INVESTIGATION OF RELATION BETWEEN EULER AND FLAT PLATE BUCKLING OF "L" SECTION STRUTS

Thesis by

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In partial fulfillment of the requirements for
the Degree of Master of Science in Aeronautical Engineering

California Institute of Technology
Pasadena, California
1934

This paper was presented on December 7, 1933 at the Annual Meeting of the American Society of Mechanical Engineers in New York City, New York, and is to be published in this form by that Society.

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The stiffener shapes used in monocoque, or for that matter any other kind of aeronautical construction, are of many different shapes and sizes. They are, however, usually made up of sheet material, formed into a great variety of shapes. They are accordingly subject to local wrinkling or buckling of this sheet under load[1]. Stiffeners of such shapes are ordinarily constructed and installed in lengths which are large in comparison to the cross-section, so that the sections are also liable to failure as long slender columns, that is as so-called "Euler columns". The stiffener thus has the possibility of buckling in two entirely different manners, i.e. as a Euler column or from local instability of the sheet of which it is composed. The question then arises as to whether these different types of buckling can be considered entirely independently of each other, or if there is a transition stage between them in which a combination buckling can take place at a lower loading than for either alone. It was to find an answer to this question, both for this case and for other cases of structures which can buckle in two or more independent ways, that this investigation was undertaken.

The tests were made with struts of equal leg "L" section, identical except for the widths of the sides. When the widths are small compared to the length such struts buckle as Euler columns, but for wide widths buckling of the sides occurs first.

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1) This type of buckling was considered by G. W. Trayer and H. W. March in N.A.C.A. Technical Report 382.

The greatest importance attaches to an accurate knowledge of the end fixity for column buckling and of the edge fixity for sheet buckling, and this type of specimen allowed all these conditions to be accurately determined. A column of this shape may be considered to be made up of two flat plates hinged along their common edges (since they can obviously buckle in such a way that the angle between the sides at the corner will remain the same, and thus there will be no moments at these edges[2]) with the other edges free. To obtain definite end conditions for the flat plate buckling, each side of the angle section was chamfered along its complete width at the ends to an angle of 60 degrees (see Fig. 1) insuring hinged edge conditions. The end blocks for supporting the specimen each had two 120 degree "V" grooves cut on one surface, the grooves intersecting at an angle of 90 degrees, to receive the ends of the specimens. In order to obtain definite end conditions for Euler buckling, a 5/8" hardened steel ball was inserted in these end blocks, making the struts "ball ended" columns, hinged, in effect, about the center of the balls. The struts were thus free to buckle in any direction but, of course, actually buckled about the axis of minimum moment of inertia of the section (perpendicular to the axis of symmetry).

As stated above, for small widths the struts buckle as Euler columns. The theory for this type of buckling is well known. It was developed by considering the bending of a column under compression as a whole, and consequently does not take into account the possibility of plate buckling nor stresses above the yield point of the material. The

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[2]L. H. Donnell: "The Problem of Elastic Stability", page 4, Transactions of American Society of Mechanical Engineers, 1933, Vol. 5, No. 4.

equation expressing Euler's theory for hinged end columns is[3)]

$$\sigma_c = \pi^2 E \left(\frac{r}{l}\right)^2$$

σ_c = critical buckling stress

E = modulus of elasticity of the material

l = length of column

r = radius of gyration of the cross-section of the column.

This theory has been checked very closely by Kármán[4)] who originated the general method of varying end conditions to counteract initial crookedness, which was used in these experiments.

For wide widths the sides buckle as flat plates. This type of failure has, to a certain extent, been neglected by engineers, as it is of more or less infrequent occurrence in general construction, where the plates used are usually thick in comparison to length and width. The stability of rectangular thin flat sheets under compression has been analyzed by Bryan for plates hinged on all four sides and by Timoshenko[5)] for other cases. The general expression for the uniformly distributed stress on the ends which will cause buckling of the plate is[2)]

$$\sigma_c = K E \left(\frac{t}{w}\right)^2$$

where

t = thickness of the plate

w = width of the plate

k = a constant, depending upon the ratio of the length to the width and upon the edge conditions.

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3) J. E. Boyd: Strength of Materials", page 236.

4) Th. von Kármán: "Mitteilungen uber Forschungsarbeiten", Ver. deuts. Ing., Berlin, 1909.

5) S. Timoshenko: "Strength of Materials", Part II, page 604.

Values of k for use with the above formula can be obtained from Fig. 7 of reference 2. For our case, of sheets hinged on three sides and free on the fourth, k is practically independent of the length-width ratio, for the proportions used in the tests, and has the value 0.4.

If all dimensions except the width are fixed, these two theories, for Euler and flat plate buckling, give two relations between σ_c and the width. These relations have been plotted on Fig. 2. It will be seen that the two curves form a sort of inverted "V". Obviously for widths up to the intersection of the two curves, the strut should theoretically fail as a Euler column, while for wider widths the strut should fail by buckling of the sides. The weight-strength ratio is, of course, proportional to σ_c and this figure shows that the maximum weight-strength ratio for this case is obtained for a width of 0.77" while for widths above or below this the ratio drops off greatly.

Nineteen satisfactory specimens were finally obtained, identical except for varying width, and all as free from initial eccentricities as is believed possible without an expenditure greatly out of proportion to the benefits to be obtained by the added refinement.

General experience is that in any buckling experiments it is impossible to make specimens sufficiently accurate to entirely eliminate eccentricities and accordingly special provisions were taken to provide a means of eliminating initial eccentricities for Euler buckling. This was done by providing an adjustment in the end blocks whereby the ball could be moved relative to the axis about which buckling takes place. The horizontal relation between the center of the ball and the specimen could be varied along the axis of the greatest moment of inertia of the specimen so that the effect of initial eccentricities could be removed (see "Method of Testing" of this section for a more complete description of apparatus). In this way very sharp buckling was obtained for Euler columns, as will be seen from Fig. 2. With these methods of construction the conditions of fixity then became known, namely for each plate, hinged at one side and at both ends and, for the specimen as a Euler column, hinged at both ends. The specimens were made of such proportions that the buckling stresses were always far below the yield point of the material, eliminating any question of the failure of the material.

The effect of initial eccentricities upon the flat plate buckling was compensated for by use of a method developed by Southwell[6], which is briefly a mathematical method of finding the probable buckling strength which a structure would have if it had no initial eccentricities, from measurements of small deflections. This method is based upon the fact that in this region the load deflection curve approximates a rectangular hyperbola having the axis of zero deflection as one asymptote and the theoretical load deflection curve for no initial eccentricities as the other asymptote. (See "Method of Testing" of this section for a more complete description of the manner of use of this method).

Plates supported on three sides and free on the fourth buckle in one half wave in the lengthwise direction. In our case this amounted to a twisting of the middle portion of the strut about a longitudinal axis which was readily measured by the use of long light metal pointers attached to the sides of the specimen.

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[6] R. V. Southwell: "On the Analysis of Experimental Observations in Problems of Elastic Stability", Proc. Roy. Soc., A, Vol. 135, 1932.

Fig. 3 shows the critical loads for the cases in which plate buckling occurred as recorded from the tests without the use of Southwell's method. It will be seen that the experimental scatter was quite large and most of the points are above the theoretical curve. Several reasons can be advanced for this. In these cases, when the ultimate buckling strength was obtained, the deflections were very large. The theory is based on the assumption of very small deflections, and it is possible that a theory of large deflections would show greater values. But a more probable explanation of this discrepancy is that when the deflections became large, it was no longer possible to keep the stress distribution along the width of the sides uniform.

By concentrating the stress at or near the corner of the angle of the specimen, the load can be carried far above the theoretical load for uniform stress distribution, due to the fact that the hinged edge is, of course, very much more stable than the free edge. By using Southwell's method based on deflection readings taken when the deflections were still small, experimental results are obtained which can justly be compared to the theory based on no initial curvatures and uniform stress distribution. The ultimate buckling loads without the use of Southwell's method were not critical and were largely a matter of judgment, it being possible to run the load up almost indefinitely if the end blocks were adjusted purposely to shift the stress towards the corners of the angle. Fig. 2 shows the critical buckling load of the specimens computed by Southwell's method from the last set of data referred to above. The scatter is greatly diminished and all points fall on or very close to the theoretical curve. The authors place a great deal more confidence in this method of obtaining the critical buckling stress than in relying simply on judgment, where it is probable that no two persons would agree exactly.

It is believed that the discrepancies occurring between the theoretical curve and the points obtained experimentally are well within the limits of experimental accuracy and that the results obtained therefore check the theory for Euler and flat plate buckling under the existing conditions.

It would appear from Fig. 2 that a very small fillet may exist in the experimental points near the apex of the inverted "V" of the theoretical curves. The authors believe, however, that the apparent fillet may be merely a coincidence due to the experimental scatter and the difficulty of overcoming initial eccentricity and hence obtaining a clear cut type of failure in this region. It is believed to be established that any relation which does exist between Euler and flat plate buckling has a very small effect and exists only in the immediate vicinity of the intersection of the two theoretical curves for the two types of buckling.

It is felt that the results of this test establish the optimum dimensions for such a shape and that the engineer can design with confidence to the stress indicated by the intersection of the two curves provided a reasonable factor of safety is included.

Method of Testing

The tests were made on a Riehle Brothers testing machine of 3,000 pounds load capacity. The machine was of the tension type, converted for compression tests by means of the cage shown in the accompanying photographs of the apparatus.

The specimens tested were all 22" in length and the width of the sides varied from .405" to 2.025". They were made of 24SRT aluminum alloy of .025" thickness, as this material and gauge is being used to a large extent by the aircraft industry in this section of the country. The width of the side was taken as the overall width minus half the thickness of the sheet.

These specimens were made in an ordinary hand brake. It was necessary to adjust the brake for each width of specimen to get an exactly 90° bend and a uniform radius at the corner; this radius was made as small as the material would stand, about 1/16". The narrower specimens were liable to come out with considerable initial curvature, due to deflection of the brake, the peening of the material along the sheared edges, etc. It was found possible to eliminate most of this by properly padding the brake with strips of paper.

Tests by the Bureau of Standards for the National Advisory Committee for Aeronautics on duralumin sheet[7)], show a marked difference in the stress strain curve taken normal or parallel to the direction of rolling. The authors felt that due to the shape of the sheet found in commercial practice, most long angle shapes would be made with the angle parallel to the direction of rolling, and consequently the specimens were made in this manner.

[7)] E. E. Lundquist: "The Compressive Strength of Duralumin Columns of Equal Angle Section", N.A.C.A. Technical Note 413, 1932.

Specially constructed end blocks, identical for both ends of specimen, definitely established the desired end conditions. A sketch of the end block is shown in Fig. 1. The block A was machined on the upper side with 120 degree "V" grooves which made an angle of exactly 90 degrees with each other. These grooves received the ends of the specimens which had been filed to an angle of 60 degrees. In filing the ends great care was taken not to file away any of the center line.

A key and keyway between A and B allowed a movement of A (and hence the end of the specimen) relative to E (the point of application of the load). This movement could be made under load by means of the wing nuts C. The motion between A and B was such that E always moved along the axis of symmetry of the specimen. This insured an equal stress distribution on each leg and left the specimen free to buckle as a Euler column about the axis of least moment of inertia, (which is perpendicular to the axis of symmetry).

To detect Euler failure of the specimens, an Ames dial gauge was used, with counterbalancing spring removed. A small metal strip was attached to the gauge point, which made point contact with the back of the angle at the middle of the length of the specimen. The gauge was also mounted with the slide making point contact at the inside of the angle, and results were the same as for the previous position. Both methods of attachment are shown in the accompanying photographs of the apparatus. The authors feel that in either position some slight rigidity was given to the specimen, but that the results were not sufficiently in error to warrant setting up an optical system for this measurement.

To detect flat plate buckling, long thin metal strips about ten inches long were attached to the free edges with paper clips (as shown in accompanying photographs of the specimens and the apparatus). These pointers magnified the movement of the free edges.

For each test, the end blocks were set so that the axis of the least moment of inertia of the specimen was in a vertical plane with the point of contact of the ball D and the base plates of the cage. The specimen and end blocks were then mounted in the testing machine, a light load applied, and the measuring devices (Ames gauge and pointers) were put in place. Both measuring devices were read as small increments of load were applied. Due to initial eccentricities or curvatures, it was found that the Ames gauge would show a deflection of the column at comparatively low loads. By adjusting the wing nuts C so as to remove the apparent eccentricity a great load could be applied without obtaining a deflection. This was repeated until finally the column would not stand a greater load without buckling. The loads at which Euler buckling occurred were for the most part very critical as the dial gauge remained perfectly steady and then for one pound increase in load would show a very large deflection. The pointers (indicating plate buckling) remained practically steady during the test of a specimen which failed as a Euler column.

Where wider widths were used and flat plate buckling occurred before Euler buckling, the actual measured results were not so significant, as there was no definite point at which this buckling occurred, due to initial eccentricities. Ranges were found for each specimen where, with the proper end adjustment, there was no Euler deflection, but where the edges showed considerable deflection with increase in load; these deflections were measured by the pointers and were used to compute the critical load by Southwell's method.

The use of this method consisted in first tabulating the deflections (δ) against the loads (P) which caused them, and from this obtaining δ/P. δ is then plotted against δ/P on ordinary cross-section paper and the slope of a straight line faired through the points so obtained is measured. Southwell proves that this slope is equal to the critical load at which buckling would theoretically take place if there were no initial curvatures or eccentricities. Table I gives the values for δ/P and P obtained for five typical runs. For specimen Number 9 the method was applied with two sets of deflections, one covering a range of loadings from 60 to 120 pounds and the other from 120 to 136 pounds; the resulting P from each was very nearly the same. Fig. 4 shows the method of plotting these values for specimen Number 12, which was typical for all of them.

Acknowledgment

The authors wish to thank Dr. Th. von Kármán, director of the Guggenheim Aeronautics Laboratory of the Institute, for the opportunity of making this research. The research was suggested by and carried out under the direction of Dr. L. H. Donnell. Acknowledgment is also due to Dr. A. L. Klein and E. E. Sechler for numerous helpful suggestions.

TABLE I

	Load	Deflection	δ / P
Specimen #17:			
	67	0	
	70	.2R	.00286
	72	.5	.00695
	74	.7	.0094
	76	1.0	.01315
	78	1.3	.0167
	80	1.8	.0225
	82	2.4	.0293
	84	3.0	.00357
Specimen #12:			
	60	0	0
	62	.2R	.00323
	64	.3	.0047
	66	.3	.00455
	68	.4	.00588
	70	.4	.00571
	72	.4	.00555
	74	.5	.00675
	76	.5	.00658
	78	.6	.0077
	80	.7	.00875
	82	.8	.00975
	84	.9	.01072
	86	1.0	.01165
	88	1.1	.0125
	90	1.2	.01335
	92	1.5	.0163
	94	1.8	.0192
	96	2.5	.0261
	98	6.5	.0664
Specimen #11:			
	80	0	0
	85	0	0
	95	.5L	.00577
	100	1.0	.0100
	105	2.0	.01905
	110	5.0	.0455
	111	6.0	.0541
	112	8.5	.0758
Specimen #1:			
	90	.1	.00111
	95	.12	.001264
	100	.13	.0013
	105	.14	.001333
	110	.17	.001545
	115	.22	.001913
	118	.25	.00212

Specimen #9:
(at 60-120# loads)

Load	Deflection	δ/P
64.5	.2	.0031
74	.6	.00812
83	1.0	.0121
90	1.4	.0157
95.5	1.8	.0189
100.5	2.2	.022
104.3	2.6	.0249
108.5	3.0	.0277
111.5	3.4	.0305
113.7	3.8	.0334

Typical data taken for determination
of critical load by Southwell's method

TABLE II

1	147.3	1.0255	2875
2	102	.6475	3145
3	65	2.0255	641
4	148	.767	3860
5	73	.6000	2435
6	52	.545	1907
7	130	.7175	3620
8	159	.8455	3760
9	138.1	.9735	2840
10	129.5	1.0355	2500
11	113.2	1.1595	1955
12	104.5	1.2855	1625
13	83.8	1.5385	1090
14	23	.4050	1135
15	112	1.0825	2075
16	111.6	1.2215	1828
17	89.5	1.4005	1280
18	75	1.7705	847
19	156	.8041	3880

Values of load, width and σ_c for each specimen

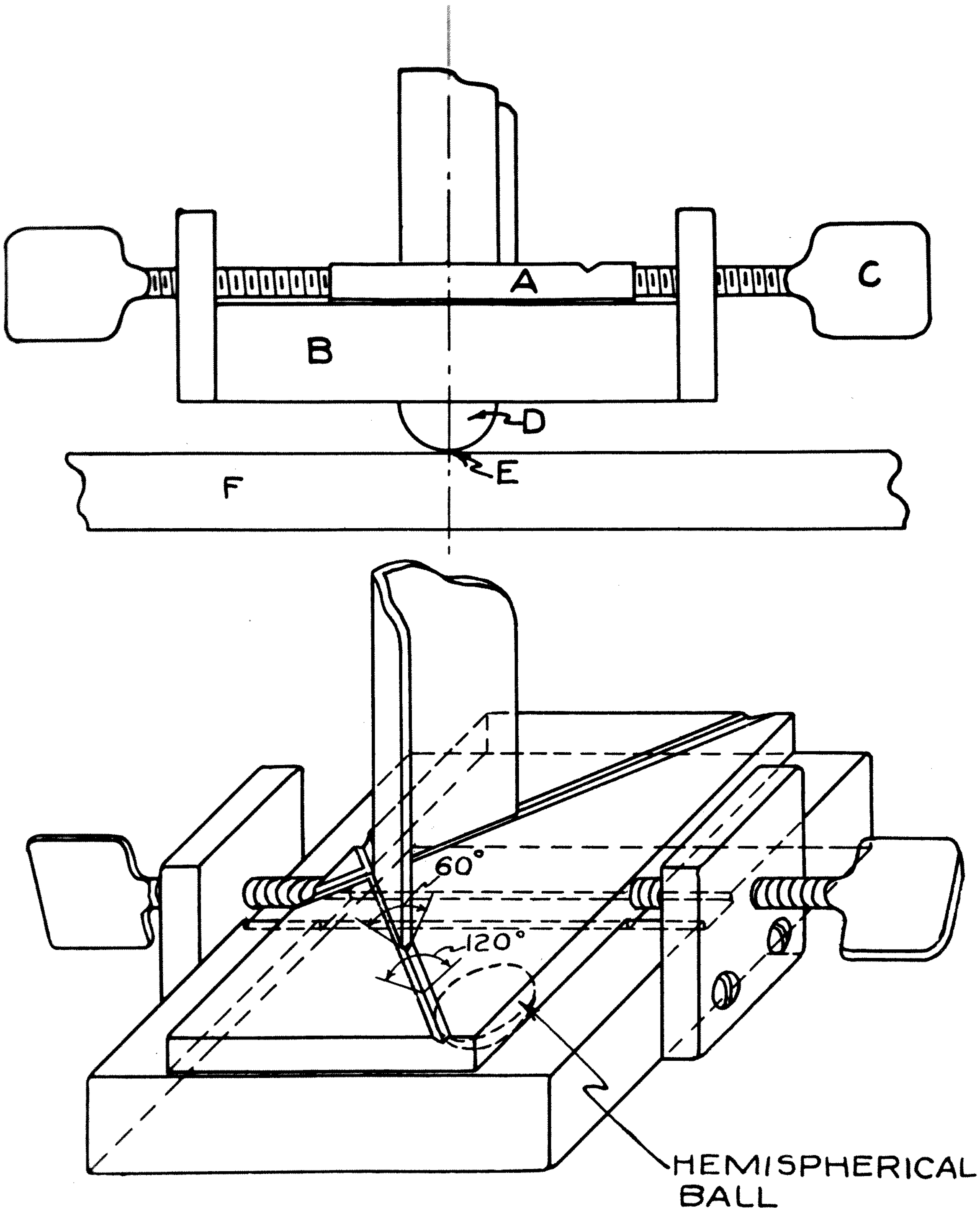
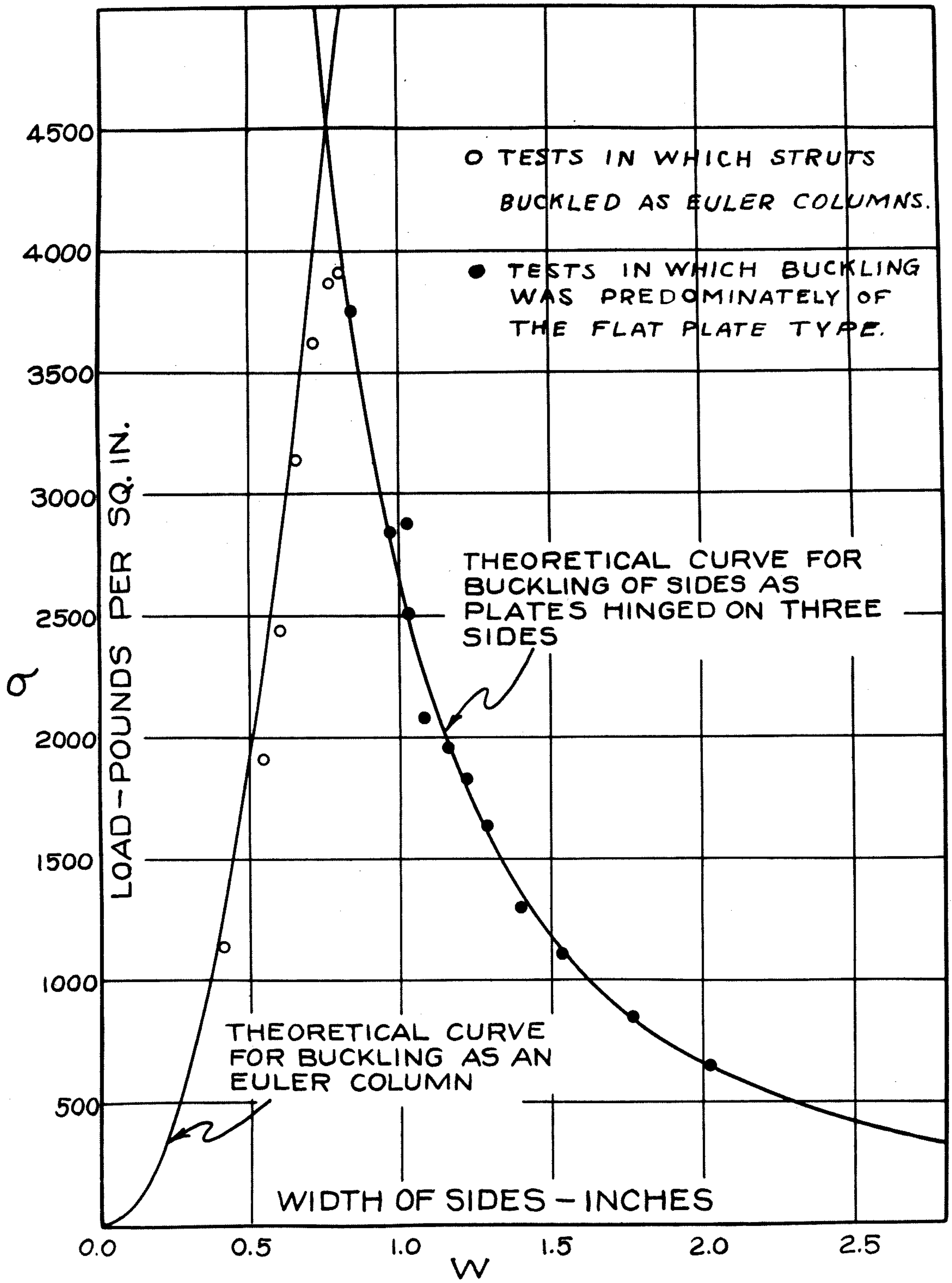


FIG. [redacted] 1.



4500
4000
3500
3000
2500
2000
1500
1000
500
σ
LOAD - POUNDS PER SQ. IN.
O TESTS IN WHICH STRUTS BUCKLED AS EULER COLUMNS.
● TESTS IN WHICH BUCKLING WAS PREDOMINATELY OF THE FLAT PLATE TYPE.
THEORETICAL CURVE FOR BUCKLING OF SIDES AS PLATES HINGED ON THREE SIDES
THEORETICAL CURVE FOR BUCKLING AS AN EULER COLUMN
WIDTH OF SIDES - INCHES
0.0
0.5
1.0
1.5
2.0
2.5
W

FIG. 2.

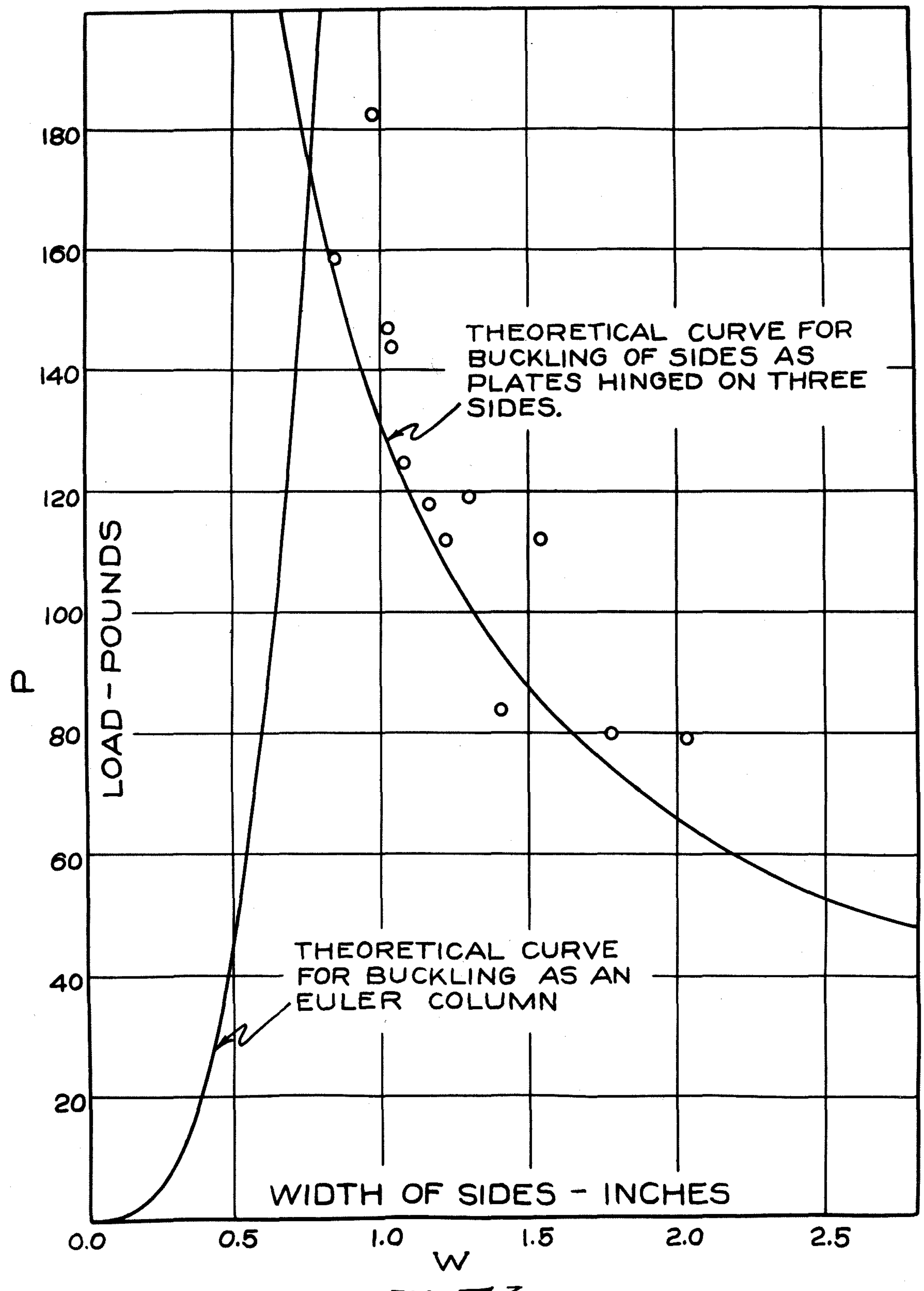
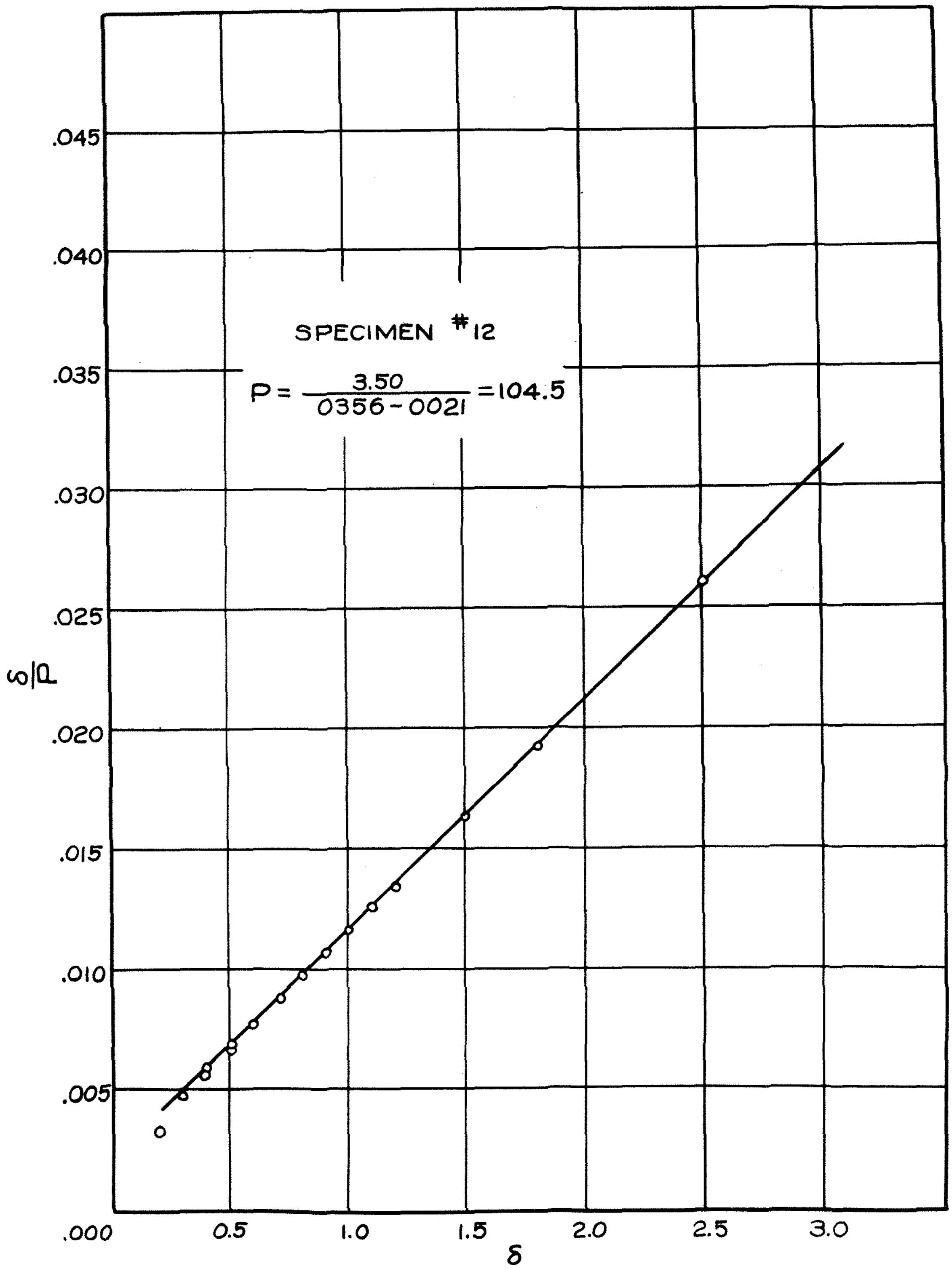


FIG. [redacted] 3.



SPECIMEN #12
P = 3.50 / 0356 - 0021 = 104.5
.045
.040
.035
.030
.025
.020
.015
.010
.005
.000
δ/P
0.5
1.0
1.5
2.0
2.5
3.0
δ

FIG. 4.

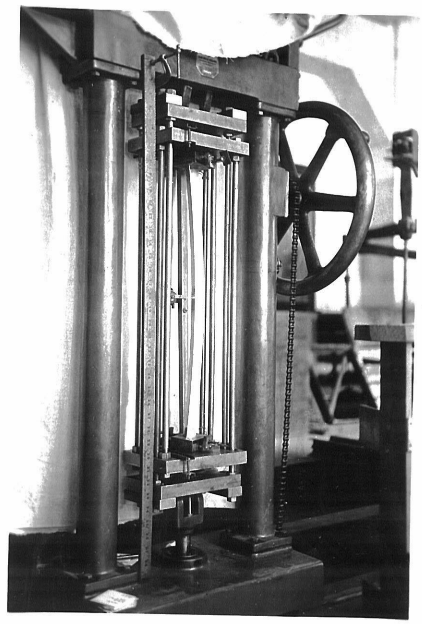


FIG. 5

General View of Apparatus Showing an Example of Euler Failure

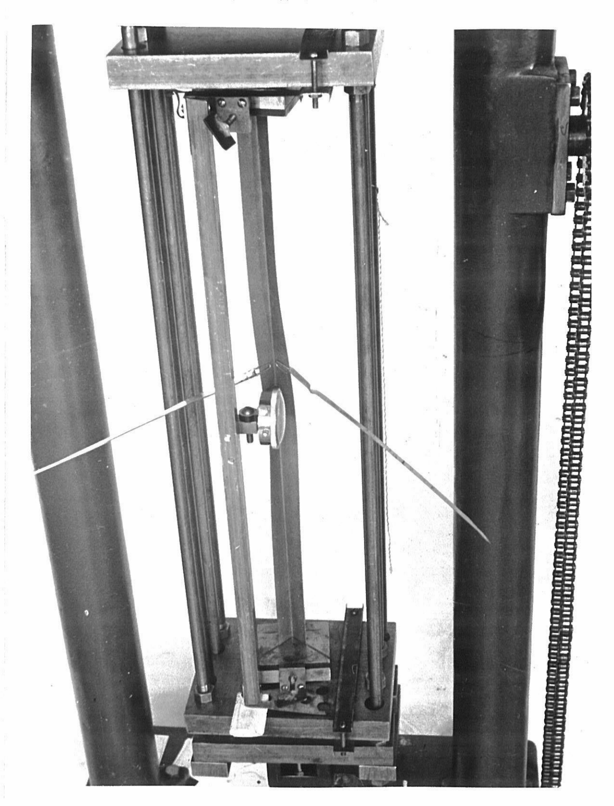


FIG. 6

View of Specimen in Machine Showing Carriage,
Method of Mounting Specimen in End Blocks and Scales
for Reading Lateral Deflections

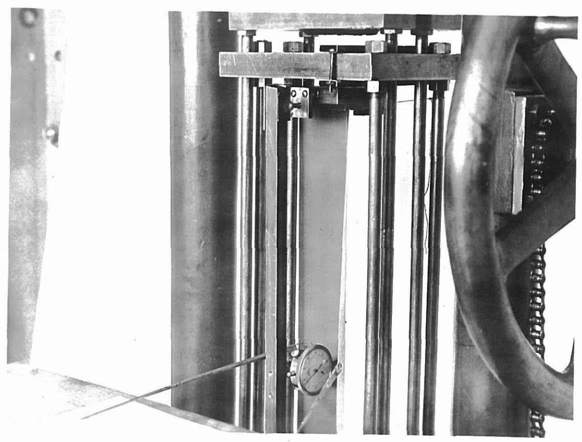


FIG. 7

General View of Apparatus Showing One Type of Flat Plate Buckling Encountered