Coupled Oscillators with Generalized Dissipation

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Aditya Ghanashyam Date

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ABSTRACT

We theoretically study a mechanical system of two coupled harmonic oscillators with arbitrary damping kernels. We consider cases where the damping is of a Markovian nature as well as the case of generalized non-Markovian damping. Previous studies had been performed for specific and equal non-Markovian damping kernels, namely an exponential and a power law kernel. We generalize this study for arbitrary and unequal damping kernels, finding that certain properties, namely the existence of a phase transition remain unchanged. This remains true for all non-zero values of the coupling strength between the modes. The study opens up new avenues for the experimental study of systems with hitherto unexplored system-bath interactions.

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Chapter 1

INTRODUCTION

A priori, all systems under observation are open systems in the sense that they always interact, i.e., exchange energy, particles, spins, etc with the environment or bath. The bath is always assumed to be in thermal equilibrium and thus has thermal fluctuations. These spontaneous fluctuations induce a stochastic element or 'noise' in the system's observables over a certain timescale. If the timescale over which the environmental influence is felt on the system is much larger than the intrinsic timescale over which the system evolves, then we may ignore the effects of the environment and regard the system as 'closed' or isolated.

The system dynamics is then conservative. When the environmental timescale is comparable or much shorter than the system's intrinsic timescale, there are two possibilities. The system-bath interaction is so strong that the probability distribution induced on the system quickly (relative to the time resolution of the observations) approaches the Boltzmann form. The system thermalizes quickly and may then be treated by the usual equilibrium statistical mechanics. In the second possibility, the approach to the thermal distribution can be tracked experimentally and the system is being observed in an out-of-equilibrium state. In both cases the system may be considered as an 'open system' . Clearly, the open/close distinction is conditional on the environmental timescale vis-a-vis the system's intrinsic timescale. For ultra precise measurements, the study of open systems becomes essential. The work of the thesis is in the context of this broad canvas.

That thermal fluctuations induce noise is amply demonstrated by Nyquist noise in electrical circuits and the phenomenon of Brownian motion. For the Brownian motion, Einstein and Smoluchowski obtained the diffusion equation for the number density of the Brownian particles and used it subsequently in their analysis. A convenient alternative dynamical model was provided by Langevin [1], focusing on individual Brownian particle. He added two terms to the equation of motion for the Brownian particle, a phenomenological *friction term* of the form $\gamma v(t)$, γ being a positive constant called friction coefficient and $v(t)$ being the instantaneous velocity of the Brownian particle, and a stochastic force contribution $\xi(t)$. Both terms arise due to rapid, incessant collisions with the molecules of the medium. The friction term represents the average effect of the molecular collisions with the Brownian particle while the stochastic term captures the rapidly fluctuating part. The medium being in thermal equilibrium, the fluctuations are Gaussian distributed and the stochastic force is also assumed to be Gaussian distributed. The molecular collisions being extremely rapid, the stochastic force is assumed to have zero mean and its statistical properties are characterized by its *autocorrelation function* $\langle \xi(t)\xi(t')\rangle_{ensemble}$ which is a function of $(t - t')$. The rapidity of collisions implies that this function is highly peaked around zero and is idealized as being proportional to $\delta(t - t')$. That the average $\langle v^2(t)\rangle_{ensemble}$ asymptotically reaches the value consistent with the equipartition theorem requires that the autocorrelation function is $\sim \gamma\delta(t - t')$ which is the original *fluctuation-dissipation theorem* for Gaussian delta-correlated noise.

The above Langevin model or equation has been generalized to allow for the possibility that the particle-medium interaction may not be just instantaneous as in a collisional model, but may depend on the particle's past history, i.e. , have a 'memory' . This is done by replacing $\gamma v(t) \to \int dt'\gamma(t - t')v(t')$. The $\gamma(t - t')$ is called a *memory kernel* and is a phenomenological characterization of the interaction between the particle (more generally called a 'system') and the medium (more generally called 'bath'). The condition of asymptotic thermalization of the system gives the corresponding modification of the autocorrelation of the stochastic force, $\gamma\delta(t - t') \to \gamma(t - t')$ and constitutes the modern form of the fluctuation-dissipation theorem [2]. This is discussed in more details in Appendix A. As a matter of terminology, the bath is said to be *Markovian* if the memory kernel is proportional to the delta function, and non-Markovian otherwise. For Markovian baths, the thermalization is almost instantaneous while the non-Markovian baths capture the more general open systems. The generalized Langevin equation or its quantum version (Heisenberg-Langevin equation) has been the main framework used in several applications [3]. We too use this framework, in this thesis with the following differences.

The system considered in the thesis consists of two harmonic oscillators, each coupled to its own non-Markovian bath, driven by an external drive and having a bilinear mutual coupling to each other. Such a system has been devised and studied in the research laboratories of Dr. Mukund Vengalattore.

The study was spurred by the discovery of extremely high mechanical quality factors of over 10^7 [4] in high stress Silicon Nitride membranes fabricated by Norcada Inc. This corresponds to more than 10 million membrane oscillations, a few seconds in actual time, before the amplitude decays significantly. These high quality factors were instrumental in demonstrating noise squeezing of thermal mechanical motion of two selected 'modes' of the membrane [5]. An appropriate theoretical modelling of the system was also provided in [6]. Further work exploring the possibility of a phase transition in such a system followed. The main result here was the discovery of a novel $U(1) \times Z_2$, above a certain threshold drive strength, upon modification of the nature of the bath the system couples to. To be particular, a new phase emerged when the system-bath interaction was changed from Markovian, corresponding to white noise and a constant dissipation rate, to non-Markovian (colored noise and time varying dissipation rate). Further details are given in [7]. These results raised the natural questions about the existence of such novel phases of the system under arbitrary system-bath interactions (generally non-Markovian). Theoretical investigation of this question is presented in this thesis.

In the second chapter, we describe the experimental arrangement and observational protocols. We discuss the genesis of the two-mode system with the optimal parameters. We describe its theoretical modeling and its validation. The crucial development has been the simulation of arbitrary baths through an *active feedback protocol*. With this, the generalized version of the two-mode system model can be analyzed experimentally.

The third chapter describes the experimental studies carried on the 2-mode system. As noted above, the system displays steady states with/without dynamical phases and a threshold response. Two basic studies were carried out: (a) determination of the relaxation times for returning to steady states after small deviations and (b) system's response to being driven across the threshold value of the drive. The first study shows that the relaxation time diverges near the threshold drive strength suggesting a critical behavior. The second study explores the occurrence of the Kibble-Zurek mechanism.

The fourth chapter containing the theoretical analysis of the two-mode system is divided into four sections. In the first section we cast the equations in a convenient form to deal with the arbitrary memory kernels in a uniform manner. This is analogous to the reduced equation of state used for gases. In this section we obtain the steady state solutions which clearly identify the possible phases of the system. It follows that the $U(1)$ phase is always present while the additional $\mathbb{Z}_2$ breaking is possible under certain conditions. These conditions are explicitly stated. The steady states single out a threshold value of the drive force which is the main focus of the subsequent analysis. In the second section, we present linearization of the equations about the steady states including the stochastic force terms. These equations are cast as an inhomogeneous, complex 2×2 matrix equation for the Fourier components of the perturbations. The ensemble averaged perturbations about any given steady state satisfy the homogeneous equation. The homogeneous equation may also be viewed as governing the dynamical, linear stability of the steady state in absence of the stochastic forces. The inhomogeneous equation is used for computing the variances or covariances of the perturbations. In the third section the stability analysis is carried out. It is conveniently grouped into stability of states below and above the threshold value of the drive. In the fourth section we present the correlations of the perturbations in terms of the covariances of the stochastic forces. We now have all the ingredients to analyze the critical behavior.

The final fifth chapter is divided into two sections. In the first we discuss and elaborate on the case of unequal memory kernels and in the second we summarize our results and make closing remarks.

Six appendices are included to present some of the detailed calculations and proofs.

Chapter 2

THE TWO-MODE SYSTEM

In this chapter, we describe the genesis of the two-mode system, the experimental set-up and the observation protocol, and its theoretical modeling. This is the model that is taken up for a detailed analysis in chapter 4. A more detailed discussion of the experiments and protocols is given in the chapter 3.

The two-mode system arose [8] in the setting up of an optomechanical platform to engineer and study quantum states of a *mesoscopic* mechanical systems. Observing quantum features at room temperature requires control of thermal noise or equivalently the dissipation rates. The mechanical system of choice is a stoichiometric silicon nitride membrane deposited onto a substrate of crystalline silicon. The fabrication process of the membrane on substrate provides flexural modes (transverse vibrations of the membrane) with the eigenfrequencies $\nu_{m,n} \sim 10^4\sqrt{m^2+n^2}, m, n$ being positive integers. With much experimentation and validation, the optimal dimensions of the membrane were found to be $5mm \times 5mm \times 100nm$. It was also found that there is a coupling between the substrate and the membrane which results in hybridization of the modes i.e. (linear combinations of modes (m,n) and (n,m)). This process affects the mechanical quality factors (Q-factors) i.e, number of oscillations before the amplitude dies down, with $m \sim n$ having higher values. Several pairs of modes were found to have their Q-values in excess of 10^7 for eigen frequencies between 1 and 1.5 MHz. A further important feature of the substrate coupling discovered was that it induces an *effective bilinear* interaction term between modes with nearby frequencies. This is a non-trivial step and was carefully established through several different observation which confirmed the predictions from this interaction term (see below) [5].

Thus, the optomechanical set up was established with a SiN membrane deposited on a substrate Si substrate having several high Q membrane's flexural modes and substrate modes mediating a specific interaction between pairs of the membrane modes, and having a frequency close to the sum of the membrane mode frequencies. This provides a high precision platform for investigating various questions relevant for optomechanical measurements, novel phases and phase transitions etc. The validation of the set up together with the physical origin of the two-mode system is described in detail in [8], [4]. Its mathematical formulation is detailed in 2.2.

Let us consider the schematic experimental arrangement and the observation protocols.

2.1 The Experimental Set-up

The system consists of a thin SiN membrane deposited on a silicon substrate, a mount for holding the membrane plus substrate in a vertical position, a piezo-electric actuator (PZT) to induce controlled mechanical vibrations in the mounted membrane and a read-out arrangement of a Michaelson interferometer with its output port connected to a six channel Zurich instrument. Each of its channel's output displays the amplitude and phase at specified channel frequencies which can be further fed to a computer for monitoring/analysis as well as for any feedback protocol. The membrane assembly and the actuator are enclosed in a high vacuum chamber ($\sim 10^{-6} Torr$) and the entire set-up is at room temperature ($\sim 300°K$). The arrangement is sketched in the diagram below.

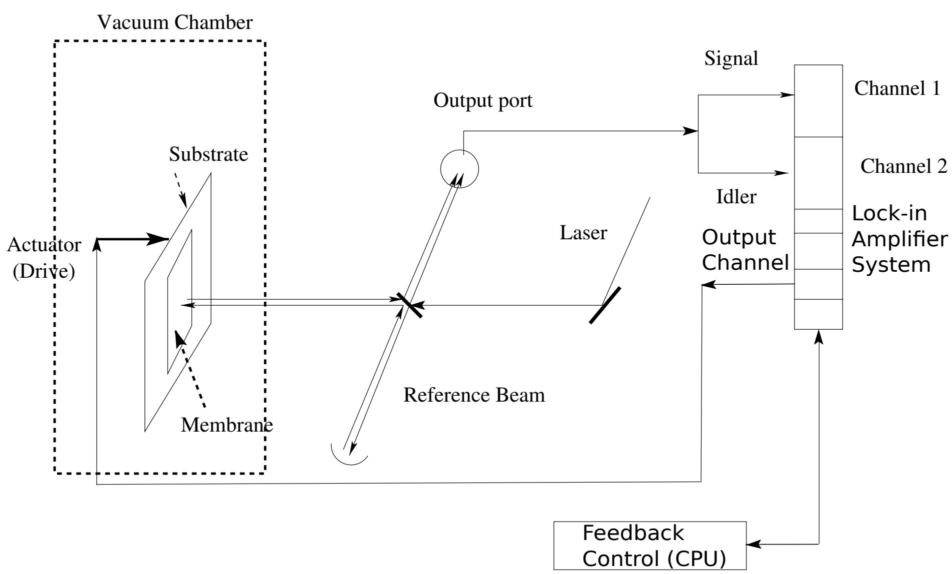

Figure 2.1: Schematic diagram of the experimental arrangement.

The PZT expands/contracts in response to applied voltage and can thus induce mechanical modulation with an AC voltage. By applying a voltage with the desired frequency components, specific modes of the membrane and the substrate can be excited with specified amplitudes. There are two types of modes-shear modes which vibrate the membrane in its plane and the flexural modes which vibrate along a direction perpendicular to the membrane plane (the transverse direction). The PZT excites the modes only through the frequency and there is no selective control to excite only the flexural or shear modes.

The transverse vibrations are picked up by one of the arms of a Michaelson interferometer and fed into the Zurich instrument. Our instrument could lock on to six different frequencies simultaneously and extract the amplitude and the phase at those selected frequencies. This information is fed into a computer for further processing as well as for controlling the voltage on the actuator when feedback is desired. In this manner the membrane system is driven in a controlled manner through the actuator while its response is monitored through the Zurich instrument. Our chosen two-mode system is tracked by monitoring its response at its two frequencies for a controlled drive stimulation.

For the observational studies discussed here, the chosen membrane frequencies are $\omega_s = 1.233$ MHz and $\omega_i = 1.466$ MHz, conventionally called the *signal mode* and *idler mode*, respectively. The substrate mode is at the frequency $\omega_P = \omega_i + \omega_s$ and is conventionally called the *pump mode*. The basic observational protocol is to track the amplitude and phase of the signal and idler modes in response to applied voltage through the PZT. There are many experimental issues that impact the precision of measurements.

One important issue to immediately address is the sensitivity of the membrane frequency to fluctuations and drifts in the ambient temperature. For instance, a temperature change of 1 K leads to a drift of the order of 500 Hz. To correct for these and to stabilize them, the following scheme is used (developed by Dr. Srivatsan Chakram and Dr. Yogesh Patil). First the thermal drifts of all the modes are measured and a high-Q *thermometer* mode is chosen whose drift reasonably correlates with that of the two modes of interest. The frequency of this thermometer mode is continuously measured and stabilized with the help of an auxiliary laser beam. This auxiliary beam is aligned such that its beam spot overlaps the interface between the membrane and surrounding substrate. The laser beam heats up the targeted spot causing a differential expansion of the membrane and substrate, thereby pulling the membrane frequencies lower or higher. The intensity of this beam is adjusted in accordance with the measured drift of the thermometer mode, thereby stabilizing the membrane mode frequencies against thermal drift. This scheme allows the frequencies to be stabilized to within 2 mHz, corresponding to temperature fluctuations less than $2\mu K$.

An additional feature available to us is the ability to modify the dynamics of the modes of interest. We exploit this to generate a damping/dissipation scheme of arbitrary choice. This is done by first measuring the bare damping rates for the modes of interest via observing their mechanical *ring-down*. Extracting the amplitude and phase information through the lock-in detection scheme allows us to cancel this bare damping rate and replace it with a damping of our choice. This is done via software and the resulting signal is fed back into the system through the PZT drive voltage (at the frequencies of the respective modes). Concomitantly a generated noise signal is also added at the mode frequencies so as to satisfy the *fluctuation-dissipation* theorem for the new damping mechanism. Further details of this feedback scheme are given in Chapter 3.

Another important ingredient in the modeling of the two-mode system is the inferred and validated interaction among the idler, signal, and pump modes. Essentially it is a manifestation of parametric excitation processes. If we denote the amplitudes of the signal, idler, and the pump modes by $x_{i,s,P}$, then the interaction is represented by the term of the form $\sim x_i x_s x_P$. As noted above, this interaction has been amply validated through different observations. The strength of the coupling g is a fixed quantity $\simeq 1 \times 10^{-6} s^{-1}$ [9]. Its existence is crucial for all the novel features revealed by the two-mode system.

To date, the following salient features have been revealed by the two-mode system qualitatively described above. For Markovian friction, $\gamma_i \neq \gamma_s$ in general, it was demonstrated that with only thermal noise and pump drive below a threshold value, the thermal fluctuations of the two modes become highly correlated and are revealed through squeezing of a certain quadrature built from both the modes. Above the threshold, a different quadrature is squeezed [5]. The steady states with pump drive above threshold display a $U(1)$ symmetry corresponding to equal and opposite shifts in the phases of the modes. When both the signal and idler modes are subjected to the same non-Markovian friction of an exponential form, $\gamma(t) \sim e^{-t/\tau}$, within certain parameter ranges there arises an additional discrete symmetry of the steady states above threshold. This corresponds to time dependent or dynamical steady states with the signal and idler modes oscillating with $e^{\pm i\Delta t}$ time dependence. In a yet another form of non-Markovian friction, again the same for both modes, with a certain a power law form for its Fourier transform, $\tilde{\gamma}(\omega)$, modification of the critical behavior when the drive threshold is crossed, is seen [9]. With the active feedback protocol in place, arbitrary friction kernels are amenable to experimental studies.

The results seen so far already raise some natural questions. Is there any systematics in the behavior of the system for different system-bath interactions of the two modes? Is the existence of a critical point universal? Is the nature of the transition universal? Are the exponents universal? Since the non-Markovian friction modifies the critical behavior, does it impact the applicability of the Kibble-Zurek ramp protocol (driving a system across its phase transition point in both directions at a certain rate) to infer some of the critical exponents? Are there further novel states and/or features with *unequal, non-Markovian memory kernels*?

To gain some handle on these questions, we turn to the theoretical model which is described next.

2.2 Theoretical model of the two-mode system

We have already discussed the system at a qualitative level. We have essentially three harmonic oscillators with a certain mutual interaction among them together with a drive and couplings to baths. Using i, s, P labels to denote the signal, the idler, and the pump modes, we write the non-interacting part of the Hamiltonian as: $H = \sum_{k=i,s,P} \frac{1}{2m_k} p_k^2 + \frac{m_k \omega_k^2}{2} q_k^2$. The ω_k are the mode frequencies given above. The "effective masses" for the membrane modes may be taken to be the same mass which is of the order of the mass of the membrane [8, 4] of about 2 picograms. These masses for the modes can always be taken to be the same by a rescaling canonical transformation. It is convenient to use the natural length and momentum scales provided by the zero point RMS fluctuations: $\Delta x_k = \sqrt{\frac{\hbar}{2m\omega_k}} =: x_{0,k}$, $\Delta p_k = \sqrt{\frac{\hbar m \omega_k}{2}} =: p_{0,k}$ and choose the dynamical variables as: $x_k := x_{0,k}(a_k^\dagger + a_k)$, $p_k := i p_{0,k}(a_k^\dagger - a_k)$. The explicit dependence of mass disappears and the $a_k, a_k^\dagger$ become dimensionless. The Hamiltonian takes the familiar form $H = \sum_{k=i,s,P} \hbar\omega_k (a_k^\dagger a_k + \frac{1}{2})$. The interaction term is taken in the form -$\hbar g \frac{x_i}{x_{0,i}} \frac{x_s}{x_{0,s}} \frac{x_P}{x_{0,P}}$. The externally controlled drive term is taken in the form $-\hbar\big(F_P e^{-i\omega_P t} a_P^\dagger + F_P^\dagger e^{i\omega_P t} a_P\big)$. The coefficients $g, F_P, F_P^\dagger$ have dimensions of *inverse time*. Suppressing the zero point energy which does not contribute to the equations of motion, the driven system's Hamiltonian is taken to be

$$H_{sys} = \sum_{k=i,s,P} \hbar\omega_k a_k^\dagger a_k - \hbar g \frac{x_i}{x_{0,i}} \frac{x_s}{x_{0,s}} \frac{x_P}{x_{0,P}} - \hbar\big(F_P e^{-i\omega_P t} a_P^\dagger + h.c.\big) \quad (2.1)$$

$$= \sum_{k=i,s,P} \hbar\omega_k a_k^\dagger a_k - \hbar g \prod_{k=i,s,P} (a_k + a_k^\dagger) - \hbar\big(F_P e^{-i\omega_P t} a_P^\dagger + h.c.\big). \quad (2.2)$$

Here h.c. denotes the Hermitian conjugate. The Heisenberg picture equations of motion are,

$$\dot{a}_k = -i\omega_k a_k + ig \prod_{l \neq k} (a_l + a_l^\dagger) + i\delta_{k,P} F_P e^{-i\omega_P t} \text{ and the adjoint equations.} \quad (2.3)$$

Here, $\delta_{k,P}$ denotes the Kronecker delta.

In the absence of the coupling g, each of the a_k has a sinusoidal time dependence with its own frequency, ω_k: $a_k(t) \sim A_k e^{i\omega_k t}$. With a weak coupling, $g \ll \omega_P (< \omega_i, \omega_s)$ turned on, we expect the amplitudes A_k's to acquire a weak time dependence. Thus we consider an ansatz: $a_k(t) = \underline{A}_k(t) e^{-i\omega_k t}$. Substitution of the ansatz leads to the equations for the slowly varying $\underline{A}_k(t)$ as

$$\dot{\underline{A}}_k = ig e^{i\omega_k t} \prod_{l \neq k} (\underline{A}^{-i\omega_l t} + \underline{A}^\dagger e^{i\omega_l t}) + i\delta_{k,P} F_P. \quad (2.4)$$

The product term shows that only for the combinations $A_s^\dagger A_P$, $A_i^\dagger A_P$ and $A_i A_s$, we have a slow time dependence which is exactly canceled if $\omega_i + \omega_s = \omega_P$. By making the observations over several cycles of the mode frequencies, the remaining combinations of the form $A_i A_P$, $A_s A_P$ etc, with fast time dependence will be washed out. Dropping these terms, we have the self consistent slow evolutions for the $\underline{A}_k$'s. This is our *rotating wave approximation*. The formal derivation using unitary transformations leads to the same result.

The equations for the slow time dependence take the form,

$$\dot{\underline{A}}_{i,s} = ig \underline{A}_{s,i}^\dagger \underline{A}_P \quad , \quad \dot{\underline{A}}_P = ig \underline{A}_i \underline{A}_s + iF_P. \quad (2.5)$$

These equations have drive but no dissipation.

To incorporate dissipation in the model, (2.5), we couple our two-mode system to a hypothetical bath (or simulated bath in our context) introducing a stochastic force with zero mean and a non-constant power spectrum. We also introduce a coupling of a general friction with memory to provide for a dissipation mechanism. The equations of motion with these modifications are called Heisenberg-Langevin equations. They could be "derived" by taking an explicit model for a bath-usually the so called independent oscillator model [10, 11, 12]-or proposed purely phenomenologically. We take the latter approach and illustrate the equation and its adaptation in our context, for a single harmonic oscillator [12].

The Heisenberg-Langevin equations of motion for the harmonic oscillator are ((q, p) denote any of the (x_k, p_k)),

$$\dot{q} = \frac{p(t)}{m} \quad , \quad \dot{p} = -m\omega^2 q(t) - \int_{t_0}^{t} dt' \underline{g}(t-t')\dot{q}(t') + \underline{F}(t) \ . \tag{2.6}$$

The $\underline{g}(t-t')$, $\underline{F}(t)$ are the modifications introduced to reflect a system-bath interaction. $\underline{F}(t)$ is the stochastic force with its statistical properties specified below while $\gamma(t) := \underline{g}(t)/m$ is a *memory kernel* representing friction with memory. The lower limit t_0 is when the system is coupled to the bath and may be taken a $-\infty$ for convenience. $\underline{g}(t) = 0$ for $t < 0$, is the condition of causality-system dynamics is influenced only by its past interaction with the bath. The above equations can admit a steady state provided the memory kernel and correlation function of the stochastic force are related in accordance with the fluctuation-dissipation theorem.

In terms of the $a, a^\dagger$ operators defined through $q(t) := q_0(a + a^\dagger), p := ip_0(a^\dagger - a)$, the equations become,

$$\dot{a}^\dagger = -\dot{a} + ip_0\frac{a^\dagger - a}{mq_0} \tag{2.7}$$

$$ip_0(\dot{a}^\dagger - \dot{a}) = -m\omega^2 q_0(a + a^\dagger) - \int_{-\infty}^{t} dt' \underline{g}(t-t')\frac{ip_0}{m}(a^\dagger - a)(t') + \underline{F}(t). \tag{2.8}$$

We have replaced the $\dot{q}$ by p/m in the integral term and also taken the lower limit of integration to be $-\infty$.

Eliminating $\dot{a}^\dagger$ and substituting for q_0, p_0 gives the equation for $\dot{a}$ as,

$$\dot{a} = -i\omega a - \frac{1}{2}\int_{-\infty}^{t} dt' \gamma(t-t')(a - a^\dagger)(t') + if(t) \ , \ \gamma(t) := \frac{\underline{g}(t)}{m} \ , \ f(t) := \frac{\underline{F}(t)}{p_0}. \tag{2.9}$$

The first term is the usual oscillator equation of motion while the next two terms are the modification due to the bath. The memory kernel $\gamma(t)$ is always real and positive for dissipation and has dimensions of T^{-2}. The scaled stochastic force f has dimensions of T^{-1}. Note that the coupling to the memory kernel has $a^\dagger(t')$. For $\gamma(t) \sim \delta(t)$, the bath is *Markovian* or memoryless since the interaction with bath depends only on the instantaneous state of the system.

The stochastic force is characterized implicitly by stipulating the thermal correlation function as:

$$\langle f(t)\rangle = 0 \ , \ \langle f(t) f^\dagger(t')\rangle \ = \ \left(\frac{\bar{n}+1}{2}\right)\gamma(t-t') \ , \ \bar{n} = \left[e^{\hbar\omega/kT} - 1\right]^{-1} \ . \quad (2.10)$$

The form of $\langle f f^\dagger\rangle$ with the same memory kernel is required for consistency with the fluctuation-dissipation theorem. A derivation is given in Appendix A.

Note: The $\langle f f^\dagger\rangle$ does not have a classical limit since its right hand side goes as $kT/\hbar\omega$. But this is because $f = \underline{F}/p_0$. The thermal average $\langle \underline{F}\,\underline{F}^\dagger\rangle \sim mkT$ in the classical limit (or for $\hbar\omega \ll kT$).

Returning to our equations (2.5) with the rotating wave approximation, and including the bath induced terms, we get

$$\dot{\underline{A}}_{i,s} \ = \ ig\underline{A}^\dagger_{s,i}\underline{A}_P - \frac{1}{2}\int_{-\infty}^{t} dt'\gamma_{i,s}(t-t')\big(\underline{A}_{i,s} - \underline{A}^\dagger_{i,s}\big)(t') + if_{i,s} \quad (2.11)$$

$$\dot{\underline{A}}_P \ = \ ig\underline{A}_i\underline{A}_s + iF_P - \frac{1}{2}\gamma_P(\underline{A}_P - \underline{A}^\dagger_P) + if_P. \quad (2.12)$$

We have taken the pump mode to couple to a Markovian bath, $\gamma(t-t') = \gamma_P\delta(t-t')$ which is justified since we are not applying a feedback at the pump mode frequency.

This describes our driven, dissipative system in rotating wave approximation and with the two modes coupled to generically non-Markovian baths. We will be looking for its steady states, analyze their stability, and obtain the correlations among perturbations about the steady states induced by the stochastic forces. This paves the way to address possible phase transitions and their characterization. In anticipation, we note that to have oscillatory steady states, we have to drop the $A^\dagger_{i,s,P}$ terms. Doing so, we write the final set of dynamical equations for our system as,

$$\dot{\underline{A}}_{i,s} = ig\underline{A}^{\dagger}_{s,i}\underline{A}_P - \frac{1}{2}\int_0^{\infty} dt'\gamma_{i,s}(t')\underline{A}_{i,s}(t-t') + if_{i,s} \;, \tag{2.13}$$

$$\dot{\underline{A}}_P = ig\underline{A}_i\underline{A}_s + iF_P - \frac{\gamma_P}{2}\underline{A}_P. \tag{2.14}$$

We have rewritten the memory kernel term by the change of variable $t' \rightarrow (t-t')$.

Remark: We have taken the quantum mechanical form of equations. However, comparing $\hbar\omega/kT$ for our system parameters, we see that the average number of quanta is $\sim [e^{10^{-7}} - 1]^{-1} \sim 10^{+7} \gg 1$. Consequently, the system is essentially *classical*. We may take the expectation values in a coherent state and replace the $a, a^{\dagger}$ (or $A, A^{\dagger}$) by complex numbers. We assume this step has been taken and take these semi-classical equations for further analysis in the next chapter.

Chapter 3

EXPERIMENTAL STUDY OF THE TWO-MODE SYSTEM

As mentioned in chapter 2, the two-mode system was discovered while studying the high quality factor modes of the SiN membrane. The existence of steady states with non-zero mode amplitudes above a threshold of the drive magnitude suggested interaction between the modes and the specific non-linear coupling was validated with extensive experimental study. To make the experimental arrangement robust, the mode frequency drift with ambient temperature had to be eliminated and was done as explained before. In effect, a physical platform was established with a stable excitation of two specified frequencies, $\omega_{i,s}$ driven at $\omega_i + \omega_s$ with a variable drive voltage and with a sensitive readout system. The Zurich instrument was capable of selecting and maintaining frequencies in the tens of MHz within $50mHz$ window. The mode amplitudes and phases could be recorded with a time resolution of milliseconds.

The experimental study naturally addresses two basic queries: (I) How long does it take for the system to return to a steady state and at what rate? (II) How does the system respond if it is driven across the threshold value of the drive? The first query identifies the relaxation time if the return is exponential. The second query may be viewed as a method of studying the nature of the transition across the threshold.

The protocol for the first query is straight forward. The two-mode system is driven at a certain drive strength close to the threshold and a kick is given to each mode, via a step voltage applied at the individual frequencies. An exponential ringdown is observed, from which a parameter τ corresponding to the relaxation time for each mode is extracted. The measurement is repeated for different values of the drive strength μ to give 3.1.

For the second query, we adapt the Kibble-Zurek protocol of linearly ramping the drive strength across the transition point and monitoring the resulting system behavior.

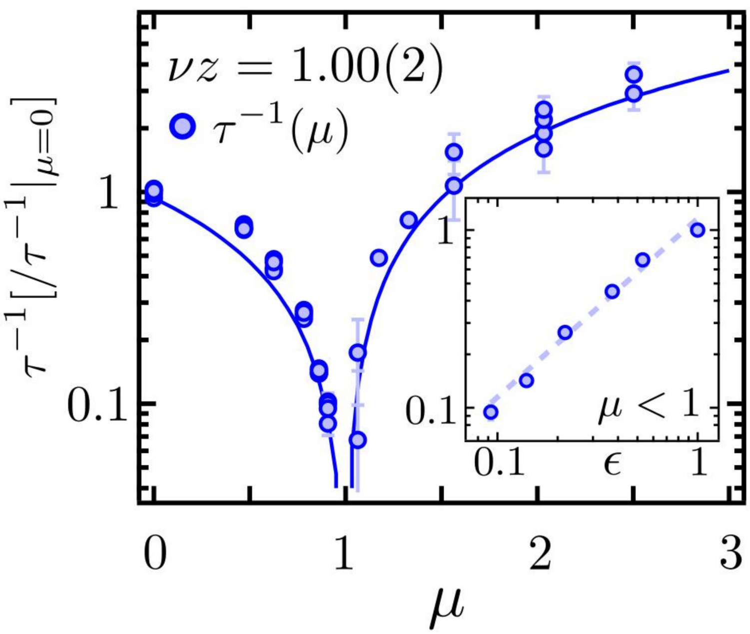

Figure 3.1: A plot of relaxation times as a function of the normalized drive. The plot is not symmetric about the threshold value[13].

According to the paradigm expounded by Kibble and Zurek, the system's response to such a ramp is determined by the ratio of two time scales - one defined by the ramp and the other is the relaxation time associated with each steady state. As we have seen previously in figure (3.1), the relaxation time diverges as we come close to the transition point. This phenomenon is also known in a general context as *critical slowing down*.

In our case, the drive strength μ defines a *unique stable steady state* except at the threshold value of $\mu^* = 1$. Below this value, the stable steady states correspond to an "unbroken" symmetry phase while above it, the stable steady states are in a "broken" symmetry phase. $\delta\phi = \phi_s - \phi_i$ between the signal and idler phases. Let us define $\epsilon(t) := [\mu(t) - \mu^*]/\mu^*$ as the distance from the transition point during the ramp. Let $\tau(\epsilon(t))$ denote the relaxation time for the stable steady state labelled by $\epsilon(t)$. The ramp defines the time scale $t_{ramp} := |\frac{\epsilon(t)}{\dot{\epsilon}(t)}|$.

The Kibble-Zurek (KZ) paradigm divides the system response into three distinct regimes demarcated by the ratio of the ramp timescale to the relaxation time as follows:

1. Adiabatic regime: $t_{ramp} > \tau$,
2. Impulse regime: $t_{ramp} < \tau$,
3. Adiabatic regime (again): $t_{ramp} > \tau$.

When the ramp timescale is much larger than the system relaxation time, the system easily follows the change in drive strength as it is ramped up. The KZ paradigm postulates that in this regime, the system responds instantaneously to the changing drive, i.e the system at all times tracks the steady state defined by the drive strength at that instance. The system evolution is thus said to the *adiabatic*.

When the ramp timescale becomes smaller than the relaxation time, the system can no longer follow the changing drive. The system dynamics is then said to be *frozen-out* with respect to the drive. In this regime, where the system can no longer track the steady state defined by the drive, the dynamics is said to be *impulsive*.

From the above description, we can extract the demarcation point denoted by $t_{freeze-out}$ as the time from the crossing of the critical point, when $t_{ramp}(\epsilon(t)) = \tau(\epsilon(t))$. From 3.1, we know that the relaxation time τ diverges close to the critical point. Let us assume the divergence to be of a power law form, i.e, $\tau(\epsilon) = \tau_0 |\epsilon|^{-\nu z}$. After a little algebra, we get:

$$t_{freeze-out} = \tau_0 \times (\tau_0 \dot{\epsilon})^{\frac{-\nu z}{1+\nu z}}. \tag{3.1}$$

From the above expression, we get a relation between the observed freeze-out time during the dynamic ramp of the drive strength, and the steady state relaxation time at various drive strengths. The power law exponent calculated via the Kibble-Zurek protocol should match with that calculated from the steady state measurement. Thus, the Kibble-Zurek paradigm provides an alternative way to calculate the critical exponents of a system without needing to perform accurate measurements near the critical point.

Another observable we could measure is the hysteresis area formed when we modify the ramp protocol to be bi-directional-first increasing drive strength past the transition point and then decreasing it back down to zero, instead of a unidirectional ramp of increasing drive strength. From the description of the adiabatic-impulse-adiabatic regimes above, it follows clearly why hysteresis should be seen when the system is driven in a closed loop. To elaborate further, during the forward ramp, the system remains frozen-out at a pre-transition amplitude until it reaches the adiabatic regime on the other side of the transition point. During the reverse ramp, the opposite happens where the amplitude remains non-zero well below the critical drive strength due to the freeze-out effects.

The hysteresis loop can be visually approximated as being triangular in shape with the area given as the base times height. The base of the triangle scales with $\epsilon_{freeze-out}$ and the system evolution from freeze-out to adiabaticity can be approximated by a straight line. Thus the area scales as the square of the scaling of $\epsilon_{freeze-out}$. From the measured scaling of $t_{freeze-out} \approx -0.5$, we get an $\epsilon_{freeze-out}$ scaling of +0.5, thereby giving an area scaling of ≈ 1, which is what we see in our experiments.

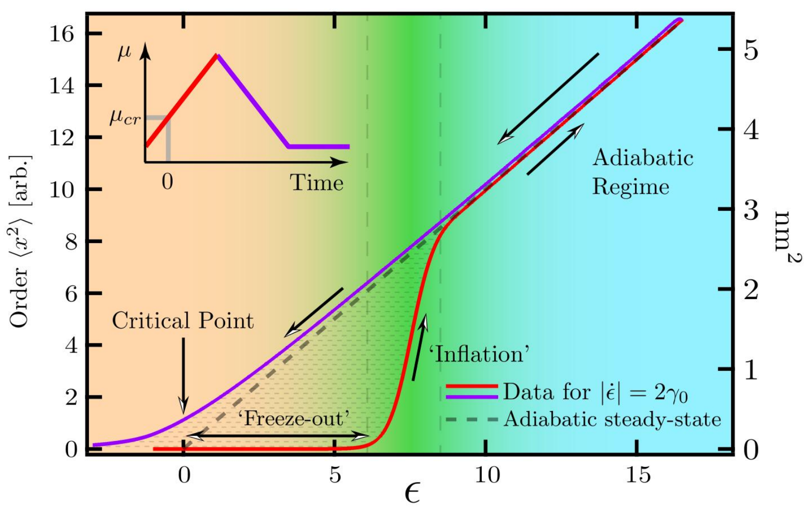

Figure 3.2: A sample plot of Kibble-Zurek ramp with the freeze-out regime clearly marked out. For the return ramp, the system continues adiabatically for longer time intervals[13].

These basic studies were first carried with physical environment being represented by two unequal Markovian baths. Once the Kibble-Zurek scalang was verified, the next step was the implementation of an *active feedback protocol* which allowed us to simulate arbitrary baths [9].

The idea is to use the recorded values of the mode amplitude and phases, fold them with some chosen memory kernels $\gamma_{i,s}(t)$, and compute the increment $\int_0^t dt' \gamma_{i,s}(t-t') A^*_{s,i} + f_{i,s}$ and add it to the drive voltage. The $f_{i,s}$ are chosen to be consistent with the fluctuation-dissipation theorem. Care is taken to *subtract* the physical, Markovian friction constants from the memory kernels. The corresponding statistical forces should also be subtracted, but they do not make much difference. In effect, we are semi-numerically integrating the equations (2.14). An entirely numerical integration will use some initial values $A^0_{i,s}$ and solve the difference equations of the form $A^{n+1}_{i,s} = A^n_{i,s} + \text{increment}_{i,s}$. Here the $A^n_{i,s}$ values are provided by the *physical system*.

The same basic queries were addressed but with the active feedback protocol. This allowed us to explore the influence of non-Markovian system-bath interactions on the dynamics. In particular where the bath was modified to be non-Markovian with a power law spectral density [E.7], it was found that the scaling exponents substantially changed from that of the Markovian case[9]. The freeze-out time now scaled as $t_{freeze-out} \propto \dot{\epsilon}^{-0.590}$ in comparison to the Markovian scaling of -0.501. This corresponds to a change in the νz exponent from 1.00 to 1.44. We note that the scaling exponent depends on the exponent of the power law spectral density s, with $t_{freeze-out}$ scaling as $\frac{-1}{1+s}$ and νz scaling as $\frac{1}{s}$. The value of s chosen in this study was 0.70.

Chapter 4

THEORETICAL ANALYSIS OF THE TWO-MODE SYSTEM

4.1 Basic Equations

We begin by recalling the basic equations arrived at in the last chapter. We make the replacement $\underline{A} \to A$ in (2.13, 2.14) and write,

$$\dot{A}_{i,s} = igA^{\dagger}_{s,i}A_P - \frac{1}{2}\int_0^{\infty} dt'\gamma_{i,s}(t')A_{i,s}(t-t') + if_{i,s} \ , \quad (4.1)$$

$$\dot{A}_P = igA_iA_s + iF_P - \frac{\gamma_P}{2}A_P \quad (4.2)$$

$$\langle f_{i,s}(t)f^{\dagger}_{i,s}(t')\rangle = \frac{\bar{n}_{i,s}+1}{2}\gamma_{i,s}(t-t') \ \text{ with, } \ \bar{n}_{i,s} := \left[e^{\hbar\omega_{i,s}/kT}-1\right]^{-1}. \quad (4.3)$$

These equations stipulate the dynamics of the slower time variations after the fast motion is separated out using the rotating wave approximation. We also assume that a semi-classical limit has been taken so that the variables $A_{i,s,P}$ denote complex numbers. The $A_{i,s,P}$ are dimensionless, the $\gamma_{i,s}$ have dimensions of T^{-2}, γ_P has dimensions T^{-1} and so do the $f_{i,s}, F_P$ and g.

We begin by looking for steady state solutions of the form: $A_{i,s}(t) := \hat{A}_{i,s}e^{-i\Delta_{i,s}t}$, $A_P(t) := \hat{A}_P$ with $\hat{A}_{i,s,P}$ being time independent and $\Delta_{i,s}$ being *real*. These are solutions in the absence of the stochastic forces.

Substitution gives,

$$-i\Delta_{i,s}\hat{A}_{i,s} = -\frac{1}{2}\hat{A}_{i,s}\gamma_{i,s}[\Delta_{i,s}] + igA^{*}_{s,i}A_Pe^{it(\Delta_i+\Delta_s)}, \quad (4.4)$$

$$0 = -\frac{\gamma_P}{2}A_P + ig\hat{A}_i\hat{A}_se^{-it(\Delta_i+\Delta_s)} + iF_P \quad (4.5)$$

where, the $\gamma_{i,s}[\Delta_{i,s}]$ denote the Fourier-Laplace transforms of the memory kernels, these being defined as: $\gamma[\omega] := \int_0^{\infty} dte^{i\omega t}\gamma(t)$. For real $\gamma(t)$, we have $\gamma^*[-\omega] = \gamma[\omega]$.

Clearly, for a steady state to be possible, we must have $\Delta_i = -\Delta_s =: \Delta \in \mathbb{R}$. Making this choice for identifying Δ, the equations become,

$$\hat{A}_i\Big(\frac{\gamma_i[\Delta]}{2} - i\Delta\Big) = ig\hat{A}_P\hat{A}^*_s \ ; \quad (4.6)$$

$$\hat{A}_s\Big(\frac{\gamma_s[-\Delta]}{2} + i\Delta\Big) = ig\hat{A}_P\hat{A}^*_i \ \leftrightarrow \ \hat{A}^*_s\Big(\frac{\gamma_s[\Delta]}{2} - i\Delta\Big) = -ig\hat{A}^*_P\hat{A}_i \ ; \quad (4.7)$$

$$\hat{A}_P = i\frac{2}{\gamma_P}(g\hat{A}_i\hat{A}_s + F_P). \quad (4.8)$$

The Fourier-Laplace transform combinations being complex in general, we write,

$$\frac{\gamma_{i,s}[\Delta]}{2} - i\Delta := \left|\frac{\gamma_{i,s}[\Delta]}{2} - i\Delta\right| e^{i\Theta_{i,s}(\Delta)}. \tag{4.9}$$

This equation defines the phases $\Theta_{i,s}$ as functions of Δ.

The reality of $\gamma_{i,s}(t)$ implies that $\gamma_{i,s}(\omega)^* = \gamma_{i,s}(-\omega)$. Using $|z|^* = |z^*|$ property of complex numbers, it follows that $\Theta_{i,s}(-\Delta) = -\Theta_{i,s}(\Delta)$.

To get a convenient form of the equations, consider scaled variables: $\hat{A}_{i,s,P} := \lambda_{i,s,P} B_{i,s,P}$, $F_P := \lambda\mu$, with all λ's being real, positive functions of Δ. Note that we have not put hats on the B's for notational convenience, but they too are time independent. The equations (4.6, 4.7, 4.8) become,

$$e^{i\Theta_i(\Delta)} B_i = i\left(\frac{g\lambda_P\lambda_s}{\lambda_i\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|}\right) B_P B_s^* \tag{4.10}$$

$$e^{i\Theta_s(\Delta)} B_s^* = -i\left(\frac{g\lambda_P\lambda_i}{\lambda_s\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|}\right) B_P^* B_i \tag{4.11}$$

$$B_P = i\left(\frac{2}{\lambda_P\gamma_P} g\lambda_i\lambda_s\right) B_i B_s + i\left(\frac{2}{\lambda_P\gamma_P}\lambda\right)\mu. \tag{4.12}$$

Setting all the brackets equal to 1, determines the λ's as,

$$\lambda_i = \frac{1}{g}\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|} \ , \ \lambda_s = \frac{1}{g}\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|}, \tag{4.13}$$

$$\lambda_P = \frac{2g}{\gamma_P}\lambda_i\lambda_s = \frac{1}{g}\sqrt{\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|} \ , \ \lambda = \frac{\gamma_P\lambda_P}{2}. \tag{4.14}$$

All $\lambda_{i,s,P}$'s are dimensionless and so are the scaled amplitudes B's. λ has dimensions of T^{-1} so that the scaled drive strength μ is dimensionless. The equations for the steady states B's, take a much simpler form,

$$B_i e^{i\Theta_i(\Delta)} = iB_P B_s^* \ , \ B_s^* e^{i\Theta_s(\Delta)} = -iB_P^* B_i \ , \ B_P = i(B_i B_s + \mu). \tag{4.15}$$

All the details of the memory kernels reside in the functions $\Theta_{i,s}(\Delta)$ and drive strength is parameterized by the dimensionless μ.

Analysis of the steady state equations

Trivial solutions: $B_i = B_s = 0, B_p = i\mu$ is an obvious solution for all μ. Below we focus on the non-trivial steady state solutions with non-zero mode amplitudes unless explicitly mentioned otherwise.

Non-trivial solutions: Taking absolute values, gives $|B_P| = 1$, $|B_i| = |B_s| =:$ $|B| \neq 0$. The $|B|$ depends on μ.

Introducing the phases as: $B_{i,s} := |B|e^{i\varphi_{i,s}}, B_P := e^{i\varphi_P}$ leads to the equations,

$$e^{i(\Theta_i+\varphi_i)} = ie^{i(\varphi_P-\varphi_s)} \leftrightarrow e^{i(\varphi_i+\varphi_s)} = ie^{i(\varphi_P-\Theta_i)} \tag{4.16}$$

$$e^{i(\Theta_s-\varphi_s)} = -ie^{i(-\varphi_P+\varphi_i)} \leftrightarrow ie^{i(\varphi_P+\Theta_s)} = e^{i(\varphi_i+\varphi_s)} \Rightarrow \tag{4.17}$$

$$e^{i(\varphi_P-\Theta_i)} = e^{i(\varphi_P+\Theta_s)} \Rightarrow \Theta_i(\Delta) = -\Theta_s(\Delta) =: \Theta(\Delta) . \tag{4.18}$$

Note that $\Theta(\Delta)$ is an odd function of Δ since $\Theta_{i,s}(\Delta)$ are.

The condition that $\Theta_i(\Delta) = -\Theta_s(\Delta)$ is a consistency condition on the two memory kernels to allow steady states with $\Delta \neq 0$ and at the same time it limits the possible values of Δ. This is made manifest by writing the condition in the equivalent form:

$$\left\{\frac{\gamma_i[\Delta]}{2} - i\Delta\right\}\left\{\frac{\gamma_s[\Delta]}{2} - i\Delta\right\} = \left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right| . \tag{4.19}$$

The condition involves *only* the memory kernels and is needed for existence of non-trivial steady states. Only real Δ solutions are relevant.

Since $\gamma_{i,s}[0] > 0$, $\Delta = 0$ is always a solution. Thus, *static states* are admissible for all memory kernels. Furthermore, for Markovian kernels, the condition is satisfied *only* for $\Delta = 0$. Reality properties of the kernels imply that if $\Delta \neq 0$ is a solution, so is $-\Delta$.

If the two memory kernels are the same, $\gamma_i(t) = \gamma_s(t) =: \gamma(t)$, then $\{\gamma[\Delta] - 2i\Delta\}$ is necessarily real and therefore $\Theta(\Delta) = 0$. This of course allows $\Delta \neq 0$. The cases treated in [7, 9] illustrate this.

An alternative form of the consistency condition useful in computations is

$$Im(\gamma_i[\Delta]) - 2\Delta = |\gamma_i[\Delta] - 2i\Delta| sin\Theta(\Delta). \tag{4.20}$$

Returning to the three equations (4.15), we see that they are reduced to two equations as,

$$e^{i\varphi_P} = i\Big(|B|^2 e^{i(\varphi_i+\varphi_s)} + \mu\Big) \text{ and } e^{i(\varphi_i+\varphi_s)} = e^{i(\varphi_P+\pi/2-\Theta} . \tag{4.21}$$

The second equation is one real equation while the first one is one complex equation and we have the four unknowns $|B|, (\varphi_i \pm \varphi_s), \varphi_P$ given the memory kernel phase $\Theta(\Delta)$ and the value of Δ. We may already notice that the combination $\varphi_i - \varphi_s$ does *not* appear in the equations and thus remains undetermined. The first of the above equation (4.21) may be written as,

$$\begin{aligned} |B|^2 &= \Big(e^{i(\varphi_P - \pi/2)} - \mu\Big) e^{-i(\varphi_i + \varphi_s)} = \Big(e^{i(\varphi_P - \pi/2)} - \mu\Big) e^{-i(\varphi_P + \pi/2 - \Theta)} \\ &= e^{-i\pi + i\Theta} - \mu e^{-i(\varphi_P + \pi/2 - \Theta)} \quad \text{or} \\ |B|^2 &= \Big\{ - cos\Theta + \mu sin(\varphi_P - \Theta)\Big\} + i\Big\{ - sin\Theta + \mu cos(\varphi_P - \Theta)\Big\} \end{aligned} \tag{4.22}$$

Equating real and imaginary part and eliminating $(\varphi_P - \Theta)$ we get the quadratic equation for $|B|^2$,

$$|B|^4 + 2cos\Theta |B|^2 + 1 - \mu^2 = 0. \tag{4.23}$$

Its solutions are:

$$|B|^2_{\pm} = -cos\Theta \pm \sqrt{\mu^2 - sin^2\Theta}\ ,\ \mu^2 \geq sin^2\Theta. \tag{4.24}$$

We also have,

$$sin(\varphi_P - \Theta) = \frac{|B|^2 + cos(\Theta)}{\mu}\ \ ,\ \ cos(\varphi_P - \Theta) = \frac{sin\Theta}{\mu}. \tag{4.25}$$

Taking the ratio gives,

$$tan(\varphi_P - \Theta) = \pm\sqrt{\frac{\mu^2}{sin^2\Theta} - 1}\ \ (\varphi_p - \Theta) \text{ is determined }. \tag{4.26}$$

The $\pm$ is correlated with the choice of $|B|^2_{\pm}$. Vanishing of $sin(\Theta)$ only means that $(\varphi_P - \Theta) \to \pm\pi/2$.

Finally, we also get,

$$\varphi_i + \varphi_s = \varphi_P - \Theta + \frac{\pi}{2}\ . \tag{4.27}$$

The non-trivial steady states being independent of $\varphi_i - \varphi_s$, means that $B_i \to B_i e^{i(\varphi_i + \theta)}, B_s \to B_s e^{i(\varphi_s - i\theta)}$ leave the set of steady states invariant. This is a $U(1)$ symmetry of these steady states. It holds quite generally for all kernels $\gamma_{i,s}$, independently of the value of Δ.

For contrast, the set of trivial steady states, is invariant under a general complex linear transformation $[B_i, B_s]^T \to [B_i, B_s]^T M^T$ where M is a non-singular complex matrix. This symmetry is much larger than the $U(1)$ symmetry of the non-trivial steady states.

When $\Delta \neq 0$, the equations determining $|B|^2_{\pm}$ and $(\varphi_P - \Theta)$ are manifestly invariant under $\Delta \leftrightarrow -\Delta$ and hence so is the sum $\varphi_i + \varphi_s$. Hence the set of non-trivial steady states is also invariant under the $\mathbb{Z}_2$ operation, for all memory kernels.

The system has *two order parameters*, $|B|$ and Δ. They demarcate the phases as:

(i) $|B| = 0$, maximal symmetry, *disordered phase*;

(ii) $|B| \neq 0, \Delta = 0$, symmetry broken to $U(1)$, $U(1)$ *phase*; and

(iii) $|B| \neq 0$ and $\Delta \neq 0$, symmetry is $U(1) \times \mathbb{Z}_2$, $U(1) \times \mathbb{Z}_2$ *phase*.

This is useful for the phase diagram considerations.

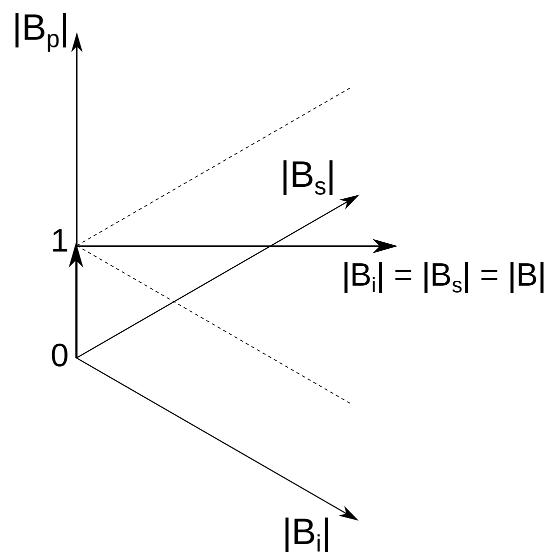

Figure 4.1: The thick lines denote the steady states. The arrows indicate the increasing value of μ.

The non-trivial steady states have a $|B|^2_{\pm} = -cos\Theta \pm \sqrt{\mu^2 - sin^2\Theta}$. It follows that,

1. $cos\Theta = 0$:

   $\mu > 1$ and there is only one admissible root $|B|^2_+ = \sqrt{\mu^2 - 1}$.

2. $cos\Theta > 0$:

   Sum of the roots of the quadratic equation (4.23) is negative. Hence at least one root must be negative. To ensure that the other root is positive, we must have $\mu^2 > 1$ (strictly) and $|B|^2_+ = -cos\Theta + \sqrt{\mu^2 - sin^2\Theta}$ as before.

3. $cos\Theta < 0$:

   Now the sum of the roots of the quadratic is positive. We can have either both the roots are positive or the second root is negative. This allows $\mu^2 > 1$ (one root negative) as well as $sin^2(\Theta) \leq \mu^2 < 1$. In the former case, we must have $|B|^2 = -cos(\Theta) + \sqrt{\mu^2 - sin^2(\Theta)}$ as before while for the latter, both the roots $|B|^2_\pm$ are positive. This means that we can have the $U(1)$ and possibly $U(1) \times \mathbb{Z}_2$ phases even for $\mu^2 < 1$. The two roots as a function of μ is plotted in the figure 4.2.

   Note: Apparently, this case is ruled out since the real part of the Fourier transform of a memory kernel is expected to be positive from the proof of the fluctuation-dissipation theorem discussed in Appendix (A). This needs to be examined experimentally and theoretically. Presently we note it as an open question.

   For the subsequent discussions, we restrict to $cos(\Theta) \geq 0$ cases only.

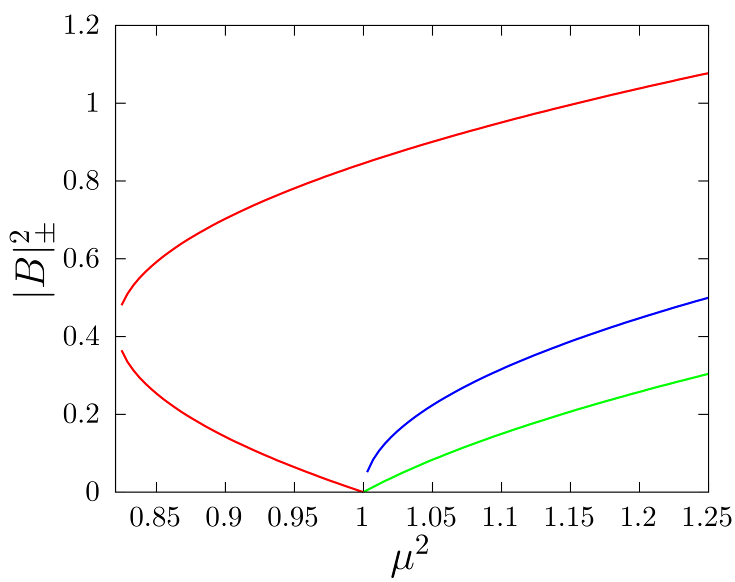

Figure 4.2: The roots $|B|^2_\pm$ are plotted as a function of μ^2. The green curve is for $\Theta = 75°$, the blue for $\Theta = 90°$ and the red curves are for $\Theta = 115°$. The lower red curve denotes the $|B|^2_-$ root. All other curves denote the $|B|^2_+$ root.

4. *In summary:*

   - $|B|^2_+ = -\cos\Theta + \sqrt{\mu^2 - sin^2\Theta}$ gives non-trivial steady states for (i) all $\mu^2 > 1$ (for $cos\Theta \geq 0$). This also means that in (4.26), only the + sign is selected.
   - $|B|^2_- = -\cos\Theta - \sqrt{\mu^2 - sin^2\Theta}$ also gives non-trivial steady states for all $sin^2\Theta < \mu^2 < 1$ for $cos\Theta < 0$;
   - The possibility of the three identifiable phases under the stated conditions, holds generically.
   - The case of unequal, non-Markovian memory kernels has not been studied experimentally. This is precisely the case where $cos(\Theta(\Delta)) \neq 1$. We return to an elaboration of this case in chapter 5.

Examples are discussed in Appendix E.

4.2 Linearization about steady states

Having obtained the steady states, the natural question is their stability properties under small deviations. This is analyzed by doing linearization about steady states. We derive the equations governing the evolution of the small perturbations from any of the steady states.

Let the amplitudes be of the form,

$$A_{i,s} = (\hat{A}_{i,s} + x_{i,s}(t))e^{-i\Delta_{i,s}t} \quad , \quad A_P = \hat{A}_P + x_P(t) \tag{4.28}$$

and also include the stochastic forces $f_{i,s}$. The perturbations $x_{i,s,P}(t)$ are small. Substituting in the basic equations (4.1) and multiplying by $e^{i\Delta_{i,s}t}$ leads to equations,

$$\dot{x}_P = -\frac{\gamma_P}{2}x_P + ig(\hat{A}_i x_s + \hat{A}_s x_i); \tag{4.29}$$

$$\dot{x}_{i,s} = i\Delta_{i,s}x_{i,s} - \frac{1}{2}\int_0^\infty dt'\gamma_{i,s}(t')x_{i,s}(t-t')e^{i\Delta_{i,s}t'} + ig(\hat{A}_P x^*_{s,i} + \hat{A}^*_{s,i}x_P) + if_{i,s}e^{i\Delta_{i,s}t}. \tag{4.30}$$

We have also used the steady state equations (4.6, 4.7, 4.8) and kept only the terms linear in the perturbations.

Note: We may note that when $\gamma_i(t) = \gamma_s(t)$, we may use the variables $x_\pm := x_i \pm x_s$ which leads to decoupled equations as done in [7]. For the general case however, such a decoupling does not occur.

Taking the Fourier transform of the equation, $\int_{-\infty}^{\infty} dt e^{i\omega t}$ (equations), leads to (Fourier transform implied by the argument),

$$
\begin{aligned}
-i\omega x_P(\omega) &= -\frac{\gamma_P}{2} x_P(\omega) + ig(\hat{A}_i x_s(\omega) + \hat{A}_s x_i(\omega)); \quad (4.31)\\
-i\omega x_{i,s}(\omega) &= i\Delta_{i,s} x_{i,s}(\omega) - \frac{1}{2}\gamma_{i,s}[\omega + \Delta_{i,s}] x_{i,s}(\omega)\\
&\quad + ig\Big(\hat{A}_P x^*_{s,i}(\omega) + \hat{A}^*_{s,i} x_P(\omega)\Big) + i f_{i,s}(\omega + \Delta_{i,s}). \quad (4.32)
\end{aligned}
$$

Note that in the above, $x^*_{s,i}(\omega)$ is the Fourier transform of $x^*_{s,i}(t)$ and *not* the complex conjugate of the Fourier transform of $x_{s,i}(t), [x_{s,i}(\omega)]^*$. Rearranging terms,

$$
\begin{aligned}
\Big(\frac{\gamma_P}{2} - i\omega\Big) x_P(\omega) &= ig\Big(\hat{A}_i x_s(\omega) + \hat{A}_s x_i(\omega)\Big); \quad (4.33)\\
\Big(\frac{\gamma_{i,s}[\omega + \Delta_{i,s}]}{2} - i(\omega + \Delta_{i,s})\Big) x_{i,s}(\omega) &= +ig\Big(\hat{A}_P x^*_{s,i}(\omega) + \hat{A}^*_{s,i} x_P(\omega)\Big)\\
&\quad + i f_{i,s}(\omega + \Delta_{i,s}). \quad (4.34)
\end{aligned}
$$

Here all $\hat{A}_{i,s,P}, \Delta$ are fixed corresponding to a particular steady state. We can use the same scaling done while going from the $\hat{A}$'s to B's using the $\lambda_{i,s,P}$ given in equations (4.13, 4.14). Thus define,

$$
\begin{aligned}
x_P &:= \lambda_P y_P = y_P \frac{1}{g}\sqrt{\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|} \quad (4.35)\\
x_i &:= \lambda_i y_i = y_i \frac{1}{g}\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|}, \quad (4.36)\\
x_s &:= \lambda_s y_s = y_s \frac{1}{g}\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|}. \quad (4.37)
\end{aligned}
$$

We also scale the force terms so as to normalize the coefficients of the ig terms to i i. e.,

$$
\begin{aligned}
f_i(\omega + \Delta) &:= g\lambda_P \lambda_s g_i(\omega + \Delta)\\
&= g_i(\omega + \Delta)\frac{1}{g}\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|} \quad (4.38)\\
f_s(\omega - \Delta) &:= g\lambda_P \lambda_i g_s(\omega - \Delta)\\
&= g_s(\omega - \Delta)\frac{1}{g}\left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|\sqrt{\frac{\gamma_P}{2}\left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right|}. \quad (4.39)
\end{aligned}
$$

In terms of the scaled variables the equations become,

$$\frac{2}{\gamma_P}\Big(\frac{\gamma_P}{2}-i\omega\Big)y_P(\omega) \;=\; i\Big[B_i y_s(\omega)+B_s y_i(\omega)\Big]; \tag{4.40}$$

$$\frac{\Big(\frac{\gamma_i[\omega+\Delta]}{2}-i(\omega+\Delta)\Big)}{\Big|\frac{\gamma_i[\Delta]}{2}-i\Delta\Big|}y_i(\omega) \;=\; i\Big(B_P y_s^*(\omega)+B_s^* y_P(\omega)+g_i(\omega+\Delta)\Big); \tag{4.41}$$

$$\frac{\Big(\frac{\gamma_s[\omega-\Delta]}{2}-i(\omega-\Delta)\Big)}{\Big|\frac{\gamma_s[\Delta]}{2}-i\Delta\Big|}y_s(\omega) \;=\; i\Big(B_P y_i^*(\omega)+B_i^* y_P(\omega)+g_s(\omega-\Delta)\Big). \tag{4.42}$$

Introduce the abbreviations:

$$\frac{1}{\Gamma_P(\omega,\Delta)} := \frac{2}{\gamma_P}\Big(\frac{\gamma_P}{2}-i\omega\Big)\ ,\ \frac{1}{\Gamma_{i,s}(\omega,\Delta)} := \frac{\Big(\frac{\gamma_{i,s}[\omega+\Delta_{i,s}]}{2}-i(\omega+\Delta_{i,s})\Big)}{\Big|\frac{\gamma_{i,s}[\Delta]}{2}-i\Delta\Big|}. \tag{4.43}$$

Reality of the memory kernels imply that $\gamma_{i,s}(-\omega)^* = \gamma_{i,s}(\omega)$. This in turn implies that,

$$(\Gamma_P(-\omega,\Delta))^* \;=\; \Gamma_P(\omega,-\Delta), \tag{4.44}$$

$$\begin{aligned}(\Gamma_{i,s}(-\omega,\Delta))^* &= \frac{\Big|\frac{\gamma_{i,s}[\Delta]}{2}-i\Delta\Big|^*}{\Big(\frac{\gamma_{i,s}[-\omega+\Delta_{i,s}]}{2}-i(-\omega+\Delta_{i,s})\Big)^*}\\ &= \frac{\Big|\Big(\frac{\gamma_{i,s}[\Delta]}{2}-i\Delta\Big)^*\Big|}{\Big(\frac{\gamma_{i,s}[\omega-\Delta_{i,s}]}{2}+i(-\omega+\Delta_{i,s})\Big)}\\ &= \frac{\Big|\frac{\gamma_{i,s}[-\Delta]}{2}-i(-\Delta)\Big|}{\Big(\frac{\gamma_{i,s}[\omega+(-\Delta_{i,s})]}{2}-i(\omega+(-\Delta_{i,s})\Big)} \;=\; \Gamma_{i,s}(\omega,-\Delta)\ .\end{aligned} \tag{4.45}$$

This will be useful shortly.

Equation (4.40) can be solved for y_P, giving

$$y_P \;=\; i\Gamma_P\Big(B_i y_s + B_s y_i\Big). \tag{4.46}$$

In the equations (4.41,4.42), eliminate y_P and also use, $B_P = i(B_i B_s+\mu)$. This eliminates the pump labeled variables (except Γ_P) completely and we get ($\varphi_\pm := \varphi_i \pm \varphi_s$),

$$\begin{aligned}\Big(1+\Gamma_i\Gamma_P|B|^2\Big)y_i + \Big(\Gamma_i\Gamma_P|B|^2e^{i\varphi_-}\Big)y_s + \Gamma_i\Big(|B|^2e^{i\varphi_+}+\mu\Big)y_s^*&\\ = i\Gamma_i g_i(\omega+\Delta)&\end{aligned} \tag{4.47}$$

$$\begin{aligned}\Big(1+\Gamma_s\Gamma_P|B|^2\Big)y_s + \Big(\Gamma_s\Gamma_P|B|^2e^{-i\varphi_-}\Big)y_i + \Gamma_s\Big(|B|^2e^{i\varphi_+}+\mu\Big)y_i^*&\\ = i\Gamma_s g_s(\omega-\Delta).&\end{aligned} \tag{4.48}$$

It is convenient to put these equations in a matrix form as:

$$\underbrace{\begin{bmatrix} 1+\Gamma_i\Gamma_P|B|^2 & \Gamma_i\Gamma_P|B|^2 e^{i\varphi_-} \\ \Gamma_s\Gamma_P|B|^2 e^{-i\varphi_-} & 1+\Gamma_s\Gamma_P|B|^2 \end{bmatrix}}_{\mathbb{M}} \begin{bmatrix} y_i \\ y_s \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & \Gamma_i\big(|B|^2 e^{i\varphi_+}+\mu\big) \\ \Gamma_s\big(|B|^2 e^{i\varphi_+}+\mu\big) & 0 \end{bmatrix}}_{\mathbb{M}'} \begin{bmatrix} y_i^* \\ y_s^* \end{bmatrix} = \begin{bmatrix} i\Gamma_i g_i \\ i\Gamma_s g_s \end{bmatrix}. \quad (4.49)$$

In the above, the argument of y, y^* *is* ω, *those of the* $\Gamma_{P,i,s}$ *are* (ω, Δ), *of* $g_{i,s}$ *is* $(\omega + \Delta_{i,s})$. These are suppressed to avoid clutter.

The determinant of $\mathbb{M}$ is

$$\det(\mathbb{M}) = 1 + \Gamma_P|B|^2(\Gamma_i + \Gamma_s) \quad (4.50)$$

which is non-zero *generically*, and its inverse is obtained immediately. Multiplying the equation (4.49) by $\mathbb{M}^{-1}$, we write,

$$\underbrace{\begin{bmatrix} y_i \\ y_s \end{bmatrix}}_{\mathbb{Y}} + \underbrace{\mathbb{M}^{-1}\mathbb{M}'}_{\mathbb{N}} \begin{bmatrix} y_i^* \\ y_s^* \end{bmatrix} = \underbrace{\mathbb{M}^{-1}\begin{bmatrix} i\Gamma_i g_i \\ i\Gamma_s g_s \end{bmatrix}}_{\mathbb{z}}. \quad (4.51)$$

We write the column matrices as $\mathbb{Y}, \mathbb{Y}^*$, the product matrix in the second term as $\mathbb{N}$ and the column matrix on the r.h.s. as $\mathbb{z}$ and take the matrix equation as, $\mathbb{Y}(\omega) + \mathbb{N}(\omega)\mathbb{Y}^*(\omega) = \mathbb{z}(\omega)$. That $\mathbb{Y}^*(\omega) \neq [\mathbb{Y}(\omega)]^*$, separating the real and the imaginary parts is not very useful. We observe that for any *complex* valued function $f(t)$,

$$(f^*)[\omega] := \int_{-\infty}^{\infty} dt e^{i\omega t} f^*(t) = \Big(\int_{-\infty}^{\infty} dt e^{-i\omega t} f(t)\Big)^* = \Big(f[-\omega]\Big)^*. \quad (4.52)$$

For *real* valued functions of course we have $f(\omega)^* = f(-\omega)$. The matrix equation can thus be written as,

$$\begin{aligned}
\mathbb{Y}(\omega) + \mathbb{N}(\omega)\mathbb{Y}(-\omega)^* = \mathbb{z}(\omega) \quad & \xrightarrow[\omega\to-\omega]{} \quad \mathbb{Y}(-\omega) + \mathbb{N}(-\omega)\mathbb{Y}(\omega)^* = \mathbb{z}(-\omega) \\
& \xrightarrow[\text{cc}]{} \quad \mathbb{Y}(-\omega)^* + \mathbb{N}(-\omega)^*\mathbb{Y}(\omega) = \mathbb{z}(-\omega)^* \\
\therefore \mathbb{N}(\omega)\mathbb{Y}(-\omega)^* \quad & = \quad -\mathbb{N}(\omega)\mathbb{N}(-\omega)^*\mathbb{Y}(\omega) + \mathbb{N}(\omega)\mathbb{z}(-\omega)^* \\
\therefore \underbrace{\Big(\mathbb{1} - \mathbb{N}(\omega)\mathbb{N}(-\omega)^*\Big)}_{\mathbb{A}(\omega)}\mathbb{Y}(\omega) \quad & = \quad \underbrace{\mathbb{z}(\omega) - \mathbb{N}(\omega)\mathbb{z}(-\omega)^*}_{\mathbb{Z}(\omega)}. \quad (4.53)
\end{aligned}$$

This is now a simple 2×2 complex matrix equation, $\mathbb{A}(\omega)\mathbb{Y}(\omega) = \mathbb{Z}(\omega)$ which is easy to manage. As discussed in Appendix B, we can always invoke the *singular* *value decomposition* theorem and write $\mathbb{A}(\omega) = \mathbb{U}_1(\omega)\Sigma(\omega)\mathbb{U}_2^\dagger(\omega)$ where Σ is a diagonal matrix with *non-negative elements* and $\mathbb{U}_{1,2}$ are unitary matrices. The matrices $\mathbb{U}_2,\ \Sigma$ are completely determined by the eigenvalues and eigenvectors of $\mathbb{A}^\dagger\mathbb{A}$ while $\mathbb{U}_1$ involves $\mathbb{A}$ explicitly. Closed form expressions for these, in terms of the matrix elements of $\mathbb{A}, \mathbb{A}^\dagger\mathbb{A}$ are also given in that appendix.

Using this, we can define $\mathbb{Y}' := \mathbb{U}_2^\dagger\mathbb{Y},\ \mathbb{Z}' := \mathbb{U}_1^\dagger\mathbb{Z}$ so that our fundamental equation for perturbations (4.53) takes the simple diagonalized form,

$$\Sigma(\omega)\mathbb{Y}'(\omega) = \mathbb{Z}'(\omega) \ \longleftrightarrow\ \sigma_\pm(\omega)y'_\pm(\omega) = z'_\pm(\omega). \tag{4.54}$$

We have denoted the diagonal matrix elements by subscripts $\pm$. The linearization about the trivial states is obtained by setting $B_i = 0 = B_s$. *We will use either of the equations (4.53, 4.54) as convenient.*

Our original equations for the perturbations $x_{i,s,P}(t)$ are really infinitely many equations since $x_{i,s,P}(t)$ couple to $x_{i,s,P}(t')$ through the memory kernels. The Fourier transforms have made this explicit since we have 2×2 matrix equation for $\mathbb{Y}(\omega)$, for *each* ω. The evolution of the perturbations is then given by the inverse Fourier transform of the $\mathbb{Y}(\omega)$.

Our linearized equation has stochastic forces, which are Gaussian distributed with zero mean. Thus taking an *ensemble average* of the equation, gives a *homogeneous* equation for the *average* $\langle\mathbb{Y}(\omega)\rangle$. Its solutions thus describe the evolution of the averages of perturbations. Thanks to dissipation, this evolution should be *transient* and averaged perturbations should vanish for a stable steady state. To see the influence of the stochastic forces, we need to look at the *fluctuations*, $\langle\mathbb{Y}(t)\mathbb{Y}^\dagger(t)\rangle$. This brings in the *inhomogeneous* equation.

In the next two sections, we consider the solutions of the homogeneous equation and the inhomogeneous equations separately as they refer to different *quantities*.

4.3 The homogeneous solutions

The homogeneous solution represents the spontaneous dynamical evolution of the system when disturbed from a steady state. To avoid clutter, we will suppress the ensemble average notation. In either view, the perturbations will die out if the steady state is stable. In general, this return can be arbitrarily complex and this is determined by the vanishing of the $\det(\mathbb{A}(\omega)) = \sigma_+(\omega)\cdot\sigma_-(\omega)$.

For example, if $\det(\mathbb{A}(\omega))$ has only isolated, simple zeros at say ω_k, then corresponding $\mathbb{Y}_k(\omega) = \hat{\mathbb{Y}}_k \delta(\omega - \omega_k)$ and its inverse Fourier transform gives $\mathbb{Y}(t) \sim \sum_k e^{-i\omega_k t} \hat{\mathbb{Y}}_k$. For stability, each of the ω_k must have negative imaginary part and the smallest imaginary part gives the inverse of the relaxation time. For more complicated zeros of the determinant, such a simple picture of relaxation time may not be available.

The vanishing determinant condition has different form for perturbations about trivial and non-trivial states.

Determinant for perturbations of trivial steady states

This one parameter family of steady states is defined by $B_i = B_s = 0$, $B_P = i\mu$, $\mu \in \mathbb{R}$. For these values, the matrices are: $\mathbb{M} = \mathbb{1}$ and $\mathbb{M}' = \begin{bmatrix} 0 & \mu\Gamma_i \\ \mu\Gamma_s & 0 \end{bmatrix}$. Thus $\mathbb{N} = \mathbb{M}'$ and the matrix $\mathbb{A}(\omega)$ is now exactly diagonal,

$$\mathbb{A}_-(\omega) = diag\left[1 - \mu^2\Gamma_i(\omega)\Gamma_s(-\omega)^*, 1 - \mu^2\Gamma_s(\omega)\Gamma_i(-\omega)^*\right] \ . \tag{4.55}$$

We have introduced the subscript $-$ to indicate the trivial steady states.

The determinant vanishes if either or both of the diagonal elements vanish. Recalling that $\Gamma[-\omega, \Delta]^* = \Gamma[\omega, -\Delta]$, we see that the second diagonal element is obtained from the first one by $\Delta \to -\Delta$. Combining the two, the *vanishing determinant condition* may thus be stated as,

$$\left\{\frac{\gamma_i[\omega \pm \Delta]}{2} - i(\omega \pm \Delta)\right\}\left\{\frac{\gamma_s[\omega \pm \Delta]}{2} - i(\omega \pm \Delta)\right\} = \mu^2 \left|\frac{\gamma_i[\Delta]}{2} - i\Delta\right| \left|\frac{\gamma_s[\Delta]}{2} - i\Delta\right|. \tag{4.56}$$

Equation with either/both signs imply vanishing of the determinant. We have also used: $|\frac{\gamma[-\Delta]}{2} + i\Delta| = |\frac{\gamma[\Delta]}{2} - i\Delta|$.

Note:

For $\omega = 0$, the equations reduce to $e^{i(\Theta_i(\Delta)+\Theta_s(\Delta))} = \mu^2$ and/or $e^{-i(\Theta_i(\Delta)+\Theta_s[\Delta])} = \mu^2$ (since $\Theta_{i,s}(-\Delta) = -\Theta_{i,s}(\Delta)$). The exponents add up to zero by the condition (4.19, 4.9). Hence either of the equations can be satisfied *only for* $\mu = 1$. Thus for all $\mu \neq 1$, $\omega = 0$ is *not* a zero of the determinant for perturbations about these trivial steady states.

While we have a closed form equation determining the zeros, the equations are quite opaque without further stipulations on the Fourier transforms of the memory kernels. The equation is illustrated for the experimentally studied three different memory kernels, in Appendix E.

As we will see from all the three examples, the trivial steady states are stable and an instability is indicated as $\mu \to 1_-$. This happens with one of the roots going to zero. In the first two examples, the determinant is a quadratic function of ω and can be expressed as $\sim (\omega - \omega_+)(\omega - \omega_-)$. As one of the roots becomes zero as $\mu \to 1_-$, the determinant has a simple zero at $\omega = 0$. In the last example of power law kernel, for $\zeta := i\omega$, we have in equation (E.10) that the determinant $\sim [2\zeta - \epsilon\gamma_0 - \gamma_{pl}\zeta^s\omega_0^{-s}]$ and again the instability is indicated by $\zeta \to 0$. However, for $s < 1$, the determinant vanishes as a *fractional power* $(1-\mu)^{1/s}$ and thus has a branch cut.

Do the two features, namely, (i) that an instability sets in as $\mu \to 1_-$ and (ii) that it is signaled by one of the roots of the determinant condition approaching zero, hold for all memory kernels? The examples already indicate that the nature of zeros of the determinant is *not generic*.

The (i) is intuitively clear looking at the structure of the steady states shown in the figure (4.1). The second feature is intimately related to the emergence of the $U(1)$ symmetry.

Determinant for perturbations of non-trivial steady states

The most noteworthy feature here is that the determinant vanishes at $\omega = 0$ for perturbations about all non-trivial steady states. The reason is easy to see.

Recall that for given memory kernels, Θ, Δ and $\mu > 1$, we have a 1-parameter family of non-trivial steady states, labeled by φ_-. A perturbation which only changes this, constitutes a zero mode $\hat{\mathbb{Y}}$. There is exactly one *zero mode* which is given explicitly as: $\hat{y}_i = iB_i\delta\varphi_i,\ \hat{y}_s = iB_s\delta\varphi_s, \delta\varphi_i + \delta\varphi_s = 0$. The zero mode satisfies $\mathbb{A}(\omega = 0)\hat{\mathbb{Y}} = 0$. This is directly related to the $U(1)$ symmetry breaking and holds *for all* $\mu > 1$. An explicit demonstration is given in Appendix C.

While there could be other zeros of the determinant, including those with zero imaginary part, we focus on the $\omega = 0$ requiring $\mu \geq 1$. The general expression for the determinant of $\mathbb{A}(\omega)$ is given in Appendix D. The behavior of the determinant for $\mu = 1 \mp \epsilon$ is given in equations (D.23, D.24).

The general equations are illustrated for the three experimentally studied memory kernels in Appendix E.

4.4 Fluctuations near the transition point, $\mu \to 1_-$

Once stochastic forces are included, the equation becomes inhomogeneous. The relevant quantities to compute now are the fluctuations. It is convenient to define two *causal Green functions*, $G_\pm(t)$ as,

$$G_\pm(t) := \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t}}{\sigma_\pm(\omega + i\epsilon)} \quad \Rightarrow \quad y'_\pm(t) = \int_{-\infty}^{t} dt' G_\pm(t - t') z'_\pm(t') \ . \quad (4.57)$$

The causality condition that $G_\pm(t)$ vanish for $t < 0$, requires that the contour be closed in the upper half plane and the integrand to be analytic there. To avoid the possible poles on the real-ω axis, we use $\omega \to \omega + i\epsilon$ in the $\sigma_\pm(\omega)$. The causality condition thus requires that $\sigma_\pm^{-1}(\omega)$ is analytic in the upper half plane of complex ω. The $y'_\pm(t), z'_\pm(t)$ denote the inverse Fourier transforms of $y'_\pm(\omega), z'_\pm(\omega)$, respectively.

We get the fluctuations as,

$$\langle y'_I(t) y'^*_J(t) \rangle = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2 G_I(t - t_1) G^*_J(t - t_2) \langle z'_I(t_1) z'^*_J(t_2) \rangle \quad (4.58)$$

$$\langle z'_I(t_1) z'^*_J(t_2) \rangle := \int_{-\infty}^{\infty} \frac{d\omega_1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega_2}{2\pi} e^{-i\omega t_1 - i\omega_2 t_2} \langle z'_I(\omega_1) z'^*_J(\omega_2) \rangle \quad (4.59)$$

$$\langle y'_I(\omega) y'^*_J(\omega') \rangle = \frac{\langle z'_I(\omega) z'^*_J(\omega') \rangle}{\sigma_I(\omega + i\epsilon) \sigma^*_J(\omega' + i\epsilon)} \quad (4.60)$$

where, $I, J = \{+, -\}$ and the correlation matrix of the z' is not necessarily *diagonal* thanks to the linear combinations involved. As seen below, the off diagonal elements of the correlation matrix of the z' are *not* time translation invariant! Here, $z'^*(\omega')$ denotes the Fourier transform of the $[z'(t)]^*$.

In the case discussed below, we will see that the $\langle z'_I(\omega_1) z'^*_I(\omega_2) \rangle \propto \delta(\omega_1 + \omega_2)$ while $\langle z'_I(\omega_1) z'^*_{J \neq J}(\omega_2) \rangle \propto \delta(\omega_1 + \omega_2 \pm 2\Delta)$. The inverse Fourier transform of the diagonal correlators thus becomes a function of $(t_1 - t_2)$ while the off diagonal correlators will have additional phase factors $e^{\pm 2i\Delta \cdot t_2}$, violating stationarity. Noting that the correlation matrix is Hermitian, the observables are its determinant and trace. In both of these observables, the $\Delta \cdot t_2$ dependence cancels and stationarity is respected.

Thus, our strategy is to compute the matrix $\langle y'_I(\omega) y'^*_J(\omega') \rangle$, take the inverse Fourier transform with respect to ω, ω' and then use the trace and determinant as stationary observables.

Note: In the above, $G^*(\omega)$ denotes the Fourier transform of $G(t)^*$ and hence $G^*(\omega) = [G(-\omega)]^*$, $z'^*(\omega) = [z'(-\omega)]^*$ and $\sigma^*_I(\omega) = \sigma_I(-\omega)^*$.

With the expression for the $\mathbb{Z}(\omega)$ matrix in (D.28), we have all the ingredients to compute the fluctuations. We present these only for the algebraically simpler case of $\mu = 1_-$.

In this case, we have (D.31),

$$\mathbb{Z}(\omega) = i \begin{bmatrix} \Gamma_i g_i + \mu \Gamma_i \Gamma_s^* g_s^* \\ \Gamma_s g_s + \mu \Gamma_i^* \Gamma_s g_i^* \end{bmatrix} =: \; i \begin{bmatrix} Z_+ \\ Z_- \end{bmatrix} . \tag{4.61}$$

This is also the case where the matrix $\mathbb{A}(\omega)$ is already diagonal and thus its singular value decomposition is not necessary. Effectively, we may take $\mathbb{U}_{1,2} = \mathbb{1}$ and $\sigma_\pm$ are just the diagonal elements: $\big(1 - \mu^2 \Gamma_i(\omega, \Delta)\Gamma_s(\omega, -\Delta)\big)$, $\big(1 - \mu^2 \Gamma_i(\omega, -\Delta)\Gamma_s(\omega, \Delta)\big)$, and $z_I' = Z_I$. Note that the diagonal elements are not necessarily real and hence the identification of $\sigma_\pm$ with the diagonal elements is only notational. Displaying all the suppressed arguments, we have

$$Z_+(\omega, \Delta) = \Gamma_i(\omega, \Delta)\big(g_i(\omega + \Delta) + \mu \Gamma_s(-\omega, \Delta)^* g_s(-\omega - \Delta)^*\big) \tag{4.62}$$

$$Z_-(\omega, \Delta) = \Gamma_s(\omega, \Delta)\big(g_s(\omega - \Delta) + \mu \Gamma_i(-\omega, \Delta)^* g_i(-\omega + \Delta)^*\big) \tag{4.63}$$

$$Z_+^*(\omega', \Delta) = \Gamma_i(-\omega', \Delta)^*\big(g_i(-\omega' + \Delta)^* + \mu \Gamma_s(\omega', \Delta) g_s(\omega' - \Delta)\big) \tag{4.64}$$

$$Z_-^*(\omega', \Delta) = \Gamma_s(-\omega', \Delta)^*\big(g_s(-\omega' - \Delta)^* + \mu \Gamma_i(\omega', \Delta) g_i(\omega' + \Delta)\big). \tag{4.65}$$

Using these, the $z_{IJ}(\omega)$ are read-off from the $\langle Z_I(\omega) Z_J^*(\omega')\rangle$. Explicitly,

$$\begin{aligned} \langle Z_+(\omega) Z_+^*(\omega')\rangle = \; & \Gamma_i(\omega, \Delta)\Gamma_i(\omega', -\Delta)\langle g_i(\omega + \Delta) g_i(-\omega' + \Delta)^*\rangle \\ & + \mu^2 \Gamma_s(\omega, -\Delta)\Gamma_s(\omega', \Delta)\langle g_s(-\omega - \Delta)^* g_s(\omega' - \Delta)\rangle \end{aligned} \tag{4.66}$$

$$\begin{aligned} \langle Z_+(\omega) Z_-^*(\omega')\rangle = \; & \mu \Gamma_i(\omega, \Delta)\Gamma_i(\omega', \Delta)\langle g_i(\omega + \Delta) g_i(\omega' + \Delta)\rangle \\ & + \mu \Gamma_s(-\omega, \Delta)^* \Gamma_s(-\omega', \Delta)^* \langle g_s(-\omega - \Delta)^* g_s(-\omega' - \Delta)^*\rangle \end{aligned} \tag{4.67}$$

$$\begin{aligned} \langle Z_-(\omega) Z_+^*(\omega')\rangle = \; & \mu \Gamma_s(\omega, \Delta)\Gamma_i(\omega', \Delta)\langle g_s(\omega - \Delta) g_s(\omega' - \Delta)\rangle \\ & + \mu \Gamma_i(-\omega, \Delta)^* \Gamma_i(-\omega', \Delta)^* \langle g_i(-\omega + \Delta)^* g_i(-\omega' + \Delta)^*\rangle \end{aligned} \tag{4.68}$$

$$\begin{aligned} \langle Z_-(\omega) Z_-^*(\omega')\rangle = \; & \Gamma_s(\omega, \Delta)\Gamma_s(\omega', -\Delta)\langle g_s(\omega - \Delta) g_s(-\omega' - \Delta)^*\rangle \\ & + \mu^2 \Gamma_i(\omega, -\Delta)\Gamma_i(\omega', \Delta)\langle g_i(-\omega + \Delta)^* g_i(\omega' + \Delta)\rangle. \end{aligned} \tag{4.69}$$

The correlators of the stochastic forces are given in eq. (F.6) in Appendix F. We observe that each of the $\langle Z Z^*\rangle$ correlator has the same factor of the delta function and we can express the correlators as,

$$\left.\begin{array}{rl}
\langle Z_+(\omega)Z_+^*(\omega')\rangle := & 2\pi Z_{++}(\omega,\Delta)\delta(\omega+\omega') \\
Z_{++}(\omega,\Delta) = & \Big[\Gamma_i(\omega,\Delta)\Gamma_i(-\omega,-\Delta)|C_i(\Delta)|^2\rho_i\gamma_i[\omega+\Delta] \\
& +\mu^2\Gamma_s(\omega,-\Delta)\Gamma_s(-\omega,\Delta)|C_s(\Delta)|^2\rho_s\gamma_s[\omega+\Delta]\Big] \\
& \\
\langle Z_+(\omega)Z_-^*(\omega')\rangle := & 2\pi Z_{+-}(\omega,\Delta)\delta(\omega+\omega'+2\Delta) \\
Z_{+-}(\omega,\Delta) = & \mu\Big[\Gamma_i(\omega,\Delta)\Gamma_i(-\omega-2\Delta,\Delta)|C_i(\Delta)|^2\rho_i\gamma_i[\omega+\Delta]+ \\
& \Gamma_s(-\omega,\Delta)^*\Gamma_s(\omega+2\Delta,\Delta)^*|C_s(\Delta)|^2\rho_s\gamma_s[\omega+\Delta]\Big] \\
& \\
\langle Z_-(\omega)Z_+^*(\omega')\rangle := & 2\pi Z_{si}(\omega,\Delta)\delta(\omega+\omega'-2\Delta) \\
Z_{-+}(\omega,\Delta) = & \mu\Big[\Gamma_s(\omega,\Delta)\Gamma_s(-\omega+2\Delta,\Delta)|C_s(\Delta)|^2\rho_s\gamma_s[\omega-\Delta]+ \\
& \Gamma_i(-\omega,\Delta)^*\Gamma_i(\omega-2\Delta,\Delta)^*|C_i(\Delta)|^2\rho_i\gamma_i[\omega-\Delta]\Big] \\
& \\
\langle Z_-(\omega)Z_-^*(\omega')\rangle := & 2\pi Z_{ss}(\omega,\Delta)\delta(\omega+\omega') \\
Z_{--}(\omega,\Delta) = & \Big[\Gamma_s(\omega,\Delta)\Gamma_s(-\omega,-\Delta)|C_s(\Delta)|^2\rho_s\gamma_s[\omega-\Delta] \\
& +\mu^2\Gamma_i(\omega,-\Delta)\Gamma_i(-\omega,\Delta)|C_i(\Delta)|^2\rho_i\gamma_i[\omega-\Delta]
\end{array}\right\} . \tag{4.70}$$

The constants $C_{i,s}(\Delta)$ are given in equation (F.4).

Chapter 5

RESULTS AND DISCUSSION

This chapter is divided into two sections. In the first, we highlight the unexplored possibility of unequal kernels. We discuss its possible realisability through tuning of the parameter $\Theta(\Delta)$, thereby providing a new approach towards criticality. In the second section, we summarize our conclusions reiterating specific points.

5.1 Case of Unequal Kernels

Having developed the general set of equations to deal with arbitrary memory kernels, we consider various types of questions and their possible experimental investigation.

The main new possibility of unequal, non-Markovian kernels is the introduction of a new parameter, $\Theta(\Delta)$ for a consistent Δ. Can this be arranged and what new possibilities does this suggest?

For this, consider two exponential kernels with different parameters. For simplicity, let them have the same strength but differing decay constants $\tau_{i,s}$. Thus let

$$\gamma_i(\omega) := \frac{2\gamma_0}{1 - i\omega\tau_i} \quad , \quad \gamma_s(\omega) := \frac{2\gamma_0}{1 - i\omega\tau_s}. \tag{5.1}$$

The factor of 2 is for convenience.

For an exponential kernel we have,

$$\begin{aligned} \frac{\gamma[\Delta]}{2} - i\Delta &= \frac{\gamma_0(1 + i\Delta\tau) - i\Delta(1 + \Delta^2\tau^2)}{(1 + \Delta^2\tau^2)} \\ &= \frac{1}{(1 + \Delta^2\tau^2)} \left[\gamma_0 + i\Delta\left\{\gamma_0\tau - 1 - \Delta^2\tau^2\right\}\right] \qquad (5.2) \\ \therefore tan\Theta(\Delta) &= \Delta\frac{\gamma_0\tau - 1 - \Delta^2\tau^2}{\gamma_0}. \qquad (5.3) \end{aligned}$$

For a steady state, we need $\Theta_i(\Delta) = -\Theta_s(\Delta)$ i.e.

$$\begin{aligned} \Delta\frac{\gamma_0\tau_i - 1 - \Delta^2\tau_i^2}{\gamma_0} &= -\Delta\frac{\gamma_0\tau_s - 1 - \Delta^2\tau_s^2}{\gamma_0} \quad \Rightarrow \qquad (5.4) \\ \Delta = 0 \text{ or } \Delta &= \pm\sqrt{\frac{\gamma_0(\tau_i + \tau_s) - 2}{\tau_i^2 + \tau_s^2}} \ , \ \gamma_0 \geq 2/(\tau_i + \tau_s). \qquad (5.5) \end{aligned}$$

We are interested in non-zero Δ and we may choose the positive square root. Substituting in $\Theta_i(\Delta)$, we get

$$
\begin{aligned}
tan\Theta_i(\Delta) &= \frac{\Delta}{\gamma_0}\frac{1}{\tau_i^2+\tau_s^2}\left[(\gamma_0\tau_i-1)(\tau_i^2+\tau_s^2)-\left\{\gamma_0(\tau_i+\tau_s)-2\right\}\tau_i^2\right] \\
&= \frac{\Delta(\tau_i-\tau_s)}{\gamma_0(\tau_i^2+\tau_s^2)}\left[\tau_i+\tau_s-\gamma_0\tau_i\tau_s\right] \;=\; -tan\Theta_s(\Delta)\;. \qquad (5.6)
\end{aligned}
$$

We see that $\Theta(\Delta) \neq 0$ can be arranged rather easily.

We can go a step further. Let us *choose* $\gamma_0 = \hat{\gamma}/(\tau_i+\tau_s)$ with the dimensionless $\hat{\gamma} > 2$ so that we can have a non-zero, real Δ. For this choice,

$$
\begin{aligned}
tan\Theta(\Delta) &= \sqrt{\frac{\hat{\gamma}-2}{\tau_i^2+\tau_s^2}}\frac{\tau_i-\tau_s}{\hat{\gamma}(\tau_i^2+\tau_s^2)}\left[\tau_i^2+\tau_s^2+\tau_i\tau_s(2-\hat{\gamma})\right] \qquad (5.7) \\
&\to \sqrt{\hat{\gamma}} \qquad\qquad (\text{for } \hat{\gamma} \gg 2). \qquad (5.8)
\end{aligned}
$$

Clearly, we can tune the value of $\Theta(\Delta)$ and arrange it to be quite large. For the mode frequencies in MHz, we may allow Δ to be in KHz which allows $\hat{\gamma} \sim 10^6$ and $\tan\Theta \sim 10^3$.

Another feature that arises with unequal memory kernels with non-zero Δ is that approaching criticality from above, $\mu = 1 + \epsilon$, the parameter ϵ appears together with $[cos\Theta]^{-1}$. In principle, we could tune the Θ parameter along with ϵ in the approach to criticality from above. This does not affect the approach from below and hence a qualitatively new behavior may be expected.

Thus, although the set of steady states remains the same even for unequal memory kernels ($cos\Theta > 0$), the critical behavior and exponents can have qualitatively new features. Thanks to the technique of active feedback protocol, these features are amenable to experimental investigations.

5.2 Summary and concluding remarks

Building on the experimentally discovered and validated set up of the two-mode system, its theoretical modelling and the development of the active feedback protocol had made it possible to raise and study several important questions:

1. Study of the out-of-equilibrium dynamics,
2. Applicability/validity of thermodynamic ideas of phase transitions and critical phenomenon in these two degrees of freedom,

3. Universality classes .

Crucial to all these features is the non-linearity of the two-mode dynamics, the nonzero value of g and the coupling to the baths. The mathematical simulation of the heat baths is also very flexible. While the different types of behaviors of zeros has already been seen and its influence on the exponents studied, there are many more opportunities to explore-can we have a steady state with the two baths at unequal temperatures. The steady state equations do seem to admit this possibility, however a real system needs to validate it.

General arguments seem to require $cos(\Theta) \geq 0$. But we can simulate kernels violating this condition. How would the system respond? This is to be explored experimentally.

In the absence of experimental studies, relying on the physical validation of the two-mode model achieved, we could attempt a numerical integration of the basic non-linear equation and explore some of these questions. This is however beyond the scope of the present work.

Appendix A

FLUCTUATION-DISSIPATION THEOREM

For a self-contained reading, and stressing certain points, we include the Fluctuation-Dissipation theorem as used in our context. We begin with the generalized Langevin equation in terms of the creation-annihilation operators of a harmonic oscillator with frequency Ω and show how the thermal average of the stochastic force bilinear, denoted by the angular brackets, is related to the memory kernel. This derivation does not use linear response theory.

$$
\begin{aligned}
\dot{a} &= -i\Omega a - \frac{1}{2}\int_{-\infty}^{t} dt'\gamma(t-t')a(t') + if(t) && \text{(A.1)}\\
\dot{a^\dagger} &= +i\Omega a^\dagger - \frac{1}{2}\int_{-\infty}^{t} dt'\gamma(t-t')a^\dagger(t') - if^\dagger(t) && \text{(A.2)}\\
\Rightarrow \langle f(t)f^\dagger(t')\rangle &= \left(\frac{\bar{n}_k+1}{2}\right)\gamma(t-t')\ ,\ \bar{n} = \left[e^{\hbar\Omega/kT}-1\right]^{-1}\ . && \text{(A.3)}
\end{aligned}
$$

The derivation has two parts. The so called *first* fluctuation-dissipation theorem relates the thermal average $\langle a(t)a^\dagger(t')\rangle$ to $\langle a(t_0)a^\dagger(t_0)\rangle$ while the *second* fluctuation-dissipation theorem relates the $\langle a(t)a^\dagger(t')\rangle$ to the thermal average of the stochastic force, $\langle f(t)f^\dagger(t')\rangle$. These steps are easily seen for the Markovian kernels, $\gamma(t) \propto \delta(t)$, since the explicit solution of the Langevin equation can be used. The theorems are valid under the usual assumptions of the stochastic force (i) having zero mean, $\langle f(t)\rangle = 0$, (ii) satisfies *stationarity property*, $\langle f(t)f(t')\rangle$ is time translation invariant, and (iii) that causality holds in the sense that dynamical variable at time t is un-correlated with stochastic force at later times. The derivation below follows closely the steps given in [14]. The standard reference is [2].

We are interested in $a(t)$ for t later than some arbitrary t_0. So we write the basic equation (A.1) at instance $t_0+t,\ t>0$.

$$
\begin{aligned}
\dot{a}(t_0+t) &= -i\Omega a(t_0+t)\\
&\quad -\frac{1}{2}\int_{t_0}^{t_0+t} dt'\gamma(t_0+t-t')a(t') + R(t_0+t,t_0), && \text{(A.4)}\\
R(t_0+t,t_0) &:= \left\{if(t_0+t) - \frac{1}{2}\int_{-\infty}^{t_0} dt'\gamma(t_0+t-t')a(t')\right\}. && \text{(A.5)}
\end{aligned}
$$

Since the friction term itself is supposed to be the "systematic part" of the stochastic force itself, the $R(t_0+t,t_0)$ now plays the role of the stochastic force for instances beyond t_0 just as $f(t)$ did for $t>-\infty$. The statement of causality now becomes $\langle R(t_0+t,t_0)a(t_0)\rangle = \langle R(t_0+t,t_0)a^\dagger(t_0)\rangle = 0$.

Multiply (A.1) by $a^\dagger(t_0)$ on the right, multiply by $e^{i\omega t}$ and integrate from 0 to ∞. On the left hand side, flipping the time derivative we can write,

$$\begin{aligned}
L.H.S. &= e^{i\omega t}a(t_0+t)a^\dagger(t_0)\Big|_0^\infty - i\omega\int_0^\infty dte^{i\omega t}a(t_0+t)a^\dagger(t_0) \\
&= 0 - a(t_0)a^\dagger(t_0) - i\omega\int_0^\infty dte^{i\omega t}a(t_0+t)a^\dagger(t_0) \;\Rightarrow \\
\langle L.H.S.\rangle &= -\langle(1+N(t_0))\rangle - i\omega\int_0^\infty dte^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle\,. \qquad \text{(A.6)}
\end{aligned}$$

The first terms is dropped by adding a small $+i\epsilon$ to ω providing the convergence factor at infinity. Here we have introduced the number operator, $a(t_0)a^\dagger(t_0) = 1+N(t_0)$. On the right hand side, the thermal average of $R(t_0+t,t_0)a^\dagger(t_0)$ vanishes by causality. The remaining terms give,

$$\begin{aligned}
\langle R.H.S.\rangle &= -i\Omega\int_0^\infty dte^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle - \frac{1}{2}\int_0^\infty dte^{i\omega t}\int_{t_0}^{t+t_0} dt' \\
&\qquad \gamma(t+t_0-t')\langle a(t')a^\dagger(t_0)\rangle + 0. \qquad \text{(A.7)} \\
&= -\int_0^\infty dte^{i\omega t}\Big\{i\Omega + \frac{\gamma[\omega]}{2}\Big\}\langle a(t_0+t)a^\dagger(t_0)\rangle. \qquad \text{(A.8)}
\end{aligned}$$

We have manipulated the second term by interchanging the two integrations, shifting $t\to t+t'$, used the definition of the Fourier-Laplace transform of the memory kernel and finally changed the dummy integration variable t' to t. From the two equations (A.6, A.8) we get,

$$\int_0^\infty dte^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle = \frac{1+\langle N(t_0)\rangle}{\Big(i(\Omega-\omega)+\frac{\gamma[\omega]}{2}\Big)}\,. \qquad \text{(A.9)}$$

This is the first fluctuation-dissipation theorem.

To relate to the thermal average of the stochastic forces, Fourier transform the equations (A.1, A.2) to get,

$$\begin{aligned}
if(\omega) &= \Big(i(\Omega-\omega)+\tfrac{\gamma[\omega]}{2}\Big)\,a(\omega) \;:=\; \xi(\omega)a(\omega) \qquad \text{(A.10)} \\
-if^\dagger(\omega) &= \Big(i(-\Omega-\omega)+\tfrac{\gamma[\omega]}{2}\Big)\,a^\dagger(\omega) \;\leftrightarrow \qquad \text{(A.11)} \\
-if(-\omega)^\dagger &= \Big(i(-\Omega-\omega)+\tfrac{\gamma[\omega]}{2}\Big)\,a(-\omega)^\dagger.
\end{aligned}$$

We have used $a^\dagger(\omega) = a(-\omega)^\dagger$, $f^\dagger(\omega) = f[-\omega]^*$. The equations are consistent since $\gamma[-\omega]^* = \gamma[\omega]$.

Consider,

$$\int_{-\infty}^{\infty} dt\ e^{i\omega t}\langle f(t_0+t)f^\dagger(t_0)\rangle = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi} e^{i\omega t - i\omega_1(t_0+t) - i\omega_2 t_0} \xi(\omega_1)\xi(-\omega_2)^*\langle a(\omega_1)a^\dagger(\omega_2)\rangle. \tag{A.12}$$

The left hand side is independent of t_0 by the stationarity assumption mentioned above. Hence the explicit t_0 dependence on the right hand side must drop out. This is possible if

$$\langle a(\omega)a^\dagger(\omega_2)\rangle \propto \delta(\omega+\omega_2) := 2\pi\alpha(\omega)\delta(\omega+\omega_2)\ . \tag{A.13}$$

And since,

$$\begin{aligned}\int_{-\infty}^{\infty} dte^{i\omega t}\langle a(t_0+t)a(t_0)^\dagger\rangle &= \int_{-\infty}^{\infty} dt\int_{-\infty}^{\infty}\frac{d\omega_1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi} e^{it(\omega-\omega_1)}e^{-it_0(\omega_1+\omega_2)}\langle a(\omega_1)a^\dagger(\omega_2)\rangle & \text{(A.14)}\\ &= \int_{-\infty}^{\infty}\frac{d\omega_2}{2\pi}e^{-it_0(\omega+\omega_2)}\langle a(\omega)a^\dagger(\omega_2)\rangle & \text{(A.15)}\\ &= \alpha(\omega)\ , & \text{(A.16)}\end{aligned}$$

the correlation function on the left hand side also satisfies the stationarity property.

Returning to equation (A.12),

$$\begin{aligned}\int_{-\infty}^{\infty} dt\ e^{i\omega t}\langle f(t_0+t)f^\dagger(t_0)\rangle &= \xi(\omega)\xi(\omega)^*\alpha(\omega)\\ &= \xi(\omega)\xi(\omega)^*\int_{-\infty}^{\infty} dt\ e^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle. \quad \text{(A.17)}\end{aligned}$$

Lastly, for both the stochastic force as well as the creation-annihilation operators, we have

$$
\begin{aligned}
\int_{-\infty}^{\infty} e^{i\omega t}\, dt \langle a(t_0+t)a^\dagger(t_0)\rangle &= \Big[\int_{-\infty}^{0} dt + \int_{0}^{\infty} dt\Big] e^{i\omega t}\langle\cdots\rangle \\
&= \int_0^\infty dt e^{-i\omega t}\langle a(t_0-t)a^\dagger(t_0)\rangle + \\
&\qquad \int_0^\infty dt e^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle \\
&= \int_0^\infty dt \Big(e^{i\omega t}\langle a(t_0)a^\dagger(t_0-t)\rangle\Big)^* + \\
&\qquad \int_0^\infty dt e^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle \\
&= 2Re\Big(\int_0^\infty dt e^{i\omega t}\langle a(t_0+t)a^\dagger(t_0)\rangle\Big) \qquad \text{(A.18)}
\end{aligned}
$$

where, in getting the equation (A.18), we have shifted $t_0 \to t_0 + t$ in the first term in the previous equation using the time translation property of the thermal average. Identically, we get

$$
\int_{-\infty}^{\infty} e^{i\omega t}\, dt\langle f(t_0+t)f^\dagger(t_0)\rangle = 2Re\Big(\int_0^\infty dt e^{i\omega t}\langle f(t_0+t)f^\dagger(t_0)\rangle\Big). \qquad \text{(A.19)}
$$

Substituting for the right hand side of equation (A.9) in equation (A.17) and using (A.19), we write

$$
\begin{aligned}
2Re\Big(\int_0^\infty dt\ e^{i\omega t}\langle f(t_0+t)f^\dagger(t_0)\rangle\Big) &= 2Re\Big(\xi(\omega)\xi(\omega)^*\frac{1+\langle N(t_0)\rangle}{\xi(\omega)}\Big) \qquad \text{(A.20)} \\
&= 2(1+\langle N(t_0)\rangle)Re\big(\xi(\omega)^*\big) \\
&= (1+\langle N(t_0)\rangle)Re\big(\gamma[\omega]\big). \qquad \text{(A.21)}
\end{aligned}
$$

The causality requirement, $\gamma(t<0) = 0 = \langle f(t<0)f(0)\rangle$ and presumed existence of the Fourier-Laplace transforms for real ω, imply that both Fourier-Laplace transforms are analytic in the upper half $\omega-$plane including the real axis. The equality of the real parts then extends to the functions themselves and we can conclude that,

$$
\langle f(t)f^\dagger(t')\rangle = \frac{\bar{n}+1}{2}\gamma(t-t') \ \ , \ \ \bar{n} := \langle N\rangle = \Big[e^{\hbar\Omega/kT} - 1\Big]^{-1}. \qquad \text{(A.22)}
$$

We have removed the reference to the arbitrary t_0 and used the thermal average of the occupation number for a harmonic oscillator. For $\hbar\Omega/kT \ll 1$, $\bar{n}+1 \to kT/\hbar\Omega$.

This is the form of the fluctuation-dissipation theorem used in the main text.

Appendix B

DIAGONAL FORM OF THE LINEARISED EQUATIONS

It is theorem in linear algebra that an arbitrary (complex) matrix A can be put in the form $A = U_1 \Sigma U_2$ where $U_{1,2}$ are unitary matrices and Σ is a diagonal matrix [15]. We will work out the closed form expressions for our case of interest, namely a 2×2 matrix.

Let $A := \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}$. Then $A^\dagger A := \begin{bmatrix} a & be^{i\theta} \\ be^{-i\theta} & c \end{bmatrix}$ where,

$$a = |\alpha|^2 + |\gamma|^2 \ , \ c = |\beta|^2 + |\delta|^2 \ , \ be^{i\theta} = \alpha^* \beta + \gamma^* \delta \ , \ a, b, c, \theta \in \mathbb{R}. \tag{B.1}$$

$A^\dagger A$ is a Hermitian matrix with non-negative eigenvalues.

If $b = 0$, the matrix $A^\dagger A$ is already diagonal with eigenvalues a, c and the orthonormalized eigenvectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Consider the generic case of $b \neq 0$. The eigenvalues of $A^\dagger A$ are given by,

$$\lambda_\pm = \frac{c + a \pm \sqrt{(c-a)^2 + 4b^2}}{2} \ , \ a + c \geq 0. \tag{B.2}$$

The corresponding orthonormalized eigenvectors can be taken as,

$$\xi_\lambda = \frac{1}{\sqrt{b^2 + (\lambda - a)^2}} \begin{bmatrix} be^{i\theta} \\ \lambda - a \end{bmatrix} \quad , \quad .\xi_\lambda^\dagger \xi_{\lambda'} = \delta_{\lambda,\lambda'}. \tag{B.3}$$

Both eigenvalues cannot be zero unless the matrix A itself vanishes. If there is a vanishing eigenvalue, we denote it as $\lambda_- = 0 \leftrightarrow \lambda_+ = (a + c)$. In this case, $b^2 = ac$ and the eigenvectors are,

$$\xi_+ = \frac{1}{\sqrt{b^2 + c^2}} \begin{bmatrix} be^{i\theta} \\ c \end{bmatrix} \quad , \quad \xi_- = \frac{1}{\sqrt{b^2 + a^2}} \begin{bmatrix} be^{i\theta} \\ -a \end{bmatrix} \quad . \tag{B.4}$$

Consider the generic case of non-zero eigenvalues and let $\lambda_+ \geq \lambda_-$. Define

$$\sigma_\pm := \sqrt{\lambda_\pm} \quad , \quad \eta_\pm := \tfrac{A\xi_\pm}{\sigma_\pm} \quad , \quad \eta_\lambda^\dagger \eta_{\lambda'} = \delta_{\lambda,\lambda'} \ ; \tag{B.5}$$

$$\Sigma := \text{diag}[\sigma_+, \sigma_-] \quad , \quad U_1 := [\eta_+, \eta_-] \quad , \quad U_2 := [\xi_+, \xi_-]. \tag{B.6}$$

The unitarity of $U_{1,2}$ follows from the orthonormality of $\xi_\pm$ and $\eta_\pm$. It follows that,

$$U_1^\dagger A U_2 = \Sigma \quad \Rightarrow \quad A = U_1 \Sigma U_2^\dagger \ . \tag{B.7}$$

The last equation is the explicit *singular value decomposition* of a 2×2 matrix. Note that the matrices U_2 and Σ are obtained explicitly in terms of the elements of the matrix $A^\dagger A$ while U_1 involves the elements of A as well.

As a simple illustration consider the standard example of a *non-diagonalizable* matrix: $A := \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Its eigenvalues are $(0, 0)$.

The matrix $A^\dagger A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Its eigenvalues are $(1, 0)$ with eigenvectors: $\xi_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\xi_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The vectors η's are: $\eta_1 = \xi_0$, $\eta_0 := \xi_1$. Thus the singular value decomposition of A is given by,

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} . \tag{B.8}$$

A key point to note is that Σ is given by eigenvalues of $A^\dagger A$ and that $A \neq 0$ already implies that both $\sigma_\pm$ *cannot* be zero.

Appendix C

ZERO MODE OF THE $U(1)$ SYMMETRY

At a fixed normalized drive, $\mu > 1$, a perturbation of the form $y_i(t) := iB_i\delta\varphi_i,\ y_s(t) := iB_s\delta\varphi_s$ with $\delta\varphi_i + \delta\varphi_s = 0$ corresponds to going to another steady state with the same $|B|_{i,s,P}, (\varphi_i + \varphi_s), \varphi_P$. Hence, its Fourier transform is proportional to $2\pi\delta(\omega)$ and non-zero. Hence, determinant of $\mathbb{A}(\omega = 0)$ must vanish.

To check this we note that for $\omega = 0$ the $\Gamma_{i,s,P}$'s take the values: $\Gamma_P = 1, \Gamma_i = e^{-i\Theta}, \Gamma_s = e^{-i\Theta}$. As this is relevant only for the non-trivial steady states, we will also use $|B|^2 e^{i\varphi_+} + \mu = -iB_P$ and $|B_P| = 1$. In this limit, the matrices go to:

$$\mathbb{M} \rightarrow \begin{bmatrix} 1 + |B|^2 e^{-i\Theta} & |B|^2 e^{-i\Theta + i\varphi_-} \\ |B|^2 e^{-i\Theta - i\varphi_-} & 1 + |B|^2 e^{-i\Theta} \end{bmatrix}, \tag{C.1}$$

$$\mathbb{M}' \rightarrow e^{-i\Theta}(-iB_P) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{C.2}$$

$$\mathbb{M}^{-1} \rightarrow \frac{1}{(1 + 2|B|^2 e^{-i\Theta})} \begin{bmatrix} 1 + |B|^2 e^{-i\Theta} & -|B|^2 e^{-i\Theta + i\varphi_-} \\ -|B|^2 e^{-i\Theta - i\varphi_-} & 1 + |B|^2 e^{-i\Theta} \end{bmatrix}, \tag{C.3}$$

$$\mathbb{N} = \mathbb{M}^{-1}\mathbb{M}' \rightarrow \left(\frac{e^{-i\Theta}(-iB_P)}{1 + 2|B|^2 e^{-i\Theta}}\right) \begin{bmatrix} -|B|^2 e^{-i\Theta + i\varphi_-} & 1 + |B|^2 e^{-i\Theta} \\ 1 + |B|^2 e^{-i\Theta} & -|B|^2 e^{-i\Theta - i\varphi_-} \end{bmatrix} \tag{C.4}$$

Since the frequency is zero, $\mathbb{N}(-\omega)^* = \mathbb{N}^*$, and we get,

$$\mathbb{N}\mathbb{N}^* \rightarrow \frac{1}{|1 + 2|B|^2 e^{-i\Theta}|^2} \times \begin{bmatrix} |B|^4 + |1 + |B|^2 e^{-i\Theta}|^2 & -(1 + |B|^2 e^{i\Theta})|B|^2 e^{-i\Theta + i\varphi_-} - (1 + |B|^2 e^{-i\Theta})|B|^2 e^{+i\Theta + i\varphi_-} \\ -(1 + |B|^2 e^{-i\Theta})|B|^2 e^{i\Theta - i\varphi_-} - (1 + |B|^2 e^{i\Theta})|B|^2 e^{-i\Theta - i\varphi_-} & |B|^4 + |1 + |B|^2 e^{-i\Theta}|^2 \end{bmatrix} \tag{C.5}$$

Noting that the diagonal elements equal $1 + 2|B|^2(|B|^2 + cos\Theta)$, the off-diagonal elements are $-2|B|^2(|B|^2 + cos\Theta)e^{\pm i\varphi_-}$ and the denominator in the prefactor is $1 + |B|^2(|B|^2 + cos\Theta)$, we write the matrix $\mathbb{A}(\omega = 0) = \mathbb{1} - \mathbb{N}\mathbb{N}^*$ as,

$$\mathbb{A}(0) = \frac{1}{|1 + 2|B|^2 e^{-i\Theta}|^2} \big(2|B|^2(|B|^2 + cos\Theta\big) \begin{bmatrix} 1 & e^{i\varphi_-} \\ e^{-i\varphi_-} & 1 \end{bmatrix} \tag{C.6}$$

$$\therefore det\mathbb{A}(0) = 0. \tag{C.7}$$

This completes the proof. Notice that the trace of $\mathbb{A}(0)$ is non-zero and hence $\mathbb{A}(0)$ necessarily has a non-zero eigenvalue. Thus there is only one non-trivial solution of the homogeneous equation $\mathbb{A}(0)\mathbb{Y}(\omega = 0) = 0$, precisely corresponding to the symmetry direction as can be verified easily.

This is true for all $\mu > 1$. For $\mu < 1$, $\mathbb{Y}(0) = 0$.

Appendix D

MATRICES $\mathbb{A}(\omega), \mathbb{Z}(\omega)$ AND THE DETERMINANT OF $\mathbb{A}(\omega)$

We calculate the determinant of the $\mathbb{A}(\omega)$ as a function of $\Gamma_{i,s,P}, |B|, \mu$ and $e^{i\varphi_\pm}$.

The matrix $\mathbb{A}(\omega)$ is defined as

$$\mathbb{A}(\omega) := \mathbb{1} - \mathbb{N}(\omega)\mathbb{N}(-\omega)^* \quad , \quad \mathbb{N}(\omega) := \mathbb{M}(\omega)^{-1}\mathbb{M}'(\omega) \ , \tag{D.1}$$

where the $\mathbb{M}, \mathbb{M}'$ are defined in equation (4.49). Recalling their definition,

$$\mathbb{M}(\omega) \ := \ \begin{bmatrix} 1+\Gamma_i\Gamma_P|B|^2 & \Gamma_i\Gamma_P|B|^2e^{i\varphi_-} \\ \Gamma_s\Gamma_P|B|^2e^{-i\varphi_-} & 1+\Gamma_s\Gamma_P|B|^2 \end{bmatrix} \tag{D.2}$$

$$\mathbb{M}'(\omega) \ := \ \begin{bmatrix} 0 & \Gamma_i\big(|B|^2e^{i\varphi_+}+\mu\big) \\ \Gamma_s\big(|B|^2e^{i\varphi_+}+\mu\big) & 0 \end{bmatrix} \tag{D.3}$$

$$\det(\mathbb{M}) \ = \ 1+\Gamma_P|B|^2(\Gamma_i+\Gamma_s) \ . \tag{D.4}$$

Let us introduce temporary notation: $\alpha := |B|^2$ and $\beta := (\mu + |B|^2e^{i\varphi_+}) = -iB_P$. Then,

$$\mathbb{M}^{-1} \ = \ \frac{1}{1+\alpha\Gamma_P(\Gamma_i+\Gamma_s)}\begin{bmatrix} 1+\alpha\Gamma_P\Gamma_s & -\alpha\Gamma_P\Gamma_ie^{i\varphi_-} \\ -\alpha\Gamma_P\Gamma_se^{-i\varphi_-} & 1+\Gamma_P\alpha\Gamma_i \end{bmatrix} \tag{D.5}$$

$$\therefore \mathbb{N}(\omega) \ = \ \frac{\beta}{1+\alpha\Gamma_P(\Gamma_i+\Gamma_s)}\begin{bmatrix} -\alpha\Gamma_P\Gamma_i\Gamma_se^{i\varphi_-} & \Gamma_i+\alpha\Gamma_P\Gamma_i\Gamma_s \\ \Gamma_s+\alpha\Gamma_P\Gamma_i\Gamma_s & -\alpha\Gamma_P\Gamma_i\Gamma_se^{-i\varphi_-} \end{bmatrix} \tag{D.6}$$

Introduce another temporary notation: $q(\omega) := \Gamma_P(\omega)\Gamma_i(\omega)\Gamma_s(\omega)$ and $r(\omega) := \Gamma_P(\omega)\big(\Gamma_i(\omega) + \Gamma_s(\omega)\big)$. Note that while α, β are state dependent but ω-independent, q, r are independent of state but depend on ω. They satisfy the properties: $q(-\omega, \Delta)^* = q(\omega, -\Delta)$ and ditto for $r(\omega)$. Furthermore, for $\omega = 0$ we have $q(0, \Delta) = e^{2i\Theta(\Delta)}$ and $r(0, \Delta) = 2e^{i\Theta(\Delta)}$. We also use the abbreviations: $\xi_{i,s} := \Gamma_{i,s} + \alpha q$ to facilitate the manipulations. With these,

$$\mathbb{N}(-\omega)^* \ = \ \frac{\beta^*}{1+\alpha r^*}\begin{bmatrix} -\alpha q^*e^{-i\varphi_-} & \xi_i^* \\ \xi_s^* & -\alpha q^*e^{+i\varphi_-} \end{bmatrix} \tag{D.7}$$

and,

$$\begin{aligned}
\mathbb{N}\mathbb{N}^* &= \frac{|\beta|^2}{|1+\alpha r|^2} \times \begin{bmatrix} -\alpha q e^{i\varphi_-} & \xi_i \\ \xi_s & -\alpha q e^{-i\varphi_-} \end{bmatrix} \begin{bmatrix} -\alpha q^* e^{-i\varphi_-} & \xi_i^* \\ \xi_s^* & -\alpha q^* e^{i\varphi_-} \end{bmatrix} \\
&= \frac{|\beta|^2}{|1+\alpha r|^2} \begin{bmatrix} \alpha^2|q|^2 + \xi_i\xi_s^* & -\alpha e^{i\varphi_-}(q\xi_i^* + q^*\xi_i) \\ -\alpha e^{-i\varphi_-}(q\xi_s^* + q^*\xi_s) & \alpha^2|q|^2 + \xi_i^*\xi_s \end{bmatrix} \qquad \text{(D.8)} \\
\mathbb{A}(\omega) &= \frac{1}{|1+\alpha r|^2}\Big\{|1+\alpha r|^2 \mathbb{1} - \\
&\qquad |\beta|^2 \begin{bmatrix} \alpha^2|q|^2 + \xi_i\xi_s^* & -\alpha e^{i\varphi_-}(q\xi_i^* + q^*\xi_i) \\ -\alpha e^{-i\varphi_-}(q\xi_s^* + q^*\xi_s) & \alpha^2|q|^2 + \xi_i^*\xi_s \end{bmatrix}\Big\} \qquad \text{(D.9)}
\end{aligned}$$

For $|\beta| = 1$ and $\omega = 0$, this matches with (C.6).

The determinant is then given by

$$\begin{aligned}
\frac{\det(\mathbb{A})(\omega)}{|1+\alpha r|^{-4}} &= \big(|1+\alpha r|^2 - |\beta|^2\alpha^2|q|^2 - |\beta|^2\xi_i\xi_s^*\big) \times \\
&\qquad \big(|1+\alpha r|^2 - |\beta|^2\alpha^2|q|^2 - |\beta|^2\xi_i^*\xi_s\big) \\
&\qquad -\alpha^2|\beta|^4\big(q\xi_i^* + q^*\xi_i\big)(q\xi_s^* + q^*\xi_s) \qquad \text{(D.10)} \\
&= (|1+\alpha r|^2 - |\beta|^2\alpha^2|q|^2)^2 + |\beta|^4|\xi_i\xi_s|^2 \\
&\quad -|\beta|^2(|1+\alpha r|^2 - |\beta|^2\alpha^2|q|^2)(\xi_i\xi_s^* + \xi_i^*\xi_s) \\
&\quad -\alpha^2|\beta|^4\big(q^2(\xi_i\xi_s)^* + (q^*)^2\xi_i\xi_s + |q|^2(\xi_i\xi_s^* + \xi_i^*\xi_s)\big). \qquad \text{(D.11)}
\end{aligned}$$

Use simplifications,

$$\begin{aligned}
\xi_i\xi_s &= (\Gamma_i + \alpha q)(\Gamma_s + \alpha q) \;=\; \alpha^2 q^2 + \frac{q}{\Gamma_P}(1+\alpha r) \qquad \text{(D.12)} \\
\therefore q^2(\xi_i\xi_s)^* &= |q|^2\left(\alpha^2|q|^2 + \frac{q}{\Gamma_P^*}(1+\alpha r)^*\right) \qquad \text{(D.13)} \\
\xi_i\xi_s^* + \xi_i^*\xi_s &= (\Gamma_i + \alpha q)(\Gamma_s + \alpha q)^* + c.c. \\
&= 2\alpha^2|q|^2 + \alpha\left(\frac{qr^*}{\Gamma_P^*} + \frac{q^*r}{\Gamma_P}\right) + \Gamma_i\Gamma_s^* + \Gamma_i^*\Gamma_s\;. \qquad \text{(D.14)}
\end{aligned}$$

After cancellation of various terms, a common factor of $|1+\alpha r|^2$ arises. This leads to,

$$\det(\mathbb{A})(\omega) = \frac{1}{|1+\alpha r|^2}\Bigg[|1+\alpha r|^2 - 4\alpha^2|\beta|^2|q|^2 + |\beta|^4\frac{|q|^2}{|\Gamma_P|^2} - \alpha|\beta|^2\left(\frac{qr^*}{\Gamma_P^*} + \frac{q^*r}{\Gamma_P}\right) - |\beta|^2\big(\Gamma_i\Gamma_s^* + \Gamma_i^*\Gamma_s\big)\Bigg] \quad \text{(D.15)}$$

$$= \frac{1}{|1+\alpha r|^2}\Big[\ \big\{1 - |\beta|^2\big(\Gamma_i\Gamma_s^* + \Gamma_i^*\Gamma_s\big) + |\beta|^4|\Gamma_i\Gamma_s|^2\big\} + \alpha\left\{(r+r^*) - |\beta|^2\left(q(\Gamma_i+\Gamma_s)^* + q^*(\Gamma_i+\Gamma_s)\right)\right\} + \alpha^2\Big\{|r|^2 - 4|\beta|^2|q|^2\Big\}\Big]. \quad \text{(D.16)}$$

The numerator is an expression quadratic in $\alpha(=|B|^2)$. Each of the coefficient has a simple form,

$$\text{Coeff. of } \alpha^0 \ : \ (1-|\beta|^2\Gamma_i\Gamma_s^*)(1-|\beta|^2\Gamma_i^*\Gamma_s) \quad \text{(D.17)}$$

$$\text{Coeff. of } \alpha^1 \ : \ (\Gamma_i+\Gamma_s)(\Gamma_P - |\beta|^2q^*) + (\Gamma_i+\Gamma_s)^*(\Gamma_P^* - |\beta|^2q) \quad \text{(D.18)}$$

$$\text{Coeff. of } \alpha^2 \ : \ (|r|+2|\beta||q|)(|r|-2|\beta||q|). \quad \text{(D.19)}$$

We note some limiting cases.

<u>$\omega \to 0$:</u>

We get $q \to e^{2i\Theta(\Delta)}$, $r \to 2e^{i\Theta(\Delta)}$. The coefficients go ever to $(1-|\beta|^2)^2$, $4cos\Theta(1-|\beta|^2)$ and $4(1-|\beta|^2)$ respectively. The determinant thus goes over to,

$$\text{Det}(\omega=0) = (1-|\beta|^2)\frac{1-|\beta|^2+4\alpha cos\Theta+4\alpha^2}{1+4\alpha cos\Theta+4\alpha^2}.$$

Clearly, this vanishes *only* for $|\beta|=1$ i.e. for $|B_P|=1 \leftrightarrow \mu > 1$.

<u>$\Delta = 0$:</u>

For $\Delta = 0$, the complex conjugation is removable since $\Gamma_s(-\omega,0)^* = \Gamma_s(\omega,0)$ etc and the expression for the determinant simplifies further.

$$\therefore \det(\mathbb{A}(\omega)) = \frac{\Big(1-|B_P|^2\Gamma_i\Gamma_s + |B|^2\Gamma_P(\Gamma_i+\Gamma_s+2|B_P|\Gamma_i\Gamma_s)\Big)}{(1+|B|^2\Gamma_P(\Gamma_i+\Gamma_s))} \times \frac{\Big(1-|B_P|^2\Gamma_i\Gamma_s + |B|^2\Gamma_P(\Gamma_i+\Gamma_s-2|B_P|\Gamma_i\Gamma_s)\Big)}{(1+|B|^2\Gamma_P(\Gamma_i+\Gamma_s))}. \quad \text{(D.20)}$$

Note: The product form makes it quite obvious that each of the factors denotes just $\sigma_\pm$, eigenvalues of $\mathbb{A}^\dagger\mathbb{A}$ up to a scale factor, e.g., the first factor equals $\lambda\sigma_+$ while the second factor equals σ_-/λ with λ an arbitrary non-zero complex number. It is also obvious that both factors *cannot* be zero simultaneously except when $|B| = 0$.

This is an exact expression valid for both trivial and non-trivial states with $\Delta = 0$. It is continuous at $\mu = 1$, but its $\mu-$derivative is discontinuous.

$\mu = 1 \mp \epsilon$:

Finally, we consider the limiting behavior of the determinant for drive near its critical value. Recall that $\alpha = |B|^2$ and $|\beta|^2 = |B_P|^2$ imply that

$$\alpha = \begin{cases} 0 & \text{for } \mu \leq 1 \\ -cos\Theta + \sqrt{\mu^2 - sin^2\Theta} & \text{for } \mu > 1 \end{cases} \tag{D.21}$$

$$|\beta|^2 = \begin{cases} \mu^2 & \text{for } \alpha = 0 \\ 1 & \text{for } \alpha \neq 0 \end{cases} . \tag{D.22}$$

Therefore, for $\mu = 1 - \epsilon$,

$$det[\mathbb{A}(\omega)]_- \simeq \Big\{(1 - \Gamma_i\Gamma_s^*)(1 - \Gamma_i^*\Gamma_s)\Big\} + \epsilon\Big[2(\Gamma_i\Gamma_s^* + \Gamma_i^*\Gamma_s)\Big]. \tag{D.23}$$

For $\mu = 1 + \epsilon$, we get $\alpha \simeq \epsilon/|cos\Theta|$ and $\frac{1}{|1+\alpha r|^2} \simeq \left(1 - \frac{\epsilon}{cos\Theta}(r + r^*)\right)$. Substitution in (D.16) gives (α^2 term is dropped),

$$\begin{aligned} det[\mathbb{A}(\omega)]_+ \simeq\ & \Big\{(1 - \Gamma_i\Gamma_s^*)(1 - \Gamma_i^*\Gamma_s)\Big\} + \\ & \frac{\epsilon}{cos\Theta}\Big[- (r + r^*)(1 - \Gamma_i\Gamma_s^*)(1 - \Gamma_i^*\Gamma_s) \\ & +(\Gamma_i + \Gamma_s)(\Gamma_P - q^*) + (\Gamma_i + \Gamma_s)^*(\Gamma_P^* - q)\Big]. \end{aligned} \tag{D.24}$$

The ω dependence is in the terms in the square brackets. The explicit $cos\Theta$ dependence shows that the $\mu-$derivative of the determinant is discontinuous across the transition point $\mu = 1$.

Calculation of the $\mathbb{Z}(\omega)$ matrix:

Recall that

$$\mathbb{Z}(\omega) := \mathbb{z}(\omega) - \mathbb{N}(\omega)\mathbb{z}(-\omega)^* \ , \ \mathbb{z}(\omega) := \mathbb{M}^{-1}(\omega)\mathbb{g}(\omega) \ , \ \mathbb{g}(\omega) := \begin{bmatrix} i\Gamma_i(\omega)g_i(\omega) \\ i\Gamma_s(\omega)g_s(\omega) \end{bmatrix}.$$

Using the matrices $\mathbb{M}^{-1}, \mathbb{N}$ given in equations (D.5,D.6), we get

$$\mathbb{z} = \frac{i}{1+\alpha r}\begin{bmatrix} \xi_i g_i - \alpha q e^{i\varphi_-} g_s \\ \xi_s g_s - \alpha q e^{-i\varphi_-} g_i \end{bmatrix} \tag{D.25}$$

$$\mathbb{N}\mathbb{z}^* = \frac{-i\beta}{|1+\alpha r|^2}\begin{bmatrix} -\alpha q e^{i\varphi_-} & \xi_i \\ \xi_s & -\alpha q e^{-i\varphi_-} \end{bmatrix}\begin{bmatrix} \xi_i^* g_i^* - \alpha q^* e^{-i\varphi_-} g_s^* \\ \xi_s^* g_s^* - \alpha q^* e^{i\varphi_-} g_i^* \end{bmatrix} \tag{D.26}$$

$$= \frac{-i\beta}{|1+\alpha r|^2}\begin{bmatrix} (\alpha^2|q|^2 + \xi_i \xi_s^*) g_s^* - \alpha g_i^* e^{i\varphi_-}(q\xi_i^* + q^*\xi_i) \\ (\alpha^2|q|^2 + \xi_i^* \xi_s) g_i^* - \alpha g_s^* e^{-i\varphi_-}(q\xi_s^* + q^*\xi_s) \end{bmatrix} \tag{D.27}$$

$$\therefore \mathbb{Z} = \frac{i}{|1+\alpha r|^2}\begin{bmatrix} Z_i \\ Z_s \end{bmatrix} \quad \text{where,} \tag{D.28}$$

$$Z_i := (1+\alpha r^*)(\xi_i g_i - \alpha q e^{i\varphi_-} g_s) + \beta\{(\alpha^2|q|^2 + \xi_i \xi_s^*) g_s^* - \alpha g_i^* e^{i\varphi_-}(q\xi_i^* + q^*\xi_i)\} \tag{D.29}$$

$$Z_s := (1+\alpha r^*)(\xi_s g_s - \alpha q e^{-i\varphi_-} g_i) + \beta\{(\alpha^2|q|^2 + \xi_i^* \xi_s) g_i^* - \alpha g_s^* e^{-i\varphi_-}(q\xi_s^* + q^*\xi_s)\}. \tag{D.30}$$

In the above, we recall, $\alpha = |B|^2,\ \beta = -iB_P = \mu + |B|^2 e^{i\varphi_+},\ q = \Gamma_i\Gamma_s\Gamma_P,\ r = \Gamma_P(\Gamma_i + \Gamma_s),\ \xi_{i,s} = \alpha q + \Gamma_{i,s}$ and the $g_{i,s}^* = g_{i,s}(-\omega + \Delta_{i,s})^*$, etc.

In general, the $Z_{i,s}$ are linear combinations of all four $g_{i,s}, g_{i,s}^*$. There is also a dependence on the $U(1)$ phase φ_-. The stochastic forces thus distinguish the non-trivial steady states labeled by φ_-.

For the explicit case discussed in section 4.4, we have $\alpha = 0,\ \beta = \mu,\ \xi_{i,s} = \Gamma_{i,s}$ which give,

$$\mathbb{Z}(\omega) = i\begin{bmatrix} \Gamma_i g_i + \mu\Gamma_i\Gamma_s^* g_s^* \\ \Gamma_s g_s + \mu\Gamma_i^*\Gamma_s g_i^* \end{bmatrix}. \tag{D.31}$$

The φ_- dependence drops out as it should.

Appendix E

ILLUSTRATIONS WITH EXAMPLES OF MEMORY KERNELS

Let us recall that arbitrary choice of memory kernels do *not* admit steady state solutions with/without a dynamical phase $\Delta := \Delta_i = -\Delta_s$. The kernels must satisfy the equation (4.19). Also, the phases of the Fourier-Laplace transform of the kernels must satisfy $\Theta(\Delta) := \Theta_i(\Delta) = -\Theta_s(\Delta)$ defined in equation (4.9). $\Theta(\Delta)$ is an odd function of Δ and satisfies $\cos(\Theta) \geq 0$. The normalized steady state solutions, (B_i, B_s, B_P), are determined in terms of these and the normalized drive μ and always satisfy $|B_i| = |B_s| =: |B|$. The solutions with $|B| = 0$ are the trivial states which may be realized for all μ while the non-trivial states with non-zero $|B|$ are possible only for $\mu > 1$. The stability properties under perturbations are determined by the zeros of the determinant given in the equation (D.16) and the correlators of the perturbations are given in terms of equation (4.70).

Markovian Kernels: These are specified as $\gamma_{i,s}(t) := \hat{\gamma}_{i,s}\delta(t),\ \hat{\gamma}_{i,s} > 0$. In the basic equations (4.1), the integration is trivially done and only $\hat{\gamma}_{i,s}$ will appear. Effectively, $\gamma_{i,s}[\Delta_{i,s}] = \hat{\gamma}_{i,s}$. The corresponding $\Theta_{i,s} = tan^{-1}(-2\Delta/\hat{\gamma}_{i,s})$. These two cannot satisfy $\Theta_i(\Delta) = -\Theta_s(\Delta)$, except when each equals zero. This implies $\Delta = 0 = \Theta(\Delta)$ for the Markovian baths.

The non-trivial steady state solutions have $|B|^2 = \mu - 1,\ \varphi_P = \pi/2,\ \varphi_i + \varphi_s = \pi$. The $\Gamma' s$ are: $\Gamma_P(\omega) = [1 - 2i\omega/\gamma_P]^{-1}$, $\Gamma_{i,s}(\omega) = [1 - 2i\omega/\hat{\gamma}_{i,s}]^{-1}$.

The vanishing determinant condition, equation (4.56) reduces to,

$$(\hat{\gamma}_i - 2i\omega)(\hat{\gamma}_s - 2i\omega) = \mu^2\hat{\gamma}_i\hat{\gamma}_s \quad \text{and putting } \lambda := -i\omega \quad \Rightarrow \tag{E.1}$$

$$\lambda_\pm = -\frac{1}{4}\left[(\hat{\gamma}_i + \hat{\gamma}_s) \mp \sqrt{(\hat{\gamma}_i - \hat{\gamma}_s)^2 + 4\mu^2\hat{\gamma}_i\hat{\gamma}_s}\right]. \tag{E.2}$$

For $\mu^2 < 1$, both roots are negative, indicating stability. As $\mu \to 1_-$, the roots become: 0 , $-(\hat{\gamma}_i + \hat{\gamma}_s)/2$, and for $\mu > 1$, one root becomes positive indicating instability.

This is an example where we have two isolated zeros of the determinant.

As noted in the equation (D.20), as $\mu \to 1_-$, we have $\sigma_+ = \sigma_- = 1 -$

$\mu^2\Gamma_i(\omega)\Gamma_s(\omega)$. All the force correlators are proportional to $\delta(\omega+\omega')$. To get the $y'_I(t)y'^*_J(t')$ correlators, we integrate the $y'_{I,J}(\omega)$ correlators: $\int d\omega e^{-i\omega t}\int d\omega'$. This sets $\omega' = -\omega$ everywhere. The singularities come from the $\{\sigma_I(\omega + i\epsilon)\sigma_J(-\omega + i\epsilon)^*\}^{-1} = \{\sigma_I(\omega + i\epsilon)\sigma_J(\omega - i\epsilon)\}^{-1}$ factors. These factors are the same for each correlator and become,

$$\Big\{\big(1-\mu^2\Gamma_i(\omega+i\epsilon)\Gamma_s(\omega+i\epsilon)\big)\big(1-\Gamma_i(\omega-i\epsilon)\Gamma_s(\omega-i\epsilon)\big)\Big\}^{-1} \quad \rightarrow$$
$$\Big(1-\mu^2\frac{\hat{\gamma}_i}{\hat{\gamma}_i-2i\omega}\frac{\hat{\gamma}_s}{\hat{\gamma}_s-2i\omega}\Big)^{-1}\Big|_{\omega+i\epsilon}\Big(1-\mu^2\frac{\hat{\gamma}_s}{\hat{\gamma}_s-2i\omega}\frac{\hat{\gamma}_i}{\hat{\gamma}_i-2i\omega}\Big)^{-1}\Big|_{\omega-i\epsilon} \quad =$$
$$\frac{(\hat{\gamma}_i-2i\omega)(\hat{\gamma}_s-2i\omega)}{-4\omega^2-2i\omega(\hat{\gamma}_i+\hat{\gamma}_s)+(1-\mu^2)\hat{\gamma}_i\hat{\gamma}_s}\Bigg|_{\omega+i\epsilon}\times$$
$$\frac{(\hat{\gamma}_i-2i\omega)(\hat{\gamma}_s-2i\omega)}{-4\omega^2-2i\omega(\hat{\gamma}_i+\hat{\gamma}_s)+(1-\mu^2)\hat{\gamma}_i\hat{\gamma}_s}\Bigg|_{\omega-i\epsilon}. \tag{E.3}$$

This has 4 poles which will be picked up by the ω integration. Thanks to the $\pm i\epsilon$ prescription, two of the poles are in the upper half plane while two are in the lower half plane. For the RMS fluctuations, $t - t' = 0$ and the contour may be closed in either half plane, picking up two of the poles. Note that for $\mu = 1$, we have poles at $\omega = \mp i\epsilon$.

The μ-dependence appears non-trivially through the pole positions. The numerator from the force correlators also has a μ-dependence which is at the most quadratic. Any critical exponents would thus be determined by the location of the singularities. Remaining factors in the integrand would just produce the coefficients of a critical power law.

As noted in Appendix D, the μ-dependence of the singular behavior is different for $\mu \to 1_+$ and so are the force correlator combinations.

Equal Kernels: We choose $\gamma_i(t) = \gamma_s(t) =: \gamma(t)$. Two specific choices of $\gamma(t)$ have been already analyzed. Clearly, we must have $\Theta_i(\Delta) = \Theta_s(\Delta)$. This is consistent with equation (4.19) provided $\Theta(\Delta) = 0$ and thus $\frac{\gamma[\Delta]}{2} - i\Delta$ is real. This also gives an equation for Δ namely, $2\Delta = Im(\gamma[\Delta])$, which is a special case of (4.20). In this special case of equal kernels, for both the examples, $\mu > 1$ and $|B|^2 = \sqrt{\mu - 1}$. The equation (4.22) implies that $\varphi_P = \frac{\pi}{2}$ and thus $\varphi_i + \varphi_s = \pi$.

The exponential memory kernel [7]: In terms of its Fourier transform it is given as,

$$\gamma[\omega] = \gamma_0\big(1 - i\omega\tau_r\big)^{-1} \;\Rightarrow\; \gamma[\Delta] = \gamma_0(1 - i\Delta\tau_r)^{-1}. \tag{E.4}$$

Our equation for Δ becomes,

$$2\Delta = \frac{1}{\gamma_0}\frac{\Delta\tau_r}{1+\Delta^2\tau_r^2} \;\Rightarrow\; \Delta = 0, \text{ or } \Delta = \pm\tau_r^{-1}\sqrt{\frac{\gamma_0\tau_r}{2}-1} \tag{E.5}$$

which reproduces the same two values given in that paper. Since only the real solutions for Δ are relevant, as noted above, we get the restriction $\gamma_0\tau_r \geq 2$.

The corresponding $\Gamma_{i,s}(\omega,\Delta)$ are given by,

$$\Gamma_i(\omega,\Delta) = \frac{\left|\frac{\gamma_0}{2(1-i\Delta\tau_r)} - i\Delta\right|}{\left(\frac{\gamma_0}{2(1-i\tau_r(\omega+\Delta))} - i(\omega+\Delta)\right)}\ ,\ \Gamma_s(\omega,\Delta) = \frac{\left|\frac{\gamma_0}{2(1-i\Delta\tau_r)} - i\Delta\right|}{\left(\frac{\gamma_0}{2(1-i\tau_r(\omega-\Delta))} - i(\omega-\Delta)\right)}.$$

Note that these are unequal for a non-zero Δ.

The vanishing determinant condition, eq.(4.56), takes the form $\{\dots\}^2 = \mu^2|\dots|^2$. Thus the $\{\dots\}$ on the l.h.s. is real and the equation becomes,

$$\begin{aligned}
\mu\left|\frac{\gamma[\Delta]}{2} - i\Delta\right| &= \frac{\gamma[\omega+\Delta]}{2} - i(\omega+\Delta). \text{ For } \Delta = 0, \text{ this becomes,} \\
\frac{\mu\gamma_0}{2} &= \frac{\gamma_0}{2}\frac{1}{1-i\omega\tau} - i\omega = \frac{\gamma_0}{2}\frac{1}{1+\lambda\tau} + \lambda\ ,\ -i\omega =: \lambda \Rightarrow \\
\lambda_\pm &= -\frac{2-\gamma_0\mu\tau}{4\tau} \pm \frac{1}{4\tau}\sqrt{(2-\gamma_0\mu\tau)^2 - 8\gamma_0\tau(1-\mu)} \\
\therefore \lambda_\pm &= \frac{\gamma_0}{4}\left[\left(\mu - \frac{2}{\gamma_0\tau}\right) \pm \sqrt{\left(\mu + \frac{2}{\gamma_0\tau}\right)^2 - \frac{8}{\gamma_0\tau}}\right].
\end{aligned} \tag{E.6}$$

The eigenvalues $\lambda_\pm$ are exactly the same as given in [7]. Detailed analysis of the phases and their emergence as the kernel parameters are varied is given in [7]. For the present purpose, it suffices to note that for $\Delta = 0$, the roots $\lambda_\pm$ are both negative for $\mu < 1$ implying stability. This too is an example of two isolated zeros of the determinant, albeit in a non-Markovian context.

The power law memory kernel [9]: In terms of its Fourier transform, given in the form, $\gamma[\omega] = \gamma'[\omega] + i\gamma''[\omega]$, with $\gamma''[\omega]$ obtained from $\gamma'[\omega]$ using Kramers-Kronig relation. The Real part is given by, $\gamma'[\omega] := \left\{\gamma_0 + \gamma_{pl}|\omega/\omega_0|^s e^{-|\omega|/\omega_c}\right\}$. The $\gamma_0, \gamma_{pl}, \omega_0, \omega_c$ and the power s are all positive constants whose values do not concern us here. The experimentally realized power is $s < 1$. We use its low frequency asymptotic form given by [9],

$$\gamma[\omega] \;=\; \{\gamma_0 + \gamma_{pl}|\omega/\omega_0|^s\} + i\left\{-\gamma_{pl}|\omega/\omega_0|^s sgn(\omega)\ tan\big(\pi s/2\big)\right\}. \tag{E.7}$$

For $\omega = 0$, the imaginary part above is zero.

Possible (real) values of Δ are obtained from

$$2\Delta = Im(\gamma[\Delta]) = -sgn(\Delta)\gamma_{pl}|\Delta/\omega_0|^s tan(\pi s/2).$$

For $s < 1$, this can be satisfied only for $\Delta = 0$. For $1 < s < 3$, the above equation does admit a non-zero Δ.

The vanishing determinant condition then becomes,

$$\mu|\gamma[0]| = \gamma[\omega_*] - 2i\omega_* \;\Rightarrow\; \gamma_{pl}\left|\frac{\omega_*}{\omega_0}\right|^s \Big(1 - isgn(\omega_*)tan(\pi s/2)\Big) = (\mu - 1)\gamma_0 + 2i\omega_* \,. \tag{E.8}$$

If ω_* is real, then the imaginary part of the equation implies that $\omega_* = 0$ is the only solution. To find complex solutions, the Fourier transform $\gamma[\omega]$ has to be continued to the complex plane. We may replace $sgn(\omega_*)$ by $sgn(Re(\omega_*))$ in the continuation. The imaginary part of the equation then implies that $Re(\omega_*) = 0$ while the real part of the equation determines the $Im(\omega_*)$ as,

$$Im(\omega_*) = -\frac{1}{2}\left((1 - \mu)\gamma_0 + \gamma_{pl}\left|\frac{\omega_*}{\omega_0}\right|^s\right). \tag{E.9}$$

Since $Re(\omega_*) = 0$, the above is an equation for $Im(\omega_*)$ alone. For $\mu < 1$, the right hand side being negative implies stability. At criticality, vanishing determinant condition itself has $\mu = 1$ and the conclusion remains the same.

Writing $\omega_* = -i\zeta,\ \zeta > 0$ and $\mu = 1 - \epsilon$, the equation for ζ becomes,

$$2\zeta - \epsilon\gamma_0 = \gamma_{pl}\zeta^s\omega_0^{-s}. \tag{E.10}$$

Graphical method of solution indicates that there is only one solution and this is an example of a single isolated zero.

In the experiment, γ_0 was chosen to be 1.00 s^{-1}, γ_{pl} to be 2.12 s^{-1} and ω_0 to be 1 s^{-1}.

Appendix F

CORRELATORS OF THE STOCHASTIC FORCES

The equations stipulating the correlations of the stochastic forces (4.1) may be written as

$$\langle f_{i,s}(t) f^{\dagger}_{i,s}(t') \rangle = \rho_{i,s} \gamma_{i,s}(t-t') \quad , \quad \rho_{i,s} := \frac{1}{2}\big[1 - e^{-\hbar\omega_{i,s}/kT}\big]^{-1} \ . \tag{F.1}$$

Defining the Fourier transforms,

$$f_{i,s}(\omega) := \int_{-\infty}^{\infty} dt e^{i\omega t} f_{i,s}(t) \quad , \quad f^{\dagger}_{i,s}(\omega) := \int_{-\infty}^{\infty} dt e^{i\omega t} f^{\dagger}_{i,s}(t)$$

it follows that,

$$\langle f_{i,s}(\omega) f^{\dagger}_{i,s}(\omega') \rangle = 2\pi \rho_{i,s} \gamma_{i,s}[\omega] \delta(\omega + \omega') \ . \tag{F.2}$$

Note that these correlators have explicit dependence on the bath temperature through the $\rho_{i,s}$ *and* on the kernel parameters through the $\gamma_{i,s}[\omega]$.

We have the $g_{i,s}$ which related to the $f_{i,s}$ as,

$$g_{i,s}(\omega) = \left(\frac{g}{\left|\frac{\gamma_{i,s}[\Delta]}{2} - i\Delta\right| \sqrt{\frac{\gamma_P}{2} \left|\frac{\gamma_{s,i}[\Delta]}{2} - i\Delta\right|}} \right) f_{i,s}[\omega] \tag{F.3}$$

$$:= C_{i,s}(\Delta) \ f_{i,s}(\omega) \tag{F.4}$$

$$\therefore \langle g_{i,s}(\omega) g^{\dagger}_{i,s}(\omega') \rangle = \langle g_{i,s}(\omega)[g_{i,s}(-\omega')]^{\dagger} \rangle$$

$$= |C_{i,s}(\Delta)|^2 \{2\pi \rho_{i,s} \gamma_{i,s}[\omega]\} \delta(\omega + \omega') \ . \tag{F.5}$$

The correlators needed in eq.(4.66) are listed below.

$$\langle g_i(\omega+\Delta) g_i(-\omega'+\Delta)^* \rangle = |C_i(\Delta)|^2 \{2\pi\rho_i\gamma_i[\omega+\Delta]\}\, \delta(\omega+\omega') \tag{F.6}$$
$$\langle g_s(-\omega-\Delta)^* g_s(\omega'-\Delta) \rangle = |C_s(\Delta)|^2 \{2\pi\rho_s\gamma_s[\omega+\Delta]\}\, \delta(\omega+\omega') \tag{F.7}$$

$$\langle g_i(\omega+\Delta) g_i(\omega'+\Delta) \rangle = |C_i(\Delta)|^2 \{2\pi\rho_i\gamma_i[\omega+\Delta]\}\, \delta(\omega+\omega'+2\Delta) \tag{F.8}$$
$$\langle g_s(-\omega-\Delta)^* g_s(-\omega'-\Delta)^* \rangle = |C_s(\Delta)|^2 \{2\pi\rho_s\gamma_s[\omega+\Delta]\}\, \delta(\omega+\omega'+2\Delta) \tag{F.9}$$

$$\langle g_i(-\omega+\Delta)^* g_i(-\omega'+\Delta)^* \rangle = |C_i(\Delta)|^2 \{2\pi\rho_i\gamma_i[\omega-\Delta]\}\, \delta(\omega+\omega'-2\Delta) \tag{F.10}$$
$$\langle g_s(\omega-\Delta) g_s(\omega'-\Delta) \rangle = |C_s(\Delta)|^2 \{2\pi\rho_s\gamma_s[\omega-\Delta]\}\, \delta(\omega+\omega'-2\Delta) \tag{F.11}$$

$$\langle g_i(-\omega+\Delta)^* g_i(\omega'+\Delta) \rangle = |C_i(\Delta)|^2 \{2\pi\rho_i\gamma_i[\omega-\Delta]\}\, \delta(\omega+\omega') \tag{F.12}$$
$$\langle g_s(\omega-\Delta) g_s(-\omega'-\Delta)^* \rangle = |C_s(\Delta)|^2 \{2\pi\rho_s\gamma_s[\omega-\Delta]\}\, \delta(\omega+\omega') \tag{F.13}$$

REFERENCES

[1] D. S. Lemons and A. Gythiel. "Paul Langevin's 1908 paper "On the Theory of Brownian Motion" ["Sur la theoriie du mouvement Brownien," C. R. Acad. Sci. (Paris) 146, 530-533 (1908)]." In: *Am. J. Phys.* 65 (1997), p. 1079.

[2] R. Kubo. "The fluctuation-dissipation theorem." In: *Rep. Prog. Phys.* 29 (1966), p. 255.

[3] W. T. Coffey, Yu. P. Kalmykov, and J. P. Waldron. *The Langevin Equation*. World Scientific Publishing Co. Pte. Ltd., 2005.

[4] C. Chakram et al. "Dissipation in ultrahigh quality factor SiN membrane resonators." In: *Phys. Rev. Lett.* 112 (2014), p. 127201.

[5] Y. S. Patil et al. "Thermomechanical two-mode squeezing in an ultrahigh Q membrane resonator." In: *Phys. Rev. Lett.* 115 (2015), p. 017202. eprint: arxiv:1410.7109.

[6] S. Chakram, Y. S. Patil, and M. Vengalattore. "Multimode phononic correlations in a nondegenerate parametric amplifier." In: *New J. Phys.* 17 (2015), p. 063018.

[7] H. F. H. Cheung, Y. S. Patil, and M. Vengalattore. "Emergent phases and novel critical behavior in a non-Markovian open quantum system." In: *Phys. Rev. A* 97 (2018), p. 052116.

[8] S. C. Sundar. "Dissipation and nonlinear mechanics in ultrahigh quality factor membrane resonators". PhD thesis. Cornell University, Jan. 2015.

[9] Y. S. Patil et al. "Critical Behavior of a Driven Dissipative System : Universality Beyond the Markovian Regime." 2018.

[10] W. B. Bowen and G. J. Milburn. *Quantum Optomechanics*. CRC Press, Taylor & Francis group, 2016.

[11] U. Weiss. *Quantum Dissipative Systems*. World Scientific, 1999.

[12] C. W. Gardiner and P. Zoller. *Quantum Noise*. Springer-Verlag, 2000.

[13] Y. S. Patil. "Open quantum systems : controlling system-bath interactions and studying their influence." PhD thesis. Cornell University, May 2018.

[14] V. Balakrishnan. "Fluctuation-dissipation theorems from the generalised Langevin equation." In: *Pramana* 12 (1979), p. 301.

[15] G. Strang. *Linear algebra and its applications.* Books/Cole Thomson Learning, 1988.