THIN CYLINDRICAL SHELLS SUBJECTED TO VARIOUS TYPES OF CONCENTRATED LOADS

By

Shao Wen Yuan

A DISSERTATION SUBMITTED IN PARTIAL FULFILMENT OF

THE REQUIREMENT FOR THE DEGREE OF

DOCTOR OF PHILOSOPHY

In The

CALIFORNIA INSTITUTE OF TECHNOLOGY

PASADENA, CALIFORNIA

1942

FORWARD

The bending of thin cylinder shells based on the general theory of elasticity is of interest not only to the mathematician but also to the engineer. The general theory of the shells has recently been developed to the point that it is now being used by engineers as a basis for the design of this type of structure.

The author wishes to acknowledge his appreciation to Professor Th. Von Kármán for suggesting the subject and his invaluable encouragement which made this paper possible. He also wishes to express his thanks to Dr. H. S. Tsien for various valuable suggestions.

TABLE OF CONTENTS

SUMMARY

It is shown that the calculation of stresses and strains in thin cylindrical shells subjected to various kinds of concentrated loads can be obtained if the specified loading function is represented by a Fourier integral in the longitudinal direction and by a Fourier series in the circumferential direction. The components of displacements are represented in like manner. In Part I an infinitely long cylinder loaded with two equal and opposite forces acting at the ends of a vertical diameter is discussed. The expression for the radial deflection in a thin cylinder of finite length was obtained from the corresponding solution for an infinitely long cylinder by using the method of images. The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference were also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The application of the solution for the radial deflection in an infinitely long cylinder to an elastically supported flat plate, or beam, in order to find the modulas of foundation is briefly discussed. Part II deals with an infinitely long cylinder subjected to two equal and opposite torque acting about the radial axis on the surface of the shell. The shearing stress-resultant was determined and the torque produced by the force-resultants is verified mathematically equal to the applied torque. A method was developed to calculate the shearing stress distribution near the s-axis because the series in the solution converges very slowly when x approaches zero.

INTRODUCTION

The subject matter of this paper is the calculation of stresses and strains in thin cylindrical shells subjected to various kinds of concentrated loads, and more specifically in infinitely long cylinders. The problem of curved plates or shells was first treated in about the year 1874 by H. Aron from the point of view of general equations of elasticity. He expressed the equations of the middle surface by means of two parameters, somewhat as Gauss did, and he adapted the problem to the method which had been used for plates. He arrived at expressions for the potential energy of the strained shell, similar to the expressions developed by Kirchhoff for plates, but the quantities that define the curvature of the middle surface were replaced by the differences of their values in the strained and unstrained state.

Lord Rayleigh proposed a theory for vibrating shells which embodied the idea that the middle surface of the vibrating body does not undergo any extension, and he determined the displacement of a point of the middle surface in accordance with this condition. Later it was shown that contrary to this condition a vibrating shell undergoes extensional strain. However, the region of this extensional strain is confined to a narrow strip near the edge of the shell, and the greater part of the shell vibrates according to Lord Rayleigh's assumptions.

The inextensional deformation of cylindrical and spherical shells was treated in detail by Lord Rayleigh in his "Theory of Sound." This type of deformation is the assumption underlying the solution of many problems of practical importance, such as the determination of stresses in thin cylindrical shells subjected to two equal and opposite forces acting at the ends of a diameter, or to internal hydrostatic pressure. The results obtained in the case of the first problem indicate that inextensional deformations correspond only to a first approximation of the complete solution, and the stresses in the proximity of the points of application of the forces are not given with sufficient accuracy. For this reason a method is needed which can deal with the case when the cylindrical shell is so loaded that its middle surface undergoes extension as well as change of curvature. Under these circumstances it is thought best to resort to the general theory* of thin plates and shells from which special cases like that of the cylindrical or spherical shell may be derived.

The application of the method of series to problems of equilibrium of elastic solid bodies was initiated by Lamé and Clapeyron. They considered the case of a body bounded by an unlimited plane to which pressure is applied according to an arbitrary law. However, previous to the discovery of the general equations of elasticity this method had already been used in problems of astronomy, acoustics, and heat conduction. In the above mentioned problem of Lamé and Clapeyron, the solutions of the differential equations of equilibrium can be expressed by definite integrals, the elements of the integrals representing the effects of singularities distributed over the surface. This class of solutions constitutes an extension of the methods

---

* See A.E.H. Love. Math. Theory of Elasticity, (1927), Pages 513-536.

introduced by Green in the Theory of the Potential. The method of singularities was first applied to the theory of elasticity by E. Betti, who set out from a reciprocal theorem of the type that is now familiar in many branches of mathematical physics. The average strain of any type that is produced in a body by given forces can be determined by a formula incidentally deduced from this theorem. Furthermore, Lord Kelvin gave the fundamental particular solution which expresses the displacement due to a force at a point in an indefinitely extended solid.

The present paper consists of two parts. In Part I an indefinitely long cylinder loaded with equal and opposite forces acting at the ends of a vertical diameter is discussed. The equations of equilibrium of an element of a cylindrical shell undergoing small displacements due to a lateral distributed external load are reduced to a single differential equation of the eighth order in the radial displacement. In this single equation all terms could be compared on a common ground and it was possible to decide which terms could be safely neglected. The specified loading function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. The Fourier coefficients and the undetermined function in the Fourier integral were determined from the loading condition which represents a concentrated load. The radial displacement is represented in a like manner with the aid of an undetermined function which was obtained by substituting both radial displacement and loading expressions in the differential equations. The definite integrals involved in the expression for radial deflection were evaluated by means of Cauchy's Theorem of Residues.

The expression for the radial deflection in a thin cylinder of finite length was obtained from the corresponding solution for an infinitely long cylinder by using the method of images. It is seen that the difference of these two radial deflections can be given by a correction factor included in the expression for a cylinder of finite length. This difference is believed to be caused by the restrained edges at the two ends of the finite cylinder.

The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference were also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The action of the couple is equivalent to that of two equal and opposite forces acting at an infinitely small distance apart. It is believed that the solution for the radial deflection in an infintely long cylinder may also be applied in order to find the modulas of foundation of an elastically supported flat plate, or beam, under a concentrated transverse load.

Part II deals with an infinitely long cylindrical shell subjected to two equal and opposite torques acting about the radial axis on the surface of the shell. The solution of this problem was achieved by replacing the torque with two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other. The components of the displacements u,v, and w and the loading functions are represented similarly as in Part I.

The shearing stress-resultant was determined from the solutions of the displacements u and v. In order to verify that the total torque produced by the force-resultants is equal to the applied torque all the force-resultants multiplied by their corresponding moment arms were summed up. The result proved to be satisfactory both in rectangular and polar coordinates.

In the numerical calculation of the shearing stress distribution near the s-axis (from the expression derived) it was found that ordinary methods failed because the series converges very slowly when x approaches zero. Hence a powerful method was developed in order to overcome this difficulty.

It is thought that this solution may have some application to the practical problem of the shearing stress distribution on the skin of a fuselage affected by the distortion of the wings.

A discussion of the curves plotted in the Appendix may be found in the Conclusion.

PART I

THIN CYLINDRICAL SHELL LOADED WITH TWO EQUAL AND OPPOSITE FORCES

1. FUNDAMENTAL EQUATIONS

The fundamental equations of a cylindrical shell under the specified loading are obtained from considerations of the equilibrium of an element cut out of the shell such as shown in Fig. 1*.

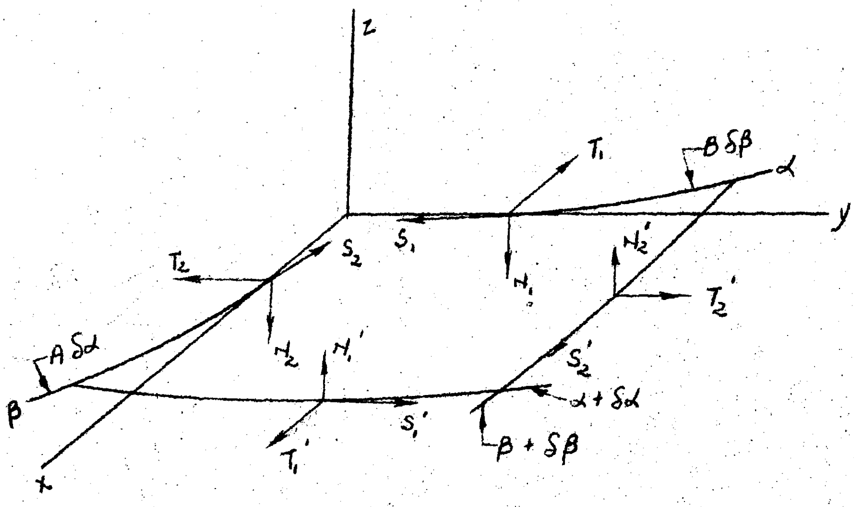


Fig. 1

Notations and conventions are the same as those commonly used in the three dimensional theory of elasticity. They are shown in Fig. 1. Symbols representing external forces have the same subscripts as used by Love in his book entitled "Mathematical Theory of Elasticity". Moments and rotations are represented by vectors corresponding to the right-hand rule. The positive

---

* A shell of double curvature.

sense of displacements, external forces and moments, and of internal forces and moments, is the same as the positive sense of the coordinate axes, if the external normal of the element concerned points in the positive direction of a coordinate axis. If it points in the negative direction, the opposite sign rule holds.

Definitions of some of the symbols follow:

h ---- Thickness of the cylindrical shell.

a ---- Radius of the cylindrical shell.

φ ---- Angle measured counter-clockwise from a line extending downwards from the center of the cylindrical shell to any point of the shell.

x, s, z ---- Longitudinal, circumferential and radial coordinates, measured axially from the normal section at the middle of cylindrical shell, and circumferentially from same generatrix.

u, v, w ---- Longitudinal, circumferential and normal displacements of points in the middle surface of the wall.

E ---- Modulus of elasticity of the material.

ν ---- Poisson's ratio.

D ---- Flexural rigidity

$$\frac{Eh^3}{12(1-\nu^2)}$$

P ---- External concentrated load.

q ---- External distributed load.

λ, n ---- Number of axial and circumferential waves.

In Fig. 1* T_1, S_1, N_1 are stress-resultant measured per unit length of the middle surface acting on the side α in the negative direction of the x, y, z, axes. The lenght of the element on which T_1, S_1, N_1 act is $B\delta\beta$. The stress-resultants acting on the side $\alpha + \delta\alpha$ are T_1', S_1', N_1'. Similarly T_2, S_2, N_2 are stress-resultants in the direction of the x, y, z axes acting on the side β on an element of length $A\delta\alpha$, and T_2', S_2', N_2' are stress-resultants acting on the side $\beta + \delta\beta$. The axes of the stress couples H_1, G_1 have the same directions as T_1, S_1 while those of H_2, G_2 have the same directions as T_2, S_2.

Since $B\delta\beta$ is the length of the side of the rectangle on which T_1 acts the stress resultant on side α in the direction of the x-axis is:

$$-T_1 B\delta\beta$$

This is so since all stress resultants and couples are measured per unit length of the middle surface of the shell. In resolving the forces that act on the side $\alpha + \delta\alpha$ parallel to the x-axis we have to allow for the change of α into $\alpha + \delta\alpha$, and for small rotations ($p_1'\delta\alpha$, $q_1'\delta\alpha$ $r_1'\delta\alpha$). Hence the components parallel to the x-axis of the force acting on side $\alpha + \delta\alpha$ are

$$T_1 B\delta\beta + N_1 B\delta\beta \cdot q_1'\delta\alpha - S_1 B\delta\beta \cdot r_1'\delta\alpha + \delta\alpha \frac{\partial}{\partial\alpha}(T_1 B\delta\beta)$$

Similarly the force parallel to x acting on side β is:

$$-S_2 A\delta\alpha$$

and that parallel to x acting on side $\beta + \delta\beta$ is:

$$S_2 A\delta\alpha + N_2 A\delta\alpha \cdot q_2'\delta\beta - T_2 A\delta\alpha\, r_2'\delta\beta + \frac{\partial}{\partial\beta}(S_2 A\delta\alpha)\delta\beta$$

---

* See A.E.H. Love. The Math. Theory of Elasticity, Page 534-536.

If X* is the x-component of the external force per unit area of the middle surface, then, since the area of the curvilinear rectangle of Fig. 1 is $AB\,\delta\alpha\,\delta\beta$, all the above enumerated forces add up to

$$\frac{\partial}{\partial\alpha}(T_1 B\,\delta\beta)\,\delta\alpha + \frac{\partial}{\partial\beta}(S_2 A\,\delta\alpha)\,\delta\beta - r_1' S_1 B\,\delta\alpha\,\delta\beta - r_2' T_2 A\,\delta\alpha\,\delta\beta$$
$$+ q_1' N_1 B\,\delta\alpha\,\delta\beta + q_2' N_2 A\,\delta\alpha\,\delta\beta + AB X'\,\delta\alpha\,\delta\beta = 0$$

Dividing through by $\delta\alpha\,\delta\beta$ one obtains

$$\frac{\partial(T_1 B)}{\partial\alpha} + \frac{\partial(S_2 A)}{\partial\beta} - (r_1' S_1 B + r_2' T_2 A) + (q_1' N_1 B + q_2' N_2 A) + AB X' = 0$$

Furthermore, in the case of a cylinder the following relations hold (see Fig. 2)

$$\alpha = x, \quad \beta = \varphi, \quad B = a$$
$$A = 1, \quad X' = 0, \quad \frac{1}{R_1} = 0$$
$$\frac{1}{R_2} = \frac{1}{a}$$

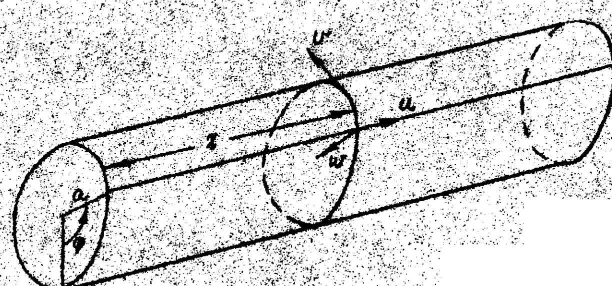


Fig. 2

and therefore: *

$$r_1' = \frac{\partial^2 v}{\partial x^2}, \qquad r_2' = \frac{\partial v}{\partial x\,\partial\varphi} - \frac{\partial w}{\partial x},$$
$$q_1' = -\frac{\partial^2 w}{\partial x^2} \qquad q_2' = -\frac{\partial^2 w}{\partial x\,\partial\varphi} + \frac{\partial v}{\partial x}$$

---

* See A.E.H. Love, Loc. cit. ante, Page 523.

It will be shown subsequently that:

$$S_1 = 6\frac{(1-\nu)}{h^2} D\left(\frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \varphi}\right)$$

Then for instance the term $r_1^2 \; s_1 B_1$ in eq. (a) above becomes

$$6\frac{1-\nu}{h^2} D\left(\frac{\partial v}{\partial x}\cdot\frac{\partial^2 v}{\partial x^2} + \frac{1}{a}\frac{\partial u}{\partial \varphi}\cdot\frac{\partial^2 v}{\partial x^2}\right) a$$

Since the displacements are small, the effect of the angular changes upon the components of the forces is of a second order of smallness and can be neglected. Therefore, after simplification Eq. (a) can be written:

$$a\frac{\partial T_1}{\partial x} + \frac{\partial S_2}{\partial \varphi} = 0$$

which is one of the simplified equations of equilibrium. In an analogous manner one may obtain the other two equations of equilibrium of forces, and the three equations of equilibrium of moments. Consequently, for the equilibrium of forces we have the following three equations

$$\left.\begin{aligned} & a\frac{\partial T_1}{\partial x} + \frac{\partial S_2}{\partial \varphi} = 0 \\ & \frac{\partial T_2}{\partial \varphi} + a\frac{\partial S_1}{\partial x} = 0 \\ & a\frac{\partial N_1}{\partial x} + \frac{\partial N_2}{\partial \varphi} + T_2 + qa = 0 \end{aligned}\right\} \qquad (1)$$

and for the equilibrium of moments

$$\left.\begin{aligned} & a\frac{\partial H_1}{\partial x} + \frac{\partial G_2}{\partial \varphi} + N_2 a = 0 \\ & a\frac{\partial G_1}{\partial x} + \frac{\partial H_2}{\partial \varphi} - N_1 a = 0 \\ & (S_1 + S_2)a = 0 \end{aligned}\right\} \qquad (2)$$

in which q is the normal pressure on the element.

Using the first approximation [1] in which certain small components of displacement and their differentials with respect to α and β as well as products of certain other small quantities are omitted, we have:

$$\left.\begin{aligned} S &= S_2 = S_1 \\ H &= -H_2 = H_1 \end{aligned}\right\} \qquad (3)$$

Substituting from (2) into (1) and making use of (3) we get:

$$\left.\begin{aligned} & a\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial \varphi} = 0 \\ & \frac{\partial T_2}{\partial \varphi} + a\frac{\partial S}{\partial x} = 0 \\ & -\frac{2}{a}\frac{\partial^2 H}{\partial \varphi \partial x} + \frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_2}{a^2 \partial \varphi^2} + \frac{T_2}{a} + q = 0 \end{aligned}\right\} \qquad (4)$$

These three equations of equilibrium combine the six equations (1) and (2). We shall transform Eq. (4) with the aid of the relation [2] between the stress resultants and the deformations:

$$T_1 = \frac{Eh}{1-\nu^2}(\epsilon_1 + \nu\epsilon_2); \quad T_2 = \frac{Eh}{1-\nu^2}(\epsilon_2 + \nu\epsilon_1)$$

$$S_1 = S_2 = S = \frac{\gamma Eh}{2(1+\nu)}$$

$$G_1 = -D(K_1 + \nu K_2); \quad G_2 = -D(K_2 + \nu K_1)$$

$$H = H_1 = -H_2 = D(1-\nu)\tau$$

---

1. See A.E.H. Love. Loc. cit. ante, Page 528.

2. Ibid., Page 530.

Where D is the flexural rigidity.

Moreover, the extensional and flexural strains in the middle surface are:*

$$\epsilon_1 = \frac{\partial u}{\partial x}; \quad \epsilon_2 = \frac{1}{a}\frac{\partial v}{\partial \varphi} - \frac{w}{a}; \quad \gamma = \frac{\partial v}{\partial x} + \frac{1}{a}\frac{\partial u}{\partial \varphi}$$

$$K_1 = \frac{\partial^2 w}{\partial x^2}; \quad K_2 = \frac{1}{a^2}\frac{\partial w}{\partial \varphi^2}; \quad \tau = \frac{\partial^2 w}{a\,\partial x \partial \varphi}$$

Hence, equations (4) can be put in the form of three equations with the three unknowns u, v, w:

$$\left.\begin{aligned}
&\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial S \partial x} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial S^2} = 0\\
&\frac{\partial^2 v}{\partial S^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial S \partial x} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{a}\frac{\partial w}{\partial S} = 0\\
&\frac{h^2}{12}\nabla^4 w - \frac{1}{a}\left(\frac{\partial v}{\partial S} - \frac{w}{a} + \nu\frac{\partial u}{\partial x}\right) - \frac{1-\nu^2}{Eh}q = 0
\end{aligned}\right\} \quad (5)$$

where $S = a\varphi$

In order to solve the simultaneous equations (5) we can apply first the operation $\frac{\partial^2}{\partial x^2}$, and then $\frac{\partial^2}{\partial S^2}$ to eq. (5:1). Solving in each case for the term containing v, and substituting these expressions in the equation obtained by applying $\frac{\partial^2}{\partial x \partial S}$ to (5:2), we obtain an equation from which v has been eliminated.

The application of $\frac{\partial^2}{\partial x^2}$ to (5:1) gives

$$\frac{\partial^4 u}{\partial x^4} + \frac{1-\nu}{2}\frac{\partial^4 u}{\partial x^2 \partial S^2} + \frac{1+\nu}{2}\frac{\partial^4 v}{\partial S \partial x^3} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} = 0 \quad (6:a)$$

The application of $\frac{\partial^2}{\partial S^2}$ to (5:1) gives

$$\frac{\partial^4 u}{\partial S^2 \partial x^2} + \frac{1-\nu}{2}\frac{\partial^4 u}{\partial S^4} + \frac{1+\nu}{2}\frac{\partial^4 v}{\partial S^3 \partial x} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x \partial S^2} = 0 \quad (6:b)$$

---

* See L.H. Donnell. NACA Rep. No. 479.

The application of $\frac{\partial^2}{\partial x \partial s}$ to (5:2) gives

$$\frac{1+\nu}{2}\frac{\partial^4 u}{\partial s^2 \partial x^2} + \frac{\partial^4 v}{\partial s^3 \partial x} + \frac{1-\nu}{2}\frac{\partial^4 v}{\partial x^3 \partial s} - \frac{1}{a}\frac{\partial^3 w}{\partial s^2 \partial x} = 0 \qquad (6:c)$$

Substitutions from eqs. (6:a) and (6:b) into eq. (6:c) lead to

$$a \nabla^4 u = \nu \frac{\partial^3 w}{\partial x^3} - \frac{\partial^3 w}{\partial x \partial s^2} \qquad (6:d)$$

Similarly, applying $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial s^2}$ to (5:2) and solving for the terms containing u, and substituting in (5:1), after applying $\frac{\partial^2}{\partial x \partial s}$ to it, we obtain an equation from which u has been eliminated.

$$a \nabla^4 v = (2+\nu)\frac{\partial^3 w}{\partial x^2 \partial s} + \frac{\partial^3 w}{\partial s^3} \qquad (6:e)$$

Now, applying $\frac{\partial}{\partial x}$ to (6:d) and $\frac{\partial}{\partial s}$ to (6:e) and substituting these two equations into eq. (5:3), after applying ∇^4 to it, we obtain an equation from which both u and v have been eliminated

$$\nabla^8 w + \frac{12(1-\nu^2)}{a^2 h^2}\frac{\partial^4 w}{\partial x^4} - \frac{1}{D}\nabla^4 q = 0 \qquad (7)$$

Eq. (7) differs from the differential equation of the flat plate only by the second term. The flat plate equation can be obtained from eq. (7) by the substitution of $a = \infty$. Consequently, this second term represents the effect of curvature in the problem of the cylindrical shell.

2. INFINITELY LONG CYLINDER LOADED WITH TWO EQUAL AND OPPOSITE FORCES

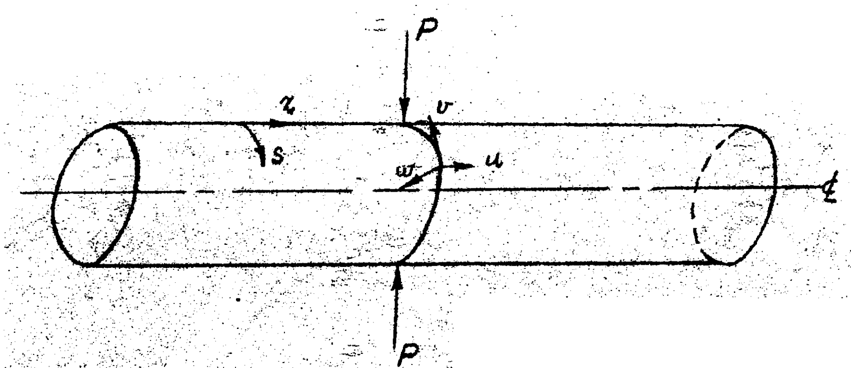


Fig. 3

The above equation will now be applied to an infinitely long thin cylinder loaded as shown in Fig. 3, i.e. by two equal and opposite compressive forces acting at the ends of a vertical diameter.

The difficulties of integrating Eq.(7) for this type of loading can be circumvented by replacing the concentrated force P with a function q of both the longitudinal and circumferential coordinates. This is possible if the function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. Since q is an even function of both x and s it can be expressed by

$$q(x,s) = \left[\frac{q_0}{2} + \sum_{n=2,4\cdots}^{\infty} q_n \cos\frac{ns}{a} \right] \int_0^{\infty} f(\lambda) \cos\frac{\lambda x}{a} \, d\lambda \qquad (8:a)$$

Furthermore, the displacement w can be expressed in a similar manner with the aid of an undetermined function w (λ):

$$w = \sum_{n=0,2,\cdots}^{\infty} \cos\frac{ns}{a} \int_0^{\infty} w(\lambda) \cos\frac{\lambda x}{a}\, d\lambda \tag{8:b}$$

It can be shown that the above expression for w satisfies the following requirements: (1) The deflection and moment are continuous. (2) The slope of the deflection curve vanishes at the point where the load is applied. (3) The deflection vanishes at infinity. Substituting Eqs. (8:a) and (8:b) in the differential equation (7), we obtain the following relations:

For $n = 0$

$$\int_0^{\infty} \left\{ w(\lambda)\left[\left(\frac{\lambda}{a}\right)^8 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4\right] - \frac{q_0}{2D} f(\lambda)\left(\frac{\lambda}{a}\right)^4 \right\} \cos\frac{\lambda x}{a}\, d\lambda = 0$$

For all values of λ , or

$$w(\lambda) = \frac{\frac{q_0}{2D} f(\lambda)\left(\frac{\lambda}{a}\right)^4}{\left(\frac{\lambda}{a}\right)^4\left[\left(\frac{\lambda}{a}\right)^4 + \frac{Eh}{a^2 D}\right]} = \frac{\frac{q_0}{2D} f(\lambda)}{\left(\frac{\lambda}{a}\right)^4 + \frac{Eh}{a^2 D}}$$

For $n = 2,\ 4 \cdots$

$$w(\lambda) = \frac{\frac{q_n f(\lambda)}{D}\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^4 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4}$$

Hence, the solution of Eq. (7) is given by

$$w = \frac{1}{2D}\int_0^{\infty} \frac{q_0 f(\lambda)}{\left(\frac{\lambda}{a}\right)^4 + \frac{Eh}{a^2 D}} \cos\frac{\lambda x}{a}\, d\lambda$$

$$+ \frac{1}{D}\sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a} \int_0^{\infty} \frac{q_n f(\lambda)\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^4 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4} \cos\frac{\lambda x}{a}\, d\lambda \tag{8:c}$$

It is next desired to find q_n and $f(\lambda)$. In order to accomplish this we must develop the functions q_n and λ from the loading condition which is shown in Fig. 4. Since the cylinder is loaded symmetrically to the generatrix and to the circle passing through the origin, only the positive direction need be considered.

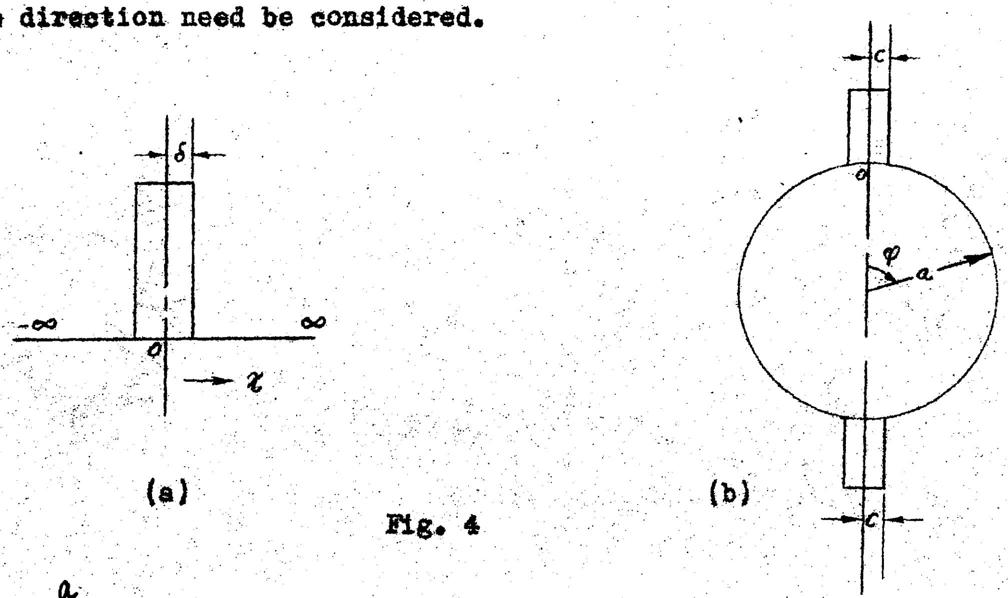


Fig. 4

From Eq. (8:a) we have

$$q\left(\frac{x}{a}\right) = \int_0^\infty f(\lambda) \cos \frac{\lambda x}{a} \, d\lambda$$

$$f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} q\left(\frac{x}{a}\right) \cos \frac{\lambda x}{a} \, d\left(\frac{x}{a}\right)$$

and

$$q\left(\frac{x}{a}\right) = 1 \quad \text{when} \quad -\delta \leq x \leq \delta$$

$$q\left(\frac{x}{a}\right) = 0 \quad \text{when} \quad x > \delta \quad \text{and} \quad -x < -\delta$$

Then

$$f(\lambda) = \frac{2}{\pi}\int_0^{\delta} \cos\frac{\lambda x}{a}\, d\left(\frac{x}{a}\right) = \frac{2}{\pi}\sin\frac{\lambda x}{a}\Big|_0^{\delta} = \frac{2}{\pi\lambda}\sin\lambda\frac{\delta}{a}$$

Similarly we can determine q_n by expanding the loading function along the circumference in a Fourier series. With

$$q_0 = \frac{1}{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z)\, dz$$

where $z = \frac{s}{a}$ and if $z = \frac{\pi}{2}$, $s = \frac{\pi}{2}a$

we get

$$q_0 = \frac{2}{\pi a}\int_{-\frac{a}{2}}^{\frac{a}{2}} q(s)\, ds = \frac{2}{\pi a}\int_{-c}^{c} q\, ds = \frac{4c}{\pi a} q$$

$$q_n = \frac{1}{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z)\cos nz\, dz = \frac{2}{\pi a}\int_{-c}^{c} q\cos n\frac{s}{a}\, ds = \frac{4q}{\pi n}\sin n\frac{c}{a}$$

Substituting q_n and $f(\lambda)$ in Eq. (8:2)

$$W = \frac{1}{2D}\int_0^{\infty} \frac{\frac{8qc}{\pi^2 a}\,\frac{\sin\lambda\frac{\delta}{a}}{\lambda}}{\left(\frac{\lambda}{a}\right)^4 + \frac{Eh}{a^2 D}} \cos\frac{\lambda x}{a}\, d\lambda$$

$$+ \frac{1}{D}\sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a}\int_0^{\infty} \frac{\frac{8q}{\pi^2 n}\sin n\frac{c}{a}\,\frac{\sin\lambda\frac{\delta}{a}}{\lambda}\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^4 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4} \cos\frac{\lambda x}{a}\, d\lambda$$

Now we consider the case of a concentrated load applied at the origin. Such a load can be obtained by making the lengths 2δ and $2c$ of the loaded portion infinitely small. Substituting

$$P = 4qc\delta \qquad \text{and} \qquad \sin\lambda\frac{\delta}{a} \approx \frac{\lambda\delta}{a}$$

$$\sin n\frac{c}{a} \approx \frac{nc}{a}$$

in the above equation, we obtain

$$W = \frac{Pa^2}{D\pi^2}\int_0^\infty \frac{\cos\lambda(\frac{x}{a})\,d\lambda}{\lambda^4 + J^2} + \frac{2Pa^2}{\pi^2 D}\sum_{n=2,4\cdots}^{\infty}\cos\frac{ns}{a}\int_0^\infty \frac{[\lambda^2+n^2]^2\cos\lambda\frac{x}{a}\,d\lambda}{[\lambda^2+n]^4 + J^2\lambda^4} \tag{9}$$

where $\qquad J^2 = \frac{Eha^2}{D} = 12(1-\nu^2)\left(\frac{a}{h}\right)^2$

In order to evaluate the definite integrals in Eq. (9) we will apply Cauchy's Theorem of residues. This method is generally found simpler than any other. Let us first consider the integral

$$\int_0^\infty \frac{\cos\lambda(\frac{x}{a})\,d\lambda}{\lambda^4 + J^2}$$

where the characteristic equation $\lambda^4 + J^2 = 0$ has four complex roots

$$\lambda = J^{\frac{1}{2}}\sqrt[4]{-1}$$

We integrate the function $\dfrac{e^{i\frac{x}{a}z}}{z^4 + J^2}$ in the complex z-plane along a closed contour, Γ, consisting of the segment of the real axis from $-R$ to R and a semicircle σ of radius R in the upper half-plane and tends to the point at infinity as $R \to \infty$. Evidently, for sufficiently large values of R, there is no singularity of the integrand on the closed contour and two singularities, namely the simple poles $\sqrt{J}\left(\frac{1+i}{\sqrt{2}}\right)$, $\sqrt{J}\left(\frac{i-1}{\sqrt{2}}\right)$, within Γ. By Cauchy's Theorem of residues, it follows that

$$I = \int_{-R}^{R} \frac{e^{i\frac{x}{a}\lambda}d\lambda}{\lambda^4+J^2} + \int_{\Gamma} \frac{e^{i\frac{x}{a}z}dz}{z^4+J^2} = 2\pi i\left[\frac{e^{-\frac{x}{a}\left(\sqrt{\frac{J}{2}}-i\sqrt{\frac{J}{2}}\right)}}{4J^{\frac{3}{2}}i\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)} + \frac{e^{-\frac{x}{a}\left(\sqrt{\frac{J}{2}}+i\sqrt{\frac{J}{2}}\right)}}{4J^{\frac{3}{2}}i\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)}\right]$$

$$= \frac{\pi}{2\sqrt{2}} J^{-\frac{3}{2}} e^{-\frac{x}{a}\sqrt{\frac{J}{2}}}\left[2\cos\sqrt{\frac{J}{2}}\frac{x}{a} + 2\sin\sqrt{\frac{J}{2}}\frac{x}{a}\right] \tag{10}$$

Since the integral $\int_{\Gamma} \frac{e^{i\frac{x}{a}z}}{z^4+J^2}$ can be proved to approach zero as $R \to \infty$, we have

$$\int_0^{\infty} \frac{\cos\lambda\frac{x}{a}\,d\lambda}{\lambda^4+J^2} = \frac{\pi}{2\sqrt{2}} J^{-\frac{3}{2}} e^{-\sqrt{\frac{J}{2}}\frac{x}{a}}\left(\cos\sqrt{\frac{J}{2}}\frac{x}{a} + \sin\sqrt{\frac{J}{2}}\frac{x}{a}\right) \tag{11}$$

The second definite integral in Eq. (9) can be evaluated in the same manner. Let us first express the rational function in the integral in the form of partial fractions

$$\frac{(\lambda^2+n^2)^2}{[\lambda^2+n]^4+J^2\lambda^4} = \frac{1}{2}\left\{\frac{1}{(\lambda^2+n^2)^2+iJ\lambda^2} + \frac{1}{(\lambda^2+n^2)^2-iJ\lambda^2}\right\}$$

$$= \frac{1}{\alpha_1^2-\alpha_2^2}\cdot\frac{1}{\lambda^2-\alpha_1^2} + \frac{1}{\alpha_2^2-\alpha_1^2}\cdot\frac{1}{\lambda^2-\alpha_2^2} + \frac{1}{\alpha_3^2-\alpha_4^2}\cdot\frac{1}{\lambda^2-\alpha_3^2} + \frac{1}{\alpha_4^2-\alpha_3^2}\cdot\frac{1}{\lambda^2-\alpha_4^2} \tag{12}$$

where $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$ and $\pm\alpha_4$ are the eight roots of the eighth degree algebraic equation in the denominator. These are:

$$
\left.
\begin{aligned}
\pm\alpha_1 &= \pm A \pm iB = \pm\frac{1}{\sqrt{2}}\left[\sqrt{(-n^2+\eta)^2+\left(-\frac{J}{2}+\phi\right)^2}-(n^2-\eta)\right]^{\frac{1}{2}} \\
&\qquad \pm\frac{i}{\sqrt{2}}\left[\sqrt{(-n^2+\eta)^2+\left(-\frac{J}{2}+\phi\right)^2}+(n^2-\eta)\right]^{\frac{1}{2}} \\
\pm\alpha_4 &= \pm A \mp iB \\
\pm\alpha_2 &= \pm C \mp iD = \pm\frac{1}{\sqrt{2}}\left[\sqrt{(n^2+\eta)^2+\left(\frac{J}{2}+\phi\right)^2}-(n^2+\eta)\right]^{\frac{1}{2}} \\
&\qquad \mp\frac{i}{\sqrt{2}}\left[\sqrt{(n^2+\eta)^2+\left(\frac{J}{2}+\phi\right)^2}+(n^2+\eta)\right]^{\frac{1}{2}} \\
\pm\alpha_3 &= \pm C \pm iD
\end{aligned}
\right\} \quad (13)
$$

where

$$\phi = \sqrt{\tfrac{1}{2}\left(R_2+\tfrac{J^2}{4}\right)}, \quad \eta = \sqrt{\tfrac{1}{2}\left(R_2-\tfrac{J^2}{4}\right)}, \quad R_2 = n^2 J\sqrt{1+\left(\tfrac{J}{4n^2}\right)^2} \quad (13:a)$$

Hence

$$
\begin{aligned}
I &= \int_{-\infty}^{\infty}\frac{(\lambda^2+n^2)^2 e^{i\lambda\frac{x}{a}}}{(\lambda^2+n^2)^4+J^2\lambda^4} = \frac{1}{2}\int_{-\infty}^{\infty}\Big[\frac{e^{i\lambda\frac{x}{a}}}{(\alpha_1^2-\alpha_2^2)(\lambda^2-\alpha_1^2)}+\frac{e^{i\lambda\frac{x}{a}}}{(\alpha_2^2-\alpha_1^2)(\lambda^2-\alpha_2^2)} \\
&\qquad +\frac{e^{i\lambda\frac{x}{a}}}{(\alpha_3^2-\alpha_4^2)(\lambda^2-\alpha_3^2)}+\frac{e^{i\lambda\frac{x}{a}}}{(\alpha_4^2-\alpha_3^2)(\lambda^2-\alpha_4^2)}\Big]d\lambda \\
&= 2\pi i\left(\frac{1}{8R_2}\right)\Big\{\frac{\alpha_2}{\alpha_1\alpha_2}(\eta-i\phi)e^{i\frac{x}{a}\alpha_1}-\frac{\alpha_1}{\alpha_1\alpha_2}(\eta-i\phi)e^{i\frac{x}{a}\alpha_2} \\
&\qquad -\frac{\alpha_4}{\alpha_3\alpha_4}(\eta+i\phi)e^{i\frac{x}{a}\alpha_3}+\frac{\alpha_3}{\alpha_3\alpha_4}(\eta+i\phi)e^{i\frac{x}{a}\alpha_4}\Big\}
\end{aligned}
\quad (14)
$$

From the relation

$$\alpha_1\alpha_2 = -n^2 = \alpha_3\alpha_4$$

we have

$$
\begin{aligned}
I &= -\frac{\pi i}{4}\frac{1}{R_2 n^2}\Big\{(\eta-i\phi)(-C+iD)e^{i\frac{x}{a}\alpha_1}-(\eta-i\phi)(A+iB)e^{i\frac{x}{a}\alpha_2} \\
&\qquad -(\eta+i\phi)(-A+iB)e^{i\frac{x}{a}\alpha_3}+(\eta+i\phi)(C+iD)e^{i\frac{x}{a}\alpha_4}\Big\}
\end{aligned}
$$

After simplifying we finally get

$$\int_0^\infty \frac{(\lambda^2+n^2)^2 \cos\frac{x}{a}\lambda\, d\lambda}{(\lambda^2+n^2)^4 + J^2\lambda^4} = \frac{\pi}{4R_2 n^2}\left\{\left[(\phi C+\eta D)\cos\frac{Ax}{a} + (\phi D - \eta C)\sin\frac{Ax}{a}\right]e^{-\frac{Bx}{a}} + e^{-D\frac{x}{a}}\left[(\phi A - \eta B)\cos\frac{Cx}{a} + (\eta A + \phi B)\sin\frac{Cx}{a}\right]\right\} \tag{15}$$

Simplifying the integrals (15) and (11) in Eq. (9), we obtain

$$W = \frac{Pa^2}{\pi D}\frac{J^{-\frac{3}{2}}}{2\sqrt{2}}\left[\cos\sqrt{\frac{J}{2}}\frac{x}{a} + \sin\sqrt{\frac{J}{2}}\frac{x}{a}\right]e^{-\sqrt{\frac{J}{2}}\frac{x}{a}} + \frac{Pa^2}{2\pi D}\sum_{n=2,4}^{\infty}\frac{\cos n\frac{s}{a}}{R_2 n^2}\left\{\left[(\phi C+\eta D)\cos\frac{Ax}{a} + (\phi D-\eta C)\sin\frac{Ax}{a}\right]e^{-\frac{Bx}{a}} + \left[(\phi A-\eta B)\cos\frac{Cx}{a} + (\eta A+\phi B)\sin\frac{Cx}{a}\right]e^{-\frac{Dx}{a}}\right\} \tag{16}$$

Alternately Eq. (16) can be written in a non-dimensional form

$$\frac{\frac{W}{h}}{\frac{P}{Eh^2}} = \frac{\sqrt[4]{3(1-\nu^2)}}{2\pi}\left(\frac{a}{h}\right)^{\frac{1}{2}}\left(\cos\sqrt{\frac{J}{2}}\frac{x}{a} + \sin\sqrt{\frac{J}{2}}\frac{x}{a}\right)e^{-\sqrt{\frac{J}{2}}\frac{x}{a}} + \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2\sum_{n=2,4\ldots}^{\infty}\frac{\cos n\frac{s}{a}}{R_2 n^2}\left\{\left[(\phi C+\eta D)\cos\frac{Ax}{a} + (\phi D-\eta C)\sin\frac{Ax}{a}\right]e^{-\frac{Bx}{a}} + \left[(\phi A-\eta B)\cos\frac{Cx}{a} + (\eta A+\phi B)\sin\frac{Cx}{a}\right]e^{-\frac{Dx}{a}}\right\} \tag{17}$$

It is seen that the first term of the above equation is very small as compared to the second term, and therefore can be neglected without any appreciable error in practical application. For a certain value of the a/h ratio D is found to be very large as compared to B. The terms containing $e^{-D\frac{x}{a}}$

can then be completely neglected, provided that x/a is not near zero. However, in the case when x/a = 0 Eq. (17) can be simplified as follows:

$$\left[\frac{\frac{w}{h}}{\frac{P}{Eh^2}}\right]_{\frac{x}{a}=0} = \frac{3\sqrt{2}\,(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos\frac{ns}{a}}{n^3} \sqrt{\frac{1+\sqrt{1+\frac{3}{4}\frac{(1-\nu^2)}{n^4}\left(\frac{a}{h}\right)^2}}{1+\frac{3}{4}\frac{(1-\nu^2)}{n^4}\left(\frac{a}{h}\right)^2}} \tag{18}$$

3. A CYLINDER OF FINITE LENGTH LOADED WITH TWO EQUAL AND OPPOSITE FORCES

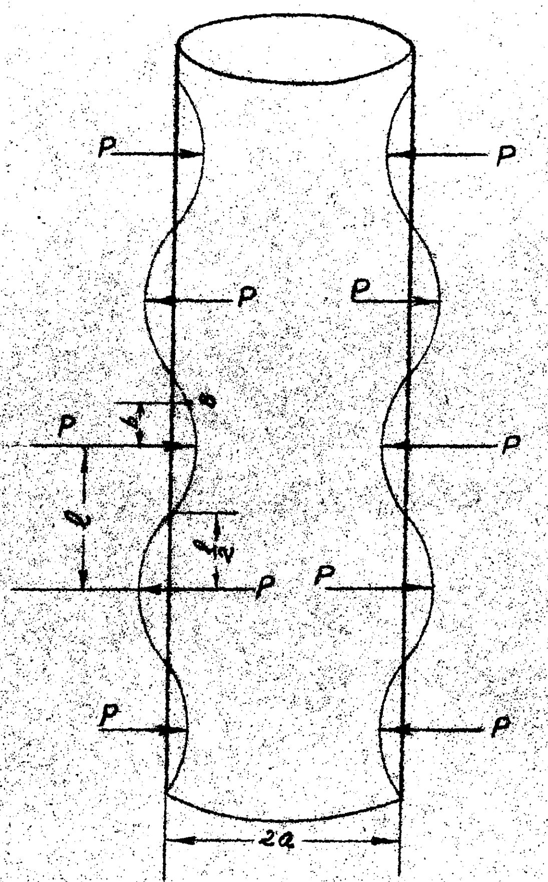


Fig. 5

The expression for the radial deflection in a thin cylinder of finite length can be obtained from equation (17) by using the method of images.* If we imagine the cylinder of finite length prolonged in both the positive and the negative x-directions, and loaded with a series of forces, P of alternating sense applied along the generatrix $(\frac{s}{a}-\phi)$ at a distance ℓ from one another (see Fig. 5), then the deflections of the infinite cylinder are evidently equal to zero at a distance $\frac{\ell}{2}$ from the applied loads P. Hence we may consider the given cylinder of length ℓ and radius a as a portion of the infinitely long cylinder loaded as shown in Fig. 5. From Eq. (17) we find that the deflection of any point, B, (at a distance b from the s-axis)

---

* This method was used by Dr. A. Nadai, see Zangen. Meth. Mech., Vol. 2, P. 1, 1922; and by M.T. Huber, see Zangen. Meth. Mech., Vol. 6, P. 228, 1926.

on the shell due to the load P acting at the center is:

$$w_a = \frac{Pa^2}{2\pi D}\sum_{n=2,4\cdots}^{\infty}\frac{\cos n\frac{s}{a}}{R_2 n}\left\{\left[(\phi C+\eta D)\cos A\frac{b}{a}+(\phi D-\eta C)\sin A\frac{b}{a}\right]e^{-B\frac{b}{a}} + \left[(\phi A-\eta B)\cos C\frac{b}{a}+(\eta A+\phi B)\sin C\frac{b}{a}\right]e^{-D\frac{b}{a}}\right. \tag{19:a}$$

The deflection produced by two adjacent forces a distance l apart is:

$$w_b = -\frac{Pa^2}{2\pi D}\sum_{n=2,4\cdots}^{\infty}\frac{\cos\frac{ns}{a}}{R_2 n}\left\{\left[(\phi C+\eta D)\cos A\frac{(l-b)}{a} + (\phi D-\eta C)\sin A\frac{(l-b)}{a}\right]e^{-B\frac{(l-b)}{a}} + \left[(\phi C+\eta D)\cos A\frac{(l+b)}{a} + (\phi D-\eta C)\sin A\frac{(l+b)}{a}\right]e^{-B\frac{(l+b)}{a}}\right. \tag{19:b}$$

Since the terms containing $e^{-D\frac{(l\pm b)}{a}}$ are all small as compared with the other terms shown above, they can be neglected without causing appreciable error. Similarly we can obtain w_c, w_d and so on. The total radial deflection at any point B is given by the sum

$$W = w_a + w_b + w_c + \cdots$$

or

$$\begin{aligned} W = \frac{Pa^2}{2\pi D}\sum_{n=2,4\cdots}^{\infty}\frac{\cos\frac{ns}{a}}{R_2 n}&\left\{\left[(\phi A-\eta B)\cos\frac{Cb}{a}+(\eta A+\phi B)\sin\frac{Cb}{a}\right]e^{-D\frac{b}{a}}\right. \\ &+(\phi C+\eta D)\left[\cos A\frac{b}{a}e^{-B\frac{b}{a}} - \cos A\frac{(l-b)}{a}e^{-B\frac{(l-b)}{a}} - \cos A\frac{(l+b)}{a}e^{-B\frac{(l+b)}{a}}\right. \\ &\left.+\cos\frac{A(2l-b)}{a}e^{-B\frac{(2l-b)}{a}} + \cos A\frac{(2l+b)}{a}e^{-B\frac{(2l+b)}{a}} - \cdots\right] \\ &+(\phi D-\eta C)\left[\sin A\frac{b}{a}e^{-B\frac{b}{a}} - \sin A\frac{(l-b)}{a}e^{-B\frac{(l-b)}{a}} - \cdots\right] \end{aligned} \tag{20}$$

It is seen that Eq. (20) can be expressed by series containing $\cos nA\frac{l}{a}\, e^{-nB\frac{l}{a}}$ and $\sin nA\frac{l}{a}\, e^{-nB\frac{l}{a}}$ terms.

$$W = \frac{Pa^2}{2\pi D}\sum_{n=2,4\ldots}^{\infty}\frac{\cos n\frac{s}{a}}{R_2 n}\Bigg\{\Big[(\phi A-\eta B)\cos C\frac{b}{a}+(\eta A+\phi B)\sin C\frac{b}{a}\Big]e^{-D\frac{b}{a}}$$
$$+(\phi c+\eta D)\Big[\cos A\frac{b}{a}e^{-B\frac{b}{a}}-2\cos A\frac{b}{a}\cosh B\frac{b}{a}\Big(\cos A\frac{l}{a}e^{-B\frac{l}{a}}-\cos\frac{2Al}{a}e^{-\frac{2Bl}{a}}$$
$$+\cos 3A\frac{l}{a}e^{-3B\frac{l}{a}}-\cdots\Big)-2\sin A\frac{b}{a}\sinh B\frac{b}{a}\Big(\sin A\frac{l}{a}e^{-B\frac{l}{a}}-\sin 2A\frac{l}{a}e^{-2A\frac{l}{a}}+\cdots\Big)\Big]$$
$$+(\phi D-\eta C)\Big[\sin A\frac{b}{a}e^{-B\frac{b}{a}}-2\cos A\frac{b}{a}\cosh B\frac{b}{a}\Big(\sin A\frac{l}{a}e^{-B\frac{l}{a}}-\sin 2A\frac{l}{a}e^{-2B\frac{l}{a}}$$
$$+\sin 3A\frac{l}{a}e^{-3B\frac{l}{a}}-\cdots\Big)+2\sin A\frac{b}{a}\sinh B\frac{b}{a}\Big(\cos A\frac{l}{a}e^{-B\frac{l}{a}}-\cos 2A\frac{l}{a}e^{-2B\frac{l}{a}}+\cdots\Big)\Big]\Bigg\} \quad (21)$$

The series in the above equation

$$\cos A\frac{l}{a}e^{-B\frac{l}{a}}-\cos 2A\frac{l}{a}e^{-2B\frac{l}{a}}+\cos 3A\frac{l}{a}e^{-3B\frac{l}{a}}-\cdots$$
$$\sin A\frac{l}{a}e^{-B\frac{l}{a}}-\sin 2A\frac{l}{a}e^{-2B\frac{l}{a}}+\sin 3A\frac{l}{a}e^{-3B\frac{l}{a}}-\cdots$$

can be expressed in the form given below. This expression is known as a geometrical series and the sum of its n terms is found with the aid of the Bionomial expansion

$$\sum_{m=1,3\cdots}^{\infty}e^{-mB\frac{l}{a}}\cos mA\frac{l}{a} = \frac{1}{2}\sum_{m=1,3\cdots}^{\infty}\Big[e^{-m\frac{l}{a}(Z_1)}+e^{-m\frac{l}{a}(Z_2)}\Big]$$

where $Z_1 = B - iA$; $Z_2 = B + iA$.

Putting $\zeta = e^{-z\frac{l}{a}}$ **we obtain**

$$\sum_{m=1,3\cdots}^{\infty} \zeta^m = \zeta + \zeta^3 + \zeta^5 + \cdots = \frac{\zeta}{1-\zeta^2}$$

$$\sum_{m=2,4\cdots}^{\infty} \zeta^m = \zeta^2 + \zeta^4 + \zeta^6 + \cdots = \frac{\zeta^2}{1-\zeta^2}$$

Consequently

$$\sum_{m=1,3\cdots}^{\infty} e^{-mB\frac{l}{a}} \cos mA\frac{l}{a} = \frac{1}{2}\left[\frac{e^{-\frac{l}{a}(B-iA)}}{1-e^{-2\frac{l}{a}(B-iA)}} + \frac{e^{-\frac{l}{a}(B+iA)}}{1-e^{-2\frac{l}{a}(B+iA)}}\right]$$

$$= \frac{1}{2}\left[\frac{\sinh\frac{l}{a}B \cdot \cos\frac{l}{a}A}{\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A}\right]$$

$$\sum_{m=2,4\cdots}^{\infty} e^{-mB\frac{l}{a}} \cos mA\frac{l}{a} = \frac{1}{2}\left[\frac{e^{-2\frac{l}{a}(B-iA)}}{1-e^{-2\frac{l}{a}(B-iA)}} + \frac{e^{-2\frac{l}{a}(B+iA)}}{1-e^{-2\frac{l}{a}(B+iA)}}\right]$$

$$= \frac{e^{-\frac{l}{a}B}\left[\sinh\frac{l}{a}B\cos^2\frac{l}{a}A - \cosh\frac{l}{a}B\sin^2\frac{l}{a}A\right]}{2\left(\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A\right)}$$

$$\sum_{m=1,3\cdots}^{\infty} e^{-mB\frac{l}{a}} \sin mA\frac{l}{a} = \frac{1}{2i}\left[\frac{e^{-\frac{l}{a}(B-iA)}}{1-e^{-2\frac{l}{a}(B-iA)}} - \frac{e^{-\frac{l}{a}(B+iA)}}{1-e^{-2\frac{l}{a}(B+iA)}}\right]$$

$$= \frac{1}{2}\left[\frac{\cosh\frac{l}{a}B\sin\frac{l}{a}A}{\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A}\right]$$

$$\sum_{m=2,4\cdots}^{\infty} e^{-mB\frac{l}{a}} \sin mA\frac{l}{a} = \frac{1}{2i}\left[\frac{e^{-2\frac{l}{a}(B-iA)}}{1-e^{-2\frac{l}{a}(B-iA)}} - \frac{e^{-2\frac{l}{a}(B+iA)}}{1-e^{-2\frac{l}{a}(B+iA)}}\right]$$

$$= \frac{e^{-\frac{l}{a}B}\cos\frac{l}{a}A\sin\frac{l}{a}A\left(\sinh\frac{l}{a}B + \cosh\frac{l}{a}B\right)}{2\left(\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A\right)}$$

Substituting the above summations in Eq. (21) we have

$$
\begin{aligned}
\frac{\frac{w}{h}}{\frac{P}{Eh^2}} = {} & \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\ldots}^{\infty} \frac{\cos n\frac{s}{a}}{R_2 n^2} \left\{ \left[(\phi C + \eta D)\cos A\frac{b}{a} + (\phi D - \eta C)\sin A\frac{b}{a} \right] e^{-B\frac{b}{a}} \right. \\
& + \left[(\phi A - \eta B)\cos C\frac{b}{a} + (\eta A + \phi B)\sin C\frac{b}{a} \right] e^{-D\frac{b}{a}} \\
& + \frac{\sinh\frac{l}{a}B\cos\frac{l}{a}A - e^{-\frac{l}{a}B}\left[\sinh\frac{l}{a}B\cos^2\frac{l}{a}A - \cosh\frac{l}{a}B\sin^2\frac{l}{a}A\right]}{\left(\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A\right)} \\
& \times \left[(\phi D - \eta C)\sin A\frac{b}{a}\sinh B\frac{b}{a} - (\phi C + \eta D)\cos A\frac{b}{a}\cosh B\frac{b}{a} \right] \\
& - \frac{\cosh\frac{l}{a}B\sin\frac{l}{a}A - e^{-\frac{l}{a}B}\cos\frac{l}{a}A\sin\frac{l}{a}A\left(\sinh\frac{l}{a}B + \cosh\frac{l}{a}B\right)}{\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A} \\
& \left. \times \left[(\phi D - \eta C)\cos A\frac{b}{a}\cosh B\frac{b}{a} + (\phi C + \eta D)\sin A\frac{b}{a}\sinh B\frac{b}{a} \right] \right\}
\end{aligned}
\tag{22}
$$

It is obvious that the first two terms of Eq. (22) are equivalent to the solution of the infinitely long cylinder given by Eq. (17). The remaining terms are evidently the correction factors due to the restrained edges at the two ends of the cylinder of finite length. The radial deflection under the applied force can be obtained by putting $b/a = 0$:

$$\frac{\frac{w}{h}}{\frac{p}{Eh^2}} = \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\cdots}^{\infty} \frac{\cos n\frac{s}{a}}{R_2 n^2}\Bigg\{(\phi C + \eta D + (\phi A - \eta B)$$

$$-(\phi C + \eta D)\frac{\sinh\frac{l}{a}B\cos\frac{l}{a}A - e^{-\frac{l}{a}B}\left[\sinh\frac{l}{a}B\cos^2\frac{l}{a}A - \cosh\frac{l}{a}\sin^2\frac{l}{a}A\right]}{\left(\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A\right)}$$

$$-(\phi D - \eta C)\frac{\cosh\frac{l}{a}B\sin\frac{l}{a}A - e^{-\frac{l}{a}B}\cos\frac{l}{a}A\sin\frac{l}{a}A\left(\sinh\frac{l}{a}B + \cosh\frac{l}{a}B\right)}{\left(\sinh^2\frac{l}{a}B\cos^2\frac{l}{a}A + \cosh^2\frac{l}{a}B\sin^2\frac{l}{a}A\right)}$$

$$-(\phi A - \eta B)\frac{\sinh\frac{l}{a}D\cos\frac{l}{a}C - e^{-\frac{l}{a}D}\left[\sinh\frac{l}{a}D\cos^2\frac{l}{a}C - \cosh\frac{l}{a}D\sin^2\frac{l}{a}C\right]}{\left(\sinh^2\frac{l}{a}D\cos^2\frac{l}{a}C + \cosh^2\frac{l}{a}D\sin^2\frac{l}{a}C\right)}$$

$$-(\eta A + \phi B)\frac{\cosh\frac{l}{a}D\sin\frac{l}{a}C - e^{-\frac{l}{a}D}\cos\frac{l}{a}C\sin\frac{l}{a}C\left[\sinh\frac{l}{a}D + \cosh\frac{l}{a}D\right]}{\left(\sinh^2\frac{l}{a}D\cos^2\frac{l}{a}C + \cosh^2\frac{l}{a}D\sin^2\frac{l}{a}C\right)}$$

(23)

4. APPLICATION OF THE SOLUTION OF THE PROBLEM OF THE INFINITELY LONG CYLINDER TO SOME SPECIFIC PROBLEMS

The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference can be analyzed by using the solution given by Eq. (17) for a single load. The action of the couple is equivalent to that of the two forces P shown in Fig. 6a, if

(a) (b)

Fig. 6

$P\Delta x$ approaches T_c while Δx approaches zero.

It is easy to see that the deflection for the case when the force P is at the point O, at a distance Δx from the origin can be obtained from w, Eq. (17), by writing $x-\Delta x$ instead of x and also $-P$ instead of P. This and the original w are then added, whereby the radial deflection for the two equal and opposite forces applied at O and O_1 respectively, is obtained in the form

$$-w_T = w(x, s) - w(x-\Delta x, s)$$

When Δx is very small, this approaches the value

$$w_T = \frac{dw(x,s)}{dx}\Delta x$$

As T_c is the moment of the applied torque and is equal to $P\Delta x$, the radial deflection due to this torque is

$$w_{T_1} = \frac{T_c}{P}\frac{dw}{dx} \qquad \text{(a)}$$

when w is the radial deflection due to the concentrated load P.

Similarly we find the radial deflection due to the couple acting along the circumferential direction, Fig. 6b

$$w_{T_2} = \frac{T_c}{P}\frac{dw}{dS} \qquad \text{(b)}$$

Substituting w from Eq. (17) in Eqs. (a) and (b) we obtain

$$\frac{\frac{w_{T_1}}{h}}{\frac{T_c}{Eh^2}} = \frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\ldots}^{\infty} \frac{\cos n\frac{s}{a}}{R_2 n^2}\left\{ e^{-B\frac{x}{a}}\cos A\frac{x}{a}\left[A(\phi D - \eta C) - B(\phi C + \eta D)\right] - e^{-B\frac{x}{a}}\sin A\frac{x}{a}\left[(\phi C + \eta D)A + B(\phi D - \eta C)\right] + e^{-D\frac{x}{a}}\cos C\frac{x}{a}\left[C(\eta A + \phi B) - D(\phi A - \eta B)\right] - e^{-D\frac{x}{a}}\sin C\frac{x}{a}\left[C(\phi A - \eta B) + (\eta A + \phi B)D\right]\right\} \qquad (24)$$

$$\frac{\frac{w_{T_2}}{h}}{\frac{T_c}{Eh^2}} = -\frac{6(1-\nu^2)}{\pi}\left(\frac{a}{h}\right)^2 \sum_{n=2,4\ldots}^{\infty} \frac{\sin n\frac{s}{a}}{R_2 n}\left\{\left[(\phi C + \eta D)\cos A\frac{x}{a} + (\phi D - \eta C)\sin A\frac{x}{a}\right] e^{-B\frac{x}{a}} + \left[(\phi A - \eta B)\cos C\frac{x}{a} + (\eta A + \phi B)\sin C\frac{x}{a}\right] e^{-D\frac{x}{a}}\right\} \qquad (25)$$

In the case when $\frac{x}{a} = 0$ we have:

$$\frac{\frac{w_{T_1}}{h}}{\frac{T_c}{Eh^2}} = \frac{\frac{w_{T_2}}{h}}{\frac{T_c}{Eh^2}} = 0 \qquad \text{at any } \frac{s}{a}$$

Hence the condition that the slope of the deflection curve must vanish under the concentrated load ($x/a = 0$) is satisfied.

The solution of an infinitely long cylinder in Eq. (17) may also be applied in order to find the modulas of foundation of an elastically supported flat plate or beam under a concentrated transverse load. From Fig. 7 we can visualize that a longitudinal strip between the nodal points of the cylinder will deform in a manner similar to a flat plate on elastic supports under a transverse load. The equation for the deflection curve of an elastically supported beam under a concentrated load consists of trigonometric and hyperbolic functions and has a form similar to Eq. (17).

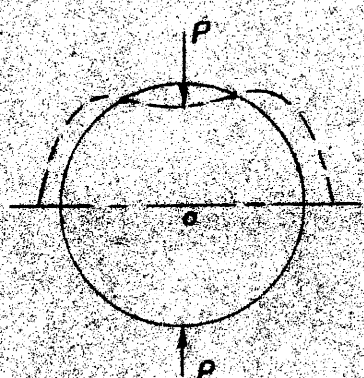


Fig. 7

Let us now compare the known maximum deflection of the cylinder under a certain concentrated load with the maximum deflection of an elastically supported beam of infinite length. From the known deflection w_c of the cylinder we can determine the modulus of foundation k of the beam by equating w_c w_b ,* where w_b is the maximum deflection of the beam:

$$w_c = w_b = \frac{P}{\sqrt{8}\left(\frac{K}{EI}\right)^{\frac{3}{4}} EI}$$

---

* Kármán and Biot. Mathematical Methods in Engineering, Chapter VII.

Then

$$k = \left(\frac{P}{\sqrt{8}\, w_c}\right)^{\frac{4}{3}} \left(\frac{1}{EI}\right)^{\frac{1}{3}}$$

since E I is the flexural rigidity of the beam, and P is the corresponding transverse load.

In the same manner the modulas of foundation k can also be determined at various other points along the beam.

PART TWO

INFINITELY LONG CYLINDER UNDER TWO EQUAL AND OPPOSITE TORQUE ACTING ABOUT THE RADIAL AXIS ON THE SURFACE OF THE SHELL

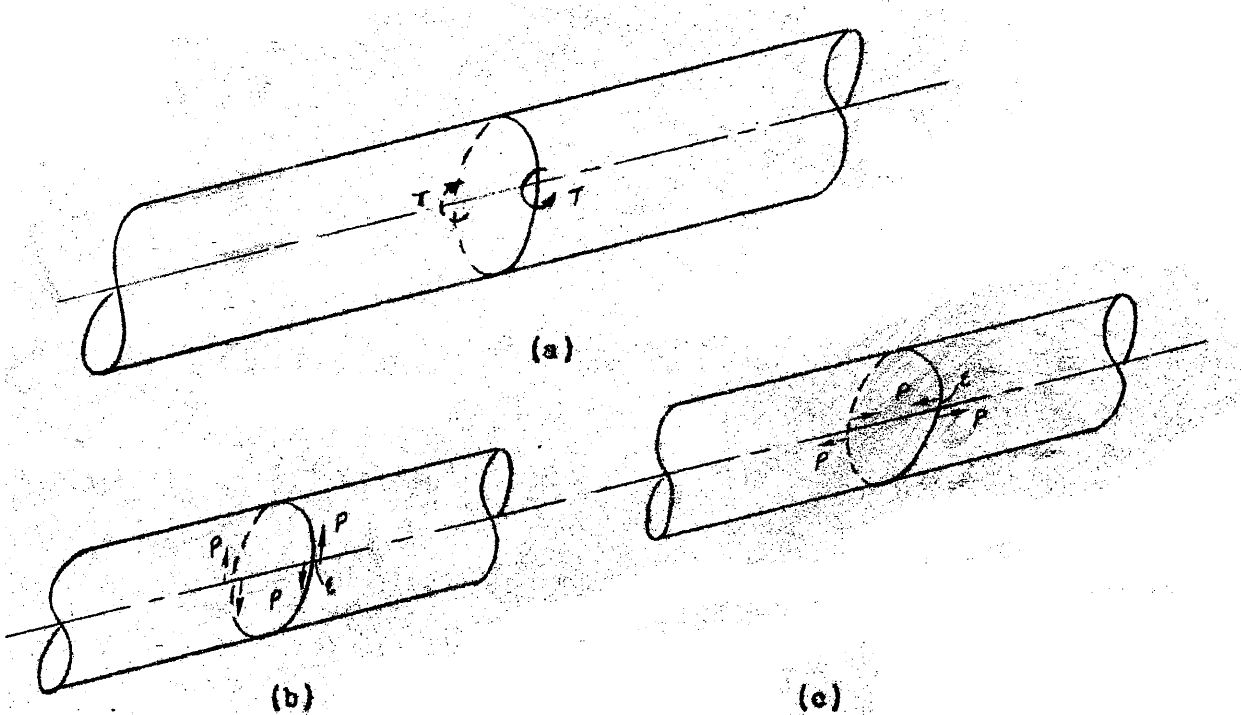


Fig. 8

The solution of the problem of concentrated torques acting on the surface of an infinitely long cylinder (Fig. 8a) can be achieved by replacing T with two equal and opposite forces acting at an infinitely small distance ξ apart (Fig. 8b and 8c). In the case of an infinite plate the shearing stresses produced by two equal and opposite forces acting perpendicularly to either axis are identical.* However, the problem we are dealing with is quite different because of the curvature effect. Instead of two equal and opposite forces acting at an infinitely small distance ξ apart in the direction of x-axis, we combine two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other.

1. Fundamental Equations

A. Load Acting Tangentially

The equations of equilibrium of a shell element under a circumferential pressure of an intensity q can be derived from considerations of the equilibrium in a manner similar to that used in the derivation of Eqs. (4) of Part One.

$$\left.\begin{aligned} & a\frac{\delta T_1}{\delta x} + \frac{\delta S}{\delta \varphi} = 0 \\ & \frac{\delta T_2}{\delta \varphi} + a\frac{\delta S}{\delta x} - q a = 0 \\ & \frac{2}{a}\frac{\delta^2 H}{\delta \varphi \delta x} + \frac{\delta^2 G_1}{\delta x^2} + \frac{\delta^2 G_2}{a^2 \delta \varphi^2} + \frac{T_2}{a} = 0 \end{aligned}\right\} \qquad (24)$$

By using the relation between the stress resultants and the deformations we can obtain the three differential equations which contain the displace-

---

* Love, A.E.H. "The Theory of Elasticity." Page 214. Cambridge, 1927.

ments u, v, and w. After simplifying we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial s\,\partial x} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2} = 0$$

$$\frac{\partial^2 v}{\partial s^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial s\,\partial x} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{a}\frac{\partial w}{\partial s} - \frac{1-\nu^2}{Eh} q = 0 \qquad (25)$$

$$\frac{h^2}{12}\nabla^4 w - \frac{1}{a}\left(\frac{\partial v}{\partial s} - \frac{w}{a} + \nu\frac{\partial u}{\partial x}\right) = 0$$

Application of the operation $\frac{\partial^2}{\partial x^2}$ to (25:1), gives

$$\frac{\partial^4 u}{\partial x^4} + \frac{1+\nu}{2}\frac{\partial^4 v}{\partial s\,\partial x^3} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1-\nu}{2}\frac{\partial^4 u}{\partial x^2\,\partial s^2} = 0 \qquad (26{:}a)$$

Application of $\frac{\partial^2}{\partial s^2}$ to (25: 1) and $\frac{\partial^2}{\partial x\,\partial s}$ to (25; 2) give respectively

$$\frac{\partial^4 u}{\partial s^2\,\partial x^2} + \frac{1+\nu}{2}\frac{\partial^4 v}{\partial s^3\,\partial x} - \frac{\nu}{a}\frac{\partial^3 w}{\partial x\,\partial s^2} + \frac{1-\nu}{2}\frac{\partial^4 u}{\partial s^4} = 0 \qquad (26{:}b)$$

$$\frac{1+\nu}{2}\frac{\partial^4 u}{\partial s^2\,\partial x^2} + \frac{\partial^4 v}{\partial x\,\partial s^3} + \frac{1-\nu}{2}\frac{\partial^4 v}{\partial x^3\,\partial s} - \frac{1}{a}\frac{\partial^3 w}{\partial s^2\,\partial x} - \frac{1-\nu^2}{Eh}\frac{\partial^2}{\partial x\,\partial s} q = 0$$

Substitution of the terms containing v in Eq. (26:a) and (26:b) into Eq. (26:c), after simplification, gives

$$a\nabla^4 u = \nu\frac{\partial^3 w}{\partial x^3} - \frac{\partial^3 w}{\partial x\,\partial s^2} - a\frac{(1+\nu)^2}{Eh}\frac{\partial^2 q}{\partial x\,\partial s} \qquad (26{:}d)$$

Similarly, applying $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial s^2}$ to Eq. (25:2) and solving for the terms containing u, and substituting in (25:1) after applying $\frac{\partial^2}{\partial x\,\partial s}$ to it, we obtain an equation from which u has been eliminated

$$a\nabla^4 v = \frac{\partial^3 w}{\partial s^3} + (2+\nu)\frac{\partial^3 w}{\partial s\,\partial x^2} + \frac{a(1+\nu)}{Eh}\left\{2\frac{\partial^2 q}{\partial x^2} + (1-\nu)\frac{\partial^2 q}{\partial s^2}\right\} \qquad (26{:}e)$$

Now, applying $\frac{\delta}{\delta x}$ to (26:d) and $\frac{\delta}{\delta s}$ to (26:e) and substituting these two equations into Eq. (25:3), after applying ∇^4 to it, we obtain an equation from which both u and v have been eliminated

$$\nabla^8 w + \frac{Eh}{a^2 D}\frac{\delta^4 w}{\delta x^4} - \frac{1}{Da}\left\{(2+\nu)\frac{\delta^3 q}{\delta^2 x\,\delta s} + \frac{\delta^3 q}{\delta s^3}\right\} = 0 \qquad (26:f)$$

B. Load Acting Longitudinally

The equations of equilibrium of an element under a longitudinal pressure of intensity q can also be obtained in the same manner.

$$a\frac{\partial T_1}{\partial x} + \frac{\partial S}{\partial \varphi} + q a = 0$$

$$\frac{\partial T_2}{\partial \varphi} + a\frac{\partial S}{\partial x} = 0 \qquad (27)$$

$$\frac{2}{a}\frac{\partial^2 H}{\partial \varphi \partial x} + \frac{\partial^2 G_1}{\partial x^2} + \frac{\partial^2 G_2}{a^2 \partial \varphi^2} + \frac{T_2}{a} = 0$$

Reduction of these equations to forms containing derivations of u, v, w gives

$$\frac{\partial^2 u}{\partial x^2} + \frac{1+\nu}{2}\frac{\partial^2 v}{\partial s \partial x} - \frac{\nu}{a}\frac{\partial w}{\partial x} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial s^2} + \frac{1-\nu^2}{Eh} q = 0$$

$$\frac{\partial^2 v}{\partial s^2} + \frac{1+\nu}{2}\frac{\partial^2 u}{\partial s \partial x} + \frac{1-\nu}{2}\frac{\partial^2 v}{\partial x^2} - \frac{1}{a}\frac{\partial w}{\partial s} = 0 \qquad (28)$$

$$\frac{h^2}{12}\nabla^4 w - \frac{1}{a}\left(\frac{\partial v}{\partial s} - \frac{w}{a} + \nu\frac{\partial u}{\partial x}\right) = 0$$

Applying $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial s^2}$ to (28:1) and solving for the terms involving v, and substituting in (28:2), after applying $\frac{\partial^2}{\partial x \partial s}$ to it, we obtain an equation from which v has been eliminated

$$a\nabla^4 u = \nu \frac{\partial^2 w}{\partial x^2} - \frac{\partial^3 w}{\partial x \partial s^2} - \frac{a(1+\nu)}{Eh}\left\{2\frac{\partial^2 q}{\partial s^2} + (1-\nu)\frac{\partial^2 q}{\partial x^2}\right\} \tag{29:a}$$

Similarly, applying $\frac{\partial^2}{\partial x^2}$ and $\frac{\partial^2}{\partial s^2}$ to Eq. (28:2) and solving for the terms involving u and substituting in (28:1), after applying $\frac{\partial^2}{\partial x \partial s}$ to it, we obtain an equation from which u has been eliminated

$$a\nabla^4 v = \frac{\partial^3 w}{\partial s} + (2+\nu)\frac{\partial^3 w}{\partial s \partial x^2} + \frac{a(1+\nu)^2}{Eh}\frac{\partial^2 q}{\partial x \partial s} \tag{29:b}$$

Now applying $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial s}$ to (29:a) and (29:b) respectively and ∇^4 to (28:3), and substituting the terms of u and v in the first two equations into the third one, we get

$$\nabla^8 w + \frac{Eh}{a^2 D}\frac{\partial^4 w}{\partial x^4} - \frac{1}{Da}\left\{\frac{\partial^3 q}{\partial x \partial s^2} - \nu\frac{\partial^3 q}{\partial x^3}\right\} = 0 \tag{29:c}$$

It is seen that the above equation reduces to the known differential equation of the flat plate if a is made infinitely large.Under such kind loading the lateral deflection of an infinitely long plate is obviously equal to zero.

2. Determination of Shearing Stress Distribution

A. S_t Due to Two Equal and Opposite Tangential Forces (Fig. 8b)

The shearing Stress resultant in the wall of a cylindrical shell is given as *

$$S_t = \frac{Eh}{2(1+\nu)} \left\{ \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} \right\} \tag{30}$$

where the displacements u and v can be determined from the differential equations (26d) and (26e).

The load distribution under consideration may be represented by an even function along the circumference and by an odd function along the generatrix. As discussed in Part One we can express the load distribution function by a combination of a Fourier Series and a Fourier Integral.

$$q(x,s) = \left[\frac{q_0}{2} + \sum_{n=2,4\cdots}^{\infty} q_n \cos\frac{ns}{a}\right] \int_0^{\infty} f(\lambda) \sin \lambda \frac{x}{a}\, d\lambda \tag{31:a}$$

The components of displacement can be written in a similar form. They must contain then three undetermined functions $u(\lambda)$, $v(\lambda)$ and $w(\lambda)$

---

* See S. Timoshenk. Theory of Plates and Shells, Pages 365 and 439.

$$U = \sum_{n=0,2\cdots}^{\infty} \sin\frac{ns}{a}\int_0^{\infty} u(\lambda)\cos\lambda\frac{x}{a}\,d\lambda \qquad (31:b)$$

$$V = \sum_{n=0,2\cdots}^{\infty} \cos\frac{ns}{a}\int_0^{\infty} v(\lambda)\sin\lambda\frac{x}{a}\,d\lambda \qquad (31:c)$$

$$W = \sum_{n=0,2\cdots}^{\infty} \sin\frac{ns}{a}\int_0^{\infty} w(\lambda)\sin\lambda\frac{x}{a}\,d\lambda \qquad (31:d)$$

The differential equations (26:d) and (26:e) can be solved by substituting Eqs. (31:a), (31:d) and (31:b) and (31:c) respectively. In order to simplify the analysis we can break up the differential equations into two parts and solve them separately. The solutions so obtained can be combined because they are all linear. The first equation (of displacement u) is obtained by putting a infinitely large. Then Eq. (26:d) becomes

$$\nabla^4 u + \frac{(1+\nu)^2}{Eh}\frac{\partial^2 q}{\partial x\,\partial s} = 0 \qquad (32:a)$$

This equation is representative of an infinitely large imaginary plate under the specified load. The second equation of displacement u can be obtained by putting q=0. It contains a relation between u and w which is

$$\nabla^4 u - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1}{a}\frac{\partial^3 w}{\partial x\,\partial s^2} = 0 \qquad (32:b)$$

The differential equations for displacement v are obtained in a similar way:

$$\nabla^4 v - \frac{1+\nu}{Eh}\left\{2\frac{\partial^2 q}{\partial x^2} + (1-\nu)\frac{\partial^2 q}{\partial s^2}\right\} = 0 \qquad (33:a)$$

$$\nabla^4 v - \frac{1}{a}\frac{\partial^3 w}{\partial s^3} - \frac{(2+\nu)}{a}\frac{\partial^3 w}{\partial s \partial x^2} = 0 \qquad (33:b)$$

Now, let us substitute Eqs. (31:a) and (31:b) in Eq. (32:a)

For $n = 0$

$$\frac{\partial^2 q}{\partial x \partial s} = 0; \qquad w(\lambda) = 0; \qquad \therefore u(\lambda) = 0$$

For $n = 2, 4, ---$

$$\sum_{n=2,4\cdots}^{\infty} \int_0^{\infty} \left\{ u(\lambda)\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 - \frac{(1+\nu)^2}{Eh} q_n f(\lambda)\left(\frac{n}{a}\right)\left(\frac{\lambda}{a}\right)\right\} \cos\frac{\lambda x}{a} \sin\frac{ns}{a}\, d\lambda = 0$$

for all values of λ, or

$$u(\lambda) = \frac{(1+\nu)^2}{Eh} q_n f(\lambda) \frac{\frac{n\lambda}{a^2}}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}$$

Hence the solution of Eq. (32:a) is given by

$$u_1 = \frac{(1+\nu)^2}{Eh} \sum_{n=2,4\cdots}^{\infty} \sin\frac{ns}{a} \int_0^{\infty} \frac{q_n f(\lambda)\frac{n\lambda}{a^2}}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2} \cos\lambda\frac{x}{a}\, d\lambda \qquad (34)$$

The functions q_n and $f(\lambda)$ in the above integral can be determined by developing them from the loading condition which is shown in Fig. 9.

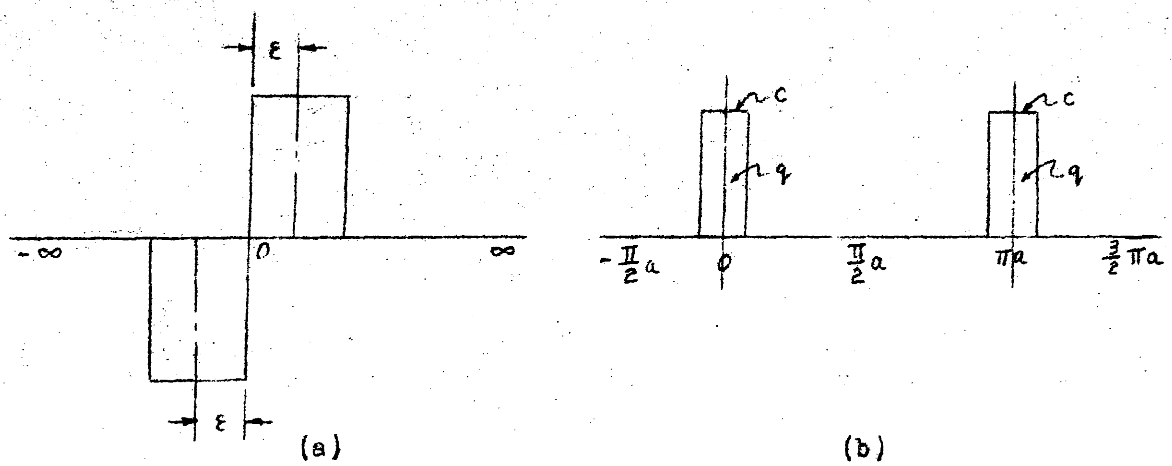


Fig. 9

The loading condition along the longitudinal direction is shown in Fig (9a) and the transformed Fourier Integral of Eq. (31:a) is given by

$$f(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} q\left(\frac{x}{a}\right) \sin\lambda\frac{x}{a}\, d\left(\frac{x}{a}\right)$$

where $q(\frac{x}{a})$ is distributed between $(-\infty, \infty)$ as follow:

$$q\left(\frac{x}{a}\right) = \pm 1 \quad \text{when } 0 < x < 2\varepsilon$$
$$0 > x > -2\varepsilon$$

$$q\left(\frac{x}{a}\right) = 0 \quad \text{when } -2\varepsilon > -x$$
$$2\varepsilon > x$$

We get

$$f(\lambda) = \frac{2}{\pi}\int_{0}^{2\varepsilon} 1 \cdot \sin\lambda\frac{x}{a}\, d\frac{x}{a} = \frac{2}{\pi\lambda}\left(\cos\frac{2\lambda\varepsilon}{a} - 1\right) = \frac{4}{\pi\lambda}\sin^2\lambda\frac{\varepsilon}{a}$$

The Fourier coefficient q_n between $(-\frac{\pi}{2}, \frac{\pi}{2})$ along the circumference is given by

$$q_n = \frac{1}{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z)\cos nz\, dz$$

The expression is valid in the interval $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ if one puts $z = \frac{s}{a}$. We get

$$q_n = \frac{1}{\frac{\pi}{2}a}\int_{-\frac{\pi}{2}a}^{\frac{\pi}{2}a} q\left(\frac{s}{a}\right)\cos n\frac{s}{a}\, ds$$

But $q(\frac{s}{a})$ is distributed in the interval $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ as follows:

$$q\left(\frac{s}{a}\right) = 0 \quad \text{when} \quad -\frac{\pi}{2}a \le s < c$$
$$\frac{\pi}{2}a \quad s \quad c$$

$$q\left(\frac{s}{a}\right) = q \quad \text{when} \quad c \ge s \ge 0$$
$$-c \le s \le 0$$

Therefore $q_n = \frac{2}{\pi a}\left[\int_{-\frac{\pi}{2}a}^{c} 0\cdot \cos n\frac{s}{a}\,ds + \int_{-c}^{c} q \cos n\frac{s}{a}\,ds + \int_{0}^{\frac{\pi}{2}a} 0\cdot \cos n\frac{s}{a}\,ds\right]$

$= \frac{4q}{\pi n}\sin\frac{nc}{a}$

Substituting $f(\lambda)$ and q_n in Eq. (34), we get

$$u_1 = \frac{16q}{\pi^2}\,\frac{(1+\nu)^2}{Eh}\sum_{n=2,4\cdots}^{\infty}\sin\frac{ns}{a}\int_0^{\infty}\frac{\sin\frac{nc}{a}\,\sin^2\frac{\lambda\varepsilon}{a}}{a^2\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2}\cos\lambda\frac{x}{a}\,d\lambda$$

Now we consider the case of a concentrated torque applied at the origin. This can be obtained by making the lengths 2ε and $2c$ of the loaded portion infinitely small.

Substituting $P = q\ (4\varepsilon c)$ and $\sin\lambda\frac{\varepsilon}{a} \approx \frac{\varepsilon\lambda}{a}$

$c \to 0$ $\quad\quad \sin n\frac{c}{a} \approx n\frac{c}{a}$

$\varepsilon \to 0$

and also Torque $= T = P \times 2\varepsilon = 8q\varepsilon^2 c$

in the above equation we have

$$u_1 = \frac{2T}{\pi^2 a}\,\frac{(1+\nu)^2}{Eh}\sum_{n=2,4\cdots}^{\infty} n\sin\frac{ns}{a}\int_0^{\infty}\frac{\lambda^2}{(\lambda^2+n^2)^2}\cos\lambda\frac{x}{a}\,d\lambda \qquad (35)$$

The definite integral in Eq. (35) can be evaluated in the same manner as explained in Part One. However, on account of the double poles, the analysis may be simplified if we use the following transformation,

$$\frac{\partial}{\partial n}\int_{-\infty}^{\infty}\frac{dz}{(z^2+n^2)} = \int_{-\infty}^{\infty}\frac{2n\,dz}{(z^2+n^2)^2} \qquad (35a)$$

Therefore we can write

$$I = \int_{-\infty}^{\infty} \frac{z^2 e^{iz\frac{x}{a}} dz}{(z^2+n^2)^2} = -\frac{1}{2n}\frac{\partial}{\partial n}\int_{-\infty}^{\infty}\frac{z^2 e^{i\frac{x}{a}z} dz}{(z^2+n^2)}$$

By Cauchy's Theorem of Residues with the aid of Jordan's Lemma we obtain

$$I = 2\pi i\left(-\frac{1}{2n}\right)\frac{\partial}{\partial n}\left\{\frac{(in)^2 e^{-\frac{x}{a}n}}{2in}\right\} = \frac{\pi}{2n}\left\{1 - n\frac{x}{a}\right\} e^{-n\frac{x}{a}}$$

or

$$\int_0^{\infty} \frac{\lambda^2 \cos \lambda \frac{x}{a}}{(\lambda^2+n^2)^2} d\lambda = \frac{\pi}{4n}\left\{1 - n\frac{x}{a}\right\} e^{-n\frac{x}{a}}$$

Substituting the above integral in Eq. (35), we get

$$u_1 = \frac{(1+\nu)^2 T}{2\pi a Eh} \sum_{n=2,4\cdots}^{\infty} \sin n\frac{s}{a}\left\{1 - n\frac{x}{a}\right\} e^{-n\frac{x}{a}} \qquad (36)$$

Differentiating Eq.(36) with respect to s we obtain:

$$\frac{\partial u_1}{\partial s} = \frac{(1+\nu) T}{2\pi a^2 Eh} \sum_{n=2,4\cdots}^{\infty} \cos \frac{ns}{a}\left\{(1+\nu)n - (1+\nu)n^2\frac{x}{a}\right\} e^{-n\frac{x}{a}} \qquad (37)$$

In order to solve Eq. (32:b) for u_2 it is necessary to find the undetermined function $w(\lambda)$ by substituting Eqs. (31:a) and (31:d) in Eq. (26:f).

For $n = 0$, $\quad w(\lambda) = 0$

For $n = 2, 4$ ---

$$\sum_{n=2,4\cdots}^{\infty}\int_0^{\infty}\left[w(\lambda)\left\{\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4\right\} - \frac{q_n f(\lambda)}{Da}\left\{(2+\nu)\frac{n}{a}\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^3\right\}\right]\sin\frac{ns}{a}\sin\frac{\lambda x}{a}\, d\lambda = 0$$

The equation has to be true for all λ

$$w(\lambda) = \frac{\frac{q_n f(\lambda)}{a D}\left[\frac{n}{a}\left\{(2+\nu)\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right\}\right]}{\left\{\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right\}^4+\frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4} = \frac{\frac{a^4}{D} q_n f(\lambda)\left[(2+\nu)\lambda^2+n^2\right]}{(\lambda^2+n^2)^4+J^2\lambda^4} \qquad (37:a)$$

Where $J^2 = 12(1-\nu^2)\left(\frac{a}{h}\right)^2$ as mentioned in Part One.

By substituting (31:b) and (31:d) in Eq. (32:b) we have the undetermined function $u(\lambda)$

For $n = 0$ $\quad w(\lambda) = 0 \quad \therefore u(\lambda) = 0$

For $n = 2, 4$ ---

$$\sum_{n=2,4\cdots}^{\infty}\int_0^{\infty}\left\{u(\lambda)\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2+\frac{w(\lambda)}{a}\left[\nu\left(\frac{\lambda}{a}\right)^3-\left(\frac{\lambda}{a}\right)\left(\frac{n}{a}\right)^2\right]\right\}\cos\frac{\lambda x}{a}\sin\frac{ns}{a}\,d\lambda = 0$$

or

$$u(\lambda) = \frac{\frac{w(\lambda)}{a}\left[\left(\frac{\lambda}{a}\right)\left(\frac{n}{a}\right)^2-\nu\left(\frac{\lambda}{a}\right)^3\right]}{\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2}$$

Substituting $w(\lambda)$ in the above form of $u(\lambda)$

$$u(\lambda) = \frac{\frac{2T\lambda^2 n a}{\pi^2 D}\left[(2+\nu)\lambda^2+n^2\right]\left[n^2-\nu\lambda^2\right]}{(\lambda^2+n^2)^2\left\{(\lambda^2+n^2)^4+J^2\lambda^4\right\}}$$

Differentiating Eq. (31:b) with respect to s we get

$$\frac{\partial u_z}{\partial s} = \sum_{n=2,4\cdots}^{\infty}\frac{n}{a}\cos\frac{ns}{a}\int_0^{\infty}u(\lambda)\cos\lambda\frac{x}{a}\,d\lambda$$

$$= -\frac{2T}{\pi^2 D}\sum_{n=2,4\cdots}^{\infty}n^2\cos\frac{ns}{a}\int_0^{\infty}\frac{\lambda^2\left[(2+\nu)\lambda^2+n^2\right]\left[\nu\lambda^2-n^2\right]\cos\lambda\frac{x}{a}}{(\lambda^2+n^2)^2\left\{(\lambda^2+n^2)^4+J^2\lambda^4\right\}}d\lambda \qquad (38)$$

With the aid of transformation (35:a) we obtain

$$I = \int_{-\infty}^{\infty} \frac{\nu(2+\nu)z^2[z^2+\xi_1][z^2-\xi_2]\, e^{iz\frac{x}{a}}}{(z^2+n^2)^2\left[(z^2+n^2)^4 + J^2 z^4\right]}\, dz$$

$$= -\frac{\nu(2+\nu)}{2n}\frac{\partial}{\partial n}\int_{-\infty}^{\infty} \frac{z^2(z^2+\xi_1)(z^2-\xi_2)\, e^{iz\frac{x}{a}}}{(z^2+n^2)\left[(z^2+n^2)^4+J^2z^4\right]}\, dz$$

where $\qquad \xi_1 = \dfrac{n^2}{2+\nu}, \qquad \xi_2 = \dfrac{n^2}{\nu}$

This is used to evaluate the integral with the double pole. Hence the integral in Eq. (38) can be evaluated by Cauchy's Theorem of Residues.

$$I = 2\pi i\, \nu(2+\nu)\left\{ -\frac{1}{2n}\frac{\partial}{\partial n}\left[\frac{-n^2(-n^2+\xi_1)(-n^2-\xi_2)\, e^{-n\frac{x}{a}}}{2in(-n^2-\alpha_1^2)(-n^2-\alpha_2^2)(-n^2-\alpha_3^2)(-n^2-\alpha_4)}\right]\right.$$

$$+ \frac{\alpha_1(\alpha_1^2+\xi_1)(\alpha_1^2-\xi_2)\, e^{i\alpha_1\frac{x}{a}}}{2(\alpha_1^2+n^2)^2(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)} + \frac{\alpha_4(\alpha_4^2+\xi_1)(\alpha_4^2-\xi_2)\, e^{i\alpha_4\frac{x}{a}}}{2(\alpha_4^2+n^2)^2(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)}$$

$$+ \frac{\alpha_2(\alpha_2^2+\xi_1)(\alpha_1^2-\xi_2)\, e^{i\alpha_1\frac{x}{a}}}{2(\alpha_2^2+n^2)^2(\alpha_2^2-\alpha_1^2)(\alpha_2^2-\alpha_3^2)(\alpha_2^2-\alpha_4^2)} + \frac{\alpha_3(\alpha_3^2+\xi_1)(\alpha_3^2-\xi_2)\, e^{i\alpha_3\frac{x}{a}}}{2(\alpha_3^2+n^2)^2(\alpha_3^2-\alpha_1^2)(\alpha_3^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)} \qquad (39)$$

where $(\pm in)^2$, $\pm\alpha_1$, $\pm\alpha_2$, $\pm\alpha_3$, and $\pm\alpha_4$ are the twelve roots of the twelfth degree algebraic equation in the denominator of the integral in equation (38), and α_s are calculated in Eq. (12).

It is next desired to determine the value of v. In order to accomplish this we can solve Eqs. (33:a) and (33:b) by substituting in (31:c), (31:a) and (3:d) respectively. First, we have

For n = 0 $\qquad \dfrac{\partial^2 q_0}{\partial x^2} = \dfrac{\partial^2 q_0}{\partial s^2} = 0 \qquad v_1 = 0$

For n = 2, 4, ---

$$\sum_{n=2,4\cdots}^{\infty} \int_0^{\infty} \left\{ v(\lambda)\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 + \frac{(1+\nu)\,f(\lambda)\,q_n}{Eh}\left[2\left(\frac{\lambda}{a}\right)^2 + (1-\nu)\left(\frac{n}{a}\right)^2\right]\right\}$$

$$\cos\frac{ns}{a}\,\sin\lambda\frac{x}{a}\,d\lambda = 0$$

or

$$v(\lambda) = -\frac{(1+\nu)}{Eh}\,q_n\,f(\lambda)\,\frac{2\left(\frac{\lambda}{a}\right)^2 + (1-\nu)\left(\frac{n}{a}\right)^2}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}$$

Substituting $v(\lambda)$ into Eq. (31:c), where $q_n f(\lambda)$ has been determined in the analysis of $u(\lambda)$, we have

$$V_1 = -\frac{1+\nu}{Eh}\,\frac{2T}{\pi^2 a^3}\sum_{n=2,4\cdots}^{\infty}\cos\frac{ns}{a}\int_0^{\infty}\frac{a^2\lambda\left[2\lambda^2 + (1-\nu)n^2\right]}{\left[\lambda^2+n^2\right]^2}\sin\frac{\lambda x}{a}\,d\lambda \qquad (40)$$

In order to evaluate the above integral we have first to simplify the rational function by putting it in the form of partial fractions, i.e.,

$$\frac{\lambda\left[2\lambda^2 + (1-\nu)n^2\right]}{(\lambda^2+n^2)^2} = \frac{Fz+G}{(\lambda^2+n^2)^2} + \frac{Qz+R}{\lambda^2+n^2}$$

Here the constants F, G, Q and R can be determined by equating the numerators. Hence

$$\frac{\lambda\left[2\lambda^2 + (1-\nu)n^2\right]}{(\lambda^2+n^2)^2} = -\frac{(1+\nu)\,n^2\lambda}{(\lambda^2+n^2)^2} + \frac{2\lambda}{\lambda^2+n^2}$$

Substituting the simplified fractions back into the integral, we obtain

$$I = -\int_{-\infty}^{\infty}\frac{(1+\nu)n^2 z}{(z^2+n^2)^2}\, e^{-i\frac{x}{a}z}\,dz + 2\int_{-\infty}^{\infty}\frac{z\, e^{iz\frac{x}{a}}}{z^2+n^2}\,dz$$

By Cauchy's Theorem of Residues and the method used in evaluating the integral in Eq. (35) we have

$$I = -2\pi i\left(\frac{1}{2n}\right)\frac{\partial}{\partial n}\left[-\frac{n(1-\nu)}{2}\,e^{-n\frac{x}{a}}\right] + 2\pi i\, e^{-n\frac{x}{a}}$$

or

$$\int_0^{\infty}\frac{\lambda[2\lambda^2+(1-\nu)n^2]}{(\lambda^2+n^2)^2}\sin\lambda\frac{x}{a}\,d\lambda = \frac{-\pi(1+\nu)}{4}n\left(\frac{x}{a}\right)e^{-n\frac{x}{a}} + \pi e^{-n\frac{x}{a}}$$

Substituting the above integral in Eq. (40)

$$v_1 = -\frac{(1+\nu)}{Eh}\,\frac{2T}{\pi a}\sum_{n=2,4\cdots}^{\infty}\cos n\frac{s}{a}\left\{1+\frac{1+\nu}{4}\,n\left(\frac{x}{a}\right)\right\}e^{-n\left(\frac{x}{a}\right)} \qquad (41)$$

Differentiating Eq. (41) with respect to x we get

$$\frac{\partial v_1}{\partial x} = \frac{(1+\nu)\,T}{2\pi a^2 Eh}\sum_{n=2,4\cdots}^{\infty}\cos n\frac{s}{a}\left\{(5+\nu)n-(1+\nu)n^2\left(\frac{x}{a}\right)\right\}e^{-n\frac{x}{a}} \qquad (45)$$

Equation (33:b) can be solved in the same manner as we solved Eq. (32:b)

For $n = 0$ $\qquad w(\lambda) = 0 \qquad v(\lambda) = 0$

For n = 2, 4 ---

$$\sum_{n=2,4\cdots}^{\infty}\int_0^{\infty}\left\{v(\lambda)\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 + \frac{w(\lambda)}{a}\left[\left(\frac{n}{a}\right)^3+(2+\nu)\left(\frac{n}{a}\right)\left(\frac{\lambda}{a}\right)^2\right]\right\}\cos\frac{\lambda x}{a}\sin\frac{\lambda x}{a}\,d\lambda = 0$$

or
$$v(\lambda) = \frac{\frac{w(\lambda)}{a}\left[\left(\frac{n}{a}\right)^2 + (2+\nu)\left(\frac{\lambda}{a}\right)^2\right]\frac{n}{a}}{\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2}$$

By substituting $w(\lambda)$ in Eq. (37:a) we have

$$v(\lambda) = \frac{-\frac{2T\lambda n^2 a}{\pi^2 D}\left[(2+\nu)\lambda^2 + n^2\right]^2}{(\lambda^2+n^2)^2\left\{\left[\lambda^2+n^2\right]^4 + J^2\lambda^4\right\}}$$

Differentiating Eq. (31:a) and substituting $v(\lambda)$ from the above form we get:

$$\frac{\partial v_2}{\partial x} = \sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a}\int_0^{\infty} v(\lambda)\left(\frac{\lambda}{a}\right)\cos\lambda\frac{x}{a}\,d\lambda$$

$$= -\frac{2T}{\pi^2 D}\sum_{n=2,4}^{\infty} n^2\cos\frac{ns}{a}\int_0^{\infty}\frac{\lambda^2\left[(2+\nu)\lambda^2+n^2\right]^2\cos\lambda\frac{x}{a}}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4+J^2\lambda^4\right]}\,d\lambda \qquad (46)$$

The above integral can be evaluated by a method similar to that used in determining u_2

$$I = \int_{-\infty}^{\infty}\frac{\lambda^2\left[(2+\nu)\lambda^2+n^2\right]^2\cos\lambda\frac{x}{a}\,d\lambda}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4+J^2\lambda^4\right]}$$

$$= 2\pi i(2+\nu)^2\left\{-\frac{1}{2n}\frac{\partial}{\partial n}\left[\frac{-n^2(-n^2+\xi_1)^2 e^{-n\frac{x}{a}}}{2in(-n^2-\alpha_1^2)(-n^2-\alpha_2^2)(-n^2-\alpha_3^2)(-n^2-\alpha_4^2)}\right]\right.$$

$$+\frac{\alpha_1(\alpha_1^2+\xi_1)^2 e^{i\alpha_1\frac{x}{a}}}{2(\alpha_1^2+n^2)^2(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)} + \frac{\alpha_4(\alpha_4^2+\xi_1)^2 e^{i\alpha_4\frac{x}{a}}}{2(\alpha_4^2+n^2)^2(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)} \qquad (47)$$

$$\left.+\frac{\alpha_2(\alpha_2^2+\xi_1)^2 e^{i\alpha_2\frac{x}{a}}}{2(\alpha_2^2+n^2)^2(\alpha_2^2-\alpha_1^2)(\alpha_2^2-\alpha_3^2)(\alpha_2^2-\alpha_4^2)} + \frac{\alpha_3(\alpha_3^2+\xi_1)^2 e^{i\alpha_3\frac{x}{a}}}{2(\alpha_3^2+n^2)^2(\alpha_3^2-\alpha_1^2)(\alpha_3^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)}\right\}$$

where α_s are calculated in Eq. (12)

Now the shearing stress resultant in Eq. (30) can be written as follows

$$S_t = S_{t_1} + S_{t_2} = \frac{Eh}{2(1+\nu)}\left\{\frac{\partial u_1}{\partial s} + \frac{\partial v_1}{\partial x} + \frac{\partial u_2}{\partial s} + \frac{\partial v_2}{\partial x}\right\}$$

where

$$S_{t_1} = \frac{Eh}{2(1+\nu)}\left\{\frac{\partial u_1}{\partial s} + \frac{\partial v_1}{\partial x}\right\}$$

$$= \frac{T}{2\pi a^2}\sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a}\left[(3+\nu)n-(1+\nu)n^2\frac{x}{a}\right]e^{-n\frac{x}{a}} \qquad (48)$$

and

$$S_{t_2} = \frac{Eh}{2(1+\nu)}\left\{\frac{\partial u_2}{\partial s} + \frac{\partial v_2}{\partial x}\right\}$$

By substituting $\frac{\partial u_2}{\partial s}$ and $\frac{\partial v_2}{\partial x}$ from Eqs. (38), (39), (46) and (47) in the above equation, after simplifying, we have

$$S_{t_2} = -\frac{T}{\pi a^2}\frac{J^2}{(1+\nu)}\sum_{n=2,4\cdots}^{\infty} i n^2 \cos\frac{ns}{a}\left\{-\frac{1}{2n}\right.$$

$$\cdot\frac{\partial}{\partial n}\left[\frac{(2+\nu)\left\{-\nu n(n^2-\xi_1)(n^2+\xi_2)-(2+\nu)n(n^2-\xi_1)^2\right\}e^{-n\frac{x}{a}}}{2i(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(\alpha_4^2+n^2)}\right] \qquad (49)$$

$$+\frac{\alpha_1^3(1+\nu)(2+\nu)(\alpha_1^2+\xi_1)e^{i\alpha_1\frac{x}{a}}}{(\alpha_1^2+n^2)^2(\alpha_1^2+\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)}+\frac{\alpha_4^3(1+\nu)(2+\nu)(\alpha_4^2+\xi_1)e^{i\alpha_4\frac{x}{a}}}{(\alpha_4^2+n^2)^2(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)}$$

$$\left.+\frac{\alpha_2^3(1+\nu)(2+\nu)(\alpha_1^2+\xi_1)e^{i\alpha_2\frac{x}{a}}}{(\alpha_2^2+n^2)^2(\alpha_2^2-\alpha_1^2)(\alpha_2^2-\alpha_3^2)(\alpha_2^2-\alpha_4^2)}+\frac{\alpha_3^3(1+\nu)(2+\nu)(\alpha_3^2+\xi_1)e^{i\alpha_3\frac{x}{a}}}{(\alpha_3^2+n)^2(\alpha_3^2-\alpha_1^2)(\alpha_3^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)}\right\}$$

Eq. (49) has a very complicated form. In order to simplify it we may take its first term which gives, after differentiating with respect to n

$$-\frac{1}{2n}\Big[\frac{-(1+\nu)(2+\nu)\left[n^3(2n)+3n^2(n^2-\frac{n^2}{2+\nu})-n^3(\frac{x}{a})(n^2-\frac{n^2}{2+\nu})\right]e^{-n\frac{x}{a}}}{i(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)}$$

$$+\frac{i(1+\nu)(2+\nu)n^3(n^2-\frac{n^2}{2+\nu})\,e^{-n(\frac{x}{a})}}{-[(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)]^2}\Big\{2n(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)$$

$$+2n(n^2+\alpha_1^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)+2n(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_4^2)$$

$$+2n(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)\Big\}$$

Substituting α_s from Eq. (12) and simplifying, we have

$$\frac{(1+\nu)}{2iJ^2n}\left\{(3+\nu)-n(\frac{x}{a})(1+\nu)\right\}e^{-n\frac{x}{a}} \tag{50}$$

Before simplifying the next four terms we introduce the following simplified notations with the aid of Eq. (12):

$$(\alpha_1^2+n^2)^2=[\eta+i(\phi-\tfrac{J}{2})]^2 \qquad (\alpha_1^2-\alpha_2^2)=2\eta+2i\phi$$
$$(\alpha_4^2+n^2)^2=[\eta-i(\phi-\tfrac{J}{2})]^2 \qquad (\alpha_1^2-\alpha_3^2)=2\eta-iJ$$
$$(\alpha_2^2+n^2)^2=[-\eta-i(\phi+\tfrac{J}{2})]^2 \qquad (\alpha_1^2-\alpha_4^2)=i(2\phi-J)$$
$$(\alpha_3^2+n^2)^2=[-\eta+i(\phi+\tfrac{J}{2})]^2 \qquad (\alpha_4^2-\alpha_2^2)=2\eta+iJ$$
$$(\alpha_4^2-\alpha_3^2)=2\eta-2i\phi$$
$$(\alpha_3^2-\alpha_2^2)=i(J+2\phi)$$

$$(\alpha_1^2+n^2)^2(\alpha_2^2+n^2)^2=-n^4J^2$$
$$(\alpha_4^2+n)^2(\alpha_3^2+n^2)^2=-n^4J^2 \tag{51}$$
$$(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)(\alpha_3^2-\alpha_2^2)=-16n^4J^2R_2$$
$$(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)(\alpha_1-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_2^2-\alpha_3^2)=-16n^4J^2R_2$$

After substituting (51) into the second and third terms of Eq. (49), we have the following form with common denominator

$$\frac{(1+\nu)(2+\nu)}{16n^8 J^4 R_2}\Big[\alpha_1\left(-n^2+\eta-i\frac{J}{2}+i\phi\right)\left(-n^2+\eta-i\frac{J}{2}+i\phi+\frac{n^2}{2+\nu}\right)e^{i\alpha_1\frac{x}{a}}$$
$$\cdot\left\{-\eta-i\left(\phi+\frac{J}{2}\right)\right\}^2\left\{2\eta+iJ\right\}\left\{2\eta-2i\phi\right\}\left\{iJ+2i\phi\right\}-\alpha_4\left(-n^2+\eta+\frac{iJ}{2}-i\phi\right)$$
$$\cdot\left(-n^2+\eta+\frac{iJ}{2}-i\phi+\frac{n}{2+\nu}\right)e^{i\alpha_4\frac{x}{a}}\left\{-\eta+i\left(\phi+\frac{J}{2}\right)\right\}^2(2\eta+2i\phi)(2\eta-iJ)(iJ+2i\phi)\Big]$$

Separating the real and imaginary parts

$$\frac{(1+\nu)(2+\nu)i(J+2\phi)}{16n^8J^4R_2}\Big[(A+iB)\Big\{\Big[(-n^2+\eta)\left(-n^2+\eta+\frac{n}{2+\nu}\right)-\left(\phi-\frac{J}{2}\right)^2+i\left(\phi-\frac{J}{2}\right)\left(-2n^2+2\eta+\frac{n}{2+\nu}\right)\Big]$$
$$\cdot\Big[(-4\eta^2+2J\phi)\left(\frac{J^2}{2}+\phi J\right)-2\eta\left(\phi+\frac{J}{2}\right)(2J\eta-4\eta\phi)-i\left(\frac{J}{2}+\phi J\right)(2J\eta-4\eta\phi)$$
$$-i2\eta\left(\phi+\frac{J}{2}\right)(4\eta^2+2J\phi)\Big]\Big\}e^{i(A+iB)\frac{x}{a}}+(A-iB)\Big\{\Big[(-n^2+\eta)\left(-n^2+\eta+\frac{n}{2+\nu}\right)$$
$$-\left(\phi-\frac{J}{2}\right)^2-i\left(\phi-\frac{J}{2}\right)\left(-2n^2+2\eta+\frac{n}{2+\nu}\right)\Big]\Big[(-4\eta^2+2J\phi)\left(\frac{J^2}{2}+\phi J\right)-2\eta\left(\phi+\frac{J}{2}\right)(2J\eta-4\eta\phi)$$
$$+i\left(\frac{J^2}{2}+\phi J\right)(2J\eta-4\eta\phi)-i2\eta\left(\phi+\frac{J}{2}\right)(4\eta^2+2J\phi)\Big]\Big\}e^{i(A-iB)\frac{x}{a}}\Big]$$

Introducing trigonometric functions and simplifying we obtain:

$$\frac{i(1+\nu)(2+\nu)(J+2\phi)^2e^{-B\frac{x}{a}}}{8n^8J^4R_2}\Big[\left(A\cos\frac{Ax}{a}-B\sin\frac{Ax}{a}\right)\Big\{\Big[n^4\left(\frac{1+\nu}{2+\nu}\right)-n^2\eta\frac{(2\nu+3)}{2+\nu}-\frac{J^2}{2}+\phi J\Big]$$
$$\cdot\Big[2R_2\phi-2JR_2-\frac{3J^2}{2}\phi+\frac{J^3}{2}\Big]-\left(\phi-\frac{J}{2}\right)\Big[8\eta^4+8J\eta\phi-2J^2\eta^2-n^4\left(\frac{3+2\nu}{2+\nu}\right)\Big(2R_2\eta-\frac{3}{2}J^2\eta+2n^2J^3\Big)\Big]\Big\}$$

$$-\left(A\sin\frac{Ax}{a}+B\cos\frac{Ax}{a}\right)\left\{\left(\phi-\frac{J}{2}\right)\left[2\eta-n^2\left(\frac{3+2\nu}{2+\nu}\right)\right]\left[2R_1\phi-2JR_1-\frac{3}{2}J^2\phi+\frac{J^3}{2}\right]\right.$$

$$\left.\left.+\left[n^4\left(\frac{1+\nu}{2+\nu}\right)-n^2\eta\left(\frac{2\nu+3}{2+\nu}\right)-\frac{J^2}{2}+\phi J\right]\left[2R_2\eta-\frac{3}{2}J^2\eta+2n^2J^2\right]\right\}\right]$$

Further simplifications will reduce the second and third terms of Eq. (49) to a final form:

$$\frac{i(1+\nu)(2+\nu)e^{-B\frac{x}{a}}}{8n^8J^4R_2}\left[\left(A\cos\frac{Ax}{a}-B\sin\frac{Ax}{a}\right)\left\{n^2\frac{(1+\nu)}{(2+\nu)}\left(4n^6J^2\phi-2n^2J^2R_2\phi+\frac{n^2}{2}J^4\phi\right)\right.\right.$$

$$\left.+2n^6J^2\eta\right\}-\left(A\sin\frac{Ax}{a}+B\cos\frac{Ax}{a}\right)\left\{n^2\left(\frac{1+\nu}{2+\nu}\right)\left(4n^6J^2\eta+2n^4J^3\phi+4n^4J^2R_2\right)\right.$$

$$\left.\left.-2n^6J^3\phi-4n^6J^2R_2\right\}\right] \tag{52}$$

Similarly the last two terms in Eq. (49) can be written, after substituting in (51)

$$\frac{i(1+\nu)(2+\nu)}{16n^8J^4R_2}\left[-\alpha_2\left(-n^2-\eta-i\frac{J}{2}-i\phi\right)\left(-n^2+\frac{n}{2+\nu}-\eta-\frac{iJ}{2}-i\phi\right)e^{i\alpha_2\frac{x}{a}}\left[\eta+i\left(\phi-\frac{J}{2}\right)\right]^2\right.$$

$$\cdot(2\eta+2i\phi)(2\eta+iJ)(2i\phi-iJ)+\alpha_3\left(-n^2-\eta+i\frac{J}{2}+i\phi\right)\left(-n^2+\frac{n}{2+\nu}-\eta+\frac{iJ}{2}+i\phi\right)$$

$$\left.\cdot e^{i\alpha_3\frac{x}{a}}\left[\eta-i\left(\phi-\frac{J}{2}\right)\right]^2(2\eta-2i\phi)(2\eta+iJ)(2i\phi-iJ)\right]$$

After further simplification, we obtain

$$\frac{i(1+\nu)(2+\nu)}{8n^8J^4R_2}\left[\left(C\cos\frac{Cx}{a}-D\sin\frac{Cx}{a}\right)\left\{n^2\left(\frac{1+\nu}{2+\nu}\right)\left[4n^6J^2\phi-2n^2J^2R_2\phi+\frac{n^2}{2}J^4\phi+2n^6J^2\eta\right\}\right.\right.$$

$$\left.+\left(D\cos\frac{Cx}{a}+C\sin\frac{Cx}{a}\right)\left\{n^2\frac{(1+\nu)}{(2+\nu)}\left[4n^6J^2\eta+2n^4J^3\phi-4n^4J^2R_2\right]-2n^6J^3\phi+4n^6J^2R_2\right\}\right] \tag{53}$$

Substituting Eqs. (50), (52) and (53) in Eq. (49), we obtain

$$
\begin{aligned}
S_{t_2} = \frac{T}{2\pi a^2} \sum_{n=2,4\cdots}^{\infty} \cos\frac{ns}{a} \Bigg\{ & -\left[(3+\nu)n - n^2(1+\nu)\frac{x}{a}\right] e^{-n\frac{x}{a}} \\
& + \frac{1}{4J\sqrt{1+j^2}} \Bigg[\left\{ \left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right) e^{-B\frac{x}{a}} + \left(C\cos\frac{Cx}{a} - D\sin\frac{Cx}{a}\right) e^{-D\frac{x}{a}} \right\} \\
& \cdot \left\{ (1+\nu)\left(4\phi - \frac{2R_2\phi}{n^4}\right) + \frac{J^2\phi}{2n^4} + 2(2+\nu)\frac{J\eta}{n^2} \right\} + \frac{4R_2}{n^2} \left\{ \left(A\sin\frac{Ax}{a}\right.\right. \\
& \left.\left. + B\cos\frac{Ax}{a}\right) e^{-B\frac{x}{a}} + \left(C\sin\frac{Cx}{a} + D\cos\frac{Cx}{a}\right) e^{-D\frac{x}{a}} \right\} - \left\{ 4(1+\nu)\eta - \frac{2J\phi}{n^2} \right\} \\
& \cdot \left\{ \left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right) e^{-B\frac{x}{a}} - \left(D\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right) e^{-D\frac{x}{a}} \right\} \Bigg] \Bigg\}
\end{aligned} \tag{54}
$$

Combining S_{t_1} and S_{t_2} from Eqs. (50) and (54), we finally have the shearing stress resultant in the alternate form

$$
\begin{aligned}
\frac{S_t}{\frac{T}{a^2}} = \frac{1}{8\pi} \sum_{n=2,4\cdots}^{\infty} \frac{\cos\frac{ns}{a}}{\sqrt{1+j^2}} \Bigg\{ & \Bigg[(2+\nu)\left(\frac{\sqrt{2J}}{n}\sqrt{\sqrt{1+j^2}-j}\right) + (1+\nu)\left(\frac{2\sqrt{2}}{\sqrt{J}}\, n\sqrt{\sqrt{1+j^2}+j}\right) \\
& - \frac{2\sqrt{1+j^2}}{n^2} + \frac{J^{\frac{3}{2}}}{\sqrt{2}\,n^3}\sqrt{\sqrt{1+j^2}+j} \Bigg] \Bigg[\left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right) e^{-B\frac{x}{a}} + \left(C\cos\frac{Cx}{a}\right. \\
& \left. - D\sin\frac{Cx}{a}\right) e^{-D\frac{x}{a}} + 4\sqrt{1+j^2} \left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right) e^{-B\frac{x}{a}} + \left(C\sin\frac{Cx}{a} + D\cos\frac{Cx}{a}\right) e^{-D\frac{x}{a}} \Bigg] \\
& - \Bigg[2(1+\nu)\sqrt{\frac{2}{J}}\, n\sqrt{\sqrt{1+j^2}-j} - \frac{\sqrt{2J}}{n}\sqrt{\sqrt{1+j^2}-j} \Bigg] \Bigg[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right) e^{-B\frac{x}{a}} - \left(D\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right) e^{-D\frac{x}{a}} \Bigg] \Bigg\}
\end{aligned} \tag{55}
$$

where $J^2 = 12(1-\nu^2)(\frac{a}{h})^2$ and $j = \frac{J}{4n^2}$

Although the radial deflection is of no great importance in this case, it may be worth investigating because it gives a clear picture of the deformation pattern. From Eq. (31:d), we obtain after substituting $w(\lambda)$ from Eq. (37:a):

$$w = \frac{eTa}{\pi^2 D} \sum_{n=2,4\cdots}^{\infty} n \sin\frac{ns}{a} \int_0^\infty \frac{\lambda[(2+\nu)\lambda^2+n^2]}{[\lambda^2+n^2]^4 + J^2\lambda^4} \sin\lambda\frac{x}{a}\, d\lambda \qquad (56)$$

The definite integral is evaluated by Cauchy's Theorem of Residues

$$\int_0^\infty \frac{\lambda[(2+\nu)\lambda^2+n^2]}{[\lambda^2+n^2]^4+J^2\lambda^4} \sin\lambda\frac{x}{a}\,d\lambda = \pi i \left\{ \frac{[(2+\nu)\alpha_1^2+n^2]\,e^{i\alpha_1\frac{x}{a}}}{2(\alpha_1^2-\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)} \right.$$

$$+ \frac{[(2+\nu)\alpha_2^2+n^2]\,e^{i\alpha_2\frac{x}{a}}}{2(\alpha_2^2-\alpha_1^2)(\alpha_2^2-\alpha_3^2)(\alpha_2^2-\alpha_4^2)} + \frac{[(2+\nu)\alpha_3^2+n^2]\,e^{i\alpha_3\frac{x}{a}}}{2(\alpha_3^2-\alpha_1^2)(\alpha_3^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)}$$

$$+ \frac{[(2+\nu)\alpha_4^2+n^2]\,e^{i\alpha_4\frac{x}{a}}}{2(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)}$$

Substituting this integral back into Eq. (56), and simplifying by using α_s in Eq. (12) we obtain an alternate form of w

$$\frac{\frac{W}{h}}{\frac{T}{Eha^2}} = \frac{(\frac{a}{h})}{8\pi} \sum_{n=2,4\cdots}^{\infty} \frac{\sin\frac{ns}{a}}{n\sqrt{1+j^2}} \left\{ 4J\sqrt{1+j^2}\left(\sin A\frac{x}{a}\,e^{-B\frac{x}{a}} - \sin C\frac{x}{a}\,e^{-D\frac{x}{a}}\right) \right.$$

$$+ \left[\frac{\sqrt{2}J^{\frac{3}{2}}}{n}\sqrt{\sqrt{1+j^2}+j} - 2\sqrt{2}(1+\nu)\sqrt{J}\,n\sqrt{\sqrt{1+j^2}-j}\right]\left(e^{-B\frac{x}{a}}\sin A\frac{x}{a} + e^{-D\frac{x}{a}}\sin C\frac{x}{a}\right) \qquad (57)$$

$$\left. + \left[\frac{\sqrt{2}J^{\frac{3}{2}}}{n}\sqrt{\sqrt{1+j^2}-j} + 2\sqrt{2}\,n(1+\nu)\sqrt{J}\sqrt{\sqrt{1+j^2}+j}\right]\left(e^{-B\frac{x}{a}}\cos A\frac{x}{a} - \cos C\frac{x}{a}\,e^{-D\frac{x}{a}}\right)\right\}$$

It is seen that the radial deflection has a pattern anti-symmetrical with respect to both the x and s axes. Moreover, we can easily prove that w equals zero at either $x = 0$ or $s = 0$, i.e.,

$$\left.\frac{\frac{w}{h}}{\frac{T}{Eha^2}}\right|_{x=0} = \frac{\frac{a}{h}}{8\pi} \sum_{n=2,4\cdots}^{\infty} \frac{\sin\frac{ns}{a}}{\sqrt{1+j^2}} \left[\frac{\sqrt{2}\, j^{\frac{3}{2}}}{n^2} \sqrt{\sqrt{1+j^2}-j} + 2\sqrt{2}(1+\nu)\sqrt{j}\sqrt{\sqrt{1+j^2}+j} \right](1-1) = 0$$

$$\left.\frac{\frac{w}{h}}{\frac{T}{Eha^2}}\right|_{s=0} = \frac{\frac{a}{h}}{8\pi} \sum_{n=2,4\cdots}^{\infty} \frac{\sin 0^\circ}{n\sqrt{1+j^2}} \left\{ \text{Refer to Eq. (57)} \right\} = 0$$

B. S_ℓ Due to Two Equal and Opposite Longitudinal Forces. Fig. (8c)

The shearing stress resultant S_ℓ can be determined in this case also from the displacements u and v in differential equations (29:a) and (29:b).

Since the load distribution in the present condition is an even function along the generatrix and an odd function along the circumference we can express the distributed load in the following form

$$q(x,s) = \sum_{n=1,3\cdots}^{\infty} q_n \sin\frac{ns}{a} \int_0^{\infty} f(\lambda) \cos\lambda\frac{x}{a}\, d\lambda \qquad (58:a)$$

It follows that the components of displacement may be written in a form containing three undetermined functions $u(\lambda)$, $v(\lambda)$ and $w(\lambda)$:

$$u = \sum_{n=1,3\cdots}^{\infty} \sin \frac{ns}{a} \int_0^{\infty} u(\lambda) \cos \lambda \frac{x}{a} \, d\lambda \tag{58:b}$$

$$v = \sum_{n=1,3\cdots}^{\infty} \cos \frac{ns}{a} \int_0^{\infty} v(\lambda) \sin \lambda \frac{x}{a} \, d\lambda \tag{58:c}$$

$$w = \sum_{n=1,3\cdots}^{\infty} \sin \frac{ns}{a} \int_0^{\infty} w(\lambda) \sin \lambda \frac{x}{a} \, d\lambda \tag{58:d}$$

It is next desired to determine the functions q_n and $f(\lambda)$. This can be accomplished by developing them from the loading condition according to the following diagrams.

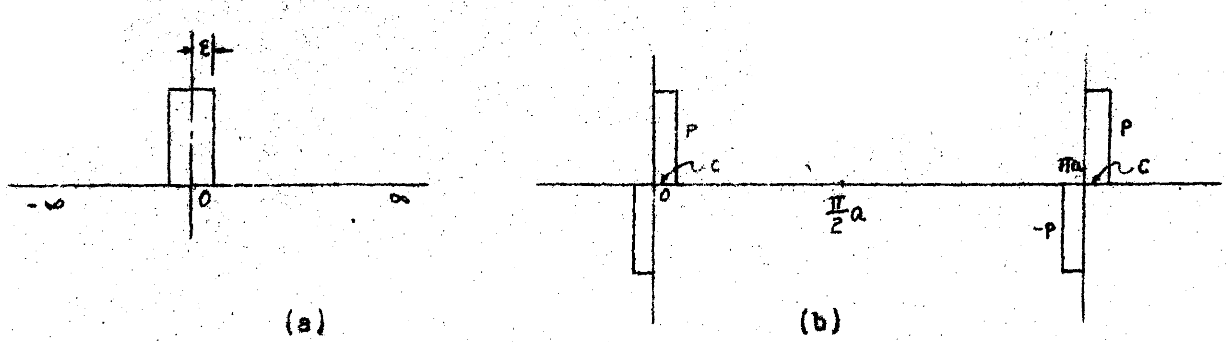


Fig. 10

The loading condition along the longitudinal direction is shown in Fig. (10a) and the transformed Fourier Integral of Eq. (58:a) is given by

$$f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} q\left(\frac{x}{a}\right) \cos \lambda \left(\frac{x}{a}\right) d\left(\frac{x}{a}\right)$$

Where $q(\frac{x}{a})$ is distributed between $(-\infty, \infty)$ as follows:

$$q\left(\frac{x}{a}\right) = 0 \qquad \text{when} \quad \begin{matrix} x > \varepsilon \\ -x < -\varepsilon \end{matrix}$$

$$q\left(\frac{x}{a}\right) = 1 \qquad \text{when} \quad \begin{matrix} 0 \geq x > -\varepsilon \\ 0 \leq x < \varepsilon \end{matrix}$$

Having the expression for the load distribution we can determine $f(\lambda)$:

$$f(\lambda) = \frac{2}{\pi}\int_0^{\varepsilon} \cos\lambda \frac{x}{a}\, d\frac{x}{a} = \frac{2}{\pi\lambda}\sin\lambda\frac{\varepsilon}{a}$$

The Fourier coefficient q_n in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ along the circumference is given by

$$q_n = \frac{1}{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z)\sin nz\, dz$$

The expression will be valid for $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ if we put $z = \frac{s}{a}$:

$$q_n = \frac{2}{\pi a}\int_{-\frac{\pi}{2}a}^{\frac{\pi}{2}a} q\left(\frac{s}{a}\right)\sin n\frac{s}{a}\, ds$$

But $q(\frac{s}{a})$ is distributed in the interval $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ as follows:

$$q\left(\frac{s}{a}\right) = 0 \qquad \text{when} \quad \begin{matrix} -\frac{\pi}{2}a < s < -c \\ \frac{\pi}{2}a > s > c \end{matrix}$$

$$q\left(\frac{s}{a}\right) = \pm q \qquad \text{when} \quad \begin{matrix} c \geq s > 0 \\ -c \leq s < 0 \end{matrix}$$

By using these load distributions we obtain

$$q_n = \frac{2}{\pi a}\left\{ \int_{-\frac{\pi}{2}a}^{-c} 0 \sin n\frac{s}{a}\,ds + \int_{-c}^{0} -q \sin n\frac{s}{a}\,ds + \int_{0}^{c} q \sin n\frac{s}{a}\,ds \right\}$$

$$= \frac{8q}{\pi n} \sin^2 \frac{nc}{2a}$$

In the case of a concentrated torque applied at the origin, we can assume the lengths 2ε and $2c$ of the loaded portion infinitely small.

Since $\quad P = q\,\dfrac{\varepsilon c}{2}$

$$\varepsilon \to 0$$
$$c \to 0$$

Therefore $\quad$ Torque $= T = P\varepsilon = q\dfrac{\varepsilon^2 c}{2}$

or

$$q_n f(\lambda) = \frac{2Tn}{\pi^2 a^3} \tag{59}$$

In the solution of Eqs. (29:a) and (29:b) for u, v a considerable simplification results from putting $a \to \infty$. Then, as before, we get a set of equations which combine u and v, respectively, with q

$$\nabla^4 u + \frac{(1+\nu)}{Eh}\left[2\frac{\partial^2 q}{\partial s^2} + (1-\nu)\frac{\partial^2 q}{\partial x^2}\right] = 0 \tag{60:a}$$

$$\nabla^4 v - \frac{(1+\nu)^2}{Eh}\frac{\partial^2 q}{\partial x \partial s} = 0 \tag{60:b}$$

Substituting Eqs. (58:a) and (58:b) in Eq. (60:a), we have

$$\sum_{n=1,3\cdots}^{\infty} \int_0^{\infty} \left\{ u(\lambda)\left[\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right]^2 + \frac{(1+\nu)}{Eh} q_n f(\lambda)\left[-2\left(\frac{n}{a}\right)^2 - (1-\nu)\left(\frac{\lambda}{a}\right)^2\right]\right\} \cos\lambda\frac{x}{a} \sin\lambda\frac{x}{a} = 0$$

or
$$u(\lambda) = \frac{2T}{\pi^2 a}\frac{(1+\nu)}{Eh}\, n\, \frac{[2n^2+(1-\nu)\lambda^2]}{[\lambda^2+n^2]^2}$$

Knowing u(λ) from the above relation we can obtain u_1 by substituting u(λ) in Eq. (58:b)

$$u_1 = \sum_{n=1,3\cdots}^{\infty} \sin n\frac{s}{a}\int_0^{\infty}\frac{\frac{2T}{\pi^2 a}\frac{(1+\nu)}{Eh}\left[2n^2+(1-\nu)\lambda^2\right]}{[\lambda^2+n^2]^2}\cos\lambda\frac{x}{a}\,d\lambda \qquad (61)$$

The above definite integral can be evaluated through the following relation

$$I = \int_{-\infty}^{\infty}\frac{[2n^2+(1-\nu)z^2]\,e^{iz\frac{x}{a}}}{[z^2+n^2]^2}\,dz$$

$$= -\frac{1}{2n}\frac{\partial}{\partial n}\int_{-\infty}^{\infty}\frac{[2n^2+(1-\nu)z^2]\,e^{iz\frac{x}{a}}}{[z^2+n^2]}\,dz$$

By Cauchy's Theorem of Residues, and with the aid of Jordan's Lemma we obtain

$$I = \frac{\pi}{2}\left[\frac{(3-\nu)}{n}+(1+\nu)\frac{x}{a}\right]e^{-n\frac{x}{a}}$$

Substituting this in Eq. (61), we have

$$u_1 = \frac{1+\nu}{Eh}\frac{T}{2\pi a}\sum_{n=1,3\cdots}^{\infty}\sin n\frac{s}{a}\left[(3-\nu)+n(1+\nu)\frac{x}{a}\right]e^{-n\frac{x}{a}} \qquad (62)$$

or

$$\frac{\partial u_1}{\partial s} = \frac{1+\nu}{Eh}\frac{T}{2\pi a^2}\sum_{n=1,3\cdots}^{\infty}\cos n\frac{s}{a}\left[(3-\nu)n+(1+\nu)n^2\frac{x}{a}\right]e^{-n\frac{x}{a}} \qquad (63)$$

In the same manner v_1 can be determined by substituting Eqs. (58:a) and (58:c) in Eq. (60:b)

$$\sum_{n=1,3\cdots}^{\infty}\int_0^{\infty}\left\{ v(\lambda)\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2 - q_n\, f(\lambda)\,\frac{(1+\nu)^2}{Eh}\,\frac{n\lambda}{a^2}\right\} \cos n\frac{s}{a}\,\sin\lambda\frac{x}{a}\,d\lambda = 0$$

or

$$V(\lambda) = \frac{\frac{(1+\nu)^2}{Eh}\, q_n\, f(\lambda)\,\frac{n\lambda}{a^2}}{\left[\left(\frac{\lambda}{a}\right)^2+\left(\frac{n}{a}\right)^2\right]^2} = \frac{(1+\nu)}{Eh}\,\frac{2T}{\pi^2 a}\,\frac{n^2\lambda}{(\lambda^2+n^2)^2}$$

From Eq. (58:c) we obtain v_1 by substituting $v(\lambda)$ in the above relation

$$v_1 = \frac{(1+\nu)^2}{Eh}\,\frac{2T}{\pi^2 a}\sum_{n=1,3\cdots}^{\infty} n^2 \cos\frac{ns}{a}\int_0^{\infty}\frac{\lambda}{(\lambda^2+n^2)^2}\sin\lambda\frac{x}{a}\,d\lambda \tag{64}$$

The definite integral above has a value

$$I = \int_{-\infty}^{\infty}\frac{z\,e^{iz\frac{x}{a}}}{(z^2+n^2)^2}\,dz = -\frac{1}{2n}\frac{\partial}{\partial n}\int_{-\infty}^{\infty}\frac{z\,e^{iz\frac{x}{a}}}{(z^2+n^2)}\,dz = \frac{\pi}{4n}\,\frac{x}{a}\,e^{-n\frac{x}{a}}$$

Substituting the above result in Eq. (64), we have

$$v_1 = \frac{(1+\nu)^2}{Eh}\,\frac{T}{2\pi a}\sum_{n=1,3\cdots}^{\infty} n\cos\frac{ns}{a}\left[\frac{x}{a}e^{-n\frac{x}{a}}\right] \tag{65}$$

and

$$\frac{\partial v_1}{\partial x} = \frac{(1+\nu)^2}{Eh}\,\frac{T}{2\pi a^2}\sum_{n=1,3\cdots}^{\infty} n\cos\frac{ns}{a}\left[1-n\frac{x}{a}\right]e^{-n\frac{x}{a}} \tag{66}$$

The second differential equations which will be used to solve u_2 and v_2 are obtained from Eqs. (29:a) and (29:b) by putting $q = 0$

$$\nabla^4 u - \frac{\nu}{a}\frac{\partial^3 w}{\partial x^3} + \frac{1}{a}\frac{\partial^3 w}{\partial x\,\partial s^2} = 0 \tag{67:a}$$

$$\nabla^4 v - \frac{1}{a}\frac{\partial^3 w}{\partial s^3} - \frac{(2+\nu)}{a}\frac{\partial^3 w}{\partial x^2\,\partial s} = 0 \tag{67:b}$$

Before solving the above equations we have to determine the undetermined function $w(\lambda)$ from Eq. (29:c). This can be achieved by substituting Eqs. (58:a) and (58:d):

$$\sum_{n=1,3\cdots}^{\infty} \int_0^{\infty} \left\{ w(\lambda)\left[\left\{\left(\frac{\lambda}{a}\right)^2 + \left(\frac{n}{a}\right)^2\right\}^4 + \frac{Eh}{a^2 D}\left(\frac{\lambda}{a}\right)^4\right] - \frac{q_n f(\lambda)}{D\, a}\left[\left(\frac{\lambda}{a}\right)\left(\frac{n^2}{a^2}\right) - \nu\left(\frac{\lambda}{a}\right)^3\right]\right\} \sin\frac{ns}{a} \sin\lambda\frac{x}{a}\, d\lambda = 0$$

for all values of λ, or

$$w(\lambda) = \frac{\frac{2Tan\lambda}{\pi^2 D}\left(n^2 - \nu\lambda^2\right)}{\left[\lambda^2 + n^2\right]^4 + J^2\lambda^4}$$

By substituting Eqs. (58:b) and (58:d) in Eq. (67:a) we obtain, as before

$$u(\lambda) = \frac{2Ta}{\pi^2 D}\, \frac{n\lambda^2\left[\nu\lambda^2 - n^2\right]\left[\nu\lambda^2 - n^2\right]}{\left[\lambda^2 + n^2\right]^2\left[(\lambda^2+n^2)^4 + J^2\lambda^4\right]}$$

Substituting this in Eq. (58:b) after differentiating it with respect to s, we have

$$\frac{\partial u_2}{\partial s} = \frac{2T}{\pi^2 D}\sum_{n=1,3\cdots}^{\infty} n^2 \cos\frac{ns}{a}\int_0^{\infty}\frac{\lambda^2\left[\nu\lambda^2 - n^2\right]^2 \cos\lambda\frac{x}{a}\, d\lambda}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4 + J^2\lambda^4\right]} \qquad (68)$$

Similarly $v(\lambda)$ is obtained by substituting Eqs. (58:a) and (58:c) in Eq. (67:b)

$$V(\lambda) = \frac{2Ta}{\pi^2 D}\, \frac{n^2\lambda\left[\nu\lambda^2 - n^2\right]\left[n^2 + (2+\nu)\lambda^2\right]}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4 + J^2\lambda^4\right]}$$

Differentiating Eq. (58:c) and using the above relation of $v(\lambda)$, we obtain

$$\frac{\partial v_2}{\partial x} = \frac{2T}{\pi^2 D}\sum_{n=1,3\cdots}^{\infty} n^2 \cos\frac{ns}{a}\int_0^{\infty}\frac{\lambda^2\left[\nu\lambda^2 - n^2\right]\left[n^2 + (2+\nu)\lambda^2\right]}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4 + J^2\lambda^4\right]} \cos\frac{\lambda x}{a}\, d\lambda \qquad (69)$$

As mentioned before the shearing stress resultant in this case can be obtained by combining the terms containing the derivatives of u and v with respect to s and x, respectively.

$$S_{l_1} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u_1}{\partial s} + \frac{\partial v_1}{\partial x}\right) = \frac{T}{2\pi a^2} \sum_{n=1,3\cdots}^{\infty} \cos n\frac{s}{a}(2n)\, e^{-n\frac{x}{a}} \tag{70:a}$$

$$S_{l_2} = \frac{Eh}{2(1+\nu)}\left(\frac{\partial u_2}{\partial s} + \frac{\partial v_2}{\partial x}\right)$$

$$= \frac{2TJ^2}{\pi^2 a^2}\nu \sum_{n=1,3\cdots}^{\infty} n^2 \cos n\frac{s}{a} \int_0^{\infty} \frac{\lambda^4\left(\lambda^2 - \frac{n^2}{\nu}\right)\cos\frac{\lambda x}{a}\, d\lambda}{(\lambda^2+n^2)^2\left[(\lambda^2+n^2)^4 + J^2\lambda^4\right]} \tag{70:b}$$

In order to evaluate the above definite integral we again apply the relation used for evaluating Eq. (35) and also the Theorem of Residues, which gives

$$\int_{-\infty}^{\infty} \frac{z^4\left(z^2 - \frac{n^2}{\nu}\right) e^{iz\frac{x}{a}}\, dz}{(z^2+n^2)^2\left[(z^2+n^2)^4 + J^2 z^4\right]}$$

$$= 2\pi i \Bigg[-\frac{1}{2n}\frac{\partial}{\partial n}\left\{\frac{n^4\left(-n^2 - \frac{n^2}{\nu}\right) e^{-n\frac{x}{a}}}{2in(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)}\right\}$$

$$+ \frac{\alpha_1^4\left(\alpha_1^2 - \frac{n^2}{\nu}\right) e^{i\alpha_1\frac{x}{a}}}{2\alpha_1(\alpha_1^2+n^2)^2(\alpha_1^2+\alpha_2^2)(\alpha_1^2-\alpha_3^2)(\alpha_1^2-\alpha_4^2)} + \frac{\alpha_2^3\left(\alpha_2^2 - \frac{n^2}{\nu}\right) e^{i\alpha_2\frac{x}{a}}}{2(\alpha_2^2+n^2)^2(\alpha_2^2-\alpha_1^2)(\alpha_2^2-\alpha_3^2)(\alpha_2^2-\alpha_4^2)} \tag{71}$$

$$+ \frac{\alpha_3^3\left(\alpha_3^2 - \frac{n^2}{\nu}\right) e^{i\alpha_3\frac{x}{a}}}{2(\alpha_3^2+n^2)^2(\alpha_3^2-\alpha_1^2)(\alpha_3^2-\alpha_2^2)(\alpha_3^2-\alpha_4^2)} + \frac{\alpha_4^3\left(\alpha_4^2 - \frac{n^2}{\nu}\right) e^{i\alpha_4\frac{x}{a}}}{2(\alpha_4^2+n^2)^2(\alpha_4^2-\alpha_1^2)(\alpha_4^2-\alpha_2^2)(\alpha_4^2-\alpha_3^2)} \Bigg]$$

where α_s are the roots of the algebraic equation in the denominator mentioned before.

The first term in expression (71) gives, after simplifying,

$$\frac{\pi}{2n}\left[\frac{\left\{2n^4+(n^2+\frac{n^2}{\nu})3n^2-\frac{x}{a}n^3(n^2+\frac{n^2}{\nu})\right\} e^{-n\frac{x}{a}}}{(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)} \right.$$

$$-\frac{n^3(n^2+\frac{n^2}{\nu})e^{-n\frac{x}{a}}}{[(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)]^2}\cdot\left\{2n(n^2+\alpha_2^2)(n^2+\alpha_3^2)(n^2+\alpha_4^2)\right.$$

$$\left.+2n(n^2+\alpha_1)(n^2+\alpha_3)(n^2+\alpha_4^2)+2n(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_4^2)+2n(n^2+\alpha_1^2)(n^2+\alpha_2^2)(n^2+\alpha_3^2)\right\}$$

Substitution of α_s from Eq. (12) results in the simplified form

$$-\frac{\pi}{2nJ^2}e^{-n\frac{x}{a}}\left[\frac{1-\nu}{\nu}+\frac{n(1+\nu)}{\nu}\frac{x}{a}\right] \tag{72}$$

In simplifying the next four terms in expression (71) we again use the notation introduced in (51). The combined form of the second and fifth terms gives

$$\frac{\pi i}{16n^8J^4R_2}\left[\alpha_1(-n^2+\eta-i\frac{J}{2}+i\phi)(-n^2+\eta-i\frac{J}{2}+i\phi-\frac{n^2}{\nu})e^{i\alpha_1\frac{x}{a}}\right.$$

$$\cdot\left\{-\eta-i(\phi+\frac{J}{2})\right\}\left\{2\eta+iJ\right\}\left\{2\eta-2i\phi\right\}\left\{iJ+2i\phi\right\}$$

$$-\alpha_4(-n^2+\eta+i\frac{J}{2}-i\phi)(-n^2+\eta+i\frac{J}{2}-i\phi-\frac{n^2}{\nu})e^{i\alpha_4\frac{x}{a}}$$

$$\cdot\left\{-\eta+i(\phi+\frac{J}{2})\right\}^2\left\{2\eta+2i\phi\right\}\left\{2\eta-iJ\right\}\left\{iJ+2i\phi\right\}$$

Separating the real and imaginary parts and introducing the trigonometric functions, we get

$$-\frac{\pi(J+2\phi)^2}{8n^8J^4R_2}\Bigg[\left(A\cos\frac{Ax}{a}-B\sin\frac{Ax}{a}\right)\Big\{-\left(n^4+\frac{n^4}{\nu}-2n^2\eta+\eta^2-\frac{n^2}{\nu}\eta-\phi^2+\phi J-\frac{J^2}{4}\right)$$

$$\cdot\left(4\eta^2J+J^2\phi-4\eta^2\phi\right)+\left(-2n^2+2\eta-\frac{n^2}{\nu}\right)\left(\phi-\frac{J}{2}\right)\left(J^2\eta-4J\eta\phi-4\eta^3\right)\Big\}$$

$$+\left(A\sin\frac{Ax}{a}+B\cos\frac{Ax}{a}\right)\Big\{\left(-2n^2+2\eta-\frac{n^2}{\nu}\right)\left(\phi-\frac{J}{2}\right)\left(4J\eta^2+2J^2\phi-4\eta^2\phi\right)$$

$$+\left(n^4+\frac{n^4}{\nu}-2n^2\eta+\eta^2-\eta\frac{n^2}{\nu}-\phi^2+\phi J-\frac{J^2}{4}\right)\left(J^2\eta-4J\eta\phi-4\eta^3\right)\Big\}\Bigg]e^{-B\frac{x}{a}}$$

Further simplifications will reduce the second and fifth terms of Eq. (71) to a final form, which is

$$-\frac{\pi e^{-B\frac{x}{a}}}{8n^8J^4R_2}\Bigg[\left(A\cos\frac{Ax}{a}-B\sin\frac{Ax}{a}\right)\Big\{n^2\left(\frac{1+\nu}{\nu}\right)\left(4n^6J^2\phi-2n^2J^2R_2\phi+\frac{1}{2}n^2J^4\phi\right)$$

$$+2n^6J^3\eta\Big\}-\left(A\sin\frac{Ax}{a}+B\cos\frac{Ax}{a}\right)\Big\{n^2\left(\frac{1+\nu}{\nu}\right)\left(4n^6J^2\eta+4n^4J^2R_2\right.$$

$$\left.+2n^4J^3\phi\right)-2n^6J^3\phi-4n^6J^2R_2\Big\}\Bigg] \tag{73}$$

Similarly the third and fourth terms in Eq. (71) are simplified as follows:

$$-\frac{\pi e^{-D\frac{x}{a}}}{8n^8J^4R_2}\Bigg[\left(C\cos\frac{Cx}{a}-D\sin\frac{Cx}{a}\right)\Big\{n^2\frac{(1+\nu)}{\nu}\left(4n^6J^2\phi-2n^2J^2R_2\phi+\frac{1}{2}n^2J^4\phi\right)$$

$$+2n^6J^3\eta+\left(D\cos\frac{Cx}{a}+C\sin\frac{Cx}{a}\right)\Big\{n^2\left(\frac{1+\nu}{\nu}\right)\left(4n^6J^2\eta+2n^4J^3\phi\right. \tag{74}$$

$$\left.-4n^4J^2R_2\right)-2n^6J^3\phi+4n^6J^2R_2\Big\}\Bigg]$$

Summarizing expressions (71), (72) and (74) and substituting the result in Eq. (70.b) we finally obtain

$$S_{l_2} = \frac{T}{2\pi a} \sum_{n=1,3\cdots}^{\infty} \cos\frac{ns}{a}\Big[-\left\{(1-\nu)n + (1+\nu)n^2\frac{x}{a}\right\} e^{-n\frac{x}{a}}$$

$$- \frac{1}{4J\sqrt{1+J^2}}\Big\{\left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right)\left[(1+\nu)\left(4\phi - \frac{2R_2\phi}{n^4} + \frac{J^2\phi}{2n^4}\right) + 2\nu\frac{J\eta}{n^2}\right] e^{-B\frac{x}{a}}$$

$$-\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)\left[(1+\nu)\left(4\eta + \frac{4R_2}{n^2} + \frac{2J\phi}{n^2}\right) - \frac{4\nu R_2}{n^2}\right] e^{-B\frac{x}{a}} \tag{75}$$

$$+\left(C\cos\frac{Cx}{a} - D\sin\frac{Cx}{a}\right)\left[(1+\nu)\left(4\phi - \frac{2R_2\phi}{n^4} + \frac{J^2\phi}{2n^4}\right) + \nu^2\frac{J\eta}{n^2}\right] e^{-D\frac{x}{a}}$$

$$+\left(D\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right)\left[(1+\nu)\left(4\eta - \frac{4R_2}{n^2} + \frac{2J\phi}{n^2}\right) - \frac{2\nu J\phi}{n^2} + \frac{4R_2\nu}{n^2}\right] e^{-D\frac{x}{a}}$$

Combining S_{l_1} and S_{l_2} from Eqs. (70:a) and (75) we obtain the stress resultant in the following alternative form:

$$\frac{S_l}{\frac{T}{a^2}} = \frac{1}{2\pi} \sum_{n=1,3\cdots}^{\infty} \cos\frac{ns}{a}\Big[\left\{(1+\nu)n - (1+\nu)n^2\frac{x}{a}\right\} e^{-n\frac{x}{a}} - \frac{1}{4J\sqrt{1+J^2}}$$

$$\cdot\Big\{\Big[e^{-B\frac{x}{a}}\left(A\cos\frac{Ax}{a} - B\sin\frac{Ax}{a}\right) + \left(C\cos\frac{Cx}{a} - D\sin\frac{Cx}{a}\right)e^{-D\frac{x}{a}}\Big]\Big[\frac{2\nu J\eta}{n^2} \tag{76}$$

$$+ (1+\nu)\left(4\phi - \frac{2R_2\phi}{n^4} + \frac{J^2\phi}{2n^4}\right)\Big] - \Big[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)e^{-B\frac{x}{a}} - \Big(D\cos\frac{Cx}{a}$$

$$+ C\sin\frac{Cx}{a}\Big)e^{-D\frac{x}{a}}\Big]\left[(1+\nu)4\eta + \frac{2J\phi}{n^2}\right] - \frac{4R_2}{n^2}\Big[\left(A\sin\frac{Ax}{a} + B\cos\frac{Ax}{a}\right)e^{-B\frac{x}{a}}$$

$$+\left(D\cos\frac{Cx}{a} + C\sin\frac{Cx}{a}\right)e^{-D\frac{x}{a}}\Big]\Big\}\Big]$$

Equations (55) and (76) are used for the calculation of the shearing stress distribution on an infinitely long thin cylinder under two equal and opposite torques acting about the radial axis on the surface of the cylinder.

3. INVESTIGATION OF STRESSES IN THE IMMEDIATE VICINITY OF THE APPLIED TORQUE

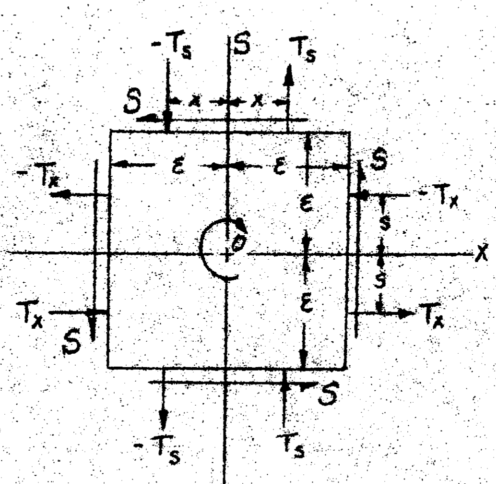


Fig. 11

In Fig. 11 an infinitely small square element is cut out of a thin cylinder in the immediate vicinity of of the applied torque. All the possible stress-resultants that may produce any torques about the element are shown in the above figure. In order to verify that the total torque produced by the force-resultants is equal to the applied torque we can summarize all the force-resultants multiplied by their corresponding moment arms. The total torque around the small square element of length 2ε may be summarized as follows

$$\text{Torque} = 2\left[\int_{-\varepsilon}^{+\varepsilon} \varepsilon S\,ds + \int_{0}^{-\varepsilon} \varepsilon S\,dx + \int_{\varepsilon}^{0} \varepsilon S\,dx\right] + 4\left[\int_{0}^{\varepsilon} -T_x\,s\,ds + \int_{0}^{\varepsilon} T_s\,x\,dx\right] \tag{77}$$

where S is the shearing stress-resultant derived in Eqs. (56) and (76), and T_x and T_s are the corresponding normal stress-resultants acting on the element in the direction of the x and s axes, respectively. They can be determined from the equations of equilibrium.

$$\frac{\partial S}{\partial s} + \frac{\partial T_x}{\partial x} = 0$$
$$\frac{\partial S}{\partial x} + \frac{\partial T_s}{\partial s} = 0 \tag{78}$$

Let us now consider the behavior of S_t in Eq. (55) when both x and s approach a very small value, i.e., $x \to s \to \epsilon$ and $\epsilon \to 0$. The parameters A, B, C, D in Eq. (15) have the following limiting values when $n \gg 0$

$$A = C = \sqrt{\frac{J}{2}}$$

$$B = D = n$$

Furthermore we can write

$$\cos A\frac{x}{a} \longrightarrow 1 \; ; \quad \sin A\frac{x}{a} \longrightarrow 0 \quad \& \quad e^{-B\frac{x}{a}} = e^{-D\frac{x}{a}} = e^{-n\frac{x}{a}}$$

Using the above notations Eq. (55) is reduced to

$$\underset{x,s \to \varepsilon}{S_t} = \frac{T}{2\pi a^2} \sum_{n=2,4\cdots}^{\infty} e^{-n\frac{x}{a}} \cos n\frac{s}{a} \left\{ \frac{(1+\nu)}{\sqrt{2}} \left[n - n^2\frac{x}{a} \right] + 2n \right\} \tag{79}$$

where the terms containing $\frac{1}{n}$ are neglected in the expression (78) because they vanish after integration in expression (77)

Since the infinite series $e^{-n\frac{x}{a}} \cos n\frac{s}{a}$ can be summed, and the sum of its n terms was found in Part I, we may write:

$$\sum_{n=2,4\cdots}^{\infty} e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{1}{2}\left[\frac{1}{e^{\frac{2}{a}(x+is)}-1} + \frac{1}{e^{\frac{2}{a}(x-is)}-1} \right]_{x+is>0}$$

The sum of n terms of $ne^{-\frac{x}{a}}\cos n\frac{s}{a}$ is found by differenting the above expression, i. e.

$$\sum_{n=2,4\cdots}^{\infty} n e^{-n\frac{x}{a}} \cos n\frac{s}{a} = -\frac{a}{2}\frac{\partial}{\partial x}\left[\frac{1}{e^{\frac{2}{a}(x+is)}-1} + \frac{1}{e^{\frac{2}{a}(x-is)}-1} \right]$$

$$= \frac{2e^{\frac{2}{a}x}\left[e^{4\frac{x}{a}} \cos 2\frac{s}{a} - 2e^{-\frac{2}{a}x} + \cos 2\frac{s}{a} \right]}{\left[e^{4\frac{x}{a}} - 2e^{2\frac{x}{a}} \cos 2\frac{s}{a} + 1 \right]^2}$$

Substituting
$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots$$

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \cdots$$

and neglecting the higher terms on account of the smallness of x and s, we obtain

$$\sum_{\substack{n=2,4\cdots \\ x\to s\to 6}}^{\infty} n e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{\left(\frac{x}{a}\right)^2 - \left(\frac{s}{a}\right)^2}{2\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^2}$$

Similarly we obtain

$$\sum_{n=2,4\cdots}^{\infty} n^2 e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{a^2}{2}\frac{\partial^2}{\partial x^2}\left[\frac{1}{e^{\frac{2}{a}(x+is)}-1} + \frac{1}{e^{\frac{2}{a}(x-is)}-1}\right]$$

$$= -4\left\{\frac{e^{2\frac{x}{a}}\left[e^{4\frac{x}{a}}\cos 2\frac{s}{a} + \cos 2\frac{s}{a} - 2e^{2\frac{x}{a}}\right]}{\left[e^{4\frac{x}{a}} - 2e^{2\frac{x}{a}}\cos 2\frac{s}{a} + 1\right]^2}\right.$$

$$\left. - \frac{2e^{4\frac{x}{a}}\left[e^{6\frac{x}{a}}\cos 2\frac{s}{a} + 3e^{2\frac{x}{a}}\cos 2\frac{s}{a} - 3e^{4\frac{x}{a}} - \cos 4\frac{s}{a}\right]}{\left[e^{4\frac{x}{a}} - 2e^{2\frac{x}{a}}\cos 2\frac{s}{a} + 1\right]^3}\right.$$

For small x and s the above expression can be reduced as follows

$$\sum_{\substack{n=2,4\cdots \\ x\to s\to 6}}^{\infty} n^2 e^{-n\frac{x}{a}} \cos n\frac{s}{a} = -\frac{\left(\frac{x}{a}\right)^2 - \left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^2} + \frac{\left(\frac{x}{a}\right)^3 - 3\frac{xs^2}{a^3}}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^3}$$

Finally we obtain a summed form of S_t by substituting the above summation in Eq. (79)

$$\underset{x\to s\to \epsilon}{S_t} = \frac{T}{2\pi a^2}\left\{\frac{1+\nu}{\sqrt{2}}\left[\frac{\left(\frac{x}{a}\right)^2 - \left(\frac{s}{a}\right)^2}{2\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^2} - \frac{\left(\frac{x}{a}\right)^4 - 3\left(\frac{x}{a}\right)^2\left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^3}\right] + \frac{\left(\frac{x}{a}\right)^2 - \left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2 + \left(\frac{s}{a}\right)^2\right]^2}\right\} \tag{80}$$

In a like manner we reduce Eq. (76) in the following expression for small x and s

$$\underset{x\to s\to 6}{S_l} = \frac{T}{2\pi a^2}\sum_{n=1,3\cdots}^{\infty} e^{-n\frac{x}{a}} \cos n\frac{s}{a}\left\{(1+\nu)\left[1-\frac{1}{\sqrt{2}}\right]\left[n - n^2\frac{x}{a}\right] + 2n\right\} \tag{81}$$

Substituting the following summations into Eq. (81)

$$\sum_{n=1,3\cdots}^{\infty} n e^{-n\frac{x}{a}} \cos n\frac{s}{a} = -\frac{a}{4}\frac{\partial}{\partial x}\left[\frac{1}{\sinh\frac{x+is}{a}} + \frac{1}{\sinh\frac{x-is}{a}}\right]$$

$$\sum_{\substack{n=1,3\cdots \\ x\to s\to\epsilon}}^{\infty} n e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{1}{2}\frac{\left(\frac{x}{a}\right)^2-\left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^2}$$

$$\sum_{n=1,3\cdots}^{\infty} n^2 e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{a^2}{4}\frac{\partial^2}{\partial x^2}\left[\frac{1}{\sinh\frac{x+is}{a}} + \frac{1}{\sinh\frac{x-is}{a}}\right]$$

$$\sum_{\substack{n=1,3\cdots \\ x\to s\to\epsilon}}^{\infty} n^2 e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \left[\frac{-4\frac{x}{a}}{\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2} + \frac{\left(\frac{x}{a}\right)^3-3\left(\frac{s}{a}\right)^2\left(\frac{x}{a}\right)}{\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^3}\right]$$

we obtain

$$\underset{x\to s\to\epsilon}{S_l} = \frac{T}{2\pi a^2}\left\{(1+\nu)\left[1-\frac{1}{\sqrt{2}}\right]\left[\frac{\left(\frac{x}{a}\right)^2-\left(\frac{s}{a}\right)^2}{2\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^2} - \frac{\left(\frac{x}{a}\right)^4-3\left(\frac{x}{a}\right)^2\left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^3}\right] + \frac{\left(\frac{x}{a}\right)^2-\left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2-\left(\frac{s}{a}\right)^2\right]^2}\right\} \tag{82}$$

The total shearing stress-resultant is

$$S = \frac{S_t+S_l}{2} = \frac{T}{2\pi a^2}\left\{(3+\nu)\left[\frac{\left(\frac{x}{a}\right)^2-\left(\frac{s}{a}\right)^2}{2\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^2}\right] - (1+\nu)\left[\frac{\left(\frac{x}{a}\right)^4-3\left(\frac{x}{a}\right)^2\left(\frac{s}{a}\right)^2}{\left[\left(\frac{x}{a}\right)^2+\left(\frac{s}{a}\right)^2\right]^3}\right]\right\} \tag{83}$$

Now the normal stress-resultants can be determined by substituting S from expression (83) into Eq. (78) and solving for T_x and T_s. Differentiating expression (83) with respect to s, we have

$$\frac{\partial S}{\partial s} = \frac{T}{2\pi}(1+\nu)\left\{\frac{1}{2}\left[\frac{(x^2+s^2)^2(-2s)-2(x^2-s^2)(x^2+s^2)(2s)}{(x^2+s^2)^4}\right] - \left[\frac{(x^2+s^2)^3(-6x^2s)-3(x^4-3x^2s^2)(x^2+s^2)^2(2s)}{(s^2+x^2)^6}\right]\right\} + \frac{T}{2\pi}\left[\frac{(x^2+s^2)^2(-2s)-2(x^2-s^2)(2s)}{(x^2+s^2)^4}\right]$$

Integrating the above expression, we get

$$T_x = -\frac{T}{2\pi}(1+\nu)\int\left[-\frac{s}{(x^2+s^2)^2}+\frac{2(2x^2s+s^3)}{(x^2+s^2)^3}+\frac{6s(x^4-3x^2s^2)}{(x^2+s^2)^4}\right]dx$$

$$-\frac{T}{2\pi}\int\left[\frac{-2s}{(x^2+s^2)^2}-\frac{4(x^2s-s^3)}{(x^2+s^2)^4}\right]dx + C$$

From the boundary condition, i.e., $s=0,\ T_x=0$, the constant of integration is found to be zero.

Hence

$$T_x = (1+\nu)\frac{T}{2\pi}\left[\frac{3xs}{(x^2+s^2)^2}-\frac{4s^3x}{(x^2+s^2)^3}\right]-\frac{T}{2\pi}\left[\frac{2xs}{(x^2+s^2)^2}\right] \tag{84}$$

Next differentiating expression (83) with respect to x, we obtain;

$$\frac{\partial S}{\partial s} = (1+\nu)\frac{T}{2\pi}\left\{\frac{x}{(x^2+s^2)^2}-\frac{(2x^3-2xs^2)+(4x^3-6s^2x)}{(x^2+s^2)^3}+\frac{6x^5-18x^3s^2}{(x^2+s^2)^4}\right\}$$

$$+\frac{T}{2\pi}\left[\frac{2x}{(x^2+s^2)^2}-\frac{4(x^3-xs^2)}{(x^2+s^2)^3}\right]$$

Integrating the above expression, we get

$$T_s = \frac{(1+\nu)T}{2\pi}\left[\frac{3xs}{(x^2+s^2)^2}-\frac{4x^3s}{(x^2+s^2)^3}\right]+\frac{T}{2\pi}\left[\frac{2sx}{(s^2+x^2)^2}\right] \tag{85}$$

Substituting expressions (83), (84) and (85) into Eq. (77) we finally have

$$\text{Torque} = \frac{T}{\pi a^2}\left\{\epsilon\int_{-\epsilon}^{\epsilon}\left[\frac{(3+\nu)\left[(\frac{x}{a})^2-(\frac{s}{a})^2\right]}{2\left[(\frac{x}{a})^2+(\frac{s}{a})^2\right]^2}-\frac{(1+\nu)\left[(\frac{x}{a})^4-3(\frac{x}{a})^2(\frac{s}{a})^2\right]}{\left[(\frac{x}{a})^2+(\frac{s}{a})^2\right]^3}\right]ds\right.$$

$$\left.+2\epsilon\int_0^{-\epsilon}\left[\frac{(3+\nu)\left[(\frac{x}{a})^2-(\frac{s}{a})^2\right]}{2\left[(\frac{x}{a})^2+(\frac{s}{a})^2\right]^2}-\frac{(1+\nu)\left[(\frac{x}{a})^4-3(\frac{x}{a})^2(\frac{s}{a})^2\right]}{\left[(\frac{x}{a})^2+(\frac{s}{a})^2\right]^3}\right]dx\right\}$$

$$+\frac{4(1+\nu)T}{2\pi}\int_0^{\epsilon}\left[-\frac{3xs^2}{(x^2+s^2)^2}+\frac{4s^4x}{(x^2+s^2)^3}\right]ds+\frac{4T}{2\pi}\int_0^{\epsilon}\frac{2xs^2}{(x^2+s^2)^2}\,ds$$

$$\frac{4(1+\nu)T}{2\pi}\int_0^{\varepsilon}\left[\frac{3x^2 s}{(x^2+s^2)^2}-\frac{4x^2 s}{(x^2+s^2)^3}\right]dx+\frac{4T}{2\pi}\int_0^{\varepsilon}\frac{2x^2 s}{(x^2+s^2)^2}\,dx$$

After integrating and putting $x \to \epsilon$ we have

$$\text{Torque} = \frac{T}{\pi a^2}\left\{\frac{(3+\nu)a^2\varepsilon}{2}\left[\frac{\varepsilon^2 s}{2\varepsilon^2(\varepsilon^2+s^2)}+\frac{s}{2(\varepsilon^2+s^2)}\right]_{-\varepsilon}^{\varepsilon}-2(1+\nu)a^2\varepsilon\left[\frac{\varepsilon^4 s}{4\varepsilon^2(s^2+\varepsilon^2)^2}\right.\right.$$

$$\left.+\frac{3\varepsilon^4 s}{8\varepsilon^4(s^2+\varepsilon^2)}+\frac{3\varepsilon^2 s}{4(s^2+\varepsilon^2)^2}-\frac{3\varepsilon^2 s}{8\varepsilon^2(s^2+\varepsilon^2)}\right]_0^{\varepsilon}+(3+\nu)a^2\left[-\frac{\varepsilon x}{2(x^2+\varepsilon^2)}-\frac{\varepsilon^3 x}{2\varepsilon^2(x^2+\varepsilon^2)}\right]_0^{-\varepsilon}$$

$$\left.-2(1+\nu)a^2\varepsilon\left[\frac{\varepsilon^2 x}{4(x^2+\varepsilon^2)^2}-\frac{5x}{8(x^2+\varepsilon^2)}+\frac{3\varepsilon^2 x}{4(x^2+\varepsilon^2)^2}-\frac{3\varepsilon^2 x}{8\varepsilon^2(x^2+\varepsilon^2)}\right]_0^{-\varepsilon}\right\}$$

$$+\frac{2(1+\nu)T}{\pi}\left[-3\varepsilon\left(-\frac{s}{2(s^2+\varepsilon^2)}+\frac{1}{2\varepsilon}\tan^{-1}\frac{s}{\varepsilon}\right)+4\varepsilon\left(\frac{\varepsilon^2 s}{4(s^2+\varepsilon^2)}-\frac{5s}{8(s^2+\varepsilon^2)}+\frac{3}{8\varepsilon}\tan^{-1}\frac{s}{\varepsilon}\right)\right]_0^{1}$$

$$+\frac{4T}{\pi}\left[-\frac{s\varepsilon}{2(s^2+\varepsilon^2)}+\frac{1}{2}\tan^{-1}\frac{s}{\varepsilon}\right]_0^{\varepsilon}+\frac{2(1+\nu)T}{\pi}\left[3\varepsilon\left(-\frac{x}{2(x^2+\varepsilon^2)}+\frac{1}{2\varepsilon}\tan^{-1}\frac{x}{\varepsilon}\right)\right.$$

$$\left.-4\varepsilon\left(\frac{\varepsilon^2 x}{4(x^2+\varepsilon^2)^2}-\frac{5x}{8(x^2+\varepsilon^2)}+\frac{3}{8\varepsilon}\tan^{-1}\frac{x}{\varepsilon}\right)\right]_0^{\varepsilon}+\frac{4T}{\pi}\left[-\frac{x\varepsilon}{2(x^2+\varepsilon^2)}+\frac{1}{2}\tan^{-1}\frac{x}{\varepsilon}\right]_0^{\varepsilon}$$

After simplifying we get

$$\text{Torque} = \frac{2T}{\pi}-\frac{2T}{\pi}+T = T$$

This is equal to the applied torque.

If we transform all the force-resultants in polar coordinates the torque produced by the tangential force-resultant can be also proven to be equal to the applied torque. The stress-resultants in polar coordinates are:

$$\left.\begin{aligned}
T_x &= \frac{(1+\nu)T}{2\pi}\left[\frac{3\sin 2\theta}{2r^2}-\frac{4\sin^3\theta\cos\theta}{r^2}\right]-\frac{T}{2\pi r^2}\sin 2\theta\\
T_s &= \frac{(1+\nu)T}{2\pi}\left[\frac{3\sin 2\theta}{2r^2}-\frac{4\cos^3\theta\sin\theta}{r^2}\right]+\frac{T}{2\pi r^2}\sin 2\theta\\
S &= \frac{(1+\nu)T}{2\pi}\left[\frac{\cos 2\theta}{2r^2}-\frac{\cos^4\theta-3\cos^2\theta\sin^2\theta}{r^2}\right]+\frac{T}{2\pi r^2}\cos 2\theta
\end{aligned}\right\}\quad (86)$$

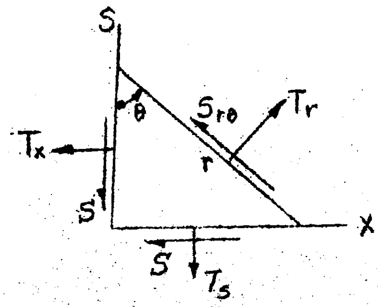


Fig. 12

From Fig. 12 it is seen that the tangential stress-resultant can be obtained with the aid of the following relations of equilibrium:

$$S_{r\theta} = T_s r \cos\theta \sin\theta - T_x r \cos\theta \sin\theta + S r(\cos^2\theta - \sin^2\theta)$$

By substituting the known quantities T_s, T_x and S from Eq. (86) into the above expression, we obtain

$$S_{r\theta} = -\frac{(1+\nu)T}{2\pi r^2}\left[\frac{1}{2}(\cos 2\theta)\right] + \frac{T}{2\pi r^2} \qquad (87)$$

The torque produced by such a tangential force-resultant acting around a small circular elementcan then be expressed by

$$\text{Torque} = 2\int_0^{\pi} S_{r\theta}\, r\, d\theta\, r$$

Substituting expression (87) in the above integral, we get

$$\text{Torque} = \frac{T}{\pi}\int_0^{\pi}\left[-\frac{1+\nu}{2}\cos 2\theta + 1\right] d\theta = T$$

This is an alternate method of verifying the summation of all the force-resultants acting on a small element will produce a torque which is equal in magnitude to the applied torque.

4. Method Used in the Calculation of Shearing Stress Distribution near the s-axis ($x \rightarrow 0$)

In numerical calculations of the shearing stress distribution near the s-axis from Eqs. (55) and (76) it is found that ordinary methods fail because the series converges very slowly when x approaches zero. Before an effective method of computation is introduced let us investigate the behavior of Eq. (55) when x is very small. As mentioned before the Parameters A, B, C and D have the following limiting values when n is very large

$$A \rightarrow C \rightarrow \text{constant}$$

$$B \rightarrow D \rightarrow n$$

Furthermore we can write if $x \rightarrow \epsilon$

$$\cos \frac{A\epsilon}{a} = \cos \frac{c\epsilon}{a} = 1$$

$$\sin \frac{A\epsilon}{a} = \sin \frac{c\epsilon}{a} \backsimeq \frac{c\epsilon}{a} = 0$$

$$e^{-D\frac{\epsilon}{a}} \rightarrow e^{-B\frac{\epsilon}{a}} \rightarrow e^{-n\frac{\epsilon}{a}}$$

Knowing the above relation we can simplify Eq. (55) as follows

$$\left.\frac{S_t}{\frac{T}{a^2}}\right|_{x \rightarrow \epsilon} = \frac{1}{2\pi} \sum_{n=2,4\cdots}^{\infty} \cos n\frac{s}{a} \Big\{ (A+C)e^{-n\frac{\epsilon}{a}} \Big[\frac{(2+\nu)}{2\sqrt{2}} \frac{\sqrt{j}}{n} \frac{\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}}$$

$$+ \frac{(1-\nu)}{\sqrt{2j}} n \frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}} - \frac{(1+\nu)}{2\sqrt{2}} \frac{1}{n} \sqrt{\sqrt{1+j^2}+j} + \frac{(1+\nu)j^{\frac{3}{2}}}{8\sqrt{2}} \frac{1}{n^3} \frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}}$$

$$+ (B+D)e^{-n\frac{\epsilon}{a}} - \Big[\frac{\sqrt{j}}{2\sqrt{2}} \frac{1}{n} \frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}} - \frac{(1+\nu)}{\sqrt{2j}} n \frac{\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}} \Big] (D-B)e^{-\frac{n\epsilon}{a}} \Big\}$$

The above expression can be further simplified if the following relation is used

$$A + C = \sqrt{\frac{J}{2}}$$

$$B - D = \sqrt{\frac{J}{2}}$$

$$B + D = \sqrt{2}n \left[\frac{\sqrt{1+\sqrt{1+j^2}}}{\sqrt{\sqrt{1+j^2}-j}} \right]$$

Hence

$$\frac{S_t}{\frac{I}{a^2}} = \frac{1}{2\pi} \sum_{n=2,4\cdots}^{\infty} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} \left\{ \frac{(2+\nu)}{4} \left(\frac{\sqrt{\sqrt{1+j^2}-j}}{n\sqrt{1+j^2}} \right) + \frac{(1+\nu)}{2} \left(\frac{n\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}} \right) \right.$$

$$- \frac{(1+\nu)J}{4} \left(\frac{\sqrt{\sqrt{1+j^2}+j}}{n} \right) + \frac{(1+\nu)J^2}{16} \left(\frac{\sqrt{\sqrt{1+j^2}+j}}{n^3\sqrt{1+j^2}} \right) + \frac{\sqrt{2}\,n\sqrt{1+\sqrt{1+j^2}}}{\sqrt{\sqrt{1+j^2}-j}} \qquad (88)$$

$$\left. - \frac{J}{4} \left(\frac{\sqrt{\sqrt{1+j^2}+j}}{n\sqrt{1+j^2}} \right) + \frac{1+\nu}{2} \left(\frac{n\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}} \right) \right\}$$

In Eq. (88) let us now consider the term

$$\sum_{n=2,4\cdots}^{\infty} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} \left[\frac{\sqrt{\sqrt{1+j^2}-j}}{n\sqrt{1+j^2}} \right] \qquad (88{:}a)$$

As n increases, j decreases in inverse proportion to the square of n, and when n reaches values of importance in practical computation the term $\frac{\sqrt{\sqrt{1+j^2}-j}}{n\sqrt{1+j^2}}$ approaches $\frac{1}{n}$. Hence we can introduce a known series $e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} \left[\frac{1}{n}\right]$ the sum of which can be calculated. The difference between the term given in (88:a) and the proposed known series allows an easy computation. It is

$$\sum_{n=2,4\cdots}^{\infty} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a}\left[\frac{\sqrt{\sqrt{1+j^2}-j}}{n\sqrt{1+j^2}}\right] = \sum_{n=2,4\cdots}^{\infty} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a}\left[\frac{\sqrt{\sqrt{1+j^2}-j}}{n\sqrt{1+j^2}} - \frac{1}{n}\right] + \sum_{n=2,4\cdots}^{\infty} \frac{1}{n} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} \tag{88:b}$$

where the first term in expression (88:b) may be computated in the ordinary manner and the second term has a sum which is found as follows:

$$\sum_{n=2,4}^{\infty} \frac{1}{n} e^{-n\frac{x}{a}} \cos n\frac{s}{a} = -\frac{1}{2a}\int\left[\frac{1}{e^{\frac{2}{a}(x+is)}-1} + \frac{1}{e^{\frac{2}{a}(x-is)}-1}\right]dx + C$$

After integration we get

$$\sum_{n=2,4\cdots}^{\infty} \frac{1}{n} e^{-n\frac{x}{a}} \cos n\frac{s}{a} = \frac{x}{a} - \frac{1}{4}\log\left|e^{\frac{4}{a}x} - 2e^{\frac{2}{a}x}\cos 2\frac{s}{a} + 1\right| + C$$

The constant of integration can be determined from the condition at $x \to \infty$. It is found to be zero after the indeterminate form is evaluated. For a very small x, we have

$$\sum_{\substack{n=2,4\cdots \\ x\to\epsilon}}^{\infty} \frac{1}{n} e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} = -\frac{1}{4}\log\left|2 - 2\cos 2\frac{s}{a}\right| \tag{88:c}$$

Similarly, we obtain

$$\sum_{\substack{n=2,4\cdots \\ x\to\epsilon}}^{\infty} n e^{-n\frac{\epsilon}{a}} \cos n\frac{s}{a} = \frac{1}{\cos 2\frac{s}{a} - 1} = -\frac{1}{2\sin^2\frac{s}{a}} \tag{88:d}$$

With the aid of summations (88:c) and (88:d) we can express Eq. (88) in the following manner

$$\frac{S_L}{\frac{I}{a^2}}\Bigg|_{x\to\epsilon} = -\left(\frac{3+\nu}{2\pi}\right)\frac{1}{2\sin^2\frac{s}{a}} - \frac{J}{32\pi}\log\left|4\sin^2\frac{s}{a}\right| + \frac{1}{2\pi}\sum_{n=2,4\ldots}^{\infty}\cos\frac{ns}{a}\left\{\frac{(1+\nu)}{16}J^2\frac{\sqrt{\sqrt{1+j^2}+j}}{n^3\sqrt{1+j^2}}\right.$$

$$+\frac{2+\nu}{4}\frac{J}{n}\left[\frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}}-1\right] - \frac{(1+\nu)}{4}\frac{J}{n}\left[\sqrt{\sqrt{1+j^2}+j}-1\right] + \frac{J}{4n}\left[\frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}}-1\right] \quad (89)$$

$$\left. + 2n\left[\frac{\sqrt{1+\sqrt{1+j^2}}}{\sqrt{2}\sqrt{\sqrt{1+j^2}-j}}-1\right] - \frac{J}{4n} + \frac{(1+\nu)}{2}n\left[\frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}} + \frac{\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}} - 2\right]\right\}$$

Eq. (76) can be expressed in the same manner after the sums of some known series are determined. These are

$$\sum_{\substack{n=1,3\cdots\\ x\to\epsilon}}^{\infty} n e^{-n\frac{x}{a}}\cos n\frac{s}{a} = -\frac{1}{2}\left[\frac{\cos\frac{s}{a}}{\sin^2\frac{s}{a}}\right] \quad (90{:}a)$$

$$\sum_{\substack{n=1,3\cdots\\ x\to\epsilon}}^{\infty} \frac{1}{n} e^{-n\frac{x}{a}}\cos n\frac{s}{a} = \frac{1}{8}\log\left\{\frac{(\cos\frac{s}{a}+1)^2}{(\cos\frac{s}{a}-1)^2}\right\} \quad (90{:}b)$$

Through substitution of expressions (90:a) and (90:b) in Eq. (76), a modified form of S_1 is obtained

$$\frac{S_L}{\frac{I}{a}}\Bigg|_{x\to\epsilon} = \frac{1}{2\pi}\left[\frac{\cos\frac{s}{a}}{\sin^2\frac{s}{a}}\right] + \frac{J}{64\pi}\log\left\{\frac{(\cos\frac{s}{a}+1)^2}{(\cos\frac{s}{a}-1)^2}\right\}$$

$$+\frac{1}{2\pi}\sum_{n=1,3\cdots}^{\infty}\cos\frac{ns}{a}\left\{-\frac{(1+\nu)}{16}J^2\frac{\sqrt{\sqrt{1+j^2}+j}}{n^3\sqrt{1+j^2}} - \frac{\nu J}{4n}\left[\frac{\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}}-1\right]\right.$$

$$+\frac{(1-\nu)}{4}\frac{J}{n}\left[\sqrt{\sqrt{1+j^2}+j}-1\right] - \frac{J}{4n}\left[\frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}}-1\right] \quad (91)$$

$$\left.+2n\left[\frac{\sqrt{1+\sqrt{1+j^2}}}{\sqrt{2}\sqrt{\sqrt{1+j^2}-j}}-1\right] - \frac{J}{4n} - \frac{1+\nu}{2}n\left[\frac{\sqrt{\sqrt{1+j^2}+j}}{\sqrt{1+j^2}} + \frac{\sqrt{\sqrt{1+j^2}-j}}{\sqrt{1+j^2}} - 2\right]\right\}$$

CONCLUSION

The results calculated from Eqs. (17) and (18) were plotted in Figs. I to IX inclusive. It is seen that the radial deflection of an infinitely long cylinder has a very long wave length along the generatrix. However, the wave length decreases with the radius over thickness ratio. It is believed that the cause of this long wave length phenomenon is due to the elastic relations along the circumference of the shell which has been explained in Part I - 4. A family of maximum deflection curves of infinitely long plates with simply supported edges was plotted in Fig. 3 with various widths of plates over radius of cylinder ratio, and it shows that the plate has greater slope than the cylinder because of the curvature restraint of the cylinder.

Deflection curves of cylindrical shells with various lengths were calculated from Eqs. (22) and (23), and the results show that the maximum radial deflection occurs at about $\frac{l}{a}=20$. Since the radial deflection of an infinitely long cylinder with $a/h=100$ becomes zero at about $\frac{x}{a}=15$ and then reverses its sign, the edges of the corresponding cylinder with finite length are so restrained that the negative deflection portion of the infinite cylinder is brought to zero at the edges of the cylinder with finite length. Hence the maximum deflection of a cylinder with $l/a=20$ is greater than the corresponding finitely long cylinder.

Shearing stress distribution along both the generatrix and the circumference was calculated from Eqs. (55) and (74), and the results show that the stress decreases very rapidly along the generatrices. When the ratio of x/a reaches to 1.0 the shearing stress is almost negligible.

APPENDIX

TABLE I

Deflection Parameter, α, of an Infinitely Long Cylinder Under Two Concentrated Loads Along the Generatrix $\frac{s}{a} = \theta^{0}$

$$\frac{a}{h} = 1000; \qquad \frac{w/a}{P/Eh^2} = \alpha$$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
n	γ	B_2			Φ	Ω	A
					$\sqrt{2}n^2 x(4)$	$\sqrt{2}n^2(3)$	
2	206.539	2730×10^3	.70711	292.1	1652	4.000	.04912
4	51.6335	2730.5×10^3	.70704	73.02	1652	16.00	.1957
6	22.9482	2732.5×10^3	.7069	32.45	1653	35.99	.4379
8	12.908	2738×10^3	.7065	18.27	1654	63.95	.7722
10	8.2614	2749.5×10^3	.7058	11.70	1655	99.82	1.191
12	5.737	2771×10^3	.7044	8.144	1658	143.4	1.690
14	4.215	2806×10^3	.7023	6.002	1664	194.7	2.258
16	3.227	2858×10^3	.698	4.617	1671	252.7	2.882
18	2.549	2933×10^3	.694	3.672	1683	318.0	3.554
20	2.065	3033×10^3	.689	3.000	1697	389.7	4.27
22	1.707	3163×10^3	.682	2.508	1717	466.8	4.87
24	1.434	3326×10^3	.671	2.136	1740	546.6	5.55
26	1.222	3516×10^3	.660	1.850	1769	631.0	6.52
28	1.053	3765×10^3	.648	1.625	1802	718.6	8.39
30	0.918	4036×10^3	.636	1.417	1804	809.0	

TABLE I -- Continued

	(8)	(9)	(10)	(11)	(12)
n	B	C	D	$\phi C + \eta D$	$\phi D - \eta C$
2	.04925	40.60	40.69	67230	67060
4	.1978	40.44	40.84	67460	66820
6	.4476	40.21	41.09	67950	66480
8	.8027	39.88	41.45	68610	66010
10	1.265	39.46	41.92	69490	65440
12	1.842	38.95	42.48	70670	64850
14	2.583	38.39	43.19	72290	64400
16	3.352	37.73	43.99	74160	63970
18	4.268	37.11	44.94	76750	63840
20	5.34	36.40	45.97	79690	63830
22	6.40	35.71	47.18	83340	64330
24	7.76	34.99	48.45	87380	65180
26	9.34	34.29	49.83	92100	66510
28	11.7	33.64	51.32	97500	67590

TABLE I -- Continued

n	(13) $A(\frac{x}{a})$	(14) $B(\frac{x}{a})$	(15) $(11)\times\cos\frac{nx}{a}$	(16) $(12)\times\sin\frac{nx}{a}$	(17) $\frac{(15)+(16)}{R_2 n^2}e^{-8\frac{x}{a}}$
$\frac{x}{a}=1$					
2	.04912	.04925	67140	3291	6.139 x 10³
4	.1957	.1978	66170	12990	1.487 x 10³
6	.4379	.4476	61540	28190	.5972 x10³
8	.7722	.8027	49140	46060	.2435 x10³
10	1.191	1.265	25760	60770	.0888 x10³
12	1.690	1.842	- 8403	64390	.0222 x10³
14	2.258	2.538	-45860	49780	.0006 x10³
16	3.554	3.352	-67950	25640	-.0021 x10³
18	4.27	4.285	-32840	-57710	-.0013 x10³
20	4.87	5.34	12510	-33060	-.0002 x10³
22	5.55	6.40	61920	-43040	
24	6.52	7.76	84920	16320	
$\frac{x}{a}=2$					
2	.09824	.0985	66910	6579	6.099 x 10³
4	.3914	.3965	62360	25490	1.359 x 10³
6	.8758	.8952	43510	51060	.3929 x10³
8	1.544	1.605	1427	65980	.07729x10³
10	2.382	2.530	- 50380	45090	-.00153x10³
12	3.38	3.684	- 68690	-15310	-.00526x10³
14	4.516	5.076	- 14090	-63180	-.00087x10³
16	7.108	6.704	50350	46950	.00016x10³

TABLE I -- Continued

	(13)	(14)	(15)	(16)	(17)
n	$A(\frac{x}{a})$	$B(\frac{x}{a})$	$(11) \times \cos \frac{Ax}{a}$	$(12) \times \sin \frac{Ax}{a}$	$\frac{(15)+(16)}{R_2 n^2} e^{-B\frac{x}{a}}$
$\frac{x}{a} = 5$					
2	.2456	.2463	65210	16300	5.835×10^{-3}
4	.9785	.989	37660	55440	$.7927 \times 10^{-3}$
6	2.189	2.238	-39340	54180	$.0161 \times 10^{-3}$
8	3.861	4.014	-51590	-43500	$-.00977 \times 10^{-3}$
10	5.955	6.325	55810	-21070	$.00031 \times 10^{-3}$
$\frac{x}{a} = 10$					
2	.4912	.4925	59280	31630	5.087×10^{-3}
4	1.957	1.978	-25410	61890	$.1155 \times 10^{-3}$
6	4.379	4.476	-22220	-62820	$-.00985 \times 10^{-3}$
8	7.722	8.027	9056	65420	$.00014 \times 10^{-3}$
$\frac{x}{a} = 15$					
2	.7368	.7387	49790	45060	4.149×10^{-3}
4	2.935	2.967	-66040	13700	$-.0616 \times 10^{-3}$
6	6.568	6.714	65230	16680	$.00102 \times 10^{-3}$
$\frac{x}{a} = 20$					
2	.9824	.985	37310	55780	3.183×10^{-3}
4	3.914	3.965	48300	46640	$.0411 \times 10^{-3}$
6	8.758	8.952	-53410	41080	$-.00015 \times 10^{-3}$

TABLE I -- Continued

n	(13)	(14)	(15)	(16)	(17)
$\frac{x}{a}=30$					
2	1.473	1.477	8560	66740	$1.614\times\bar{10}^{3}$
4	5.871	5.934	61790	-26770	.0021x "
$\frac{x}{a}=40$					
2	1.965	1.970	-25820	61920	.4611x "
4	7.828	7.912	1754	60750	.00057x"
$\frac{x}{a}=50$					
2	2.456	2.463	-52040	42460	-.0747x "
$\frac{x}{a}=60$					
2	2.947	2.955	-65950	12940	-.253 x "
$\frac{x}{a}=70$					
2	3.438	3.447	-64270	-19580	-.244 x "
$\frac{x}{a}=80$					
2	3.930	3.940	-47400	-47540	-.168 x "
$\frac{x}{a}=90$					
2	4.421	4.432	-19300	-64240	-.0910x "
$\frac{x}{a}=100$					
2	4.912	4.925	13310	-65720	-.0348x "

$\frac{x}{a}$	Σ	$\frac{w}{h} / \frac{p}{Eh^2}$
0		174.1×10^{2}
1		149.0x "
2	$7.921\times\bar{10}^{3}$	137.7x "
5	6.634x "	115.3x "
10	5.193x "	90.25x"
15	4.088x "	71.05x"
20	3.224x "	56.03x "
30	1.616x "	28.09x"
48.25	.4617x"	8.02x"
50	-.0747x"	- 1.30x"
60	-.253 x"	- 4.40X"
70	-.244 x"	- 4.24x"
80	-.168 x"	- 2.92x"
90	-.091 x"	- 1.58x"
100	-.035 x"	- .61 x"
40		

TABLE II

Deflection Parameter, α, of an Infinitely Long Cylinder Under Two Concentrated Loads Along the Generatrix, $S/a = 0$

$$\frac{a}{h} = 10^2 \qquad \frac{w}{h} \Big/ \frac{P}{Eh^2} = \alpha$$

x/a	α	x/a	α
0	5.519×10^2	15.0	0.01767×10^2
0.5	4.489 "	20.0	-0.1363 "
1.0	4.037 "	25.0	-0.09328 "
2.0	3.399 "	30.0	-0.03267 "
3.0	2.907 "	35.0	-0.00208 "
5.0	2.145 "	40.0	-0.00537 "
10.0	0.716 "	50.0	0.00129 "

TABLE II

Deflection Parameter, α, of an Infinitely Long Cylinder Under Two Concentrated Loads Along the Generatrix $\frac{s}{a} = 0°$

$$\frac{a}{h} = 100; \quad \left(\frac{w}{h} \Big/ \frac{P}{Eh^2}\right) = \alpha$$

	(1)	(2)	(3)	(4)	(5)	(6)	(7)
n	$g = \frac{82.619}{n^2}$	R_2	$\sqrt{g\sqrt{1+g^2} - g^2}$	$\sqrt{g\sqrt{1+g^2} + g^2}$	η	ϕ	A
2	20.65300	273.4x10[2]	.706900	29.22000	3.9988	165.3	.1536
4	5.16334	276.05x10[2]	.703840	7.33592	15.9260	166.0	.5899
6	2.29482	297.6 "	.691579	3.31823	35.2100	168.9	1.2260
8	1.29083	345.4 "	.664459	1.94268	60.1400	175.8	1.9270
10	0.82613	428.7 "	.623800	1.32400	88.2100	187.2	2.5810
12	0.57370	548.7 "	.576600	0.99520	117.4000	202.6	3.1090
14	0.42150	702.9 "	.528900	0.79680	146.6000	220.9	3.5360
16	0.32270	889.1 "	.484800	0.66570	175.5000	241.0	3.8750
18	0.25490	1105.0 "	.445000	0.57270	203.9000	262.4	4.1610
20	0.20650	1342.0 "	.409000	0.50230	231.4000	284.2	4.3390
22	0.17070	1615.0 "	.378400	0.44900	259.0000	307.3	4.5340
24	0.14340	1922.0 "	.352400	0.40680	287.1000	331.4	4.7060
26	0.12220	2245.0 "	.329000	0.37160	314.5000	355.2	4.8410
28	0.10530	2606.0 "	.308000	0.34200	341.5000	379.3	4.9500
30	0.09178	2986.0 "	.289000	0.31710	367.9000	403.7	5.0410
32	0.08068	3394.0 "	.273000	0.29600	395.3000	428.6	5.1460
34	0.07146	3827.0 "	.258000	0.27700	421.8000	452.9	5.2110
36	0.06374	4291.0 "	.245000	0.26100	449.1000	478.4	5.2960
38	0.05721	4782.0 "	.233000	0.24600	475.8000	502.3	5.3390
40	0.05163	5292.0 "	.054500	0.23300	502.4000	527.3	5.3870

TABLE II -- Continued

n	(8) B	(9) C	(10) D	(11) $\phi C + \eta D$	(12) $\phi D - \eta C$
2	0.1574	12.700	13.00	2151	2098
4	0.6493	12.260	13.50	2250	2046
6	1.5150	11.640	14.37	2472	2017
8	2.7520	10.930	15.61	2860	2087
10	4.2940	10.510	17.14	3479	2282
12	6.0210	9.749	18.88	4191	2681
14	7.8670	9.296	20.72	5090	3214
16	9.7690	8.869	22.59	6101	3888
18	11.7200	8.708	24.57	7295	4672
20	13.6900	8.473	26.51	8542	5573
22	15.6800	8.292	28.49	9927	6607
24	17.6300	8.156	30.48	11454	7766
26	19.6100	8.050	32.48	13069	9005
28	21.5800	7.940	34.45	14772	10356
30	23.6100	7.646	36.40	16477	11882
32	25.5900	7.936	38.49	18610	13350
34	27.6000	8.044	40.53	20730	14960
36	29.5800	7.969	42.52	22900	16760
38	31.5700	7.980	44.52	25190	18560
40	33.5400	8.003	46.55	27610	20520

TABLE II -- Continued

	(13) $A\frac{z}{a}$	(14) $B\frac{z}{a}$	(15) $\cos A\frac{z}{a}$	(16) $\sin A\frac{z}{a}$	(17) $e^{-B\frac{z}{a}}$	(18) (11)x(15)	(19) (12)x(16)
$\frac{x}{a} = 0.5$							
2	.07680	.07870	.99710	.07672	.924300	2145.0	160.9
4	.29490	.32460	.95680	.29060	.722800	2153.0	594.6
6	.61300	.75750	.81790	.57530	.468800	2022.0	1160.0
8	.96350	1.37600	.57060	.82120	.252600	1632.0	1714.0
10	1.29100	2.14700	-.27620	.96110	.116800	960.9	2193.0
12	1.55400	3.01100	.02679	.99960	.049300	112.3	2680.0
14	1.76800	3.93400	-.19550	.98070	.019600	-995.1	3152.0
16	1.93700	4.88400	-.35770	.93380	.007550	-2182.0	3631.0
18	2.08100	5.86000	-.48800	.87280	.002850	-3560.0	4078.0
20	2.17000	6.84500	-.56360	.82600	.001060	-4814.0	4573.0
22	2.26700	7.84000	-.64100	.76750	.000393	-6363.0	5071.0
24	2.35300	8.81400	-.70460	.70960	.000148	-8070.0	5511.0
$\frac{x}{a} = 1.0$							
2	.15360	.15740	.98820	.15300	.854400	2126.0	321.0
4	.58990	.64930	.83090	.55630	.522400	1869.0	1138.0
6	1.22600	1.51500	.33800	.94110	.219800	835.5	1898.0
8	1.92700	2.75200	-.34930	.93700	.063800	-999.0	1955.0
10	2.58100	4.29400	-.84670	.53200	.013600	-2946.0	1214.0
12	3.10900	6.02100	-.99990	.00330	.002430	-4190.0	7.5
14	3.53600	7.86700	-.92340	-.38390	.000383	-4700.0	-876.0

TABLE II -- Continued

n	$A\frac{x}{a}$	$B\frac{x}{a}$	$\cos A\frac{x}{a}$	$\sin A\frac{x}{a}$	$e^{-B\frac{x}{a}}$	(11)x(15)	(12)x(16)
$\frac{x}{a} = 2$							
2	.3072	.3148	.9532	.3024	.729900	2050.0	634.4
4	1.1800	1.2990	.3809	.9246	.272800	857.0	1892.0
6	2.4520	3.0300	-.7712	.6363	.048320	-1906.0	1284.0
8	3.8540	5.5040	-.7571	-.6533	.004060	-2165.0	-1363.0
10	5.1620	8.5880	.3493	-.9370	.000186	1215.0	-2138.0
$\frac{x}{a} = 3$							
2	.4608	.4722	.8955	.4446	.623600	1926.0	932.8
4	1.7700	1.9480	-.1975	.9803	.142600	- 444.3	1858.0
6	3.6780	4.5450	-.9986	-.0536	.010600	-2468.0	- 108.1
8	5.7810	8.2560	.8766	-.4812	.000259	2507.0	-1004.0
x = 5							
2	.7680	.7870	.7193	.6947	.455200	1547.0	1457.0
4	2.9490	3.2460	-.9814	.1918	.038900	-2208.0	392.4
6	6.1300	7.5750	.9999	-.0153	.000513	2472.0	- 30.8
x = 10							
2	1.5360	1.5740	.03479	.9994	.20720	74.83	2097.0
4	5.8990	6.4930	.92720	-.3746	.00151	2086.00	- 766.4

TABLE II -- Continued

n	$A\frac{x}{d}$	$B\frac{x}{d}$	$\cos A\frac{x}{d}$	$\sin A\frac{x}{d}$	$e^{-B\frac{x}{d}}$	(11)×(15)	(12)×(16)
x = 15							
2	2.304	2.361	-.6689	.74330	.094320	-1441.00	1559.0
4	8.848	9.739	-.8381	.54550	.000059	-1886.00	1116.0
x = 20							
2	3.072	3.148	-.9975	.06994	.042900	-2146.00	146.7
4	11.800	12.990	.7179	-.69610		1615.00	
x = 25							
2	3.840	3.935	-.7661	-.64270	.019600	-1648.00	-1348.0
x = 30							
2	4.608	4.722	-.1046	-.99450	.008900	- 225.00	-2086.0
x = 35							
2	5.376	5.509	.6161	-.78770	.004050	1325.00	-1652.0
x = 40							
2	6.144	6.296	.9904	-.13850	.001840	2130.00	- 290.6
x = 50							
2	7.680	7.870	-.0158	.99990	.000382	33.98	2098.0

Table III

Maximum Radial Deflection Parameter of an Infinitely Long Cylinder With Different Radius/Thickness Ratio

a/h	$0.20456(a/h)^{\frac{1}{2}}$	$1.2289(a/h)^2$ (3)	Σ (4)	(3) x (4)	$\frac{w/h}{P/Eh^2}$
50	1.4465	3.072×10^3	0.06287	1.931×10^2	1.945×10^2
100	2.0456	12.290 x "	0.04474	5.499 x "	5.519 x "
200	2.[illegible]929	49.160 x "	0.03151	15.490 x "	15.520 x "
300	3.5431	110.600 x "	0.02607	28.830 x "	28.870 x "
500	4.5729	307.200 x "	0.02006	61.620 x "	61.670 x "
1000	6.4670	1229.000 x "	0.01416	174.000 x "	174.100 x "
1500	7.9203	2765.000 x "	0.01160	320.700 x "	320.800 x "
2000	9.1457	4916.000 x "	0.009972	490.200 x "	490.300 x "
5000	14.4648	30720.000 x "	0.006322	1942.000 x "	1942.000 x "

TABLE III

Maximum Radial Deflection Parameter of an Infinitely Long Cylinder With Different Radius/Thickness Ratio

	$\frac{a}{h} = 50$		$\frac{a}{h} = 100$		$\frac{a}{h} = 200$	
n	$1+\frac{.6825}{n^4}\left(\frac{a}{h}\right)^2$		$1+\frac{.6825}{n^4}\left(\frac{a}{h}\right)^2$		$1+\frac{.6825}{n^4}\left(\frac{a}{h}\right)^2$	
2	107.6	.04064	427.56	.02815	1707.24	.01968
4	7.663	.01096	27.66	.00743	107.64	.00508
6	2.316	.00483	6.266	.00346	22.06	.00215
8	1.416	.00243	2.666	.00194	7.665	.00137
10	1.171	.00133	1.683	.00117	3.730	.00089
12	1.082	.00079	1.329	.00074	2.316	.00059
14	1.044	.00051	1.178	.00049	1.711	.00042
16	1.026	.00034	1.104	.00033	1.417	.00031
18	1.016	.00024	1.066	.00024	1.260	.00022
20	1.011	.00018	1.043	.00017	1.171	.00017
22	1.007	.00013	1.029	.00013	1.116	.00013
24	1.005	.00010	1.021	.00010	1.082	.00010
26	1.004	.00008	1.015	.00008	1.060	.00008
28	1.003	.00007	1.011	.00007	1.045	.00007
30	1.002	.00005	1.008	.00005	1.034	.00006
32	1.002	.00005	1.007	.00005	1.026	.00005
34	1.001	.00004	1.005	.00004	1.021	.00004
36		.00010		.00010		.00010
$\sum_{n=2,4\cdots}^{\infty}$		0.06287		0.04474		0.03151

TABLE III Continued

	$\frac{a}{h} = 300$		$\frac{a}{h} = 500$		$\frac{a}{h} = 1000$	
n	$1 + \frac{.6825}{n^4}(\frac{a}{h})^2$		$1 + \frac{.6825}{n^4}(\frac{a}{h})^2$		$1 + \frac{.6825}{n^4}(\frac{a}{h})^2$	
2	3840.04	.01601	10665	.01236	42657	.008717
4	240.94	.00427	667.50	.00313	2667	.002196
6	48.39	.00188	132.65	.00142	527.6	.000986
8	15.99	.00109	42.66	.00082	167.6	.000564
10	7.143	.00072	18.06	.00054	69.24	.000367
12	3.962	.00050	9.228	.00038	33.92	.000260
14	2.599	.00037	5.441	.00029	18.77	.000195
16	1.937	.00027	3.604	.00022	11.42	.000151
18	1.585	.00021	2.625	.00017	7.501	.000121
20	1.384	.00016	2.066	.00014	5.265	.000098
22	1.262	.00012	1.728	.00011	3.913	.000081
24	1.185	.00010	1.514	.00009	3.057	.000069
26	1.134	.00008	1.374	.00007	2.494	.000058
28	1.100	.00006	1.278	.00006	2.111	.000049
30	1.076	.00005	1.211	.00005	1.843	.000041
32	1.059	.00004	1.163	.00004	1.651	.000036
34	1.046	.00004	1.128	.00004	1.511	.000031
36		.00010	1.102	.00003	1.406	.000026
38			1.082	.00003	1.327	.000023
40			1.067	.00002	1.267	.000020
42				.00005		.000065
$\sum_{n=2,4\cdots}^{\infty}$		0.02607		0.02006		0.01416

TABLE III Continued

	$\frac{a}{h} = 1500$		$\frac{a}{h} = 2000$		$\frac{a}{h} = 5000$	
n	$1+\frac{.6825}{n^2}\left(\frac{a}{h}\right)^2$		$1+\frac{.6825}{n^2}\left(\frac{a}{h}\right)^2$		$1+\frac{.6825}{n^2}\left(\frac{a}{h}\right)^2$	
2	95797	.007116	170625	.006156	10664×10^2	.003893
4	5988.25	.001788	10665	.001528	66651	.000975
6	1183.66	.000811	2107.46	.000658	13160	.000434
8	375.2	.000455	667.5	.000391	4166.6	.000244
10	154.3	.000295	273.9	.000254	1707.2	.000158
12	74.92	.000208	132.7	.000178	823.8	.000100
14	40.90	.000155	72.06	.000133	445.1	.000083
16	24.39	.000120	42.65	.000103	261.4	.000063
18	15.60	.000096	27.00	.000081	163.5	.000050
20	10.58	.000079	18.62	.000066	107.7	.000040
22	7.543	.000066	12.65	.000056	73.83	.000034
24	5.619	.000056	9.228	.000048	52.43	.000029
26	4.354	.000048	6.974	.000041	38.34	.000025
28	3.494	.000041	5.441	.000036	28.78	.000021
30	2.892	.000036	4.370	.000031	22.06	.000019
32	2.462	.000031	3.603	.000028	17.27	.000018
34	2.147	.000028	3.043	.000025	11.16	.000016
36	1.192	.000025	2.625	.000023		.000120
38	1.735	.000021	2.309	.000020		
40	1.598	.000019	2.066	.000018		
42		.000100		.000100		
$\sum_{n=2,4\cdots}^{\infty}$		0.01160		0.009972		0.006322

TABLE IV

Deflection Parameter, ϕ, of an Infinitely Long Cylinder Under Two Concentrated Loads Along the Circumference

$(\frac{x}{a})=0$

$\frac{a}{h}=10^2$ $\qquad$ $\frac{w}{h}\frac{P}{Eh^2}=\phi$

$\frac{x}{a}$	$\varphi = 90°$	$\varphi = 67\frac{1}{2}°$	$\varphi = 45°$	$\varphi = 38°$	$\varphi = 22\frac{1}{2}°$	$\varphi = 10°$	$\varphi = 5°$	$= 0°$
0	-2.825×10^2	-2.302×10^2	-0.742×10^2	$-.037\times10^2$	1.882×10^2	3.986×10^2	4.95×10^2	5.519×10^2
0.5	-2.80 x "	-2.28 x "	-0.712x "	0.023x "	2.146 x "	3.904 x "	4.336 x "	4.489 x "
1.0	-2.802x "	-2.238x "	- .609x "	0.199x "	2.277 x "	3.641 x "	3.936 x "	4.037 x "
2.0	-2.815x "	-2.202x "	- .294x "	.498x "	2.205 x "	3.147 x "	3.335 x "	3.399 x "
3.0	-2.749x "	-2.006x "	- .0787x"	.619x "	2.006 x "	2.720 x "	2.859 x "	2.907 x "
5.0	-2.20 x "	-1.536x "	.0276x"	.55 x "	1.536 x "	2.021 x "	2.113 x "	2.145 x "
10	-0.714x "	-0.506x "	-	.172x "	0.506 x "	.673 x "	.705 x "	0.7157 x "
15	- .0177x"	- .0125x"	-	.0043x"	.0125x "	.0167x "	.0177x "	0.0177 x "
20	.136x "	.0963x"	-	- .0328x"	- .0963x "	- .128 x "	-0.134 x "	-0.136 x "
25	.0933x"	.066 x"	-	- .0225x"	- .0660x "	- .0875x "	-0.0918x "	- .0933 x "

TABLE V

Reflection Parameter , α, of an Infinitely Long Cylinder Under Two Concentrated Loads Along the Circumference

$(\frac{x}{a}) = 0$ $\frac{a}{h} = 10^3$ $\frac{w}{h} / \frac{P}{Eh^2} = \alpha$

$\frac{x}{a}$	$\varphi = 90°$	$\varphi = 67\frac{1}{2}°$	$\varphi = 45°$	$\varphi = 38°$	$\varphi = 22\frac{1}{2}°$	$\varphi = 10°$	$\varphi = 5°$	$\varphi = 0°$
0	-88.23 x 10^2	-71.82 x10^2	-22.35 x10^2	0.09 x10^2	59.97 x10^2	119.1 x10^2	147.0 x10^2	174.1x10^2
1.0	-88.19 x "	-71.27 x "	-22.04 x "	-0.226x "	62.74 x "	125.6 x "	142.8 x "	149.0x "
2	-87.91 x "	-71.48 x "	-22.18 x "	.982x "	68.77 x "	121.4 x "	133.4 x "	137.7x "
5	-88.08 x "	-70.61 x "	-13.95 x "	12.10 x "	70.94 x "	106 x "	112.9 x "	115.3x "
10	-86.24 x "	-62.64 x "	- 2.005x "	19.74 x "	62.64 x "	84.54 x "	88.79x "	90.25x"
15	-73.2 x "	-50.97 x "	1.07 x "	18.39 x "	50.97 x "	66.95 x "	70.02x "	71.05x"
20	-54.61 x "	-39.12 x "	- .714x "	12.75 x "	39.12 x "	52.54 x "	55.16x "	56.03x"
30	-28.02 x "	-19.83 x "	- .036x "	6.752x "	19.82 x "	26.4 x "	27.65x "	28.09x"
40	- 8.003x "	- 5.666x "	- .0099x"	1.93 x "	5.666x "	7.538x "	7.901x"	8.02x"
50	1.298x "	.918x "	0	- .3140x"	- .918x "	- 1.22 x "	-1.278x"	-1.3 x"
60	4.397x "	3.109x "	0	-1.064 x"	-3.109x "	- 4.131x "	-4.331x"	-4.4 x"

TABLE VI

Deflection Parameter, α, of Cylindrical Shells

with Different /a Ratio Along the Generatrix s/a=0

$$\frac{a}{h} = 10^2 \qquad \frac{w}{h} \Big/ \frac{p}{Eh^2} = \alpha$$

ℓ/a \ $\frac{b}{a}/\frac{\ell}{a}$	0	0.1	0.2	0.3	0.4	0.5
2	1.595×10^2	0.9703×10^2	0.6827×10^2	0.4265×10^2	0.1954×10^2	0
5	2.483x "	1.5130 "	1.0150 "	0.6338 "	0.3234 "	0
10	3.888 "	2.4430 "	1.7140 "	1.0900 "	0.5212 "	0
15	5.423 "	3.6210 "	2.6650 "	1.7990 "	0.9093 "	0
20	5.803 "	3.7490 "	2.7370 "	1.8330 "	0.9316 "	0
30	5.588 "	3.7490 "	2.7370 "	1.8330 "	0.9316 "	0
∞	5.519 "	-	-	-	-	-

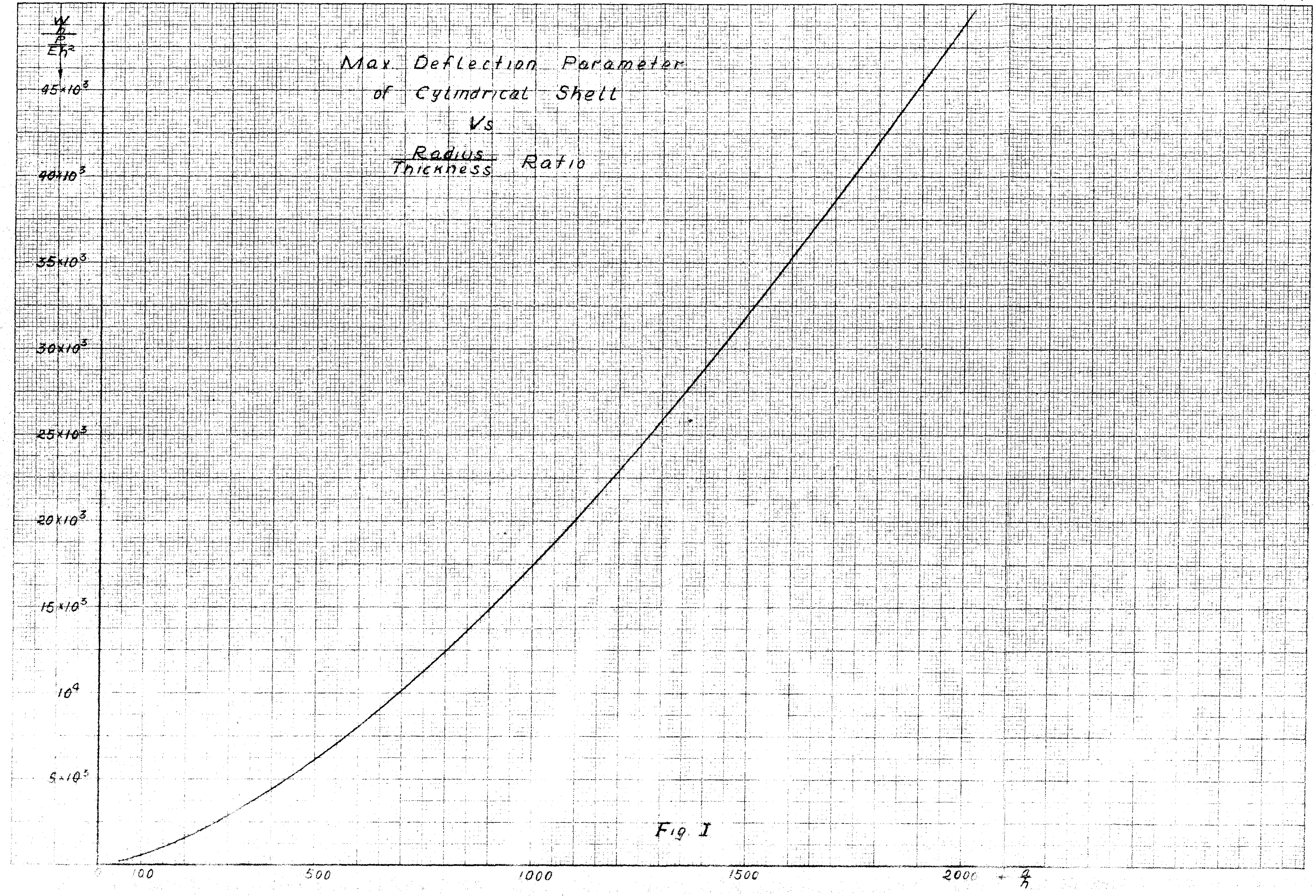
TABLE VII

Shearing Stress Distribution

Along the Circumference, $x/a=0$, and the Generatrix, $s/a=0$

$a/h=100$

x/a	0.5	1.0	2.0	3.0	5.0	10.0
$S/\frac{T}{a^2}$	0.645	0.238	0.085	0.043	0.030	0.006
s/a	$\frac{\pi}{2}$	$5\frac{\pi}{12}$	$\frac{\pi}{3}$	$\frac{\pi}{4}$	$\frac{\pi}{6}$	$\frac{\pi}{12}$
$S/\frac{T}{a^2}$	-4.44	-2.27	0.72	4.68	9.66	16.58



$\frac{W/h}{p/Eh^2}$
45×10³
40×10³
35×10³
30×10³
25×10³
20×10³
15×10³
10⁴
5×10³
100
500
1000
1500
2000
a/h
Max Deflection Parameter
of Cylindrical Shell
Vs
Radius/Thickness Ratio

Fig. I

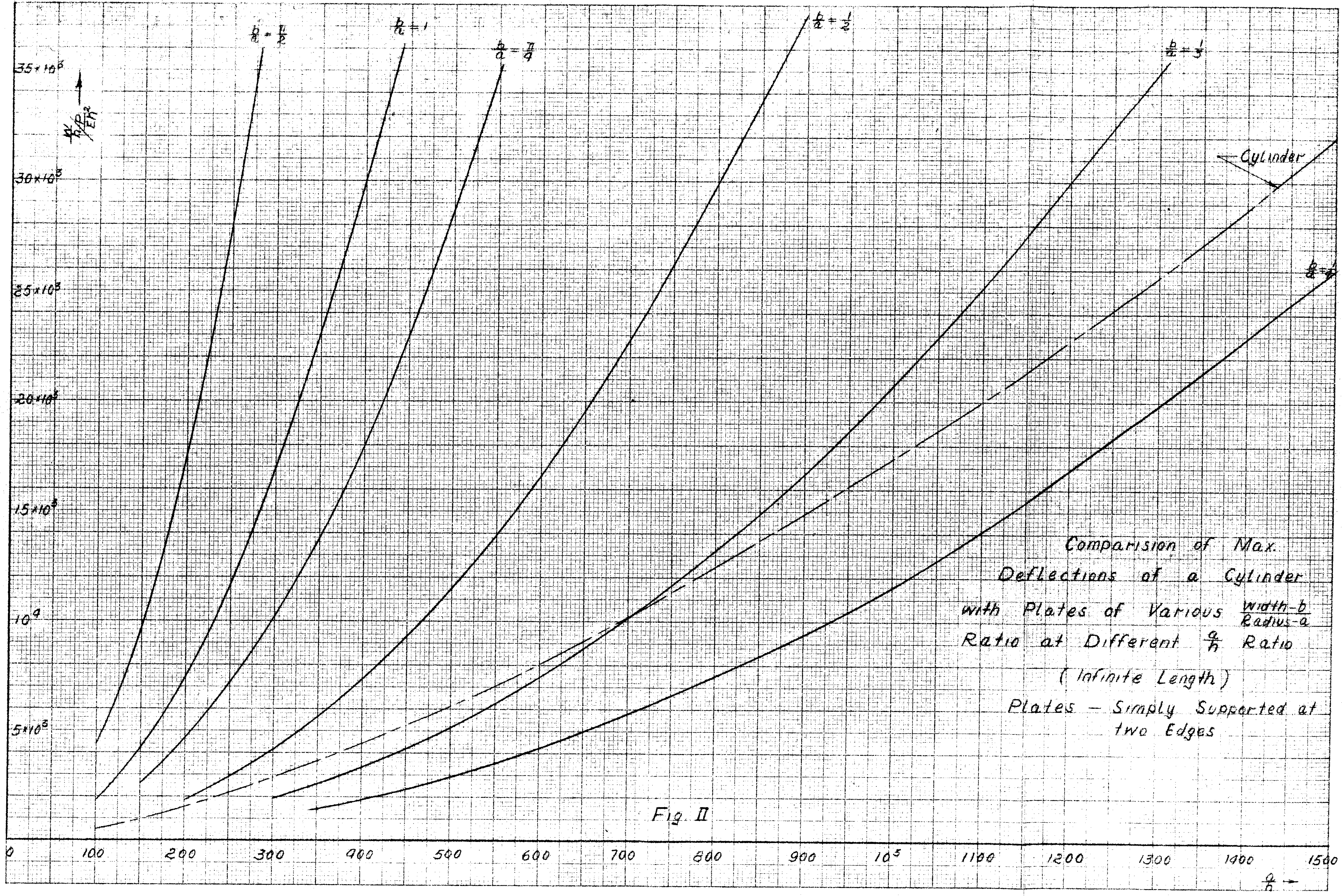

$\frac{b}{a} = \frac{\pi}{2}$
$\frac{b}{a} = 1$
$\frac{b}{a} = \frac{\pi}{4}$
$\frac{b}{a} = \frac{1}{2}$
$\frac{b}{a} = \frac{1}{3}$
$\frac{b}{a} = \frac{1}{4}$
Cylinder
35×10³
30×10³
25×10³
20×10³
15×10³
10⁴
5×10³
0
100
200
300
400
500
600
700
800
900
10³
1100
1200
1300
1400
1500
$\frac{a}{h}$
Comparision of Max.
Deflections of a Cylinder
with Plates of Various $\frac{\text{Width-b}}{\text{Radius-a}}$
Ratio at Different $\frac{a}{h}$ Ratio
(Infinite Length)
Plates – Simply Supported at
two Edges

Fig. II

Deflection Curves of Circular Cylindrical Shells Along the Generatrix (Infinite Length)

$s/a = 0$

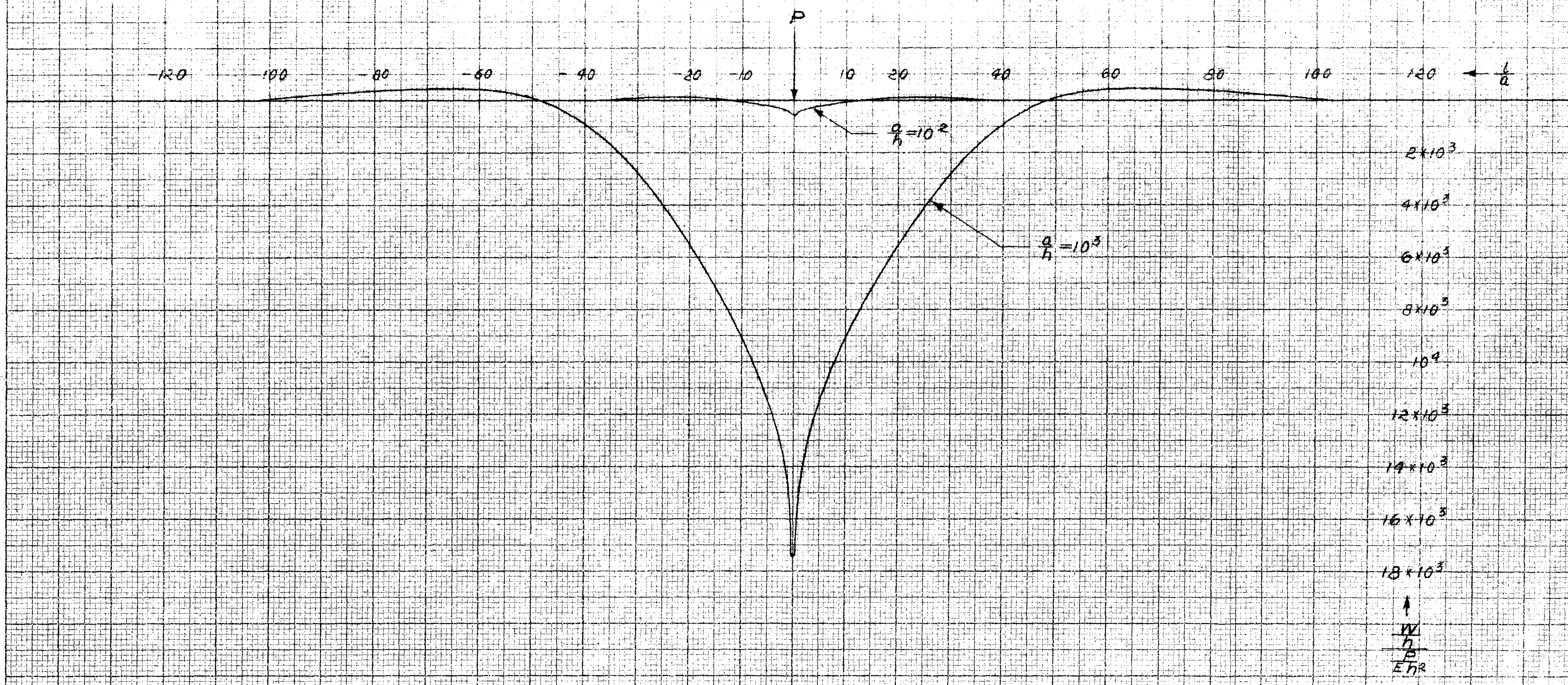
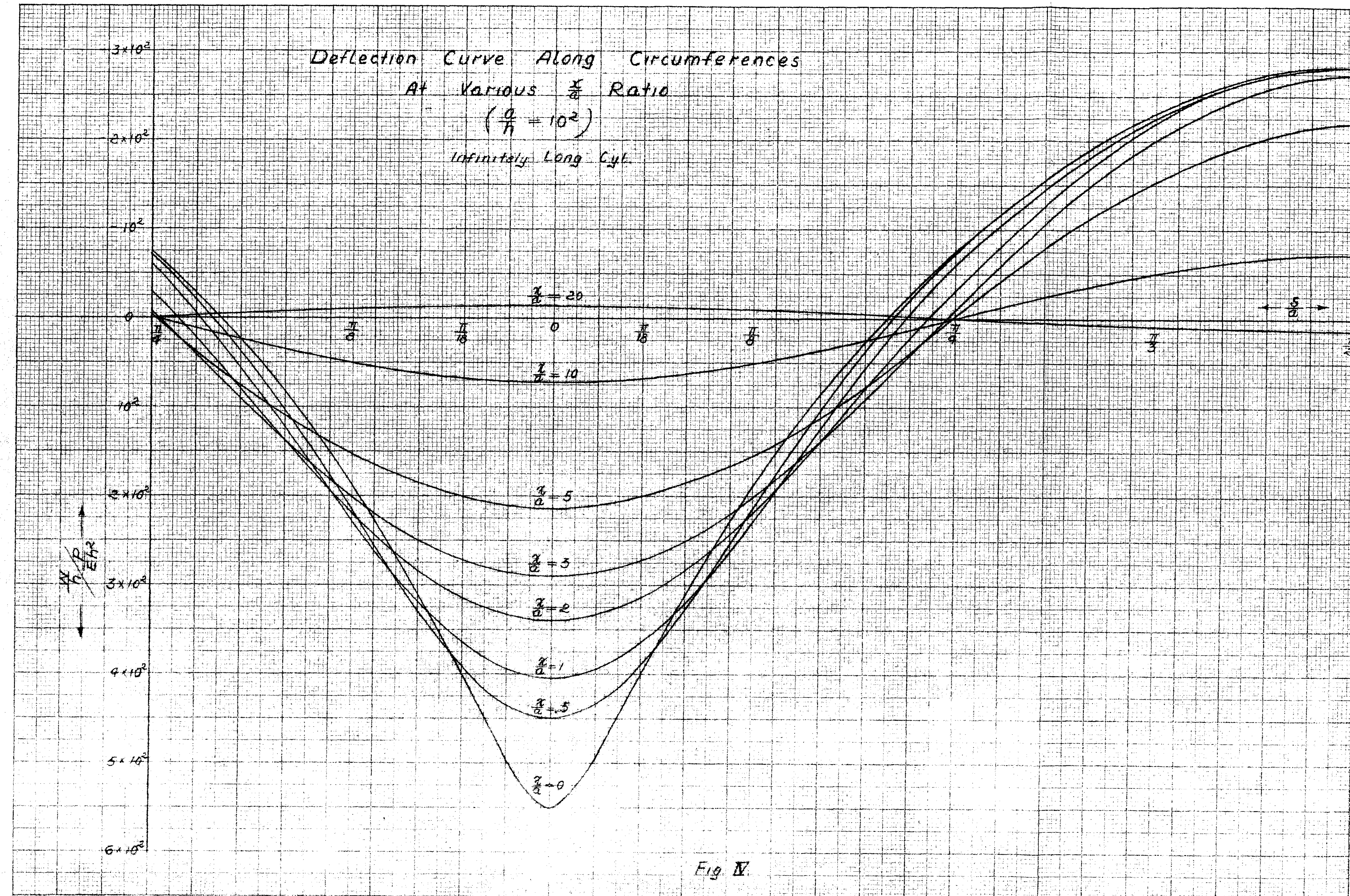


Fig. III



Deflection Curve Along Circumferences
At Various $\frac{x}{a}$ Ratio
$\left(\frac{a}{h} = 10^2\right)$
Infinitely Long Cyl.
3×10^2
2×10^2
-10^2
0
10^2
2×10^2
3×10^2
4×10^2
5×10^2
6×10^2
$\frac{w}{h}\Big/\frac{P}{Eh^2}$
$\frac{\pi}{4}$
$\frac{\pi}{8}$
$\frac{\pi}{16}$
0
$\frac{\pi}{16}$
$\frac{\pi}{8}$
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{\pi}{2}$
$\frac{s}{a}$
$\frac{x}{a} = 20$
$\frac{x}{a} = 10$
$\frac{x}{a} = 5$
$\frac{x}{a} = 3$
$\frac{x}{a} = 2$
$\frac{x}{a} = 1$
$\frac{x}{a} = .5$
$\frac{x}{a} = 0$

Fig. IV

Deflection Curve Along Circumferences At Various $\frac{x}{a}$

$\left(\frac{a}{h} = 10^3\right)$

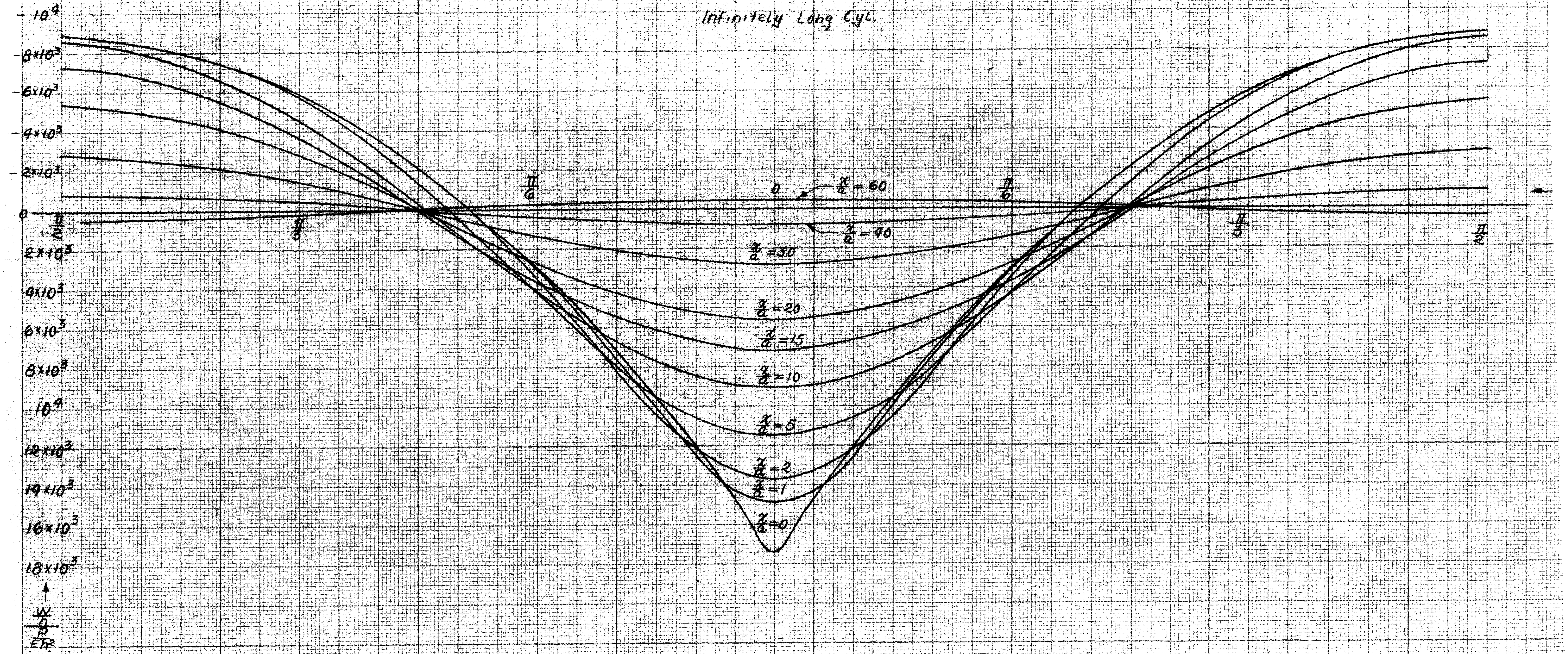


Fig. V

Deflection Curves Along the Generatrices At Various Angles

$\left(\frac{a}{h}=10^2\right)$

∞ cyl.

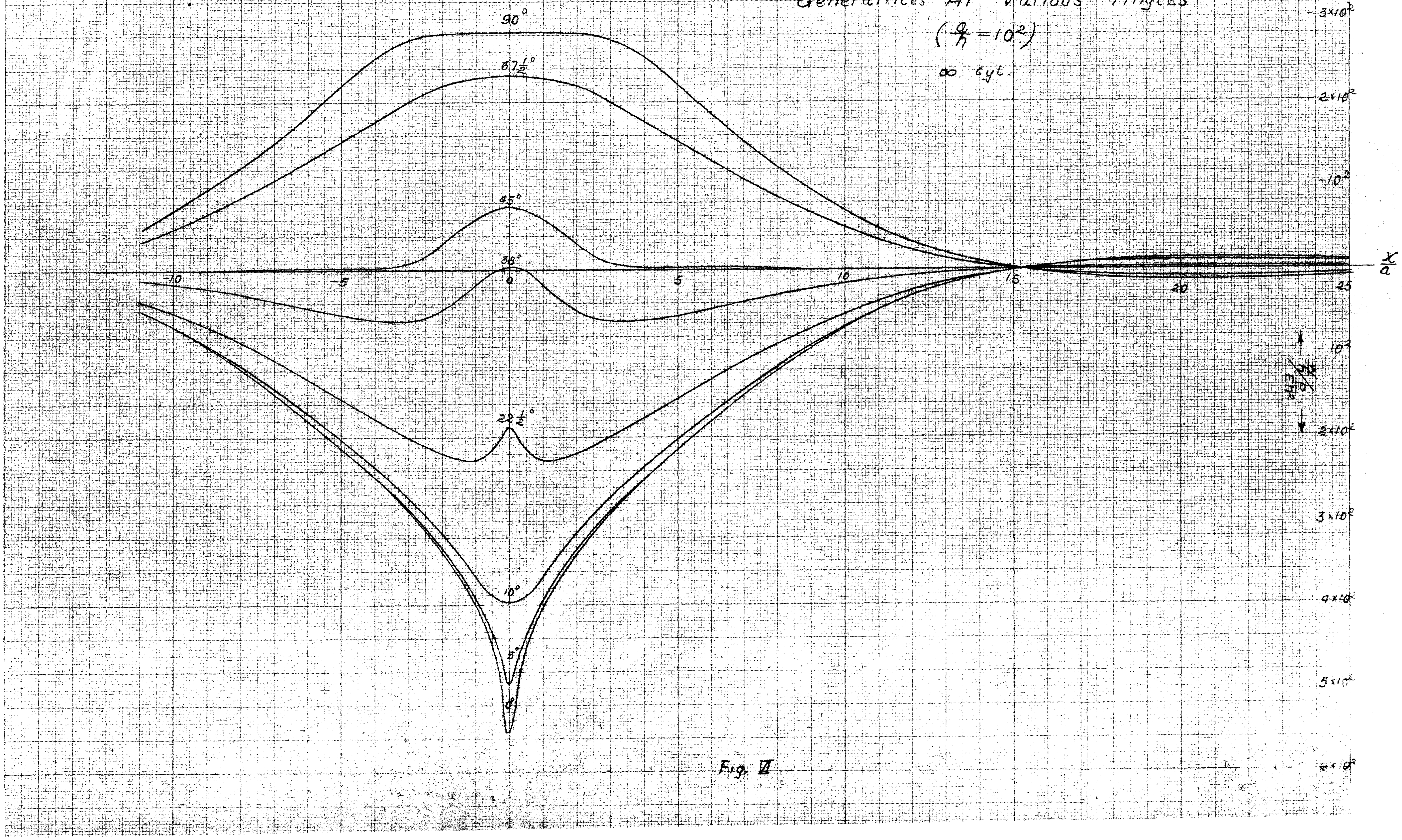


Fig. VI

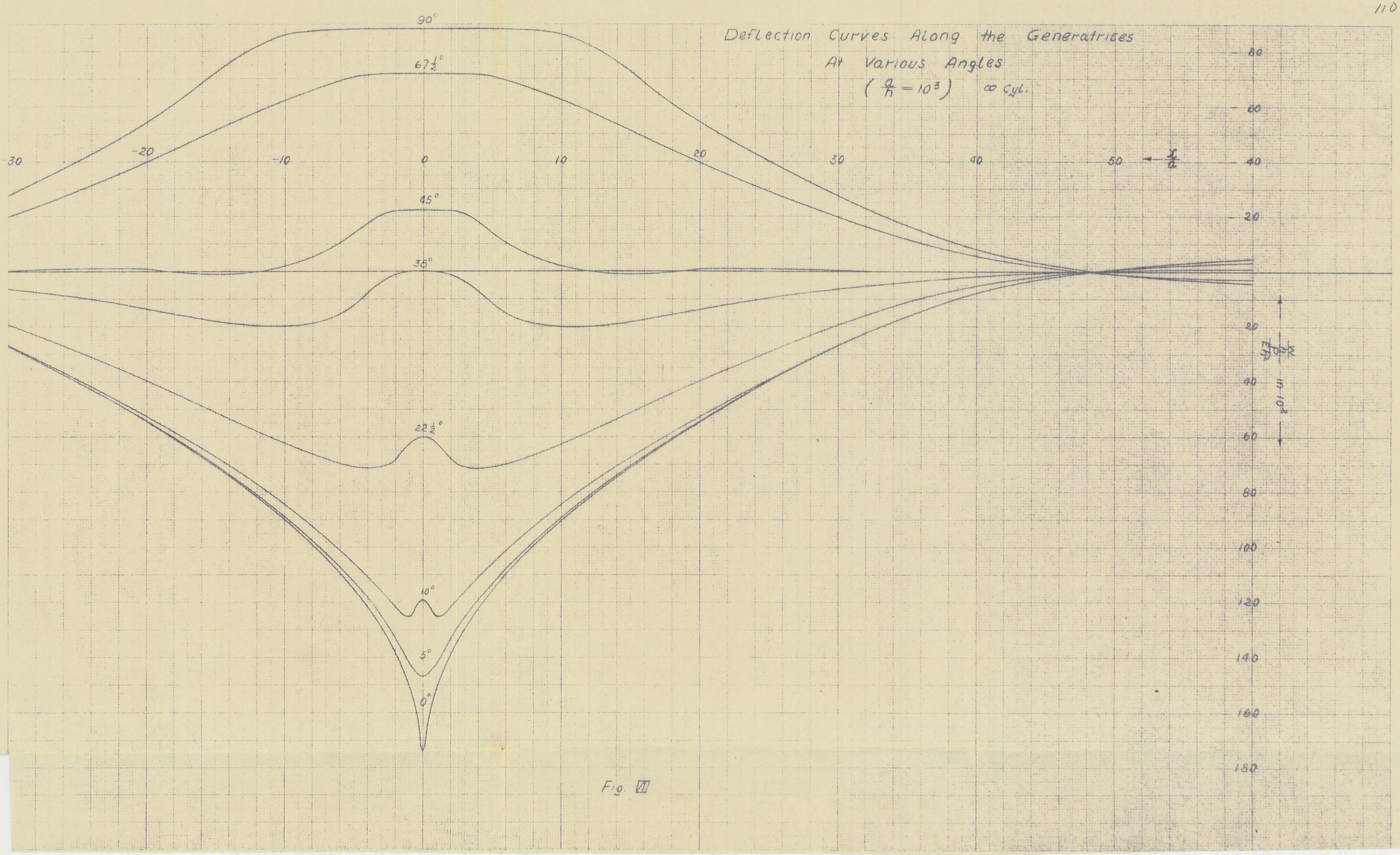

Deflection Curves Along the Generatrices
At Various Angles
$(\frac{a}{h} = 10^3)$ ∞ Cyl.
90°
67½°
45°
30°
22½°
10°
5°
0°
-30
-20
-10
0
10
20
30
40
50
$\frac{x}{a}$
80
60
40
20
20
40
60
80
100
120
140
160
180
$\frac{w}{\frac{p}{Eh^2}}$ in 10^3

Fig. VI

Shape of the Deflection Surface Represented
by the Contour Lines

Infinite Long Thin Cylinder with $\frac{a}{h} = 10^2$

$\left(\lambda = \frac{w}{h}\frac{p}{E h \nu}\right)$

Fig. VIII

Shape of the Deflection Surface Represented
by the Contour Lines

Infinite Long Thin Cylinder with $\frac{a}{h} = 10^3$

$\left(\lambda = \frac{w}{h} \Big/ \frac{p}{Eh^2}\right)$

Fig IX

Deflection Curves of Cylindrical Shells with Different ($\frac{l}{a}$) Ratio Along the Generatrix

$\frac{a}{h} = 10^2$

$\frac{\xi}{a} = 0$

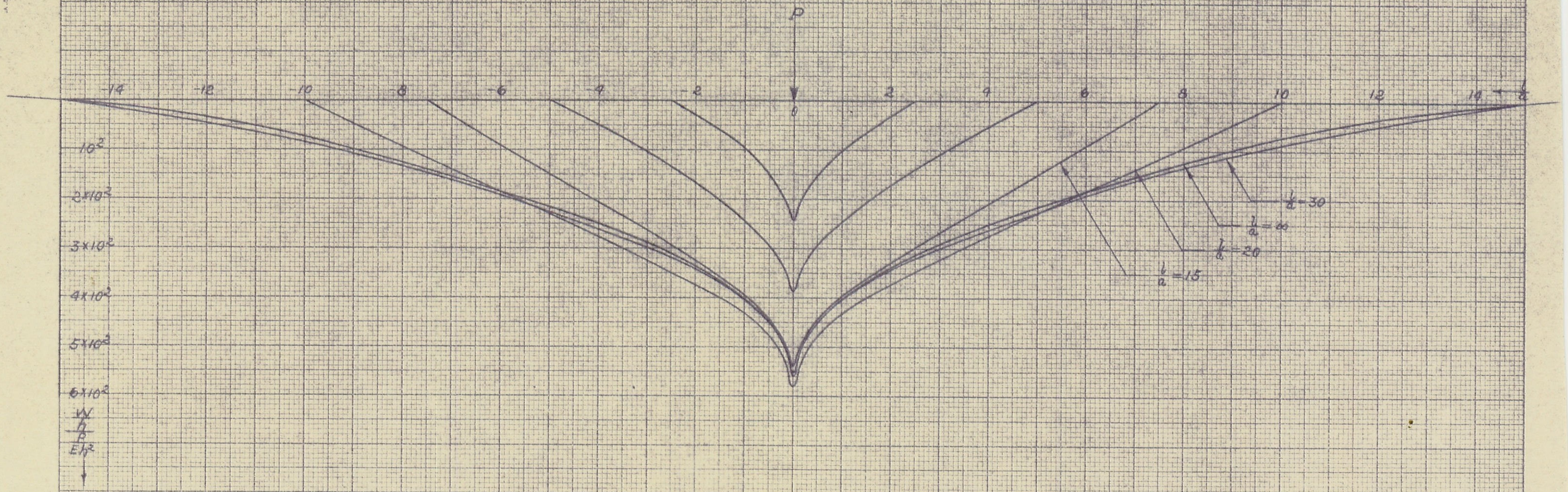


Fig. X

Deflection Curves of Cylindrical Shells with Different ($\frac{l}{a}$) Ratio Along the Generatrix

$\frac{a}{h} = 10^2$

$\frac{s}{a} = 0$

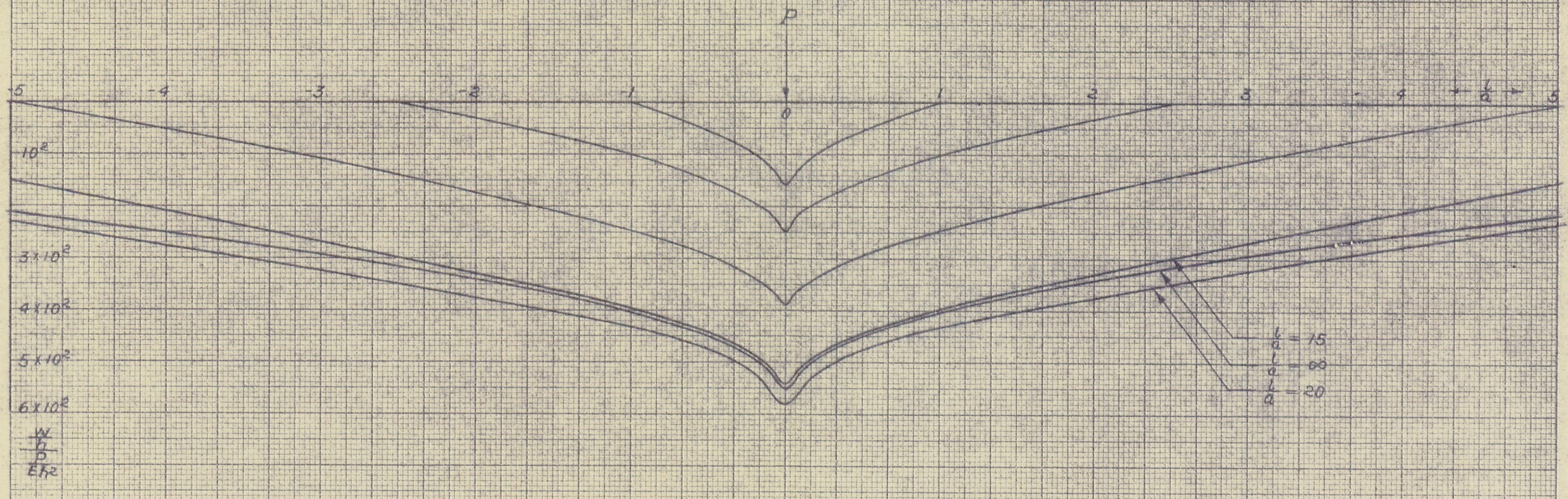


Fig. XI

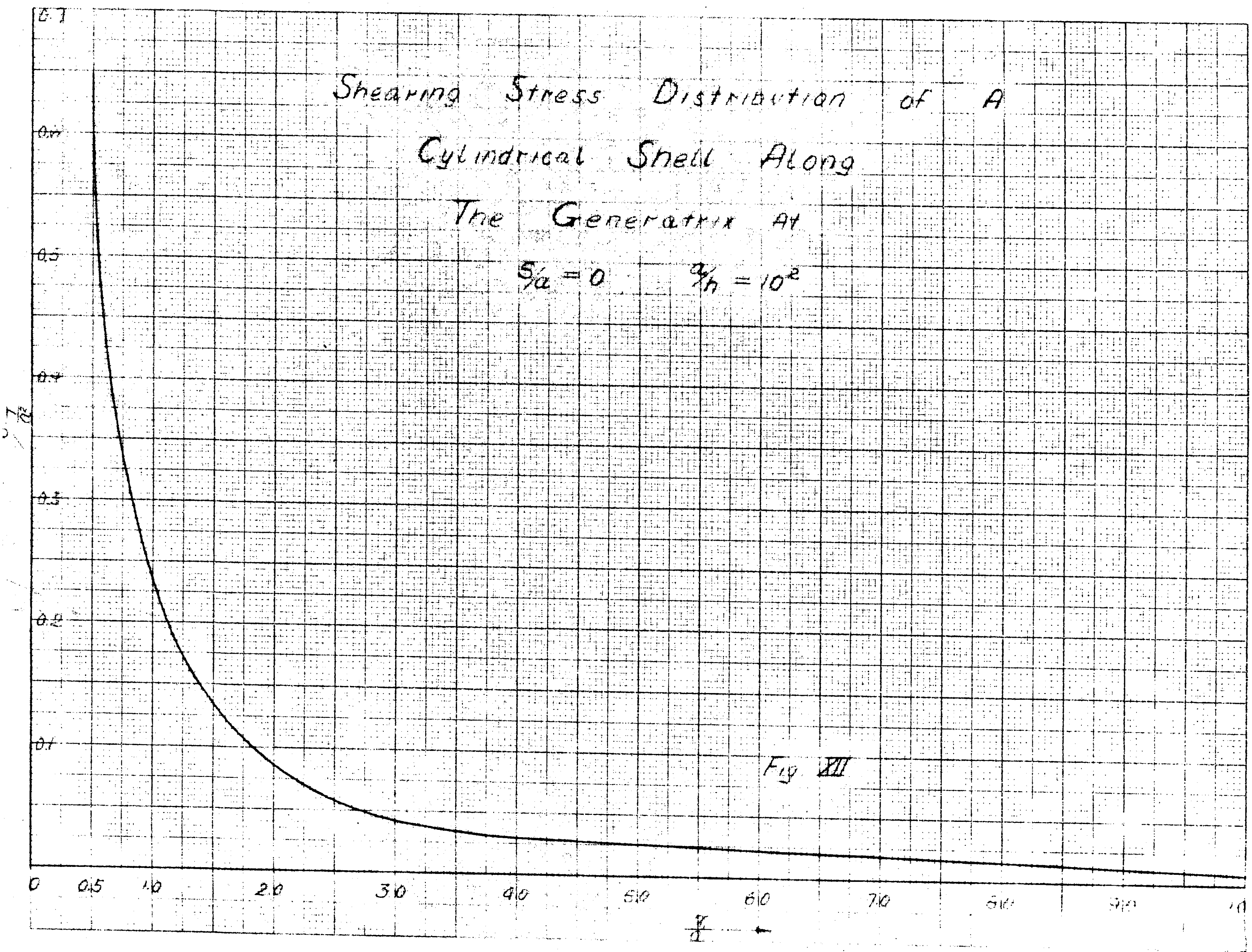

Shearing Stress Distribution of A
Cylindrical Shell Along
The Generatrix At
S/a = 0   a/h = 10^2
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.5
1.0
2.0
3.0
4.0
5.0
6.0
7.0
8.0
9.0
10
x/a

Fig. XII

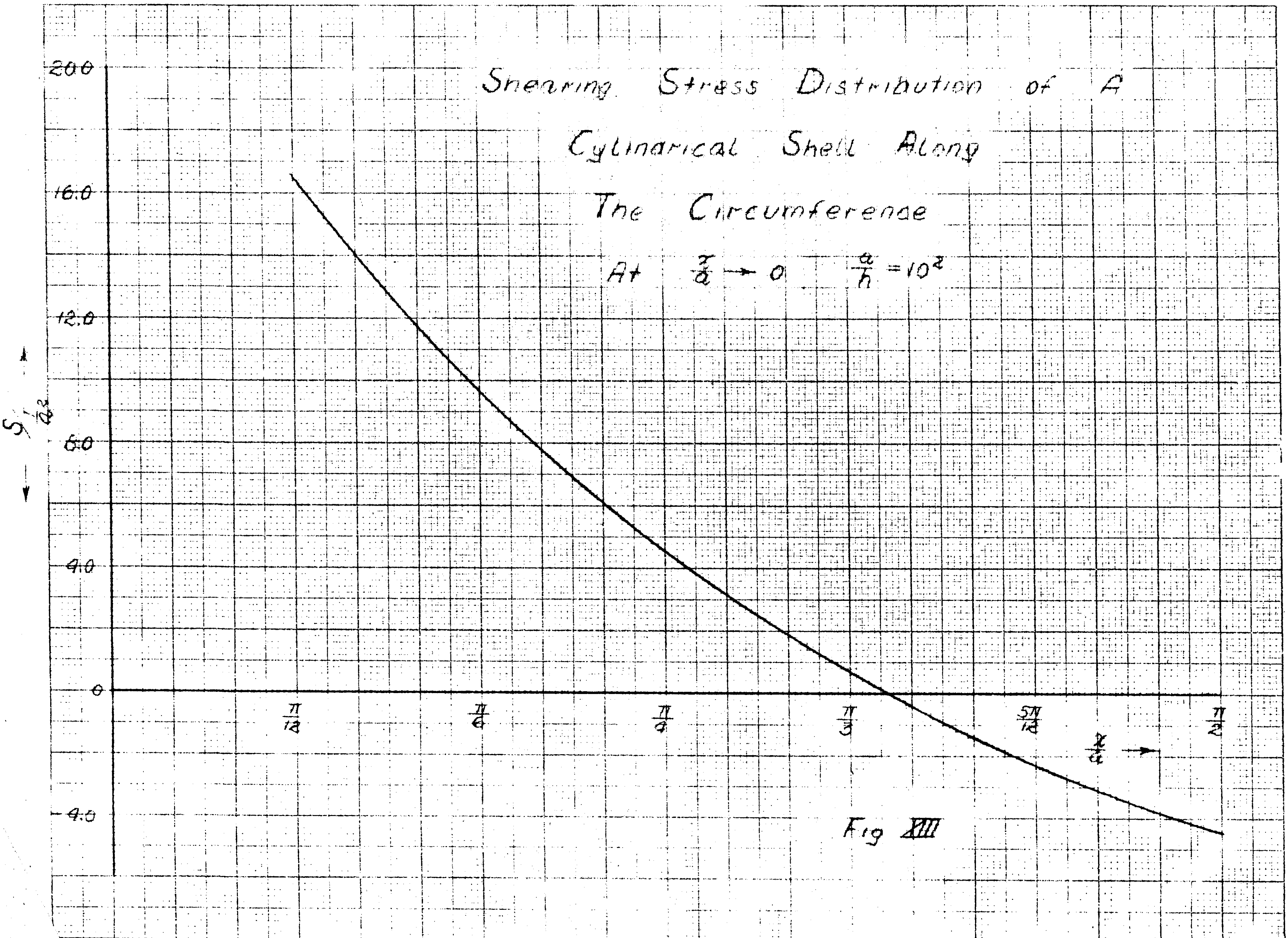

Shearing Stress Distribution of A
Cylindrical Shell Along
The Circumference
At $\frac{x}{a} \to 0$ $\frac{a}{h} = 10^2$
20.0
16.0
12.0
8.0
4.0
0
4.0
S_r / a^2
$\frac{\pi}{12}$
$\frac{\pi}{6}$
$\frac{\pi}{4}$
$\frac{\pi}{3}$
$\frac{5\pi}{12}$
$\frac{\pi}{2}$
$\frac{x}{a}$

Fig XIII

REFERENCES

A. E. H. Love, The Mathematical Theory of Elasticity, (1927), Pages 515 - 536, 565 - 575.

Th. Von Kármán, M. A. Biot, Mathematical Methods in Engineering, Chapter VIII.

S. Timoshenko, Theory of Elasticity, (1934), Chapter II.

S. Timoshenko, Theory of Plates and Shells, (1940), Chapter XI.

H. Lamb, Proc. London Math. Soc., Vol. 21, (1891), P. 119.

L. H. Donnell, Stability of Thin-Walled Tubes under Torsion, P.12, N.A.C.A. Report No. 479, 1933

L. H. Donnell, A Discussion of Thin Shell Theory, Proceedings of the Fifth Int'l Congress of Applied Mechanics, 1938.

E. Reissner, A New Derivation of the Equations for the Deformation of Elastic Shell, Amer. Jour. of Math., 63, P.177, 1941.

W. Flügge, Statik und Dynamik der Schalen, J. Springer, Berlin, 1934.

C. B. Biezeno, R. Grammel, Technische Dynamik, J. Springer, Berlin 1939.

K. Miesel, Ingenieur-Archiv, Vol. 1, P.29, 1929.

U. Finsterwalder, Ingenieur-Archiv, Vol. 4, P.43, 1933.

A. Nadai, Z. Angew Math. Mech., Vol. 6, P.1, 1922

M. T. Huber, Z. Angew Math. Mech., Vol. 6, P.228

E. C. Titchmarsh, Introduction to the Theory of Fourier Integrals, Chapter I.

W. E. Byerly, An Elementary Treatise on Fourier Series and Spherical, Cylindrical and Ellipsoidal Harmonies.

H. S. Carslaw, Theory of Fourier Series and Integrals.

E. T. Whittaker, G. N. Watson, A Course of Modern Analysis, Chapter VI.