Twisted Heisenberg Central Extensions and the Affine ADE Basic Representation

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Abstract

We study various aspects of the representation theory of loop groups, all with the aim of giving geometric constructions, parameterized by conjugacy classes of the Weyl group, of the basic representation of the affine Lie algebras associated to a simply laced simple Lie algebra as a restriction isomorphism on dual sections of the level 1 line bundle on the affine Grassmannian. Along the way, we obtain various results on the structure of loop tori, the definition of a notion of a Heisenberg Central extension as an alternative for twisted modules over the lattice vertex algebra and the determination of their representation theory, some computations on central extensions of a torus over a field by $\boldsymbol{K}_2$, and a new proof of the classification of the conjugacy classes of the Weyl group by parabolic induction.

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Chapter 1

Introduction

This thesis gives a geometric realization of the representation-theoretic determination of the structure of the basic representation V of the affine Lie algebra $\hat{\mathfrak{g}}$ associated to a simple Lie algebra $\mathfrak{g}$ of type ADE in [KP85], where a different construction is given for each conjugacy class of the Weyl group W of $\mathfrak{g}$. In the case of $E8$, that is 112 constructions, hence the name of the paper [KP85]. We call the representation-theoretic work the 'twisted FKS isomorphism'. The classical (non-twisted) FKS isomorphism refers to the work of [FK81, Seg81] that do this for the conjugacy class $\{1\} \subseteq W$.

There are two parts to the thesis. The twisted FKS isomorphism was constructed in [KP85] using representation-theoretic methods of twisted vertex operators, one family for each conjugacy class c of W. In the first part of the project, we give a group-theoretic interpretation, where we replace the use of twisted vertex operators with a central extension, which we call the **(twisted) Heisenberg central extension**, $L\hat{\mathcal{T}}$ of the algebraic loop group $L\mathcal{T}$ of the torus $\mathcal{T}$ over $F = \mathbb{C}((t))$ defined by Galois descent by the action of $w \in W$ on the character lattice. We determine the group theoretic structure of $L\hat{\mathcal{T}}$ and in doing so, deduce its category of representations, in a way somewhat resembling some Lie-theoretic works of [BK04, Lep85]. The study of $L\hat{\mathcal{T}}$ passes through the study of central extensions of $\mathcal{T}$ by the sheaf $\boldsymbol{K}_2$ of [BD01] and we make some independent homological algebra computations of central extensions of $\mathcal{T}$ by $\boldsymbol{K}_2$. Along the way, we completely deduce the structure of an arbitrary loop torus $L\mathcal{T}$, in particular proving a basic fact that the Kottwitz map to the group of connected components admits a homomorphic section.

In the second part of the thesis, we apply our results on representations of Heisenberg central extensions to give a geometric construction of the twisted FKS isomorphism. Suppose G is the simply connected group of type ADE with Lie algebra $\mathfrak{g}$. There exists a line bundle $\mathcal{L}$ on the affine Grassmannian Gr_G whose dual global sections is identified with the basic representation V of $\hat{\mathfrak{g}}$. We construct for every $w \in W$ and lift $\sigma \in G$ of w, a subspace $\mathcal{S}(\sigma) \subseteq \mathrm{Gr}_G$ with the property that the dual global sections of the restriction $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee$ has the structure of an $\hat{L\mathcal{T}}$ module and the dual restriction map $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L})^\vee$ is an isomorphism.

The proof we give is completely geometric for many conjugacy classes of W, those we call **homogeneous**. They include 9 out of 30 elliptic conjugacy classes in $E8$ and all of the conjugacy classes of $w \in W$ that Lie in a subgroup of type A. For the non-homogeneous conjugacy classes, we needed to use some representation theoretic results of [KP85]. A fully geometric proof for all conjugacy classes of W is reduced to an explicit computation on the number of torsion points of the connected components flag variety of the integral closure of a torus in $\mathcal{T}$ in $G \times_{\mathbb{C}} F$ whose conjugacy class is classified by the conjugacy class of W.

In the original paper [Zhu09] motivating this problem, it was not known what is a correct subspace to use to obtain a twisted version of the geometric FKS isomorphism. It was only conjectured that for the a geometric twisted FKS isomorphism, an affine Springer fiber in Gr_G could be used instead of $\mathcal{S}(\sigma)$ to obtain a dual restriction isomorphism of dual global sections of $\mathcal{L}$. We did not end up pursuing the investigation of affine Springer fibers, because we have found $\mathcal{S}(\sigma)$ to be a more natural space to study for the nature of the problem. However we know that $\mathcal{S}(\sigma)$ is contained in an affine Springer fiber, although this discussion is omitted from the thesis.

Along the way, we found a new, geometric, proof of the classification of conjugacy classes of W by parabolic induction of [GP00] that originally used combinatorial methods. The conjugacy classes of W have also been studied by different combinatorial methods in the original paper [Car72] that was used in the paper constructing the twisted FKS isomorphism in [KP85].

Chapter 2

Twisted Heisenberg Central Extensions

2.1 Lie Algebra Preliminaries and Summary of Main Results

2.1.1 Motivation and Summary of Main Results

For a simple and simply connected complex group G with Lie algebra $\mathfrak{g}$, there exists a $\mathbb{G}_m$-central extension $\hat{LG_F}$ of the loop group of the base change of G to $F = \mathbb{C}((t))$, such that Lie$\hat{LG_F}$ is (formal version of) the affine Lie algebra $\hat{\mathfrak{g}}$ associated to $\mathfrak{g}$. We elaborate this in 3.

We eventually wish to study a representation of $\hat{LG_F}$ by restricting the action to the sub-central extension $\hat{L\mathcal{T}}$ for a maximal torus $\mathcal{T} \subseteq G_F$ that may not be split. The purpose of this chapter is to describe axiomatically define what kind of central extension $\hat{L\mathcal{T}}$ is and determine its representation theory. We call such central extensions **(twisted) Heisenberg central extensions**. We determine the group theoretic structure of $\hat{L\mathcal{T}}$ and in doing so, deduce its representation theory. We find that the category of representations is semisimple, with every irreducible object induced from a certain distinguished finite subgroup $\hat{\Sigma} \subseteq \hat{L\mathcal{T}}$ which we call the **principal finite Heisenberg subgroup**. It is obtained by restricting the central extension to an embedding of the torsion subgroup Galois coinvariants of $\Sigma := \mathbb{X}_*(T)_{\mathrm{Gal}(F),\mathrm{tor}}$ into

$L\mathcal{T}$. All of these generalize the study of the case when $\mathcal{T}$ is split in [Bei06].

The definition is indirect, passing through some computations of study of central extension of tori over a field by the sheaf $\boldsymbol{K}_2$ in the sense of [BD01]. The motivation is that there exists a central extension $\mathcal{E}$ of G_F by $\boldsymbol{K}_2$ such that the $\mathbb{C}$ points $L\hat{G}_F(\mathbb{C})$ is obtained by taking F points $\mathcal{E}(F)$ and then the pushout by the tame symbol $K_2(F) \to \mathbb{C}^\times$. We define a Heisenberg central extension $\hat{L\mathcal{T}}$ of an arbitrary torus $\mathcal{T}$ over F as one whose $\mathbb{C}$ points is obtained in the same way from a central extension of $\mathcal{T}$ by $\boldsymbol{K}_2$, but further require the Lie algebra to be (a formal version of) the Heisenberg Lie algebra studied in the representation theory literature, e.g., [FLM88, Kac90, KP85, BK04, Lep85].

We begin with preliminaries on certain Heisenberg Lie algebras and their representations, loop groups and intermediate models that will be also assumed in the entire remainder of the work. Along the way, we completely determine the structure of $L\mathcal{T}$ for an arbitrary torus $\mathcal{T}$ over F as a direct product of various other kinds of abelian groups objects, e.g., non-reduced, unipotent, discreet, torus. It involves showing a new basic fact that likely has many other applications, that the Kottwitz homomorphism from $L\mathcal{T}$ to the group of connected components admits a homomorphic section even when $\mathcal{T}$ is not necessarily split nor an induced torus. Finally, we also present some explicit computations about central extensions of $\mathcal{T}$ by $\boldsymbol{K}_2$ we use to compute $\hat{L\mathcal{T}}$.

2.1.2 Heisenberg Lie Algebras and Fock Space

We set some definitions regarding Heisenberg Lie algebras and their formal completions, and review their representation theory.

Definition 2.1: *[FLM88, 1.7] A* ***Heisenberg type Lie algebra*** *is a Lie algebra $\mathfrak{l}$ with an one-dimensional center that is equal to its derived subalgebra, i.e.,*

$$Z(\mathfrak{l}) = \mathfrak{l}' = [\mathfrak{l}, \mathfrak{l}] = \mathbb{C}K$$

for some nonzero $K \in \mathfrak{l}$. If a specific choice of K is specified, we call it the ***canonical***

central element.

The Heisenberg type Lie algebras that we study are usually given explicitly in terms of generators and relations. They all turn out to be isomorphic and have the same representation theory. For a $\mathbb{Z}$-graded Lie algebra $\mathfrak{l} = \oplus_{i\in\mathbb{Z}}\mathfrak{l}_i$, define the positive part $\mathfrak{l}_+ = \oplus_{i\geq 1}\mathfrak{l}_i$ and negative part $\mathfrak{l}_- = \oplus_{i\leq -1}\mathfrak{l}_i$.

Theorem 2.1: *[FLM88, 1.7] Suppose $\mathfrak{l}$ is a Heisenberg type Lie algebra of countable dimension admitting a $\mathbb{Z}$-grading with finite dimensional components. Define the Lie algebra $\mathfrak{H}$, defined by a basis $\{e_i, f_i, K : i \geq 1\}$ with relations*

$$[e_i, f_i] = K \text{ for } i \geq 1$$

$$[e_i, e_j] = [f_i, f_j] = [K, e_i] = [K, e_j] = 0 \text{ for } i \neq j, i \geq 1, j \geq 1$$

Equip $\mathfrak{H}$ with the grading $\mathfrak{H}_0 := \mathbb{C}K, \mathfrak{H}_i = \mathbb{C}e_i$ and $\mathfrak{H}_{-i} = \mathbb{C}f_i$, $i \geq 1$. Then there is an isomorphism of Lie algebras $\mathfrak{l} \approxeq \mathfrak{H}$ preserving the positive and negative graded parts.

Definition 2.2: *A **Heisenberg Lie algebra** is a countable dimensional Heisenberg type Lie algebra Lie algebra $\mathfrak{l}$ that admits a $\mathbb{Z}$ grading with finite dimensional components, together with a decomposition $\mathfrak{l} = \mathfrak{l}_- \oplus Z(\mathfrak{l}) \oplus \mathfrak{l}_+$ into the **positive, central, and negative parts,** respectively, and a choice of a canonical central element $K \in Z(\mathfrak{l})$.*

Corollary 2.1: *of 2.1. Let $\mathfrak{l}$ be the Heisenberg Lie Algebra. Then the commutator map*

$$\mathfrak{l}_+ \oplus \mathfrak{l}_- \to Z(\mathfrak{l})$$

$$(X, Y) \mapsto [X, Y]$$

is a perfect pairing, when we identify $Z(\mathfrak{l}) = \mathbb{C}K \approxeq \mathbb{C}$ by $K \mapsto 1$.

For any decomposition $\mathfrak{l}_+ = \oplus_{i\geq 1}\mathfrak{l}_i$ into finite dimensional components as a vector space, there is a canonical induced decomposition $\mathfrak{l}_- = \oplus_{i\geq 1}\mathfrak{l}_{-i}$ such that $\mathfrak{l}_i, \mathfrak{l}_{-i}$ are in perfect pairing by the commutator. This gives rise to a well-defined Lie bracket:

$$\mathfrak{l}_- \oplus \prod_{i\geq 1} \mathfrak{l}_i \to Z(\mathfrak{l})$$

extending the Lie bracket $\mathfrak{l}_- \oplus \mathfrak{l}_+ \to Z(\mathfrak{l})$, since any $X \in \mathfrak{l}_-$ is a finite sum of $X_{-i} \in \mathfrak{l}_{-i}$.

Definition 2.3: *A* ***formal completion*** *of a Heisenberg Lie algebra* $\mathfrak{l}$ *is defined to be*

$$\bar{\mathfrak{l}} := \mathfrak{l}_- \oplus Z(\mathfrak{l}) \oplus \prod_{i\geq 1} \mathfrak{l}_i$$

for some decomposition $\mathfrak{l}_+ = \oplus_{i\geq 1}\mathfrak{l}_i$ *as a vector space with Lie bracket induced from that in* $\mathfrak{l}$*, i.e.,* $Z(\mathfrak{l})$ *is central,* $\prod_{i\geq 1}\mathfrak{l}_i$ *is abelian and the bracket* $\mathfrak{l}_- \oplus \prod_{i\geq 1} \mathfrak{l}_i \to Z(\mathfrak{l})$ *canonically extends that on* $\mathfrak{l}_- \oplus \mathfrak{l}_+ \to Z(\mathfrak{l})$*.*

Define the ***negative, neutral, and positive parts*** *of* $\mathfrak{l}$*, respectively:*

$$\bar{\mathfrak{l}}_- := \mathfrak{l}_-$$

$$\bar{\mathfrak{l}}_0 := Z(\mathfrak{l})$$

$$\bar{\mathfrak{l}}_+ := \prod_{i\geq 1} \mathfrak{l}_i$$

A ***formal Heisenberg Lie algebra*** *is the formal completion of a Heisenberg Lie algebra, together with the data of the decomposition into its negative, neutral, and positive parts.*

Definition 2.4: *A finite dimensional representation of* $\overline{\mathfrak{l}_+}$ *is a finite dimensional representation that factors through the quotient* $\overline{\mathfrak{l}_+} \to \oplus_{i=1}^N \mathfrak{l}_i$ *for some* N*.*

Remark 2.1: *Under this notion of a finite dimensional representation of* $\overline{\mathfrak{l}_+}$*, restriction by the inclusion* $\mathfrak{l}_+ \hookrightarrow \overline{\mathfrak{l}_+}$ *induces an equivalence between finite dimensional representations of* $\overline{\mathfrak{l}_+}$ *and finite dimensional representations of* $\mathfrak{l}_+$*. This is because any finite dimensional representation of* $\mathfrak{l}_+$ *factors through some finite dimensional quotient of the form* $\oplus_{i=1}^N \mathfrak{l}_i$ *for some* N*, and thus canonically extends to an action of* $\overline{\mathfrak{l}_+}$ *by having* $\prod_{i\geq N+1} \mathfrak{l}_i$ *act trivially.*

Definition 2.5: *Let $\mathfrak{l}$ be a Heisenberg Lie algebra and $\bar{\mathfrak{l}}$ the formal completion. For $p \in \mathbb{C}$ with $p \neq 0$, a level p representation of $\mathfrak{l}$, resp $\bar{\mathfrak{l}}$, is a representation V of $\mathfrak{l}$, resp. $\bar{\mathfrak{l}}$, such that:*

1. *The canonical central element $K \in \mathfrak{l}$ acts by multiplication by p.*
2. *$V = \cup_i V_i$ restricts to a countable union of finite dimensional representations V_i of $\mathfrak{l}_+$, resp $\bar{\mathfrak{l}}_+$.*

Remark 2.2: *Since our notion of a finite dimensional representation ensures $\mathfrak{l}_+$ and $\overline{\mathfrak{l}_+}$ have the finite dimensional representations, $\mathfrak{l}$ and $\bar{\mathfrak{l}}$ have the same level p representations, i.e., every level p representation of $\mathfrak{l}$ has an induced canonical structure of a level p representation of $\bar{\mathfrak{l}}$ and vice versa.*

Definition 2.6: *The level p Fock space π_p is the representation of $\mathfrak{l}$ defined by*

$$\pi_p := \mathrm{Ind}_{\mathbb{C}K \oplus \mathfrak{l}_+}^{\mathfrak{l}} \mathbb{C}$$

where $\mathfrak{l}_+$ acts trivially and K acts by multiplication by 1.

Recall from every Heisenberg Lie algebra is isomorphic to the Heisenberg Lie algebra $\mathfrak{H} = (\oplus_{i \geq 1} \mathbb{C} f_i) \oplus \mathbb{C}K \oplus (\oplus_{i \geq 1} \mathbb{C} e_i)$ in 2.1 as a Lie algebra by an isomorphism preserving the positive and negative parts. We have the following characterization of the level p representations of $\mathfrak{H}$:

Theorem 2.2: *[Kac90, 9.13] Suppose V is representation of $\mathfrak{H}$ with the property that K acts by multiplication by p for $p \neq 0$ and for every $v \in V$, there exists $N > 0$ such that any tensor product of N or more elements of $e_i : i \geq 1$ in $U(\mathfrak{H}_+) = \mathrm{Sym}(\mathfrak{H}_+) = \mathbb{C}[e_i : i \geq 1]$ acts trivially. Then V is isomorphic to a direct sum of π_p.*

This allows us to determine the level p representation of $\mathfrak{l}$:

Theorem 2.3: *Let $\mathfrak{l}$ be a Heisenberg Lie algebra. Fix a presentation $\mathfrak{l} \approxeq \mathfrak{H}$ preserving the positive and negative parts. Any level p representation of $\mathfrak{l}$, when considered as a representation of $\mathfrak{H}$, satisfies the condition of 2.2. Therefore the category of level p*

representations of $\mathfrak{l}$*, resp,* $\bar{\mathfrak{l}}$*, is semisimple with exactly one unique irreducible object* π_p *up to isomorphism. We conclude that there is an equivalence of abelian categories*

$$\textit{Level 1 representations of } \mathfrak{l} \approxeq \textit{vector spaces}$$

$$V \mapsto \Omega(V) := V^{\mathfrak{l}_+}$$

$$\mathrm{Ind}_{\mathbb{C}K \oplus \mathfrak{l}_+}^{\mathfrak{l}} M \leftarrow\!\shortmid M$$

where $\Omega(V)$ *is denoted the* ***vacuum space*** *and* K *acts on* M *by multiplication by* p*.*

PROOF: Let V be a level p representation of $\mathfrak{l}$. Let $v \in V$. Then there exists $W \subseteq V$ a finite dimensional $\mathfrak{H}_+$-stable subspace such that $v \in W$. Since $\mathfrak{H}_+$ is commutative, $\mathrm{Sym}(\mathfrak{H}_+) = \mathbb{C}[e_i : i \geq 1]$. Write the action of $\mathfrak{H}_+$ on W as an algebra homomorphism

$$\mathrm{Sym}(\mathfrak{H}_+) \xrightarrow{\rho} \mathrm{End}W.$$

Since W is finite dimensional over $\mathbb{C}$, so is $\mathrm{End}W$ and $\mathrm{Sym}(\mathfrak{H}_+)/\ker\rho$. Let $J \subseteq \mathrm{Sym}(\mathfrak{H}_+)$ be a finite set of linearly independent polynomials such that

$$\mathrm{Sym}(\mathfrak{H}_+) = \langle J \rangle \oplus \ker\rho$$

as a vector space. Let

$$N = \max\{\deg f : f \in J\}$$

where $\deg f$ is the number of variables in the term of f, as a polynomial in $\mathbb{C}[e_i : i \geq 1]$, that is the product of the largest number of variables. Whenever $n > N$, any monomial f product of n terms in $\{e_i : i \geq 1\}$ will be linearly independent to J and thus $f = g + h$ for $g \in \langle J \rangle$, $h \in \ker \rho$ and $h \neq 0$. Since f is a monomial, $g = 0$, and we conclude that $f \in \ker \rho$. The result follows. ■

Remark 2.3: *For us, the main focus will be on the case level* $p = 1$ *representations of a Heisenberg Lie algebra.*

2.2 Definition and Representation Theory

2.2.1 Weil Restriction, Loop groups, Jet Groups, Integral Models

Let us fix some notation and recall basic facts. Let $T' \to T$ be a morphism of schemes. For a functor X' over $(\mathrm{Sch}/T')^{\mathrm{op}}$, the **restriction of scalars**, denoted by $\mathrm{Res}_{T'/T}X'$ is the functor $(\mathrm{Sch}/T)^{\mathrm{op}} \to \mathrm{Set}$ given by

$$\mathrm{Res}_{T'/T}X'(S) = X'(S \times_T T') = X'(S_{T'})$$

for every scheme $S \to T$ over T. When $T' \to T$ is a morphism of affine schemes given by a ring map $A \to B$, define

$$\mathrm{Res}_{B/A}X' := \mathrm{Res}_{\mathrm{Spec}B/\mathrm{Spec}A}X'.$$

The assignment $X' \mapsto \mathrm{Res}_{T'/T}X'$ is functorial in X'. If $\mathcal{G}$ is a group valued functor over T', then $\mathrm{Res}_{T'/T}\mathcal{G}$ is canonically a group-valued functor over T. We often wish to apply the construction to the case when X' is an algebro-geometric object. In some special cases, the functor $\mathrm{Res}_{T'/T}X'(S)$ is also representable by an algebro-geometric object. For example, we have

Example 2.1: *Suppose $T' \to T$ is finite locally free and X' is quasi-projective over T'. Then by [BLR90, 7.6/4],* $\mathrm{Res}_{T'/T}X'$ *is representable by a scheme over T. If furthermore X', T', T are affine with X' finite type over T' then* $\mathrm{Res}_{T'/T}X'$ *is affine and finite type over T, [CGP15, Proposition A.5.2].*

For us, the main examples of the type above are when $T' \to T$ is finite étale. In this chapter, we are primarily concerned with the case when $T' \to T$ is given by a finite extension of fields E/F. In this case, $\mathrm{Res}_{E/F}X'$ called the **Weil restriction**. In chapter 3, we also consider the case when $T' \to T$ is a finite étale cover of curves.

Definition 2.7: *Let k be a field, $\mathcal{O} = k[[t]]$ and $F = k((t))$. Let $\mathcal{X}^+$ be a scheme over $\mathcal{O}$ and $\mathcal{X}$ be a scheme over F. Define the* ***jet space*** *as the functor for every k-algebra R*

$$L^+\mathcal{X}^+(R) := \mathcal{X}^+(R[[t]])$$

and the ***loop space*** *as*

$$L\mathcal{X} := \mathcal{X}(R((t))).$$

If $\mathcal{X}^+ = \mathcal{G}^+$, resp. $\mathcal{X} = \mathcal{G}$, are group schemes over $\mathcal{O}$, resp. F, we call $L^+\mathcal{G}^+$, resp. $L\mathcal{G}$ the ***jet group****, resp. the* ***loop group****.*

Remark 2.4: *It is not the case that $L^+\mathcal{X}^+ = \mathrm{Res}_{\mathcal{O}/k}\mathcal{X}^+$ nor $L\mathcal{X} = \mathrm{Res}_{F/k}\mathcal{X}$, because for example $R[[t]]$ is the topological tensor product $R\hat{\otimes}_k\mathcal{O}$ and not $R\otimes_k\mathcal{O}$ and similarly so for $R((t))$.*

We have the following facts about $L^+\mathcal{X}^+$ and $L\mathcal{X}$ from [PR08, Section 1]:

1. Both $L^+\mathcal{X}^+$ and $L\mathcal{X}$ are k-spaces, i.e., sheaves of sets for the fpqc topology on the category of k-algebras.

2. If $\mathcal{X}^+ = \mathrm{Spec}(A)$ is affine and finite type over $\mathcal{O}$, $L^+\mathcal{X}^+$ is representable by an affine scheme over k.

3. If $\mathcal{X} = \mathrm{Spec}(B)$ is affine over F, $L\mathcal{X}$ is representable by a strict ind-scheme over k, i.e., an ind-scheme where the transition morphisms are closed embeddings.

In particular, $L\mathcal{G}$ is a group ind-scheme, i.e., a group object in the category of ind-schemes over k. Warning however $L\mathcal{G}$ is not necessarily representable by inductive limit of group schemes over k. When $\mathcal{P}, \mathcal{G}$ satisfy various properties, e.g., affine, reductive or semisimple, $L^+\mathcal{P}$ and $L\mathcal{G}$ will satisfy additional various other properties that will be recalled as necessary from [PR08].

Definition 2.8: *[Yu15, 2.1] Suppose $\mathcal{G}$ is an affine group scheme over F of finite type. We say a scheme $\mathcal{G}^+$ over $\mathcal{O}$ is an* ***integral model*** *of $\mathcal{G}$ over $\mathcal{O}$ if $\mathcal{G}^+$ is affine finite type over $\mathcal{O}$ with generic fiber equal to $\mathcal{G}$.*

A smooth integral model is an integral model that is also smooth over $\mathcal{O}$. For convenience, we record some key facts about integral models from [Yu15, 2.2, 2.3, 2.6, 2.7] that we may refer to by name and not by direct citation:

1. (Uniqueness principle): Suppose $\mathcal{X}$ is smooth over F. If $\mathcal{X}^+, \mathcal{X}^{+'}$ are two smooth integral models of $\mathcal{X}$ with the same $\mathcal{O}$-points, then $\mathcal{X}^+ = \mathcal{X}^{+'}$.

2. (Closure principle): Let $\mathcal{X}^+$ be an integral model of $\mathcal{X}$ and $\mathcal{Z} \subseteq \mathcal{X}$ a closed subscheme. Then the scheme theoretic closure $\overline{\mathcal{Z}} \subseteq \mathcal{X}^+$ is an integral model of $\mathcal{Z}$ uniquely characterized by the property that it represents the following functor on the category of flat $\mathcal{O}$-algebras:
$$R \mapsto \mathcal{Z}(R \otimes_{\mathcal{O}} F) \cap \mathcal{X}^+(R).$$

3. (Extension principle): Suppose $\mathcal{X}^+, \mathcal{Y}^+$ are integral models of $\mathcal{X}, \mathcal{Y}$, respectively, with $\mathcal{X}^+$ is smooth. Then any morphism $\mathcal{X} \to \mathcal{Y}$ over F mapping the image of $\mathcal{X}^+(\mathcal{O})$ in $\mathcal{X}(F)$ to the image of $\mathcal{Y}^+(\mathcal{O})$ in $\mathcal{Y}(F)$ extends uniquely to a morphism $\mathcal{X}^+ \to \mathcal{Y}^+$ over $\mathcal{O}$.

We say a group scheme $\mathcal{G}^+$ over $\mathcal{O}$ is an integral model of a group scheme $\mathcal{G}$ over F if $\mathcal{G}^+$ is an integral model of $\mathcal{G}$ and the multiplication map $\mathcal{G}^+ \times \mathcal{G}^+ \to \mathcal{G}^+$ over $\mathcal{O}$ extends the multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ over F. The following basic lemma is used:

Lemma 2.1: *Suppose $\mathcal{G}^+$ is a smooth group scheme over $\mathcal{O}$ and is an integral model of a group scheme $\mathcal{G}$ over F, and $\mathcal{H} \subseteq \mathcal{G}$ is a closed subgroup scheme. Then the scheme theoretic closure $\overline{\mathcal{H}} \subseteq \mathcal{G}^+$ of $\mathcal{H}$ in $\mathcal{G}^+$ is a subgroup scheme of $\mathcal{G}^+$.*

Proof: Applying the functor of points characterization of $\overline{\mathcal{H}}$ in the closure principle with $R = \mathcal{O}$ gives the subgroup structure on $\mathcal{O}$-points, with $\overline{\mathcal{H}}(\mathcal{O}) = \mathcal{H}(F) \cap \mathcal{G}^+(\mathcal{O})$. By the extension principle, it extends to a unique multiplication map of schemes $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \overline{\mathcal{H}}$. The uniqueness in the extension principle applied to the compositions $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \mathcal{G}^+ \times \mathcal{G}^+ \to \mathcal{G}^+$ and $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \overline{\mathcal{H}} \to \mathcal{G}^+$ give compatibility of $\overline{\mathcal{H}} \times \overline{\mathcal{H}} \to \overline{\mathcal{H}}$ with the multiplication map $\mathcal{G}^+ \times \mathcal{G}^+ \to \mathcal{G}^+$. ∎

In studying loop groups, we frequently appeal to the Lie algebra to study the connected component. It is defined as follows.

Definition 2.9: *Let G be a group functor over k. The Lie algebra is the k-vector space defined by*

$$\mathrm{Lie}G := \ker(G(k[\varepsilon]) \to G(k))$$

where $k[\varepsilon] = k[t]/t^2$ is the ring of dual numbers and the map $k[\varepsilon] \to k$ is $\varepsilon \mapsto 0$.

Example 2.2: *Let $E = k((u))$ be a cyclic extension of F with $u^m = t$, m invertible in k. Then*

$$\mathrm{Lie}L\mathrm{Res}_{E/F}\mathbb{G}_{m,E} = \{f \in \mathbb{C}[\varepsilon]((t)) : f = \sum a_i \varepsilon u^i\} = \mathbb{C}((u)).$$

The identification of $\mathrm{Lie}L\mathrm{Res}_{E/F}\mathbb{G}_{m,E}$ *with $\mathbb{C}((u))$ is canonical with $\varepsilon \mapsto 1$. In general,* $\mathrm{Lie}L\mathrm{Res}_{E/F}T_E = \mathfrak{t}((u))$ *for a split torus T over k with Lie algebra* $\mathfrak{t}$.

2.2.2 Integral Models and Structure of their Loop Groups of Tori

We recall geometry of loop groups of tori and the jet groups of their integral models, prove a lemma about the epimorphism property of the norm map, and a fundamental new fact that the Kottwitz map from the loop group of an arbitrary torus to the group of connected components admits a homomorphic section. This allows us to determine the structure of the loop group of a torus.

For this subsection, let k be algebraically closed, $F = k((t))$ and $\mathcal{O}_F = k[[t]]$. Fix torus $\mathcal{T}$ over F. Let E/F be an extension such that $\mathcal{T}_E$ is split of rank d. Put $E = k((u))$ for some $u^m = t$ where $m = [E : F]$. Put $\mathcal{O}_E = k[[u]]$. Let $\Gamma = \mathrm{Gal}(E/F)$ be the finite cyclic Galois group with $\nu \in \Gamma$ a choice of a generator. Then $\mathcal{T}_E$ has an action of Γ over E. Let $Y = \mathbb{X}_*(\mathcal{T}) = \mathrm{Hom}(\mathbb{G}_{m,E}, \mathcal{T}_E)$ be the absolute cocharacter lattice of Y, equipped with the action of Γ. Recall the action of Γ on Y is given by for all $\lambda \in Y$, $\gamma \in \Gamma$ and $x \in \mathbb{G}_{m,E}(R)$

$$\gamma.\lambda(x) := \gamma(\lambda(\gamma^{-1}(x))).$$

By Galois descent, we have

$$\mathcal{T} = (\mathrm{Res}_{E/F}\mathcal{T}_E)^{\Gamma}.$$

Definition 2.10: *[Ngo10, 3.8] The* ***Néron model*** $\mathcal{T}^{\flat}$ *of* $\mathcal{T}$ *is the smooth integral model of* $\mathcal{T}$ *such that* $\mathcal{T}^{\flat}(\mathcal{O}_F)$ *is the maximal bounded subgroup of* $\mathcal{T}(F)$*.*

By the uniqueness principle of integral models, $\mathcal{T}^{\flat}$ exists and is unique up to unique isomorphism. The Néron model is constructed in [Ngo10, 3.8] as follows. Fix an identification $\mathcal{T}_E \approxeq T_E$ for a split torus T over $\mathbb{Z}$. Consider the integral model $\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}T_{\mathcal{O}_E}$ of $\mathrm{Res}_{E/F}\mathcal{T}_E$, which is smooth by [BLR90, 7.6 Prop 5]. By the extension principle, the action of Γ on $\mathrm{Res}_{E/F}\mathcal{T}_E$ over F extends to an action on $\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}T_{\mathcal{O}_E}$ over $\mathcal{O}$. Then

$$\mathcal{T}^{\flat} = (\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}T_{\mathcal{O}_E})^{\Gamma}.$$

Since $\mathcal{T}^{\flat}$ is a closed subscheme of an affine scheme over $\mathcal{O}_F$, it is affine over $\mathcal{O}_F$. By [Edi92, 2.2], $\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}T_E$ is smooth and by [Edi92, 3.4], $\mathcal{T}^{\flat}$ is smooth.

Definition 2.11: *1. The* ***connected Néron model*** $\mathcal{T}^{\flat,0}$ *is the neutral component of* $\mathcal{T}^{\flat}$ *in the sense of [Yu15, 1.2], i.e., the open subscheme of* $\mathcal{T}^{\flat}$ *consisting of the generic fiber and the connected component of the special fiber.*

2. An ***intermediate integral model*** $\mathcal{T}^{\#}$ *is an open subgroup scheme of* $\mathcal{T}^{\flat}$ *containing* $\mathcal{T}^{\flat,0}$*.*

All intermediate integral models of $\mathcal{T}$ are smooth over $\mathcal{O}_F$.

Lemma 2.2: *Any intermediate integral model of* $\mathcal{T}$ *is affine over* F*.*

Proof: Let $\mathcal{T}^{\#}$ be an intermediate integral model of $\mathcal{T}$. Let s be the special point of $\mathrm{Spec}\mathcal{O}_F$ and $Z = \mathcal{T}^{\sharp}_s$. Then Z consists of (finite) union of connected components of $\mathcal{T}_s^\flat$, which is affine because $\mathcal{T}_s^\flat$ is, and therefore Z is affine. By the uniqueness principle and the description of the dilation in [Yu15, 2.7], $\mathcal{T}^\#$ is the dilation of Z in $\mathcal{T}^\flat$. Also according to [Yu15, 2.7], the dilation on an affine scheme is affine. The result follows. ■

The torus $\mathrm{Res}_{E/F}\mathcal{T}_E$ has an action of Γ over F coming from the action of Γ on $\mathcal{T}_E$. There is also a norm map $N : \mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ given by $x \mapsto \sum_{\gamma\in\Gamma} \gamma.x$ for all $x \in \mathrm{Res_{E/F}}\mathcal{T}_\mathrm{E}(R)$ for every F-algebra R.

Lemma 2.3: *The categorical quotient* $\mathrm{Res}_{E/F}\mathcal{T}_E/\Gamma$ *is canonically a torus and canonically identified with* $\mathcal{T}$. *The categorical quotient map* $\mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ *of schemes over* F *factors as*

$$\mathrm{Res}_{E/F}\mathcal{T}_E \xrightarrow{N} (\mathrm{Res}_{E/F}\mathcal{T}_E)^\Gamma = \mathcal{T}$$

and the norm map $N : \mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ *is surjective.*

PROOF: The functor

$$\{F\text{- diagonalizable groups}\} \to \{\Gamma\text{-modules finite over } \mathbb{Z}\}^{\mathrm{op}}$$

$$\mathcal{S} \mapsto \mathbb{X}^*(\mathcal{S})$$

is an equivalence of abelian categories by [Poo17, 5.5.10]. On the other hand, the functor

$$\{\Gamma\text{-modules finite over } \mathbb{Z}\}^{\mathrm{op}} \to \{\Gamma\text{-modules finite over}\mathbb{Z}\}$$

$$M \mapsto M^\vee := \mathrm{Hom}_\Gamma(M, \mathbb{Z})$$

restricts to an additive equivalence on the subcategory $\{\Gamma\text{-modules finite and free over } \mathbb{Z}\}^{\mathrm{op}}$ of $\{\Gamma\text{-modules finite over } \mathbb{Z}\}^{\mathrm{op}}$. Therefore the restriction of the composition is

$$\{F\text{−tori}\} \to \{\Gamma\text{-modules finite and free over } \mathbb{Z}\}.$$

$$\mathcal{S} \mapsto \mathbb{X}_*(\mathcal{S})$$

is an additive equivalence of categories and is exact on short exact sequences of tori. An inverse is $M \mapsto \mathrm{Hom}_F(\underline{M^\vee}_F, \mathbb{G}_{m,F})^\Gamma$ where $\underline{A}_F$ denotes the constant group scheme over F associated to the abstract group A.

Let us first show $\mathrm{Res}_{E/F}\mathcal{T}_E/\Gamma = \mathcal{T}$. Let $Y = \mathbb{X}_*(\mathcal{T})$ be the Γ-module corresponding to $\mathcal{T}$. The F-torus $\mathrm{Res}_{E/F}\mathcal{T}_E$ is an induced torus, meaning that the cocharacter lattice has a $\mathbb{Z}$ basis permuted by Γ, in fact $\mathbb{X}_*(\mathrm{Res}_{E/F}\mathcal{T}_E) = Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$ as Γ-modules. We warn that $\mathbb{X}_*(\mathrm{Res}_{E/F}\mathcal{T}_E)$ has **two** actions of Γ. There is one latent action by the virtue that $\mathrm{Res}_{E/F}\mathcal{T}_E$ is an F-torus, and there is a second Γ action intertwining with the first coming from the action of Γ on $\mathrm{Res}_{E/F}\mathcal{T}_E$ over F by group homomorphisms. The categorical quotient of $Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$ in the category of finite $\mathbb{Z}[\Gamma]$-modules for the second action of Γ (where Γ acts on the $\mathbb{Z}[\Gamma]$ factor canonically and trivially on Y) is Y with the quotient map $Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \to Y$ given by the augmentation map $\varepsilon(\gamma) = 1$ for all $\gamma \in \Gamma$. We get a corresponding categorical quotient map $\mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ for Γ in the category of diagonalizable group schemes over F. Since Γ acts on $\mathrm{Res}_{E/F}\mathcal{T}_E$ by group homomorphisms, $\mathrm{Res}_{E/F}\mathcal{T}_E/\Gamma$, which exists as an F-scheme, has the canonical structure of a group scheme. Since $\mathrm{Res}_{E/F}\mathcal{T}_E$ is diagonalizable, so is $\mathrm{Res}_{E/F}\mathcal{T}_E/\Gamma$. We conclude that $\mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ is also the quotient by Γ in the larger category of schemes over F.

Now we show the categorical quotient map factors as $\mathrm{Res}_{E/F}\mathcal{T}_E \xrightarrow{N} (\mathrm{Res}_{E/F}\mathcal{T}_E)^\Gamma \to \mathcal{T}$. Let $n = \sum_{\gamma\in\Gamma} \gamma \in \mathbb{Z}[\Gamma]$ be the norm element. There is a factorization of the augmentation map $\varepsilon : Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \to Y$ as

$$Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \xrightarrow{N} Y \otimes_{\mathbb{Z}} \mathbb{Z}n = (Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma])^\Gamma = Y$$

where the first map is the norm map $\lambda \mapsto \sum_{\gamma\in\Gamma} \gamma.\lambda$ which is seen to be equal to the map defined by for all $\lambda \in Y$ and $\gamma \in \Gamma$,

$$\lambda \otimes \gamma \mapsto \lambda \otimes n.$$

The second equality $Y \otimes_{\mathbb{Z}} \mathbb{Z}n = (Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma])^\Gamma$ is given by the vanishing $\mathrm{H}^i(\Gamma, Y \otimes_{\mathbb{Z}}$

$\mathbb{Z}[\Gamma]) = 0$ for all $i > 0$, which is shown by tensoring the periodic resolution of $\mathbb{Z}$ as the trivial Γ-module

$$\cdots \xrightarrow{N} \mathbb{Z}[\Gamma] \xrightarrow{\nu-1} \mathbb{Z}[\Gamma] \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with the free, thus projective, $\mathbb{Z}[\Gamma]$-module $Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma]$, see [Bro94, III.1. example 2, ex 1.c].

The third identification map $Y = Y \otimes_{\mathbb{Z}} \mathbb{Z}n$, which is induced by the universal property of Y as the Γ-quotient, is given by $\lambda \mapsto \lambda \otimes n$. The torus corresponding to $(Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma])^{\Gamma}$ is $(\mathrm{Res}_{E/F}\mathcal{T}_E)^{\Gamma} = \mathcal{T}$ because it is the image of N and all terms of the short exact sequence

$$1 \to \mathrm{Ker}N \to Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma] \xrightarrow{N} (Y \otimes_{\mathbb{Z}} \mathbb{Z}[\Gamma])^{\Gamma} \to 0$$

are free as $\mathbb{Z}$ modules. This also shows $N : \mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ is surjective. ∎

Remark 2.5: *Consider $\mathcal{T}_E$ as a group scheme over F by the composition of the projection $\mathcal{T}_E \to \mathcal{T}$ with $\mathcal{T} \to \mathrm{Spec}F$. Then $\mathcal{T}_E$ also has a Γ-action with categorical quotient map $\mathcal{T}_E \to \mathcal{T}$. However, $\mathcal{T}_E \neq \mathrm{Res}_{E/F}\mathcal{T}_E$. For example, $\mathrm{Res}_{E/F}\mathcal{T}_E$ is dimension $m \cdot \mathrm{rank}(\mathcal{T}) = md$ while $\mathcal{T}_E$ has the same underlying topological space as $\mathcal{T}$. Moreover, if E/F is not finite, $\mathrm{Res}_{E/F}\mathcal{T}_E$ may not be representable by scheme over F while $\mathcal{T}_E$ will always be a scheme over F.*

This gives us the following lemma that will be foundational to our computations.

Lemma 2.4: *Suppose $m = [E : F]$ is invertible in k. Then the norm map $N : L\mathrm{Res}_{E/F}\mathcal{T}_E \to L\mathcal{T}$ is both surjective and surjective on k points.*

Proof: Applying [PR08, 1.a.3] to the Artinian k-algebra $R = k$, taking k-points of loop groups of the surjection $N : \mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ gives that the k points of the norm map $N : L\mathrm{Res}_{E/F}\mathcal{T}_E(k) \to L\mathcal{T}(k)$ is surjective. According to [PR08, 5.1], each connected component of $L\mathcal{T}$ contains a k point. Hence to show N is surjective, it suffices to show N is surjective on the neutral component $L\mathcal{T}^0$. This can be done by showing surjectivity on the Lie algebras. Let $\mathfrak{l} = \mathrm{Lie} L\mathrm{Res}_{E/F}\mathcal{T}_E$ be the Lie algebra over k. Then $\mathrm{Lie}L\mathcal{T} = \mathfrak{l}^\Gamma$. The induced norm map on Lie algebras is

$$N : \mathfrak{l} \to \mathfrak{l}^\Gamma$$

$$X \mapsto \sum_{\gamma\in\Gamma} \gamma.X.$$

Since m is invertible in k and a section $\mathfrak{l}^\Gamma \to \mathfrak{l}$ is given by $X \mapsto \frac{1}{m}X$. The result follows. ■

Lemma 2.5: *A surjective morphism of group-valued sheaves $f : G \to H$ on a site is an epimorphism in the category of sheaves of sets.*

Proof: To be precise, let K be the kernel of f as a group valued functor, so f is automatically a sheaf. Since g is surjective it induces a canonical isomorphism between G and the sheafification of the presheaf G/K defined by

$$R \mapsto G(R)/K(R).$$

The induced map of sheaves $f : G \to G/K$ is surjective on R points, and hence is an epimorphism. Let A be a sheaf, $g, g' : H \to A$ be morphisms of sheaves such that $g \circ f = g' \circ f$. Consider the commutative diagram

$$\begin{array}{ccccc} & & G/K & & \\ & \nearrow_{f} & \downarrow_{\varphi} & & \\ G & \underset{f}{\rightarrow} & H & \overset{g}{\underset{g'}{\rightrightarrows}} & A \end{array}$$

where φ is the canonical map to the sheafification. By the epimorphism property of $f : G \to G/K$, $g \circ \varphi = g' \circ \varphi$. Taking sheafification of $g \circ \varphi, g' \circ \varphi$, we obtain $g = g'$ as desired. ■

For a Γ-module A, denote A_Γ to be the quotient module of coinvariants. The valuation map $k((u))^\times \to \mathbb{Z}$ extends to a morphism of group ind schemes val_E :

$L\mathbb{G}_{m,E} \to \mathbb{Z}$, [PR08, 5.1]. Define a map

$$L\mathrm{Res}_{E/F}\mathcal{T}_E \overset{\pi_0'}{\to} Y$$

by fixing an identification $\mathcal{T}_E \approxeq \mathbb{G}_{m,E}^d$, which induces identifications $L\mathrm{Res}_{E/F}\mathcal{T}_E = L\mathcal{T}_E \approxeq L\mathbb{G}_{m,E}^d$ and $Y \approxeq \mathbb{Z}^d$, and composing with $(\mathrm{val}_E)^d : L\mathbb{G}_{m,E}^d \to \mathbb{Z}^d$.

Definition 2.12: *Using that N is an epimorphism, the* ***Kottwitz homomorphism*** *$\pi_0 : L\mathcal{T} \to Y_\Gamma$ is the morphism of group ind schemes defined to be the unique map making the diagram of Γ-equivariant maps commute:*

$$\begin{array}{ccc} L\mathrm{Res}_{E/F}\mathcal{T}_E & \overset{\pi_0'}{\to} & Y \\ {\scriptstyle N}\downarrow & & \downarrow \\ L\mathcal{T} & \overset{\pi_0}{\to} & Y_\Gamma \end{array}$$

where $Y \to Y_\Gamma$ is the canonical projection.

Remark 2.6: *This definition of the Kottwitz homomorphism differs from the original one in [Kot97] on k-points, but it agrees with a remark given by a referee at the end of the paper.*

Since val_E is surjective, π_0 is surjective. We have the following property of the Kottwitz homomorphism, which explains the choice of notation π_0:

Theorem 2.4: *[PR08, 5.1] Both π_0 and π_0' are locally constant for the Zariski topology of and induces an isomorphism of the target with the group ind scheme of connected components of the source, i.e., $\ker\pi_0' = (L\mathrm{Res}_{E/F}\mathcal{T}_E)^0$ and $\ker\pi_0 = L\mathcal{T}^0$.*

Remark 2.7: *If $\mathcal{T} = \mathrm{Res}_{E/F}T_E$ is an induced torus, then π_0 can be constructed directly as π_0', i.e., as $(\mathrm{val}_E)^d$.*

To study $L\mathcal{T}$, and central extensions of $L\mathcal{T}$, we study their Lie algebra and connected components. We have the following fundamental theorem. Let T be the split torus over k with the same absolute character lattice of $\mathcal{T}$, so that $T \subseteq L\mathrm{Res}_{E/F}\mathcal{T}_E$.

Lemma 2.6: *1. For every $f \in \mathbb{G}_{m,k}(E) = k((u))^\times$ of valuation 1, the Kottwitz homomorphism has a section over Y, i.e., there exists a map $\mathrm{ev}_f : Y \to L\mathrm{Res}_{E/F}\mathcal{T}_E$, called the **evaluation map**, such that $\pi_0 \circ N \circ \mathrm{ev}_f$ is the projection $Y \to Y_\Gamma$. In particular, $N \circ \mathrm{ev}_f : Y \to L\mathcal{T}$ hits every connected component of $L\mathcal{T}$.*

2. For $\lambda \in Y$, putting $f = u$ gives

$$N \circ \mathrm{ev}_u((1-\nu)\lambda) = (N.\lambda)(\zeta^{-1}).$$

3. There exists a map $\upsilon : Y \to T^{\Gamma,0} \subseteq L^+\mathcal{T}^{\flat,0}$ such that $\mathrm{ev}_u \cdot \upsilon$ is trivial on $(1-\nu)Y$, and therefore descends to a homomorphic section

$$s_u \coloneqq \mathrm{ev}_u \cdot \upsilon : Y_\Gamma \to L\mathcal{T}$$

for π_0. Consequently it induces a direct product decomposition

$$L\mathcal{T} \approxeq L\mathcal{T}^0 \times Y_\Gamma.$$

Remark 2.8: *The section ev_f over Y is known to exist and a common tool, for example, in [Zhu14, 3.4]. However, it does not seem to be known that an honest homomorphic section $Y_\Gamma \to L\mathcal{T}$ exists, or at least the author has not found a reference. The determination $L\mathcal{T} \approxeq L\mathcal{T}^0 \times Y_\Gamma$ for an arbitrary torus $\mathcal{T}$ over F is a fundamental fact that should have many other applications.*

Proof: Since Y is discreet, it suffices to define ev_f on k points. By definition of π_0', for each $\lambda \in Y = \mathrm{Hom}(\mathbb{G}_{m,E}, \mathcal{T}_E)$, the element $\lambda(f) \in L\mathrm{Res}_{E/F}\mathcal{T}_E(k)$ lies above the connected component of $L\mathrm{Res}_{E/F}\mathcal{T}_E$ corresponding to λ. Then define the homomorphic section for π_0' as

$$\mathrm{ev}_f : Y \to L\mathrm{Res}_{E/F}\mathcal{T}_E(k)$$

$$\lambda \mapsto \lambda(f)$$

This shows (1).

We compute for $\lambda \in Y$ and $x \in \mathbb{G}_{m,E}(k)$,

$$(N \circ s_u)((1-\nu)\lambda) = N((1-\nu)\lambda(f))$$

$$= \prod_{i=1}^{m} \frac{\nu^i \lambda(\zeta^{-i}u)}{\nu^{i+1}\lambda(\zeta^{-i}u)}$$

$$= \frac{\nu\lambda(\zeta^{-1}u)\cdots\nu^m\lambda(\zeta^{-m}u)}{\nu^2\lambda(\zeta^{-1}u)\cdots\nu^{m+1}\lambda(\zeta^{-m}u)}$$

$$= \frac{\nu\lambda(\zeta^{-1}u)\cdots\nu^m\lambda(\zeta^{-m}u)}{\nu\lambda(u)\cdots\nu^m\lambda(\zeta^{-(m-1)}u)}$$

$$= \nu\lambda(\zeta^{-1})\cdots\nu^m\lambda(\zeta^{-1})$$

$$= (N.\lambda)(\zeta^{-1})$$

This shows (2).

Now remark that the element $(N.\lambda)(\zeta^{-1})$ lies in $T^{\Gamma,0}$ because the image of the norm map $N : T \to T^{\Gamma}$ must be connected as T is. Choose a retraction $r : T \to T^{\Gamma,0}$ for the injection $T^{\Gamma,0} \hookrightarrow T$. It exists because it corresponds to a section for the surjection $\mathbb{X}^*(T) \twoheadrightarrow \mathbb{X}^*(T^{\Gamma,0})$ that exists because $\mathbb{X}^*(T^{\Gamma,0})$ is free. Put υ as

$$Y \to T^{\Gamma,0}$$

$$\lambda \mapsto r((\sum_{i=1}^{m} i\nu^i.\lambda)(\zeta)).$$

Then

$$\upsilon(1-\nu).\lambda = r((\sum_{i=1}^{m} i\nu^i.\lambda - \sum_{i=1}^{m} i\nu^{i+1}.\lambda)(\zeta)).$$

Since

$$\sum_{i=1}^{m} i\nu^i.\lambda - \sum_{i=1}^{m} i\nu^{i+1}.\lambda =$$

$$= \nu.\lambda + 2\nu^2.\lambda + \cdots + (m-1)\nu^{m-1}.\lambda + m\nu^m.\lambda$$

$$-(\nu^2.\lambda + 2\nu^3.\lambda + \cdots + (m-1)\nu^m.\lambda + m\nu^{m+1}.\lambda)$$

$$= N.\lambda - m\nu.\lambda,$$

we have

$$\upsilon(1-\nu).\lambda = r((N.\lambda)(\zeta)/(m\nu.\lambda)(\zeta)) = r((N.\lambda)(\zeta)) = (N.\lambda)(\zeta)$$

because $(m\nu.\lambda)(\zeta) = 1$ and $(N.\lambda)(\zeta)$ already lies in $T^{\Gamma,0}$. Thus

$$\mathrm{ev}_u((1-\nu).\lambda) \cdot \upsilon((1-\nu).\lambda) = (N.\lambda)(\zeta^{-1})(N.\lambda)(\zeta) = 1.$$

This shows (3) and the result follows. ■

The group structure and geometry of $L\mathcal{T}$ and the subgroup $L^+\mathcal{T}^\sharp$ can be studied explicitly. Let us define certain special subgroups of $L\mathcal{T}$ other than $L^+\mathcal{T}^\sharp$. Begin with the loop group of the multiplicative group. We have from [OZ16, 2.1] a group theoretic decomposition

$$L\mathbb{G}_{m,E} \approxeq \hat{\mathbb{W}} \times \mathbb{G}_{m,\mathbb{C}} \times \mathbb{Z} \times \mathbb{W}$$

where $\mathbb{W}$ is the groups scheme of big Witt vectors defined by

$$\mathbb{W}(R) = \{1 + \sum_{i=1}^{\infty} b_i t_i : b \in R\}$$

and $\hat{\mathbb{W}}$is the group ind-scheme of formal Witt vectors. It is defined by

$$\hat{\mathbb{W}}(R) = \{1 + \sum_{i=-1}^{-n} c_i t^i : n \in \mathbb{Z}_{>0}, c_i \in R \text{ and is nilpotent}\}$$

$$\hat{\mathbb{W}}(R) = \lim_{\to \{\epsilon_i\}} \mathrm{Spec}\mathbb{Z}[c_{-1}, c_2 \cdots]/I_{\{\epsilon_i\}}$$

with the limit taken over $\{\epsilon_i\}$ in the countable direct sum $\mathbb{N}^{\oplus \mathbb{N}}$ and $I_{\{\varepsilon_i\}}$ is the ideal generated by $c_i^{\epsilon_i+1} : i < 0$.

Remark 2.9: *The underlying topological space of $\hat{\mathbb{W}}$ is a single point.*

We have that $\mathbb{W}$ is a projective limit affine spaces, identified as

$$\mathbb{W} \approxeq \lim_{\leftarrow i} \mathbb{A}^i$$

$$1 + \sum_{i=1}^{\infty} b_i t_i \mapsto \lim_{\leftarrow i} (b_1, \cdots, b_i).$$

The decomposition $L\mathbb{G}_{m,E} \approxeq \hat{\mathbb{W}} \times \mathbb{G}_{m,k} \times \mathbb{Z} \times \mathbb{W}$ is obtained as follows. For R such that Spec$R$ is connected, for each $f \in L\mathbb{G}_{m,E}(R) = R((u))^{\times}$, there is a unique decomposition, for some integer $N > 0$,

$$f = \prod_{i<0}^{i>-N} (1 - a_i u^i) \cdot a_0 \cdot u^{v_E(f)} \cdot \prod_{i>0}^{\infty} (1 - a_i u^i)$$

where $a_i : i < 0$ are nilpotent in R, $a_0 \in R^{\times}$ and $a_i : i > 0$ are arbitrary in R. Define

$$f_- := \prod_{i<0}^{i>-N} (1 - a_i u^i)$$

$$f_0 = a_0 \cdot u^{v_E(f)}$$

$$f_+ = \prod_{i>0}^{\infty} (1 - a_i u^i).$$

Then the map $L\mathbb{G}_{m,E}(R) \overset{\approxeq}{\to} \hat{\mathbb{W}}(R) \times \mathbb{G}_{m,k}(R) \times \mathbb{Z} \times \mathbb{W}(R)$ (note Spec$R$ is connected, so $\mathbb{Z} = \mathbb{Z}(R)$) is given by

$$f \mapsto (f_-, a_0, v(f), f_+)$$

From the description, we also have

$$(L\mathbb{G}_{m,E})_{\text{red}} = L^+\mathbb{G}_{m,\mathcal{O}_E} = \mathbb{G}_{m,\mathbb{C}} \times \mathbb{Z} \times \mathbb{W}.$$

Definition 2.13: *For the split torus $\mathcal{T}_E$ identified with $\mathbb{G}_{m,E}^d$ over E, so $\mathcal{T}^\flat = \mathcal{T}^{\flat,0}$ and $L^+\mathcal{T}^{\flat,0} = \mathbb{G}_{m,k}^d \times \mathbb{W}^d$, put*

$$L^-\mathcal{T}_E = \hat{\mathbb{W}}^d$$

$$L^{++}\mathcal{T}_E^{\flat,0} = \mathbb{W}^d \subseteq L^{+}\mathcal{T}_E^{\flat,0}$$

$$L^{++,-}\mathcal{T}_E^{\flat,0} = L^{++}\mathcal{T}_E^{\flat,0} \times L^{-}\mathcal{T}_E.$$

Lemma 2.7: *The subspaces T, $L^{++}\mathcal{T}_E^{\flat,0}$, and $L^{-}\mathcal{T}_E$ are stable under Γ. We conclude that there is a Γ-equivariant decomposition of the neutral component*

$$L\mathcal{T}_E^0 = L^{-}\mathcal{T}_E \times T \times L^{++}\mathcal{T}^{\flat,0}.$$

Proof: The space $L^{++}\mathcal{T}_E^{\flat,0}$ is Γ invariant because it is the kernel of the Γ-equivariant morphism $L^{+}\mathcal{T}_E^{\flat,0} \to T$ induced by applying $\mathcal{T}_E^{\flat,0}$ to the evaluation map $R[[u]] \mapsto R$ given by $u \mapsto 0$. The space T is Γ-invariant because $T(R) = \mathcal{T}_E^{\flat,0}(R) \subseteq \mathcal{T}_E^{\flat,0}(R[[u]])$. Finally to show $L^{-}\mathcal{T}_E$ is Γ-invariant, define the R subalgebra

$$\mathcal{N}R = R[(c - a_i u^i) : c \in R, a_i \in R \text{ is nilpotent}, i < 0] \subseteq R((t)).$$

Then $L^{-}\mathcal{T}_E(R) \subseteq \mathcal{T}_E(\mathcal{N}R)$. Then the result follows by observing that $L^{-}\mathcal{T}_E(R) \subset \mathcal{T}_E(\mathcal{N}R)$ is the kernel of the map $\mathcal{T}_E(\mathcal{N}R) \to T(R)$ induced by $u^{-1} \mapsto 0$. ∎

Lemma 2.8: *We have a decomposition*

$$L\mathcal{T}^0 \approxeq L^{-}\mathcal{T}_E^{\Gamma} \times T^{\Gamma} \times L^{++}\mathcal{T}_E^{\Gamma}$$

and

$$L^{+}\mathcal{T}^{\flat,0} = T^{\Gamma,0} \times L^{++}\mathcal{T}_E^{\Gamma}$$

and for any intermediate integral $\mathcal{T}^{\sharp}$ of $\mathcal{T}$, there exists a subgroup $H \subseteq T^{\Gamma}$ containing $T^{\Gamma,0}$ such that

$$L^{+}\mathcal{T}^{\sharp} = H \times L^{++}\mathcal{T}_E^{\Gamma}.$$

Proof: This follows by the uniqueness principle of integral models, the characterization of $\mathcal{T}^{\flat,0}$, the fact that $T^{\Gamma,0} \times L^{++}\mathcal{T}_E^{\Gamma}(k)$ is the maximal connected bounded subgroup of $L\mathcal{T}(k)$, and the fact that $L^{+}\mathcal{T}^{\sharp}$ contains $L^{+}\mathcal{T}^{\flat,0}$. ∎

Definition 2.14: *Put*

$$L^{-}\mathcal{T} = L^{-}\mathcal{T}_E^{\Gamma}$$

$$L^{++}\mathcal{T}^{\sharp} = H \times L^{++}\mathcal{T}_E^{\Gamma}$$

where $T^{\Gamma,0} \subseteq H \subseteq T^{\Gamma}$ *is as in the above lemma, and*

$$L^{++,-}\mathcal{T}^{\sharp} = L^{++}\mathcal{T}^{\sharp} \times L^{-}\mathcal{T}.$$

For a subgroup ind scheme $P \subseteq L\mathcal{T}$*, the **maximal semisimple subgroup*** S *is the minimal* S *such that the reduced locus of the quotient group ind scheme* $[P/S]$ *is pro-unipotent. The **maximal torus** is the connected component of* P*.*

From the above characterizations of $L\mathcal{T}$, resp. $L^{+}\mathcal{T}^{\sharp}$, the maximal torus of $L^{+}\mathcal{T}^{\sharp}$ or $L\mathcal{T}$ is $T^{\Gamma,0}$ and the maximal semisimple subgroup is H, resp. T^{Γ}.

In summary, we have:

Corollary 2.2: *The section* s_u *from 2.6 induces a decomposition of group ind schemes:*

$$L\mathcal{T} \approxeq L^{-}\mathcal{T} \times T^{\Gamma,0} \times L^{+}\mathcal{T}^{\flat,0} \times Y_{\Gamma}$$

where $L^{-}\mathcal{T}$*,* $T^{\Gamma,0}$*,* $L^{++}\mathcal{T}^{\flat,0}$*,* Y_{Γ} *is the nilpotent part, torus part, pro-unipotent part, and discreet part respectively.*

This recalls a similar decomposition for commutative finite type group schemes over k, except there is no nilpotent part.

2.2.3 Quillen K-groups and Tame Symbol

Let us fix some notation and review concepts regarding algebraic K theory.

Definition 2.15: *[BD01, ch. 0] For every integer* $i \geq 0$*, let* K_i *be the functor from commutative rings to abelian groups, where* $K_i(R)$ *is the ith Quillen* K*-group, as constructed in [Wei13]. Let* K_* *be the direct sum of* K_i*, where each* $K_*(R)$ *is a graded* *abelian group with grading induced by i. For a fixed base scheme S regular of finite type over a field, let $\boldsymbol{K}_{i,S}$, resp. $\boldsymbol{K}_{*,S}$, be the sheafification the presheaf K_i, resp. K_*, on the big Zariski site S_{Zar} of S. When S is affine, let $K_{i,S}$, resp $K_{*,S}$ be the restriction of K_i, resp. K_* to the category of S-algebras, and we may replace S with $A = \Gamma(S, \mathcal{O}_S)$ in the subscripts.*

Example 2.3: *[BD01, 1.4] For $i = 0$, the sheaf $\boldsymbol{K}_{i,S}$ is the constant group scheme $\mathbb{Z}$. For $i = 1$, the sheaf $\boldsymbol{K}_{i,S}$ is $\mathcal{O}^\times$ mapping each scheme $U \to S$ over S to $\Gamma(U, \mathcal{O}_U)^\times$, the units of the ring of global sections of U.*

There is the structure of a (non-commutative) ring on $K_*(R)$ respecting the grading via the K-theory pairing, denoted by $\cdot$. For us, the focus is on $i = 0, 1, 2$ and the restriction of $\cdot$ to $K_i \times K_j \to K_{i+j}$ for $i + j \leq 2$.

By [Wei13, III.1.1.1], for any ring R, $R^\times$ is a direct summand of $K_1(R)$.

Definition 2.16: *For $r, s \in R^\times$ denote by $\{r, s\} := r \cdot s \in K_2(R)$, i.e., $\{-, -\}$ is the restriction of $\cdot$ to the image of $R^\times \times R^\times$ in $K_*(R) \times K_*(R)$. It is skew-symmetric, [Wei13, 5.12.1].*

For a ring homomorphism $R \to S$, the functoriality of K_i give homomorphisms $K_i(R) \to K_i(S)$. If furthermore S is finitely generated and projective as an R-module, there exists a **transfer map** demoted by $\mathrm{tr} : K_i(S) \to K_i(R)$, [Wei13, IV.6.3.2]. Here are some basic examples:

1. [Wei13, II,III] Suppose $R = F$ is a field. Then

$$K_0(F) = \mathbb{Z}$$

$$K_1(F) = F^\times$$

$$K_2(F) = F^\times \otimes_{\mathbb{Z}} F^\times / \langle a \otimes (1 - a) : a \in F \backslash \{1, 0\} \rangle$$

   the relations $a \otimes (1 - a) = 1$ are called the **Steinberg relations**.

2. [Wei13, III. 1.7.1] Suppose E/F is a finite field extension. Then the transfer map $\mathrm{tr} : K_1(E) \to K_1(F)$ is the norm map $N : E^\times \to F^\times$, where we recall that for E/F not necessarily Galois, $N(x)$ is defined as the determinant of the F-linear map of multiplication by x on E.

3. [Wei13, III.6.1.3] For any field extension E/F, the kernel of the map $K_2(F) \to K_2(E)$ is torsion. If E/F is finite, then the kernel of $K_2(F) \to K_2(E)$ is annihilated by $m = [E : F]$.

4. [Wei13, III.6.1.2] Let F be a field and X be a smooth geometrically irreducible variety over F with a F-rational point. Then $K_2(F)$ injects into $K_2(F(X))$ as a direct summand [BD01, 2.1].

Let F be a field with a discreet valuation $v : F^\times \to \mathbb{Z}$ and residue field k. For $a \in R := v^{-1}(0)$, define $\overline{a}$ to be the image of a in k.

Definition 2.17: *The* ***tame symbol*** *is a map* $\{-,-\}_{\mathrm{tame}_F} : F^\times \times F^\times \to k^\times$ *defined as follows:*

$$\{r,s\}_{\mathrm{tame}_F} = (-1)^{v(r)v(s)}\overline{(\frac{s^{v(r)}}{r^{v(s)}})}$$

where we note that the term $\frac{s^{v(r)}}{r^{v(s)}}$ *lies in* R *because it has valuation* $v(s)v(r) - v(r)v(s) = 0$. *According to [Wei13, III.6.3],* $\{-,-\}_{\mathrm{tame}_F}$ *is bimultiplicative and satisfies the Steinberg relations, and thus descends to a unique map*

$$K_2(F) \to k^\times.$$

Denote this map by $\{-\}_{\mathrm{tame}_F}$.

Remark 2.10: *The transpose* $(r,s) \mapsto \{s,r\}_{\mathrm{tame}_F}$ *of the tame symbol also satisfies the Steinberg relations because the set of symbols* $\{a \otimes (1-a) : a \in F^\times \backslash \{0,1\}\}$ *is stable under transposition. We may also denote it by* $\{r,s\}_{\mathrm{tame}_F}$.

Example 2.4: *If* $F = k((t))$ *with the valuation induced by powers of* t, *the tame symbol is given as follows. For* $f, g \in F^\times$, *write* $f = a_0 t^{v(f)} \cdot \prod_{i\geq 1}^{\infty}(1 - a_i t^i)$ *and*

$g = b_0 t^{v(f)} \cdot \prod_{i\geq 1}^{\infty}(1-b_i t^i)$ *with* $a_i, b_i \in k$ *as in [OZ16, 2.7]. Then*

$$\{f,g\}_{\mathrm{tame}_F} = (-1)^{v(f)v(g)} \frac{a_0^{v(g)}}{b_0^{v(f)}}.$$

Let $E = k((u))$ *be the cyclic extension of* F *with* $u^m = t$ *with valuation over* F *induced by powers of* u. *Then for* $f, g \in F^\times \subseteq E^\times$, *we can either take the tame symbol by considering* f, g *as elements of* F *or take the tame symbol by considering* f, g *as elements of* E. *Letting* v_F *be the valuation for* F *and* v_E *be the valuation for* E. *We have for* $f, g \in F^\times$,

$$v_E(f) = v_F(f)^m.$$

Using the fact that $m^2 \equiv m$ *mod 2 for any integer* m, *we have*

$$\{f,g\}_{\mathrm{tame}_E} = \{f,g\}^m_{\mathrm{tame}_F}$$

2.2.4 Definition of Heisenberg Central Extensions, Bilinear Forms, Commutator Pairing

For this subsection, let $k = \mathbb{C}$, $F = k((t))$ and $\mathcal{T}$ be a torus over F. Let E/F be an extension such that $\mathcal{T}_E$ is split of rank d. Put $E = k((u))$ for some $u^m = t$ where $m = [E : F]$. Let $\Gamma = \mathrm{Gal}(E/F)$ be the finite cyclic Galois group with ν a choice of generator. Let $X = \mathrm{Hom}_E(\mathcal{T}_E, \mathbb{G}_{m,E})$, resp $Y = \mathrm{Hom}_E(\mathbb{G}_{m,E}, \mathcal{T}_E)$, be the (absolute) character lattice, resp (absolute) cocharacter lattice. Let $N = \sum_{\gamma\in\Gamma}\gamma.(-)$ be the norm map on an abelian group or sheaf with a Γ-action. For a base scheme S regular of finite type over a field, recall from 2.2.3 the presheaf $K_{2,S}$ and the sheaf $\boldsymbol{K}_{2,S}$ of abelian groups on the big Zariski site of S. For Z a set or presheaf of sets, let $\tau : Z \times Z \to Z \times Z$ be the transposition map $(y_1, y_2) \mapsto (y_2 y_1)$, and $\mathrm{pr}_i : Z^n \to Z$ be the projection to the ith-coordinate, for $i = 1, \cdots, n$. If Z is a group-valued, let $\mu : Z \times Z \to Z$ be the multiplication map.

In this subsection, we define the notion of a Heisenberg central extension $\hat{L\mathcal{T}}$ of $L\mathcal{T}$ by $\mathbb{G}_{m,k}$. Our definition is indirect, passing through the notion of a central extension of

$\mathcal{T}$ by $\boldsymbol{K}_{2,F}$ studied in [BD01]. However, we are able to deduce an explicit formula for the k-points of the commutator for $\hat{L\mathcal{T}}$, by explicitly describing the commutator for a central extension of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$. The commutator is enough to study group-theoretic structure and representation theory of $\hat{L\mathcal{T}}$ in 2.2.5. Some computations about central extensions of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$ are deferred to the later subsection 2.3 because the techniques are disjoint from the main application here. In the case when $\mathcal{T}$ is split, we show our definition of a Heisenberg central extension agrees with the definition in [Bei06, 1.4].

Definition 2.18: *1. Let $\tilde{\mathcal{T}(F)}$ be a central extension of $\mathcal{T}(F)$ by $K_2(F)$ as an abstract group:*

$$1 \to K_2(F) \to \tilde{T(F)} \to T(F) \to 1.$$

Pushing out by the tame symbol $\{-\}_{\mathrm{tame}_F} : K_2(F) \to k^\times$ 2.4, we obtain a central extension

$$1 \to k^\times \to \tilde{T(F)} \to T(F) \to 1$$

which we denote by $\tilde{\mathcal{T}(F)}^{\mathrm{tame}_F}$.

2. A central extension of a sheaf G of groups on S_{Zar} by a sheaf $\mathcal{A}$ of abelian groups on S_{Zar} is a group-valued sheaf $\mathcal{E}$ on S_{Zar} fitting in to a short exact sequence

$$1 \to \mathcal{A} \to \mathcal{E} \to G \to 1$$

in the category of group valued sheaves on S_{Zar} such that $\mathcal{A}$ lies in the center of $\mathcal{E}$.

Remark 2.11: *If G is abelian, then there is a well-defined commutator map $G \times G \to \mathcal{A}$ and it is bimultiplicative for $\mathbb{Z}$, defined by gluing local set-theoretic commutators as in [BD01, 0.N.4].*

According to [BD01, 1.4], when $\mathcal{A} = \boldsymbol{K}_{2,S}$ and S is the spectrum of a field and G to is a group scheme of finite type over F, taking S points preserves exactness give rise to central extension of $G(S)$ by $K_2(S)$ as abstract groups:

$$1 \to K_2(S) \to \mathcal{E}(S) \to G(S) \to 1.$$

Recall from 2.14 the subgroup $L^{++,-}\mathcal{T}^{\flat,0} \subseteq L\mathcal{T}$ that is the product of the pro-unipotent part and the nilpotent part of $L\mathcal{T}$.

Definition 2.19: *A Heisenberg central extension $\hat{L\mathcal{T}}$ is a $\mathbb{G}_{m,k}$-central extension of $L\mathcal{T}$ such that:*

1. The induced central extension of Lie algebras of the restriction $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$.

$$0 \to \mathbb{C} \to \mathrm{Lie}L^{++,\hat{-}}\mathcal{T}^{\flat,0} \to \mathrm{Lie}L^{++,0}\mathcal{T}^{\flat,0} \to 0$$

is a formal Heisenberg Lie algebra with positive part $\mathrm{Lie}L^{+}\mathcal{T}^{\flat,0}$, i.e., the center of $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ is $\mathbb{G}_{m,k}$ and equals the commutator.

2. Define the ***connected Heisenberg subgroup*** *of $\hat{L\mathcal{T}}$ to be $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$.*

3. There exists a central extension $\mathcal{E}$ of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$ such that the commutator of $\hat{L\mathcal{T}}(k)$ is the commutator of $\mathcal{E}(F)^{\mathrm{tame},F}$, i.e., the commutator of the k points of $\hat{L\mathcal{T}}$ is obtained by pushout by the tame symbol of the commutator of the F-points of a central extension of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$.

Remark 2.12: *In private communication, Xinwen Zhu has informed the author of their unpublished work that gives a way to intrinsically define a Heisenberg central extension without using algebraic K-theory. We did not know how to pursue this approach because our study relies heavily on the work of [BD01].*

Since $\mathbb{G}_{m,k}$ is a scheme, any Heisenberg central extension $\hat{L\mathcal{T}}$ is group ind scheme by pullback of a group ind scheme structure on $L\mathcal{T}$. We also have that the projection $\hat{L\mathcal{T}} \to L\mathcal{T}$ induces an isomorphism $\pi_0(\hat{L\mathcal{T}}) \approxeq \pi_0(L\mathcal{T})$ on connected components.

Remark 2.13: *So far we have only defined the notion of a Heisenberg central extension, but have not demonstrated their existence. Suppose G is a split, simple, and simply-connected group over F. By [BD01, 12.10], for such G there exists a central extension $\mathcal{E}$ of G by $\boldsymbol{K}_{2,F}$ such that $\hat{LG}(k) = \mathcal{E}(F)^{\mathrm{tame}_F}$. It is shown in chapter 3 that for any possibly non-split maximal torus $\mathcal{T} \subseteq G$, the restriction $\hat{L\mathcal{T}}$ is a Heisenberg central extension. This is the intended application.*

The rest of this subsection will describe the commutator pairing of a central extension $\mathcal{E}$ of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$. Such extensions were proven to satisfy Galois descent properties in [BD01, sec. 7], although the commutator pairing was not explicitly given.

Suppose S is a field and Z is an irreducible affine variety over S. Elements $C \in K_2(S(Z))$ give rise to partially defined morphisms $Z(S) \times Z(S) \to K_2(S)$ as follows. Write C in terms of Steinberg symbols $C = \sum\{f_i, g_i\}$ where f_i, g_i are units in the function field $S(Z)$. Each f_i, g_i is invertible on some open maximal open domain of definition $U_i \subseteq Z$ and define maps $U_i \to \mathbb{G}_{m,S}$. Taking the intersection U of the U_i, C defines a map $U \times U \to K_{2,S}$ where for every F-algebra R, the map is

$$U(R) \times U(R) \to K_2(R)$$

$$(x, y) \mapsto \prod\{f(x_i), g(y_i)\}.$$

When C is taken to satisfy a cocycle condition, such partial maps $Z \times Z \to K_{2,S}$ are called **generic cocycles** in [BD01, 0.4]. When C lies inside the image of the multiplication map $\mathcal{O}_Z^\times \otimes_\mathbb{Z} \mathcal{O}_Z^\times = \mathrm{Hom}(Z, \mathbb{G}_{m,S}) \otimes_\mathbb{Z} \mathrm{Hom}(Z, \mathbb{G}_{m,S}) \to K_2(S(Z))$, each f_i, g_i are globally defined on Z and C defines global maps

$$Z \times Z \to K_{2,S}.$$

Definition 2.20: *For $a \in \mathcal{O}_Z^\times \otimes_\mathbb{Z} \mathcal{O}_Z^\times$, the* ***associated map*** *$\varphi_a : Z^2 \to K_{2,S}$ is the morphism $Z \times Z \to K_{2,S}$ defined as above.*

Remark 2.14: *[BD01, 1.4] Suppose Z is any scheme over S, identified with the representable functor $h_Z = \mathrm{Hom}_{S_{\mathrm{Zar}}}(-, Z)$ into sets. By the Yoneda Lemma, for any presheaf $\mathcal{A}$ on S_{Zar}, there is a natural isomorphism*

$$\mathrm{Hom}_S(Z, \mathcal{A}) = \mathrm{H}^0(Z, \mathcal{A}) = \mathcal{A}_Z(Z).$$

In particular, if $Z = \mathrm{Spec}B$ is affine, morphisms $Z \to K_{2,S}$ correspond to elements of the group $K_2(B)$.

Let $\mathrm{Bilin}(Y)$ be the set of bilinear forms on Y, $\mathrm{Alt}(Y) \subseteq \mathrm{Bilin}(Y)$ be the subset of alternating bilinear forms, and $\mathrm{ESBilin} \subseteq X \otimes_{\mathbb{Z}} X$ the subset of even symmetric bilinear forms.

Lemma 2.9: *The following sequence is short exact*

$$0 \to \mathrm{Alt}(Y) \hookrightarrow \mathrm{Bilin}(Y) \stackrel{\mathrm{Id}+\tau^*\mathrm{Id}}{\to} \mathrm{ESBilin}(Y) \to 0$$

where the first map is the canonical inclusion.

PROOF: The map $\mathrm{Id} + \tau^*\mathrm{id} : \mathrm{Bilin}(Y) \to \mathrm{ESBilin}(Y)$ has kernel precisely $\mathrm{Alt}(Y)$ by the definition of $\mathrm{Alt}(Y)$. It suffices to show that it is surjective. Recall the following identifications as in [BD01, 3.5].

$$X \otimes_{\mathbb{Z}} X \approxeq \mathrm{Bilin}(Y)$$

$$x_1 \otimes x_2 \mapsto \{(y_1, y_2) \mapsto x_1(y_1)x_2(y_2)\}$$

$$X \wedge X \approxeq \mathrm{Alt}(Y)$$

$$x_1 \wedge x_2 \mapsto \{(y_1, y_2) \mapsto x_2(y_1)x_1(y_2) - x_1(y_1)x_2(y_2)\}$$

Under the above identification, the short exact sequence

$$0 \to X \wedge X \stackrel{x_1\wedge x_2 \mapsto x_2\otimes x_1 - x_1\otimes x_2}{\to} X \otimes_{\mathbb{Z}} X \stackrel{x_1\otimes x_2 \mapsto x_1x_2}{\to} \mathrm{Sym}^2 X \to 0$$

becomes a short exact sequence

$$0 \to \mathrm{Alt}(Y) \hookrightarrow \mathrm{Bilin}(Y) \to \mathrm{Sym}^2 X \to 0$$

where the first map is the canonical inclusion. In particular, it induces an isomorphism $\mathrm{Sym}^2 X \approxeq \mathrm{ESBilin}(Y)$. The result follows. ■

Remark 2.15: *The canonical inclusion* $\mathrm{ESBilin}(Y) \hookrightarrow \mathrm{Bilin}(Y)$ *is **not** a section for* $\mathrm{Bilin}(Y) \stackrel{Id+\tau^* Id}{\to} \mathrm{ESBilin}(Y)$. *The composition* $\mathrm{ESBilin}(Y) \hookrightarrow \mathrm{Bilin}(Y) \stackrel{Id+\tau^* Id}{\to}$

ESBilin(Y) *is multiplication by* 2. *However, since* $\mathrm{Sym}^2 X \subseteq \mathrm{Sym} X = \mathbb{Z}[X]$, $\mathrm{Sym}^2 X$ *is torsion free and finite over* $\mathbb{Z}$ *and hence free, so some section exists.*

Remark 2.16: *This constructed identification* $\mathrm{Sym}^2 X \approxeq \mathrm{ESBilin}(Y)$ *is given by* $x_1x_2 \mapsto \{(y_1, y_2) \mapsto x_1(y_1)x_2(y_2) + x_2(y_1)x_1(y_2)\}$ *and agrees with composing the usual identifications in [BD01, 3.5] as follows. Let* Quad(Y) *be the group of quadratic forms* $Y \to \mathbb{Z}$. *The identification* $\mathrm{Sym}^2 X \overset{\approxeq}{\to} \mathrm{ESBilin}(Y)$ *induced by the above is the composition of the classical identifications*

$$\mathrm{Sym}^2 X \approxeq \mathrm{Quad}(Y)$$

$$x_1x_2 \mapsto \{y \mapsto x_1(y)x_2(y)\}$$

$$\mathrm{Quad}(Y) \approxeq \mathrm{ESBilin}(Y)$$

$$q \mapsto \{(y_1, y_2) \mapsto q(y_1 + y_2) - q(y_1) - q(y_2)\}$$

Definition 2.21: *Let* $Z, \mathcal{A}$ *be presheaves of sets on a site and suppose* $\mathcal{A}$ *takes values in groups.* $f : Z \times Z \to \mathcal{A}$ *is a morphism. The* ***commutator*** *of* f *is the map*

$$(z_1, z_2) \mapsto f(z_1, z_2)f(z_2, z_1)^{-1}$$

,i.e., the map $f + (\tau^* f)^{-1}$.

The definition above applied to the cocycle map of a central extension of an abelian sheaf by another abelian sheaf coincides with the usual definition of the commutator.

Lemma 2.10: *Let* S *be a field,* Z *be an irreducible affine variety over* S, *and* $X = Hom(Z, \mathbb{G}_{m,S})$. *Let* Ψ *be the composition*

$$X \otimes_{\mathbb{Z}} X \overset{pr_1^* \otimes pr_2^*}{\to} X^2 \otimes_{\mathbb{Z}} X^2 \hookrightarrow \mathcal{O}_{Z^2}^{\times} \otimes_{\mathbb{Z}} \mathcal{O}_{Z^2}^{\times}.$$

Let $C \in X \otimes_{\mathbb{Z}} X$ *and* $B = C + \tau^* C$. *Then the commutator of the associated map* $\varphi_{\Psi(C)} : Z^2 \to K_{2,S}$ *to* $\Psi(C)$ *is the same as the associated map* $\varphi_{\Psi(B)} : Z^2 \to K_{2,S}$ *to* $\Psi(B)$.

PROOF: It suffices to show this when $C = x_1 \otimes x_2$ for $x_1, x_2 \in X$. By unwinding the definition of Ψ and the associated map, the associated map $Z^2 \to K_{2,S}$ of $\Psi(x_1 \otimes x_2)$ is defined by the property that for $(z_1, z_2) \in Z(R) \times Z(R)$,

$$(z_1, z_2) \mapsto \{x_1(z_1), x_2(z_2)\} \in K_2(R).$$

Using the antisymmetric property of the Steinberg symbol, the commutator is defined by

$$(z_1, z_2) \mapsto \{x_1(z_1), x_2(z_2)\}\{x_1(z_2), x_2(z_1)\}^{-1} = \{x_1(z_1), x_2(z_2)\}\{x_2(z_1), x_1(z_2)\}.$$

On the other hand, the associated map $Z^2 \to K_2$ to $\Psi(x_1 \otimes x_2 + \tau^*(x_1 \otimes x_2)) = \Psi(x_1 \otimes x_2 + x_2 \otimes x_1)$ is defined by

$$(z_1, z_2) \mapsto \{x_1(z_1), x_2(z_2)\}\{x_2(z_1), x_1(z_2)\}.$$

The two are the same. ∎

Now we can state the computation of the cocycle and for central extension of a split torus by $\boldsymbol{K}_{2,F}$ and the classification in [BD01, 3.9.3, 3.14].

Theorem 2.5: *Suppose S is a field and T is a split torus over S with character lattice X. Then for any central extension $\mathcal{E}$ of T by $\boldsymbol{K}_2$, there exists $C \in X \otimes_{\mathbb{Z}} X$ such that $\mathcal{E}$ is equivalent to the central extension of T by $\boldsymbol{K}_{2,S}$ defined by the cocycle $\varphi_{\Psi(C)}$: $T \times T \to K_{2,S}$ (with values already lying in the presheaf $K_{2,S}$ instead of the sheaf $\boldsymbol{K}_{2,S}$) associated to the image of C, with Ψ as in 2.10.Two elements $C, C' \in X \otimes X$ give equivalent central extensions of T by $\boldsymbol{K}_{2,S}$ iff they map to the same element under the map $X \otimes_{\mathbb{Z}} X \to \mathrm{Sym}^2 X$.*

Combining this with the interpretation of the map $X \otimes_{\mathbb{Z}} X \to \mathrm{Sym}^2 X$ as $\mathrm{Bilin}(Y) \overset{\mathrm{Id}+\tau^*\mathrm{Id}}{\to} \mathrm{ESBilin}(Y)$ in the proof of 2.9 and with 2.10, we re-phrase the classification to derive an explicit expression for the commutator:

Corollary 2.3: *Preserve the notation of the above. The isomorphism class of a central extension of T by $\boldsymbol{K}_{2,S}$ is determined by its commutator and the set of isomorphism classes is in canonical bijection with* $\mathrm{ESBilin}(Y)$ *as follows. Identify* $X \otimes X \approxeq \mathrm{Bilin}(Y)$ *naturally and consider* $\mathrm{ESBilin}(Y)$ *naturally as a subset of* $\mathrm{Bilin}(Y)$ *as in 2.9. For an even symmetric bilinear form B, a representative central extension is constructed by the cocycle $\varphi_{\Psi(C)} : T^2 \to K_{2,S}$ associated to $\Psi(C)$ for any C such that $C + \tau^* C = B$ and its commutator is $\varphi_{\Psi(B)}$.*

Remark 2.17: *The evaluation map $\mathbb{G}_{m,S} \times Y \to T$ defined by $(f, \lambda) \mapsto \lambda(S)$ is surjective, hence an epimorphism. Hence a given morphism $T \to \mathcal{F}$ to another sheaf $\mathcal{F}$ is determined by its pre-composition with the evaluation map $\mathbb{G}_{m,S} \times Y \to T$, i.e., by testing at all the one-parameter subgroups.*

Definition 2.22: *For $\lambda \in Y$ and $f \in \mathbb{G}_{m,S}(R)$, denote by $\lambda \otimes f$ for the element $\lambda(f) \in T(R)$.*

Lemma 2.11: *Preserve the notation of above. Let $C \in X \otimes_{\mathbb{Z}} X$, identified canonically with a bilinear form on Y as in 2.9. Then the map $\varphi_{\Psi(C)} : T^2 \to K_{2,S}$ associated to $\Psi(C)$ is uniquely defined by property that for every $\lambda, \mu \in Y$, the pre-composition*

$$\mathbb{G}_{m,S} \times \mathbb{G}_{m,S} \stackrel{\lambda \times \mu}{\to} T \times T \to K_{2,S}$$

is defined by for all $f, g \in \mathbb{G}_{m,S}(R)$,

$$(f, g) \mapsto (\lambda(f), \mu(g)) \mapsto \{f, g\}^{C(\lambda,\mu)} \in K_2(R).$$

Proof: Let I be a set with $|I| = d$, the rank of T. Let $\{y_i : i \in I\}$ be a basis for Y over $\mathbb{Z}$ and $\{x_i : i \in I\}$ be the dual basis for X, considered as the set of group homomorphisms $T \to \mathbb{G}_{m,S}$. Write $C = \sum_{i,j \in I} c_{i,j} x_i \otimes x_j$. Denote $\langle -, - \rangle : X \times Y \to \mathbb{Z}$ for the canonical pairing. We compute the composition above for each $\lambda, \mu \in Y$,

$f, g \in \mathbb{G}_{m,S}(R)$:

$$(f,g) \mapsto \prod_{i,j \in I} c_{i,j}.\{(x_i \circ \mathrm{pr}_1)(\lambda(f), \mu(g)), (x_j \circ \mathrm{pr}_2)(\lambda(f), \mu(g))\}$$

$$= \prod_{i,j \in I} c_{i,j}.\{f^{\langle x_i, \lambda \rangle}, g^{\langle x_j, \mu \rangle}\}$$

$$= \{f, g\}^{\sum_{i,j \in I} c_{i,j} \langle x_i, \lambda \rangle \langle x_j, \mu \rangle}$$

$$= \{f, g\}^{C(\lambda, \mu)}.$$

The result follows. ■

Definition 2.23: *Let Z be a scheme over S and $e : S \to Z$ a distinguished section. Let $\mathcal{A}$ be a presheaf of abelian groups on S_{Zar}. A morphism $f : Z \to \mathcal{A}$ of presheaves is **normalized at** e if the $f \circ e$ is the trivial map $S \to \mathcal{A}$. If the point e is clear, we may simply say f is **normalized**. For any subset $M \subseteq \mathrm{H}^0(Z, \mathcal{A})$ we denote by M^{norm} the subset of normalized elements of M.*

Remark 2.18: *If $M \subseteq \mathrm{H}^0(Z, \mathcal{A})$ is a subgroup, M^{norm} is also a subgroup given as the kernel of the evaluation map*

$$M \to \mathrm{H}^0(S, \mathcal{A})$$

$$f \mapsto f \circ e.$$

We have a Galois descent property of normalized global sections for $\boldsymbol{K}_{2,F}$:

Lemma 2.12: *Let X be a smooth geometrically irreducible variety over F with a distinguished F-rational point e, also considered as a Γ-invariant E-rational point. The base change map induces an isomorphism*

$$\mathrm{H}^0(X, \boldsymbol{K}_{2,F})^{\mathrm{norm}} \xrightarrow{\cong} \mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})^{\mathrm{norm},\Gamma} := (\mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}})^{\Gamma}.$$

Consequently if a normalized morphism $f_E : X_E \to \boldsymbol{K}_{2,E}$ is Γ-invariant, it descends uniquely to a normalized morphism $f : X \to \boldsymbol{K}_{2,F}$.

Remark 2.19: *The meaning of unique descent of f_E to f is the following. For a map $f : X \to \boldsymbol{K}_{2,F}$, consider the commutative square*

$$\begin{array}{ccc} \mathrm{Res}_{E/F} X_E & \xrightarrow{f|_E} & \mathrm{Res}_{E/F} \boldsymbol{K}_{2,E} \\ \uparrow & & \uparrow \\ X & \xrightarrow{f} & \boldsymbol{K}_{2,F} \end{array}$$

obtained by for every F-algebra R evaluation the map $X \xrightarrow{f} \boldsymbol{K}_{2,F}$ at the ring map $R \to R \otimes_F E$. Then the statement of 2.12 is that there is a unique normalized f such that $f|_E = f_E$.

Proof: By [BD01, 2.4.(i)] We have that the base change map induces an isomorphism

$$\mathrm{H}^0(X, \boldsymbol{K}_{2,F})/K_2(F) \xrightarrow{\cong} (\mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})/K_2(E))^\Gamma$$

where $K_2(F) \to \mathrm{H}^0(X, \boldsymbol{K}_{2,F})$ embeds as the subgroup of constant global sections and similarly so for $K_2(E) \to \mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})$. Evaluation at e gives a section for each of these embeddings. Since $e \in X(E)$ is Γ-stable, the canonical isomorphism $\mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}} \to \mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})/K_2(E)$ is a map of Γ-modules. The result then follows by considering the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^0(X, \boldsymbol{K}_{2,F})^{\mathrm{norm}} & \to & \mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}} \\ \downarrow & & \downarrow \\ \mathrm{H}^0(X, \boldsymbol{K}_{2,F})/K_2(F) & \to & \mathrm{H}^0(X_E, \boldsymbol{K}_{2,E})/K_2(E) \end{array}$$

where the horizontal maps are base change, and the vertical maps are the canonical isomorphisms. ∎

Definition 2.24: *Let $\mathcal{E}$ be a central extension of a group G over F by $\boldsymbol{K}_{2,F}$, considered as a multiplicative $\boldsymbol{K}_{2,F}$-torsor, and $p : G_E \to G$ be the projection map. Pulling back $\mathcal{E}$ to G_E gives central extension of G_E by $p^*\boldsymbol{K}_{2,F}$. Then taking the pushout by the map $p^*\boldsymbol{K}_{2,F} \to \boldsymbol{K}_{2,E}$ gives a central extension of G_E by $\boldsymbol{K}_{2,E}$. We denote it by $\mathcal{E}_E$, called the* ***base change*** *$\mathcal{E}$ to E. The operation of base change is functorial.*

We are now equipped to state the main computation of this subsection, of the commutator for a central extension of the torus $\mathcal{T}$ over F by $\boldsymbol{K}_2$.

Theorem 2.6: *For any central extension of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$ over F, there exists an invariant even symmetric bilinear form $B \in \mathrm{ESBilin}(Y)^\Gamma \subseteq X \otimes X$ such that the central extension is equivalent to a central extension $\mathcal{E}$ with commutator $C_\mathcal{E} : \mathcal{T}^2 \to \boldsymbol{K}_{2,F}$ (with values lying in the presheaf $K_{2,F}$) satisfying:*

1. The base change $\mathcal{E}_E$ has commutator $C_{\mathcal{E}_E}$ (with values lying in the presheaf $K_{2,E}$) equal to $\varphi_{\Psi(B)}$ to $\Psi(B)$, where Ψ is the composition

$$X \otimes_\mathbb{Z} X \overset{pr_1^* \otimes pr_2^*}{\to} X^2 \otimes_\mathbb{Z} X^2 \hookrightarrow \mathcal{O}^\times_{\mathcal{T}^2_E} \otimes_\mathbb{Z} \mathcal{O}^\times_{\mathcal{T}^2_E}$$

from 2.10. Furthermore, $\varphi_{\Psi(B)}$ is Γ-equivariant and the resulting map $\mathcal{T} \to \boldsymbol{K}_{2,F}$ given by applying the descent for normalized global sections in 2.12 is precisely the commutator for $\mathcal{E}$.

2. As a morphism $\mathcal{T}^2 \to \boldsymbol{K}_{2,F}$ whose restriction to the category of E-algebras is $C_{\mathcal{E}_E}$, $C_\mathcal{E}$ is uniquely determined by the composition

$$\mathrm{Res}_{E/F}\mathcal{T}^2_E \overset{N\times N}{\to} \mathcal{T}^2 \overset{C_\mathcal{E}}{\to} \boldsymbol{K}_{2,F} \to \mathrm{Res}_{E/F}\boldsymbol{K}_{2,E}$$

where N denotes the norm map. The above composition is computed by: for $\lambda, \mu \in Y$ and $f, g \in \mathbb{G}_{m,E}$, the element

$$(\lambda \otimes f, \mu \otimes g) \in \mathrm{Res}_{E/F}\mathcal{T}^2_E$$

maps to the mage of the element

$$\prod_{i,j=1}^m \{\nu^i.f, \nu^j.g\}^{B(\nu^i.\lambda, \nu^j\mu)} \in \mathrm{Res}_{E/F}K_{2,E}$$

in $\mathrm{Res}_{E/F}\boldsymbol{K}_{2,E}$.

Remark 2.20: *Any commutator for a central extension of an abelian sheaf by an-*

other is automatically normalized. In particular $C_{\mathcal{E}}(e) = 1 \in K_2(F)$ *at the unit* $e \in \mathcal{T}^2(F)$.

PROOF: Let us discuss (1). The existence of such a Γ-invariant even symmetric bilinear for $B \in \mathrm{Sym}^2 X$ determining the commutator for $\mathcal{E}_E$ is proved in the later Section 2.11. Now observe that for any Γ-invariant bilinear form B, the element $\Psi(B) \in K_2(\mathcal{T}_E^2)$ is Γ-invariant. This shows that $\varphi_{\Psi(B)} \in \mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}}$ is Γ-invariant and the descent lemma of 2.12 applies to $\varphi_{\Psi(B)}$. The fact that the resulting normalized map $\mathcal{T}^2 \to \boldsymbol{K}_{2,F}$ is precisely the commutator $C_{\mathcal{E}}$ is proven in later 2.22 using 2.13 that shows this is true generically. This shows (1).

Let us prove (2). Observe that $\Psi(X \otimes_{\mathbb{Z}} X)$ consists of elements normalized at e; this follows by applying the computation 2.11 to $f, g = 1$ any any $\lambda, \mu \in Y$. Therefore $\Psi((X \otimes_{\mathbb{Z}} X)^\Gamma)$ maps to $(K_2(E(\mathcal{T}_E^2)^{\mathrm{reg,norm}})^\Gamma$. In particular, $\varphi_{\Psi(B)}$ is normalized and Γ-invariant. By the descent lemma 2.12, for all $B \in (X \otimes_{\mathbb{Z}} X)^\Gamma$, the commutator $C_{\mathcal{E}}$, as a morphism $\mathcal{T}_E \to \boldsymbol{K}_{2,F}$, is uniquely determined by the property that it is normalized and fits into the commutative diagram

$$\begin{array}{ccccc}
\mathrm{Res}_{E/F}\mathcal{T}_E^2 & \overset{\varphi_{\Psi(B)}}{\to} & \mathrm{Res}_{E/F}K_{2,E} & \to & \mathrm{Res}_{E/F}\boldsymbol{K}_{2,E} \\
\uparrow & & & & \uparrow \\
\mathcal{T}^2 & & \overset{C_{\mathcal{E}}}{\to} & & \boldsymbol{K}_{2,F}
\end{array}$$

Since $C_{\mathcal{E}}$ is a commutator and already normalized, the normalization condition is redundant. Since the norm map $N : \mathrm{Res}_{E/F}\mathcal{T}_E \to \mathcal{T}$ is an epimorphism, the composition $\mathcal{T}^2 \to \mathrm{Res}_{E/F}\boldsymbol{K}_{2,E}$ from the bottom left to top right is uniquely determined by its precomposition with $\mathrm{Res}_{E/F}\mathcal{T}_E^2 \overset{N\times N}{\to} \mathcal{T}^2$. It remains to compute it. We have for $\lambda, \mu \in Y$ and $f, g \in \mathbb{G}_{m,E}$, $(\lambda \otimes f, \mu \otimes g) \in \mathrm{Res}_{E/F}\mathcal{T}_E^2$ maps to $\varphi_{\Psi(B)}(N(\lambda \otimes f), N(\mu \otimes g))$. By 2.11 and bimultiplicative property $\varphi_{\Psi(B)}$, we have

$$\varphi_{\Psi(B)}(N(\lambda \otimes f), N(\mu \otimes g)) = \prod_{i,j=1}^{N} \varphi_{\Psi(B)}(\nu^i.\lambda \otimes \nu^i.f, \nu^j.\mu \otimes \nu^j.g)$$

$$= \prod_{i,j=1}^{N} \{\nu^i.f, \nu^j.g\}^{B(\nu^i.\lambda, \nu^j\mu)}.$$

Then (2) is proved. ■

Let us conclude by showing our definition of a Heisenberg central extension is consistent with the definition of a Heisenberg central extension of a split torus given in [Bei06, 1.4]. The tame symbol $F^\times \times F^\times \to k^\times$ extends to a morphism of group ind-schemes

$$\{-,-\}_{\text{c.c.}} L\mathbb{G}_{m,F} \times L\mathbb{G}_{m,F} \to \mathbb{G}_{m,k}$$

defined in [OZ16, 2.2] called the **Contou-Carrerè symbol**. It descends to a map $LK_{2,F} \times LK_{2,F} \to \mathbb{G}_{m,k}$. By [OZ16, 2.4], it has the property that the induced map on tangent spaces at 1 is the map

$$\mathbb{C}((t)) \oplus \mathbb{C}((t)) \to \mathbb{C}$$

$$(f,g) \mapsto \operatorname{res}(f\mathrm{d}g).$$

Definition 2.25: *Let T be a (split) torus over k and B an even symmetric bilinear form on Y. A Heisenberg Central extension $\hat{LT}_F$ in the sense of [Bei06, 1.4] associated to B is a $\mathbb{G}_{m,k}$-central extension equipped with a splitting $L^+T_F \to \hat{LT}_F$ with the property that the commutator satisfies for all $\lambda, \mu \in Y$, $f, g \in L\mathbb{G}_{m,F}$,*

$$(\lambda \otimes f, \mu \otimes g) \mapsto \{\lambda(f), \mu(g)\}_{\text{c.c.}}^{B(\lambda,\mu)}.$$

Lemma 2.13: *Suppose B is non-degenerate. Then a Heisenberg central extension of LT_F in the sense of [Bei06, 1.4] associated to B is a Heisenberg central extension as in 2.19.*

Proof: Let $\hat{LT}_F$ be a Heisenberg central extension in the sense of [Bei06, 1.4]. Since the Contou-Carrerè symbol extends the tame symbol on k-points, by 2.10 the computation of the formula of the commutator for a central extension of T by $\boldsymbol{K}_2$ associated to B, it remains only to show that if B is non-singular, then the induced central extension $\mathrm{Lie}L^{\widehat{+,-}}T_F^{\flat,0} = \mathrm{Lie}L^{\widehat{+,-}}T_{\mathcal{O}_F}$ is a formal Heisenberg Lie algebra with positive part $\mathfrak{t}[[t]]$ where $\mathfrak{t} = \mathrm{Lie}T$, identified with $Y \otimes_{\mathbb{Z}} \mathbb{C}$. Extend B to a bilinear form on $\mathfrak{t}$ by $\mathbb{C}$-bilinearity. By [Bei06, 1.5(ii)] $\hat{\mathfrak{t}} \coloneqq \mathrm{Lie}L^{\widehat{+,-}}T_F^{\flat,0}$ is isomorphic to the formal completion of the central extension of $\mathfrak{t}_{+,-}[t^{\pm 1}] \coloneqq t^{-1}\mathfrak{h}[t^{-1}] \oplus t\mathfrak{h}[t]$ presented as

$$0 \to \mathbb{C}K \to \hat{\mathfrak{t}} \to \mathfrak{t}_{+,-}[t^{\pm 1}] \to 0$$

where $K \in Z(\hat{\mathfrak{t}})$ is some central element, with commutator

$$(X \otimes f, Y \otimes g) \mapsto B(X,Y)\mathrm{Res} f \mathrm{d}g K.$$

Since B is nonzero, the commutator pairing is nontrivial. Since B is non-degenerate, the center of $\hat{\mathfrak{t}}$ is $\mathbb{C}K$. We conclude that the center of $\hat{\mathfrak{t}}$ is one-dimensional and equal to its commutator. The result follows. ∎

2.2.5 Group-Theoretic Decomposition of Heisenberg Central Extensions

In this section, we determine the group theoretic structure of a Heisenberg central extension, using the decomposition of a loop torus in 2.2.

Let $k = \mathbb{C}$, $F = k((t))$ and $\mathcal{T}$ be a torus over F. Let E/F be an extension such that $\mathcal{T}_E$ is split of rank d. Put $E = k((u))$ for some $u^m = t$ where $m = [E:F]$. Let $\Gamma = \mathrm{Gal}(E/F)$ be the finite cyclic Galois group with ν a choice of generator. Fix a primitive mth root of unity ζ defined by the property that $\nu(u)/u = \zeta_m$. Let μ_r denote the group of rth roots of unity in $k^\times$ for each integer $r \geq 1$. Let $X = \mathrm{Hom}_E(\mathcal{T}_E, \mathbb{G}_{m,E})$, resp. $Y = \mathrm{Hom}_E(\mathbb{G}_{m,E}, \mathcal{T}_E)$, be the (absolute) character, resp. cocharacter, lattices with their canonical Γ action. Let $N = \sum_{\gamma \in \Gamma} \gamma.(-)$ be the norm map on an abelian group or sheaf with a Γ action. For a central extension $\mathcal{P}$ of an abelian group, resp. presheaf by another abelian group, resp. presheaf, denote $C_{\mathcal{P}}$ the commutator map. Let T be the split torus over k with character lattice X.

For this section, fix $\hat{L\mathcal{T}}$ to be a Heisenberg central extension, defined in 2.19. Let $\mathcal{E}$ be a central extension of $\mathcal{T}$ by $\boldsymbol{K}_2$ such that $C_{\hat{L\mathcal{T}(k)}} = C_{\mathcal{E}(k)^{\text{tame},F}}$is pushout of $\mathcal{E}(k)$ by the tame symbol over F. Let B be the Γ-invariant even symmetric bilinear form determining the commutator for $\mathcal{E}$ in 2.6.

Definition 2.26: *For a morphism $H \to L\mathcal{T}$ of group functors, define $\hat{H}$ to be the restriction of the central extension to H.*

The purpose of this subsection is to establish two theorems regarding the group-theoretic structure of $\hat{L\mathcal{T}}$ which will be used to determine the representation theory of $\hat{L\mathcal{T}}$ in the next subsection 2.2.6:

Theorem 2.7: *There exists a splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$.*

Now recall the decomposition from 2.2:

$$L\mathcal{T} = L^-\mathcal{T}_E^\Gamma \times T^{\Gamma,0} \times L^{++}\mathcal{T}_E^\Gamma \times Y_\Gamma$$

where $T^{\Gamma,0}$ is the maximal torus of $L^+\mathcal{T}^{\flat,0}$.

Theorem 2.8: *Suppose B is non-degenerate. For a fixed splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$, there exists an isomorphism (given the choice of homomorphic section $s_u : Y_\Gamma \to L\mathcal{T}(k)$ for the Kottwitz homomorphism):*

$$\hat{L\mathcal{T}} \approxeq (L^{++,\hat{,}-}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0}))/\mathbb{G}_{m,k}$$

where

1. The restriction $\hat{Y}_\Gamma$ of $\hat{L\mathcal{T}}$ to Y_Γ by s_u has cocycle taking values in the group μ_r roots of unity for some integer $r \geq 0$.

2. The embedding $\mathbb{G}_{m,k} \to L^{++,\hat{,}-}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})$ is given by the diagonal

$$x \mapsto (x, x^{-1})$$

into the product of the central $\mathbb{G}_{m,k}$ of $L^{++,\hat{,}-}\mathcal{T}^{\flat,0}$ and the central $\mathbb{G}_{m,k}$ of $\hat{Y}_\Gamma$.

3. $L^{++}\hat{\mathcal{T}}^{\flat,0}$ *is trivial as a central extension, i.e.,* $L^{++}\hat{\mathcal{T}}^{\flat,0}(k) \approxeq \mathbb{G}_{m,k} \times L^{++}\mathcal{T}^{\flat,0}$

We use the morphism $s_u = (N \circ \mathrm{ev}_u) \cdot \upsilon : Y_\Gamma \to L\mathcal{T}(k)$ 2.6. Recall N is the norm map, ev_u is defined by the property that $\lambda \mapsto \lambda(u) \in L\mathcal{T}_E(k)$, and υ is some map $Y_\Gamma \to T^{\Gamma,0}$.

Definition 2.27: *For a central extension* $\hat{\mathcal{A}}$ *of* $\mathcal{A}$ *by some other group or group-valued sheaf, for* $x \in \hat{\mathcal{A}}$ *denote by* $\overline{x}$ *the image of* x *in* $\mathcal{A}$*. In the case when* $\mathcal{A}$ *is a group and* $\hat{\mathcal{A}} \to \mathcal{A}$ *is surjective, for* $a \in \mathcal{A}$ *denote by* $\hat{a}$ *any lift of* a *in* $\hat{\mathcal{A}}$*.*

Proposition 2.1: *There exists a surjection* $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 \to L\hat{\mathcal{T}}$*, where and the action of* $L\hat{\mathcal{T}}^0$ *on* $\hat{Y}$ *is defined by for* $r \in L\hat{\mathcal{T}}^0$, $s \in \hat{Y}$,

$$r.s = rs_u(\overline{s})r^{-1} = C_{L\hat{\mathcal{T}}}(\overline{r}, s_u(\overline{s})).\hat{s}$$

where $C_{L\hat{\mathcal{T}}}(\overline{r}, s_u(\overline{s})) \in \mathbb{G}_{m,k}$ *is considered an element of the central* $\mathbb{G}_{m,k}$ *of* $\hat{Y}$*.*

Remark 2.21: *In fact, for any subgroups* $H, P \subseteq L\mathcal{T}$*, multiplication by the commutator as above gives a well-defined action of* P *on* $\hat{H}$*. This action lifts the conjugation action of* $\hat{P}$ *on* $\hat{H}$*.*

Proof: Since $L\mathcal{T}$ is abelian, conjugation by $L\hat{\mathcal{T}}^0$ preserves $\hat{H}$ for any subgroup $H \subseteq L\mathcal{T}$. Put $H = s_u(Y_\Gamma)$. The action above is then the pullback of the conjugation action of $L\hat{\mathcal{T}}^0$ on H by the map and thus well-defined.

The morphism $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 \to L\hat{\mathcal{T}}$ is defined as follows. Forgetting the group structure, the underlying in-scheme space of $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0$ is $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0$. Then define $L\hat{\mathcal{T}}^0 \rtimes \hat{Y} \to L\hat{\mathcal{T}}$ as the composition

$$\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 \overset{s_u \times \iota}{\to} L\hat{\mathcal{T}} \times L\hat{\mathcal{T}} \overset{\mu}{\to} L\hat{\mathcal{T}}$$

where $\iota : L\hat{\mathcal{T}}^0 \to L\hat{\mathcal{T}}$ is the canonical inclusion, and μ is the multiplication map. Such a map is flat as a morphism of ind-schemes. It is a group homomorphism because the restriction to $\hat{Y_\Gamma} \times 1$ and $1 \times L\hat{\mathcal{T}}^0$ both are group homomorphisms and the definition action of $L\hat{\mathcal{T}}^0$ on $\hat{Y_\Gamma}$ is such that $\hat{Y_\Gamma} = 1 \times \hat{Y_\Gamma}$ acts by its image under $\mu \circ ((N \circ s_u) \times \iota)$. To check that the map is surjective, it suffices to check surjection on the neutral component $L\hat{\mathcal{T}}^0$ and that it hits every connected component of $L\hat{\mathcal{T}}$. The first follows because it restricts to an isomorphism on $L\hat{\mathcal{T}}^0 \times 1$. The second follows by the fact that s_u is a section for the Kottwitz homomorphism 2.6. The proposition is proved.■

The main theorems 2.7 and 2.8 of this subsection are proven by refining the surjection $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 \to L\hat{\mathcal{T}}$ by showing the following:

1. 2.2 The commutator $C_{L\hat{\mathcal{T}}}$ is trivial on $L^+\mathcal{T}^{\flat,0} \times L^+\mathcal{T}^{\flat,0}$ and $L^{++}\hat{\mathcal{T}}^{\flat,0}$ is a direct sum $L^{++}\hat{\mathcal{T}}^{\flat,0}(k) \approxeq \mathbb{G}_{m,k} \times L^{++}\mathcal{T}^{\flat,0}$. Hence a splitting $L^+\mathcal{T}^{\flat,0} \to L\hat{\mathcal{T}}$ exists.

2. 2.3 For any splitting $L^+\mathcal{T}^{\flat,0} \to L\mathcal{T}$, we have $L\hat{\mathcal{T}}^0 = L^{++,\hat{-}}\mathcal{T}^{\flat,0} \times T^{\Gamma,0}$ and $L^{++,-}\mathcal{T}^{\flat,0}$ acts trivially on $\hat{Y_\Gamma}$. The splitting induces a canonical isomorphism $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 = L^{+\hat{+},-}\mathcal{T}^{\flat,0} \times (\hat{Y_\Gamma} \rtimes T^{\Gamma,0})$.

3. 2.4. The action of $T^{\Gamma,0}$ on $\hat{Y_\Gamma}$ is computed and found identical to the one in [BK04, sec. 4.3] and lifts to an action of $T^{\Gamma,0}$ on $\hat{Y}$, where $\hat{Y}$ is the restriction of $\hat{Y_\Gamma}$by the projection $Y \to Y_\Gamma$. Explicitly, the action of $\exp h \in T^{\Gamma,0}$ for $h \in \mathfrak{t}^\Gamma$ on $\hat{\lambda} \in \hat{Y}$ for $\lambda \in Y$ is

$$\exp(h).\hat{\lambda} = e^{B(h,\lambda)}.\hat{\lambda}$$

   where $\exp : \mathfrak{t}^\Gamma \to T^{\Gamma,0}$ is the exponential map and $e^{(-)} : \mathbb{C} \to \mathbb{G}_{m,k}$ is the exponential function.

4. 2.5 The kernel of $\hat{Y_\Gamma} \rtimes L\hat{\mathcal{T}}^0 \to L\hat{\mathcal{T}}$ is the diagonal embedding $x \to (x, x^{-1})$ of $\mathbb{G}_{m,k}$ into the product of the central $\mathbb{G}_{m,k}$ of $L\hat{\mathcal{T}}^0$ and the central $\mathbb{G}_{m,k}$ of $\hat{Y_\Gamma}$.

5. 2.19 Up to equivalence of central extensions, the restriction $\hat{Y_\Gamma}$ of $L\hat{\mathcal{T}}$ to Y_Γ by s_u has cocycle taking values in the group μ_r roots of unity for some integer $r \geq 0$.

Remark 2.22: *A central μ_{2m} extension of the discreet group Y_Γ and its commutator also features as foundational in the work of [Lep85]. In [Lep85, BK04], that central* *extension is defined as an axiomatic starting point for Lie-theoretic computations. In contrast, we interpret $\hat{Y}_\Gamma$ geometrically as coming from a section for the projection of $L\mathcal{T}$ to the group of connected components. We expect that $\hat{Y}_\Gamma$ should agree with the μ_{2m}-central extension of Y_Γ in [Lep85, BK04] possibly after modifying the section s_u, but have not investigated this. This is because we did not need to explicitly compute $\hat{Y}_\Gamma$ to deduce similar looking results on the representation theory of $L\hat{\mathcal{T}}$ in 2.2.6.*

Let us first make some preparations that reduce these computations to computations about the k-points.

Definition 2.28: *A **pro-scheme** is a projective limit of schemes together with limit structure. An **ind-pro scheme** is an inductive limit of pro-schemes together with its limit structure. Let $H = \lim_{\leftarrow i} H_i$, $H' = \lim_{\leftarrow i} H'_i$ be a pro-schemes over k. A **pro-morphism** $f : H \to H'$ is a morphism over k that factors through morphisms $f_i : H_i \to H'_i$. Now suppose $H = \lim_{\to i} H_i$, $H' = \lim_{\to i} H'_i$ are ind-pro-schemes where H_i, H'_i are pro-schemes. A **ind-pro-morphism** is a map $f : H \to H'$ that factors through morphisms $f_i : H_i \to H'_i$ that are pro-morphisms.*

Lemma 2.14: *Suppose H, H' are projective limits of finite type and reduced affine schemes and $f : H \to H'$ is a pro-morphism. Then f is determined by the induced map on k points $f : H(k) \to H'(k)$.*

Proof: Let $H = \lim_{\leftarrow i} H_i$, $H' = \lim_{\leftarrow i} H'_i$ where H_i and H'_i are finite type, reduced and affine. Then H, H' are affine and since each H_i, H'_i are affine, $H(k) = \lim_{\leftarrow i} H_i(k)$ and $H'(k) = \lim_{\leftarrow} H'_i(k)$. The result then follows from the well known fact that over an algebraically closed field, the functor of taking k points on the category reduced and finite type schemes is faithful. ∎

Lemma 2.15: *Suppose f is an ind-pro automorphism of $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ that fixes the central $\mathbb{G}_m$ and restricts to the identity on k-points. Then f is trivial.*

Proof: Since $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ is connected, it suffices to show Lie$f$ is trivial. Since f restricts to the identity on k points, f restricts to the identity on the $L^{++}\mathcal{T}^{\flat,0}$, which is a projective limit of finite type reduced affine schemes, by 2.14. Thus Lie$f$ restricts to the identity on Lie$L^{++}\mathcal{T}^{\flat,0}$. Now let $\mathfrak{l} \subseteq$ Lie$L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ be the Heisenberg Lie algebra such that Lie$L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ is the formal completion of $\mathfrak{l}$, where the positive part $\mathfrak{l}_+$ lies in Lie$L^{++}\mathcal{T}^{\flat,0}$. Since $\mathfrak{l}_-$ and $\mathfrak{l}_+$ are in perfect pairing by the commutator by 2.1, Lie$f$ is a Lie algebra morphism and Lie$f$ fixes both the center $Z(\mathfrak{l})$ and $\mathfrak{l}_+$, we conclude that Lie$f$ fixes $\mathfrak{l}_-$ as well, and therefore also $\mathfrak{l}$. The result follows as Lie$f$ is continuous. ■

Now let us make some reductions that allow us to evaluate the commutator of $\mathcal{E}(k)$ over E instead of F on connected sub ind pro groups of $L\mathcal{T}(k)$.

Definition 2.29: *Define $C_{\mathcal{E}(k),\mathrm{tame}_E}$ to be the composition*

$$\mathcal{T}(F) \times \mathcal{T}(F) \overset{C_{\mathcal{E}(k)}}{\to} K_2(F) \to K_2(E) \overset{\mathrm{tame}_E}{\to} k^\times.$$

Lemma 2.16: *We have as maps $L\mathcal{T}(k) \times L\mathcal{T}(k) \to k^\times$,*

$$C_{\mathcal{E}(k),\mathrm{tame}_E} = C^m_{\hat{L\mathcal{T}}(k)}$$

,i.e., $C_{\mathcal{E}(k),\mathrm{tame}_E}$ is the mth multiple of the commutator for $\hat{L\mathcal{T}}(k)$.

Proof: We have from 2.4 a commutative diagram

$$\begin{array}{ccc} F^\times \otimes_{\mathbb{Z}} F^\times & \overset{\mathrm{tame}_F}{\to} & k^\times \\ \downarrow & & \downarrow_{(-)^m} \\ E^\times \otimes_{\mathbb{Z}} E^\times & \overset{\mathrm{tame}_E}{\to} & k^\times \end{array}$$

where $(-)^m$ denotes the map of raising to the mth power. Since the tame symbol satisfies the Steinberg relations, it descends to the commutative diagram

$$\begin{array}{ccccc} \mathcal{T}(F) \times \mathcal{T}(F) & \overset{C_{\mathcal{E}(k)}}{\to} & K_2(F) & \overset{\mathrm{tame}_F}{\to} & k^\times \\ & & \downarrow & & \downarrow_{(-)^m} \\ & & K_2(E) & \overset{\mathrm{tame}_E}{\to} & k^\times \end{array}$$

the result follows. ■

Now recall $L\mathcal{T}^0_{\mathrm{red}} = L^+\mathcal{T}^{\flat,0}$ from 2.14.

Lemma 2.17: *Suppose $H \subseteq L\mathcal{T}^0(k) = L^+\mathcal{T}(k)$ is a connected sub (ind) pro group and $r \in L\mathcal{T}(k)$. If $C_{\mathcal{E}(k),\mathrm{tame}_E}(H,r) = \{1\}$, then $C_{\hat{L\mathcal{T}(k)}}(H,r) = \{1\}$, i.e., if the pushout of the commutator over E with r is trivial on H, then the commutator of the pushout over F with r is trivial on H.*

Proof: Suppose $C_{\mathcal{E}(k),\mathrm{tame}_E}(H,r) = \{1\}$. Using the above lemma that $C_{\mathcal{E}(k),\mathrm{tame}_E} = C^m_{\hat{L\mathcal{T}(k)}}$, we have that $C_{\hat{L\mathcal{T}(k)}}(H,r)$ takes values in the group μ_m, which is discreet. Since $x \mapsto C_{\hat{L\mathcal{T}(k)}}(x,r)$ is continuous and sends the identity element in H to 1, it follows that it must send H to $\{1\}$. ■

Now apply the computation of commutator main theorem from 2.6 to the F-points. We obtain

$$C_{\mathcal{E}(k),\mathrm{tame}_E}(N(\lambda \otimes f), N(\mu \otimes g)) = \prod_{i,j=1}^{m} \{\nu^i.f, \nu^j.g\}^{B(\nu^i.\lambda,\nu^j\mu)}_{\mathrm{tame}_E} \in k^\times.$$

Proposition 2.2: *2.7 The commutator $C_{\hat{L\mathcal{T}}}$ is trivial on $L^+\mathcal{T}^{\flat,0} \times L^+\mathcal{T}^{\flat,0} \subseteq L\mathcal{T}$, and there exists a partial splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ of $\hat{L\mathcal{T}}$ as a central extension over $L^+\mathcal{T}^{\flat,0} \subseteq L\mathcal{T}$.*

Proof: Since the restriction of $C_{\hat{L\mathcal{T}}}$ to $L^+\mathcal{T}^{\flat,0} \times L^+\mathcal{T}^{\flat,0}$ is a pro-morphism, by 2.14 it suffices to show that $C_{\hat{L\mathcal{T}}}$ is trivial on the k-points of $L^+\mathcal{T}^{\flat,0} \times L^+\mathcal{T}^{\flat,0}$. Applying 2.17 to the connected subgroup $H = L^+\mathcal{T}^{\flat,0}(k)$ with r ranging over elements of H, it suffices to show that $C_{\mathcal{E}(k),\mathrm{tame}_E}$ is trivial on $L^+\mathcal{T}^{\flat,0}(k) \times L^+\mathcal{T}^{\flat,0}(k)$.

To this end, recall from 2.4 that the norm map $N : L\mathcal{T}_E(k) \to L\mathcal{T}(k)$ is surjective on k points. By the properties of the Kottwitz homomorphism 2.6, N restricts to a surjection

$$(1-\nu)Y \otimes (L\mathbb{G}_{m,E})(k) \xrightarrow{N} (L\mathcal{T}^0)(k) = L\mathcal{T}^{\flat,0}(k)$$

$$(1-\nu)\lambda \otimes f \mapsto N((1-\nu)\lambda \otimes f)$$

where the equality on the right hand side is given by 2.14. For $\lambda, \mu \in Y$ and $f, g \in L\mathbb{G}_{m,E}(k)$, it suffices to compute the following quantity and show that it is equal to 1:

$$C_{\mathcal{E}(k),\mathrm{tame}_E}(N_{E/F}((1-\nu)\lambda\otimes f), N_{E/F}((1-\nu)\mu\otimes g)) = \prod_{i,j=1}^{m}\{\nu^i.f, \nu^j.g\}_{\mathrm{tame}_E}^{B(\nu^i(1-\nu).\lambda,\nu^j(1-\nu).\mu)}.$$

Put $f = a_0 u^{v_E(f)} \cdot f_+$ and $g = b_0 u^{v_E(g)} \cdot g_+$ for $f_+, g_+ \in L^{++}\mathcal{T}^{\flat,0}(k)$ as in 2.14. Then using the formula for the tame symbol 2.4,

$$\begin{aligned}\{\nu^i.f, \nu^j.g\}_{\mathrm{tame}_E} &= \{a_0\zeta^i u^{v_E(f)}, b_0\zeta^j u^{v_E(g)}\}_{\mathrm{tame}_E}\\ &= (-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}\zeta^{iv_E(f)v_E(g)}}{b_0^{v_E(g)}\zeta^{jv_E(f)v_E(g)}}\\ &= (-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}}\zeta^{(i-j)v_E(f)v_E(g)}.\end{aligned}$$

Plugging in to the formula for $C_{\mathcal{E}(k),\mathrm{tame}_E}$, we obtain

$$C_{\mathcal{E}(k),\mathrm{tame}_E}(N_{E/F}((1-\nu)\lambda\otimes f), N_{E/F}((1-\nu)\mu\otimes g)) =$$

$$\prod_{i,j=1}^{m}((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}}\zeta^{(i-j)v_E(f)v_E(g)})^{B(\nu^i(1-\nu).\lambda,\nu^j(1-\nu).\mu)}.$$

By Γ-invariance of B, this is

$$= \prod_{i,j=1}^{m}((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}}\zeta^{(i-j)v_E(f)v_E(g)})^{B(\nu^{i-j}(1-\nu).\lambda,(1-\nu).\mu)}.$$

Rearrange the double product $\prod_{i,j=1}^{m}$ to a product $\prod_{k=1}^{m}\prod_{i-j\equiv k \bmod m}$. Since for every k, there are exactly m pairs (i,j) with $i, j \in \{0, \cdots, m-1\}$ with $i - j \equiv k \bmod m$, the above quantity is

$$= \prod_{k=1}^{m}((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}}\zeta^{kv_E(f)v_E(g)})^{mB(\nu^k(1-\nu).\lambda,(1-\nu).\mu)}.$$

The ζ-factor is annihilated by the multiple of m, giving

$$= \prod_{k=1}^{m}((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}})^{mB(\nu^k(1-\nu).\lambda,(1-\nu).\mu)}$$

$$= ((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}})^{m\sum_{k=1}^{m}B(\nu^k(1-\nu).\lambda,(1-\nu).\mu)}.$$

By bilinearity of B,

$$= ((-1)^{v_E(f)v_E(g)}\frac{a_0^{v_E(g)}}{b_0^{v_E(g)}})^{mB(\sum_{k=1}^{m}\nu^k(1-\nu).\lambda,(1-\nu).\mu)}.$$

Since the element $\sum_{k=1}^{m}\nu^k(1-\nu).\lambda \in Y$ is Γ-invariant and B is Γ-invariant, $B(\sum_{k=1}^{m}\nu^k(1-\nu).\lambda,(1-\nu).\mu) = 0$, and the above term is 1. Hence $L^{++}\hat{\mathcal{T}}^{\flat,0}$ is abelian. Since $L^{++}\mathcal{T}^{\flat,0}$ is pro-unipotent while $\mathbb{G}_{m,k}$ is semisimple, $L^{++}\hat{\mathcal{T}}^{\flat,0}$ is split and the result follows. ■

Lemma 2.18: *$L^{++}\hat{\mathcal{T}}^{\flat,0}(k)$ lies in the center of $L\hat{\mathcal{T}}(k)$.*

Proof: Since $L^{++}\mathcal{T}^{\flat,0}(k)$ is connected, 2.17, it suffices to show for each $r \in L\mathcal{T}(k)$ and $s \in L^{++}\mathcal{T}^{\flat,0}$ that $C_{\mathcal{E}(k),\mathrm{tame}_E}(s,r) = 1$.

From the Γ-invariant decomposition

$$L^{+}\mathcal{T}_E^{\flat,0} = T \times L^{++}\mathcal{T}_E$$

$$L^{+}\mathcal{T}^{\flat,0} = T^{\Gamma,0} \times L^{++}\mathcal{T}_E^{\Gamma}$$

as in 2.14, we have that $L^{++}\mathcal{T}^{\flat,0}(k)$ is precisely the image of $L^{++}\mathcal{T}_E(k)$ under the norm map. According to the decomposition of $L\mathcal{T}_E$ in 2.2.2, elements of $L^{++}\mathcal{T}_E^{\flat,0}(k)$ are precisely of the form $\mu \otimes g$ for $\mu \in Y$ and $g \in L^{++}\mathbb{G}_{m,E}(k)$. According to the properties of the Kottwitz homomorphism 2.6, elements of $L^{++}\mathcal{T}^{\flat,0}(k)$ are precisely of the form $N((1-\nu)\mu \otimes g)$ for $\mu \in Y$ and $g \in L^{++}\mathbb{G}_{m,E}(k)$. By the description of $L^{++}\mathbb{G}_{m,E}$ in 2.2.2, for all such g, $v_E(g) = 0$. Hence it suffices to show $C_{\mathcal{E}(k),\mathrm{tame}_E}(N(\lambda\otimes$

$f), N((1-\nu)\mu \otimes g)) = 1$ for $\lambda, \mu \in Y$, $f \in L\mathbb{G}_{m,E}(k)$ and $g \in L^{++}\mathbb{G}_{m,E}$. We have

$$C_{\mathcal{E}(k),\mathrm{tame}_E}(N(\lambda \otimes f), N((1-\nu)\mu \otimes g)) = \prod_{i,j=1}^{m} \{\nu^i f, \nu^j.g\}_{\mathrm{tame}_E}^{B(\nu^i.\lambda, \nu^j(1-\nu)\mu)}.$$

Since $v_E(g) = 0$, we have

$$\{\nu^i f, \nu^j.g\} = (-1)^{v_E(f)v_E(g)} \frac{a_0^{v_E(g)}}{1^{v_E(f)}} = 1$$

where $f = a_0 \cdot u^{v_E(f)} \cdot f_+$ as in 2.2.2. The result follows. ∎

Proposition 2.3: *$L^{++,-}\mathcal{T}^{\flat,0}$ acts trivially on $\hat{Y}_\Gamma$ and $T^{\Gamma,0}$ acts trivially on $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$. We conclude that for every choice of splitting $L^{++}\mathcal{T}^{\flat,0} \to L\hat{\mathcal{T}}$ inducing an inclusion $T^{\Gamma,0} \to L\hat{\mathcal{T}}^0$,*

$$L\hat{\mathcal{T}}^0 = L^{++,\hat{-}}\mathcal{T}^{\flat,0} \times T^{\Gamma,0}$$

and

$$\hat{Y}_\Gamma \rtimes L\hat{\mathcal{T}}^0 = L^{+\hat{+},-}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})$$

Proof: The result follows from 2.15 and the above lemma 2.18 that $L^{++}\mathcal{T}^{\flat,0}(k)$ lies in the center of $L\hat{\mathcal{T}}(k)$. ∎

Proposition 2.4: *The action of $T^{\Gamma,0}$ on $\hat{Y}_\Gamma$ lifts to an action on $\hat{Y}$. For $h \in \mathfrak{t}^\Gamma = \mathrm{Lie}T^{\Gamma,0}$ and $\lambda \in Y$, we have*

$$C_{L\hat{\mathcal{T}}}(\exp(h), \lambda) = e^{B(h,\lambda)}$$

where $\exp : \mathfrak{t}^\Gamma \to T^{\Gamma,0}$ is the exponential map, and $e : \mathbb{C} \to \mathbb{G}_{m,k}$ is the exponential function. Hence the action of $T^{\Gamma,0}$ on $\hat{Y}$ is identical to the one in [BK04, sec. 4.3].

Proof: By the description of the section s_u from 2.6, $s_u((1-\nu)Y) \subseteq T^{\Gamma,0}$. Since $L^{+}\hat{\mathcal{T}}^{\flat,0}$ is split from 2.7 and $T^{\Gamma,0} \subseteq L^{+}\mathcal{T}^{\flat,0}$, $C_{L\hat{\mathcal{T}}}$ is trivial on $T^{\Gamma,0} \times s_u((1-\nu)Y)$ and therefore lifts to an action of $T^{\Gamma,0}$ on $\hat{Y}$ by

$$x.\lambda \coloneqq x.s_u(\lambda)$$

where we consider s_u as a map $Y \to L\mathcal{T}$ by pre-composition with the projection $Y \to Y_\Gamma$.

Now fix $\lambda \in Y$ and recall $s_u = (N \circ \mathrm{ev}_u) \cdot v$ for some map $v : Y \to T^{\Gamma,0}$. Since $C_{\hat{L\mathcal{T}}}$ is trivial on $T^{\Gamma,0} \times T^{\Gamma,0}$, for $\lambda \in Y$ the action of $T^{\Gamma,0}$ on $s_u(\lambda)$ equals the action of $T^{\Gamma,0}$ on $(N \circ \mathrm{ev}_u)(\lambda)$. The map $x \mapsto C_{\hat{L\mathcal{T}}}(X, (N \circ \mathrm{ev}_u)(\lambda))$ is a group homomorphism between the connected groups $T^{\Gamma,0} \to \mathbb{G}_{m,k}$ and thus is determined by the induced map on Lie algebras $\mathfrak{t}^\Gamma \to \mathbb{C}$. In particular, it suffices to compute the mth multiple of $T^{\Gamma,0} \to \mathbb{G}_{m,k}$ and show that it gives $\exp(h) \mapsto e^{mB(h,\lambda)}$, by dividing the induced map on Lie algebras by m. To this end, write $\mathfrak{h}^\Gamma = Y^\Gamma \otimes_\mathbb{Z} \mathbb{C}$ and $h = \lambda \otimes \log(a)$ for $\lambda \in Y^\Gamma$ and $a \in \mathbb{G}_{m,k}$ so $\exp(h) = \lambda \otimes a = \lambda(a) \in T^{\Gamma,0}$. Then for $\mu \in Y$, by 2.16,

$$m.C_{\hat{L\mathcal{T}}}(\exp(h), (N \circ \mathrm{ev}_u)(\lambda)) = C_{\mathcal{E}(k),\mathrm{tame}_E}(\exp(h), N(\mathrm{ev}_u(\mu)))$$

$$= C_{\mathcal{E}(k),\mathrm{tame}_E}(\lambda \otimes a, N(\mu \otimes u))$$

$$= \{a, \zeta^i u\}_{\mathrm{tame}_E}^{\sum_{i=1}^m B(\lambda,\nu^i.\mu)}.$$

Since λ is Γ-invariant and B is Γ-invariant, $\sum_{i=1}^m B(\lambda, \nu^i.\mu) = \sum_{i=1}^m B(\lambda, \mu) = mB(\lambda, \mu)$. We also have $\{a, \zeta^i u\}_{\mathrm{tame}_E} = \frac{a}{1} = a$. Since $e^{B(\lambda \otimes \log(a), \mu)} = a^{B(\lambda,\mu)}$, the result follows. ∎

Proposition 2.5: *The surjection $\hat{Y}_\Gamma \rtimes \hat{L\mathcal{T}}^0 \to L\mathcal{T}$ has kernel precisely the image of $\mathbb{G}_m \hookrightarrow \hat{L\mathcal{T}}^0 \rtimes \hat{Y}_\Gamma$ under the map $x \mapsto (x, x^{-1})$ into the product of the central $\mathbb{G}_{m,k}$ of $\hat{Y}_\Gamma$ and the central $\mathbb{G}_{m,k}$ of $\hat{L\mathcal{T}}^0$.*

Proof: The kernel of $\hat{L\mathcal{T}}^0 \rtimes \hat{Y}_\Gamma \to L\mathcal{T}$ lies in the neutral component $\hat{L\mathcal{T}}^0 \times \mathbb{G}_{m,k}$. Since the map is multiplication, it the kernel is precisely the set

$$\{(x, y) \in \mathbb{G}_{m,k} \times \mathbb{G}_{m,k} : xy = 1\}$$

where the left $\mathbb{G}_{m,k}$ is the central $\mathbb{G}_{m,k}$ of $\hat{L\mathcal{T}}$ and the right $\mathbb{G}_{m,k}$ is the central $\mathbb{G}_{m,k}$ of $\hat{Y}_\Gamma$. This is the injective image of $\mathbb{G}_{m,k}$ as described. ∎

Lemma 2.19: *Up to equivalence as central extensions, the image of the cocycle for $\hat{Y_\Gamma}$ lies in μ_r for some $r \geq 0$.*

PROOF: As a central extension, $\hat{Y_\Gamma}$ is determined by the pullback $\hat{Y}$ from the projection $Y \to Y_\Gamma$. By the [Bro94, V.6 ex. 5], there is a short exact sequence

$$0 \to \mathrm{Ext}(Y, \mu_r) \to \mathrm{H}^2(Y, \mu_r) \to \mathrm{Hom}(\wedge^2 Y, \mu_r) \to 0$$

where the first map is the inclusion of abelian central extensions into all central extensions and the second map takes a cocycle for a central extension to the associated commutator. Since $\mathrm{Ext}(Y, A) = 0$ for any abelian group A as Y is free,

$$\mathrm{H}^2(Y, \mu_r) = \mathrm{Hom}(\wedge^2 Y, \mu_r)$$

and the central extension $\hat{Y}$ is in fact completely determined by its commutator. The image of the commutator for $\hat{Y}$ in $\mathbb{G}_{m,k}$ must be finitely generated because $Y \otimes Y$ is. Hence the cocycle for $\hat{Y}$ lies in μ_r for some integer $r \geq 0$, therefore the cocycle for Y_Γ satisfies the same. ■

Remark 2.23: *Constructions in [BK04, Lep85, sec. 4.5, resp. 4] required the choice of a cocycle for the commutator for $\hat{Y}$, and they turn out to be immaterial.*

We conclude that by 2.3 that $s_u : Y_\Gamma \to L\mathcal{T}(k)$ and a choice of splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ defines a presentation:

$$\hat{Y}_\Gamma \ltimes \hat{L\mathcal{T}}^0 = L^{+\hat{+,-}}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \ltimes T^{\Gamma,0})$$

Taking the quotient by $\mathbb{G}_{m,k}$ in the embedding 2.5, we thus have proved the main result of this subsection 2.8.

2.2.6 Representations of Heisenberg Central Extensions

We define the notion of a representation of a Heisenberg central extension $\hat{L\mathcal{T}}$. We show the category of representations is semisimple and describe the irreducible objects in terms of representations of a distinguished finite subgroup of $\hat{L\mathcal{T}}$.

Let $k = \mathbb{C}$, $F = k((t))$ and $\mathcal{T}$ be a torus over F. Let E/F be an extension such that $\mathcal{T}_E$ is split of rank d. Put $E = k((u))$ for some $u^m = t$ where $m = [E : F]$. Let $\Gamma = \mathrm{Gal}(E/F)$ be the finite cyclic Galois group with ν a choice of generator. Fix a primitive mth root of unity ζ defined by the property that $\nu(u)/u = \zeta$. Let μ_r denote the group of rth roots of unity in $k^\times$ for each integer $r \geq 1$. Let $X = \mathrm{Hom}_E(\mathcal{T}_E, \mathbb{G}_{m,E})$, resp. $Y = \mathrm{Hom}_E(\mathbb{G}_{m,E}, \mathcal{T}_E)$, be the (absolute) character, resp. cocharacter, lattices with their canonical Γ action.

Let $\hat{L\mathcal{T}}$ be a Heisenberg central extension, as defined in 2.19. Let B be the Γ-invariant even symmetric bilinear form associated to $\hat{L\mathcal{T}}$in 2.6. Fix a choice of splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ from 2.7, and obtain the induced decomposition

$$\hat{L\mathcal{T}} \approxeq (\hat{L^{++,-}\mathcal{T}^{\flat,0}} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0}))/\mathbb{G}_{m,k}$$

from 2.8. Recall from definition that $\bar{\mathfrak{l}} := \mathrm{Lie}\hat{L^{++,-}\mathcal{T}^{\flat,0}}$ is a formal Heisenberg Lie algebra, the formal completion of a Heisenberg Lie algebra $\mathfrak{l}$ that has positive part $\mathfrak{l}_+$ with $\bar{\mathfrak{l}}_+ = \mathrm{Lie}L^{++}\mathcal{T}^{\flat,0}$. For each $p \in \mathbb{C}\backslash\{0\}$, recall the notion of a level p representation of a Heisenberg Lie algebra in 2.3 and the fact that the category of level p Heisenberg Lie algebras as semisimple with exactly one irreducible object up to isomorphism

$$\pi_p = \mathrm{Ind}^{\mathfrak{l}}_{\mathbb{C}\oplus\mathfrak{l}_+}\mathbb{C}$$

where $\mathfrak{l}_+$ acts by 0 and $\mathbb{C}$ acts by 1.

Definition 2.30: *A level p representation of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ is a representation V such that the central $\mathbb{G}_{m,k}$ of $\hat{Y}_\Gamma$ acts on V by the character $x \mapsto x^p$ and the action of $T^{\Gamma,0}$ on V is diagonalizable, i.e., V splits into a direct sum of weight spaces for $T^{\Gamma,0}$.*

For the remainder of this subsection, we will be most interested in the case $p = 1$.

Definition 2.31: *A representation of a Heisenberg Central extension $\hat{L\mathcal{T}}$ equipped with a splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ is a representation V such that:*

1. The induced representation of the formal Heisenberg Lie algebra $\bar{\mathfrak{l}} = \mathrm{Lie}\hat{L\mathcal{T}}$ *is a level 1 representation in the sense of 2.3.*

2. Under the decomposition induced by the splitting $L^{+}\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$*,*

$$\hat{L\mathcal{T}} \approxeq (L^{++,\hat{}-}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0}))/\mathbb{G}_{m,k}$$

the induced action of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ *on* V *is a level* p *representation for* $p = 1$ *in the sense of 2.30.*

It will turn out that a representation V of $\hat{L\mathcal{T}}$is determined by a distinguished subspace $\Omega(V)$ defined as follows.

Definition 2.32: *The* ***vacuum space*** $\Omega(V)$ *of a representation* V *of* $\hat{L\mathcal{T}}$ *relative to a splitting* $L^{+}\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ *is the fixed point space*

$$\Omega(V) = V^{L^{++}\mathcal{T}^{\flat,0}}.$$

Hence $\Omega(V)$ *is functorial in* V*.*

Recall from 2.19 $\hat{Y}_\Gamma$ has cocycle with values in μ_r for some integer $r \geq 0$. Hence $\hat{Y}_\Gamma$ is the pushout by the inclusion $\mu_r \hookrightarrow \mathbb{G}_{m,k}$ of a μ_r-central extension of $\hat{Y}_\Gamma$ that is also a canonically sub-central extension of $\hat{Y}_\Gamma$ by the inclusion $\mu_r \hookrightarrow \mathbb{G}_m$. Let us introduce the associated distinguished finite subgroup:

Definition 2.33: *Let* $\Sigma := Y_{\Gamma,\mathrm{tor}}$*. The* ***principal finite Heisenberg group*** $\hat{\Sigma}$ *of* $\hat{Y}_\Gamma$ *is the restriction to* Σ *of the* μ_r *sub-central extension of* $\hat{Y}_\Gamma$*. For an integer* $p \geq 0$ ***level p representation*** *for* $\hat{\Sigma}$ *is a representation such that the central* μ_r *acts by the character* $x \mapsto x^p$*.*

Remark 2.24: $\hat{\Sigma}$ *does not depend on a choice of splitting* $L^{+}\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ *but may depend on the choice of section* $Y_\Gamma \to \hat{L\mathcal{T}}$ *to the Kottwitz homomorphism.*

Lemma 2.20: *(analogous to [BK04, sec. 4.4]) Let* $\hat{\Sigma}$ *be the finite Heisenberg subgroup of* $\hat{Y}_\Gamma$*. Suppose* B *is nondegenerate. Then the centralizer of* $T^{\Gamma,0}$ *in* $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$

is

$$\hat{\Sigma} \times T^{\Gamma,0} \subseteq \hat{Y}_\Gamma \rtimes T^{\Gamma,0}$$

In particular $T^{\Gamma,0}$ acts trivially on $\hat{\Sigma}$.

PROOF: Let $\hat{Y}$ be the restriction of $\hat{Y}_\Gamma$ by $Y \to Y_\Gamma$. Recall from 2.4 that for $h \in \mathfrak{h}^\nu$ and $\lambda \in Y$, the action of $T^{\Gamma,0}$ on $\hat{Y}$ is defined by the commutator

$$C_{\hat{L\mathcal{T}}}(\exp(h), \lambda) = e^{B(h,\lambda)}$$

Since B is nondegenerate, it induces a perfect pairing $\mathfrak{h}^\Gamma \times \mathfrak{h}^\Gamma \to \mathbb{C}$. The result follows from the fact that the preimage of $Y_{w,\text{tor}} = \Sigma$ in Y lies in the orthogonal complement $(\mathfrak{h}^\Gamma)^\perp$ defined by B. ■

Remark 2.25: *A representation of $\hat{\Sigma} \times T^{\Gamma,0}$, where the restriction to $\hat{\Sigma}$ is a level p representation and the restriction to $T^{\Gamma,0}$ is diagonalizable, is the same thing as an $\mathbb{X}_*(T^{\Gamma,0})$-graded level p representation of $\hat{\Sigma}$.*

Representations of groups, such as $\hat{\Sigma}$, that are central extensions of finite abelian groups are well understood and classified by characters of the center. We record the result here:

Lemma 2.21: *[BK04, 4.5.3] The category of level 1 representations of $\hat{\Sigma}$ is semisimple with finitely many irreducible objects classified by characters of the center. They all have the same dimension d satisfying*

$$d^2 = |\hat{\Sigma}/Z(\hat{\Sigma})|$$

Definition 2.34: *The **defect** value of of $\hat{L\mathcal{T}}$ is the number d as above; the dimension of an irreducible representation of the principal finite Heisenberg group $\hat{\Sigma}$ associated to $\hat{L\mathcal{T}}$.*

We first reduce the problem of determining the representation theory of $\hat{L\mathcal{T}}$ to that of level 1 representations of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$.

Theorem 2.9: *Fix a splitting* $L^{++}\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$*. Then* $\Omega(V)$ *is stable under* $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ *and there is a canonical equivalence of categories*

$$\text{Representations of } \hat{L\mathcal{T}} \to \text{Level 1 representations of } \hat{Y}_\Gamma \rtimes T^{\Gamma,0}$$

$$V \mapsto \Omega(V)$$

with an inverse given by

$$\mathrm{Ind}^{\hat{L\mathcal{T}}}_{(L^{++}\hat{\mathcal{T}}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0}))/\mathbb{G}_{m,k}} M \leftarrow\!\shortmid M$$

where $L^{++}\mathcal{T}^{\flat,0} \hookrightarrow L^{++}\hat{\mathcal{T}}^{\flat,0}$ *acts on* U *trivially and the central* $\mathbb{G}_{m,k}$ *of* $L^{++}\hat{\mathcal{T}}^{\flat,0}$ *acts by the identity character.*

Proof: The fact that $\Omega(V)$ is stable under $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ follows because $L^{++}\hat{\mathcal{T}}^{\flat,0}$ commutes with $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$. Hence the functors in both directions are well-defined. It suffices to show they are mutual inverse.

A representation of $\hat{L\mathcal{T}}$ is canonically equivalent to a representation of $L^{++,-}\hat{\mathcal{T}}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})$ where both the central $\mathbb{G}_{m,k}$ of $\hat{L\mathcal{T}}^0$ and the central $\mathbb{G}_{m,k}$ of $\hat{Y}_\Gamma$ act by the identity character. Let $\mathfrak{l}$ be the Heisenberg Lie algebra whose formal completion is $\bar{\mathfrak{l}} = \mathrm{Lie}L^{+\hat{+},-}\mathcal{T}^{\flat,0}$. Now let $\pi = \pi_1$ be the level $p = 1$ Fock space from 2.2 as a module for $\mathfrak{l}$. Then

$$\pi = \mathrm{Lie}(\mathrm{Ind}^{\mathrm{Lie}L^{+\hat{+},-}\mathcal{T}^{\flat,0}}_{L^{++}\hat{\mathcal{T}}^{\flat,0}}\mathbb{C})$$

,i.e., π is integrable, where the central $\mathbb{G}_{m,k}$ of $L^{++}\hat{\mathcal{T}}^{\flat,0}$ acts by the identity character and $L^{++}\mathcal{T}^{\flat,0}$ acts trivially. Observe that also

$$\pi = \mathrm{Lie}(\mathrm{Ind}^{L^{++,-}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})}_{L^{++}\hat{\mathcal{T}}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})}\mathbb{C})$$

where $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ acts trivially on $\mathbb{C}$. Therefore the canonical functorial isomorphism from 2.2

$$\mathrm{Lie}V \approxeq \mathrm{Lie}(\mathrm{Ind}^{L^{++,\hat{-}}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})}_{L^{++}\hat{\mathcal{T}}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})}\Omega(V)) \approxeq \pi \otimes \Omega(V)$$

$$M \cong \Omega(\mathrm{Ind}_{\mathrm{Lie}L^{++}\hat{\mathcal{T}}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})}^{\mathrm{Lie}L^{++,\hat{-}}\mathcal{T}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})} M)$$

as $\mathfrak{l}$-modules, where $\mathfrak{l}$ acts on π and $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ also acts on $\Omega(V)$, integrates to a canonical functorial equivalence of $L^{++,\hat{-}}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{\Gamma,0})$ modules

$$V \cong \mathrm{Ind}_{L^{++}\hat{\mathcal{T}}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})}^{L^{++,\hat{-}}\mathcal{T}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})} \Omega(V)$$

$$M \cong \Omega(\mathrm{Ind}_{L^{++}\hat{\mathcal{T}}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})}^{L^{++,\hat{-}}\mathcal{T}^{\flat,0}\times(\hat{Y}_\Gamma\rtimes T^{\Gamma,0})} M).$$

The result follows. ■

Theorem 2.10: *Suppose B is non-degenerate, inducing an injection $B : \mathbb{X}_*(T^{\Gamma,0}) \hookrightarrow \mathbb{X}_*(T^{\Gamma,0})^\vee \cong \mathbb{X}^*(T^{\Gamma,0})$ with the second isomorphism being canonical. The category of level 1 representations of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$ is semisimple, and any such representation is induced from a representation of $\hat{\Sigma} \times T^{\Gamma,0}$ and is an equivalence of categories*

Level 1 representations of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0} \cong \mathbb{X}^*(T^{\Gamma,0})/B(\mathbb{X}_*(T^{\Gamma,0}))$-graded level 1 representations of $\hat{\Sigma}$

$$\oplus_{\lambda\in B(\mathbb{X}_*(T^{\Gamma,0}))} U_{\lambda_0} \otimes \mathbb{C}_\lambda = \mathrm{Ind}_{\hat{\Sigma}\times T^{\Gamma,0}}^{\hat{Y}_\Gamma\rtimes T^{\Gamma,0}} U_\alpha \leftarrow U_{\alpha+B(\mathbb{X}_*(T^{\Gamma,0}))}$$

for $\alpha \in \mathbb{X}^*(T^{\Gamma,0})$.

Proof: This proof follows [BK04, 4.4] where $\hat{Y}_\Gamma$ was replaced by some other central extension given explicitly terms of a formula. We only check here the same proof goes through without needing to know the exact commutator for $\hat{Y}_\Gamma$ and that 2.4 and 2.20 are enough.

Since B is nondegenerate, it induces an injection of the quotient of by torsion of the coinvariants $Y_{\Gamma,\mathrm{cotor}} \hookrightarrow Y_{\Gamma,\mathrm{cotor}} \otimes \mathbb{C} = \mathfrak{t}^\Gamma \overset{B}{\hookrightarrow} \mathfrak{t}^{\Gamma,\vee}$. Let Ω be a representation of $\hat{Y}_\Gamma \rtimes T^{\Gamma,0}$. For $\alpha \in \mathbb{X}_*(T^{\Gamma,0})$ let Ω_α denote the α-weight space. For $\lambda \in Y_\Gamma$ denote $\overline{\lambda}$ the image in $Y_{\Gamma,\mathrm{cotor}}$ and $\hat{\lambda}$ any lift in $\hat{Y}_\Gamma$.

We first claim the following auxiliary properties. Fix $\alpha \in \mathbb{X}^*(T^{\Gamma,0})$ such that Ω_α is nonzero.

(i) For $\lambda \in Y_\Gamma$,

$$\hat{\lambda}.\Omega_\alpha = \Omega_{B(\bar{\lambda})+\alpha}.$$

In particular, each weight space Ω_α is stable under $\hat{\Sigma}$.

(ii) The space

$$\hat{Y_\Gamma}.\Omega_\alpha = \oplus_{\bar{\lambda}\in Y_{\Gamma,\mathrm{cotor}}}\Omega_{B(\bar{\lambda})+\alpha} = \mathrm{Ind}_{\hat{\Sigma}\times T^{\Gamma,0}}^{\hat{Y_\Gamma}\rtimes T^{\Gamma,0}} U_\alpha \subseteq \Omega$$

is stable under $\hat{Y_\Gamma} \rtimes T^{\Gamma,0}$.

(iii) The submodule$\hat{Y_\Gamma}.\Omega_\alpha$ of $\hat{Y_\Gamma} \rtimes T^{\Gamma,0}$ is irreducible iff each or any $\Omega_{B(\bar{\lambda})+\alpha}$ is irreducible for $\hat{\Sigma}$.

Let's prove (i), (ii), (iii). For (i), the result follows form the computation of the commutator action of $T^{\Gamma,0}$ on $\hat{Y_\Gamma}$ of 2.4. For (ii), it is enough to observe that $\hat{Y_\Gamma}.\Omega_\alpha$ is stable under the subgroups$\hat{Y_\Gamma}$ and $T^{\Gamma,0}$ that together generate $\hat{Y_\Gamma} \rtimes T^{\Gamma,0}$. For (iii), suppose some $\Omega_{B(\bar{\lambda})+\alpha}$ is irreducible for $\hat{\Sigma}$. Then by (i) and the fact that all representations of $\hat{\Sigma}$ have the same dimension, every $\Omega_{B(\bar{\lambda})+\alpha}$ is irreducible for $\hat{\Sigma}$. Any submodule $V \subseteq \hat{Y_\Gamma}.\Omega_\alpha$ must have a compatible weight space decomposition, each weight space a submodule for $\hat{\Sigma}$. Since every $\Omega_{B(\bar{\lambda})+\alpha}$ is irreducible for $\hat{\Sigma}$, $V = \hat{Y_\Gamma}.\Omega_\alpha$. Conversely suppose $\hat{Y_\Gamma}.\Omega_\alpha$ is irreducible for $\hat{Y_\Gamma} \rtimes T^{\Gamma,0}$. Let $v, w \in \Omega_{B(\bar{\lambda})+\alpha}$ be nonzero vectors. By assumption there exists $x \in \hat{Y_\Gamma} \rtimes T^{\Gamma,0}$ such that $xv = w$ and it suffices to show the first co-ordinate of x lies in $\hat{\Sigma}$. This follows from (i).

Let's now use (i), (ii), (iii) to prove the theorem. Let us show Ω is completely reducible. For any $\alpha \in \mathbb{X}^*(T^{\Gamma,0})$ such that Ω_α is nonzero, by (iii) the sub-module $\hat{Y_\Gamma}.\Omega_\alpha$ is completely reducible because each $\Omega_{B(\bar{\lambda})+\alpha}$ is completely reducible for $\hat{\Sigma}$. Choosing α among a set of representatives of $\mathbb{X}^*(T^{\Gamma,0})/B(\mathbb{X}_*(T^{\Gamma,0}))$, we conclude that Ω is completely reducible. This shows the category of level 1 representations of $\hat{Y_\Gamma} \rtimes T^{\Gamma,0}$ is semisimple. The fact the functor given above is an equivalence of categories follows from (ii) and (ii) since every irreducible component of Ω must be stable under $\hat{Y_\Gamma}$. The result follows. ∎

Remark 2.26: *2.10 is a generalization of [Bei06, 1.9] which classifies the represen-*

tations of $\hat{L\mathcal{T}}$ *in the case when* $\mathcal{T}$ *is split, so* $T^{\Gamma,0} = T$, $\hat{\Sigma}$ *is the trivial group, and the category of* $\mathbb{X}^*(T^{\Gamma,0})$*-graded representaions of* $\hat{\Sigma}$ *becomes the category of* $Y^\vee/B(Y)$*-graded vector spaces.*

Remark 2.27: *Two different choices of splitting* $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ *are determined by a morphism* $L^+\mathcal{T}^{\flat,0} \to \mathbb{G}_{m,k}$, *by dividing the two splittings. Since* $L^{++}\mathcal{T}^{\flat,0}$ *is pro-unipotent, such a morphism is determined by the restriction to* $T^{\Gamma,0}$, *i.e., by a character for* $T^{\Gamma,0}$. *Since the* $T^{\Gamma,0}$*-weight spaces for an irreducible representation of* $\hat{L\mathcal{T}}$, *the defect* d *has an intrinsic definition as the dimension of a single weight space of the vacuum space of any irreducible representation of* $\hat{L\mathcal{T}}$ *for the inclusion* $T^{\Gamma,0} \hookrightarrow \hat{L\mathcal{T}}$ *induced by any choice of splitting* $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$. *In particular* d *is independent of the choice of splitting* $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$.

According to 2.10, representations of $\hat{L\mathcal{T}}$ induced from a representation of $\hat{\Sigma}$ that is a single weight space for $T^{\Gamma,0}$ are especially easy to understand and they are the intended application for the next chapter 3. We record the corollary here:

Corollary 2.4: *Suppose an* $\mathbb{X}^*(T^{\Gamma,0})/B(\mathbb{X}_*(T^{\Gamma,0}))$*-graded representation* U *of* $\hat{\Sigma}$ *has only one graded component. Then*

$$U \approxeq \bigoplus_{\lambda \in \mathbb{X}_*(T^{\Gamma,0})} U_\lambda$$

where for each λ, $T^{\Gamma,0}$ *acts on* U_λ *by* $B(\lambda) \in \mathbb{X}^*(T^{\Gamma,0})$ *and* U_λ *is stable under* $\hat{\Sigma}$. *In particular, if a given representation* V *of* $\hat{L\mathcal{T}}$ *is given as*

$$V \approxeq \bigoplus_{\lambda \in \mathbb{X}_*(T^{\Gamma,0})} \pi \otimes U_\lambda$$

where $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ *acts on* π, $T^{\Gamma,0}$ *acts on* U_λ *by* $B(\lambda)$ *and each* U_λ *is stable under* $\hat{\Sigma}$, *then we have a characterization of the vacuum space*

$$\Omega(V) = \bigoplus_{\lambda \in \mathbb{X}_*(T^{\Gamma,0})} U_\lambda$$

and V is irreducible under $L\hat{\mathcal{T}}$ iff any U_λ is irreducible as a $\hat{\Sigma}$ representation iff every U_λ is irreducible as a $\hat{\Sigma}$ representation.

Furthermore any morphism $V \to W$ where W is another $L\hat{\mathcal{T}}$ representation is injective iff it is injective on any of the U_λ.

2.3 Classification and Generic Cocycles for Central Extensions of a Torus by $\boldsymbol{K}_2$

Let F be any field, not necessarily $\mathbb{C}((t))$, and $\mathcal{T}$ be a torus over F. Let E/F be a Galois extension such that $\mathcal{T}_E$ is split. This section concerns some computations about the category of central extensions of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$. We essentially reproduce the techniques in [BD01] but slightly modify them to give us the explicit computations. The techniques are disjoint from the remainder of this paper and only the results will be used. The results are listed below and the proof is given in later subsections. Suppose $\mathcal{T}$ is a torus over F and E/F is a Galois extension where $\mathcal{T}_E$ is split. Let $\Gamma = \mathrm{Gal}(E/F)$. Let $X = \mathrm{Hom}(\mathcal{T}_E, \mathbb{G}_{m,E})$, resp. $Y = \mathrm{Hom}(\mathbb{G}_{m,E}, \mathcal{T}_E)$, be the (absolute) character, resp. cocharacter lattice, both as Γ-modules.

Recall the sheaf $\boldsymbol{K}_{2,F}$ and the presheaf $K_{2,F}$ on the big Zariski site $\mathrm{Spec}(F)_{\mathrm{Zar}}$ from 2.2.3. For a group scheme G over S and an abelian sheaf $\mathcal{A}$ on S, let $\mathrm{CExt}(G, \mathcal{A})$ be the Picard category of central extensions of G by $\mathcal{A}$.

We have the following characterization of the group of isomorphism classes of $\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F})$, where the group structure is induced by the sum of cocycles.

Theorem 2.11: *We have a short exact sequence of abelian groups*

$$0 \to \mathrm{H}^1(\Gamma, X \otimes_{\mathbb{Z}} E^\times) \xrightarrow{\iota} \pi_0\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F}) \xrightarrow{\rho} \mathrm{AdmSym}^2 X^\Gamma \to 0$$

where $\mathrm{AdmSym}^2 X \subseteq \mathrm{Sym}^2 X$ is a certain subset of 'admissible' quadratic forms, realized as the kernel of a connecting morphism in a long exact sequence

$$\mathrm{Sym}^2 X^\Gamma \to \mathrm{H}^2(\Gamma, X \otimes_{\mathbb{Z}} E^\times)$$

where:

1. *For* $\mathcal{E} \in \pi_0\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F})$*, the map* ρ *gives the commutator for the base change* $\mathcal{E}_E$ *as in 2.3.*

2. *The group* $\mathrm{H}^1(\Gamma, X \otimes_{\mathbb{Z}} E^\times)$ *consists of Galois descent data for the trivial multiplicative* $\boldsymbol{K}_{2,E}$ *torsor* $\mathcal{E}_{0,E}$ *on* $\mathcal{T}_E$*, where* $X \otimes_{\mathbb{Z}} E^\times$ *is identified with* $\mathrm{Aut}(\mathcal{E}_{0,E})$*.*

We also have the following description of the generic cocycle for a central extension of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$. The statement of the computation uses a Galois descent property for $K_{2,F}$ of function fields:

Theorem 2.12: *[CT83, Theorem B] in characteristic* 0*, [Sus87] in any characteristic, for a geometrically integral* F*-variety* Z *with* F*-rational point and a Galois extension* E/F*,*

$$\mathrm{H}^1(\Gamma, K_2(E(Z))/K_2(E)) = 0$$

and the natural map $K_2(F(Z)) \to K_2(E(Z))$ *induces an isomorphism*

$$K_2(F(Z))/K_2(F) \approxeq K_2(E(Z))/K_2(E).$$

Theorem 2.13: *Let* $\mathcal{E}$ *be a central extension of* $\mathcal{T}$ *by* $\boldsymbol{K}_{2,F}$ *and* $\mathcal{E}_E$ *the base change to* E*. Let* $c \in \mathrm{H}^0(\mathcal{T}_E^2, \boldsymbol{K}_{2,E})^{\mathrm{norm}}$ *be a cocycle for* $\mathcal{E}_E$ *and* $\{f_\gamma\}_{\gamma\in\Gamma} \in C^1(\Gamma, \mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}})$ *be a 1-cochain (where each* $f_\gamma \in \mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\mathrm{norm}}$*) that defines the descent datum for* $\mathcal{E}_E$*. Let* d *be the image in* $K_2(E(\mathcal{T}_E^2))/K_2(E)$ *of the restriction of* c *to the generic point of* $\mathcal{T}_E^2$*. Using* $H^1(\Gamma, K_2(E(\mathcal{T}_E)/K_2(E)) = 0$ *from 2.12, we have that* d *is a coboundary determined by some* $n \in K_2(E(\mathcal{T}_E)/K_2(E)$*. Then the element*

$$d - (\mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*)(n) \in K_2(E(\mathcal{T}_E)/K_2(E)$$

is Γ*-invariant and gives the corresponding generic cocycle for* $\mathcal{E}$ *under the isomorphism* $K_2(F(\mathcal{T}))/K_2(F) \approxeq K_2(E(\mathcal{T}_E))/K_2(E)$ *from 2.12.*

This implies the following fact that we have used in the proof of 2.6:

Lemma 2.22: *The commutator* $C_{\mathcal{E}} : \mathcal{T}^2 \to \boldsymbol{K}_{2,F}$ *a central extension* $\mathcal{E}$ *of* $\mathcal{T}$ *by* $\boldsymbol{K}_{2,F}$ *factors through the presheaf* $K_{2,F}$ *and is* $\varphi_{\Psi(B)}$ *as in 2.3 where* B *is the bilinear form* $\rho([\mathcal{E}])$ *from 2.11.*

Remark 2.28: *This allows us to directly define a Heisenberg central extension* $\hat{L\mathcal{T}}$*, without reference to the Heisenberg Lie algebra* $Lie\hat{L^{++,-}}\mathcal{T}^{\flat,0}$*, by starting with the central extension of* $\mathcal{T}$ *by the presheaf* $K_{2,F}$ *and taking the pushout of the induced central extension of* $L\mathcal{T}$ *by* $LK_{2,F}$ *via the Contou-Carrerè symbol [OZ16, 2.2]* $LK_{2,F} \to \mathbb{G}_{m,F}$*.*

Proof: Suppose $\mathcal{E}_E$ has commutator $\varphi_{\Psi(B)}$. Let $C \in X \otimes X$ such that $B = C + \tau^* C$ such that the d in 2.13 is given by the image of $\Psi(C)$ in $K_2(E(\mathcal{T}_E^2))/K_2(E)$, where Ψ is as in 2.3. Now observe that the element $(\mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*)(n)$ is invariant under transposition because $\tau^*\mu = \mu$ (because $\mathcal{T}$ is abelian) and $\tau^*\mathrm{pr}_1 = \mathrm{pr}_2^*$, $\tau^*\mathrm{pr}_2 = \mathrm{pr}_1$. Hence the commutator associated to $d - (\mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*)(n)$ is $d - \tau^* d$. By 2.10, the generic commutator for $\mathcal{E}$ is also given by $d + \tau^* d$ and thus also $\Psi(B)$. According to [BD01, 8.8], a 2-cocycle $\mathcal{T}^2 \to \boldsymbol{K}_{2,F}$ is determined by its restriction to the generic point. Since any commutator associated to a cocycle is also a 2-cocycle, as proved in the lemma 2.23 below, this shows that $C_{\mathcal{E}}$ is the composition of $\Psi(B)$ with the natural map $K_{2,F} \to \boldsymbol{K}_{2,F}$. This is what was desired. ∎

Lemma 2.23: *Let* H*,* $\mathcal{A}$ *be presheaves of abelian groups and* $c : H^2 \to \mathcal{A}$ *a 2-cocycle. Then* $-\tau^* c$ *is also a two-cocycle and therefore so is the commutator* $c - \tau^* c$ *associated to* c*.*

Proof: We recall that the cocycle condition is that for all b_1, b_2, b_3 in H,

$$c(b_2, b_3) - c(b_1 b_2, b_3) + c(b_1, b_2 b_3) - c(b_1, b_2) = 0 \text{ for all } b_2, b_2, b_3.$$

Substituting c for $-\tau^* c$ gives

$$-(c(b_3, b_2) - c(b_3, b_1 b_2) + c(b_2 b_3, b_1) - c(b_2, b_1))$$

$$= c(b_2, b_1) - c(b_2 b_3, b_1) + c(b_3, b_1 b_2) - c(b_3, b_2).$$

Applying the transformation $(b_1, b_2, b_3) \mapsto (b_3, b_2, b_1)$ and commutativity of H gives us that the above element is equal to

$$= c(b_2, b_3) - c(b_1 b_2, b_3) + c(b_1, b_2 b_3) - c(b_1, b_2)$$

for all $b_1, b_2, b_3 \in H$. The above element is zero since c is a 2-cocycle. The result follows. ∎

2.3.1 Multiplicative Torsors, Galois Descent, Homological Algebra

We review some of the foundations in [BD01, sec. 1] and re-frame them in terms of derived categories for our computations. The main purpose is a rephrasing of [BD01, 1.9.(2)] and [BD01, 2.4], which were both only stated for the case of $\boldsymbol{K}_{2,F}$ torsors over a scheme X over S, in a more general setting of group actions on chain complexes. This allows us to apply them to the setting of central extensions of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$.

Definition 2.35: *A* ***Picard groupoid*** *is a symmetric monoidal category where every object is invertible. Let $\mathcal{C}$ be a Picard groupoid and β the symmetry constraint. We say a Picard groupoid $\mathcal{C}$ is* ***strictly commutative*** *if for all $X \in \mathcal{C}$, the symmetry constraint $\beta_{X,X} : X \otimes X \approxeq X \otimes X$ is the identity map. A* ***Picard functor*** *is a morphism between Picard groupoids preserving the tensor structure.*

Let S be a base scheme. Let $\mathcal{A}$ be a sheaf of abelian groups on the big Zariski site S_{Zar}. Denote by $\mathcal{A}_X$ the restriction of $\mathcal{A}$ to X. Let X be a scheme over S.

Definition 2.36: *By an $\mathcal{A}$-torsor on X we mean an $\mathcal{A}_X$-torsor on X. The strictly commutative groupoid of* ***$\mathcal{A}$-torsors*** *on X is denoted by*

$$\mathrm{Tors}(X, \mathcal{A})$$

The tensor structure is denoted by $+$ called the ***sum of torsors****, where for $\mathcal{P}, \mathcal{Q} \in$ $\mathrm{Tors}(X, \mathcal{A})$, $\mathcal{P}+\mathcal{Q}$ is defined to be the pushout of the $\mathcal{A} \oplus \mathcal{A}$ torsor $\mathcal{P} \oplus \mathcal{Q}$ under the sum* *map $\mathcal{A} \oplus \mathcal{A} \to \mathcal{A}$. The* ***trivial $\mathcal{A}$-torsor*** *on X is denoted $\mathcal{E}_{0,X}$. The commutativity and associativity constraints are inherited from those for the operation $\oplus$.*

Remark 2.29: *The reason that $\mathrm{Tors}(X, \mathcal{A})$ is strictly commutative is that given $\mathcal{P}, \mathcal{Q} \in \mathrm{Tors}(X, \mathcal{A})$ with a cover $U \to X$ trivializing $\mathcal{P}, \mathcal{Q}$, we have that $(\mathcal{P} + \mathcal{Q})|_U$ is the sheafification of the sheaf of sections of $\mathcal{P}|_U \oplus \mathcal{Q}|_U$ quotiented by the relation $(r, s) = (r', s')$ if $r + s = s + r$. Hence the symmetry constraint, induced by swapping the order of addition, $\mathcal{P} + \mathcal{Q} \approxeq \mathcal{Q} + \mathcal{P}$ is the identity and there is a direct equality $\mathcal{P} + \mathcal{Q} = \mathcal{Q} + \mathcal{P}$ as $\mathcal{A}$-torsors over X. This condition is actually stronger than strict commutativity, which requires this to hold only for $\mathcal{Q} = \mathcal{P}$.*

Suppose now X has a distinguished section $e : S \to X$.

Definition 2.37: *The strictly commutative Picard groupoid of* ***pointed*** *$\mathcal{A}$-torsors on X is denoted by*

$$\mathrm{Tors}_e(X, \mathcal{A})$$

and is the category of pairs $(\mathcal{P}, \beta)$ where $\mathcal{P} \in \mathrm{Tors}(X, \mathcal{A})$ and $\beta : e^*\mathcal{P} \approxeq \mathcal{E}_{0,S}$ is a trivialization. A ***morphism of pointed torsors*** *between $(\mathcal{P}, \beta) \to (\mathcal{Q}, \eta)$ is a morphism of $\mathcal{A}$-torsors $\mathcal{P} \to \mathcal{Q}$ inducing a commutative triangle*

$$\begin{array}{ccc} e^*\mathcal{P} & \to & e^*\mathcal{Q} \\ {}_{\beta}\downarrow & \swarrow_{\eta} & \\ \mathcal{E}_{0,S} & & \end{array}$$

The sum structure $+$ on $\mathrm{Tors}_e(X, \mathcal{A})$ is defined by $(\mathcal{P}, \beta) + (\mathcal{Q}, \eta) \coloneqq (\mathcal{P} + \mathcal{Q}, \beta + \eta)$ using the canonical isomorphism $\mathcal{E}_{0,S} + \mathcal{E}_{0,S} \approxeq \mathcal{E}_{0,S}$.

Now suppose $X = G$ is a group scheme over S with multiplication map $\mu : G \times G \to G$ over S and $e : S \to G$ is the unit section. Let $\mathrm{pr}_i : G^n \to G$ be the projection to the ith co-ordinate, for $i = 1, \cdots, n$.

Definition 2.38: *The strictly commutative Picard groupoid of* ***multiplicative*** $\mathcal{A}$*-****torsors*** *on* G*, or* ***central extensions of*** G ***by*** $\mathcal{A}$ *is denoted by*

$$\mathrm{CExt}(G, \mathcal{A})$$

and is defined to be the category of pairs $(\mathcal{P}, \beta)$ *where* $\mathcal{P} \in \mathrm{Tors}(\mathcal{A}, G)$ *and*

$$\beta : \mathrm{pr}_1^*\mathcal{P} + \mathrm{pr}_2^*\mathcal{P} \cong \mu^*\mathcal{P}$$

is an isomorphism. We refer to β *as the* ***multiplicative structure*** *of the central extension* $\mathcal{P}$*. A morphism* $(\mathcal{P}, \beta) \to (\mathcal{Q}, \eta)$ *is defined to be a morphism* $\mathcal{P} \to \mathcal{Q}$ *in* $\mathrm{Tors}(\mathcal{A}, G)$ *inducing a commutative diagram*

$$\begin{array}{ccc} pr_1^*\mathcal{P} + pr_2^*\mathcal{P} & \xrightarrow{\beta} & \mu^*\mathcal{P} \\ \downarrow & & \downarrow \\ pr_1^*\mathcal{Q} + pr_2^*\mathcal{Q} & \xrightarrow{\eta} & \mu^*\mathcal{Q} \end{array}$$

of $\mathcal{A}$*-torsors on* $G \times G$*. The sum of two central extensions is defined to be* $(\mathcal{P}, \beta) + (\mathcal{Q}, \eta) := (\mathcal{P} + \mathcal{Q}, \beta + \eta)$ *where it is checked that* $\beta + \eta$ *defines a multiplicative structure on* $\mathcal{P} + \mathcal{Q}$*. According to [BD01, 1.5] a multiplicative structure of an* $\mathcal{A}$*-torsor on* G *gives rise, by restriction to* e*, a canonical pointed structure with respect to* e*, i.e., there is a canonical forgetful map*

$$\mathrm{CExt}(G, \mathcal{A}) \to \mathrm{Tors}_e(G, \mathcal{A}).$$

We call the image of a central extension $(\mathcal{P}, \beta)$ *of* G *by* $\mathcal{A}$ *under this map to be the* ***underlying pointed*** *torsor, and similarly we call* $\mathcal{P}$ *so for the* ***underlying*** $\mathcal{A}$*-****torsor****.*

Remark 2.30: *The correspondence between central extensions of* G *by* $\mathcal{A}$ *as group-values sheaves on* S_{Zar} *and multiplicative* $\mathcal{A}$ *torsors on* G *is spelled out in [BD01, 1.4].*

Let us now recall a theorem [AGV+73, tome 3, XVIII] that allows us to study small strictly commutative Picard categories using homological algebra:

Theorem 2.14: *The category of small strictly commutative Picard categories, with morphisms Picard functors, is equivalent to* $D^{[0,1]}(\mathrm{Ab})$*, the derived category of chain complexes of abelian groups with cohomology supported in degrees* 0 *and* 1*. An equivalence is given by restriction of the following morphism*

$$D(\mathrm{Ab}) \to \textit{small strictly commutative Picard categories}$$

by $D^{[0,1]}(\mathrm{Ab}) \to D(\mathrm{Ab})$*. It associates a chain complex* K^* *a small strictly commutative Picard category* $\mathcal{C}(K^*)$ *is defined as follows:*

1. *The objects of* $\mathcal{C}(K^*)$ *are* $Z(K^1)$*; the* 1*-cycles.*
2. *For* $a, b \in Z(K^1)$*, a morphism* $f : a \to b$ *is an element* $f \in K^0$ *such that* $\mathrm{d}f = b - a$*, where* d *is the differential map of* K^**.*
3. *The composition of morphisms in* $\mathcal{C}(K^*)$ *is induced by the group structure on* K^0*.*
4. *The addition bifunctor in* $\mathcal{C}(K^*)$ *is induced by the addition structures on* K^1 *and in* K^0*.*
5. *The associativity and symmetry constraints are trivial, induced by the abelian group identities of elements of* K^1 *and of* K^0*.*

Definition 2.39: *Suppose* $\mathcal{C}$ *is a strictly commutative Picard category and* $\mathcal{C} = \mathcal{C}(K^*)$ *for a chain complex* K^* *as above. We say* $\mathcal{C}$ *is* ***presented*** *by* K^**.*

In particular, a quasi-isomorphism $K^* \overset{\text{quis}}{\approxeq} L^*$ between two chain complexes K^* and L^* induces an equivalence of categories $\mathcal{C}(K^*) \approxeq \mathcal{C}(L^*)$.

Definition 2.40: *Suppose* $\mathcal{C}$ *is a small strictly commutative Picard category and* K^* *is a chain complex of abelian groups.*

1. *We say* K^* ***incarnates*** $\mathcal{C}$ *if there is a chain complex* K^* *and an equivalence of Picard categories* $\mathcal{C} \approxeq \mathcal{C}(K^*)$*.*

2. The ***group of isomorphism classes*** *of $\mathcal{C}$ is defined to be the set $\pi_0\mathcal{C}$ with group structure induced by the sum in $\mathcal{C}$.*

3. The ***automorphisms of the trivial object*** *is denoted by $\pi_1\mathcal{C}$.*

Hence by definition, of $\mathcal{C} = \mathcal{C}(K^*)$, then $\pi_0\mathcal{C} = \mathrm{H}^1(K^*)$ and $\pi_1\mathcal{C} = \mathrm{H}^0(K^*)$. Observe that the 0s and 1s are switched. For all $\mathcal{P} \in \mathcal{C}$, the functor $(-) \otimes \mathcal{P}$ induces an isomorphism $\pi_1\mathcal{C} \approxeq \mathrm{Aut}_{\mathcal{C}}(\mathcal{P})$.

Remark 2.31: *Given a chain complex incarnating* $\mathrm{Tors}(X, \mathcal{A})$*, it is described in [BD01, 1.9] how to obtain another chain complex incarnating* $\mathrm{Tors}_e(X, \mathcal{A})$ *and* $\mathrm{CExt}(G, \mathcal{A})$ *when $X = G$.*

Let us now study the notion of descent of a strictly commutative Picard category by action of a group and connections with group cohomology.

Definition 2.41: *Suppose $\mathcal{C}$ is a category and Γ a group. A right Γ-**action** on $\mathcal{C}$ is an anti-homomorphism of monoids*

$$\Gamma \to \mathrm{End}(\mathcal{C})$$

$$\gamma \mapsto \gamma^*$$

So in for example, $e^* = \mathrm{Id}_{\mathcal{C}}$ and $(\gamma\delta)^* = \delta^* \circ \gamma^*$ for $\gamma, \delta \in \Gamma$ and the identity $e \in \Gamma$.

Remark 2.32: *Usually the requirement is not that $(\gamma\delta)^* = \delta^* \circ \gamma^*$ holds as a strict equality, but instead one asks for a system of natural isomorphisms of functors $(\gamma\delta)^* \approxeq \delta^* \circ \gamma^*$ for $\gamma, \delta \in \Gamma$ satisfying some compatibility constraints. However, for our applications we are only concerned with categories of the form $\mathcal{C}(K^*)$ for some chain complex K^* of Γ-modules and the Γ-action on $\mathcal{C}$ is inherited from the Γ action on K^*. For such Γ-actions, the identification $(\gamma\delta)^* = \delta^* \circ \gamma^*$ is a strict equality.*

Example 2.5: *1. Suppose $\Gamma \subseteq \mathrm{Aut}(X)$. The category* $\mathrm{Tors}(X, \mathcal{A})$ *has a right Γ-action by $\gamma \mapsto \gamma^*$ where γ^* is the usual pullback of torsors by γ.*

2. If $e \in X$ is a distinguished S-point and Γ preserves e, i.e., $\gamma \circ e = e$ for all $\gamma \in \Gamma$, then Γ acts canonically on $\mathrm{Tors}_e(X, \mathcal{A})$ by pullback of torsors and the trivialization at e.

3. If $X = G$ is a group scheme over S with unit e and Γ acts on G by group automorphisms, then Γ acts canonically on $\mathrm{CExt}(G, \mathcal{A})$ by pullback of torsors and the multiplicative structure.

Suppose $\mathcal{C}$ is a category equipped with a right action of Γ.

Definition 2.42: *The category $\mathcal{C}^\Gamma$ of Γ-**descent datum**, or **homotopy fixed points**, is the category of pairs*

$$(\mathcal{P}, \{f_{\gamma,\mathcal{P}}\}_{\gamma\in\Gamma})$$

*where $\mathcal{P} \in \mathcal{C}$ and $\{f_{\gamma,\mathcal{P}}\}_{\gamma\in\Gamma}$, called the **descent datum** for $\mathcal{P}$, is a family of isomorphisms*

$$f_{\gamma,\mathcal{P}} : \mathcal{P} \overset{\cong}{\to} \gamma^*\mathcal{P}$$

*that satisfy the **cocycle condition**: For all $\gamma, \delta \in \Gamma$ and $\mathcal{P} \in \mathcal{C}$,*

$$f_{\gamma\delta,\mathcal{P}} = \delta^* f_{\gamma,\mathcal{P}} \circ f_{\delta,\mathcal{P}}$$

as maps

$$\mathcal{P} \overset{f_{\delta,\mathcal{P}}}{\to} \delta^*\mathcal{P} \overset{\delta^* f_{\gamma,\mathcal{P}}}{\to} \delta^*\gamma^*\mathcal{P} = (\gamma\delta)^*\mathcal{P}.$$

A morphism $(\mathcal{P}, \{f_{\gamma,\mathcal{P}}\}_{\gamma\in\Gamma}) \to (\mathcal{Q}, \{g_{\gamma,\mathcal{Q}}\}_{\gamma\in\Gamma})$ is a morphism $\mathcal{P} \to \mathcal{Q}$ inducing a commutative diagram

$$\begin{array}{ccc} \mathcal{P} & \overset{f_\gamma}{\to} & \gamma^*\mathcal{P} \\ \downarrow & & \downarrow \\ \mathcal{Q} & \overset{g_\gamma}{\to} & \gamma^*\mathcal{Q} \end{array}$$

for each $\gamma \in \Gamma$.

Remark 2.33: *If $\mathcal{C}$ is a strictly commutative Picard category with sum $+$ and Γ acts on $\mathcal{C}$ by Picard functors, then $\mathcal{C}^\Gamma$ is also a strictly commutative Picard category with* *sum*

$$(\mathcal{P}, \{f_{\gamma,\mathcal{P}}\}_{\gamma\in\Gamma}) + (\mathcal{Q}, \{g_{\gamma,\mathcal{Q}}\}_{\gamma\in\Gamma}) = (\mathcal{P}+\mathcal{Q}, \{f_{\gamma,\mathcal{P}} + g_{\gamma,\mathcal{Q}}\}_{\gamma\in\Gamma})$$

where it is observed that $\{f_{\gamma,\mathcal{P}} + g_{\gamma,\mathcal{Q}}\}_{\gamma\in\Gamma}$ *is a descent datum for* $\mathcal{P}+\mathcal{Q}$*.*

Example 2.6: *Suppose* $S = \mathrm{Spec}F$ *is a field and* E/F *is a field extension and* $\Gamma \subseteq \mathrm{Aut}(E/F)$*. For* $\mathcal{C}$ *being either of* $\mathrm{Tors}(X,\mathcal{A})$*,* $\mathrm{Tors}_e(X,\mathcal{A})$*, or* $\mathrm{CExt}(G,\mathcal{A})$*. Then* Γ *acts on* $\mathcal{C}$ *by pullback by Picard functors and* $\mathcal{C}^\Gamma$ *is the usual category of descent datum over* E*.*

Suppose Γ is a group and $K^* \in D(\mathrm{mod}-\Gamma)$, where $\mathrm{mod}-\Gamma$ is the category of right Γ-modules. Then $\mathcal{C}(K^*)$ has a canonical right Γ-action induced by the action of Γ on K^*.

The following theorem [BD01, 1.9.(2)] was stated and proved for the special case of a chain complex K^* incarnating $\mathrm{Tors}(X,\mathcal{A})$. However, the same proof goes forward for general K^*. We present it below for completeness.

Theorem 2.15: *The category of descent data* $\mathcal{C}(K^*)^\Gamma$ *is incarnated by the total* Hom *complex* $R\mathrm{Hom}^{**}(\mathbb{Z}[0],K) \in D(\mathrm{Ab})$ *where* $\mathbb{Z}[0]$ *is the complex concentrated in degree* 0 *with term* $\mathbb{Z}$ *equipped with the trivial* Γ*-action. To be precise, there is a canonical equivalence of Picard categories*

$$\mathcal{C}(K^*)^\Gamma \approxeq \mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0],K)).$$

In particular, $\pi_1\mathcal{C}(K^*) \approxeq \mathbb{H}^0(\Gamma,K^*)$ *and* $\pi_0\mathcal{C}(K^*) \approxeq \mathbb{H}^1(\Gamma,K^*)$ *where* $\mathbb{H}^i(\Gamma,-)$ *are the ith hypercohomology groups.*

Proof: Let us compute $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0],K))$ and see that it is canonically equivalent to $\mathcal{C}(K^*)^\Gamma$. Let $\mathrm{B}\mathbb{Z}_* \to \mathbb{Z}$ be the bar resolution as Γ-modules and apply the derived functor $\mathrm{RHom}(-,K^*) : \mathrm{D}(\mathrm{mod}-\Gamma) \to \mathrm{D}(\mathrm{Ab})$ to $\mathrm{B}\mathbb{Z}_*$. We obtain the total complex of the double complex $\mathrm{Hom}^{**}(B\mathbb{Z},K) = \{C^q(\Gamma,K^p)\}_{pq}$ where $C^q(\Gamma,-)$ is the group homology q-cochains. The first few terms of $\mathrm{Hom}^{**}(B\mathbb{Z}, K)$ are

$$\begin{array}{ccccc}
\vdots & & \cdots & & \\
\mathrm{d}_v^{01} \uparrow & & \vdots & & \\
C^1(\Gamma, K^0) & \overset{\mathrm{d}_h^{01}}{\to} & C^1(\Gamma, K^1) & \cdots & \\
\mathrm{d}_v^{00} \uparrow & & -\mathrm{d}_v^{10} \uparrow & & \vdots \\
K^0 & \overset{\mathrm{d}_h^{00}}{\to} & K^1 & \overset{\mathrm{d}_h^{10}}{\to} & \cdots
\end{array}$$

Unraveling the definitions, $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0], K))$ is defined to be the category of pairs $(\{f_\gamma\}_{\gamma\in\Gamma}, E)$ where $E \in K^1$ and $\{f_\gamma\}_{\gamma\in\Gamma} \in C^1(\Gamma, K^0)$ such that:

1. $\mathrm{d}_h^{10}(E) = 0 \Leftrightarrow E \in Z^1(L)$: this is the condition precisely that E is an object of $\mathcal{C}(K^*)$.

2. $\mathrm{d}_h^{01}(\{f_\gamma\}_{\gamma\in\Gamma}) - \mathrm{d}_v^{10}(E) = 0$ where $\mathrm{d}_h^{01}(\{f_\gamma\}_{\gamma\in\Gamma})$ is the 1-cocycle $\gamma \mapsto \mathrm{d}_h^{00}(f_\gamma)$ and $\mathrm{d}_v^{10}(E)$ is the 1-cocycle $\gamma \mapsto \gamma^*E - E$. This condition is precisely that for $\gamma \in \Gamma$, f_γ is an arrow $E \to \gamma^*E$.

3. $\mathrm{d}_v^{01}(\{f_\gamma\}_{\gamma\in\Gamma}) = 0$ where $\mathrm{d}_v^{01}(\{f_\gamma\}_{\gamma\in\Gamma})$ is the 2-cocycle

$$(\gamma, \delta) \mapsto \delta^* f_\gamma - f_{\gamma\delta} + f_\delta.$$

Given part (2) above, this is precisely the condition that $f_{\gamma\delta} : E \to (\gamma\delta)^*E = \delta^*\gamma^*E$ is equal to the composition

$$E \overset{f_\delta}{\to} \delta^*E \overset{\delta^* f_\gamma}{\to} \delta^*\gamma^*E.$$

This defines an association between objects of $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0], K))$ and objects of $\mathcal{C}(K^*)^\Gamma$.

Let us define the association between morphisms. A morphism in $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0], K))$ between $(\{f_\gamma\}_{\gamma\in\Gamma}, E)$ and $(\{g_\gamma\}_{\gamma\in\Gamma}, F)$ is by definition an element $a \in K^0$ such that $\mathrm{d}_h^{00}a = F - E$ and $\mathrm{d}_v^{00}a = \{g_\gamma\}_{\gamma\in\Gamma} - \{f_\gamma\}_{\gamma\in\Gamma}$ where $\mathrm{d}_v^{00}a$ is the 1 cocycle $\gamma \mapsto \gamma^*a - a$. The first condition is that a is a morphism $E \to F$ in $\mathcal{C}(K^*)$. The second condition is precisely that for all $\gamma \in \Gamma$, the square commutes

$$\begin{array}{ccc} E & \stackrel{a}{\to} & F \\ f_\gamma \downarrow & & g_\gamma \downarrow \\ \gamma^* E & \stackrel{\gamma^* a}{\to} & \gamma^* F \end{array}$$

This is the condition that the map a in $\mathcal{C}(K^*)$ respects the descent datum for E and F. This defines the association between morphisms of $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0], K))$ and morphisms of $\mathcal{C}(K^*)^\Gamma$.

Finally, the sum stricture on $\mathcal{C}(R\mathrm{Hom}^{**}(\mathbb{Z}[0], K))$ comes from that in $K^1 \oplus C^1(\Gamma, K^0)$ and thus coincides with that of $\mathcal{C}(K^*)^\Gamma$. The result follows. ∎

Definition 2.43: *The* ***neutral component*** *of a Picard groupoid $\mathcal{C}$, denoted $\mathcal{C}^0$, is defined to be the full subcategory of objects isomorphic to the unit.*

Suppose $\mathcal{C}$ is a strictly commutative Picard category with sum $+$. Suppose Γ is a group acting on $\mathcal{C}$ on the right by Picard functors. There is a canonical induced action of Γ on $\pi_0\mathcal{C}$ and $\pi_1\mathcal{C}$ and the action of Γ on $\mathcal{C}$ restricts to an action on $\mathcal{C}^0$.

Lemma 2.24: *1. The forgetful map $\mathcal{C}^\Gamma \to \mathcal{C}$ induces a group homomorphism $\pi_0(\mathcal{C}^\Gamma) \to \pi_0(\mathcal{C})^\Gamma$.*

2. There is a short exact sequence

$$0 \to \pi_0(\mathcal{C}^{0,\Gamma}) \to \pi_0(\mathcal{C}^\Gamma) \to Q \to 0$$

where $\mathcal{C}^{0,\Gamma} := (\mathcal{C}^0)^\Gamma$ and Q is the image of $\pi_0(\mathcal{C}^\Gamma) \to \pi_0(\mathcal{C})^\Gamma$.

PROOF: For all $\mathcal{P} \in \mathcal{C}$ under the image of the forgetful map $\mathcal{C}^\Gamma \to \mathcal{C}$, we have for all $\gamma \in \Gamma$, there exists an isomorphism $\mathcal{P} \approxeq \gamma^*\mathcal{P}$. Therefore the class $[\mathcal{P}] \in \pi_0\mathcal{C}$ is Γ-invariant. This proves 1.

The kernel of $\pi_0(\mathcal{C}^\Gamma) \to \pi_0(\mathcal{C})^\Gamma$ consists of classes of pairs $(\mathcal{P}, \{f_\gamma\}_{\gamma\in\Gamma})$ such that $[P] = 0$ in $\pi_0(\mathcal{C})$, i.e., such that $(\mathcal{P}, \{f_\gamma\}_{\gamma\in\Gamma}) \in (\mathcal{C}^0)^\Gamma$. This shows 2. ∎

Suppose $\mathcal{C} = \mathcal{C}(K^*)$ for some chain complex K^* equipped with a right Γ-action. We obtain the following homological algebra interpretation of the above lemma and generalization of [BD01, 2.4]:

Theorem 2.16: *Suppose $\mathcal{C} = \mathcal{C}(K^*)$ is presented by a chain complex $K^* \in D^{[0,1]}(\text{mod}-\Gamma)$ where K^* as no nonzero negative degree terms. Equip $\mathcal{C}$ with the right Γ-action induced by the action of Γ on K^*. Let Q be the image of the induced map $\pi_0(\mathcal{C}^\Gamma) \to \pi_0(\mathcal{C})^\Gamma$. Then:*

1.In the derived category, K^* fits into a distinguished triangle

$$\pi_1\mathcal{C}[0] \to K^* \to \pi_0\mathcal{C}[-1] \to .$$

where for an abelian group A we mean by $A[n]$ the complex concentrated in degree n with term A.

2. The short exact sequence

$$0 \to \pi_0(\mathcal{C}^{0,\Gamma}) \to \pi_0(\mathcal{C}^\Gamma) \to Q \to 0$$

is canonically identified as a truncation of a long exact sequence of hypercohomology of the distinguished triangle of (1), and Q is isomorphic to the kernel of a connecting morphism in a long exact sequence.

Proof: 1. Since K^* is concentrated in nonnegative degrees, there is an injection of chain complexes

$$\iota : Z(K^0)[0] \hookrightarrow K^*$$

where $Z(K^0) = \mathrm{H}^0(K^*) = \pi_1\mathcal{C}$ is the group of 0-cycles of K^*. Taking the mapping cone $C(\iota)$ of the above morphism, we obtain a distinguished triangle

$$\pi_1\mathcal{C}[0] \to K^* \to C(\iota) \to$$

The long exact sequence of cohomology groups gives $C(\iota) \overset{\text{quis}}{\approx} \pi_0\mathcal{C}[-1]$.

2. Apply the characterization 2.15 and take the long exact sequence $\mathbb{H}^*(\Gamma, -)$ of hypercohomology groups on the distinguished triangle $\pi_1\mathcal{C}[0] \to K^* \to \pi_0\mathcal{C}[-1] \to$. Since $\mathbb{H}^0(\Gamma, C(\iota)) = 0$, we obtain

$$(\pi_1\mathcal{C})^\Gamma = \mathbb{H}^0(\Gamma, K^*) \approxeq \pi_1(\mathcal{C}^\Gamma)$$

and an exact sequence of abelian groups

$$0 \to \mathbb{H}^1(\Gamma, \pi_1\mathcal{C}) \to \mathbb{H}^1(\Gamma, K^*) \approxeq \pi_0(\mathcal{C}^\Gamma) \to \mathbb{H}^1(\Gamma, \pi_0\mathcal{C}[-1])] \approxeq (\pi_0\mathcal{C})^\Gamma \overset{\partial}{\to}$$

where ∂ is the connecting morphism

$$\mathbb{H}^1(\Gamma, \pi_0\mathcal{C}[-1])] \overset{\partial}{\to} \mathrm{H}^1(\Gamma, \pi_1\mathcal{C}).$$

The definition of the map $\pi_1\mathcal{C}[0] \to K^*$ as inclusions of automorphisms of the zero element identifies $\pi_0(\mathcal{C}^{0,\Gamma}) \approxeq \mathrm{H}^1(\Gamma, \pi_1\mathcal{C})$ and identifies the first map with the injection $\pi_0(\mathcal{C}^{0,\Gamma}) \to \pi_0(\mathcal{C}^\Gamma)$. It follows that Q is identified as the kernel of ∂. ■

2.3.2 Applications to Central Extensions of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$

Let F be a field and $\mathcal{T}$ be a torus over F. Let E/F be a Galois extension such that $\mathcal{T}_E$ is split. We now apply the results of the previous subsection to make explicit computations about $\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F})$. We adopt the convention that actions on schemes are left actions, and actions on rings are right actions. For example for $\Gamma \subseteq \mathrm{Aut}(E/F)$ acting on E, we use the right action $a.\gamma := \gamma^{-1}(a)$ for $\gamma \in \Gamma$ and $a \in E$. For an affine scheme $Z = \mathrm{Spec}A$, write $K_i(Z) := K_i(A)$. Let $Y = \mathrm{Hom}_E(\mathbb{G}_{m,E}, \mathcal{T}_E)$ be the (absolute) cocharacter lattice and $X = \mathrm{Hom}_E(\mathcal{T}_E, \mathbb{G}_{m,E})$ be the (absolute) character lattice, and consider $X \subseteq \Gamma(\mathcal{T}_E, \mathcal{O}_{\mathcal{T}_E})$. Hence $\mathcal{O}_{\mathcal{T}_E} = E[X]$ as a ring.

We have the following explicit description of $\mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{i,E})$ for $i = 0, 1, 2$, given by combining [BD01, 3.3.1] with the fundamental theorems in [Wei13, 6.3] relating $K_i(R)$ with $K_i(R[t^{\pm 1}])$ for all i for a regular noetherian ring:

Lemma 2.25: *We have:*

0. The map $E \to \mathcal{O}_{\mathcal{T}_E} \approxeq E[X]$ *induces an isomorphism*

$$\mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{0,E}) = K_0(\mathcal{T}_E) = \mathbb{Z}.$$

1. The inclusion $X \otimes E^\times \hookrightarrow E[X]^\times \hookrightarrow K_1(\mathcal{T}_E)$ *is an isomorphism, hence*

$$\mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{1,E}) = K_1(\mathcal{T}_E) = X \otimes_{\mathbb{Z}} E^\times.$$

2. Let $\{-\} : R^\times \otimes R^\times \to K_2(R)$ *the Steinberg symbol map* $a \otimes b \mapsto \{a, b\}$*. Then we have*

$$\mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E}) = K_2(\mathcal{T}_E) = K_2E \oplus \langle \{X \otimes_{\mathbb{Z}} E^\times\}, \{X \otimes_{\mathbb{Z}} X\}\rangle$$

where $\{-\}$ *restricts to an injection* $X \otimes_{\mathbb{Z}} E^\times \to K_2(\mathcal{T}_2)$ *and the only further relations are* $\{x, x\} = \{x, -1\}$ *for all* $x \in X$*.*

Remark 2.34: *In particular, a morphism* $\mathcal{T}_E \to K_{i,E}$ *is completely determined by its composition with the natural map* $K_{i,E} \to \boldsymbol{K}_{i,E}$*.*

Example 2.7: *Recall the notion of a normalized morphism* $\mathcal{T}_E^i \to, \boldsymbol{K}_{2,\mathrm{E}}$ *from 2.23. We have for all* $i \geq 0$*, and explicit description of the normalized global sections*

$$\mathrm{H}^0(\mathcal{T}_E^i, \boldsymbol{K}_{2,E})^{\mathrm{norm}} = \langle \{X^i \otimes_{\mathbb{Z}} E^\times\}, \{X^i \otimes_{\mathbb{Z}} X^i\}\rangle \subseteq \mathrm{H}^0(\mathcal{T}_E, K_{2,E})$$

where $\langle \{X^i \otimes_{\mathbb{Z}} E^\times\}, \{X^i \otimes_{\mathbb{Z}} X^i\}\rangle$ *is the subgroup given by replacing* $\mathcal{T}$ *with* $\mathcal{T}^i$ *in 2.25.*

We have the following theorem about incarnating the category $\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ from [BD01, sec. 3]:

Theorem 2.17: *1.* $\mathrm{H}^i(\mathcal{T}_E, \boldsymbol{K}_{j,E}) = 0$ *for* $i > 0$ *and* $j \geq 0$*. Consequently by [BD01, 1.9.(i)] every* $\mathcal{E} \in \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ *has underlying trivial* $\boldsymbol{K}_{2,E}$*-torsor.*

2. Combined with [BD01, 1.9.vii], the category $\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ *is incarnated by the normalized global sections complex*

$$D^* := \mathrm{H}^0(\mathcal{T}_E^*, \boldsymbol{K}_{2,F})[-1] = K_2(\mathcal{T}_E^*)^{\mathrm{norm}}[-1]$$

(we use $[-1]$ to denote the left shift) and the differentials are obtained as follows. Consider the simplicial scheme

$$\mathrm{B}\mathcal{T}_E := \{\cdots \mathcal{T}_E^3 \overset{\rightarrow}{\underset{\rightarrow}{\vdots}} \mathcal{T}_E^2 \overset{\rightarrow}{\underset{\rightarrow}{\rightarrow}} \mathcal{T}_E \rightarrow e\}$$

with face maps ∂_i. Then apply the functor $\mathrm{H}^0(-, \boldsymbol{K}_{2,F})[-1]$ and take the alternating sum $\sum_i (-1)^i \partial_i$ of the face maps. Explicitly, for $p = 1$, the differential is

$$pr_1^* - \mu^* + pr_2^* : K_2(\mathcal{T}_E)^{\mathrm{norm}} \rightarrow K_2(\mathcal{T}_E^2)^{\mathrm{norm}}$$

$$f \mapsto \{(x, y) \mapsto f(x) - f(xy) + f(y)\}.$$

Furthermore, the incarnation is Γ-equivariant, i.e., there exists a Γ-equivariant equivalence of categories $\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E}) \approxeq \mathcal{C}(D^*)$.

3. We have canonically

$$\pi_1 \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E}) \approxeq X \otimes_{\mathbb{Z}} E^{\times}$$

by the canonical association between the automorphism group of the trivial torsor with the global sections $\mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ and the inclusion $\{-\} : X \otimes_{\mathbb{Z}} E^{\times} \rightarrow \mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ from 2.25.

The isomorphism classes of $\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ are given by

$$\pi_0 \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E}) \approxeq \mathrm{Sym}^2 X$$

classified by the commutator $\mathcal{T}_E \rightarrow \boldsymbol{K}_{2,E}$ as in 2.3.

We now apply the results of 2.16 to compute $\pi_0 \mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F})$ and the generic cocycle $\mathcal{T}^2 \rightarrow \boldsymbol{K}_{2,F}$. Let D^* be the normalized global sections complex from 2.17. For each $i \geq 0$, the neutral point $e \in \mathcal{T}_E^i(E)$ is Γ-invariant for the diagonal action. Consequently Γ acts by pullback on $D^i = \mathrm{H}^0(\mathcal{T}_E^i, \boldsymbol{K}_{2,E})^{\mathrm{norm}} = K_2(\mathcal{T}_E^i)^{\mathrm{norm}}$ and also on $\mathcal{C}(D^*)$.

Remark 2.35: *As* Γ*-chain complexes,* D^* *is isomorphic to* $\mathrm{H}^0(\mathcal{T}_E^{i+1}, K_{2,E})/K_2(E)$*.*

Let $\mathcal{E}$ be a central extension of a group G over F by $\boldsymbol{K}_{2,F}$. The base change $\mathcal{E}_E$ inherits a canonical descent data from $p^*\mathcal{E}$ that we denote $\{f_{\mathrm{can},\gamma}\}_{\gamma\in\Gamma}$. By [BD01, sec. 7], for any reductive group G over F, $\mathrm{CExt}(G, \boldsymbol{K}_{2,F})$ satisfies Galois descent, i.e., the base change functor

$$\mathrm{CExt}(G, \boldsymbol{K}_{2,F}) \to \mathrm{CExt}(G_E, \boldsymbol{K}_{2,E})^{\Gamma}$$

$$(\mathcal{E}, \beta) \mapsto (\mathcal{E}_E, \beta_E, \{f_{\mathrm{can},\gamma}\}_{\gamma\in\Gamma})$$

is an equivalence of Picard categories. We have a proof of the first main theorem:

Proof: [of 2.11] Apply the theorem 2.16 to the normalized global sections chain complex D^*. Use $\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F}) \approxeq \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\Gamma}$ together with $\mathcal{C}(D^*) \approxeq \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ and obtain the precise short exact sequence as required, where $Q = \mathrm{AdmSym}^2 X$ is the kernel of the connecting morphism. The descriptions of ι and ρ come from the categorical interpretations in 2.16 of Q as the image of

$$\pi_0(\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\Gamma}) \to \pi_0(\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E}))^{\Gamma} = (\mathrm{Sym}^2 X)^{\Gamma}$$

and $\mathrm{H}^1(\Gamma, X \otimes_{\mathbb{Z}} E^{\times})$ as the kernel. ■

Let us now move to computing the generic cocycle for a central extension $\mathcal{E}$ of $\mathcal{T}$ by $\boldsymbol{K}_{2,F}$:

Proof: [of 2.13] Let D^* be the complex of normalized co-chains incarnating $\mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ from 2.17. Let D^*_{gen} be the complex obtained by restricting each term D^i to the generic point and then taking mod $K_2(E)$, i.e.,

$$D^i_{\mathrm{gen}} = K_2 L(\mathcal{T}_E^{i+1})/K_2 E.$$

According to [BD01, 8.6], the restriction map $D^* \to D^*_{\mathrm{gen}}$ is a quasi-isomorphism.

Now consider $D^* \to D^*_{\text{gen}}$ as a morphism of complex of right Γ-modules, and apply the functor $\mathrm{RHom}(\mathrm{B}\mathbb{Z}_*, -)$ to $D^* \to D^*_{\text{gen}}$. Let E_r^{pq} be the spectral sequence for the double complex $\mathrm{Hom}^{**}(\mathrm{B}\mathbb{Z}, D_{\text{gen}})$ associated to the column filtration. The zero page E_0^{pq} is

$$\begin{array}{ccccc} \uparrow & & \uparrow & & \\ C^1(\Gamma, K_2E(\mathcal{T}_E)/K_2E) & \to & C^1(\Gamma, K_2E(\mathcal{T}_E^2)/K_2E) & \to & \\ \mathrm{d}_v^{00} \uparrow & & \uparrow & & \\ K_2L(\mathcal{T}_E)/K_2E & \overset{-\mathrm{d}_h^{00}}{\to} & K_2L(\mathcal{T}_E^2)/K_2E & & \vdots \end{array}$$

where $\mathrm{d}_h^{00} = \mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*$. Using $\mathrm{H}^1(\Gamma, K_2E(\mathcal{T}_E^i)/K_2E) = 0$ for all $i \geq 0$ of 2.12, taking vertical cohomology gives the first page E_1^{pq}:

$$\begin{array}{ccccc} \cdots & & \cdots & & \\ 0 & \to & 0 & \to & \\ & & & & \\ (K_2L(\mathcal{T}_E)/K_2E)^\Gamma & \to & (K_2L(\mathcal{T}_E^2)/K_2E)^\Gamma & \to & \end{array}$$

We see E_1^{pq} already achieves the limit of E_r^{pq}. Hence by [BD01, cor. 8.8] the category $\mathrm{CExt}(\mathcal{T}, \boldsymbol{K}_{2,F})$ is incarnated by the bottom row which is also $D^{\Gamma,*}_{\text{gen}}$, the sub-complex of Γ invariant elements of D^*_{gen}.

Let $\mathcal{E}$ be a central extension of $\mathcal{T}$by $\boldsymbol{K}_{2,F}$ and $\mathcal{E}_E$ the base change. Let $c \in \mathrm{H}^0(\mathcal{T}_E^2, \boldsymbol{K}_{2,E})^{\text{norm}}$ be a cocycle for $\mathcal{E}_E \in \mathrm{CExt}(\mathcal{T}_E, \boldsymbol{K}_{2,E})$ and $\{f_\gamma\}_{\gamma\in\Gamma} \in C^1(\Gamma, \mathrm{H}^0(\mathcal{T}_E, \boldsymbol{K}_{2,E})^{\text{norm}})$ be a 1-cochain that defines the descent datum for $\mathcal{E}_E$. Let $\{g_\gamma\}_{\gamma\in\Gamma}$ be the image of the restriction of $\{f_\gamma\}_{\gamma\in\Gamma}$ to η in $C^1(\Gamma, K_2E(\mathcal{T}_E)/K_2E)$ and d be the image of the restriction of $c|_{\eta^2}$ in $K_2L(\mathcal{T}_E^2)/K_2E$. Hence the pair $(\{g_\gamma\}_{\gamma\in\Gamma}, d)$ lies in the first anti-diagonal of E_0^{pq} maps to 0 in $C^1(\Gamma, K_2E(\mathcal{T}_E^2)/K_2E)$. Let $n \in K_2L(\mathcal{T}_E)/K_2E$ be be such that $\mathrm{d}_v^{00}(n) = \{g_\gamma\}_{\gamma\in\Gamma}$. Commutativity of the bottom left square of E_0^{pq} ensures that the element

$$d - \mathrm{d}_h^{00}(n) = d - (\mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*)(n) \in K_2L(\mathcal{T}_E^2)/K_2E$$

also maps to 0 in $C^1(\Gamma, K_2E(\mathcal{T}_E^2)/K_2E)$ and thus lies in $(K_2L(\mathcal{T}_E^2)/K_2E)^\Gamma$, i.e.,

$d - (\mathrm{pr}_1^* - \mu^* + \mathrm{pr}_2^*)(n)$ i Γ-invariant. Applying the horizontal differential kills the $\mathrm{d}_h^{00}(n)$ term and shows $d - \mathrm{d}_h^{00}(n)$ also satisfies the cocycle condition. The statement of convergence $E_r^{pq} \Rightarrow \mathrm{RHom}(\mathrm{B}\mathbb{Z}_*, D^*_{\mathrm{gen}})$ gives that the natural map $D^*_{\mathrm{gen}} \to \mathrm{RHom}(\mathrm{B}\mathbb{Z}_*, D^*_{\mathrm{gen}})$ is a quasi-isomorphism, and shows $d - \mathrm{d}_h^{00}(n)$ is the generic cocycle for $\mathcal{E}$. ■

Chapter 3

From Conjugacy Classes in the Weyl Group to Geometric Constructions of the Affine ADE Basic Representation

3.1 The Split Geometric FKS Isomorphism and Statement of Main Results

Let us motivate the problem and state the results. First let us recall the classical (split) FKS isomorphism and its geometric realization in terms of vertex algebras. Vertex algebras and (twisted modules) for vertex algebras never appear mathematically in the mathematics of our work, sine we replace them with representations of Heisenberg central extensions, but it helps to mention them for illustration and motivation.

Let G be a simple algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$, equipped with the normalized invariant bilinear form B. Let $\hat{\mathfrak{g}}$ be the affine Lie algebra associated to $\mathfrak{g}$, e.g., as defined in [Zhu09, 0.1]. It is an infinite dimensional Lie algebra, given as a one dimensional central extension of the loop algebra $\mathfrak{g}[t^{\pm 1}]$ of $\mathfrak{g}$. There is a distinguished representation, called the basic representation, V^1 of $\hat{\mathfrak{g}}$ that is the study of the representation theoretic literature [FK81, Seg81, Lep85]. It is the quotient of a Verma module V_1 of $\hat{\mathfrak{g}}$. The space V_1 has the structure of a vertex algebra, constructed in [FBZ04, 1.3] such that V^1 is a quotient vertex algebra. The motivation to study the vertex algebra structure is that there is a natural equivalence of categories between vertex algebra modules for V_1 and Lie algebra representations of $\hat{\mathfrak{g}}$ [FBZ04, 5.6].

The classical FKS isomorphism of [FK81, Seg81] is the determination of the vertex algebra structure of V^1 when $\mathfrak{g}$ is simply laced, i.e., type ADE, in terms of a simpler vertex algebra that can be described explicitly. It is as follows. Assume G is simply connected and $T \subseteq G$ is a maximal torus. Let $Y = \mathbb{X}_*(T)$ be the cocharacter lattice, equal to the coroot lattice and $\mathfrak{t} = \mathrm{Lie}T$. Let $\hat{\mathfrak{t}}$ be the restriction of the central extension $\hat{\mathfrak{g}}$ to $\mathfrak{t}[t^{\pm 1}] \subseteq \mathfrak{g}[t^{\pm 1}]$. Then $\hat{\mathfrak{t}}$ has a unique level 1 representation π_λ for each $\lambda \in \mathfrak{t}^\vee$, called the Fock space with highest weight λ for $\mathfrak{t}$. The restriction of B to Y is an even nondegenerate integer valued bilinear form inducing an embedding $B : Y \hookrightarrow Y^\vee = \mathbb{X}_*(T)$ and associated to the data $(B, Y, \hat{\mathfrak{h}})$ is a vertex algebra V_Y called the Lattice vertex algebra, with underlying $\hat{\mathfrak{h}}$ module

$$V_Y = \oplus_{\lambda \in Y} \pi_{B(\lambda)}.$$

The vertex algebra structure on V_Y is known and explicitly described in [FBZ04, 5.4]. The classical FKS isomorphism is:

Theorem 3.1: *[FK81, Seg81] When $\mathfrak{g}$ is type ADE, there exists an isomorphism of vertex algebras*

$$V^1(\hat{\mathfrak{g}}) \approxeq V_Y.$$

This isomorphism $V^1(\hat{\mathfrak{g}}) \approxeq V_Y$ was defined Lie-theoretically. However, it has been geometrically realized in [Zhu09] as a restriction problem about line bundles on an infinite dimensional variety. It is as follows. On one hand, there is a Borel-Weil type theorem for Kac-Moody groups [Kum02, 8.3.12] which states there is a certain line bundle $\mathcal{L}$ on the affine Grassmannian Gr_G and an isomorphism of $\hat{\mathfrak{g}}$ modules

$$V^1(\hat{\mathfrak{g}}) \approxeq \Gamma(\mathrm{Gr}_G, \mathcal{L})^\vee.$$

It is an affine analogue for the finite type Borel-Weil theorem replacing $\hat{\mathfrak{g}}$ with a finite dimensional reductive Lie algebra and $\mathcal{L}$ with a line bundle on the finite dimensional flag variety. On the other hand, the space $\mathrm{Gr}_T \subseteq \mathrm{Gr}_G$ is zero dimensional, non reduced, with connected components in bijection with Y, and all connected components isomorphic. The dual ring of regular functions of the component corresponding to λ is naturally identified with $\pi_{B(\lambda)}$, giving rise to an identification

$$\Gamma(\mathrm{Gr}_T, \mathcal{L}|_{\mathrm{Gr}_T})^\vee \approxeq \oplus_{\lambda \in Y} \pi_{B(\lambda)}$$

which has the same shape of the lattice vertex algebra V_Y. Indeed a vertex algebra structure on $\Gamma(\mathrm{Gr}_T, \mathcal{L}|_{\mathrm{Gr}_T})^\vee$ and on $\Gamma(\mathrm{Gr}_G, \mathcal{L})^\vee$ was constructed geometrically in [Zhu09] using Beilinson and Drinfeld's correspondence between vertex algebras and chiral algebras, which are algebro-geometric objects. The geometric FKS isomorphism theorem is thus:

Theorem 3.2: *[Zhu09] For G of type ADE, the natural restriction induces an isomorphism*

$$\Gamma(\mathrm{Gr}_T, \mathcal{L}|_{\mathrm{Gr}_T})^\vee \approxeq \Gamma(\mathrm{Gr}_G, \mathcal{L})^\vee.$$

Our goal here is to prove a geometric realization of a twisted version of the FKS isomorphism in [KP85] that generalizes the classical FKS isomorphism. The twisted FKS isomorphism is as follows.

Instead of ordinary modules for vertex algebras, we consider twisted modules. Let $w \in W$ be a Weyl group element, acting on Y, giving rise to an automorphism of V_Y. A certain twisted module, $M(w)$ is defined for V_Y whose underlying vector space is described [Lep85] as follows. The map B induces an injection of the quotient of by torsion of the coinvariants $Y_{w,\mathrm{cotor}} \hookrightarrow Y_{w,\mathrm{cotor}} \otimes \mathbb{C} = \mathfrak{t}^w \overset{B}{\hookrightarrow} \mathfrak{t}^{w,\vee}$. Then

$$M(w) = \oplus_{\lambda \in Y_{w,\mathrm{cotor}}} \pi(w)_{B(\lambda)} \otimes U$$

where $\pi(w)_{B(\lambda)}$ is a twisted version of the Fock space and the unique irreducible module for a twisted version $\hat{\mathfrak{t}(w)}$ of $\hat{\mathfrak{t}}$ containing $\mathfrak{t}^w$, such that $\mathfrak{t}^w$ acts by $B(\lambda)$. Furthermore, U is finite dimensional and a representation for a finite group $\hat{\Sigma}$ that is a central extension of the torsion subgroup $\Sigma = Y_{w,\mathrm{tor}}$ by some roots of unity. The twisted module structure on $M(w)$ for V_Y is explicitly described in [KP85]. Remark that such a decomposition is similar to the decomposition of representations of a Heisenberg central extension we proved in 2.2.6.

On the other hand, a lift σ of w acting on $\mathfrak{g}$ is chosen and used to defined a twisted module structure on $V^1(\hat{\mathfrak{g}})$ for the vertex algebra V_Y. The twisted FKS isomorphism theorem is:

Theorem 3.3: *When G is type ADE, there is an isomorphism of twisted modules for V_Y:*

$$M(w) = \oplus_{\lambda \in Y_{w,\text{cotor}}} \pi(w)_{B(\lambda)} \otimes U \approxeq V^1(\hat{\mathfrak{g}}).$$

Our primary project is to realize the twisted FKS isomorphism as another restriction problem of sections of the line bundle $\mathcal{L}$ on Gr_G. To be precise, for any lift σ of w in G_{ad}, we define a subspace space $\mathcal{S}(\sigma) \subseteq \text{Gr}_G$ that is a replacement for Gr_T in the classical FKS isomorphism, such that:

Theorem 3.4: *Suppose G is type ADE. The restriction induces an isomorphism*

$$\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \approxeq \Gamma(\text{Gr}_G, \mathcal{L}_{\text{Gr}_G})^\vee.$$

The space $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee$ is identified with $M(w)$ in the following sense. The space $S(\sigma)$ has connected components in bijection with $Y_{w,\text{cotor}}$ and all components are isomorphic. Each component has underlying reduced locus a finite dimensional flag variety of a connected reductive group, namely the fixed points M^σ where $M = Z_G(T^{w,0})$ is the centralizer of the connected component of the torus fixed points. The space U is identified

$$U = \Gamma(S(\sigma)^0_{\text{red}}, \mathcal{L}|_{S(\sigma)^0_{\text{red}}})^\vee$$

having an action of both the finite group $\hat{\Sigma}$ and the algebraic group M^σ, in fact is a minuscule representation for M^σ. Finally, the nilpotent thickenings of $S(\sigma)^0_{\text{red}}$ in $S(\sigma)^0$ is controlled by an infinitesimal thickening of a single point whose dual regular functions is identified with $\pi(w)_{B(\lambda)}$. This gives an identification $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \approxeq \oplus_{\lambda \in Y_{w,\text{cotor}}} \pi(w)_{B(\lambda)} \otimes U$. To do this, we also find a maximal torus $\mathcal{T} \subseteq G_F$ such that both $\Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ and $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee$ is also a representation for a Heisenberg central extension $\hat{L\mathcal{T}}$ of $L\mathcal{T}$. We then study the map $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ using the representation theory of Heisenberg central extensions we developed 2.2.6. In this way, we replaced the use of twisted modules for V_Y with representations of $\hat{L\mathcal{T}}$.

Some notable features of our study are:

1. We only show $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \hookrightarrow \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ is injective in general for all $w \in W$ and all types for G, but use preexisting representation theoretic results in [KP85] that $\Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ is irreducible for $\hat{L\mathcal{T}}$ in type ADE to deduce the isomorphism. Nonetheless, it demonstrates correct identification of the subspace $\mathcal{S}(\sigma)$ is given, where previously it was not known what suitable subspace is for a restriction problem in the twisted case, only that the affine Springer fibers were proposed [Zhu09].

2. But a fully geometric proof of $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \approxeq \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ is given for many $w \in W$ without the representation theoretic results of [KP85]. These $w \in W$ are precisely the homogeneous elements, defined in 3.7, and do occupy many of the elements, including all of the elements in type A and more generally whenever w lies in a parabolic subgroup of type A.

3. A conditional proof of $\Gamma(\mathcal{S}(\sigma), \mathcal{L}|_{\mathcal{S}(\sigma)})^\vee \approxeq \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ in the remaining cases is given, on a precise numerical computation regarding the closure $\mathcal{T}^\sharp$ over $G_{\mathcal{O}_F}$ of a certain maximal torus $\mathcal{T} \subseteq G_F$ associated to w and a lift σ of w, namely that the number of torsion elements of $\pi_0 \mathcal{F}_{\mathcal{T}^\sharp}$ equals the defect $d([w])$, a quantity also introduced explicitly in [KP85, 10.1] and recalled in 2.34, that depends only on the action of w on Y and on B. Computer assisted computations have shown it to be true in type D_6, D_8, but they are omitted from this thesis and we leave this computation as an open problem.

4. The subspace $\mathcal{S}(\sigma)$ contains $\mathcal{F}_{\mathcal{T}^\sharp}$. Conditional on the above computation, we show that $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}|_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \approxeq \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ which gives another candidate sub-

space the geometric twisted FKS isomorphism, and in fact a more elegant formulation: for every conjugacy class of maximal tori in G_F, there exists a representative $\mathcal{T}$ such that the inclusion of the image of $L\mathcal{T}$ in Gr_G induces an isomorphism on global sections of $\mathcal{L}$.

5. As an auxiliary result, a new, geometric, proof of the classification of the conjugacy classes of W by parabolic induction is given, where before in [GP00] it was done using combinatorial methods and Coxeter theory when W is generalized to any finite Coxeter group.

3.2 Closure of Maximal Tori in G_F Over $G_{\mathcal{O}_F}$

For an element σ of a group, we say σ is order m if $\sigma^m = 1$ and do not require m to be minimal. The minimal order of σ is denoted $|\sigma|$. The image of $i \in \mathbb{Z}$ under the map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ will be denoted $\bar{i}$. For a group G, let G_{ad} denote the adjoint group of G, defined to be the image of G under the adjoint morphism $G \to \mathrm{Aut}_k(G)$ or $G/Z(G)$. For any subgroup $H \subseteq G$ where G is unambiguously present, we denote by H_{ad} the image of H in G_{ad} and not the adjoint group of H.

3.2.1 Parahoric Group Schemes and Affine Flag Varieties

Let us recall some notions, introduce the relevant examples, and prove some basic lemmas regarding parahoric subgroups, parahoric group schemes. Let H be a connected reductive group over $F = \mathbb{C}[[t]]$ and put $k = \mathbb{C}$. By a **parahoric subgroup** of $H(F)$, we mean in the sense of [Yu15, 7.1], i.e., the connected component of the stabilizer of a facet in the Bruhat-Tits building $B(H_{\mathrm{ad}}(K))$ of the adjoint group. There is a metric on $H(F)$ and Parahoric subgroups are all bounded. In a similar fashion to the theory of parabolic subgroups in a reductive group over k, every parahoric subgroup is $H(K)$-conjugate **standard parahoric subgroup** defined by the choice of an alcove in $B(H_{\mathrm{ad}}(K))$. Each standard parahoric subgroup is associated a strict subset of the affine Dynkin diagram I_{aff} of G in an inclusion preserving manner. A minimal parahoric subgroup is called an **Iwahori subgroup**, which is the affine analogue of the Borel subgroup of a reductive group over k. Each standard parahoric subgroup strictly containing the standard Iwahori subgroup is associated to a subset $S \subseteq I_{\mathrm{aff}}$ is generated by the parahoric subgroups associated to a collection $\mathcal{V}$ of subsets of I_{aff} that union to S. In particular, there is a finite set of minimal standard parahoric subgroups that strictly contain the standard Iwahori subgroup. Each is generated by the standard Iwahori subgroup and a standard **affine root subgroup**.

Definition 3.1: *Let $\mathcal{P}$ be an affine group scheme over $\mathcal{O}_F$ with connected generic fiber. The **neutral component** of $\mathcal{P}$ is the open subgroup scheme consisting of the generic fiber and the neutral component of the special fiber.*

We do not use the details of Bruhat-Tits theory, and are only concerned with one particular example of parahoric subgroups explicitly described below. Suppose G is connected reductive over k with base change G_F and σ is an automorphism of G of order m. Let E/F be the degree m extension with $E = k((u))$, $u^m = t$. Let $\nu \in \Gamma \coloneqq \mathrm{Gal}(E/F)$ be a choice of generator and ζ a primitive mth root of unity such that $\nu(u)/u = \zeta$. Put $\mathcal{O}_E = k[[u]]$ so Γ preserves $\mathcal{O}_E$. Set

$$H = \mathrm{Res}_{E/F}(G \times E)^{\sigma\times\nu^{-1}}.$$

Then H is a reductive group over F obtained by Galois descent on the group G_E over E with $\nu \in \Gamma$ acting by $\sigma \times \nu^{-1}$. Then as in the argument of [PR08, 7.a] combined with the main results [PY02] that follow through for an arbitrary $\sigma \in \mathrm{Aut}_k(G)$, the neutral component of the group

$$(G \times \mathcal{O}_E)^{\sigma\times\nu^{-1}}(\mathcal{O}_E) \subseteq (G \times E)^{\sigma\times\nu^{-1}}(E) = H(F)$$

is a parahoric subgroup of $H(F)$. For us, we are primarily concerned with the case when $\sigma \in G_{\mathrm{ad}}$ is an inner automorphism.

Let us prove a basic lemma that when the derived subgroup $G' = [G, G]$ is simply connected, and σ is inner, we do not need to take the connected component.

Lemma 3.1: *Suppose G' is simply connected and σ is inner. Then the $\mathcal{O}_F$ points of the $\mathcal{O}_F$-group scheme*

$$\mathcal{P} := \mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}(G \times \mathcal{O}_E)^{\sigma\times\nu^{-1}}$$

is connected.

PROOF: First we claim G^σ is connected. Let $H = Z(G)^0$ be the connected component of the center so $G = HG'$ where H is a torus and G' is simply connected. Let $h \in H, g \in G'$ and suppose $hg \in G^\sigma$. Since σ is inner and centralizes H,

$$\sigma(hg) = h(\sigma.g) = hg$$

$$\Rightarrow g \in G'^{,\sigma}$$

so $G^\sigma \subseteq HG'^{,\sigma}$. The other direction is trivial, and we have a presentation

$$G^\sigma = HG'^{,\sigma}.$$

,By a theorem of Steinberg, [OV90, 4.8.9], since G' is simply connected, $G'^{,\sigma}$ is connected reductive. We conclude that the continuous multiplication map $H \times G'^{,\sigma} \to G^\sigma$ is surjective with connected source, and thus G^σ is connected.

Now to show $\mathcal{P}(\mathcal{O}_F) = G(\mathcal{O}_E)^{\sigma\times\nu^{-1}}$ is connected, consider the reduction map mod u, $\mathcal{O}_E \to k$, where the uniformizer u is an eigenvector for ν with eigenvalue ζ. Therefore the reduction map is invariant for Γ. Then the induced reduction map $G(\mathcal{O}_E) \to G(k)$ is equivariant for the action on the left by $\sigma \times \nu^{-1}$ and the action on the right by σ and hence restricts to a surjection

$$G(\mathcal{O}_E)^{\sigma\times\nu^{-1}} \to G^\sigma.$$

Since the image is connected and the kernel is pro-unipotent (and hence connected) we conclude that the source $G(\mathcal{O}_E)^{\sigma\times\nu^{-1}}$ is connected. The result follows. ■

For every parahoric subgroup P of $H(F)$, there exits a smooth integral model $\mathcal{P}$ of

H over $\mathcal{O}_F$ with the property that $\mathcal{P}(\mathcal{O}_F) = P$; the construction is reviewed in [Yu15, sec. 7]. By the uniqueness principle of smooth integral models, $\mathcal{P}$ is unique with respect to this property. For example, if H, G, σ is as in above, the parahoric group scheme associated to the parahoric subgroup $(G \times \mathcal{O}_E)^{\sigma \times \nu^{-1}}(\mathcal{O}_E)$ is the connected component of $\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}(G \times \mathcal{O}_E)^{\sigma \times \nu^{-1}}$. Such a model of H over $\mathcal{O}_F$ is indeed smooth by [Edi92, 2.2] and [Edi92, 3.4].

Example 3.1: *If $\sigma = \mathrm{Id}_G$ is trivial, then $\mathcal{P} = G_{\mathcal{O}_F}$ is the standard hyperspecial parahoric group scheme associated to the subset subset $I \subseteq I_{\mathrm{aff}}$ consisting of the finite Dynkin diagram of G.*

Example 3.2: *If $G = T$ is a torus so $H \coloneqq \mathcal{T}$ is a torus over F, then $\mathcal{P} = \mathcal{T}^{\flat,0}$, the connected Néron model. In fact, this is the only parahoric group scheme of $\mathcal{T}$.*

Now let us fix notation and recall the basic theory of affine flag varieties following [PR08]. For an affine finite type flat group scheme $\mathcal{G}$ over $\mathcal{O}_F$ with generic fiber H, the **affine flag variety** is the fpqc sheaf

$$\mathcal{F}_{\mathcal{G}} \coloneqq [LH/L^+\mathcal{G}].$$

If $\mathcal{G}$ is smooth over $\mathcal{O}_F$, then $\mathcal{F}_{\mathcal{G}}$ is representable by an ind-scheme over k that is ind-finite type, [PR08, 1.4]. The examples of $\mathcal{G}$ that we are concerned with are parahoric group schemes associated to a reductive group over F and intermediate integral models $\mathcal{T}^\sharp$ of of a torus $\mathcal{T}$ over F. Some basic properties of affine flag varieties are recalled below from [PR08, sec.0, 1], [Zhu17, 1.2.11]:

1. If $\mathcal{G}$ is connected, then $L^+\mathcal{G}$ is connected. If $\mathcal{G}$ is smooth over $\mathcal{O}_F$ then $L^+\mathcal{G}$ is reduced. The map $LH \to \mathcal{F}_{\mathcal{G}}$ induces an isomorphism on connected components, and there is an isomorphism induced by the Kottwitz homomorphism

$$\pi_0(LH) \approxeq \pi_1(H)_I$$

where $\pi_1(H)$ is the algebraic fundamental group of G over the separable closure of F, equipped with the Galois action of $I \coloneqq \mathrm{Gal}(F^{\mathrm{sep}}/F)$. In particular, if

H is simply connected, then LH and the flag variety of any parahoric group scheme of H is connected.

2. If H is semisimple, then both LH and $\mathcal{F}_H$ are reduced.

3. Suppose $H = \mathcal{T}$ is a torus split over E with the Γ-module $Y = \mathrm{Hom}(\mathbb{G}_{m,E}, \mathcal{T}_E)$. Suppose $\mathcal{G} = \mathcal{T}^\sharp$ is an intermediate integral model. By our study of subgroups of $L\mathcal{T}$ in 2.2.2, $L\mathcal{T}^\sharp$ and $\mathcal{F}_{\mathcal{T}^\sharp}$ are both not reduced. We also have $\mathcal{F}_{\mathcal{T}^\sharp}$ is discreet, with the Kottwitz homomorphism inducing an identification of $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})$ with a quotient of $Y_\Gamma = \pi_1(L\mathcal{T})$.

4. If $\mathcal{P}$ is smooth fiberwise connected affine group scheme over $\mathcal{O}_F$ then the affine flag variety $\mathcal{F}_\mathcal{P}$ is ind-projective iff $\mathcal{P}$ is a parahoric group scheme of H. This recalls the classical definition of a parabolic subgroup of a reductive group as one where the associated (non-affine) flag variety is projective.

Now suppose $H = G_F$ is the base change of a reductive group G already defined over k and $\mathcal{P} = G_{\mathcal{O}_F}$. This special case is important and is given a name to match the literature, e.g., [Zhu09].

Definition 3.2: *The **affine Grassmannian** of a reductive group G over k is*

$$\mathrm{Gr}_G \coloneqq \mathcal{F}_{G_{\mathcal{O}_F}}.$$

Additionally, for simplicity, define

$$LG \coloneqq LG_F$$

$$L^+G \coloneqq L^+G_{\mathcal{O}_F}.$$

*We refer to $G(\mathcal{O}_F)$ as the **canonical standard hyperspecial parahoric subgroup** and $G_{\mathcal{O}_F}$ as the canonical standard hyperspecial parahoric group scheme.*

3.2.2 Conjugacy Classes of W and Conjugacy Classes of Maximal Tori in G_F

Fix $k = \mathbb{C}$ and $F = \mathbb{C}((t))$ and $\mathcal{O}_F = \mathbb{C}[[t]]$. Let G be a connected reductive group over k (not over F) and $T \subseteq G$ a fixed choice of a (split) maximal torus over k. Let $W = N_G(T)/Z_G(T)$ denote the Weyl group of G and $[W]$ the set of conjugacy classes. Tori over F in the base change G_F may not necessarily be split; we denote them by $\mathcal{T}$. Let $\mathfrak{g} = \mathrm{Lie}G$ and $\mathfrak{t} = \mathrm{Lie}T$.

The set $[W]$ is of independent interest and has been studied by combinatorial methods in [Car72] and [GP00, 3.2.12], the latter in the more when W is replaced with any finite Coxeter group. Recently, many interesting geometric object have come to be parameterized by $[W]$ and fundamental questions were asked on how to lift elements of W to $N_G(T)$, for example in [AH17, AHN20]. The set $[W]$ has even appeared in the original study of the twisted FKS isomorphism in [KP85]. In this subsection we study the relation between G_F-conjugacy classes of maximal tori in G_F and $[W]$. We first review the correspondence between conjugacy classes of maximal tori in G_F and $[W]$. We review the notion of an **elliptic** conjugacy class (of both Weyl group elements and Tori in G_F), and introduce the notion of the **principal Levi** subgroup of G associated to $w \in W$. Using the principal Levi and and the conjugacy classes of tori, we give a new and geometric, proof of the classification of conjugacy classes of W by parabolic induction in [GP00, 3.2.12] in the case when the Coxeter group is a Weyl group of a reductive group. Our study of $[W]$ and of lifting elements of W to $N_G(T)$ prepares for the study of choosing representatives of conjugacy classes of maximal tori in G_F in following subsections.

The following theorem was proved in [KL88, sec. 1].

Theorem 3.5: *The set $[W]$ is in natural bijection with conjugacy classes of maximal tori in G_F.*

Definition 3.3: *For a class $c \in [W]$ we say a maximal tori $\mathcal{T} \subseteq G_F$ is of **type** c if the conjugacy class of $\mathcal{T}$ corresponds to c.*

For illustration, we give another proof.

PROOF: For $w \in W$, a maximal torus of type $[w] \in [W]$ is described as follows. Let $\sigma \in N_G(T)$ be a finite order lift of order m. The fact that a finite order lift exists with minimal order $|w|$ or $2|w|$ is proven in [AH17, Theorem C], with an explicit description when which case of occurs. Let E/F be the degree m extension with $E = \mathbb{C}((u))$, $u^m = t$ and $\nu \in \Gamma := \mathrm{Gal}(E/F)$ a chosen generator and ζ a primitive mth root of unity such that $\nu(u)/u = \zeta$. Then a maximal torus $\mathcal{T}$ of type $[w]$ together with its embedding into G_F is defined as the composition

$$\mathcal{T} := (\mathrm{Res}_{E/F} T \times_k E)^{w \times \nu^{-1}}$$

$$\subseteq (\mathrm{Res}_{EF} G \times_k E)^{\sigma \times \nu^{-1}} \overset{f}{\cong} G_F$$

where f is some fixed chosen isomorphism. It exists due to a theorem of Steinberg [Ste65, 1.9] that $\mathrm{H}^1(R, H) = 0$ whenever R is a perfect field of cohomological dimension ≤ 1 and H is connected and affine over R so there are no inner forms of $H_{\overline{R}}$ over R.

A map from conjugacy classes of maximal tori in G_F to $[W]$ is given as in [SS70, 2.7] as follows. Let $\mathcal{T} \subseteq G_F$ be a maximal torus split over E with $\Gamma = \mathrm{Gal}(E/F), \nu$ as above acting on G_E. Since both T_E and $\mathcal{T}_E$ are split in G_E, they are conjugate by some $g \in G(E)$, i.e.,

$$g.T_E = \mathcal{T}_E.$$

Then the element

$$\sigma := g^{-1}\nu(g)$$

normalizes T_E and this lies in $N_{G_E}(T_E)$. Then the class $[w] \in [W]$ associated to the conjugacy class of $\mathcal{T}$ is the image of σ in

$$(N_{G_E}(T_E)/Z_{G_E}(T_E))(E)$$

$$\subseteq (N_G(T)/Z_G(T))(\overline{F}) = (N_G(T)/Z_G(T))(k)$$

where $\overline{F}$ is any algebraically closed field containing E. The bottom equality follows from the fact that the Weyl group is constant as a group scheme. The map is well-defined to be constant on the conjugacy class of $\mathcal{T}$ by [SS70, 2.7].

These two operations are inverse to each other. ∎

Remark 3.1: *The isomorphism*

$$(\mathrm{Res}_{EF} G \times_k E)^{\sigma \times \nu^{-1}} \overset{f}{\cong} G_F$$

in the proof above is not given explicitly, but the main point of our computations is to provide such an explicit isomorphism.

Now let us classify the conjugacy classes of $[W]$ by parabolic induction using the interpretation of $[W]$ as the set of conjugacy classes of maximal tori of G_F. Let $\Psi = \Psi(G, T)$ be the root system of G with respect to T and Δ be a chosen base of simple roots and B be the corresponding choice of Borel subgroup. Let $S \subseteq W$, $S = S(\Delta)$ be the set of simple reflections corresponding relative to the choice of Δ. So (W, S) forms a Coxeter system.

Definition 3.4: *For $J \subseteq S$, the **parabolic subgroup** of W corresponding to J is the subgroup $W_J \subseteq W$ generated by J.*

The reason for the terminology is that W_J is the Weyl group of the Levi factor of the parabolic subgroup $P_J \subseteq G$ corresponding the roots J, or to be precise the image of a Levi factor of a cover of P_J.

Definition 3.5: *An element $w \in W$ is called **elliptic** if either of the equivalent conditions hold:*

1. [GP00, 3.1.1] For every proper subset $J \subseteq S$, w does not lie in W_J.

2. For some (or for all) semisimple group G with maximal torus T such that $W(G, T) = W$, the group

$$T^w = \mathrm{Spec}\mathbb{C}[\mathbb{X}^*(T)_w]$$

is finite, or equivalently $\mathbb{X}^*(T)_w$ is finite, or equivalently $\mathfrak{t}^w = 0$, where $\mathfrak{t} = \mathrm{Lie}T$.

A conjugacy class $c \in [W]$ is called an ***elliptic class*** *if any (or equivalently if all) $w \in c$ is elliptic. A maximal torus, resp. conjugacy class of maximal tori, $\mathcal{T} \subseteq G_F$ is called an* ***elliptic torus*** *if the type of $\mathcal{T}$, resp. any torus in the class, in $[W]$ is an elliptic class.*

Lemma 3.2: *Suppose G is semisimple. Then a conjugacy class of W is elliptic iff the corresponding conjugacy class of tori in G_F by 3.5 consists of anisotropic tori.*

PROOF: A torus $\mathcal{T}$ over a field F has no split subtori over F iff $\mathbb{X}^*(\mathcal{T})$ has no quotient modules for the Galois group Γ that are positive rank over $\mathbb{Z}$. This happens iff $\mathbb{X}^*(\mathcal{T})_\Gamma$ is finite. Apply our case the case $F = \mathbb{C}((t))$ and Γ is topologically generated by one element ν acting on $\mathbb{X}^*(\mathcal{T})$ by w. This happens iff $T^w = \mathrm{Spec}(\mathbb{C}[\mathbb{X}^*(\mathcal{T}]_\Gamma)$ is finite. The result follows. ■

The notions of a parabolic subgroup of W and an elliptic conjugacy class are identical when (W, S) is generalized to with a finite Coxeter system. Elliptic classes of W are of special interest, because they allow for an inductive classification of $[W]$. Roughly speaking, every maximal torus is elliptic for a parabolic subgroup of W.

Theorem 3.6: *[GP00, 3.2.12] combined with [GP00, 3.2.11]: Suppose (W, S) is a finite Coxeter system. Let $\mathcal{P}$ be the set of pairs (J, d) where $J \subseteq S$ and d is an elliptic conjugacy class for the parabolic subgroup W_J. Let $\sim$ be the equivalence relation on $\mathcal{P}$ defined by the property that $(J, d) \sim (J', d')$ iff there exist $x \in W$ such that $x.J = J'$ and $x.d = d'$. Then the following map gives a well-defined bijection*

$$\mathcal{P}/\sim \xrightarrow{\approx} [W]$$

$$(J, d) \mapsto W.d$$

where $W.d$ denotes the orbit of the subset $d \subseteq W_J \subseteq W$ under the action of W.

Remark 3.2: *The actual map given in [GP00, 3.2.12] is in the other direction $[W] \to P/\sim$ and more complicated but combining with [GP00, 3.2.11], we obtain the inverse map given above in 3.6.*

Let us now give an alternative proof of 3.6 in the case when W is a Weyl group.

Definition 3.6: *For $w \in W$, the* ***principal Levi*** *is*

$$M_w := Z_G(T^{w,0}) \subseteq G$$

where $T^{w,0} \subseteq T^w$ is the neutral component, i.e the maximal torus, of T^w.

Example 3.3: *Suppose G is semisimple.*

1. *If $w = 1$, $M_w = \{e\}$ is the trivial subgroup of G.*
2. *If $w \in W$ is elliptic, then$M_w = G$.*

Some basic properties of the principal Levi are:

Remark 3.3: *Fix $w \in W = N_G(T)/Z_G(T)$. We have the following basic properties:*

1. *M_w contains all possible lifts of w to $N_G(T)$.*
2. *For every sub-torus $T' \subseteq T^{w,0}$, we have $M_w \subseteq Z_G(T')$.*
3. *M_w is minimal in the sense that if $T' \subseteq M$ is a torus and $T^{w,0} \subsetneq T'$ then $Z_G(T') \subsetneq M$ contains no lift of w.*
3. *T' is strictly contained in $T^{w,0}$, then $Z_G(T')$ does not contain any lift of w.*
4. *Conjugating w by W gives a $N_G(T)$ conjugate of M_w. Conversely conjugating M_w by $N_G(T)$ gives the principal Levi of a W-conjugate of w.*

Now interpret $[W]$ as the conjugacy classes of maximal tori in G_F as in 3.5. Define the map

$$\Psi : [W] \to \mathcal{P}$$

as follows. Let $\mathcal{T} \subseteq G_F$ be a maximal torus, $\mathcal{T}^a$ the maximal anisotropic subtorus and $\mathcal{T}^s$ the maximal split subtorus. Then the cocharacter lattice is the quotient by torsion

$$\mathbb{X}^*(\mathcal{T}^s) = \mathbb{X}^*(T)_{w,\mathrm{tor}} = \mathbb{X}^*(T^{w,0}).$$

It follows that $\mathcal{T}^s = (T^{w,0})_F$ and thus the centralizer $Z_{G_F}(\mathcal{T}^s) = M_{w,F} := (M_w)_F$ is *the base change of the principal Levi. Put*

$$M := M_w.$$

We furthermore have $\mathcal{T} \subseteq Z_{G_F}(\mathcal{T}^s)$ *and by 3.5 the conjugacy class of* $\mathcal{T}$ *in* M_F *is in bijection with a conjugacy class of the subgroup* $W(M_F, T_F) = W(M, T) \subseteq W$. *Let* $J \subseteq S$ *be any subset such that* M *is* G*-conjugate to the standard Levi subgroup* $M_J \subseteq G$. *Such a conjugation must send the maximal torus* $T \subseteq M$ *to the maximal torus* $T \subseteq M_J$ *and we conclude that* M *is* $N_G(T)$*-conjugate* M_J. *This induces a* W*-conjugation* $W(M, T)$ *to* W_J. *Define* d *to be the conjugacy class of* W_J *associated to the conjugacy class of* $W(M, T)$ *associated to* $\mathcal{T}$ *and put*

$$\Psi([\mathcal{T}]) = (J, d).$$

The following steps show that Ψ *induces a well-defined bijection* $[W] \approxeq \mathcal{P}/\sim$, *giving a proof of 3.6 in the case when* W *is a Weyl group.*

Lemma 3.3: d *is elliptic for* W_J.

PROOF: We have

$$\mathcal{T} = \mathcal{T}^a \mathcal{T}^s.$$

Since M_F is the centralizer of $\mathcal{T}^s$ in G_F, $\mathcal{T}^s = Z(M_F)^0$ is the connected component of the center. Therefore $(\mathcal{T}^a)_{\text{ad}} \subset (M_F)_{\text{ad}}$ is the maximal torus of $(M_F)_{\text{ad}} = (M_{\text{ad}})_F$. The map $M \to M_{\text{ad}}$ induces a natural bijection $W(M, T) \to W(M_{\text{ad}}, T_{\text{ad}})$ so $d \subseteq W_J$ is associated to a G-conjugate of the maximal torus $(\mathcal{T}^a)_{\text{ad}}$ in M_{ad}. Since $(\mathcal{T}^a)_{\text{ad}}$ is still anisotropic, the lemma boils down to showing the correspondence 3.5 maps a conjugacy class of anisotropic tori to an elliptic conjugacy class of the Weyl group. This is shown in 3.2. ∎

Lemma 3.4: Ψ *is well-defined as a map* $[W] \to \mathcal{P}/\sim$, *i.e., the equivalence class of* (J, d) *is independent of the choice of representative of* $[\mathcal{T}]$.

PROOF: Let $\mathcal{T}, \mathcal{T}'$ be conjugate tori in G_F by some $g \in G(F)$. Let J, resp. J', be subsets of S such that M_J, resp. $M_{J'}$, is conjugate to $Z_{G_F}(\mathcal{T}^s)$, resp. $Z_{G_F}(\mathcal{T}'^s)$. Then g restricts to a conjugation sending $\mathcal{T}^s$ to $\mathcal{T}'^s$, and therefore also sends $Z_{G_F}(\mathcal{T}^s)$ to $Z_{G_F}(\mathcal{T}'^s)$. It follows that $M_{J,F}$ is conjugate to $M_{J',F}$ by some $h \in G(F)$. By the main theorem [Sol20, 1.b], h can be chosen to normalize the split maximal torus T_F of G_F. Letting w be the image of h in $W = W(G_F, T_F) = W(G, T)$. Then $w.J$ is a choice of a set of simple reflections for $W_{J'}$. Hence there exists $w_2 \in W_{J'} \subseteq W$ such that

$$w_2 w, J = J'.$$

Let $d \in [W_J]$, resp. $d' \in [W_{J'}]$ correspond to the conjugacy class of the torus $\mathcal{T}$, resp. $\mathcal{T}'$, in $M_{J,F}$, resp. $M_{J',F}$. Since w restricts to an isomorphism $W_J \approxeq W_{J'}$, $w.d$ is a single conjugacy class in W_J'. Hence

$$w_2(w.d) = w.d.$$

Furthermore, $w.d = d'$ because h maps $[\mathcal{T}]$ to $[\mathcal{T}']$. It follows that

$$w_2 w.d = d'$$

and therefore (J, d) is equivalent to (J', d') by the element $w_2 w \in W$ under the definition of the equivalence relation $\sim$ on $\mathcal{P}$. The result follows. ∎

Lemma 3.5: Ψ *is bijective; in fact we explicitly describe an inverse.*

PROOF: The inverse map $\Phi : (\mathcal{P}/\sim) \to [W]$ is defined as follows. Let us define it on $\mathcal{P}$ and show it factors through $\sim$. Let $(J, d) \in \mathcal{P}$. Then d corresponds to a conjugacy class of maximal torus $\mathcal{T}$ in $M_{J,F}$ such that $\mathcal{T}_{\text{ad}} \subseteq M_{J,F,\text{ad}}$ is anisotropic by3.2. Define

$$\Phi((J, d)) = [\mathcal{T}]$$

to be the conjugacy class of W associated to the conjugacy class of $\mathcal{T}$, but considered as a maximal torus of G_F (it remains maximal in G_F because recall any Levi subgroup has the same rank as the original group). Φ factors through $\sim$ because if $(J, d) \sim (J', d')$, then by definition $d \subseteq W_J, d' \subseteq W_{J'}$ are subsets of the same W-conjugacy class in W and the $M_{J,F}$-conjugacy class of tori in $M_{J,F}$ associated to d is $G(F)$-conjugate to the associated $M_{J',F}$-conjugacy class of tori in $M_{J',F}$ associated to d'. The composition $[W] \xrightarrow{\Psi} (\mathcal{P}/\sim) \xrightarrow{\Phi} [W]$ is the identity by the construction of the maps. Let us show $(\mathcal{P}/\sim) \xrightarrow{\Phi} [W] \xrightarrow{\Psi} (\mathcal{P}/\sim)$ is the identity. Let $J \subseteq S$ and $d \in [W_J]$ be elliptic for W_J and corresponds to a class of anisotropic tori in M_J by 3.2. We can write

$$M_J = Z_G(H)$$

where the (split) torus $H \subseteq G$ is the neutral component of the center of M. Let $\mathcal{T} \subseteq (M_J)_F$ be any maximal torus. Then $H_F \subseteq \mathcal{T}^s$ and $\mathcal{T}_{\text{ad}}$ is a quotient of $\mathcal{T}$ by a diagonalizable group with neutral component containing H_F. Hence $\mathcal{T}_{\text{ad}}$ is generated by a quotient of $\mathcal{T}^a$ by a finite subgroup and a quotient of $\mathcal{T}^s$. Hence $\mathcal{T}_{\text{ad}}$ is anisotropic iff $\mathcal{T}^s$ maps to the trivial subgroup in $\mathcal{T}_{\text{ad}}$. This holds iff $H_F = \mathcal{T}^s$ and $M_J = Z_{G_F}(\mathcal{T}^s)$. This shows $(\mathcal{P}/\sim) \xrightarrow{\Phi} [W] \xrightarrow{\Psi} (\mathcal{P}/\sim)$ is the identity. The result follows. ■

3.2.3 Representatives of Conjugacy Classes of Maximal Tori in G_F with Large Closure in $G_{\mathcal{O}_F}$

In this subsection, we study the problem of choosing representatives $\mathcal{T}$ of G_F-conjugacy classes of maximal tori in G_F with the property the Zariski closure $\overline{\mathcal{T}}$ of $\mathcal{T}$ in $G_{\mathcal{O}_F}$ has large $\mathcal{O}_F$-points, in the sense that $\overline{\mathcal{T}}(\mathcal{O}_F)$ contains the maximal connected bounded subgroup of $\mathcal{T}(F)$. This means precisely that $\overline{\mathcal{T}}$ is an intermediate integral model of $\mathcal{T}$, i.e., it contains the connected Néron model. We define the notion of a **homogeneous** conjugacy class of W, which include all conjugacy classes of a parabolic subgroup of type A, that have the property that $\mathcal{T}$ can be chosen such that $\overline{\mathcal{T}}(\mathcal{O}_F)$ is even larger to be the maximal bounded subgroup of $\mathcal{T}(F)$, i.e., $\overline{\mathcal{T}} = \mathcal{T}^\flat$. Homogeneous conjugacy classes are import for later study because restriction to their case simplifies the geometric FKS isomorphism. It turns out that in order to study this problem, it is necessary to study the problem of lifting elements of W to $N_G(T)$, and we review some results about this from [AHN20] that we use.

For a triple (G, B, H) where G is a reductive group over k, H a choice of maximal torus, and B a Borel subgroup containing H, let us set the notation and recall some notions regarding the spherical building of (G, B, H), for example from [OV90, ch. 4]:

1. $\mathfrak{g} = \mathrm{Lie}G$, $\mathfrak{b} = \mathrm{Lie}B$, $\mathfrak{h} = \mathrm{Lie}H$.

2. The exponential map is denoted $\exp : \mathfrak{h} \to H$.

3. The adjoint action of G_{ad} on itself is denoted Ad, the action of G_{ad} on $\mathfrak{g}$ is denoted ad.

4. d is the rank of G_{ad}, and of the derived subgroup of G.

5. $\Psi = \Psi(G, H)$ the root system of G relative to H.

6. $\Delta = \Delta(G, B, H) \subseteq \Psi$ the choice of simple roots induced by B. For a fixed ordering of Δ, denote the elements $\Delta = \{\alpha_1, \cdots, \alpha_d\}$. We may write $\Psi = \Psi(G, B, H)$ to mean the roots system including the choice of Δ.

7. $\mathcal{A} = \mathcal{A}(G, B, H) \subseteq \mathfrak{h}_{\mathrm{ad}}$, is the **(closed)** fundamental alcove of the triple $(G_{\mathrm{ad}}, B_{\mathrm{ad}}, H_{\mathrm{ad}})$. Note we use $\mathfrak{h}_{\mathrm{ad}}$ instead of $\mathfrak{h}$. The general literature uses the interior of $\mathcal{A}$ formally defined as the intersection of the dominant halves $\alpha^{-1}(\mathbb{R}_+)$ of $\mathfrak{h}_{\mathrm{ad}}$ for all $\alpha \in \Psi$, but for our convention, we include the walls.

8. When G_{ad} is simple, let $\theta \in \Psi$ be the highest root.

9. Equip $\mathfrak{h}$ with a normalized invariant bilinear form $\langle -, - \rangle$ so that the length of the highest root for each simple factor of $\mathfrak{g}$ is 2.

10. $\omega_1^\vee, \cdots, \omega_d^\vee \subseteq \mathfrak{h}$ are the fundamental co-weights relative to Δ and $\langle -, - \rangle$. Identify the fundamental coweights with the vertices I of the Dynkin diagram of G_{ad}.

11. Let $P = \mathbb{Z}\langle \omega_1^\vee, \cdots, \omega_d^\vee \rangle$ be the co-weight lattice. For any point $v \in P$, there is a unique P-translate of v in $\mathcal{A}$ that lies inside I_{aff}. Call this element the **type** of v.

12. I_{aff} is the set of vertices of the affine Dynkin diagram of G. In particular, if G is simple, then $I_{\text{aff}} = I \cup \{\omega_0^\vee\}$ consists of the finite Dynkin diagram and one extra vertex $\omega_0^\vee \in \mathcal{A}$ denoted the **affine vertex**. For general reductive G, I_{aff}, resp, I, is a disjoint union of the vertices of the affine Dynkin diagram, resp. Dynkin diagram, of each simple component of G' or G_{ad}.

13. The set $\mathcal{A}$ has the structure of a simplex with vertices I_{aff}.

14. $\pi_1(G) = P/Q$ where Q is the coroot lattice, and in the natural map $I_{\text{aff}} \to \pi_1(G)$ is a bijection. It induces an action of $\pi_1(G)$ on I_{aff} by translation. A vertex $v \in I_{\text{aff}}$ is called **hyperspecial** if it lies in the orbit of some affine vertex under the action of $\pi_1(G)$. For example, if G is type A, every vertex is hyperspecial. In general, the set of hyperspecial vertices of I_{aff} is the union of the hyperspecial vertices of each connected component of I_{aff}.

15. $\mathfrak{g}_{\alpha_i}$ is the root space of $\mathfrak{g}$ corresponding to α_i.

The main tool we use, recalled below, is Kac's numerical classification conjugacy classes of finite order inner automorphisms of a simple Lie algebra, or equivalently conjugacy classes of finite order elements of G_{ad} when G is simple. Let $\sigma \in G_{\text{ad}}$ be an inner automorphism G of order m. Let E/F be the degree m extension with $E = k((u))$, $u^m = t$. Let $\nu \in \Gamma := \text{Gal}(E/F)$ be a choice of generator and ζ a primitive mth root of unity such that $\nu(u)/u = \zeta$. Put $\mathcal{O}_E = k[[u]]$ so Γ preserves $\mathcal{O}_E$.

Theorem 3.7: *[Kac90, 8.1] and [OV90, 4.7.8] Suppose G is simple of rank d. Let $H \subseteq G_{\text{ad}}$be a maximal torus containing σ, $\mathfrak{h} = \text{Lie}H$ and $B \subseteq G_{\text{ad}}$ a Borel subgroup such that $\sigma = \exp(\lambda)$ for some $\lambda \in \mathcal{A}(G_{\text{ad}}, B, H)$. Write*

$$\theta = a_1\alpha_1 + \cdots + a_d\alpha_d$$

where $a_i \geq 0$ and unique with respect to this property. Then we have the following:

*1. There exists a unique sequence of non-negative relatively prime integers $(s_0, \cdots, s_n)$, called the **Kac labels** of σ where each s_i labels $\omega_i^\vee \in I_{\mathrm{aff}}$, such that*

$$s_0 + s_1 a_1 + \cdots + s_d a_d = m$$

and

$$m\lambda = s_1\omega_1^\vee + \cdots + s_d\omega_d^\vee.$$

In particular, for all i, $X \in \mathfrak{g}_{\alpha_i}$, the action of σ on X is given by

$$\mathrm{ad}(\sigma).x = \zeta^s x.$$

2. Conjugacy classes of finite order elements of G_{ad} are classified by the sequences of integers $s_0, \cdots, s_d$ as above up to the permutation of $\pi_1(G_{\mathrm{ad}})$ induced by the action on I_{aff}.

We can use 3.7 to give the explicit isomorphism $(\mathrm{Res}_{EF} G \times_k E)^{\sigma\times\nu^{-1}} \approxeq G_F$ promised in 3.5 for a connected reductive G. Let us begin with the case when G is simple. Recall that for a maximal torus H of G, the exponential map induces a natural identification of center of G with $P/\mathbb{X}_*(T)$. So in particular if G is adjoint type, every coweight is a cocharacter.

For a k-algebra R, define $\mathfrak{g}_R = \mathfrak{g} \otimes_k R$.

Lemma 3.6: *Let G be simple and $\sigma = \exp(\lambda) \in G_{\mathrm{ad}}$ be an order m inner automorphism of the form described in 3.7 together with the choice of maximal torus H and Borel subgroup B. Define the loop $u^{m\lambda} \in H_{\mathrm{ad}}(E)$ by considering $m\lambda = s_1\omega_1^\vee + \cdots + s_n\omega_n^\vee$ as an element of $\mathbb{X}_*(T_{\mathrm{ad}}) = \mathrm{Hom}(\mathbb{G}_m, H_{\mathrm{ad}})$ and $u \in \mathbb{G}_m(\mathcal{O}_E)$ and defining*

$$u^{m\lambda} := (m\lambda)(u).$$

Then the morphism of group schemes over E given by $\mathrm{Ad}(u^{m\lambda}) : G_E \to G_E$ induces *an isomorphism*

$$G_F = \mathrm{Res}_{E/F}(G \times_k E)^{1\times\nu^{-1}} \overset{\mathrm{Ad}(u^{m\lambda})}{\to} (G \times_k E)^{\sigma\times\nu^{-1}}.$$

Proof: The first equality is from Galois descent. Both sides are connected because they are isomorphic a posteriori by Steinberg's theorem [Ste65, 1.9] that $\mathrm{H}^1(F, G) = 0$. Therefore it suffices to check $\mathrm{ad}(u^{m\lambda}) : \mathfrak{g}_E \to \mathfrak{g}_E$ restricts to a bijection of the respective Lie algebras. The remainder of the following computation is similar to [Kac90, 8.5]. Decompose

$$\mathfrak{g} = \oplus_{\alpha\in\Psi}\mathfrak{g}_\alpha$$

into roots spaces relative to $\mathfrak{h}$. By the description of the action of σ on $\mathfrak{g}_\alpha$ in 3.7, the eigenspace decomposition $\mathfrak{g} = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_{\bar{i}}$ for $\mathrm{ad}(\sigma)$ is compatible with the root space decomposition in the sense that

$$\mathfrak{g}_{\bar{i}} = \oplus_{\alpha\in\Psi:\overline{\langle m\lambda,\alpha\rangle}=\bar{i}}\mathfrak{g}_\alpha.$$

Decompose $E = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}u^iF$ as an F-vector space spanned by $1, u, u^2, \cdots$. By inspection of the definition of the root subgroups of G_E, the action of $\mathrm{ad}(u^{m\lambda})$ on $\mathfrak{g}_E = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g} \otimes u^{\bar{i}}F$ restricts to an isomorphism

$$\mathfrak{g}_\alpha \otimes u^{\bar{i}} \overset{\cong}{\to} \mathfrak{g}_\alpha \otimes u^{\overline{i+\langle m\lambda,\alpha\rangle}}$$

$$X \otimes u^{\bar{i}} \mapsto X \otimes u^{\overline{i+\langle m\lambda,\alpha\rangle}}.$$

Since $\nu^{-1}(u)/u = \zeta^{-1}$,

$$\mathrm{Lie}_F(\mathrm{Res}_{E/F}(G \times_k E)^{1\times\nu^{-1}}) = (\mathfrak{g} \otimes_k E)^{\mathrm{ad}(\sigma\otimes\nu^{-1})} = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_{\bar{i}} \otimes u^{\bar{i}}F.$$

Putting the above two together, it follows that $\mathrm{ad}(u^{m\lambda})$ restricts to an isomorphism

$$\mathrm{Lie}_F G_F = \mathfrak{g} \otimes F \overset{\mathrm{ad}(u^{m\lambda})}{\to} \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_{\bar{i}} \otimes u^{\bar{i}}F = \mathrm{Lie}_F(\mathrm{Res}_{E/F}(G \times_k E)^{1\times\nu^{-1}}).$$

The result follows. ■

We extend the above lemma to the case when G is connected reductive below. Remark that if $G' = \prod_i G_i$ is a decomposition of the derived subgroup into simple factors, there is an induced decomposition of the adjoint group $G_{\mathrm{ad}} = \prod G_{i,\mathrm{ad}}$ as the product of the adjoint groups of each simple factor.

Lemma 3.7: *Let G be connected reductive and $G' = \prod_i G_i$ be a decomposition of the derived subgroup into simple factors. Let $\sigma \in G_{\mathrm{ad}}$ with $\sigma = \prod \sigma_i$ and $\sigma_i \in G_{i,\mathrm{ad}}$ acting only on G_i. Suppose for each i, $\sigma_i = \exp(\lambda_i)$ for $\lambda_i \in \mathfrak{h}_i$ in the from described in 3.7 together with the choice of a maximal torus H_i and Borel subgroup B_i containing H_i in $G_{i,\mathrm{ad}}$. Then the loop $x \coloneqq \prod u^{m\lambda_i} \in \prod H_i(E)$ where each $u^{m\lambda_i}$ is as defined in 3.6 as the property that $\mathrm{Ad}(x) : G_E \to G_E$ induces an isomorphism*

$$G_F = (G \times_k E)^{1 \times \nu^{-1}} \approxeq (G \times_k E)^{\sigma \times \nu^{-1}}.$$

Proof: Both sides are connected because $\mathrm{H}^1(F, G) = 0$, so it suffices to show that the adjoint action on the Lie algebras over F is an isomorphism. Let $H = Z(G)^0$ be the connected component of the center. Then we have $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus \mathfrak{g}_i$and thus

$$\mathfrak{g}_E = \mathfrak{h}_E \bigoplus \mathfrak{g}_{i,E}$$

where $\mathrm{ad}(x)$ acts by $1 \otimes \nu^{-1}$ on $\mathfrak{h}_E$ and by $\mathrm{ad}(u^{m\lambda_i})$ separately on each factor $\mathfrak{g}_{i,E}$. We thus have

$$\mathfrak{g}_E^{\mathrm{ad}(\sigma \times \nu^{-1})} = \mathfrak{h}_F \oplus \bigoplus \mathfrak{g}_{i,E}^{\mathrm{ad}(\sigma \times \nu^{-1})}.$$

Applying 3.6 to each simple factor $\mathfrak{g}_i$ gives that $\mathrm{ad}(x)$ induces an isomorphism $\mathfrak{g}_{i,F} \approxeq \mathfrak{g}_{i,E}^{\mathrm{ad}(\sigma \times \nu^{-1})}$ separately on each factor. The result follows. ■

Our study uses some results on lifting Weyl group elements and their relation to Kac labels. Namely given $w \in W$, a choice of a finite order lift $\sigma \in N_G(T)$ gives rise to Kac labels determining the conjugacy class of $\sigma_{\mathrm{ad}} \in G_{\mathrm{ad}}$ by 3.7. This operation is independent of conjugacy class of w for an elliptic w, up to $\pi_1(G_{\mathrm{ad}})$:

Theorem 3.8: *[AHN20, 1.1.3 and 6.8] Suppose G is semisimple and $w \in W = W(G,T)$ is elliptic. Then:*

1. Any two lifts of w to $N_{G_{\mathrm{ad}}}(T_{\mathrm{ad}})$ are conjugate in G_{ad}. Consequently there is a well-defined association from elliptic conjugacy classes of W with Kac labels of I_{aff} given as in 3.7 by taking the Kac co-ordinates of each simple factor of the conjugacy class of a lift.

2. If a given lift $\sigma \in G_{\mathrm{ad}}$ of w is of the form $\exp(m\lambda)$ for $m\lambda \in \mathcal{A}(G_{\mathrm{ad}}, B, H)$ for maximal torus $H \subseteq G_{\mathrm{ad}}$ (not necessarily equal to T) and a choice of Borel B containing H as in 3.7, then $m\lambda \in P_{\mathrm{ad}}$, as an element of the coweight lattice of G_{ad} relative to H_{ad}, is invariant under the action of $\pi_1(G_{\mathrm{ad}})$. Therefore there is a well-defined Kac labeling of $[w]$.

This allows us to define a special kind of conjugacy class of W that is an important special case of later study:

Definition 3.7: *Suppose G is semisimple, $T \subseteq G$ a maximal torus, and $W = W(G,T)$.*

*1. A finite order element $\sigma \in G_{\mathrm{ad}}$ is **homogeneous** if the Kac labels of σ from 3.7 have value $\neq 0$ on every hyperspecial vertex of I_{aff}. An elliptic element $w \in W$ is defined to be homogeneous iff any lift is. An elliptic class if homogeneous iff any lift is. This is well-defined by 3.8.*

*2. Suppose $c \in [W]$ is not necessarily elliptic. By the classification 3.6 of $[W]$ by parabolic induction, there exists a choice of simple reflections S and $J \subseteq S$ such that $w \in W_J$ is elliptic as an element of W_J. Define the **Kac labels** of c to be the Kac labels on the affine Dynkin diagram $I_{J,\mathrm{aff}}$ of W_J (not of I_{aff}) of c as an elliptic class of $[W_J]$. Such an operation is also well-defined by [AHN20, 8.3].*

*3. A (not necessarily elliptic) element $w \in W$ **homogeneous** if for any choice of simple reflections $S \subseteq W$ and $J \subseteq S$ such that $w \in W_J$ is elliptic for W_J, w is homogeneous as an element of W_J. A class $c \in [W]$ is homogeneous if any lift is.*

Example 3.4: *Let G be simple and $(s_0, \cdots, s_d)$ be the Kac labels of $\sigma \in G_{\mathrm{ad}}$ as in 3.7. Recall s_0 is the label of the affine vertex, so if σ is homogeneous, then $s_0 \neq 0$.*

The converse is usually not true as there are usually more hyperspecial vertices than the affine vertex.

The principal Levi, or rather its adjoint group, plays a role in visualizing the Kac labels of a not necessarily elliptic class $c \in W$, consistent with the above definition. It is as follows. Let G be semisimple with maximal torus T and $W = W(G,T)$. Let $c \in [W]$ and $w \in c$. Let $M := M_w = Z_G((T^{w,0}))$ be the principal Levi, which contains all lifts of w. Let $\sigma \in M$ be an arbitrary lift. Then $\sigma_{\text{ad}} \in M_{\text{ad}}$ is elliptic for its action on T_{ad} in the sense that the fixed points are finite since $T^{w,0} \subseteq Z(M)$. Then the Kac labels of c is the Kac label of σ_{ad} in M_{ad} and c is homogeneous iff σ_{ad} is.

Some numerical examples are collected below :

1. For any Weyl group W, the **Coxeter class** cox $\in [W]$, defined as the conjugacy class consisting of the products of any chosen set of simple reflections in any order, is elliptic. The Kac labels of cox are $s_i = 1$ for any vertex of I_{aff}, as shown in computations of the various cases for classical groups in [RLYG12] and for all of the exceptional groups in [AHN20, sec. 9]. Hence cox is homogeneous.

2. When W is type A, cox is the only elliptic class, e.g., [AHN20, sec. 6]. Since every element of W is elliptic for some parabolic subgroup and every parabolic subgroup of a type A group is also type A, if $w \in W$ is contained in any parabolic subgroup of type A, then w is automatically homogeneous.

3. There exists many non-Coxeter homogeneous classes, even if we require furthermore them to be elliptic. But the homogeneous conjugacy classes occupy a significant portion of all conjugacy classes. For example, according to the computer computations of Kac co-ordinates for the exceptional groups in [AHN20, sec. 9], as tabulated in the table below:

Remark 3.4: *An interesting coincidence is the following: the number of elliptic homogeneous conjugacy classes in type E agree with the number of 'primitive' classes in the study of the twisted FKS isomorphism in [KP85, sec. 10]. Primitive classes are*

Group	G_2	F_4	E_6	E_7	E_8
Homogeneous Elliptic classes	2	5	3	5	9
Elliptic Classes	3	9	5	12	30

Table 3.1: Number of Homogeneous Elliptic Conjugacy Classes in the Exceptional Types

of those w such that $\det(1-w) = \det A$ *where A is the Cartan matrix of W and* det *is taken for the action of w on $\mathfrak{h}_{\text{ad}}$. The author wonders whether primitive conjugacy classes are the same as elliptic homogeneous classes, but has not investigated this question further.*

Let us prepare with some discussion of graded structures on Lie algebras. For Lie algebra $\mathfrak{l}$ and a set I, an **I-graded** structure on $\mathfrak{l}$ is a vector space decomposition $\mathfrak{l} = \oplus_{i\in I}\mathfrak{l}_i$ where $\mathfrak{l}_i \subseteq \mathfrak{l}$ are subspaces. A **graded subalgebra** $\mathfrak{m} \subseteq \mathfrak{l}$ is a subalgebra of the form $\mathfrak{m} = \oplus_{i\in I}\mathfrak{m}_i$ where each $\mathfrak{m}_i \subseteq \mathfrak{l}_i$ is a subspace.

Definition 3.8: *For a $\mathbb{Z}$-grading on $\mathfrak{l}$ and $i_0 \in \mathbb{Z}$, set*

$$\mathfrak{l}_{>i_0} := \oplus_{i>i_0}\mathfrak{l}_i$$

$$\mathfrak{l}_{<i_0} := \oplus_{i<i_0}\mathfrak{l}_i$$

and similarly so for $\mathfrak{l}_{\geq i_0}$, resp $\mathfrak{l}_{\leq i_0}$, where the $>$, resp. $<$, is replaced with $\geq$, resp. $\leq$, in the above sum.

Remark 3.5: *Usually I is required to have a commutative sum structure and the grading of $\mathfrak{l}$ is required to be compatible with the sum on I in the sense that $[\mathfrak{l}_i, \mathfrak{l}_j] \subseteq \mathfrak{l}_{i+j}$. But we do not impose it on I here.*

For a Lie algebra $\mathfrak{p}$ over k and a power series field $M = k((s))$, it is not convenient to put a graded structure on the (formal) loop algebra $\mathfrak{p}_M = \mathfrak{g} \otimes M$ according to powers to s because the inclusion $\bigoplus_i \mathfrak{p} \otimes s^i \subseteq \mathfrak{p}_M = \mathfrak{p} \otimes k((s))$ is strict. But the subalgebra $\bigoplus_i \mathfrak{p} \otimes s^i$, called the **polynomial loop algebra**, is dense in $\mathfrak{p}_E$, and we can directly define the grading on it. Indeed, polynomial loop algebras are a replacement for formal loop algebras in the representation theory literature [Lep85, KP85, BK04].

Definition 3.9: *For an indeterminate s over k, denote the* ***polynomial loop algebra*** *by $\mathfrak{l}_{k[s^{\pm 1}]}$ with $\mathbb{Z}$-grading defined by*

$$\mathfrak{l}_{k[s^{\pm 1}],i} = \mathfrak{l} \otimes s^i$$

and Lie bracket induced by the bracket in $\mathfrak{l}_{k((s))}$, i.e., $[X \otimes f, Y \otimes g] = [X, Y] \otimes fg$.

The previous computations in 3.7 on formal loop algebras also hold for polynomial loop algebras as below. Suppose G, σ is as in 3.7.

1. The Galois group Γ preserves $k[u^{\pm 1}] \subseteq E$. The fixed point algebra $\mathfrak{g}_{k[u^{\pm 1}]}^{\mathrm{ad}(\sigma \times \nu^{-1})}$ is $\bigoplus_i \mathfrak{g}_i \otimes u^i$.

2. For any $\mathbb{Z}$-graded subalgebra $\mathfrak{p} \subseteq \mathfrak{g}_{k[t^{\pm 1}]}$, the image of $\mathfrak{p}$ under the map $\mathrm{ad}(u^{m\lambda})$ is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{g}_E$. But $\mathrm{ad}(u^{m\lambda})$ does **not** preserve the grading, for example the image of the zero-degree part $\mathfrak{g} \otimes t^0$ can have components in many different degrees.

3. The subalgebra $\mathfrak{g}_{k[t]} \subseteq \mathfrak{g}_{k[u^{\pm 1}]}$ has closure $\mathfrak{g}_{\mathcal{O}_F}$ in $\mathfrak{g}_F$ where $\mathfrak{g}_{\mathcal{O}_F} = \mathrm{Lie} L^+ G$ is the Lie algebra positive loop group of the parahoric group scheme $G_{\mathcal{O}_F}$associated to the hyperspecial parahoric subgroup $G(\mathcal{O}_F) \subseteq G(F)$. We will use the subalgebra $\mathfrak{g}_{k[t]}$ to compute the closure of maximal tori $\mathcal{T} \subseteq G_F$ in $G_{\mathcal{O}_F}$.

Definition 3.10: *Let $H \subseteq G$ be a maximal torus and $B \subseteq G$ a Borel subgroup, with Lie algebras $\mathfrak{h}$ and $\mathfrak{b}$, respectively. The* ***(polynomial) standard Iwahori subalgebra*** *of $\mathfrak{g}_{k[t^{\pm 1}]}$is the $\mathbb{Z}$ graded subalgebra*

$$I(\mathfrak{g}_{k[u^{\pm 1}]}) := \mathfrak{b} \otimes t^0 \oplus \bigoplus_{i>0} \mathfrak{g} \otimes t^i$$

with grading defined by

$$I(\mathfrak{g}_{k[u^{\pm 1}]})_i = 0 \text{ for } i < 0$$

$$I(\mathfrak{g}_{k[u^{\pm 1}]})_0 = \mathfrak{b} \otimes t^0$$

$$I(\mathfrak{g}_{k[u^{\pm 1}]})_i = \mathfrak{g} \otimes t^i \text{ for } i > 0.$$

Remark 3.6: *The closure of $I(\mathfrak{g}_{R_F})$ in $\mathfrak{g}_F$ is $\mathrm{Lie}L^+\mathcal{I}$ where $\mathcal{I}$ is the smooth integral model of G_F corresponding to the standard Iwahori subgroup of $G(F)$. We have that*

$$\mathrm{Lie}L^+\mathcal{I} = \mathfrak{b} \otimes t^0 \oplus \prod_{i>0} \mathfrak{g} \otimes t^i$$

is the preimage of $\mathfrak{b}$ under the reduction map $\mathfrak{g}_{\mathcal{O}_F} \to \mathfrak{g}$ given by $t \mapsto 0$,since the corresponding statement at the level of groups is true for $L^+\mathcal{I}$ at the level of groups [Zhu17, ex 1.2.9]. Therefore

$$I(\mathfrak{g}_{R_F}) = \mathrm{Lie}L^+\mathcal{I} \cap \mathfrak{g}_{k[u^{\pm 1}]}$$

and it $I(\mathfrak{g}_{k[u^{\pm 1}]})$ is also the preimage of $\mathfrak{b}$ under the reduction map $\mathfrak{g}_{k[u]} \to \mathfrak{g}$.

Suppose G is simple. Suppose $H \subseteq G$ is a maximal torus and $B \subseteq G$ is a Borel subgroup containing G. Let

$$\mathfrak{g} = \mathfrak{g}^- \oplus \mathfrak{h} \oplus \mathfrak{g}^+$$

be the induced Cartan decomposition, where $\mathfrak{h} = \mathrm{Lie}H$ and $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{g}^+ = \mathrm{Lie}B$. So $\mathfrak{g}^+$,resp. $\mathfrak{g}^-$, is the sum of the root spaces of the positive roots, resp. negative roots and $\mathfrak{h} = \mathfrak{g}_{\mathbf{0}}$ is the weight space for the zero vector $\mathbf{0} \in \mathfrak{h}^\vee$. Let $\Psi = \Psi(G, H)$ be the root system of G relative to H.

Definition 3.11: *Suppose G is simple. Let σ be a finite order inner automorphism of G of the form $\exp(\lambda)$ of 3.7 together with maximal torus $H \subseteq G_{\mathrm{ad}}$ and Borel subgroup B containing H. The* ***refined eigenspace grading*** *on $\mathfrak{g}$ to be the finitely supported $\mathbb{Z}$-grading defined by*

$$\mathfrak{g}_i = \oplus_{\alpha \in \Psi \cup \{\mathbf{0}\} : \langle \alpha, m\lambda \rangle = i} \mathfrak{g}_\alpha.$$

It refines the ***eigenspace grading*** *of $\mathfrak{g} = \oplus_{\bar{i} \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{g}_{\bar{i}}$ in the sense that $\mathfrak{g}_i \subseteq \mathfrak{g}_{\bar{i}}$.*

The subalgebras $\mathfrak{g}^-$ and $\mathfrak{g}^+$ are graded subalgebra for both gradings. Furthermore, since $m\lambda$ is dominant, i.e., a positive combination of fundamental coweights, the refined eigenspace grading on $\mathfrak{g}^+$, resp. $\mathfrak{g}^-$, is supported in positive, resp. negative degrees. We have the following about the relationship between the eigenspace grading on $\mathfrak{g}, \mathfrak{g}^+$ and $\mathfrak{g}^-$:

$$\mathfrak{g}_{<0} = \mathfrak{g}^-_{<0}$$

$$\mathfrak{g}_0 = \mathfrak{g}^-_0 \oplus \mathfrak{h} \oplus \mathfrak{g}^+_0$$

$$\mathfrak{g}_{>0} = \mathfrak{g}^+_{>0}.$$

The motivation for the notion of a homogeneous conjugacy class comes from the following more detailed relationship between the eigenspace grading and refined eigenspace grading, which depends on the value of Kac label s_0 of the affine vertex.

Lemma 3.8: *Suppose G is simple with Lie algebra $\mathfrak{g}$ and $\sigma \in G_{\mathrm{ad}}, \lambda \in \mathcal{A}$ be of the form in 3.7 together with the choice of maximal torus $H \subseteq G_{\mathrm{ad}}$ containing σ and Borel subgroup B containing H. Then*

$$\mathfrak{g}_{\bar{i}} = \mathfrak{g}^-_{i-m} \oplus \mathfrak{g}^+_i \textit{ for } \bar{i} \neq 0$$

$$\mathfrak{g}_{\bar{0}} = \mathfrak{g}^-_{-m} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^+_m = \mathfrak{g}^-_{-m} \oplus \mathfrak{g}^-_0 \oplus \mathfrak{h} \oplus \mathfrak{g}^+_0 \oplus \mathfrak{g}^+_m$$

where $\mathfrak{g}^-_{-m}$ and $\mathfrak{g}^+_m$ are both nontrivial iff $s_0 = 0$.

Proof: Let $\Psi = \Psi^+ \sqcup \Psi^-$ be the induced decomposition of the roots of $(\mathfrak{g}, \mathfrak{h})$ into the positive roots Ψ^+ and negative roots Ψ^-. Let $\alpha_1, \cdots, \alpha_d$ be the set of simple roots and θ be the highest root. Since $m\lambda$ is dominant, for every $\alpha \in \Psi$, $\langle m\lambda, \alpha\rangle \geq 0$ iff $\alpha \in \Psi^+$ and $\langle m\lambda, \alpha\rangle \leq 0$ iff $\alpha \in \Psi^-$ and $\langle m\lambda, \alpha\rangle$ attains its maximum, resp. minimum, value on θ, resp. $-\theta$. By the relations

$$m\lambda = s_1\omega_1^\vee + \cdots + s_d\omega_d^\vee$$

$$\theta = a_1\alpha_1 + \cdots + s_d\alpha_d$$

$$s_0 + s_1a_1 + \cdots s_da_d = m \text{ where } s_i \geq 0$$

we have

$$\langle m\lambda, \theta\rangle = s_1a_1 + \cdots + s_da_d \leq m$$

with equality holding iff the Kac co-ordinate s_0 of σ at the affine vertex equals 0. This occurs iff σ is homogeneous. Therefore for all $\alpha \in \Psi$,

$$-m \leq \langle m\lambda, \alpha\rangle \leq m$$

$$\langle m\lambda, \Psi^+\rangle \geq 0$$

$$\langle m\lambda, \Psi^-\rangle \leq 0.$$

The result follows. ∎

The following computes the image of the standard Iwahori subalgebra of $\mathfrak{g}_{k[t^{\pm 1}]}$ and the subalgebra $\mathfrak{g}_{k[t]}$ under the automorphisms $u^{m\lambda}$ of $\mathfrak{g}_E$ the form described in 3.7.

Theorem 3.9: *Suppose G is simple with Lie algebra $\mathfrak{g}$ and $\sigma \in G_{\mathrm{ad}}, \lambda \in \mathcal{A}$ be of the form in 3.7 together with the choice of maximal torus $H \subseteq G_{\mathrm{ad}}$ containing σ and Borel subgroup B containing H. Let $\mathfrak{g} = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{g}_{\bar{i}}$ be the eigenspace grading for σ and $\mathfrak{g} = \oplus_{i\in\mathbb{Z}}\mathfrak{g}_i$ be the refined eigenspace grading for σ. We have:*

1. The subalgebra $I' := \mathrm{ad}(u^{m\lambda})(I(\mathfrak{g}_F))$ of $\mathfrak{g}_E^{\mathrm{ad}(\sigma\times\nu^{-1})}$ with $\mathbb{Z}$-grading induced by the grading on $\mathfrak{g}_E$ has graded components described as follows:

$$I'_i = 0 \textit{ for } i < 0$$

$$I'_0 = (\mathfrak{h} \oplus \mathfrak{g}_0^+ \oplus \mathfrak{g}_{-m}^-) \otimes u^0 \subseteq \mathfrak{g}_{\bar{0}} \otimes u^0$$

$$I'_i = \mathfrak{g}_{\bar{i}} \otimes u^i \textit{ for } i > 0.$$

2. The image $\mathfrak{g}'_{k[t]} := \mathrm{ad}(u^{m\lambda})(\mathfrak{g}_{k[t]})$ of $\mathfrak{g}_E^{\mathrm{ad}(\sigma\times\nu^{-1})}$ with $\mathbb{Z}$-grading induced by the *grading on $\mathfrak{g}_E$ is described as follows:*

$$\mathfrak{g}'_{k[t],i} = \mathfrak{g}_i^- \otimes u^i \textit{ for } i < 0$$

$$\mathfrak{g}'_{k[t],0} = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-m}^-) \otimes u^0$$

$$\mathfrak{g}'_{k[t],i} = \mathfrak{g}_{\bar{i}} \otimes u^i \textit{ for } i > 0.$$

PROOF: Apply the computation of the map $\mathrm{ad}(u^{m\lambda})$ in the proof of 3.6 and the relationship between the eigenspace grading and refined eigenspace grading 3.8. We obtain that for each i, $\mathrm{ad}(u^{m\lambda})(\mathfrak{g} \otimes u^i)$ is supported in degrees $i-m, \cdots, i+m$ and

$$\mathrm{ad}(u^{m\lambda})(\mathfrak{h} \otimes u^i) = \mathfrak{h} \otimes u^i$$

$$\mathrm{ad}(u^{m\lambda})(\mathfrak{g}^+ \otimes u^i) = \bigoplus_{j=0}^{m} \mathfrak{g}_j^+ \otimes u^{i+j}$$

$$\mathrm{ad}(u^{m\lambda})(\mathfrak{g}^- \otimes u^i) = \bigoplus_{j=0}^{m} \mathfrak{g}_j^- \otimes u^{i-j}$$

Applying these observations allows us to compute I' and $\mathfrak{g}'_{\mathcal{O}_F}$ directly as follows.

1. Since $\mathrm{ad}(u^{m\lambda})(I(\mathfrak{g}_F)_0) = \mathrm{ad}(u^{m\lambda})(\mathfrak{g}_{\geq 0} \otimes u^0)$ is supported in degrees $0, \cdots, m$ and for $i > 0$, $\mathrm{ad}(u^{m\lambda})(I(\mathfrak{g}_F)_{im})$ is supported in degrees $i \geq 0$, we have

$$I'_i = 0 \text{ for } i < 0.$$

The degree 0 part of I' must lie in the image of the degrees $0, m$ part of $I(\mathfrak{g}_F)$ since for $i \geq 2$, the support of $\mathrm{ad}(u^{m\lambda})(I(\mathfrak{g}_F)_{im})$ lies in degrees ≥ 1. We find

$$I'_0 = \mathrm{ad}(u^{m\lambda})(I(\mathfrak{g}_F)_0 \otimes u^0 \oplus \mathfrak{g}_{-m}^- \otimes u^m)$$

$$= (\mathfrak{h} \oplus \mathfrak{g}_0^+ \oplus \mathfrak{g}_{-m}^-) \otimes u^0 \subseteq \mathfrak{g}_{\bar{0}} \otimes u^0$$

as desired. By a similar reasoning, for $i \geq 1$, the degree im part of I' lies in the image of the degrees $(i-1)m, im, (i+1)m$ part of $I(\mathfrak{g}_F)$. Hence for such i,

$$I'_{im} = \mathrm{ad}(u^{m\lambda})(\mathfrak{g}_m^+ \otimes u^{(i-1)m} \oplus \mathfrak{g}_0 \otimes u^{im} \oplus \mathfrak{g}_{-m}^- \otimes u^{(i+1)m})$$

$$= \mathfrak{g}_{\bar{0}} \otimes u^{im}.$$

Now let $j = 1, \cdots, m-1$ and $i \geq 0$. The degree $im+j$ part of I' lies in the image of the degree $im, (i+1)m$ part of $I(\mathfrak{g}_F)$. For such i, j

$$I'_{im+j} = \mathrm{ad}(u^{m\lambda})(\mathfrak{g}_j^+ \otimes u^{im} \oplus \mathfrak{g}_{j-m}^- \otimes u^{(i+1)m})$$

$$= \mathfrak{g}_{\bar{j}} \otimes u^{im+j}.$$

This completes the description for all the graded pieces of I'.

2. The computation for $\mathfrak{g}'_{k[t]}$ is similar. For $i < 0$, $\mathfrak{g}'_{k[t],i}$ must lie in the image under $\mathrm{ad}(u^{m\lambda})$ of $\mathfrak{g}_{k[t],0}$, which gives us for such i,

$$\mathfrak{g}'_{k[t],i} = \mathfrak{g}_i^- \otimes u^i$$

as desired. Next, $\mathfrak{g}'_{k[t],0}$ must lie in the image of the degree $0, m$ part of $\mathfrak{g}_{k[t]}$, which gives us

$$\mathfrak{g}_{k[t],0} = \mathrm{ad}(u^{m\lambda})(\mathfrak{g}_0 \otimes u^0 \oplus \mathfrak{g}_{-m}^- \otimes u^{-1}) = (\mathfrak{g}_0 \oplus \mathfrak{g}_{-m}^-) \otimes u^0$$

as desired. Finally, the case for $\mathfrak{g}'_{k[t],i}$ for $i > 0$ is identical to that of I'_i because for such i, $\mathfrak{g}_{k[t],i} = I(\mathfrak{g}_F)_i$. The result follows. ∎

We now state the main result of this section.

Theorem 3.10: *Suppose G is simply connected (not necessarily simple). Let $c \in [W]$ be a conjugacy class, $w \in c$ a representative, and $M = Z_G(T^{w,0})$ the principal Levi.*

Then for each order m lift σ of w in M_{ad}, there exists a torus $\mathcal{T}$ of type c in G_F, and a choice of maximal torus $H \subseteq M$ (different from T) and Borel subgroup $B \subseteq M$ containing H, such that

1. For any standard Parahoric group scheme $\mathcal{P}$ of M_F, the Zariski closure of $\mathcal{T}$ in $\mathcal{P}$ is an intermediate integral model $\mathcal{T}^{\#}$ containing the connected Néron model $\mathcal{T}^{\flat,0}$.

2. Furthermore, there exists a standard parahoric group scheme $\mathcal{P}(\sigma, M)$ of M_F such that the closure of $\mathcal{T}$ in $\mathcal{P}(\sigma, M)$ is the full Néron model.

3. If furthermore c is homogeneous, then σ can be chosen so that $\mathcal{P}(\sigma, M) = M_{\mathcal{O}_F} \subseteq G_{\mathcal{O}_F}$ is the canonical standard hyperspecial parahoric group scheme of M_F.

Definition 3.12: *The maximal torus $\mathcal{T}$ associated to the lift σ of $w \in W$ is called a **principal maximal torus of type** $[w]$ (it is not unique with respect to $[w]$ or even w, but it is uniquely defined by σ) and the parahoric group scheme $\mathcal{P}(\sigma, M)$ is called the **principal** parahoric associated to σ, or a principal parahoric associated to $[w]$ or w. Put $P(\sigma, M) = \mathcal{P}(\sigma, M)$ and call it the **principal parahoric subgroup.***

The proof occupies the remainder of this subsection. We first quickly define the principal maximal torus and the principal parahoric and then show that they satisfy the required properties.

For the remainder of this section, suppose G is simply connected, let $c \in W$, $w \in c$, $M = M_w$ be the principal Levi, and $\sigma \in M_{\text{ad}}$ be a finite order lift of order m. Let $H \subseteq G_{\text{ad}}$ be a maximal torus and $B \subseteq G_{\text{ad}}$ be a Borel subgroup containing H such that $\sigma \in H$ is $\exp\lambda$ for some $\lambda \in \mathcal{A}(G_{\text{ad}}, B, H)$, the fundamental alcove. Let E/F be the degree m extension with uniformizer $u \in E$ with $u^m = t$ and $\nu \in \Gamma := \text{Gal}(E/F)$ a chosen generator and ζ a primitive mth root of unity such that $\nu(u)/u = \zeta$.

A principal maximal torus $\mathcal{T}$ of type c together with its embedding in M_F is defined to be

$$\mathcal{T} := \text{Res}_{E/F}(T \times_k E)^{w \times \nu^{-1}}$$

$$\hookrightarrow \text{Res}_{E/F}(M \times_k E)^{\sigma \times \nu^{-1}} \overset{\text{Ad}(u^{-m\lambda})}{\to} M_F$$

where the element $u^{m\lambda} \in M_E$ is as in 3.7, where it was shown that $\text{Ad}(u^{-m\lambda})$ is an isomorphism. This is identical to the step of the proof 3.5 mapping a conjugacy class of W to conjugacy classes of maximal tori in a reductive group, except now the map $\text{Ad}(u^{-m\lambda})$ is given explicitly.

The principal parahoric group scheme is defined as follows. According to [Con14, 6.5.2.IV], the derived group of a Levi subgroup of a complex simply connected group is simply connected. Therefore M has simply connected derived subgroup. Therefore by 3.1 $\mathcal{P} := \mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}(M \times \mathcal{O}_E)^{\sigma\times\nu^{-1}}$ has connected $\mathcal{O}_F$-points and is a parahoric group scheme of $\mathrm{Res}_{E/F}(M \times_k E)^{\sigma\times\nu^{-1}}$. Since $\mathrm{Ad}(u^{-m\lambda})$ is an isomorphism, $\mathrm{Ad}(u^{-m\lambda})(\mathcal{P}(\mathcal{O}_E)) \subseteq M(F)$ is a parahoric subgroup of $M(F)$. Define the principal parahoric group scheme $\mathcal{P}(\sigma, M)$ to be the parahoric group scheme corresponding to $\mathrm{Ad}(u^{-m\lambda})(\mathcal{P}(\mathcal{O}_E))$. By the extension principle, $\mathrm{Ad}(u^{-m\lambda})$ extends to a unique isomorphism $\overline{\mathrm{Ad}(u^{-m\lambda})} : \mathcal{P} \to \mathcal{P}(\sigma, M)$ over $\mathcal{O}_F$. When c is homogeneous, i.e., if $c \in W$ is a homogeneous class and σ is an arbitrary lift of any $w \in c$, we show in 3.11 below that property (2) of 3.10 will still hold if we redefine $\mathcal{P}(c, M) := M_{\mathcal{O}_F}$.

It remains to show that for these definitions of $\mathcal{T}$ and $P(\sigma, M)$, the 3 conditions of 3.10 are satisfied. This is done in the following 3 lemmas. Let $\mathfrak{m} = \mathrm{Lie}M$ and $\mathfrak{t} = \mathrm{Lie}T$.

To set notation, let $\mathfrak{t} = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{t}_{\bar{i}}$ be the eigenspace decomposition for $\mathrm{ad}(w)$ acting on $\mathfrak{t}$. Let $M_{\mathrm{ad}} = \prod M_i$ be a decomposition into simple components M_i and $\mathfrak{m}_i = \mathrm{Lie}M_i$. We have

$$\mathfrak{m} = \mathfrak{t}_{\bar{0}} \bigoplus \mathfrak{m}_i$$

and write $\sigma = \prod \sigma_i$ where $\sigma_i \in M_i$. Then each $\mathrm{ad}(\sigma_i)$ acts separately on $\mathfrak{m}_i$ and trivially on $\mathfrak{t}_{\bar{0}}$. The triple (M_{ad}, T, B) induces by intersection triples (M_i, T_i, B_i) where $T_i \subseteq M_i$ is a maximal torus and $T_i \subseteq B_i \subseteq M_i$ is a Borel subgroup. Let $\lambda = \prod \lambda_i$ where $\lambda_i \in \mathcal{A}(M_i, T_i, B_i)$. Then $\mathrm{ad}(u^{-m\lambda}) = \oplus\mathrm{ad}(u^{-m\lambda_i})$.

Lemma 3.9: *For the standard Parahoric group scheme $\mathcal{P}$ of M_F, the Zariski closure of $\mathcal{T}$ in $\mathcal{P}$ is an intermediate integral model $\mathcal{T}^{\#}$ containing the connected Néron model $\mathcal{T}^{\flat,0}$.*

Proof: Let $I(M_F)$ be the standard Iwahori subgroup of M_F relative to T, B. Since it is contained in every standard parahoric subgroup of M_F, it suffices to check $\mathcal{T}^{\flat,0}(\mathcal{O}_F) = L^+\mathcal{T}^{\flat,0}(k) \subseteq I(M_F) = LM_F(k)$. Since $L^+\mathcal{T}^{\flat,0}$ is connected by properties of the Kottwitz homomorphism, it suffices to check the containment on Lie algebras.

We have that Lie$L^+\mathcal{T}^{\flat,0}$ is the image of Lie$L^+T_{\mathcal{O}_E}^{\mathrm{ad}(w\times\nu^{-1})}$ under $\mathrm{ad}(u^{-m\lambda})$ in $\mathfrak{m}_F$. Since $\mathfrak{t}_{k[u]}^{\mathrm{ad}(w\times\nu^{-1})}$ is dense in Lie$L^+\mathcal{T}^{\flat,0}$, it suffices to show $\mathrm{ad}(u^{-m\lambda})$ sends $\mathfrak{t}_{k[u]}^{\mathrm{ad}(w\times\nu^{-1})}$ to the polynomial standard Iwahori subalgebra $I(\mathfrak{m}_F)$ of $\mathfrak{m}_F$. We have a decomposition

$$I(\mathfrak{m}_F) = \mathfrak{t}_{\overline{0},k[t]} \bigoplus I(\mathfrak{m}_{i,F}).$$

The map $\mathrm{ad}(u^{-m\lambda})$ acts trivially on $\mathfrak{t}_{\overline{0},k[t]}$, which already lies in $I(\mathfrak{m}_F)$. It then suffices to show that each $\mathrm{ad}(u^{-m\lambda_i})$ sends $\mathfrak{t}_{i,k[t]}^{\mathrm{ad}(w\times\nu^{-1})}$ to $I(\mathfrak{m}_{i,F})$, or equivalently $\mathrm{ad}(u^{m\lambda_i})$ sends $I(\mathfrak{m}_{i,F})$ to a subalgebra containing $\mathfrak{t}_{i,k[t]}^{\mathrm{ad}(w\times\nu^{-1})}$.

Now fix i and for convenience put $\mathfrak{p} = \mathfrak{m}_i$, $\mathfrak{l} = \mathfrak{t}_i$. We have as in the proof of 3.7,

$$\mathfrak{l}_{k[t]}^{\mathrm{ad}(w\times\nu^{-1})} = \bigoplus_{i\geq 0} \mathfrak{l}_{\overline{i}} \otimes u^i$$

while for $j > 0$,

$$\mathrm{ad}(u^{m\lambda_i})(I(\mathfrak{m}_{i,F}))_j = \mathfrak{p}_{\overline{j}} \otimes u^i.$$

Hence for $j > 0$, $\mathfrak{l}_{k[t],j}^{\mathrm{ad}(w\times\nu^{-1})} \subseteq \mathrm{ad}(u^{m\lambda_i})(I(\mathfrak{m}_{i,F}))_j$. It remains to show the same for the case $j = 0$. This follows from the fact that

$$\mathfrak{l}_{\overline{0}} = 0$$

because w is elliptic when considered as an element of $W(M_{\mathrm{ad}}, T_{\mathrm{ad}}) \subseteq W$, as explained in the classification 3.6 of conjugacy classes of W by parabolic induction.

The result follows. ■

Lemma 3.10: *The closure of $\mathcal{T}$ in $\mathcal{P}(\sigma, M)$ is the full Néron model $\mathcal{T}^\flat$ of $\mathcal{T}$, i.e., by the closure principle it suffices to show $\mathcal{T}^\flat(\mathcal{O}_F) = P(\sigma, M) \cap \mathcal{T}(F)$.*

Proof: By the construction of the Néron model, we have $\mathcal{T}^\flat$ is the image under $\overline{\mathrm{Ad}(u^{-m\lambda})}$ of

$$\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}(T \times_k \mathcal{O}_E)^{w\times\nu^{-1}}.$$

On the other hand, by definition $\mathcal{P}(\sigma, M)$ is the image of $\overline{\mathrm{Ad}(u^{-m\lambda})}$ of

$$\mathrm{Res}_{\mathcal{O}_E/\mathcal{O}_F}(M \times_k \mathcal{O}_E)^{w\times\nu^{-1}}.$$

Hence $\mathcal{T}^\flat(\mathcal{O}_F) \subseteq P(\sigma, M)$. We have that $\mathcal{T}^\flat(\mathcal{O}_F) \subseteq \mathcal{T}(F)$ is automatic. The fact that the containment $\mathcal{T}^\flat(\mathcal{O}_F) \subseteq P(\sigma, M) \cap \mathcal{T}(F)$ is actually equality follows from the fact that $P(\sigma, M)$ bounded in $M(F)$ while $\mathcal{T}^\flat(\mathcal{O}_F)$ is the maximal bounded subgroup of $\mathcal{T}(F)$. ∎

Lemma 3.11: *If $\sigma \in M_{\mathrm{ad}}$ is homogeneous, we have*

$$P(\sigma, M) \subseteq M(\mathcal{O}_F)$$

and we still have $\mathcal{T}^\flat(\mathcal{O}_F) = M(\mathcal{O}_F) \cap \mathcal{T}(F)$. Consequently by the closure principal, $\mathcal{P}(\sigma, M)$ can be replaced with $M_{\mathcal{O}_F}$ and 3.9 will still hold.

Proof: First we claim that showing $P(\sigma, M) \subseteq M(\mathcal{O}_F)$ is sufficient. We have already shown in 3.10 $\mathcal{T}^\flat(\mathcal{O}_F) = P(\sigma, M) \cap \mathcal{T}(F)$, hence $\mathcal{T}^\flat(\mathcal{O}_F) \subseteq M(\mathcal{O}_F) \cap \mathcal{T}(F)$. The containment is then an equality because $M(\mathcal{O}_F)$ is also bounded in $M(F)$ while $\mathcal{T}^\flat(\mathcal{O}_F)$ is the maximal bounded subgroup of $\mathcal{T}(F)$.

To show $L^+\mathcal{P}(\sigma, M)(k) = P(\sigma, M) \subseteq M(\mathcal{O}_F) = L^+M(k)$, it suffices to check the containment on Lie algebras $\mathrm{Lie}L^+\mathcal{P}(\sigma, M) \subseteq \mathrm{Lie}L^+M$ since $L^+\mathcal{P}(\sigma, M)$ is connected. We have that $\mathrm{Lie}L^+\mathcal{P}(\sigma, M)$ is the image under $\mathrm{ad}(u^{-m\lambda})$ of $\mathrm{Lie}L^+M_{\mathcal{O}_E}^{\sigma\times\nu^{-1}}$. Since $\mathfrak{m}_{k[u]}^{\mathrm{ad}(\sigma\times\nu^{-1})}$ is dense in $\mathrm{Lie}L^+M_{\mathcal{O}_E}^{\sigma\times\nu^{-1}}$, it suffices to show that $\mathrm{ad}(u^{-m\lambda})(\mathfrak{m}_{k[u]}^{\mathrm{ad}(\sigma\times\nu^{-1})}) \subseteq \mathfrak{m}_{k[t]}$, or equivalently $\mathfrak{m}_{k[u]}^{\mathrm{ad}(\sigma\times\nu^{-1})} \subseteq \mathrm{ad}(u^{m\lambda})(\mathfrak{m}_{k[t]})$. We have

$$\mathfrak{m}_{k[t]} = \mathfrak{t}_{\overline{0},k[t]} \bigoplus \mathfrak{m}_{i,k[t]}$$

$$\mathfrak{m}_{k[u]}^{\sigma\times\nu^{-1}} = \mathfrak{t}_{\overline{0},k[u]}^{\mathrm{ad}(1\times\nu^{-1})} \bigoplus \mathfrak{m}_{i,k[u]}^{\mathrm{ad}(\sigma_i\times\nu^{-1})} = \mathfrak{t}_{\overline{0},k[t]} \bigoplus \mathfrak{m}_{i,k[u]}^{\mathrm{ad}(\sigma_i\times\nu^{-1})}.$$

Since $\mathrm{ad}(u^{m\lambda})$ acts trivially on $\mathfrak{t}_{\overline{0},k[t]}$, it suffices to show that $\mathfrak{m}_{i,k[u]}^{\mathrm{ad}(\sigma_i\times\nu^{-1})} \subseteq \mathrm{ad}(u^{m\lambda_i})(\mathfrak{m}_{i,k[t]})$. Fix i and for convenience put $\mathfrak{p} = \mathfrak{m}_i$ for the simple factor and $\mu = \lambda_i$. Let

$\mathfrak{p} = \oplus_{\bar{i}\in\mathbb{Z}/m\mathbb{Z}}\mathfrak{p}_{\bar{i}}$ be the eigenspace decomposition of $\mathfrak{p}$ for σ. We have

$$\mathfrak{m}_{i,k[u]}^{\mathrm{ad}(\sigma_i\times\nu^{-1})} = \bigoplus_{i\geq 0}\mathfrak{p}_{\bar{i}}\otimes u^i.$$

By the computation for the case of a simple Lie algebra 3.9, for $i > 0$

$$\mathrm{ad}(u^{m\kappa})(\mathfrak{p}_{k[t]})_i = \bigoplus_{i\geq 0}\mathfrak{p}_{\bar{i}}\otimes u^i.$$

Therefore it suffices to show that $\mathfrak{p}_{\bar{0}}\otimes u^0 \subseteq \mathrm{ad}(u^{m\mu})(\mathfrak{p}_{k[t]})_0$. We have by 3.9,

$$\mathrm{ad}(u^{m\mu})(\mathfrak{p}_{k[t]})_0 = (\mathfrak{p}_0 \oplus \mathfrak{p}^-_{-m})\otimes u^0.$$

On the other hand, by the relations between the eigenspace and refined eigenspace gradings 3.8,

$$\mathfrak{p}_{\bar{0}}\otimes u^0 = (\mathfrak{p}^+_m \oplus \mathfrak{p}_0 \oplus \mathfrak{p}^-_{-m})\otimes u^0.$$

Here is the crucial part where the homogeneity of σ is used. By definition $\sigma = \prod\sigma_i$ is homogeneous iff each σ_i is homogeneous. Let θ be the highest root of $\Psi(M_{\mathrm{ad}}, B_i, T_i)$. Homogeneity of σ_i means precisely in that the Kac label of σ_i has $s_0 = 0$. From the proof of 3.8, and it holds iff the inequality $\langle m\lambda,\theta\rangle \leq m$ is strict. Since $|\langle m\lambda,\alpha\rangle|$ for $\alpha\in\Psi$ achieves its maximum at $\alpha=\theta$, it follows that $\mathfrak{p}^+_m = \mathfrak{p}^-_{-m} = 0$. We conclude that $\mathfrak{p}_{\bar{0}}\otimes u^0 = \mathrm{ad}(u^{m\mu})(\mathfrak{p}_{k[t]})_0$.

The result follows. ■

Remark 3.7: *In the proof of 3.11, we have shown that for $i\geq 0$*

$$\mathrm{ad}(u^{m\mu})(\mathfrak{p}_{k[t]})_i = (\mathfrak{p}_{k[u]}^{\mathrm{ad}(\kappa\times\nu^{-1})})_i.$$

But it is not necessarily the case that $\mathrm{ad}(u^{m\mu})(\mathfrak{p}_{k[t]}) = \mathfrak{p}_{k[u]}^{\mathrm{ad}(\mu\times\nu^{-1})}$ because the right hand side has no negative graded components, while by the computation 3.9, the left hand side has for $i<0$:

$$\mathrm{ad}(u^{m\kappa})(\mathfrak{p}_{k[t]})_i = \mathfrak{p}^-_{i}\otimes u^i.$$

3.3 The Geometric Twisted FKS Isomorphism

For this entire section, let G be a simple and simply connected algebraic group over $k = \mathbb{C}$. Sometimes we may restrict further to the case when G is type ADE. Fix a maximal torus $T \subseteq G$ and put the Weyl group $W = W(G,T)$. Put $F = k((t))$ and $\mathcal{O}_F = k[[t]]$. Put $Y = \mathbb{X}_*(T)$ and $X = \mathbb{X}^*(T)$.

Let B be the normalized invariant bilinear form on $\mathfrak{g}$. The restriction of B to $Y \subseteq \mathfrak{t} = Y \otimes_{\mathbb{Z}} \mathbb{C}$ has the property that the length of a long coroot is $\sqrt{2}$. Since G is semisimple, B is nondegenerate. When G is simply-laced, i.e., type ADE, i.e., every coroot is the same length, they are all long by convention. Since G is simply connected, Y equals the coroot lattice as G. Therefore when G is type ADE, B is even on Y.

3.3.1 Affine Lie Algebras and Representations, Affine Borel-Weil Theorem, Statement of Results

Definition 3.13: *The (formal)* ***affine Lie algebra*** $\hat{\mathfrak{g}_F}$ *is the one-dimensional central extension*

$$0 \to \mathbb{C}K \to \hat{\mathfrak{g}_F} \to \mathfrak{g}_F \to 0$$

where K is a fixed choice of a nonzero central element and the commutator is

$$(X \otimes f, Y \otimes g) \mapsto B(X,Y)\mathrm{Res} f \mathrm{d} g K.$$

The ***(polynomial)*** *affine Lie algebra is the restriction* $\hat{\mathfrak{g}_{k[t^{\pm 1}]}}$ *of the central extension to the polynomial loop group* $\mathfrak{g}_{k[t^{\pm 1}]}$.

By the formula for the commutator, the central extension $\hat{\mathfrak{g}_F}$ is split over $\mathfrak{g}_{\mathcal{O}_F}$.

We have that

$$\mathfrak{g}_F = \lim_{\to i \in \mathbb{Z}_{\leq 0}} \mathfrak{g}_{t^i k[[t]]} = \lim_{\to i \in \mathbb{Z}_{\leq 0}} \lim_{\leftarrow j \in \mathbb{N}} \mathfrak{g}_{t^i k[t]/(t^j)}$$

is an ind-pro vector space where the Lie bracket $\mathfrak{g}_F \times \mathfrak{g}_F \to \mathfrak{g}_F$ is an ind-pro morphism of vector spaces. Taking pullback, the same is true for $\hat{\mathfrak{g}_F}$.

Definition 3.14: *Put $R = k((t))$ or $R = k[t^{\pm 1}]$. A representation of $\hat{\mathfrak{g}_R}$ is a representation on a countable dimensional vector space V such that there exists an ind-structure $V = \bigcup_{i \in \mathbb{N}} V_i$ where*

1. V_i is finite dimensional and stable under $\hat{\mathfrak{g}_R}$.

2. For each $i \in \mathbb{N}$ and $j \in \mathbb{Z}_{\leq 0}$, the vector space map $\hat{\mathfrak{g}_{t^j R}} \to End(V_i)$ factors through $\mathfrak{g}_{t^j R/(t^l)} \to \mathrm{End}(V_i)$ for some $l \in \mathbb{N}$.

The restriction of a representation of $\hat{\mathfrak{g}_F}$ to $\hat{\mathfrak{g}}_{k[t^{\pm 1}]}$ gives a representation $V = \cup V_i$ of $\hat{\mathfrak{g}}_{k[t^{\pm 1}]}$. Conversely every representation $V = \bigcup_{i \in \mathbb{N}} V_i$ of $\hat{\mathfrak{g}_{k[t^{\pm 1}]}}$ extends uniquely to a representation of $\hat{\mathfrak{g}_F}$, using the fact that $\mathfrak{g}_{t^j R/(t^l)}$ is the same regardless if $R = k[t^{\pm 1}]$ or $R = F$.

Remark 3.8: *The representation theoretic literature uses $\hat{\mathfrak{g}_{k[t^{\pm 1}]}}$ in studying their representations, while the Lie algebras of the algebro-geometric objects we study are $\hat{\mathfrak{g}_F}$. Since they have the same representations, we no longer take the care to separate them in citations of the literature.*

Definition 3.15: *For $l \in \mathbb{C}$, the level l* ***vacuum representation*** *of $\hat{\mathfrak{g}_F}$ is*

$$V_l(\hat{\mathfrak{g}_F}) = \mathrm{Ind}_{\mathfrak{g}_{\mathcal{O}_F} \oplus \mathbb{C}K}^{\hat{\mathfrak{g}_F}} \mathbb{C}$$

where K acts by multiplication by l and $\mathfrak{g}_{\mathcal{O}_F}$ acts trivially. When $l \in \mathbb{Z}_+$, $V_l(\hat{\mathfrak{g}_F})$ has a unique irreducible quotient [Zhu09, 0.1.1] that we call the level l ***integrable representation*** *and denote by $V^l(\hat{\mathfrak{g}_F})$.*

Suppose $e \in \mathrm{Gr}_G(k)$ be any point, usually taken to be the point corresponding to the identity coset of L^+G. The Picard groupoid $\mathrm{Pic}^e(\mathrm{Gr}_G)$ of line bundles rigidified at e is defined to be the Picard groupoid of pairs $(\mathcal{L}, \varphi)$ where $\mathcal{L}$ is a line bundle on Gr_G and $\varphi : e^*\mathcal{L} \approxeq k$ is a trivialization at e. It is known that since G is simple and simply connected, $\mathrm{Pic}^e(\mathrm{Gr}_G) \approxeq \mathbb{Z}$ is discreet [Zhu09, 2.4.2].

Definition 3.16: *The* ***level 1 line bundle*** *$\mathcal{L}_G$ or $\mathcal{O}(1)$ on Gr_G is the line bundle corresponding to the ample generator of $\mathrm{Pic}^e(\mathrm{Gr}_G)$. For any morphism of ind schemes*

$\mathcal{S} \to \mathrm{Gr}_G$, denote the restriction

$$\mathcal{L}_{\mathcal{S}} \coloneqq \mathcal{L}_G|_{\mathcal{S}}$$

line bundle on $\mathcal{S}$. Now recall that LG acts on Gr_G, induced from the left action of LG on itself.

As in [Zhu17, 2.5.1], there is a well-defined central extension, called the (formal) ***Kac-Moody central extension***

$$1 \to \mathbb{G}_m \to \hat{LG} \to LG \to 1$$

where for every k-algebra R,

$$\hat{LG}(R) = \{(g, \varphi) : g \in LG(\hat{R}) \text{ and } \varphi : g^*\mathcal{L}_G \approxeq \mathcal{L}_G\}$$

Then $\hat{LG}$ acts on $\mathcal{L}_G$ and also on Gr_G by projection to LG, such that $\mathcal{L}_G$ is an equivariant line bundle on Gr_G for $\hat{LG}$.

Theorem 3.11: *[Zhu17, 2.5.2] and [Kum02, ch. 8] We have*

$$Lie\hat{LG} = \hat{\mathfrak{g}_F}.$$

Therefore the (topological) dual global sections $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k})^\vee$ is a representation of $\hat{\mathfrak{g}_F}$ for every $k \in \mathbb{Z}$. We have the following starting point: the affine Borel-Weil theorem, which connects the problem of the (twisted) FKS isomorphism with algebraic geometry of $\mathcal{L}_G$ on Gr_G:

Theorem 3.12: *[Zhu17, 2.5.5] and [Kum02, 8.3.12]: For every $k \in \mathbb{Z}_{>0}$ there is an isomorphism of $\hat{LG}$-modules*

$$\Gamma(\mathrm{Gr}_G, \mathcal{L}_G^{\otimes k})^\vee \approxeq V^k(\hat{\mathfrak{g}_F}).$$

Remark 3.9: *The original formulation in [Kum02, 8.3.12] is a priori different from* *the one above, given in terms of the theory of Kac-Moody groups, more closely related to $\mathfrak{g}_{k\hat{[t^{\pm 1}]}}$ than $\hat{\mathfrak{g}_F}$, and their flag varieties. However, the two notions of flag varieties for affine Kac-Moody groups and affine flag varieties for formal loop groups are shown to coincide in [PR08].*

Let us now formally state our results. The original geometric FKS isomorphism of [Zhu09, 0.2.2] is that the dual restriction map $\Gamma(\mathrm{Gr}_T, \mathcal{L}_T)^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$ is an isomorphism. The question asked in [Zhu09, 0.3.3] is whether or not the same is true when Gr_T is replaced by an affine Springer fiber, something which Gr_T is one example of. We do not investigate affine Springer fibers, but instead we answer the question for an alternative subspace that replaces the affine Springer fiber. To be precise, construct subspaces $\mathcal{S}(\sigma) \subseteq \mathrm{Gr}_G$ for each lift σ of an element of a conjugacy class $c \in [W]$ such that $\Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$ is an isomorphism.

Let $c \in [W]$, $w \in c$ and $M = Z_G(T^{w,0})$ be the principal Levi, and $\sigma \in M$ a lift of w that is finite order with order m. Let $\mathcal{T} \subseteq G_F$ be the principal representative maximal torus of type c and $\mathcal{P}(c, M)$ the principal parahoric group scheme containing $\mathcal{T}^\flat$ from 3.9. So in particular, we set by definition when c is homogeneous 3.7 that $\mathcal{P}(c, M) = M_{\mathcal{O}_F}$. By 3.9 there is a choice of Borel subgroup $B_M \subseteq M$ (not to be confused with the bilinear form B) such that we have that $\mathcal{T}^{\flat,0}(\mathcal{O}_F) \subseteq I(M_F, B_M)$ is the Iwahori subgroup of M_F relative to B_M. Let $\mathcal{I}(M_F, B_M)$ be the corresponding Iwahori group scheme of M_F. Since

$$\mathrm{Lie} L^+\mathcal{I}(M_F, B_M) = \mathrm{Lie} B_M \otimes t^0 \bigoplus_{i \geq 1} \mathfrak{m} \otimes t^i \subseteq \mathfrak{g}_{\mathcal{O}_F},$$

we have that $I(M_F, B_M) \subseteq G(\mathcal{O}_F)$. Therefore $\mathcal{T}^{\flat,0}(\mathcal{O}_F) \subseteq G(\mathcal{O}_F)$.

Definition 3.17: *For a maximal torus $\mathcal{Z} \subseteq G_F$, define $\mathcal{Z}^\#$ to be the closure of $\mathcal{Z}$ in $G_{\mathcal{O}_F}$. So by the closure principle, $\mathcal{Z}^\sharp$ is characterized by the property that*

$$\mathcal{Z}^\sharp(\mathcal{O}_F) = G(\mathcal{O}_F) \cap \mathcal{Z}(F)$$

and when $\mathcal{Z}^{\flat,0}(\mathcal{O}_F) \subseteq G(\mathcal{O}_F)$, for example if $\mathcal{Z}$ is a principal maximal torus of type

$c \in W$, then $\mathcal{Z}^\sharp$ is an intermediate integral model of $\mathcal{Z}$.

By the extension principle for smooth integral models, there is an induced canonical map $\mathcal{T}^\sharp \to G_{\mathcal{O}_F}$ over $\mathcal{O}_F$. Then the inclusion $L\mathcal{T} \to LG$ induces an inclusion of flag varieties $\mathcal{F}_{\mathcal{T}^\sharp} \hookrightarrow \mathrm{Gr}_G$ as the scheme theoretic image.

Definition 3.18: *For c, w, σ as above, the* ***principal subspace***

$$\mathcal{S}(\sigma) \subseteq \mathrm{Gr}_G$$

is the image of the orbit

$$\mathcal{O} := L\mathcal{T}.L^+\mathcal{P}(\sigma, M) \subseteq LG$$

of $L\mathcal{T}$ under the ***right*** *action of $L^+\mathcal{P}(\sigma, M)$ on LG defined by the composition of the inclusions $L^+\mathcal{P}(c, M) \hookrightarrow LM \hookrightarrow LG$, i.e., it is the fpqc quotient*

$$S(\sigma) = [\mathcal{O}/(L^+G \cap \mathcal{O})] \subseteq \mathrm{Gr}_G.$$

Remark 3.10: *If c is a homogeneous conjugacy class, then $\mathcal{P}(\sigma, M) = M_{\mathcal{O}_F}$, $\mathcal{T}^\sharp = \mathcal{T}^\flat$ and*

$$\mathcal{S}(\sigma) = \mathcal{F}_{\mathcal{T}^\sharp} = \mathcal{F}_{\mathcal{T}^\flat}$$

simplifies to the image of $L\mathcal{T}$ in Gr_G.

Observe that since $\mathcal{T}^\flat(\mathcal{O}_F) = P(\sigma, M) \cap \mathcal{T}(F)$ as in the proof of 3.10, the map $L\mathcal{T} \to LM$ induces a natural inclusion $\mathcal{F}_{\mathcal{T}^\sharp} \hookrightarrow \mathcal{S}(\sigma)$. So $\mathcal{S}(\sigma)$ is stable under the left action of $L\mathcal{T}$. For any subspace stable under the left action of $L\mathcal{T}$, the space of dual sections of the restriction of $\mathcal{L}_G$ has an action of the restriction $\hat{L\mathcal{T}}$ of $\hat{LG}$ to $L\mathcal{T}$. This applies to $\mathcal{F}_{\mathcal{T}^\sharp}$ and $\mathcal{S}(\sigma)$.

Our main result of this section is:

Theorem 3.13: *Suppose G is not necessarily of type ADE but still simple and simply* *connected. Consider the maps of $L\hat{\mathcal{T}}$ modules*

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee.$$

Then the first map is nonzero and the second map is injective.

The proof is given in the next subsection. This gives the geometric twisted FKS isomorphism, but conditional on the representation-theoretic result of [KP85, 11]:

Corollary 3.1: *Using the main theorem [KP85, 11] and the Borel-Weil theorem that if G is furthermore type ADE then $\Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee \approxeq V^1(\hat{\mathfrak{g}_F})$ is irreducible for $L\hat{\mathcal{T}} \subseteq L\hat{G}$, we conclude that the map*

$$\Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$$

is an isomorphism.

Even though we do not give fully geometric proof, this already verifies that the principal subspace $\mathcal{S}(\sigma)$ is a correct candidate for a geometric realization of the twisted FKS isomorphism, where before it was not known what subspaces to consider and it was only conjectured that an affine Springer fiber is another possible candidate. We give a fully geometric proof when c is a homogeneous conjugacy class in 3.4.

It follows from the following statement that we prove in the next section 3.4 by deducing from the split case of $c = [1], \sigma = 1$ and $\mathcal{T} = T_F$ by cohomology and base change and global methods. For $\mu \in Y = \mathrm{Hom}(\mathbb{G}_m, T)$ evaluate μ on F-points and let s_μ be the image in Gr_G of the element $\mu(t) \in G(F)$. Let $\mathrm{Gr}_{G,\mu} \subseteq \mathrm{Gr}_G$ be the left L^+G orbit of $s_\mu \in \mathrm{Gr}_G$ and $\overline{\mathrm{Gr}_{G,\mu}}$. Such an orbit closure is called a **Schubert Variety**. Then we have by [Zhu09, 2.3.5] that

$$\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee = \bigcup_{\mu \in Y} \Gamma(\overline{\mathrm{Gr}_{G,\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G,\mu}}})^\vee.$$

We prove in the next section 3.4:

Theorem 3.14: *Suppose G is type ADE. Assume the nontrivial premise that the map*

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$$

is injective. (By 3.13 this holds when $S(\sigma) = \mathcal{F}_{\mathcal{T}^\sharp}$, such as when c is homogeneous). Then there exists a collection of subspaces $\mathcal{F}_{\mathcal{T}^\sharp,\mu} \subseteq \mathcal{F}_{\mathcal{T}^\sharp}$ for $\mu \in Y$ such that for each $\mu \in Y$,

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee \to \Gamma(\overline{\mathrm{Gr}_{G,\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G,\mu}}})^\vee$$

is an isomorphism and

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee = \bigcup_{\mu \in Y} \Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee.$$

We conclude that

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \overset{\cong}{\twoheadrightarrow} \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$$

is also surjective (i.e., an isomorphism using 3.13) and both maps

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$$

are isomorphisms.

If such a statement is true for all the conjugacy classes c, for some lift σ of some element $w \in c$ and the principal maximal tori $\mathcal{T}$, it will provide **two** candidate subspaces for the geometric twisted FKS isomorphism, namely both $\mathcal{F}_{\mathcal{T}^\sharp}$ and $\mathcal{S}(\sigma)$. The two spaces are the same when c is homogeneous, although from the examples discussed in 3.2.3, the homogeneous conjugacy classes are a significant number in type E and are all the classes when w lies in a parabolic subgroup of type A.

We are able to provide an exact numerical condition on the number of torsion points of $\pi_0\mathcal{F}_{\mathcal{T}^\sharp}$ that would imply $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$ is injective, namely:

Theorem 3.15: *If $\pi_0\mathcal{F}_{\mathcal{T}^\sharp,\mathrm{tor}} = d(c)$, where $d(c)$ is the defect of c as defined 2.34 (a quantity depending only on G and c) then $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$ is injective.*

This is explained in the next section. If we can find $w \in c$ and a lift σ of w such that $\pi_0 \mathcal{F}_{\mathcal{T}^\sharp,\mathrm{tor}} = d(c)$, this would also give a full geometric proof of the geometric twisted FKS isomorphism without relying on the representation theoretic result of $[KP85, 11]$. We leave this as an open problem. Computer assisted computations have shown it to be true in type D_6, D_8, but they are omitted from this thesis.

It gives the following more elegant formulation of the geometric FKS isomorphism, which we presently only know for homogeneous conjugacy classes:

Proposition 3.1: ***(this is a conjecture we propose, not proven)*** *Suppose G is simple, simply connected, and simply laced. For an arbitrary conjugacy class of maximal tori in G_F, there exists a representative $\mathcal{T}$ such that the inclusion of the image of $L\mathcal{T}$ in Gr_G induces an isomorphism on global sections of $\mathcal{L}_G$.*

Remark 3.11: *We will show that $\hat{L\mathcal{T}}$ is a Heisenberg central extension in the sense of 2.19. Fix a splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$, which exists by 2.7, although we show $\hat{LG}$ actually splits over L^+G and choose a compatible splitting for $L^+\mathcal{T}^{\flat,0}$ by restriction from $L^+\mathcal{T}^{\flat,0} \hookrightarrow L^+G$. The principal part of the vacuum space*

$$U = (\Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee)^{L^+\mathcal{T}^{\flat,0}, T^{w,0}}$$

determines $\Gamma(\mathrm{Gr_G}, \mathcal{L}_G)^\vee$ as a representation of $\hat{L\mathcal{T}}$ an a priori is only known to be representation of the principal finite Heisenberg group $\hat{\Sigma} \subseteq \hat{L\mathcal{T}}$, from our study of the representation theory of $\hat{L\mathcal{T}}$ 2.2.6. However, we find furthermore that U also is a minuscule representation an algebraic group,M^σ. This is a priori consistent with the remark [KP85, 15.E] which states that U is a trivial or minuscule representation of the larger group G^σ, when σ is chosen to be some specific lift of w. However, the set of possible dimensions of minuscule representation of G^σ should not be the same as the set of possible dimensions of minuscule representations of M^σ, and we think M^σ is the correct group to consider.

3.3.2 Proof of Main Theorem

Preserve all setups and notation of the previous section.

Let $B_M \subseteq M$ be the choice of Borel subgroup containing σ given by 3.7. Extend B_M to a Borel subgroup B_G of G. This is not to be confused with the bilinear form B. For $H = M$ or G, let $I_H \subseteq H(F)$ be the corresponding standard Iwahori subgroup of $H(F)$ and $\mathcal{I}_H$ the corresponding Iwahori group scheme for H_F. By a standard parahoric subgroup, we mean one containing this chosen standard Iwahori subgroup. Remark that neither B_M nor B_G necessarily contain T. We have $I_M = I_G \cap M(F)$.

The main feature of our proof is to pull back $\mathcal{L}$ to the full flag variety $\mathcal{F}_{\mathcal{I}_G}$ and consider the preimage of $\mathcal{S}(c)$ in $\mathcal{F}_{\mathcal{I}_G}$.

Let us first prove two basic lemmas.

Lemma 3.12: *There exist a splitting $L^+G \to \hat{LG}$.*

Proof: We have $\mathrm{Lie} L^+G = \mathfrak{g}_{\mathcal{O}_F} = \mathfrak{g} \oplus \mathfrak{g}_{t\mathcal{O}_F}$ as Lie algebras where $\mathfrak{g}_{t\mathcal{O}_F}$ is pro-unipotent. Since G is simply connected, this upgrades to an isomorphism $L^+G \approxeq G \times L^{++}G$. By the formula 3.7, $\hat{\mathfrak{g}_F}$ is split over $\mathfrak{g}_{\mathcal{O}_F}$. Since $\mathfrak{g}$ is simple and $\mathfrak{g}_{t\mathcal{O}_F}$ is pro-unipotent, we conclude that $\mathfrak{g}_{\hat{\mathcal{O}_F}} \approxeq \mathbb{C} \oplus \mathfrak{g} \oplus \mathfrak{g}_{t\mathcal{O}_F}$. Therefore $\hat{LG}$ is isogeneous to $\mathbb{G}_m \times G \times L^{++}G$. Since $L^{++}G$ and G are both simply-connected, the kernel of an isogeny $\mathbb{G}_m \times G \times L^{++}G \to \hat{LG}$ must lie in $\mathbb{G}_m$. Any isogeny from $\mathbb{G}_m$ is $\mathbb{G}_m$ itself, by a quotient by some group of rth roots of unity for some r. We conclude that $\hat{LG} \approxeq \mathbb{G}_m \times G \times L^{++}G$ and the result follows. ∎

For the remainder of this subsection, fix a choice of splitting $L^+G \to \hat{LG}$. This restricts by $L^+\mathcal{T}^{\flat,0} \hookrightarrow L^+G$ to a choice of splitting $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$. Additionally fix the choice of section $s_u : Y_\Gamma \to L\mathcal{T}$ for the Kottwitz homomorphism 2.6, giving us the full structure theory and representation theory of $\hat{L\mathcal{T}}$ in 2.2.5 and 2.2.6.

Definition 3.19: *Set*

$$\hat{\mathfrak{t}_F}(\sigma) := \mathrm{Lie} L^{++,\hat{-}}\mathcal{T}^{\flat,0}.$$

The second basic lemma:

Lemma 3.13: *The central extension $\hat{L\mathcal{T}}$ is a Heisenberg central extension in the sense of 2.19. Here $\mathcal{T} \subseteq G_F$ can be any maximal torus, not necessarily a principal maximal torus.*

Proof: By [BD01, 12.10], there exists a central extension $\mathcal{E}$ of G by $\boldsymbol{K}_{2,F}$ such that $\hat{LG}(k) = \mathcal{E}(F)^{\mathrm{tame}_F}$. In fact $\mathcal{E}$ is determined by the bilinear form B on Y. It follows that

$$\hat{L\mathcal{T}}(k) = \mathcal{E}|_{\mathcal{T}}(F)^{\mathrm{tame}_F}.$$

It remains to check the other condition that $\hat{L\mathcal{T}}$ needs to satisfy in the definition 2.19, namely that $\hat{\mathfrak{t}_F(\sigma)}$ is a Heisenberg Lie algebra, i.e., the center is one dimensional and equals the commutator. To this end, it suffices to check any nonzero multiple of $\hat{\mathfrak{t}_F(\sigma)}$ as a central extension is a Heisenberg Lie algebra. For a central extension $\hat{\mathfrak{l}}$ of a Lie algebra by $\mathbb{C}$, denote the mth multiple by $m \cdot \hat{\mathfrak{l}}$. Let E/F be a degree m extension such that $\mathcal{T}_E$ is split and $\nu \in \mathrm{Gal}(E/F)$ a generator and ζ a chosen primitive mth root of unity so that $\nu(u)/u = \zeta$. Recall $\mathcal{T}$ is the image of

$$\mathcal{Z} := \mathrm{Res}_{E/F}(T \times_k E)^{w \times \nu^{-1}} \subseteq \mathrm{Res}_{E/F} G_E$$

under $\mathrm{Ad}(u^{-m\lambda})$ where λ is the Kac labels of σ as in the proof of 3.10, where $u^{-m\lambda} \in G(E)$. Let $\hat{\mathfrak{g}}_E$ be the affine Lie algebra as in 3.13 with F replaced with E. Let $\mathfrak{t} = \oplus_{\bar{i} \in \mathbb{Z}/m\mathbb{Z}} \mathfrak{t}_{\bar{i}}$ is the eigenspace decomposition of $\mathfrak{t}$ for w. Then the restriction of $\hat{\mathfrak{g}_E}$ to $\mathfrak{g}_F \hookrightarrow \mathfrak{g}_E$ is $m \cdot \hat{\mathfrak{g}_F}$. The adjoint action by the commutator LG on $\hat{LG}$ is an action by automorphisms as $\mathbb{G}_m$-central extensions. Therefore $m \cdot \hat{\mathfrak{t}_F(\sigma)}$ is isomorphic to the restriction of $\hat{\mathfrak{g}_E}$ to

$$\mathrm{Lie} L^{++,-}\mathcal{Z}^{\flat,0} = \oplus_{\bar{i} \neq 0} \mathfrak{t}_{\bar{i}} \otimes u^i$$

where the computation of $\mathrm{Lie}L^{++,-}\mathcal{Z}^{\flat,0}$ is as in the proof of 3.6. By the formula 3.13, and the $\mathfrak{g}$-invariance of B, $\hat{\mathrm{Lie}L^{++,-}\mathcal{Z}^{\flat,0}}$ is given by the presentation for $\bar{i}, \bar{j} \neq 0$ and $X \in \mathfrak{t}_{\bar{i}}, Y \in \mathfrak{t}_{\bar{j}}$,

$$(X \otimes u^i, Y \otimes u^j) \mapsto B(X,Y)\mathrm{Res} u^i \mathrm{d} u^j K$$

$$= B(X,Y) i\delta_{i+j,0} K$$

where $1_{(-)}$denotes the Boolean indicator symbol for the expression $(-)$. Since B is w-invariant, for $i, j \in \mathbb{Z}$ and $X \in \mathfrak{t}_{\bar{i}}, Y \in \mathfrak{t}_{\bar{j}}$,

$$B(X,Y) = B(w.X, w.Y) = \zeta^{i+j} B(X,Y).$$

This shows that the restriction of B to $\mathfrak{t}_{\bar{i}} \oplus \mathfrak{t}_{\bar{j}}$ is zero unless $\overline{i+j} = \bar{0}$. Since B is also non-degenerate, the restriction of B to $\mathfrak{t}_{\bar{i}} \oplus \mathfrak{t}_{-\bar{i}}$ is a perfect pairing for every $\bar{i} \in \mathbb{Z}/m\mathbb{Z}$. This implies that the center is $\mathbb{C}K$, i.e, one dimensional and equal to the commutator. The result follows. ■

Remark 3.12: *The argument that B restricts to a perfect pairing $\mathfrak{t}_{\bar{i}} \oplus \mathfrak{t}_{-\bar{i}} \to \mathbb{C}$ is similar to [Kac90, 8.1.a].*

This shows that our results on the representation theory of $\hat{L\mathcal{T}}$ in 2.2.6 apply to help us understand the maps 3.13.

Since $P(\sigma, M) \cap \mathcal{T}(F) = \mathcal{T}^{\flat,0}(\mathcal{O}_F)$ and $L^+\mathcal{P}(c, M)$ is reduced, we have $L^+\mathcal{P}(\sigma, M) \cap L\mathcal{T} = L^+\mathcal{T}^\flat$ and

$$\mathcal{F}_{\mathcal{T}^\flat} = [(L\mathcal{T}.L^+\mathcal{P}(c, M))/L^+\mathcal{P}(\sigma, M)]$$

is expressed as the quotient of the orbit of $L\mathcal{T}$ under the right action of $L^+\mathcal{P}(c, M)$. Consider the quotient by the smaller space $L^+\mathcal{I}_M$

$$\mathcal{X}(\sigma) = [(L\mathcal{T}.L^+\mathcal{P}(c, M))/L^+\mathcal{I}_M].$$

We have two projection maps p, π:

$$\mathcal{F}_{\mathcal{T}^\flat} \xleftarrow{p} \mathcal{X}(\sigma) \xrightarrow{\pi} \mathcal{S}(\sigma).$$

Define the intersection

$$Q(\sigma, M) := P(\sigma, M) \cap G(\mathcal{O}_F)$$

.Using $P(\sigma, M) \subseteq M(F)$, we have

$$Q(\sigma, M) = P(\sigma, M) \cap G(\mathcal{O}_F) \cap M(F)$$

$$= P(\sigma, M) \cap M(\mathcal{O}_F).$$

So $Q(\sigma, M)$ is the intersection of two standard parahoric subgroups of M_F, and therefore itself is also a standard parahoric subgroup. Let $\mathcal{Q}(\sigma, M)$ be the corresponding parahoric group scheme of M_F over $\mathcal{O}_F$. Then both p, π are flat and projective with fibers isomorphic to the (projective) Schubert varieties $P(c, M)/I_M$ and $P(c, M)/Q(c, M)$, respectively.

Lemma 3.14: *The natural map is an isomorphism*

$$\mathcal{L}_{\mathcal{S}(\sigma)} \cong \pi_* \mathcal{L}_{\mathcal{X}(\sigma)}.$$

we thus obtain a natural isomorphism

$$\Gamma(\mathcal{X}(\sigma), \mathcal{L}_{\mathcal{X}(\sigma)})^\vee \cong \Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee.$$

PROOF: The ind scheme $\mathcal{S}(\sigma)$ is ind finite type because it is a closed sub ind scheme of Gr_G. Therefore π is ind-proper because it is the pullback of $\mathcal{F}_{\mathcal{I}_G} \to \mathrm{Gr}_G$, which is ind-proper by [PR08, 8.e.1], by the closed embedding $\mathcal{S}(\sigma) \hookrightarrow \mathrm{Gr}_G$.

Write $\mathcal{S}(\sigma) = \lim_{\to i} \mathcal{S}(\sigma)_i$ as the limit of finite type closed subschemes $\mathcal{S}(\sigma)_i$ and let $\mathcal{X}(\sigma) = \lim_{\to i} \mathcal{X}(\sigma)_i$ be the induced presentation of finite type closed subschemes $\mathcal{X}(\sigma)_i$ by pullback. Then it suffices to show that the natural maps are isomorphisms

$$\mathcal{L}_{\mathcal{S}(\sigma)_i} \cong \pi_* \mathcal{L}_{\mathcal{X}(\sigma)_i}.$$

This follows from the fact that $\mathcal{X}(\sigma)_i \to \mathcal{S}(\sigma)_i$ is proper, flat, finite type whose geometric fibers are isomorphic to $P(\sigma, M)/Q(\sigma, M)$, which is connected and reduced, and [Vak17, ex. 28.1.I and 28.1.H].

Let $\hat{\Sigma} \subseteq \hat{L\mathcal{T}}$ be the principal finite Heisenberg subgroup defined in 2.33 with $\Sigma = Y_{w,\mathrm{tor}}$. The Kottwitz homomorphism $L\mathcal{T} \to Y_w$ induces a bijection $\pi_0(L^+\mathcal{T}^\flat) \to Y_{w,\mathrm{tor}}$ because the latter is the maximal compact subgroup of Y_w. Therefore $\hat{\Sigma} \subseteq \hat{L^+\mathcal{T}^\flat} \subseteq L^+\hat{\mathcal{P}(\sigma, M)}$ and the subspaces $P(\sigma, M)/I_M \subseteq \mathcal{X}(\sigma)$ and $P(\sigma, M)/Q(\sigma, M) \subseteq S(\sigma)$ are both stable under the right action of Σ. ■

Definition 3.20: *To save notation in the following proofs, put*

$$P = P(\sigma, M), \mathcal{P} = \mathcal{P}(\sigma, M)$$

$$Q = Q(\sigma, M), \mathcal{Q} = \mathcal{Q}(\sigma, M).$$

Therefore the dual sections

$$\Gamma(P/Q, \mathcal{L}_{P/Q})^\vee$$

$$\Gamma(P/I_M, \mathcal{L}_{P/I_M})^\vee$$

have a natural action of $\hat{\Sigma}$.

Lemma 3.15: *The dimensions of* $\Gamma(P/Q, \mathcal{L}_{P/Q})^\vee$ *and* $\Gamma(P/I_M, \mathcal{L}_{P/I_M})^\vee$ *are each a positive multiple of the defect* $d(c)$ *as defined in 2.34.*

Proof: Follows from the characterization of representations of $\hat{\Sigma}$ 2.34. ■

Lemma 3.16: *The natural map* $M^\sigma \hookrightarrow M \hookrightarrow M(F)$ *induces an isomorphism between the space* P/I_M *with the (finite type) full flag variety of the connected reductive group* M^σ *(connectedness of* M^σ *follows from 3.1 and [Con14, 6.5.2.IV]). Thus by the finite type Borel-Weil theorem,* $\Gamma(P/I_M, \mathcal{L}_{P/I_M})^\vee$ *is an irreducible representation of* M^σ *by* $M^\sigma \hookrightarrow LG \hookrightarrow \hat{LG}$*, and* $T^{w,0} \subseteq Z(M^w)$ *acts on it by a character.*

Furthermore, this representation is minuscule, i.e., corresponds to a hyperspecial vertex of the Dynkin diagram of M^σ*.*

PROOF: The map $\mathrm{ad}(u^{-m\lambda})$ of 3.7 identifies the Levi factor of $\mathrm{Lie}L^+\mathcal{P}$ with $\mathfrak{m}_{\overline{0}}$, so the reductive quotient of the reduction of P at $t=0$ is M^σ. Since I_M is the Iwahori subgroup of P, P/I_M must be full finite flag variety M^σ/B of M^σ.

Now the line bundle $\mathcal{L}_{\mathcal{F}_{\mathcal{I}_G}}$ is characterized by the property that the restriction to the copy of $\mathbb{P}^1$ given by the image of a standard affine root subgroup is $\mathcal{O}(1)$ for the affine root subgroup and trivial for the others [PR08, 10.1]. Restricting the maps $P/I_M \subseteq \mathcal{F}_{\mathcal{I}_M} \subseteq \mathcal{F}_{\mathcal{I}_G}$, that $\mathbb{P}^1$ that corresponds to the affine root of G maps to the $\mathbb{P}^1$ corresponding to a hyperspecial vertex of M^σ. The result follows. ■

Now recall the decomposition

$$\hat{L\mathcal{T}} = (\hat{L^{++,-}\mathcal{T}^{\flat,0}} \times (\hat{Y}_\Gamma \rtimes T^{w,0}))/\mathbb{G}_{m,k}$$

from 2.8 given $L^+\mathcal{T}^{\flat,0} \to \hat{L\mathcal{T}}$ and $s_u : Y_\Gamma \to L\mathcal{T}$, where $\hat{\Sigma} \hookrightarrow \hat{L\mathcal{T}}$ factors through $\hat{\Sigma} \hookrightarrow \hat{Y_\Gamma}$, $T^{w,0}$ acts trivially on $\hat{\Sigma}$, and the centralizer of $T^{w,0}$ in $\hat{Y}_\Gamma \rtimes T^{w,0}$ is $\hat{\Sigma} \times T^{w,0}$.

Lemma 3.17: *As a representation of $\hat{L\mathcal{T}}$, we have*

$$\Gamma(\mathcal{F}_{\mathcal{T}^\flat}, p_*\mathcal{L}_{\mathcal{X}(\sigma)})^\vee \approxeq \mathrm{Ind}_{(\hat{L^{++}\mathcal{T}^{\flat,0}} \times (\hat{\Sigma} \times T^{w,0}))/\mu_r}^{\hat{L\mathcal{T}}} U$$

in the classification of representations of $\hat{L\mathcal{T}}$ from 2.2.6 where

$$U = \Gamma(P/I_M, \mathcal{L}_{P/I_M})^\vee$$

is equipped with the action of $\hat{\Sigma} \subseteq \hat{Y_w}$, forms a single weight space for $T^{w,0}$, $\hat{L^{++}\mathcal{T}^{\flat,0}} = \mathbb{G}_m \times L^{++}\mathcal{T}^{\flat,0}$ has the $\mathbb{G}_m$ factor acting by the identity character, and $L^{++}\mathcal{T}^{\flat,0}$ acts trivially.

PROOF: We first claim $p_*\mathcal{L}_{\mathcal{X}(\sigma)}$ is free and finite rank as a quasicoherent sheaf on $\mathcal{F}_{\mathcal{T}^\flat}$. The map p is the pullback by the inclusion $\mathcal{F}_{\mathcal{T}^\flat} \hookrightarrow \mathcal{F}_{\mathcal{P}}$ of the map $\mathcal{F}_{\mathcal{I}_M} \to \mathcal{F}_{\mathcal{P}}$, which is ind-proper by [PR08, 8.e.1]. Therefore p is ind-proper. Let $\mathcal{F}_{\mathcal{T}^\flat} = \lim_{\to i} \mathcal{F}_{\mathcal{T}^\flat,i}$ be a presentation as an ind-scheme of closed sub-schemes $\mathcal{F}_{\mathcal{T}^\flat,i}$ and $\mathcal{X}(\sigma) = \lim_{\to i} \mathcal{X}(\sigma)_i$ be the induced presentation as an ind-scheme of closed subschemes $\mathcal{X}(\sigma)_i$ by pullback. Let $p_i : \mathcal{X}(\sigma)_i \to \mathcal{F}_{\mathcal{T}^\flat,i}$ be the restriction of p to $\mathcal{X}(\sigma)_i$. Since each $\mathcal{F}_{\mathcal{T}^\flat,i}$ is zero-dimensional by properties of the Kottwitz homomorphism, it suffices to show that $p_{i,*}\mathcal{L}_{\mathcal{X}(\sigma)_i}$ is locally free of finite rank where the rank is independent of i. Each p_i is flat and thus $\mathcal{L}_{\mathcal{X}(\sigma)_i}$ is flat over $\mathcal{F}_{\mathcal{T}^\flat,i}$ by [Sta18, Tag 01U6]. The geometric fibers of p_i are all isomorphic to the Schubert variety P/I_M, which by 3.14 has the property that

$$\mathrm{H}^1(P/I_M, \mathcal{L}_{P/I_M}) = 0$$

because the line bundle $\mathcal{L}_{P/I_M}$ is ample on the Schubert variety P/I_M and by [Kum02, 8.1.8]. By cohomology and base change, [Gro63, 3.2.1], $p_{i,*}\mathcal{L}_{\mathcal{X}(\sigma)_i}$ is locally free with rank equal to the dimension $\Gamma(P/I_M, \mathcal{L}_{P/I_M})$. This is independent of i as required.

The characterization

$$\Gamma(\mathcal{F}_{\mathcal{T}^\flat}, p_*\mathcal{L}_{\mathcal{X}(\sigma)})^\vee \approxeq \mathrm{Ind}_{(L^+\hat{+}\mathcal{T}^{\flat,0}\times(\hat{\Sigma}\times T^{w,0}))/\mu_r}^{L\hat{\mathcal{T}}} U$$

follows from the fact that each geometric fiber of p is isomorphic to P/I_M and the map $\mathcal{F}_{\mathcal{T}^{\flat,0}} \to \mathcal{F}_{\mathcal{T}^\flat}$ is finite and free of finite rank, and so induces an isomorphism on connected components. Since the action of $L^+\mathcal{T}^{\flat,0}$ on $\mathcal{L}_{\mathcal{S}(\sigma)}$ is trivial, so is its action on $p_*\mathcal{L}_{\mathcal{S}(c)}$. The fact that U is one single weight space for $T^{w,0}$ follows from 3.16. ■

Combining 3.17 and 3.14, we obtain

Corollary 3.2: *As a representation of $L\hat{\mathcal{T}}$, we have*

$$\Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \approxeq \mathrm{Ind}_{(L^+\hat{+}\mathcal{T}^{\flat,0}\times(\hat{\Sigma}\times T^{w,0}))/\mu_r}^{L\hat{\mathcal{T}}} U$$

where

$$U = \Gamma(P/Q, \mathcal{L}_{P/Q})^\vee$$

is a single weight pace for $T^{w,0}$.

We remark that the space P/Q is still a finite flag variety for M^σ. It happens that $\mathcal{L}_{P/I_M}$ and $\mathcal{L}_{P/Q}$ have the same global sections.

Remark 3.13: *In terms of the representation theory of* $\hat{L\mathcal{T}} \cong (L^{++,\hat{-}}\mathcal{T}^{\flat,0} \times (\hat{Y}_\Gamma \rtimes T^{w,0}))/\mathbb{G}_{m,k}$ *of 2.2.6, we have a vector space decomposition*

$$\Gamma(\mathcal{S}(c), \mathcal{L}_{\mathcal{S}(c)})^\vee \cong \pi(w) \otimes \mathbb{C}[Y_{w,\text{cotor}}] \otimes U$$

where $L^{++,\hat{-}}\mathcal{T}^{\flat,0}$ *acts by its Heisenberg Lie algebra on the Fock space* $\pi(w)$*, identified with the dual ring of regular functions of the connected component* $\mathcal{F}^0_{\mathcal{T}^{\flat,0}}$*. The composition*

$$Y_w \to Y_w \otimes \mathbb{C} = \mathfrak{t}^w \xrightarrow{B} \mathfrak{t}^{w,\vee}$$

with the second map is $X \mapsto B(X, -)$*, induces an embedding*

$$Y_{w,\text{cotor}} \hookrightarrow \mathfrak{t}^{w,\vee}$$

which we denote by $\lambda \mapsto \overline{\lambda}$*. Then* $\mathfrak{t}^{w,0}$ *acts on* $\mathbb{C}[Y_{w,\text{cotor}}] = \oplus_{\lambda \in Y_{w,\text{cotor}}} \mathbb{C}_{\overline{\lambda}}$ *as a direct sum of the corresponding weight spaces, as shown in [BK04, 4.7].*

We now can prove the main theorem of this section.

PROOF: (of 3.13) Apply the characterization 3.2 and 3.13. To show that $\Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\text{Gr}_G, \mathcal{L}_G)^\vee$ is injective, by the representation theory of $\hat{L\mathcal{T}}$ 2.9 applied to those induced from representations of $\hat{\Sigma}$ that form a single weight space for $T^{\Gamma,0}$ 3.16, it suffices to show injectivity of the induced map on the vacuum spaces

$$\text{Ind}_{\hat{\Sigma} \times T^{w,0}}^{\hat{Y}_\Gamma \rtimes T^{w,0}} U = \mathbb{C}[Y_{w,\text{cotor}}] \otimes U \to \Gamma(\text{Gr}_G, \mathcal{L}_G)^\vee$$

where $U = \Gamma(P/Q, \mathcal{L}_{P/Q})^\vee$. Now crucially since U is one distinct weight space for $\mathfrak{t}^w$, each summand of

$$U \otimes \mathbb{C}[Y_{w,\text{cotor}}] = \oplus_{\lambda \in Y_{w,\text{cotor}}} \mathbb{C}_{\overline{\lambda}} \otimes U$$

is a single weight space for $\mathfrak{t}^w$ with the weights of different summands are distinct. Thus it suffices to show that the induced map on each summand

$$\mathbb{C}_{\overline{\lambda}} \otimes U \to \Gamma(\text{Gr}_G, \mathcal{L}_G)^\vee$$

is injective, for the images of different summands will have trivial intersection. Since both $U \otimes \mathbb{C}[Y_{w,\mathrm{cotor}}]$ and $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$ are equivariant for $\hat{Y_w}$, it suffices to show injectivity for one factor and assume $\lambda = 0$. We thus finally reduce to showing the map

$$U \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$$

is injective. By [Kum02, 8.1.23] and [PR08, 8.8, 8.1] this is shown if we can show that P/Q is closed under the left action of I_G, for it would realize U as an affine Demazure submodule of $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee \cong V^1(\hat{\mathfrak{g}})$.

To show P/Q that is closed under the left action of I_G, recall the Cartan decomposition as follows. According to [PR08, 8.a, 8.1], the inclusion $M_F \hookrightarrow G_F$ induces an inclusion of affine Weyl groups $W_{M,\mathrm{aff}} \hookrightarrow W_{G,\mathrm{aff}}$ that is compatibly identified with the map of double cosets

$$I_M \backslash M(F)/I_M \to I_G \backslash G(F)/I_G.$$

therefore $I_M \backslash M(F)/I_M \to I_G \backslash G(F)/I_G$ is injective. Therefore left I_M orbits $M(F)/I_M \subseteq G_F/I_G$ are closed under the left action of I_G. Therefore the I_M orbit P/Q is closed under the left action of I_G as desired.

Finally, the map $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{T}^\sharp})^\vee \to -(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee$ is nonzero, because the one dimensional subspace of dual sections $\Gamma(e, \mathcal{L}_e)^\vee$ corresponding to the neutral point $e \in \mathcal{F}_{\mathcal{T}^\sharp}(k)$ is the Demazure submodule of $\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee \cong V^1(\hat{\mathfrak{g}})$ associated to the Schubert variety of the neutral point of Gr_G. ■

3.4 Global Flag Varieties and Geometric Proof for Homogeneous Conjugacy Classes

The main purpose of this final section is to prove 3.14. The method is to deduce it from the case when $\mathcal{T} = T_F$ of [Zhu09] by cohomology and base change in a family over the curve $C = \mathbb{A}^1$ whose completed local ring at 0 is $\mathcal{O}_F$. Similar kinds of ideas have appeared in [BH20] when the split group G_F over F is replaced by a ramified group but, $\mathcal{T}$ is a torus corresponding to a purely outer automorphism of the base change of the group to the algebraic closure. We expect our techniques to directly follow through for the setting of [BH20], under an analogous notion of a principal tori in a ramified group and an analogous notion of homogeneous conjugacy classes for twisted Weyl groups. Indeed, the results of [AHN20] we used were already fully extended to the case of twisted Weyl groups.

This section is designed to be read after the previous one. Preserve all of the notation and setup of the previous Section 3.3. Recall the maps

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$$

of 3.14. Since $\mathcal{T}^\sharp$ is an intermediate integral model of $\mathcal{T}$, $\mathcal{F}_{\mathcal{T}^\sharp}$, is zero-dimensional and the Kottwitz homomorphism induces an identification of the reduced locus with the connected components

$$\mathcal{F}_{\mathcal{T}^\sharp,\mathrm{red}} \approxeq \pi_0(\mathcal{F}_{\mathcal{T}^\sharp}).$$

Identify $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})_{\mathrm{tor}}$, which happens to be the maximal compact subgroup of $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})$, with its preimage in $\mathcal{F}_{\mathcal{T}^\sharp,\mathrm{red}}$. The Kottwitz homomorphism identifies $\pi_0(\mathcal{F}_{\mathcal{T}^{\flat,0}}) \approxeq Y_w$ and thus also identifies $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})_{\mathrm{tor}}$ with the quotient $Y_{w,\mathrm{tor}} = \Sigma$ by the connected components of $Q(c, M) \cap \mathcal{T}(F)$. Similar to the proof of 3.17, there is a characterization as $\hat{L\mathcal{T}}$-modules.

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee = \mathrm{Ind}^{\hat{L\mathcal{T}}}_{(L^+\hat{+}\mathcal{T}^{\flat,0} \times (\hat{\Sigma} \times T^{w,0}))/\mu_r} U$$

where

$$U = \Gamma(\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})_{\mathrm{tor}}, \mathcal{L}_{\pi_0(\mathcal{F}_{\mathcal{T}^\sharp,\mathrm{tor}})})^\vee$$

is a representation of $\hat{\Sigma}$ because $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})_{\mathrm{tor}}$ is stable under the action of Σ. Furthermore U is a single weight space for $T^{w,0}$ because it maps to $\Gamma(P/I_M, \mathcal{L}_{P/I_M})^\vee$ which is a single weight space for $T^{w,0}$ as proven in 3.17. We have also already shown in the proof of 3.13 that $U \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$ is nonzero.

Therefore according to the representation theory of $\hat{L\mathcal{T}}$ in 2.2.6 and as in the proof of 3.17, the nonzero map $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr}_\mathrm{G}, \mathcal{L}_G)^\vee$ is injective iff U is irreducible as a representation of $\hat{\Sigma}$ iff $\dim U = d(c)$, the defect of 2.34. This proves 3.15, the full consequence we put here for convenience:

Theorem 3.16: *Suppose for each conjugacy class $c \in [W]$ there exists $w \in W$ and a lift σ of W such that $\pi_0(\mathcal{F}_{\mathcal{T}^\sharp})_{\mathrm{tor}} = \pi_1(P(\sigma, M)/Q(\sigma, M))$ equals the defect $d(c)$ of c, which is given explicitly in 2.34 in terms of only the lattice action of the conjugacy class of w and the bilinear form B. Then both dual restriction maps*

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathcal{S}(\sigma), \mathcal{L}_{\mathcal{S}(\sigma)})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$$

are isomorphisms of $\hat{L\mathcal{T}}$ modules.

As explained in the previous Section 3.3, we presently only know this for homogeneous c.

3.4.1 Global Analogues of the Tori, Loop Groups, Flag Varieties, Schubert Varieties and the Line bundle

Let us introduce global versions of $\mathcal{T}, \mathrm{Gr}_G$, $\mathcal{L}$, etc. Briefly recall the setup as in [Zhu14] as below. Let C be a smooth curve over k, although we will only use $\mathbb{A}^1$ or $\mathbb{A}^1 \backslash \{0\}$. Let $\mathcal{G}_C$ be a smooth affine group scheme over C, where we use the subscript C to emphasize the global nature. Let R be a k-algebra and $y : \mathrm{Spec}R \to C$ be an R-point. Denote by $\Gamma_y \subseteq C_R$ the closed subscheme of the graph of y. Let $\hat{\Gamma}_y$ be the affine scheme given by the relative spectrum of the ring of regular functions along the formal completion of C_R along Γ_y. Let $\hat{\Gamma}_y^\circ$ denote the complement of the natural closed immersion $\Gamma_y \hookrightarrow \hat{\Gamma}_y$.

Definition 3.21: *The **global jet group**, or **global positive loop group**, is the functor*

$$L^+\mathcal{G}_C(R) = \{(y, \beta) : y \in C(R), \beta \in \mathcal{G}_C(\hat{\Gamma}_g)\}.$$

The ***global loop group*** *is the functor*

$$L\mathcal{G}_C(R) = \{(y, \beta) : y \in C(R), \beta \in \mathcal{G}_C(\hat{\Gamma}^{\circ}_y)\}.$$

The ***global flag variety*** *is the fpqc quotient*

$$\mathcal{F}_{\mathcal{G}_C} = [L\mathcal{G}_C / L^+\mathcal{G}_C].$$

When $\mathcal{G}_C = G_C = G \times_k C$*, put*

$$\mathrm{Gr}_{G_C} := \mathcal{F}_{G_C}$$

and call it the ***global affine Grassmannian****.*

They are all functors on k*-algebras.*

Remark that unlike their local counterparts, the global loop groups are not defined by Weil restriction. It is known by [Zhu14, 3.1, 3.3] that $L^+\mathcal{G}_C$ is a formally smooth but not necessarily finite type scheme over C and $L\mathcal{G}_C, \mathcal{F}_{\mathcal{G}_C}$ are formally smooth ind schemes over C where $\mathcal{F}_{\mathcal{G}_C}$ is ind-proper. The fibers of $L^+\mathcal{G}_C, L\mathcal{G}_C$, and $\mathcal{F}_{\mathcal{G}_C}$ at closed points $x \in C$ are their local counterparts. To be precise, let $\mathcal{O}_x$ be the completed local ring at x and F_x the fraction field. Then there are natural isomorphisms

$$(L\mathcal{G}_C)_x \approxeq L(\mathcal{G}_{C,F_x})$$

$$(L^+\mathcal{G}_C)_x \approxeq L^+(\mathcal{G}_{C,\mathcal{O}_x})$$

$$(\mathcal{F}_{\mathcal{G}_C})_x \approxeq \mathcal{F}(\mathcal{G}_{C,\mathcal{O}_x})$$

where for an R-point of C, $\mathcal{G}_{C,R}$ means the restriction of $\mathcal{G}_C$ by $\mathrm{Spec}R \to C$.

Now for the remainder of this section, put $C = \mathbb{A}^1 = \mathrm{Spec}k[t]$ and $\overset{\circ}{C} = C \backslash \{0\}$. Let us extend $\mathcal{T}^\sharp$ to a global group scheme $\mathcal{T}^\sharp_C$ whose fiber at $\mathcal{O}_0$ is $\mathcal{T}^\sharp$ and fiber at $\mathcal{O}_x$ for $x \neq 0$ is non-canonically isomorphic to T_x. First, let us extend 3.7 to a global version replacing F with $\overset{\circ}{C}$:

Theorem 3.17: *Preserve the notation of* $\sigma, \lambda, B_{\mathrm{ad}}, H_{\mathrm{ad}}$ *of 3.7. Let* $\tilde{\overset{\circ}{C}} = \mathrm{Spec}k[u^{\pm 1}]$

be the degree m Galois cover with Galois group Γ where we canonically extend ν to an automorphism of $\tilde{\mathring{C}}$. Consider $m\lambda \in \mathbb{X}_*(H_{\mathrm{ad}}) = \mathrm{Hom}(\mathbb{G}_m, H_{\mathrm{ad}})$ and let $u^{m\lambda} \in H_{\mathrm{ad}}(k[u^{\pm 1}]) = H_{\mathrm{ad}}(\tilde{\mathring{C}})$ be the image of $u \in k[u^{\pm 1}]^\times = \mathbb{G}_m(k[u^{\pm 1}])$ under $m\lambda$. Then the map $\mathrm{Ad}(u^{m\lambda}) : M_{\tilde{\mathring{C}}} \to M_{\tilde{\mathring{C}}}$ induces an isomorphism of closed subgroup schemes of $\mathrm{Res}_{\tilde{\mathring{C}}} M_{\tilde{\mathring{C}}}$ over $\mathring{C}$:

$$M_{\mathring{C}} = (\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} M \times_k \tilde{\mathring{C}})^{\mathrm{id}\times\nu^{-1}} \overset{\mathrm{Ad}(u^{m\lambda})}{\to} (\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} M \times_k \tilde{\mathring{C}})^{\sigma\times\nu^{-1}}.$$

Proof: First we show it over the generic fiber $\eta = \mathrm{Spec}(k(t))$. Then it follows identically to the proof of 3.7 with the field $F = k((t))$ replaced with $k(t)$; all of the steps remain valid.

It remains to show why it is sufficient to check on the generic fiber. To this end, since $M_{\mathring{C}}$ and $(\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} M \times_k \tilde{\mathring{C}})^{\sigma\times\nu^{-1}}$ are closed sub group schemes of $\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} M \times_k \tilde{\mathring{C}}$, it suffices to show that the generic fibers are dense. According to [Edi92, 3.4], both are smooth, so in particular reduced. Therefore by [Sta18, Tag 0CC1] it suffices to show that each is irreducible. It suffices to show that each is connected in addition to being smooth. This follows from the fact that both are fiber bundles over the connected base $\mathring{C}$ with connected fibers that are each isomorphic to M. ■

The extension of $\mathcal{T}^\sharp$ to C is now as follows. Notably for our work here, we do not use a global version of $\mathcal{P}(\sigma, M)$, i.e., a group scheme over C whose fiber at the completed local ring at $x = 0$ is $\mathcal{P}(\sigma, M)$ and at $x \neq 0$ is isomorphic $M_{\mathcal{O}_F}$, but it could also be defined if desired.

Theorem 3.18: *There exists a subgroup scheme $\mathcal{T}_C^{\#} \subseteq G_C$ over C such that for each closed point $x \in C$, we have the fiber at the completed local ring $\mathcal{O}_x$ is given as follows. Let F_x be the fraction field of $\mathcal{O}_x$.*

1. *For $x = 0$,*
$$(\mathcal{T}_C^\sharp)_{\mathcal{O}_x} = \mathcal{T}^\sharp.$$

2. *For $x \neq 0$, $(\mathcal{T}_C^\sharp)_{\mathcal{O}_x}$ is a split maximal torus of $G_{\mathcal{O}_x}$.*

Proof: The idea is to define on the open curve and glue with $\mathcal{T}^{\#}$. Define the torus $\mathcal{T}_{\mathring{C}} \subset G_{\mathring{C}}$ to be the composition

$$\mathcal{T}_{\mathring{C}} := (\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} T \times_k \tilde{\mathring{C}})^{w\times\nu^{-1}} \hookrightarrow (\mathrm{Res}_{\tilde{\mathring{C}}/\mathring{C}} M \times_k \mathring{C})^{\sigma\times\nu^{-1}}$$

$$\overset{\mathrm{Ad}(u^{-m\lambda})}{\to} M_{\mathring{C}} \hookrightarrow G_{\mathring{C}}$$

where $\mathrm{Ad}(u^{-m\lambda})$ is the isomorphism from 3.17. Then for all closed points $x \in \mathring{C}$, $(\mathcal{T}_{\mathring{C}})_{\mathcal{O}_x}$ is a split maximal torus of $G_{\mathring{C}}$.

Restricting $\mathcal{T}_{\mathring{C}}$ to $F = k((t)) \supset k(t)$ gives the torus of type $\mathcal{T}$ of 3.10. Finally, applying the descent lemma [Hei10, lem. 5] along the fpqc cover $C = \mathring{C} \cup \mathrm{Spec}(\mathcal{O}_F)$ gives the desired global group scheme $\mathcal{T}_C^\sharp$ and a glued morphism $\mathcal{T}_C^\sharp \to G_C$. Consequently, $\mathcal{T}_C^\sharp$ can also be described as the Zariski closure of $\mathcal{T}_{\mathring{C}}$ in G_C. ∎

Applying the functors $L(-), L^+(-), \mathcal{F}_{(-)}$, we obtain subgroup schemes $L\mathcal{T}_C \hookrightarrow LG_C$, $L^+\mathcal{T}_C \hookrightarrow L^+G_C$ and a morphism $\mathcal{F}_{\mathcal{T}_C^\sharp} \to \mathrm{Gr}_{G_C}$.

Lemma 3.18: *The map $\mathcal{F}_{\mathcal{T}_C^\sharp} \to \mathrm{Gr}_{G_C}$ is a closed embedding.*

Proof: It is a monomorphism by [Sta18, tag 01L1], the fact that the pullback to each point of C is a monomorphism and the fact that every point of $\mathcal{F}_{\mathcal{T}_C^\sharp}$ lies over either the closed point or generic point of C. The result follows from the fact that both sides are ind-proper over C. ∎

In conclusion, we have constructed a closed embedding of global flag varieties $\mathcal{F}_{\mathcal{T}_C^\sharp} \to \mathrm{Gr}_{G_C}$ such that the fibers over closed points $x \in C$ are described by:

- For $x = 0$, $(\mathcal{F}_{\mathcal{T}_C^\sharp})_x \to (\mathrm{Gr}_{G_C})_x$ is equal to the embedding of local flag varieties $\mathcal{F}_{\mathcal{T}^\sharp} \hookrightarrow \mathrm{Gr}_G$ of 3.13.
- For $x \neq 0$, after fixing an identification $(\mathrm{Gr}_{G_C})_x \approxeq \mathrm{Gr}_G$ and using the conjugacy of maximal split tori in $G(F)$, the map $(\mathcal{F}_{\mathcal{T}_C^\sharp})_x \to (\mathrm{Gr}_{G_C})_x$ is $LG(k)$-conjugate to the inclusion $\mathrm{Gr}_T \hookrightarrow \mathrm{Gr}_G$.

Let us now define the global level 1 line bundle $\mathcal{L}_C$, global Schubert varieties of Gr_{G_C} and recall a refinement of the geometric split FKS isomorphism of [Zhu09] in terms of the global Schubert varieties.

Let $\mathcal{L}$ be a line bundle on Gr_{G_C}. Then for each closed point $x \in C$, the restriction of $\mathcal{L}$ to $(\mathrm{Gr}_{G_C})_x$ is isomorphic to some integer c power of the ample generator of $\mathrm{Pic}((\mathrm{Gr}_{G_C})_x) \approxeq \mathbb{Z}$. It is proven in [Zhu14, 4.1] that c is constant as function of x, and it is called the **central charge** of $\mathcal{L}$.

The group scheme G_C has the property that all fibers at $x \in C$ are semisimple. Therefore by [Zhu14, 4.1.1], the relative Picard group $\mathrm{Pic}(\mathrm{Gr}_{G_C}/C)$ is a constant étale sheaf on C isomorphic to $\overline{\mathbb{Z}}$. By [Zhu09, 1.1.9], we can choose a generator and a representative line bundle $\mathcal{L}_{G_C}$ with central charge 1, and we call it the **global level 1** line bundle on Gr_{G_C}. In particular, the restriction of $\mathcal{L}_{G_C}$ to each closed point $x \in C$ is the level 1 line bundle on $(\mathrm{Gr}_{G_C})_x$. Consider the global split torus $T_C \subseteq G_C$ and naturally identify the cocharacter lattices at each point $\mathbb{X}_*((\mathcal{T}_C)_{F_x}) \approxeq Y$ using the structure map $k \hookrightarrow K_x$. Then according to [Zhu14, 3.4], for each $\mu \in Y$, there exists a section $s_\mu : C \to LT_C$ with the property that for any closed point $x \in C$, $s_\mu(x) \in (LT_C)_x(k) = T(F_x)$ maps to μ under the Kottwitz homomorphism $LT_C \to Y$.

Definition 3.22: *The left L^+G_C orbit of s_μ in Gr_{G_C} is denoted $\mathrm{Gr}_{G_C,\mu}$. The closure $\overline{\mathrm{Gr}_{G_C,\mu}}$ is called the **global Schubert variety** associated to μ.*

Then the fiber at 0 of $\mathrm{Gr}_{G_C,\mu}$, resp. $\overline{\mathrm{Gr}_{G_C,\mu}}$, is the local version $\mathrm{Gr}_{G,\mu}$, resp. $\overline{\mathrm{Gr}_{G,\mu}}$. By [Zhu14, theorem 3], the fibers of $\overline{\mathrm{Gr}_{G_C,\mu}}$ at each closed point $x \in C$ (not just at $x = 0$) are isomorphic to $\overline{\mathrm{Gr}_{G,\mu}}$. Two Schubert varieties (either global or local) associated to $\lambda, \mu \in Y$ are the same iff they lie in the same W-orbit and the Schubert varieties give rise to all the positive loop group orbit closures [Zhu09, 1.1.7]. A set of representative μ for each orbit can be chosen by taking the subset $Y_+ \subseteq Y$ of dominant cocharacters with respect to some choice of Borel subgroup B_G of G.

Fix such a choice B_G. We have the following from [Zhu09, 1.1.4]. $\lambda \leq \mu$ in Y iff $\overline{\mathrm{Gr}_{G,\lambda}} \subseteq \overline{\mathrm{Gr}_{G,\mu}}$. Define

$$\mathrm{Gr}_{T,\mu} := \overline{\mathrm{Gr}_{G,\mu}} \cap \mathrm{Gr}_T.$$

Then also

$$\mathrm{Gr}_G = \bigcup_{\mu \in Y} \overline{\mathrm{Gr}_{G,\mu}}$$

$$\mathrm{Gr}_T = \bigcup_{\mu \in Y} \mathrm{Gr}_{T,\mu}.$$

By [Zhu09, 2.3.5] for $\lambda \leq \mu$, the map $\Gamma(\overline{\mathrm{Gr}_{G,\lambda}}, \mathcal{L}_{\mathrm{Gr}_{G,\mu}})^\vee \to \Gamma(\overline{\mathrm{Gr}_{G,\mu}}, \mathcal{L}_{\mathrm{Gr}_{G,\mu}})^\vee$ is injective. Thus the map $\Gamma(\mathrm{Gr}_{T,\lambda}, \mathcal{L}_{\mathrm{Gr}_{T,\lambda}})^\vee \to \Gamma(\mathrm{Gr}_{T,\mu}, \mathcal{L}_{\mathrm{Gr}_{T,\mu}})^\vee$ is also injective because $\mathrm{Gr}_{T,\lambda}, \mathrm{Gr}_{T,\mu}$ are zero dimensional. We therefore have

$$\Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee = \bigcup_{\mu \in Y} \Gamma(\overline{\mathrm{Gr}_{G,\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G,\mu}}})^\vee$$

$$\Gamma(\mathrm{Gr}_T, \mathcal{L}_T)^\vee = \bigcup_{\mu \in Y} \Gamma(\mathrm{Gr}_{T,\mu}, \mathcal{L}_{\mathrm{Gr}_{T,\mu}})^\vee.$$

We record the main theorem [Zhu09, 0.0.2]:

Theorem 3.19: *For all $\mu \in Y$, the map*

$$\Gamma(\mathrm{Gr}_{T,\mu}, \mathcal{L}_{\mathrm{Gr}_{T,\mu}})^\vee \to \Gamma(\overline{\mathrm{Gr}_{G,\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G,\mu}}})^\vee$$

is an isomorphism. In particular, both have the same dimension.

Consequently, $\Gamma(\mathrm{Gr}_T, \mathcal{L}_T)^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_G)^\vee$ is also an isomorphism, which recovers the geometric FKS isomorphism for $\mathcal{T} = T$.

3.4.2 Proof of Main Theorem

We prove 3.14. Preserve the setup of the global objects of the previous subsection.

Put

$$\mathcal{F}_{\mathcal{T}^\circ_C,\mu} := \mathcal{F}_{\mathcal{T}^\circ_C} \cap \overline{\mathrm{Gr}_{G_C,\mu}}.$$

Define the global closed subscheme $\mathcal{F}_{\mathcal{T}^\sharp_C,\mu} \subseteq \mathcal{F}_{\mathcal{T}^\sharp_C}$ to be the scheme theoretic closure

$$\mathcal{F}_{\mathcal{T}^\sharp_C,\mu} := \overline{\mathcal{F}_{\mathcal{T}^\circ_C,\mu}} \subseteq \mathcal{F}_{\mathcal{T}^\sharp_C}.$$

It factors through $\overline{\mathrm{Gr}_{G_C,\mu}}$. Since $\mathcal{F}_{\mathcal{T}^\sharp_C} \subseteq \mathrm{Gr}_{G_C}$is closed embedding, $\mathcal{F}_{\mathcal{T}^\sharp_C,\mu}$ is a finite type closed subscheme of Gr_{G_C} over C. Then for a closed point $x \in C$, the fiber at $x \neq 0$ of $\mathcal{F}_{\mathcal{T}^\sharp_C,\mu}$ is $LG(k)$-conjugate to the inclusion $\mathrm{Gr}_{T,\mu} \subseteq \mathrm{Gr}_G$ and factors through $\overline{\mathrm{Gr}_{G,\mu}}$.

Define the space $\mathcal{F}_{\mathcal{T}^\sharp,\mu} \subseteq \mathcal{F}_{\mathcal{T}^\sharp} \subseteq \mathrm{Gr}_G$ as required in 3.14 to be the fiber at 0:

$$\mathcal{F}_{\mathcal{T}^\sharp,\mu} = (\mathcal{F}_{\mathcal{T}^\sharp_C,\mu})_0.$$

Lemma 3.19: *The schemes $\mathcal{F}_{\mathcal{T}^\sharp,\mu}$ and $\overline{\mathrm{Gr}_{G,\mu}}$ are flat over C.*

Proof: Apply [Zhu14, 6.1.4] and obtain that $\overline{\mathrm{Gr}_{G,\mu}}$ is Cohen-Macaulay. Since C is regular and $\overline{\mathrm{Gr}_{G,\mu}} \to C$ is proper, thus closed, it is also flat by equidimensionality of the fibers and miracle flatness [Gro65, 6.1.5].

The finite type scheme $\mathcal{F}_{\mathcal{T}_{\mathring{C}},\mu}$ is zero dimensional over $\mathring{C}$. Therefore all closed points of $\mathcal{F}_{\mathcal{T}_{\mathring{C}}}$ are zero dimensional and by [Sta18, Tag 021N], $\mathcal{F}_{\mathcal{T}_{\mathring{C}},\mu}$ is Cohen-Macaulay. By the equidimensionality of the fibers, $\mathcal{F}_{\mathcal{T}_{\mathring{C}},\mu}$ is flat over $\mathring{C}$. Since $\mathring{C} = \mathrm{Spec} k[t^{\pm 1}]$ is a PID, we can use the flatness criterion of [Eis95, 6.3]. The closure $\mathcal{F}_{\mathcal{T}^\sharp_C,\mu}$ in the flat $\overline{\mathrm{Gr}_{G_C,\mu}}$ scheme over C is the spectrum of an algebra over $k[t^{\pm 1}]$ that is torsion-free. The result follows. ■

We now give the final proof:

Proof: (of 3.14). Since both $\mathcal{F}_{\mathcal{T}^\sharp,\mu}$ and $\overline{\mathrm{Gr}_{G,\mu}}$ are flat over C, the line bundles $\mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}}$ and $\mathcal{L}_{\overline{\mathrm{Gr}_{G,\mu}}}$ are both flat over C, by [Sta18, Tag 01U2]. Fix a closed point $x \in C$. Since $(\mathcal{F}_{\mathcal{T}^\sharp,\mu})_x$ is zero dimensional, $\mathcal{L}_{(\mathcal{F}_{\mathcal{T}^\sharp,\mu})_x}$ is trivial and

$$\mathrm{H}^1((\mathcal{F}_{\mathcal{T}^\sharp_C,\mu})_x, \mathcal{L}_{(\mathcal{F}_{\mathcal{T}^\sharp_C,\mu})_x}) = 0$$

Since $\mathcal{L}_{(\overline{\mathrm{Gr}_{G_C,\mu}})_x}$ is ample and $(\overline{\mathrm{Gr}_{G_C,\mu}})_x$ is a Schubert variety, by [Kum02, 8.1.8],

$$\mathrm{H}^1((\overline{\mathrm{Gr}_{G_C,\mu}})_x, \mathcal{L}_{(\overline{\mathrm{Gr}_{G_C,\mu}})_x}) = 0.$$

By upper semi-continuity, we conclude that there is the same vanishing of H^1 for both when x is the generic point of C as well. By cohomology and base change, the dimensions $\dim \Gamma((\mathcal{F}_{\mathcal{T}^\sharp_C,\mu})_x, \mathcal{L}_{(\mathcal{F}_{\mathcal{T}^\sharp_C,\mu})_x})$ and $\dim \Gamma((\overline{\mathrm{Gr}_{G_C,\mu}})_x, \mathcal{L}_{(\overline{\mathrm{Gr}_{G_C,\mu}})_x})$ are each independent of $x \in C$. By the split case 3.19, and the $\hat{LG}$ equivariance of $\mathcal{L}_G$, their dimensions coincide when $x \neq 0$. Therefore the dimensions coincide at $x = 0$ as well. Therefore taking the fiber at 0 and the dual, the map

$$\Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee \to \Gamma(\overline{\mathrm{Gr}_{G_\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G_\mu}}})^\vee$$

is a map of vector spaces of the same dimension. For $\lambda \leq \mu$, $\mathcal{F}_{\mathcal{T}^\sharp,\lambda} \subseteq \mathcal{F}_{\mathcal{T}^\sharp,\mu}$. Since both are zero dimensional, the maps $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\lambda}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\lambda}})^\vee \to \Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee$ are injective. Since $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ is injective by assumption, we conclude that $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee \to \Gamma(\overline{\mathrm{Gr}_{G_\mu}}, \mathcal{L}_{\overline{\mathrm{Gr}_{G_\mu}}})^\vee$ is also injective, and therefore an isomorphism. Now put $X = \bigcup_{\mu\in Y} \Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee$ and consider the composition

$$\Gamma(X, \mathcal{L}_X)^\vee \to \Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee.$$

The composition is an isomorphism because $\Gamma(\mathrm{Gr}_T, \mathcal{L}_T)^\vee = \bigcup_{\mu\in Y} \Gamma(\mathrm{Gr}_{T,\mu}, \mathcal{L}_{\mathrm{Gr}_{T,\mu}})^\vee$. Therefore the same is true for both left and right arrows. We conclude that $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee \to \Gamma(\mathrm{Gr}_G, \mathcal{L}_{\mathrm{Gr}_G})^\vee$ is an isomorphism as desired. ∎

Remark 3.14: *By showing that* $\Gamma(\mathcal{F}_{\mathcal{T}^\sharp}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp}})^\vee = \bigcup_{\mu\in Y} \Gamma(\mathcal{F}_{\mathcal{T}^\sharp,\mu}, \mathcal{L}_{\mathcal{F}_{\mathcal{T}^\sharp,\mu}})^\vee$, *we have shown that* $\mathcal{F}_{\mathcal{T}^\sharp} = \bigcup_{\mu\in Y} \mathcal{F}_{\mathcal{T}^\sharp,\mu}$ *(since they are zero dimensional) and thus* $\mathcal{F}_{\mathcal{T}^\sharp_C} = \bigcup_{\mu\in Y} \mathcal{F}_{\mathcal{T}^\sharp_C,\mu}$. *This gives a highly indirect proof, conditional factors known only when* $\mathcal{T}$ *is a principal torus of the type of a homogeneous conjugacy class in a group* G_F *when* G *is type ADE, that:*

Corollary 3.3: *Assume the hypothesis of 3.14. Then* $\mathcal{F}_{\mathcal{T}^\sharp_C}$ *is ind-flat over* C.

Remark 3.15: *Ind-flatness of a global flag variety seems like a basic property that should not need such an indirect proof. We wonder to what extent ind-flatness of* $\mathcal{F}_{\mathcal{T}^\sharp_C}$ *holds when* $\mathcal{T}^\sharp_C$ *is more general group scheme over* C, *perhaps a Bruhat-Tits group scheme in the sense that it is generically reductive and the restriction to each* *completed local ring at a closed point is a parahoric group scheme of the further restriction to the fraction field of the completed local ring.*

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