THIN CYLINDRICAL SHELLS SUBJECTED TO VARIOUS TYPES OF CONCENTRATED LOADS

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The bending of thin cylinder shells based on the general theory of elasticity is of interest not only to the mathematician but also to the engineer. The general theory of the shells has recently been developed to the point that it is now being used by engineers as a basis for the design of this type of structure.

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SUMMARY

It is shown that the calculation of stresses and strains in thin cylindrical shells subjected to various kinds of concentrated loads can be obtained if the specified loading function is represented by a Fourier integral in the longitudinal direction and by a Fourier series in the circumferential direction. The components of displacements are represented in like manner. In Part I an infinitely long cylinder loaded with two equal and opposite forces acting at the ends of a vertical diameter is discussed. The expression for the radial deflection in a thin cylinder of finite length was obtained from the corresponding solution for an infinitely long cylinder by using the method of images. The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference were also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The application of the solution for the radial deflection in an infinitely long cylinder to an elastically supported flat plate, or beam, in order to find the modulus of foundation is briefly discussed. Part II deals with an infinitely long cylinder subjected to two equal and opposite torque acting about the radial axis on the surface of the shell. The shearing stress-resultant was determined and the torque produced by the force-resultants is verified mathematically equal to the applied torque. A method was developed to calculate the shearing stress distribution near the z-axis because the series in the solution converges very slowly when \( x \) approaches zero.
INTRODUCTION

The subject matter of this paper is the calculation of stresses and strains in thin cylindrical shells subjected to various kinds of concentrated loads, and more specifically in infinitely long cylinders. The problem of curved plates or shells was first treated in about the year 1874 by H. Aron from the point of view of general equations of elasticity. He expressed the equations of the middle surface by means of two parameters, somewhat as Gauss did, and he adapted the problem to the method which had been used for plates. He arrived at expressions for the potential energy of the strained shell, similar to the expressions developed by Kirchhoff for plates, but the quantities that define the curvature of the middle surface were replaced by the differences of their values in the strained and unstrained state.

Lord Rayleigh proposed a theory for vibrating shells which embodied the idea that the middle surface of the vibrating body does not undergo any extension, and he determined the displacement of a point of the middle surface in accordance with this condition. Later it was shown that contrary to this condition a vibrating shell undergoes extensional strain. However, the region of this extensional strain is confined to a narrow strip near the edge of the shell, and the greater part of the shell vibrates according to Lord Rayleigh's assumptions.

The inextensional deformation of cylindrical and spherical shells was treated in detail by Lord Rayleigh in his "Theory of Sound." This type of
deformation is the assumption underlying the solution of many problems of practical importance, such as the determination of stresses in thin cylindrical shells subjected to two equal and opposite forces acting at the ends of a diameter, or to internal hydrostatic pressure. The results obtained in the case of the first problem indicate that inextensional deformations correspond only to a first approximation of the complete solution, and the stresses in the proximity of the points of application of the forces are not given with sufficient accuracy. For this reason a method is needed which can deal with the case when the cylindrical shell is so loaded that its middle surface undergoes extension as well as change of curvature. Under these circumstances it is thought best to resort to the general theory of thin plates and shells from which special cases like that of the cylindrical or spherical shell may be derived.

The application of the method of series to problems of equilibrium of elastic solid bodies was initiated by Lamé and Clapeyron. They considered the case of a body bounded by an unlimited plane to which pressure is applied according to an arbitrary law. However, previous to the discovery of the general equations of elasticity this method had already been used in problems of astronomy, acoustics, and heat conduction. In the above mentioned problem of Lamé and Clapeyron, the solutions of the differential equations of equilibrium can be expressed by definite integrals, the elements of the integrals representing the effects of singularities distributed over the surface. This class of solutions constitutes an extension of the methods

introduced by Green in the Theory of the Potential. The method of singularities was first applied to the theory of elasticity by E. Betti, who set out from a reciprocal theorem of the type that is now familiar in many branches of mathematical physics. The average strain of any type that is produced in a body by given forces can be determined by a formula incidentally deduced from this theorem. Furthermore, Lord Kelvin gave the fundamental particular solution which expresses the displacement due to a force at a point in an indefinitely extended solid.

The present paper consists of two parts. In Part I an indefinitely long cylinder loaded with equal and opposite forces acting at the ends of a vertical diameter is discussed. The equations of equilibrium of an element of a cylindrical shell undergoing small displacements due to a lateral distributed external load are reduced to a single differential equation of the eighth order in the radial displacement. In this single equation all terms could be compared on a common ground and it was possible to decide which terms could be safely neglected. The specified loading function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. The Fourier coefficients and the undetermined function in the Fourier integral were determined from the loading condition which represents a concentrated load. The radial displacement is represented in a like manner with the aid of an undetermined function which was obtained by substituting both radial displacement and loading expressions in the differential equations. The definite integrals involved in the expression for radial deflection were evaluated by means of Cauchy's Theorem of Residues.

The expression for the radial deflection in a thin cylinder of finite
length was obtained from the corresponding solution for an infinitely long cylinder by using the method of images. It is seen that the difference of these two radial deflections can be given by a correction factor included in the expression for a cylinder of finite length. This difference is believed to be caused by the restrained edges at the two ends of the finite cylinder.

The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference were also analyzed by using the corresponding solution for the radial deflection under a concentrated load. The action of the couple is equivalent to that of two equal and opposite forces acting at an infinitely small distance apart. It is believed that the solution for the radial deflection in an infinitely long cylinder may also be applied in order to find the modulus of foundation of an elastically supported flat plate, or beam, under a concentrated transverse load.

Part II deals with an infinitely long cylindrical shell subjected to two equal and opposite torques acting about the radial axis on the surface of the shell. The solution of this problem was achieved by replacing the torque with two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other. The components of the displacements \( u, v \), and \( w \) and the loading functions are represented similarly as in Part I.

The shearing stress-resultant was determined from the solutions of the displacements \( u \) and \( v \). In order to verify that the total torque produced by the force-resultants is equal to the applied torque all the force-resultants multiplied by their corresponding moment arms were summed up. The result proved to be satisfactory both in rectangular and polar coordinates.
In the numerical calculation of the shearing stress distribution near the s-axis (from the expression derived) it was found that ordinary methods failed because the series converges very slowly when x approaches zero. Hence a powerful method was developed in order to overcome this difficulty.

It is thought that this solution may have some application to the practical problem of the shearing stress distribution on the skin of a fuselage affected by the distortion of the wings.

A discussion of the curves plotted in the Appendix may be found in the Conclusion.
PART I

THIN CYLINDRICAL SHELL LOADED

WITH TWO EQUAL AND OPPOSITE FORCES

1. FUNDAMENTAL EQUATIONS

The fundamental equations of a cylindrical shell under the specified
loading are obtained from considerations of the equilibrium of an element
cut out of the shell such as shown in Fig. 1*.

Fig. 1

Notations and conventions are the same as those commonly used in the three
dimensional theory of elasticity. They are shown in Fig. 1. Symbols represent-
ing external forces have the same subscripts as used by Love in his book
entitled "Mathematical Theory of Elasticity". Moments and rotations are
represented by vectors corresponding to the right-hand rule. The positive

* A shell of double curvature.
sense of displacements, external forces and moments, and of internal forces
and moments, is the same as the positive sense of the coordinate axes, if
the external normal of the element concerned points in the positive direction
of a coordinate axis. If it points in the negative direction, the opposite
sign rule holds.

Definitions of some of the symbols follow:

h ---- Thickness of the cylindrical shell.
a ---- Radius of the cylindrical shell.
\( \varphi \) ---- Angle measured counter-clockwise from a line extending downwards
from the center of the cylindrical shell to any point of the shell.

x, s, z ---- Longitudinal, circumferential and radial coordinates, meas-
ered axially from the normal section at the middle of cylindrical
shell, and circumferentially from same generatrix.

u, v, w ---- Longitudinal, circumferential and normal displacements of
points in the middle surface of the wall.

E ---- Modulus of elasticity of the material.

\( \nu \) ---- Poisson’s ratio.

D ---- Flexural rigidity

\[
D = \frac{E h^3}{12(1-\nu^2)}
\]

P ---- External concentrated load.

q ---- External distributed load.

\( \lambda, \mu \) ---- Number of axial and circumferential waves.
In Fig. 1* $T_1$, $S_1$, $N_1$ are stress-resultant measured per unit length of the middle surface acting on the side $\alpha$ in the negative direction of the $x$, $y$, $z$ axes. The length of the element on which $T_1$, $S_1$, $N_1$ act is $B\delta\beta$. The stress-resultants acting on the side $\alpha + \delta\alpha$ are $T_1'$, $S_1'$, $N_1'$. Similarly $T_2$, $S_2$, $N_2$ are stress-resultants in the direction of the $x$, $y$, $z$ axes acting on the side $\beta$ on an element of length $A\delta\alpha$, and $T_2'$, $S_2'$, $N_2'$ are stress-resultants acting on the side $\beta + \delta\beta$. The axes of the stress couples $H_1$, $G_1$ have the same directions as $T_1$, $S_1$ while those of $H_2$, $G_2$ have the same directions as $T_2$, $S_2$.

Since $B\delta\beta$ is the length of the side of the rectangle on which $T_1$ acts the stress resultant on side $\alpha$ in the direction of the $x$-axis is:

$$-T_1 B\delta\beta$$

This is so since all stress resultants and couples are measured per unit length of the middle surface of the shell. In resolving the forces that act on the side $\alpha + \delta\alpha$ parallel to the $x$-axis we have to allow for the change of $\alpha$ into $\alpha + \delta\alpha$, and for small rotations ($p', \delta\alpha', q', \delta\alpha'$, $r', \delta\alpha'$). Hence the components parallel to the $x$-axis of the force acting on side $\alpha + \delta\alpha$ are

$$T_1 B\delta\beta + N_1 B\delta\beta \cdot q' \delta\alpha - S_1 B\delta\beta \cdot r' \delta\alpha + \delta\alpha \frac{\partial}{\partial \alpha} (T_1 B\delta\beta)$$

Similarly the force parallel to $x$ acting on side $\beta$ is:

$$-S_2 A\delta\alpha$$

and that parallel to $x$ acting on side $\beta + \delta\beta$ is:

$$S_2 A\delta\alpha + N_2 A\delta\alpha \cdot q' \delta\beta - T_2 A\delta\alpha \cdot r' \delta\beta + \delta\beta \frac{\partial}{\partial \beta} (S_2 A\delta\alpha) \delta\beta$$

If $X^*$ is the $x$-component of the external force per unit area of the middle surface, then, since the area of the curvilinear rectangle of Fig. 1 is $AB \delta \alpha \delta \beta$, all the above enumerated forces add up to

$$
\frac{\partial}{\partial \alpha} (T B \delta \beta) \delta \alpha + \frac{\partial}{\partial \beta} (S_z A \delta \alpha) \delta \beta - r_1 S_z B \delta \alpha \delta \beta - r_2 T_z A \delta \alpha \delta \beta + q'_N N_z A \delta \alpha \delta \beta + AB X' \delta \alpha \delta \beta = 0
$$

Dividing through by $\delta \alpha \delta \beta$ one obtains

$$
\frac{\partial(T B)}{\partial \alpha} + \frac{\partial(S_z A)}{\partial \beta} - (r_1 S_z B + r_2 T_z A) + (q'_N N_z A) + AB X' = 0.
$$

Furthermore, in the case of a cylinder the following relations hold (see Fig. 2)

$$
\alpha = x, \quad \beta = \varphi, \quad B = \alpha
$$

$$
A = 1, \quad X' = 0, \quad \frac{1}{R_i} = 0
$$

$$
\frac{1}{R_i} = \frac{1}{\alpha}
$$

and therefore:

$$
\begin{align*}
q'_1 &= \frac{\partial^2 n}{\partial x^2} \quad & \quad r_1' &= \frac{\partial}{\partial x} \frac{\partial v}{\partial \varphi} - \frac{\partial w}{\partial \varphi}, \\
q'_2 &= \frac{\partial^2 n}{\partial x^2} \quad & \quad q'_2 &= \frac{\partial^2 w}{\partial x \partial \varphi} + \frac{\partial v}{\partial x}.
\end{align*}
$$

It will be shown subsequently that:

\[ S_1 = 6 \frac{(1 - \nu)}{h^2} D \left( \frac{\partial^2 v}{\partial x^2} + \frac{1}{a} \frac{\partial u}{\partial \phi} \frac{\partial^2 v}{\partial x^2} \right) \]

Then for instance the term \( \frac{3}{2} s_1 b_i \) in eq. (a) above becomes

\[ 6 \frac{1 - \nu}{h^2} D \left( \frac{\partial^2 v}{\partial x^2} \frac{\partial^2 v}{\partial x^2} + \frac{1}{a} \frac{\partial u}{\partial \phi} \frac{\partial^2 v}{\partial x^2} \right) a \]

Since the displacements are small, the effect of the angular changes upon the components of the forces is of a second order of smallness and can be neglected. Therefore, after simplification Eq. (a) can be written:

\[ a \frac{\partial T}{\partial x} + a \frac{\partial S}{\partial \phi} = 0 \]

which is one of the simplified equations of equilibrium. In an analogous manner one may obtain the other two equations of equilibrium of forces, and the three equations of equilibrium of moments. Consequently, for the equilibrium of forces we have the following three equations:

\[ \begin{align*}
    a \frac{\partial T_1}{\partial x} + a \frac{\partial S_1}{\partial \phi} &= 0 \\
    \frac{\partial T_2}{\partial \phi} + a \frac{\partial S_2}{\partial x} &= 0 \\
    a \frac{\partial N_1}{\partial x} + a \frac{\partial N_2}{\partial \phi} + T_2 + qa &= 0
\end{align*} \]

and for the equilibrium of moments

\[ \begin{align*}
    a \frac{\partial H_1}{\partial x} + a \frac{\partial S_2}{\partial \phi} + N_2 a &= 0 \\
    a \frac{\partial S_1}{\partial x} + a \frac{\partial H_2}{\partial \phi} - N_1 a &= 0 \\
    (S_1 + S_2) a &= 0
\end{align*} \]
in which \( q \) is the normal pressure on the element.

Using the first approximation \(^1\) in which certain small components of displacement and their differentials with respect to \( \alpha \) and \( \beta \) as well as products of certain other small quantities are omitted, we have:

\[
\begin{align*}
S &= S_2 = S_1 \\
H &= -H_2 = H_1
\end{align*}
\]

Substituting from (2) into (1) and making use of (3) we get:

\[
\begin{align*}
\alpha \frac{\partial T_x}{\partial x} + \frac{\partial S}{\partial \varphi} &= 0 \\
\frac{\partial T_z}{\partial \varphi} + a \frac{\partial S}{\partial x} &= 0 \\
-\frac{\alpha}{a} \frac{\partial^2 H}{\partial \varphi \partial x} + \frac{\partial^2 G_z}{\partial x^2} + \frac{\partial^2 G_z}{\alpha^2 \partial \varphi^2} + \frac{T_z}{a} + q &= 0
\end{align*}
\]

These three equations of equilibrium combine the six equations (1) and (2).

We shall transform Eq. (4) with the aid of the relation \(^2\) between the stress resultents and the deformations:

\[
\begin{align*}
T_x &= \frac{Eh}{1 - \nu^2} (\epsilon_x + \nu \epsilon_z) \\
T_z &= -\frac{Eh}{1 - \nu^2} (\epsilon_z + \nu \epsilon_x) \\
S_1 &= S_2 = S = \frac{r Eh}{2(1 + \nu)} \\
G_i &= -D(K_i + \nu K_z) \\
G_z &= -D(K_z + \nu K_i) \\
H &= H_1 = -H_2 = D(1 - \nu) T
\end{align*}
\]

2. Ibid., Page 530.
Where D is the flexural rigidity.

Moreover, the extensional and flexural strains in the middle surface are:

\[ \varepsilon_1 = \frac{\partial u}{\partial x}, \quad \varepsilon_2 = \frac{1}{a} \frac{\partial v}{\partial \phi}, \quad \gamma = \frac{\partial v}{\partial x} + \frac{1}{a} \frac{\partial u}{\partial \phi} \]

\[ K_1 = \frac{\partial^2 w}{\partial x^2}, \quad K_2 = \frac{1}{a^2} \frac{\partial^2 w}{\partial \phi^2}, \quad \tau = \frac{\partial^2 w}{a \partial x \partial \phi} \]

Hence, equations (4) can be put in the form of three equations with the three unknowns \( u, v, w \):

\[
\begin{align*}
\frac{\partial^4 u}{\partial x^4} + \frac{1 + \nu}{2} \frac{\partial^4 v}{\partial x^2 \partial \phi^2} - \frac{\nu}{a} \frac{\partial^2 w}{\partial x^2} + \frac{1 - \nu}{2} \frac{\partial^4 u}{\partial x^4} &= 0 \\
\frac{\partial^4 v}{\partial x^4} + \frac{1 + \nu}{2} \frac{\partial^4 u}{\partial x^2 \partial \phi^2} + \frac{1 - \nu}{2} \frac{\partial^4 v}{\partial x^4} - \frac{1}{a} \frac{\partial^2 w}{\partial \phi^2} &= 0 \\
\frac{1}{2} \nabla^2 w - \frac{1}{a} \left( \frac{3 \partial v}{\partial \phi} - \frac{w}{a} + \nu \frac{\partial u}{\partial x} \right) - \frac{1 - \nu}{\frac{a}{E} h} q &= 0
\end{align*}
\]

where \( S = a \phi \).

In order to solve the simultaneous equations (5) we can apply first the operation \( \frac{\partial^4}{\partial x^4} \), and then \( \frac{\partial^4}{\partial x^2 \partial \phi^2} \) to eq. (5:1). Solving in each case for the term containing \( v \), and substituting these expressions in the equation obtained by applying \( \frac{\partial^4}{\partial x^2 \partial \phi^2} \) to (5:2), we obtain an equation from which \( v \) has been eliminated.

The application of \( \frac{\partial^4}{\partial x^4} \) to (5:1) gives

\[
\frac{\partial^4 u}{\partial x^4} + \frac{1 - \nu}{2} \frac{\partial^4 u}{\partial x^2 \partial \phi^2} + \frac{1 + \nu}{2} \frac{\partial^4 v}{\partial x^4} - \frac{\nu}{a} \frac{\partial^2 w}{\partial x^2} = 0
\]  

(5:3)

The application of \( \frac{\partial^4}{\partial x^2 \partial \phi^2} \) to (5:1) gives

\[
\frac{\partial^4 u}{\partial \phi^2 \partial x^4} + \frac{1 - \nu}{2} \frac{\partial^4 u}{\partial \phi^2 \partial x^2} + \frac{1 + \nu}{2} \frac{\partial^4 v}{\partial \phi^2 \partial x^4} - \frac{\nu}{a} \frac{\partial^2 w}{\partial x \partial \phi^2} = 0
\]  

(5:4)

The application of \( \frac{\partial^3}{\partial x \partial s^3} \) to (5.2) gives
\[
\frac{1 + \nu}{2} \frac{\partial^4 u}{\partial s^4 \partial x^2} + \frac{\partial^4 v}{\partial s^4 \partial x} + \frac{1 - \nu}{2} \frac{\partial^4 v}{\partial x^2 \partial s^4} - \frac{1}{\alpha} \frac{\partial^3 w}{\partial s^3 \partial x} = 0
\] (6.6)

Substitutions from eqs. (6.6) and (6.6) into eq. (6.6) lead to
\[
\alpha \nabla^4 u = \nu \frac{\partial^3 w}{\partial x^3 \partial s} - \frac{\partial^3 v}{\partial x^2 \partial s^2}
\] (6.7)

Similarly, applying \( \frac{\partial^3}{\partial x^3} \) and \( \frac{\partial^3}{\partial s^3} \) to (5.3) and solving for the terms containing \( u \), and substituting in (5.1), after applying \( \frac{\partial^3}{\partial x^3 \partial s} \) to it, we obtain an equation from which \( u \) has been eliminated.
\[
\alpha \nabla^4 v = (2 + \nu) \frac{\partial^3 w}{\partial x^3 \partial s} + \frac{\partial^3 w}{\partial s^3}
\] (6.8)

Now, applying \( \frac{\partial}{\partial x} \) to (6.7) and \( \frac{\partial}{\partial s} \) to (6.8) and substituting these two equations into eq. (5.3), after applying \( \nabla^2 \) to it, we obtain an equation from which both \( u \) and \( v \) have been eliminated.
\[
\nabla^6 w + \frac{12(1 - \nu^2)}{\alpha^2 n^4} \frac{\partial^4 w}{\partial x^4} - \frac{1}{D} \nabla^2 q = 0
\] (7)

Eq. (7) differs from the differential equation of the flat plate only by the second term. The flat plate equation can be obtained from eq. (7) by the substitution of \( \alpha = \infty \). Consequently, this second term represents the effect of curvature in the problem of the cylindrical shell.
2. INFINITELY LONG CYLINDER LOADED

WITH TWO EQUAL AND OPPOSITE FORCES

The above equation will now be applied to an infinitely long thin cylinder loaded as shown in Fig. 3, i.e., by two equal and opposite compressive forces acting at the ends of a vertical diameter.

The difficulties of integrating Eq. (7) for this type of loading can be circumvented by replacing the concentrated force $P$ with a function $q$ of both the longitudinal and circumferential coordinates. This is possible if the function is represented by a Fourier integral in the longitudinal direction, and by a Fourier series in the circumferential direction. Since $q$ is an even function of both $x$ and $s$ it can be expressed by

$$q(x, s) = \left[ \frac{q_s}{2} + \sum_{n=2, 4, \ldots} q_n \cos \frac{ns}{a} \right] \int_0^\infty f(\lambda) \cos \frac{\lambda x}{a} \, d\lambda \quad (8:a)$$
Furthermore, the displacement \( w \) can be expressed in a similar manner with the aid of an undetermined function \( w(\lambda) \):

\[
w = \sum_{n=0}^{\infty} \cos \frac{na}{\lambda} \int_0^\infty w(\lambda) \cos \frac{dx}{\lambda} d\lambda
\]

(8.b)

It can be shown that the above expression for \( w \) satisfies the following requirements: (1) The deflection and moment are continuous. (2) The slope of the deflection curve vanishes at the point where the load is applied. (3) The deflection vanishes at infinity. Substituting Eqs. (8.a) and (8.b) in the differential equation (7), we obtain the following relations:

For \( n = 0 \)

\[
\int_0^\infty \left\{ w(\lambda) \left[ \left( \frac{\lambda}{a} \right)^2 + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4 \right] \right. - \frac{q_0}{2D} f(\lambda) \left( \frac{\lambda}{a} \right)^4 \left\} \cos \frac{dx}{\lambda} d\lambda = 0
\]

For all values of \( \lambda \), or

\[
w(\lambda) = \frac{\frac{q_0}{2D} f(\lambda) \left( \frac{\lambda}{a} \right)^4}{\left( \frac{\lambda}{a} \right)^4 \left[ \left( \frac{\lambda}{a} \right)^2 + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4 \right] + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4} = \frac{\frac{q_0}{2D} f(\lambda)}{\left( \frac{\lambda}{a} \right)^4 \left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n\rho}{a} \right)^2 \right] + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4}
\]

For \( n = 2, 4 \ldots \)

\[
w(\lambda) = \frac{\frac{q_n f(\lambda)}{D} \left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n\rho}{a} \right)^2 \right]^2}{\left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n\rho}{a} \right)^2 \right]^4 + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4}
\]

Hence, the solution of Eq. (7) is given by

\[
w = \left\{ \frac{1}{2D} \int_0^\infty \frac{q_0 f(\lambda)}{\left( \frac{\lambda}{a} \right)^4 + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4} \cos \frac{dx}{\lambda} d\lambda + \frac{1}{D} \sum_{n=2}^{\infty} \cos \frac{na}{\lambda} \int_0^\infty \frac{q_n f(\lambda) \left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n\rho}{a} \right)^2 \right]^2}{\left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n\rho}{a} \right)^2 \right]^4 + \frac{Eh}{a_D} \left( \frac{\lambda}{a} \right)^4} \cos \frac{dx}{\lambda} d\lambda \right\}
\]

(8.c)
It is next desired to find \( q_n \) and \( f(\lambda) \). In order to accomplish this we must develop the functions \( q_n \) and \( \lambda \) from the loading condition which is shown in Fig. 4. Since the cylinder is loaded symmetrically to the generatrix and to the circle passing through the origin, only the positive direction need be considered.

![Diagram](a)

From Eq. (8:b) we have

\[
q \left( \frac{x}{a} \right) = \int_{0}^{\infty} f(\lambda) \cos \frac{\lambda x}{a} \, d\lambda \\
f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} q \left( \frac{X}{a} \right) \cos \frac{\lambda X}{a} \, d\left( \frac{X}{a} \right)
\]

and

\[
q \left( \frac{x}{a} \right) = 1 \quad \text{when} \quad -\delta \leq x \leq \delta \\
q \left( \frac{x}{a} \right) = 0 \quad \text{when} \quad x > \delta \quad \text{and} \quad -x < -\delta
\]
Then
\[ f(\lambda) = \frac{\pi}{\alpha} \int_0^\lambda \cos \frac{\lambda x}{\alpha} \, d\lambda = \frac{\pi}{\alpha} \sin \frac{\lambda x}{\alpha} \bigg|_0^\lambda = \frac{\pi}{\alpha \lambda} \sin \lambda \frac{x}{\alpha} \]

Similarly we can determine \( q_{in} \) by expanding the loading function along the circumference in a Fourier series. With
\[ q_{in} = \frac{1}{\pi} \int_{-\pi}^{\pi} q(z) \, dz \]

where \( z = \frac{x}{\alpha} \) and if \( z = \frac{\pi}{2} \), \( s = \frac{\pi}{2} \alpha \)

we get
\[ q_{in} = \frac{\pi}{\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z) \cos n n \, dz = \frac{\pi}{\alpha} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(z) \cos n \frac{s}{\alpha} \, dz = \frac{4q}{\pi n} \sin \frac{n \pi}{\alpha} \]

Substituting \( q_{in} \) and \( f(\lambda) \) in Eq. (8:2)
\[ W = \frac{1}{2D} \int_{\lambda=0}^{\lambda=0} \frac{8q \cos \frac{\lambda x}{\alpha}}{\pi^2 a} \frac{\sin \frac{\lambda x}{\alpha}}{\lambda} \frac{Eh}{a^4 D} \cos \frac{\lambda x}{\alpha} \, d\lambda \]

\[ + \frac{1}{D} \sum_{n=2,4,\ldots}^{n=\infty} \cos \frac{n \pi s}{\alpha} \int_{\lambda=0}^{\lambda=0} \frac{8q \sin \frac{n \pi s}{\alpha}}{\pi^2 a} \frac{\sin \frac{n \pi s}{\alpha}}{\lambda} \left[ \left( \frac{n \pi s}{\alpha} \right)^2 + \left( \frac{n \pi s}{\alpha} \right)^4 \right]^2 \frac{Eh}{a^4 D} \left( \frac{\lambda}{a} \right)^4 \cos \frac{\lambda x}{\alpha} \, d\lambda \]
Now we consider the case of a concentrated load applied at the origin. Such a load can be obtained by making the lengths 2Δ and 2c of the loaded portion infinitely small. Substituting
\[ \rho = 4 \varphi c \delta \quad \text{and} \quad S_{IN} \frac{\lambda}{a} \approx \frac{\lambda}{a} \]
\[ S_{IN} \frac{n c}{a} \approx \frac{n c}{a} \]
in the above equation, we obtain

\[ W = \frac{P a^2}{D \pi^2} \int_0^{\infty} \frac{\cos n \lambda z d \lambda}{\lambda^2 + J^2} + \frac{2P a^2}{\pi^2 D} \sum_{n=2,4,6, \ldots} \frac{n s}{a} \int_0^{\infty} \frac{[\lambda^2 + n^2] \cos \lambda a z d \lambda}{[\lambda^2 + n^2]^2 + J^2 \lambda^4} \quad (9) \]

where
\[ J^2 = \frac{E h a^2}{D} = 12 (1 - \nu^2) \left( \frac{a}{h} \right)^2 \]

In order to evaluate the definite integrals in Eq. (9) we will apply Cauchy's Theorem of residues. This method is generally found simpler than any other. Let us first consider the integral

\[ \int_0^{\infty} \frac{\cos \lambda z d \lambda}{\lambda^2 + J^2} \]

where the characteristic equation \( \lambda^2 + J^2 \lambda = 0 \) has four complex roots
\[ \lambda = J^{\frac{1}{2}} \frac{1}{J^{\frac{1}{2}}} \]

We integrate the function \( \frac{e^{i \frac{1}{2} z}}{z^2 + J^2} \) in the complex z-plane along a closed contour \( \gamma \), consisting of the segment of the real axis from \(-R\) to \(R\) and a semicircle \( \sigma \) of radius \(R\) in the upper half-plane and tends to the point at infinity as \( R \to \infty \). Evidently, for sufficiently large values of \( R \), there is no singularity of the integrand on the closed contour and two
singularities, namely the simple poles $\sqrt{1 - \frac{i}{J^2}}$, $\sqrt{\frac{j}{J^2}}$, within $\Gamma$. By Cauchy's Theorem of residues, it follows that

$$I = \int_{-\infty}^{\infty} \frac{e^{i\frac{x}{J^2}} d\lambda}{\lambda + \frac{1}{J^2}} + \int_{R} e^{i\frac{\lambda}{J^2}} d\lambda = 2 \pi i \left[ \frac{e^{i\frac{\lambda}{J^2}} (1 + i\frac{1}{J^2})}{4\pi i (\frac{\lambda}{J^2} + i\frac{1}{J^2})} + \frac{e^{i\frac{\lambda}{J^2}} (1 + i\frac{1}{J^2})}{4\pi i (\frac{\lambda}{J^2} - i\frac{1}{J^2})} \right]$$

$$= \frac{\pi}{2\sqrt{\alpha}} J^{\frac{3}{2}} e^{-\frac{1}{2\alpha}} \left[ 2 \cos \frac{\theta}{2} \frac{x}{\alpha} + 2 \sin \frac{\theta}{2} \frac{x}{\alpha} \right] \quad (10)$$

Since the integral $\int_{R} e^{i\frac{\lambda}{J^2}}$ can be proved to approach zero as $R \to \infty$, we have

$$\int_{-\infty}^{\infty} \frac{\cos \frac{\lambda}{J^2} d\lambda}{\lambda^2 + \frac{1}{J^2}} = \frac{\pi}{2\sqrt{\alpha}} J^{\frac{3}{2}} e^{-\frac{1}{2\alpha}} \left( \cos \frac{\theta}{2} \frac{x}{\alpha} + \sin \frac{\theta}{2} \frac{x}{\alpha} \right) \quad (11)$$

The second definite integral in Eq. (9) can be evaluated in the same manner. Let us first express the rational function in the integral in the form of partial fractions

$$\frac{(\lambda^2 + \alpha^2)^2}{(\lambda^2 + \alpha^2)^2 + \beta^2 \lambda^2} = \frac{1}{2} \left\{ \frac{1}{(\lambda^2 + \alpha^2)^2 + \beta^2 \lambda^2} + \frac{1}{(\lambda^2 + \alpha^2)^2 - \beta^2 \lambda^2} \right\}$$

$$= \frac{1}{\alpha^2 - \alpha_1^2} + \frac{1}{\lambda^2 - \alpha_1^2} + \frac{1}{\lambda^2 - \alpha_2^2} + \frac{1}{\lambda^2 - \alpha_3^2} + \frac{1}{\lambda^2 - \alpha_4^2} \quad (12)$$

where $\pm \alpha_1$, $\pm \alpha_2$, $\pm \alpha_3$, and $\pm \alpha_4$ are the eight roots of the eighth degree algebraic equation in the denominator. These are:
\[\pm \alpha_1 = \pm A \pm iB = \mp \frac{1}{12} \left[ \left( (-n^2 + \eta)^2 + \left( -\frac{z}{2} + \phi \right)^2 \right) \right]^{\frac{1}{2}}\]
\[\pm \frac{1}{12} \left[ \left( (-n^2 + \eta)^2 + \left( -\frac{z}{2} + \phi \right)^2 \right) + (n^2 + \eta) \right]^{\frac{1}{2}}\]
\[\pm \alpha_3 = \pm C \pm iD = \pm \frac{1}{12} \left[ \left( (n^2 + \eta)^2 + \left( \frac{z}{2} + \phi \right)^2 \right) - (n^2 + \eta) \right]^{\frac{1}{2}}\]
\[\pm \frac{1}{12} \left[ \left( (n^2 + \eta)^2 + \left( \frac{z}{2} + \phi \right)^2 \right) + (n^2 + \eta) \right]^{\frac{1}{2}}\]

where
\[\phi = \frac{\lambda \eta}{L (R_z + \frac{Z}{2})}, \quad \eta = \frac{\lambda \eta}{L (R_z - \frac{Z}{2})}, \quad R_z = n^2 \int \left( 1 + \frac{z}{4n^2} \right)^2.\]

Hence
\[I = \int_{-\infty}^{\infty} \left( \lambda^2 + n^2 \right) \lambda e^{i\lambda \frac{z}{2}} = \frac{1}{2} \int \left[ \frac{e^{i\lambda^2 \frac{Z}{2}}}{(\alpha_1^2 - \alpha_2^2)(\lambda^2 - \alpha_1^2)} + \frac{e^{i\lambda^2 \frac{Z}{2}}}{(\alpha_3^2 - \alpha_4^2)(\lambda^2 - \alpha_3^2)} \right] d\lambda\]
\[= 2 \pi \frac{\lambda \eta}{L R_z} \left\{ \frac{\alpha_1^2}{\alpha_2 \alpha_3^2} (\eta - i\phi) e^{i\frac{Z}{2} \alpha_2} - \frac{\alpha_2^2}{\alpha_3 \alpha_4^2} (\eta - i\phi) e^{i\frac{Z}{2} \alpha_4} \right.\]
\[\left. - \frac{\alpha_3^2}{\alpha_4 \alpha_1^2} (\eta + i\phi) e^{i\frac{Z}{2} \alpha_1} + \frac{\alpha_4^2}{\alpha_1 \alpha_3^2} (\eta + i\phi) e^{i\frac{Z}{2} \alpha_3} \right\}\]

From the relation
\[\alpha_1 \alpha_2 = -n^2 = \alpha_3 \alpha_4\]

we have
\[I = -\frac{\pi \eta}{4} \frac{i}{R_z n^2} \left\{ (\eta - i\phi)(-C + iD) e^{i\frac{Z}{2} \alpha_1} - (\eta - i\phi)(A + iB) e^{i\frac{Z}{2} \alpha_2} \right.\]
\[\left. - (\eta + i\phi)(-A + iB) e^{i\frac{Z}{2} \alpha_3} + (\eta + i\phi)(C + iD) e^{i\frac{Z}{2} \alpha_4} \right\}\]
After simplifying we finally get

\[
\int_0^{\infty} \frac{(\lambda^2 + \eta^2)^t \cos \frac{A x}{a} d\lambda}{(\lambda^2 + \eta^2)^t + \eta^2 \lambda^2} = \frac{\pi}{4R_x \eta^t} \left\{ \left[ (\phi C + \eta D) \cos \frac{A x}{a} + (\phi D - \eta C) \sin \frac{A x}{a} \right] e^{-D \frac{A x}{a}} + e^{-D \frac{A x}{a}} \left[ (\phi A - \eta B) \cos \frac{C x}{a} + (\eta A + \phi B) \sin \frac{C x}{a} \right] \right\}
\]  

(15)

Simplifying the integrals (15) and (11) in Eq. (9), we obtain

\[
W = \frac{P a^t}{2 \pi D} \left[ \cos \sqrt{\frac{A x}{a}} \sin \sqrt{\frac{A x}{a}} \right] e^{-D \frac{A x}{a}} \\
+ \frac{P a^t}{2 \pi D} \sum_{n=2}^{\infty} \frac{\cos n \frac{A x}{a}}{R_x n^t} \left\{ \left[ (\phi C + \eta D) \cos \frac{A x}{a} + (\phi D - \eta C) \sin \frac{A x}{a} \right] e^{-D \frac{A x}{a}} + \left[ (\phi A - \eta B) \cos \frac{C x}{a} + (\eta A + \phi B) \sin \frac{C x}{a} \right] e^{-D \frac{C x}{a}} \right\}
\]  

(16)

Alternately Eq. (16) can be written in a non-dimensional form

\[
\frac{W}{D} = \frac{3(1-\nu^4)}{2 \pi} \left( \frac{a}{b} \right)^t \left[ \cos \sqrt{\frac{A x}{a}} \sin \sqrt{\frac{A x}{a}} \right] e^{-D \frac{A x}{a}} \\
+ \frac{6(1-\nu^4)}{\pi} \left( \frac{a}{b} \right)^t \sum_{n=2}^{\infty} \frac{\cos n \frac{A x}{a}}{R_x n^t} \left\{ \left[ (\phi C + \eta D) \cos \frac{A x}{a} \right] e^{\frac{D x}{a}} + \left[ (\eta A - \phi B) \cos \frac{C x}{a} \right] e^{-D \frac{C x}{a}} \right\}
\]  

(17)

It is seen that the first term of the above equation is very small as compared to the second term, and therefore can be neglected without any appreciable error in practical application. For a certain value of the \( a/h \) ratio \( D \) is found to be very large as compared to \( B \). The terms containing \( e^{-D \frac{A x}{a}} \)
can then be completely neglected, provided that \( x/a \) is not near zero.

However, in the case when \( x/a = 0 \) Eq. (17) can be simplified as follows:

\[
\frac{W}{h} \frac{P}{Eh^2} \bigg|_{\frac{x}{a} = 0} = \frac{3 \mu^2}{\pi} \left( \frac{a}{h} \right)^2 \sum_{n=4}^{\infty} \frac{\cos \frac{n \pi}{a}}{n^3} \left[ 1 + \frac{\frac{3}{4} \frac{(1-\nu^2)}{n^2} \left( \frac{a}{h} \right)^2}{\left( 1 + \frac{\frac{3}{4} \frac{(1-\nu^2)}{n^2} \left( \frac{a}{h} \right)^2 \right)} \right] (18)
\]
3. A CYLINDER OF FINITE LENGTH LOADED

WITH TWO EQUAL AND OPPOSITE FORCES

The expression for the radial deflection in a thin cylinder of finite length can be obtained from equation (17) by using the method of images. If we imagine the cylinder of finite length prolonged in both the positive and the negative x-directions, and loaded with a series of forces, $P$ of alternating sense applied along the generatrix $(\frac{a}{2} - c)$ at a distance $\ell$ from one another (see Fig. 5), then the deflections of the infinite cylinder are evidently equal to zero at a distance $\frac{\ell}{2}$ from the applied loads $P$.

Hence we may consider the given cylinder of length $\ell$ and radius $a$ as a portion of the infinitely long cylinder loaded as shown in Fig. 5. From Eq. (17) we find that the deflection of any point, $B$, (at a distance $b$ from the $x$-axis)

* This method was used by Dr. A. Neder, see Zangen. Meth. Mech., Vol. 2, P. 1, 1922; and by M.T. Hiber, see Zangen. Meth. Mech., Vol. 6, P. 226, 1928.
on the shell due to the load \( P \) acting at the center is:

\[
\begin{align*}
\omega_a &= \frac{Pa^2}{2\pi D} \sum_{n \geq 2} \cos \frac{\eta S}{R_n} \left\{ \left[ (\phi_c + \eta D) \cos A \frac{C b}{R_n} + (\eta D - \eta c) \sin A \frac{C b}{R_n} \right] e^{-\frac{D b}{a}} \right. \\
&\left. + \left[ (\phi a - \eta B) \cos C \frac{b}{a} + (\eta A + \phi B) \sin C \frac{b}{a} \right] e^{-\frac{D b}{a}} \right\} \\
\omega_b &= \frac{-Pa^2}{2\pi D} \sum_{n \geq 2} \cos \frac{\eta S}{R_n} \left\{ \left[ (\phi c + \eta D) \cos A \frac{l - b}{a} \right. \\
&\left. + (\phi a - \eta B) \sin A \frac{l - b}{a} \right] e^{-\frac{D (l - b)}{a}} + \left[ (\phi c + \eta D) \cos A \frac{l + b}{a} \\
&\left. + (\phi a - \eta B) \sin A \frac{l + b}{a} \right] e^{-\frac{D (l + b)}{a}} \right\}
\end{align*}
\]

(19:a)

The deflection produced by two adjacent forces a distance \( l \) apart is:

\[
\begin{align*}
\omega_b &= \frac{-Pa^2}{2\pi D} \sum_{n \geq 2} \cos \frac{\eta S}{R_n} \left\{ \left[ (\phi c + \eta D) \cos A \frac{l - b}{a} \right. \\
&\left. + (\phi a - \eta B) \sin A \frac{l - b}{a} \right] e^{-\frac{D (l - b)}{a}} + \left[ (\phi c + \eta D) \cos A \frac{l + b}{a} \\
&\left. + (\phi a - \eta B) \sin A \frac{l + b}{a} \right] e^{-\frac{D (l + b)}{a}} \right\}
\end{align*}
\]

(19:b)

Since the terms containing \( e^{-\frac{D (l \pm b)}{a}} \) are all small as compared with the other terms shown above, they can be neglected without causing appreciable error. Similarly we can obtain \( \omega_c, \omega_d \) and so on. The total radial deflection at any point \( B \) is given by the sum

\[
W = \omega_a + \omega_b + \omega_c + \cdots
\]

or

\[
W = \frac{Pa^2}{2\pi D} \sum_{n \geq 2} \cos \frac{\eta S}{R_n} \left\{ \left[ (\phi a - \eta B) \cos A \frac{C b}{a} + (\eta A + \phi B) \sin A \frac{C b}{a} \right] e^{-\frac{D b}{a}} \right. \\
&\left. + (\phi c + \eta D) \cos A \frac{l - b}{a} e^{\frac{D (l - b)}{a}} + (\phi c + \eta D) \cos A \frac{l + b}{a} e^{\frac{D (l + b)}{a}} \right\} \\
&\left. + (\phi D - \eta C) \sin A \frac{b}{a} e^{\frac{D b}{a}} + (\phi D - \eta C) \sin A \frac{l - b}{a} e^{\frac{D (l - b)}{a}} \right\} \\
&\left. + (\phi D - \eta C) \sin A \frac{l + b}{a} e^{\frac{D (l + b)}{a}} \right\}
\]

(20)
It is seen that Eq. (20) can be expressed by series containing \( \cos \frac{nA}{a} e^{-\frac{nB}{a}} \) and \( \sin \frac{nA}{a} e^{-\frac{nB}{a}} \) terms.

\[
W = \frac{p a^2}{\pi D} \sum_{n=1,3,5,...}^\infty \frac{\cos \frac{nA}{a}}{R_i n} \left\{ \left[ (\phi A - \Omega B) \cos \frac{B}{a} + (\Omega A + \phi B) \sin \cos \frac{D}{a} \right] e^{-\frac{B}{a}} \\
+ \left( \phi C - \Omega D \right) \left[ \cos \frac{A}{a} e^{-\frac{B}{a}} - 2 \cos \frac{A}{a} \cos \frac{B}{a} \left( \cos \frac{A}{a} e^{-\frac{B}{a}} - \cos \frac{A}{a} e^{-\frac{2B}{a}} \\
+ \cos \frac{3A}{a} e^{-\frac{3B}{a}} - \ldots \right) - 2 \sin \frac{A}{a} \sin \frac{B}{a} \left( \sin \frac{A}{a} e^{-\frac{B}{a}} - \sin \frac{A}{a} e^{-\frac{2B}{a}} \\
+ \sin \frac{3A}{a} e^{-\frac{3B}{a}} - \ldots \right) \right] + 2 \sin \frac{A}{a} \sin \frac{B}{a} \left( \cos \frac{A}{a} e^{-\frac{B}{a}} - \cos \frac{A}{a} e^{-\frac{2B}{a}} - \ldots \right) \right\}
\]

(21)

The series in the above equation

\[
\cos \frac{A}{a} e^{-\frac{B}{a}} - \cos \frac{2A}{a} e^{-\frac{2B}{a}} + \cos \frac{3A}{a} e^{-\frac{3B}{a}} - \ldots \\
\sin \frac{A}{a} e^{-\frac{B}{a}} - \sin \frac{2A}{a} e^{-\frac{2B}{a}} + \sin \frac{3A}{a} e^{-\frac{3B}{a}} - \ldots
\]

can be expressed in the form given below. This expression is known as a geometrical series and the sum of its \( n \) terms is found with the aid of the Binomial expansion

\[
\sum_{m=1,3}^\infty e^{mB/a} \cos \frac{mA}{a} = \frac{1}{2} \sum_{m=1}^\infty \left[ e^{mB/a(z)} + e^{-mB/a(z)} \right]
\]

where \( Z_1 = B - iA \); \( Z_2 = B + iA \).
Putting $\sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{\sin(x)}{x} \right) = e^{\sin(x)/x}$ we obtain

$$\sum_{m=1}^{\infty} J^m = J + J^3 + J^5 + \cdots = \frac{J}{1 - J^2}$$

$$\sum_{m=2,4,\ldots} J^m = J^2 + J^4 + J^6 + \cdots = \frac{J^2}{1 - J^2}$$

Consequently

$$\sum_{m=1,3,\ldots} \lim_{B \to 0} e^{-mB^{\frac{1}{2}}} \cos mA \frac{1}{a} = \frac{1}{2} \left[ \frac{e^{-\frac{1}{2}(B-iA)}}{1 - e^{-2\frac{1}{2}(B-iA)}} + \frac{e^{-\frac{1}{2}(B+iA)}}{1 - e^{-2\frac{1}{2}(B+iA)}} \right]$$

$$= \frac{1}{2} \left[ \frac{\sinh \frac{1}{2} B \cos \frac{1}{2} A}{\sinh \frac{1}{2} B \cos \frac{1}{2} A + \cosh \frac{1}{2} B \sin \frac{1}{2} A} \right]$$

$$\sum_{m=2,4,\ldots} \lim_{B \to 0} e^{-mB^{\frac{1}{2}}} \cos mA \frac{1}{a} = \frac{1}{2} \left[ \frac{\frac{1}{2}(B-iA)}{1 - e^{-2\frac{1}{2}(B-iA)}} + \frac{\frac{1}{2}(B+iA)}{1 - e^{-2\frac{1}{2}(B+iA)}} \right]$$

$$= \frac{e^{\frac{1}{2}B} \left( \sinh \frac{1}{2} B \cos \frac{1}{2} A - \cosh \frac{1}{2} B \sin \frac{1}{2} A \right)}{2 \left( \sinh \frac{1}{2} B \cos \frac{1}{2} A + \cosh \frac{1}{2} B \sin \frac{1}{2} A \right)}$$

$$\sum_{m=1,3,\ldots} \lim_{B \to 0} e^{-mB^{\frac{1}{2}}} \sin mA \frac{1}{a} = \frac{1}{2i} \left[ \frac{\cosh \frac{1}{2} B \sin \frac{1}{2} A}{\sinh \frac{1}{2} B \cos \frac{1}{2} A + \cosh \frac{1}{2} B \sin \frac{1}{2} A} \right]$$

$$\sum_{m=2,4,\ldots} \lim_{B \to 0} e^{-mB^{\frac{1}{2}}} \sin mA \frac{1}{a} = \frac{1}{2i} \left[ \frac{\frac{1}{2}(B-iA)}{1 - e^{-2\frac{1}{2}(B-iA)}} + \frac{\frac{1}{2}(B+iA)}{1 - e^{-2\frac{1}{2}(B+iA)}} \right]$$

$$= \frac{e^{-\frac{1}{2}B} \cos \frac{1}{2} A \sin \frac{1}{2} A \left( \sinh \frac{1}{2} B + \cosh \frac{1}{2} B \right)}{2 \left( \sinh \frac{1}{2} B \cos \frac{1}{2} A + \cosh \frac{1}{2} B \sin \frac{1}{2} A \right)}$$
Substituting the above summations in Eq. (21) we have

\[ \frac{W}{P} = \frac{b(1 - \mu^2)}{\pi} \left( \frac{a}{h} \right)^2 \sum_{n=2}^{\infty} \frac{\cos \eta_n\frac{5}{2}}{R_n \eta_n^2} \left\{ \left[ (\phi C + \eta D) \cos A \frac{b}{a} + (\phi D - \eta C) \sin A \frac{b}{a} \right] e^{-\frac{b}{2}n^2} + \left[ (\phi A - B) \cos C \frac{b}{a} + (\eta A + B) \sin C \frac{b}{a} \right] e^{-\frac{b}{2}n^2} \right\} \]

\[ + \frac{\text{Sinh} \frac{b}{2} B \cos \frac{b}{2} A - e^{-\frac{b}{2}B} \left[ \text{Sinh} \frac{b}{2} B \cos \frac{b}{2} A - \text{Cosh} \frac{b}{2} B \sin \frac{b}{2} A \right]}{\left( \text{Sinh} \frac{b}{2} A \cos \frac{b}{2} \frac{b}{a} A + \text{Cosh} \frac{b}{2} B \sin \frac{b}{2} A \right)} \times \left[ (\phi D - \eta C) \sin A \frac{b}{a} \sinh B \frac{b}{a} - (\phi C + \eta D) \cos A \frac{b}{a} \cosh B \frac{b}{a} \right] \]

\[ - \frac{\text{Cosh} \frac{b}{2} B \sin \frac{b}{2} A - e^{-\frac{b}{2}B} \cos \frac{b}{2} A \sin \frac{b}{2} A \left[ \text{Sinh} \frac{b}{2} B + \text{Cosh} \frac{b}{2} B \right]}{\left( \text{Sinh} \frac{b}{2} B \cos \frac{b}{2} \frac{b}{a} A + \text{Cosh} \frac{b}{2} B \sin \frac{b}{2} A \right)} \times \left[ (\phi D - \eta C) \cos A \frac{b}{a} \cosh B \frac{b}{a} + (\phi C + \eta D) \sin A \frac{b}{a} \sinh B \frac{b}{a} \right] \]
\[
\frac{W}{Eh^3} = \frac{6(1-v^2)}{\pi} \left( \frac{a}{h} \right)^2 \sum_{n \neq 2,4} \frac{\cos \frac{\pi n}{h}}{R_n h^3} \left\{ (\phi_c + \eta_D + (\phi_A - \eta_B) \right.

- (\phi_c + \eta_D) \frac{\sinh \frac{1}{2}B \cos \frac{1}{2}A - e^{-\frac{1}{2}D} \left( \sinh \frac{1}{2}B \cos \frac{1}{2}A - \cosh \frac{1}{2} \sin \frac{1}{2}A \right)}{\left( \sinh \frac{1}{2}B \cos \frac{1}{2}A + \cosh \frac{1}{2}B \sin \frac{1}{2}A \right)} \\

- (\phi_d - \eta_c) \frac{\cosh \frac{1}{2}B \sin \frac{1}{2}A - e^{-\frac{1}{2}D} \cos \frac{1}{2}A \sin \frac{1}{2}A \left( \sinh \frac{1}{2}B + \cosh \frac{1}{2}B \right)}{\left( \sinh \frac{1}{2}B \cos \frac{1}{2}A + \cosh \frac{1}{2}B \sin \frac{1}{2}A \right)} \\

- (\phi_c - \eta_B) \frac{\sinh \frac{1}{2}D \cos \frac{1}{2}C - e^{-\frac{1}{2}D} \left( \sinh \frac{1}{2}D \cos \frac{1}{2}C - \cosh \frac{1}{2} \sin \frac{1}{2}C \right)}{\left( \sinh \frac{1}{2}D \cos \frac{1}{2}C + \cosh \frac{1}{2}D \sin \frac{1}{2}C \right)} \\

- (\eta_A + \phi_B) \frac{\cosh \frac{1}{2}D \sin \frac{1}{2}C - e^{-\frac{1}{2}D} \cos \frac{1}{2}C \sin \frac{1}{2}C \left( \sinh \frac{1}{2}D + \cosh \frac{1}{2}D \right)}{\left( \sinh \frac{1}{2}D \cos \frac{1}{2}C + \cosh \frac{1}{2}D \sin \frac{1}{2}C \right)} \\

\right\} 
\]
4. APPLICATION OF THE SOLUTION OF THE

PROBLEM OF THE INFINITELY LONG CYLINDER

TO SOME SPECIFIC PROBLEMS

The cases of a couple acting on an infinitely long cylinder in the direction of either the generatrix or the circumference can be analyzed by using the solution given by Eq. (17) for a single load. The action of the couple is equivalent to that of the two forces $P$ shown in Fig. 6a, if

$P \Delta x$ approaches $T_c$ while $\Delta x$ approaches zero.

It is easy to see that the deflection for the case when the force $P$ is at the point $O$, at a distance $\Delta x$ from the origin can be obtained from $w$, Eq. (17), by writing $x-\Delta x$ instead of $x$ and also $-P$ instead of $P$. This and the original $w$ are then added, whereby the radial deflection for the two equal and opposite forces applied at $O$ and $O_1$ respectively, is obtained in the form

$$-w_1 = w(x, s) - w(x-\Delta x, s).$$
When $\Delta x$ is very small, this approaches the value

$$W_T = \frac{dW(x, s)}{dx} \Delta x$$

As $T_c$ is the moment of the applied torque and is equal to $P \Delta x$, the radial deflection due to this torque is

$$W_{T_i} = \frac{T_c}{P} \frac{dW}{dx} \tag{a}$$

where $W$ is the radial deflection due to the concentrated load $P$.

Similarly we find the radial deflection due to the couple acting along the circumferential direction, Fig. 6b

$$W_{T_2} = \frac{T_c}{P} \frac{dW}{ds} \tag{b}$$

Substituting $W$ from Eq. (17) in Eqs. (a) and (b) we obtain

$$\frac{W_{T_1}}{\frac{h}{2 \pi}} = \frac{6(1-\nu^2)}{\pi} \left(\frac{a}{h}\right)^2 \sum_{n=2 \pi}^{\infty} \frac{\cos \eta \frac{x}{a}}{R \eta^2} \left\{ e^{-\frac{B}{2}} \cos \frac{A}{x} \left[A(\phi D - \eta C) - B(\phi C + \eta D)\right] - e^{\frac{B}{2}} \sin A \frac{x}{a} \left[(\phi C + \eta D)A + B(\phi D - \eta C)\right] + e^{-\frac{D}{2}} \cos C \frac{x}{a} \left[(\eta A + \phi B) - D(\phi A - \eta B)\right] \right\}$$

$$\frac{W_{T_2}}{\frac{h}{2 \pi}} = -\frac{6(1-\nu^2)}{\pi} \left(\frac{a}{h}\right)^2 \sum_{n=2 \pi}^{\infty} \frac{\sin \eta \frac{x}{a}}{R \eta^2} \left\{ (\phi C + \eta D) \cos A \frac{x}{a} \right\}$$

$$+ (\phi D - \eta C) \sin A \frac{x}{a} \right\} e^{-\frac{B}{2}} + \left[(\phi A - \eta B) \cos C \frac{x}{a} \right\} e^{-\frac{D}{2}} \right\}$$

$$+ (\eta A + \phi B) \sin C \frac{x}{a} \right\} e^{-\frac{D}{2}} \right\}$$

(24)
In the case when \( \frac{x}{a} = 0 \) we have:

\[
\frac{W_R}{h} = \frac{W_L}{h} = 0 \quad \text{at any } s,
\]

Hence the condition that the slope of the deflection curve must vanish under the concentrated load \( (x/a = 0) \) is satisfied.

The solution of an infinitely long cylinder in Eq. (17) may also be applied in order to find the modulus of foundation of an elastically supported flat plate or beam under a concentrated transverse load. From Fig. 7 we can visualize that a longitudinal strip between the nodal points of the cylinder will deform in a manner similar to a flat plate on elastic supports under a transverse load. The equation for the deflection curve of an elastically supported beam under a concentrated load consists of trigonometric and hyperbolic functions and has a form similar to Eq. (17).

Let us now compare the known maximum deflection of the cylinder under a certain concentrated load with the maximum deflection of an elastically supported beam of infinite length. From the known deflection \( w_c \) of the cylinder we can determine the modulus of foundation \( k \) of the beam by equating \( w_c = w_b \), where \( w_b \) is the maximum deflection of the beam:

\[
w_c = w_b = \frac{P}{18 \left( \frac{k}{EI} \right) h^3 EI}
\]

---

* Karman and Biot. Mathematical Methods in Engineering, Chapter VII.
Then

\[ k = \left( \frac{P}{\frac{1}{15} w_c} \right) \left( \frac{1}{EI} \right)^{\frac{4}{3}} \]

since \( EI \) is the flexural rigidity of the beam, and \( P \) is the corresponding transverse load.

In the same manner the modulus of foundation \( k \) can also be determined at various other points along the beam.
PART TWO

INFINITELY LONG CYLINDER UNDER TWO

EQUAL AND OPPOSITE TORQUE ACTING ABOUT

THE RADIAL AXIS ON THE SURFACE OF THE SHELL

Fig. 8

The solution of the problem of concentrated torques acting on the surface of an infinitely long cylinder (Fig. 8a) can be achieved by replacing T with two equal and opposite forces acting at an infinitely
small distance \( \xi \) apart (Fig. 8b and 8c). In the case of an infinite plate the shearing stresses produced by two equal and opposite forces acting perpendicularly to either axis are identical. However, the problem we are dealing with is quite different because of the curvature effect. Instead of two equal and opposite forces acting at an infinitely small distance \( \xi \) apart in the direction of \( x \)-axis, we combine two double forces with moment, the moments being about the same axis and of the same sign, and the directions of the forces being at right angles to each other.

1. Fundamental Equations

A. Load Acting Tangentially

The equations of equilibrium of a shell element under a circumferential pressure of an intensity \( q \) can be derived from considerations of the equilibrium in a manner similar to that used in the derivation of Eqs. (4) of Part One.

\[
\begin{align*}
\frac{a}{\xi} \frac{\partial T_\xi}{\partial \xi} + \frac{\partial S}{\partial \phi} &= 0 \\
\frac{\partial T_\xi}{\partial \phi} + a \frac{\partial S}{\partial \xi} - qa &= 0 \\
\frac{\xi}{2} \frac{\partial^2 H}{a \partial \phi^2} + \frac{\partial G_x}{\partial \xi} + \frac{\partial^2 G_{x\phi}}{a \partial \phi^2} + \frac{T_\xi}{a} &= 0
\end{align*}
\]

By using the relation between the stress resultants and the deformations we can obtain the three differential equations which contain the displace-

---

ments u, v, and w. After simplifying we get

\[
\frac{\partial^2 u}{\partial x^2} + \frac{1 + \sqrt{\gamma}}{\alpha} \frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x \partial y} + \frac{1 - \sqrt{\gamma}}{\alpha} \frac{\partial^2 u}{\partial y^2} = 0
\]

\[
\frac{\partial^2 u}{\partial y^2} + \frac{1 + \sqrt{\gamma}}{\alpha} \frac{\partial u}{\partial y} - \frac{\partial^2 u}{\partial x \partial y} + \frac{1 - \sqrt{\gamma}}{\alpha} \frac{\partial^2 u}{\partial x^2} = 0
\]

\[
\frac{1}{\alpha} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) = 0
\]

Application of the operation \( \frac{\partial}{\partial x^2} \) to \( (25:1) \), gives

\[
\frac{\partial^4 u}{\partial x^4} + \frac{1 + \sqrt{\gamma}}{\alpha} \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 u}{\partial x \partial y} + \frac{1 - \sqrt{\gamma}}{\alpha} \frac{\partial^4 u}{\partial y^4} = 0
\]

Application of \( \frac{\partial}{\partial y^2} \) to \( (25:1) \) and \( \frac{\partial^2}{\partial x \partial y} \) to \( (25:2) \) give respectively

\[
\frac{\partial^2 u}{\partial y^2} + \frac{1 + \sqrt{\gamma}}{\alpha} \frac{\partial^2 u}{\partial y \partial z} - \frac{\partial^2 u}{\partial y \partial z} + \frac{1 - \sqrt{\gamma}}{\alpha} \frac{\partial^4 u}{\partial z^4} = 0
\]

\[
\frac{1}{\alpha} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{1}{\alpha} \left( \frac{\partial^2 u}{\partial y \partial z} \right) + \frac{1}{\alpha} \left( \frac{\partial^4 u}{\partial z^4} \right) + \frac{1}{\alpha} \left( \frac{\partial^2 u}{\partial x \partial z} \right) - \frac{1}{\alpha} \left( \frac{\partial^2 u}{\partial x^2} \right) = 0
\]

Substitution of the terms containing v in Eq. (26:a) and (26:b) into Eq. (26:c), after simplification, gives

\[
a \nabla^4 u = \nu \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - \alpha \frac{(1 + \sqrt{\gamma})^2}{Eh} \frac{\partial^2 u}{\partial y^2}
\]

Similarly, applying \( \frac{\partial^2}{\partial x^2} \) and \( \frac{\partial^2}{\partial y^2} \) to Eq. (25:2) and solving for the terms containing u, and substituting in (25:1) after applying \( \frac{\partial^2}{\partial x \partial y} \) to it, we obtain an equation from which u has been eliminated

\[
a \nabla^4 u = \frac{\partial^2 u}{\partial y^2} + (2 + \nu) \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial (1 + \sqrt{\gamma})^2}{Eh} \left( \frac{\partial^2 u}{\partial x^2} + (1 - \nu) \frac{\partial^2 u}{\partial x \partial y} \right)
\]
Now, applying $\frac{\partial}{\partial x}$ to (25:d) and $\frac{\partial}{\partial s}$ to (25:e) and substituting these two equations into Eq. (25:3), after applying $\nabla^4$ to it, we obtain an equation from which both $u$ and $v$ have been eliminated.

$$\nabla^4 W + \frac{Eh}{a^2 D} \frac{\partial^4 W}{\partial x^4} - \frac{1}{Da} \left( \nabla^2 + \nabla^3 \right) \left( \frac{\partial^3}{\partial x^3} \frac{\partial}{\partial s} + \frac{\partial^4}{\partial s^4} \right) = 0$$

(25:f)
B. Load Acting Longitudinally

The equations of equilibrium of an element under a longitudinal pressure of intensity \( q \) can also be obtained in the same manner.

\[
\begin{align*}
\alpha \frac{\delta T}{\delta x} + \frac{\delta S}{\delta \gamma} + q a &= 0, \\
\frac{\delta T}{\delta \gamma} + a \frac{\delta S}{\delta x} &= 0, \\
\frac{2}{a} \frac{\delta^2 H}{\delta x^2} + \frac{\delta^2 G}{\delta x^2} + \frac{\delta^2 G_2}{\delta x^2} + \frac{T}{a} &= 0
\end{align*}
\]  

(27)

Reduction of these equations to forms containing derivations of \( u, v, w \) gives

\[
\begin{align*}
\frac{\delta^2 u}{\delta x^2} + \frac{L + V}{E} \frac{\delta^2 u}{\delta s \delta x} - \frac{V}{E} \frac{\delta^2 u}{\delta x^2} + \frac{1 - V^2}{E} \frac{\delta^2 u}{\delta s^2} + \frac{1 - \psi}{E} \psi &= 0, \\
\frac{1}{\psi} \frac{\delta v}{\delta s} + \frac{1 + V}{E} \frac{\delta v}{\delta s} + \frac{1 - V}{E} \frac{\delta v}{\delta x} - \frac{1}{2} \frac{\delta w}{\delta s} &= 0, \\
\frac{h^2}{a^2} \Delta^4 \sigma - \frac{1}{\alpha} \left( \frac{\delta v}{\delta s} - \frac{\psi}{\delta s} + \nu \frac{\delta u}{\delta x} \right) &= 0
\end{align*}
\]

(28)

Applying \( \frac{\delta^2}{\delta x^2} \) and \( \frac{\delta^2}{\delta s^2} \) to (28:1) and solving for the terms involving \( v \), and substituting in (28:2), after applying \( \frac{\delta^2}{\delta x \delta s} \) to it, we obtain an
equation from which \( \nu \) has been eliminated

\[
\alpha \, \nabla^4 \, u = \sqrt{\frac{E\, h}{\alpha^2 \, D}} \frac{s^2 \omega}{\delta x} - \frac{\alpha (1 + \nu)}{E\, h} \left\{ \frac{\delta^2 w}{\delta x^2} + (1 - \nu) \frac{\delta^3 \eta}{\delta x^3} \right\} \tag{29:a}
\]

Similarly, applying \( \frac{\delta^2}{\delta x^2} \) and \( \frac{\delta^2}{\delta s^2} \) to Eq. (28:2) and solving for the terms involving \( u \) and substituting in (28:1), after applying \( \frac{\delta^2}{\delta x \, \delta s} \) to it, we obtain an equation from which \( u \) has been eliminated

\[
\alpha \, \nabla^4 \, u = \frac{\delta^2 \omega}{\delta s^2} + (2 + \nu) \frac{\delta^2 \omega}{\delta s \, \delta x} + \frac{\alpha (1 + \nu)}{E\, h} \frac{\delta^3 \eta}{\delta x \, \delta s} \tag{29:b}
\]

Now applying \( \frac{\delta}{\delta \gamma} \) and \( \frac{\delta}{\delta s} \) to (29:a) and (29:b) respectively and \( \nabla^4 \) to (28:3), and substituting the terms of \( u \) and \( \nu \) in the first two equations into the third one, we get

\[
\nabla^8 \omega + \frac{E\, h}{\alpha^2 \, D} \frac{s^4 \omega}{\delta x^4} - \frac{1}{D \, a^2} \left\{ \frac{\delta^2 \eta}{\delta x \, \delta s^2} - \nu \frac{\delta^3 \eta}{\delta x^3} \right\} = 0 \tag{29:c}
\]

It is seen that the above equation reduces to the known differential equation of the flat plate if \( a \) is made infinitely large. Under such kind loading the lateral deflection of an infinitely long plate is obviously equal to zero.
2. Determination of Shearing Stress Distribution

A. $S_t$ Due to Two Equal and Opposite Tangential Forces (Fig. 8b)

The shearing stress resultant in the wall of a cylindrical shell is given as*

$$S_t = \frac{Eh}{2(1-\nu)} \left\{ \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} \right\}$$

(30)

where the displacements $u$ and $v$ can be determined from the differential equations (26d) and (26e).

The load distribution under consideration may be represented by an even function along the circumference and by an odd function along the generatrix. As discussed in Part One we can express the load distribution function by a combination of a Fourier Series and a Fourier Integral,

$$q(x, s) = \left[ \frac{2}{\pi} + \sum_{n=1,2,\ldots} q_n \cos \frac{ns}{a} \right] \int_0^\infty f(\lambda) S_{IN} \wedge \frac{x}{a} d\lambda$$

(31:a)

The components of displacement can be written in a similar form. They must contain then three undetermined functions $u(\lambda), v(\lambda)$ and $w(\lambda)$

* See S. Timoshenko. Theory of Plates and Shells, Pages 365 and 439.
\[ U = \sum_{n=0,2,\ldots}^\infty S_n \frac{n^3}{a} \int_0^\infty u(\lambda) \cos \lambda \frac{x}{a} d\lambda \]  
(31:b)

\[ V = \sum_{n=0,2,\ldots}^\infty \cos \frac{n^3}{a} \int_0^\infty v(\lambda) S_n \lambda \frac{\lambda^2}{a} d\lambda \]  
(31:c)

\[ W = \sum_{n=0,2,\ldots}^\infty \sin \frac{n^3}{a} \int_0^\infty w(\lambda) S_n \lambda \frac{x}{a} d\lambda \]  
(31:d)

The differential equations (26:d) and (26:e) can be solved by substituting Eqs. (31:a), (31:d) and (31:b) and (31:c) respectively. In order to simplify the analysis we can break up the differential equations into two parts and solve them separately. The solutions so obtained can be combined because they are all linear. The first equation (of displacement u) is obtained by putting a infinitely large. Then Eq. (36:d) becomes

\[ \nabla^4 u + \frac{(1+\nu)^2}{E h} \frac{\partial^2 \varphi}{\partial x \partial s} = 0 \]  
(32:a)

This equation is representative of an infinitely large imaginary plate under the specified load. The second equation of displacement u can be obtained by putting q = 0. It contains a relation between u and w which is

\[ \nabla^4 u - \frac{\nu}{a} \frac{\partial^2 w}{\partial x^2} + \frac{1}{a} \frac{\partial^2 w}{\partial x \partial s} = 0 \]  
(32:b)

The differential equations for displacement v are obtained in a similar way:

\[ \nabla^4 v - \frac{1+\nu}{E h} \left\{ 2 \frac{\partial^2 q}{\partial x^2} + (1-\nu) \frac{\partial^2 q}{\partial s^2} \right\} = 0 \]  
(33:a)
\[ \sqrt{\nu} - \frac{1}{a} \frac{\delta^2 w}{\delta s^2} - \left( \frac{2 + \nu}{a} \right) \frac{\delta^2 w}{\delta s \delta x^2} = 0 \quad (33:b) \]

Now, let us substitute Eqs. (31:a) and (31:b) in Eq. (32:a)

For \( n = 0 \)

\[ \frac{\partial^2 q}{\partial x \partial s} = 0; \quad w(\lambda) = 0; \quad \therefore u(\lambda) = 0 \]

For \( n = 2, 4, \ldots \)

\[ \sum_{n=2,4,\ldots} \int_{\alpha}^{\infty} \left\{ u(\lambda) \left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n}{a} \right)^2 \right] - \frac{(1 + \nu)}{E_h} q_n f(\lambda) \left( \frac{\lambda}{a} \right) \left( \frac{\lambda}{a} \right) \right\} \cos \frac{\lambda x}{a} \sin \frac{n \pi s}{a} d\lambda = 0 \]

for all values of \( \lambda \), or

\[ u(\lambda) = \frac{(1 + \nu)}{E_h} q_n f(\lambda) \frac{\frac{n \pi s}{a}}{\left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n}{a} \right)^2 \right]^2} \]

Hence the solution of Eq. (32:a) is given by

\[ u = \frac{(1 + \nu)}{E_h} \sum_{n=2,4,\ldots} \int_{\alpha}^{\infty} q_n f(\lambda) \frac{\frac{n \pi s}{a}}{\left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n}{a} \right)^2 \right]^2} \cos \frac{\lambda x}{a} d\lambda \quad (34) \]

The functions \( q_n \) and \( f(\lambda) \) in the above integral can be determined by developing them from the loading condition which is shown in Fig. 9.

![Fig. 9](image-url)
The leading condition along the longitudinal direction is shown in Fig (9a) and the transformed Fourier Integral of Eq. (31a) is given by

\[ f(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} q(\frac{x}{a}) S_{1N} \lambda \frac{\alpha}{a} d\left(\frac{x}{a}\right) \]

where \( q(\frac{x}{a}) \) is distributed between \((-\infty, \infty)\) as follow:

\[ q\left(\frac{x}{a}\right) = \begin{cases} 1 & \text{when } 0 < x < 2\pi e \\ 0 & \text{when } 0 > x > -2\pi e \\ 2\pi e & \text{when } -2\pi e > x \end{cases} \]

We get:

\[ f(\lambda) = \frac{2\pi}{\pi} \int_{-\infty}^{\infty} S_{1N} \lambda \frac{\alpha}{a} d\left(\frac{x}{a}\right) = \frac{2\pi}{\pi} \left( \cos \frac{2\lambda e}{\alpha} - 1 \right) = \frac{4}{\pi} \frac{\alpha}{\pi} S_{1N} \lambda \frac{\alpha}{a} \]

The Fourier coefficient \( q_n \) between \((-\frac{\pi}{2}, \frac{\pi}{2})\) along the circumference is given by

\[ q_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q(x) \cos nx dx \]

The expression is valid in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) if one puts \( x = \frac{s}{a} \). We get

\[ q_n = \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} q\left(\frac{x}{a}\right) \cos \frac{s a}{2} ds \]

But \( q\left(\frac{x}{a}\right) \) is distributed in the interval \((-\frac{\pi}{2}, \frac{\pi}{2})\) as follows:

\[ q\left(\frac{x}{a}\right) = 0 \quad \text{when} \quad -\frac{\pi}{2} a \leq s < c \]

\[ q\left(\frac{x}{a}\right) = q \quad \text{when} \quad -\frac{\pi}{2} a < s \leq c \]

\[ q\left(\frac{x}{a}\right) = q \quad \text{when} \quad c \leq s \leq 0 \]

\[ -c \leq s \leq 0 \]
Therefore \( q = \frac{2}{\pi a} \left[ \int_{-\frac{a}{2}}^{\frac{a}{2}} 0 \cdot \cos n \frac{\lambda}{a} d\lambda + \int_{-\frac{a}{2}}^{\frac{a}{2}} q \cos n \frac{\lambda}{a} d\lambda + \int_{0}^{\frac{a}{2}} 0 \cdot \cos n \frac{\lambda}{a} d\lambda \right] \)

\[ = \frac{4q}{\pi n} \sin n \frac{c}{a} \]

Substituting \( f(\lambda) \) and \( q \) in Eq. (34), we get

\[ u_i = \frac{16q}{\pi^2} \frac{(1 + \nu)^2}{E h} \sum_{n=1,3,5,\ldots}^{\infty} \sin n \frac{c}{a} \int_{0}^{\infty} \frac{\sin n \frac{c}{a} \sin \frac{\lambda c}{a}}{\left( (\frac{\lambda c}{a})^2 + (\frac{c}{a})^2 \right)^{\frac{3}{2}}} \cos \frac{\lambda}{a} d\lambda \]

Now we consider the case of a concentrated torque applied at the origin. This can be obtained by making the lengths \( 2a \) and \( 2c \) of the loaded portion infinitely small.

Substituting \( P = q (4c) \) and \( \sin \frac{2c}{a} \approx \frac{2c}{a} \)

\[ \begin{align*}
  c & \to 0 \\
  \varepsilon & \to 0 \\
  & \sin \frac{\varepsilon}{a} \approx \frac{\varepsilon}{a}
\end{align*} \]

and also \( \text{Torque} = T = Fx = 8q \varepsilon c \)

in the above equation we have

\[ u_i = \frac{2T}{\pi^2 a} \frac{(1 + \nu)^2}{E h} \sum_{n=1,3,5,\ldots}^{\infty} \sin \frac{n \varepsilon}{a} \int_{0}^{\infty} \frac{\lambda^3}{\left( (\frac{\lambda c}{a})^2 + (\frac{\varepsilon}{a})^2 \right)^{\frac{3}{2}}} \cos \frac{\lambda}{a} d\lambda \]  \( (35) \)

The definite integral in Eq. (35) can be evaluated in the same manner as explained in Part One. However, on account of the double poles, the analysis may be simplified if we use the following transformation,

\[ \frac{\partial}{\partial n} \int_{-\infty}^{\infty} \frac{d\varepsilon}{(\varepsilon^2 + n^2)} = \int_{-\infty}^{\infty} \frac{2n d\varepsilon}{(\varepsilon^2 + n^2)^3} \]  \( (35:a) \)
Therefore we can write

\[ I = \int_{-\infty}^{\infty} \frac{e^{iz^2}}{(z^2 + n^2)^{1/2}} \, dz = \frac{1}{2n} \int_{-\infty}^{\infty} \frac{e^{iz^2}}{(z^2 + n^2)} \, dz \]

By Cauchy's Theorem of Residues with the aid of Jordan's Lemma we obtain

\[ I = 2\pi i \left( -\frac{i}{2n} \right) \frac{\partial}{\partial n} \left\{ \frac{(in)^2 e^{-\frac{x}{a}}}{2in} \right\} = \frac{\pi}{2n} \left\{ 1 - n \frac{x}{a} \right\} e^{-\frac{x}{a}} \]

or

\[ \int_{0}^{\infty} \frac{\lambda \cos \frac{\lambda}{\sqrt{\lambda + n^2}}}{(\lambda + n^2)^{1/2}} \, d\lambda = \frac{\pi}{4n} \left\{ 1 - n \frac{x}{a} \right\} e^{-\frac{x}{a}} \]

Substituting the above integral in Eq. (35), we get

\[ u_1 = \frac{(1 + \nu)^{1/2}}{2\pi a E_h} \sum_{n=1,4,\ldots} S_{1n} \frac{n^2}{a} \left\{ 1 - n \frac{x}{a} \right\} e^{-\frac{x}{a}} \quad (36) \]

Differentiating Eq. (36) with respect to \( a \) we obtain:

\[ \frac{\partial u_1}{\partial a} = \frac{(1 + \nu)^{1/2}}{2\pi a E_h} \sum_{n=1,4,\ldots} \cos \frac{n^2}{a} \left\{ (1 + \nu)n - (1 + \nu)n^2 \frac{x}{a} \right\} e^{-\frac{x}{a}} \quad (37) \]

In order to solve Eq. (38:b) for \( u_2 \) it is necessary to find the undetermined function \( w(\lambda) \) by substituting Eqs. (31:a) and (31:d) in Eq. (38:2).

For \( n = 0 \), \( w(\lambda) = 0 \)

For \( n = 2, 4 \ldots \)

\[ \sum_{n=1,4,\ldots} \int_{0}^{\infty} \left\{ \omega(\lambda) \left[ \left( \frac{\lambda}{a} \right)^2 + \left( \frac{n}{a} \right)^2 \right]^2 + \frac{E_h}{aD} \left( \frac{\lambda}{a} \right)^4 \right\} \frac{\partial f(\lambda)}{\partial a} \left[ \left( \frac{2 + \nu}{a} \left( \frac{\lambda}{a} \right) + (\frac{\lambda}{a})^3 \right) \right] \sin \frac{n\lambda}{a} \sin \frac{\Delta x}{a} \, d\lambda = 0 \]
The equation has to be true for all \( \lambda \)

\[
\omega(\lambda) = \frac{n f(\lambda)}{a D} \left[ \frac{[2 + \nu]\left( \frac{\lambda}{a} \right)^2 + \left( \frac{n}{a} \right)^2]}{\left( \frac{\lambda}{a} \right)^4 + \left( \frac{n}{a} \right)^2} \right] + \frac{E_k}{a D} \left( \frac{\lambda}{a} \right)^4 = \frac{n f(\lambda)}{a D} \left[ 2 + \nu + \left( \frac{n}{a} \right)^2 \right] \left( \lambda^2 + n^2 \right)^4 + J^4 \lambda^4
\]  

(37a)

Where \( J^2 = 12(1 - \nu \varepsilon^2) \left( \frac{E_k}{D} \right)^2 \) as mentioned in Part One.

By substituting (31b) and (31d) in Eq. (32b) we have the undetermined function \( u(\lambda) \)

For \( a = 0 \) \( w(\lambda) = 0 \) \( \therefore u(\lambda) = 0 \)

For \( a = 3 \) ---

\[
\sum_{n=2,4, \ldots} \int_0^\infty \left\{ u(\lambda) \left[ \left( \frac{\lambda}{a} \right)^4 + \left( \frac{n}{a} \right)^2 \right]^2 + \frac{w(\lambda)}{a} \left[ \nu \left( \frac{\lambda}{a} \right)^3 - \left( \frac{\lambda}{a} \right) \left( \frac{n}{a} \right)^2 \right] \right\} \cos \frac{\lambda^2}{a} \sin \frac{n}{a} d\lambda = 0
\]

or

\[
u(\lambda) = \frac{w(\lambda)}{a} \left[ \frac{\left( \frac{\lambda}{a} \right)(\frac{n}{a})^2 - \nu \left( \frac{\lambda}{a} \right)^3}{\left( \frac{\lambda}{a} \right)^4 + \left( \frac{n}{a} \right)^2} \right]^2
\]

Substituting \( w(\lambda) \) in the above form of \( u(\lambda) \)

\[
u(\lambda) = \frac{\pi T \lambda^2 n a}{\pi^2 D} \left[ \left( 2 + \nu \right) \lambda^2 + n^2 \right] \left[ n^2 - \nu \lambda^2 \right] \left( \lambda^2 + n^2 \right)^2 + J^4 \lambda^4 \]

Differentiating Eq. (31b) with respect to \( a \) we get

\[
\frac{3 u_2}{\partial S} = \sum_{n=2,4, \ldots} \frac{n}{a} \cos \frac{n}{a} \int_0^\infty u(\lambda) \cos \frac{\lambda x}{a} d\lambda
\]

\[
= -\frac{\pi T}{2 \pi D} \sum_{n=2,4, \ldots} n^2 \cos \frac{n}{a} \int_0^\infty \frac{\lambda^2 \left[ \left( 2 + \nu \right) \lambda^2 + n^2 \right] \left[ n^2 - \nu \lambda^2 \right] \cos \frac{\lambda x}{a} d\lambda}
\]

(38)
With the aid of transformation (35:a) we obtain

\[
I = \int_{-\infty}^{\infty} \frac{\nu(z+\nu)z^2(z^2+\bar{\tau}_1)(z^2-\bar{\tau}_2)}{(z^n+n^2)^2 + j^2 z^4} \, dz
\]

\[
= -\frac{\nu(z+\nu)}{2n} \frac{\partial}{\partial n} \int_{-\infty}^{\infty} \frac{z^2(z^2+\bar{\tau}_1)(z^2-\bar{\tau}_2)}{(z^n+n^2)^2 + j^2 z^4} \, dz
\]

where

\[
\bar{\tau}_1 = \frac{n^2}{2+n}, \quad \bar{\tau}_2 = \frac{n^2}{\nu}
\]

This is used to evaluate the integral with the double pole. Hence the integral in Eq. (38) can be evaluated by Cauchy's Theorem of Residues.

\[
I = 2 \pi i \nu(z+\nu) \left\{ -\frac{1}{2n} \frac{\partial}{\partial n} \left[ \frac{-n^2(-n^2+\bar{\tau}_1)(-n^2+\bar{\tau}_2) e^{-\frac{n^2}{\nu}}}{2 \sin(-n^2-x^2)(-n^2-x^2)(n^2-x_0)(n^2-x_0)} \right] \right. 
\]

\[
+ \frac{x_0(x_0^2+\bar{\tau}_1)(x_0^2-\bar{\tau}_2) e^{\frac{x_0^2}{\nu}}}{2(x_0^2+n^2)(x_0^2-x_1)(x_0^2-x_2)(x_0^2-x_3)(x_0^2-x_4)} + \frac{x_0^2(x_0^2+\bar{\tau}_1)(x_0^2-\bar{\tau}_2) e^{\frac{x_0^2}{\nu}}}{2(x_0^2+n^2)(x_0^2-x_1)(x_0^2-x_2)(x_0^2-x_3)(x_0^2-x_4)} 
\]

\[
+ \frac{x_0(x_0^2+\bar{\tau}_1)(x_0^2-\bar{\tau}_2) e^{\frac{x_0^2}{\nu}}}{2(x_0^2+n^2)(x_0^2-x_1)(x_0^2-x_2)(x_0^2-x_3)(x_0^2-x_4)} + \frac{x_0^2(x_0^2+\bar{\tau}_1)(x_0^2-\bar{\tau}_2) e^{\frac{x_0^2}{\nu}}}{2(x_0^2+n^2)(x_0^2-x_1)(x_0^2-x_2)(x_0^2-x_3)(x_0^2-x_4)} 
\]

(39)

where \((\pm in)^2, \pm x_1, \pm x_2, \pm x_3, \text{ and } \pm x_4\) are the twelve roots of the twelfth degree algebraic equation in the denominator of the integral in equation (38), and \(x_0\) are calculated in Eq. (12).

It is next desired to determine the value of \(\nu\). In order to accom-
plish this we can solve Eqs. (33:a) and (33:b) by substituting in (31:c), (31:e) and (31:d) respectively. First, we have

For \( n = 0 \)

\[
\frac{\partial^2 q_n}{\partial x^2} - \frac{\partial^2 q_n}{\partial z^2} = 0 \quad v = 0
\]

For \( n = 2, 4, \ldots \)

\[
\sum_{n=2}^{\infty} \int_{a}^{b} \left( v(\lambda) \left[ \frac{\partial^2}{\partial x^2} + \left( \frac{n}{a} \right)^2 \right] \right) + \frac{(1 + \nu) f(\lambda)}{Eh} \left[ 2 \left( \frac{\Delta}{a} \right)^2 + (1 - \nu) \left( \frac{n}{a} \right)^2 \right] \frac{\cos \frac{na}{a} \cdot S_{\lambda \lambda} \frac{\lambda}{a} d \lambda}{a} = 0
\]

or

\[
v(\lambda) = -\frac{(1 + \nu)}{Eh} q_n f(\lambda) \frac{\Delta^2 + (1 - \nu) \left( \frac{n}{a} \right)^2}{\left[ \left( \frac{\Delta}{a} \right)^2 + \left( \frac{n}{a} \right)^2 \right]^2}
\]

Substituting \( v(\lambda) \) into Eq. (31:c), where \( q_n f(\lambda) \) has been determined in the analysis of \( u(\lambda) \), we have

\[
v = -\frac{1 + \nu}{Eh} \frac{\Delta^2}{n^4 a^3} \sum_{n=2}^{\infty} \cos \frac{na}{a} \int_{0}^{a} \cos \frac{\lambda \Delta}{a} \left[ \frac{2 \lambda^2 + (1 - \nu) n^2}{\left( \lambda^2 + n^2 \right)^2} \right] S_{\lambda \lambda} \frac{\Delta}{a} d \lambda \quad (40)
\]

In order to evaluate the above integral we have first to simplify the rational function by putting it in the form of partial fractions, i.e.,

\[
\frac{\lambda \left[ 2 \lambda^2 + (1 - \nu) n^2 \right]}{(\lambda^2 + n^2)^2} = \frac{Fz + G}{(\lambda^2 + n^2)^2} + \frac{Qz + R}{\lambda^2 + n^2}
\]

Here the constants \( F, G, Q \) and \( R \) can be determined by equating the numerators. Hence

\[
\frac{\lambda \left[ 2 \lambda^2 + (1 - \nu) n^2 \right]}{(\lambda^2 + n^2)^2} = -\frac{(1 + \nu) n^2 \lambda}{(\lambda^2 + n^2)^2} + \frac{2 \lambda}{\lambda^2 + n^2}
\]
Substituting the simplified fractions back into the integral, we obtain

\[ I = - \int_{-\infty}^{\infty} \frac{(1+\nu) \eta^2}{(z^2 + \eta^2)^{1/2}} e^{-iz} \, dz + 2 \int_{-\infty}^{\infty} \frac{z e^{iz}}{z^2 + \eta^2} \, dz \]

By Cauchy's Theorem of Residues and the method used in evaluating the integral in Eq. (35) we have

\[ I = -2\pi i \left( \frac{1}{2\eta} \right) \frac{\partial}{\partial \eta} \left[ -\eta \left( 1 - \frac{1}{2} \right) e^{-n \frac{\eta}{2}} \right] + 2\pi i \, e^{-n \frac{\eta}{2}} \]

or

\[ \int_0^\infty \frac{\lambda \left( \lambda^2 + (1-\nu) \eta^2 \right)}{(\lambda^2 + \eta^2)^{1/2}} S_{n \lambda} \, d\lambda = \frac{\pi(1+\nu)}{\eta} \left( n \frac{\eta}{2} \right) e^{-n \frac{\eta}{2}} + \pi e^{-n \frac{\eta}{2}} \]

Substituting the above integral in Eq. (40)

\[ V = -\frac{(1+\nu)}{E_h} \frac{c T}{\pi a} \sum_{\eta = z_\lambda}^{\infty} \cos n \frac{\eta}{2} \left[ (1+\nu) \eta - (1+\nu) \eta \left( n \frac{\eta}{2} \right) \right] e^{-n \frac{\eta}{2}} \]

(41)

Differentiating Eq. (41) with respect to \( \eta \) we get

\[ \frac{\partial V}{\partial \eta} = -\frac{(1+\nu)}{E_h} \frac{T}{\pi a^2} \sum_{\eta = z_\lambda}^{\infty} \cos n \frac{\eta}{2} \left[ (1+\nu) \eta - (1+\nu) \eta \left( n \frac{\eta}{2} \right) \right] e^{-n \frac{\eta}{2}} \]

(45)

Equation (33:b) can be solved in the same manner as we solved Eq. (32:b)

For \( n = 0 \)

\( w(\lambda) = 0 \)

\( v(\lambda) = 0 \)

For \( n = 2, 4 \) ---

\[ \sum_{\eta = z_\lambda}^{\infty} \int_0^\infty \left[ v(\lambda) \left[ \left( \frac{\eta}{2} \right)^2 + \left( \frac{\eta}{2} \right)^2 \right] + w(\lambda) \left[ \left( \frac{\eta}{2} \right)^2 + \left( 2 + v \right) \left( \frac{\eta}{2} \right) \left( \frac{\lambda}{a} \right)^2 \right] \right] \cos \frac{\lambda \pi}{h} S_{n \lambda} \frac{\lambda \pi}{h} \, d\lambda = 0 \]
or

\[ v(\lambda) = \frac{\omega(\lambda) \left[ \left( \frac{n^2}{k^2} + (z + v)(\frac{\lambda^2}{2})^3 \right) \right]}{\left[ (\frac{k^2}{\lambda})^2 + (\frac{\gamma}{\lambda})^2 \right]^2} \]

By substituting \( w(\lambda) \) in Eq. (37a) we have

\[ v(\lambda) = \frac{-\frac{\infty}{D}}{\left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}}} \left\{ \left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} \right\} \left[ \left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} + J^2 \lambda^2 \right] \]

Differentiating Eq. (31:a) and substituting \( v(\lambda) \) from the above form we get:

\[ \frac{3V_0}{x} = \sum_{n=2}^{\infty} \cos \left( \frac{n\lambda}{a} \right) \int_{0}^{\infty} v(\lambda) \cos \frac{n\lambda}{a} d\lambda \]

\[ = -\frac{\infty}{D} \sum_{n=2}^{\infty} n^2 \cos \frac{n\lambda}{a} \int_{0}^{\infty} \frac{\lambda^2 \left[ \left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} \cos \frac{n\lambda}{a} \right]}{\left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} + J^2 \lambda^2} d\lambda \]

The above integral can be evaluated by a method similar to that used in determining \( u_0 \)

\[ I = \int_{0}^{\infty} \frac{\lambda^2 \left[ \left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} \cos \frac{n\lambda}{a} \right]}{\left( \frac{\lambda^2}{n^2} + n^2 \right)^{\frac{3}{2}} + J^2 \lambda^2} d\lambda \]

\[ = \frac{\infty}{2} \left\{ \sum_{1}^{\infty} \frac{\alpha_i \left( \alpha_i^2 + \frac{\gamma}{\lambda} \right)^2 e^{i\alpha_i^2 \sqrt{\lambda}}} {2(i^2 + n^2)^2 \left( \frac{x_i^2 - x_j^2}{x_i^2 - x_j^2} \right) \left( x_i^2 - x_j^2 \right) \left( x_i^2 - x_k^2 \right) \left( x_i^2 - x_k^2 \right)} \right\} \]

\[ + \left\{ \frac{\alpha_i \left( \alpha_i^2 + \frac{\gamma}{\lambda} \right)^2 e^{i\alpha_i^2 \sqrt{\lambda}}} {2(i^2 + n^2)^2 \left( \frac{x_i^2 - x_j^2}{x_i^2 - x_j^2} \right) \left( x_i^2 - x_j^2 \right) \left( x_i^2 - x_k^2 \right) \left( x_i^2 - x_k^2 \right)} \right\} \]

\[ = \left\{ \frac{\alpha_i \left( \alpha_i^2 + \frac{\gamma}{\lambda} \right)^2 e^{i\alpha_i^2 \sqrt{\lambda}}} {2(i^2 + n^2)^2 \left( \frac{x_i^2 - x_j^2}{x_i^2 - x_j^2} \right) \left( x_i^2 - x_j^2 \right) \left( x_i^2 - x_k^2 \right) \left( x_i^2 - x_k^2 \right)} \right\} \]

\[ + \left\{ \frac{\alpha_i \left( \alpha_i^2 + \frac{\gamma}{\lambda} \right)^2 e^{i\alpha_i^2 \sqrt{\lambda}}} {2(i^2 + n^2)^2 \left( \frac{x_i^2 - x_j^2}{x_i^2 - x_j^2} \right) \left( x_i^2 - x_j^2 \right) \left( x_i^2 - x_k^2 \right) \left( x_i^2 - x_k^2 \right)} \right\} \]

where \( \alpha_i \) are calculated in Eq. (12)
Now the shearing stress resultant in Eq. (30) can be written as follows

\[ S_i = S_{t_1} + S_{t_2} = \frac{Eh}{2(1+\nu)} \left\{ \frac{\partial u_i}{\partial s} + \frac{\partial v_i}{\partial x} + \frac{\partial u_i}{\partial s} + \frac{\partial v_i}{\partial x} \right\} \]

where

\[ S_{t_1} = \frac{Eh}{2(1+\nu)} \left\{ \frac{\partial u_i}{\partial s} + \frac{\partial v_i}{\partial x} \right\} \]

\[ = \frac{T}{2\pi a^2} \sum_{n=1}^{\infty} \frac{n^2 \cos \frac{n\pi}{a}}{(3+\nu)(1+\nu)} e^{-n^2\frac{x}{a}} \]

and

\[ S_{t_2} = \frac{Eh}{2(1+\nu)} \left\{ \frac{\partial u_i}{\partial s} + \frac{\partial v_i}{\partial x} \right\} \]

By substituting \( \frac{\partial u_i}{\partial s} \) and \( \frac{\partial v_i}{\partial x} \) from Eqs. (38), (39), (46) and (47) in the above equation, after simplifying, we have

\[ S_{t_i} = -\frac{T}{\pi a} \sum_{n=1}^{\infty} \frac{n^2 \cos \frac{n\pi}{a}}{(3+\nu)(1+\nu)} \left\{ \frac{1}{2n} \right\} \]

\[ = \frac{\alpha_i^2 (1+\nu)(2+\nu)(\alpha_i^2 + \beta_i^2)}{(\alpha_i^2 + n^2)(\alpha_i^2 + \alpha_i^2)(\alpha_i^2 + \alpha_i^2)} e^{i\alpha_i^2 \frac{x}{a}} + \frac{\alpha_i^4 (1+\nu)(2+\nu)(\alpha_i^2 + \beta_i^2)}{(\alpha_i^2 + n^2)(\alpha_i^2 + \alpha_i^2)(\alpha_i^2 + \alpha_i^2)} \]

Eq. (49) has a very complicated form. In order to simplify it
we may take its first term which gives, after differentiating with respect to \( n \)
\[
- \frac{1}{2n} \left[ \frac{-(1+\nu) (2+\nu) \left( n^4 (2\lambda n^2 + 3n^2 - \frac{n^2}{2+\nu}) - n^2 (\frac{4}{3}) (n^2 - \frac{n^2}{2+\nu}) \right)}{i (n^2 - \alpha_n^2) (n^2 - \alpha_n^2) (n^2 - \alpha_n^2) (n^2 + \alpha_n^2)} \right]
\]
\[
+ \frac{i (1+\nu) (2+\nu) n^3 (n^2 - \frac{n^2}{2+\nu})}{-i (n^2 - \alpha_n^2) (n^2 + \alpha_n^2) (n^2 + \alpha_n^2) (n^2 + \alpha_n^2)} \left\{ 2n (n^2 - \alpha_n^2) (n^2 - \alpha_n^2) (n^2 - \alpha_n^2) + 2n (n^2 - \alpha_n^2) (n^2 + \alpha_n^2) (n^2 + \alpha_n^2) + 2n (n^2 + \alpha_n^2) (n^2 - \alpha_n^2) (n^2 + \alpha_n^2) \right\} \]
\]

Substituting \( \alpha_n \) from Eq. (12) and simplifying, we have
\[
\frac{(1+\nu)}{2J^2 n} \left\{ (3+\nu) - n (\frac{\lambda}{n}) (1+\nu) \right\} e^{-\frac{\nu}{n}} \] (50)

Before simplifying the next four terms we introduce the following simplified notations with the aid of Eq. (12):

\[
(\alpha_n^2 + n^2)^2 = [\eta + i (\phi - \frac{\pi}{2})]^2 \quad (\alpha_n^2 - \alpha_n^2) = 2\eta + 2i\phi
\]
\[
(\alpha_n^2 + n^2)^2 = [\eta - i (\phi - \frac{\pi}{2})]^2 \quad (\alpha_n^2 - \alpha_n^2) = 2\eta - 2i\phi
\]
\[
(\alpha_n^2 + n^2)^2 = [-\eta + i (\phi + \frac{\pi}{2})]^2 \quad (\alpha_n^2 - \alpha_n^2) = 2\eta - iJ
\]
\[
(\alpha_n^2 + n^2)^2 = [-\eta - i (\phi + \frac{\pi}{2})]^2 \quad (\alpha_n^2 - \alpha_n^2) = 2\eta + iJ
\]

(51)
After substituting (51) into the second and third terms of Eq. (49), we have the following form with common denominator:

\[
\left[ (1+\nu)(\xi+\phi)(J+2\phi) \right] \frac{1}{\tilde{\xi} - \nu} \left[ (\lambda + i\phi) \left[ \left( -\eta^2 + \eta \gamma \right) + \eta \left( \eta \gamma + \frac{n}{z_\nu} \right) - n - \left( \frac{J}{z_\nu} \right) - \gamma \left( \frac{J}{z_\nu} \right) \right] \right.
\]

\[
\left. - \right. (\eta + 2\eta \gamma)(\eta \gamma + \frac{n}{z_\nu}) \right] \exp \left[ i \left( \eta + \frac{J}{z_\nu} \right) \left( \eta \gamma + \frac{n}{z_\nu} \right) \right]
\]

Separating the real and imaginary parts

\[
\left[ (1+\nu)(\xi+\phi)(J+2\phi) \right] \frac{1}{\tilde{\xi} - \nu} \left[ (\lambda + i\phi) \left[ \left( -\eta^2 + \eta \gamma \right) + \eta \left( \eta \gamma + \frac{n}{z_\nu} \right) - n - \left( \frac{J}{z_\nu} \right) - \gamma \left( \frac{J}{z_\nu} \right) \right] \right.
\]

\[
\left. - \right. (\eta + 2\eta \gamma)(\eta \gamma + \frac{n}{z_\nu}) \right] \exp \left[ i \left( \eta + \frac{J}{z_\nu} \right) \left( \eta \gamma + \frac{n}{z_\nu} \right) \right]
\]

Introducing trigonometric functions and simplifying we obtain:

\[
\frac{i(1+\nu)(\xi+\phi)(J+2\phi)}{\tilde{\xi} - \nu} \left[ \left( \lambda \cos \frac{\alpha}{\nu} - \beta \sin \frac{\alpha}{\nu} \right) \left( \eta^2 - \eta \gamma \right) - \eta \left( \frac{J}{z_\nu} \right) - \gamma \left( \frac{J}{z_\nu} \right) \right]
\]

\[
\cdot \left[ 2R_\phi - 2J_\gamma \left( \frac{J}{z_\nu} \right) - \left( \frac{J}{z_\nu} \right) \right] \left[ \eta \gamma + 2J_\gamma \gamma - n \left( \frac{3+2\nu}{z_\nu} \right) \right]
\]
\[-(A \sin \frac{A\alpha}{\alpha} + B \cos \frac{A\alpha}{\alpha}) \left\{ (\frac{1}{2} - \frac{1}{2} \left[ 2 \phi - n^3 \left( \frac{3 + 2\nu}{2 + \nu} \right) \right] \left[ 2 R_\phi - 2 R_\eta - \frac{3}{2} J^3 \phi + \frac{3}{2} J^3 \phi \right] \right\} + \left[ n^3 \left( \frac{i + \nu}{2 + \nu} \right) - n^3 \phi \left( \frac{2 + \nu}{2 + \nu} \right) - \frac{3}{2} J^3 \phi \right] \left[ 2 R_\phi - \frac{3}{2} J^3 \phi + 2 n^3 J^3 \right] \right\} \]

Further simplifications will reduce the second and third terms of Eq. (49) to a final form:

\[
\frac{i(i + \nu)(2 + \nu)}{\beta \eta^3 J^3 R_\phi} \left[ (A \cos \frac{A\alpha}{\alpha} - B \sin \frac{A\alpha}{\alpha}) \left\{ n^3 \left( \frac{i + \nu}{2 + \nu} \right) \left( 4 n^3 J^3 \phi - 4 n^3 J^3 R_\phi + \frac{n^3}{2} J^3 \phi \right) \right\} + \left[ 2 n^3 J^3 \phi - 4 n^3 J^3 R_\phi \right] \right] \]

Equation (52)

Similarly the last two terms in Eq. (49) can be written, after substituting in (51):

\[
\frac{i(i + \nu)(2 + \nu)}{16 \eta^3 J^3 R_\phi} \left[ \alpha (-n^3 - \eta - i \frac{\tau}{2} - i \phi)(-n^3 + \frac{n}{2 + \nu} - \eta - i \frac{\tau}{2} - i \phi) e^{\frac{\alpha\tau}{2}} \left[ \eta + i(\phi - \frac{\tau}{2}) \right]^2 \cdot (2 \eta + 2i \phi)(2 \eta + i \phi)(2 \eta + \frac{n}{2 + \nu} - \eta + i \frac{\tau}{2} + i \phi) \cdot e^{\frac{\alpha\tau}{4}} \left[ \eta - i(\phi - \frac{\tau}{2}) \right]^2 (2 \eta + 2i \phi)(2 \eta + i \phi)(2 \eta + i \phi)(2 \eta + i \phi)(2 \eta + \frac{n}{2 + \nu} - \eta + i \frac{\tau}{2} + i \phi) \right]
\]

After further simplification, we obtain

\[
\frac{i(i + \nu)(2 + \nu)}{\beta \eta^3 J^3 R_\phi} \left[ (C \cos \frac{C\alpha}{\alpha} - D \sin \frac{C\alpha}{\alpha}) \left\{ n^3 \left( \frac{i + \nu}{2 + \nu} \right) \left[ 4 n^3 J^3 \phi - 2 n^3 J^3 R_\phi + \frac{n^3}{2} J^3 \phi + 2 n^3 J^3 \phi \right] \right\} + \left[ (D \cos \frac{C\alpha}{\alpha} + C \sin \frac{C\alpha}{\alpha}) \left\{ n^3 \left( \frac{i + \nu}{2 + \nu} \right) \left[ 4 n^3 J^3 \phi + 2 n^3 J^3 \phi - 4 n^3 J^3 R_\phi \right] - 2 n^3 J^3 \phi + 4 n^3 J^3 R_\phi \right\} \right] \right]
\]

Equation (53)
Substituting Eqs. (50), (52) and (53) in Eq. (49), we obtain

$$S_{ t_2 } = \frac{T}{2\pi a^3} \sum_{n=1}^{\infty} \cos \frac{n \alpha}{\alpha} \left\{ -[(3+n)\eta - n^2 (1+\nu) \frac{J_4}{J_2}] e^{-n^2 \xi} \right\}$$

$$+ \frac{1}{4 J_4 / J_2} \left\{ \left[ (A \cos \frac{A x}{\alpha} - B \sin \frac{A x}{\alpha}) e^{-B \xi} + (C \cos \frac{C x}{\alpha} - D \sin \frac{C x}{\alpha}) e^{-D \xi} \right] \right\}$$

$$\cdot \left\{ (1+\nu) \left[ 2 \frac{J_2}{J_4} + \frac{J_4}{J_2} + 2 (2+\nu) \frac{J_2}{n^2} \right] + \frac{4 R_4}{J_4} \left( \frac{A \sin \frac{A x}{\alpha}}{\alpha} \right) \right\}$$

$$+ B \cos \frac{A x}{\alpha} e^{-B \xi} + (C \sin \frac{C x}{\alpha} + D \cos \frac{C x}{\alpha}) e^{-D \xi} \right\} - \left\{ 4 (1+\nu) \left[ \frac{2 J_2}{n^4} \right] \right\}$$

$$\cdot \left\{ (A \sin \frac{A x}{\alpha} + B \cos \frac{A x}{\alpha}) e^{-B \xi}- (D \cos \frac{C x}{\alpha} + C \sin \frac{C x}{\alpha}) e^{-D \xi} \right\} \right\}$$

Combining $S_{ t_1 }$ and $S_{ t_2 }$ from Eqs. (50) and (54), we finally have the shear-stress resultant in the alternate form

$$S_{ t_1 } = - \frac{1}{8\pi} \sum_{n=1}^{\infty} \cos \frac{n \alpha}{\alpha} \left\{ \left[ (2+\nu) \left( \frac{J_2}{J_4} \frac{J_4}{J_2} \right) \right] + (1+\nu) \left( \frac{J_2}{J_4} \frac{J_4}{J_2} \right) \right\}$$

$$- \frac{2 \sqrt{1+\nu} J_4}{n^2} + \frac{J_4}{J_2} \left[ \left( \frac{A \cos \frac{A x}{\alpha} - B \sin \frac{A x}{\alpha}) e^{-B \xi} + (C \cos \frac{C x}{\alpha} \right) \right]$$

$$- D \sin \frac{C x}{\alpha} e^{-D \xi} + B \cos \frac{A x}{\alpha} e^{-B \xi} + (C \sin \frac{C x}{\alpha} + D \cos \frac{C x}{\alpha}) e^{-D \xi} \right\}$$

$$- \left\{ 2 (1+\nu) \left( \frac{J_2}{J_4} \frac{J_4}{J_2} \right) - \left( \frac{J_2}{J_4} \frac{J_4}{J_2} \right) \right\} \left[ \left( A \sin \frac{A x}{\alpha} + B \cos \frac{A x}{\alpha} \right) e^{-B \xi} - (D \cos \frac{C x}{\alpha} + C \sin \frac{C x}{\alpha}) e^{-D \xi} \right]$$
where \( J^2 = 12(1 - \nu^2)(\frac{a}{h})^2 \) and \( j = \frac{J}{4\pi} \)

Although the radial deflection is of no great importance in this case, it may be worth investigating because it gives a clear picture of the deformation pattern. From Eq. (31;d), we obtain after substituting \( w(\lambda) \) from Eq. (37:a):

\[
W = \frac{cTa}{H'D} \sum_{n=1,3,5,...} \infty \sin \frac{\lambda}{a} \left[ \frac{(2 + \nu) \lambda^2 + n^2}{(\lambda^2 + n^2)^2 + 4\lambda^2} \right] S_{IN} \frac{\lambda^2}{a} d\lambda
\]

The definite integral is evaluated by Cauchy's Theorem of Residues

\[
\int_{0}^{\infty} \frac{\lambda[(2 + \nu) \lambda^2 + n^2]}{(\lambda^2 + n^2)^2 + 4\lambda^2} S_{IN} \lambda^2 d\lambda = \pi i \left[ \frac{(2 + \nu) \alpha_1^2 + n^2}{2(\alpha_1^2 - \alpha_2^2)(\alpha_2^2 - \alpha_3^2)(\alpha_3^2 - \alpha_4^2)} \right] + \left[ \frac{(2 + \nu) \alpha_1^2 + n^2}{2(\alpha_4^2 - \alpha_1^2)(\alpha_2^2 - \alpha_3^2)(\alpha_3^2 - \alpha_4^2)} \right]
\]

Substituting this integral back into Eq. (56), and simplifying by using \( \alpha_3 \) in Eq. (12) we obtain an alternate form of \( W \)

\[
\frac{W}{\eta} = \frac{c}{\pi} \sum_{n=1,3,5,...} \infty \sin \frac{\lambda}{a} \left[ 4j[1+j]^2 (\sin A x \frac{a}{d} e^{-B x \frac{a}{d}} - \sin C x \frac{a}{d} e^{-D x \frac{a}{d}}) \right]
\]

\[
+ \left[ \frac{\pi j^2}{n} \left[ 1 + j^2 + 2j(1 + \nu) \int \sin (1 + j^2 + j) \left( e^{-B x \frac{a}{d}} \sin A x \frac{a}{d} + e^{-B x \frac{a}{d}} \sin C x \frac{a}{d} \right) \right] \right]
\]

\[
+ \left[ \frac{\pi j^2}{n} \left[ 1 + j^2 - j + 2j(1 + \nu) \int \sin (1 + j^2 + j) \left( e^{-B x \frac{a}{d}} \cos A x \frac{a}{d} - \cos C x \frac{a}{d} \right) \right] \right]
\]

- 50 -
It is seen that the radial deflection has a pattern anti-symmetrical with respect to both the \( x \) and \( s \) axes. Moreover, we can easily prove that \( w \) equals zero at either \( x = 0 \) or \( s = 0 \), i.e.,

\[
\frac{W}{\pi \frac{h}{3}} = \frac{a}{8 \pi} \sum_{n=1,4,7,\ldots}^{\infty} \frac{\sin \frac{n\pi}{a}}{n} \left[ \frac{j^2}{(1 + j^2)^2} \sum_{j=1}^{\infty} \frac{1}{(1 + j^2)^2} - j + 2 (1 + j^2) \sum_{j=1}^{\infty} \frac{1}{(1 + j^2)^2} \right] (1 - 1) = 0
\]

\[
\frac{W}{\pi \frac{h}{3}} = \frac{a}{8 \pi} \sum_{n=1,4,7,\ldots}^{\infty} \frac{\sin \frac{n\pi}{a}}{n} \left\{ \text{Refer to Eq. (57)} \right\} = 0
\]

**B. \( S_y \) Due to Two Equal and Opposite Longitudinal Forces. Fig. (8c)**

The shearing stress resultant \( S_y \) can be determined in this case also from the displacements \( u \) and \( v \) in differential equations (29:a) and (29:b).

Since the load distribution in the present condition is an even function along the generatrix and an odd function along the circumference we can express the distributed load in the following form

\[
q(x, s) = \sum_{n=1,3,\ldots}^{\infty} q_n \sin \frac{n\pi}{d} \int_{0}^{\infty} f(\lambda) \cos \lambda \frac{x}{d} d\lambda
\]

(58:a)

It follows that the components of displacement may be written in
a form containing three undetermined functions \( u(\lambda) \), \( v(\lambda) \) and \( w(\lambda) \):

\[
\begin{align*}
\text{(58:b)} \\
u &= \sum_{n=1}^{\infty} s_{1n} \frac{n\pi}{a} \int_{0}^{\infty} u(\lambda) \cos \frac{\lambda}{a} d\lambda \\
v &= \sum_{n=1}^{\infty} c_{1n} \frac{n\pi}{a} \int_{0}^{\infty} v(\lambda) s_{1n} \frac{\lambda}{a} d\lambda \\
w &= \sum_{n=1}^{\infty} s_{2n} \frac{n\pi}{a} \int_{0}^{\infty} w(\lambda) s_{1n} \frac{\lambda}{a} d\lambda \\
\end{align*}
\]

It is next desired to determine the functions \( a_n \) and \( f(\lambda) \).

This can be accomplished by developing them from the loading condition according to the following diagrams.

![Diagram](attachment:image_url)

**Fig. 10**

The loading condition along the longitudinal direction is shown in Fig. 10(a) and the transformed Fourier Integral of Eq. (58:a) is given by

\[
\begin{align*}
f(\lambda) &= \frac{1}{\pi} \int_{-\infty}^{\infty} q(\frac{\lambda}{a}) \cos \lambda(\frac{\lambda}{a}) \frac{\lambda}{a} d(\frac{\lambda}{a}) \\
\end{align*}
\]
Where $q\left( \frac{x}{a} \right)$ is distributed between $(-\infty, \infty)$ as follows:

\[
q\left( \frac{x}{a} \right) = \begin{cases} 
0 & \text{when } x > \varepsilon \\
& \text{and } -x < -\varepsilon \\
1 & \text{when } 0 \geq x > -\varepsilon \\
& \text{and } 0 \leq x < \varepsilon 
\end{cases}
\]

Having the expression for the load distribution we can determine $f(\lambda)$:

\[
f(\lambda) = \frac{2}{\pi} \int_{0}^{\varepsilon} \cos \lambda \frac{x}{a} d\frac{x}{a} = \frac{2}{\pi \lambda} S_{\text{in}} \lambda \frac{\varepsilon}{a}
\]

The Fourier coefficient $q_n$ in the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$ along the circumference is given by

\[
q_n = \frac{1}{\pi} \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} q(z) S_{\text{in}} n z dz
\]

The expression will be valid for $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ if we put $z = \frac{s}{a}$:

\[
q_n = \frac{2}{\pi a} \int_{\frac{-\pi}{2}a}^{\frac{\pi}{2}a} q\left( \frac{s}{a} \right) S_{\text{in}} n \frac{s}{a} ds
\]

But $q\left( \frac{s}{a} \right)$ is distributed in the interval $(-\frac{\pi}{2}a, \frac{\pi}{2}a)$ as follows:

\[
q\left( \frac{s}{a} \right) = \begin{cases} 
0 & \text{when } -\frac{\pi}{2} a < s < -c \\
& \text{and } \frac{\pi}{2} a > a > c \\
\frac{1}{q} & \text{when } c \leq s < 0 \\
& \text{and } -c \leq s < 0
\end{cases}
\]
By using these load distributions we obtain

\[ q_n = \frac{2}{\pi a} \left( \int_0^\pi \sin n \frac{x}{a} \cos \frac{x}{a} \, dx + \int_0^\pi -q \sin n \frac{x}{a} \, dx + \int_0^\pi \frac{\partial}{\partial \pi} \sin n \frac{x}{a} \, dx \right) \]

\[ = \frac{2q}{\pi n} \sin n \frac{\pi}{a} \]

In the case of a concentrated torque applied at the origin, we can assume the lengths \( 2\ell \) and \( 2c \) of the loaded portion infinitely small.

Since

\[ F = q \frac{\ell c}{2} \]

\[ \ell \to 0 \]

\[ c \to 0 \]

Therefore Torque = \( T = FC = q \frac{\ell^2 c}{2} \)

or

\[ q_n f(\lambda) = \frac{2 q_n}{\ell^2 c} \]

(59)

In the solution of Eqs. (39:a) and (39:b) for \( u, v \) a considerable simplification results from putting \( a \to \infty \). Then, as before, we get a set of equations which combine \( u \) and \( v \), respectively, with \( q_n \)

\[ \nabla^4 u + \frac{(1+\nu)}{Eh} \left[ 2 \frac{\partial^2}{\partial s^2} + (1-\nu) \frac{\partial^2}{\partial x^2} \right] = 0 \]  

(60:a)

\[ \nabla^4 v - \frac{(1+\nu)}{Eh} \frac{\partial^2}{\partial x^4} = 0 \]  

(60:b)

Substituting Eqs. (59:a) and (59:b) in Eq. (60:a), we have

\[ \sum_{n=1,2,\ldots}^{\infty} \left\{ u(\lambda) \left[ 2 \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial s} \right)^2 \right] + \frac{(1+\nu)}{Eh} q_n f(\lambda) \left[ -\frac{\partial}{\partial x} - (1-\nu) \left( \frac{\partial}{\partial s} \right)^2 \right] \right\} \cos \lambda \frac{x}{a} \sin \lambda \frac{s}{a} = 0 \]
or
\[ u(\lambda) = \frac{e^T (1+\nu)}{2\pi \alpha \pi h} \left( \frac{2n^2 \lambda^2}{[\lambda^2 + n^2]^2} \right) \cos \frac{\pi \lambda}{\lambda} \frac{n}{\lambda} d\lambda \]

Knowing \( u(\lambda) \) from the above relation we can obtain \( u \) by substituting \( u(\lambda) \) in Eq. (58:b)

\[ u, = \sum_{n=1,2,\ldots} \sin n \frac{\pi \lambda}{\lambda} \left( \frac{2n+ (1+n) \lambda^2}{[\lambda^2 + n^2]^2} \right) \cos \frac{\pi \lambda}{\lambda} \frac{n}{\lambda} d\lambda \]  

(61)

The above definite integral can be evaluated through the following relation

\[ I = \int_{-\infty}^{\infty} \frac{[2n^2 + (1+n) \lambda^2] e^{iz \frac{x}{\lambda}}}{[z^2 + n^2]} \, dz. \]

By Cauchy's Theory of Residues, and with the aid of Jordan's Lemma we obtain

\[ I = \frac{\pi}{2} \left( \frac{3-n}{n} + (1+\nu) \frac{k}{\lambda} \right) e^{-n \frac{x}{\lambda}}. \]

Substituting this in Eq. (61), we have

\[ u_i = \frac{1+\nu}{Eh} \frac{T}{2\pi \alpha} \left( \sum_{n=1,2,\ldots} \sin n \frac{\pi \lambda}{\lambda} \left( [3-n] + n(1+\nu) \frac{k}{\lambda} \right) e^{-n \frac{x}{\lambda}} \right) \]  

(62)

or

\[ \frac{du_i}{dS} = \frac{1+\nu}{Eh} \frac{T}{2\pi \alpha} \left( \sum_{n=1,2,\ldots} \cos n \frac{\pi \lambda}{\lambda} \left( [3-n] n + (1+\nu) n \frac{k}{\lambda} \right) e^{-n \frac{x}{\lambda}} \right) \]  

(63)

In the same manner \( v \), can be determined by substituting Eqs. (58:a)
and (58:c) in Eq. (60:b)

\[
\sum_{n=1,3,\ldots}^{\infty} \int_{0}^{\infty} \left\{ n(L) \left[ \left( \frac{\lambda}{a} \right)^{2} + \left( \frac{n}{a} \right)^{2} \right] - q_n f(\lambda) \frac{(1+\nu)^{2}}{Eh} \frac{n\lambda}{a^{3}} \right\} \cos \frac{n}{a} S \sin \lambda \frac{\lambda}{a} \, d\lambda = 0
\]

or

\[
v(\lambda) = \frac{(1+\nu)^{2}}{Eh} q_n f(\lambda) \frac{n\lambda}{a^{3}} = \frac{(1+\nu) \frac{2T}{\pi a} n^{5} \lambda}{(\lambda^{2} + n^{2})^{2}}
\]

From Eq. (58:a) we obtain \( v \) by substituting \( v(\lambda) \) in the above relation

\[
v = \frac{(1+\nu)^{2}}{Eh} \frac{2T}{\pi a} \sum_{n=1,3,\ldots}^{\infty} n^{5} \cos \frac{n}{a} \int_{0}^{\infty} \frac{\lambda}{(\lambda^{2} + n^{2})^{2}} S \sin \lambda \frac{\lambda}{a} \, d\lambda
\]

The definite integral above has a value

\[
I = \int_{0}^{\infty} \frac{ze^{x^{2}}}{(z^{2} + n^{2})^{2}} \, dz = - \frac{1}{2} \frac{\partial}{\partial n} \int_{0}^{\infty} \frac{ze^{x^{2}}}{(z^{2} + n^{2})} \, dz = \frac{\pi}{4n a} e^{-n^{2}/a^{2}}
\]

Substituting the above result in Eq. (64), we have

\[
v = \frac{(1+\nu)^{2}}{Eh} \frac{T}{2\pi a} \sum_{n=1,3,\ldots}^{\infty} n \cos \frac{n}{a} \left[ \frac{1}{a} e^{-n^{2}/a^{2}} \right]
\]

and

\[
\frac{\partial v}{\partial x} = \frac{(1+\nu)^{2}}{Eh} \frac{T}{2\pi a} \sum_{n=1,3,\ldots}^{\infty} n \cos \frac{n}{a} \left[ -n \frac{1}{a} \right] e^{-n^{2}/a^{2}}
\]

The second differential equations which will be used to solve \( u_2 \) and \( v_2 \) are obtained from Eqs. (29:a) and (29:b) by putting \( q = 0 \)

\[
\nabla^{4} u - \frac{\nu}{\beta} \frac{\partial^{4} w}{\partial x^{4}} + \frac{1}{\beta} \frac{\partial^{4} w}{\partial x^{2} \partial s^{2}} = 0
\]

\[
\nabla^{4} v - \frac{1}{\beta} \frac{\partial^{4} w}{\partial s^{4}} - \frac{(2+\nu)}{\beta} \frac{\partial^{4} w}{\partial x^{2} \partial s^{2}} = 0
\]
Before solving the above equations we have to determine the undetermined function \( w(\lambda) \) from Eq. (29:c). This can be achieved by substituting Eqs. (58:a) and (58:d):

\[
\sum_{n=1}^{\infty} -\int_{0}^{\infty} \left\{ w(\lambda) \left[ \left( \frac{\lambda}{a} \right)^{n} + \left( \frac{\lambda}{a} \right)^{-n} \right] + \frac{Eh}{aD} \left( \frac{\lambda}{a} \right)^{4} \right\} - \frac{2}{D} \frac{f(\lambda)}{a} \left[ \left( \frac{\lambda}{a} \right)^{n} + \left( \frac{\lambda}{a} \right)^{-n} \right] \left[ \lambda^{n} \right] \right\} d\lambda = 0
\]

for all values of \( \lambda \), or

\[
w(\lambda) = \frac{2 T a n \lambda}{n^{2} D} \frac{(n^{2} - \nu \lambda^{2})}{\lambda^{n^{2} + j^{2} \lambda}}
\]

By substituting Eqs. (58:b) and (58:d) in Eq. (67:a) we obtain, as before

\[
u = \frac{2 T a}{n^{2} D} \frac{n^{2} \left( \nu \lambda^{2} - n^{2} \right)}{\lambda^{n^{2} + j^{2} \lambda}}
\]

Substituting this in Eq. (58:b) after differentiating it with respect to \( s \), we have

\[
\frac{\lambda u_{s}}{s} = \frac{2 T a}{n^{2} D} \sum_{n=1}^{\infty} n^{2} \cos \frac{n s}{a} \int_{0}^{\infty} \lambda^{n^{2} + j^{2} \lambda} \frac{\nu \lambda^{2} - n^{2}}{\lambda^{n^{2} + j^{2} \lambda}} \frac{d\lambda}{\lambda^{n^{2} + j^{2} \lambda}}
\]

Similarly \( v(\lambda) \) is obtained by substituting Eqs. (58:a) and (58:c) in Eq. (67:b)

\[
v(\lambda) = \frac{2 T a}{n^{2} D} \frac{n^{2} \lambda^{n^{2} + \nu \lambda^{2}}}{\lambda^{n^{2} + j^{2} \lambda}}
\]

Differentiating Eq. (58:c) and using the above relation of \( v(\lambda) \), we obtain

\[
\frac{\lambda v_{s}}{s} = \frac{2 T a}{n^{2} D} \sum_{n=1}^{\infty} n^{2} \cos \frac{n s}{a} \int_{0}^{\infty} \lambda^{n^{2} + j^{2} \lambda} \frac{\nu \lambda^{2} - n^{2}}{\lambda^{n^{2} + j^{2} \lambda}} \frac{d\lambda}{\lambda^{n^{2} + j^{2} \lambda}}
\]

(69)
As mentioned before, the shearing stress resultant in this case can be obtained by combining the terms containing the derivatives of \( u \) and \( v \) with respect to \( s \) and \( x \), respectively.

\[
S_{i_1} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial u_1}{\partial s} + \frac{\partial v_1}{\partial x} \right) = \frac{T}{2\pi a^2} \sum_{n=1,3,\ldots}^{\infty} \cos \frac{n}{a}(2n) e^{-n^2 \lambda^2} \quad (70:a)
\]

\[
S_{i_2} = \frac{Eh}{2(1+\nu)} \left( \frac{\partial u_2}{\partial s} + \frac{\partial v_2}{\partial x} \right) = \frac{2TJ^2}{\pi a^2} \nu \sum_{n=1,3,\ldots}^{\infty} n^2 \cos \frac{n}{a} \int_0^\infty \frac{\lambda^4(\lambda^2 - n^2) \cos \frac{\lambda a}{3} d\lambda}{(\lambda^2 + n^2)^{(a^2 + n^2)^2 + j^2 \lambda^2}} \quad (70:b)
\]

In order to evaluate the above definite integral we again apply the relation used for evaluating Eq. (35) and also the Theorem of Residues, which gives

\[
\int_{-\infty}^{\infty} \frac{e^{iz} e^{iz \lambda^2}}{(z^2 + n^2)^2} dz
\]

\[
= 2\pi i \left\{ -\frac{1}{2n} \left( \frac{n^4(-n^2 - \frac{n^3}{a^2}) e^{-n^2 \lambda^2}}{2i n(n^2 + \alpha_i^2)(n^2 + \alpha_i^4)} \right) + \frac{\alpha_i^4(\alpha_i^2 - \frac{n^3}{a^2}) e^{i\alpha_i^2 \lambda^2}}{2(\alpha_i^2 + n^2)(\alpha_i^2 - \alpha_i^4)(\alpha_i^2 - \alpha_i^4)} \quad (71)
\]

\[
+ \frac{\alpha_i^3(\alpha_i^2 - \frac{n^3}{a^2}) e^{i\alpha_i \lambda^2}}{2(\alpha_i^2 + n^2)(\alpha_i^2 - \alpha_i^4)(\alpha_i^2 - \alpha_i^4) - \frac{\alpha_i^3}{\alpha_i^3(\alpha_i^2 - n^2)} e^{i\alpha_i \lambda^2}} + \frac{\alpha_i^3(\alpha_i^2 - \frac{n^3}{a^2}) e^{i\alpha_i \lambda^2}}{2(\alpha_i^2 + n^2)(\alpha_i^2 - \alpha_i^4)(\alpha_i^2 - \alpha_i^4)} \right\}
\]
where $\alpha_s$ are the roots of the algebraic equation in the denominator mentioned before.

The first term in expression (71) gives, after simplifying,

\[
\frac{\prod}{2n} \left\{ \frac{2n + (n^* + \frac{n^*}{a}) 3n^* - \frac{n^*}{a} n^*(n^* + \frac{n^*}{a})}{(n^* + \alpha^*_1)(n^* + \alpha^*_2)(n^* + \alpha^*_3)} e^{-n^*_2} \right. \\
\left. \cdot \frac{-n^*(n^* + \frac{n^*}{a}) e^{-n^*_2}}{[(n^* + \alpha^*_1)(n^* + \alpha^*_2)(n^* + \alpha^*_3)]^2} \right\}^{2n(n^* + \alpha^*_1)(n^* + \alpha^*_2)(n^* + \alpha^*_3) + 2n(n^* + \alpha^*_1)(n^* + \alpha^*_3) + 2n(n^* + \alpha^*_2)(n^* + \alpha^*_3)} \\
+ 2n(n^* + \alpha^*_1)(n^* + \alpha^*_2)(n^* + \alpha^*_3) + 2n(n^* + \alpha^*_1)(n^* + \alpha^*_3) + 2n(n^* + \alpha^*_2)(n^* + \alpha^*_3)}
\]

Substitution of $\alpha_s$ from Eq. (12) results in the simplified form

\[
-\frac{\prod}{2n^*} e^{-n^*_2} \left[ \frac{1 + \frac{n^* + \nu}{\nu}} \right]^{-2n^*} \\
(72)
\]

In simplifying the next four terms in expression (71) we again use the notation introduced in (51). The combined form of the second and fifth terms gives

\[
\frac{\prod}{16n^*\nu R^4} \left\{ (n^* + \eta - i \frac{J}{2} + i \phi)(-n^* + \eta - i \frac{J}{2} + i \phi - \frac{n^*}{a^*}) e^{i\omega\phi} \right. \\
\cdot \left\{ (n^* + \eta - i \frac{J}{2} + i \phi)(-n^* + \eta - i \frac{J}{2} + i \phi - \frac{n^*}{a^*}) e^{i\omega\phi} \right\} \\
- (n^* + \eta + i \frac{J}{2} - i \phi)(-n^* + \eta + i \frac{J}{2} - i \phi - \frac{n^*}{a^*}) e^{i\omega\phi} \\
\left\{ (n^* + \eta + i \frac{J}{2} - i \phi)(-n^* + \eta + i \frac{J}{2} - i \phi - \frac{n^*}{a^*}) e^{i\omega\phi} \right\}
\]

Separating the real and imaginary parts and introducing the trigonometric functions, we get
\[- \frac{\pi (J + 2 \mu)^2}{8 n^6 J^4 R_e} \left\{ \left( A \cos \frac{A_k}{\alpha} - B \sin \frac{A_k}{\alpha} \right) \left\{ \left( \eta + \frac{n^4}{\nu} - 2 n^2 \eta + \eta - \frac{n^4}{\nu} \phi + \phi J - \frac{J^2}{4} \right) \right\} \\
+ (4 \eta^2 J + J^2 \phi - 4 \eta^2 \phi) + (-2 n^4 + 2 \eta^2 - \frac{n^4}{\nu}) \left( \phi - \frac{J}{2} \right) \left( J^2 \eta - 4 J \eta \phi - 4 \eta^3 \phi \right) \right\} \\
+ \left( A \sin \frac{A_k}{\alpha} + B \cos \frac{A_k}{\alpha} \right) \left\{ \left( -2 n^4 + 2 \eta^2 - \frac{n^4}{\nu} \right) \left( \phi - \frac{J}{2} \right) \left( 4 J \eta^2 + 2 J^2 \phi - 4 \eta^3 \phi \right) \right\} \\
+ \left( n^4 + \frac{n^4}{\nu} - 2 n^2 \eta + \eta^2 - \eta \frac{n^4}{\nu} \phi + \phi J + \frac{J^2}{4} \right) \left( J^2 \eta - 4 J \eta \phi - 4 \eta^3 \phi \right) \right\} e^{\frac{-6 \phi}{n^5 J^4 R_e}} \]

Further simplifications will reduce the second and fifth terms of Eq. (71) to a final form, which is

\[- \frac{\pi e^{-6 \phi}}{8 n^6 J^4 R_e} \left\{ \left( A \cos \frac{A_k}{\alpha} - B \sin \frac{A_k}{\alpha} \right) \left\{ n^2 \left( \frac{1 + \mu}{\nu} \right) \left( 4 n^4 J^2 \phi - 2 n^2 J^2 R_e \phi + \frac{J}{2} n^4 J^4 \phi \right) \right\} \\
+ 2 n^2 J^2 \eta \right\} \left( A \sin \frac{A_k}{\alpha} + B \cos \frac{A_k}{\alpha} \right) \left\{ n^2 \left( \frac{1 + \mu}{\nu} \right) \left( 4 n^4 J^2 \phi + 4 n^4 J^2 R_e \right) \\
+ 2 n^2 J^2 \phi \right\} - 2 n^2 J^2 \phi - 4 n^4 J^2 R_e \right\} \]

Similarly the third and fourth terms in Eq. (71) are simplified as follows:

\[- \frac{\pi e^{-6 \phi}}{8 n^6 J^4 R_e} \left\{ \left( C \cos \frac{C_k}{\alpha} - D \sin \frac{C_k}{\alpha} \right) \left\{ n^2 \left( \frac{1 + \mu}{\nu} \right) \left( 4 n^4 J^2 \phi - 2 n^2 J^2 R_e \phi + \frac{J}{2} n^4 J^4 \phi \right) \right\} \\
+ 2 n^2 J^2 \eta + \left( D \cos \frac{C_k}{\alpha} + C \sin \frac{C_k}{\alpha} \right) \left\{ n^2 \left( \frac{1 + \mu}{\nu} \right) \left( 4 n^4 J^2 \phi + 2 n^4 J^2 \phi \right) \\
- 4 n^4 J^2 R_e \right\} - 2 n^2 J^2 \phi + 4 n^4 J^2 R_e \right\} \]

Summarizing expressions (71), (72) and (74) and substituting the result in Eq. (70.5) we finally obtain
\[ S_{1z} = \frac{T}{2\pi a} \sum_{n=1}^{\infty} \cos \frac{ns}{a} \left\{ \left( (1-\nu)n + (1+\nu)n^2 \phi \frac{x}{a} \right) e^{-n^2 \frac{\alpha}{a}} - \frac{1}{4 J_{1+j}^2} \left[ (A \cos \frac{A_k}{a} - B \sin \frac{A_k}{a}) \left( (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{t \phi}{2n^2}) + 2 \nu \frac{J_1 \phi}{n^2} \right) e^{-B \frac{\phi}{a}} 
right. \\
+ (A \sin \frac{A_k}{a} + B \cos \frac{A_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{J_1 \phi}{n^2}) - 4 \nu K_0 \frac{\phi}{n^2} \right] e^{-B \frac{\phi}{a}} \\
\left. + (C \cos \frac{C_k}{a} - D \sin \frac{C_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{2 J_1 \phi}{n^2}) - \frac{4 \nu K_0 \phi}{n^2} \right] e^{-D \frac{\phi}{a}} \\
+ (D \cos \frac{C_k}{a} + C \sin \frac{C_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{2 J_1 \phi}{n^2}) + \frac{4 \nu K_0 \phi}{n^2} \right] e^{-D \frac{\phi}{a}} \right\} \]

Combining \( S_{1z} \) and \( S_{2z} \) from Eqs. (70; a) and (75) we obtain the stress resultant in the following alternative form:

\[ \frac{S_{1z}}{a^2} = \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos \frac{ns}{a} \left\{ \left( (1+\nu)n - (1+\nu)n^2 \phi \frac{x}{a} \right) e^{-n^2 \frac{\alpha}{a}} - \frac{1}{4 J_{1+j}^2} \left[ (A \cos \frac{A_k}{a} - B \sin \frac{A_k}{a}) \left( (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{t \phi}{2n^2}) + 2 \nu \frac{J_1 \phi}{n^2} \right) e^{-B \frac{\phi}{a}} 
\right. \\
\left. + (A \sin \frac{A_k}{a} + B \cos \frac{A_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{J_1 \phi}{n^2}) - 4 \nu K_0 \frac{\phi}{n^2} \right] e^{-B \frac{\phi}{a}} \\
+ (C \sin \frac{C_k}{a}) e^{-D \frac{\phi}{a}} \right\} \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{2 J_1 \phi}{n^2}) - \frac{4 \nu K_0 \phi}{n^2} \right] e^{-B \frac{\phi}{a}} \\
\left. (A \cos \frac{A_k}{a} - B \sin \frac{A_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{J_1 \phi}{n^2}) + 2 \nu \frac{J_1 \phi}{n^2} \right] e^{-B \frac{\phi}{a}} \\
\right. \\
\left. + (D \cos \frac{C_k}{a} + C \sin \frac{C_k}{a}) \left[ (1+\nu)(4 \phi - \frac{2 R_0 \phi}{n^2} + \frac{2 J_1 \phi}{n^2}) + \frac{4 \nu K_0 \phi}{n^2} \right] e^{-D \frac{\phi}{a}} \right\} \]

Equations (55) and (76) are used for the calculation of the shearing stress distribution on an infinitely long thin cylinder under two equal and opposite torques acting about the radial axis on the surface of the cylinder.
3. INVESTIGATION OF STRESSES IN THE IMMEDIATE VICINITY OF THE APPLIED TORQUE

In Fig. 11 an infinitely small square element is cut out of a thin cylinder in the immediate vicinity of the applied torque. All the possible stress-resultants that may produce any torques about the element are shown in the above figure. In order to verify that the total torque produced by the force-resultants is equal to the applied torque we can summarize all the force-resultants multiplied by their corresponding moment arms. The total torque around the small square element of length $2\varepsilon$ may be summarized as follows

\[
\text{Torque} = \varepsilon \left[ \int_{-\varepsilon}^{\varepsilon} \varepsilon S \, ds + \int_{-\varepsilon}^{\varepsilon} \varepsilon S \, dx + \int_{-\varepsilon}^{\varepsilon} \varepsilon S \, dx \right]
+ 4 \left[ \int_{-\varepsilon}^{\varepsilon} T_x \, ds + \int_{-\varepsilon}^{\varepsilon} T_s \, dx \right].
\]

(77)

where $S$ is the shearing stress-resultant derived in Eqs. (56) and (78), and $T_x$ and $T_s$ are the corresponding normal stress-resultants acting on the element in the direction of the $x$ and $s$ axes, respectively. They can be determined from the equations of equilibrium.

\[
\frac{\partial S}{\partial s} + \frac{\partial T_x}{\partial x} = 0
\]

\[
\frac{\partial S}{\partial x} + \frac{\partial T_s}{\partial s} = 0
\]

(78)
Let us now consider the behavior of \( S_\chi \) in Eq. (55) when both \( x \) and \( s \) approach a very small value, i.e., \( x \to s \to \varepsilon \) and \( \varepsilon \to 0 \). The parameters \( A, B, C, D \) in Eq. (15) have the following limiting values when \( n \gg 0 \):

\[
A = C = \frac{\sqrt{\pi}}{2}c
\]

\[
B = D = n
\]

Furthermore, we can write

\[
\cos A \frac{x}{\varepsilon} \to 1, \quad S_{\varepsilon} A \frac{x}{\varepsilon} \to 0 \quad \& \quad e^{-B A} = e^{-B A} = e^{-n A}
\]

Using the above notations, Eq. (55) is reduced to

\[
S_{\varepsilon} = \frac{1}{\varepsilon \pi n^2} \sum_{n=1,4,9,\ldots} e^{-n^2 \frac{x}{\varepsilon}} \cos n \frac{x}{\varepsilon} \left\{ \frac{(1+u)}{2} \left[ n - n \frac{2 \pi x}{\varepsilon} \right] + \varepsilon n \right\}
\]

(79)

where the terms containing \( \frac{1}{n} \) are neglected in the expression (78) because they vanish after integration in expression (77).

Since the infinite series \( e^{-n^2 \frac{x}{\varepsilon}} \cos n \frac{x}{\varepsilon} \) can be summed, and the sum of its \( n \) terms was found in Part I, we may write:

\[
\sum_{n=1,4,9,\ldots} e^{-n^2 \frac{x}{\varepsilon}} \cos n \frac{x}{\varepsilon} = \frac{1}{\varepsilon} \left[ \frac{1}{e^{\frac{x}{\varepsilon} (x+is) - 1} + e^{\frac{x}{\varepsilon} (x-is) - 1}} \right]_{x \to \infty, s > 0}
\]

The sum of \( n \) terms of \( n e^{-n^2 \frac{x}{\varepsilon}} \cos n \frac{x}{\varepsilon} \) is found by differentiating the above expression, i.e.,

\[
\sum_{n=1,4,9,\ldots} n e^{-n^2 \frac{x}{\varepsilon}} \cos n \frac{x}{\varepsilon} = -\frac{\partial}{\partial x} \left[ \frac{1}{e^{\frac{x}{\varepsilon} (x+is) - 1} + e^{\frac{x}{\varepsilon} (x-is) - 1}} \right]_{x \to \infty, s > 0}
\]

\[
= \frac{\varepsilon e^{\frac{x}{\varepsilon}} \left[ e^{\frac{x}{\varepsilon}} \cos 2 \frac{x}{\varepsilon} - 2 e^{-\frac{x}{\varepsilon}} + \cos 2 \frac{x}{\varepsilon} \right]}{\left[ e^{\frac{x}{\varepsilon}} - 2 e^{2 \frac{x}{\varepsilon}} \cos 2 \frac{x}{\varepsilon} + 1 \right]^2}
\]

Substituting

\[
\cos 2 = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots
\]

\[
e^{2} = 1 + \frac{x^2}{1!} + \frac{x^3}{2!} + \cdots
\]
and neglecting the higher terms on account of the smallness of \( x \) and \( s \),
we obtain

\[
\sum_{n=2,4,...; \quad x \to s \to e}^{\infty} n e^{-n\frac{x}{a}} \cos n \frac{s}{a} = \frac{(\frac{s}{a})^{2} - (\frac{x}{a})^{2}}{2[(\frac{s}{a})^{2} + (\frac{x}{a})^{2}]^{2}}
\]

Similarly we obtain

\[
\sum_{n=2,4,...; \quad x \to s \to e}^{\infty} n^{2} e^{-n\frac{x}{a}} \cos n \frac{s}{a} = \frac{a^{2} \frac{d}{dx}}{2} \left[ \frac{1}{e^{\frac{x}{a}(x+s)} - 1} + \frac{1}{e^{\frac{x}{a}(x-s)} - 1} \right]
\]

\[
= -4 \left\{ \frac{e^{\frac{x}{a}}(e^{\frac{x}{a}} \cos \frac{s}{a} + \cos \frac{s}{a} - 2 e^{\frac{s}{a}})}{[e^{\frac{x}{a}} - 2 e^{\frac{s}{a}} \cos \frac{s}{a} + 1]^{3}} \right. \\
- \frac{2 e^{\frac{x}{a}}(e^{\frac{x}{a}} \cos \frac{s}{a} + 3 e^{\frac{s}{a}} \cos \frac{s}{a} - 3 e^{\frac{s}{a}} - \cos \frac{s}{a})}{[e^{\frac{x}{a}} - 2 e^{\frac{s}{a}} \cos \frac{s}{a} + 1]^{3}} \right\}
\]

For small \( x \) and \( s \) the above expression can be reduced as follows

\[
\sum_{n=2,4,...; \quad x \to s \to e}^{\infty} n^{2} e^{-n\frac{x}{a}} \cos n \frac{s}{a} = -\left(\frac{s}{a}\right)^{5} - \left(\frac{x}{a}\right)^{5} + \frac{\left(\frac{s}{a}\right)^{3} - 3 \frac{x a^{2}}{a^{2}}}{[(\frac{s}{a})^{2} + (\frac{x}{a})^{2}]^{3}} \right.
\]

\[
= \frac{\left(\frac{s}{a}\right)^{5} - \left(\frac{x}{a}\right)^{5}}{[(\frac{s}{a})^{2} + (\frac{x}{a})^{2}]^{3}} \right) + \frac{\left(\frac{s}{a}\right)^{3} - \left(\frac{x}{a}\right)^{3}}{[(\frac{s}{a})^{2} + (\frac{x}{a})^{2}]^{3}} \right)
\]

Finally we obtain a summed form of \( S_{t} \) by substituting the above summation in Eq. (79)

\[
S_{t} = \frac{T}{2 \pi a^{z}} \left\{ \frac{1 + \nu}{12} \left[ \frac{\left(\frac{s}{a}\right)^{6} - \left(\frac{x}{a}\right)^{6}}{2[(\frac{s}{a})^{4} + (\frac{x}{a})^{4}]^{2}} - \frac{(\frac{s}{a})^{4} - 3 (\frac{s}{a})^{2}(\frac{x}{a})^{2}}{[(\frac{s}{a})^{4} + (\frac{x}{a})^{4}]^{3}} \right] \right\}
\]

(80)

In a like manner we reduce Eq. (78) in the following expression for small \( x \) and \( s \)

\[
S_{t} = \frac{T}{2 \pi a^{z}} \sum_{n=1,3,...}^{\infty} e^{-n\frac{x}{a}} \cos n \frac{s}{a} \left[ (1 + \nu) \left[ 1 - \frac{1}{12} \right] \left[ n - n \frac{a}{x} \right] + 2 n \right]
\]

(81)
Substituting the following summations into Eq. (61)

\[
\sum_{n=1,3,5,\ldots}^\infty n e^{-n^2 \frac{x}{a}} \cos n \frac{s}{a} = -\frac{a}{4} \frac{\partial}{\partial x} \left[ \frac{1}{\sinh \frac{x+is}{a}} + \frac{1}{\sinh \frac{x-is}{a}} \right]
\]

\[
\sum_{n=1,3,5,\ldots}^\infty n e^{-n^2 \frac{x}{a}} \cos n \frac{s}{a} = \frac{1}{2} \frac{(\frac{x}{a})^2 - (\frac{s}{a})^2}{[(\frac{x}{a})^2 + (\frac{s}{a})^2]^2}
\]

\[
\sum_{n=1,3,5,\ldots}^\infty n^2 e^{-n^2 \frac{x}{a}} \cos n \frac{s}{a} = \frac{a^2}{4} \frac{\partial^2}{\partial x^2} \left[ \frac{1}{\sinh \frac{x+is}{a}} + \frac{1}{\sinh \frac{x-is}{a}} \right]
\]

\[
\sum_{n=1,3,5,\ldots}^\infty n^4 e^{-n^2 \frac{x}{a}} \cos n \frac{s}{a} = \left[ \frac{-4 \frac{x}{a}}{(\frac{x}{a})^2 + (\frac{s}{a})^2} + \frac{\left(\frac{x}{a}\right)^3 - 3(\frac{x}{a})^2(\frac{s}{a})}{[(\frac{x}{a})^2 + (\frac{s}{a})^2]^3} \right]
\]

we obtain

\[
S_t = \frac{T}{2\pi a^2} \left\{ (1+\nu) \left[ 1 - \frac{1}{2} \right] \left[ \frac{(\frac{x}{a})^2 - (\frac{s}{a})^2}{2 \left[ (\frac{x}{a})^2 + (\frac{s}{a})^2 \right]^2} - \frac{(\frac{x}{a})^4 - 3(\frac{x}{a})^2(\frac{s}{a})^2}{[(\frac{x}{a})^2 + (\frac{s}{a})^2]^3} \right] + \frac{(\frac{x}{a})^4 - (\frac{s}{a})^4}{[(\frac{x}{a})^2 - (\frac{s}{a})^2]^2} \right\}
\]  

(82)

The total shearing stress-resultant is

\[
S = \frac{S_k + S_s}{2} = \frac{T}{2\pi a^2} \left\{ (3+\nu) \left[ \frac{(\frac{x}{a})^2 - (\frac{s}{a})^2}{2 \left[ (\frac{x}{a})^2 + (\frac{s}{a})^2 \right]^2} - (1+\nu) \left[ \frac{(\frac{x}{a})^4 - 3(\frac{x}{a})^2(\frac{s}{a})^2}{[(\frac{x}{a})^2 + (\frac{s}{a})^2]^3} \right] \right\}
\]  

(83)

Now the normal stress-resultants can be determined by substituting \( S \) from expression (83) into Eq. (78) and solving for \( T_z \) and \( T_{z_s} \). Differentiating expression (83) with respect to \( s \), we have

\[
\frac{\partial S}{\partial s} = \frac{T}{2\pi} (1+\nu) \left\{ \frac{1}{2} \left[ \frac{(x^2 + s^2)^2 (-s) - 2 (x^2 - s^2)(x^2 + s^2)(2s)}{(x^2 + s^2)^2} \right]
\]

\[
- \left[ \frac{(x^2 + s^2)^2 (-6 x s^5) - 3 (x^4 - 3 x^2 s^2)(x^2 + s^2)^2 (2s)}{s^2 (x^2)} \right] \}
\]

\[
+ \frac{T}{2\pi} \left[ \frac{(x^2 + s^2)^3 (-2 s) - 2 (x^2 - s^2)(2s)}{(x^2 + s^2)^4} \right]
\]

- 65 -
Integrating the above expression, we get

\[ T_x = -\frac{T}{2\pi} (1+\nu) \left[ -\frac{S}{(x^3 + s^3)} + \frac{2(2x^4s + s^4)}{(x^3 + s^3)^2} + \frac{6s^3(x^6 - 3x^3s^3)}{(x^3 + s^3)^3} \right] dx + C \]

\[ -\frac{T}{2\pi} \left[ \frac{-2S}{(x^3 + s^3)^2} - \frac{4(x^5 - s^3)}{(x^3 + s^3)^3} \right] dx + C \]

From the boundary condition, i.e., \( S = 0 \), \( T_x = 0 \), the constant of integration is found to be zero.

Hence

\[ T_x = (1+\nu) \frac{T}{2\pi} \left[ \frac{3xs}{(x^3 + s^3)^2} - \frac{4s^3x}{(x^3 + s^3)^3} \right] \]

\[ -\frac{T}{2\pi} \left[ \frac{-2S}{(x^3 + s^3)^2} - \frac{4(x^5 - s^3)}{(x^3 + s^3)^3} \right] \]

Equation (84)

Next differentiating expression (83) with respect to \( x \), we obtain:

\[ \frac{3S}{ds} = (1+\nu) \frac{T}{2\pi} \left[ \frac{x^2}{(x^3 + s^3)^2} - \frac{(2x^2 - 2xs^2) + (4x^4 - 6s^2x)}{(x^3 + s^3)^3} + \frac{6x^2 - 12x^3s^2}{(x^3 + s^3)^3} \right] \]

\[ + \frac{T}{2\pi} \left[ \frac{2x}{(x^3 + s^3)^2} - \frac{4(x^5 - x^3s^2)}{(x^3 + s^3)^3} \right] \]

Integrating the above expression, we get

\[ T_s = (1+\nu) \frac{T}{2\pi} \left[ \frac{3xs}{(x^3 + s^3)^2} - \frac{4s^3x}{(x^3 + s^3)^3} \right] + \frac{T}{2\pi} \left[ \frac{2xs}{(x^3 + s^3)^2} \right] \]

Equation (85)

Substituting expressions (83), (84) and (85) into Eq. (77) we finally have

\[ \text{Torque} = \frac{T}{\pi a^2} \left\{ \varepsilon \int_0^\varepsilon \left[ \frac{(3 + \nu)[(\frac{\xi}{\alpha})^3 - (\frac{\xi}{\alpha})^5]}{\varepsilon \left[ (\frac{\xi}{\alpha})^3 + (\frac{\xi}{\alpha})^5 \right]^3} - \frac{(1+\nu)\left[ (\frac{\xi}{\alpha})^3 - 3(\frac{\xi}{\alpha})^5 \left( \frac{\xi}{\alpha} \right)^3 \right]}{(\frac{\xi}{\alpha})^3 + (\frac{\xi}{\alpha})^5} \right] dS \]

\[ + 2\varepsilon \int_0^\varepsilon \left[ \frac{(3 + \nu)[(\frac{\xi}{\alpha})^3 - (\frac{\xi}{\alpha})^5]}{\varepsilon \left[ (\frac{\xi}{\alpha})^3 + (\frac{\xi}{\alpha})^5 \right]^3} - \frac{(1+\nu)\left[ (\frac{\xi}{\alpha})^3 - 3(\frac{\xi}{\alpha})^5 \left( \frac{\xi}{\alpha} \right)^3 \right]}{(\frac{\xi}{\alpha})^3 + (\frac{\xi}{\alpha})^5} \right] dx \]

\[ + 4(1+\nu) \frac{T}{2\pi} \int_0^\varepsilon \left[ \frac{3xs^2}{(x^3 + s^3)^2} + \frac{4s^3x}{(x^3 + s^3)^3} \right] dS + \frac{4T}{2\pi} \int_0^\varepsilon \left( \frac{2xs^2}{(x^3 + s^3)^2} \right) dS \]
After integrating and putting \( x = \varepsilon \) we have

\[
\text{Torque} = \frac{T}{\pi a^2} \left\{ \left[ \frac{3x^5}{(x^2 + \varepsilon^2)^{5/2}} - \frac{4x^5}{(x^2 + \varepsilon^2)^{3/2}} \right] + \frac{4T}{2\pi} \int_0^\varepsilon \frac{2x^5}{(x^2 + \varepsilon^2)^{3/2}} \, dx \right\} \varepsilon - \left[ \frac{3x^4}{8(5^2 + \varepsilon^2)} + \frac{3x^4}{8(5^2 + \varepsilon^2)} - \frac{3x^4}{8(5^2 + \varepsilon^2)} \right] \varepsilon + (3 + \nu) a^2 \left[ \frac{3x^3}{4(x^2 + \varepsilon^2)^{5/2}} + \frac{3x^3}{4(x^2 + \varepsilon^2)^{5/2}} - \frac{3x^3}{4(x^2 + \varepsilon^2)^{5/2}} \right] \varepsilon - \left[ \frac{3x^2}{4(x^2 + \varepsilon^2)^{5/2}} \right] \varepsilon \left( - \frac{3x^2}{4(x^2 + \varepsilon^2)^{5/2}} + \frac{3x^2}{4(x^2 + \varepsilon^2)^{5/2}} \right) \varepsilon\right\}

After simplifying we get

\[
\text{Torque} = \frac{2T}{\pi} - \frac{2T}{\pi} + T = T
\]

This is equal to the applied torque.

If we transform all the force-resultants in polar coordinates the torque produced by the tangential force-resultant can be also proven to be equal to the applied torque. The stress-resultants in polar coordinates are:

\[
\begin{align*}
T_x &= \frac{(1 + \nu)T}{2\pi} \left[ \frac{3\sin^2 \theta}{2r^2} - \frac{4\sin^3 \theta \cos \theta}{r^2} \right] - \frac{T}{2\pi r^2} \sin \theta \\
T_s &= \frac{(1 + \nu)T}{2\pi} \left[ \frac{3\sin^2 \theta}{2r^2} - \frac{4\cos^3 \theta \sin \theta}{r^2} \right] + \frac{T}{2\pi r^2} \sin \theta \\
S &= \frac{(1 + \nu)T}{2\pi} \left[ \frac{\cos^2 \theta - 3\cos^2 \theta \sin^2 \theta}{r^2} \right] + \frac{T}{2\pi r^2} \cos \theta
\end{align*}
\]

(86)
From Fig. 12 it is seen that the tangential stress-resultant can be obtained with the aid of the following relations of equilibrium:

\[ S_{\theta} = T_{s} \cos \theta \sin \theta - T_{x} \cos \theta \sin \theta \cos \theta - 3 \sin \theta \]

Fig. 12

By substituting the known quantities \( T_{s}, T_{x}, \) and \( S \) from Eq. (66) into the above expression, we obtain

\[ S_{rr \theta} = -\left( \frac{1 + \nu}{2} \right) \frac{T}{9 \pi r^3} \left[ \frac{1}{2} \cos \theta \right] + \frac{T}{2 \pi r^2} \]  \( \text{(67)} \)

The torque produced by such a tangential force-resultant acting around a small circular element can then be expressed by

\[ \text{Torque} = 2 \int_{0}^{\pi} S_{rr \theta} r \, d\theta = \tau \]

Substituting expression (67) in the above integral, we get

\[ \text{Torque} = \frac{T}{\pi} \int_{0}^{\pi} \left[ \frac{1 + \nu}{2} \cos \theta + 1 \right] \, d\theta = \tau \]

This is an alternate method of verifying the summation of all the force-resultants acting on a small element will produce a torque which is equal in magnitude to the applied torque.
4. Method Used in the Calculation of

Shearing Stress Distribution near
the s-axis (x → 0)

In numerical calculations of the shearing stress distribution near the s-axis from Eqs. (55) and (76) it is found that ordinary methods fail because the series converges very slowly when x approaches zero. Before an effective method of computation is introduced let us investigate the behavior of Eq. (55) when x is very small. As mentioned before, the Parameters A, B, C and D have the following limiting values when n is very large

A → C → constant
B → D → n

Furthermore we can write if x → 0

\[
\begin{align*}
\cos \frac{Ae}{a} &= \cos \frac{C}{a} = 1 \\
\sin \frac{Ae}{a} &= \sin \frac{C}{a} = \frac{C}{a} = 0 \\
e^{-D \frac{e}{a}} &\to e^{-\frac{e}{a}} \to e^{-n \frac{e}{a}}
\end{align*}
\]

Knowing the above relation we can simplify Eq. (55) as follows

\[
\begin{align*}
\frac{S_*}{T^2} = & \frac{1}{2\pi} \sum_{n=2k}^{\infty} \cos n \frac{\epsilon}{a} \left\{ (A+C) e^{-n \frac{e}{a}} \left[ \frac{(1+\nu) \sqrt{1+ \frac{j}{j_1} - j}}{2j_2} \frac{1}{\sqrt{j_1 j} \sqrt{j_1 + j^2}} \right. \\
+ & \left. \frac{(1+\nu)}{j_2} \frac{1}{n} \frac{1}{\sqrt{j_1 j}} \sqrt{j_1 + j^2} \right] \right. \\
+ (B+D) e^{-n \frac{e}{a}} & \left. \left[ \frac{\sqrt{1+ \frac{j}{j_1} + j}}{2j_2} \frac{1}{n} \frac{1}{\sqrt{j_1 j} \sqrt{j_1 + j^2}} \right. \\
- & \left. \frac{(1+\nu)}{j_2} \frac{1}{n} \frac{1}{\sqrt{j_1 j} \sqrt{j_1 + j^2}} \right] \right\}
\end{align*}
\]
The above expression can be further simplified if the following relation is used

\[ A + C = \frac{3}{\sqrt{2}} \]

\[ B - D = \frac{3}{\sqrt{2}} \]

\[ B + D = \sum_{n=1,4,\ldots} \left( \frac{nJ}{nJ + j} \right) \]

Hence

\[ \sum_{N} \frac{1}{\alpha^2} \sum_{n=1,4,\ldots} e^{-\frac{n^2}{\alpha^2}} \cos \frac{nJ}{\alpha} \left\{ \frac{1+\nu}{4} \left( \frac{nJ}{nJ + j} \right) + \frac{1+\nu}{2} \left( \frac{nJ}{nJ + j} \right) \right\} \]

\[ \left\{ \frac{1}{4} \left( \frac{nJ}{nJ + j} \right) + \frac{1+\nu}{2} \left( \frac{nJ}{nJ + j} \right) \right\} \]

\[ (88) \]

In Eq. (88) let us now consider the term

\[ \sum_{n=1,4,\ldots} e^{-\frac{n^2}{\alpha^2}} \cos \frac{nJ}{\alpha} \left[ \frac{\sqrt{1+J^2} - j}{nJ + j} \right] \]  \hspace{1cm} (88:a)

As \( n \) increases, \( j \) decreases in inverse proportion to the square of \( n \), and when \( n \) reaches values of importance in practical computation the term \( \frac{\sqrt{1+J^2} - j}{nJ + j} \) approaches \( \frac{1}{n} \). Hence we can introduce a known series \( e^{-\frac{n^2}{\alpha^2}} \cos \frac{nJ}{\alpha} \left[ \frac{1}{n} \right] \) the sum of which can be calculated. The difference between the term given in (88:a) and the proposed known series allows an easy computation. It is
\[
\sum_{n=2A}^{\infty} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} \left[ \frac{1}{n} \right]_{1+J^2} = \sum_{n=2A}^{\infty} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} \left[ \frac{1}{n} \right]_{1+J^2} - \frac{1}{n}
\]

(88:b)

\[+
\sum_{n=2A}^{\infty} \frac{i}{n} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha}
\]

where the first term in expression (88:b) may be computed in the ordinary manner and the second term has a sum which is found as follows:

\[
\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} = -\frac{1}{\alpha} \int \left[ \frac{1}{e^{\frac{1}{\alpha}(x+i\xi)} - 1} + \frac{1}{e^{\frac{1}{\alpha}(x-i\xi)} - 1} \right] d\xi + C
\]

After integration we get

\[
\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} = \frac{x}{\alpha} - \frac{1}{4} \log \left| e^{\frac{1}{\alpha}x} - 2 e^{\frac{1}{\alpha}x} \cos 2 \frac{x}{\alpha} + 1 \right| + C
\]

The constant of integration can be determined from the condition at \( x \to \infty \). It is found to be zero after the indeterminate form is evaluated. For a very small \( x \), we have:

\[
\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} = -\frac{1}{4} \log \left| 2 - 2 \cos 2 \frac{x}{\alpha} \right|
\]

(88:c)

Similarly, we obtain

\[
\sum_{n=1}^{\infty} n e^{-\frac{n^2}{\alpha}} \cos n \frac{x}{\alpha} = \frac{1}{\cos 2 \frac{x}{\alpha} - 1} = -\frac{1}{2 \sin^2 \frac{x}{\alpha}}
\]

(88:d)

With the aid of summations (88:c) and (88:d) we can express Eq. (88) in the
\[
\frac{S_{1}}{\delta} = -\left(\frac{3+\nu}{2}\right) - \frac{1}{2} \sum_{n=1}^{\infty} \frac{\cos n\delta}{2} \cos \frac{n\delta}{\delta} + J \frac{1}{4} \sum_{n=1}^{\infty} \frac{\cos n\delta}{2} \cos \frac{n\delta}{\delta} \frac{(1+\nu) J}{16} \frac{\sqrt{J+1} + J}{J+1} + \frac{1}{4n} \left[ \frac{\sqrt{J+1} + J}{J+1} - 1 \right] \\
+ \frac{1+\nu}{4} \left[ \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 1 \right] - \frac{1+\nu}{4} \left[ \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 1 \right] + \frac{J}{4n} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] \\
+ 2n \left[ \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 1 \right] - \frac{J}{4n} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] - \frac{1+\nu}{2} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} + \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 2 \right] \right] 
\]

Eq. (76) can be expressed in the same manner after the sums of some known series are determined. These are

\[
\sum_{n=1}^{\infty} n e^{-\frac{n^2}{\delta}} \cos \frac{n\delta}{\delta} = -\frac{1}{2} \left[ \frac{\cos \frac{\delta}{\delta}}{2} \right] \quad (90a)
\]

\[
\sum_{n=1}^{\infty} \frac{1}{n} e^{-\frac{n^2}{\delta}} \cos \frac{n\delta}{\delta} = \frac{1}{2} \log \left\{ \frac{(\cos \frac{\delta}{\delta} + i)^2}{(\cos \frac{\delta}{\delta} - i)^2} \right\} \quad (90b)
\]

Through substitution of expressions (90a) and (90b) in Eq. (76), a modified form of \( S_{1} \) is obtained

\[
\frac{S_{1}}{\delta} = \frac{1}{2} \left[ \frac{\cos \frac{\delta}{\delta}}{2} \right] + J \frac{1}{4} \log \left\{ \frac{(\cos \frac{\delta}{\delta} + 1)^2}{(\cos \frac{\delta}{\delta} - 1)^2} \right\} \\
+ \frac{1}{2\pi} \sum_{n=1}^{\infty} \cos n\delta \left\{ \frac{(1+\nu) J}{16} \frac{\sqrt{J+1} + J}{n^2} - \frac{1+\nu}{4n} \left[ \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 1 \right] \\
+ \frac{1+\nu}{4} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] - \frac{1+\nu}{4} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] + \frac{J}{4n} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] \\
+ 2n \left[ \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 1 \right] - \frac{J}{4n} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} - 1 \right] - \frac{1+\nu}{2} \left[ \frac{\sqrt{J+1} + J}{\sqrt{J+1}} + \frac{\sqrt{J+1} - J}{\sqrt{J+1}} - 2 \right] \right\} \quad (91)
\]
CONCLUSION

The results calculated from Eqs. (17) and (18) were plotted in Figs. I to IX inclusive. It is seen that the radial deflection of an infinitely long cylinder has a very long wave length along the generatrix. However, the wave length decreases with the radius over thickness ratio. It is believed that the cause of this long wave length phenomenon is due to the elastic relations along the circumference of the shell which has been explained in Part I - 4. A family of maximum deflection curves of infinitely long plates with simply supported edges was plotted in Fig. 3 with various widths of plates over radius of cylinder ratio, and it shows that the plate has greater slope than the cylinder because of the curvature restraint of the cylinder.

Deflection curves of cylindrical shells with various lengths were calculated from Eqs. (22) and (23), and the results show that the maximum radial deflection occurs at about \( \frac{x}{a} = 20 \). Since the radial deflection of an infinitely long cylinder with \( a/h = 100 \) becomes zero at about \( \frac{x}{a} = 15 \) and then reverses its sign, the edges of the corresponding cylinder with finite length are so restrained that the negative deflection portion of the infinite cylinder is brought to zero at the edges of the cylinder with finite length. Hence the maximum deflection of a cylinder with \( 1/a = 20 \) is greater than the corresponding infinitely long cylinder.

Shearing stress distribution along both the generatrix and the circumference was calculated from Eqs. (35) and (74), and the results show that the stress decreases very rapidly along the generatrices. When the ratio of \( x/a \) reaches to 1.0 the shearing stress is almost negligible.
APPENDIX
### TABLE I

Deflection Parameter, $\alpha$, of an
Infinitely Long Cylinder Under Two Concentrated Loads Along the Generatrix $s = \theta^0$

\[
\frac{a}{h} = 100, \quad \frac{W}{Eh} = \alpha
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TABLE II

Deflection parameter, $\xi$, of an
Infinitely Long Cylinder Under Two Concentrated
Loads Along the Generatrix, $S_\xi = 0$

\[ \frac{a}{h} = 10^2 \quad \frac{w}{h/P} = \xi \]

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- 79 -
TABLE II
Deflection Parameter, \( \alpha \), of an
Infinitely Long Cylinder Under Two Concentrated
Loads Along the Generatrix \( \theta = 0^\circ \)
\( \alpha = 100 \left( \frac{n}{p} \right) \left( \frac{H}{H} \right) \)

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| 14    | 0.45150  | 702.9
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| 18    | 0.45490  | 1105.0
| 20    | 0.50650  | 1342.0
| 22    | 0.45070  | 1615.0
| 24    | 0.38430  | 1922.0
| 26    | 0.12220  | 2245.0
| 28    | 0.10530  | 2606.0
| 30    | 0.09178  | 2986.0
| 32    | 0.08068  | 3394.0
| 34    | 0.07146  | 3827.0
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| 40    | 0.05163  | 5292.0

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Table III

Maximum Radial Deflection Parameter of an Infinitely Long Cylinder

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<th>a/h</th>
<th>$0.20456(a/h)^{3/2}$</th>
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### Table III

**Maximum Radial Deflection Parameter of an Infinitely Long Cylinder**

*With Different Radius/Thickness Ratio*

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<td>$1 + \frac{6a^2}{n^2} (\frac{h}{a})^2$</td>
<td>$1 + \frac{6a^2}{n^2} (\frac{h}{a})^2$</td>
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\[ \sum_{n=2.4}^{\infty} \approx 0.06287 \quad 0.04474 \quad 0.03151 \]
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<th>( \frac{a}{h} = 500 )</th>
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$$\sum_{n=2.9}^{\infty} = 0.01160$$

* - 38 -
TABLE IV

Deflection Parameter $\zeta$, of an

Infinitely Long Cylinder Under Two

Concentrated Loads Along the Circumference

\[
\frac{4}{3} \left( \frac{a^2}{\pi E I} \right)
\]

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TABLE VII

Shearing Stress Distribution
Along the Circumference, \( x/a = 0 \), and the Generatrix, \( s/a = 0 \)
\( a/h = 100 \)

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</table>

<table>
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<tr>
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<th>5( \frac{\pi}{12} )</th>
<th>( \frac{\pi}{3} )</th>
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Comparison of Max. Deflections of a Cylinder with Plates of Various Width-to-Radius Ratios at Different Width-to-Radius Rations (Infinite Length)

Plates - Simply Supported at two Edges

Fig. II
Deflection Curve Along Circumferences
At Various $\frac{\alpha}{h}$
($\frac{\alpha}{h} = 10^3$)

Infinitely Long Cyl
Deflection Curves Along the Generatrices At Various Angles

\( n = 10^3 \)

inf. cyl.
Deflection Curves of Cylindrical Shells

with Different ($\frac{h}{R}$) Ratio Along the Generatrix

\[
\frac{h}{R} = 10^6
\]

\[
\frac{h}{R} = 0
\]

Fig. X
Deflection Curves of Cylindrical Shells with Different $t/h$ Ratios Along the Generatrix

\[ \frac{t}{h} = 10^2 \]

\[ \frac{c}{h} = 0 \]

Fig. XI
Shearing Stress Distribution Along a Cylindrical Shell at the Generator at $\frac{Z}{a} = 0$, $\frac{Z}{a} = 1$.
Shearing Stress Distribution of a Cylindrical Shell Along the Circumference

At $\frac{\theta}{\alpha} \rightarrow 0$, $\frac{\alpha}{h} = 10^2$

Fig XIII
REFERENCES


Th. Von Karman, M. A. Biot, Mathematical Methods in Engineering, Chapter VIII.

S. Flaubeneko, Theory of Elasticity, (1934), Chapter II.

S. Timoshenko, Theory of Plates and Shells, (1940), Chapter XI.


E. J. Titchmarsh, Introduction to the Theory of Fourier Integrals, Chapter I.


R. J. Jaralaw, Theory of Fourier Series and Integrals.

L. T. Whittaker, J. N. Watson, A Course of Modern Analysis, Chapter VI.