a richer theory of integral transforms. Namely, for every lisse sheaf $\mathcal{L} \in \mathrm{D}_{\mathrm{lis}}(X, \mathbb{Q}_\ell)$ and $\mathcal{K}$ as before, we get a functor

$$S_{\mathcal{L},\mathcal{K}} \colon \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X), \overline{\mathbb{Q}_\ell}) \to \mathrm{D}_{\mathrm{ctf}}(\mathrm{Bun}_G(X)$$

given by

$$S_{\mathcal{L},\mathcal{K}}(\mathcal{F}) = (h_r)_! \big(h_l^*(\mathcal{L} \boxtimes \mathcal{F}) \otimes \mathcal{K}\big).$$

The connection to arithmetic of the function field comes in through the equivalence

$$\mathrm{D}_{\mathrm{lis}}(X, \mathbb{Q}_\ell) \simeq \mathrm{D}^b(\mathrm{Rep}(\pi_1^{\acute{e}t}(X), \overline{\mathbb{Q}_\ell}).$$

The geometric Lanlgands program, which informed the work of Lafforgue suggests that it would be necessary to consider multi-point modifications. Namely, for every finite set I we consider the moduli of modification of a bundle at I points:

$$\mathrm{Hecke}_{G,I}(R) = \Big\{\big((x_i)_{i\in I}, \mathcal{E}_0, \mathcal{E}_1, \alpha \colon \mathcal{E}_0 \to \mathcal{E}_1\big)\Big| x_i \in X_R,\ \mathcal{E}_0|_{X\setminus\{x_i\}_{i\in I}} \simeq \mathcal{E}_1|_{X\setminus\{x_i\}_{i\in I}}\Big\}$$

which give rise to the compatible family of correspondences:

$$\begin{tikzcd} & \mathrm{Hecke}_I(X) \arrow[dl, "r"'] \arrow[dr, "p"] & \\ X^I \times \mathrm{Bun}_G(X) & & \mathrm{Bun}_G(X) \end{tikzcd}$$

However, unlike the case of $|I| = 1$, there is no way to relate lisse sheaves on X^I with the fundamental group $\pi_1^{\acute{e}t}(X)$. This is because in general, the natural map

$$\pi_1^{\acute{e}t}(X^I) \to \pi_1^{\acute{e}t}(X)^I$$

is not an isomorphism. This issue was solved by Drinfeld in the seminal paper Vladimir Gershonovich Drinfeld, 1980, with the realization that the prospective presented above did not bring into account the Frobenius symmetry. Namely, note that rather than just one Frobenius morphism on X^I, we should take into account the *partial Frobenius morphisms*. For every $i \in I$ we have a map

$$\varphi_i \colon X^I \to X^I$$

given by the Frobenius map on the i-th component and the identity on the other components. These morphism commute and satisfy

$$\varphi_1 \circ \cdots \circ \varphi_n = \mathrm{Frob}_{X^I}$$

where Frob_X is the q-Frobenius on X^I and where n is the size of i.

Drinfeld realized that we should require this symmetry on the representations as well. Namely, instead of considering all representations of $\pi_1^{\acute{e}t}(X^I)$, we should consider the representations V of $\pi_1^{\acute{e}t}(X^I)$ equipped with pair-wise commuting morphisms:

$$\psi_i : V \to V$$

as $\pi_1^{\acute{e}t}(X^I)$-representations, such that the total composition $\psi_1, \ldots, \psi_n$ is equivalent to the action of Frob_{X^I}. Informally, Drinfeld established an equivalence between representations of $\pi_1^{\acute{e}t}(X^I)$ with a partial Frobenius equivariant structure as above and representations of $\pi_1^{\acute{e}t}(X)^I$. Our first main result is a generalization of Drinfeld's lemma to the entire derived category of constructible sheaves.

2.2 Drinfeld's lemma and the Künneth formula for Weil sheaves

To state the partial Frobenius equivariance conditions and our results, it is convenient to shift our prespective and formulate the theory in terms of Weil sheaves and Weil prestacks. We give an informal account here, the full details are in Chapter 6.

We will use the formalism of prestacks and constructible sheaves on them, for a review of these, see Section 7.2. Fix $\mathbb{F} = \overline{\mathbb{F}_q}$. Instead of working with schemes over $\mathbb{F}_q$ and keeping track of the partial and absolute Frobenius maps, we find it preferable to work with geometric object over $\mathbb{F}$. We can do this by associating to a scheme the formal quotient of the base change to the algebraic closure $\mathbb{F}$ by Frobenius. Namely, let Let X be a scheme over $\mathbb{F}_q$ and denote by $X_{\mathbb{F}}$ it's base change to $\mathbb{F}$. Let

$$\phi_X \colon X_{\mathbb{F}} \longrightarrow X_{\mathbb{F}}$$

be defined by $\phi_X = \mathrm{Frob}_X \times \mathrm{id}_{\mathrm{Spec}\,\mathbb{F}}$.

Definition 2.2.1. *Let X be a scheme over $\mathbb{F}_q$. The Weil prestack is defined as*

$$X^{\mathrm{Weil}} := \mathrm{colim}\left(X \times_{\mathbb{F}_q} \mathbb{F} \underset{\mathrm{id}}{\overset{\phi_X}{\rightrightarrows}} X \times_{\mathbb{F}_q} \mathbb{F} \right) \in \mathrm{PreStk}_{\mathbb{F}},$$

i.e., it is the prestack sending $R \in \mathrm{CAlg}_{\mathbb{F}}$ to the colimit

This description works well with products. Namely, for schemes X_1, X_2 over $\mathbb{F}_q$ we have

$$X_1 \times_{\mathbb{F}_q} X_2 \times_{\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F} = X_{1,\mathbb{F}} \times_{\mathbb{F}} X_{2,\mathbb{F}}$$

We can consider the product of the corresponding Weil prestacks:

$$X^{\mathrm{Weil}} = X_1^{\mathrm{Weil}} \times_{\mathbb{F}} X_2^{\mathrm{Weil}}.$$

For every prestack, we have a well defined ∞-category of constructible sheaves. The main result of the first part of the thesis can be formulated as follows:

Theorem 2.2.2. *Let X_1, X_2 be schemes of finite type over $\mathbb{F}_q$. The external product functor $(M_1, M_2) \mapsto M_1 \boxtimes M_2$ induces an equivalence of categories*

$$\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\Lambda} \mathrm{D}_{\mathrm{cons}}(X_2^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}} \times_k \cdots \times_k X_n^{\mathrm{Weil}}, \Lambda\big)$$

For Λ a discrete ℓ-torsion ring, an algebraic extension of $\mathbb{Q}_\ell$ and its ring of integers.

Remark 2.2.3. *For lisse sheaves, this holds if Λ is a torsion ring or an extension of $\mathbb{Z}_\ell$, and always holds for lisse if $X_1, \ldots, X_n$ are geometrically unibranch.*

In particular, we have an equivalence

$$\mathrm{D}\big(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Weil}(X), \overline{\mathbb{Q}_\ell})\big)^{\otimes I} \to \mathrm{D}_{\mathrm{lis}}\big((X^{\mathrm{Weil}})^I, \overline{\mathbb{Q}_\ell}\big) \tag{2.2}$$

2.3 Application to the moduli of bundles and shtukas

We can now come back to the situation of the moduli stack of bundles and review the ideas behind the construction of Lafforgue from the perspective of Weil prestacks. First, it was realized by Drinfeld, and later Lafforgue, that in order to take the Frobenius structure into account, we should not only consider bundles but the moduli of *shtukas*. A full account of this work is well beyond the scope of this introduction, but we will try to convey the main geometrical constructions. First, for every finite set S, we can consider the prestack of shtukas with $|S|$-legs defined by the moduli problem whose groupoid of points for every $R \in \mathrm{CAlg}_{\mathbb{F}}$ is given by

$$\mathrm{Sht}_{G,S}(X)(R) = \left\{ \begin{matrix} ((x_i)_{i \in S}, \mathcal{E}, \gamma) \mid (x_i) \in (X^{\mathrm{Weil}})^{|S|}, \mathcal{E} \in \mathrm{Bun}_G(X_R), \\ \gamma\colon \mathcal{E}|_{X_R \setminus x_i} \simeq \Phi_X^*(\mathcal{E})|_{X_R \setminus x_i} \end{matrix} \right\}$$

We can then define a corresponding Hecke stack which parameterize modifications of two shtukas. These depend on three finite sets S, T, I and given by

$$\mathrm{Hecke}_{G,S,T,I}(X)(R) = \left\{ \begin{matrix} ((x_i)_{i \in I}, \mathcal{S}_0, \mathcal{S}_1, \alpha) \mid (x_i) \in \big(X^{\mathrm{Weil}}\big)^I, \\ \mathcal{S}_0 \in \mathrm{Sht}_{G,S}(X)(R), \mathcal{S}_1 \in \mathrm{Sht}_{G,T}(X)(R) \\ \alpha\colon \mathcal{E}_0|_{X_R \setminus x_i} \simeq \mathcal{E}_1|_{X_R \setminus x_i} \end{matrix} \right\}$$

where $\mathcal{E}_0$, $\mathcal{E}_1$ are the underlying G-bundles corresponding to the shtukas $\mathcal{S}_0$, $\mathcal{S}_1$, respectively. We then have the Hecke correspondences

$$\begin{tikzcd} & \mathrm{Hecke}_{G,S,T,I}(X) \arrow[dl, "h_l"'] \arrow[dr, "h_r"] & \\ (X^{\mathrm{Weil}})^I \times \mathrm{Sht}_{G,S}(X) & & \mathrm{Sht}_{G,T}(X) \end{tikzcd}$$

Using the equivalence

$$\mathrm{D}_{\mathrm{lis}}((X^{\mathrm{Weil}})^I) \simeq \mathrm{D}^b(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Weil}(X)^I, \overline{\mathbb{Q}_\ell}))$$

we can get a family of integral transform functors which generalize the family used in Lafforgue's work. Namely, for every representation $V \in \mathrm{D}^b(\mathrm{Rep}_{\mathrm{cts}}(\mathrm{Weil}(X)^I, \overline{\mathbb{Q}_\ell}))$ and a complex $\mathcal{K} \in \mathrm{D}_{\mathrm{cons}}(\mathrm{Hecke}_{G,S,T,I}(X, \overline{\mathbb{Q}_\ell}))$ we have the functor

$$S_{S,T,I,\mathcal{K},V} \colon \mathrm{D}_{\mathrm{cons}}(\mathrm{Sht}_{G,S}(X), \overline{\mathbb{Q}_\ell}) \to \mathrm{D}_{\mathrm{cons}}(\mathrm{Sht}_{G,T}(X), \overline{\mathbb{Q}_\ell})$$

given by

$$S_{S,T,I,\mathcal{K},V}(\mathcal{F}) = (h_r)_!\big(h_l^*(\mathcal{L} \boxtimes \mathcal{F}) \otimes \mathcal{K}\big).$$

These functors are the key component in constructing the "spectral action", see (Zhu, 2021).

Chapter 3

THE CATEGORICAL TRACE IN THE CATEGORICAL LOCAL LANLGANDS PROGRAM

3.1 Loop groups

We consider the situation where F a non-archimedean local field, e.g. $\mathbb{Q}_p, \mathbb{F}_q((t))$, and $W_F \subseteq \Gamma_F$ its Weil group and Galois group. The goal of the local Langlands program is to understand the smooth representations of the group $G(F)$. Fix a prime ℓ not equal to p. We denote by $\mathrm{Rep}(G(F), \overline{\mathbb{Q}}_\ell)$ the category of smooth representations of $G(F)$ with coefficients in $\overline{\mathbb{Q}}_\ell$. We denote by $\mathrm{D}(\mathrm{Rep}(G(F), \overline{\mathbb{Q}}_\ell))$ the unbounded derived category of $\mathrm{Rep}(G(F), \overline{\mathbb{Q}}_\ell)$. In the categorical variant of the local Langlands program, one wishes to understand the category $\mathrm{D}(\mathrm{Rep}(G(F), \overline{\mathbb{Q}}_\ell))$. In this chapter, we will review some of the ways to use algebraic geometry to achieve better understanding of this category.

As with the global correspondence, the rich connection to algebraic geometry arises from the realization of some of the structures involved as sets of rational points of an algebraic-geometric object. We begin by describing the corresponding structures for $G(F)$ itself.

For the sake of simplicity, we will assume F is of equal characteristic. Namely, of the form $\mathbb{F}_q((t))$ for a finite field $\mathbb{F}_q$. However, the definitions and results can be modified to include the mixed-characteristic case as well.

We recall the arc-group and loop group associated to G, these are given by the group-functors sending a $\mathbb{F}_q$ algebra R to the groups:

$$L^+G(R) = G(R[[t]]), \quad LG(R) = G(R((t))).$$

The functor L^+G is represented by a perfect affine scheme. In particular,

$$G(\mathbb{F}_q) = G(\mathbb{F}_q[[t]]), \quad LG(\mathbb{F}_q) = G(\mathbb{F}_q((t))).$$

Let LG/L^+G be the associated affine Grassmannian. It is represented by an ind-projective perfect ind-scheme. We will denote by $LG_{\mathbb{F}}$ and $L^+G_{\mathbb{F}}$ the base change of the loop and arc-group to the fixed algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$.

In the global case, the connection from algebraic object (i.e. sheaves) into the automorphic objects was facilitated by the function-sheaf dictionary. The function sheaf dictionary can be seen as a way to take the "trace of the Frobenius" of an object. However, in the local theory, our target object is not a function space, but a category. This suggests the need to have a categorical way of taking the trace. This is exactly what is achieved by the categorical trace constructions. These ideas originated by Drinfeld, Gaitsgory, and others. See (Dennis Gaitsgory, 2016) for a review. We will see how these ideas work in the setting of unipotent representations.

3.2 Convolution Patterns and the Iwahori-Hecke category

The abstract results of the second part of the thesis are motivated by the understanding of unipotent representations. It is the subcategory of smooth representations of G generated by the compact induction from an Iwahori subgroup.

Fix a Borel subgroup $B \subseteq G$ and recall that the corresponding Iwahori subgroup $I \subseteq G(\mathbb{F}_q[[t]])$ is the inverse image of $B(\mathbb{F}_q)$ under the surjection

$$G(\mathbb{F}_q[[t]]) \to G(\mathbb{F}_q).$$

We can then consider the compactly induced representation $c\text{-ind}_I^{G(F)}(\overline{\mathbb{Q}_\ell})$. The endomorphism algebra:

$$\mathrm{H}_I = \mathrm{End}_{\mathrm{Rep}(G(F))}(c\text{-ind}_I^{G(F)}, c\text{-ind}_I^{G(F)}),$$

is known as the Iwahori-Hecke algebra. Up to a choice of Haar measure on $G(F)$, there is a canonical isomorphism:

$$\mathrm{H}_I = I \backslash G(\mathbb{F}_q((t)))/I.$$

The composition action on H_I corresponds to the convolution operation on the double coset space.

A complex $V \in \mathrm{D}(\mathrm{Rep}(G), \overline{\mathbb{Q}_\ell})$ is called *unipoitent* if it is in the full subcategory generated by $c\text{-ind}_I^{G(F)}$ by colimits and retracts. Equivalently, V is unipotent if the cohomologies $H^i(V)$ are unipotent representations in the sense of (Lusztig, 1995). The full subcategory of unipotent representations is equivalent to the derived category of left-H_I modules. This equivalence is obtained via:

$$V \mapsto \mathrm{Hom}_{\mathrm{D}(\mathrm{Rep}(G), \overline{\mathbb{Q}_\ell})}(c\text{-ind}_I^{G(F)}, V).$$

We can use algebraic geometry to study the category of unipotent representations. We denote by $\pi_+ \colon L^+G \to G$ the map corresponding to sending t to zero. This subgroup can also be lifted to an algebraic geometric object. Namely, we define the functor Iw by

$$\mathrm{Iw}(R) = \{g \in G(R[[t]]) \mid \pi_+(g) \in B(R)\}.$$

Then the Iwahori-Hecke algebra H_I identifies with the $\mathbb{F}_q$-points of the Iwahori-Hecke stack. This stack can be described by the the fpqc-quotient

$$\mathrm{IwHecke} \simeq [\mathrm{Iw}\backslash LG/\mathrm{Iw}].$$

The category of constructible sheaves $\mathrm{D}_{\mathrm{cons}}(\mathrm{IwHecke}_{I,\mathbb{F}}, \overline{\mathbb{Q}}_\ell)$ on the base change to the algebraic closure carries a monoidal structure given by convolution.

In order to get a sense of how this works, it would be useful to consider the general situation of convolution monoidal structures. In general, these arise from a "nice-enough" morsphism $f\colon (X, \phi_X) \to (Y, \phi_Y)$. In such a situation, $\mathcal{D}_{\mathrm{cons}}(X \times_Y X)$ has a convolution monoidal structure, given by

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\Delta"] \arrow[d, "m"] & (X \times_Y X) \times (X \times_Y X) \\ X \times_Y X & \end{tikzcd}$$

Namely, the monoidal operation is given by:

$$\mathcal{F} \star \mathcal{G} := m_* \Delta^! (\mathcal{F} \boxtimes \mathcal{G}), \quad \mathcal{F}, \mathcal{G} \in \mathcal{D}(X \times_Y X)$$

In our case, we will take X to be the classifying stack $B(\mathrm{Iw}_{\mathbb{F}})$ and Y to be the classifying stack $B(LG_{\mathbb{F}})$ of the loop group. For many purposes, the morphism $B(\mathrm{Iw}_{\mathbb{F}}) \to B(LG_{\mathbb{F}})$ behaves like a proper morphism. For more details, see (Zhu, 2021) and (Bouthier, Kazhdan, and Varshavsky, 2020). Under the identifications

$$X \times_Y X \simeq \mathrm{Iw}\backslash LG/\mathrm{Iw}, \; X \times_Y X \times_Y X \simeq \mathrm{Iw}\backslash LG \times^{\mathrm{Iw}} LG/\mathrm{Iw}$$

the general convolution monoidal structure identifies with the familiar description of convolution on Iwahori-Hecke algebras.

As we did with the function sheaf dictionary, we can relate this monoidal category to the category of unipotent representations by taking the "trace of the Frobenius". Namely, the Frobenius map on Iw and LG give corresponding maps on $\mathrm{Iw}_{\mathbb{F}}$ and $LG_{\mathbb{F}}$ and a corresponding map on the classifying spaces.

3.3 The Categorical trace construction

Given a Λ-linear monoidal ∞-category $\mathcal{A}$ together with a monoidal endomorphism $\phi\colon \mathcal{A} \to \mathcal{A}$, its categorical trace is given by the Hochschild homology

$$\mathrm{Tr}(\mathcal{A}, \phi) := \mathrm{HH}(\mathcal{A}, {}^{\phi}\mathcal{A}) = \mathcal{A} \otimes_{\mathcal{A}\otimes\mathcal{A}^{\mathrm{rev}}} {}^{\phi}\mathcal{A}$$

Roughly speaking, it is determined by the following universal property: a functor $\mathrm{Tr}(\mathcal{A}, \phi) \to C$ is equivalent to a functor $\mathcal{F}\colon \mathcal{A} \to C$ equipped with equivalences

$$F(a \otimes b) \simeq F(b \otimes \phi(a)), \quad a, b \in \mathrm{Ob}(\mathcal{A})$$

together with all the necessary higher coherence data. In particular, there is a tautological functor

$$\mathrm{Tr}_{\phi}\colon \mathcal{A} \to \mathrm{Tr}(\mathcal{A}, \phi)$$

sending an object a to its *universal ϕ-twisted trace*.

The trace $\mathrm{Tr}(\mathcal{A}, \phi)$ always exists for general reasons, and it can be computed as the using the cyclic bar resolution. Namely, as the homotopy colimit of the simplicial object

$$\mathrm{HH}_n(\mathcal{A}, {}^{\phi}\mathcal{A}) = \mathcal{A}^{\otimes(n+1)} = \mathcal{A} \otimes \cdots \otimes \mathcal{A}, \quad n \geq 0$$

which has boundary maps

$$d_i(a_0 \otimes \ldots, a_{i-1} \otimes a_i \otimes \cdots \otimes a_n) = \begin{cases} a_0 \otimes \cdots \otimes a_{i-1}a_i \otimes \cdots \otimes a_n & i \neq 0 \\ a_1 \otimes a_2 \otimes \cdots \otimes a_n\phi(a_0) & i = 0 \end{cases}$$

In general, it is difficult to compute such homotopy colimits, however, under certain conditions it is possible to say more in the case of convolution patterns as in the previous section.

For monoidial structures arising from convooution patters, it is sometimes possible to realize the ϕ-twisted trace as a full subcategory of the category of sheaves of a certain geometric object. Namely, consider the "fixed point object"

$$\begin{tikzcd} \mathcal{L}_{\phi}(Y) \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\mathrm{id}_Y \times \phi_Y"] & Y \end{tikzcd}$$

We have a correspondence

$$\begin{tikzcd} \mathcal{L}_{\phi}(Y) \times_Y X \arrow[r] \arrow[d] & X \times_Y X \\ \mathcal{L}_{\phi}(Y) & \end{tikzcd}$$

Using the identification $X \times_Y \mathcal{L}_\phi(Y) \simeq X \times_{Y\times Y} Y$. The main result of the second part of the thesis can be informally formulated as follows:

Theorem 3.3.1. *Proposition (Ben Zvi-Nadler, H.-Zhu) Under suitable assumptions, there is a canonical commutative diagram*

$$\begin{tikzcd} \mathcal{D}(X \times_Y X) \arrow[r] \arrow[d] & \mathcal{D}(X \times_Y \mathcal{L}_\phi(Y)) \arrow[d] \\ \mathrm{Tr}(\mathcal{D}(X \times_Y X), \phi) \arrow[r, hook] & \mathcal{D}(\mathcal{L}_\phi(Y)) \end{tikzcd}$$

with the bottom horizontal arrow fully faithful

Let's try to apply this paradigm to our case of $B(\mathrm{Iw}_{\mathbb{F}}) \to B(LG_{\mathbb{F}})$ and ϕ given by the Frobenius which we denote by σ. In this case we have the fixed point object

$$\begin{tikzcd} LG/\mathrm{Ad}_\sigma(LG) \arrow[r] \arrow[d] & BLG \arrow[d, "\Delta"] \\ BLG \arrow[r, "\mathrm{id}\times\sigma"] & BLG \end{tikzcd}$$

and we can also identify:

$$X \times_Y X \simeq LG_{\mathbb{F}}/\mathrm{Ad}_\sigma(\mathrm{Iw}_{\mathbb{F}})$$

This means, given the correct definitions of categories of sheaves on the corresponding geometric objects, that our main result gives a fully faithful functor

$$tr(\mathrm{D}_{\mathrm{cons}}(\mathrm{Iw}\backslash LG/\mathrm{Iw}, \overline{\mathbb{Q}_\ell}), \phi) \to \mathrm{D}(LG/\mathrm{Ad}_\sigma(LG), \overline{\mathbb{Q}_\ell}).$$

These definitions and its implication to the study of the category of unipotent representations are the subject of a joint project with Xinwen Zhu and are beyond the scope of this thesis.

Part II

A Categorical Künneth Formula For Weil Sheaves

Chapter 4

RECOLLECTIONS ON INFINITY CATEGORIES

Throughout this part, Λ denotes a unital, commutative ring. We briefly collect some notation pertaining to ∞-categories from (Lurie, 2017; Lurie, 2009). As in (Lurie, 2009, Section 5.5.3), Pr^{L} denotes the ∞-category of presentable ∞-categories with colimit-preserving functors. It contains the subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories.

4.1 Monoidal aspects

The category Pr^{L} carries the Lurie tensor product (Lurie, 2017, Section 4.8.1). This tensor product induces one on the full subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories (Lurie, 2017, Proposition 4.8.2.18). For our commutative ring Λ, the ∞-category Mod_Λ of chain complexes of Λ-modules, up to quasi-isomorphism, is a commutative monoid in $\mathrm{Pr}^{\mathrm{St}}$ with respect to this tensor product. This structure includes, in particular, the existence of a functor

$$\mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$$

which, after passing to the homotopy categories is the classical *derived* tensor product on the unbounded derived category of Λ-modules.

We define $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ to be the category of modules, in $\mathrm{Pr}^{\mathrm{St}}$, over Mod_Λ. Noting that modules over Mod_Λ are in particular modules over Sp, the ∞-category of spectra, $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ can be described as the ∞-category consisting of *stable* presentable ∞-categories together with a Λ-linear structure, such that functors are continuous and Λ-linear. Therefore $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ carries a symmetric monoidal structure, whose unit is Mod_Λ. We will also denote by $\mathrm{Pr}^{\mathrm{St}}_\omega$ the category of compactly generated presentable with functors that send compact objects to compact objects (equivalently, those whose right adjoint is continuous).

In order to express monoidal properties of ∞-categories consisting, say, of bounded complexes, recall from (Lurie, 2017, Corollary 4.8.1.4 joint with Lemma 5.3.2.11) or (Ben-Zvi, Francis, and Nadler, 2010, Proposition 4.4) the symmetric monoidal structure on the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ of idempotent complete stable ∞-categories and exact functors: it is characterized by

$$D_1 \otimes D_2 := \big(\mathrm{Ind}(D_1) \otimes \mathrm{Ind}(D_2)\big)^{\omega} \tag{4.1}$$

that is, the compact objects in the Lurie tensor product of the Ind-completions. With respect to these monoidal structures, the Ind-completion functor (taking values in compactly generated presentable ∞-categories with the Lurie tensor product) and the functor forgetting the compact generated-ness:

$$\mathrm{Cat}_{\infty}^{\mathrm{Ex}}(\mathrm{Idem}) \xrightarrow{\mathrm{Ind}} \mathrm{Pr}_{\omega}^{\mathrm{St}} \hookrightarrow \mathrm{Pr}^{\mathrm{St}} \tag{4.2}$$

are both symmetric monoidal (Lurie, 2017, Lemmas 5.3.2.9, 5.3.2.11).

The subcategory of compact objects in Mod_{Λ} is given by perfect complexes of Λ-modules (Lurie, 2017, Proposition 7.2.4.2.). It is denoted Perf_{Λ}. Under the equivalence in Equation (4.2), the category $\mathrm{Perf}_{\Lambda} \subseteq \mathrm{Cat}_{\infty}^{\mathrm{Ex}}(\mathrm{Idem})$ corresponds to Mod_{Λ}. Moreover, Perf_{Λ} is a commutative monoid in $\mathrm{Cat}_{\infty}^{\mathrm{Ex}}(\mathrm{Idem})$, so that we can consider its category of modules, denoted as $\mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$. This category inherits a symmetric monoidal structure denoted by $D_1 \otimes_{\mathrm{Perf}_{\Lambda}} D_2$.

Any stable ∞-category D is canonically enriched over the category of spectra Sp. We write $\mathrm{Hom}_D(-,-)$ for the mapping spectrum. Any category in $\mathrm{Pr}_{\Lambda}^{\mathrm{St}}$ is canonically enriched over Mod_{Λ}, so that we refer to $\mathrm{Hom}_D(-,-) \in \mathrm{Mod}_{\Lambda}$ as the mapping complex. For example, for $M, N \in \mathrm{Mod}_{\Lambda}$, then $\mathrm{Hom}_{\mathrm{Mod}_{\Lambda}}(M,N)$ is commonly also denoted by $\mathrm{Hom}(M,N)$. Its n-th cohomology is the Hom-group $\mathrm{Hom}(M, N[n])$ in the classical derived category.

4.2 Fixed points of infinity categories

A basic structure in Drinfeld's lemma is the equivariance datum for the partial Frobenii. In this section, we assemble some abstract results where such ∞-categorical constructions are carried out.

Definition 4.2.1. *Let $\phi\colon D \to D$ be an endofunctor in* $\mathrm{Cat}_{\infty}^{\mathrm{Ex}}(\mathrm{Idem})$. *The category of ϕ*-fixed points *is*

$$D^{\phi=\mathrm{id}} := \mathrm{Fix}(D,\phi) := \lim\left(D \underset{\mathrm{id}_D}{\overset{\phi}{\rightrightarrows}} D\right).$$

Recall that for a symmetric monoidal ∞-category D, a commutative monoid object $\Lambda \in \mathrm{CAlg}(D)$, the forgetful functors $\mathrm{CAlg}(D) \to D$ and $\mathrm{Mod}_{\Lambda}(D) \to D$ preserve limits (Lurie, 2017, Corollary 3.2.2.5, Corollary 4.2.3.3). In particular, if D is in addition Λ-linear, that is, an object in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$, and ϕ is also Λ-linear, then $\mathrm{Fix}(D,\phi)$ admits a natural Λ-linear structure as well.

Because of these facts, we will usually not specify where the limit above is formed. Note that all functors

$$\mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem}) \underset{\cong}{\xrightarrow{\mathrm{Ind}}} \mathrm{Pr}^{\mathrm{St}}_{\omega} \xrightarrow{(*)} \mathrm{Pr}^{\mathrm{St}} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}_{\infty}} \tag{4.3}$$

except for the forgetful functor marked $(*)$ preserve limits, see (Lurie, 2017, Corollary 4.2.3.3) and (Lurie, 2009, Proposition 5.5.3.13) for the rightmost two functors.

To give a concrete example of that failure in our situation, note that $\mathrm{Fix}(D,\mathrm{id}_D) = \mathrm{Fun}(B\mathbb{Z}, D)$, that is, objects are pairs (M,α) consisting of some $M \in D$ and some automorphism $\alpha\colon M \cong M$.

Now consider $D = \mathrm{Vect}^{\mathrm{f.d}}_{\Lambda}$, the (abelian) category of finite-dimensional vector spaces over a field Λ. The natural functor

$$\mathrm{Ind}\left(\lim\left(\mathrm{Vect}^{\mathrm{f.d}}_{\Lambda} \rightrightarrows \mathrm{Vect}^{\mathrm{f.d}}_{\Lambda}\right)\right) \to \lim\left(\mathrm{Ind}\big(\mathrm{Vect}^{\mathrm{f.d}}_{\Lambda}\big) \rightrightarrows \mathrm{Ind}\big(\mathrm{Vect}^{\mathrm{f.d}}_{\Lambda}\big)\right) = \lim\left(\mathrm{Vect}_{\Lambda} \rightrightarrows \mathrm{Vect}_{\Lambda}\right)$$

is fully faithful, but *not* essentially surjective: given an automorphism α of an infinite-dimensional vector space M, there need not be a filtration $M = \bigcup M_i$ by finite-dimensional subspaces M_i that is compatible with α.

Fixed point categories inherit t-structures as follows:

Lemma 4.2.2. *Let* $\phi\colon D \to D$ *be a functor in* $\mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem})$*. Suppose* D *carries a t-structure such that* ϕ *is t-exact. Then* $\mathrm{Fix}(D,\phi)$ *carries a unique t-structure such that the evaluation functor is t-exact. There is a natural equivalence*

$$\mathrm{Fix}(D^{\heartsuit},\phi) \xrightarrow{\cong} \mathrm{Fix}(D,\phi)^{\heartsuit}.$$

Proof. Let us abbreviate $\widetilde{D} := \mathrm{Fix}(D,\phi)$. For $\bullet$ being either "≤ 0" or "≥ 0", we put $\widetilde{D}^{\bullet} := \mathrm{Fix}(D^{\bullet},\phi)$, which is a (non-stable) ∞-category. This is clearly the only choice for a t-structure making ev a t-exact functor. It satisfies the claim about the hearts of the t-structure by definition.

We need to show that it is a t-structure. Being a limit of full subcategories, the categories $\widetilde{D}^{\bullet}$ are full subcategories of $\widetilde{D}$. Since ϕ, being t-exact, commutes with $\tau^{\le 0}_D$ and $\tau^{\ge 0}_D$, these two functors also yield truncation functors for $\widetilde{D}$. For $M \in \widetilde{D}^{\le 0}$, $N \in \widetilde{D}^{\ge 1}$ (we use cohomological conventions), we have

$$\mathrm{Hom}_{\widetilde{D}}(M,N) \;=\; \lim\left(\mathrm{Hom}_D(M,N) \rightrightarrows \mathrm{Hom}_D(M,N)\right),$$

where on the right hand side M, N denote the underlying objects in D. Since $M \in D^{\leq 0}$, $N \in D^{\geq 1}$, we have $\mathrm{H}^i\mathrm{Hom}_D(M,N) = 0$ for $i = -1, 0$. Thus, $\mathrm{H}^0\mathrm{Hom}_{\widetilde{D}}(M,N) = 0$ as well. □

4.2.1 can be generalized as follows: Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. For example, for $n = 1$, this amounts to giving $D = \varphi(*) \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ and an equivalence $\phi = \varphi(1)\colon D \to D$. For $n = 2$, such a datum corresponds to giving D, equivalences $\phi_1, \phi_2\colon D \xrightarrow{\cong} D$ together with an equivalence $\phi_1 \circ \phi_2 \xrightarrow{\cong} \phi_2 \circ \phi_1$. So we define the ∞-category of *simultaneous fixed points* as

$$\mathrm{Fix}(D, \phi_1, \ldots, \phi_n) := \lim \varphi \in \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}).$$

Remark 4.2.3. *The statement of 4.2.2 carries over verbatim assuming that D has a t-structure and all ϕ_i are t-exact, noting that $B\mathbb{Z}^n = (S^1)^n$ is a finite simplicial set.*

Lemma 4.2.4. *Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ be a diagram. Denote $D = \varphi(*)$ and $\phi_i = \varphi(e_i)$ for the i-th standard vector $e_i \in \mathbb{Z}^n$. The functor*

$$\mathrm{Fix}\big(D, \phi_1, \ldots, \phi_n\big) \to \mathrm{Fix}\big(\mathrm{Ind}(D), \phi_1, \ldots, \phi_n\big)$$

induced from the inclusion $D \subset \mathrm{Ind}(D)$ is fully faithful and takes values in compact objects. In particular, it yields a fully faithful functor

$$\mathrm{Ind}\big(\mathrm{Fix}(D, \phi_1, \ldots, \phi_n)\big) \to \mathrm{Fix}\big(\mathrm{Ind}(D), \phi_1, \ldots, \phi_n\big).$$

Proof. Let $M \in \mathrm{Fix}(D, \phi_1, \ldots, \phi_n)$ and denote its underlying object in D by the same symbol. For every $N \in \mathrm{Fix}\big(\mathrm{Ind}(D), \phi_1, \ldots, \phi_n\big)$, we have a limit diagram of mapping complexes

$$\mathrm{Hom}_{\mathrm{Fix}(\mathrm{Ind}(D))}(M,N) \cong \mathrm{Fix}\big(\mathrm{Hom}_{\mathrm{Ind}(D)}(M,N), \phi_1, \ldots, \phi_n\big).$$

Since filtered colimits commute with finite limits in the ∞-category of anima (a.k.a. spaces) (Lurie, 2009, Proposition 5.3.3.3.), we see that M is compact in $\mathrm{Fix}(\mathrm{Ind}(D))$ because M is so in $\mathrm{Ind}(D)$. □

Lemma 4.2.5. *Let $\varphi_i\colon B\mathbb{Z} \to \mathrm{Cat}_{\infty,\Lambda}^{\mathrm{Ex}}(\mathrm{Idem})$, $i = 1, \ldots, n$ be given. Denote $D_i = \varphi_i(*)$, $\phi_i = \varphi_i(1)$ and $\widetilde{D}_i = \mathrm{Ind}(D_i)$. Then there is a canonical equivalence*

$$\mathrm{Fix}\big(\widetilde{D}_1, \phi_1\big) \otimes_{\mathrm{Mod}_\Lambda} \ldots \otimes_{\mathrm{Mod}_\Lambda} \mathrm{Fix}\big(\widetilde{D}_n, \phi_n\big) \xrightarrow{\cong} \mathrm{Fix}\left(\widetilde{D}_1 \otimes_{\mathrm{Mod}_\Lambda} \ldots \otimes_{\mathrm{Mod}_\Lambda} \widetilde{D}_n, \phi_1, \ldots, \phi_n\right).$$

Proof. The categories $\mathrm{Fix}(\widetilde{D}_i, \phi_i)$ are compactly generated: the forgetful functor $U : \mathrm{Fix}(\widetilde{D}_i, \phi_i) \to \widetilde{D}_i = \mathrm{Ind}(D_i)$ preserves colimits, so its left adjoint L preserves compact objects. Moreover, U is conservative, so that the objects $L(d_i)$, for $d_i \in D_i$, form a family of compact generators. Then, we use that any compactly generated category in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda}$ is dualizable (Lurie, 2018, Remark D.7.7.6 (1)) so that tensoring with it preserves limits. $\square$

Chapter 5

LISSE AND CONSTRUCTIBLE SHEAVES

In order to state and prove the categorical Künneth formula for Weil sheaves, we use the framework for lisse and constructible sheaves provided by (Hemo, Richarz, and Scholbach, 2021). For the convenience of the reader, we collect here some basics of the formalism.

Throughout, Λ denotes a condensed ring, for example any T1-topological ring such as discrete rings, algebraic extensions $E/\mathbb{Q}_\ell$ or their ring of integers O_E. In the synopsis below, we refer to the latter choices of Λ as the *standard coefficient rings*. We write Λ_* for the underlying ring. Let $\mathrm{D}(X, \Lambda)$ be the derived category of sheaves of Λ-modules on the proétale site $X_{\text{proét}}$.

Definition 5.0.1. *For every scheme X and every condensed ring Λ, there are the full subcategories*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda) \tag{5.1}$$

By definition, the left hand category of lisse sheaves *consists of the dualizable objects in the right-most category. An object (henceforth referred to as a sheaf) M in the right hand category is* constructible*, if on any affine $U \subset X$ there is a finite stratification into constructible locally closed subschemes $U_i \subset U$ such that $M|_{U_i}$ is lisse, that is, dualizable. Finally, an* ind-lisse *(respectively,* ind-constructible*) sheaf is a filtered colimit, in the category* $\mathrm{D}(X, \Lambda)$*, of lisse (respectively, constructible) sheaves. The corresponding full subcategories of* $\mathrm{D}(X, \Lambda)$ *are denoted by*

$$\mathrm{D}_{\mathrm{indlis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \subset \mathrm{D}(X, \Lambda).$$

For the standard coefficient rings Λ above and quasi-compact quasi-separated (qcqs) schemes X, that definition of lisse and constructible sheaves agrees with the classical ones, see loc. cit. for details.

(i) Via the natural functor $\mathrm{Mod}_{\Lambda_*} \to \mathrm{D}(X, \Lambda)$, $M \mapsto \underline{M} \otimes_{\underline{\Lambda_*}} \Lambda_X$, the category $\mathrm{D}(X, \Lambda)$ is an object in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. The functor restricts to a functor $\mathrm{Perf}_{\Lambda_*} \to \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, and the categories $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ are objects of the category $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$. In particular, all categories listed in Equation (5.1) are stable idempotent complete Λ_*-linear ∞-categories.

(ii) The extension-by-zero functor along any constructible locally closed immersion and quasi-compact étale morphisms preserves constructibility.

(iii) The functors $X \mapsto \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ and $X \mapsto \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ satisfy proétale hyperdescent, and the functor $X \mapsto \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, resp. $X \mapsto \mathrm{D}_{\mathrm{indlis}}(X, \Lambda)$ satisfies hyperdescent for quasi-compact étale, resp. finite étale covers.

(iv) If $\Lambda = \operatorname{colim} \Lambda_i$ is a filtered colimit of condensed rings and X is qcqs, then the natural functors

$$\operatorname{colim} \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

are equivalences.

The categories enjoy the following properties:

(v) If X is qcqs, then any constructible sheaf is bounded with respect to the t-structure on $\mathrm{D}(X, \Lambda)$.

(vi) For X locally Noetherian (and much more generally), the t-structure on $\mathrm{D}(X, \Lambda)$ restricts to one on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ and $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ provided that Λ is t-admissible in the sense of (Hemo, Richarz, and Scholbach, 2021). The topological condition on the condensed structure of Λ is satisfied for all the standard coefficient rings listed above.

(vii) For X locally Noetherian (and again more generally), a sheaf is lisse if and only if it is proétale locally the constant sheaf associated to a perfect complex of Λ_*-modules.

(viii) Let X be a qcqs scheme. If the Λ-cohomological dimension is uniformly bounded for all proétale affines $U = \lim_i U_i$ over X, then $\mathrm{Ind}(\mathrm{D}_{\mathrm{cons}}(X, \Lambda)) = \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ and likewise for ind-lisse sheaves. If X is of finite type over $\mathbb{F}_q$ or a separably closed field, this condition holds for any of the above standard rings. For discrete p-torsion rings, algebraic extensions $E/\mathbb{Q}_p$ and their ring of integers O_E, this holds for arbitrary qcqs schemes in characteristic p.

For schemes $X_1, \ldots, X_n$ over a fixed base scheme S (for example, the spectrum of a field) and a condensed ring Λ, we denote the external product in the usual way:

$$\begin{aligned} \boxtimes \colon \mathrm{D}(X_1, \Lambda) \times \ldots \times \mathrm{D}(X_n, \Lambda) &\longrightarrow \mathrm{D}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right), \\ (M_1, \ldots, M_n) &\longmapsto M_1 \boxtimes \ldots \boxtimes M_n := p_1^*(M_1) \otimes_{\Lambda_X} \ldots \otimes_{\Lambda_X} p_n^*(M_n). \end{aligned}$$

Here $p_i\colon X := X_1 \times_S \ldots \times_S X_n \to X_i$ are the projections. This functor induces the functor

$$\boxtimes\colon \mathrm{D}(X_1,\Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}(X_n,\Lambda) \to \mathrm{D}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right), \tag{5.2}$$

in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$. Here we regard $\mathrm{D}(X_i,\Lambda)$ as objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$, like in Item i in the synopsis above. The external tensor product of constructible sheaves is again constructible, and hence induces a functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{cons}}(X_1,\Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n,\Lambda) \to \mathrm{D}_{\mathrm{cons}}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right), \tag{5.3}$$

in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$ and likewise for the categories of ind-constructible, resp. (ind-)lisse sheaves.

Chapter 6

CONSTRUCTIBLE WEIL SHEAVES

In this section, we introduce the categories

$$\mathrm{D}_{\mathrm{lis}}\big(X^{\mathrm{Weil}},\Lambda\big) \subseteq \mathrm{D}_{\mathrm{cons}}\big(X^{\mathrm{Weil}},\Lambda\big) \subseteq \mathrm{D}\big(X^{\mathrm{Weil}},\Lambda\big)$$

consisting of lisse, resp. constructible, resp. all Weil sheaves. These are the categories featuring in the categorical Künneth formula (2.2.2).

Throughout this section, X is a scheme over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Unless the contrary is mentioned, we impose no conditions on X. Moreover, Λ is a condensed ring. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, and denote by $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec}\mathbb{F}$ the base change. Denote by ϕ_X (resp. $\phi_{\mathbb{F}}$) the endomorphism of $X_{\mathbb{F}}$ that is the q-Frobenius on X (resp. $\operatorname{Spec}\mathbb{F}$) and the identity on the other factor.

Let

$$\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}},\Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}},\Lambda) \subset \mathrm{D}(X_{\mathbb{F}},\Lambda)$$

be the categories of lisse, resp. constructible, resp. all proétale sheaves of Λ-modules on $X_{\mathbb{F}}$ (5.0.1). These categories are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$, that is, Λ_*-linear stable idempotent complete symmetric monoidal ∞-categories where $\Lambda_* = \Gamma(*,\Lambda)$ is the underlying ring.

6.1 The Weil-proétale site

The Weil-étale topology for schemes over finite field is introduced in (Lichtenbaum, 2005) see also (Geisser, 2004). Our approach for the proétale topology is slightly different:

Definition 6.1.1. *The Weil-proétale site of X, denoted by $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$, is the following site: Objects in $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ are pairs (U,φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$ equipped with an endomorphism $\varphi\colon U \to U$ of $\mathbb{F}$-schemes such that the map $U \to X_{\mathbb{F}}$ intertwines φ and ϕ_X. Morphisms in $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ are given by equivariant maps, and a family $\{(U_i,\varphi_i) \to (U,\varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$.*

Note that $X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ admits small limits formed componentwise as

$$\lim(U_i,\varphi_i) = (\lim U_i, \lim \varphi_i).$$

In particular, there are limit-preserving maps of sites

$$(X_{\mathbb{F}})_{\text{proét}} \to X^{\text{Weil}}_{\text{proét}} \to X_{\text{proét}} \tag{6.1}$$

given by the functors (in the opposite direction) $U \leftarrow\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\shortmid U$. We denote by $\mathrm{D}(X^{\text{Weil}}, \Lambda)$ the unbounded derived category of sheaves of Λ_X-modules on $X^{\text{Weil}}_{\text{proét}}$. The maps of sites (6.1) induce functors

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\text{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{6.2}$$

whose composition is the usual pullback functor along $X_{\mathbb{F}} \to X$.

Remark 6.1.2. *The functor* $\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\text{Weil}}, \Lambda)$ *is not an equivalence in general. This relates to the difference between continuous representations Galois versus Weil groups. See, however, Proposition 6.6.1 for filtered colimits of finite discrete rings* Λ*.*

We have the following basic functoriality: Let $j\colon U \to X$ be a weakly étale morphism and consider the corresponding object $(U_{\mathbb{F}}, \phi_U)$ of $X^{\text{Weil}}_{\text{proét}}$. Then the slice site $(X^{\text{Weil}}_{\text{proét}})_{/(U_{\mathbb{F}}, \phi_U)}$ is equivalent to $U^{\text{Weil}}_{\text{proét}}$. This gives a functor $(X_{\text{proét}})^{\text{op}} \to \mathrm{Pr}^{\text{St}}_{\Lambda}$, $U \mapsto \mathrm{D}(U^{\text{Weil}}, \Lambda)$ which is a hypercomplete sheaf of Λ_*-linear presentable stable categories.

Also, we obtain an adjunction

$$j_!\colon \mathrm{D}(U^{\text{Weil}}, \Lambda) \rightleftarrows \mathrm{D}(X^{\text{Weil}}, \Lambda)\colon j^*$$

that is compatible with the $((j_{\mathbb{F}})_!, (j_{\mathbb{F}})^*)$-adunction under (6.2). The category $\mathrm{D}(X^{\text{Weil}}, \Lambda)$ is equivalent to the category of ϕ_X-equivariant sheaves on $X_{\mathbb{F}}$, as we will now explain.

For each $i \geq 0$, consider the object $(X_i, \Phi_i) \in X^{\text{Weil}}_{\text{proét}}$ with $X_i = \mathbb{Z}^{i+1} \times X_{\mathbb{F}}$ the countably disjoint union of $X_{\mathbb{F}}$, the map $X_i \to X_{\mathbb{F}}$ given by projection and the endomorphism $\Phi_i\colon X_i \to X_i$ given by $(\underline{n}, x) \mapsto (\underline{n} - (1, \ldots, 1), \phi_X(x))$ on sections.

The inclusion $\mathbb{Z}^i \to \mathbb{Z}^{i+1}$, $\underline{n} \mapsto (0, \underline{n})$ induces a map of schemes $X_{i-1} \to X_i$ where $X_{-1} := X_{\mathbb{F}}$. By pullback, we get a limit-preserving map of sites

$$(X_{i-1})_{\text{proét}} \to \left(X^{\text{Weil}}_{\text{proét}}\right)_{/(X_i, \Phi_i)} \tag{6.3}$$

Lemma 6.1.3. *For each* $i \geq 0$*, the map* (6.3) *induces an equivalence on the associated* 1*-topoi.*

Proof. As universal homeomorphisms induce equivalences on proétale 1-topoi (Bhatt and Scholze, 2015, Lemma 5.4.2), we may assume that X is perfect. In this case, the sites (6.3) are equivalent because ϕ_X is an isomorphism. Explicitly, an inverse is given by sending an object $U \in (X_{i-1})_{\text{proét}}$ to the object $V = \bigsqcup_{\underline{n} \in \mathbb{Z}^{i+1}} V_{\underline{n}}$, $V_{\underline{n}} \to \{\underline{n}\} \times X_{\mathbb{F}}$ defined by

$$V_{\underline{n}} = U_{(n_2-n_1,\dots,n_{i+1}-n_1)} \times_{X_{\mathbb{F}},\phi_X^{n_1}} X_{\mathbb{F}},$$

and with endomorphism $\varphi\colon V \to V$ defined by the maps $V_{\underline{n}} = V_{\underline{n}-(1,\dots,1)} \times_{X_{\mathbb{F}},\phi_X} X_{\mathbb{F}} \to V_{\underline{n}-(1,\dots,1)}$. □

Weil sheaves admit the following presentation as the ϕ_X^*-fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$, see 4.2.1:

Proposition 6.1.4. *The last functor in* (6.2) *induces an equivalence*

$$\mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \cong \lim \left(\mathrm{D}(X_{\mathbb{F}}, \Lambda) \overset{\phi_X^*}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F}}, \Lambda) \right). \tag{6.4}$$

Remark 6.1.5. *Objects in Equation* (6.4) *are pairs* (M, α) *where* $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ *and* α *is an isomorphism* $M \cong \phi_X^* M$. *Note that the composition* $\phi_X \circ \phi_{\mathbb{F}}$ *is the absolute q-Frobenius of* $X_{\mathbb{F}}$. *In particular, it induces the identity on proétale topoi, see (Bhatt and Scholze, 2015, Lemma 5.4.2). Therefore, replacing* ϕ_X^* *by* $\phi_{\mathbb{F}}^*$ *in Equation* (6.4) *yields an equivalent category.*

Proof of Proposition 6.1.4. The structural morphism $(X_0, \Phi_0) \to (X_{\mathbb{F}}, \phi_X)$ is a cover in $X^{\mathrm{Weil}}_{\text{proét}}$. Its Čech nerve has objects $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\text{proét}}$, $i \geq 0$ as above. By descent, there is an equivalence

$$\mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{Tot}\left(\mathrm{D}\big((X^{\mathrm{Weil}}_{\text{proét}})_{/(X_\bullet, \Phi_\bullet)}, \Lambda\big) \right). \tag{6.5}$$

Under Lemma 6.1.3, the cosimplicial 1-topos associated with $(X^{\mathrm{Weil}}_{\text{proét}})_{/(X_\bullet, \Phi_\bullet)}$ is equivalent to the cosimplicial 1-topos associated with the action of ϕ_X^* on $(X_{\mathbb{F}})_{\text{proét}}$. The equivalence Equation (6.5) then becomes

$$\mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \lim_{B\mathbb{Z}} \mathrm{D}(X_{\mathbb{F}}, \Lambda),$$

for the diagram $B\mathbb{Z} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$ corresponding to the endomorphism ϕ_X^* of $\mathrm{D}\big(X_{\mathbb{F}}, \Lambda\big)$. That is, $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the homotopy fixed points of $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$ with respect to the action of ϕ_X^*, which is our claim. □

6.2 Weil sheaves on products

The discussion of the previous section generalizes to products of schemes as follows. Let $X_1, \dots, X_n$ be schemes over $\mathbb{F}_q$, and denote by $X := X_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} X_n$ their product. For every $1 \leq i \leq n$, we have a morphism $\phi_{X_i} \colon X_{i,\mathbb{F}} \to X_{i,\mathbb{F}}$ as in the previous section. We use the notation ϕ_{X_i} to also denote the corresponding map on $X_{\mathbb{F}} = X_{1,\mathbb{F}} \times_{\mathbb{F}} \dots \times_{\mathbb{F}} X_{n,\mathbb{F}}$ which is ϕ_{X_i} on the i-th factor and the identity on the other factors.

We define the site $(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}})_{\text{proét}}$ whose underlying category consists of tuples $(U, \varphi_1, \dots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\text{proét}}$ and pairwise commuting endomomorphisms $\varphi_i \colon U \to U$ such that the following diagram commutes

$$\begin{tikzcd} U \arrow[r, "\varphi_i"] \arrow[d] & U \arrow[d] \\ X_{\mathbb{F}} \arrow[r, "\phi_{X_i}"] & X_{\mathbb{F}}, \end{tikzcd}$$

for all $1 \leq i \leq n$. As before, we denote by $\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ the corresponding derived category of Λ-sheaves.

Using a similar reasoning as in the previous section, we can identify this category of sheaves with the homotopy fixed points

$$\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{Fix}\big(\mathrm{D}(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \dots, \phi_{X_n}^*\big) \tag{6.6}$$

of the commuting family of the functors $\phi_{X_i}^*$, see Remark 4.2.3. Explicitly, for $n = 2$, this is the homotopy limit of the diagram

$$\begin{tikzcd} \mathrm{D}\big(X_{\mathbb{F}}, \Lambda\big) \arrow[r, shift left, "\phi_{X_1}^*"] \arrow[r, shift right, "\mathrm{id}"'] \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\phi_{X_2}^*"] & \mathrm{D}\big(X_{\mathbb{F}}, \Lambda\big) \arrow[d, shift right, "\mathrm{id}"'] \arrow[d, shift left, "\phi_{X_2}^*"] \\ \mathrm{D}\big(X_{\mathbb{F}}, \Lambda\big) \arrow[r, shift left, "\phi_{X_1}^*"] \arrow[r, shift right, "\mathrm{id}"'] & \mathrm{D}\big(X_{\mathbb{F}}, \Lambda\big). \end{tikzcd}$$

Roughly speaking, objects in the category $\mathrm{D}(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}}, \Lambda)$ are given by tuples $(M, \alpha_1, \dots, \alpha_n)$ with $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ and with pairwise commuting equivalences $\alpha_i \colon M \cong \phi_{X_i}^* M$. That is, equipped with a collection of equivalences $\phi_{X_j}^*(\alpha_i) \circ \alpha_j \simeq \phi_{X_i}^*(\alpha_j) \circ \alpha_i$ for all i, j satisfying higher coherence conditions.

6.3 Partial-Frobenius stability

For schemes $X_1, \dots, X_n$ over $\mathbb{F}_q$, we denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product together with the partial Frobenii $\mathrm{Frob}_{X_i} \colon X \to X$, $1 \leq i \leq n$.

To give a reasonable definition of lisse and constructible Weil sheaves, we need to understand the relation between partial-Frobenius invariant constructible subsets in X and constructible subsets in the single factors X_i:

Definition 6.3.1. *A subset $Z \subset X$ is called partial-Frobenius invariant if for all $1 \leq i \leq n$ we have* $\mathrm{Frob}_{X_i}(Z) = Z$.

The composition $\mathrm{Frob}_{X_1} \circ \cdots \circ \mathrm{Frob}_{X_n}$ is the absolute q-Frobenius on X and thus induces the identity on the topological space underlying X. Therefore, in order to check that $Z \subset X$ is partial-Frobenius invariant, it suffices that, for any fixed i, the subset Z is Frob_{X_j}-invariant for all $j \neq i$. This remark, which also applies to $X_{\mathbb{F}} = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$, will be used below without further comment.

We first investigate the case of two factors with one being a separably closed field. This eventually rests on Drinfeld's descent result (V. G. Drinfeld, 1987, Proposition 1.1) for coherent sheaves:

Lemma 6.3.2. *Let X be a qcqs $\mathbb{F}_q$-scheme, and let $k/\mathbb{F}_q$ be a separably closed field. Denote by $p \colon X_k \to X$ the projection. Then $Z \mapsto p^{-1}(Z)$ induces a bijection*

$$\{\textit{constructible subsets in } X\} \leftrightarrow \{\textit{partial-Frobenius invariant, constructible subsets in } X_k\}$$

Proof. The injectivity is clear because p is surjective. It remains to check the surjectivity. Without loss of generality we may assume that k is algebraically closed, and replace Frob_X by Frob_k which is an automorphism. Given that $Z \mapsto p^{-1}(Z)$ is compatible with passing to complements, unions and localizations on X, we are reduced to proving the bijection for constructible closed subsets Z and for X affine over $\mathbb{F}_q$. By Noetherian approximation (6.3.4), we reduce further to the case where X is of finite type over $\mathbb{F}_q$ and still affine. Now we choose a locally closed embedding $X \to \mathbb{P}^n_{\mathbb{F}_q}$ into projective space. A closed subset $Z' \subset X_k$ is ϕ_k-invariant if and only if its closure inside $\mathbb{P}^n_k$ is so. Hence, it is enough to consider the case where $X = \mathbb{P}^n_{\mathbb{F}_q}$ is the projective space. Let Z' be a closed Frob_k-invariant subset of X_k. When viewed as a reduced subscheme, the isomorphism ϕ_k restricts to an isomorphism of Z'. In particular, $\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_{X_k}$-module equipped with an isomorphism $\mathcal{O}_{Z'} \cong \phi_k^* \mathcal{O}_{Z'}$. Hence, Drinfeld's descent result (V. G. Drinfeld, 1987, Proposition 1.1) (see also (Kedlaya, 2019, Section 4.2) for a recent exposition) yields $Z' = Z_k$ for a unique closed subscheme $Z \subset X$. □

The following proposition generalizes the results (Lau, 2004, Lemma 9.2.1) and (V. Lafforgue, 2018, Lemme 8.12) in the case of curves.

Proposition 6.3.3. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} X_n$. Then any partial-Frobenius invariant constructible closed subset $Z \subset X$ is a finite set-theoretic union of subsets of the form $Z_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} Z_n$, for appropriate constructible closed subschemes $Z_i \subset X_i$.*

In particular, any partial-Frobenius invariant constructible open subscheme $U \subset X$ is a finite union of constructible open subschemes of the form $U_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} U_n$, for appropriate constructible open subschemes $U_i \subset X_i$.

Proof. By induction, we may assume $n = 2$. By Noetherian approximation (Lemma 6.3.4), we reduce to the case where both X_1, X_2 are of finite type over $\mathbb{F}_q$. In the following, all products are formed over $\mathbb{F}_q$, and locally closed subschemes are equipped with their reduced subscheme structure. Let $Z \subset X_1 \times X_2$ be a partial-Frobenius invariant closed subscheme. The complement $U = X_1 \times X_2 \setminus Z$ is also partial-Frobenius invariant.

In the proof, we can replace X_1 (and likewise X_2) by a stratification in the following sense: Suppose $X_1 = A' \sqcup A''$ is a set-theoretic stratification into a closed subset A' with open complement A''. Once we know $Z \cap A' \times X_2 = \bigcup_j Z'_{1j} \times Z'_{2j}$ and $Z \cap A'' \times X_2 = \bigcup_j Z''_{1j} \times Z''_{2j}$ for appropriate closed subschemes $Z'_{1j} \subset A'$, $Z''_{1j} \subset A''$ and $Z'_{2j}, Z''_{2j} \subset X_2$, we have the set-theoretic equality

$$Z = \bigcup_j Z'_{1i} \times Z'_{2j} \cup \bigcup_j \overline{Z''_{1j}} \times Z''_{2j},$$

where $\overline{Z''_{1j}} \subset X_1$ denotes the scheme-theoretic closure. Here we note that taking scheme-theoretic closures commutes with products because the projections $X_1 \times X_2 \to X_i$ are flat, and that the topological space underlying the scheme-theoretic closure agrees with the topological closure because all schemes involved are of finite type.

The proof is now by Noetherian induction on X_2, the case $X_2 = \emptyset$ being clear (or, if the reader prefers the case where X_2 is zero dimensional reduces to Lemma 6.3.2). In the induction step, we may assume, using the above stratification argument, that both X_i are irreducible with generic point η_i. We let $\overline{\eta}_i$ be a geometric generic point over η_i, and denote by $p_i\colon X_1 \times X_2 \to X_i$ the two projections. Both p_i are faithfully flat of finite type and in particular open, so that $p_i(U)$ is open in X_i. We have a set-theoretic equality

$$Z = \big((X_1 \setminus p_1(U)) \times X_2\big) \cup \big(X_1 \times (X_2 \setminus p_2(U))\big) \cup \big(Z \cap p_1(U) \times p_2(U)\big).$$

Once we know $Z \cap p_1(U) \times p_2(U) = \bigcup_j Z_{1j} \times Z_{2j}$ for appropriate closed $Z_{ij} \subset p_i(U)$, we are done. We can therefore replace X_i by $p_i(U)$ and assume that both $p_i \colon U \to X_i$ are surjective.

The base change $U \times_{X_2} \overline{\eta}_2$ is a $\phi_{\overline{\eta}_2}$-invariant subset of $X_1 \times \overline{\eta}_2$. By Lemma 6.3.2, it is thus of the form $U_1 \times \overline{\eta}_2$ for some open subset $U_1 \subset X_1$. There is an inclusion (of open subschemes of $X_1 \times \eta_2$): $U \times_{X_2} \eta_2 \subset U_1 \times \eta_2$. It becomes a set-theoretic equality, and therefore an isomorphism of schemes, after base change along $\overline{\eta}_2 \to \eta_2$. By faithfully flat descent, this implies that the two mentioned subsets of $X_1 \times \eta_2$ agree. We claim $U_1 = X_1$. Since the projection $U \to X_2$ is surjective, in particular its image contains η_2, so that U_1 is a non-empty subset, and therefore open dense in the irreducible scheme X_1. Let $x_1 \in X_1$ be a point. Since the projection $U \to X_1$ is surjective, $U \cap (\{x_1\} \times X_2)$ is a non-empty open subscheme of $\{x_1\} \times X_2$. So it contains a point lying over (x_1, η_2). We conclude $X_1 \times \eta_2 \subset U$.

We claim that there is a non-empty open subset $A_2 \subset X_2$ such that

$$X_1 \times A_2 \subset U \text{ or, equivalently, } X_1 \times (X_2 \setminus A_2) \supset X_1 \times X_2 \setminus U.$$

The underlying topological space of $V = X_1 \times X_2 \setminus U$ is Noetherian and thus has finitely many irreducible components V_j. The closure of the projection $\overline{p_2(V_j)} \subset X_2$ does not contain η_2, since $X_1 \times \eta_2 \subset U$. Thus, $A_2 := \bigcap_j X_1 \setminus \overline{p_2(V_j)}$ satisfies our requirements.

Now we continue by Noetherian induction applied to the stratification $X_2 = A_2 \sqcup (X_2 \setminus A_2)$: We have $Z \cap X_1 \times A_2 = \emptyset$, so that we may replace X_2 by the proper closed subscheme $X_2 \setminus A_2$. Hence, the proposition follows by Noetherian induction. □

The following lemma on Noetherian approximation of partial Frobenius invariant subsets is needed for the reduction to finite type schemes:

Lemma 6.3.4. *Let $X_1, \dots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} X_n$. Let $X_i = \lim_j X_{ij}$ be a cofiltered limit of finite type $\mathbb{F}_q$-schemes with affine transition maps, and write $X = \lim_j X_j$, $X_j := X_{1j} \times_{\mathbb{F}_q} \dots \times_{\mathbb{F}_q} X_{nj}$ (see Let $Z \subset X$ be a constructible closed subset. Then the intersection*

$$Z' = \bigcap_{i=1}^{n} \bigcap_{m \in \mathbb{Z}} \mathrm{Frob}_{X_i}^m(Z)$$

is partial Frobenius invariant, constructible closed and there exists an index j and a partial Frobenius invariant closed subset $Z'_j \subset X_j$ such that $Z' = Z'_j \times_{X_j} X$ as sets.

We note that each Frob_{X_i} induces a homeomorphism on the underlying topological space of X so that Z' is well-defined. This lemma applies, in particular, to partial Frobenius invariant constructible closed subsets $Z \subset X$ in which case we have $Z = Z'$.

Proof. As Z is constructible, there exists an index j and a constructible closed subscheme $Z_j \subset X_j$ such that $Z = Z_j \times_{X_j} X$ as sets. We put $Z'_j = \cap_{i=1}^n \cap_{m\in\mathbb{Z}} \mathrm{Frob}^m_{X_{ij}}(Z_j)$. As X_j is of finite type over $\mathbb{F}_q$, the subset Z'_j is still constructible closed. As partial Frobenii induce bijections on the underlying topological spaces, one checks that $\mathrm{Frob}^m_{X_{ij}}(Z_j) \times_{X_j} X = \mathrm{Frob}^m_{X_i}(Z)$ as sets for all $m \in \mathbb{Z}$. Thus, $Z' = Z'_j \times_{X_j} X$ which, also, is constructible closed because $X \to X_j$ is affine. □

6.4 Lisse and constructible Weil sheaves

In this subsection, we define the subcategories of lisse and constructible Weil sheaves and establish a presentation similar to (6.4). Let $X_1, \ldots, X_n$ be schemes over $\mathbb{F}_q$, and denote $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Let Λ be a condensed ring.

Definition 6.4.1. *Let $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$.*

1. *The Weil sheaf M is called lisse if it is dualizable. (Here dualizability refers to the symmetric monoidal structure on $\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$, given by the derived tensor product of Λ-sheaves on the Weil-proétale topos.)*

2. *The Weil sheaf M is called constructible if for any open affine $U_i \subset X_i$ there exists a finite subdivision into constructible locally closed subschemes $U_{ij} \subseteq U_i$ such that each restriction $M|_{U_{1j}^{\mathrm{Weil}} \times \ldots \times U_{nj}^{\mathrm{Weil}}} \in \mathrm{D}\big(U_{1j}^{\mathrm{Weil}} \times \ldots \times U_{nj}^{\mathrm{Weil}}, \Lambda\big)$ is lisse.*

The full subcategories of $\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ consisting of lisse, resp. constructible Weil sheaves are denoted by

$$\mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big).$$

Both categories are idempotent complete stable $\Gamma(X, \Lambda)$-linear symmetric monoidal ∞-categories.

From the presentation (6.6), we get that a Weil sheaf M is lisse if and only if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is lisse. So (6.6) restricts to an equivalence

$$\mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\left(\mathrm{D}_{\mathrm{lis}}\big(X_{\mathbb{F}}, \Lambda\big), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\right). \tag{6.7}$$

The same is true for constructible Weil sheaves by the following proposition:

Proposition 6.4.2. *A Weil sheaf $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ is constructible if and only if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is constructible. Consequently, Equation* (6.6) *restricts to an equivalence*

$$\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\left(\mathrm{D}_{\mathrm{cons}}\big(X_{\mathbb{F}}, \Lambda\big), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\right). \tag{6.8}$$

Proof. Clearly, if M is constructible, so is $M_{\mathbb{F}}$ by 6.4.1. Let $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ such that $M_{\mathbb{F}}$ is constructible. We may assume that all X_i are affine. We claim that there is a finite subdivision $X_{\mathbb{F}} = \sqcup X_\alpha$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_\alpha}$ is lisse and such that each X_α is partial Frobenius invariant.

Assuming the claim we finish the argument as follows. By 6.3.3, any open stratum $U = X_{j_0} \subset X_{\mathbb{F}}$ is a finite union of subsets of the form $U_{1,\mathbb{F}} \times_{\mathbb{F}} \ldots \times_{\mathbb{F}} U_{n,\mathbb{F}}$ and the restriction of M to each of them is lisse. In particular, the complement $X_{\mathbb{F}} \backslash U$ is defined over $\mathbb{F}_q$ and arises as a finite union of schemes of the form $X' = X_1' \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n'$ for suitable qcqs schemes X_i' over $\mathbb{F}_q$. Intersecting each $X_{\mathbb{F}}'$ with the remaining strata $\sqcup_{j \neq j_0} X_j$, we conclude by induction on the number of strata.

It remains to prove the claim. We start with any finite subdivision $X_{\mathbb{F}} = \sqcup X_j'$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_j'}$ is lisse. Pick an open stratum X_{j_0}', and set

$$X_{j_0} = \bigcup_{i=1}^{n} \bigcup_{m \in \mathbb{Z}} \phi_{X_i}^m (X_{j_0}'). \tag{6.9}$$

This is a constructible open subset of $X_{\mathbb{F}}$ by 6.3.4 applied to its closed complement. Furthermore, $M_{\mathbb{F}}|_{X_{j_0}}$ is lisse by its partial Frobenius equivariance, noting that $\phi_{X_i}^*$ induces equivalences on proétale topoi to treat the negative powers in (6.9). As before, $X_{\mathbb{F}} \backslash X_{j_0}$ is defined over $\mathbb{F}_q$. So replacing X_j', $j \neq j_0$ by $X_j' \cap (X_{\mathbb{F}} \backslash X_{j_0})$, the claim follows by induction on the number of strata. ☐

In the case of a single factor $X = X_1$, the preceding discussion implies

$$\mathrm{D}_\bullet\big(X^{\mathrm{Weil}}, \Lambda\big) \cong \lim \left(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \underset{\mathrm{id}}{\overset{\phi_X^*}{\rightrightarrows}} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)\right), \tag{6.10}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

6.5 Relation with the Weil groupoid

In this subsection, we relate lisse Weil sheaves with representations of the Weil groupoid. Throughout, we work with étale fundamental groups as opposed to their proétale variants in order to have Drinfeld's lemma available, see Section 7.4. The two concepts differ in general, but agree for geometrically unibranch (for example, normal) Noetherian schemes, see (Bhatt and Scholze, 2015, Lemma 7.4.10).

For a Noetherian scheme X, let $\pi_1(X)$ be the *étale fundamental groupoid* of X as defined in (*Revêtements étales et groupe fondamental (SGA 1)* 2003, Exposé V, §7 and §9). Its objects are geometric points of X, and its morphisms are isomorphisms of fiber functors on the finite étale site of X. This is an essentially small category. The automorphism group in $\pi_1(X)$ at a geometric point $x \to X$ is profinite. It is denoted $\pi_1(X, x)$ and called the *étale fundamental group* of (X, x). If X is connected, then the natural map $B\pi_1(X, x) \to \pi_1(X)$ is an equivalence for any $x \to X$. If X is the disjoint sum of schemes X_i, $i \in I$, then $\pi_1(X)$ is the disjoint sum of the $\pi_1(X_i)$, $i \in I$. In this case, if $x \to X$ factors through X_i, then $\pi_1(X, x) = \pi_1(X_i, x)$.

Definition 6.5.1. *Let $X_1, \ldots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and write $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. The Frobenius-Weil groupoid is the stacky quotient*

$$\mathrm{FWeil}(X) = \pi_1(X_{\mathbb{F}})/\langle \phi_{X_1}^{\mathbb{Z}}, \ldots, \phi_{X_n}^{\mathbb{Z}} \rangle, \tag{6.11}$$

where we use that the partial Frobenii ϕ_{X_i} induce automorphisms on the finite étale site of $X_{\mathbb{F}}$.

For $n = 1$, we denote $\mathrm{FWeil}(X) = \mathrm{Weil}(X)$. Even if X is connected, its base change $X_{\mathbb{F}}$ might be disconnected in which case the action of ϕ_X permutes some connected components. Therefore, fixing a geometric point of $X_{\mathbb{F}}$ is inconvenient, and the reason for us to work with fundamental groupoids as opposed to fundamental groups. The automorphism groups in $\mathrm{Weil}(X)$ carry the structure of locally profinite groups: indeed, if X is connected, then $\mathrm{Weil}(X)$ is, for any choice of a geometric point $x \to X_{\mathbb{F}}$, equivalent to the classifying space of the Weil group $\mathrm{Weil}(X, x)$ from (Deligne, 1980, Définition 1.1.10).

Recall that this group sits in an exact sequence of topological groups

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{Weil}(X, x) \to \mathrm{Weil}(\mathbb{F}/\mathbb{F}_q) \simeq \mathbb{Z}, \tag{6.12}$$

where $\pi_1(X_{\mathbb{F}}, x)$ carries its profinite topology and $\mathbb{Z}$ the discrete topology. The topology on the morphism groups in $\mathrm{Weil}(X)$ obtained in this way is independent from the choice of $x \to X_{\mathbb{F}}$. The image of $\mathrm{Weil}(X,x) \to \mathbb{Z}$ is the subgroup $m\mathbb{Z}$ where m is the degree of the largest finite subfield in $\Gamma(X, O_X)$. In particular, we have $m = 1$ if $X_{\mathbb{F}}$ is connected. Let us add that if $x \to X_{\mathbb{F}}$ is fixed under ϕ_X, then the action of ϕ_X on $\pi_1(X_{\mathbb{F}}, x)$ corresponds by virtue of the formula $\phi_X^* = (\phi_{\mathbb{F}}^*)^{-1}$ to the action of the geometric Frobenius, that is, the inverse of the q-Frobenius in $\mathrm{Weil}(\mathbb{F}/\mathbb{F}_q)$.

Likewise, for every $n \geq 1$, the stabilizers of the Frobenius-Weil groupoid are related to the partial Frobenius-Weil groups introduced in (V. G. Drinfeld, 1987, Proposition 6.1) and (V. Lafforgue, 2018, Remarque 8.18). In particular, there is an exact sequence

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{FWeil}(X,x) \to \mathbb{Z}^n,$$

for each geometric point $x \to X_{\mathbb{F}}$. This gives $\mathrm{FWeil}(X)$ the structure of a locally profinite groupoid.

Let Λ be either of the following coherent topological rings: a coherent discrete ring, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for some prime ℓ, or its ring of integers $O_E \supset \mathbb{Z}_\ell$. For a topological groupoid W, we will denote by $\mathrm{Rep}_\Lambda(W)$ the category of continuous representations of W with values in finitely presented Λ-modules and by $\mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W) \subset \mathrm{Rep}_\Lambda(W)$ its full subcategory of representations on finite projective Λ-modules. Here finitely presented Λ-modules M carry the quotient topology induced from the choice of any surjection $\Lambda^n \to M$, $n \geq 0$ and the product topology on Λ^n.

Lemma 6.5.2. *In the situation above, the category* $\mathrm{Rep}_\Lambda(W)$ *is* Λ_**-linear and abelian. In particular, its full subcategory* $\mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$ *is* Λ_**-linear and additive.*

Proof. Let W_{disc} be the discrete groupoid underlying W, and denote by $\mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ the category of W_{disc}-representations on finitely presented Λ-modules. Evidently, this category is Λ_*-linear. It is abelian since Λ is coherent. We claim that $\mathrm{Rep}_\Lambda(W) \subset \mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ is a Λ_*-linear full abelian subcategory. If Λ is discrete (and coherent), then every finitely presented Λ-module carries the discrete topology and the claim is immediate, see also (Stacks, Tag 0A2H). For $\Lambda = E, O_E$, one checks that every map of finitely presented Λ-modules is continuous, every surjective map is a topological quotient and every injective map is a closed embedding. For the latter, we use that every finitely presented Λ-module can be written as a countable filtered colimit of compact Hausdorff spaces along injections, and that every injection of compact Hausdorff spaces is a closed embedding. This implies the claim. □

We apply this for W being either of the locally profinite groupoids $\pi_1(X)$, $\pi_1(X_{\mathbb{F}})$ or $\mathrm{FWeil}(X)$. Note that restricting representations along $\pi_1(X_{\mathbb{F}}) \to \mathrm{FWeil}(X)$ induces an equivalence of Λ_*-linear abelian categories

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \cong \mathrm{Fix}\left(\mathrm{Rep}_\Lambda\big(\pi_1(X_{\mathbb{F}})\big), \phi_{X_1}, \ldots, \phi_{X_n}\right), \tag{6.13}$$

and similarly for the Λ_*-linear additive category $\mathrm{Rep}^{\mathrm{f.p}}_\Lambda\big(\mathrm{FWeil}(X)\big)$.

Definition 6.5.3. *For an integer $n \geq 0$, we write $\mathrm{D}^{\{-n,n\}}_{\mathrm{lis}}(X, \Lambda)$ for the full subcategory of $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ of objects M such that M and its dual $M^\vee$ lie in degrees $[-n, n]$ with respect to the t-structure on $\mathrm{D}(X, \Lambda)$.*

Lemma 6.5.4. *In the situation above, there is a natural functor*

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \to \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)^\heartsuit, \tag{6.14}$$

that is fully faithful. Moreover, the following properties hold if Λ is either finite discrete or $\Lambda = \mathcal{O}_E$ for $E \supset \mathbb{Q}_\ell$ finite:

1. *An object M lies in the essential image of Equation* (6.14) *if and only if its underlying sheaf $M_{\mathbb{F}}$ is locally on $(X_{\mathbb{F}})_{\mathrm{proét}}$ isomorphic to $\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_{X_{\mathbb{F}}}$ for some finitely presented Λ_*-module N.*
2. *The functor* (6.14) *restricts to an equivalence of Λ_*-linear additive categories*
$$\mathrm{Rep}^{\mathrm{f.p}}_\Lambda\big(\mathrm{FWeil}(X)\big) \xrightarrow{\cong} \mathrm{D}^{\{0,0\}}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big).$$
3. *If Λ_* is regular (so that Λ is t-admissible, cf. Chapter 5 Item vi), then Equation* (6.14) *restricts to an equivalence of Λ_*-linear abelian categories*
$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)^\heartsuit.$$

If all X_i, $i = 1, \ldots, n$ are geometrically unibranch, then (1), (2) *and* (3) *hold for general coherent topological rings Λ as above.*

Proof. There is a canonical equivalence of topological groupoids $\pi_1(X_{\mathbb{F}}) \cong \widehat{\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})}$ with the profinite completion of the proétale fundamental groupoid, see (Bhatt and Scholze, 2015, Lemma 7.4.3). It follows from (Bhatt and Scholze, 2015, Lemmas 7.4.5, 7.4.7) that restricting representations along $\pi_1^{\mathrm{proét}}(X_{\mathbb{F}}) \to \pi_1(X_{\mathbb{F}})$ induces full embeddings

$$\mathrm{Rep}_\Lambda\big(\pi_1(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{Rep}_\Lambda\big(\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit, \tag{6.15}$$

that are compatible with the action of ϕ_{X_i} for all $i = 1, \dots, n$. So we obtain the fully faithful functor (6.14) by passing to fixed points, see (6.13), (6.7) and Lemma 4.2.2 (see also 4.2.3).

Part Item 1 describes the essential image of $\mathrm{Rep}_\Lambda\big(\pi_1^{\text{proét}}(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^{\heartsuit}$. So if Λ is finite discrete or profinite, then the first functor in (6.15) is an equivalence, and we are done. Part Item 2 is immediate from Item 1, noting that an object in the essential image of Equation (6.15) is lisse if and only if its underlying module is finite projective. Likewise, part Item 3 is immediate from Item 1, using Item vii. Here we need to exclude rings like $\Lambda = \mathbb{Z}/\ell^2$ in order to have a t-structure on lisse sheaves.

Finally, if all X_i are geometrically unibranch, so is $X_{\mathbb{F}}$ which follows from the characterization (Stacks, Tag 0BQ4). In this case, we get $\pi_1(X_{\mathbb{F}}) \cong \pi_1^{\text{proét}}(X_{\mathbb{F}})$ by (Bhatt and Scholze, 2015, Lemma 7.4.10). This finishes the proof. □

6.6 Weil-étale versus étale sheaves

We end this section with the following description of Weil sheaves with (ind-)finite coefficients. Note that such a simplification in terms of ordinary sheaves is not possible for $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say.

Proposition 6.6.1. *Let X be a qcqs $\mathbb{F}_q$-scheme. Let Λ be a finite discrete ring or a filtered colimit of such rings. Then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \to \mathrm{D}_{\mathrm{lis}}\big(X^{\mathrm{Weil}}, \Lambda\big), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \to \mathrm{D}_{\mathrm{cons}}\big(X^{\mathrm{Weil}}, \Lambda\big),$$

are equivalences.

Proof. Throughout, we repeatedly use that filtered colimits commute with finite limits in Cat_∞. Using compatibility of $\mathrm{D}_{\mathrm{cons}}$ with filtered colimits in Λ (Chapter 5 Item iv), we may assume that Λ is finite discrete. By the comparison result with the classical bounded derived category of constructible sheaves, we can identify the categories $\mathrm{D}_\bullet(X, \Lambda)$, resp. $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$ for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$ with full subcategories of the derived category of étale Λ-sheaves $\mathrm{D}(X_{\acute{e}t}, \Lambda)$, resp. $\mathrm{D}(X_{\mathbb{F},\acute{e}t}, \Lambda)$. Write $X = \lim X_i$ as a cofiltered limit of finite type $\mathbb{F}_q$-schemes X_i with affine transition maps (Stacks, Tag 01ZA). Using the continuity of étale sites (Stacks, Tag 03Q4), there are natural equivalences

$$\operatorname{colim} \mathrm{D}_\bullet(X_i, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_\bullet(X_i^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \quad (6.16)$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Hence, we can assume that X is of finite type over $\mathbb{F}_q$.

To show full faithfulness, we claim more generally that the natural map

$$\mathrm{D}(X_{\acute{e}t}, \Lambda) \to \lim \left(\mathrm{D}(X_{\mathbb{F},\acute{e}t}, \Lambda) \underset{\mathrm{id}}{\overset{\phi_X^*}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F},\acute{e}t}, \Lambda) \right) =: \mathrm{D}\big(X_{\acute{e}t}^{\mathrm{Weil}}, \Lambda\big)$$

is fully faithful. As Λ is torsion, this is immediate from (Geisser, 2004, Corollary 5.2) applied to the inner homomorphisms between sheaves.

Let us add that this induces fully faithful functors

$$\mathrm{D}^+(X_{\acute{e}t}, \Lambda) \to \mathrm{D}^+\big(X_{\acute{e}t}^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big) \tag{6.17}$$

on bounded below objects, see (Bhatt and Scholze, 2015, Proposition 5.2.6 (1)).

It remains to prove essential surjectivity. Using a stratification as in 6.4.1, it is enough to consider the lisse case. Pick $M \in \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda)$. It is enough to show that M lies is in the essential image of Equation (6.17), noting that the functor detects dualizability. As M is bounded, this will follow from showing that for every $j \in \mathbb{Z}$, the cohomology sheaf $\mathrm{H}^j(M) \in \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^{\heartsuit}$ is in the essential image of Equation (6.17).

Fix $j \in \mathbb{Z}$. As M is lisse, the underlying sheaf $\mathrm{H}^j(M)_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)^{\heartsuit}$ is proétale-locally constant (Chapter 5 Item vii) and valued in finitely presented Λ-modules. By Lemma 6.5.4 Item 1, it comes from a representation of $\mathrm{Weil}(X)$. Restriction of representations along $\mathrm{Weil}(X) \to \pi_1(X)$ fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda\big(\pi_1(X)\big) \arrow[r, "\cong"] \arrow[d] & \mathrm{Rep}_\Lambda\big(\mathrm{Weil}(X)\big) \arrow[d] \\
\mathrm{D}(X_{\acute{e}t}, \Lambda)^{\heartsuit} \arrow[r] & \mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big)^{\heartsuit},
\end{tikzcd}$$

where the upper horizontal arrow is an equivalence since Λ is finite. In particular, the object $\mathrm{H}^j(M)$ is in the essential image of the fully faithful functor Equation (6.17).

$\square$

Chapter 7

THE CATEGORICAL KÜNNETH FORMULA

7.1 The main result

We continue with the notation of Chapter 6. In particular, $\mathbb{F}_q$ denotes a finite field of characteristic $p > 0$. Recall from Chapter 4 the tensor product of Λ_*-linear idempotent complete stable ∞-categories. The external tensor product of sheaves $(M_1, \ldots, M_n) \mapsto M_1 \boxtimes \ldots \boxtimes M_n$ as in Equation (5.2) induces a functor

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big), \quad (7.1)$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Throughout, we consider the following situation. In Remark 7.1.3 we explain the compatibility of Equation (7.1) with certain (co-)limits in the schemes X_i and coefficients Λ, which allows to relax these assumptions on X and Λ somewhat.

Situation 7.1.1. *The schemes $X_1, \ldots, X_n$ are of finite type over $\mathbb{F}_q$, and Λ is the condensed ring associated with one of the following topological rings:*

(a) a finite discrete ring of prime-to-p-torsion;

(b) the ring of integers $\mathcal{O}_E$ of an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\bar{\mathbb{Z}}_\ell$);

(c) an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ (for example $\bar{\mathbb{Q}}_\ell$);

(d) a finite discrete p-torsion ring that is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$.

Theorem 7.1.2. *In Situation 7.1.1, the functor Equation* (7.1) *is an equivalence in each of the following cases:*

1. $\bullet = \mathrm{cons}$ *and Λ is as in Item a, Item b or Item c;*
2. $\bullet = \mathrm{lis}$ *and Λ is as in Item a, Item b, Item d or as in Item c if all X_i, $i = 1, \ldots, n$ are geometrically unibranch (for example, normal).*

In the p-torsion free cases Item a, Item b and Item c, the full faithfulness is a direct consequence of the Künneth formula applied to the $X_{i,\mathbb{F}}$. In the p-torsion case Item d, we use Artin–Schreier theory instead. It would be interesting to see whether this part can be extended to constructible sheaves using the mod-p-Riemann–Hilbert correspondence as in, say, (Bhatt and Lurie, 2019). In all cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations.

Before turning to the proof of Theorem 7.1.2, we record the following compatibility of the functor Equation (7.1) with (co-)limits. This can be used to reduce the case of an (infinite) algebraic extension $E \supset \mathbb{Q}_\ell$ in cases Item b and Item c above to the case where $E \supset \mathbb{Q}_\ell$ is finite. In the sequel we will therefore assume E is finite in these cases. Remark 7.1.3 can further be used to extend Theorem 7.1.2 to qcqs $\mathbb{F}_q$-schemes X_i and finite discrete rings like $\mathbb{Z}/m$ for any integer $m \geq 1$ in cases Item a and Item d.

Remark 7.1.3 (Compatibility of (7.1) with certain (co-)limits)**.** *Throughout, we repeatedly use that filtered colimits commute with finite limits in* $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$*: the forgetful functors* $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem}) \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty}(\mathrm{Idem}) \to \mathrm{Cat}_\infty$ *create these (co)limits (Lurie, 2017, Theorem 1.1.4.4), (Lurie, 2009, Corollary 4.4.5.21), and the statement holds in any compactly generated ∞-category, such as* Cat_∞ *(Bhatt and Mathew, 2021, Example 3.6(3)). We will also throughout use that in all the stable ∞-categories encountered below the tensor product preserves colimits and in particular finite limits.*

1. *Filtered colimits in Λ. First off, extension of scalars along any map of condensed rings $\Lambda \to \Lambda'$ induces a commutative diagram in* $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda_*}(\mathrm{Idem})$*:*

$$\begin{tikzcd} \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \dots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \arrow[r] \arrow[d] & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \dots \times X_n^{\mathrm{Weil}}, \Lambda) \arrow[d] \\ \mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda') \otimes_{\mathrm{Perf}_{\Lambda'_*}} \dots \otimes_{\mathrm{Perf}_{\Lambda'_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda') \arrow[r] & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \dots \times X_n^{\mathrm{Weil}}, \Lambda') \end{tikzcd}$$

   It follows from the compatibility of $\mathrm{D}_{\mathrm{cons}}$ *with filtered colimits in Λ (Chapter 5 Item iv) that both sides of Equation* (7.1) *are compatible with filtered colimits in Λ.*

2. *Finite products in Λ. Let $\Lambda = \prod \Lambda_i$ be a finite product of condensed rings. For any scheme X, the natural map* $\mathrm{D}_\bullet(X, \Lambda) \to \prod \mathrm{D}_\bullet(X, \Lambda_i)$ *is an equivalence for* $\bullet \in \{\emptyset, \mathrm{lis}, \mathrm{cons}\}$*, and likewise for Weil sheaves if X is defined over $\mathbb{F}_q$. As $\Lambda_* = \prod \Lambda_{i,*}$, we see that* (7.1) *is compatible finite products in the coefficients.*

3. *Limits in X_i for discrete Λ. Assume that Λ is finite discrete, see Situation 7.1.1 Item a, Item d. Let $X_1, \ldots, X_n$ be qcqs $\mathbb{F}_q$-schemes. Write each X_i as a cofiltered limit $X_i = \lim X_{ij}$ of finite type $\mathbb{F}_q$-schemes X_{ij} with affine transition maps (Stacks, Tag 01ZA). As Λ is finite discrete, we can use the continuity of étale sites as in Equation* (6.16) *to show that the natural map*

$$\operatorname*{colim}_j \mathrm{D}_\bullet\big(X_{1j}^{\mathrm{Weil}} \times \ldots \times X_{nj}^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big),$$

is an equivalence for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. *Thus, Equation* (7.1) *is compatible with cofiltered limits of finite type $\mathbb{F}_q$-schemes with affine transition maps.*

7.2 A formulation in terms of prestacks

Before turning to the proof, we point out a formulation of the results of the previous subsection in terms of symmetric monoidality of a certain sheaf theory. This formulation makes the connection with constructions in the geometric approaches to the Langlands program (Dennis Gaitsgory, Kazhdan, et al., 2022; Zhu, 2021; V. Lafforgue and Zhu, 2019) more manifest. Readers not familiar with prestacks and formulations of sheaf theories on them can safely skip this section. The categories of constructible, resp. lisse Λ-sheaves assemble into a lax symmetric monoidal functor

$$\mathrm{D}_{\bullet,\Lambda}\colon (\mathrm{Sch}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})\ (\bullet = \mathrm{lis\ or\ cons}). \tag{7.2}$$

Namely, as a functor it sends a scheme X to the category of constructible, resp. lisse Λ-sheaves on X, and a morphism $f\colon X \to Y$ to the functor $f^*\colon \mathrm{D}_\bullet(Y, \Lambda) \to \mathrm{D}_\bullet(X, \Lambda)$. These are objects, resp. maps in the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) := \mathrm{Mod}_{\mathrm{Perf}_\Lambda}(\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}))$, cf. Section 4.1 for notation. The lax monoidal structure is given by the external tensor product of sheaves:

$$\boxtimes\colon \mathrm{D}_\bullet(X_{\mathrm{proét}}, \Lambda) \otimes_{\mathrm{Perf}_\Lambda} \mathrm{D}_\bullet(Y_{\mathrm{proét}}, \Lambda) \to \mathrm{D}_\bullet((X \times_{\mathbb{F}} Y)_{\mathrm{proét}}, \Lambda).$$

That is, we consider the category of schemes as symmetric monoidal with respect to the fiber product over $\mathbb{F}$, and the external tensor product is natural on X and Y in the appropriate sense, see (Dennis Gaitsgory and Lurie, 2019, Section 3.1), (Dennis Gaitsgory and Rozenblyum, 2017b, Section III.2) for details and precise statements. This functor $\boxtimes$ often fails to be an equivalence, so $\mathrm{D}_{\bullet,\Lambda}$ is not symmetric monoidal. The assertion of Theorem 7.1.2 is that this issue is resolved by replacing sheaves with Weil sheaves.

In order to formulate 7.1.2 as the monoidality of a certain functor, we need to replace the category of schemes by a category of objects that model Weil sheaves. We will represent these by taking the appropriate formal quotient by the partial Frobenius automorphism. Such formal quotients can be taken in the category of prestacks.

We denote by $\mathrm{PreStk}_{\mathbb{F}}$ the category of (accessible) functors from the category $\mathrm{CAlg}_{\mathbb{F}}$ of commutative algebras over $\mathbb{F}$ to the ∞-category Ani of Anima. The functor of taking points embeds the category of schemes fully faithfully into $\mathrm{PreStk}_{\mathbb{F}}$.

We denote by

$$\mathrm{D}_{\bullet,\Lambda}\colon (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \tag{7.3}$$

the functor obtained by right Kan extension (Lurie, 2009, §4.3.2) along the inclusion $(\mathrm{Sch}^{\mathrm{fp}}_{\mathbb{F}})^{\mathrm{op}} \subset (\mathrm{PreStk}_{\mathbb{F}})^{\mathrm{op}}$. Concretely, (Lurie, 2018, Proposition 6.2.1.9, Proposition 6.2.3.1), given a prestack Y which can be written as a colimit of schemes Y_α over some indexing category A we have a canonical equivalence

$$\mathrm{D}_\bullet(Y,\Lambda) \cong \lim_{\alpha} \mathrm{D}_\bullet(Y_\alpha,\Lambda). \tag{7.4}$$

This limit is formed in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$; recall from around (4.3) that the Ind-completion functor to $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}) \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$ does *not* preserve (even finite) limits.

With this general sheaf theory in place, we can restrict our attention to the class of prestacks that is relevant to the derived Drinfeld lemma.

Definition 7.2.1. *Let X be a scheme over $\mathbb{F}_q$. The Weil prestack is defined as*

$$X^{\mathrm{Weil}} := \mathrm{colim}\left(X \times_{\mathbb{F}_q} \mathbb{F} \underset{\mathrm{id}}{\overset{\phi_X}{\rightrightarrows}} X \times_{\mathbb{F}_q} \mathbb{F} \right) \in \mathrm{PreStk}_{\mathbb{F}},$$

i.e., it is the prestack sending $R \in \mathrm{CAlg}_{\mathbb{F}}$ to the colimit

$$X^{\mathrm{Weil}}(R) = \mathrm{colim}\left(X(R) \underset{\mathrm{id}}{\overset{\phi_X}{\rightrightarrows}} X(R) \right). \tag{7.5}$$

We denote by $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ the smallest full monoidal subcategory of $\mathrm{PreStk}_{\mathbb{F}}$ containing the Weil prestacks of finite type schemes $X/\mathbb{F}_q$. Equivalently, this is the full subcategory consisting of finite products of the form $X_1^{\mathrm{Weil}} \times \cdots \times X_n^{\mathrm{Weil}}$.

Lemma 7.2.2. *Let $X_1, \ldots, X_n$ be schemes over $\mathbb{F}_q$. There is a canonical equivalence*

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}) \xrightarrow{\cong} \mathrm{Fix}\big(\mathrm{D}_\bullet(X_{\mathbb{F}},\Lambda), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\big) \tag{7.6}$$

Proof. Let $\Phi\colon \mathrm{B}\mathbb{Z}^n \to \mathrm{PreStk}_{\mathbb{F}}$ be the functor corresponding to the commuting automorphisms ϕ_{X_i}. Then the claim follows immediately from the identification of $X_1^{\mathrm{Weil}} \times_{\mathbb{F}} \cdots \times_{\mathbb{F}} X_n^{\mathrm{Weil}}$ with the colimit of Φ (as an object in $\mathrm{PreStk}_{\mathbb{F}}$). □

Theorem 7.2.3. *Suppose $\bullet$ and Λ are as in 7.1.2. Then the restriction of* $\mathrm{D}_{\bullet,\Lambda}$ *to Weil prestacks, i.e., the following composite*

$$\mathrm{D}_{\bullet,\Lambda} \colon (\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}})^{\mathrm{op}} \subset \mathrm{PreStk}_{\mathbb{F}} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem}), \tag{7.7}$$

is symmetric monoidal.

Proof. As was noted above, the functor in Equation (7.2) is lax symmetric monoidal. By (Torii, 2022, Proposition 2.7), the Kan extension in Equation (7.3) is still lax symmetric monoidal. To check its restriction to the (symmetric monoidal) subcategory $\mathrm{Sch}^{\mathrm{fp}}_{\mathrm{Weil}}$ is symmetric monoidal it suffices to show that the lax monoidal maps are in fact isomorphisms. This is precisely the content of Theorem 7.1.2. □

7.3 Full faithfulness

In this section, we prove that the functor Equation (7.1) is fully faithful under the conditions of Theorem 7.1.2. We first consider the p-torsion free cases:

Proposition 7.3.1. *Let $X_1, \dots, X_n$ and Λ be as in Situation 7.1.1 Item a, Item b or Item c. Then the functor Equation* (7.1) *is fully faithful for* $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.

Proof. In a nutshell, this is an instance of the Künneth formula: for constructible sheaves on $X_{i,\mathbb{F}}$ (as opposed to X_i^{Weil}), this interpretation of the Künneth formula appears already in (Dennis Gaitsgory, Kazhdan, et al., 2022, Section A.2). Throughout, we drop Λ from the notation. It is enough to verify that for all $M_i, N_i \in \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}})$ the natural map

$$\bigotimes_{i=1}^{n} \mathrm{Hom}_{\mathrm{D}(X_i^{\mathrm{Weil}})}(M_i, N_i) \to \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}})}(M_1 \boxtimes \ldots \boxtimes M_n, N_1 \boxtimes \ldots \boxtimes N_n) \tag{7.8}$$

is an equivalence. As Equation (7.8) is functorial in the objects and compatible with shifts, it suffices, by Definition 6.4.1, to consider the case where M_i, $i = 1, \dots, n$ is the extension by zero of a lisse Weil Λ-sheaf on some locally closed subscheme $Z_i \subset X_i$. Using the adjunction

$$(\iota_i)_! : \mathrm{D}_{\mathrm{cons}}(Z_i^{\mathrm{Weil}}) \rightleftarrows \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}}) : (\iota_i)^!,$$

and the dualizability of lisse sheaves, we reduce to the case $M_i = \Lambda_{X_i}$, $i = 1, \dots, n$. That is, Equation (7.8) becomes a map of cohomology complexes.

By Proposition 6.1.4, we have

$$\mathrm{R}\Gamma\big(X_i^{\mathrm{Weil}}, N_i\big) = \mathrm{Fib}\left(\mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i) \xrightarrow{\phi_{X_i}^* - \mathrm{id}} \mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i)\right).$$

A similar computation holds for the mapping complexes in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}})$, see Equation (6.6). Such finite limits commute with the tensor product in Mod_Λ.

Thus, Section 7.3 reduces to the Künneth formula

$$\mathrm{R}\Gamma\big(X_{1,\mathbb{F}}, N_1\big) \otimes \ldots \otimes \mathrm{R}\Gamma\big(X_{n,\mathbb{F}}, N_n\big) \xrightarrow{\cong} \mathrm{R}\Gamma\big(X_{1,\mathbb{F}} \times_{\mathbb{F}} \ldots \times_{\mathbb{F}} X_{n,\mathbb{F}}, N_1 \boxtimes \ldots \boxtimes N_n\big),$$

where we use that the X_i are of finite type and the coprimality assumptions on Λ, see (Stacks, Tag 0F1P). □

Next, we consider the p-torsion case:

Proposition 7.3.2. *Let $X_1, \ldots, X_n$ and Λ be as in Situation 7.1.1 Item d. Then the functor* (7.1) *is fully faithful for* $\bullet = \mathrm{lis}$.

Proof. As in the proof of Proposition 7.3.1, we need to show that the map

$$\bigotimes_{i=1}^{n} \mathrm{R}\Gamma\big(X_i^{\mathrm{Weil}}, N_i\big) \to \mathrm{R}\Gamma\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, N_1 \boxtimes \ldots \boxtimes N_n\big) \tag{7.9}$$

is an equivalence for any $N_i \in \mathrm{D}_{\mathrm{lis}}(X_i^{\mathrm{Weil}})$. Using Zariski descent for both sides, we may assume that each X_i is affine. As Λ is finite discrete (see also the discussion around (6.16)), the invariance of the étale site under perfection reduces us to the case where each X_i is perfect. The proof now proceeds by several reduction steps: 1) reduce to $N_i = \Lambda_{X_i}$; 2) reduce to $\Lambda = \mathbb{Z}/p$; 3) reduce to $q = p$ being a prime. The last step 4) is then an easy computation.

Step 1): *We may assume $N_i = \Lambda_{X_i}$.* In order to show Equation (7.9) is a quasi-isomorphism, it suffices to show this after applying $\tau^{\leq r}$ for arbitrary r. The complexes N_i are bounded (Chapter 5 Item v). By shifting them appropriately, we may assume $r = 0$. Note that $\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \cong \mathrm{R}\Gamma(X_i, N_i)$, see Proposition 6.6.1. By right exactness of the tensor product, we have $\tau^{\leq 0}\left(\bigotimes_i \mathrm{R}\Gamma(X_i, N_i)\right) = \bigotimes_i \tau^{\leq 0}\mathrm{R}\Gamma(X_i, N_i)$. Using the compatibility with the classical notion of constructible sheaves, there is an étale covering $U_i \to X_i$ such that $N_i|_{U_i}$ is perfect-constant. Let $U_{i,\bullet}$ be the Čech nerve of this covering. By étale descent, we have

$$\mathrm{R}\Gamma(X_i, N_i) = \lim_{[j]\in\Delta} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

For each $r \in \mathbb{Z}$, there is some j_r such that

$$\tau^{\leq r} \lim_{[j] \in \Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \lim_{[j] \in \Delta, j \leq j_r} \tau^{\leq r} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

This can be seen from the spectral sequence (note that it is concentrated in degrees $j \geq 0$ and degrees $j' \geq r$ for some r, since the complexes N_i are bounded from below)

$$\mathrm{H}^{j'}(U_{i,j}, N_i) \Rightarrow \mathrm{H}^{j'+j} \lim_{j \in \Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \mathrm{H}^{j'+j}(X_i, N_i).$$

As the tensor product in Equation (7.9) commutes with *finite* limits, we may thus assume that each N_i is perfect-constant. Another dévissage reduces us to the case $N_i = \Lambda_{X_i}$, the constant sheaf itself.

Step 2): *We may assume* $\Lambda = \mathbb{Z}/p$. By assumption, Λ is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$. We immediately reduce to $\Lambda = \mathbb{Z}/p^m$. For any perfect affine scheme $X = \operatorname{Spec} R$ in characteristic $p > 0$, we claim that $\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \cong \mathrm{R}\Gamma(X, \mathbb{Z}/p^r)$. Assuming the claim, we finish the reduction step by tensoring Equation (7.9) with the short exact sequence of $\mathbb{Z}/p^m$-modules $0 \to \mathbb{Z}/p^{m-1} \to \mathbb{Z}/p^m \to \mathbb{Z}/p \to 0$, using that finite limits commutes with tensor products.

It remains to prove the claim. The Artin–Schreier–Witt exact sequence of sheaves on $X_{\acute{e}t}$ yields

$$\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) = [W_m(R) \stackrel{F-\mathrm{id}}{\to} W_m(R)].$$

Now we use that $W_m(R) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \stackrel{\cong}{\to} W_r(R)$ compatibly with F, which holds since R is perfect. This shows the claim, and we have accomplished Step 2).

Step 3): *We may assume q is prime.* Recall that $q = p^r$ is a prime power. In order to reduce to the case $r = 1$, let $X_i' := X_i$, but now regarded as a scheme over $\mathbb{F}_p$. We have $X'_{i,\mathbb{F}} = \bigsqcup_{i=1}^r X_{i,\mathbb{F}}$. The Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is generated by the p-Frobenius, which acts by permuting the components in this disjoint union. Thus, we have $\mathrm{D}((X_i')^{\mathrm{Weil}}) = \mathrm{D}(X_i^{\mathrm{Weil}})$. The same reasoning also applies to several factors X_i^{Weil}, so we may assume our ground field to be $\mathbb{F}_p$.

Step 4): Set $R := \bigotimes_{i,\mathbb{F}_p} R_i$, $R_{\mathbb{F}} := R \otimes_{\mathbb{F}_p} \mathbb{F}$. We write ϕ_i for the p-Frobenius on R_i and also for any map on a tensor product involving R_i, by taking the identity on the remaining tensor factors. By Artin–Schreier theory, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, \mathbb{Z}/p) \stackrel{6.6.1}{=} \mathrm{R}\Gamma(X_i, \mathbb{Z}/p) = [R_i \stackrel{\phi_i - \mathrm{id}}{\to} R_i],$$

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_{\mathbb{F}} \ldots \times_{\mathbb{F}} X_{n,\mathbb{F}}, \mathbb{Z}/p) = [R_{\mathbb{F}} \stackrel{\phi - \mathrm{id}}{\to} R_{\mathbb{F}}],$$

where ϕ is the absolute p-Frobenius of $R_{\mathbb{F}}$. Thus, the right hand side in (7.9) is the homotopy orbits of the action of $\mathbb{Z}^{n+1}$ on $R_{\mathbb{F}}$, whose basis vectors act as $\phi_1, \ldots, \phi_n$ and ϕ. Note that ϕ is the composite $\phi_{\mathbb{F}} \circ \phi_1 \circ \cdots \circ \phi_n$, where $\phi_{\mathbb{F}}$ is the Frobenius on $\mathbb{F}$. Thus, the previously mentioned $\mathbb{Z}^{n+1}$-action on $R_{\mathbb{F}}$ is equivalent to the one where the basis vectors act as $\phi_1, \ldots, \phi_n$ and $\phi_{\mathbb{F}}$. We conclude our claim by using that $[R_{\mathbb{F}} \overset{\mathrm{id}-\phi_{\mathbb{F}}}{\to} R_{\mathbb{F}}]$ is quasi-isomorphic to $R[0]$. □

7.4 Drinfeld's lemma

The essential surjectivity in Theorem 7.1.2 is based on the following variant of Drinfeld's lemma (V. G. Drinfeld, 1980, Theorem 2.1) (see also (L. Lafforgue, 1997, IV.2, Theorem 4), (Lau, 2004, Theorem 8.1.4), (V. Lafforgue, 2018, Lemme 8.11), and (Kedlaya, 2019, Theorem 4.2.12), (Heinloth, 2018, Lemma 6.3), (Scholze and Weinstein, 2020, Theorem 16.2.4) for expositions). Its formulation is close to (Lau, 2004, Theorem 8.1.4), and in this form is a slight extension of (V. Lafforgue, 2018, Lemme 8.2) for $\mathbb{Z}_\ell$-coefficients and (Xue, 2020b, Lemma 3.3.2) for $\mathbb{Q}_\ell$-coefficients. We will drop the coefficient ring Λ from the notation whenever convenient.

Let $X_1, \ldots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and denote $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Recall the Frobenius–Weil groupoid $\mathrm{FWeil}(X)$, see Definition 6.5.1. The projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ onto the single factors induce a continuous map of locally profinite groupoids

$$\mu \colon \mathrm{FWeil}(X) \to \mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n). \tag{7.10}$$

Theorem 7.4.1 (Drinfelds's lemma)**.** *Let Λ be as in Situation 7.1.1. Restriction along the map Equation* (7.10) *induces an equivalence*

$$\mathrm{Rep}_\Lambda\big(\mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n)\big) \xrightarrow{\cong} \mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big), \tag{7.11}$$

between the abelian categories of continuous representations on finitely presented Λ-modules.

Proof. For all objects $x \in \mathrm{FWeil}(X)$, that is, all geometric points $x \to X_{\mathbb{F}}$, passing to the automorphism groups induces a commutative diagram of locally profinite groups

$$\begin{tikzcd}
1 \arrow[r] & \pi_1(X_{\mathbb{F}}, x) \arrow[r] \arrow[d] & \mathrm{FWeil}(X, x) \arrow[r] \arrow[d, "\mu_x"] & \mathbb{Z}^n \arrow[d, equal] \\
1 \arrow[r] & \prod_{i=1}^n \pi_1(X_{i,\mathbb{F}}, x) \arrow[r] & \prod_{i=1}^n \mathrm{Weil}(X_i, x) \arrow[r] & \mathbb{Z}^n.
\end{tikzcd}$$

The left vertical arrow is surjective (Stacks, Tag 0BN6)0385. Thus μ_x is surjective as well and hence (7.11) is fully faithful. For essential surjectivity, it remains to show that any continuous representation $\mathrm{FWeil}(X,x) \to \mathrm{GL}(M)$ on a finitely presented Λ-module M factors through μ_x. The key input is Drinfeld's lemma: it implies that μ_x induces an isomorphism on profinite completions. Therefore, it is enough to apply Lemma 7.4.2 below with $H := \mathrm{FWeil}(X,x) \to \mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n) =: G$ and $K := \pi_1(X_{\mathbb{F}}, x)$. This completes the proof of (7.11). □

The following lemma formalizes a few arguments from (Xue, 2020b, §3.2.3), and we reproduce the proof for the convenience of the reader:

Lemma 7.4.2 (Drinfeld, Xue)**.** *Let Λ be as in 7.1.1. Let $\mu\colon H \to G$ be a continuous surjection of locally profinite groups that induces an isomorphism on profinite completions. Assume that there exists a compact open normal subgroup $K \subset H$ containing* $\ker \mu$ *such that H/K is finitely generated and injects into its profinite completion. Then μ induces an equivalence*

$$\mathrm{Rep}_\Lambda(G) \cong \mathrm{Rep}_\Lambda(H)$$

between their categories of continuous representations on finitely presented Λ-modules.

Proof. The case where Λ is finite discrete is obvious, and hence so is the case $\Lambda = O_E$ for some finite field extension $E \supset \mathbb{Q}_\ell$. The case $\Lambda = E$ is reduced to $\Lambda = \mathbb{Q}_\ell$. As μ is surjective, it remains to show that every continuous representation $\rho\colon H \to \mathrm{GL}(M)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space factors through G, that is, $\ker \mu \subset \ker \rho$. One shows the following properties:

1. The group $\ker \mu$ is the intersection over all open subgroups in K which are normal in H.
2. The group $\ker \rho \cap K$ is a closed normal subgroup in H such that $K/\ker \rho \cap K \cong \rho(K)$ is topologically finitely generated.

These properties imply $\ker \mu \subset \ker \rho \cap K$ as follows: For a finite group L, let $U_L := \cap \ker(K \to L)$ where the intersection is over all continuous morphisms $K \to L$ that are trivial on $\ker \rho \cap K$. Because of the topologically finitely generatedness in (2), this is a finite intersection so that U_L is open in K. Also, it is normal in H, and hence $\ker\mu \subset U_L$ by (1). On the other hand, it is evident that $\ker\rho \cap K = \cap_L U_L$ because K is profinite.

For the proof of (1) observe that $\ker\mu$ agrees with the kernel of $H \to H^\wedge \cong G^\wedge$ by our assumption on the profinite completions. Using $\ker\mu \subset K$ and the injection $H/K \to (H/K)^\wedge$ implies (1).

For (2) it is evident that $\ker\rho \cap K$ is a closed normal subgroup in H. Since K is compact, its image $\rho(K)$ is a closed subgroup of the ℓ-adic Lie group $\mathrm{GL}(M)$, hence an ℓ-adic Lie group itself. The final assertion follows from (Serre, 1964, théorème 2). $\square$

For the overall goal of proving essential surjectivity in Theorem 7.1.2, we need to investigate how representations of product groups factorize into external tensor products of representations. In view of Lemma 6.5.2 and its proof, it is enough to consider representations of abstract groups, disregarding the topology. This is done in the next section.

7.5 Factorizing representations

In this subsection, let Λ be a Dedekind domain (Stacks, Tag 034X). Thus, since Λ is Noetherian and of projective dimension ≤ 1, any submodule N of a finite projective Λ-module M is again finite projective.

Given any group W, we write $\mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$ for the category of W-representations on finite projective Λ-modules.

As in (Curtis and Reiner, 2006, Sections 73.8, 75), we say that such a W-representation M is *fp-simple* if any subrepresentation $0 \neq N \subset M$ has maximal rank. By induction on the rank, every non-zero representation in $\mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$ admits a non-zero fp-simple subrepresentation. The proof of the following lemma is left to the reader. It parallels (Curtis and Reiner, 2006, Theorem 75.6).

Lemma 7.5.1. *A representation $M \in \mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$ is fp-simple if and only if $M \otimes_\Lambda \mathrm{Frac}(\Lambda)$ is fp-simple (hence, simple).*

The following proposition will serve in the proof of Theorem 7.1.2 using Theorem 7.4.1, where we will need to decompose representations of a product of Weil groups into decompositions of the individual Weil groups.

Proposition 7.5.2. *Let $W = W_1 \times W_2$ be a product of two groups. Let $M \in \mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$ be fp-simple. Fix a W_1-subrepresentation $M_1 \subset M$ that is fp-simple. Consider the W_2-representation $M_2 := \mathrm{Hom}_{W_1}(M_1, M)$ and the associated evaluation map*

$$\mathrm{ev}\colon M_1 \boxtimes M_2 \to M.$$

1. *If Λ is an algebraically closed field, then* ev *is an isomorphism and M_2 is simple.*
2. *If Λ is a perfect field, then* ev *is a split surjection and M_2 is semi-simple.*
3. *If Λ is a Dedekind domain of Krull dimension* 1 *with perfect fraction field, then there is a short exact sequence*

$$0 \to M \oplus \ker(\mathrm{ev}) \to M_1 \boxtimes M_2 \to T \to 0, \tag{7.12}$$

where T is Λ-torsion.

Proof. Note that ev is a map in $\mathrm{Rep}_\Lambda^{\mathrm{f.p}}(W)$. Its image has maximal rank by the fp-simplicity of M. Thus, if Λ is a field, then it is surjective.

In case Item 1, we claim that ev is an isomorphism. The following argument was explained to us by Jean-François Dat: for injectivity, observe that $M_1 \boxtimes M_2 = M_1^{\oplus \dim M_2}$ as W_1-representations. Hence, if the kernel of ev is non-trivial, then it contains M_1 as an irreducible constituent. Therefore, it suffices to prove that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is injective. Since Λ is algebraically closed, we have $\mathrm{End}_{W_1}(M_1) = \Lambda$ by Schur's lemma. Hence, the composition

$$M_2 = \mathrm{Hom}_{W_1}(\mathrm{End}_{W_1}(M_1), M_2) \cong \mathrm{Hom}_{W_1}(M_1, M_1 \boxtimes M_2) \to \mathrm{Hom}_{W_1}(M_1, M) = M_2$$

is the identity. This shows that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is an isomorphism.

In case Item 2, we claim that $M_1 \boxtimes M_2$ is semi-simple, and hence that M appears as a direct summand. Using (Bourbaki, 2012, Section 13.4 Corollaire) applied to the group algebras it is enough to show that M_1 and M_2 are absolutely semi-simple. Since Λ is perfect, any finite-dimensional representation is semi-simple if and only if it is absolutely semi-simple, see (Bourbaki, 2012, Section 13.1). Hence, it remains to check that $M_{2,\bar\Lambda} = M_2 \otimes_\Lambda \bar\Lambda$ is semi-simple where $\bar\Lambda/\Lambda$ is an algebraic closure. The module $M_{2,\bar\Lambda} = \mathrm{Hom}_{W_1}(M_{1,\bar\Lambda}, M_{\bar\Lambda})$ splits as a direct sum according to the simple constituents $\bar M_1 \subset M_{1,\bar\Lambda}$ and $\bar M \subset M_{\bar\Lambda}$. Finally, each $\bar M_2 = \mathrm{Hom}_{W_1}(\bar M_1, \bar M)$ is either simple or vanishes: if there exists a non-zero W_1-equivariant map $\bar M_1 \to \bar M$, then it must be injective by the simplicity of $\bar{M}_1$. As $\bar{\Lambda}$ is algebraically closed, the proof of Item 1 shows that $\bar{M} \cong \bar{M}_1 \boxtimes \bar{M}_2$ so that $\bar{M}_2$ must be simple because $\bar{M}$ is so. This shows that M_2 is absolutely semi-simple as well.

In case Item 3, abbreviate $\Lambda' := \mathrm{Frac}\Lambda$, $M' := M \otimes_\Lambda \Lambda'$ and so on. We will repeatedly use that $(-) \otimes_\Lambda \Lambda'$ preserves and detects fp-simplicity of representations, see Lemma 7.5.1. By Item 2, the evaluation map $\mathrm{ev}' := \mathrm{ev} \otimes \Lambda'$ admits a Λ'-linear section $\tilde{i}\colon M' \to (M_1 \boxtimes M_2)'$. As M' is finitely presented, there is some $0 \neq \lambda \in \Lambda$ such that $\lambda\tilde{i}$ arises by scalar extension of a map $i\colon M \to M_1 \boxtimes M_2$. By construction, the map $i \oplus \mathrm{incl}\colon M \oplus \ker(\mathrm{ev}) \to M_1 \boxtimes M_2$ is an isomorphism after tensoring with Λ'. So its cokernel is Λ-torsion, and it is injective as both modules at the left are projective (hence Λ-torsion free). This finishes the proof of the proposition. □

7.6 Essential surjectivity

In this section, we prove the essential surjectivity asserted in Theorem 7.1.2. Throughout, we freely use the full faithfulness proven in Proposition 7.3.1 and Proposition 7.3.2.

Recall that $X_1, \ldots, X_n$ are finite type $\mathbb{F}_q$-schemes, and write $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Let Λ be either a finite discrete ring, a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ or its ring of integers O_E. Note that this covers all cases from Situation 7.1.1.

First, we show that it suffices to prove containment in the essential image étale locally:

Lemma 7.6.1. *Let $U_i \to X_i$ be quasi-compact étale surjections for $i = 1, \ldots, n$. Then the following properties hold:*

1. *An object $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ belongs to the full subcategory*
$$\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$$
if and only if its restriction $M|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ belongs to the full subcategory
$$\mathrm{D}_{\mathrm{cons}}\big(U_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(U_n^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}\big(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}, \Lambda\big).$$
2. *Assume that all $U_i \to X_i$ are finite étale. Then Item 1 holds for the categories of lisse sheaves.*

Proof. The only if direction in part Item 1 is clear. Conversely, assume that $M|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of the external tensor product. By étale descent, we have an equivalence:

$$\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{Tot}\left(\mathrm{D}\big(U_{1,\bullet}^{\mathrm{Weil}} \times \ldots \times U_{n,\bullet}^{\mathrm{Weil}}, \Lambda\big)\right).$$

In particular, we get an equivalence $|(j_\bullet)_! \circ j_\bullet^* M| \xrightarrow{\sim} M$ where $j_\bullet := j_{1,\bullet} \times \ldots \times j_{n,\bullet}$ with $j_{i,\bullet} \colon U_{i,\bullet} \to X_i$ for $i = 1, \ldots, n$. For each $m \geq 0$, the object $j_m^* M$ lies in

$$\mathrm{D}_{\mathrm{cons}}\big(U_{1,m}^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(U_{n,m}^{\mathrm{Weil}}, \Lambda\big).$$

It follows from Chapter 5 Item ii that these subcategories are preserved under $(j_m)_!$. So we see

$$(j_m)_! j_m^*(M) \in \mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$$

for all $m \geq 0$. For every $m \geq 0$, let M_m denote the realization of the m-th skeleton of the simplicial object $(j_\bullet)_! \circ j_\bullet^* M$ so that we have a natural equivalence $\operatorname{colim} M_m \xrightarrow{\cong} M$ in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. We claim that M is a retract of some M_m, and hence lies in $\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$ by idempotent completeness. To prove the claim, note that the sheaf $M_{\mathbb{F}} \in \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)$ underlying M is compact in the category of ind-constructible sheaves $\mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$, see Chapter 5 Item viii. As taking partial Frobenius fixed points is a finite limit, so commutes with filtered colimits, we see that the natural map of mapping complexes

$$\operatorname{colim} \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, M_m) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, \operatorname{colim} M_m)$$

is an equivalence. In particular, the inverse equivalence $M \xrightarrow{\cong} \operatorname{colim} M_m$ factors through some M_m, presenting M as a retract of M_m. This proves the claim, and hence Item 1.

For Item 2, note that if $U_i \to X_i$ are finite étale, then the functors $(j_m)_!$ preserve the lisse categories, see Chapter 5 Item ii. In particular, for every $m \geq 0$ the object $(j_m)_! j_m^*(M)$ is lisse and so is M_m. We conclude using compactness as before. $\square$

Using Lemma 4.2.4 and Chapter 5 Item viii, the fully faithful functor Equation (7.1) uniquely extends to a fully faithful functor

$$\mathrm{Ind}\big(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)\big) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \tag{7.13}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.

We use this in the following variant of Lemma 7.6.1.

Lemma 7.6.2. *The statements Item 1 and Item 2 of Lemma 7.6.1 hold for the functor Equation* (7.13) *with* $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. *Namely, to check that an object lies in the essential image of Equation* (7.13)*, one can pass to a quasi-compact étale cover if* $\bullet = \mathrm{cons}$*, and to a finite étale cover if* $\bullet = \mathrm{lis}$.

Proof. This is immediate from the proof of Lemma 7.6.1: Arguing as above and using étale descent for ind-constructible, resp. ind-lisse sheaves (Chapter 5 Item iii), we see that $M \cong \operatorname{colim} M_m$ with

$$M_m \in \operatorname{Ind}\big(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \operatorname{Ind}\big(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)\big)$$

for all $m \geq 0$ and $\bullet = \mathrm{cons}$, resp. $\bullet = \mathrm{lis}$. As the essential image of Equation (7.13) is closed under colimits, M lies in the corresponding subcategory as well. □

Now we have enough tools to prove the categorical Künneth formula alias derived Drinfeld's lemma:

Proof of 7.1.2. In view of Proposition 7.3.1 and Proposition 7.3.2, it remains to show the essential surjectivity of the external tensor product functor on Weil sheaves Equation (7.1) under the assumptions in Theorem 7.1.2. Part Item 1, the case of constructible sheaves, is reduced to part Item 2, the case of lisse sheaves, by taking a stratification as in Definition 6.4.1 Item 2 and using the full faithfulness already proven. Here we note that by refining the stratification witnessing the constructibility if necessary, we can even assume all strata to be smooth, so in particular geometrically unibranch. Hence, it remains to prove part Item 2, that is, the essential surjectivity of the fully faithful functor

$$\boxtimes\colon \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big), \tag{7.14}$$

when either Λ is finite discrete as in cases Item a, Item d in Theorem 7.1.2 Item 2, or $\Lambda = \mathcal{O}_E$ for a finite field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ as in case Item b, or $\Lambda = E$ and the X_i are geometrically unibranch as in the remaining case Item c. In fact, the latter two cases are easier to handle due to the presence of natural t-structures on the categories of lisse sheaves (Chapter 5 Item vi). So we will distinguish two cases below: 1) $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch; 2) Λ is finite discrete.

Now pick $M \in \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. By Chapter 5 Item v, M is bounded in the standard t-structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. So M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$. As M is lisse, Lemma 6.5.4 Item 1 shows in both cases 1) and 2) that each $\mathrm{H}^j(M)$ comes from a continuous representation on a finitely presented Λ-module in

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \overset{7.4.1}{\cong} \mathrm{Rep}_\Lambda(W), \tag{7.15}$$

where we denote $W := W_1 \times \ldots \times W_n$ with $W_i := \mathrm{Weil}(X_i)$.

Throughout, we repeatedly use that the functor (7.14) is fully faithful, commutes with finite (co-)limits and shifts, and that its essential image is closed under retracts (as the source category is idempotent complete, by definition) and contains all perfect-constant sheaves.

Case 1): *Assume $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch.* In this case, we have a t-structure on lisse Weil sheaves so that each $\mathrm{H}^j(M)$ belongs to $\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit$. By induction on the length of M, using the full faithfulness of Equation (7.14), we reduce to the case where M is abelian, that is, a continuous W-representation on a finitely presented Λ-module. The external tensor product induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}_\Lambda(W_1) \times \ldots \times \mathrm{Rep}_\Lambda(W_n) \arrow[r, "\boxtimes"] \arrow[d, "\cong"] & \mathrm{Rep}_\Lambda(W) \arrow[d, "\cong"] \\
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)^\heartsuit \times \ldots \times \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit \arrow[r, "\boxtimes"] & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit,
\end{tikzcd}$$

where the vertical equivalences are given by Lemma 6.5.4. Note that M splits into a direct sum $M_{\mathrm{tor}} \oplus M_{\mathrm{f.p}}$ where the finitely presented Λ-module underlying M_{tor} is Λ-torsion and $M_{\mathrm{f.p}}$ is projective. So we can treat either case separately. Using that the essential image of Equation (7.14) is closed under extensions (by full faithfulness) and retracts, the finite projective case is reduced to the fp-simple case and, by Proposition 7.5.2, to the finite torsion case. Note that the W_i-representations constructed in, say Lemma 7.5.1 (7.12), are obtained from $M_{\mathrm{f.p}}$ by taking subquotients and tensor products, so are automatically continuous. Next, as the Λ-module underlying M_{tor} is finite torsion, the Λ-sheaf M_{tor} is perfect-constant along some finite étale cover. So we conclude by 7.6.1 Item 2.

Case 2): *Assume Λ is finite discrete as above.* In a nutshell, the argument is similar to the last step in case 1), but a little more involved due to the absence of natural t-structures on the categories of lisse sheaves in general, see Chapter 5 Item vi. More precisely, in the special case, where Λ is a finite field, the argument of case

1) applies, but not so if $\Lambda = \mathbb{Z}/\ell^2$, say. So, instead, we extend Equation (7.14) by passing to Ind-completions to a commutative diagram

$$\begin{tikzcd}
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda) \arrow[r, "\boxtimes"] \arrow[d] & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}},\Lambda) \arrow[d] \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda)) \arrow[r, "\mathrm{Ind}(\boxtimes)"] & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}},\Lambda)),
\end{tikzcd}$$

of full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}},\Lambda)$, see the discussion around Equation (7.13). Note that the fully faithful embedding Equation (7.13) factors through $\mathrm{Ind}(\boxtimes)$. Both vertical arrows are the inclusion of the subcategories of compact objects by idempotent completeness of the involved categories and Equation (4.1). Thus, if M lies in the essential image of $\mathrm{Ind}(\boxtimes)$, then it is a retract of a finite colimit of objects in the essential image of $\boxtimes$, so lies itself in this essential image. As M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, it suffices to show

$$\mathrm{H}^j(M) \in \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}},\Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}},\Lambda)),$$

for all $j \in \mathbb{Z}$. So fix j and denote $N := \mathrm{H}^j(M)$ viewed as a continuous W-representation on a finitely presented Λ-module. As Λ is finite, N comes from a continuous representation of $\pi_1(X_1)\times\ldots\pi_1(X_n)$ on which some open subgroup acts trivially. Hence, there exist finite étale surjections $U_i \to X_i$ such that the subgroup $\pi_1(U_1)\times\ldots\times\pi_1(U_n)$ acts trivially on N. In particular, $N|_{U_1^{\mathrm{Weil}}\times\ldots\times U_n^{\mathrm{Weil}}}$ is constant, and hence lies in the essential image of the functor

$$\mathrm{Mod}_R \cong \mathrm{Ind}(\mathrm{Perf}_R) \to \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}},\Lambda)),$$

where $R := \Gamma(\pi_0(U_1)\times\ldots\times\pi_0(U_n),\Lambda)$. As the sets $\pi_0(U_i)$ are finite discrete, each $R_i := \Gamma(\pi_0(U_i),\Lambda)$ is a finite free Λ_*-algebra, and we have $R \cong R_1 \otimes_{\Lambda_*} \ldots \otimes_{\Lambda_*} R_n$. Thus, the external tensor product induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}_{R_1} \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Mod}_{R_n} \arrow[r, "\cong"] \arrow[d] & \mathrm{Mod}_R \arrow[d] \\
\mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}},\Lambda)) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_n^{\mathrm{Weil}},\Lambda)) \arrow[r, "\mathrm{Ind}(\boxtimes)"] & \mathrm{Ind}(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}},\Lambda)),
\end{tikzcd}$$

where the upper horizontal arrow is an equivalence. So $N|_{U_1^{\mathrm{Weil}}\times\ldots\times U_n^{\mathrm{Weil}}}$ lies in the essential image of $\mathrm{Ind}(\boxtimes)$, and we conclude by Lemma 7.6.2 applied to the finite étale covers $U_i \to X_i$ and $\bullet = \mathrm{lis}$. $\square$

Chapter 8

IND-CONSTRUCTIBLE WEIL SHEAVES

In this section, we introduce the full subcategories

$$\mathrm{D}_{\mathrm{indlis}}\big(X^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}_{\mathrm{indcons}}\big(X^{\mathrm{Weil}}, \Lambda\big)$$

of $\mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big)$ consisting of ind-objects of lisse, resp. constructible sheaves equipped with partial Frobenius action. That is, the partial Frobenius only preserves the ind-system of objects, but not necessarily each member. We will define analogous categories for a product of schemes. Similarly to the lisse, resp. constructible case, there is a fully faithful functor

$$\mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_{\mathrm{indcons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big),$$

which, however, will not be an equivalence in general, see Remark 8.1.6. Nevertheless, we can identify a class of objects that lie in the essential image and that include many cases of interest such as the shtuka cohomology studied in (V. Lafforgue, 2018; V. Lafforgue and Zhu, 2019; Xue, 2020b; Xue, 2020c).

8.1 Ind-constructible Weil sheaves

Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and fix an algebraic closure $\mathbb{F}$. Let $X_1, \ldots, X_n$ be schemes of finite type over $\mathbb{F}_q$. Let Λ be a condensed ring associated with the one of the following topological rings: a discrete coherent torsion ring (for example, a discrete finite ring), an algebraic field extension $E \supset \mathbb{Q}_\ell$, or its ring of integers O_E. We write $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$, and denote by $X_{i,\mathbb{F}} := X_i \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ and $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$ the base change. Recall that under these assumptions, by Chapter 5 Item viii, we have a fully faithful embedding

$$\mathrm{Ind}\big(\mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)\big) \xrightarrow{\cong} \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{8.1}$$

and likewise for (ind-)lisse sheaves.

Definition 8.1.1. *An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called ind-lisse, resp. ind-constructible if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is ind-lisse, resp. ind-constructible in the sense of Definition 5.0.1.*

We denote by

$$\mathrm{D}_{\mathrm{indlis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$$

the resulting full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of ind-lisse, resp. ind-constructible objects. Both categories are naturally commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ (see the notation from Chapter 4), that is, presentable stable Λ_*-linear symmetric monoidal ∞-categories where $\Lambda_* := \Gamma(*, \Lambda)$ is the ring underlying Λ.

It is immediate from Definition 8.1.1 that the equivalence (6.6) restricts to an equivalence

$$\mathrm{D}_{\bullet}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\left(\mathrm{D}_{\bullet}(X_{\mathbb{F}}, \Lambda), \phi^*_{X_1}, \ldots, \phi^*_{X_n}\right)$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$.

Remark 8.1.2. *Note that that we have a fully faithful embedding of $D_{\mathrm{cons}}(X^{\mathrm{Weil}})$ into $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}})$ whose image consists of compact objects. However, the latter category is not generated by this image. Indeed, even in the case of a point, the ind-cons category consists of Λ-modules with an action of an endomorphism, whereas the image of the embedding consists of Λ-modules with an action of an automorphism. This automorphism does not have to fix any finitely generated submodule, which would be the case for any objects generated by the image of the constructible Weil complexes.*

Our goal in this chapter is to obtain a categorical Künneth formula for the categories of ind-lisse, resp. ind-constructible Weil sheaves. In order to state the result, we need the following terminology. Under our assumptions on Λ, each cohomology sheaf $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$ for $M \in \mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda)$ is naturally a continuous representation of the proétale fundamental groupoid $\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})$ on a finitely presented Λ-module, see 6.5.4. Further, the projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ induce a full surjective map of topological groupoids

$$\pi_1^{\mathrm{proét}}(X_{\mathbb{F}}) \to \pi_1^{\mathrm{proét}}(X_{1,\mathbb{F}}) \times \ldots \times \pi_1^{\mathrm{proét}}(X_{n,\mathbb{F}}). \tag{8.2}$$

Definition 8.1.3. *Let $M \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$.*

1. *The sheaf M is called split lisse if it is lisse and the action of $\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M)$ factors through (8.2) for all $j \in \mathbb{Z}$.*

2. *The sheaf M is called split constructible if it is constructible and there exists a finite subdivision into locally closed subschemes $X_{i,\alpha} \subseteq X_i$ such that for each $X_\alpha = \prod_i X_{i,\alpha} \subseteq X$, each restriction $M|_{X_\alpha}$ is split lisse.*

Definition 8.1.4. *An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called ind-(split lisse), resp. ind-(split constructible) if the underlying object $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is a colimit of split lisse, resp. split constructible objects.*

As the category $\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda)$, $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$ is cocomplete, every ind-(split lisse) object is ind-lisse, and likewise, every ind-(split constructible) object is ind-constructible.

Theorem 8.1.5. *Assume that Λ is either a finite discrete ring of prime-to-p torsion, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$, or its ring of integers O_E. Then the functor induced by the external tensor product*

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \quad (8.3)$$

is fully faithful for $\bullet \in \{\mathrm{indlis},\ \mathrm{indcons}\}$. For $\bullet = \mathrm{indlis}$, resp. $\bullet = \mathrm{indcons}$ the essential image contains the ind-(split lisse), resp. ind-(split constructible) objects.

Proof. For full faithfulness, it is enough to consider the case $\bullet = \mathrm{indcons}$. Using Lemma 4.2.5, it remains to show that the functor

$$\bigotimes_i \mathrm{D}_{\mathrm{indcons}}(X_{i,\mathbb{F}}, \Lambda) \cong \mathrm{Ind}\left(\bigotimes_i \mathrm{D}_{\mathrm{cons}}(X_{i,\mathbb{F}}, \Lambda)\right) \to \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \qquad (8.4)$$

is fully faithful. In view of Equation (8.1), this is immediate from the Künneth formula for constructible Λ-sheaves as explained in Section 7.3.

To identify objects in the essential image, we note that the fully faithful functors Equation (8.3) and Equation (8.4) induce a Cartesian diagram (see Lemma 4.2.5):

$$\begin{tikzcd} \bigotimes_i \mathrm{D}_\bullet(X_i^{\mathrm{Weil}}, \Lambda) \arrow[r] \arrow[d] & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \arrow[d] \\ \bigotimes_i \mathrm{D}_\bullet(X_{i,\mathbb{F}}, \Lambda) \arrow[r] & \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda), \end{tikzcd} \qquad (8.5)$$

for $\bullet \in \{\mathrm{indlis},\ \mathrm{indcons}\}$. Thus, it is enough to show that the object $M_{\mathbb{F}}$ underlying an ind-split object M lies in the image of the lower horizontal arrow. Since this essential image is closed under colimits, it remains to show it contains the split lisse objects for $\bullet = \mathrm{indlis}$, resp. the split constructible objects for $\bullet = \mathrm{indcons}$.

By the full faithfulness of Equation (8.4), the split constructible case reduces to the split lisse case, see also the proof of 7.1.2 in Section 7.6. So assume $\bullet$ = indlis and let $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ be split lisse. As each cohomology sheaf $\mathrm{H}^j(M_{\mathbb{F}})$, $j \in \mathbb{Z}$ is at least ind-lisse, an induction on the cohomological length of $M_{\mathbb{F}}$ reduces us to show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies in the essential image. By definition, being split lisse implies that the action of $\pi_1^{\text{proét}}(X_{\mathbb{F}})$ on $\mathrm{H}^j(M_{\mathbb{F}})$ factors through $\pi_1^{\text{proét}}(X_{1,\mathbb{F}}) \times \ldots \times \pi_1^{\text{proét}}(X_{n,\mathbb{F}})$. Then the arguments of Section 7.6 show that $\mathrm{H}^j(M_{\mathbb{F}})$ lies is in the essential image of the lower horizontal arrow in Equation (8.5). We leave the details to the reader. □

Remark 8.1.6. *The functor Equation* (8.3) *is not essentially surjective in general. To see this, note that the functor* $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$ *admits a left adjoint F that adds a free partial Frobenius action. Explicitly, for an object* $M \in \mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$ *the object $F(M)$ has underlying sheaf $F(M)_{\mathbb{F}}$ given by a countable direct sum of copies of M. If M was not originally in the image of the external tensor product, then $F(M)$ will not be either. This is, however, the only obstacle for essential surjectivity: as noted in the proof of Theorem 8.1.5, the diagram Equation* (8.5) *is Cartesian.*

8.2 Cohomology of shtuka spaces

Finally, let us mention a key application of Theorem 8.1.5. Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Let $N \subset X$ be a finite subscheme, and denote its complement by $Y = X \backslash N$. Let $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ be an algebraic field extension containing a fixed square root of q. Let $\mathcal{O}_E$ be its ring of integers and denote by k_E the residue field. Let Λ be any of the topological rings $E, \mathcal{O}_E, k_E$. Let G be a split (for simplicity) reductive group over $\mathbb{F}_q$. We denote by $\widehat{G}$ the Langlands dual group of G considered as a split reductive group over Λ.

In the seminal works (V. G. Drinfeld, 1980; L. Lafforgue, 2002) ($G = \mathrm{GL}_n$) and (V. Lafforgue, 2018; V. Lafforgue and Zhu, 2019) (general reductive G) on the Langlands correspondence over global function fields, the construction of the $\mathrm{Weil}(Y)$-action on automorphic forms of level N is realized using the cohomology sheaves of moduli stacks of shtukas, defined in (Varshavsky, 2004) and (V. Lafforgue, 2018, Section 2). As explained in (V. Lafforgue and Zhu, 2019; Dennis Gaitsgory, Kazhdan, et al., 2022; Zhu, 2021), the output of the geometric construction of Lafforgue can be encoded as a natural transformation

$$\mathrm{H}_{N,I} \colon \mathrm{Rep}_\Lambda^{\mathrm{f.p}}\big(\widehat{G}^I\big) \to \mathrm{Rep}_\Lambda^{\mathrm{cts}}\big(\mathrm{Weil}(Y)^I\big), \quad I \in \mathrm{FinSet} \tag{8.6}$$

of functors FinSet $\to$ Cat from the category of finite sets to the category of 1-categories. Here the functor $\mathrm{Rep}_{\Lambda}^{\mathrm{f.p}}(\widehat{G}^{\bullet})$ assigns to a finite set I the category of algebraic representations of $\widehat{G}^{I}$ on finite free Λ-modules, and $\mathrm{Rep}_{\Lambda}^{\mathrm{cts}}(\mathrm{Weil}(Y)^{\bullet})$ the category of continuous representations of $\mathrm{Weil}(Y)^{I}$ in Λ-modules. In both cases, the transition maps are given by restriction of representations.

Let us recall some elements of its construction. For a finite set I, (Varshavsky, 2004) and (V. Lafforgue, 2018, Section 2) define the ind-algebraic stack $\mathrm{Cht}_{N,I}$ classifying I-legged G-shtukas on X with full level-N-structure. The morphism sending a G-shtuka to its legs

$$\mathfrak{p}_{N,I} \colon \mathrm{Cht}_{N,I} \to Y^{I} \tag{8.7}$$

is locally of finite presentation. For every $W \in \mathrm{Rep}_{\Lambda}^{\mathrm{f.p}}(\widehat{G}^{I})$, there is the normalized Satake sheaf $\mathcal{F}_{N,I,W}$ on $\mathrm{Cht}_{N,I}$, see (V. Lafforgue, 2018, Définition 2.14). Base changing to $\mathbb{F}$ and taking compactly supported cohomology, we obtain the object

$$\mathcal{H}_{N,I}(W) := (\mathfrak{p}_{N,I,\mathbb{F}})_{!}(\mathcal{F}_{N,I,W,\mathbb{F}}) \in \mathrm{D}_{\mathrm{indcons}}\big(Y_{\mathbb{F}}^{I}, \Lambda\big)$$

see (V. Lafforgue, 2018, Définition 4.7) and (Xue, 2020a, Definition 2.5.1). Under the normalization of the Satake sheaves, the degree 0 cohomology sheaf

$$\mathrm{H}_{N,I}(W) := \mathrm{H}^{0}(\mathcal{H}_{I}(W)) \in \mathrm{D}_{\mathrm{indcons}}\big(Y_{\mathbb{F}}^{I}, \Lambda\big)^{\heartsuit}$$

corresponds to the middle degree compactly supported intersection cohomology of $\mathrm{Cht}_{N,I}$. Using the symmetries of the moduli stacks of shtukas, the sheaf $\mathrm{H}_{N,I}(W)$ is endowed with a partial Frobenius equivariant structure (L. Lafforgue, 2002, §6). So we obtain objects

$$\mathrm{H}_{N,I}(W) \in \mathrm{D}_{\mathrm{indcons}}\big((Y^{\mathrm{Weil}})^{I}, \Lambda\big)^{\heartsuit} \tag{8.8}$$

Next, using the finiteness (Xue, 2020b) and smoothness (Xue, 2020c, Theorem 4.2.3) results, the classical Drinfeld's lemma (Theorem 7.4.1) applies to give objects $\mathrm{H}_{N,I}(W) \in \mathrm{Rep}_{\Lambda}^{\mathrm{cts}}(\mathrm{Weil}(Y)^{I})$. The construction of the natural transformation (8.6) encodes the functoriality and fusion satisfied by the objects $\{\mathrm{H}_{N,I}(W)\}$ for varying I and W.

However, in order to analyze construction Equation (8.6) further, it is desirable to upgrade the natural transformation of functors Equation (8.6) to the derived level. Namely, to have construction for the complexes $\{\mathcal{H}_{I}(W)\}_{I,W}$ and not just for their cohomology sheaves, compare with (Zhu, 2021). Such an upgrade is possible using the derived version of Drinfeld's lemma, as given in the following proposition. A further study of this construction will appear in future work of the first named author (T. H.).

Proposition 8.2.1. *For $\Lambda \in \{E, \mathcal{O}_E, k_E\}$ and any $W \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, the shtuka cohomology* (8.8) *lies in the essential image of the fully faithful functor*

$$\mathrm{D}_{\mathrm{indlis}}(Y^{\mathrm{Weil}}, \Lambda)^{\otimes I} \to \mathrm{D}_{\mathrm{indcons}}\big((Y^{\mathrm{Weil}})^I, \Lambda\big) \tag{8.9}$$

Proof. By (Xue, 2020c, Theorem 4.2.3), the ind-constructible sheaf $\mathrm{H}_{N,I}(W)$ is ind-lisse. By (Xue, 2020b, Proposition 3.2.15), the action of $\mathrm{FWeil}(Y^I)$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\mathrm{Weil}(Y)^I$. In particular, the action of $\pi_1(X^I_{\mathbb{F}})$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\pi_1(X_{\mathbb{F}})^I$. So it is ind-(split lisse) in the sense of Definition 8.1.4, and we are done by Theorem 8.1.5. □

Remark 8.2.2. *One can upgrade the above construction in a homotopy coherent way to show that the whole complex $\mathcal{H}_{N,I}(W)$ lies in $\mathrm{D}_{\mathrm{indcons}}\big((Y^{\mathrm{Weil}})^I, \Lambda\big)$. If $N \neq \varnothing$ so that $\mathcal{H}_{N,I}(W)$ is known to be bounded, then Proposition 8.2.1 implies that $\mathcal{H}_{N,I}(W)$ lies in the essential image of Equation* (8.9).

Part III

The Categorical Trace and Convolution Patterns

Chapter 9

MORE RECOLLECTIONS ON ∞-CATEGORIES

We review some more required categorical preliminaries mainly following (Lurie, 2009) (Lurie, 2017) and (Dennis Gaitsgory and Rozenblyum, 2017a), in the (simplified) set-up suitable for our purpose.

Let Cat denote the $(\infty,1)$-category of small categories, and $\mathcal{C}$at the category of all (not necessarily small) $(\infty,1)$-categories. For a(n ordinary) commutative ring Λ with unit, let $\mathrm{Lincat}^{\mathrm{sm}}_{\Lambda}$ denote the ∞-category of Λ-linear small idempotently complete stable categories with functors being Λ-linear exact functors, and let $\mathrm{Lincat}_{\Lambda}$ denote the $(\infty,1)$-category of all presentable Λ-linear stable categories with functors being continuous functors. For $C \in \mathrm{Lincat}_{\Lambda}$, and $X, Y \in C$, we write $\mathrm{Hom}_C(X,Y) \in \mathrm{Mod}_{\Lambda}$ representing $\mathrm{Map}_{\mathrm{Mod}_{\Lambda}}(M, \mathrm{Hom}_C(X,Y)) = \mathrm{Map}_C(M \otimes X, Y)$. Then $\mathrm{Map}_C(X,Y) = \tau^{\leq 0}\mathrm{Hom}_C(X,Y)$.

Definition 9.0.1. *Let*

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ \downarrow{\scriptstyle v} & & \downarrow{\scriptstyle u} \\ \mathcal{D} & \xrightarrow{g} & \mathcal{D}' \end{array}$$

be a commutative square in $\mathrm{Lincat}_{\Lambda}$. *That is, we are given a specified equivalence* $u \circ g \simeq g \circ V$. *Then we say that the square above is left adjointable if* f *and* g *admit left adjoints* f^L *and* g^L, *and the Beck-Chevalley map* $\beta\colon g^L \circ u \to v \circ f^L$ *given by*

$$g^L \circ u \to g^L \circ u \circ f \circ f^L \simeq g^L \circ g \circ v \circ f^L \to v \circ f^L$$

is an equivalence. If f *and* g *has right adjoints* f^R *and* g^R *then we say that the square is right adjointable is the map* $\gamma\colon v \circ f^R \to g^R \circ u$, *obtained by a dual construction to the one above, is an equivalence.*

Let S be a small infinity category. We denote by $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathrm{Lincat}_{\Lambda})$ (resp. $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Lincat}_{\Lambda})$) the subcategory of $\mathrm{Fun}(S, \mathrm{Lincat}_{\Lambda})$ consisting of functors $F\colon S \to \mathrm{Lincat}$ so that for all arrows $s \to s'$ in S the functor $F(s) \to F(s')$ admits a continuous right adjoint (resp. a left adjoint) and morphisms $\alpha\colon F \to F'$

such that for every $s \to s'$ the square

$$\begin{tikzcd} F(s) \arrow[r] \arrow[d] & F(s') \arrow[d] \\ F'(s) \arrow[r] & F'(s') \end{tikzcd}$$

is left (resp. right) adjointable. We recall (Lurie, 2017, Corollary 4.7.4.18.):

Proposition 9.0.2. *The ∞-categories* $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathrm{Lincat}_\Lambda)$, $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Lincat}_\Lambda)$ *are presentable and the inclusion functors* $\mathrm{Fun}^{\mathrm{LAd}}(S, \mathrm{Lincat}_\Lambda) \subseteq \mathrm{Fun}(S, \mathrm{Lincat}_\Lambda)$ *and* $\mathrm{Fun}^{\mathrm{RAd}}(S, \mathrm{Lincat}_\Lambda) \subseteq \mathrm{Fun}(S, \mathrm{Lincat}_\Lambda)$ *preserve small limits.*

In particular, if we have an indexing category $\mathcal{I}$ and diagrams $C, \mathcal{D} \colon \mathcal{I} \to \mathrm{Lincat}_\Lambda$ with a natural transformation $\phi \colon C \to \mathcal{D}$, which is equivalent to having a functor $F \colon \mathcal{I} \to \mathrm{Fun}(\Delta^1, \mathrm{Lincat}_\Lambda)$, then F factors through $\mathrm{Fun}^{\mathrm{LAd}}(\Delta^1, \mathrm{Lincat}_\Lambda)$ (resp. $\mathrm{Fun}^{\mathrm{RAd}}(\Delta^1, \mathrm{Lincat}_\Lambda)$) if for every vertex i of $\mathcal{I}$, the functor $\phi_i \colon C_i \to \mathcal{D}_i$ has a left (right) adjoint $\psi_i \colon \mathcal{D}_i \to C_i$ and for all arrows $i \to j$ in $\mathcal{I}$ the commutative square

$$\begin{tikzcd} C_i \arrow[r] \arrow[d] & \mathcal{D}_i \arrow[d] \\ C_j \arrow[r] & \mathcal{D}_j \end{tikzcd}$$

is left (resp. right) adjointable. Denote $C = \lim_i C_i$ and $\mathcal{D} = \lim_i \mathcal{D}_i$. Then the proposition says that under the conditions above the functor $\phi \colon C \to \mathcal{D}$ induced on the limits has a left (resp. right) adjoint ψ and for all $i \in \mathcal{I}$ the composition $\mathcal{D} \xrightarrow{\psi} C \to C_i$ is equivalent to the composition $\mathcal{D} \to \mathcal{D}_i \xrightarrow{\psi_i} C_i$.

Let $F \colon S \to \mathrm{Lincat}_\Lambda$ be a diagram. For an arrow $\varphi \colon s \to s'$, we have that the corresponding functor $F(\varphi) \colon F(s) \to F(s')$ preserves colimits morphism and therefore by (Lurie, 2009, Corollary 5.5.2.9) admits a right adjoint $F^R(\varphi)$. By passing to right adjoints we get a diagram $F^R \colon S^{op} \to \mathcal{C}\mathrm{at}$. By (Lurie, 2009, §5.5.3) the morphism

$$\operatorname*{colim}_{s \in S} F(s) \to \lim_{s \in S^{op}} F^R(s) \tag{9.1}$$

determined by the maps right adjoint to $\mathrm{ins}_s \colon F(s) \to \mathrm{colim}_S F$ is an equivalence, where the left is computed in Lincat_Λ and then is mapped to $\mathcal{C}\mathrm{at}$ and the right is computed in $\mathcal{C}\mathrm{at}$. In addition, if all $F^R(\varphi)$ are continuous, then the r.h.s. can also be computed in Lincat_Λ. Denote by ev_s the right adjoint of ins_s. It follows from adjunction that for every object $c \in \lim_S F$, the natural map

$$\operatorname*{colim}_{s \in S}(\mathrm{ins}_s \circ \mathrm{ev}_s(c)) \to c \tag{9.2}$$

is an equivalence in $\mathrm{colim}_S F$.

Remark 9.0.3.

1. *Assume that for each $\varphi\colon s \to s'$ the functor $F(\varphi)\colon F(s) \to F(s')$ preserves compact objects (equivaletnly, $F(\varphi)$ preserves compact objects, the right adjoint $F^R(\varphi)$ is continuous). Then the functors* $\mathrm{ins}_s\colon F(s) \to \mathrm{colim}_S F$ *also preserve compact objects.*

2. *Assume in addition that each of the categories $F(s)$ is compactly generated, that is, $F(s) \simeq \mathrm{Ind}(F^0(s))$. Then the colimit* $\mathrm{colim}_S F$ *is also compactly generated, with compact objects given by retracts of objects coming from $F^0(s)$ for $s \in S$. In particular, if S is filtered, we have*

$$\operatorname*{colim}_S F \simeq \mathrm{Ind}(\operatorname*{colim}_{s\in S} F^0(s))$$

with the colimit taken in the category of small infinity categories.

3. *If S is filtered and the morphisms in the image of F have continuous right adjoints, then for an object s in S the composition* $\mathrm{ev}_s \circ \mathrm{ins}_s\colon F(s) \to \mathrm{colim}_S F \simeq \lim_{S^{op}} F^R \to F(s)$ *is equivalent to the colimit*

$$\mathrm{ev}_s \circ \mathrm{ins}_s \simeq \operatorname*{colim}_{\varphi\colon s\to s'} F^R(\varphi) \circ F(\varphi).$$

Using the equivalence (9.1) and Proposition 9.0.2 it is also possible to get that adjointability preserved under taking colimits (Lurie, 2017, Proposition 4.7.4.19).

Proposition 9.0.4 (Lurie)**.** *Let S, T be small ∞-categories and let $F\colon S \times T \to$* Lincat_Λ *be a functor. Assume that for all $s \to s'$ in S and $t \to t'$ in T the square*

$$\begin{tikzcd} F(s,t) \arrow[r] \arrow[d] & F(s',t) \arrow[d] \\ F(s,t') \arrow[r] & F(s',t') \end{tikzcd}$$

is right adjointable. Then there is an extension $\overline{F}\colon S^{\triangleright} \times T \to$ Lincat_Λ *of F such that:*

1. *For each $t \in T$, the diagram $\overline{F}\colon S^{\triangleright} \times \{t\} \to$* Lincat_Λ *is a colimit diagram in* Lincat_Λ.

2. *For all $s \to s'$ in $S^{\triangleright}$ and $t \to t'$ in T the square*

$$\begin{tikzcd} F(s,t) \arrow[r] \arrow[d] & F(s',t) \arrow[d] \\ F(s,t') \arrow[r] & F(s',t') \end{tikzcd}$$

is right adjointable.

We will also need (cohomological) descent. In this setting we are usually dealing with a functor $C^\bullet \colon \Delta \to \mathcal{C}\mathrm{at}$, also called a cosimplicial ∞-category, and we would like to say something about the totalization

$$\mathrm{Tot}(C^\bullet) = \lim_{[n]\in\Delta} C^n$$

of $C^\bullet$. Usually the totalization is difficult to compute. However, under certain left adjointability conditions it is possible to deduce that the evaluation functor $\mathrm{Tot}(C^\bullet) \to C^0$ is *monadic*. Recall that for an ∞-category $\mathcal{D}$ a *monad on* $\mathcal{D}$ is an associative algebra object T in the monoidal category $\mathrm{Fun}(\mathcal{D},\mathcal{D})$. Informally, this means that we are gives a multiplication map $T \circ T \to T$ and a unit map $\mathrm{id}_{\mathcal{D}} \to T$ which satisfy associativity and unit conditions up to coherent homotopy. For example, if $G \colon \mathcal{E} \to \mathcal{D}$ is a functor between ∞-categories which admits a left adjoint F, the composition $T = G \circ F$ has the structure of a monad on $\mathcal{D}$ with identity given by the unit map $\mathrm{id}_{\mathcal{D}} \to G \circ F$ of the adjunction and composition map induced by the co-unit $F \circ G \to \mathrm{id}_{\mathcal{E}}$ via

$$T \circ T = (G \circ F) \circ (G \circ F) \simeq G \circ (F \circ G) \circ F \to G \circ F.$$

Given a monad T on $\mathcal{D}$ one can consider the category $\mathrm{LMod}_T(\mathcal{D})$ of left modules over T. Informally, this category consists of objects A in $\mathcal{D}$ equipped with a map $T(A) \to A$ giving an action of the algebra T on A and morphisms giving by morphisms preserving that structure. The forgetful functor

$$G \colon \mathrm{LMod}_T(\mathcal{D}) \to \mathcal{D}$$

has a left adjoint given by the free construction $A \mapsto T(A)$. An adjunction $F \colon \mathcal{D} \rightleftarrows \mathcal{E} \colon G$ is called monadic if $\mathcal{E}$ is equivalent to $\mathrm{LMod}_T(\mathcal{D})$ for $T = G \circ F$ and G given by the forgetful functor.

For that purpose, we recall (Lurie, 2017, Theorem 4.7.5.2, Corollary 4.7.5.3).

Theorem 9.0.5 (Lurie)**.** *Let $\mathcal{C}^\bullet\colon \Delta \to \mathcal{C}\mathrm{at}$ be a cosimplicial ∞-category. Assume that for any $\alpha\colon [m] \to [n]$ in Δ, the induced diagram*

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d] & \mathcal{C}^{m+1} \arrow[d] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

is left adjointable. We denote the left adjoint of $d^0\colon \mathcal{C}^n \to \mathcal{C}^{n+1}$ by $F(n)$. Let $\mathcal{C} = \mathrm{Tot}(\mathcal{C}^\bullet)$. Then

1. *The functor $G\colon \mathcal{C} \to \mathcal{C}^0$ admits a left adjoint F.*
2. *The diagram*

$$\begin{tikzcd} \mathcal{C} \arrow[r, "G"] \arrow[d, "G"] & \mathcal{C}^0 \arrow[d, "d^1"] \\ \mathcal{C}^0 \arrow[r, "d_0"] & \mathcal{C}^1 \end{tikzcd}$$

   is left adjointable. That is, the canonical map $F(0) \circ d^1 \to G \circ F$ is an equivalence.
3. *The adjunction $F\colon \mathcal{C}^0 \rightleftarrows \mathcal{C}\colon G$ is monadic. That is, $\mathcal{C}$ is equivalent to the category of left modules $\mathrm{LMod}_T(\mathcal{C}^0)$ with $T = F(0) \circ d^1 \simeq G \circ F$.*

Corollary 9.0.6 (Lurie)**.** *Let $\mathcal{C}^\bullet\colon \Delta_+ \to \mathcal{C}\mathrm{at}$ be an augmented cosimplicial ∞-category. Denote $G\colon \mathcal{C}^{-1} \to \mathcal{C}^0$. Assume that*

1. *The category $\mathcal{C}^{-1}$ admits geometric realizations of G-split simplicial objects that are preserved by G.*
2. *for any $\alpha\colon [m] \to [n]$ in Δ_+ the diagram*

$$\begin{tikzcd} \mathcal{C}^m \arrow[r, "d^0"] \arrow[d] & \mathcal{C}^{m+1} \arrow[d] \\ \mathcal{C}^n \arrow[r, "d^0"] & \mathcal{C}^{n+1} \end{tikzcd}$$

   is left adjointable.

Then the canonical map $\phi\colon \mathcal{C}^{-1} \to \mathrm{Tot}(\mathcal{C}^\bullet)$ admits a fully faithful left adjoint. If, in addition $\mathcal{C}^{-1} \to \mathcal{C}^0$ is conservative, ϕ is an equivalence.

Remark 9.0.7. *There is also a dual (co-monadic) version replacing "left adjoint" with "right adjoint", and "realizations of G-split simplicial objects" with "totalizations of G-split cosimplicial objects" in the statement above.*

9.1 Dual categories

Recall (e.g. (Lurie, 2018, §D.7)) that every object $C \in \mathrm{Lincat}_\Lambda$ which is compactly generated is dualizable. If $C = \mathrm{Ind}(C_0)$ for some $C_0 \in \mathrm{Cat}_\Lambda$ we can identify the dual $C^\vee$ of C with $\mathrm{Ind}(C^{op})$. Explicitly, the evaluation map $C^\vee \otimes C \to \mathrm{Mod}_\Lambda$ is given by the unique continuous extension of the functor given by the unique continuous extension of the functor

$$\begin{aligned} \mathrm{Ind}(C^{op}) \otimes C &\to \mathrm{Mod}_\Lambda \\ (\mathcal{F}, \mathcal{G}) &\to \mathrm{Hom}_C(\mathcal{F}, \mathcal{G}). \end{aligned}$$

Let $\mathcal{A} = \mathrm{Ind}(\mathcal{A}_0)$ and $\mathcal{B} = \mathrm{Ind}(\mathcal{B}_0)$ be objects of Lincat_Λ which are compactly generated and let $F\colon \mathcal{A} \to \mathcal{B}$ be a continuous functor that preserves compact objects. We have a tautological functor

$$F_0^{op} : \mathcal{A}_0^{op} \to \mathcal{B}_0^{op}.$$

Taking its ind-extension, we get a functor

$$F^o : \mathcal{A}^\vee \to \mathcal{B}^\vee \tag{9.3}$$

which is called the *conjugate* functor to F. The construction

$$\mathcal{A} \mapsto \mathcal{A}^\vee, \quad F \mapsto F^o$$

can be upgraded to an endofunctor on the subcategory of Lincat_Λ consisting of compactly generated Λ-linear categories and continuous functors which preserve compact objects.

Remark 9.1.1. *In terms of the duality on* Lincat_Λ*, the functor F^o is equivalent to the left adjoint of $F^\vee$. Equivalently, the functor F^o is the dual of the right adjoint to F, see (Dennis Gaitsgory, 2012, §1.5).*

Let $\mathcal{I} \to \mathrm{Lincat}_\Lambda$ be a cofiltered diagram of Λ-linear categories such that of each i the category $i \mapsto \mathcal{A}_i$ compactly generated and all transition functors $\mathcal{A}_i \to \mathcal{A}_j$ preserve compact objects. Denote $\mathcal{A} = \mathrm{colim}_{i \in \mathcal{I}} \mathcal{A}_i$. It follows from (Dennis Gaitsgory and Rozenblyum, 2017a, Proposition 6.3.4) that the natural map

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{A}_i^\vee \to \mathcal{A}^\vee \tag{9.4}$$

obtained by passing to conjugate functors, is an equivalence.

9.2 Relative tensor product

Let us first review the general formalism of relative tensor products. Let $\mathcal{R}$ be a monoidal category with $\mathbf{1}_{\mathcal{R}}$ its unit. Let $\mathrm{Alg}(\mathcal{R})$ denote the category of associative algebra objects in $\mathcal{R}$. Let $\mathrm{LMod}(\mathcal{R})$ (resp. $\mathrm{RMod}(\mathcal{R})$) the category of left (resp. right) module objects in $\mathcal{R}$. I.e., objects in $\mathrm{LMod}(\mathcal{R})$ (resp. $\mathrm{RMod}(\mathcal{R})$) consist of pairs (A, M) with $A \in \mathrm{Alg}(\mathcal{R})$ and M a left (resp. right) A-module. For $A \in \mathrm{Alg}(\mathcal{R})$ we denote by $\mathrm{LMod}_A(\mathcal{R}) = \mathrm{LMod}(\mathcal{R}) \times_{\mathrm{Alg}(\mathcal{R})} \{A\}$ (resp. $\mathrm{RMod}_A(\mathcal{R}) = \{A\} \times_{\mathrm{Alg}(\mathcal{R})} \mathrm{LMod}(\mathcal{R})$).

Similarly, let $\mathrm{BMod}(\mathcal{R})$ denote the category of bimodule objects in $\mathcal{R}$. For $A, B \in \mathrm{Alg}(\mathcal{R})$ we denote by ${}_A\mathrm{BMod}_B = \{A\} \times_{\mathrm{Alg}(\mathcal{R})} \mathrm{BMod}(\mathcal{R}) \times_{\mathrm{Alg}(\mathcal{R})} \{B\}$ the category of A-B-bimodules. An A-A-bimodule is also called as an A-bimodule. For example, A itself can be regarded as A-bimodule via the left and the right multiplication. See (Lurie, 2017, §4.3) for detailed discussions. Given associative algebra objects $A, B, C \in \mathrm{Alg}(R)$ and bimodules $M \in {}_A\mathrm{BMod}_B$ and $N \in {}_B\mathrm{BMod}_C$, the relative tensor product $M \otimes_B N$, if exists, is the unique object (up to equivalence) in ${}_A\mathrm{BMod}_C$, corepresenting the functor sending $X \in {}_A\mathrm{BMod}_C$ to the space of B-bilinear A-C-bimodule maps $M \otimes N \to X$ (in appropriate homotopy sense, see (Lurie, 2017, Definition 4.4.2.3)). On the other hand, there is the two-sided bar construction

$$\mathrm{LMod}(\mathcal{R}) \times_{\mathrm{Alg}(\mathcal{R})} \mathrm{RMod}(\mathcal{R}) \to ({}_A\mathrm{BMod}_C)^{\Delta^{\mathrm{op}}}, \quad (M, N) \mapsto \mathrm{Bar}_B(M, N)_\bullet,$$

where $\mathrm{Bar}_B(M, N)_\bullet$ is a simplicial object in the category of A-C-bimodules, given informally as

$$\mathrm{Bar}_B(M, N)_n = M \otimes B^n \otimes N$$

with boundary maps induced by the multiplication on B and actions on M and N, and degeneracy maps given by insertions of the unit of B. See (Lurie, 2017, Notation 4.4.2.4, Construction 4.4.2.7). If $A = B = C$ and $M = N = A$, we simply denote $\mathrm{Bar}_A(A, A)_\bullet$ by $\mathrm{Bar}(A)_\bullet$, called the *bar construction* of the bimodule A.

We do not know in general whether $M \otimes_B N$ is always given by the geometric realization of $\mathrm{Bar}_B(M, N)_\bullet$ as soon as the latter exists. This is the case if $\mathcal{R}$ admits geometric realizations and such that the monoidal product $\otimes\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ preserves geometric realizations in each variable, by (Lurie, 2017, Theorem 4.4.2.8). It is also the case in the following two examples.

Example 9.2.1. *Assume that $M = M_0 \otimes B$ with M_0 a left A-module (resp. $N = B \otimes N_0$ with N_0 a right C-module). Then $M \otimes_B N$ exists and is represented by $M_0 \otimes N$*

(resp. $M \otimes N_0$). To see this, we follow the argument of (Lurie, 2017, Proposition 5.2.2.6). If $\mathcal{R}$ admits geometric realizations and such that the monoidal product $\otimes\colon \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ preserves geometric realizations in each variable, then $M \otimes_B N$ exists and is isomorphic to $M_0 \otimes N$ (resp. $M \otimes N_0$) by (Lurie, 2017, Proposition 4.4.3.14, 4.4.3.16). The general situation reduces this case via Yoneda embedding.

Example 9.2.2. *Using a similar argument, one can also prove the relative tensor product exists in the following situation. Let C be a (small) category admitting finite limits. Let* pt *denote the final object. Then it C^{op} has the coCartesian symmetric monoidal structure: for $X, Y \in C$, the tensor product $X \otimes Y$ in C^{op} is the finite product $X \times Y$ in C. We note that every object X is a commutative algebra object in C^{op}, with the multiplication given by the diagonal map $\Delta_X : X \to X \times X$ in C and the unit given by the structural map $p_X : X \to \mathrm{pt}$. In addition, every morphism $f : X \to Y$ in C gives a commutative algebra homomorphism in C^{op}. Now given two morphisms $a : M \to X, b : N \to X$ in C, we regard M, N as X-modules in C^{op}. We claim that $M \otimes_X N$ exists and is representable by $M \times_X N$. Namely, the two-sided bar complex as the cosimplicial object in C is*

$$\Delta \to C : M \times N \underset{\mathrm{id}\times b\times\mathrm{id}}{\overset{\mathrm{id}\times a\times\mathrm{id}}{\rightrightarrows}} M \times X \times N \substack{\longrightarrow \\ \longrightarrow \\ \longrightarrow} M \times X \times X \times N \cdots$$

We will consider the embedding $C^{\mathrm{op}} \to \mathrm{Ind}(C^{\mathrm{op}})$, where $\mathrm{Ind}(C^{\mathrm{op}})$ denotes the ind-completion of C^{op}, equipped with the induced symmetric monoidal structure so that the tensor product preserves colimits in each variable. Again, by the argument of (Lurie, 2017, Proposition 5.2.2.6), it is enough to show that the geometric realization of this simplicial object $\Delta^{\mathrm{op}} \to \mathrm{Ind}(C^{\mathrm{op}})$ is represented by $M \times_X N$. By (Lurie, 2009, Lemma 6.1.4.7), its geometric realization can be computed as the colimit of the truncated colimit diagram $(\Delta_{\leq 1})^{\mathrm{op}} \to \mathrm{Ind}(C^{\mathrm{op}})$, which in turn is the limit of $M \times N \rightrightarrows M \times X \times N$ in C. But this is exactly $M \times_X N$.

Chapter 10

THE FORMALISM OF CORRESPONDENCES

In this subsection, we review the formalism of correspondences, as first appeared in (Liu and Zheng, 2012, §6.1) and (Dennis Gaitsgory and Rozenblyum, 2017a, Chapter 7). There are mainly two (closely related) usages of this formalism in the paper. First, it provides a convenient framework to discuss convolution pattern arising from algebraic geometry and representation theory and therefore is useful for our study of (geometric) trace. Second, it encodes various sheaf theories in algebraic geometry in a concise way, as first observed by Lurie.

10.1 Category of correspondences

Let C be an ∞-category that admits finite limits and coproducts. The category C will play the role of the category of geometric objects. We denote by pt the final object in C. Let $vert, horiz$ be two classes of morphisms in C, each of which contains all equivalences in C and is stable under composition and base change. Note that the classes $vert, horiz$ are stable under products. That is, if $f_i\colon X_i \to Y_i, i = 1, 2$ are in $vert$ (resp. $horiz$) then the product $f_1 \times f_2\colon X_1 \times X_2 \to Y_1 \times Y_2$ is a member of $vert$ (resp. $horiz$). We denote by C_{vert} (resp. C_{horiz}) the 1-full subcategory of C consisting of morphisms in $vert$ (resp. $horiz$).

Let $\mathrm{Corr}(C)_{vert;horiz}$ denote the category of correspondences. Informally, objects of $\mathrm{Corr}(C)_{vert;horiz}$ are the same as those of C and morphisms from X to Y are given by diagrams

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "f"] & X \\ Y & \end{tikzcd} \tag{10.1}$$

with $g \in horiz$ and $f \in vert$. We sometimes just write such diagram by $Y \xleftarrow{f} Z \xrightarrow{g} X$ for short. In addition, to emphasize that such a morphism in $\mathrm{Corr}(C)_{vert;horiz}$ is a correspondence rather than an actual map, we sometimes write it as $X \overset{f\circ g^{-1}}{\dashrightarrow} Y$ or simply as $X \dashrightarrow Y$. The composition of the correspondences $Y \leftarrow S \to X$ and $Z \leftarrow T \to Y$ is given by the correspondence $Z \leftarrow W \to X$ which is formed by taking the pullback:

$$\begin{tikzcd}
W \arrow[r] \arrow[d] & S \arrow[r] \arrow[d] & X \\
T \arrow[r] \arrow[d] & Y & \\
Z & &
\end{tikzcd}.$$

Given $g\colon Y \to X$ in *horiz* we will sometimes identify it with the correspondence $Y \xleftarrow{\mathrm{id}} Y \xrightarrow{g} X$ and we refer to such morphisms as *horizontal*. Similarly, given a morphism $f\colon X \to Y$ in $vert$ we will identify it with the correspondence $Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X$ and refer to such morphisms of $\mathrm{Corr}(C)_{vert;horiz}$ as *vertical*. If $vert, horiz$ are all morphisms, we simply write $\mathrm{Corr}(C)_{vert;horiz}$ by $\mathrm{Corr}(C)$. We refer to (Dennis Gaitsgory and Rozenblyum, 2017a, §7.1) and (Liu and Zheng, 2012, §6.1) for precise definition of it as an ∞-category.

Remark 10.1.1. *The category* $\mathrm{Corr}(C)_{vert;horiz}$ *admits an* $(\infty, 2)$*-categorical enhancement* $\mathrm{Corr}(C)^{adm}_{vert;horiz}$ *of the category of correspondences, depending on a certain class* adm *of morphisms of* C*. A* 2*-morphism in* $\mathrm{Corr}(C)^{adm}_{vert;horiz}$ *is given by a diagram*

$$\begin{tikzcd}
S' \arrow[dr, "r"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& S \arrow[r] \arrow[d] & X \\
& Y &
\end{tikzcd}$$

with $r \in adm$*. See (Dennis Gaitsgory and Rozenblyum, 2017a, §7.1.1.2) for details.*

As C admits finite limits it is a symmetric monoidal category under the Cartesian monoidal structure. This induces a symmetric monoidal structure on $\mathrm{Corr}(C)_{vert;horiz}$ constructed in (Dennis Gaitsgory and Rozenblyum, 2017a, Chapter 9) (see also (Liu and Zheng, 2014, §6.1)). Namely, the tensor product of objects X, Y in $\mathrm{Corr}(C)_{vert;horiz}$ is their product $X \times Y$ as objects of C. Note that in general $X \times Y$ is not the product of X and Y in $\mathrm{Corr}(C)_{vert;horiz}$. For this reason, sometimes we write $X \otimes Y$ to emphasize we regard $X \times Y$ as the tensor product of X and Y in $\mathrm{Corr}(C)_{vert;horiz}$. We note that the 1-full subcategories C_{vert} and $(C_{horiz})^{\mathrm{op}}$ are symmetric monoidal subcategories.

In particular, it makes sense to talk about associative and commutative algebra objects in $\mathrm{Corr}(C)_{vert;horiz}$.

Example 10.1.2. *Every object $X \in C$ with the diagonal map $\Delta_X : X \to X \times X$ and the structural map $\pi_X : X \to \mathrm{pt}$ belonging to horiz has a natural commutative algebra structure in $(C_{horiz})^{\mathrm{op}}$ with the multiplication given by Δ_X and the unit given by π_X. In addition, if X and Y are two objects satisfying the above properties and $f : X \to Y$ is a morphism belonging to horiz, then we have a commutative algebra homomorphism in $(C_{horiz})^{\mathrm{op}}$ from Y to X induced by f. In particular, X is a (left) Y-module.*

As $(C_{horiz})^{\mathrm{op}} \to \mathrm{Corr}(C)_{vert;horiz}$ is a symmetric monoidal subcategory, we obtain the corresponding (maps between) commutative algebra objects in $\mathrm{Corr}(C)_{vert;horiz}$. If in addition $f \in C_{vert}$, then $f : X \to Y$ is naturally a morphism of Y-modules from X to Y on $\mathrm{Corr}(C)_{vert;horiz}$.

10.2 Monads and modules in the category of correspondences

We will be interested in a particular class of algebra objects and their bimodules arising from *monads* in $\mathrm{Corr}(C)$. We review the description of algebras in terms of Segal objects and note how these constructions generalize to describe bimodules.

Consider the $(\infty, 2)$-category $\mathrm{Corr}(C)^{all}$. For every $X, X' \in C$ we have a category $\mathbf{Hom}_{\mathrm{Corr}(C)}(X', X)$. If $X = X'$ this category is denoted by $\mathbf{End}_{\mathrm{Corr}(C)}(X)$. It carries a canonical monoidal structure induced by composition. In general, the category $\mathbf{Hom}_{\mathrm{Corr}(C)}(X', X)$ is equipped with a left action of $\mathbf{End}_{\mathrm{Corr}(C)}(X)$ and a right action of $\mathbf{End}_{\mathrm{Corr}(C)}(X')$.

10.3 Monads and Segal objects

First recall the definition of Segal objects (also known as category objects) in an ∞-category.

Definition 10.3.1. *A simplicial object $X_\bullet \colon \Delta^{op} \to C$ is called a Segal object if for every $n \geq 1$, the map*

$$X_n \to X_1 \times_{X_0} X_1 \times_{X_0} \cdots \times_{X_0} X_1$$

induced by the maps $\delta_i \colon [1] \cong \{i, i+1\} \subset [n]$ for $i = 0, 1, \ldots, n-1$, is an equivalence.

Remark 10.3.2. *If C is an ordinary category, a Segal object is fully determined by the objects X_0, X_1, the boundary maps $d_1, d_0 \colon X_1 \to X_0$, $d_1 \colon X_2 \to X_1$ and the degeneracy map $s \colon X_0 \to X_1$. These define a category object of C in the usual sense. Namely, X_0 is the class of objects, X_1 the morphism objects, the morphisms*

$d_1, d_0 \colon X_1 \to X_0$ as source and target. The composition is given by the morphism $d_1 \colon X_2 \to X_1$ and unit by $d_1 \colon X_2 \to X_1$.

Example 10.3.3. *Let $f \colon X \to Y$ be a morphism in $\mathcal{C}$. The Čech nerve $X_\bullet \to Y$ of f (see (Lurie, 2009, §6.1.2)), where*

$$X_n = \overbrace{X \times_Y X \times \cdots \times_Y X}^{n+1},$$

is easily seen to be a Segal object of $\mathcal{C}$. Indeed, it is even a groupoid object, in the sense of (Lurie, 2009, Definition 6.1.2.7)). This will be our main example.

Example 10.3.4. *A Segal object $X_\bullet$ with $X_0 = \mathrm{pt}$ is a monoid object (in the sense of (Lurie, 2009, p. 4.1.2)). Giving such a monoid object is equivalent to giving an associative algebra object in $\mathcal{C}$ by (Lurie, 2009, Proposition 4.1.2.10, 4.1.2.11).*

This last example admits the following generalization. Let $X \in \mathcal{C}$ and consider the monoidal category $\mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X)$. Then (Dennis Gaitsgory and Rozenblyum, 2017a, Proposition 9.4.1.5, Theorem 9.4.4.2) gives:

Theorem 10.3.5. *1. There is a natural lax monoidal functor*

$$\mathrm{Sp} \colon \mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X) \to \mathrm{Corr}(\mathcal{C}).$$

2. There is a canonical equivalence between Segal objects $X_\bullet$ in $\mathcal{C}$ with $X_0 = X$ and associative algebra objects in $\mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X)$.

Roughly speaking, the functor Sp from (1) sends $X \leftarrow Z \to X$ to the object $Z \in \mathcal{C}$ with the lax structure given by horizontal arrow $Z \times_X Z \to Z \times Z$.

To construct the equivalence from (2), let $A \in \mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X)$. Then X itself is an A-A-bimodule of A in $\mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X)$. Then we have the two-sided Bar complex $\mathrm{Bar}_A(X, X)_\bullet$ as a simplicial object in $\mathbf{End}_{\mathrm{Corr}(\mathcal{C})}(X)$. Its image under $\underline{\mathrm{Sp}}$ gives a simplicial object in $\mathcal{C}$. which is the desired Segal object in $\mathcal{C}$. Conversely, a segal object $X_\bullet$ in $\mathcal{C}$ gives rise to an algebra object in $X_1 \in \mathrm{Corr}(\mathcal{C})$ with multiplication given by the correspondence

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \simeq X_2 \arrow[r, "d_0 \times d_2"] \arrow[d, "d_1"] & X_1 \times X_1 \\ X_1 \end{tikzcd}$$

and the unit given by the correspondence

$$\begin{tikzcd} X_0 \arrow[r, "\pi_{X_0}"] \arrow[d, "s"] & \mathrm{pt} \\ X_1 & \end{tikzcd}$$

Remark 10.3.6. *Assume that the Segal object $X_\bullet$ is such that:*

- *All morphisms of the simplicial object $X_\bullet$ are in C_{vert};*
- *The diagonal map $\Delta_{X_0}\colon X_0 \to X_0 \times X_0$ and the structural map $\pi_{X_0} : X_0 \to \mathrm{pt}$ are in C_{horiz}.*

Then the associative algebra object X_1 can be realized as an associative algebra object of the monoidal category $\mathrm{Corr}(C)_{vert;horiz}$.

10.4 The action on bimodules

Given $X_0, X_0' \in C$ and an associative algebra objects $X_1 \in \mathrm{Alg}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0))$, $X_1' \in \mathrm{Alg}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0'))$ the lax-monoidal functor from Theorem 10.3.5 (1) induces a functor

$$\mathrm{Sp}\colon {}_{X_1}\mathrm{BMod}_{X_1'}(\mathbf{Hom}_{\mathrm{Corr}(C)}(X_0', X_0)) \to {}_{X_1}\mathrm{BMod}_{X_1'}(\mathrm{Corr}(C)).$$

Letting $X_0 = X_0'$ and $X_1 = X_1'$, we have a lax monoidal functor

$$\mathrm{Sp}_{X_1}\colon {}_{X_1}\mathrm{BMod}_{X_1}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0)) \to {}_{X_1}\mathrm{BMod}_{X_1}(\mathrm{Corr}(C)).$$

The unit of the monoidal category $\mathrm{BMod}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0))$ is sent by Sp_{X_1} to the algebra X_0 with multiplication given by the horizontal arrow $X_0 \to X_0 \times X_0$ (it corresponds to the Segal object given by the constant simplicial object X_0). Moreover, it is clear that as an algebra, X_0^{rev} is isomorphic to X_0. Then Theorem 10.3.5 (1) upgrades to a lax monoidal functor

$$\mathrm{Sp}_{X_0}\colon \mathbf{End}_{\mathrm{Corr}(C)}(X_0) \to {}_{X_0}\mathrm{BMod}_{X_0}(\mathrm{Corr}(C)) \tag{10.2}$$

Explicitly, an object $Q \in \mathbf{End}_{\mathrm{Corr}(C)}(X_0)$ has left and right actions given by the two maps $X_0 \leftarrow Q \to X_0$. The simple structure of such modules gives a simple description of the relative tensor product over such bimodules in the category of correspondences.

Remark 10.4.1. *Let $Q \in {}_{X_1}\mathrm{BMod}_{X_1}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0))$ with X_1, X_0 as in Remark 10.3.6. If the action maps $X_1 \times_{X_0} Q \to Q$ and $Q \times_{X_0} X_1 \to Q$ are in C_{vert}, then the corresponding module under* (10.2) *is an object of* $\mathrm{Corr}(C)_{vert;horiz}$.

Chapter 11

SHEAF THEORIES

11.1 The notion of a sheaf theory

We fix C as before. By definition, a sheaf theory of C is a lax symmetric monoidal functor

$$\mathcal{D}\colon \mathrm{Corr}(C)_{vert;horiz} \to \mathrm{Lincat}_{\Lambda}. \tag{11.1}$$

For a horizontal morphism $Y \xleftarrow{\mathrm{id}} Y \xrightarrow{g} X$ we will denote the corresponding functor by $g^{\star}\colon \mathcal{D}(X) \to \mathcal{D}(Y)$. For a vertical morphism $Y \xleftarrow{f} X \xrightarrow{\mathrm{id}} X$ we denote the corresponding functor by $f_{\dagger}\colon \mathcal{D}(X) \to \mathcal{D}(Y)$. Then for a general correspondence $Y \xleftarrow{f} Z \xrightarrow{g} X$ the associated functor is equivalent to $f_{\dagger} \circ g^{\star}$.

Let us spread out some structures encoded by such a functor.

1. We can pass to right adjoints. For $(g : X \to Y) \in C_{horiz}$, let $g_{\star}$ be the (not necessarily continuous) right adjoint of $g^{\star}$, and for $(f : X \to Y) \in C_{vert}$, let $f^{\dagger}$ be the (not necessarily continuous) right adjoint of $f_{\dagger}$.

2. The functoriality of $\mathcal{D}$ encodes a "base change theorem". Namely, let

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd} \tag{11.2}$$

   be a pullback square in C. Then if $f \in vert$ and $g \in horiz$, we have that $f' \in vert$, $g' \in horiz$ and part of the data of the functor $\mathcal{D}$ is to give an equivalence

$$(f')_{\dagger} \circ (g')^{\star} \simeq g^{\star} \circ f_{\dagger}. \tag{11.3}$$

3. The lax symmetric monoidal structure of $\mathcal{D}$ provides a functor

$$\boxtimes\colon \mathcal{D}(X) \otimes \mathcal{D}(Y) \to \mathcal{D}(X \times Y), \quad X, Y \in C \tag{11.4}$$

   together with all necessary higher coherence conditions. Let X be as in Example 10.1.2. This induces a symmetric monoidal structure on the category $\mathcal{D}(X)$. Informally, the symmetric monoidal structure is given by the composition

$$\mathcal{D}(X)\otimes\mathcal{D}(X)\xrightarrow{\boxtimes}\mathcal{D}(X\times X)\xrightarrow{\Delta_X^\star}\mathcal{D}(X),\quad \mathcal{F},\mathcal{G}\mapsto\mathcal{F}\otimes\mathcal{G}:=\Delta_X^\star(\mathcal{F}\boxtimes\mathcal{G}). \tag{11.5}$$

We let $\Lambda_X\in\mathcal{D}(X)$ denote the unit object with respect to this symmetric monoidal structure, which corresponds to the functor

$$\mathrm{Mod}_\Lambda\to\mathcal{D}(\mathrm{pt})\xrightarrow{\pi_X^\star}\mathcal{D}(X). \tag{11.6}$$

This symmetric monoidal structure is closed. That is, for every pair of objects $\mathcal{F}_1,\mathcal{F}_2\in\mathcal{D}(X)$ there is an object $\underline{\mathrm{Hom}}(\mathcal{F}_1,\mathcal{F}_2)$ such that for every $\mathcal{G}\in\mathcal{D}(X)$ there is a canonical equivalence

$$\mathrm{Map}_{\mathcal{D}(X)}\big(\mathcal{G},\underline{\mathrm{Hom}}(\mathcal{F}_1,\mathcal{F}_2)\big)\simeq\mathrm{Map}_{\mathcal{D}(X)}\big(\mathcal{G}\otimes\mathcal{F}_1,\mathcal{F}_2\big). \tag{11.7}$$

Note we have

$$\underline{\mathrm{Hom}}(\mathcal{F}_1\otimes\mathcal{F}_2,\mathcal{F}_3)=\underline{\mathrm{Hom}}(\mathcal{F}_1,\underline{\mathrm{Hom}}(\mathcal{F}_2,\mathcal{F}_3)). \tag{11.8}$$

4. Let $f\colon X\to Y$ in C_{horiz} as in Example 10.1.2. Then $f^\star:\mathcal{D}(Y)\to\mathcal{D}(X)$ is a symmetric monoidal functor. In particular, it endows $\mathcal{D}(X)$ with a structure of a $\mathcal{D}(Y)$ monoidial category. If in addition $f\in C_{vert}$ as well, then $f_\dagger\colon\mathcal{D}(X)\to\mathcal{D}(Y)$ is a morphism of $\mathcal{D}(Y)$-modules. In particular, for $\mathcal{F}\in\mathcal{D}(X)$, $\mathcal{G}\in\mathcal{D}(Y)$ we have a canonical equivalence

$$f_\dagger(\mathcal{F})\otimes\mathcal{G}\simeq f_\dagger(\mathcal{F}\otimes f^\star(\mathcal{G})). \tag{11.9}$$

which encodes a "projection formula" for $f_\dagger$ and $f^\star$. Along with the co-unit of the adjunction $(f_\dagger,f^\dagger)$, it gives, for every $\mathcal{F},\mathcal{G}\in\mathcal{D}(Y)$, a natural map

$$f^\dagger(\mathcal{G})\otimes f^\star(\mathcal{F})\to f^\dagger(\mathcal{G}\otimes\mathcal{F}) \tag{11.10}$$

adjoint to

$$f_\dagger(f^\dagger(\mathcal{G})\otimes f^\star(\mathcal{F}))\simeq f_\dagger f^\dagger(\mathcal{G})\otimes\mathcal{F}\to\mathcal{G}\otimes\mathcal{F},$$

In particular, one has the natural transformation of functors

$$f^\dagger(\Lambda_Y)\otimes f^\star\to f^\dagger \tag{11.11}$$

In addition, we have

$$f^\dagger\underline{\mathrm{Hom}}(\mathcal{F},\mathcal{G})\simeq\underline{\mathrm{Hom}}(f^\star\mathcal{F},f^\dagger\mathcal{G}),\quad f_\star\underline{\mathrm{Hom}}(\mathcal{F},f^\dagger\mathcal{G})\simeq\underline{\mathrm{Hom}}(f_\dagger\mathcal{F},\mathcal{G}). \tag{11.12}$$

5. Assume that $p : X \to \mathrm{pt}$ belongs to $vert$. Let

$$\mathbb{D}_X : \mathcal{D}(X)^{\mathrm{op}} \to \mathcal{D}(X), \quad \mathbb{D}_X\mathcal{F} := \underline{\mathrm{Hom}}(\mathcal{F}, p_X^{\dagger}\Lambda_{\mathrm{pt}}). \tag{11.13}$$

   Then for $f : X \to Y$ in $vert \cap horiz$, the isomorphisms in (11.12) specialize to isomorphisms $\mathbb{D}_X f^{\star} = f^{\dagger}\mathbb{D}_Y$ and $f_{\star}\mathbb{D}_X = \mathbb{D}_Y f_{\dagger}$.

Remark 11.1.1. *In many examples of a sheaf theory, the functor 11.4 is fully faithful and admits a continuous right adjoint. Indeed, fully faithfulness is equivalent to a Künneth type formula.*

Remark 11.1.2. *Let $S \in C$ be as in Example 10.1.2. Then there is the (non-full) embedding*

$$\mathrm{Corr}(C_{/S})_{vert;horiz} \to \mathrm{Corr}(C)_{vert;horiz}$$

which is a lax symmetric monoidal functor. Therefore, one can restrict $\mathcal{D}$ along this embedding to obtain a sheaf theory on $C_{/S}$, denoted by $\mathcal{D}_{/S}$. The lax symmetric monoidal structure is provided by

$$\mathcal{D}(X) \otimes \mathcal{D}(Y) \to \mathcal{D}(X \times Y) \xrightarrow{\Delta_S^{\star}} \mathcal{D}(X \times_S Y)$$

Remark 11.1.3. *Let $X \in C$ such that both π_X and Δ_X are in $C_{vert} \cap C_{horiz}$, then X is self-dual in* $\mathrm{Corr}(C)$. *Then the evaluation and the unit maps are given by*

$$\mathrm{pt} \xleftarrow{\pi_X} X \xrightarrow{\Delta_X} X \times X, \quad X \times X \xleftarrow{\Delta_X} X \xrightarrow{\pi_X} \mathrm{pt}$$

It follows that if the sheaf theory $\mathcal{D}$ is symmetric monoidal (rather than just lax symmetric monoidal), then $\mathcal{D}(X)$ is self-dual.

11.2 Additional base change theorems

The sheaf theory $\mathcal{D}$ in practice often satisfies certain additional base change theorems for certain class of morphisms. The general setting is as follows.

Let $C_{sm} \subseteq C_{horiz}$ and $C_{prop} \subseteq C_{vert}$ be another two classes of morphisms of C stable under composition and base change and containing equivalences. We make the following assumptions on these classes:

(a). For $f \in C_{prop}$, the functor $f^{\dagger}$ is continuous.

(b). For $g \in C_{sm}$, the functor $g_{\star}$ is continuous.

We make the following assumptions on various compatibilities between these adjoints. Consider the Cartesian square (11.2). Then consider the following assumptions.

Assumptions 11.2.1. *The adjoint pairs* $((-)^\star, (-)_\star)$, $((-)_\dagger, (-)^\dagger)$ *satisfy the following conditions:*

(1) If $f \in C_{prop}$ and $g \in C_{vert}$ the natural map $f'_\dagger \circ g'^\dagger \to g^\dagger \circ f_\dagger$ is an equivalence.

(2) If $f \in C_{sm}$ and $g \in C_{horiz}$ then the natural map $f^\star \circ g_\star \to g'_\star \circ f'^\star$ is an equivalence.

(3) If $f \in C_{prop}$ and $g \in C_{sm}$ the natural map $g'^\star \circ f^\dagger \to f'^\dagger \circ g^\star$ is an equivalence.

(4) If $f \in C_{prop}$ and $g \in C_{horiz}$ the natural map $f_\dagger \circ g'_\star \to g_\star \circ f'_\dagger$ is an equivalence.

(5) For $f_i \colon X_i \to Y_i$ in C_{prop} with $i = 1, 2$, and $\mathcal{F}_i \in \mathcal{D}(Y_i)$ the natural map

$$f_1^\dagger(\mathcal{F}_1) \boxtimes f_2^\dagger(\mathcal{F}_2) \to (f_1 \times f_2)^\dagger(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

is an equivalence.

(6) For $f_i \colon X_i \to Y_i$ in C_{sm} with $i = 1, 2$, and $\mathcal{F}_i \in \mathcal{D}(X_i)$ the natural map

$$(f_1)_\star(\mathcal{F}_1) \boxtimes (f_2)_\star(\mathcal{F}_2) \to (f_1 \times f_2)_\star(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$$

is an equivalence.

We make some additional assumptions. They are satisfied in certain sheaf theories, but not always.

1. *Let $f \in C_{vert}$. For $g \in C_{prop}$, the natural map $f_\dagger \circ g^\dagger \to g'^\dagger \circ f'_\dagger$ is an equivalence. For $g \in C_{sm}$, the natural map $f_\dagger \circ g'_\star \to g_\star \circ f'_\dagger$ is an equivalence.*

2. *Let $f \in C_{horiz}$. For $g \in C_{prop}$, the natural map $f'^\star \circ g^\dagger \to g'^\dagger \circ f^\star$ is an equivalence. For $g \in C_{sm}$, the natural map $f^\star \circ g'_\star \to g_\star \circ f^\star$ is an equivalence.*

(IX) For $f \in C_{prop} \cap C_{horiz}$, $f^\dagger \cong f^\star$;

(X) For $f \in C_{vert} \cap C_{sm}$, $f_\star \cong f_\dagger[n]$ for some $n_f \in \mathbb{Z}$.

Example 11.2.2. *We have the following basic example of a sheaf theory. Let $\mathcal{D} : C^{\mathrm{op}} \to \mathrm{Lincat}_\Lambda$ be a lax symmetric monoidal functor. Then let $horiz = all$ and let $vert$ consist of the following class of morphisms: $f : X \to Y$ belongs to $vert$ if the functor $\mathcal{D}(f) : \mathcal{D}(Y) \to \mathcal{D}(X)$ admits a continuous left adjoint $\mathcal{D}(f)^L$ and for every diagram as* (11.2)*, the induced Beck-Chevalley map is an equivalence*

$$\mathcal{D}(f')^L \circ \mathcal{D}(g) \cong \mathcal{D}(g') \circ \mathcal{D}(f)$$

Then $\mathcal{D}$ extends to a sheaf theory $\mathcal{D} : \mathrm{Corr}(C)_{vert;horiz} \to \mathrm{Lincat}_\Lambda$, such that for $g \in horiz$, $g^\star = \mathcal{D}(g)$ and $f \in vert$, $f_\dagger = \mathcal{D}(f)^L$. In this case, we can let $prop = vert$. Then Assumptions 11.2.1 1 4 5 automatically hold.

11.3 Category of cohomological correspondences

The sheaf theory $\mathcal{D}$ can also be (largely) encoded as a symmetric monoidal category $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$, usually called the category of cohomological correspondences. Namely, let $\mathcal{D}$ be a sheaf theory on C as above. Applying the symmetric monoidal Grothendieck construction ((Lurie, 2017, Proposition 2.4.3.16)) to the lax symmetric monoidal functor

$$\mathrm{Corr}(C)_{vert;horiz} \to \mathrm{Lincat}_\Lambda \to C\mathrm{at},$$

one obtains a coCartesian fibration $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz} \to \mathrm{Corr}(C)_{vert;horiz}$. Objects of $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$ consist of pairs $(X, \mathcal{F})$ where $X \in C$ and $\mathcal{F} \in \mathcal{D}(X)$, and morphisms been $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ consist of pairs (C, u), where C a correspondence $Y \xleftarrow{\overleftarrow{c}} C \xrightarrow{\overrightarrow{c}} X$, with $\overleftarrow{c}$ in $vert$ and $\overrightarrow{c}$ in $horiz$, and $u : \overrightarrow{c}^\star \mathcal{F} \to \overleftarrow{c}^\dagger \mathcal{G}$.

There is a natural symmetric monoidal structure on $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$, with the symmetric monoidal structure given by $(X, \mathcal{F}) \otimes (Y, \mathcal{G}) = (X \times Y, \mathcal{F} \boxtimes \mathcal{G})$. The unit is given by $(\mathrm{pt}, \Lambda_{\mathrm{pt}})$. We refer to (Xiao and Zhu, 2017, Appendix A) and (Lu and Zheng, 2020) for more detailed discussions of constructions in this category.

We recall some results about dualizable objects in $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$. We make the following assumption throughout the rest of the section.

Assumptions 11.3.1. *Assume that $horiz$ consist of all morphisms in C and the class $vert$ has the following property: for a sequence of morphisms $Z_1 \to Z_2 \to Z_3$ in C, if $Z_1 \to Z_3$ belongs to $vert$, so is $Z_1 \to Z_2$.*

Assume that the structural map $\pi_X : X \to \mathrm{pt}$ belongs to $vert$. Then as argued in (Lu and Zheng, 2020, Lemma 2.8), for $(X, \mathcal{F})$, $(Y, \mathcal{G})$ and $(Z, \mathcal{H}) \in \mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$, we have

$$\mathrm{Map}\big((Z,\mathcal{H})\otimes(X,\mathcal{F}),(Y,\mathcal{G})\big)=\mathrm{Map}\big((Z,\mathcal{H}),(X\times Y,\underline{\mathrm{Hom}}((p_X)^\star\mathcal{F},(p_Y)^\dagger\mathcal{G}))\big),$$

where $\underline{\mathrm{Hom}}$ is the bifunctor from (11.7) (for $X\times Y$), and $p_X : X\times Y\to X, p_Y : X\times Y\to Y$ are the two natural projections. That is, the internal hom between $(X,\mathcal{F})$ and $(Y,\mathcal{G})$ exists and is represented by $(X\times Y,\underline{\mathrm{Hom}}(p_X^\star\mathcal{F},p_Y^\dagger\mathcal{G}))$. The following lemma then follows by general facts about symmetric monoidal categories.

Lemma 11.3.2. *1. An object $(X,\mathcal{F})\in\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$ is dualizable if and only if the natural map*

$$\mathcal{F}\boxtimes\mathbb{D}_X\mathcal{F}\to\underline{\mathrm{Hom}}((p_1)^\star\mathcal{F},(p_2)^\dagger\mathcal{F})$$

is an isomorphism, where $p_i : X^2\to X,\ i=1,2$ are two projections.

2. If $(X,\mathcal{F})$ is dualizable, then its dual is $(X,\mathbb{D}_X\mathcal{F})$, with the evaluation map given by the tautological map

$$e_{(X,\mathcal{F})}:(\Delta_X)^\star(\mathcal{F}\boxtimes\mathbb{D}_X\mathcal{F})=\mathcal{F}\otimes\mathbb{D}_X\mathcal{F}\to(\pi_X)^\dagger\Lambda_{\mathrm{pt}},\tag{11.14}$$

and with the unit map is given by the dual of $e_{(X,\mathcal{F})}$. (Note that $(X\times X,\mathcal{F}\boxtimes\mathbb{D}_X\mathcal{F})$ is self-dual in $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$.)

Lemma 11.3.3. *Let $(X,\mathcal{F})$ be a dualizable object in $\mathrm{Corr}^{\mathcal{D}}(C)_{vert;horiz}$. If $(\Delta_X)_\dagger$ sends compact objects to compact objects, and if Λ_X is compact in $\mathcal{D}(X)$, then $\mathcal{F}\in\mathcal{D}(X)$ is compact.*

Proof. Note that $(\Delta_X)^\dagger$ and $\mathrm{Hom}_{\mathcal{D}(X)}(\Lambda_X,-)$ commute with colimits. Therefore,

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F},\operatorname*{colim}_i\mathcal{G}_i)&=\mathrm{Hom}_{\mathcal{D}(X)}(\Lambda_X,(\Delta_X)^\dagger(\mathbb{D}_X\mathcal{F}\boxtimes(\operatorname*{colim}_i\mathcal{G}_i)))\\&=\operatorname*{colim}_i\mathrm{Hom}_{\mathcal{D}(X)}(\Lambda_X,(\Delta_X)^\dagger(\mathbb{D}_X\mathcal{F}\boxtimes\mathcal{G}_i))=\operatorname*{colim}_i\mathrm{Hom}_{\mathcal{D}(X)}(\mathcal{F},\mathcal{G}_i)\end{aligned}$$

□

Chapter 12

THE CATEGORICAL TRACE

12.1 Hochschild homology

Let $\mathcal{R}$ be a symmetric monoidal category. The trace of an endomorphism of a dualizable object in $\mathcal{R}$ is a classical notion. Namely, if X is a dualizable object in $\mathcal{R}$ equipped with an endomorphism $f : X \to X$, its trace $\mathrm{tr}(X, f) \in \mathrm{End}(\mathbf{1}_{\mathcal{R}})$ is given by the composition

$$\mathbf{1}_{\mathcal{R}} \xrightarrow{u} X^{\vee} \otimes X \xrightarrow{\mathrm{id}_{X^{\vee}} \otimes f} X^{\vee} \otimes X \cong X \otimes X^{\vee} \xrightarrow{e} \mathbf{1}_{\mathcal{R}}. \tag{12.1}$$

On the other hand, in this article, we will also need a difference type of trace construction, known as the categorical trace. Let us first review the general formalism. We refer to Section 9.2 for a review of bimodules and relative tensor products.

Let A and B be two associative algebras in $\mathcal{R}$. By (Lurie, 2017, Proposition 4.6.3.11)[1], an A-B-bimodule can also be regarded as a left $(A \otimes B^{\mathrm{rev}})$-module or a right $(B \otimes A^{\mathrm{rev}})$-module, where A^{rev} (resp. B^{rev}) is the algebra A (resp. B) with the multiplication reversed. For an associative algebra A, and an A-A-bimodule F, the Hochschild homology of F, if exists, is defined as

$$\mathrm{Tr}(A, F) = A \otimes_{A \otimes A^{\mathrm{rev}}} F \in \mathcal{R}. \tag{12.2}$$

We write

$$[-]_F : F \to \mathrm{Tr}(A, F) \tag{12.3}$$

for the natural morphism, sometimes called the universal trace morphism.

On the other hand, there always exists the Hochschild complex of F defined as

$$\mathrm{HH}(A, F)_{\bullet} = \mathrm{Bar}(A)_{\bullet} \otimes_{A \otimes A^{\mathrm{rev}}} F = A^{\otimes \bullet} \otimes F, \tag{12.4}$$

regarded as a simplicial object $\Delta^{\mathrm{op}} \to \mathcal{R}$. Explicitly, on the level of simplicies and morphisms, for every $n \geq 0$ we have an equivalence

$$\mathrm{HH}(A, F)_n \simeq A^{\otimes n} \otimes F. \tag{12.5}$$

[1] Note that assumption $(\star)$ of *loc. cit.* is not essential, as explained before (Lurie, 2017, Notation 4.6.3.3).

Under this identification, for $0 < i < n$ the face map d_i^{HH} is given by the multiplication map applied to the i-th and the $(i+1)$-th factors in $A^{\otimes n}$, and the face map d_0^{HH} is given by multiplying the first factor in $A^{\otimes n}$ to Q from the right and the face map d_n^{HH} is given by multiplying the n-th factor in $A^{\otimes n}$ to Q from the left. If $\mathcal{R}$ admits geometric realizations and the tensor product preserves geometric realizations in each variable, then the Hochschild homology of F exists and can be computed as the geometric realization of the Hochschild complex.

Remark 12.1.1. *Associated to $\mathcal{R}$, there is a symmetric monoidal 2-category* $\mathrm{Morita}(\mathcal{R})$ *whose objects are associative algebras in $\mathcal{R}$ and whose morphism categories are given by categories of bimodules:*

$$\mathrm{Map}_{\mathrm{Morita}(\mathcal{R})}(A, B) = {}_B\mathrm{BMod}_A$$

and compositions are given by the relative tensor products (assuming relative tensor products exist in $\mathcal{R}$). Then every A-bimodule F gives an endomorphism of A in $\mathrm{Morita}(\mathcal{R})$. *In addition, every algebra A is a dualizable object in* $\mathrm{Morita}(\mathcal{R})$. *Under the equivalence ${}_A\mathrm{BMod}_A \cong {}_{\mathbf{1}_\mathcal{R}}\mathrm{BMod}_{A^{\mathrm{rev}}\otimes A} \cong {}_{A\otimes A^{\mathrm{rev}}}\mathrm{BMod}_{\mathbf{1}_\mathcal{R}}$, the natural A-bimodule structure on A itself gives unit and evaluation maps*

$$A^e \in {}_{A\otimes A^{\mathrm{rev}}}\mathrm{BMod}_{\mathbf{1}_\mathcal{R}}, \quad A^c \in {}_{\mathbf{1}_\mathcal{R}}\mathrm{BMod}_{A^{\mathrm{rev}}\otimes A}, \tag{12.6}$$

which identify the dual of A (in $\mathrm{Morita}(\mathcal{R})$*) as A^{rev}. Then the Hochschild homology of (A, F) is nothing but the trace F in the sense of* (12.1)*, regarded as an endomorphism of A in* $\mathrm{Morita}(\mathcal{R})$. *This justifies our choice of notations. However, we will not systematically explore this approach in this article. On the one hand, we do not want to systematically review the formalism of $(\infty, 2)$-categories. On the other hand, in our applications where $\mathcal{R}$ will mainly be* Lincat_Λ*, we implicitly need some* 3*-category structure on* $\mathrm{Morita}(\mathrm{Lincat}_\Lambda)$.

Example 12.1.2. *If the A-bimodule $F = M \otimes N$, where M is a left A-module and N a right A-module. Then*

$$\mathrm{Tr}(A, F) = A \otimes_{A\otimes A^{\mathrm{rev}}} (M \otimes N) \cong N \otimes_A M.$$

In fact, $\mathrm{HH}(A, F)_\bullet \cong \mathrm{Bar}_A(N, M)_\bullet$.

Example 12.1.3. *Of particular importance in this paper is the following type of bimodules. Let ϕ be an endomorphism of the algebra A. For an A-bimodule F we will denote by ${}^\phi F$ the bimodule obtained by the same action on the right but with* *a pre-composition with ϕ for the left action. In this case we will also denote the Hochschild homology of the bimodule ${}^{\phi}A$ by* $\mathrm{Tr}(A,\phi)$*. That is,*

$$\mathrm{Tr}(A,\phi) \simeq A \otimes_{A\otimes A^{\mathrm{rev}}} {}^{\phi}A. \tag{12.7}$$

In addition, $\phi = \mathrm{id}$*, we simply write* $\mathrm{Tr}(A,\mathrm{id})$ *by* $\mathrm{Tr}(A)$.

Now let $\mathcal{R} = \mathrm{Lincat}_\Lambda$. Then an algebra object A in Lincat_Λ is a presentable Λ-linear monoidal category with monoidal product commutes with colimits separately in each variable. Let F be an A-bimodule category. In this case, $\mathrm{Tr}(A,F)$ always exists, and is sometimes called the categorical trace of (A,F). Note that it is a presentable Λ-linear category.

12.2 Functoriality of Hochschild homology

Let $A, B \in \mathrm{Alg}(\mathcal{R})$ be associative algebras and let $M \in {}_A\mathrm{BMod}_B$. Recall (Lurie, 2017, §4.6.2) that a left dual of M is given by an object $N \in {}_B\mathrm{BMod}_A$ together with a unit (or co-evaluation)

$$u : B \to N \otimes_A M \tag{12.8}$$

which is a morphism in ${}_B\mathrm{BMod}_B$, and a co-unit (or evaluation)

$$e : M \otimes_B N \to A \tag{12.9}$$

which is a morphism in ${}_A\mathrm{BMod}_A$, such that the compositions

$$M \simeq M \otimes_B B \xrightarrow{\mathrm{id}\otimes u} M \otimes_B N \otimes_A M \xrightarrow{e\otimes \mathrm{id}} A \otimes_A M \simeq M \tag{12.10}$$

$$N \simeq B \otimes_B N \xrightarrow{u\otimes \mathrm{id}} N \otimes_A M \otimes_B N \xrightarrow{\mathrm{id}\otimes e} N \otimes_A A \simeq N. \tag{12.11}$$

are equivalent to the identities on M and N. Specializing to $B = \mathbf{1}_{\mathcal{R}}$, we obtain the notion of a left dual of a left A-module.

Remark 12.2.1. *By (Lurie, 2017, Proposition 4.6.2.1) such data are equivalent to giving a right adjoint to the functor*

$$M \otimes_B (-) \colon \mathrm{LMod}_B(\mathcal{R}) \to \mathrm{LMod}_A(\mathcal{R}).$$

Namely, right adjoint is given by $N \otimes_A (-)$.

Now let $F_A \in {}_A\mathrm{BMod}_A$ and $F_B \in {}_B\mathrm{BMod}_B$ be two bimodules. Assume that we are given left dualizable $M \in {}_A\mathrm{BMod}_B$ together with a morphism of bimodules

$$\alpha \colon M \otimes_B F_B \to F_A \otimes_A M. \tag{12.12}$$

Then we can associate to (M, α) a morphism in $\mathcal{R}$

$$\operatorname{Tr}(M, \alpha) \colon \operatorname{Tr}(B, F_B) \to \operatorname{Tr}(A, F_A), \tag{12.13}$$

given by

$$\begin{aligned}\operatorname{Tr}(B, F_B) = B \otimes_{B \otimes B^{\mathrm{rev}}} F_B &\xrightarrow{u \otimes \mathrm{id}} (N \otimes_A M) \otimes_{B \otimes B^{\mathrm{rev}}} F_B \simeq A \otimes_{A \otimes A^{\mathrm{rev}}} (M \otimes_B F_B \otimes_B N) \\ &\xrightarrow{\mathrm{id} \otimes \alpha \otimes \mathrm{id}} A \otimes_{A \otimes A^{\mathrm{rev}}} (F_A \otimes_A M \otimes_B N) \xrightarrow{\mathrm{id} \otimes \mathrm{id} \otimes e} A \otimes_{A \otimes A^{\mathrm{rev}}} F_A = \operatorname{Tr}(A, F_A),\end{aligned}$$

where the isomorphism

$$(N \otimes_A M) \otimes_{B \otimes B^{\mathrm{rev}}} F_B \simeq A \otimes_{A \otimes A^{\mathrm{rev}}} (M \otimes_B F_B \otimes_B N)$$

can be established by the same way as in Example 12.1.2.

In the particular case when $B = F_B = \mathbf{1}_{\mathcal{R}}$ is the unit object of $\mathcal{R}$, M is just a left A-module, and α is a map $M \to F_A \otimes_A M$. Then the above definition of $\operatorname{Tr}(M, \alpha)$ is simplified as

$$\begin{aligned}\mathbf{1}_{\mathcal{R}} &\xrightarrow{u} N \otimes_A M \simeq A \otimes_{A \otimes A^{\mathrm{rev}}} (M \otimes N) \\ &\xrightarrow{\mathrm{id} \otimes (\alpha \otimes \mathrm{id})} A \otimes_{A \otimes A^{\mathrm{rev}}} (F_A \otimes_A M \otimes N) \xrightarrow{\mathrm{id} \otimes e} A \otimes_{A \otimes A^{\mathrm{rev}}} F_A = \operatorname{Tr}(A, F_A).\end{aligned} \tag{12.14}$$

In this case, we also denote $\operatorname{Tr}(M, \alpha)$ as $[M, \alpha]_F$, thought as a point in the space $\operatorname{Map}(\mathbf{1}_{\mathcal{R}}, \operatorname{Tr}(A, F_A))$.

Example 12.2.2. *Let $M = A$, regarded as a left A-module. Then M admits a left dual given by $N = A$ regarded as the right A-module. The unit and evaluation maps are*

$$\mathbf{1}_{\mathcal{R}} \xrightarrow{\mathbf{1}_A} A \cong A \otimes_A A, \quad A \otimes A \xrightarrow{m} A.$$

Giving a left A-module morphism $\alpha : M \to F_A \otimes_A M$ is equivalent to giving a map $\alpha_0 : \mathbf{1}_{\mathcal{R}} \to F_A$. Then $[A, \alpha]_{F_A} : \mathbf{1}_{\mathcal{R}} \to \operatorname{Tr}(A, F_A)$ is canonically equivalent to $[-]_{F_A} \circ \alpha_0 : \mathbf{1}_{\mathcal{R}} \to A \to F_A$.

In the particular case $\mathcal{R} = \mathrm{Lincat}_\Lambda$, giving $\alpha_0 : \mathbf{1}_{\mathcal{R}} \to F_A$ is equivalent to giving an object $X \in F_A$. We denote the corresponding left A-module morphism $M \to F_A \otimes_A M$ by α_X. Then we have a canonical isomorphism in $\operatorname{Tr}(A, F_A)$

$$[A, \alpha_X]_{F_A} \cong [X]_{F_A}.$$

Now suppose $\mathcal{R} = \mathrm{Lincat}_\Lambda$ and A and B two algebras in Lincat_Λ. Then we say an A-B-bimodule M is *fully dualizable* if the maps (12.8) and (12.9) admit *continuous* right adjoint u^R and e^R respectively, and u^R (resp. e^R) is a B-bimodule (resp. A-bimodule) homomorphism. Suppose M is fully dualizable, and suppose α also admits continuous right adjoint α^R, and suppose α^R is also an A-B-bimodule homomorphism, then $\mathrm{Tr}(M, \alpha)$ admits continuous right adjoint $\mathrm{Tr}(M, \alpha)^R$.

Example 12.2.3. *We consider the situation as in Example 12.2.2 with* $\mathcal{R} = \mathrm{Lincat}_\Lambda$. *We call A a rigid monoidal category if $M = A$ as a left A-module is fully dualizable. This is equivalent to the following conditions:*

1. *The unit object $\mathbf{1}_A$ is compact;*

2. *The multiplication functor $m : A \otimes A \to A$ admits a continuous right adjoint m^R;*

3. *The functor $m^R : A \to A \otimes A$ is a functor of A-bimodules.*

So the above definition of rigidity is exactly the same as the one from (Dennis Gaitsgory and Rozenblyum, 2017a, §1.9.1). If A is also compactly generated this agrees with the more familiar defintion requiring that compact objects of A admit both left and right duals, see (Lurie, 2018, Definition D.7.4.1)) and (Dennis Gaitsgory and Rozenblyum, 2017a, Lemma 1.9.1.5).

Now, let $X \in F_A$, giving $\alpha_X : M \to F_A \otimes_A M$. Then α_X admits a continuous right adjoint $(\alpha_X)^R$ if and only if X is a compact object in F_A, in which case $(\alpha_X)^R$ is automatically a left A-module homomorphism as it is a right-lax functor, see (Dennis Gaitsgory and Rozenblyum, 2017a, Lemma 9.3.6.). It follows that $[X]_{F_A} = [A, \alpha_X]_{F_A}$, regarded as a functor $\mathrm{Mod}_\Lambda \to \mathrm{Tr}(A, F_A)$ *admits a continuous right adjoint. That is, $[X]_{F_A}$ is a compact object in* $\mathrm{Tr}(A, F_A)$. *This can also be deduced from Corollary 12.3.3 below.*

12.3 Categorical traces of rigid monoidal categories

We let $\mathcal{R} = \mathrm{Lincat}_\Lambda$. Categorical traces of rigid monoidal categories have particularly nice formal properties. Namely, the rigidity implies (in fact is equivalent to) the monadicity of Hochschild complex.

Lemma 12.3.1. *Assume that A is rigid and F is an A-bimodule. Then for every $\alpha : [n] \to [m]$, the diagram*

$$\begin{tikzcd} \mathrm{HH}(A,F)_{m+1} \arrow[r] \arrow[d, "d_0^{\mathrm{HH}}"'] & \mathrm{HH}(A,F)_{n+1} \arrow[d, "d_0^{\mathrm{HH}}"] \\ \mathrm{HH}(A,F)_m \arrow[r] & \mathrm{HH}(A,F)_n \end{tikzcd} \tag{12.15}$$

is right-adjointable. I.e., the cosimplicial object $\mathrm{HH}(A,F)^\bullet$ obtained from $\mathrm{HH}(A,F)_\bullet$ by passing to the right adjoints satisfies the Beck-Chevalley conditions. In particular,

$$|\mathrm{HH}(A,F)_\bullet| \cong \mathrm{Tot}(\mathrm{HH}(A,F)^\bullet) \cong \mathrm{LMod}_T(F),$$

with T the monad given to $d_0^{\mathrm{HH}} \circ (d_1^{\mathrm{HH}})^R$.

Proof. To prove the first statement, we may assume that α is either a coboundary map or a coface map. In the case that α is a coboundary map, the induced map $\mathrm{HH}(A,F)_m = A^{\otimes m} \otimes F \to \mathrm{HH}(A,F)_n = A^{\otimes n} \otimes F$ is a right A-module morphism with respect to the right A-action on the last factor F. As such morphism is induced by inserting the unit at an appropriate position, it admits a continuous right adjoint, which by (Dennis Gaitsgory and Rozenblyum, 2017a, Lemma 1.9.3.6) is an A-module morphism. This exactly means that the commutative diagram in the lemma is right-adjointable when α is a coboundary map.

Next, assume that $\alpha = d_i$ is a coface map. As explained in (Dennis Gaitsgory and Rozenblyum, 2017a, Lemma 1.9.3.2), for a left A-module M, the action map $a : A \otimes M \to M$ admits a continuous right adjoint a^R given by

$$M \cong \mathrm{Mod}_\Lambda \otimes M \xrightarrow{(m^R \circ \mathbf{1}_A) \otimes \mathrm{id}_M} A \otimes A \otimes M \xrightarrow{\mathrm{id}_A \otimes a} A \otimes M$$

and moreover the functor a^R is a morphism of A-modules (a-priori it is only lax). Similar statements hold for right A-modules. From these facts, one sees that the commutative diagram is right-adjointable for $\alpha = d_i$. Indeed, since each of the functors only these claim reduce to the diagrams with $[n] \leq 1$. and the requirement of right adjointability is exactly the fact that $F \to A \otimes F$ and $A \to A \otimes A$ are morphisms of A-modules.

That $|\mathrm{HH}(A,F)_\bullet| \cong \mathrm{Tot}(\mathrm{HH}(A,F)^\bullet)$ follows from (9.1) and the last equivalence follows from the first assertion and Theorem 9.0.5. □

The following statements directly follow from the proof.

Corollary 12.3.2. *Assume that A is rigid. Let $F_1 \to F_2$ be a fully faithful functor of A-bimodules. Then the induced functor*

$$\mathrm{Tr}(A, F_1) \to \mathrm{Tr}(A, F_2)$$

is fully faithful.

Proof. By Lemma 12.3.1 the functor between the trace categories can be realized as a limit of functors between Hochschild complexes, so to get fully faithfulness it is enough to have level-wise fully faithfulness of the functors $A^{\otimes n} \otimes_\Lambda F_1 \to A^{\otimes n} \otimes_\Lambda F_2$ for all $n \geq 0$. Each of these can be realized via a bar construction with Mod_Λ so it is enough to show that As A is rigid, it is canonically self-dual as an object of $\mathcal{R}$, it is enough to show that the functor $\mathrm{LFun}_\Lambda(A, F_1) \to \mathrm{LFun}_\Lambda(A, F_2)$ between categories of Λ-linear functors, is fully faithful. Fully faithfulness is then known (see for example (Gepner, Haugseng, and Nikolaus, n.d., Lemma 5.2.) $\square$

Corollary 12.3.3. *An A-bimodule functor $F_1 \to F_2$ induces a functor between cosimplicial objects $\mathrm{HH}(A, F_1)^\bullet \to \mathrm{HH}(A, F_2)^\bullet$. In addition, the following diagram is right adjointable*

$$\begin{tikzcd} F_1 \arrow[r, "{[-]_{F_1}}"] \arrow[d] & \mathrm{Tr}(A, F_1) \arrow[d] \\ F_2 \arrow[r, "{[-]_{F_2}}"] & \mathrm{Tr}(A, F_2) \end{tikzcd}$$

If $F_1 \to F_2$ admits a continuous right adjoint (as plain Λ-linear presentable stable categories), then the following diagram is right adjointable

$$\begin{tikzcd} F_1 \arrow[r] \arrow[d, "{[-]_{F_1}}"'] & F_2 \arrow[d, "{[-]_{F_2}}"] \\ \mathrm{Tr}(A, F_1) \arrow[r] & \mathrm{Tr}(A, F_2) \end{tikzcd}$$

Remark 12.3.4. *Assume that A is rigid. Let M be a left A-module. Let $\mathrm{Mod}_\Lambda \to N \otimes_A M$ and $M \otimes N \to A$ be the duality datum for M as a left A-module. Then it follows easily from the above corollary that N is the dual of M in Lincat_Λ with the duality datum given by*

$$\mathrm{Mod}_\Lambda \to N \otimes_A M \xrightarrow{[-]^R} N \otimes M, \quad M \otimes N \to A \xrightarrow{\mathrm{Hom}(\mathbf{1}_A, -)} \mathrm{Mod}_\Lambda,$$

where $[-]^R$ denotes the continuous right adjoint of $[-]_{M \otimes N}$.

One can also easily deduce the following corollary, which has appeared in (D. Gaitsgory et al., 2019, Theorem 3.8.5).

Corollary 12.3.5. *Assume that A is rigid, equipped with a monoidal endomorphism $\phi : A \to A$. Then*

$$\mathrm{tr}(A, \phi) \cong \mathrm{End}_{\mathrm{Tr}(A,\phi)}([\mathbf{1}_A]_{\phi A})$$

as Λ-algebras. Let M be a left A-module equipped with an A-module homomorphism $\alpha : M \to {}^{\phi}A \otimes_A M = {}^{\phi}M$. Assume that M is dualizable in Lincat_Λ. *Then under the above isomorphism,* $\mathrm{Hom}_{\mathrm{Tr}(A,\phi)}([\mathbf{1}_A]_{\phi A}, [M,\alpha]_{\phi A})$ *is isomorphic to* $\mathrm{tr}(M, \phi)$ *as modules.*

Proof. Applying the monad from Lemma 12.3.1 to $\mathbf{1}_A$, we see that $\mathrm{End}_{\mathrm{Tr}(A,\phi)}([\mathbf{1}_A]_{\phi A})$ is isomorphic to $\mathrm{Hom}_{\phi A}(\mathbf{1}_A, T(\mathbf{1}_A))$ as algebras, which is given by

$$\mathrm{Mod}_\Lambda \xrightarrow{\mathbf{1}_A} A \xrightarrow{m^R} {}^{\phi}A \otimes A \overset{\mathrm{sw}}{\cong} A \otimes {}^{\phi}A \xrightarrow{m} A \xrightarrow{\mathbf{1}_A^R} \mathrm{Mod}_\Lambda.$$

By (Dennis Gaitsgory and Rozenblyum, 2017a, §9.2.1), the pair $\mathrm{Mod}_\Lambda \xrightarrow{m^R \circ \mathbf{1}_A} A \otimes A$ and $A \otimes A \xrightarrow{\mathbf{1}_A^R \circ m} \mathrm{Mod}_\Lambda$ form a duality datum of A (regarded as object in Lincat_Λ). It follows that $\mathrm{Hom}_{\phi A}(\mathbf{1}_A, T(\mathbf{1}_A)) \cong \mathrm{tr}(A, \phi)$. The case of module is similar. □

The following description of hom spaces between certain objects in $\mathrm{Tr}(A, \phi)$ is useful in practice.

Corollary 12.3.6. *Assume that A is rigid and is compactly generated, with $\{c_i\}$ being a set of compact generators. Then for $X, Y \in A$ with X compact in A,*

$$\mathrm{Hom}_{\mathrm{Tr}(A,\phi)}([X]_{\phi A}, [Y]_{\phi A}) \cong \operatorname*{colim}_{C \otimes C_{/m^R(Y)}} \mathrm{Hom}_A(X, c_j \otimes \phi(c_i)),$$

where $C \otimes C \subset A \otimes A$ denotes the full subcategory spanned by $\{c_i \boxtimes c_j\}_{i,j}$.

Note that a morphism $c_i \boxtimes c_j \to m^R(Y)$ in $A \otimes A$ is equivalent to a morphism $c_i \otimes c_j \to Y$ in A. So informally, this corollary says that every morphism $[X]_{\phi A} \to [Y]_{\phi A}$ in $\mathrm{Tr}(A, \phi)$ can be represented as a pair of morphisms $(X \to c_j \otimes \phi(c_i), c_i \otimes c_j \to Y)$ in A (compare with (Zhu, 2016b, §3.1)).

Proof. We have

$$\mathrm{Hom}_{\mathrm{Tr}(A,\phi)}([X]_{\phi A}, [Y]_{\phi A}) \cong \mathrm{Hom}_{\phi A}(X, m \circ \mathrm{sw} \circ m^R(Y)).$$

As A is compactly generated, so is $A \otimes A$ with a set of compact generators given by $\{c_i \boxtimes c_j\}_{i,j}$. Then $m^R(Y) = \mathrm{colim}_{c_i \boxtimes c_j \to Y} c_i \boxtimes c_j$. As X is compact, the corollary follows. $\square$

Remark 12.3.7. *In fact, the above corollary admits a more economic form. Namely, suppose we write* $m^R(\mathbf{1}_A) \cong \mathrm{colim}_i(c_{i,1} \boxtimes c_{i,2})$ *as a filtered colimit of compact objects in $A \otimes A$. Then as m^R is a right A-module homomorphism, we have* $m^R(Y) \cong \mathrm{colim}_i(c_{i,1} \boxtimes (c_{i,2} \otimes Y))$. *Therefore,*

$$\mathrm{Hom}_{\mathrm{Tr}(A,\phi)}([X]_{\phi A}, [Y]_{\phi A}) \cong \operatorname*{colim}_i \mathrm{Hom}_A(X, c_{i,2} \otimes Y \otimes \phi(c_{i,1})).$$

As mentioned above, if A is rigid, then it is dualizable as object in Lincat_Λ. It follows from (Dennis Gaitsgory and Rozenblyum, 2017a, Proposition 9.5.3) that A^c and A^e from (12.6) admit left duals, denoted by S_A and T_A respectively, which we can identify with A-bimodules under the equivalence ${}_{A^{\mathrm{rev}} \otimes A}\mathrm{BMod}_{\mathbf{1}} \cong {}_A\mathrm{BMod}_A$ (resp. ${}_{\mathbf{1}}\mathrm{BMod}_{A \otimes A^{\mathrm{rev}}} \cong {}_A\mathrm{BMod}_A$). By (Lurie, 2017, Remark 4.6.5.4), there is an automorphism σ_A of A (as a monoidal category), usually called the Serre functor of A, such that $S_A = {}^{\sigma_A}A$ as A-bimodules.

Example 12.3.8. *We recall that a pivotal category is a rigid monoidal category equipped with an isomorphism* $\mathrm{id}_A \cong \varphi_A$. *If A is compactly generated, this means that for $a \in A$ compact, $a^{\vee,L}$ and $a^{\vee,R}$ are functorially isomorphic.*

Assume that A is pivotal. Assume that Q is an A-bimodule, dualizable as a Λ-linear category, with ${}^\vee Q$ its left dual. Then $\mathrm{Tr}(A, Q)$ *is dualizable with* $\mathrm{Tr}(A, Q^\vee)$ *is dual. The unit is given by*

$$\mathrm{Mod}_\Lambda \xrightarrow{u} A \xrightarrow{[-]_A} \mathrm{Tr}(A, A) \to \mathrm{Tr}(A, {}^\vee Q \otimes_A Q) \to \mathrm{Tr}(A, Q) \otimes \mathrm{Tr}(A, {}^\vee Q).$$

Chapter 13

GEOMETRIC TRACES IN SHEAF THEORY

Following ideas of (Ben-Zvi and Nadler, 2009) (Ben-Zvi, Nadler, and Preygel, 2017) to develop a method to calculate the (twisted) categorical trace of monoidal categories arising from convolution pattern in algebraic geometry. As mentioned before, Compared with the work of *loc. cit.*, we will first calculate a geometric version of categorical trace. Then we will compare the geometric version with the usual version in favorable cases. Our approach allows us to bypass integral transform of sheaf theories, which usually do not hold in the ℓ-adic setting.

13.1 Geometric Hochschild homology

We use the formalism of category of correspondences and sheaf theory as in Chapter 10.

Let A be an associative algebra object in $\mathrm{Corr}(C)_{vert,horiz}$, and let M be a left A-module object in $\mathrm{Corr}(C)_{vert,horiz}$. As $\mathcal{D}$ is a lax symmetric monoidal functor, $\mathcal{D}(A)$ is an algebra object in Lincat_Λ and $\mathcal{D}(M)$ is a $\mathcal{D}(A)$-module object in Lincat_Λ, with multiplication and action maps given by

$$\mathcal{D}(A) \otimes \mathcal{D}(A) \xrightarrow{\boxtimes} \mathcal{D}(A \times A) \to \mathcal{D}(A)$$
$$\mathcal{D}(A) \otimes \mathcal{D}(M) \xrightarrow{\boxtimes} \mathcal{D}(A \times M) \to \mathcal{D}(M)$$

Similarly, if F is an A-bimodule, then $\mathcal{D}(F)$ is an $\mathcal{D}(A)$-bimodule. Then one can form its Hochschild homology (a.k.a categorical trace) of $(\mathcal{D}(A), \mathcal{D}(F))$

$$\mathrm{Tr}(\mathcal{D}(A), \mathcal{D}(F)) = \mathcal{D}(A) \otimes_{\mathcal{D}(A)\otimes\mathcal{D}(A)^{\mathrm{rev}}} \mathcal{D}(F) \in \mathrm{Lincat}_\Lambda.$$

In practice, however, we need to consider a variant $\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F))$, which we call the geometric trace of $\mathcal{D}(F)$. Namely, we consider the Yoneda embedding

$$\mathrm{Corr}(C)_{vert;horiz} \to \mathcal{P}(\mathrm{Corr}(C)_{vert;horiz}),$$

where $\mathcal{P}(\mathrm{Corr}(C)_{vert;horiz})$ is the category of presheaves on $\mathrm{Corr}(C)_{vert;horiz}$ equipped with the induced symmetric monoidal structure, which by definition preserves colimits in each variable (see (Lurie, 2017, Corollary 4.8.1.12). Then we have the Hochschild homology of the A-bimodule F in $\mathcal{P}(\mathrm{Corr}(C)_{vert;horiz})$

$$\mathrm{Tr}(A, F) := |\mathrm{HH}(A, F)_\bullet| \in \mathcal{P}(\mathrm{Corr}(C)_{vert;horiz}).$$

By the universal property of $\mathcal{P}(\mathrm{Corr}(C)_{vert;horiz})$, the functor $\mathcal{D} : \mathrm{Corr}(C)_{vert;horiz} \to \mathrm{Lincat}_\Lambda$ extends to a continuous functor $\mathcal{D} : \mathcal{P}(\mathrm{Corr}(C)_{vert;horiz}) \to \mathrm{Lincat}_\Lambda$. Then we define the *geometric trace* of $\mathcal{D}(F)$ as

$$\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F)) := \mathcal{D}(\mathrm{Tr}(A, F)).$$

Explicitly, $\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(F), \mathcal{D}(A))$ can be computed in the following way. We first apply the functor $\mathcal{D}$ to the standard Hochschild complex (12.4) (which now is a simplicial object in $\mathrm{Corr}(C)_{vert,horiz}$) to obtain a simplicial object $\mathcal{D}(\mathrm{HH}_\bullet(A, F))$ in Lincat_Λ. Then the geometric trace $\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F))$ is the geometric realization of this simplicial object in Lincat_Λ

$$\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F)) \cong |\mathcal{D}(\mathrm{HH}(A, F)_\bullet)|. \tag{13.1}$$

We emphasize that $\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F))$ depends not only on $\mathcal{D}(F)$, but on the A-bimodule F itself (and of course the functor $\mathcal{D}$).

In particular, for A equipped with an algebra endomorphism $\phi : A \to A$ we have the A-bimodule $F = {}^\phi A$ in $\mathrm{Corr}(C)_{vert,horiz}$ as before. We write

$$\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \phi) = \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}({}^\phi A)).$$

Remark 13.1.1. *As $\mathcal{D}$ is equipped with a lax monoidal structure we get a natural comparison functor*

$$\mathrm{Tr}(\mathcal{D}(A), \mathcal{D}(F)) \simeq |\mathcal{D}(A)^{\otimes\bullet} \otimes \mathcal{D}(F)| \to |\mathcal{D}(A^{\otimes\bullet} \otimes F)| = \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(A), \mathcal{D}(F)) \tag{13.2}$$

from the usual trace of $\mathcal{D}(F)$ to the geometric trace. This functor is not an equivalence in general. Of course, if for each n, the functor $\mathcal{D}(A)^{\otimes n} \otimes \mathcal{D}(Q) \to \mathcal{D}(A^n \times Q)$ is an equivalence, then the comparison map (13.2) *is an equivalence. We will see later that this functor is an equivalence in many more cases of interest.*

13.2 Fixed point objects and geometric traces of convolution categories

We specialize the previous constructions to the situation appearing in our applications. Let $X \in C$ satisfying conditions as in Example 10.1.2 (that is, $\Delta_X : X \to X \times X$ and $\pi_X : X \to \mathrm{pt}$ belong to C_{horiz}). Let $f : X \to Y$ in C_{vert} and assume that the relative diagonal map $\Delta_{X/Y} : X \to X \times_Y X$ belongs to $vert$. Let $X_\bullet \to Y$ denote the Čech nerve of f. From Example 10.3.3 and Remark 10.3.6, it gives rise to an object $\mathrm{Alg}(\mathbf{End}_{\mathrm{Corr}(C)}(X))$ which induces

$$X_1 := X \times_Y X$$

with the structure of an associative algebra object in $\mathrm{Corr}(C)_{vert;horiz}$. The multiplication and unit maps are given by

$$\begin{tikzcd} X \times_Y X \times_Y X \arrow[r, "\mathrm{id}\times\Delta_X\times\mathrm{id}"] \arrow[d, "\mathrm{id}\times f\times\mathrm{id}"] & (X \times_Y X) \times (X \times_Y X) \\ X \times_Y X & \end{tikzcd}, \qquad \begin{tikzcd} X \arrow[r] \arrow[d, "\Delta_{X/Y}"] & \mathrm{pt} \\ X \times_Y X & \end{tikzcd},$$

Let $Z \in C$ equipped with two morphisms $g_i \colon Z \to Y,\ i = 1, 2$ in C and let

$$Q = X \times_Y Z \times_Y X = Z \times_{Y\times Y} (X \times X).$$

Then via the construction of Section 10.4 the object Q admits the structure of an $(X \times_Y X)$-bimodule in $\mathrm{Corr}(C)_{vert;horiz}$. In particular, the left action is given by the diagram

$$\begin{tikzcd} X \times_Y X \times_Y Z \times_Y X \arrow[r] \arrow[d, "\mathrm{id}\times f\times\mathrm{id}\times\mathrm{id}"] & (X \times_Y X) \times (X \times_Y Z \times_Y X) \\ X \times_Y Z \times_Y X & \end{tikzcd}$$

and we have a similar diagram for the right action. Consider the following diagram

$$\begin{tikzcd} X \times_{Y\times Y} Z \arrow[r, "\delta_0=(\Delta_X\times\mathrm{id}_Z)"] \arrow[d, "q=(f\times\mathrm{id}_Z)"'] & (X \times X) \times_{Y\times Y} Z \\ Y \times_{Y\times Y} Z, & \end{tikzcd} \tag{13.3}$$

which induces a functor $q_\dagger \circ (\delta_0)^\star : \mathcal{D}(X \times_Y Z \times_Y X) \to \mathcal{D}(Y \times_{Y\times Y} Z)$.

Proposition 13.2.1. *The following diagram is commutative*

$$\begin{tikzcd} \mathcal{D}(X \times_Y Z \times_Y X) \arrow[r, "(\delta_0)^\star"] \arrow[d] & \mathcal{D}(X \times_{Y\times Y} Z) \arrow[dd, "q_\dagger"] \\ \mathrm{Tr}(\mathcal{D}(X \times_Y X), \mathcal{D}(X \times_Y Z \times_Y X)) \arrow[d] & \\ \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X \times_Y X), \mathcal{D}(X \times_Y Z \times_Y X)) \arrow[r] & \mathcal{D}(Y \times_{Y\times Y} Z). \end{tikzcd}$$

Assume that the sheaf theory $\mathcal{D}$ satisfies Assumptions 11.2.1 1-3, and assume that:

1. $\Delta_X : X \to X \times X \in C_{sm}$,
2. $f \colon X \to Y \in C_{prop}$,

3. $\Delta_{X/Y}\colon X \to X \times_Y X$ *is in* C_{prop}.

Then the bottom horizontal functor of the above diagram is fully faithful, with the essential image is generated under colimits by the image of $q_\dagger \circ \delta_0^\star$.

The proof of the proposition will be given at the end of Section 13.3. We note that there is no assumption on $(g_1, g_2) : Z \to Y \times Y$.

We specialize Proposition 13.2.1 to the following two cases. First, assume we are given morphisms $\phi_X\colon X \to X$ and $\phi_Y\colon Y \to Y$ in C intertwined by f, that is, equipped with an equivalence $f \circ \phi_X \simeq \phi_Y \circ f$. We will usually abuse notation and denote both maps by ϕ if it is clear from context. We let $Z = Y$ with the map $g_1 = \mathrm{id}$ and $g_2 = \phi$. In this case, $Z \times_{Y\times Y} Y$ is nothing but the ϕ-fixed point object $\mathcal{L}_\phi(Y)$, defined by the pullback

$$\begin{tikzcd} \mathcal{L}_\phi(Y) \arrow[r] \arrow[d, "p_\phi"] & Y \arrow[d, "\Delta_Y"] \\ Y \arrow[r, "\mathrm{id}\times\phi"] & Y \times Y. \end{tikzcd} \tag{13.4}$$

We assume in addition that ϕ_X is an equivalence. In this case the $(X \times_Y X)$-module $X \times_Y Z \times_Y X$ is isomorphic to the ϕ-twisted module ${}^\phi(X \times_Y X)$, with the isomorphism sending $(x, z, x') \in X \times_Y Z \times_Y X$ to $(\phi(x), x') \in {}^\phi(X \times_Y X)$. Then (13.3) becomes

$$\begin{tikzcd} X \times_Y \mathcal{L}_\phi(Y) \arrow[r, "\delta_0"] \arrow[d, "q"'] & X \times_Y X \\ \mathcal{L}_\phi(Y) & \end{tikzcd}$$

Corollary 13.2.2. *Under the same assumption as in Proposition 13.2.1 and given* ϕ_X, ϕ_Y *as above, there is a canonical factorization*

$$\begin{tikzcd} \mathcal{D}(X \times_Y X) \arrow[r, "(\delta_0)^\star"] \arrow[d] & \mathcal{D}(X \times_{X\times X} (X \times_Y X)) \arrow[d, "q_\dagger"] \\ \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X \times_Y X), \phi) \arrow[r] & \mathcal{D}(\mathcal{L}_\phi Y) \end{tikzcd}$$

with the lower horizontal arrow is fully faithful. The essential image is generated under colimits by the image of $q_\dagger \circ \delta_0^\star$.

Another case we need to consider is $Z = W_1 \times W_2$ with $g_i : W_i \to Y$ two maps in C. In this case,

$$Z \times_{Y\times Y} Y = W_1 \times_Y W_2, \quad Z \times_{Y\times Y} (X \times X) = (W_1 \times_Y X) \times (X \times_Y W_2),$$

We denote:

$$\mathcal{D}(W_1 \times_Y X) \otimes^{\mathrm{geo}}_{\mathcal{D}(X\times_Y X)} \mathcal{D}(X \times_Y W_2) := \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X \times_Y X), \mathcal{D}(Z \times_{Y\times Y} (X \times X)), \tag{13.5}$$

which is the geometric analogue of the relative tensor product.

Corollary 13.2.3. *Under the same assumption as in Proposition 13.2.1, we have a canonical square*

$$\begin{tikzcd} \mathcal{D}((W_1 \times_Y X) \times (X \times_Y W_2)) \arrow[r, "(\mathrm{id}_{W_1}\times\Delta_X\times\mathrm{id}_{W_2})^\star"] \arrow[d] & \mathcal{D}(W_1 \times_Y X \times_Y W_2) \arrow[d, "(\mathrm{id}_{W_1}\times f\times\mathrm{id}_{W_2})_\dagger"] \\ \mathcal{D}(W_1 \times_Y X) \otimes^{\mathrm{geo}}_{\mathcal{D}(X\times_Y X)} \mathcal{D}(X \times_Y W_2) \arrow[r] & \mathcal{D}(W_1 \times_Y W_2) \end{tikzcd}$$

with the bottom functor fully faithful. The essential image is generated under colimits by the image of $(\mathrm{id}_{W_1} \times f \times \mathrm{id}_{W_2})_\dagger \circ (\mathrm{id}_{W_1} \times \Delta_X \times \mathrm{id}_{W_2})^\star$.

Again, there is no assumption on g_1 and g_2.

13.3 The geometric trace and relative resolutions

Now we prove Proposition 13.2.1. In fact, (to save notations) we will prove a slightly general statement. We consider the geometric trace for pair $\mathrm{BMod}(\mathrm{Corr}(C)_{vert;horiz})$ arising from an associative algebra $X_1 \in \mathrm{Alg}\big(\mathbf{End}_{\mathrm{Corr}(C)}(X_0)\big)$ and a bi-module object $Q \in {}_{X_1}\mathrm{BMod}_{X_1}(\mathbf{End}_{\mathrm{Corr}(C)}(X_0))$. Roughly speaking, these are pairs (X_1, Q) seen as objects in the category $\mathrm{BMod}(\mathrm{Corr}(C)_{vert;horiz})$ consisting of an algebra $X_1 \in \mathrm{Alg}(\mathrm{Corr}(C)_{vert;horiz})$ whose multiplication and unit maps are of the form

$$\begin{tikzcd} X_1 \times_{X_0} X_1 \arrow[r, "\eta"] \arrow[d, "m"] & X_1 \times X_1 \\ X_1 & \end{tikzcd}, \quad \begin{tikzcd} X_0 \arrow[r, "\pi_{X_0}"] \arrow[d, "u"] & \mathrm{pt} \\ X_1 & \end{tikzcd}, \tag{13.6}$$

and an X_1-bimodule $Q \in {}_{X_1}\mathrm{BMod}_{X_1}(\mathrm{Corr}(C)_{vert;horiz})$ whose action maps are of the form

$$\begin{tikzcd} X_1 \times_{X_0} Q \arrow[r, "\xi_l"] \arrow[d, "a_l"] & X_1 \times Q \\ Q & \end{tikzcd}, \quad \begin{tikzcd} Q \times_{X_0} X_1 \arrow[r, "\xi_r"] \arrow[d, "a_r"] & Q \times X_1 \\ Q & \end{tikzcd}. \tag{13.7}$$

Here we require $\Delta_{X_0} : X_0 \to X_0 \times X_0$ and $\pi_{X_0} : X_0 \to \mathrm{pt}$ belong to C_{horiz} (so $\eta, \xi_l, \xi_r \in C_{horiz}$ as well) and $m, u, a_l, a_r \in C_{vert}$. See Remark 10.4.1 for more details and references.

We can then consider the geometric trace

$$\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)) = \mathcal{D}(\mathrm{Tr}(X_1,Q)) \cong |\mathcal{D}(\mathrm{HH}(X_1,Q)_\bullet)|$$

defined in the previous section. On the other hand, the extra structure on the algebra and module allows one to construct a variant of the geometric trace.

In the monoidal category $\mathrm{End}_{\mathrm{Corr}(C)_{vert;horiz}}(X_0)$ we consider the Bar complex of the algebra object X_1, which we denoted by $\mathrm{Bar}^{X_0}(X_1)_\bullet$. Under the lax monoidal functor $\mathrm{End}_{\mathrm{Corr}(C)_{vert;horiz}}(X_0) \to \mathrm{Corr}(C)_{vert;horiz}$, it gives a simplicial object in $\mathrm{Corr}(C)_{vert;horiz}$ (in fact in C_{vert}), denoted by the same notation, which can be written as

$$\mathrm{Bar}^{X_0}(X_1)_\bullet \cong X_\bullet \times_{X_0\times X_0} (X_1\times X_1),$$

where the two maps $X_n \to X_0$ corresponds to $\{0\} \subset \{0,1,\ldots,n\}$ and $\{n\} \subset \{0,1,\ldots,n\}$ respectively and $X_1\times X_1 \to X_0\times X_0$ is given by (d_1,d_0). The action of $(X_1\times X_1)\times Q \to Q$ by right and left multiplication gives

$$\mathrm{Bar}^{X_0}(X_1)_\bullet \otimes Q = X_\bullet \times_{X_0\times X_0} (X_1\times X_1)\times Q \to X_\bullet \times_{X_0\times X_0} Q =: \mathrm{HH}^{X_0}(X_1,Q)_\bullet$$

which is $(X_1\otimes X_1)$-bilinear and therefore induces

$$\mathrm{Bar}^{X_0}(X_1)_\bullet \otimes_{X_1\otimes X_1} Q \to \mathrm{HH}^{X_0}(X_1,Q)_\bullet$$

The lax monoidal functor $\mathrm{End}_{\mathrm{Corr}(C)_{vert;horiz}}(X_0) \to \mathrm{Corr}(C)_{vert;horiz}$ also induces a natural map of simplicial objects

$$\mathrm{Bar}(X_1)_\bullet \to \mathrm{Bar}^{X_0}(X_1)_\bullet$$

in $\mathrm{Corr}(C)_{vert;horiz}$. It follows that we obtain a map of simplicial objects in $\mathrm{Corr}(C)_{vert;horiz}$

$$\delta_\bullet : \mathrm{HH}(X_1,Q)_\bullet = \mathrm{Bar}(X_1)_\bullet \otimes_{X_1\otimes X_1} Q \to \mathrm{Bar}^{X_0}(X_1)_\bullet \otimes_{X_1\otimes X_1} Q \to \mathrm{HH}^{X_0}(X_1,Q)_\bullet, \tag{13.8}$$

which is given on each level $n\geq 0$ by the horizontal arrow

$$X_n \times_{X_0\times X_0} Q \xleftarrow{\mathrm{id}} X_n \times_{X_0\times X_0} Q \xrightarrow{\delta_n} X_1^n \times Q.$$

Now we define the *X_0-relative Hochschild homology* of Q as

$$\mathrm{Tr}^{X_0}(X_1,Q) = |\mathrm{HH}^{X_0}(X_1,Q)_\bullet| \in \mathcal{P}(\mathrm{Corr}(C)_{vert;horiz}),$$

and define the X_0-*relative geometric trace* of $\mathcal{D}(Q)$ as the geometric realization in Lincat_Λ

$$\mathrm{Tr}^{X_0}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)) := \mathcal{D}(\mathrm{Tr}^{X_0}(X_1,Q)) \cong |\mathcal{D}(\mathrm{HH}^{X_0}(X_1,Q)_\bullet)|.$$

Then (13.8) gives a functor

$$\delta^\star\colon \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)) \to \mathrm{Tr}^{X_0}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)),$$

which fits into a commutative diagram

$$\begin{tikzcd}
\mathcal{D}(Q) \arrow[r, "(\delta_0)^\star"] \arrow[d] & \mathcal{D}(X_0 \times_{X_0\times X_0} Q) \arrow[dd] \\
\mathrm{Tr}(\mathcal{D}(X_1),\mathcal{D}(Q)) \arrow[d] & \\
\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)) \arrow[r, "\delta^\star"] & \mathrm{Tr}^{X_0}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q)).
\end{tikzcd} \tag{13.9}$$

Proposition 13.3.1. *Assume that the sheaf theory $\mathcal{D}$ satisfies Assumptions 11.2.1 1-3. In addition, in the notations of* (13.6) *and* (13.7) *assume that*

1. $m\colon X_1 \times_{X_0} X_1 \to X_1$ *and* $u\colon X_0 \to X_1$ *are in* C_{prop},
2. $a_l\colon X_1 \times_{X_0} Q \to Q$ *and* $a_r\colon Q \times_{X_0} X_1 \to Q$ *are in* C_{prop},
3. *the diagonal* $\Delta_{X_0}\colon X_0 \to X_0 \times X_0$ *is in* C_{sm},
4. $\pi_{X_0}: X_0 \to \mathrm{pt}$ *is in* C_{horiz}.

Then the functor $\delta^\star$ from (13.9) *is fully faithful. The essential image is generated under colimits by the image of* $\mathcal{D}(Q) \xrightarrow{(\delta_0)^\star} \mathcal{D}(X_0\times_{X_0\times X_0} Q) \to \mathrm{Tr}^{X_0}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(Q))$.

Proof. Passing to right adjoints gives a natural transformation

$$(\delta_\bullet)_\star\colon \mathcal{D}(\mathrm{HH}^{X_0}(X_1,Q)_\bullet) \to \mathcal{D}(\mathrm{HH}(X_1,Q)_\bullet)$$

of cosimplicial categories. To prove the left adjoint $\delta^\star$ is fully faithful we will use Theorem 9.0.5 and Corollary 9.0.6. The first step is to verify each of the cosimplicial categories satisfies the Beck-Chevalley conditions.

Lemma 13.3.2. *Under the assumptions of Proposition 13.3.1, the cosimplicial categories obtained from the simplicial categories* $\mathcal{D}(\mathrm{HH}(X_1,Q)_\bullet)$ *and* $\mathcal{D}(\mathrm{HH}^{X_0}(X_1,Q)_\bullet)$ *by passing to right adjoints satisfy the Beck-Chevalley conditions.*

Proof. For the cosimplicial category $\mathcal{D}(\mathrm{HH}^{X_0}(X_1, Q)_\bullet)$, the face maps corresponding to $0 \mapsto 0 \in [n]$ are given by the right adjoints $(d_0)^\dagger$ of $(d_0)_\dagger$. All the morphisms m, a_l, a_r are in C_{prop} by assumption and therefore all maps of the simplicial object $\mathcal{D}(\mathrm{HH}^{X_0}(X_1, Q)_\bullet)$ are in C_{prop} as well. Then from Assumptions 11.2.11 all the necessary Beck-Chevalley maps are equivalences.

It is left to deal with $\mathcal{D}(\mathrm{HH}_\bullet(X_1, Q))$. For every map $\alpha\colon [m] \to [n]$, we have the diagram

$$\begin{tikzcd} X_1^m \times Q & X_1^{m+1} \times Q \arrow[l, dashed, "d_m^0"'] \\ X_1^n \times Q \arrow[u, dashed] & X_1^{n+1} \times Q \arrow[l, dashed, "d_n^0"'] \arrow[u, dashed] \end{tikzcd} \tag{13.10}$$

in $\mathrm{Corr}(C)_{vert;horiz}$ (see (10.1) and after for notations). We need to show that the induced diagram

$$\begin{tikzcd} \mathcal{D}(X_1^m \times Q) \arrow[r, "d_m^0"] \arrow[d] & \mathcal{D}(X_1^{m+1} \times Q) \arrow[d] \\ \mathcal{D}(X_1^n \times Q) \arrow[r, "d_n^0"] & \mathcal{D}(X_1^{n+1} \times Q) \end{tikzcd} \tag{13.11}$$

is left adjointable. Clearly, it is enough to consider the case when α is either a co-face or a co-degeneracy map.

In the case of co-face map, we may assume that $\alpha = d_n : [n] \to [n+1]$, that is, $\alpha(i) = i$ for all i. (The proof is similar in all other cases.) Then the diagram (13.10) is explicitly given by

$$\begin{tikzcd} X_1^n \times Q & X_1^n \times Q \times_{X_0} X_1 \arrow[l, "a_{r,n}"'] \arrow[r, "\xi_{r,n}"] & X_1^{n+1} \times Q \\ X_1^n \times X_1 \times_{X_0} Q \arrow[u, "a_{l,n}"] \arrow[d, "\xi_{l,n}"] & X_1^n \times X_1 \times_{X_0} Q \times_{X_0} X_1 \arrow[l, "\tilde{a}_r"'] \arrow[r, "\tilde{\xi}_r"] \arrow[u, "\tilde{a}_l"] \arrow[d, "\tilde{\xi}_l"] & X_1^{n+1} \times X_1 \times_{X_0} Q \arrow[u, "a_{l,n+1}"] \arrow[d, "\xi_{l,n+1}"] \\ X_1^{n+1} \times Q & X_1^{n+1} \times Q \times_{X_0} X_1 \arrow[l, "a_{r,n+1}"'] \arrow[r, "\xi_{r,n+1}"] & X_1^{n+2} \times Q, \end{tikzcd}$$

as a diagram in C, where $a_{l,n} = \mathrm{id}_{X_1^n} \times a_l$, etc. Note that all squares are Cartesian in C.

We have $d_k^0 = (\xi_{l,k})_\star \circ (a_{l,k})^\dagger$ (for $k = n, n+1$) with the left adjoint $(a_{l,k})_\dagger \circ \xi_{l,k}^\star$, and the vertical arrows in (13.11) are given by $(\xi_{l,k})_\star \circ (a_{l,k})^\dagger$. Left adjointability of 13.11 then means that the natural map

$$(a_{l,n+1})_\dagger \circ (\xi_{r,n+1})^\star \circ (\xi_{l,n+1})_\star \circ (a_{l,n+1})^\dagger \to (\xi_{l,n})_\star \circ (a_{l,n})^\dagger \circ (a_{r,n})_\dagger \circ (\xi_{r,n})^\star$$

is an equivalence. It is enough to show that the Beck-Chevalley maps

$$(\tilde{a}_r)_\dagger \circ (\tilde{a}_l)^\dagger \to (a_{l,n})^\dagger \circ (a_{r,n})_\dagger, \quad (\xi_{r,n+1})^\star \circ (\xi_{l,n+1})_\star \to (\tilde{\xi}_l)_\star \circ (\tilde{\xi}_r)^\star \tag{13.12}$$

$$(\tilde{\xi}_r)^\star \circ (a_{l,n+1})^\dagger \to (\tilde{a}_l)^\dagger \circ (\xi_{r,n})^\star, \quad (a_{r,n+1})_\dagger \circ (\tilde{\xi}_l)_\star \to (\xi_{l,n})_\star \circ (\tilde{a}_r)_\dagger \tag{13.13}$$

are equivalences. Note that the maps $a_{l,k}, a_{r,k}, \tilde{a}_l, \tilde{a}_r$ are in C_{prop}. So the left equivalence of (13.12) holds by Assumptions 11.2.11. As the maps $\xi_{l,k}, \xi_{r,k}, \tilde{\xi}_l, \tilde{\xi}_r$ belong to C_{sm}, the right equivalence of (13.12) holds by Assumptions 11.2.12 and the equivalences (13.13) hold by Assumptions 11.2.1 3 and 4.

In the case of co-degeneracy map, we can assume $\alpha = s_0 \colon [n+1] \to [n]$ given by $s(0) = 0$ and $s(i) = i - 1$ for $1 \le i \le n$. As in the previous case, the diagram (13.10) is explicitly given as

$$\begin{tikzcd}
X_1^{n+1} \times Q & X_1^{n+1} \times Q \times_{X_0} X_1 \arrow[l, "a_{r,n+1}"'] \arrow[r, "\xi_{r,n+1}"] & X_1^{n+2} \times Q \\
X_1^n \times X_0 \times Q \arrow[u, "u_n"] \arrow[d, "\pi_n"] & X_1^n \times X_0 \times Q \times_{X_0} X_1 \arrow[l, "\tilde{a}_r"'] \arrow[r, "\tilde{\xi}_r"] \arrow[u, "\tilde{u}"] \arrow[d, "\tilde{\pi}"] & X_1^{n+1} \times X_0 \times Q \arrow[u, "u_{n+1}"] \arrow[d, "\pi_{n+1}"] \\
X_1^n \times Q & X_1^n \times Q \times_{X_0} X_1 \arrow[l, "a_{r,n}"'] \arrow[r, "\xi_{r,n}"] & X_1^{n+1} \times Q,
\end{tikzcd}$$

where all squares are Cartesian. As before, in order to show the corresponding adjointability equivalence it is enough to show the corresponding commutativity in each square. As the map $u \colon X_0 \to X_1$ is in C_{prop}, the upper two squares of the diagram can be handled as in the co-face map case. For the remaining squares, we need the maps

$$(\xi_{r,n})^\star \circ (\pi_{n+1})_\star \to (\tilde{\pi})_\star \circ (\tilde{\xi}_r)^\star, \quad (a_{r,n})_\dagger \circ (\tilde{\pi})_\star \to (\pi_n)_\star \circ (\tilde{a}_r)_\dagger$$

to be equivalences. As $\pi_{X_0} : X_0 \to \mathrm{pt}$ is in C_{horiz} the same is true for $\pi_k, \tilde{\pi}$ so the equivalences hold by Assumptions 11.2.1, 2 and 4. □

We continue to prove Proposition 13.3.1. Passing to the right adjoint of (13.9) gives

$$\begin{tikzcd}
\mathcal{D}(Q) & \mathcal{D}(X_0 \times_{X_0 \times X_0} Q) \arrow[l, "(\delta_0)^\star"'] \\
\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1), \mathcal{D}(Q)) \arrow[u] & \mathrm{Tr}^{X_0}_{\mathrm{geo}}(\mathcal{D}(X_1), \mathcal{D}(Q)). \arrow[l, "\delta^\star"'] \arrow[u]
\end{tikzcd}$$

with horizontal arrows monadic (by Lemma 13.3.2). Let T denote the monad corresponding to the cosimplicial category $\mathcal{D}(\mathrm{HH}^{X_0}(X_1, Q)_\bullet)$ and by V the monad corresponding to $\mathcal{D}(\mathrm{HH}(X_1, Q)_\bullet)$. Then to show that $\delta^\star$ is fully faithful it is enough to show that the natural map

$$V \to (\delta_0)^\star \circ T \circ (\delta_0)_\star,$$

is an equivalence. The monad T is given by $(d_0)_\dagger \circ (d_1)^\dagger$ with

$$d_1, d_0 \colon X_1 \times_{X_0 \times X_0} Q \to X_0 \times_{X_0 \times X_0} Q.$$

Recall that the map d_1 is induced by the left action of X_1 on Q in $\mathbf{End}_{\mathrm{Corr}(C)}(X_0)$ by

$$X_1 \times_{X_0 \times X_0} Q \simeq X_0 \times_{X_0 \times X_0} (X_1 \times_{X_0} Q) \to X_0 \times_{X_0 \times X_0} Q$$

Likewise, the map d_0 is induced by the right action via

$$X_1 \times_{X_0 \times X_0} Q \simeq X_0 \times_{X_0 \times X_0} (Q \times_{X_0} X_1) \to X_0 \times_{X_0 \times X_0} Q.$$

The monad V is given by

$$V \simeq (a_r)_\dagger \circ (\xi_r)^\star \circ (\xi_l)_\star \circ (a_l)^\dagger.$$

These maps fit into a commutative diagram diagram in C:

$$\begin{tikzcd}
X_1 \times Q & X_1 \times_{X_0} Q \arrow[l, "\xi_l"'] \arrow[r, "a_l"] & Q \\
Q \times_{X_0} X_1 \arrow[u, "\xi_r"] \arrow[d, "a_r"] & X_1 \times_{X_0 \times X_0} Q \arrow[l, "\chi"'] \arrow[r, "d_1"] \arrow[u, "\zeta"] \arrow[d, "d_0"] & X_0 \times_{X_0 \times X_0} Q \arrow[u, "\delta_0"] \\
Q & X_0 \times_{X_0 \times X_0} Q \arrow[l, "\delta_0"'] &
\end{tikzcd} \tag{13.14}$$

such that the two upper squares are Cartesian. Then it is enough to show that the natural maps

$$(\xi_r)^\star \circ (\xi_l)_\star \to \chi_\star \circ \zeta^\star, \quad \zeta^\star \circ (a_l)^\dagger \to (d_1)^\dagger \circ (\delta_0)^\star, \quad (a_r)_\dagger \circ \chi_\star \to (\delta_0)_\star \circ (d_0)_\dagger$$

are equivalences, which hold by Assumptions 11.2.1, 2, 3 and 4, respectively, and the fact that $\xi_l, \xi_r, \zeta, \chi \in C_{sm}$ and $a_l, a_r, d_0, d_1 \in C_{prop}$. □

Now we specialize the above discussions to the case $X_1 = X \times_Y X$ and $Q = X \times_Y Z \times_Y Z$ as in Proposition 13.2.1. In this case, the relative Hochschild complex has a simple interpretation. Consider the fiber product

$$\begin{tikzcd} Y \times_{Y\times Y} Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_Y"] \\ Z \arrow[r, "g = g_1 \times g_2"] & Y \times Y \end{tikzcd}$$

in C and consider the map $q = f \times \mathrm{id}_Z : X \times_{Y\times Y} Z \to Y \times_{Y\times Y} Z$.

Lemma 13.3.3. *There is a canonical equivalence of simplicial objects in* C*.*

$$\mathrm{HH}^X(X \times_Y X, X \times_Y Z \times_Y X)_\bullet \simeq X_\bullet \times_{Y\times Y} Z$$

where the right hand side is the Čech nerve of $q : X \times_{Y\times Y} Z \to Y \times_{Y\times Y} Z$*. Under the identification, the map* δ_0 *from* (13.8) *is the horizontal map in* (13.3).

Proof. The construction of the left hand side is natural in X and applying it to the identity map $Y \to Y$ gives the right hand side. Thus, $f : X \to Y$ induces an augmentation $\mathrm{HH}^X(X \times_Y X, X \times_Y Z \times_Y X)_\bullet$ of the corresponding simplicial object. In order to identify this augmented simplicial object with the Čech nerve of q, can use the characterization (Lurie, 2009, Proposition 6.1.2.11) as it is easy to check that the necessary squares are pullbacks. □

Proof of Proposition 13.2.1. Only fully faithfulness requires a proof. Consider the augmented simplicial category associated to the Čech nerve of q. As $f \in C_{prop}$ so is the map $X \times_{Y\times Y} Y \to Y \times_{Y\times Y} Z$. Using Lemma 13.3.3 we identify the Čech nerve of this map with the relative Hochschild complex. By passing to right adjoints and using Corollary 9.0.6 we get a fully faithful functor

$$\left|\mathcal{D}(\mathrm{HH}^X(X \times_Y X, Z \times_{Y\times Y} (X \times X))_\bullet)\right| \to \mathcal{D}(Z \times_{Y\times Y} Y). \tag{13.15}$$

Composition with the fully faithful functor from Proposition 13.3.1 gives the desired functor. The essential image of (13.15) is generated by the image of $q_\dagger$ so the description of the essential image follows. □

We record the following functorality for later purpose. Let $(f : X \to Y) \in C_{vert}$, with $\Delta_X, \pi_X \in C_{horiz}$ and $\Delta_{X/Y} \in C_{vert}$. Let $Z \leftarrow C \to Z'$ be a morphism in $\mathrm{Corr}(C_{/Y\times Y})_{vert;horiz}$, i.e. all Z, Z', C are equipped with morphisms to $Y \times Y$ and $C \to Z$ and $C \to Z'$ are $(Y \times Y)$-morphisms in C.

Let $X_\bullet$ be as above and let $Q' = X \times_Y Z' \times_Y X$ and $Q = X \times_Y Z \times_Y X$. Then the following diagram is commutative

$$\begin{tikzcd}
\mathcal{D}(Q') \arrow[r] \arrow[d] & \mathrm{Tr}(\mathcal{D}(X_1), \mathcal{D}(Q')) \arrow[r] \arrow[d] & \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1), \mathcal{D}(Q')) \arrow[r] \arrow[d] & \mathcal{D}(Y \times_{Y\times Y} Z') \arrow[d] \\
\mathcal{D}(Q) \arrow[r] & \mathrm{Tr}(\mathcal{D}(X_1), \mathcal{D}(Q)) \arrow[r] & \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1), \mathcal{D}(Q)) \arrow[r] & \mathcal{D}(Y \times_{Y\times Y} Z).
\end{tikzcd} \tag{13.16}$$

Remark 13.3.4. *When we take C to be the category of (nice) algebraic stacks over $\mathbb{C}$ and the sheaf theory $\mathcal{D}$ to be* D*-modules, one always has* $\mathrm{Tr}_{\mathrm{geo}}(\mathrm{D}(X \times_Y X), \phi) = \mathrm{Tr}(\mathrm{D}(X \times_Y X), \phi)$ *as* $\mathrm{D}(X) \otimes \mathrm{D}(Y) \cong \mathrm{D}(X \times Y)$*. Therefore, Corollary 13.2.2 recovers (Ben-Zvi and Nadler, 2009, Theorem 6.6). In loc. cit., instead of directly considering* $\mathrm{D}(\mathrm{HH}^X(X \times_Y X, X \times_Y X)_\bullet)$*, the authors used the relative bar resolution for the monoidal category* $\mathrm{D}(X \times_Y X)$ *and then used integral transforms to embed each level in the resulting simplicial object of this relative resolution fully faithfully into the corresponding level of* $\mathrm{D}(\mathrm{HH}^X(X \times_Y X, X \times_Y X)_\bullet)$*. Our method bypasses using the integral transforms, which might fail in other sheaf theoretic content. See Section 13.4 for discussions.*

13.4 Comparison between geometric and ordinary traces

In practice, we need to compare the geometric trace defined and studied as above with the ordinary traces reviewed in Section 12.1. The easiest situation has been discussed in Remark 13.1.1. On the other hand, the monadicity of the simplicial objects in Lemma 13.3.2 can be used to compare the usual trace and the geometric trace in other situations. Recall notations in (13.6).

Proposition 13.4.1. *Assume that the sheaf theory* $\mathcal{D} : \mathrm{Corr}(C)_{vert;horiz} \to \mathrm{Lincat}_\Lambda$ *satisfies the following condition: for every $X, Y \in C$, the exterior tensor product*

$$\boxtimes : \mathcal{D}(X) \otimes \mathcal{D}(Y) \to \mathcal{D}(X \times Y)$$

is fully faithful and admits a continuous right adjoint $\boxtimes^R$ (see Remark 11.1.1.) In addition, Assumptions 11.2.1 1-6 hold for $\mathcal{D}$.

Let $X_\bullet$, Q as in the statement of Proposition 13.3.1. Assume that

1. *the unit object $\Lambda_{X_0} \in \mathcal{D}(X_0)$ (for the symmetric monoidal structure of $\mathcal{D}(X_0)$ as in Section 11.1) is compact;*

2. *the image of Λ_{X_0} under the functor $\eta_\star \circ m^\dagger \circ u_\dagger$ belongs to $\mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \subset \mathcal{D}(X_1 \times X_1)$.*

Then $\mathcal{D}(X_1)$ is rigid and the comparison map $\mathrm{Tr}(\mathcal{D}(X_1), \mathcal{D}(Q)) \to \mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1), \mathcal{D}(Q))$ *(see* (13.2)*) is an equivalence.*

Proof. We first show that $\mathcal{D}(X_1)$ is rigid. We need to check the conditions of Example 12.2.3. The unit $\mathbf{1}_{X_1}$ is equivalent to $u_\dagger(\Lambda_{X_0})$, which is compact as Λ_{X_0} is compact. The multiplication map is given by the composition $m_\dagger \circ \eta^\star \circ \boxtimes$, with the continuous right adjoint given by $\boxtimes^R \circ \eta_\star \circ m^\dagger$. It remains show that that this right adjoint is a $\mathcal{D}(X_1)$-bimodule homomorphism. To see that it is a left $\mathcal{D}(X_1)$-module morphism (the case of right $\mathcal{D}(X_1)$-module structure is similar), consider the following diagram

$$\begin{tikzcd}
\mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \arrow[r, "\mathrm{id}\otimes(\eta_\star \circ m^\dagger)"] \arrow[d, "\boxtimes"'] & \mathcal{D}(X_1) \otimes \mathcal{D}(X_1 \times X_1) \arrow[r, "\mathrm{id}\otimes\boxtimes^R"] \arrow[d, "\boxtimes"] & \mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \arrow[d] \\
\mathcal{D}(X_1 \times X_1) \arrow[r, "(\mathrm{id}\times\eta)_\star \circ (\mathrm{id}\otimes m)^\dagger"] \arrow[d, "m_\dagger \circ \eta^\star"'] & \mathcal{D}(X_1 \times X_1 \times X_1) \arrow[d, "(m\times\mathrm{id})_\dagger \circ (\eta\times\mathrm{id})^\star"] & \mathcal{D}(X_1 \times X_1) \otimes \mathcal{D}(X_1) \arrow[d] \\
\mathcal{D}(X_1) \arrow[r, "\eta_\star \circ m^\dagger"] & \mathcal{D}(X_1 \times X_1) \arrow[r, "\boxtimes^R"] & \mathcal{D}(X_1) \otimes \mathcal{D}(X_1).
\end{tikzcd}$$

By Assumptions 11.2.1 5-6, the left upper square is commutative. By Lemma 13.3.2, the left lower square is commutative. In other words, the functor $\eta_\star \circ m^\dagger$ is a left $\mathcal{D}(X_1)$-module homomorphism. Then Assumption (2) of the proposition implies that the essential images of the functor $\boxtimes^R \circ \eta_\star \circ m^\dagger$ belong to $\mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \subset \mathcal{D}(X_1 \times X_1)$. Therefore, the outer square of the above diagram is commutative. (However we do not claim the right square is commutative.)

Next we show that the comparison map (13.2) is an equivalence. Recall that it is induced by the morphism of simplicial objects

$$\mathrm{HH}(\mathcal{D}(X_1), \mathcal{D}(Q))_\bullet = \mathcal{D}(X_1)^{\otimes\bullet} \otimes \mathcal{D}(Q) \to \mathcal{D}(X_1^\bullet \times Q) = \mathcal{D}(\mathrm{HH}(X_1, Q)_\bullet).$$

The 0-th objects of both simplicial objects are given by $\mathcal{D}(Q)$. We use d_m^i to denote the ith face map $d_i : \mathcal{D}(\mathrm{HH}(X_1, Q)_{m+1}) \to \mathcal{D}(\mathrm{HH}(X_1, Q)_m)$.

By Lemma 12.3.1, the co-simplicial object $\mathcal{D}(X_1)^{\otimes\bullet} \otimes \mathcal{D}(Q)$ obtained by passing to right adjoints on $\mathrm{HH}(\mathcal{D}(X_1), \mathcal{D}(Q))_\bullet$, satisfies the Beck-Chevalley conditions, and that the resulting monad on $\mathcal{D}(Q)$ is given by $d_0 \circ \boxtimes \circ \boxtimes^R \circ (d_1)^R$. By Lemma 13.3.2, $\mathcal{D}(\mathrm{HH}(X_1, Q)_\bullet)$ also satisfies the Beck-Chevallay conditions, and that the resulting monad on $\mathcal{D}(Q)$ is given by $d_0 \circ (d_1)^R$. It remains to show that the two monads are identified which will imply the proposition. Note that $(d_1)^R$ is given by the composition of the bottom functors in the following commutative diagram

$$\begin{tikzcd}
 & & \mathcal{D}(X_1) \otimes \mathcal{D}(X_1) \otimes \mathcal{D}(Q) \arrow[r, "\mathrm{id}\otimes((a_l)_\dagger\circ(\xi_l)^\star\circ\boxtimes)"] \arrow[d, "\boxtimes\otimes\mathrm{id}"] & \mathcal{D}(X_1) \otimes \mathcal{D}(Q) \arrow[d, equal] \\
\mathcal{D}(Q) \arrow[r, "\mathbf{1}_{X_1}\otimes\mathrm{id}"] \arrow[d, equal] & \mathcal{D}(X_1) \otimes \mathcal{D}(Q) \arrow[ur, "(\boxtimes^R\circ\eta_\star\circ m^\dagger)\otimes\mathrm{id}"] \arrow[r, "(\eta_\star\circ m^\dagger)\otimes\mathrm{id}"] \arrow[d, "\boxtimes"] & \mathcal{D}(X_1 \times X_1) \otimes \mathcal{D}(Q) \arrow[d, "\boxtimes"] & \mathcal{D}(X_1) \otimes \mathcal{D}(Q) \arrow[d, "\boxtimes"] \\
\mathcal{D}(Q) \arrow[r, "(u\times\mathrm{id})_\dagger\circ(\pi_{X_0}\times\mathrm{id})^\star"] & \mathcal{D}(X_1 \times Q) \arrow[r, "(\eta\times\mathrm{id})_\star\circ(m\times\mathrm{id})^\dagger"] & \mathcal{D}(X_1 \times X_1 \times Q) \arrow[r, "(\mathrm{id}\times a_l)_\dagger\circ(\mathrm{id}\times\xi_l)^\star"] & \mathcal{D}(X_1 \times Q).
\end{tikzcd}$$

We remark that the commutativity of the triangle follows from Assumption (2), and the commutativity of the square below the triangle follows from Assumptions 11.2.1 5-6. It follows that $\boxtimes \circ \boxtimes^R \circ (d_1)^R = (d_1)^R$. In particular, the two monads are identified. $\square$

Corollary 13.4.2. *Assume that the sheaf theory $\mathcal{D}$ is as in* **??**. *Let X, Y, Z be as in Proposition 13.2.1. Assume that Λ_X is compact and its image under the functor $(\mathrm{id} \times \Delta_X \times \mathrm{id})_\star \circ (\mathrm{id} \times f \times \mathrm{id})^\dagger \circ (\Delta_{X/Y})_\dagger$ belongs to $\mathcal{D}(X \times_Y X) \otimes \mathcal{D}(X \times_Y X)$, then the canonical map*

$$\mathrm{Tr}(\mathcal{D}(X \times_Y \times X), \mathcal{D}(X \times_Y Z \times_Y X)) \to \mathcal{D}(Y \times_{Y\times Y} Z)$$

is fully faithful. If in addition, $Z = W_1 \times W_2$ as in Corollary 13.2.3, then the canonical map

$$\mathcal{D}(W_1 \times_Y X) \otimes_{\mathcal{D}(X\times_Y X)} \mathcal{D}(X \times_Y W_2) \to \mathcal{D}(W_1 \times_Y W_2).$$

is fully faithful.

Remark 13.4.3. *If the sheaf theory $\mathcal{D}$ satisfies ???, i.e. for $f \in C_{prop} \cap C_{horiz}$, $f^\dagger = f^\star$, and for $g \in C_{sm} \cap C_{vert}$, $g_\star = g_\dagger$ then under the assumption that $f \in C_{prop} \cap C_{horiz}$ and $\Delta_X \in C_{sm} \cap C_{vert}$, we have*

$$(\mathrm{id} \times \Delta_X \times \mathrm{id})_\star \circ (\mathrm{id} \times f \times \mathrm{id})^\dagger \circ (\Delta_{X/Y})_\dagger(\Lambda_X) \cong (\Delta_{X_1})_\dagger(\pi_{X_1})^\dagger\Lambda.$$

Proof. The first statement is a combination of Proposition 13.2.1 and **??**. As $\mathcal{D}(X\times_Y W_1) \otimes \mathcal{D}(W_2 \times_Y X) \to \mathcal{D}(X \times_Y W_1 \times W_2 \times_Y X)$ is a fully faithful $\mathcal{D}(X_1)$-bimodule homomorphism, the second statement follows from the first and Corollary 12.3.2. $\square$

Example 13.4.4. *Consider the case $Z = W \times X$ for some $g\colon W \to Y$. Then we have a split augmented simplicial object*

$$\mathrm{HH}(X \times_Y X, (W \times X) \times_{Y \times Y} (X \times X))_\bullet \to W \times_Y X$$

with the last map given by the action of $X \times_Y X$ on $W \times_Y X$. In this case, we reduce to the tautological equivalence

$$\mathcal{D}(W \times_X X) \otimes_{\mathcal{D}(X \times_Y X)} \mathcal{D}(X \times_Y X) \xrightarrow{\sim} \mathcal{D}(W \times_Y X).$$

13.5 Functoriality of categorical traces in geometric setting

Next we discuss functoriality of categorical traces arising from convolution patterns.

We start with the following observation. Let $f : X \to Y$ be as in Section 13.2. Let $Z \dashrightarrow Z'$ be a morphism in $\mathrm{Corr}(\mathcal{C}_{/Y \times Y})_{vert;horiz}$, given as $Z' \leftarrow C \to Z$.

Lemma 13.5.1. *Assumptions are as in Corollary 13.4.2 and assume that the sheaf theory $\mathcal{D}$ satisfies Assumptions 11.2.1 1-2. Then the following diagram is right adjointable.*

$$\begin{tikzcd}
\mathrm{Tr}(\mathcal{D}(X \times_Y X), \mathcal{D}(X \times_Y Z \times_Y X)) \arrow[r] \arrow[d] & \mathcal{D}(Y \times_{Y \times Y} Z) \arrow[d] \\
\mathrm{Tr}(\mathcal{D}(X \times_Y X), \mathcal{D}(X \times_Y Z' \times_Y X)) \arrow[r] & \mathcal{D}(Y \times_{Y \times Y} Z').
\end{tikzcd}$$

Proof. As $\mathcal{D}(X \times_Y X)$ is rigid, using Lemma 12.3.1 it is enough to show that the following diagram is right adjointable

$$\begin{tikzcd}
\mathcal{D}(X \times_Y Z \times_Y X) \arrow[r] \arrow[d] & \mathcal{D}(Y \times_{Y \times Y} Z) \arrow[d] \\
\mathcal{D}(X \times_Y Z' \times_Y X) \arrow[r] & \mathcal{D}(Y \times_{Y \times Y} Z').
\end{tikzcd}$$

But this follows from our assumption of $\mathcal{D}$ (in particular Assumptions 11.2.1 1-2). $\square$

Next we discuss duality of modules arising from the convolution patterns. Let $(f : X \to Y, g : Z \to Y \times Y)$ and let $(f' : X' \to Y', g' : Z' \to Y' \times Y')$ be as in Section 13.2.. Let $W \to Y \times Y'$ be a morphism. Let

$$X_1 = X \times_Y X, \quad Q = X \times_Y Z \times_Y X, \quad X'_1 = X' \times_{Y'} X', \quad Q' = X' \times_{Y'} Z' \times_{Y'} X',$$

and let
$$M = X \times_Y W \times_{Y'} X'.$$
We would like to know when $\mathcal{D}(M)$ is dualizable as a $\mathcal{D}(X_1)$-$\mathcal{D}(X_1')$-bimodule, with the dual of given by $\mathcal{D}(N)$ where $N = X' \times_{Y'} W \times_Y X$. We will assume that

- $W \to Y'$ and $W \to W \times_{Y'} W$ belong to C_{horiz};
- $W \to Y$ and $W \to W \times_Y W$ belongs to C_{vert}.

Then we have the morphism $u_{\text{geo}} : Y' \dashrightarrow W \times_Y W$ in $\text{Corr}(C)_{vert;horiz}$ given by $W \times_Y W \leftarrow W \to Y'$ and $e_{\text{geo}} : W \times_{Y'} W \dashrightarrow Y$ given by $Y \leftarrow W \to W \times_{Y'} W$. They induce

$$\begin{tikzcd} & \mathcal{D}(X_1') \arrow[d, "\mathcal{D}(\mathrm{id}\times u_{\text{geo}}\times \mathrm{id})"] \\ \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \arrow[r] & \mathcal{D}(X' \times_{Y'} W \times_Y W \times_{Y'} X'). \end{tikzcd} \tag{13.17}$$

and

$$\begin{tikzcd} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[r] \arrow[dr, "e"'] & \mathcal{D}(X \times_Y W \times_{Y'} W \times_Y X) \arrow[d, "\mathcal{D}(\mathrm{id}\times e_{\text{geo}}\times \mathrm{id})"] \\ & \mathcal{D}(X_1). \end{tikzcd} \tag{13.18}$$

Here e is defined to be the composition.

Lemma 13.5.2. *If the vertical morphism in* (13.17) *factors through a $\mathcal{D}(X_1')$-bimodule morphism*

$$u : \mathcal{D}(X_1') \to \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M)$$

(e.g. if $\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \to \mathcal{D}(X' \times_{Y'} W \times_Y W \times_{Y'} X')$ is an equivalence), then u and e from (13.18) *give the duality datum of $\mathcal{D}(M)$ as a $\mathcal{D}(X_1)$-$\mathcal{D}(X_1')$-module.*

Proof. Write $R = X' \times_{Y'} W \times_Y W \times_{Y'} X'$ and $S = X \times_Y W \times_{Y'} W \times_Y X$ for simplicity.

Note that (using (13.16)) we have the following commutative diagram

$$
\begin{tikzcd}
\mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[r, equal] \arrow[dd, "u\otimes\mathrm{id}"'] & \mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[r, "\cong"] \arrow[d] & \mathcal{D}(N) = \mathcal{D}(X' \times_{Y'} Y' \times_{Y'} W \times_Y X) \arrow[dd, "\mathcal{D}(\mathrm{id}\times u_{\mathrm{geo}}\times\mathrm{id}\times\mathrm{id})"] \\
 & \mathcal{D}(R) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[dr] & \\
\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[ur] \arrow[dr] \arrow[dd, "\mathrm{id}\otimes e"'] & & \mathcal{D}(X' \times_{Y'} W \times_Y W \times_{Y'} W \times_Y X) \arrow[dd, "\mathcal{D}(\mathrm{id}\times\mathrm{id}\times e_{\mathrm{geo}}\times\mathrm{id})"] \\
 & \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(S) \arrow[ur] \arrow[d] & \\
\mathcal{D}(N) \otimes_{\mathcal{D}(X)} \mathcal{D}(X) \arrow[r, equal] & \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(X_1) \arrow[r, "\cong"] & \mathcal{D}(N) = \mathcal{D}(X' \times_{Y'} Y' \times_{Y'} W \times_Y X)
\end{tikzcd}
$$

The composition of functors in the right column is isomorphic to the identity functor by the base change isomorphism (11.2). It follows that (12.11) in the current setting holds. The same reasoning implies that (12.10) in the current setting also holds. ◻

In practice, the assumption in Lemma 13.5.2 may not hold. But under some certain technical assumptions, we can still understand the duality datum.

Lemma 13.5.3. *Suppose the sheaf theory $\mathcal{D}$ satisfies assumptions as in Corollary 13.4.2, and let $f : X \to Y$ and $f' : X' \to Y'$ be as in Corollary 13.4.2 (so in particular $\mathcal{D}(X_1)$ and $\mathcal{D}(X_1')$ are rigid). Suppose $\mathcal{D}(M) \otimes \mathcal{D}(T) \to \mathcal{D}(M \times T)$ is an equivalence for every $T \in \mathcal{C}$. Then*

$$u : \mathcal{D}(X_1') \to \mathcal{D}(X' \times_{Y'} W \times_Y W \times_{Y'} X') \to \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M),$$

where the last functor is the right adjoint of the vertical morphism in (13.17) *and e from* (13.18) *form a duality datum.*

Proof. As in the proof of Lemma 13.5.2, it is enough to establish the following commutative diagram

$$
\begin{tikzcd}
\mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[r, equal] \arrow[dd, "u\otimes\mathrm{id}"'] & \mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[r, "\cong"] \arrow[d] & \mathcal{D}(N) \arrow[dd] \\
 & \mathcal{D}(R) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[dl, "(**)"'] \arrow[dr] & \\
\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N) \arrow[dr] \arrow[dd, "\mathrm{id}\otimes e"'] & & \mathcal{D}(X' \times_{Y'} W \times_Y W \times_{Y'} W \times_Y X) \arrow[dl, "(**)"] \arrow[dd] \\
 & \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(S) \arrow[d] & \\
\mathcal{D}(N) \otimes_{\mathcal{D}(X)} \mathcal{D}(X) \arrow[r, equal] & \mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(X_1) \arrow[r, "\cong"] & \mathcal{D}(N),
\end{tikzcd}
$$

where the arrows labelled by $(**)$ are right adjoint of the corresponding arrows in the diagram from the proof of Lemma 13.5.2.

Only the commutativity of the middle parallelogram and the lower right trapzoid requires justification. For the middle parallelogram, first consider the commutative diagram

$$\begin{tikzcd}
\mathcal{D}(N\times M\times N) \arrow[r] \arrow[d] & \mathcal{D}(R\times N) \arrow[d] \\
\mathcal{D}(N\times S) \arrow[r] & \mathcal{D}(X'\times_{Y'} W\times_Y W\times_{Y'} W\times_Y X).
\end{tikzcd}$$

with horizontal morphisms are induced by the correspondence $Y\leftarrow X\rightarrow X\times X$ and vertical morphisms induced by $Y'\leftarrow X'\rightarrow X'\times X'$. As $f, f'\in C_{prop}$ and $\Delta_X, \Delta_{X'}\in C_{sm}$, the above diagram is right adjointable by the same proof as in Lemma 13.3.2. Under our assumption that $\mathcal{D}(M)\otimes\mathcal{D}(T)\rightarrow\mathcal{D}(M\times T)$ is an equivalence for every $T\in C$, we may replace $\mathcal{D}(N\times M\times N)$ by $\mathcal{D}(N)\otimes\mathcal{D}(M)\otimes\mathcal{D}(N)$, $\mathcal{D}(R\times N)$ by $\mathcal{D}(R)\otimes\mathcal{D}(N)$ and $\mathcal{D}(N\times S)$ by $\mathcal{D}(N)\otimes\mathcal{D}(S)$. Then as $\mathcal{D}(X_1')$ and $\mathcal{D}(X_1)$ are rigid, using Lemma 12.3.1, we obtain the commutativity of the parallelogram.

Similarly, the lower right trapzoid is commutative by Lemma 13.5.1 and that $\mathcal{D}(N)\otimes_{\mathcal{D}(X_1)}\mathcal{D}(S)\cong\mathcal{D}(N)\otimes^{\mathrm{geo}}_{\mathcal{D}(X_1)}\mathcal{D}(S):=\mathrm{Tr}_{\mathrm{geo}}(\mathcal{D}(X_1),\mathcal{D}(N\times S))$. □

Now suppose we are given a $\mathcal{D}(X_1)$-$\mathcal{D}(X_1')$-bimodule homomorphism

$$\alpha:\mathcal{D}(M)\otimes_{\mathcal{D}(X_1')}\mathcal{D}(Q')\rightarrow\mathcal{D}(Q)\otimes_{\mathcal{D}(X_1)}\mathcal{D}(M).$$

Then as explained above, under certain dualizability assumption of $\mathcal{D}(M)$, there is a functor

$$\mathrm{Tr}(\mathcal{D}(M),\alpha):\mathrm{Tr}(\mathcal{D}(X_1'),\mathcal{D}(Q'))\rightarrow\mathrm{Tr}(\mathcal{D}(X_1),\mathcal{D}(Q)).$$

On the other hand, suppose we are given a correspondence

$$\alpha_{\mathrm{geo}}:W\times_{Y'}Z'\dashrightarrow Z\times_Y W$$

in $\mathrm{Corr}(C_{/Y\times Y'})_{vert;horiz}$. One can form the correspondence

$$C(W,\alpha_{\mathrm{geo}}):Y'\times_{Y'\times Y'}Z'\dashrightarrow Y\times_{Y\times Y}Z$$

given by the composition

$$\begin{aligned}Y'\times_{Y'\times Y'}Z' &\overset{u_{\mathrm{geo}}\times\mathrm{id}}{\dashrightarrow}(W\times_Y W)_{Y'\times Y'}Z'\cong Y\times_{Y\times Y}(W\times_{Y'}Z'\times_{Y'}W)\\ &\overset{\mathrm{id}\times\alpha_{\mathrm{geo}}\times\mathrm{id}}{\dashrightarrow}Y\times_{Y\times Y}(Z\times_Y W\times_{Y'}W)\overset{\mathrm{id}\times\mathrm{id}\times e_{\mathrm{geo}}}{\dashrightarrow}Y\times_{Y\times Y}Z. \qquad (13.19)\end{aligned}$$

The sheaf theory $\mathcal{D}$ then induces a functor

$$\mathcal{D}(C(W,\alpha_{\mathrm{geo}})) : \mathcal{D}(Y' \times_{Y'\times Y'} Z') \to \mathcal{D}(Y \times_{Y\times Y} Z).$$

We would like to relate $\mathrm{Tr}(\mathcal{D}(M),\alpha)$ with the above functor under certain assumptions.

Assumptions 13.5.4. *(I) We assume that the following diagram is commutative*

$$\begin{tikzcd} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(Q') \arrow[r] \arrow[d, "\alpha"'] & \mathcal{D}(X \times_Y W \times_{Y'} Z' \times_{Y'} X') \arrow[d, "\mathcal{D}(\mathrm{id}\times\alpha_{\mathrm{geo}}\times\mathrm{id})"] \\ \mathcal{D}(Q) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \arrow[r] & \mathcal{D}(X \times_Y Z \times_Y W \times_{Y'} X'). \end{tikzcd} \tag{13.20}$$

(II) We assume that the following diagram is commutative

$$\begin{tikzcd} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(Q') \arrow[d, "\alpha"'] & \mathcal{D}(X \times_Y W \times_{Y'} Z' \times_{Y'} X') \arrow[l] \arrow[d, "\mathcal{D}(\mathrm{id}\times\alpha_{\mathrm{geo}}\times\mathrm{id})"] \\ \mathcal{D}(Q) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) & \mathcal{D}(X \times_Y Z \times_Y W \times_{Y'} X'). \arrow[l] \end{tikzcd} \tag{13.21}$$

where the horizontal arrows are right adjoint of the natural ones.

Remark 13.5.5. *Note that Assumptions 13.5.4 holds in the case $Z' = Y'$ with $g_1' = \phi_{Y'} : Y' \to Y$, $Z = Y$ with $g_2 = \phi_Y : Y \to Y$ and there is $\phi_W : W \to W$ compatible with $\phi_{Y'}$ and ϕ_Y.*

Proposition 13.5.6. *Under the assumption in Lemma 13.5.2 and Assumptions 13.5.4 I, then the following diagram is commutative*

$$\begin{tikzcd} \mathrm{Tr}(\mathcal{D}(X_1'),\mathcal{D}(Q')) \arrow[r, "\mathrm{Tr}(\mathcal{D}(M),\alpha)"] \arrow[d] & \mathrm{Tr}(\mathcal{D}(X_1),\mathcal{D}(Q)) \arrow[d] \\ \mathcal{D}(Y' \times_{Y'\times Y'} Z') \arrow[r, "\mathcal{D}(C(W,\alpha_{\mathrm{geo}}))"] & \mathcal{D}(Y \times_{Y\times Y} Z). \end{tikzcd}$$

Under the assumption in Lemma 13.5.3 and Assumptions 13.5.4 II, the following diagram is commutative

$$\begin{tikzcd} \mathrm{Tr}(\mathcal{D}(X_1'),\mathcal{D}(Q')) \arrow[r, "\mathrm{Tr}(\mathcal{D}(M),\alpha)"] \arrow[d] & \mathrm{Tr}(\mathcal{D}(X_1),\mathcal{D}(Q)) \\ \mathcal{D}(Y' \times_{Y'\times Y'} Z') \arrow[r, "\mathcal{D}(C(W,\alpha_{\mathrm{geo}}))"] & \mathcal{D}(Y \times_{Y\times Y} Z) \arrow[u] \end{tikzcd}$$

Proof. The first case follows from the following commutative diagram

$$\begin{tikzcd}
\mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[r] \arrow[d, "u\otimes 1"'] & \mathcal{D}(Y' \times_{Y'\times Y'} Z') \arrow[d, "\mathcal{D}(u_{\mathrm{geo}}\times\mathrm{id})"] \\
(\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M)) \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[r] \arrow[d, "\cong"'] & \mathcal{D}((W \times_Y W) \times_{Y'\times Y'} Z') \arrow[d, "\cong"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(Q') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) \arrow[r] \arrow[d, "1\otimes\alpha\otimes 1"'] & \mathcal{D}(Y \times_{Y\times Y} (W \times_{Y'} Z' \times_{Y'} W)) \arrow[d, "\mathcal{D}(\mathrm{id}\times\alpha_{\mathrm{geo}}\times\mathrm{id})"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(Q) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) \arrow[r] \arrow[d, "1\otimes 1\otimes e"'] & \mathcal{D}(Y \times_{Y\times Y} (Z \times_Y W \times_{Y'} W)) \arrow[d, "\mathcal{D})\mathrm{id}\times\mathrm{id}\times e_{\mathrm{geo}})"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} \mathcal{D}(Q) \arrow[r] & \mathcal{D}(Y \times_{Y\times Y} Z)
\end{tikzcd} \tag{13.22}$$

The second case follows from a similar diagram

$$\begin{tikzcd}
\mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[r] \arrow[d, "u\otimes 1"'] & \mathcal{D}(Y' \times_{Y'\times Y'} Z') \arrow[d, "\mathcal{D}(u_{\mathrm{geo}}\times\mathrm{id})"] \\
(\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M)) \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[d, "\cong"'] & \mathcal{D}((W \times_Y W) \times_{Y'\times Y'} Z') \arrow[l] \arrow[d, "\cong"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(Q') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) \arrow[d, "1\otimes\alpha\otimes 1"'] & \mathcal{D}(Y \times_{Y\times Y} (W \times_{Y'} Z' \times_{Y'} W)) \arrow[l] \arrow[d, "\mathcal{D}(\mathrm{id}\times\alpha_{\mathrm{geo}}\times\mathrm{id})"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(Q) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M) \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) \arrow[d, "1\otimes 1\otimes e"'] & \mathcal{D}(Y \times_{Y\times Y} (Z \times_Y W \times_{Y'} W)) \arrow[l] \arrow[d, "\mathcal{D}(\mathrm{id}\times\mathrm{id}\times e_{\mathrm{geo}})"] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} \mathcal{D}(Q) & \mathcal{D}(Y \times_{Y\times Y} Z), \arrow[l]
\end{tikzcd} \tag{13.23}$$

where the horizontal left arrows are obtained by the corresponding horizontal right arrows in (13.22) by passing to the right adjoint. We need to justify the commutativity of this diagram. First we have the commutativity of the following diagram

$$\begin{tikzcd}
 & \mathcal{D}(X_1') \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[r] \arrow[dl, "u\otimes 1"'] \arrow[d] & \mathcal{D}(Y' \times_{Y'\times Y'} Z') \arrow[d, "\mathcal{D}(u_{\mathrm{geo}}\times\mathrm{id})"] \\
(\mathcal{D}(N) \otimes_{\mathcal{D}(X_1)} \mathcal{D}(M)) \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') & \mathcal{D}(R) \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \arrow[l] & \mathcal{D}((W \times_Y W) \times_{Y'\times Y'} Z') \arrow[l]
\end{tikzcd}$$

Indeed, the left triangle is commutative as we are in the case as in Lemma 13.5.3, and the right square is commutative as the natural functor $\mathcal{D}(R) \otimes_{\mathcal{D}(X_1')\otimes\mathcal{D}(X_1')^{\mathrm{rev}}} \mathcal{D}(Q') \to \mathcal{D}((W \times_Y W) \times_{Y'\times Y'} Z')$ is fully faithful by Corollary 13.4.2. This justifies the commutativity of the top square in (13.23).

For the commutativity of the third square in (13.23), by our assuption it is enough to show that

$$\begin{tikzcd}
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(X \times_Y W \times_{Y'} Z' \times_{Y'} X') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) \arrow[d] & \mathcal{D}(Y \times_{Y\times Y} (W \times_{Y'} Z' \times_{Y'} W)) \arrow[l] \arrow[d] \\
\mathcal{D}(X_1) \otimes_{\mathcal{D}(X_1)\otimes\mathcal{D}(X_1)^{\mathrm{rev}}} (\mathcal{D}(X \times_Y Z \times_Y W \times_{Y'} X') \otimes_{\mathcal{D}(X_1')} \mathcal{D}(N)) & \mathcal{D}(Y \times_{Y\times Y} (Z \times_Y W \times_{Y'} W)) \arrow[l]
\end{tikzcd}$$

is commutative. Under the assumption that $\mathcal{D}(M) \otimes \mathcal{D}(T) \to \mathcal{D}(M \times T)$ is an equivalence for any $T \in C$, we can use Lemma 13.5.1 twice to conclude.

Similar argument also shows that the last square in (13.23) is commutative. □

Example 13.5.7. *Take $Y' = \mathrm{pt}$ and $W = X$. Then the naive class identifies with the element $\mathcal{L}_\phi(f)_\dagger(e_X)$ of $\mathcal{D}(\mathcal{L}_\phi(Y))$ considered as a functor from* Mod_Λ.

Example 13.5.8. *If $Y \to \mathrm{pt}$ is in C_{horiz} we can take $Z = Y$ and $g = \mathrm{id}_Y$. Then* $(\Delta_{Y/Y})_\dagger = \mathrm{id}$. *Then the geometric class in $\mathcal{D}(\mathcal{L}_\phi(Y))$ is equivalent to $\mathcal{P}_{\mathrm{Tr}_{\mathrm{geo}}}(e_{\mathcal{L}_\phi(Y)})$.*

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