Theory of Mathematical Optimization for Delegated Portfolio Management

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Zijian Tao

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ABSTRACT

We study the optimization problem of finding closed convex sets $\Gamma \subseteq \mathbb{R}^d$ containing the origin that minimize

$$\mathcal{F}(\Gamma) = \sum_{i=1}^{k} w_i \left| \frac{\theta_i}{2} - p_\Gamma(\theta_i) \right|^2,$$

where $w_1, \ldots, w_k > 0$, $\theta_1, \ldots, \theta_k \in \mathbb{R}^d$ are given, and $p_\Gamma(\theta_i)$ are the closest points in Γ to θ_i, $i = 1, \ldots, k$. This problem is motivated by the topic of delegated portfolio management in finance. In Chapter 2, we will explore this connection. To approach the problem, we first prove existence of a solution for the general problem. To further study properties of the solution, we next introduce the semidefinite programming relaxation, for which we have a first-order characterization of optimality. We then explore the question of exactness of this relaxation, which turns out to be equivalent to the notion of localizability: the shape optimization problem embedded in higher dimensions must have solutions in the original dimension. Finally, we present special cases for which localizability holds.

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Chapter 1

INTRODUCTION

In this paper we study the following optimization problem. Let $\theta_1, \ldots, \theta_k$ be points in Euclidean space $\mathbb{R}^d$. Given a closed convex set Γ and a point $\theta \in \mathbb{R}^d$, let $p_\Gamma(\theta)$ be the unique closest point in Γ to θ. One wishes to find, among all closed convex sets containing the origin, one that minimizes the objective functional

$$\mathcal{F}(\Gamma) = \sum_{i=1}^{k} w_i \left| \frac{\theta_i}{2} - p_\Gamma(\theta_i) \right|^2,$$

where $w_i > 0$ are positive weights. Optimization problems of this type, where the decision variable is a set, is often known as shape optimization. Examples of such problems can be found in [2], [7], and [10].

Our approach to the problem consists of three steps. First, we establish a general existence theorem for optimality via techniques analogous to the direct method of calculus of variations. Actually characterizing the solution, on the other hand, is nontrivial. To the best of our knowledge, there is no applicable theory such as "shape calculus" that permits a first-order characterization of the optimal solution. This difficulty necessitates our second step. We switch the decision variable from the convex set Γ to the closest points p_i in Γ to θ_i. The geometric constraints of the shape optimization problem will be recast as a family of quadratic inequalities involving p_i. The problem now simplifies to a quadratic program with quadratic constraints (QCQP). Unfortunately, the constraints are nonconvex, making the problem intractable. In the third step, we convexify the problem using semidefinite programming (SDP) relaxation. This relaxed problem is much more tractable. In particular, it is possible to obtain a first-order characterization of solutions, as well as compute a numerical solution to arbitrary accuracy in polynomial time. This SDP relaxation provides a non-trivial lower bound for the optimal value of the complete problem. The issue that remains is whether the relaxation can recover an exact solution to the original quadratic program. This question turns out to be equivalent to the following problem about localizability. It asks whether the solution to the optimization problem would change when $\theta_1, \ldots, \theta_k$ are embedded in a higher dimensional space $\mathbb{R}^{d'}$, and Γ is allowed to vary over all closed convex sets in $\mathbb{R}^{d'}$ containing the origin. We establish localizability in certain special cases, including (i) when $\theta_1, \ldots, \theta_k$ are linearly independent, and (ii) when $\theta_1, \ldots, \theta_k$ lie on a line and are sufficiently far from the origin. The general problem, however, remains open.

Our optimization problem has its roots in the finance of delegated portfolio management. Specifically, consider a mutual fund facing a set $\{1, \ldots, k\}$ of potential investors. The mutual fund is a monopoly: investors do not have to invest their money, but if they choose to, they must invest through the mutual fund. The market consists of several risky assets, and all investors have the same mean-variance utility function with the same risk aversion. Investors are heterogeneous, characterized by their initial endowment in the risky assets (henceforth referred to as types, denoted θ). Knowing the distribution of investor types, the mutual fund designs a menu of funds to be offered, each of which charges a fee. Here, we assume fees are linear, and that investors are free to allocate their wealth across multiple funds. Thus, if fund x_1 is available with fee π_1, x_2 is available with fee π_2, then for $q_1, q_2 \geq 0$. $q_1 x_1 + q_2 x_2$ must be available with fee $q_1 \pi_1 + q_2 \pi_2$. The mutual fund's problem is to decide the set of funds to offer and their associated fees in a way that maximizes the aggregate fee collected. This is the optimal fund menus problem (OFM). As it turns out, OFM is equivalent to minimizing $\mathcal{F}(\Gamma)$. In Chapter 1, we will prove a correspondence theorem that establishes the precise relation between the two problems. Under this formulation, the aforementioned notion of localizability also has important implications. Financially, if the mutual fund is allowed to introduce assets with new sources of risk, would the optimal menu entail investors (whose goal is to hedge their original risk exposures) taking on these new risks? The answer is "no" if and only if localizability holds.

Our paper contributes to the literature of mathematical programming, especially semidefinite programming (SDP). Semidefinite programming is a subclass of convex optimization problems which generalizes linear programming. It is widely applicable to problems in engineering and applied sciences. Many nonconvex problems admit an SDP relaxation, which produces an approximate optimal solution of the original problem, and in some cases, an exact solution. The classic example is [8], where SDP relaxation is used to find a near optimal solution to the max cut problem. Examples of SDP relaxation providing an exact solution to the original problem can be found in [15], [9], [4], [3] and [14]. In particular, certain techniques in our paper are inspired by [14] in which the authors study the sensor network localization (SNL) problem. The problem involves finding a feasible configuration of points in a Euclidean space of a given dimension, subject to constraints on the pairwise distances. The authors relax certain constraints of SNL to form an SDP, and show that this relaxation is exact if and only if adding additional dimensions does not change the optimal configuration. In our paper, we use SDP to relax the optimal fund menu problem, and show that the relaxation is exact if and only if adding zero-return assets to which the investors have no exposure does not change the optimal menu. However, we currently do not know if SDP relaxation always provides an exact solution to the optimal fund menus problem. There is no general criterion in mathematical programming that guarantees exactness. The aforementioned examples all exploit very special structures in their respective problems. At this point, we can only establish exactness for our problem for special cases. On the other hand, numerical experiments suggest that exactness might hold in general, prompting further work in this direction.

Our paper also contributes to the literature of screening and asset bundling. It can be seen as a generalization of the model proposed in [6], where a monopolistic mutual fund manager faces investors with different beliefs in the return of one asset. In their paper, the dual approach is used. That is, rather than optimize directly over fund menus, they optimize over the space of indirect utility functions induced by all possible fund menus. Such indirect utility functions are not arbitrary, and must satisfy certain incentive compatibility constraints. To solve the optimization problem, the authors temporarily drop the incentive compatibility constraints on the indirect utility function, solve a calculus of variations problem, and show that at optimum, the constraint does not bind. In contrast, the heterogeneity of investors in our paper involves differing initial exposures to possibly more than one asset. While we also adopt the general dual approach by considering all indirect utility functions induced by menus, our method of study is significantly different. For our model, incentive compatibility requires that an indirect utility function be of the form squared distance to a convex region. The exclusion of this constraint will, in general, not give a feasible solution to the complete problem. Our model also provides a linear pricing analog to the ones studied in [11], [12], and [5], in which a monopoly is allowed to use nonlinear pricing to screen customers with heterogeneous preferences across one or multiple dimensions. In these models, the space of all feasible indirect utility functions forms a convex cone, making the problem amenable to conic optimization techniques. In contrast, the linear pricing constraint in our paper gives rise to a nonconvex problem. Hence, the techniques developed in the aforementioned papers do not directly apply here.

The structure of this paper is as follows. Chapter 2 describes the problem, the financial model that motivates it, and establishes a precise correspondence between the two. Chapter 3 adapts the direct method of calculus of variations to prove general existence of an optimal solution. Chapter 4 seeks to characterize the solution. In Section 4.1, we derive the quadratic programming formulation of the problem. In Section 4.2, we study the SDP relaxation and its associated properties. Section 4.3 studies the relation between the relaxation and the complete problem, which leads to the notion of localizability. Finally, Section 4.4 explores particular cases in which the SDP relaxation provides an exact solution to the complete problem.

Chapter 2

PROBLEM FORMULATION

2.1 The Shape Optimization Problem

Consider a finite set of points $\theta_1, \ldots, \theta_k$ in Euclidean space $\mathbb{R}^d$, and let $w_1, \ldots, w_k > 0$ be positive weights. For any closed convex set Γ and any point $\theta \in \mathbb{R}^d$, there exists a unique point in Γ that is closest to θ, which we denote by $p_\Gamma(\theta)$. Among all closed convex sets Γ containing the origin, we wish to find one that minimizes

$$\mathcal{F}(\Gamma) = \sum_{i=1}^{k} w_i \left| \frac{\theta_i}{2} - p_\Gamma(\theta_i) \right|^2.$$

While we are mainly concerned with the case of finitely many points, the problem can be formulated more generally. Given a subset Ω of $\mathbb{R}^d$ and a finite measure μ on Ω, one may consider the problem

$$\begin{aligned} \text{maximize}_\Gamma \quad & \mathcal{F}(\Gamma) := \int_\Omega \left| \frac{\theta}{2} - p_\Gamma(\theta) \right|^2 d\mu(\theta) \\ \text{s.t.} \quad & \Gamma \subseteq \mathbb{R}^d \text{ closed and convex}, 0 \in \Gamma. \end{aligned} \tag{2.1}$$

Note that for the finite case, $\Omega = \{\theta_1, \ldots, \theta_k\}$, and $\mu = \sum_{i=1}^k w_i \delta_{\theta_i}$.

Our central question concerns localizability, which we define now.

Definition. We say that (Ω, μ) is **localizable** if it satisfies the following condition:

Let $d' \geq d$ be arbitrary, and let Γ be a closed convex set in $\mathbb{R}^{d'}$ with $0 \in \Gamma$. Let $p(\theta) = p_\Gamma(\theta)$ be the closest point to θ in Γ. If $\int_\Omega \left| \frac{\theta}{2} - p_\Gamma(\theta) \right|^2 d\mu(\theta) = s^*$, where s^* is the optimal value for Problem (2.1) then $p_\Gamma(\theta) \in \mathbb{R}^d$ for all $\theta \in \Omega$.

Essentially, localizability guarantees that the optimal solution does not change when the problem is embedded in higher dimensions. As will be discussed in Chapter 3, this property is key to establishing uniqueness and first-order characterization of the solution.

For the remainder of this chapter, we will describe the financial problem of optimal fund menus that motivated the study of (2.1) and establish their correspondence.

2.2 The Financial Model

We consider a one-period model. The market consists of the riskless asset with return r, and d risky assets with excess return (over the risk-free rate) $\epsilon \in \mathbb{R}^d$. The excess return $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$ is a random vector. The coordinates $\epsilon_1, \ldots, \epsilon_d$ are independent, each having variance 1. We let $\zeta = (\zeta_1, \ldots, \zeta_d) := \mathbb{E}[\epsilon]$.

The market is populated by a monopolistic risk-neutral mutual fund manager and heterogeneous risk-averse investors. Each investor is associated with a type $\theta \in \mathbb{R}^d$. An investor with type $\theta = (\theta_1, \ldots, \theta_d)$ initially owns θ_i dollars worth of the i-th risky asset. All investors have mean-variance utility given by

$$\mathbb{E}[w_1] - \frac{1}{2}\text{var}[w_1],$$

where w_1 is the terminal wealth. The set of all investor types is $\Omega \subseteq \mathbb{R}^d$, and the distribution of types is μ—a finite measure on Ω. We assume μ is known to the mutual fund manager.

A monopolistic mutual fund manager can offer a menu of funds. Each fund is described by (x, π), where $x = (x_1, \ldots, x_d)$, x_i indicating the dollar amount invested in the i-th risky asset, and π is the fee charged to investors. We assume that investors are free to allocate their wealth to multiple funds, and that the fees are subject to linear pricing. It's not possible, however, to take short position in a fund. Hence, if funds (x_1, π_1) and (x_2, π_2) are available, then for $q_1, q_2 \geq 0$, $(q_1 x_1 + q_2 x_2, q_1\pi_1 + q_2\pi_2)$ must also be available. To capture these properties, we define a fund menu as follows.

Definition. A fund menu is a closed convex cone in $\mathcal{M} \subseteq \mathbb{R}^d \times \mathbb{R}_+$ that is closed upward.

Consider an investor of type $\theta = (\theta_1, \ldots, \theta_d)$. Suppose, in addition to the risky assets, she owns an additional C amount of cash (this amount turns out to be irrelevant for utility maximization purpose). Thus, her initial wealth is $w_0 = C + \theta_1 + \cdots + \theta_d$. If she buys one unit of fund (x, π), her terminal wealth is

$$w_1 = w_0(1 + r) + (\theta + x) \cdot \epsilon - \pi.$$

She derives utility

$$\begin{aligned}\mathbb{E}[w_1] - \frac{1}{2}\text{var}[w_1] &= \mathbb{E}[w_0(1+r) + (\theta+x)\cdot\epsilon - \pi] - \frac{1}{2}\text{var}[w_0(1+r) + (\theta+x)\cdot\epsilon - \pi] \\ &= w_0(1+r) + (\theta+x)\cdot\zeta - \pi - \frac{1}{2}|\theta+x|^2 \\ &= \left(w_0(1+r) + \theta\cdot\zeta - \frac{1}{2}|\theta|^2\right) + \left(x\cdot\zeta - \pi - \frac{1}{2}|x|^2 - x\cdot\theta\right).\end{aligned}$$

The investor's goal is to choose a fund (x,π) in the menu that maximizes $\mathbb{E}[w_1] - \frac{1}{2}\text{var}[w_1]$. Note that only the part $x\cdot\zeta - \pi - \frac{1}{2}|x|^2 - x\cdot\theta$ is relevant for the investor's utility maximization problem. Henceforth, we shall write $U_\theta(x,\pi) = x\cdot\zeta - \pi - \frac{1}{2}|x|^2 - x\cdot\theta$. With this notation, the type θ investor's problem is

$$\text{maximize}_{(x,\pi)\in\mathcal{M}} \quad U_\theta(x,\pi). \tag{2.2}$$

Now we consider the fund manager's problem. Loosely, the manager seeks to design a fund menu that maximizes the aggregate fee collected across all investors. More precisely, let $(x_{\mathcal{M}}(\theta), \pi_{\mathcal{M}}(\theta)) \in \arg\max_{(x,\pi)\in\mathcal{M}} U_\theta(x,\pi)$. We will show in the next section that the maximizer is unique. Thus, the manager's fee collected from type θ is well-defined, given by $\pi_{\mathcal{M}}(\theta)$. The aggregate fee collected across all investors is $\int_{\mathcal{M}} \pi_{\mathcal{M}}(\theta)d\mu(\theta)$. Thus the manager's problem is

$$\text{maximize}_{\mathcal{M}} \int_\Omega \pi_{\mathcal{M}}(\theta)d\mu(\theta). \tag{2.3}$$

Remark 1. We impose the assumption that the asset returns are independent with unit variance to simplify computation. In general, suppose there are k risky assets $x_1,\ldots,x_d$ whose returns have covariance matrix Σ. Let $\Sigma = ZZ^T$ be the Cholesky decomposition. Provided Σ is nonsingular, we may create d new assets $y_1,\ldots,y_d$ with uncorrelated, unit variance returns, given by $y_i = \sum_{j=1}^d a_{ij}x_j$, where $A = Z^{-1}$. A fund investing $(q_1,\ldots,q_d)$ in $(x_1,\ldots,x_d)$ at fee rate π is equivalent to one investing $(p_1,\ldots,p_d)$ in $(\xi_1,\ldots,\xi_d)$ with the same fee, where $\begin{pmatrix} p_1 \\ \vdots \\ p_d \end{pmatrix} = Z^T \begin{pmatrix} q_1 \\ \vdots \\ q_d \end{pmatrix}$.

Remark 2. Instead of differing in initial risk exposure, we can also consider a model in which investor type is characterized by his subjective evaluations of the assets' expected returns $\eta = (\eta_1,\ldots,\eta_d) \in \mathbb{R}^d$. The utility of an investor with type η investing quantity x at fee π is

$$\eta\cdot x - \pi - \frac{1}{2}|x|^2.$$

The manager's problem is to design a menu $\mathcal{M}$ that maximizes

$$\int_\Omega \pi_{\mathcal{M}}(\eta) d\mu(\eta),$$

where $(x(\eta), \pi(\eta)) \in \arg\max_{(x,\pi)\in\mathcal{M}} \left\{\eta \cdot x - \pi - \frac{1}{2}|x|^2\right\}$. In particular, the model of [6] is of this form, with μ being the uniform distribution on the line segment $\{(\eta_1, \eta_2) \in \mathbb{R}^2 \mid \eta_1 = \xi, 0 \le \eta_2 \le \theta_H\}$, where ξ and θ_H are constants. Here, ξ is the true mean return rate of the index asset, and θ_H is the most optimistic investor's subjective estimate of the non-index asset's return.

This is mathematically equivalent to the main model where the heterogeneity is initial risk exposure. They are related by a change of variable $\theta = \zeta - \eta$. The model of [6] corresponds to uniform distribution on the segment $\{(\theta_1, \theta_2) \in \mathbb{R}^2 \mid \theta_1 = \zeta_1 - \xi, \zeta_2 - \theta_H \le \theta_2 \le \zeta_2\}$.

Remark 3. One could try to apply the method of [6] to the model of the present paper. Define the indirect utility function

$$u_{\mathcal{M}}(\theta) := \sup_{(x,\pi)\in\mathcal{M}} \left\{x \cdot \zeta - \pi - \frac{1}{2}|x|^2 - x \cdot \theta\right\}.$$

By a direct calculation, we can express the manager's aggregate fee collected as

$$\int_\Omega \{-u_{\mathcal{M}}(\theta) - \frac{1}{2}|\nabla u_{\mathcal{M}}(\theta)|^2 + (\theta - \zeta) \cdot \nabla u_{\mathcal{M}}(\theta)\} d\mu(\theta). \tag{2.4}$$

Moreover, $u_{\mathcal{M}}$ is convex, and satisfies the eikonal equation $2u_{\mathcal{M}}(\theta) = |\nabla u_{\mathcal{M}}(\theta)|^2$. The aggregate fee can be rewritten as

$$\int_\Omega \{-|\nabla u_{\mathcal{M}}(\theta)|^2 + (\theta - \zeta) \cdot \nabla u_{\mathcal{M}}(\theta)\} d\mu(\theta). \tag{2.5}$$

In the special case when $d = 2$, θ_1 is constant, and θ_2 is uniformly distributed in $[\theta_L, \theta_H]$, let $v_{\mathcal{M}}(\theta_2) = u_{\mathcal{M}}(\theta_1, \theta_2)$. From the eikonal equation, we find $\frac{\partial u_{\mathcal{M}}}{\partial \theta_1} = \sqrt{2v_{\mathcal{M}}(\theta_2) - [\dot{v_{\mathcal{M}}}(\theta_2)]^2}$. The aggregate fee is

$$\frac{1}{\theta_H - \theta_L} \int_{\theta_L}^{\theta_H} \{(\theta_2 - \zeta_2)\dot{v_{\mathcal{M}}}(\theta_2) - 2v_{\mathcal{M}}(\theta_2) + (\theta_1 - \zeta_1)\sqrt{2v_{\mathcal{M}}(\theta_2) - [\dot{v_{\mathcal{M}}}(\theta_2)]^2}\} d\theta.$$

This coincides with the manager's value function appearing in [6], after making a change of variable (see Remark 2). In this special case, we may proceed by calculus of variations over all possible functions $v_{\mathcal{M}}$.

In the general case, however, we cannot use calculus of variations to maximize (2.5). Calculus of variations does not take into account the eikonal constraint

$2u_{\mathcal{M}}(\theta) = |\nabla u_{\mathcal{M}}(\theta)|^2$. One cannot temporarily drop this constraint, and hope that the optimal solution of the relaxed problem automatically satisfies it. In fact, this is never true.

$$-|\nabla u_{\mathcal{M}}(\theta)|^2 + (\theta - \zeta) \cdot \nabla u_{\mathcal{M}}(\theta) = -\left|\nabla u_{\mathcal{M}}(\theta) - \frac{\theta - \zeta}{2}\right|^2 + \frac{|\theta - \zeta|^2}{4}$$

so (2.5) is maximized when $\nabla u_{\mathcal{M}}(\theta) = \frac{\theta-\zeta}{2}$. This implies $u_{\mathcal{M}}(\theta) = \frac{|\theta-\zeta|^2}{4} + C$, which does not satisfy $2u_{\mathcal{M}}(\theta) = |\nabla u_{\mathcal{M}}(\theta)|^2$.

2.3 The Correspondence between Shape Optimization and Optimal Fund Menus

In this section, we give a characterization of fund menus in terms of their non-participation regions. This reduces the problem of finding the optimal menu $\mathcal{M}$ to one of finding an optimal convex region.

Definition. Given a fund $(x, \pi) \in \mathbb{R}^d \times \mathbb{R}_+$, we define its non-participation half-space by $\mathbb{H}(x, \pi) = \{\theta \in \mathbb{R}^d \mid (\zeta - \theta) \cdot x - \pi \leq 0\}$. More generally, given a fund menu $\mathcal{M} \subseteq \mathbb{R}^d \times \mathbb{R}_+$, we define the non-participation region induced by this menu to be $\varphi(\mathcal{M}) = \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x, \pi)$.

Remark 4. $\mathbb{H}(x, \pi)$ is a closed half-space in $\mathbb{R}^d$ containing ζ (unless $x = 0$, in which case, $\mathbb{H}(x, \pi) = \mathbb{R}^d$). Since $\varphi(\mathcal{M})$ is an intersection of closed half-spaces, all of which containing ζ, $\varphi(\mathcal{M})$ is a closed convex set in $\mathbb{R}^d$ containing the point ζ.

The non-participation region is so-called because it represents all types $\theta \in \mathbb{R}^d$ for which it is optimal to not invest in the fund menu.

Theorem 1. *There is a bijection*

$$\{\textit{fund menus } \mathcal{M}\} \leftrightarrow \{\textit{closed convex sets } \Gamma \subseteq \mathbb{R}^d \textit{ containing } \zeta\}$$

given by

$$\mathcal{M} \xrightarrow{\varphi} \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x, \pi)$$

$$\{(x, \pi) \in \mathbb{R}^d \times \mathbb{R}_+ \mid \Gamma \subseteq \mathbb{H}(x, \pi)\} \xleftarrow{\psi} \Gamma.$$

Moreover, given a fund menu $\mathcal{M}$, *let* $\Gamma = \varphi(\mathcal{M})$. $(x_{\mathcal{M}}(\theta), \pi_{\mathcal{M}}(\theta)) = \textit{arg max}_{(x,\pi)\in\mathcal{M}} U_\theta(x, \pi)$ *is unique, given by*

$$x_{\mathcal{M}}(\theta) = p_\Gamma(\theta) - \theta$$
$$\pi_{\mathcal{M}}(\theta) = (\zeta - p_\Gamma(\theta)) \cdot (p_\Gamma(\theta) - \theta),$$

where $p_\Gamma(\theta)$ *denotes the (unique) closest point to* θ *in* Γ.

Lemma 2. *Let* $\Gamma = \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x,\pi)$. *If* (x,π) *satisfies* $\Gamma \subseteq \mathbb{H}(x,\pi)$, *then* $(x,\pi) \in \mathcal{M}$.

Proof. Let $\mathcal{N} = \{(x,\pi) \in \mathbb{R}^d \times \mathbb{R} \mid \Gamma \subseteq \mathbb{H}(x,\pi)\}$. It's clear that $\mathcal{M} \subseteq \mathcal{N}$. It remains to show that $\mathcal{N} \subseteq \mathcal{M}$.

Suppose $(x,\pi) \in \mathcal{N}\backslash\mathcal{M}$. Since $\mathcal{M}$ is a closed convex cone, there exists a nonzero $(v,-t) \in \mathbb{R}^d \times \mathbb{R}$ such that $(v,-t)\cdot(x,\pi) > 0 \geq \sup_{(x',\pi')\in\mathcal{M}}(v,-t)\cdot(x',\pi')$. Since $\mathcal{M}$ is closed upward, we must have $t \geq 0$. (Otherwise, $(v,-t)\cdot(x',\pi') \longrightarrow +\infty$ when we take $\pi' \longrightarrow +\infty$.) If $t > 0$, then let $\widetilde{v} = \frac{v}{t}$, $\widetilde{\theta} = \zeta - \widetilde{v}$. Then $(\widetilde{v},-1)\cdot(x,\pi) > 0 \geq \sup_{(x',\pi')\in\mathcal{M}}(\widetilde{v},-1)\cdot(x',\pi')$. We deduce that $\widetilde{\theta} \in \Gamma$, and $(x,\pi) \notin \mathcal{N}$. If $t = 0$, then $v \neq 0$, and $v \cdot x > 0 \geq \sup_{(x',\pi')\in\mathcal{M}} v \cdot x'$. Let $\widetilde{v} = \alpha v$, where $\alpha > 0$ is large so that $\widetilde{v}\cdot x > \pi$. Since $\mathcal{M} \subseteq \mathbb{R}^d \times \mathbb{R}_+$, $\widetilde{v}\cdot x - \pi > 0 \geq \sup_{(x',\pi')} \widetilde{v}\cdot x' \geq \sup_{(x',\pi')} \widetilde{v}\cdot x' - \pi'$. Once again, put $\widetilde{\theta} = \zeta - \widetilde{v}$, and we see that $\widetilde{\theta} \in \Gamma$, but $(x,\pi) \notin \mathcal{N}$. □

Lemma 3. *If a subset* Γ *of* $\mathbb{R}^d$ *is closed, convex, and contains* ζ, *then there exists a fund menu* $\mathcal{M}$ *such that* $\Gamma = \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x,\pi)$.

Proof. Suppose Γ is closed, convex, and contains ζ. Let $\mathcal{M} = \{(x,\pi) \in \mathbb{R}^d \times \mathbb{R} \mid \Gamma \subseteq \mathbb{H}(x,\pi)\}$. It's clear that $\mathcal{M}$ satisfies the requirements of a fund menu, and $\Gamma \subseteq \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x,\pi)$. To show that $\Gamma = \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x,\pi)$, suppose $\theta_0 \notin \Gamma$. Then $\zeta - \theta_0$ and $\zeta - \Gamma$ are disjoint. By the separating hyperplane theorem, there exists $(x_0,\pi_0) \in \mathbb{R}^d \times \mathbb{R}$ such that

$$\sup_{\theta\in\Gamma}(\zeta - \theta)\cdot x_0 - \pi_0 \leq 0 < (\zeta - \theta_0)\cdot x_0 - \pi_0.$$

This implies $(x_0,\pi_0) \in \mathcal{M}$, and $\theta_0 \notin \mathbb{H}(x_0,\pi_0)$. Consequently, $\theta_0 \notin \bigcap_{(x,\pi)\in\mathcal{M}} \mathbb{H}(x,\pi)$. □

Lemma 4. *Let* $\theta_0 \in \mathbb{R}^d$. *Then* $\frac{[(\zeta-\theta_0)\cdot x - \pi]_+}{|x|}$ *is the distance from* θ_0 *to* $\mathbb{H}(x,\pi)$, *provided* $x \neq 0$.

Proof. If $(\zeta - \theta_0)\cdot x - \pi \leq 0$, then $\theta_0 \in \mathbb{H}(x,\pi)$, and the distance to $\mathbb{H}(x,\pi)$ is zero. In this case, we have $\frac{[(\zeta-\theta_0)\cdot x - \pi]_+}{|x|} = 0$, as desired.

Consider the case $(\zeta - \theta_0)\cdot x - \pi > 0$. We find the closest point p in $\mathbb{H}(x,\pi)$ to θ_0. It is clear that p must be on the boundary of $\mathbb{H}(x,\pi)$. Consider the problem

$$\begin{aligned} \text{minimize} \quad & |p - \theta_0|^2 \\ \text{s.t.} \quad & (\zeta - p)\cdot x - \pi = 0. \end{aligned}$$

Solving the Lagrange multiplier system

$$2(p - \theta_0) = \lambda x$$
$$(\zeta - p) \cdot x - \pi = 0,$$

we find solution $p = \theta_0 + \frac{(\zeta-\theta_0)\cdot x - \pi}{|x|^2} x$. The distance from θ_0 to $\mathbb{H}(x, \pi)$ is $|\theta_0 - p| = \frac{|(\zeta-\theta_0)\cdot x-\pi|}{|x|} = \frac{[(\zeta-\theta_0)\cdot x-\pi]_+}{|x|}$. □

Lemma 5. *Let $\mathcal{M}$ be a fund menu. Then* $\sup_{(x,\pi)\in\mathcal{M}} \frac{[(\zeta-\theta)\cdot x-\pi]_+}{|x|} = dist(\theta, \Gamma)$*, where* $\Gamma = \bigcap \mathbb{H}(x, \pi)$*. Moreover, if* $dist(\theta, \Gamma) > 0$*, then there is a unique* (x^*, π^*) *(up to scalar multiple) such that* $dist(\theta, \Gamma) = dist(\theta, \mathbb{H}(x^*, \pi^*))$*.*

Proof. If $\mathrm{dist}(\theta, \Gamma) = 0$, then $\theta \in \overline{\Gamma} = \Gamma$. For any $(x, \pi) \in \mathcal{M}$, $\theta \in \Gamma \subseteq \mathbb{H}(x, \pi)$, so $\mathrm{dist}(\theta, \mathbb{H}(x, \pi)) = 0$.

If $\mathrm{dist}(\theta, \Gamma) > 0$, let p^* be the unique closest point in Γ to θ. Let $x^* = p^* - \theta$, $\pi^* = (\zeta - p^*) \cdot (p^* - \theta)$. Then $\mathbb{H}(x^*, p^*)$ is the supporting halfspace of Γ at p^* with outward normal $\theta - p^*$. It's clear that $\Gamma \subseteq \mathbb{H}(x^*, \pi^*)$. By Lemma 2, $(x^*, \pi^*) \in \mathcal{M}$. For any $(x, \pi) \in \mathcal{M}$, $p^* \in \mathbb{H}(x, \pi)$, so $\mathrm{dist}(\theta, \Gamma) = |\theta - p^*| \geq \mathrm{dist}(\theta, \mathbb{H}(x, \pi))$. Moreover, the inequality is strict unless $\mathbb{H}(x, \pi) = \mathbb{H}(x^*, \pi^*)$. □

Proof of theorem 1. It's clear that $\psi(\varphi(\mathcal{M})) \supseteq \mathcal{M}$. By Lemma 2, $\psi(\varphi(\mathcal{M})) \subseteq \mathcal{M}$, so $\psi \circ \varphi = \mathrm{id}$. In particular, φ is injective. Lemma 3 shows that φ is surjective. Consequently, φ, ψ establish a bijection.

Now fix a type θ, and consider Problem (2.2). If $\theta \in \Gamma$, then $(\zeta - \theta) \cdot x - \pi \leq 0$ for all $(x, \pi) \in \mathcal{M}$, and

$$U_\theta(x, \pi) = (\zeta - \theta) \cdot x - \pi - \frac{1}{2}|x|^2 \leq 0 \quad \text{for all } (x, \pi) \in \mathcal{M},$$

with equality if and only if $x = 0$ and $\pi = 0$. In this case, the unique maximizer is $x = 0, \pi = 0$.

If $\theta \notin \Gamma$, put $V_\theta(x, \pi) = \frac{[(\zeta-\theta)\cdot x-\pi]_+^2}{2|x|^2}$. We claim that $U_\theta(x, \pi) \leq V_\theta(x, \pi)$, with equality if and only if $[(\zeta - \theta) \cdot x - \pi]_+ = |x|^2$. Indeed,

$$\begin{aligned} U_\theta(x, \pi) &\leq \max_{q\geq 0} U_\theta(qx, q\pi) \\ &= \max_{q\geq 0} \left\{ q(\zeta - \theta) \cdot x - q\pi - \frac{1}{2}q^2|x|^2 \right\} \\ &= \frac{[(\zeta - \theta) \cdot x - \pi]_+^2}{2|x|^2} \\ &= V_\theta(x, \pi). \end{aligned}$$

Maximum on the right-hand side is achieved if and only if $q = \frac{[(\zeta-\theta)\cdot x-\pi]_+}{|x|^2}$. But equality holds if and only if the right-hand maximizer is $q = 1$. Hence, we have proven the claim.

Note that $(x, \pi) \in \mathcal{M}$ maximizes $V_\theta(x, \pi)$ if and only if it maximizes $\sqrt{2V_\theta(x, \pi)} = \frac{[(\zeta-\theta)\cdot x-\pi]_+}{|x|}$. By Lemma 4 and Lemma 5,

$$\arg\max_{(x,\pi)\in\mathcal{M}} V_\theta(x, \pi) = \arg\max_{(x,\pi)\in\mathcal{M}} \frac{[(\zeta-\theta)\cdot x-\pi]_+}{|x|} = \{\alpha(x^*, \pi^*) \mid \alpha > 0\},$$

where $x^* = p_\Gamma(\theta) - \theta$, $\pi^* = (\zeta - p_\Gamma(\theta)) \cdot (p_\Gamma(\theta) - \theta)$. Finally,

$$\begin{aligned} &\arg\max_{(x,\pi)\in\mathcal{M}} U_\theta(x, \pi) \\ &= \arg\max_{(x,\pi)\in\mathcal{M}} V_\theta(x, \pi) \cap \{(x, \pi) \in \mathcal{M} \mid [(\zeta-\theta)\cdot x-\pi]_+ = |x|^2\} \\ &= \{(x^*, \pi^*)\}. \end{aligned}$$

□

Theorem 1 allows us reformulate the manager's problem (2.3) into the following shape optimization problem:

$$\begin{aligned} \text{maximize}_\Gamma \quad & \int_\Omega (\zeta - p_\Gamma(\theta)) \cdot (p_\Gamma(\theta) - \theta) d\mu(\theta) \\ \text{s.t.} \quad & \Gamma \subseteq \mathbb{R}^d \text{ closed and convex}, \zeta \in \Gamma. \end{aligned} \tag{2.6}$$

Without loss of generality, by shifting the origin of $\mathbb{R}^d$ to ζ if necessary, we may assume that $\zeta = 0$ and focus on the problem

$$\begin{aligned} \text{maximize}_\Gamma \quad & \mathcal{F}(\Gamma) := \int_\Omega (-p_\Gamma(\theta)) \cdot (p_\Gamma(\theta) - \theta) d\mu(\theta) \\ \text{s.t.} \quad & \Gamma \subseteq \mathbb{R}^d \text{ closed and convex}, 0 \in \Gamma. \end{aligned} \tag{2.7}$$

Note that $-p_\Gamma(\theta) \cdot (p_\Gamma(\theta) - \theta) = -\left|\frac{\theta}{2} - p_\Gamma(\theta)\right|^2 + \frac{|\theta|^2}{4}$, so (2.7) is equivalent to

$$\begin{aligned} \text{minimize}_\Gamma \quad & \int_\Omega \left|\frac{\theta}{2} - p_\Gamma(\theta)\right|^2 d\mu(\theta) \\ \text{s.t.} \quad & \Gamma \subseteq \mathbb{R}^d \text{ closed and convex}, 0 \in \Gamma. \end{aligned} \tag{2.8}$$

Chapter 3

EXISTENCE OF SOLUTION

Proposition 6. *Assume Ω is bounded. Then there exists a closed convex set Γ containing the origin such that $\mathcal{F}(\Gamma)$ is minimized.*

Proof. Let $p^* = \inf_{\Gamma \text{ closed convex}, 0\in\Gamma} \mathcal{F}(\Gamma)$. Then there exists a sequence of closed convex sets Γ_n containing 0 such that $\mathcal{F}(\Gamma_n) \to p^*$. Since $0 \in \Gamma_n$, we must have $(\theta - p_{\Gamma_n}(\theta)) \cdot (0 - p_{\Gamma_n}(\theta)) \leq 0$. Hence,

$$\begin{aligned} |p_{\Gamma_n}(\theta)|^2 &= |p_{\Gamma_n}(\theta)|^2 - p_{\Gamma_n}(\theta) \cdot \theta + p_{\Gamma_n}(\theta) \cdot \theta \\ &= p_{\Gamma_n}(\theta) \cdot (p_{\Gamma_n}(\theta) - \theta) + p_{\Gamma_n}(\theta) \cdot \theta \\ &\leq |p_{\Gamma_n}(\theta)||\theta|. \end{aligned}$$

We see that $|p_{\Gamma_n}(\theta)| \leq |\theta|$. Now let $R = \sup_{\theta\in\Omega} |\theta|$, and $K_n = \Gamma_n \cap \overline{B}(0, R)$. Then each K_n is a closed convex set containing 0, $p_{\Gamma_n}(\theta) = p_{K_n}(\theta)$ for all θ, hence $\mathcal{F}(K_n) \to p^*$. By the Blaschke selection theorem, there exists a subsequence K_{n_k} and a closed convex set K such that $\Delta_H(K_{n_k}, K) \to 0$. Here, $\Delta_H(X, Y) = \max\{\sup_{x\in X} \operatorname{dist}(x, Y), \sup_{y\in Y} \operatorname{dist}(y, X)\}$ is the Hausdorff distance. By passing to subsequence, we may assume $\Delta_H(K_n, K) \to 0$.

Claim 1: $0 \in K$.

Proof. $0 \in K_n$ for each k, so $\operatorname{dist}(0, K) \leq \Delta_H(K_n, K)$. We find $\operatorname{dist}(0, K) = 0$. Since K is closed, $0 \in K$. □

Claim 2: For any $\theta \in \mathbb{R}^d$, $\operatorname{dist}(\theta, K_n) \to \operatorname{dist}(\theta, K)$.

Proof. There exists a $p_n \in K_n$ such that $|p_n - p_K(\theta)| \leq \Delta_H(K_n, K)$. So

$$\begin{aligned} \operatorname{dist}(\theta, K_n) &\leq |\theta - p_n| \\ &\leq |\theta - p_K(\theta)| + |p_K(\theta) - p_n| \\ &\leq \operatorname{dist}(\theta, K) + \Delta_H(K_n, K). \end{aligned}$$

On the other hand, for each n there exists $q_n \in K$ such that $|q_n - p_{K_n}(\theta)| \leq \Delta_H(K_n, K)$. So

$$\begin{aligned} \operatorname{dist}(\theta, K) &\leq |\theta - q_n| \\ &\leq |\theta - p_{K_n}(\theta)| - |p_{K_n}(\theta) - q_n| \\ &\leq \operatorname{dist}(\theta, K_n) + \Delta_H(K_n, K). \end{aligned}$$

We find

$$|\operatorname{dist}(\theta, K_n) - \operatorname{dist}(\theta, K)| \leq \Delta_H(K_n, K) \to 0.$$

□

Claim 3: For any $\theta \in \mathbb{R}^n$, $p_{K_n}(\theta) \to p_K(\theta)$.

Proof. Suppose, for contradiction, this does not hold. Then there exists some $\varepsilon > 0$ such that $|p_{K_n}(\theta) - p_K(\theta)| \geq \varepsilon$, after passing to some subsequence. If $\theta \in K$, then $p_K(\theta) = \theta$. Since $|p_{K_n}(\theta) - \theta| > \varepsilon$, $K_n \cap B(\theta, \varepsilon) = \emptyset$. But then $\Delta_H(K_n, K) \geq \operatorname{dist}(p_K(\theta), K_n) \geq \varepsilon$. If $\theta \notin K$, then $p_K(\theta) \neq \theta$.

$$\begin{aligned} &(\theta - p_K(\theta)) \cdot (p_{K_n}(\theta) - p_K(\theta)) \\ =& \frac{1}{2}|p_K(\theta) - p_{K_n}(\theta)|^2 - \frac{1}{2}\Big(|\theta - p_{K_n}(\theta)|^2 - |\theta - p_K(\theta)|^2\Big) \\ \geq& \frac{\varepsilon^2}{2} - \frac{1}{2}\Big(\operatorname{dist}(\theta, K_n) - \operatorname{dist}(\theta, K)\Big). \end{aligned}$$

Since $\operatorname{dist}(\theta, K_n) \to \operatorname{dist}(\theta, K)$, $\frac{1}{2}|\operatorname{dist}(\theta, K_n) - \operatorname{dist}(\theta, K)| < \varepsilon^2/4$ for sufficiently large n. Hence,

$$(\theta - p_K(\theta)) \cdot (p_{K_n}(\theta) - p_K(\theta)) \geq \frac{\varepsilon^2}{4}$$

for sufficiently large n. Let $\mathbb{H}$ be the supporting halfspace of K at $p_K(\theta)$ with normal $\theta - p_K(\theta)$. We find

$$\Delta_H(K_n, K) \geq \operatorname{dist}(p_{K_n}(\theta), K) \geq \operatorname{dist}(p_{K_n}(\theta), \mathbb{H}) \geq \frac{\varepsilon^2}{4|\theta - p_K(\theta)|}.$$

This is a contradiction. □

Finally, we now know that $\left|\frac{\theta}{2} - p_{K_n}(\theta)\right|^2 \to \left|\frac{\theta}{2} - p_K(\theta)\right|^2$ pointwise. Moreover, $|p_{K_n}(\theta)| \leq R$, and $|\theta| \leq R$. So $\left|\frac{\theta}{2} - p_{K_n}(\theta)\right|^2 \leq \frac{9}{4}R^2$. Since $\mu(\Omega) < +\infty$, we conclude

$$p^* = \lim \int_\Omega \left|\frac{\theta}{2} - p_{K_n}(\theta)\right|^2 d\mu(\theta) = \int_\Omega \left|\frac{\theta}{2} - p_K(\theta)\right|^2 d\mu(\theta)$$

by the dominated convergence theorem. □

The proof of Proposition 6 can be extended to unbounded regions Ω, provided μ decays sufficiently fast.

Theorem 7. *Let $\Omega \subseteq \mathbb{R}^d$ be arbitrary, and assume μ is a measure on Ω such that $\int_\Omega |\theta|^2 d\mu(\theta) < +\infty$. Then there exists a closed convex set Γ containing the origin such that $\mathcal{F}(\Gamma)$ is minimized.*

Proof. Let $p^* = \inf_{\Gamma \text{ closed convex}, 0\in\Gamma} \mathcal{F}(\Gamma)$. Then there exists a sequence of closed convex sets Γ_n containing 0 such that $\mathcal{F}(\Gamma) \to p^*$. Since $\{\Gamma_n \cap \overline{B}(0,1)\}_{n=1}^\infty$ is a bounded sequence of convex sets containing 0, the Blaschke selection theorem implies that there exists a subsequence $n_{1,1}, n_{1,2}, n_{1,3}, \ldots$ such that $\Gamma_{n_{1,j}} \cap \overline{B}(0,1) \to K_{1,\infty}$ in Hausdorff distance, where $K_{1,\infty}$ is a closed convex set containing 0, and contained in $\overline{B}(0,1)$. Since $\{\Gamma_{n_{1,j}} \cap \overline{B}(0,2)\}_{j=1}^\infty$ is a bounded sequence of convex sets containing 0, using the Blaschke selection theorem again, we obtain a further subsequence $n_{2,1}, n_{2,2}, n_{2,3}, \ldots$ such that $\Gamma_{n_{2,j}} \cap \overline{B}(0,1) \to K_{1,\infty}$ and $\Gamma_{n_{2,j}} \cap \overline{B}(0,2) \to K_{2,\infty}$, where $K_{2,\infty}$ is a closed convex set containing 0, and contained in $\overline{B}(0,2)$. By the Cantor diagonalization method, we find a subsequence $n_{i,i}$ such that $\Gamma_{n_{i,i}} \cap \overline{B}(0,j) \to K_{j,\infty}$ for all $j = 1, 2, 3, \ldots$.

Claim 1: Suppose X, Y are compact convex sets containing 0, and $\Delta_H(X,Y) \leq \varepsilon$. Then for any $i > 0$, $\Delta_H(X \cap \overline{B}(0,i), Y \cap \overline{B}(0,i)) \leq 2\varepsilon$.

Proof. Let $x \in X \cap \overline{B}(0,i)$, and let $y \in Y$ so that $|x-y| = \mathrm{dist}(x,Y)$. If $y \in \overline{B}(0,i)$, then $\mathrm{dist}(x, Y \cap \overline{B}(0,i)) \leq \varepsilon$. Consider the case $y \notin \overline{B}(0,i)$. Then $\varepsilon \geq |x-y| \geq |y| - |x|$, so $|y| \leq |x| + \varepsilon \leq i + \varepsilon$. Put $y = y' + h$, where $y' = i\frac{y}{|y|}$, and $h = y - y'$. Then $|h| \leq \varepsilon$, and $|y'| = i$. Note that since $0 \in Y \cap \overline{B}(0,i)$ and $Y \cap \overline{B}(0,i)$ is convex, $y' \in Y \cap \overline{B}(0,i)$. So $|x - y'| \leq |x-y| + |h| \leq 2\varepsilon$. In either case, we find $\sup_{x\in X\cap\overline{B}(0,i)} \mathrm{dist}(x, Y \cap \overline{B}(0,i)) \leq 2\varepsilon$. Switching the roles of X and Y, we conclude that $\Delta_H(X \cap \overline{B}(0,i), Y \cap \overline{B}(0,i)) \leq 2\varepsilon$. □

Claim 2: $K_{i,\infty} = K_{j,\infty} \cap \overline{B}(0,i)$ for all $i < j$.

Proof. We know that $\Gamma_{n_{k,k}} \cap \overline{B}(0,j) \to K_{j,\infty}$. By Claim 1, $\Gamma_{n_{k,k}} \cap \overline{B}(0,j) \cap \overline{B}(0,i) \to K_{j,\infty} \cap \overline{B}(0,i)$. But $\Gamma_{n_{k,k}} \cap \overline{B}(0,j) \cap \overline{B}(0,i) = \Gamma_{n_{k,k}} \cap \overline{B}(0,i) \to K_{i,\infty}$. So $K_{i,\infty} = K_{j,\infty} \cap \overline{B}(0,i)$. □

Claim 3: Let $K_\infty = \bigcup_{i=1}^\infty K_{i,\infty}$. Then K_∞ is closed, convex, and contains 0.

Proof. It's clear that K_∞ is convex and contains 0. To show that K_∞ is closed, suppose $\{p_i\}_{i=1}^{\infty}$ is a sequence in K_∞ and $p_i \to p_\infty$. Without loss of generality, we may assume that $p_1, p_2, p_3, \ldots, p_\infty \in \overline{B}(0, n)$ for some n. Since $K_{n,\infty} = K_\infty \cap \overline{B}(0, n)$, we must have $p_i \in K_{n,\infty}$. But $K_{n,\infty}$ is closed, so the limit $p_\infty \in K_{n,\infty} \subseteq K_\infty$. □

Claim 4: Denote $f_\Gamma(\theta) = \left|\frac{\theta}{2} - p_\Gamma(\theta)\right|^2$. Then $f_{\Gamma_{i,i}}(\theta) \to f_{K_\infty}(\theta)$ pointwise.

Proof. Fix θ. For any closed convex set Γ containing 0, we must have $|p_\Gamma(\theta)| \leq |\theta|$. Therefore, for sufficiently large n, $p_{\Gamma_{n_{i,i}} \cap \overline{B}(0,n)}(\theta) = p_{\Gamma_{n_{i,i}}}(\theta)$ for all $i = 1, 2, 3, \ldots,$ and $p_{K_{n,\infty}}(\theta) = p_{K_\infty}(\theta)$. Since $\Gamma_{n_{i,i}} \cap \overline{B}(0, n) \to K_{n,\infty}$, we have

$$f_{\Gamma_{n_{i,i}}}(\theta) = f_{\Gamma_{n_{i,i}} \cap \overline{B}(0,n)}(\theta) \to f_{K_{n,\infty}}(\theta) = f_{K_\infty}(\theta).$$

□

Finally, $|f_{\Gamma_{n_{i,i}}}(\theta)| \leq \frac{9}{4}|\theta|^2$. By the dominated convergence theorem,

$$p^* = \lim_{i \to \infty} \int_\Omega f_{\Gamma_{n_{i,i}}}(\theta) d\mu(\theta) = \int_\Omega f_{K_\infty}(\theta) d\mu(\theta).$$

□

Chapter 4

MATHEMATICAL OPTIMIZATION FOR THE DISCRETE PROBLEM

4.1 The Quadratically Constrained Quadratic Program

To the best of our knowledge, no existing theory is applicable to the specific shape optimization problem (2.8). To make the problem more tractable, we convert it into a quadratically constrained quadratic program. The key to this conversion is the following result:

Proposition 8. *Let $\{\theta_i\}_{i\in I}, \{p_i\}_{i\in I}$ be two collections of points in $\mathbb{R}^d$, where I is an arbitrary index set. There exists a closed convex set $\Gamma \subseteq \mathbb{R}^d$ containing 0 such that $p_i = p_\Gamma(\theta_i)$ for all $i \in I$ if and only if*

$$
\begin{aligned}
&(\theta_i - p_i)\cdot(p_j - p_i) \le 0 \quad \text{for all } i, j \in I, i \neq j,\\
&(\theta_i - p_i)\cdot(-p_i) \le 0 \quad \text{for all } i \in I.
\end{aligned}
$$

Proof. ($\Leftarrow$) Suppose for all $i, j \in I$, we have $(\theta_i - p_i)\cdot(p_j - p_i) \le 0$, and $(\theta_i - p_i)\cdot(-p_i) \le 0$. For each $i \in I$, define

$$\mathbb{H}_i = \{\theta \in \mathbb{R}^d \mid (\theta_i - p_i)\cdot(\theta - p_i) \le 0\}.$$

This is a halfspace with normal vector $\theta_i - p_i$ (or the full space $\mathbb{R}^d$ in the case $p_i = \theta_i$). It's clear that $\operatorname{dist}(\theta_i, \mathbb{H}_i) = |\theta_i - p_i|$. Since $(\theta_i - p_i)\cdot(0 - p_i) \le 0$, we deduce that $0 \in \mathbb{H}_i$.

Let $\Gamma := \bigcap_{i\in I}\mathbb{H}_i$. This is a closed convex set that contains 0. It remains to show that for each $i \in I$, $p_i = p_\Gamma(\theta_i)$. Fix $i \in I$. Then for any $j \neq i$, $(\theta_j - p_j)\cdot(p_i - p_j) \le 0$. This implies that $p_i \in \mathbb{H}_j$. Since j is arbitrary, $p_i \in \bigcap_{j\in I}\mathbb{H}_j = \Gamma$. We find

$$
\begin{aligned}
\operatorname{dist}(\theta_i, \Gamma) &\le |\theta_i - p_i|\\
&= \operatorname{dist}(\theta_i, \mathbb{H}_i)\\
&\le \operatorname{dist}(\theta_i, \Gamma).
\end{aligned}
$$

All inequalities above become equalities, and we deduce that $\operatorname{dist}(\theta_i, \Gamma) = |\theta_i - p_i|$. Since $p_i \in \Gamma$, p_i is the closest point in Γ to θ_i. That is, $p_i = p_\Gamma(\theta_i)$.

(⇒) Assume Γ is a closed convex set containing 0 such that $p_i = p_\Gamma(\theta_i)$ for $i \in I$. Since $0 \in \Gamma$, $(\theta_i - p_i) \cdot (0 - p_i) \leq 0$, hence $(\theta_i - p_i) \cdot (-p_i) \leq 0$. Next, fix a pair i, j with $i \neq j$. Since $p_i = p_\Gamma(\theta_i)$, it must be that for any $q \in \Gamma$, $(\theta_i - p_i) \cdot (q - p_i) \leq 0$. In particular, since $p_j = p_\Gamma(\theta_j) \in \Gamma$, we have $(\theta_i - p_i) \cdot (p_j - p_i) \leq 0$. □

Thus, rather than seek an optimal convex set Γ, we merely need to find the optimal points p_i corresponding to each $\theta_i \in \Omega$. Our shape optimization Problem (2.8) becomes

$$\begin{aligned} \text{minimize}_{p:\Omega\to\mathbb{R}^d} \quad & \int_\Omega \left|\frac{\theta}{2} - p(\theta)\right|^2 d\mu(\theta) \\ \text{s.t.} \quad & (\theta - p(\theta)) \cdot (p(\theta') - p(\theta)) \leq 0 \quad \text{for all } \theta, \theta' \in \Omega, \\ & (\theta - p(\theta)) \cdot (-p(\theta)) \leq 0 \quad \text{for all } \theta \in \Omega. \end{aligned} \tag{4.1}$$

As a corollary, we identify a class of problems for which Problem (2.7) has trivial solution.

Corollary 9. *If Ω is a subset of a sphere in $\mathbb{R}^d$ centered at 0 with some radius $r > 0$, then the optimal value of Problem (2.8) is 0.*

Proof. Let $p(\theta) = \frac{\theta}{2}$ for $\theta \in \Omega$. Then $\theta - p(\theta) = \frac{\theta}{2}$. $(\theta - p(\theta)) \cdot (-p(\theta)) = -\frac{|\theta|^2}{4} \leq 0$. For any $\theta' \in \Omega$,

$$\begin{aligned} (\theta - p(\theta)) \cdot (p(\theta') - p(\theta)) &= \frac{1}{4}\theta \cdot (\theta' - \theta) \\ &\leq \frac{1}{4}[|\theta||\theta'| - |\theta|^2] \\ &= 0. \end{aligned}$$

We see that $p(\theta) = \frac{\theta}{2}$ is feasible, and $\int_\Omega \left|\frac{\theta}{2} - p(\theta)\right|^2 d\mu(\theta) = 0$. □

Henceforth, we will restrict attention to the case when μ is finitely supported (that is, $\Omega = \{\theta_1, \ldots, \theta_k\} \subseteq \mathbb{R}^d$, $\mu = \sum_{i=1}^k w_i \delta_{\theta_i}$, $w_i > 0$). In this case, Problem (4.1) takes the following form:

$$\begin{aligned} \text{minimize}_{p_1,\ldots,p_k\in\mathbb{R}^d} \quad & \sum_{i=1}^k w_i \left|\frac{\theta_i}{2} - p_i\right|^2 \\ \text{s.t.} \quad & (\theta_i - p_i) \cdot (p_j - p_i) \leq 0, \quad i, j = 1, \ldots, k, i \neq j, \\ & (\theta_i - p_i) \cdot (-p_i) \leq 0, \quad i = 1, \ldots, k. \end{aligned} \tag{4.2}$$

Quadratically constrained quadratic programs (QCQPs) are widely studied in the optimization literature. Unfortunately, a general nonconvex QCQP is NP-hard. Problem (4.2), in particular, is nonconvex. This is because a constraint of the form $(\theta_i - p_i) \cdot (p_j - p_i) \leq 0$ is nonconvex in $p_1, \ldots, p_k$. For nonconvex QCQPs, relaxation methods exist to obtain approximate solutions, as well as provide bounds on the optimal value.

In the next section, we will study the semidefinite programming relaxation problem (4.2). To motivate this relaxation, we convert problem (4.2) into matrix form.

For two $n \times n$ matrices A, B, we define the Frobenius inner product $A \bullet B = \mathrm{tr}(A^T B) = \sum_{i=1}^n \sum_{j=1}^n a_{ij} b_{ij}$.

Let $X = \begin{pmatrix} p_1 & \ldots & p_k \end{pmatrix} \in \mathbb{R}^{d \times k}$. Let $\mathbf{e}_i = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$, where -1 appears at the i-th entry, and let $\mathbf{e}_{ij} = \begin{pmatrix} 0 \\ \vdots \\ -1 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \in \mathbb{R}^k$, where -1 appears at the i-th entry, and 1 appears at the j-th entry. Then $\theta_i - p_i = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix}$, $p_j - p_i = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{e}_{ij} \end{pmatrix}$,

$-p_i = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{e}_i \end{pmatrix}$, $\frac{\theta_i}{2} - p_i = \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} \frac{\theta_i}{2} \\ \mathbf{e}_i \end{pmatrix}$. We have

$$\begin{aligned}
(\theta_i - p_i) \cdot (p_j - p_i) &= \left(\begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \right)^T \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{e}_{ij} \end{pmatrix} \\
&= \begin{pmatrix} \theta_i^T & \mathbf{e}_i^T \end{pmatrix} \begin{pmatrix} I \\ X^T \end{pmatrix} \begin{pmatrix} I & X \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{e}_{ij} \end{pmatrix} \\
&= \begin{pmatrix} \theta_i^T & \mathbf{e}_i^T \end{pmatrix} \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \begin{pmatrix} 0 \\ \mathbf{e}_{ij} \end{pmatrix} \\
&= \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_{ij}^T \end{pmatrix} \right).
\end{aligned}$$

Similarly,

$$(\theta_i - p_i) \cdot (-p_i) = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_i^T \end{pmatrix} \right),$$

and

$$\sum_{i=1}^k w_i \left| \frac{\theta_i}{2} - p_i \right|^2 = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\sum_{i=1}^k w_i \begin{pmatrix} \frac{\theta_i}{2} \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} \frac{\theta_i^T}{2} & \mathbf{e}_i^T \end{pmatrix} \right).$$

Problem (4.2) may now be written as

$$\begin{aligned}
\text{minimize}_{X \in \mathbb{R}^{d \times k}} \quad & \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\sum_{i=1}^k w_i \begin{pmatrix} \frac{\theta_i}{2} \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} \frac{\theta_i^T}{2} & \mathbf{e}_i^T \end{pmatrix} \right) \\
\text{s.t.} \quad & \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_{ij}^T \end{pmatrix} \right) \leq 0, \quad i, j = 1, \ldots, k, i \neq j, \\
& \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix} \bullet \left(\begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_i^T \end{pmatrix} \right) \leq 0, \quad i = 1, \ldots, k.
\end{aligned} \tag{4.3}$$

Put $\widetilde{C} = \sum_{i=1}^k w_i \begin{pmatrix} \frac{\theta_i}{2} \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} \frac{\theta_i^T}{2} & \mathbf{e}_i^T \end{pmatrix}$, $\widetilde{A}_{ij} = \begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_{ij}^T \end{pmatrix}$, $\widetilde{B}_i = \begin{pmatrix} \theta_i \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} 0 & \mathbf{e}_i^T \end{pmatrix}$, $Y = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix}$. Since Y is symmetric, $Y \bullet \widetilde{A}_{ij} = Y \bullet \left(\frac{\widetilde{A}_{ij} + \widetilde{A}_{ij}^T}{2} \right)$, $Y \bullet \widetilde{B}_i = Y \bullet \left(\frac{\widetilde{B}_i + \widetilde{B}_i^T}{2} \right)$, $Y \bullet \widetilde{C} = Y \bullet \left(\frac{\widetilde{C} + \widetilde{C}^T}{2} \right)$. Let $A_{ij} = \frac{\widetilde{A}_{ij} + \widetilde{A}_{ij}^T}{2}$, $B_i = \frac{\widetilde{B}_i + \widetilde{B}_i^T}{2}$, $C = \frac{\widetilde{C} + \widetilde{C}^T}{2}$. In semidefinite programming, it is customary to symmetrize the coefficient matrices. Hence, we write

(4.3) in equivalent symmetric form:

$$\begin{aligned} \text{minimize}_{X\in\mathbb{R}^{d\times k}} \quad & \begin{pmatrix} I & X \\ X^T & X^TX \end{pmatrix} \bullet C \\ \text{s.t.} \quad & \begin{pmatrix} I & X \\ X^T & X^TX \end{pmatrix} \bullet A_{ij} \le 0, \quad i,j = 1,\dots,k, i \ne j, \\ & \begin{pmatrix} I & X \\ X^T & X^TX \end{pmatrix} \bullet B_i \le 0, \quad i = 1,\dots,k. \end{aligned} \tag{4.4}$$

A direct calculation shows that C, A_{ij}, B_i have the following forms:

$$C = \begin{pmatrix} \sum_{i=1}^k w_i \frac{\theta_i\theta_i^T}{4} & -w_1\frac{\theta_1}{2} & \dots & -w_i\frac{\theta_i}{2} & \dots & -w_k\frac{\theta_k}{2} \\ -w_1\frac{\theta_1^T}{2} & w_1 & & & & \\ \vdots & & \ddots & & & \\ -w_i\frac{\theta_i^T}{2} & & & w_i & & \\ \vdots & & & & \ddots & \\ -w_k\frac{\theta_k^T}{2} & & & & & w_k \end{pmatrix},$$

where $\sum_{i=1}^k w_i \frac{\theta_i\theta_i^T}{4}$ is $d \times d$, $-w_i\frac{\theta_i}{2}$ appears in the $(d+i)$-th column, $-w_i\frac{\theta_i^T}{2}$ appears in the $(d+i)$-th row.

$$A_{ij} = \begin{pmatrix} 0_{d\times d} & \dots & -\frac{\theta_i}{2} & \dots & \frac{\theta_j}{2} & \dots \\ \vdots & & & & & \\ -\frac{\theta_i^T}{2} & & 1 & & -\frac{1}{2} & \\ \vdots & & & & & \\ \frac{\theta_j}{2} & & -\frac{1}{2} & & 0 & \\ \vdots & & & & & \end{pmatrix},$$

where $-\frac{\theta_i}{2}$ appears in the $(d+i)$-th column, $\frac{\theta_j}{2}$ appears in the $(d+j)$-th column, $-\frac{\theta_i^T}{2}$ appears in the $(d+i)$-th row, and $\frac{\theta_j^T}{2}$ appears in the $(d+j)$-th row.

$$B_i = \begin{pmatrix} 0_{d\times d} & \dots & -\frac{\theta_i}{2} & \dots \\ \vdots & & & \\ -\frac{\theta_i^T}{2} & & 1 & \\ \vdots & & & \end{pmatrix},$$

where $-\frac{\theta_i}{2}$ appears in the $(d+i)$-th column, and $-\frac{\theta_i^T}{2}$ appears in the $(d+i)$-th row.

4.2 The SDP Relaxation

For any $n > 0$, let S^n be the set of real symmetric $n \times n$ matrices. For a symmetric matrix Y, we write $Y \succeq 0$ if Y is positive semidefinite, and $Y \succ 0$ if it is positive definite.

Recall from the last section that we reformulated (2.1) as (4.4):

$$\begin{aligned} \text{minimize}_{Z \in S^{d+k}} \quad & Z \bullet C \\ \text{s.t.} \quad & Z \bullet A_{ij} \leq 0, \quad i, j = 1, \ldots, k, i \neq j, \\ & Z \bullet B_i \leq 0, \quad i = 1, \ldots, k, \\ & Z = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix}. \end{aligned}$$

Observe that $Z = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix}$ is a positive semidefinite matrix of rank d. In fact, we have the following equvalence:

Proposition 10. *Let* $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \in S^{d+k}$*, with* $Z \succeq 0$*. Then* Z *has rank* d *if and only if* $Y = X^T X$*.*

Proof. If $Y = X^T X$, then $Z = U^T U$, where $U = \begin{pmatrix} I & X \end{pmatrix}$. Since U has d independent rows, rank $U = d$. Hence, rank $U^T U =$ rank $U = d$.

Conversely, assume $Z \succeq 0$ has rank d. Then Z admits decomposition of the form $Z = U^T U$, where $U \in \mathbb{R}^{d \times (d+k)}$. Write $U = \begin{pmatrix} M & N \end{pmatrix}$, where $M \in \mathbb{R}^{d \times d}$, $N \in \mathbb{R}^{d \times k}$. Then

$$\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} = Z = U^T U = \begin{pmatrix} M^T M & M^T N \\ N^T M & N^T N \end{pmatrix}.$$

In particular, $M^T M = I$, so M is an orthogonal matrix. Also, $X = M^T N, Y = N^T N$. It follows that

$$Y = N^T N = N^T M M^T N = (M^T N)^T (M^T N) = X^T X.$$

$\square$

From Proposition 10, we see that the condition $Z = \begin{pmatrix} I & X \\ X^T & X^T X \end{pmatrix}$ is equivalent to $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0$ for some $Y \in S^k$, and rank $Z = d$. The semidefinite programming relaxation of QCQP is obtained by removing the rank constraint:

$$
\begin{aligned}
\text{minimize}_{Z \in S^{d+k}} \quad & Z \bullet C \\
\text{s.t.} \quad & Z \bullet A_{ij} \leq 0, \quad i, j = 1, \ldots, k, i \neq j, \\
& Z \bullet B_i \leq 0, \quad i = 1, \ldots, k, \\
& Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \geq 0.
\end{aligned}
\tag{4.5}
$$

Since (4.5) is a relaxation of (4.4), the optimal value of (4.5) is a lower bound for the optimal value of (4.4).

We sometimes need to specify the dependence of problem (4.5) on the input data $\theta = (\theta_1, \ldots, \theta_k)$ and $w = (w_1, \ldots, w_k)$. We will refer to it as Problem (4.5) with parameters (θ, w).

In the next section, we will explore further relation between (4.5) and the (4.4). For the remainder of this section, we will discuss some known facts about semidefinite programming in general.

Preliminaries on Semidefinite Programming

Semidefinite programming is a generalization of linear programming. It has become a popular method in optimization due to two reasons: 1. A wide range of engineering problems can be formulated in terms of semidefinite programming. 2. Methods exist to solve this class of problems, both theoretically (strong duality) and numerically (interior point method).

We will briefly discuss duality theory for semidefinite programming. The standard form of semidefinite program (SDP) is

$$
\begin{aligned}
s^* = \inf_{X \in S^n} \; & X \bullet C \\
\text{s.t.} \quad & X \bullet A_i = a_i, \quad i = 1, \ldots, m, \\
& X \geq 0,
\end{aligned}
\tag{4.6}
$$

where $C, A_i \in S^n$. The dual problem is

$$
\begin{aligned}
d^* = \sup_{y_1, \ldots, y_m \in \mathbb{R}} \; & \sum_{i=1}^{m} y_i a_i \\
\text{s.t.} \quad & C - \sum_{i=1}^{m} y_i A_i \geq 0.
\end{aligned}
\tag{4.7}
$$

The following theorem concerning strong duality of SDP can be found in [1].

Theorem 11. *Assume there exists X feasible for (4.6) such that $X \succ 0$, and that there exist $y_1, \dots, y_m \in \mathbb{R}$ such that $C - \sum_{i=1}^m y_i A_i \succ 0$. Then both (4.6) and (4.7) attain their optimal values, and $s^* = d^*$. Moreover, a feasible primal dual pair (X, y) is optimal if and only if $X \bullet (C - \sum_{i=1}^m y_i A_i) = 0$.*

Our problem (4.5) is not in standard form because it contains inequality constraints. We need to adapt the preceding strong duality theorem to our case of interest.

Corollary 12. *Consider a semidefinite program with both equality and inequality constraints:*

$$\begin{aligned} s^* = \inf_{X \in S^n} & X \bullet C \\ s.t. \quad & X \bullet A_i = a_i, \quad i = 1, \dots, m, \\ & X \bullet B_j \leq b_j, \quad j = 1, \dots, k, \\ & X \succeq 0, \end{aligned} \tag{4.8}$$

and its dual problem

$$\begin{aligned} d^* = & \sup_{y_1,\dots,y_m \in \mathbb{R}, z_1,\dots,z_k \leq 0} \sum_{i=1}^m y_i a_i + \sum_{j=1}^k z_j b_j \\ s.t. \quad & C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j \succeq 0. \end{aligned} \tag{4.9}$$

Assume there exists $X \succ 0$ such that $X \bullet A_i = a_i$, $X \bullet B_j < b_j$, and there exist $y_1, \dots, y_m \in \mathbb{R}$, $z_1, \dots, z_k < 0$ such that $C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j \succ 0$. Then both (4.8) and (4.9) attain their optimal values, and $s^* = d^*$. Moreover, a feasible primal dual pair (X, y, z) is optimal if and only if $X \bullet (C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j) = 0$ and $\sum_{j=1}^k z_j (b_j - X \bullet B_j) = 0$.

Proof. Put $\widetilde{C} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$, $\widetilde{A_i} = \begin{pmatrix} A_i & 0 \\ 0 & 0 \end{pmatrix}$, $\widetilde{B_j} = \begin{pmatrix} B_j & 0 \\ 0 & E_j \end{pmatrix} \in S^{n+k}$, where E_j is $k \times k$ with 1 at the jj-th entry, zero everywhere else.

$$\begin{aligned} s^* = \inf_{\widetilde{X} \in S^{n+k}} & \widetilde{X} \bullet \widetilde{C} \\ \text{s.t.} \quad & \widetilde{X} \bullet \widetilde{A_i} = a_i, \quad i = 1, \dots, m, \\ & \widetilde{X} \bullet \widetilde{B_j} = b_j, \quad j = 1, \dots, k, \\ & \widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & \operatorname{diag}(t_1, \dots, t_k) \end{pmatrix} \succeq 0. \end{aligned} \tag{4.10}$$

The requirement that certain off-diagonal entries of $\widetilde{X}$ be zero can be stated as equality constraints: $\widetilde{X} \bullet \widetilde{F}_{ij} = 0$, where $\widetilde{F}_{ij} = \frac{F_{ij}+F_{ij}^T}{2}$, and F_{ij} has a 1 at the ijth entry, zero everywhere else. Hence, (4.10) is an SDP in standard form. Observe that X^* is optimal for problem (4.8) if and only if $\begin{pmatrix} X^* & 0 \\ 0 & \operatorname{diag}(t_1^*, \ldots, t_k^*) \end{pmatrix} \succeq 0$, where $t_j^* = b_j - X^* \bullet B_j$, is optimal for (4.10).

The dual of (4.10) is

$$\begin{aligned} d^* = & \sup_{y_1,\ldots,y_m,z_1,\ldots,z_k,f_{ij}\in\mathbb{R}} \sum_{i=1}^{m} y_i a_i + \sum_{j=1}^{k} z_j b_j \\ \text{s.t.} \quad & \begin{pmatrix} C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j & 0 \\ 0 & \operatorname{diag}(-z_1, \ldots, -z_k) \end{pmatrix} - \sum_{i,j} f_{ij} F_{ij} \succeq 0. \end{aligned} \tag{4.11}$$

Observe that (y^*, z^*) is optimal for (4.9) if and only if there exist $f_{ij}^* \in \mathbb{R}$ such that (y^*, z^*, f^*) is optimal for (4.11).

By assumption, there exists $X \succ 0$ such that $X \bullet A_i = a_i$, $X \bullet B_j < b_j$. Let $t_j = b_j - X \bullet B_j$. Then $\widetilde{X} = \begin{pmatrix} X & 0 \\ 0 & \operatorname{diag}(t_1, \ldots, t_k) \end{pmatrix} \succ 0$, and $\widetilde{X} \bullet \widetilde{A}_i = a_i$, $\widetilde{X} \bullet \widetilde{B}_j = b_j$. Also by assumption, there exist $y_1, \ldots, y_m \in \mathbb{R}$, $z_1, \ldots, z_k < 0$ such that $C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j \succ 0$. Letting $f_{ij} = 0$, we obtain (y, z, f) such that $\begin{pmatrix} C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j & 0 \\ 0 & \operatorname{diag}(-z_1, \ldots, -z_k) \end{pmatrix} - \sum f_{ij} F_{ij} \succ 0$. By theorem 11, (4.10) and (4.11) satisfy strong duality. Moreover, a feasible primal dual pair $(\widetilde{X}, y, z, f)$ is optimal if and only if

$$\begin{pmatrix} X & 0 \\ 0 & \operatorname{diag}(t_1, \ldots, t_k) \end{pmatrix} \bullet \left\{ \begin{pmatrix} C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j & 0 \\ 0 & \operatorname{diag}(-z_1, \ldots, -z_k) \end{pmatrix} - \sum f_{ij} F_{ij} \right\} = 0$$

or equivalently, $X \bullet (C - \sum_{i=1}^m y_i A_i - \sum_{j=1}^k z_j B_j) = 0$ and $\sum_{j=1}^k z_j t_j = 0$. □

Strong Duality of (4.5)

The dual of (4.5) is

$$\begin{aligned} \text{maximize} \quad & \operatorname{tr}(V) \\ \text{s.t.} \quad & C - \sum_{i \neq j} y_{ij} A_{ij} - \sum_{i=1}^{k} z_i B_i - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \succeq 0, \\ & y_{ij} \leq 0, z_i \leq 0, V \in S^d. \end{aligned} \tag{4.12}$$

Lemma 13. *Problem (4.12) is strictly feasible. That is, there exist $y_{ij} < 0$, $z_i < 0$ and $V \in S^d$ such that $C - \sum_{i \neq j} y_{ij} A_{ij} - \sum_{i=1}^k z_i B_i - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \succ 0$.*

Proof. For any $V \in S^d$,

$$C - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^k w_i \frac{\theta_i \theta_i^T}{4} - V & \Xi \\ \Xi^T & \operatorname{diag}(w_1, \dots, w_k) \end{pmatrix}$$

where $\Xi = \begin{pmatrix} -w_1 \frac{\theta_1}{2} & \dots & -w_k \frac{\theta_k}{2} \end{pmatrix}$. Let $V = -\sum_{i=1}^k \frac{\theta_i \theta_i^T}{4} - \mu I$, so $C - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \mu I & \Xi \\ \Xi^T & \operatorname{diag}(w_1, \dots, w_k) \end{pmatrix}$. For sufficiently large μ, $\mu I - \Xi \operatorname{diag}(w_1^{-1}, \dots, w_k^{-1}) \Xi^T \succ 0$. By the Schur complement criterion, $C - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} \succ 0$. For sufficiently small ε, $-\varepsilon < y_{ij}, z_i < 0$ guarantees $C - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix} - \sum_{i \neq j} y_{ij} A_{ij} - \sum_{i=1}^k z_i B_i \succ 0$. □

Lemma 14. *Assume $0 \notin \{\theta_1, \dots, \theta_k\}$. Then problem (4.5) is strictly feasible. That is, there exists $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succ 0$ such that $Z \bullet A_{ij} < 0$, $Z \bullet B_i < 0$ for all i, j.*

Proof. In $\mathbb{R}^{d+1}$, let Γ be the ball centered at $(0, \dots, 0, 1)^T$ with radius 1. Γ is a closed convex set in $\mathbb{R}^{d+1}$ containing 0. Let $(p_i, q_i) = p_\Gamma(\theta_i)$, where $p_i \in \mathbb{R}^d$, $q_i \in \mathbb{R}$.

Let $X = \begin{pmatrix} p_1 & \dots & p_k \end{pmatrix}$, $X' = \begin{pmatrix} q_1 & \dots & q_k \end{pmatrix}$, $Y = X^T X + (X')^T X'$, $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}$. Then $Z \bullet A_{ij} = ((\theta_i, 0) - (p_i, q_i)) \cdot ((p_j, q_j) - (p_i, q_i))$, $Z \bullet B_i = ((\theta_i, 0) - (p_i, q_i)) \cdot (-(p_i, q_i))$. Let $z = (0, \dots, 0, 1)$, $v_i = (\theta_i, 0) - z$. It's clear from geometry that $(p_i, q_i) = z + \frac{v_i}{|v_i|}$, and $|v_i| = |(\theta_i, 1)| > 1$. Moreover,

$$\begin{aligned} &((\theta_i, 0) - (p_i, q_i)) \cdot ((p_j, q_j) - (p_i, q_i)) \\ &= \left(z + v_i - \left(z + \frac{v_i}{|v_i|}\right)\right) \cdot \left(z + \frac{v_j}{|v_j|} - \left(z + \frac{v_i}{|v_i|}\right)\right) \\ &= (|v_i| - 1) \frac{v_i}{|v_i|} \cdot \left(\frac{v_j}{|v_j|} - \frac{v_i}{|v_i|}\right) \\ &= (|v_i| - 1) \left(\frac{v_i \cdot v_j}{|v_i||v_j|} - 1\right). \end{aligned}$$

By Cauchy-Schwarz, $v_i \cdot v_j \leq |v_i||v_j|$. Moreover, v_i, v_j are not collinear. Indeed, if $v_i = \alpha v_j$, then $(\theta_i - \alpha\theta_j, 0) = (1-\alpha)z$. For the last coordinate to be zero, we must have $\alpha = 1$. But then $\theta_i - \theta_j = 0$, which is a contradiction. Thus, $v_i \cdot v_j < |v_i||v_j|$. We deduce that

$$((\theta_i, 0) - (p_i, q_i)) \cdot ((p_j, q_j) - (p_i, q_i)) = (|v_i| - 1)\Big(\frac{v_i \cdot v_j}{|v_i||v_j|} - 1\Big) < 0.$$

Similarly,

$$\begin{aligned} ((\theta_i, 0) - (p_i, q_i)) \cdot (0 - (p_i, q_i)) &= \Big(z + v_i - \Big(z + \frac{v_i}{|v_i|}\Big)\Big) \cdot \Big(0 - \Big(z + \frac{v_i}{|v_i|}\Big)\Big) \\ &= (|v_i| - 1)\frac{v_i}{|v_i|} \cdot \Big(-z - \frac{v_i}{|v_i|}\Big) \\ &< (|v_i| - 1)\Big(|z| - 1\Big) \\ &= 0. \end{aligned}$$

We conclude that $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} \succeq 0$, with $Z \bullet A_{ij} < 0$, $Z \bullet B_i < 0$ for all i, j. By the Schur complement criterion, $Y - X^T X \succeq 0$. Put $\widetilde{Z} = \begin{pmatrix} I & X \\ X^T & Y + \varepsilon I \end{pmatrix}$. Then for sufficiently small ε, $\widetilde{Z} \bullet A_{ij} < 0$, $\widetilde{Z} \bullet B_i < 0$ for all i, j. Moreover, $Y + \varepsilon I - X^T X \succ 0$, so $\widetilde{Z} \succ 0$. □

Remark 5. Any feasible value of the dual problem (4.12) is a lower bound for the optimal value of Problem (4.4).

4.3 Relation between SDP and QCQP

In this section, we investigate the question of when the solution of the relaxed problem (4.5) gives a solution of the complete problem (4.4). In the case when relaxation fails to solve the complete problem, we provide an economic interpretation for this relaxation gap.

Recall that (4.5) relaxes (4.4) by dropping the constraint $Y = X^T X$, or equivalently (by Proposition 10), the rank $\begin{pmatrix} I & X \\ X^T & Y \end{pmatrix} = d$ constraint. We say that Problem (4.5) with parameters (θ, w) is **exact** if every optimal solution $Z = \begin{pmatrix} I & X \\ X^T & Y \end{pmatrix}$ satisfies rank $Z = d$.

Theorem 15. *Put* $\Omega = \{\theta_1, \ldots, \theta_k\}$, $\mu = \sum_{i=1}^k w_i \delta_{\theta_i}$. *Problem (4.5) with parameters* (θ, w) *is exact if and only if* (Ω, μ) *is localizable.*

Proof. Suppose (Ω, μ) is localizable. Consider $Z = \begin{pmatrix} I_d & X \\ X^T & Y \end{pmatrix}$ optimal for (4.5), with rank $Z = d + r > d$. Since $Z \geq 0$, $Y - X^T X \geq 0$. Moreover, since rank $Z > d, Y - X^T X \neq 0$. In fact, by the Guttman rank additivity formula (see [16]), rank $(Y - X^T X) = r$. Hence, there exists $X' \in \mathbb{R}^{r \times k}$ such that $Y = X^T X + (X')^T X'$. Put $X = (p_1, \ldots, p_k)$, $X' = (p'_1, \ldots, p'_k)$, and let $\widetilde{p_i} = \begin{pmatrix} p_i \\ p'_i \end{pmatrix}$, $\widetilde{\theta_i} = \begin{pmatrix} \theta_i \\ 0 \end{pmatrix}$. Then

$$\begin{aligned}
(\widetilde{\theta_i} - \widetilde{p_i}) \cdot (\widetilde{p}_j - \widetilde{p_i}) &= (\theta_i - p_i) \cdot (p_j - p_i) + (-p'_i) \cdot (p'_j - p'_i) \\
&= \theta_i^T (p_j - p_i) + \mathbf{e}_i^T X^T X \mathbf{e}_{ij} + \mathbf{e}_i^T (X')^T X' \mathbf{e}_{ij} \\
&= \theta_i^T X \mathbf{e}_{ij} + \mathbf{e}_i^T Y \mathbf{e}_{ij} \\
&= Z \bullet \left(\begin{pmatrix} 0 \\ \mathbf{e}_{ij} \end{pmatrix} \begin{pmatrix} \theta_i^T & \mathbf{e}_i^T \end{pmatrix} \right) \\
&= Z \bullet A_{ij} \\
&\leq 0,
\end{aligned}$$

$$\begin{aligned}
(\widetilde{\theta_i} - \widetilde{p_i}) \cdot (-\widetilde{p_i}) &= (\theta_i - p_i) \cdot (-p_i) + |p'_i|^2 \\
&= \theta_i^T (-p_i) + \mathbf{e}_i^T X^T X \mathbf{e}_i + \mathbf{e}_i^T (X')^T X' \mathbf{e}_i \\
&= \theta_i^T X \mathbf{e}_i + \mathbf{e}_i^T Y \mathbf{e}_i \\
&= Z \bullet \left(\begin{pmatrix} 0 \\ \mathbf{e}_i \end{pmatrix} \begin{pmatrix} \theta_i^T & \mathbf{e}_i^T \end{pmatrix} \right) \\
&= Z \bullet B_i \\
&\leq 0,
\end{aligned}$$

and

$$\begin{aligned}
\sum_{i=1}^{k} w_i \left|\frac{\widetilde{\theta_i}}{2} - p_i\right|^2 &= \sum_{i=1}^{k} w_i \left(\left|\frac{\theta_i}{2} - p_i\right|^2 + |p_i'|^2\right) \\
&= \sum_{i=1}^{k} w_i \left(\frac{|\theta_i|^2}{4} - 2\frac{\theta_i^T}{2} p_i + |p_i|^2 + |p_i'|^2\right) \\
&= \sum_{i=1}^{k} w_i \left(\frac{|\theta_i|^2}{4} - 2\frac{\theta_i^T}{2} X\mathbf{e}_i + \mathbf{e}_i^T X^T X \mathbf{e}_i + \mathbf{e}_i^T (X')^T X' \mathbf{e}_i\right) \\
&= \sum_{i=1}^{k} w_i \left(\frac{|\theta_i|^2}{4} - 2\frac{\theta_i^T}{2} X\mathbf{e}_i + \mathbf{e}_i^T Y \mathbf{e}_i\right) \\
&= \sum_{i=1}^{k} Z \bullet w_i \left(\begin{pmatrix} \frac{\theta_i}{2} \\ \mathbf{e}_i \end{pmatrix} \left(\tfrac{\theta_i^T}{2}; \mathbf{e}_i^T\right) \right) \\
&= Z \bullet C.
\end{aligned}$$

By Proposition 8, there exists a convex set Γ in $\mathbb{R}^{d+r}$ such that $0 \in \Gamma$, and $\widetilde{p_i} = p_\Gamma(\widetilde{\theta_i})$. By localizability, $\widetilde{p_i} \in \mathbb{R}^d$, so $p_i' = 0$. But this implies $X' = 0$, and rank $Z = d$, contradiction.

Conversely, assume (4.5) is exact. Consider a higher dimension $d' = d + r > d$, and a convex region $\Gamma \subseteq \mathbb{R}^{d'}$ containing 0. Put $\widetilde{\theta_i} = \begin{pmatrix} \theta_i \\ 0 \end{pmatrix}$. Let $\widetilde{p_i} = p_\Gamma(\widetilde{\theta_i}) = \begin{pmatrix} p_i \\ p_i' \end{pmatrix}$, where $p_i \in \mathbb{R}^d$, $p_i' \in \mathbb{R}^r$. Assume Γ is optimal:

$$\sum_{i=1}^{k} w_i \left|\frac{\widetilde{\theta_i}}{2} - \widetilde{p_i}\right|^2 = s^*.$$

Let $X = \begin{pmatrix} p_1 & \dots & p_k \end{pmatrix}$, $X' = \begin{pmatrix} p_1' & \dots & p_k' \end{pmatrix}$, and $Y = X^T X + (X')^T X'$, $Z = \begin{pmatrix} I_d & X \\ X^T & Y \end{pmatrix}$. As before, a direct calculation shows that Z is feasible for (4.5), with $Z \bullet C = \sum_{i=1}^k w_i \left|\frac{\widetilde{\theta_i}}{2} - \widetilde{p_i}\right|^2 = s^*$. Since (4.5) is exact, rank $Z = d$. It follows that $Y = X^T X$, and $X' = 0$. We conclude $\widetilde{p_i} = p_\Gamma(\theta_i) \in \mathbb{R}^d$. □

Remark 6. Theorem 15 provides an interpretation of the relaxation gap when Problem (4.5) fails to be exact. When such a gap exists, one can exceed the optimal value of problem (2.8) by embedding the problem in higher dimensions. $Y - X^T X$ is accounted for by the last $d' - d$ coordinates of $p_\Gamma(\theta_1), \dots, p_\Gamma(\theta_k)$ of the higher-dimensional optimum.

Remark 7. We may also view the nonexactness of Problem (4.5) in terms of the original optimal fund menus problem. Suppose, in addition to the d existing assets, the fund manager is able to offer r additional artificial assets whose returns (in excess of the risk free rate) are independent with mean 0, and that investors have no exposure to these additional assets. The corresponding shape optimization problem is precisely

$$\begin{aligned} \text{minimize}_\Gamma \quad & \sum_{i=1}^k w_i \left| \frac{\theta_i}{2} - p_\Gamma(\theta_i) \right|^2 \\ \text{s.t.} \quad & \Gamma \subseteq \mathbb{R}^{d+r} \text{ closed and convex}, 0 \in \Gamma. \end{aligned} \tag{4.13}$$

When exactness/localizability fails, the optimal $p_\Gamma(\theta_1), \dots, p_\Gamma(\theta_k)$ do not lie in $\mathbb{R}^d$. The last r coordinates of $p_\Gamma(\theta_i)$ represent the amount invested in the r artificial assets by type θ_i.

Since investors have no initial exposure to them, the zero mean return artificial assets can only decrease their utility. If all assets were offered separately, no one would invest in the artificial ones. Thus, failure of regularity represents an opportunity for the fund manager to extract more profit by bundling assets that are of no value to investors.

We conclude this section with a sufficient condition for uniqueness of Problem (4.4).

Proposition 16. *If Problem (4.5) with parameters (θ, w) is exact, then both (4.5) and (4.4) have a unique solution.*

Proof. Since (4.5) is exact, by Theorem (15), any optimal solution has rank d, and is of the form $Z = \begin{pmatrix} I_d & X \\ X^T & X^T X \end{pmatrix}$. Let $Z_1 = \begin{pmatrix} I_d & X_1 \\ X_1^T & X_1^T X_1 \end{pmatrix}$, $Z_2 = \begin{pmatrix} I_d & X_2 \\ X_2^T & X_2^T X_2 \end{pmatrix}$ be two solutions. Since (4.5) is a convex problem, $\frac{1}{2}Z_1 + \frac{1}{2}Z_2$ is also optimal. By assumption,

$$\frac{1}{2}Z_1 + \frac{1}{2}Z_2 = \begin{pmatrix} I_d & \frac{1}{2}X_1 + \frac{1}{2}X_2 \\ \frac{1}{2}X_1^T + \frac{1}{2}X_2^T & \frac{1}{2}X_1^T X_1 + \frac{1}{2}X_2^T X_2 \end{pmatrix}$$

has rank d. Consequently,

$$\begin{aligned} \frac{1}{2}X_1^T X_1 + \frac{1}{2}X_2^T X_2 &= (\frac{1}{2}X_1 + \frac{1}{2}X_2)^T (\frac{1}{2}X_1 + \frac{1}{2}X_2) \\ &= \frac{1}{4}X_1^T X_1 + \frac{1}{4}X_1^T X_2 + \frac{1}{4}X_2^T X_1 + \frac{1}{4}X_2^T X_2. \end{aligned}$$

Subtracting $\frac{1}{2}X_1^T X_1 + \frac{1}{2}X_2^T X_2$ from both sides, we find

$$\begin{aligned} 0 &= -\frac{1}{4}X_1^T X_1 + \frac{1}{4}X_1^T X_2 + \frac{1}{4}X_2^T X_1 - \frac{1}{4}X_2^T X_2 \\ &= -\frac{1}{4}(X_1 - X_2)^T (X_1 - X_2). \end{aligned}$$

This implies $X_1 = X_2$. Hence, the solution of (4.5) is unique.

Due to exactness, any solution $p_1, \dots, p_k$ of (4.4) corresponds to an optimal solution $Z = \begin{pmatrix} I_d & X \\ X^T & X^T X \end{pmatrix}$ of (4.5) via $X = \begin{pmatrix} p_1 & \dots & p_k \end{pmatrix}$. Consequently, the solution of (4.4) is also unique. □

4.4 Special Cases

The following proposition shows that when there is only one asset, and all investors share similar level of exposure to it, then the type distribution is regular.

Proposition 17. *Assume $0 < \theta_1 < \cdots < \theta_k \in \mathbb{R}$, and $\frac{\theta_k}{2} \le \theta_1$. Then for any $w_1, \dots, w_k > 0$, $(\{\theta_1, \dots, \theta_k\}, \sum_{i=1}^k w_i \delta_{\theta_i})$ is localizable.*

Proof. For convenience, put $\zeta_i = \frac{\theta_i}{2}$. Suppose Γ is a convex region containing 0 such that $\theta_1, \dots, \theta_k$ are projected to $p_1, \dots, p_k \in \mathbb{R}^d$, with $p_1, \dots, p_{r-1} \in \mathbb{R}$, $p_r \notin \mathbb{R}$. Since $\dim \operatorname{span}(\mathbb{R} \cup \{p_r\}) = 2$, we may assume (after change of coordinates) that $p_r = (p_r', q_r) \in \mathbb{R}^2$, where $q_r \neq 0$. Note that by Proposition 8, $p_1, \dots, p_k$ satisfy

$$\begin{aligned} (\theta_i - p_i) \cdot (p_j - p_i) &\le 0 \quad \text{for all } i \neq j, \\ (\theta_i - p_i) \cdot (-p_i) &\le 0 \quad \text{for all } i. \end{aligned}$$

Let $\mathbb{H}_r = \{x \in \mathbb{R}^d \mid (x - p_r) \cdot (\theta_r - p_r) \le 0\}$. Then $p_1, \dots, p_k \in \Gamma \subseteq \mathbb{H}_r$. Let $B_i = \left\{x \in \mathbb{R}^d \,\middle|\, \left|x - \frac{p_r+\theta_i}{2}\right| \le \left|\frac{p_r-\theta_i}{2}\right|\right\}$ be the Euclidean ball in $\mathbb{R}^d$ with diameter $[p_r, \theta_i]$. Since $(p_r - p_i) \cdot (\theta_i - p_i) \le 0$, we must have $p_i \in B_i$. Indeed,

$$\begin{aligned} \left|p_i - \frac{p_r + \theta_i}{2}\right|^2 &= |p_i|^2 - p_i \cdot (p_r + \theta_i) + \left|\frac{p_r + \theta_i}{2}\right|^2 \\ &= |p_i|^2 - p_i \cdot p_r - p_i \cdot \theta_i + \left|\frac{p_r + \theta_i}{2}\right|^2 \\ &= \Big(|p_i|^2 - p_i \cdot p_r - p_i \cdot \theta_i + p_r\theta_i\Big) + \Big(- p_r\theta_i + \left|\frac{p_r + \theta_i}{2}\right|^2\Big) \\ &= (p_r - p_i) \cdot (\theta_i - p_i) + \left|\frac{p_r - \theta_i}{2}\right|^2 \\ &\le \left|\frac{p_r - \theta_i}{2}\right|^2. \end{aligned}$$

We have shown that $p_i \in \mathbb{H}_r \cap B_i$ for $i > r$.

We claim that $|\zeta_i - p_i| \geq \inf_{z \in \mathbb{H}_r \cap B_i} |\zeta_i - z| = |\zeta_i - p_r|$ for $i > r$. To see this, consider the problem

$$\begin{aligned} &\text{minimize} \quad |\zeta_i - z|^2 \\ \text{s.t.} \quad &\left| z - \frac{p_r + \theta_i}{2} \right|^2 \leq \left| \frac{p_r - \theta_i}{2} \right|^2, \\ &(\theta_r - p_r) \cdot (z - p_r) \leq 0. \end{aligned}$$

The Lagrangian is

$$\begin{aligned} &L(z, \lambda_1, \lambda_2) \\ &= |\zeta_i - z|^2 + \lambda_1 \left(\left| z - \frac{p_r + \theta_i}{2} \right|^2 - \left| \frac{p_r - \theta_i}{2} \right|^2 \right) + \lambda_2 (\theta_r - p_r) \cdot (z - p_r) \\ &= |\zeta_i - z|^2 + \lambda_1 (p_r - z)(\theta_i - z) + \lambda_2 (\theta_r - p_r) \cdot (z - p_r) \\ &= (1 + \lambda_1)|z|^2 + \Big(-2\zeta_1 - \lambda_1 (p_r + \theta_i) + \lambda_2 (\theta_r - p_r) \Big) \cdot z \\ &\quad + |\zeta_i|^2 + \lambda_1 p_r \cdot \theta_i - \lambda_2 \theta_r \cdot p_r, \end{aligned}$$

where $\lambda_1, \lambda_2 \geq 0$.

$$\nabla_z L(z, \lambda_1, \lambda_2) = \lambda_1 \left(-p_r - \theta_i + 2z \right) + \lambda_2 \left(-p_r + \theta_r \right) + 2z - 2\zeta_i.$$

It's easy to verify that $z = p_r$, $\lambda_1 = \frac{2(\theta_r - \zeta_i)}{\theta_i - \theta_r}$, $\lambda_2 = \frac{2(\theta_i - \zeta_i)}{\theta_i - \theta_r}$ satisfy $\nabla_z L(z, \lambda_1, \lambda_2) = 0$. Note that $z = p_r$ is primal feasible. Also, since $\zeta_i \leq \theta_r < \theta_i$, $\lambda_1, \lambda_2 \geq 0$ must be dual feasible. We found a primal-dual pair satisfying the Karush-Kuhn-Tucker condition. Hence, p_r is the minimizer.

Define $\widetilde{p}_1, \ldots, \widetilde{p}_k$ by $\widetilde{p}_i = p_i$ for $i = 1, \ldots, r-1$, and $\widetilde{p}_i = p_r$ for $i = r, \ldots, k$. We claim that they satisfy

$$\begin{aligned} (\theta_i - \widetilde{p}_i) \cdot (\widetilde{p}_j - \widetilde{p}_i) &\leq 0 \quad \text{for all } i \neq j, \\ (\theta_i - \widetilde{p}_i) \cdot (-\widetilde{p}_i) &\leq 0 \quad \text{for all } i. \end{aligned}$$

By construction, it suffices to check

$$\begin{aligned} (\theta_j - p_r) \cdot (p_i - p_r) &\leq 0 \quad \text{for } i < r < j, \\ (\theta_j - p_r) \cdot (-p_r) &\leq 0 \quad \text{for } j > r. \end{aligned}$$

To prove the first inequality, fix $i < r < j$. First note that $p_i \leq \theta_i$. This follows directly from $(\theta_i - p_i) \cdot (-p_i) \leq 0$. Next, we have $p_i \leq p_r'$ (recall that p_r' is the first coordinate of $p_r \in \mathbb{R}^2$). Indeed, suppose, to the contrary, $p_i > p'_r$. Then $p'_r < p_i \le \theta_i \le \theta_r$, and $|\theta_r - p_r|^2 = |\theta_r - p'_r|^2 + |q_r|^2 > |\theta_r - p_i|^2$. p_i would be a closer point in Γ to θ_r than p_r, contradiction. Since $i < r < j$, $p_i \le p'_r$, $\theta_r \le \theta_j$, and

$$(\theta_j - \theta_r) \cdot (p_i - p_r) = (\theta_j - \theta_r)(p_i - p'_r) \le 0.$$

We already know

$$(\theta_r - p_r) \cdot (p_i - p_r) \le 0,$$

so adding the two inequalities gives

$$(\theta_j - p_r) \cdot (p_i - p_r) \le 0.$$

For the second inequality, fix $j > r$. We know that $(\theta_r - p_r) \cdot (-p_r) = (\theta_r - p'_r)(-p'_r) + |q_r|^2 \le 0$. From this, we deduce that $p'_r \ge 0$ (if $p'_r < 0$, then from $(\theta_r - p'_r)(-p'_r) \le 0$ we find $\theta_r - p'_r \le 0$, so $0 \le \theta_r \le p'_r < 0$, contradiction). Since $\theta_j \ge \theta_r$ and $p'_r \ge 0$,

$$(\theta_j - \theta_r) \cdot (-p_r) \le 0.$$

We already know

$$(\theta_r - p_r) \cdot (-p_r) \le 0,$$

so adding the two inequalities gives

$$(\theta_j - p_r) \cdot (-p_r) \le 0.$$

Now define $\overline{p}_1, \dots, \overline{p}_k$ as follows: $\overline{p}_i = \widetilde{p}_i = p_i$ for $i < r$, $\overline{p}_i = \begin{pmatrix} p'_r \\ 0 \end{pmatrix}$ for $i \ge r$. It's easy to verify that they satisfy

$$(\theta_i - \overline{p}_i) \cdot (\overline{p}_j - \overline{p}_i) \le 0 \quad \text{for all } i \ne j,$$
$$(\theta_i - \overline{p}_i) \cdot (-\overline{p}_i) \le 0 \quad \text{for all } i.$$

By Proposition 8, there exists a convex region $\overline{\Gamma} \subseteq \mathbb{R}^d$ containing 0 such that $\overline{p}_i = p_{\overline{\Gamma}}(\theta_i)$ for $i = 1, \dots, k$. Moreover,

$$\sum_{i=1}^{k} w_i |\zeta_i - p_i|^2 \ge \sum_{i=1}^{r-1} w_i |\zeta_i - p_i|^2 + \sum_{i=r}^{k} w_i |\zeta_i - p_r|^2$$
$$> \sum_{i=1}^{k} w_i |\zeta_i - \overline{p}_i|^2.$$

It follows that the convex set Γ we started with cannot be optimal. $\square$

The situation covered by Proposition 17 is one-dimensional. We next study a special case that goes beyond one dimension. We show that when $\theta_1, \ldots, \theta_k$ are linearly independent, (4.5) is exact. It results from the convenient structure of the dual problem (4.12). We will use the following lemma:

Lemma 18. *Suppose y_{ij}^*, z_i^*, V^* are optimal for the dual problem (4.12). Let $S^* = C - \sum_{i\neq j} y_{ij}^* A_{ij} - \sum_{i=1}^k z_i^* B_i - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$. If rank $S^* \geq k$, then (4.5) is exact.*

Proof. Suppose Z^* is optimal for (4.5). Then $Z^* \bullet S^* = 0$. Since Z^*, S^* are both positive semidefinite, we have the matrix product $Z^* S^* = 0$. Since Z^*, S^* are $(d + k) \times (d + k)$, rank Z^* + rank $S^* \leq d + k$. Since rank $S^* \geq k$, it must be that rank $Z^* \leq d$. By Proposition 10, Problem (4.5) is exact. □

With Lemma 18, it suffices to show that rank $S^* \geq k$ whenever we wish to show that the SDP relaxation is exact. We now present the special case of interest.

Proposition 19. *Suppose $\theta_1, \ldots, \theta_k$ are linearly independent. Then for any $w_1, \ldots, w_k > 0$, problem (4.5) with parameters (θ, w) is exact.*

Proof. Suppose y_{ij}^*, z_i^*, V^* are optimal for (4.12). Let $S^* = C - \sum_{i\neq j} y_{ij}^* A_{ij} - \sum_{i=1}^k z_i^* B_i - \begin{pmatrix} V & 0 \\ 0 & 0 \end{pmatrix}$. Then S^* takes the form

$$S^* = \begin{pmatrix} \sum_{i=1}^k w_i \frac{\theta_i \theta_i^T}{4} - V^* & f_1 & f_2 & \ldots & f_k \\ f_1^T & e_{11} & e_{12} & \ldots & e_{1k} \\ f_2^T & e_{21} & e_{22} & \ldots & e_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ f_k^T & e_{k1} & e_{k2} & \ldots & e_{kk} \end{pmatrix},$$

where

$$f_i = -w_i \frac{\theta_i}{2} + \sum_{j\neq i} y_{ij}^* \frac{\theta_i}{2} + z_i^* \frac{\theta_i}{2} - \sum_{j \neq i} y_{ji}^* \frac{\theta_j}{2}, \quad i = 1, \ldots, k,$$
$$e_{ii} = w_i - \sum_{j\neq i} y_{ij}^* - z_i^*, \quad i = 1, \ldots, k,$$
$$e_{ij} = -\frac{1}{2} y_{ij}^* - \frac{1}{2} y_{ji}^*, \quad i \neq j.$$

We show that $f_1, f_2, \dots, f_k$ are linearly independent. Write

$$f_i = \Big(-w_i + \sum_{j\neq i} y^*_{ij} + z^*_i\Big)\frac{\theta_i}{2} - \sum_{j\neq i} y^*_{ji}\frac{\theta_j}{2}.$$

Since $\theta_1, \dots, \theta_k$ are linearly independent, it suffices to show that

$$\begin{pmatrix} -w_1 + \sum_{j\neq 1} y^*_{1j} + z^*_1 & -y^*_{12} & \dots & -y^*_{1k} \\ -y^*_{21} & -w_2 + \sum_{j\neq 2} y^*_{2j} + z^*_2 & \dots & -y^*_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ -y^*_{k1} & -y^*_{k2} & \dots & -w_k + \sum_{j\neq k} y^*_{kj} + z^*_k \end{pmatrix}$$

is nonsingular. Since $y^*_{ij} \leq 0, z^*_i \leq 0$, this matrix is strictly diagonally dominant, hence indeed nonsingular.

Having established linear independence of $f_1, \dots, f_k$, we have shown that rank $S^* \geq k$. By Lemma 18, Problem (4.5) is exact. □

This result can be partially extended to the case when $d \geq k$.

Lemma 20. *Let $v(\theta, w)$ be the optimal value of problem (4.5) with parameters (θ, w). Then for any pair $\theta, \theta' \in \mathbb{R}^{d\times k}$,*

$$|v(\theta, w) - v(\theta', w)| \leq \frac{9}{4}\sum_{i=1}^{k} w_i[2\max\{|\theta_i|, |\theta'_i|\}|\theta_i - \theta'_i| + |\theta_i - \theta'_i|^2]. \tag{4.14}$$

In particular, if $\theta^l \in \mathbb{R}^{d\times k}$ converges to θ, then $\lim_{l\to\infty} v(\theta^l, w) = v(\theta, w)$.

Proof. Put $\zeta_i = \frac{\theta_i}{2}$. By Theorem 15, there exists $r \geq 0$ and a closed convex set $\Gamma \subseteq \mathbb{R}^{d+r}$ such that $0 \in \Gamma$ and $v(\theta, w) = \sum_{i=1}^k w_i|\zeta_i - p_\Gamma(\theta_i)|^2$. The map $p_\Gamma : \mathbb{R}^d \to \mathbb{R}^d$ is contractive, that is, $|p_\Gamma(x) - p_\Gamma(y)| \leq |x - y|$ (see [13]). Consequently,

$$\begin{aligned} |\zeta'_i - p_\Gamma(\theta'_i)| &= |\zeta_i - p_\Gamma(\theta_i) - [\zeta_i - p_\Gamma(\theta_i)] + [\zeta'_i - p_\Gamma(\theta'_i)]| \\ &\leq |\zeta_i - p_\Gamma(\theta_i)| + |\zeta_i - \zeta'_i| + |p_\Gamma(\theta_i) - p_\Gamma(\theta'_i)| \\ &\leq |\zeta_i - p_\Gamma(\theta_i)| + \frac{1}{2}|\theta_i - \theta'_i| + |\theta_i - \theta'_i|, \end{aligned}$$

and

$$\begin{aligned} &|\zeta'_i - p_\Gamma(\theta'_i)|^2 \\ &\leq |\zeta_i - p_\Gamma(\theta_i)|^2 + 3|\zeta_i - p_\Gamma(\theta_i)||\theta_i - \theta'_i| + \frac{9}{4}|\theta_i - \theta'_i|^2 \\ &\leq |\zeta_i - p_\Gamma(\theta_i)|^2 + \frac{9}{2}|\theta_i||\theta_i - \theta'_i| + \frac{9}{4}|\theta_i - \theta'_i|^2. \end{aligned}$$

Upon summation over i, we find

$$\begin{aligned} v(\theta', w) &\leq \sum_{i=1}^{k} w_i |\zeta_i' - p_\Gamma(\theta_i')|^2 \\ &\leq v(\theta, w) + \frac{9}{4} \sum_{i=1}^{k} w_i [2|\theta_i||\theta_i - \theta_i'| + |\theta_i - \theta_i'|^2]. \end{aligned}$$

Inequality (4.14) follows from reversing the roles of θ and θ'. □

Corollary 21. *When $d \geq k$, Problem (4.5) admits a solution of rank d.*

Proof. Since $d \geq k$, there exists a sequence $(\theta_1^l, \dots, \theta_k^l) \to (\theta_1, \dots, \theta_k)$ such that for each l, $\theta_1^l, \dots, \theta_k^l$ are linearly independent. Let C^l, A_{ij}^l, B_i^l be the coefficient matrices for Problem (4.5) with input $(\theta_1^l, \dots, \theta_k^l)$.

Let $Z^l = \begin{pmatrix} I_d & p_1^l & \dots & p_k^l \\ (p_1^l)^T & y_{11}^l & \dots & y_{1k}^l \\ \vdots & \vdots & \ddots & \vdots \\ (p_k^l)^T & y_{k1}^l & \dots & y_{kk}^l \end{pmatrix}$ be the optimal solution for Problem (4.5) with input θ^l. By Proposition 19, rank $Z^l = d$, and there exists a closed convex set $\Gamma^l \subseteq \mathbb{R}^d$ such that $0 \in \Gamma^l$, and $p_i^l = p_{\Gamma^l}(\theta_i^l)$. We know that $|p_i^l| \leq |\theta_i^l|$. Moreover, since rank $Z^l = d$, we must have $|y_{ij}^l| = |p_i^l \cdot p_j^l| \leq |\theta_i^l||\theta_j^l|$. This analysis shows that the sequence $\{Z^l\}_{l=1}^{\infty}$ is bounded. After passing to a subsequence, Z^l converges to some matrix Z. Note that Z is feasible for Problem (4.5) with input θ, and that rank $Z = d$. Also, $\lim_{l \to \infty} Z^l \bullet C^l = Z \bullet C$. By Lemma 20, Z is optimal.

□

Remark 8. This result is weaker than Proposition 19 because it does not imply uniqueness of solution for (4.5). It guarantees that an optimal solution of rank d exists, but there may be additional solutions with higher rank.

Stated in terms of the fund menu problem, if there are at least as many assets as the number of types, then the manager cannot (strictly) increase aggregate fee by introducing additional independent assets with zero mean return.

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Appendix A

GAMMA CONVERGENCE

We provide justification for focusing on the finite case of minimizing $\sum_{i=1}^{k} w_i \left|\frac{\theta_i}{2} - p_\Gamma(\theta_i)\right|^2$.

Consider a bounded region $\Omega \subseteq \mathbb{R}^d$ and a finite measure μ. We may discretize μ by partitioning $\mathbb{R}^d$ into dyadic cubes and forming a weighted sum of point-masses at the centers. Formally, let $\mathcal{D}_n$ be the collection of dyadic cubes in $\mathbb{R}^d$ of length 2^{-n}. For each $Q_i \in \mathcal{D}_n$, let θ_i be the center of Q_i. Define

$$\mu_n := \sum_{Q_i \in \mathcal{D}_n} \mu(\Omega \cap Q_i)\delta_{\theta_i} \tag{A.1}$$

Let C be the collection of closed convex subsets of $\mathbb{R}^d$ containing the origin, and let $\mathcal{F}_n(\Gamma) := \int \left|\frac{\theta}{2} - p_\Gamma(\theta)\right|^2 d\mu_n$. Loosely speaking, as the dyadic cube partition gets finer, the discretized problem $\text{minimize}_{\Gamma\in C}\mathcal{F}_n(\Gamma)$ "approximates" the target problem $\text{minimize}_{\Gamma\in C}\mathcal{F}(\Gamma)$ better. The precise formulation of such approximation (often known as gamma-convergence in calculus of variations) is given by the following result:

Proposition 22. *Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, and μ is a finite measure on Ω. Define μ_n as in (A.1). Then there exists a sequence of minimizers $\{\Gamma_n\}$ for problems $\text{minimize}_{\Gamma\in C}\mathcal{F}_n(\Gamma)$ having an accumulation point in Δ_H. Moreover, any such accumulation point is an optimal solution of $\text{minimize}_{\Gamma\in C}\mathcal{F}(\Gamma)$.*

Lemma 23. *Suppose $\{\Gamma_n\}$ is a sequence in C such that $\Gamma_n \to \Gamma$ in Δ_H. Then $\mathcal{F}_n(\Gamma_n) \to \mathcal{F}(\Gamma)$.*

Proof. $|\mathcal{F}_n(\Gamma_n) - \mathcal{F}(\Gamma)| \le |\mathcal{F}_n(\Gamma_n) - \mathcal{F}(\Gamma_n)| + |\mathcal{F}(\Gamma_n) - \mathcal{F}(\Gamma)|$. By Proposition 6, the second term on the right tends to 0 as $n \to \infty$.

Now consider the first term. On each dyadic cube Q_i, $\int_{Q_i} \left|\frac{\theta}{2} - p_{\Gamma_n}(\theta)\right|^2 d\mu_n = \int_{Q_i} \left|\frac{\theta_i}{2} - p_{\Gamma_n}(\theta_i)\right|^2 d\mu$. Since $|p_{\Gamma_n}(\theta)| \le |\theta|$, and $|p_{\Gamma_n}(\theta_i) - p_{\Gamma_n}(\theta)| \le |\theta_i - \theta|$,

$$\left| \int_{Q_i} \left|\frac{\theta_i}{2} - p_{\Gamma_n}(\theta_i)\right|^2 d\mu - \int_{Q_i} \left|\frac{\theta}{2} - p_{\Gamma_n}(\theta)\right|^2 d\mu \right| \lesssim \int_{Q_i} (|\theta_i| + |\theta|)|\theta_i - \theta| d\mu.$$

By the dominated convergence theorem, as $n \to \infty$, $\sum_{Q_i\in\mathcal{D}_n} \int_{Q_i} (|\theta_i| + |\theta|)|\theta_i - \theta| d\mu \to 0$. □

Proof of Proposition 22. Since Ω is bounded, all μ_n are supported in a large enough ball, and we may assume that Γ_n are contained in this ball. By the Blaschke selection theorem, we may assume that $\{\Gamma_n\}$ contains a subsequence that converges in the Hausdorff distance. Suppose Γ is any accumulation point of $\{\Gamma_n\}$. By Lemma 23, $\mathcal{F}_n(\Gamma_n) \to \mathcal{F}(\Gamma)$. Moreover, if $\Gamma' \in \mathcal{C}$, then

$$\mathcal{F}(\Gamma') = \lim \mathcal{F}_n(\Gamma') \geq \lim \mathcal{F}_n(\Gamma_n) = \mathcal{F}(\Gamma).$$

Consequently, Γ is a minimizer of $\mathcal{F}(\cdot)$. □

Appendix B

NUMERICAL RESULTS

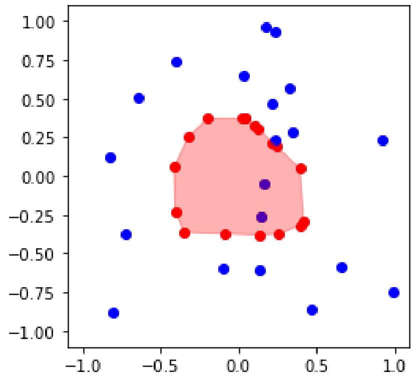
We examine the numerical solutions of Problem (4.5) in several cases of interest. Each case consists of points $\theta_1, \dots, \theta_k$ in $\mathbb{R}^2$ with uniform weight $w_1 = \cdots = w_k = 1$. We plot the types $\theta_1, \dots, \theta_k$ (blue), the optimal convex set Γ (shaded region), as well as the projection points $p_\Gamma(\theta_1), \dots, p_\Gamma(\theta_k)$.

In addition to the plots, we will be interested in the numerical value of $\lambda_{d+1}(S^*)$—the $(d+1)$-st smallest eigenvalue of the optimal dual slack matrix

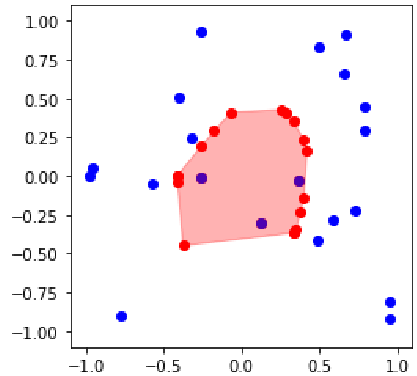
$$S^* = C - \sum_{i \neq j} y_{ij}^* A_{ij} - \sum_{i=1}^{k} z_i^* B_i - \begin{pmatrix} V^* & 0 \\ 0 & 0 \end{pmatrix}$$

for Problem (4.12). A strictly positive value implies rank $S^* \geq k$. By Lemma (18), it allows us to verify, a posteriori, that the SDP numerical solution is exact for the complete problem (4.4).

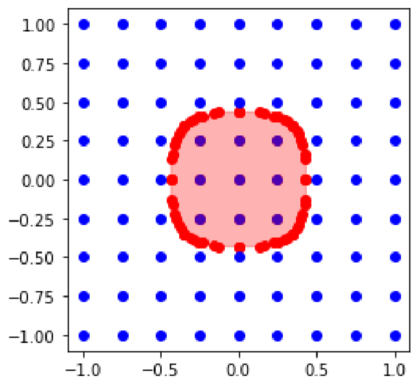
The numerical solution is implemented with Python package CVXPY.



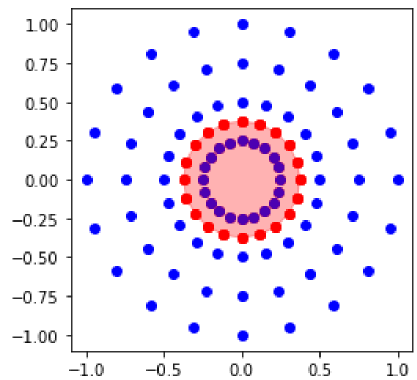
(a) random, $\lambda_3(S^*) = 0.27$



(b) random, $\lambda_3(S^*) = 0.30$



(c) grid, $\lambda_3(S^*) = 0.54$



(d) radial, $\lambda_3(S^*) = 0.55$

Figure B.1: Numerical solutions for random/nonrandom types