Quantum statistical mechanics, noncommutative geometry, and the boundary of modular curves

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ABSTRACT

The Bost-Connes system [BC95] is a $\mathcal{C}^*$-dynamical system whose partition function, KMS states, and symmetries are related to the explicit class field theory of $\mathbb{Q}$. In particular, its zero-temperature KMS states, when evaluated on certain points in an arithmetic sub-algebra, yield the generators of $\mathbb{Q}^{cycl}$. The Bost-Connes system can be viewed in terms of a geometric picture of 1-dimensional $\mathbb{Q}$-lattices. The GL_2-system [CM04] is an extension of this idea to the setting of 2-dimensional $\mathbb{Q}$-lattices. A specialization of the GL_2-system introduced in [CMR06] is related in a similar way to the explicit class field theory of imaginary quadratic extensions.

Inspired by the philosophy of Manin's real multiplication program, we define a boundary version of the GL_2-system. In this viewpoint we see $P^1(\mathbb{R})$ under a certain $\mathrm{PGL}_2(\mathbb{Z})$ action (which is related to the shift of the continued fraction expansion) as a moduli space characterizing degenerate elliptic curves. These degenerate elliptic curves can be realized as non-commutative 2-tori. This moduli space of the non-commutative tori is interpreted as an "invisible" boundary of the moduli space of elliptic curves. In fact, we define a family of such boundary GL_2 systems indexed by a choice of continued fraction algorithm. We analyze their partition functions, KMS states, and ground states. We also define an arithmetic algebra of unbounded multipliers in analogy with the GL_2 case. We show that the ground states when evaluated on points in the arithmetic algebra give pairings of the limiting modular symbols of [MM02] with weight-2 cusp forms.

We also begin the project of extending this picture to the higher weight setting by defining a higher-weight limiting modular symbol. We use as a starting point the Shokurov modular symbols [Sho81a], which are constructed using Kuga varieties over the modular curves. We subject these modular symbols to a limiting procedure. We then show, using the coding space setting of [KS07b], that these limiting modular symbols can be written as a Birkhoff ergodic average everywhere.

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Both authors contributed equally. The author order is alphabetical. The contents of Chapter 3 of this thesis is based upon the paper.

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INTRODUCTION

Hilbert's 12th problem asks for a description of abelian extensions of number fields in terms of arithmetic data. It has been solved explicitly only in two cases, that of the rational numbers $\mathbb{Q}$ and the imaginary quadratic extensions of $\mathbb{Q}$. The latter uses a geometric theory of elliptic curves and modular forms. The elliptic curves are characterized by the moduli space of the upper half plane $\mathbb{H}$ quotiented by the action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations. Manin's real multiplication program [Man04] proposes a viewpoint of studying real quadratic extensions using geometric objects which are obtained via degenerations of complex tori (elliptic curves). These degenerate objects do not have good topological quotients but they can be interpreted using methods from noncommutative geometry as noncommutative 2-tori. In this way we can see the real boundary of the upper half plane (quotiented by an action of $\mathrm{GL}_2(\mathbb{Z})$ related to the continued fraction expansion) as a moduli space characterizing these objects. We interpret it as the "invisible" boundary of the space of elliptic curves.

The modular symbols introduced in [Man72] are a useful tool in computing with modular forms. The modular symbol associated to a pair of cusps is elements of the homology group $H_1(X_G, \mathbb{R})$ where X_G is some modular curve. There is a perfect pairing between modular symbols and weight-2 cusp forms. In [MM02], the limiting modular symbols were introduced in order to extend the picture of modular symbols to the invisible boundary of modular curves. The limiting modular symbols are defined via a limiting procedure and are known to exist almost everywhere. They can be expressed in terms of continued fraction expansions and in particular they are non-vanishing at real quadratic points, which have periodic continued fraction expansion.

There is a rich interplay between quantum statistical mechanics and the class field theory of number fields via the Bost-Connes type systems. These are quantum statistical dynamical systems whose thermodynamic properties, such as partition functions, equilibrium states (i.e. KMS states), and symmetries, are related to properties of number fields. The original Bost-Connes system corresponds to the number field $\mathbb{Q}$, while a later extension corresponds to the imaginary quadratic extensions. The first aim of this work is to construct a further extension that takes into account the invisible boundary of Manin's real multiplication program in such a way that the resulting $\mathcal{C}^*$-dynamical system naturally connects to the limiting modular symbols.

The Bost-Connes system is a $\mathcal{C}^*$-dynamical system introduced in 1995 by Jean Benoit Bost and Alain Connes [BC95]. The underlying $\mathcal{C}^*$-algebra was originally constructed as a Hecke algebra, but it can be seen as the groupoid $\mathcal{C}^*$-algebra of the space of 1-dimensional $\mathbb{Q}$-lattices up to a commensurability relation. This gives the Bost-Connes system a geometric interpretation. Formally, a $\mathbb{Q}$-lattice is a pair (Λ, ϕ) where Λ is a lattice in $\mathbb{R}$ and ϕ is a homomorphism $\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}\Lambda/\Lambda$. Two $\mathbb{Q}$-lattices are commensurable if $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and $\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$. The groupoid $\mathcal{C}^*$-algebra is constructed by taking the algebra of continuous compactly supported functions on the space, and endowing it with an appropriate involution, convolution product, and a $\mathcal{C}^*$-norm. The Bost Connes algebra is then given a dynamics of the form

$$\sigma_t(f)(\Lambda, \Lambda') = |\Lambda/\Lambda'|^{it} f(\Lambda, \Lambda')$$

where $|\Lambda/\Lambda'|$ is the ratio of the lattice covolumes. The system has an associated arithmetic sub-algebra, which can be thought of as an algebra of classical points. The Bost-Connes system is important because of its arithmetic properties. The partition function given by the dynamics is the Riemann zeta function. The symmetries of the system are automorphisms of the algebra given by $\hat{\mathbb{Z}}^* = \mathrm{GL}_1(\mathbb{A}_f)/\mathbb{Q}^* = Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$. The β-KMS states exhibit spontaneous symmetry breaking: they are unique at high temperature, but are non-unique at low temperature, with a critical temperature at $\beta = 1$. The extremal KMS states at low temperature are parameterized by the embeddings $\mathbb{Q}^{cycl} \to \mathbb{C}$. Ground states are obtained as weak limits of the low-temperature states. The ground states, when evaluated on points in the arithmetic sub-algebra, yield generators of the maximal abelian extension of $\mathbb{Q}$. All of this ties the Bost-Connes system and the arithmetic sub-algebra to the explicit class field theory of $\mathbb{Q}$.

Connes and Marcolli later extended this picture by considering 2-dimensional $\mathbb{Q}$-lattices, leading to a $\mathcal{C}^*$-dynamical system called the GL_2-system [CM06a]. An arithmetic algebra was also constructed, in this setting as an algebra of unbounded multipliers on the GL_2 algebra. The system has partition function $\zeta(\beta)\zeta(\beta - 1)$ and a symmetry group consisting of both automorphisms and endomorphisms of the algebra given by $\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^* \simeq Aut(F)$, where F is the modular field. The β-KMS states exhibit symmetry breaking at two critical temperatures, $\beta = 1$ and $\beta = 2$, with the extremal zero-temperature states being parameterized by the invertible $\mathbb{Q}$-lattices, which can be seen equivalently as the set $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$. The ground states, when evaluated on points in the arithmetic algebra, yield generators of specializations of the modular field to points in the upper half plane. This picture was subsequently extended again by Connes, Marcolli, and Ramachandran, based upon the geometric object of K-lattices [CMR06]. For an imaginary quadratic field $K = \mathbb{Q}(\tau)$, let $\mathcal{O} = \mathbb{Z} + \tau\mathbb{Z}$ be its ring of integers. A K-lattice is a pair (Λ, ϕ) where Λ is a finitely generated $\mathcal{O}$-module in $\mathbb{C}$ satisfying $\Lambda \otimes_{\mathcal{O}} K \simeq K$ and $\phi : K/\mathcal{O} \to K\Lambda/K$ is a morphism of $\mathcal{O}$-modules. The resulting $\mathcal{C}^*$-dynamical system is known as the CM system, due to its connection with complex multiplication. The extremal zero-temperature KMS states evaluated on arithmetic points are related to $\mathbb{A}^*_{K,f}/K^*$ for an imaginary quadratic field K. The CM system is thus connected to the explicit class field theory for imaginary quadratic extensions. The CM system can be viewed as a specialization of the GL_2 system, since a K-lattice can be viewed as a 2-dimensional $\mathbb{Q}$-lattice.

The construction using K-lattices has been extended by Laca, Larsen, and Neshveyen in [LLN09] to all number fields. The construction yields some of the desired properties: the correct partition function, KMS-states, symmetries, and symmetry breaking behavior. However, the evaluation of the ground states on an arithmetic algebra to obtain generators of a maximal abelian extension has not been obtained in these models. This is not surprising, since doing so would shed light on the solution to Hilbert's 12th problem for an arbitrary number field.

In this work, we take a different approach. Instead of using the K-lattices, we view $P^1(\mathbb{R})$ as an "invisible" boundary of the $\mathbb{H}$, with points in $\mathbb{R}$ representing psuedolattices which can be viewed as degenerations of complex tori as suggested by Manin's real multiplication program. We construct a boundary version of the GL_2-system by incorporating the boundary $P^1(\mathbb{R})$ directly, with an action of the shift operator which implements the shift on the continued fraction expansion.

In fact, we construct a family of quantum-statistical mechanical systems, indexed by $\mathbb{N}$ where the parameter corresponds to a choice of continued fraction expansion algorithm. We consider the countable family of N-continued fraction expansions given by

$$[a_0; a_1, a_2, a_3, ...]_N = a_0 + \cfrac{N}{a_1 + \frac{N}{a_2 + \frac{N}{a_3 + ...}}}$$

with $a_i \geq N$ when $N \geq 1$ and $a_i \geq |N| + 1$ when $N \leq -1$. This continued fraction expansion has an associated shift on the boundary $p^1(\mathbb{R})$. For each N-continued fraction expansion, we introduce an algebra associated to the boundary $\mathbb{P}^1(\mathbb{R})$ with the action of a certain subsemigroup of $\mathrm{GL}_2(\mathbb{Q})$ depending on the choice of N. In the case that $N = 1$ this semigroup is contained in $\mathrm{GL}_2(\mathbb{Z})$ and in the case that $N = -1$ it is contained in $\mathrm{SL}_2(\mathbb{Z})$. In the $N = \pm 1$ cases, the associated algebra can be interpreted as a boundary algebra of the GL_2 system. While we have no such interpretation when $|N| > 1$, considering the whole family of systems leads to some interesting observations about the structure of the KMS states. We also define an arithmetic algebra of unbounded multipliers, completely analogous to the standard GL_2 case.

The boundary $\mathcal{C}^*$-algebras we construct have a semi-direct product structure $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$ where the $\mathcal{B}_{\partial,N}$ part of the algebra is a modified GL_2-system depending on a choice of finite index subgroup G of $\mathrm{GL}_2(\mathbb{Z})$, and the Red_N is a Cuntz-Krieger-Toeplitz type algebra generated by isometries $S_{N,k}$ related to the shift of the continued fraction expansion. The dynamics are given by $\sigma_{N,t}(f)(g,\rho,x,s) = |\det(g)|^{it}(f)(g,\rho,x,s)$ on the GL_2 part of the system, and by $\sigma_{N,t}(S_{N,k}) = k^{it} S_{N,k}$ on the Red_N part of the system. We analyze the partition function and the structure of the KMS states.

The partition function for the system $(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$ is

$$Z(\beta) = \begin{cases} \dfrac{\zeta(\beta)\zeta(\beta-1)\prod_{p\ prime\,:\,p|N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)}{1+\sum_{n=1}^{N-1} n^{-\beta} - \zeta(\beta)} & \text{if } N > 1 \\ \dfrac{\zeta(\beta)\zeta(\beta-1)\prod_{p\ prime\,:\,p|N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)}{1+\sum_{n=1}^{|N|} n^{-\beta} - \zeta(\beta)} & \text{if } N \le -1 \end{cases}$$

where $\zeta(\beta)$ the Riemann zeta function. In the $N = 1$ case there is no partition function.

The KMS_β states of the QSM system $(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$ can be characterized as follows. When $N = 1$, there are no KMS_β states for any temperature β. When $N \le -1$ or $N > 1$, the system has a critical temperature $\beta_{N,c} \in (1,2)$. We then have the following.

1. When $\beta < \beta_{N,c}$ there are no β-KMS states.
2. When $\beta > \beta_{N,c}$, there is one β-KMS state for every β-KMS state of the modified GL_2 system corresponding to $\mathcal{B}_{\partial,N}$.
3. In particular, when $\beta > 2$, the β-KMS states correspond to a unique β-KMS on the Red_N part of the system paired with one of a set of Gibbs states on the $\mathcal{B}_{\partial,N}$ part of the system, parameterized by (ρ,x,s) with $\rho \in M_2(\hat{\mathbb{Z}})$ invertible.

The figure below compares the classification of the KMS states of the standard Bost-Connes system and of the GL_2 system with the newly constructed boundary-GL_2-systems.

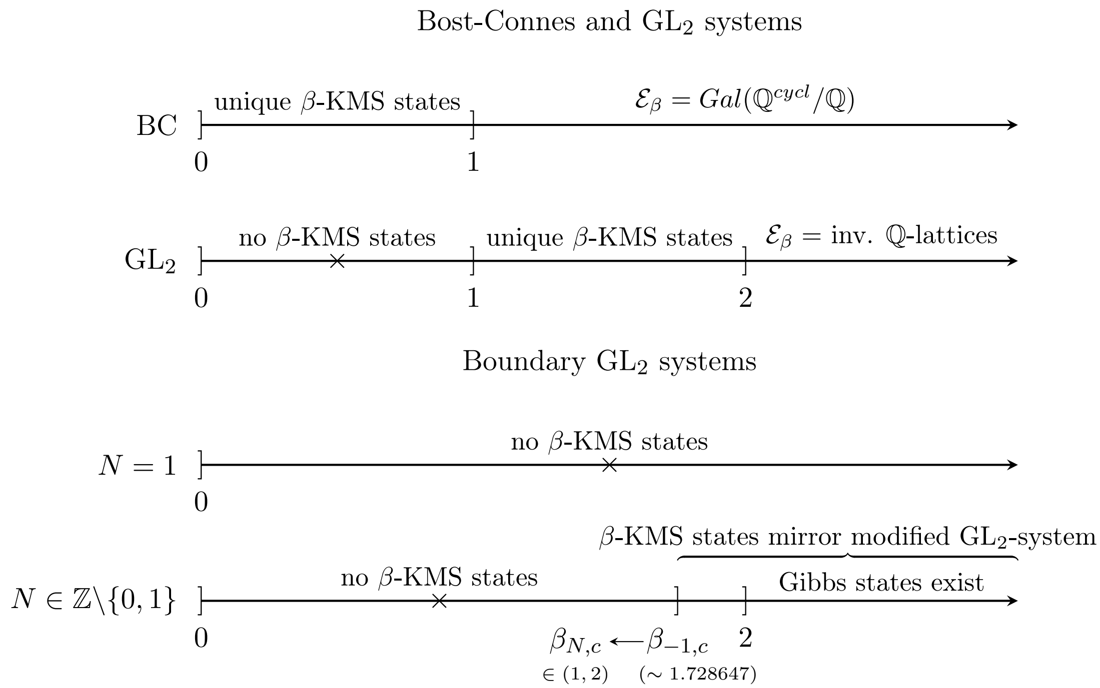

FIGURE 1.1: KMS STATES OF THE BOST-CONNES, GL_2, AND BOUNDARY-GL_2-SYSTEMS

The precise structure of the KMS states in the region $\beta \in (\beta_{N,c}, 2)$ remains an open problem. The standard GL_2-system has been studied in this temperature region by Laca, Larsen, and Neshveyev in [LLN07]. Their analysis of the behavior when $\beta \in (1,2)$ is not directly applicable in our case, because for $N = 1$, the Red_N part of the system has no KMS states at any temperature β. However, an interesting problem for future work is to see whether a similar analysis can be applied to the modified GL_2-system.

Even in the $N = 1$ case where we have no low-temperature KMS states, we can still define ground states, in the sense that they satisfy a KMS-∞ condition. They take the form of a projection onto the kernel of the Hamiltonian in various representations of the system, and they are consistent with the 0-temp states obtained as limits of the KMS states in the cases $N \neq 1$. The ground states are parametrized by $(\rho, x, s) \in M_z(\hat{\mathbb{Z}}) \times [0,1] \times \mathcal{P}_N$ such that ρ is invertible and $\mathcal{P}_N$ is a coset space that accounts for the choice of G in the modified-GL_2 part of the system. Here $[0,1] \times \mathcal{P}_N$ plays the role of $\mathbb{H}$ in the standard GL_2-system.

Having established the ground states for the $N = 1$ boundary-GL_2-system, we proceed with a study of their evaluation on the points in the arithmetic algebra. In particular, we relate these evaluations to a pairing between the cusp forms and the limiting modular symbols. This result is the main advantage of the boundary-

GL_2 construction presented here. The table below summarizes the properties of the Bost-Connes, GL_2-, and boundary-GL_2-systems.

	B-C	GL_2	N-Boundary-GL_2
Partition function	$\zeta(\beta)$	$\zeta(\beta)\zeta(\beta-1)$	$\frac{\zeta(\beta)\zeta(\beta-1)\prod_{p\mid N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)}{1+\sum_{n=1}^{N-1} n^{-\beta}-\zeta(\beta)}$
Critical temperatures	$\beta=1$	$\beta=1,2$	$\beta=2$, $\beta=\beta_{N,c}\in(1,2)$
Low-temp. KMS states parameterized	embeddings $\mathbb{Q}^{cycl}\to\mathbb{C}$	$(\rho,\tau)\in M_2(\hat{\mathbb{Z}})\times\mathbb{H}$ s.t. ρ invertible $\sim$(invertible $\mathbb{Q}$-lattices)	$(\rho,x,s)\in M_2(\hat{\mathbb{Z}})\times[0,1]\times\mathcal{P}_N$ s.t. ρ invertible
Ground states evaluated on arith. algebra	generators of $\mathbb{Q}^{cycl}$	generators of F_τ (the modular field)	pairing of limiting modular symbol and cusp form
Symmetries	$\mathbb{A}_f^*/\mathbb{Q}^*\simeq Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$	$\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^*\simeq Aut(F)$	?

Figure 1.2: Quantum statistical mechanical properties of the Bost-Connes, GL_2, and Boundary-GL_2 systems

The modular symbols of [Man72] are of weight-2, in the sense that they have a perfect pairing with the cusp forms of weight-2. However, in [Sho81a], Shokurov gave a construction for modular symbols of higher weight. This is done by using a nonsingular projective variety called a Kuga variety, which has natural projection Φ onto the modular curve X_G. The modular symbols of weight w are then elements of the relative homology $H_1(X_G,\{\text{cusps}\},(R_1\Phi^*\mathbb{Q})^w)$ where $(R_1\Phi^*\mathbb{Q})^w$ is the symmetric tensor power of the sheaf $R_1\Phi^*\mathbb{Q}=G\otimes_{\mathbb{Q}}\mathbb{Q}$. The modular symbols of Shokurov with parameter w have a pairing with the cusp forms of weight $w+2$.

We show that a higher-weight limiting modular symbol can be defined using a limiting procedure on the Shokurov modular symbols, and that these exist almost everywhere. Rather than using the method in [Mar03], we use a similar technique to that applied in [KS07b]. We pass to a coding space for geodesics related to the Farey tessellation. This approach allows us to obtain an expression for the limiting modular symbol in terms of continued fraction expansions that holds everywhere, whereas the technique of [Mar03] leaves out an exceptional set of measure 0 and Hausdorff dimension 1. It is expected that the boundary GL_2 system developed here will also extend to the Kuga variety setting, and that the evaluation of the ground states on the arithmetic algebra will yield the relations between periods of Hecke eigencusp forms described in [Man73]. This is left to a future work.

Chapter 2

BACKGROUND

This thesis sits at the intersection of operator algebras, number theory, and non-commutative geometry. The central idea of the research program of which this work is one part is to use techniques from operator algebras to construct certain C^*-dynamical systems whose partition functions, equilibrium states, and symmetries are related to important objects in the study of the class field theory of number fields. Some readers may be familiar with operator algebra techniques but not as familiar with the geometric/number theoretical setting, or vice versa. The background section is thus divided into three distinct parts. The first part (Section 2.1) discusses the geometric/number theory setting we are interested in studying, the second part (Section 2.2) provides an overview of some operator algebra techniques that will be used, and the third part (Section 2.3) provides a survey of previous work in this research program. Readers are invited to read or skip Sections 2.1 and 2.2 as is appropriate to their areas of expertise.

2.1 Number fields and geometry

This work is motivated by questions in the study of number fields, and particularly the study of real quadratic extensions. Explicit solutions to Hilbert's 12th problem have been worked out only in the case of $\mathbb{Q}$ and imaginary quadratic extensions, the latter using the geometric theory of elliptic curves. Considerable work has been done in Manin's real multiplication program to develop an analogous non-commutative geometric setting corresponding to the real quadratic extensions.

2.1.1 Class Field Theory

The general goals of class field theory are to describe, for K a finite extension of $\mathbb{Q}$, properties of abelian extensions of K in terms of the arithmetic of K. We begin by elaborating what we mean by the "arithmetic of K".

Let $\mathcal{O}_K$ be the ring of algebraic integers of K. A fractional ideal $\mathfrak{a}$ of K is a finitely-generated $\mathcal{O}_K$-module with generators in K. We endow the set of fractional ideals with multiplication as follows. If $\mathfrak{a}$ is generated by $\alpha_1, ..., \alpha_n$ and $\mathfrak{b}$ is generated by $\beta_1, ..., \beta_m$, we let $\mathfrak{ab}$ be the fractional ideal generated by $\{\alpha_i\beta_j\}_{i=1,...,n,j=1,...m}$. With this multiplication, the set of fractional ideals forms a group with the identity elements given by $\mathcal{O}_K = (1)$. We call this group A_K. The arithmetic of K is the study of A_K and related objects such as its subgroups and ideals.

In terms of the arithmetic of K, class field theory aims to describe:

1. the abelian extensions L of K,
2. the Galois group $Gal(L/K)$, and
3. the decomposition of a prime ideal from K to L (i.e. provide a reciprocity law).

The first two points can be equivalently stated as describing K^{ab} the maximal abelian extension, and its Galois group $Gal(K^{ab}/K)$. Addressing these first two questions for all number fields is known as Hilbert's 12th problem. Explicit solutions to Hilbert's 12th problem have been worked out for only two cases: $\mathbb{Q}$ and imaginary quadratic extensions $\mathbb{Q}(\sqrt{-d})$. In the case of $\mathbb{Q}$, the solution was given in 1896.

Proposition 2.1.1 (Kronecker-Weber Theorem)**.** *Every abelian extension of* $\mathbb{Q}$ *is contained in a cyclotomic extension* $\mathbb{Q}(\zeta_n)$ *where* ζ_n *is a primitive* n^{th} *root of unity.*

The Galois group is given by $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = \mathbb{Z}/n\mathbb{Z}$. Equivalently, the maximal abelian extension is $\mathbb{Q}^{ab} = \mathbb{Q}^{cycl}$, the field obtained by adjoining all roots of unity, and the Galois group is $Gal(\mathbb{Q}^{cycl}/\mathbb{Q}) \simeq \lim_{\leftarrow} \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}^*$.

In the case of imaginary quadratic extensions, the solution is more difficult and requires the machinery of elliptic curves. The case of real quadratic extensions is still open. It is known, however, that the maximal abelian extension can be expressed in terms of the idèle class group.

We define the ring of adèles of K to be

$$\mathbb{A}_K = \prod_P \hat{K}_P$$

where $\hat{K}_P$ is the completion of K at the prime P of $\mathcal{O}_K$, with the additional constraint that for an element given by the tuple (a_P), all but finitely many of the a_P satisfy $a_P \in (\hat{\mathcal{O}_K})_P$. The group of units of $\mathbb{A}_K$ is the group of idèles $\mathbb{I}_K$. $K^\times$ embeds in $\mathbb{I}_K$ and we define the idèle class group to be

$$C_K = \mathbb{I}_K/K^\times.$$

We define a norm map, for a field extension L/K and L and K number fields.

$$\mathcal{N} : \mathbb{A}_L \to \mathbb{A}_K$$
$$\prod_Q \alpha_Q \mapsto \prod_P \prod_{Q|P} N_{\hat{K}_P}^{\hat{L}_Q}(\alpha_P)$$

where Q runs over the primes of L and P runs over the primes of P and $N_{K'}^{L'}$ is the norm map

$$N_K^L : L \to K$$
$$\alpha \mapsto \det(l_\alpha)$$

where $l_\alpha : L \to L$ is multiplication by α on the left. The norm map descends to a map

$$\mathcal{N} : C_L \to C_K.$$

For a finite index field extension L/K we define the norm subgroup

$$\mathcal{N}_L := \mathcal{N}(C_L) \subset C_K.$$

It is not trivial to prove, but turns out to be the case that $\mathcal{N}_L$ is a closed subgroup of finite index in C_K. We have the following result which generalizes the Kronecker-Weber theorem.

Proposition 2.1.2. *Let L/K be a finite field extension of number fields. There is a natural isomorphism*

$$\Theta : Gal(L/K)^{ab} \to C_K/\mathcal{N}_L. \tag{2.1}$$

2.1.2 Imaginary quadratic fields and elliptic curves

The theory of elliptic curves over the complex numbers is used to explicitly solve Hilbert's 12th problem in the case of imaginary quadratic extensions. We recall some basic facts of this theory.

An elliptic curve over $\mathbb{C}$ is the set of points

$$E = \{(x, y) : y^2 = 4x^3 - ax - b\} \cup \{\infty\}$$

where $a, b \in \mathbb{C}$ satisfy that the discriminant $a^3 - 27b^2 \neq 0$, so that the curve is non-singular. The elliptic curve is also endowed with an addition which turns it into a group. We will see below that every elliptic curve is isomorphic to a complex torus, and the addition on the elliptic curve is the one which is compatible with that on the torus. We will not describe it explicitly here.

Every complex torus $\mathbb{C}/\Lambda$, where $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ is a rank 2 lattice in $\mathbb{C}$, is isomorphic to an elliptic curve over $\mathbb{C}$. This isomorphism is given by means of the Weierstrass $\wp$-functions

$$\wp_\Lambda(u) = \frac{1}{u^2} + \sum_{\omega\in\Lambda} \left((u-\omega)^{-2} + \omega^{-2}\right) \tag{2.2}$$

which have Laurent expansions

$$\begin{aligned} \wp_\Lambda(u) &= \frac{1}{u^2} + \sum_{n=2}^{\infty}(2n-1)G_{2n}(\Lambda)u^{2n-2}, \\ G_{2n}(\Lambda) &= \sum_{\omega\in\Lambda} \omega^{-2n}. \end{aligned} \tag{2.3}$$

Let

$$g_2(\Lambda) = 60G_4(\Lambda), \quad g_3(\Lambda) = 140G_6(\Lambda). \tag{2.4}$$

For a given lattice Λ, we define the elliptic curve

$$E_\Lambda : y^2 = 4x^3 - g_2(\Lambda)x - g_3(\Lambda). \tag{2.5}$$

Then the map

$$\begin{aligned} \mathbb{C}/\Lambda &\to E_\Lambda \\ u &\mapsto (\wp_\Lambda(u), \wp'_\Lambda(u)) \end{aligned} \tag{2.6}$$

is an isomorphism.

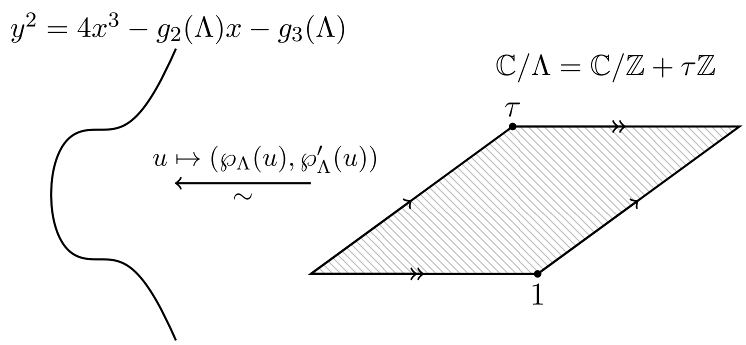

Figure 2.1: Isomorphism between an elliptic curve and a complex torus

In fact, it is the case that for every elliptic curve, there is a complex torus isomorphic to it. We define the j-function on the upper half plane $\mathbb{H}$ by

$$j(\tau) = 1728\frac{g_2(\tau)^3}{g_2(\tau)^3 - 27g_3(\tau)^2} \tag{2.7}$$

where $g_i(\tau) = g_i(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ for $\tau = \omega_1/\omega_2$. The map

$$j : \Gamma\backslash\mathbb{H} \to \mathbb{C}$$

where $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ acts by fractional linear transformations, is a bijection (see e.g. [Lan27] §3.) We can use this fact about the j-function to define, for an arbitrary elliptic curve E, a complex torus which is isomorphic to E. Given two complex numbers c_2 and c_3 satisfying $c_2^3 - 27c_3^2 \neq 0$, there is $\tau \in \mathbb{H}$ satisfying

$$j(\tau) = 1728\frac{c_2^3}{c_2^3 - 27c_3^2}.$$

From this we construct a lattice L_τ

$$L_\tau = \begin{cases} w_1(\tau\mathbb{Z} + \mathbb{Z}) & \text{if } c_2 = 0 \\ w_2(\tau\mathbb{Z} + \mathbb{Z}) & \text{if } c_2 \neq 0 \end{cases}$$

where w_1 satisfies $w_1^{-6} g_3(\tau\mathbb{Z} + \mathbb{Z}) = c_3 \neq 0$ and w_2 satisfies $w_2^{-4} g_2(\tau\mathbb{Z} + \mathbb{Z}) = c_2$. The elliptic curve associated to this lattice is $E_{\Lambda_\tau} : y^2 = 4x^3 - c_2 x - c_3$.

The j-function parameterizes the isomorphism classes of complex tori i.e. $j(\tau) = j(\tau')$ if and only if $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z}) \simeq \mathbb{C}/(\mathbb{Z}+\tau'\mathbb{Z})$, and consequently also parameterizes the isomorphism classes of elliptic curves. The moduli space characterizing elliptic curves is given by the one point compactification

$$(\Gamma\backslash\mathbb{H}) \cup \{\infty\} = \Gamma\backslash(\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})),$$

with each point in the space representing an isomorphism class of elliptic curves.

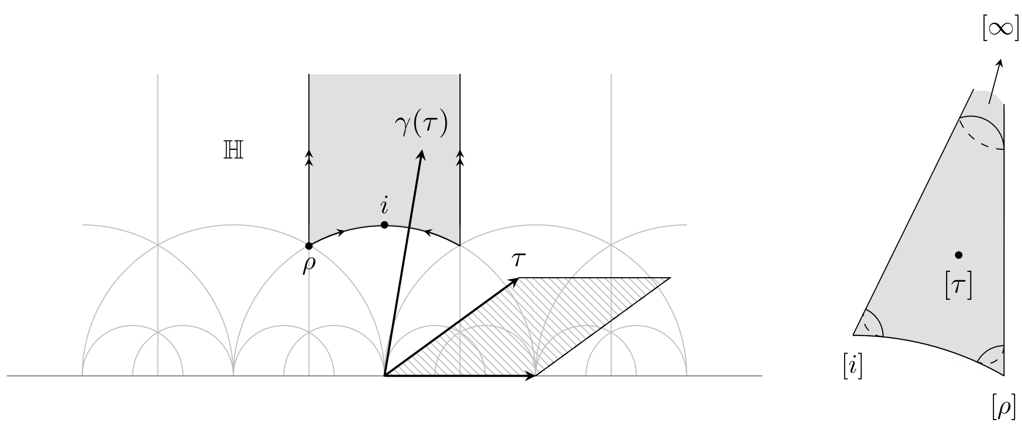

Figure 2.2: A fundamental domain for the $\mathrm{SL}_2(\mathbb{Z})$ action on $\mathbb{H}$, and the moduli space of elliptic curves with cusp added

We will also be interested in the modular curves, which are constructed as quotients of $\mathbb{H}$ by certain subgroups of the modular group.

Definition 2.1.3. The *principal congruence subgroup* of level n is

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, d \equiv 1 \text{ and } b, c \equiv 0 \bmod (n) \right\}.$$

In particular, $\Gamma(1) = \mathrm{SL}_2(\mathbb{Z})$. A subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ is a *congruence subgroup* if it contains $\Gamma(n)$ for some n. The minimal such n is called the level of Γ.

Two important congruence subgroups are called the *Hecke congruence subgroups*:

$$\Gamma_0(n) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod n \right\},$$
$$\Gamma_1(n) = \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) : \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \mod n \right\}.$$

Let Γ be a congruence subgroup. Then the quotient $\Gamma/\mathbb{H}$ is a Riemann surface. As before, we compactify this space by adding in the cusps to get the compact Riemann surface

$$\overline{X}_\Gamma = \Gamma/(\mathbb{H} \cup P^1(\mathbb{Q})) = \Gamma/\mathbb{H} \cup \{\text{cusps}\}.$$

The modular field

A *modular function* of level n is a function on $\mathbb{H}$ that is invariant under $\Gamma(n)$, the principal congruence subgroup of level n, and is meromorphic at the cusps.

Definition 2.1.4. We denote by F_n the field of modular functions $f(\tau)$ of level n that are rational over the cyclotomic field $\mathbb{Q}(\zeta_n)$, i.e. such that the expansion in powers of $q^{1/n} = e^{2\pi i\tau/n}$ has coefficients in $\mathbb{Q}(e^{2\pi i/n})$. The field $F = \cup_{n\in\mathbb{N}} F_n$ is called the *modular field.*

The Galois group $Gal(\mathbb{Q}^{cycl}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$ acts on the coefficients of the q-expansion of the functions in F, which determines a homomorphism

$$\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F). \tag{2.8}$$

The fields F_n are generated by certain functions related to the Weierstrass $\wp$-functions. Let $\Lambda(\omega_1, \omega_2) = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$. For $a \in \mathbb{Q}^2 \backslash \mathbb{Z}^2$, let

$$f_a(\omega_1/\omega_2) = \frac{g_2(\Lambda(\omega_1, \omega_2))g_3(\Lambda(\omega_1, \omega_2))}{\Delta(\Lambda(\omega_1, \omega_2))} \wp_{\Lambda(\omega_1,\omega_2)} \left(a \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \right) \tag{2.9}$$

where

$$\Delta(\Lambda) = g_2(\Lambda)^3 - 27 g_3(\Lambda)^2.$$

Note that we can always substitute the lattice $\Lambda(1, \tau)$ for some $\tau \in \mathbb{H}$ for the the lattice $\Lambda(\omega_1, \omega_2)$, so that we can see the function f_a as

$$f_a(\tau) = \frac{g_2(\Lambda(1, \tau)g_3(\Lambda(1, \tau))}{\Delta(\Lambda(1, \tau))} \wp_{\Lambda(1,\tau)} \left(a \begin{pmatrix} 1 \\ \tau \end{pmatrix} \right)$$

for $\tau \in \mathbb{H}$.

Now, for $n \in \mathbb{N}$, the field

$$\mathbb{Q}(j, f_a : a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2)$$

coincides with F_n ([Shi71] Proposition 6.9) Using this description, it can be shown that F_n is a Galois extension of $\mathbb{Q}(j)$ which contains the primitive n^{th} roots of unity ([Shi71] Theorem 6.6)

The automorphism group of the modular field F has been determined by Shimura ([Shi71] §6.6):

$$\mathrm{GL}_2(\mathbb{A}_f)/\mathbb{Q}^* \xrightarrow{\sim} Aut(F). \tag{2.10}$$

This can be seen as a non-commutative analogue of the class field isomorphism

$$\Theta : \mathbb{A}_f^*/\mathbb{Q}_+^* \xrightarrow{\sim} Gal(\mathbb{Q}^{cycl}/\mathbb{Q}). \tag{2.11}$$

Certain points $\tau \in \mathbb{H}$ are called *generic*, and will be of interest when we look at the 0-temperature states of the GL_2-system.

Definition 2.1.5. We say that $\tau \in \mathbb{H}$ is generic (for n) if the specialization

$$\{j, f_a : a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2\} \to \{j(\tau), f_a(\tau) : a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2\}$$

induces an isomorphism of fields

$$F_n = \mathbb{Q}(j, f_a : a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2) \to \mathbb{Q}(j(\tau), f_a(\tau) : a \in n^{-1}\mathbb{Z}^2 \setminus \mathbb{Z}^2).$$

It is known that such generic points exist ([Shi71] Lemma 6.5.)

Complex multiplication

Certain points in the moduli space have a property called complex multiplication. These points are related to the class field theory of imaginary quadratic extensions.

All the endomorphisms of a complex torus $\mathbb{C}/\Lambda$ are given by multiplication by some complex number λ satisfying $\lambda(\Lambda) \subset \Lambda$. The set of endomorphisms of a complex torus always includes $\mathbb{Z}$ whose elements just scale and/or flip the lattice. However, the set of endomorphisms may be more interesting and include additional elements. We say that a complex torus $\mathbb{C}/\Lambda$ has *complex multiplication* if the set of endomorphisms is larger than $\mathbb{Z}$, i.e.

$$End(\mathbb{C}/\Lambda) = \{\lambda \in \mathbb{C} : \lambda(\Lambda) \subset \Lambda\} \neq \mathbb{Z}.$$

An important fact is that the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ has complex multiplication if and only if $\mathbb{Q}[\tau]$ is an imaginary quadratic field. Furthermore, in this case, $j(\tau)$ is an algebraic integer. We now state the result linking this geometric picture to the class field theory of imaginary quadratic fields.

Proposition 2.1.6. *Let K be an imaginary quadratic field and $\mathcal{O}_K$ be its ring of units. Let $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ be a complex torus satisfying*

$$End(\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})) = \mathcal{O}_K.$$

The maximal abelian extension of K is generated by the set $\{f(\tau) : f \in F\}$ where F is the modular field.

The action of $Gal(K^{ab}/K)$ is induced by the action of $Aut(F)$.

2.1.3 Real quadratic fields and non-commutative tori

Manin's real multiplication program [Man04] proposes the development of an analogous theory where elliptic curves are replaced by "degenerate" elliptic curves (non-commutative tori).

The main idea is to consider the limiting object obtained when we take the complex torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ and allow the generator τ of the lattice to tend towards a real irrational point on $\mathbb{R}$. The lattice $\Lambda_\tau = \mathbb{Z}+\tau\mathbb{Z}$ becomes a pseudolattice $L_\theta = \mathbb{Z}+\theta\mathbb{Z} \subset \mathbb{R}$. The resulting topological quotient of $\mathbb{R}$ by the pseudolattice will be bad, but we can still define a limiting object by using standard methods from non-commutative geometry. Instead of considering the equivalence relation of the quotient directly, we will consider the non-commutative algebra of functions defined on the graph of the equivalence relation with the convolution product

$$f_1 \star f_2(x,y) = \sum_{x\sim y\sim z} f_1(x,z) f_2(y,z).$$

In particular, when we have a discrete group G acting on a compact topological space X, the algebra will be given by the crossed-product algebra

$$C(X) \rtimes_\alpha G$$

where we define the action $\alpha_g(f)(x) = f(g^{-1}(x))$ for all $f \in C(X)$ and $g \in G$. The non-commutative multiplication in the algebra is given by

$$(fU_g)(f'U_{g'}) = f\alpha_g(f')U_{gg'}.$$

To apply this idea to our setting of complex tori, we first identify the torus $\mathbb{C}/(\mathbb{Z}+\tau\mathbb{Z})$ with $\mathbb{C}^*/(q^{\mathbb{Z}})$ where $q = e^{2\pi i\tau}$ via the exponential map.

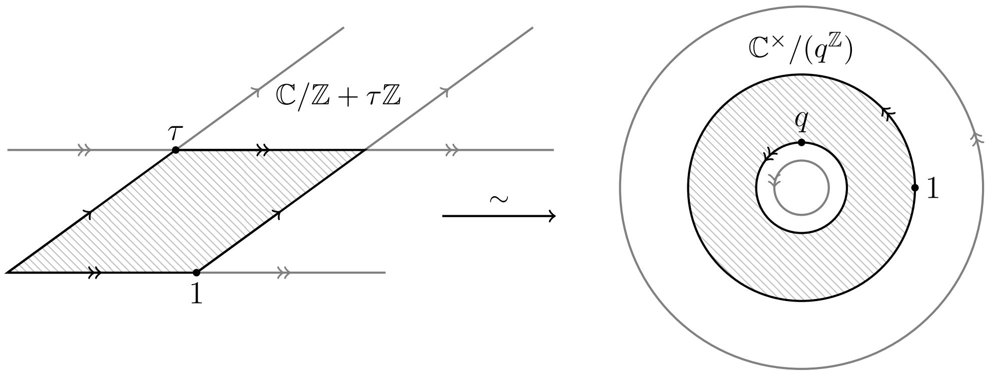

Figure 2.3: Uniformization of a complex torus

The identification of the boundary of the torus corresponds to gluing together the two boundary circles of the annulus with a twist by $e^{2\pi i\tau}$. The limit that $\tau \to \theta \in \mathbb{R} - \mathbb{Q}$ corresponds to the limit $|q| \to 1$, with a twist on the boundary S^1 by $e^{2\pi i\theta}$. We obtain the noncommutative algebra

$$C(S^1) \rtimes_\theta \mathbb{Z}$$

where θ acts on S^1 by a rotation of $2\pi\theta$. This algebra is called the noncommutative 2-torus, and is denoted by $\mathbb{T}_\theta$.

There are two notions of isomorphism that we may want to use when considering non-commutative tori. The first takes the point of view of directly replacing lattices in $\mathbb{C}$ by pseudeolattices. Formally, a pseudolattice is a quadruple (L, V, j, s) where L is a rank-two free abelian group, V is a one-dimensional complex vector space, $j : L \to V$ is an injective homomorphism whose image lies in the real line, and s is a choice of orientation of the real line. A strict isomorphism of pseudolattices is a commutative diagram

$$\begin{array}{ccc} L' & \xrightarrow{j'} & V' \\ \varphi \downarrow & & \downarrow \psi \\ L & \xrightarrow[j]{} & L \end{array}$$

where φ is a group homomorphism and ψ is linear map that takes the orientation s to s'. Every strict isomorphism class of pseudolattices has a representative element with $j : \mathbb{Z}^2 \to \mathbb{C}$ such that $j(0, 1) = 1$ and $j(1, 0) = \theta$ for an irrational real number θ. We denote this element by L_θ. Two psuedolattices L_θ and $L_{\theta'}$ are strictly isomorphic if θ and θ' lie in the same $\mathrm{PGL}_2(\mathbb{Z})$ orbit. Then the moduli space characterizing the isomorphism classes of pseduolattices is

$$\mathrm{PGL}_2(\mathbb{Z})\backslash(P^1(\mathbb{R}) - \{\mathrm{cusp}\})$$

where the cusp is the orbit of the rational numbers. The action of $\mathrm{PGL}_2(\mathbb{Z})$ on $\mathbb{R}$ is related to the continued fraction expansion by the classical result that two irrational numbers are in the same $\mathrm{PGL}_2(\mathbb{Z})$ orbit if and only if their continued fraction expansion has the same tail.

On the other hand, in the setting of noncommutative tori, Morita equivalence is the correct notion of isomorphism to employ if we want to imitate the framework of complex multiplication. We say that a point $\theta \in \mathbb{R}$ has real multiplication if $\mathbb{T}_\theta$ has non-trivial Morita self equivalences. Then $\mathbb{T}_\theta$ has real multiplication if and only if $\mathbb{Q}[\theta]$ is a real quadratic field. Two noncommutative tori $\mathbb{T}_\theta$ and $\mathbb{T}_{\theta'}$ are Morita equivalent if and only if θ and θ' are in the same $\mathrm{SL}_2(\mathbb{Z})$ orbit in $\mathbb{R}$. We can consider the moduli space of the noncommutative tori, which characterizes their Morita equivalence classes. This is also a noncommutative space, given by the action of $\mathrm{SL}_2(\mathbb{Z})$ on $P^1(\mathbb{R})$. It is again described by a crossed product algebra

$$C(P^1(\mathbb{R})) \rtimes \mathrm{SL}_2(\mathbb{Z}).$$

2.1.4 Modular forms, Hecke operators, and modular symbols

An important tool in the study of modular curves is the modular forms. The modular forms are a class of functions defined on $\mathbb{H}$, which capture the geometry of the underlying space.

Definition 2.1.7. A *modular form of weight k* is a function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following properties.

1. f is holomorphic,
2. f is holomorphic at infinity (i.e. as $\mathrm{Im}(z) \to \infty$, $|f(z)|$ is majorized by a polynomial in $\max\{1, \mathrm{Im}(z)^{-1}\}$), and
3. for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$

$$f(z) = (cz + d)^k f(\gamma \cdot z)$$

where γ acts by fractional linear transformation.

We say that f is a *cusp form* if it satisfies the stronger growth condition

2'. f decays rapidly at infinity (i.e. $|f(z)|$ is majorized by $\mathrm{Im}(z)^{k/2}$ as $\mathrm{Im}(z) \to \infty$).

Similarly, if Γ is a congruence subgroup then a *modular form of weight* k for Γ (resp. cusp form of weight k) satisfies the same above three conditions except that in (3) $\mathrm{SL}_2(\mathbb{Z})$ is replaced with Γ and the growth condition (2) (resp (2')) holds for all $(cz+d)^{-1}f(\gamma \cdot z)$ where $\gamma \in \Gamma$. The growth condition gives that the Fourier expansion of a modular form given by

$$f(z) = \sum_{n \in \mathbb{Z}} a_n e^{2\pi i n z}$$

satisfies $a_n = 0$ for all $n < 0$. For a cusp form we have instead that $a_n = 0$ for all $n \leq 0$. We denote by M_k the space of modular forms of weight k, and by $S_k \subset M_k$ the space of cusp forms of weight k.

Equivalently, modular forms can be seen as functions on lattices satisfying a certain homogeneity property. Let $\tilde{f}$ be a function satisfying

$$\tilde{f}(\lambda\Lambda) = \lambda^{-k}\tilde{f}(\Lambda)$$

for all lattices Λ and $\lambda \in \mathbb{C}^*$. Then for $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$ we have

$$\tilde{f}(\lambda\omega_1\mathbb{Z} + \lambda\omega_2\mathbb{Z}) = \lambda^{-k}\tilde{f}(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}).$$

From this equation we see that $\omega_2^{-k}\tilde{f}(\omega_1\mathbb{Z} + \omega_2\mathbb{Z})$ depends only on ω_1/ω_2 and so there exists some f defined on $\mathbb{H}$ such that

$$\tilde{f}(\omega_1\mathbb{Z} + \omega_2\mathbb{Z}) = \omega_2^{-k}f(\omega_1/\omega_2).$$

We also see that $\tilde{f}$ is invariant under the $\mathrm{SL}_2(\mathbb{Z})$ action on (ω_1, ω_2) and so f satisfies

$$f(z) = (cz+d)^{-k}f\left(\gamma \cdot z\right)$$

for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

The space M_k of modular forms of weight k is finite dimensional and admits a collection of commuting linear operators called the *Hecke operators*. They are defined by

$$(T_n f)(z) = \sum_{\gamma \in \chi_n} (\det \gamma)^{k-1}(cz+d)^{-k}f(\gamma \cdot z)$$

where

$$\chi_n = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : g \geq 1, ad = n, 0 \leq b < d \right\}.$$

In the lattice picture, the Hecke operator T_n can be seen taking as input a function $\tilde{f}$ and outputting a function which averages $\tilde{f}$ over index-n sublattices (for details see [Ser77] §VII 5.1). The Hecke operators satisfy relations

$$\begin{aligned} T_n T_m &= T_{nm} && \text{if } (n,m) = 1 \\ T_{p^n} &= T_{p^{n-1}} T_p - p^{k-1} T_{p^{n-2}} && \text{for } p \text{ prime.} \end{aligned}$$

There is an alternate definition of the Hecke operators in terms of the Hecke algebra of double cosets (see e.g., [Miy06] §2.7.) This type of Hecke algebra is used in the construction of the Bost-Connes system in section 2.3.

Definition 2.1.8. Let Γ be a congruence subgroup and $\Delta \subset \mathrm{GL}_2(\mathbb{Q})^+$ be a semigroup. We define the Hecke algebra by

$$R(\Gamma, \Delta) := \left\{ \sum_{\alpha \in \Delta} a_\alpha \Gamma \alpha \Gamma : a_\alpha \in \mathbb{Z} \text{ and } a_\alpha = 0 \text{ for all but finitely many } \alpha \right\}$$

with the multiplication

$$\Gamma \alpha \Gamma \cdot \Gamma \beta \Gamma = \sum_{\gamma \in \Delta} c_\gamma \Gamma \gamma \Gamma$$

where c_γ counts the number of (i,j) such that $\Gamma \alpha_i \beta_j = \Gamma \gamma$ for $\Gamma \alpha \Gamma = \bigsqcup_i \Gamma \alpha_i$.

This multiplication turns out to be well-defined (independent of the decomposition $\Gamma \alpha \Gamma = \bigsqcup_i \Gamma \alpha_i$). We define the Hecke operators as elements of $R(\Gamma_0(n), \Delta_0(n))$ where $\Gamma_0(n)$ is the Hecke congruence subgroup and

$$\Delta_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) : c \equiv 0 \bmod n, (a,n) = 1, ab - bc > 0 \right\}$$

as follows.

$$T(n) = \sum_{\det \alpha = n} \Gamma_0(n) \alpha \Gamma_0(n)$$

where the sum is taken over all the double cosets $\Gamma_0(n) \alpha \Gamma_0(n)$ such that $\det(\alpha) = n$.

The modular symbols are a useful tool for computing with modular forms. We define here the modular symbols of weight 2 introduced in [Man72], which will have a pairing with the weight-2 cusp forms. A more general construction of higher weight modular symbols has been given in [Sho81a], and is summarized in Chapter 4.

We fix some modular curve X_G for a modular group G. The modular symbol associated to points α, β in $P^1(\mathbb{Q})$ is a real homology class in $H_1(X_G, \mathbb{R})$, constructed as follows. Consider $C_{\alpha,\beta}$ the oriented geodesic going from α to β in $\mathbb{H}$. Let

$\varphi : \mathbb{H} \cup P^1(\mathbb{Q}) \to X_G$ be the quotient map. Because α and β are cusps, the image $\varphi(C_{\alpha,\beta})$ is closed on X_G. We defined the modular symbol $\{\alpha, \beta\}_G$ by

$$\int_{\{\alpha,\beta\}_G} \omega := \int_{\alpha}^{\beta} \varphi^*(\omega) = \int_{\varphi(C_{\alpha,\beta})} \omega$$

for ω a differential form on X_G.

The modular symbols are related to the weight-2 cusp forms as follows. Consider some cusp form $f \in S_2(G)$. The function f doesn't descend to a function on X_G because it isn't G-invariant, but the one-form fdz does. We have the invariance,

$$f(\gamma \cdot z)d(\gamma \cdot z) = f\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 f(z) \frac{ac-bd}{(cz+d)^2} dz = f(z)dz$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \subset \mathrm{SL}_2(\mathbb{Z})$. We then obtain a pairing

$$\langle , \rangle : S_2(G) \times H_1(X_G, \mathbb{Z}) \to \mathbb{C}$$

by integrating along the image in X_G of the geodesic in $\mathbb{H}$ connecting α and β

$$\langle f, \{\alpha, \beta\}_G \rangle = \int_{\alpha}^{\beta} f(z)dz.$$

We extend the pairing to a pairing $\langle , \rangle : S_2(G) \times H_1(X_G, \mathbb{R}) \to \mathbb{C}$ by linearity. This pairing is perfect and it identifies the dual $S_2(G)^*$ with $H_1(X_G, \mathbb{Z})$.

The modular symbols have several basic properties, which all follow easily from the definition:

- $\{\alpha, \beta\}_G = -\{\beta, \alpha\}_G$
- $\{\alpha, \beta\}_G = \{\alpha, \gamma\}_G + \{\gamma, \beta\}_G$
- $\{g\alpha, g\beta\}_G = \{\alpha, \beta\}_G$ for all $g \in G$

Because of the second property, it suffices to consider modular symbols of the form $\{0, \alpha\}_G$. A common technique used when working with modular symbols is to decompose them using the continued fraction expansion of α. Recall that every real number can be represented as a continued fraction expansion of the form

$$[a_0; a_1, a_2, a_3, ...] := a_0 + \cfrac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\cdots}}}}$$

and in particular rational numbers have a finite continued fraction expansion while quadratic irrationalities have a periodic continued fraction expansion. The k^{th} approximant is the rational number

$$\frac{p_k}{q_k} = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_k}}}}.$$

The numbers p_k and q_k satisfy recurrence relations

$$p_k = a_k p_{k-1} + p_{k-2}$$
$$q_k = a_k q_{k-1} + q_{k-2}$$

where $p_0 = 0$ and $q_0 = 1$. It follows that $p_k q_{k-1} - p_{k-1} q_k = (-1)^{k-1}$. The matrix

$$g_k(\alpha) = \begin{pmatrix} p_k(\alpha) & p_{k-1}(\alpha) \\ q_k(\alpha) & q_{k-1}(\alpha) \end{pmatrix}$$

is thus in $\mathrm{GL}_2(\mathbb{Z})$. For $\alpha = [a_1, a_2, ..., a_n]$ rational, we can write the modular symbol as a finite sum:

$$\{0, \alpha\}_G = \sum_{k=1}^{n} \left\{ \frac{p_{k-1}}{q_{k-1}}, \frac{p_k}{q_k} \right\}_G = \sum_{k=1}^{n} \{g_k(0), g_k(i\infty)\}_G \,.$$

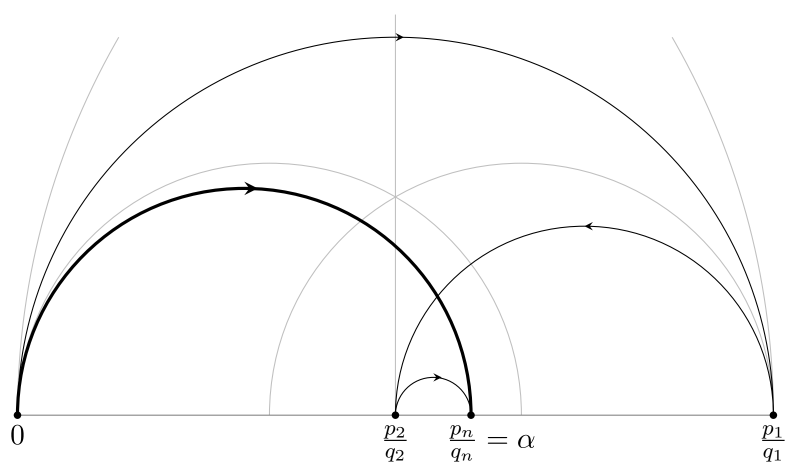

Figure 2.4: Approximating a path in $\mathbb{H}$ for a modular symbol by continued fractions

This corresponds to approximating the image of the path $C_{0,\alpha}$ in the modular curve by geodesics whose endpoints approach α.

2.2 $\mathcal{C}^*$-dynamical systems and KMS states

Before we describe the Bost-Connes type systems, which are $\mathcal{C}^*$-dynamical systems with connections to number fields, we give an overview of $\mathcal{C}^*$-dynamical systems in general. We describe the mathematical formalism used to model the states, observables, and time evolution of infinitely extended quantum systems, and devote some time to defining the equilibrium states appropriately in this setting. We also discuss the geometric structure of the space of equilibrium states, which will turn out to be a simplex. The structure of the equilibrium states is crucial to the Bost-Connes picture.

The standard mathematical formalism for a dynamical system in quantum mechanics consists of a triple $(\mathcal{H}, H, \rho_0)$ where $\mathcal{H}$ is a finite-dimensional Hilbert space, H is a self-adjoint operator in $\mathcal{B}(\mathcal{H})$, and ρ is a positive semi-definite trace-one self-adjoint operator in $\mathcal{B}(\mathcal{H})$. The pure states of a system are viewed as norm-one elements of $\mathcal{H}$. Observables are self-adjoint elements A of $\mathcal{B}(\mathcal{H})$ with some spectral decomposition

$$A = \sum_i \lambda_i P_i$$

where P_i is the projection on the λ_i eigenspace. The possible observed values of the measurement described by A are the eigenvalues λ_i of A, and each λ_i is observed for a fixed pure state $\psi \in \mathcal{H}$ with probability $(\psi, P_i\psi)$.

Mixed states are modeled by positive semi-definite trace-one self-adjoint operators on $\mathcal{H}$. This is motivated by considering a statistical ensemble of pure states ψ_j each occurring with probability p_j. The probability of measuring the eigenvalue λ_i of A is then given by

$$\operatorname{tr}(\rho A) = \sum_j p_j(\psi_j, P_i\psi_j)$$

where $\rho(\cdot) = \sum_j p_j(\psi_j, \cdot)\psi_j$. Note that ρ is positive and trace-one because the p_i's give a probability distribution on the states ψ_i. The operator ρ_0 represents the initial mixed state of the system.

The time evolution of the system, in the absence of measurement, is described by a special observable called the Hamiltonian, denoted by H. Since H is self-adjoint it induces a continuous one-parameter group of unitary transformations on the mixed states, or equivalently on the observables by

$$\tau^t(\rho) = e^{-itH}\rho e^{itH} \text{ or } \tau^t(A) = e^{itH}Ae^{-itH}.$$

In the case of infinitely-extended quantum systems, it can often be the case that certain states we wish to consider fail to be traceclass. In this situation, a different formalism is used. We move from considering the Hilbert space as the fundamental object to studying $\mathcal{B}(\mathcal{H})$ directly.

Now, we consider a triple $(\mathcal{A}, \tau^t, \omega)$ where $\mathcal{A}$ is a generic $\mathcal{C}^*$-algebra playing the role of $\mathcal{B}(\mathcal{H})$, τ^t is a strongly-continuous one parameter group of automorphisms on $\mathcal{A}$, and ω is a positive linear functional of norm 1 on $\mathcal{A}$ playing the role of the state. This point of view is supported by the structure theorem for $\mathcal{C}^*$-algebras, which states that every $\mathcal{C}^*$-algebra is isomorphic to a norm-closed, self-adjoint (i.e. closed under taking adjoints) algebra of bounded operators on a Hilbert space ([BR87] Theorem 2.1.10).

2.2.1 States

In the finite dimensional setting, the states of a system are given by density matrices. In the more general $\mathcal{C}^*$-algebraic setting, this needs to be modified as there may be cases when the trace against a density matrix does not converge. Given a density matrix ρ over a (finite dimensional) Hilbert space $\mathcal{H}$, we can construct ω_ρ, a linear functional on $\mathcal{B}(\mathcal{H})$. For $A \in \mathcal{B}(\mathcal{H})$,

$$\omega_\rho(A) := \mathrm{tr}(\rho A).$$

We have that ω_ρ is positive, in the sense that it maps positive operators to positive numbers. This follows by considering the eigenvalues of ρA, which are all positive since both ρ and A are positive. Furthermore, we may put a norm on the space of linear functionals defined by

$$||\omega|| = \sup_{||A||=1} |\omega(A)|.$$

This norm can equally well be defined on the space of linear functionals over an abstract $\mathcal{C}^*$-algebra. Since ρ is trace-one, it follows that $||\omega_\rho|| = 1$.

Importantly, these positive linear functionals which are induced by the density matrices are objects which still make sense in the $\mathcal{C}^*$-algebra setting. Motivated by this discussion, we give the following definition. A linear functional ω over a $\mathcal{C}^*$-algebra $\mathcal{A}$ is *positive* if $\omega(A^*A) \geq 0$ for all $A \in \mathcal{A}$. A positive linear functional ω with $||\omega|| = 1$ is called a *state*. Conveniently, the positivity condition automatically gives continuity.

A natural question to ask is whether all states defined in this way can be recovered by tracing against an appropriate density matrix. This is not the case. However, for a fixed state we can construct a certain "local" representation such that in the representation the state is represented by a density matrix.

It is a fact that the set of states on a $\mathcal{C}^*$-algebra is convex. It is then natural to define the pure states as the extremal points of the set of states. This mirrors the classical case when we view the set of mixed states, which are probability measures, as a subset of the hyperplane in $\mathbb{R}^n$. The set is convex in the euclidean geometry with the extremal points being the pure probability measures. The difference is that in the classical setting there is a unique way to write each mixed state as a linear combination of pure states: the set of states is a simplex. However, in the quantum case we do not have this unique decomposition.

2.2.2 Measurement

$\mathcal{C}^*$-algebras have a very nice spectral theory that will allow us to generalize the formalism for measurement from the finite-dimensional setting. First we use the version of the spectral theorem that allows us to "take functions of" observables, called the functional calculus version of the spectral theorem.

Theorem 2.2.1 (Spectral Theorem, Functional Calculus Version)**.** *Let A be a self-adjoint element of a $\mathcal{C}^*$-algebra $\mathcal{A}$, and let $\mathcal{C}^*(A)$ be the sub-algebra of $\mathcal{A}$ generated by A. Then there exists a map $\Phi_A : C(\sigma(A)) \to \mathcal{C}^*(A)$, where $C(\sigma(A))$ is the space of continuous functions on the spectrum, $\sigma(A)$, satisfying*

1. *Φ_A is an isomorphism ,*
2. *Φ_A is an isometry (in particular it is continuous) with the sup norm on $C(\sigma(A))$,*
3. *$\Phi_A(1) = \mathbb{1}$, where 1 is the map $\mathbb{R} \ni x \mapsto 1 \in \mathbb{R}$,*
4. *$\Phi_A(id) = A$, where id is the map $\mathbb{R} \ni x \mapsto x \in \mathbb{R}$.*

Furthermore, $\sigma(\Phi_A(f)) = f(\sigma(A))$. This is sometimes called the Spectral Mapping Theorem.

The element $\Phi_A(f)$ for a continuous function f should be interpreted as "taking the function" of the element A. We denote it by $f(A)$.

For a fixed state $\omega \in \mathcal{A}^*$, and observable $A \in \mathcal{A}$, we now let $\mu_{A,\omega}$ be the unique Borel measure on $\sigma(A)$ obtained from the Riesz-Markov theorem, satisfying

$$\omega(f(A)) = \int_{\sigma(A)} f(x) d\mu_{A,\omega}(x).$$

This is called the spectral measure associated to A and ω. The possible results of a measurement described by A for a system in state ω are given by $\sigma(A)$, and the probability distribution is given as follows. For a Borel subset $E \subset \sigma(A)$, the probability of measuring a value in E when the system is in the state ω is $\mu_{A,\omega}(E)$.

2.2.3 Dynamics

Recall that in the finite dimensional setting, our dynamics on the observables was given by a Hamiltonian H

$$\tau^t(A) = e^{itH} A e^{-itH}.$$

We can compute the derivative to obtain the differential equation

$$\frac{d}{dt}\tau^t(A) = iH\tau^t(A) - \tau^t(A)(iH) = i[H, \tau^t(A)]$$

with initial condition $\tau^0(A) = A$. A solution to this is

$$\tau^t(A) = e^{it[H,\cdot]}A$$

where we view $\delta_H := i[H, \cdot] : A \mapsto i[H, A]$ as a bounded linear operator on the space $\mathcal{B}(\mathcal{H})$, noting that in this setting $\mathcal{B}(\mathcal{H})$ is a finite dimensional Banach space. δ_H is self-adjoint and is called the generator of τ_t. Importantly, δ_H is in the $\mathcal{C}^*$-algebra of operators rather than the underlying Hilbert space, so this is the formulation we will use when moving to the general setting.

First let us consider what happens if the Hilbert space is infinite dimensional. We define dynamics using unitary operators, which preserve the inner product structure.

Definition 2.2.2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space. A dynamics on $\mathcal{B}(\mathcal{H})$ is a map $\tau^t(A) = U(t)AU(t)^*$ where $U(t)$ is a one-paramter strongly-continuous unitary group. In other words,

1. for each t, $U(t)$ is unitary,
2. for $s, t > 0$, $U(t + s) = U(t)U(s)$,
3. $U(0) = \mathbb{1}$,
4. for each A, $t \mapsto U(t)AU(t)^*$ is continuous.

We can apply Stone's theorem to our dynamics, which gives a correspondence between one-parameter strongly-continuous unitary groups $U(t)$ and self-adjoint (but not necessarily bounded) operators A.

$$U(t) \leftrightarrow e^{itA}$$

The exponential is well-defined using a power series if A is bounded or using spectral theory if A is unbounded [RS80]. Therefore, while we can still write our dynamics in the form $\tau^t(A) = e^{itH}Ae^{-itH}$, H may be unbounded and hence not an element of the algebra $\mathcal{B}(\mathcal{H})$. From a physical point of view, this corresponds to an infinitely extended system having infinite energy, but it will pose problems when we move to the $\mathcal{C}^*$-algebraic formalism, where we do not have access to the Hilbert space.

We can rewrite the definition of a dynamics in terms of the operator algebra by using the fact that unitary operators preserve inner products, and hence conjugation by unitaries preserves operator norm.

Definition 2.2.3. A dynamics on a $\mathcal{C}^*$-algebra $\mathcal{A}$ is a map

$$\mathbb{R} \ni t \mapsto \tau^t \in Aut(\mathcal{A})$$

which satisfies

1. $\tau^{t+s} = \tau^t \circ \tau^s$,
2. $t \mapsto \tau^t(A)$ is continuous for all $A \in \mathcal{A}$ (strong continuity).

For a $\mathcal{C}^*$-dynamics, which is a one-parameter strongly-continuous automorphism group on a Banach space, we define the generator as the possibly unbounded operator $\delta : \mathcal{D}(\delta) \to \mathcal{A}$:

$$\mathcal{D}(\delta) = \{A \in \mathcal{A} : \lim_{t\to 0} \frac{1}{t}(\tau^t(A) - A) \text{ exists}\},$$
$$\delta(A) = \lim_{t\to 0} \frac{1}{t}(\tau^t(A) - A) \text{ for } A \in \mathcal{D}(\delta).$$

The following theorem says that the generator does indeed generate the dynamics (see e.g. [Rud91] Theorem 13.35).

Theorem 2.2.4 (Hille-Yosida)**.** *Let $t \mapsto \tau^t$ be a one-parameter strongly continuous group of automorphisms on a Banach space X with generator δ. Then the domain of δ, $\mathcal{D}(\delta)$, is dense and δ is closed. Furthermore if $A \in \mathcal{D}(\delta)$ then*

$$\big(t \mapsto \tau^t(A)\big) \in C^0(\mathbb{R}, \mathcal{D}(\delta)) \cap C^1(\mathbb{R}, X)$$

and δ satisfies the differential equation

$$\frac{d}{dt}\tau^t(A) = \delta\tau^t(A) = \tau^t\delta(A).$$

For this reason we use the formal notation $\tau^t = e^{t\delta}$. Note that if we have a Hamiltonian dynamics on a finite dimensional Hilbert space given by Hamiltonian H, then we may compute the generator δ in this last more general sense and find that $\delta = \delta_H = i[H, \cdot]$.

2.2.4 Gibbs states and KMS states

In statistical mechanics, we often study equilibrium states. An equilibrium state is both invariant under the dynamics of the system, and has a certain stability property. We express this stability property precisely in terms of the entropy of the system. In the finite-dimensional case the von Neumann entropy is the functional $\mathcal{S} : \{\rho \in \mathcal{B}(\mathcal{H}) | \mathrm{tr}(\rho) = 1, \rho \geq 0\} \to \mathbb{R}$ defined by

$$\mathcal{S}(\rho) = -\mathrm{tr}(\rho \log \rho) = -\sum_i \lambda_i \log(\lambda_i)$$

where $\rho = \sum_i \lambda_i P_{\lambda_i}$ is the spectral decomposition. Note that $\log(\lambda_i)$ is defined since ρ is positive. If we take ρ to be a density matrix defined by $\sum_i p_i(\psi_i, \cdot)\psi_i$, then the von Neumann entropy is the same as the classical (Gibbs) entropy of the probability distribution $(p_1, ..., p_n)$.

An equilibrium state is one which maximizes the entropy among all states with a fixed energy $E \in [E_{min}, E_{max}]$ where E_{min} and E_{max} are the smallest and largest eigenvalues of the Hamiltonian. In other words, it is a state ρ_0 such that $\mathrm{tr}(H\rho_0) = E$ and

$$\max\{S(\rho) : \rho \text{ a state, } \mathrm{tr}(H\rho) = E\} = S(\rho_0).$$

In the finite-dimensional setting, the equilibrium states are completely described by Gibbs states.

Definition 2.2.5. For a finite dimensional quantum system $\mathcal{H}$ with Hamiltonian H, the Gibbs state at inverse temperature β is

$$\rho_\beta := \frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}.$$

Note that this is a positive operator with trace one.

For each fixed $E \in [E_{min}, E_{max}]$ there is a unique inverse temperature $\beta \in [-\infty, \infty]$ such that $\rho_\beta(H) = E$. For this value of β,

$$\max\{S(\rho) : \rho \text{ a state, } \mathrm{tr}(H\rho) = E\} = S(\rho_\beta)$$

and the unique maximizer is ρ_β. (See e.g. [Jak+12] for the proof.) The definition of the Gibbs state in the finite dimensional setting relied on using the trace: $\mathrm{tr}(e^{-\beta H})$. When we move to the infinite dimensional setting, this trace does not always converge, and therefore, the Gibbs state may not exist.

We wish to give a definition of the equilibrium states for the general $\mathcal{C}^*$-algebra setting. In order to do this, we will first derive a condition on states which, in the Hilbert space setting, is equivalent to the state being Gibbs. This condition will be written in terms of the linear functional induced by tracing against a density matrix.

We have a Hamiltonian dynamics acting on $\mathcal{B}(\mathcal{H})$ given by $\tau^t(A) = e^{itH}Ae^{-itH}$. For fixed A, B we can extend the function $\mathbb{R} \ni t \mapsto \omega(A\tau^t(B))$ to a strip in the complex plane, just by applying the functional calculus to the Hamiltonian. Then the Gibbs state is equivalent to an approximate commutation property [Jak+12].

Proposition 2.2.6. *Let $\mathcal{H}$ be a finite dimensional Hilbert space. The state $\omega \in \mathcal{B}(\mathcal{H})^*$ is a Gibbs state at inverse temperature β if and only if*

$$\omega(A\tau^{i\beta}(B)) = \omega(BA)$$

for all $A, B \in \mathcal{B}(\mathcal{H})$. This condition is called the KMS condition.

Proof. Suppose ω_β is the Gibbs state.

$$\omega_\beta(A\tau^{i\beta}(B)) = \mathrm{tr}(\rho_\beta Ae^{-\beta H}Be^{\beta H}) = \mathrm{tr}(\frac{e^{-\beta H}}{\mathrm{tr}(e^{-\beta H})}Ae^{-\beta H}Be^{\beta H}) = \mathrm{tr}(\rho_\beta BA) = \omega_\beta(BA)$$

The KMS condition follows from the cyclicity of the trace. This implication actually holds as long as the Gibbs state exists, even if H is not finite-dimensional. The traceclass operators are an ideal in the space of bounded operators, and so the above calculation still makes sense.

Now suppose ω is a state satisfying the KMS condition. Since we are in the finite dimensional case, we have some density matrix ρ such that $\omega(\cdot) = \mathrm{tr}(\rho\ \cdot)$. The KMS condition can then be written as

$$\mathrm{tr}(\rho BA) = \mathrm{tr}(\rho Ae^{-\beta H}Be^{\beta H}) \qquad \forall A, B \in \mathcal{B}(\mathcal{H}).$$

Let $X = e^{\beta H}\rho$ and $Y = Ae^{-\beta H}$. Then we have

$$\mathrm{tr}(XBY) = \mathrm{tr}(XYB) \quad \forall B, Y \in \mathcal{B}(\mathcal{H}).$$

It follows that $[X, B] = 0$ for all B and hence $X = \alpha\mathbb{1}$ for some scalar α. Hence $\rho = \alpha e^{-\beta H}$. Normalizing so that $\mathrm{tr}(\rho) = 1$ gives the Gibbs state. □

Remark 2.2.1. The KMS condition can be rewritten as

$$\omega(\tau^t(B)A) = \omega(A\tau^{t+i\beta}(B)) \quad \forall t \in \mathbb{R}.$$

In this form we can see it as a boundary condition of the function

$$F_{A,B}(z) := \omega(A\tau^z(B))$$

on the strip $0 \le \mathrm{Im}(z) \le \beta$: $F_{A,B}(t + i\beta) = \omega(\tau^t(B)A)$.

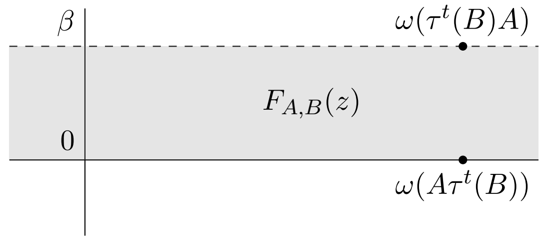

FIGURE 2.5: THE KMS CONDITION AS A BOUNDARY CONDITION ON A STRIP

We now wish to generalize the KMS condition to the operator algebra setting. First we need to verify that, for a general $\mathcal{C}^*$ dynamics, we can extend the function $t \mapsto \omega(A\tau^t(B))$ analytically to a strip in the complex plane. Indeed, we can show that for a fixed $\mathcal{C}^*$-dynamics τ^t, such an analytic extension is possible for a norm-dense sub-algebra called $\mathcal{A}_\tau$.

Definition 2.2.7. Let $t \mapsto \tau^t$ be a strongly continuous, one-parameter group of $*$-automorphisms of a $\mathcal{C}^*$-algebra $\mathcal{A}$. An element $A \in \mathcal{A}$ is analytic for τ^t if there exists a strip $I_\lambda = \{z \in \mathbb{C} : |\text{Im}(z)| < \lambda\}$ and a function $f : I_\lambda \to \mathcal{A}$ such that

1. for $t \in \mathbb{R}$, $f(t) = \tau^t(A)$ and
2. for each $\omega \in \mathcal{A}^*$, the function $z \mapsto \omega(f(z))$ is analytic.

This pointwise analyticity condition is actually equivalent to a stronger notion of analyticity in the space $\mathcal{A}$ itself.

One can show that the sub-algebra of entire analytic elements (for a given dynamics τ^t), which we denote by $\mathcal{A}_\tau$, is dense in the norm of $\mathcal{A}$ (see e.g. [BR87] Prop 2.5.22). We then give the KMS condition for $\mathcal{C}^*$-algebras.

Definition 2.2.8. Let $(\mathcal{A}, \tau)$ be a $\mathcal{C}^*$-dynamical system. A state ω is a (τ, β)-KMS state if

$$\omega(A\tau^{i\beta}(B)) = \omega(BA)$$

for all A, B in some norm-dense, τ-invariant sub-algebra $\mathcal{B}_\tau \subset \mathcal{A}_\tau$. We often suppress the τ in our notation when it is clear from context which dynamics we are referring to.

The set of β-KMS states for some fixed inverse temperature β, denoted by K_β is a simplex (see e.g. [BR96] Theorem 5.3.30). We denote the set of extremal β-KMS states by $\mathcal{E}_\beta$. These extremal KMS states can be thought of as pure phases.

2.2.5 Ground states

There are two possible choices for how to define the equilibrium states of a $\mathcal{C}^*$-dynamical system at 0-temperature. One may consider weak limits of low temperature KMS states (we refer to these as ∞-KMS states) or one may use a more general definition, which allows for 0-temperature states to exist even when there are no low-temperature KMS states. We refer to the latter as ground states.

Definition 2.2.9. ω is a ground state for $(\mathcal{A}, \tau)$ if for any $A, B \in \mathcal{A}$, there exists a function $F_{A,B}$ which is continuous on $Im(z) \geq 0$ and analytic and bounded on $Im(z) > 0$ such that

$$F_{A,B}(t) = \omega(A\tau_t(B))$$

for all $t \in \mathbb{R}$.

This corresponds to taking $\beta \to \infty$ in the strip of Remark 2.2.1. We refer to 0-temperature states as defined by weak limits of low-temperature states as ∞-KMS states.

Definition 2.2.10. Let $(\mathcal{A}, \tau)$ be a $\mathcal{C}^*$-dynamical system and $\{\omega_\alpha\}$ a net of states on $\mathcal{A}$ such that

$$\lim_\alpha \omega_\alpha(A) = \omega(A)$$

for all $A \in \mathcal{A}$. Let ω_α be a β_α-KMS state for $\beta_\alpha \in \mathbb{R}$ such that

$$\lim_\alpha \beta_\alpha = \infty.$$

Then ω is called an ∞-KMS state.

It is a fact that if ω is an ∞-KMS state, then it is automatically a ground state (see e.g. [BR96] Prop 5.3.23). However, the converse is not true. For example, if the time evolution is trivial, then all states are ground states, but only tracial states (those satisfying $\omega(AB) = \omega(BA)$) are ∞-KMS states. It can be convenient to consider the ∞-KMS states because they form a simplex and therefore we can consider the set of extremal states $\mathcal{E}_\infty$.

2.2.6 Symmetries and symmetry-breaking

An important phenomenon to study in quantum statistical mechanics is symmetry breaking. This occurs when there is some underlying group G of symmetries on a $\mathcal{C}^*$-algebra $\mathcal{A}$ which commutes with the time evolution. In certain temperature ranges, the induced action on the equilibrium states is trivial. However, in other temperature ranges, the choice of an equilibrium state φ breaks the symmetry group into smaller subgroups

$$G_\varphi = \{g \in G : g^*(\varphi) = \varphi\}$$

where g^* is the induced action of g on the states. This often occurs when there is some critical inverse temperate β_c such that above this temperature ($\beta < \beta_c$) there is a unique β-KMS state, and below this temperature the β-KMS states are no longer unique. We will consider symmetries that are both automorphisms and endomorphisms of $\mathcal{A}$.

- **Automorphisms:** We consider subgroups $G \subset Aut(\mathcal{A})$ which commute with the time evolution

  $$g\sigma_t = \sigma_t g \quad \forall g \in G, t \in \mathbb{R}.$$

  There is an induced action of G on the set of KMS states K_β and on the extremal KMS states $\mathcal{E}_\beta$.

- **Endomorphisms:** We consider subgroups $G \subset End(\mathcal{A})$ that commute with the time evolution, so that

  $$\rho\sigma_t = \sigma_t\rho \quad \forall \rho \in G, t \in \mathbb{R}.$$

  Let $e_\rho = \rho(\mathbb{1})$. If we have an extremal KMS state $\varphi \in \mathcal{E}_\beta$ such that $\varphi(e_\rho) \neq 0$ then we may define a pullback

  $$\rho^*(\varphi) = \frac{1}{\varphi(e_\rho)}\varphi \circ \rho.$$

  More care is needed when defining an action of G in this way on $\mathcal{E}_\infty$. There are cases where $\varphi(e_\rho) = 0$ but one can still define an interesting action on $\mathcal{E}_\infty$ by first "warming up" the states, acting by ρ, and then "cooling down" (see e.g. [Mar04] for details).

2.3 Bost-Connes type systems

In [BC95], a $\mathcal{C}^*$-dynamical system called the Bost-Connes system was introduced. It was initially constructed as a Hecke algebra, but it has a geometric interpretation as a coordinate algebra of 1-dimensional $\mathbb{Q}$-lattices up to a certain equivalence relation. A sub-algebra of the Bost-Connes algebra, called the arithmetic sub-algebra was also constructed. The Bost-Connes system is related to the number field $\mathbb{Q}$ and various important objects in its study. The partition function given by the Hamiltonian which generates the dynamics is the Riemann-zeta function. The Galois group $Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$ acts as a symmetry group on the extremal KMS states (pure phases) and with respect to this group of symmetries, the system exhibits spontaneous symmetry breaking at a critical temperature. Finally, the system has 0-temperature equilibrium states, and these states, when evaluated on points in the arithmetic sub-algebra, give a set of algebraic numbers which generate the maximal abelian extension $\mathbb{Q}^{cycl}$.

Later, in [CM04], this picture was extended to the geometric setting of 2-dimensional $\mathbb{Q}$-lattices. A $\mathcal{C}^*$-dynamical system called the GL_2-system (again based on a Hecke algebra) was constructed, as well as an arithmetic sub-algebra of, in this case, the algebra of unbounded multipliers on the Hecke algebra.

A construction of a Bost-Connes type system for an arbitrary number field K is given in [LLN09], which has the desired partition function, symmetries and behaviour of the KMS states. However, it is still unknown whether these systems have the important property that the ∞-KMS states, when evaluated on an arithmetic algebra, generate the maximal abelian extension K^{ab}. These dynamical systems are initially constructed using the class field theory data, but a geometric object (the K-lattices) are introduced which allow for a description of the systems that does not rely on the class field theory data. The present work takes a different tack. We instead define a boundary version of the GL_2 system directly by viewing points in $\mathbb{R}$ as parameterizing pseudolattices (degenerate elliptic curves) and connect the ground states of this system evaluated on the arithmetic algebra to the limiting modular symbols. Therefore, in this section we shall focus on the construction of the GL_2 system, which is most relevant to our construction.

2.3.1 Bost-Connes system

We begin by describing the $\mathcal{C}^*$-dynamical system constructed by Bost and Connes in [BC95] and its connections to properties of the number field $\mathbb{Q}$. The KMS states, the ground states, and the symmetries of the system will be of particular interest.

Construction of the Bost-Connes system

The original construction given by Bost and Connes in 1995 proceeds by first constructing a Hecke algebra, then moving to a regular representation of the algebra on a Hilbert space, and finally obtaining a canonical time evolution in the induced von Neumann algebra using Tomita-Takeaski theory. For a discrete group Γ and subgroup Γ_0 we define the Hecke algebra $\mathcal{H}(\Gamma,\Gamma_0)$ to be the algebra of complex-valued functions with finite support on the double coset space $\Gamma_0\backslash\Gamma/\Gamma_0$ equipped with the convolution product

$$(f_1 * f_2)(\gamma) = \sum_{\gamma_1\in\Gamma_0\backslash\Gamma} f_1(\gamma\gamma_1^{-1})f_2(\gamma_1) \quad \forall\gamma\in\Gamma. \tag{2.12}$$

Here we view f_1, f_2 as functions on Γ which are invariant with respect to multiplication on the left and right by Γ_0 and which have finite support in $\Gamma_0\backslash\Gamma/\Gamma_0$. It is also equipped with the involution

$$f^*(\gamma) = \overline{f(\gamma^{-1})} \quad \forall\gamma\in\Gamma_0\backslash\Gamma/\Gamma_0.$$

At this point we impose an additional condition that Γ_0 is an almost normal subgroup of Γ, which means that the orbits of the left action of Γ_0 on Γ/Γ_0 are finite. Under this condition, we turn the Hecke algebra into a $\mathcal{C}^*$-algebra by completing it in an appropriate representation. The map

$$\begin{aligned} \lambda : \mathcal{H}(\Gamma,\Gamma_0) &\to l^2(\Gamma_0\backslash\Gamma) \\ f &\mapsto \lambda_f \end{aligned}$$

defined by the formula

$$(\lambda_f\xi)(\gamma) = \sum_{\gamma_1\in\Gamma_0\backslash\Gamma} f(\gamma\gamma_1^{-1})\xi(\gamma_1) \quad \forall\gamma\in\Gamma_0\backslash\Gamma, \xi\in l^2(\Gamma_0\backslash\Gamma) \tag{2.13}$$

is a representation, which we call the regular representation. We denote by $C_r^*(\Gamma,\Gamma_0)$ the norm closure of $\mathcal{H}(\Gamma,\Gamma_0)$ in $l^2(\Gamma_0\backslash\Gamma)$.

Also under the condition that Γ_0 is an almost normal subgroup of Γ, we define a dynamics on $C_r^*(\Gamma,\Gamma_0)$ by taking the unique strongly-continuous one-parameter automorphism group σ_t such that

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{-it} f(\gamma) \tag{2.14}$$

where $R(\gamma)$ and $L(\gamma)$ are the cardinalities of the image of $\Gamma_0\gamma\Gamma_0$ in $\Gamma_0\backslash\Gamma$ and Γ/Γ_0 respectively. This time evolution is obtained in the regular representation via Tomita-Takesaki theory as the canonical time evolution associated to the state

$$\varphi(f) = (e, \lambda(f)e)_{l^2(\Gamma_0\backslash\Gamma)}$$

where e is a separating and cyclic vector given by the left coset $\Gamma_0 \mathbb{1} \in \Gamma_0 \backslash \Gamma$ (see Prop 4 of [BC95]).

Definition 2.3.1. The *Bost-Connes system* is the $\mathcal{C}^*$-algebra $\mathcal{A}_1 = C_r^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ and its associated dynamics σ_t of Equation 2.14 where

$$P_{\mathbb{Q}}^+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a, b \in \mathbb{Q}, a > 0 \right\},$$
$$P_{\mathbb{Z}}^+ = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathbb{Z} \right\}.$$

Note that $P_{\mathbb{Z}}^+$ is an almost normal subgroup of $P_{\mathbb{Q}}^+$ (Lemma 13 of [BC95]), so that this definition makes sense.

A presentation for the algebra $C_r^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$ is also obtained. We first observe that $C_r^*(\Gamma, \Gamma_0)$ has a linear basis given by $\{\epsilon_X\}$ where $X \in \Gamma_0 \backslash \Gamma / \Gamma_0$ is a double coset.

Proposition 2.3.2 ([BC95] Prop. 18, simplified in [LR99] Lemma 2.7)**.** *For $n \in \mathbb{N}$, let*

$$\mu_n = n^{-1/2} \epsilon_{X_n}, \quad \textit{where } X_n = P_{\mathbb{Z}}^+ \begin{pmatrix} 1 & 0 \\ 0 & n \end{pmatrix} P_{\mathbb{Z}}^+.$$

For $\gamma \in \mathbb{Q}/\mathbb{Z}$, let

$$e(\gamma) = \epsilon_{X^\gamma}, \quad \textit{where } X^\gamma = P_{\mathbb{Z}}^+ \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.$$

The elements μ_n for $n \in \mathbb{N}$ and $e(\gamma)$ for $\gamma \in \mathbb{Q}/\mathbb{Z}$ generate $C_r^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$, and the following relations give a presentation.

- $\mu_n^* \mu_n = 1 \quad \forall n \in \mathbb{N}$
- $\mu_{nm} = \mu_n \mu_m \quad \forall n, m \in \mathbb{N}$
- $e(0) = 1, e(\gamma)^* = e(-\gamma)$ *and* $e(\gamma_1 + \gamma_2) = e(\gamma_1) e(\gamma_2) \quad \forall \gamma, \gamma_1, \gamma_2 \in \mathbb{Q}/\mathbb{Z}$
- $\mu_n e(\gamma) \mu_n^* = \frac{1}{n} \sum_{\delta \in \mathbb{Q}/\mathbb{Z} : n\delta = \gamma} e(\delta) \quad \forall n \in \mathbb{N}, \gamma \in \mathbb{Q}/\mathbb{Z}$

In this presentation, the time evolution acts by

$$\sigma_t(\mu_n) = n^{it} \mu_n, \qquad \sigma_t(e(\gamma)) = e(\gamma). \tag{2.15}$$

By means of this presentation, one can show that the Bost-Connes algebra is isomorphic to the semigroup crossed product

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$$

where $\mathbb{N}$ is the semigroup under multiplication [LR99]. Here $C^*(\mathbb{Q}/\mathbb{Z})$ is the group C^*-algebra.

We will also consider an arithmetic sub-algebra of the Bost-Connes algebra, which we think of as a sub-algebra of certain classical points on which we will evaluate the ground states of the Bost-Connes system. It is defined as follows.

Definition 2.3.3. The *arithmetic sub-algebra* of the Bost-Connes system, denoted by $\mathcal{A}_{1,\mathbb{Q}}$, is the algebra over $\mathbb{Q}$ generated by the elements $e(\lambda)$ for $\lambda \in \mathbb{Q}/\mathbb{Z}$ and μ_n and μ_n^* for $n \in \mathbb{N}$ in the presentation of Proposition 2.3.2.

The arithmetic sub-algebra can equivalently be described in the Hecke algebra picture as the compactly-supported $\mathbb{Q}$-valued functions on $\Gamma_0\backslash\Gamma$ with the convolution product 2.12.

Geometric interpretation

There is a geometric picture corresponding to the Bost-Connes system in terms of 1-dimensional $\mathbb{Q}$-lattices under a commensurability relation. We present here the general description of $\mathbb{Q}$-lattices (as we will make use of the 2-dimensional case later) before specializing to the 1-dimensional case.

Definition 2.3.4. An *n-dimensional* $\mathbb{Q}$*-lattice* consists of a pair (Λ, ϕ) where $\Lambda \subset \mathbb{R}^n$ is a lattice and ϕ is a labelling of its torsion points given by

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}\Lambda/\Lambda$$

where ϕ is a group homomorphism. In the special case that ϕ is an isomorphism, we say that (Λ, ϕ) is *invertible*.

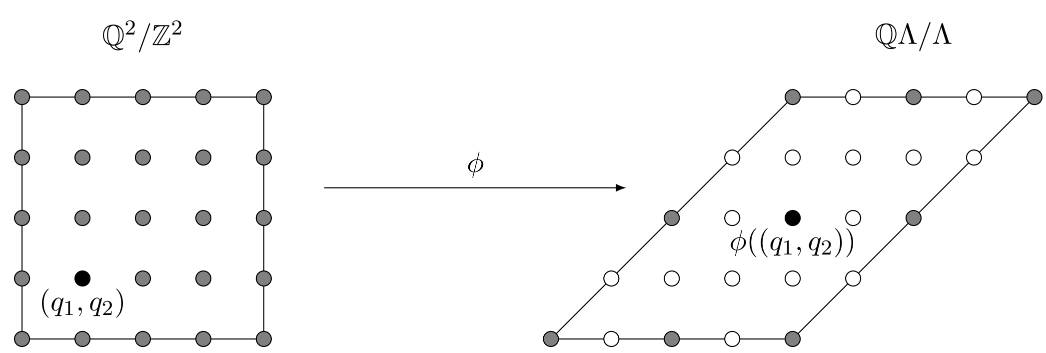

Figure 2.6: Non-invertible 2-dimensional $\mathbb{Q}$-lattice.

We denote by ϕ_N the restriction of the map ϕ to the N-torsion points,

$$\phi_N : \left(\frac{1}{N}\mathbb{Z}/\mathbb{Z}\right)^n \to \frac{1}{N}\Lambda/\Lambda. \tag{2.16}$$

We say that (Λ, ϕ) is *divisible by* N if $\phi_N = 0$.

Definition 2.3.5. We say that two $\mathbb{Q}$-lattices (Λ_1, ϕ_1) and (Λ_2, ϕ_2) are commensurable, denoted by $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$, if

$$\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2 \quad \text{and} \quad \phi_1 = \phi_2 \pmod{\Lambda_1 + \Lambda_2}. \tag{2.17}$$

Commensurability of $\mathbb{Q}$-lattices is an equivalence relation (see e.g. [CM08] Lemma 3.18). We denote by $\mathcal{L}_n$ the set of commensurability classes of n-dimensional $\mathbb{Q}$-lattices.

In the 1-dimensional case, a $\mathbb{Q}$-lattice can be written in the form $(\Lambda, \phi) = (\lambda\mathbb{Z}, \lambda\rho)$ where $\lambda > 0$ is some scaling factor and

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}.$$

The 1-dimensional $\mathbb{Q}$-lattices, up to scaling by $\mathbb{R}^+$, are completely determined by the choice of $\rho \in \hat{\mathbb{Z}}$. We will construct a non-commutative $\mathcal{C}^*$-algebra describing the commensurability classes of 1-d $\mathbb{Q}$-lattices up to scaling. First we take the algebra of coordinates of the space of all the 1-d $\mathbb{Q}$-lattices up to scaling, which is $C(\hat{\mathbb{Z}})$. The commensurability relation is implemented by the action of $\mathbb{N}$ on the coordinate space of $\mathbb{Q}$-lattices by the maps

$$\alpha_n(f)(\Lambda, \phi) = \begin{cases} f(n\Lambda, \phi) & \text{if } (\Lambda, \phi) \text{ is divisible by n} \\ 0 & \text{if } (\Lambda, \phi) \text{ is not divisible by n} \end{cases}. \tag{2.18}$$

This corresponds in $C(\hat{\mathbb{Z}})$ to the action of $\mathbb{N}$ by the maps

$$\alpha_n(f)(\rho) = f(n^{-1}\rho) \quad \forall \rho \in n\hat{\mathbb{Z}}. \tag{2.19}$$

From this, it follows that the algebra of coordinates of commensurability classes of 1-d $\mathbb{Q}$-lattices up to scaling, which we denote by $C(\mathcal{L}_1/\mathbb{R}^+)$, is isomorphic to $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$. Pontryagin duality can then be used to identify $C(\hat{\mathbb{Z}})$ and $C^*(\mathbb{Q}/\mathbb{Z})$, giving us that $C(\mathcal{L}_1/\mathbb{R}^+) = C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$, which is isomorphic to the original Bost-Connes algebra $\mathcal{A}_1$. In the $\mathbb{Q}$-lattice picture, the time evolution of the Bost-Connes system is given on $C(\mathcal{L}_1/\mathbb{R}^+)$ by

$$\sigma_t(f)(L_1, L_2) = |L_1/L_2|^{it} f(L_1, L_2) \tag{2.20}$$

where $L_i = (\Lambda_i, \phi_i)$ is a $\mathbb{Q}$-lattice, we view elements of $C(\mathcal{L}_1/\mathbb{R}^+)$ as functions on pairs of commensurable $\mathbb{Q}$-lattices invariant under scaling

$$f(\lambda L_1, \lambda, L_2) = f(L_1, L_2) \quad \forall \lambda \in \mathbb{R}^+,$$

and the quantity $|L_1/L_2|$ is the ratio of the co-volumes of the lattices

$$|L_1/L_2| = \frac{\mathrm{covol}(\Lambda_1)}{\mathrm{covol}(\Lambda_2)}.$$

The arithmetic sub-algebra can be viewed in the lattice picture as the sub-algebra of $C(\mathcal{L}_1/\mathbb{R}^+)$ generated by certain weight-0 functions on the space of $\mathbb{Q}$ lattices. Let $a \in \mathbb{Q}/\mathbb{Z}$. Then we define a weight-0 function by

$$e_a(\Lambda, \phi) = c(\Lambda) \sum_{y \in \Lambda + \phi(a)} y^{-1} \tag{2.21}$$

where $c(\Lambda)$ is a multiple of $\mathrm{covol}(\Lambda)$ determined by the formula

$$2\pi i c(\mathbb{Z}) = 1. \tag{2.22}$$

The arithmetic sub-algebra $\mathcal{A}_{1,\mathbb{Q}}$ is generated by e_a for $a \in \mathbb{Q}/\mathbb{Z}$ and the μ_n, μ_n^* for $n \in \mathbb{N}$.

KMS states and symmetries

The Bost-Connes system is interesting because of the connection between its thermodynamic behaviour and properties of the number field $\mathbb{Q}$. In particular, the system has a group of symmetries given by the Galois group $Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$.

Observe that each element $\alpha \in Gal(\mathbb{Q}^{cycl}, \mathbb{Q})$ has an associated representation $\pi_\alpha : \mathcal{A}_1 \to l^2(\mathbb{N}^*)$ (Prop 24 [BC95]). Let ϵ_k for $k \in \mathbb{N}^*$ be the canonical basis for $l^2(\mathbb{N}^*)$. In this basis, π_α is given by

$$\begin{aligned} \pi_\alpha(\mu_n)\epsilon_k &= \epsilon_{nk} & \forall n, k \in \mathbb{N}^* \\ \pi_\alpha(e(\gamma))\epsilon_k &= \alpha(\exp 2\pi i k\gamma)\epsilon_k & \forall k \in \mathbb{N}^*, \gamma \in \mathbb{Q}/\mathbb{Z}. \end{aligned} \tag{2.23}$$

Note that there is a canonical extension of π_α to the full $\mathcal{C}^*$-algebra $\mathcal{A}_1$. In the representation, the time evolution is given by the Hamiltonian

$$H\epsilon_n = \log(n)\epsilon_n \tag{2.24}$$

i.e. $e^{itH}\pi_\alpha(x)e^{-tH} = \pi_\alpha(\sigma_t(x))$ for all $x \in \mathcal{A}_1$ and $t \in \mathbb{R}$. In the representation, the partition function is the Riemann-zeta function.

$$Z(\beta) = \mathrm{tr}(e^{-\beta H}) = \sum_{n \in \mathbb{N}} n^{-\beta} = \zeta(\beta)$$

We will now describe the KMS states of the system, using these representations.

Proposition 2.3.6 (Theorem 5, Theorem 25 [BC95]). *For $0 < \beta \leq 1$, there is a unique β-KMS state, denoted by φ_β, on the BC system $(\mathcal{A}_1, \sigma_t)$. On the $C^*(\mathbb{Q}/\mathbb{Z})$ part of the algebra, the restriction of the state is given by the function on $\mathbb{Q}/\mathbb{Z}$*

$$\varphi(e(a/b)) = b^{-\beta} \prod_{p \text{ prime }, p|b} (1 - p^{\beta-1})(1 - p^{-1})^{-1} \tag{2.25}$$

where $\frac{a}{b} \in \mathbb{Q}/\mathbb{Z}$ such that $a, b \in \mathbb{Z}$ are relatively prime, and $b > 0$.

For $\beta > 1$, the extremal β-KMS states, $\mathcal{E}_\beta$, are parameterized by embeddings $\chi : \mathbb{Q}^{cycl} \to \mathbb{C}$. For each $\alpha \in Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$, there is a β-KMS state given by

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} Tr(\pi_\alpha(x) e^{-\beta H}) \quad \forall x \in \mathcal{A}_1 \tag{2.26}$$

The map $\alpha \mapsto \varphi_{\beta,\alpha}$ is a homeomorphism of $Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$ and $\mathcal{E}_\beta$.

The restriction of the state $\varphi_{\beta,\alpha}$ to the $C^*(\mathbb{Q}/\mathbb{Z})$ part of the algebra is given by

$$\varphi_{\beta,\alpha}(e(a/b)) = \frac{1}{\zeta(\beta)} \sum_{n=1}^{\infty} n^{-\beta} \chi_\alpha(\zeta_{a/b}^n) \tag{2.27}$$

where $\chi_\alpha : \mathbb{Q}^{cycl} \to \mathbb{C}$ is the embedding corresponding to the element $\alpha \in Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$, and $\zeta_{a/b}$ is a root of unity.

There is a critical temperature $\beta = 1$. Above this temperature ($\beta \leq 1$) the KMS states are unique. However, for low temperature ($\beta > 1$) there are many β-KMS states, and they are described by embeddings of the the cyclotomic field in $\mathbb{C}$. In addition, $Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$ acts on the set of β-KMS states by composition, and the system exhibits spontaneous symmetry-breaking at $\beta = 1$ with respect to this action.

We note that $\varphi_{\beta,\alpha}$ are Gibbs states in the representation π_α. In the limit $\beta \to \infty$ we obtain extremal ∞-KMS states of the form

$$\varphi_{\infty,\alpha}(x) = (\pi_\alpha(x)\epsilon_1, \epsilon_1)_{\ell^2(\mathbb{N}^*)} \tag{2.28}$$

each one corresponding to $\alpha \in Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$. The Galois group also acts on the ∞-KMS states by composition. We end by describing how the ∞-KMS states behave when evaluated on the arithmetic subalgebra $\mathcal{A}_{1,\mathbb{Q}}$.

Proposition 2.3.7. *Let $\varphi_{\infty,\alpha} \in \mathcal{E}_\infty$ be an ∞-KMS state on $(\mathcal{A}_1, \sigma_t)$. Then $\varphi_{\infty,\alpha}(\mathcal{A}_{1,\mathbb{Q}}) \subset \mathbb{Q}^{cycl}$. Furthermore, the class field isomorphism*

$$\Theta : Gal(\mathbb{Q}^{cycl}/\mathbb{Q}) \to \hat{\mathbb{Z}}^*$$

intertwines the Galois action on $\varphi_{\infty,\alpha}(\mathcal{A}_{1,\mathbb{Q}})$ with the action of $\hat{\mathbb{Z}}^*$ on $\mathcal{A}_{1,\mathbb{Q}}$:

$$\gamma\varphi_{\infty,\alpha}(x) = \varphi_{\infty,\alpha}(\Theta(\gamma)x)$$

where $\gamma \in Gal(\mathbb{Q}^{cycl}/\mathbb{Q})$ and $x \in \mathcal{A}_{1,\mathbb{Q}}$.

2.3.2 GL_2-system

A 2-dimensional version of the Bost-Connes system was given by Connes and Marcolli in [CM04], based upon the geometric picture of $\mathbb{Q}$-lattices. It utilizes 2-dimensional $\mathbb{Q}$-lattices rather than 1-dimensional $\mathbb{Q}$-lattices. The idea is to consider an algebra of coordinates $C(\mathcal{L}_2/\mathbb{C}^*)$ on the space of commensurability classes of 2-d $\mathbb{Q}$-lattices up to scaling. As before, we begin by characterizing all the $\mathbb{Q}$-lattices by some parameters including a scaling parameter. In the 2-dimensional case, every $\mathbb{Q}$-lattice can be written in the form

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \tau\mathbb{Z}), \lambda\rho) \tag{2.29}$$

where $\lambda \in \mathbb{C}^*$ is some scaling factor, $\tau \in \mathbb{H}$, and $\rho \in \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2) = M_2(\hat{\mathbb{Z}})$. Recall that the lattices $\mathbb{Z} + \tau\mathbb{Z}$ correspond to elliptic curves, and the quotient under the $\mathrm{SL}_2(\mathbb{Z})$ action of the parametrizing points $\tau \in \mathbb{H}$ gives the modular curve, which encodes the isomorphism classes of elliptic curves. The thermodynamic properties of the GL_2-system are related to the imaginary quadratic fields similarly to how the thermodynamic properties of the Bost-Connes system are related to the number field $\mathbb{Q}$. We will go through a careful construction of the GL_2-system, comparing it to the Bost-Connes system, and using an underlying picture of 2-dimensional $\mathbb{Q}$-lattices. We will then summarize results surrounding the GL_2-system's KMS states, phase transitions, ground states, and symmetries.

Construction of the GL_2-system

We will proceed by first considering $\mathcal{R}_2$, the groupoid of the set of 2-d $\mathbb{Q}$-lattices (not up to scaling) with the equivalence relation of commensurability, and then describe how the scaling action of $\mathbb{C}^*$ acts on $\mathcal{R}_2$. We begin by obtaining an equivalent description of the groupoid $\mathcal{R}_2$.

Let G_2 be the groupoid of pairs (g, ρ) where $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(\hat{\mathbb{Z}})$, and $g\rho \in M_2(\hat{\mathbb{Z}})$, with composition given by

$$(g_1, \rho_1) \circ (g_2, \rho_2) = (g_1 g_2, \rho_2)$$

if $g_2\rho_2 = \rho_1$. The associated $\mathcal{C}^*$-algebra $C^*(G_2)$ is the groupoid $\mathcal{C}^*$-algebra introduced in [Ren80], which has the convolution product

$$f_1 \star f_2(g, \rho) = \sum_s f_1(gs^{-1}, s\rho) f_2(s, \rho) \tag{2.30}$$

and involution

$$f^*(g, \rho) = \overline{f(g^{-1}, g\rho)}.$$

This $\mathcal{C}^*$-algebra can be equivalently viewed as the semigroup crossed product $C(M_2(\hat{\mathbb{Z}})) \rtimes M_2(\mathbb{Z})^+$. Comparing this to the 1-d case where we had $C(\mathcal{L}_1/\mathbb{R}^+)$ is isomorphic to $C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$, we see that $M_2(\hat{\mathbb{Z}})$ is playing the role of $\hat{\mathbb{Z}}$. However, G_2 by itself is actually not an equivalent description of $\mathcal{R}_2$. To obtain the correct full description of $\mathcal{R}_2$, we take a crossed product with $\mathrm{GL}_2(\mathbb{R})$. Let

$$\begin{aligned} \psi : G_2 &\to GL_2^+(\mathbb{R}) \\ (g, \rho) &\mapsto g \end{aligned} \tag{2.31}$$

be the homomorphism obtained from the inclusion $\mathrm{GL}_2^+(\mathbb{Q}) \subset \mathrm{GL}_2^+(\mathbb{R})$.

Definition 2.3.8. For a groupoid G, a group H, and a homomorphism $\psi : G \to H$, we define the *cross product groupoid* $G \times_\psi H$ to be the set $G \times H$ with objects $G^{(0)} \times H$, range and source maps

$$r(g, h) = (r(g), \psi(g)h), \qquad s(g, h) = (s(g), h),$$

and composition

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 \circ g_2, h_2).$$

Let $\tilde{G}_2 = G_2 \times_\psi \mathrm{GL}_2(\mathbb{R})$ be the cross product groupoid. The set of objects of $\tilde{G}_2$ is

$$\tilde{G_2}^{(0)} = \{(g, \rho, \alpha) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}) : g\rho \in M_2(\hat{\mathbb{Z}})\}. \tag{2.32}$$

We can take the quotient of $\tilde{G}_2$ by the free action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ given by

$$(\gamma_1, \gamma_2)(g, \rho, \alpha) = (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 \alpha) \tag{2.33}$$

to obtain a groupoid S_2 which is Morita equivalent to $\tilde{G}_2$. There is an isomorphism of locally compact groupoids between the quotient S_2 and $\mathcal{R}_2$ given by

$$\Phi(g, \rho, \alpha) = ((\alpha^{-1} g^{-1} \Lambda_0, \alpha^{-1}\rho), (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)) \quad \forall (g, \rho, \alpha) \in S_2 \tag{2.34}$$

where Λ_0 is the lattice $\Lambda_0 = \mathbb{Z} + i\mathbb{Z}$ ([CM04] Proposition 1.22). Having obtained this equivalent description of $\mathcal{R}_2$, we now describe how the action of scaling by $\mathbb{C}^*$ on the $\mathbb{Q}$-lattices behaves. The scaling action of $\mathbb{C}^*$ on $\mathcal{R}_2$ is not free, so the quotient $\mathcal{R}_2/\mathbb{C}^*$ will not be a groupoid. However, we will still be able to define a convolution $\mathcal{C}^*$-algebra. Observe that we can view $\mathbb{C}$ as a subgroup of $\mathrm{GL}_2^+(\mathbb{R})$ via the identification

$$\begin{aligned} \mathbb{C} &\to \mathrm{GL}_2^+(\mathbb{R}) \\ a + ib &\mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \end{aligned} \tag{2.35}$$

and one can identify $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{C}^*$ with the upper half plane $\mathbb{H}$ by

$$\begin{aligned} \mathrm{GL}_2^+(\mathbb{R}) &\to \mathbb{H} \\ \alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \alpha(i) = \frac{ai+b}{ci+d}. \end{aligned} \tag{2.36}$$

Consider the space $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ and the action of $\mathrm{GL}_2^+(\mathbb{Q})$ on it by

$$\gamma(\rho, \tau) = \left(\gamma\rho, \frac{a\tau+b}{c\tau+d}\right) \tag{2.37}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(\hat{\mathbb{Z}})$, and $\tau \in \mathbb{H}$. Let Z be the space defined as the quotient of

$$\mathcal{U} = \{(g, \rho, \tau) : g \in \mathrm{GL}_2^+(\mathbb{Q}), \rho \in M_2(\hat{\mathbb{Z}}), \tau \in \mathbb{H}, g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{2.38}$$

by the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ given by

$$(\gamma_1, \gamma_2)(g, \rho, \tau) = (\gamma_1 g \gamma_2^{-1}, \gamma_2(\rho, \tau)) \quad \forall \gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{Z}). \tag{2.39}$$

This quotient can be identified with the space of commensurability classes of $\mathbb{Q}$-lattices up to scaling, using the isomorphism of equation 2.34 and the lift of the quotient map in 2.36 to first obtain an isomorphism

$$\begin{aligned} \theta : \mathrm{SL}_2(\mathbb{Z})\backslash(M_2(\hat{\mathbb{Z}}) \times \mathbb{H}) &\to X = \{\text{2-d } \mathbb{Q}\text{-lattices }\}/\mathbb{C}^* \\ (\rho, \tau) &\mapsto (\mathbb{Z} + \tau\mathbb{Z}, \phi_\rho) \end{aligned} \tag{2.40}$$

where ϕ_ρ is the map $\phi_\rho(x) = \rho_1(x) - \tau\rho_2(x)$ for $x \in \mathbb{Q}^2/\mathbb{Z}^2$. Here we use the notation $\rho_i(x) = \sum_j \rho_{ij}(x_j)$ for $x = (x_1, x_2) \in \mathbb{Q}^2/\mathbb{Z}^2$.

From this, we obtain the desired isomorphism

$$\begin{aligned} \tilde{\theta} : Z &\to \mathcal{R}_2/\mathbb{C}^* \\ (g, \rho, \tau) &\mapsto (\lambda_{g,\tau}\theta(g(\rho, \tau)), \theta(\rho, \tau)) \end{aligned}$$

where $\lambda_{g,\tau} = \det(g)^{-1}(c\tau + d)$ for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

The quotient space Z is not a groupoid, so we cannot simply take the groupoid $\mathcal{C}^*$-algebra of Z. However, we can still define a coordinate $\mathcal{C}^*$-algebra on Z. Let $\tilde{\mathcal{A}}_2 = C_c(Z)$ be the space of continuous functions with compact support on Z. We can view an element $f \in \tilde{\mathcal{A}}_2$ as a function on $\mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ satisfying

$$f(\gamma g, \rho, \tau) = f(g, \rho, \tau), \quad f(g\gamma, \rho, \tau) = f(g, \gamma(\rho, \tau)) = f\left(g, \gamma\rho, \frac{a\tau+b}{c\tau+d}\right)$$

for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$, $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(\hat{\mathbb{Z}})$, and $\tau \in \mathbb{H}$. We define a Hecke algebra via the convolution product

$$(f_1 \star f_2)(g, \rho, \tau) = \sum_{h \in \mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2^+(\mathbb{Q})} f_1(gh^{-1}, h(\rho, \tau)) f_2(h, \rho, \tau) \tag{2.41}$$

and the involution

$$f^*(g, \rho, \tau) = \overline{f(g^{-1}, g(\rho, \tau))}. \tag{2.42}$$

Finally, we need to complete $\tilde{\mathcal{A}}_2$ to a $\mathcal{C}^*$-algebra. Let

$$G_\rho = \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\}. \tag{2.43}$$

For each lattice (Λ, ϕ), which corresponds to a pair $(\rho, \tau) \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ by 2.29, we construct a representation $\pi_{\rho,\tau}$ of $\tilde{\mathcal{A}}_2$ into the Hilbert space $\mathcal{H}_\rho = l^2(\mathrm{SL}_2(\mathbb{Z}) \backslash G_\rho)$ by

$$(\pi_{\rho,\tau}(f)\xi)(g) = \sum_{h \in \mathrm{SL}_2(\mathbb{Z}) \backslash G_\rho} f(gh^{-1}, h(\rho, \tau)) \xi(h) \tag{2.44}$$

for $f \in \tilde{\mathcal{A}}_2$ and $\xi \in \mathcal{H}_\rho$.

Remark 2.3.1. The representations 2.44 are related to the commensurability classes of $\mathbb{Q}$-lattices. For each $x \in X = \{\text{2-d } \mathbb{Q}\text{-lattices }\}/\mathbb{C}^*$, let $c(x) \subset X$ be the commensurability class of x. Let p be the quotient map $p : M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \to X$ associated to 2.40. The map

$$\begin{aligned} G_\rho &\to X \\ g &\mapsto p(g(\rho, \tau)) \end{aligned}$$

induces a surjection from $\mathrm{SL}_2(\mathbb{Z}) \backslash G_\rho$ onto $c(x_{\rho,\tau})$ where $p(\rho, \tau) = x_{\rho,\tau}$. However, this map is not injective in general.

We obtain a $\mathcal{C}^*$-algebra $\mathcal{A}_2$ by completing $\tilde{\mathcal{A}}_2$ in the norm given by

$$||f|| = \sup_{(\rho,\tau) \in M_2(\hat{\mathbb{Z}}) \times \mathbb{H}} ||\pi_{\rho,\tau}(f)||_{l^2(\mathrm{SL}_2(\mathbb{Z}) \backslash G_\rho)} \tag{2.45}$$

([CM04] Prop 1.23.)

Definition 2.3.9. The GL_2-system is the $\mathcal{C}^*$-algebra $\mathcal{A}_2$ together with the $\mathcal{C}^*$ dynamics

$$\sigma_t(f)(g, \rho, \tau) = (\det g)^{it} f(g, \rho, \tau).$$

Arithmetic algebra

As in the Bost-Connes case, we will also construct an arithmetic algebra, on whose elements we will later evaluate the 0-temperature KMS states. In the GL_2 setting, the appropriate choice of arithmetic algebra turns out to be an algebra unbounded multipliers of $\mathcal{A}_2$ rather than a sub-algebra of $\mathcal{A}_2$. The construction is detailed below.

Let f be a continuous function on Z and for $g \in \mathrm{GL}_2^+(\mathbb{Q})$, $\rho \in M_2(\hat{\mathbb{Z}})$ and $z \in \mathbb{H}$, let

$$f_{g,\rho}(z) = f(g, \rho, z)$$

so that $f_{g,\rho} \in C(\mathbb{H})$ and consider the canonical projection $p_n : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/n\mathbb{Z})$. We say that f is of level n if

$$f_{g,\rho} = f_{g,p_n(\rho)} \quad \forall g \in \mathrm{GL}_2^+(\mathbb{Q}),\ \rho \in M_2(\hat{\mathbb{Z}}).$$

Note that if f is of level n, then f is completely determined by the functions

$$f_{g,m} \quad \forall\, g \in \mathrm{GL}_2^+(\mathbb{Q}),\ m \in M_2(\mathbb{Z}/n\mathbb{Z}).$$

The arithmetic algebra is defined as follows.

Definition 2.3.10. A function $f \in C(Z)$, is in the arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ if

1. f is finitely supported in $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathrm{GL}_2^+(\mathbb{Q})$.
2. f is of level n for some finite n with

$$f_{g,m} \in F \quad \forall\, g \in \mathrm{GL}_2^+(\mathbb{Q}),\ m \in M_2(\mathbb{Z}/n\mathbb{Z})$$

   where F is the modular field of definition 2.1.4.
3. f satisfies

$$f_{g,\alpha(u)m} = \mathrm{cycl}(u) f_{g,m}$$

   for all diagonal $g \in \mathrm{GL}_2^+(\mathbb{Q})$ and $u \in \hat{\mathbb{Z}}^*$, where cycl is the homomorphism of equation 2.8 and

$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

The algebra $\mathcal{A}_{2,\mathbb{Q}}$ has the convolution product given by 2.41.

Note that the modular field $F_n \subset F$ contains a primitive n^{th} root of unity. Therefore, requiring only condition (2) without condition (3) would have allowed $\mathcal{A}_{2,\mathbb{Q}}$ to contain $\mathbb{Q}^{cycl}$. This would preclude the symmetry group (which will be introduced in the next section) from acting on the set of zero-temperature states evaluated on points in $\mathcal{A}_{2,\mathbb{Q}}$, as desired. Condition (3) is added to resolve this issue.

The arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$ has a presentation in terms of the Eisenstein series

$$\begin{aligned} E_{2k,a}(\rho,\tau) &= \frac{1}{\pi^{2k}} \sum_{y\in\Lambda+\phi(a)} y^{-2k} \\ X_a(\rho,\tau) &= \frac{1}{\pi^2}\left(\sum_{y\in\Lambda+\phi(a)} y^{-2} + \sum_{y\in\Lambda} y^{-2}\right) \end{aligned} \tag{2.46}$$

where $(\Lambda,\phi) = \theta(\tau,\rho)$ is given by the isomorphism 2.40 (see [CM04] proof of Theorem 1.39 and Prop 1.41). This is similar to the original Bost-Connes system, which had a presentation in terms of Eisenstein series in the 1-d $\mathbb{Q}$-lattice setting, normalized to be 0-weight (c.f. equation 2.21).

KMS states and symmetries

As in the original Bost-Connes system, the GL_2-system exhibits symmetry breaking, except in this case there are two phase transitions at $\beta = 1$ and $\beta = 2$. There is interesting action of a certain symmetry group on the low- and 0-temperature states. We begin by summarizing the structure of the KMS states. First, note that in a representation $\pi_{\rho,\tau}$, the dynamics σ_t is implemented by a Hamiltonian

$$(H_\rho\xi)(h) = \log(\det h)\xi(h) \quad \forall h \in G_\rho \tag{2.47}$$

so that $\pi_{\rho,\tau}(\sigma_t(A)) = e^{itH_\rho}\pi_{\rho,\tau}(A)e^{-itH_\rho}$ for all $A \in \mathcal{A}_2$.

In the special case that $(\Lambda,\phi) \sim (\rho,\tau)$ is an invertible $\mathbb{Q}$-lattice,

$$\mathcal{H}_\rho \simeq l^2(\mathrm{SL}_2(\mathbb{Z})\backslash M_2^+(\mathbb{Z}))$$

and the Hamiltonian is given by

$$H_\rho\epsilon_m = \log(\det m)\epsilon_m$$

where ϵ_m for $m \in \mathrm{SL}_2(\mathbb{Z})\backslash M_2^+(\mathbb{Z})$ is the standard basis for $l^2(\mathrm{SL}_2(\mathbb{Z})\backslash M_2^+(\mathbb{Z}))$. The partition function is then

$$Z(\beta) = \mathrm{tr}(e^{-\beta H_\rho}) = \sum_{m\in\mathrm{SL}_2(\mathbb{Z})\backslash M_2^+(\mathbb{Z})} (\det m)^{-\beta} = \sum_{k=1}^{\infty}\sigma(k)k^{-\beta} = \zeta(\beta)\zeta(1-\beta) \tag{2.48}$$

where $\sigma(k) = \sum_{d|k} d$.

Proposition 2.3.11 ([CM04] Cor 1.32 and Theorem 1.26, [LLN07] Theorem 4.1)**.** *The β-KMS states of the GL_2-system $(\mathcal{A}_2,\sigma_t)$ can be classified as follows.*

- *For $\beta \leq 1$, there are no β-KMS states.*
- *For $1 < \beta \leq 2$, there is a unique β-KMS state.*
- *For $\beta > 2$, each invertible lattice $l = (\Lambda, \phi) \sim (\rho, \tau)$ gives an extremal β-KMS state*

$$\varphi_{\beta,l}(f) = \frac{1}{\zeta(\beta)\zeta(1-\beta)} \sum_{m \in SL_2(\mathbb{Z})\backslash M_2^+(\mathbb{Z})} f(1, m(\rho,\tau))(\det m)^{-\beta}$$

and in fact the map $l \mapsto \varphi_{l,\beta}$ is a bijection from the space of invertible $\mathbb{Q}$-lattices up to scaling and $\mathcal{E}_\beta$, the space of extremal β-KMS states.

In the weak limit as $\beta \to \infty$ we obtain extremal ∞-KMS which restrict to $C_c(X) \subset \mathcal{A}_2$ (recalling that X and Z are identified in 2.40) as

$$\varphi_{\infty,l}(f) = f(l) \quad \forall f \in C_c(X). \tag{2.49}$$

These ∞-KMS states, when evaluated on the arithmetic algebra $\mathcal{A}_{2,\mathbb{Q}}$, generate specializations of the modular field.

Proposition 2.3.12 ([CM04] Theorem 1.39)**.** *Let $l = (\rho, \tau)$ be an invertible $\mathbb{Q}$-lattice and $\varphi_{\infty,l} \in \mathcal{E}_\infty$ be the corresponding ∞-KMS state. Then $\varphi_{l,\infty}(\mathcal{A}_{2,\mathbb{Q}}) \subset \mathbb{C}$ generates the specialization at τ, F_τ, of the modular field.*

Unlike the original Bost-Connes system, the GL_2-system has a symmetry group that consists of both automorphisms and endomorphisms. In this case, the symmetry group is

$$S = \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_F).$$

The group $\mathrm{GL}_2(\mathbb{A}_F)$ satisfies

$$\mathrm{GL}_2(\mathbb{A}_F) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}})$$

and we shall see that the action of the $\mathrm{GL}_2(\hat{\mathbb{Z}})$ part corresponds to the automorphisms of the system, while the action of the $\mathrm{GL}_2^+(\mathbb{Q})$ corresponds to endomorphisms. We describe the automorphisms first by noting that $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts from the right on the $\mathbb{Q}$-lattices by

$$(\Lambda, \phi) \cdot \gamma = (\Lambda, \phi \circ \gamma) \qquad \forall \gamma \in \mathrm{GL}_2(\hat{\mathbb{Z}}).$$

This action preserves commensurability classes of $\mathbb{Q}$-lattices. The corresponding action on $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ commutes with the left action of $\mathrm{GL}_2(\mathbb{Q})^+$ on $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ given by 2.37. We define automorphisms of $\mathcal{A}_2$ by

$$\theta_\gamma(f)(g, \rho, \tau) = f(g, \rho \circ \gamma, \tau) \tag{2.50}$$

for $f \in \mathcal{A}_2$, and $\gamma \in \mathrm{GL}_2(\hat{\mathbb{Z}})$.

The $\mathrm{GL}_2(\mathbb{Q})^+$ part of the symmetry group acts as endomorphisms as follows. For $m \in M_2(\mathbb{Z})^+$ define $\tilde{m} = \det(m)m^{-1}$. We define an endomorphism on $\mathcal{A}_2$ by the formula

$$\theta_m(f)(g, \rho, \tau) = \begin{cases} f(g, \rho \circ \tilde{m}^{-1}, \tau) & \text{if } \rho \in M_2(\hat{\mathbb{Z}})\tilde{m} \\ 0 & \text{otherwise} \end{cases}. \tag{2.51}$$

The map, $\theta_m(f)$ commutes with the dynamics σ_t, and the action on $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ given $m : (\rho, \tau) \mapsto (\rho \circ \tilde{m}^{-1}, \tau)$ commutes with the left action of $\mathrm{GL}_2(\mathbb{Q})^+$.

Chapter 3

THE BOUNDARY-GL_2-SYSTEM, EQUILIBRIUM STATES, AND LIMITING MODULAR SYMBOLS

In this chapter we outline a construction of a “boundary algebra” for the GL_2-system, endowed with an induced time evolution, that is built on the noncommutative boundary of modular curves described in [MM02] and on limiting modular symbols.

The GL_2-system of [CM06b] is based on a convolution algebra involving Hecke operators and possibly degenerate level structures on modular curves. More precisely, in [CM06b] one considers functions on the set

$$\{(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{Z}) \times \mathbb{H} \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\}$$

that are *invariant* under the action of $\mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ by

$$(\gamma_1, \gamma_2) : (g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z).$$

This algebra is then endowed with a time evolution and covariant representations. A first difficulty in extending the system constructed in this way to the boundary is that on $\mathbb{P}^1(\mathbb{R}) = \partial\mathbb{H}$ the action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformation has dense orbits, hence it no longer makes sense to consider functions that are invariant under this action. Thus, what we consider here as an alternative is to retain the $\mathrm{SL}_2(\mathbb{Z})$ invariance as above in the variables (g, ρ), while replacing the action of $\mathrm{SL}_2(\mathbb{Z})$ on $z \in \mathbb{H}$ by a different action for which, instead of requiring invariance, we introduce as part of the algebra generators that implement that action (which amounts to taking a quotient in the noncommutative sense). The action considered is built out of the partial inverses of the shift of the continued fraction expansion. (Regarding the resulting decoupling of this action and the $\mathrm{GL}_2^+(\mathbb{Q})$-action, see the comments below about isogenies.)

We first extend the original definition of the GL_2-system of [CM06b] to other subgroups of $\mathrm{GL}_2(\mathbb{Q})$ that include the case of $\mathrm{SL}_2(\mathbb{Z})$ and $\mathrm{GL}_2(\mathbb{Z})$. The main requirement for such subgroups $\Gamma \subset \mathrm{GL}_2(\mathbb{Q})$ is to have an associated Hecke algebra $\Xi = \Gamma\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma$. In order to account for some additional structure considered in the setting of limiting modular symbols in [MM02], we also introduce the choice of a finite index subgroup $G \subset \Gamma$ and the coset spaces $\mathbb{P}_\alpha = \Gamma\alpha G/G$.

We focus in particular on a choice of $\Gamma = \Gamma_N$, dependent on an integer $N \in \mathbb{Z} \smallsetminus \{0\}$, consisting of matrices in $\mathrm{GL}_2(\mathbb{Q})$ with determinant in the subgroup of $\mathrm{GL}_1(\mathbb{Q})$

generated by the prime factors of N and -1. The main motivation for this choice is that these subgroups contain certain semigroups associated to an N-dependent family of continued fraction algorithms that we will use in the construction of the boundary systems. This family includes a $\mathrm{GL}_2(\mathbb{Z})$-version of the original GL_2-system as a special case. Moreover, the partition function for these systems has a natural interpretation as the zeta function of the GL_2-system (the zeta function of $\mathbb{P}^1$) with a finite number of Euler factors removed.

We then construct two families of noncommutative algebras (both dependent on the integer parameter N). The first is a family of "bulk algebras" that generalize the GL_2-system of [CM06b], involving the Hecke algebra Ξ_N with a (partially defined) action on the level structure $\rho \in M_2(\hat{\mathbb{Z}})$, the union $\mathcal{P}_N$ of coset spaces $\mathbb{P}_\alpha$, and the half planes $\mathbb{H}^\pm$. The other is a corresponding family of "boundary algebra" that are semigroup crossed products of the algebra of continuous functions from a "disconnection" of the interval $[0,1]$ (in the sense of [Spi93], see also [MM08]) to the restriction of the bulk algebra to the coordinates $(g,\rho,s) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathcal{P}_N$, by a semigroup $\mathrm{Red}_N \subset \Gamma_N$ that implements a family of continued fraction algorithms parameterized by N.

In this construction, in the special case where $N = 1$, the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{H}^\pm$ of the first system is replaced in the second one by the action of the shift T of the usual GL_2-continued fraction algorithm on $[0,1]$. This action on $[0,1]$ is equivalent to the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$, so it is interpreted here as a way to describe pushing the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{H}^\pm$ to the boundary $\mathbb{P}^1(\mathbb{R})$. As mentioned above, we no longer require invariance with respect to this action, and we introduce isometries S_k, associated to the partial inverses of T, to implement the action at the level of the algebra. Note that, by exchanging the $\mathrm{GL}_2(\mathbb{Z})$ action with the semigroup action implemented by the S_k, this action on $[0,1]$ becomes decoupled from the partial action of $\mathrm{GL}_2(\mathbb{Q})$, unlike what happens on the upper half plane. In terms of the original interpretation of the GL_2-system as implementing the commensurability relation on lattices with possibly degenerate level structure, in this boundary setting what remains of the commensurability relation affects the level structures (both through the action on $M_2(\mathbb{Z})$ and on the coset space $\mathcal{P}$) but does not change the pseudolattice $\mathbb{Z}\theta + \mathbb{Z} \subset \mathbb{R}$. The reason behind this choice is the lack (at present) of a good theory of isogeny for noncommutative tori, unlike the notion of isomorphism realized by the bimodules implementing Morita equivalence. This means that, at the level of the noncommutative boundary of the modular curve (which should be thought of as the moduli space of noncommutative tori with level structure), we see the commensurability relation as a relation on level structures. Both the semigroup and the Hecke operators simultaneously act on the cosets in $\mathcal{P}$, with commuting actions.

A more elaborate model of the boundary algebra would require developing a good setting for nontrivial isogenies of noncommutative tori. This can in principle be done by considering a larger class of bimodules that are not imprimitivity bimodules associated to Morita equivalences (for instance, the bimodules constructed in Proposition 5.7 of [LM21]), and select among these the ones that correspond to a good notion of isogenies. While this approach is certainly feasible, it is outside of the narrower scope of the present paper, and will be considered elsewhere.

The case $N = -1$ corresponds to the $\mathrm{SL}_2(\mathbb{Z})$-continued fraction algorithm. The other algebras in our family, for other values of N, do not have the same direct interpretation in terms of the geometry of modular curves as the $N = \pm 1$ cases, because the semigroup Red_N of the continued fraction algorithm sits inside the group Γ_N but will no longer necessarily have, in general, the same orbit structure. The main reason to consider this entire family of algebras is because the structure of KMS states of the resulting quantum statistical mechanical systems becomes more transparent when viewed over this whole family with varying parameter N.

We show that the structure of KMS states depends on that of the Cuntz–Krieger–Toeplitz type system generated by the isometries implementing the continued fraction algorithm and on the generalization of the GL_2-system. For all $N \neq 1$ there is a critical inverse temperature $\beta_{N,c}$ in the interval $(1, 2)$ with the property that no KMS exist for $\beta < \beta_{N,c}$. Above this critical temperature there are as many extremal KMS states as there are for our generalized GL_2-system. In particular, for all $\beta > 2$ all the KMS states are Gibbs states and are parameterized by the representations $\pi_{\rho,x,s}$ with an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. For $\beta \to \infty$ these Gibbs states converge weakly to KMS_∞ states given by evalutation at a point $(1, \rho, x, s)$. In the special case $N = 1$ the KMS states at finite β disappear entirely, due to the fact that in this case the Cuntz–Krieger–Toeplitz part has no KMS states, while only the ground states remain and satisfy a weaker form of the KMS condition. The SL_2-case with $N = -1$ is, in this respect, the nicer in this family of algebras, as it has both the geometric interpretation in terms of modular curves and a nicer structure of KMS states with a convergent partition function in the low-temperature range $\beta > 2$ and Gibbs states converging to the ground states as the temperature goes to zero.

The ground states are the only ones that we need in order to investigate the pairing with limiting modular symbols. So for that purpose we can restrict to the case $N = 1$, which more closely reflects the setting of [MM02]. We introduce a class of "boundary arithmetic elements". These are obtained by first constructing a "boundary value map" which is a linear map from the bulk to the boundary algebra associated to the choice of a cusp form. The same map can be applied to the arithmetic algebra of the bulk system (which as in the original GL_2-case is an algebra of unbounded multipliers consisting of modular functions and Hecke operators). The sub-algebra generated by the images is the arithmetic algebra of the boundary system. The image of the boundary value map consists of elements obtained by a procedure of averaging along geodesics. The evaluation of the ground states on these boundary values therefore agrees with the pairing of cusp forms with limiting modular symbols. This construction reflects the fact that, while the abelian class field theory of imaginary quadratic fields arises from evaluation of modular functions at complex multiplication points in the upper half plane, the corresponding geometry of real multiplication is expected to depend on a suitable averaging along geodesics with endpoints at conjugate quadratic irrationalities in a real quadratic field.

3.1 The modified GL_2-system

In this section we recall the basic properties of the GL_2-quantum statistical mechanical system of [CM06b], in a version that accounts for the choice of a finite index subgroup of $\mathrm{GL}_2(\mathbb{Z})$ and for a more general class of subgroups of $\mathrm{GL}_2(\mathbb{Q})$ that include $\mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$ as special cases.

3.1.1 The GL_2-quantum statistical mechanical system

The GL_2-quantum statistical mechanical system constructed in [CM06b] as a generalization of the Bost–Connes system of [BC95] has algebra of observables given by the non-commutative C^*-algebra describing the "bad quotient" of the space of 2-dimensional $\mathbb{Q}$-lattices up to scaling by the equivalence relation of commensurability. This algebra is made dynamical by a time evolution defined by the determinant of the $\mathrm{GL}_2^+(\mathbb{Q})$ matrix that implements commensurability. There is an arithmetic algebra of unbounded multipliers on the C^*-algebra of observable, which is built in a natural way out of modular functions and Hecke operators (see [CM06b] and Chapter 3 of [CM08]). The KMS-states for the time evolution have an action of $\mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_{\mathbb{Q},f})$ by symmetries, which include both automorphisms and endomorphisms of the C^*-dynamical system. The KMS states at zero temperature, defined as weak limits of KMS-states at positive temperature, are evaluations of modular functions at points in the upper half plane and the induced action of symmetries on KMS-states recovers the Galois action of the automorphisms of the modular field. In the case of imaginary quadratic fields, the associated Bost–Connes system, constructed in [HP05] and [LLN09], can be seen as a specialization of the GL_2-quantum statistical mechanical system of [CM06b] at 2-dimensional $\mathbb{Q}$-lattices that are 1-dimensional $\mathbb{K}$-lattices, with $\mathbb{K}$ the imaginary quadratic field, and to CM points in the upper half plane.

Our goal here is to adapt this construction to obtain a specialization of the GL_2-system to the boundary $\mathbb{P}^1(\mathbb{R})$ and a further specialization for real quadratic fields. Our starting point will be the same algebra of the GL_2-system of [CM06b], hence we start by reviewing briefly that construction in order to use it in our setting. It is convenient, for the setting we consider below, to extend the construction of the GL_2-system recalled above to the case where we replace $\mathrm{GL}_2^+(\mathbb{Q})$ acting on the upper half plane $\mathbb{H}$ with $\mathrm{GL}_2(\mathbb{Q})$ acting on $\mathbb{H}^\pm$ (the upper and lower half planes) and we consider a fixed finite index subgroup G of $\mathrm{GL}_2(\mathbb{Z})$, where the latter replaces $\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \mathrm{GL}_2^+(\mathbb{Q}) \cap \mathrm{GL}_2(\hat{\mathbb{Z}})$ in the construction of the GL_2-system. We can formulate the resulting quantum statistical mechanical system in the following way. We can consider two slightly different versions of the convolution algebra.

Definition 3.1.1. The involutive algebra $\mathcal{A}_G^c$ is given by complex valued functions on

$$\mathcal{U}^\pm = \{(g, \rho, z) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H}^\pm \mid g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{3.1}$$

that are invariant under the action of $G \times G$ by $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z)$, and that have finite support in $g \in G\backslash \mathrm{GL}_2(\mathbb{Q})$, depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through the projection onto some finite level $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ and have compact support in the variable $z \in \mathbb{H}^\pm$. The convolution product on $\mathcal{A}_G^c$ is given by

$$(f_1 \star f_2)(g, \rho, z) = \sum_{h \in G\backslash \mathrm{GL}_2(\mathbb{Q}) \,:\, h\rho \in M_2(\hat{\mathbb{Z}})} f_1(gh^{-1}, h\rho, h(z)) g_2(h, \rho, z) \tag{3.2}$$

and the involution is $f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$. The algebra $\mathcal{A}_G^c$ is endowed with a time evolution given by $\sigma_t(f)(g, \rho, z) = |\det(g)|^{it} f(g, \rho, z)$.

Let Ξ denote the coset space $\Xi = \mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{Q})/\mathrm{GL}_2(\mathbb{Z})$ and let $\mathbb{Z}\Xi$ denote the free abelian group generated by the elements of Ξ. For simplicity of notation we write $\Gamma = \mathrm{GL}_2(\mathbb{Z})$. The following facts are well known from the theory of Hecke operators. For any double coset $T_\alpha = \Gamma\alpha\Gamma$ in Ξ, there are finitely many $\alpha_i \in \Gamma\alpha\Gamma$ such that $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$. Thus, one can define a product on Ξ by setting

$$T_\alpha T_\beta = \sum_\gamma c_{\alpha\beta}^\gamma T_\gamma \tag{3.3}$$

where for $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$ and $\Gamma\beta\Gamma = \sqcup_j \Gamma\beta_j$, the coefficient $c_{\alpha\beta}^\gamma$ counts the number of pairs (i, j) such that $\Gamma\alpha_i\beta_j = \Gamma\gamma$. The ring structure on $\mathbb{Z}\Xi$ determined by the product (3.3) can equivalently be described by considering finitely supported functions $f : \Xi \to \mathbb{Z}$ with the associative convolution product

$$(f_1 \star f_2)(g) = \sum_h f_1(gh^{-1}) f_2(h) \tag{3.4}$$

where the sum is over the cosets Γh with $h \in \mathrm{GL}_2(\mathbb{Q})$ or equivalently over $\Gamma\backslash\mathrm{GL}_2(\mathbb{Q})$. The Hecke operators are built in this form into the algebra of the GL_2-system, through the dependence on the variable $g \in \Gamma\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma$, see the discussion in [CM08], Proposition 3.87.

Coset spaces

We introduce here a variant $\mathcal{A}^c_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ of the GL_2-algebra, where an additional variable is introduced that accounts for the choice of the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ through the coset spaces $\mathbb{P}_\alpha = \mathrm{GL}_2(\mathbb{Z})\alpha G/G$, for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$, which include for $\alpha = 1$ the coset space $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$.

Lemma 3.1.2. *Let $G \subset GL_2(\mathbb{Z})$ be a finite index subgroup such that $\alpha G\alpha^{-1} \cap GL_2(\mathbb{Z})$ is also a finite index subgroup, for all $\alpha \in GL_2(\mathbb{Q})$. Consider the double coset $GL_2(\mathbb{Z})\alpha G$, with the left action of $GL_2(\mathbb{Z})$ and the right action of G. The orbit spaces $\mathbb{P}_\alpha = GL_2(\mathbb{Z})\alpha G/G$ are finite. The algebra $\mathbb{Z}\Xi$ of Hecke operators acts on the module $\mathbb{Z}\mathcal{P}$ with $\mathcal{P} = \cup_\alpha \mathbb{P}_\alpha$.*

Proof. We show that the map $\mathrm{GL}_2(\mathbb{Z}) \to \mathrm{GL}_2(\mathbb{Z})\alpha G$ given by multiplication $\gamma \mapsto \gamma\alpha$ induces a bijection between $\mathrm{GL}_2(\mathbb{Z})/(\alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z}))$ and $\mathrm{GL}_2(\mathbb{Z})\alpha G/G$, hence the orbit space $\mathbb{P}_\alpha = \mathrm{GL}_2(\mathbb{Z})\alpha G/G$ is finite. For $\ell = \alpha g\alpha^{-1} \in \alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z})$ and $\gamma \in \mathrm{GL}_2(\mathbb{Z})$ we have $\gamma\ell\alpha = \gamma\alpha g \sim \gamma\alpha$ in $\Gamma\alpha G/G$ so the map is well defined on equivalence classes. It is injective since two $\gamma, \gamma' \in \Gamma$ with the same image differ by $\gamma'\gamma^{-1} = \alpha\ell\alpha^{-1}$ in $\alpha G\alpha^{-1} \cap \Gamma$ and it is also surjective by construction. The action of the algebra $\mathbb{Z}\Xi$ of Hecke operators is given by the usual multiplication of cosets. We write $\Gamma\alpha\Gamma = \sqcup_i \Gamma\alpha_i$ for finitely many $\alpha_i \in \Gamma\alpha\Gamma$ and $\Gamma\beta G = \sqcup_j \Gamma\beta_j$ for finitely many $\beta_j \in \Gamma\beta G$. The product is then given by

$$\Gamma\alpha\Gamma \cdot \Gamma\beta G = \sum_\gamma c^\gamma_{\alpha\beta} \Gamma\gamma G$$

where c_γ counts the number of pairs (i, j) such that $\Gamma\alpha_i\beta_j = \Gamma\gamma$. □

The condition that $\alpha G\alpha^{-1} \cap \mathrm{GL}_2(\mathbb{Z})$ is also a finite index subgroup, for all $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ is certainly satisfied, for instance, when G is a congruence subgroup.

In terms of generators T_α in the Hecke algebra $\mathbb{Z}\Xi$ and an element $\sum_s a_s\delta_s$ in $\mathbb{Z}\mathcal{P}$, we write the action of $\mathbb{Z}\Xi$ on the module $\mathbb{Z}\mathcal{P}$ as

$$T_\alpha \sum_s a_s\delta_s = \sum_s a_s T_\alpha\delta_s = \sum_s a_s \sum_i c^i_{\alpha,s}\delta_{s_i} = \sum_i (\sum_s a_s c^i_{\alpha,s})\delta_{s_i}. \tag{3.5}$$

We can equivalently write elements of $\mathbb{Z}\mathcal{P}$ as finitely supported functions $\xi : \mathcal{P} \to \mathbb{Z}$, and elements of $\mathbb{Z}\Xi$ as finitely supported functions $f : \Xi \to \mathbb{Z}$, and write the action in the equivalent form

$$(f \star \xi)(s) = \sum_h f(gh^{-1})\xi(hs) \tag{3.6}$$

where the sum is over cosets Γh and for $\xi(s) = \sum_\sigma a_\sigma \delta_\sigma(s)$ we write $\xi(hs)$ as

$$\xi(hs) := \sum_i (\sum_\sigma a_\sigma c^i_{h,\sigma})\delta_{s_i}(s). \tag{3.7}$$

More general subgroups and coset spaces

We will also want to consider a more general family of double coset spaces in order to consider all possible N-continued fraction expansions as described at the beginning of Section 3.2. For $N \in \mathbb{Z}\backslash\{0\}$, Let Ξ_N denote the coset space $\Xi_N = \Gamma_N\backslash\mathrm{GL}_2(\mathbb{Q})/\Gamma_N$, where for $|N| > 1$,

$$\Gamma_N = \{g \in \mathrm{GL}_2(\mathbb{Q}) | \det(g) \in \mathcal{G}_N\}$$

where $\mathcal{G}_N$ is the subgroup of $\mathbb{Q}^\times$ generated by -1 and the prime factors of N.

In the case of $N = 1$ we take $\Gamma_1 := \mathrm{GL}_2(Z)$ and $\Xi_1 = \Xi$, as before. When $N = -1$, we take $\Gamma_{-1} := SL_2(\mathbb{Z})$.

We may also consider a finite index subgroup G of Γ_N and associated orbit spaces $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. The discussion in Lemma 3.1.2 remains the same, where now $c^\gamma_{\alpha\beta}$ counts the number of pairs (i, j) such that $\Gamma_N \alpha_i \beta_j = \Gamma_N \gamma$. We suppress the N subscript when it is clear from context.

To be more concrete, we illustrate here some explicit examples.

The $\mathbf{SL}_2(\mathbb{Z})$ case

In the case of the algebra of the GL_2-system of [CM06b], for an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, the relevant Hecke algebra is $\mathbb{Z}\Xi_{-1} = \mathcal{H}(\Gamma_{-1}, \mathcal{M})$ where $\Gamma_{-1} = \mathrm{SL}_2(\mathbb{Z})$ and the subsemigroup $\mathcal{M} = M_2^+(\mathbb{Z})$ of $\mathrm{GL}_2^+(\mathbb{Q})$.

In this case (see Chapter 4 of [Kri90]) $\mathcal{H}(\Gamma_{-1}, \mathcal{M})$, as an algebra over $\mathbb{Z}$, is generated by the Hecke operators

$$T(\ell) = \sum_{\alpha \in \Gamma_{-1}\backslash\mathcal{M}(\ell)/\Gamma_{-1}} \Gamma_{-1}\alpha\Gamma_{-1} = \sum_{ad=\ell,\, a|d} T(a, d),$$

with $\mathcal{M}(\ell) = \{\alpha \in \mathcal{M} \mid \det(\alpha) = \ell\}$ and

$$T(a, d) = \Gamma_{-1} \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \Gamma_{-1},$$

subject to the relations, for $k, \ell \in \mathbb{N}$,

$$T(\ell)T(k) = \sum_{d|\gcd\{k,\ell\}} d\, T(d,d)\, T(\frac{k\ell}{d^2}).$$

Equivalently, the Hecke algebra $\mathcal{H}(\Gamma_{-1}, \mathcal{M})$ splits into primary components

$$\mathcal{H}(\Gamma_{-1}, \mathcal{M}) = \otimes_p \mathcal{H}(\Gamma_{-1}, \mathcal{M})_p,$$

over the set of primes p, where $\mathcal{H}(\Gamma_{-1}, \mathcal{M})_p = \mathbb{Z}[T(p), T(p,p)]$.

This description of the Hecke algebra is obtained directly from the following properties of right cosets and double cosets (Chapter 4 of [Kri90]). Given $\alpha \in \mathcal{M}$, the right coset $\Gamma_{-1}\alpha$ contains a unique representative of the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{with } a, d \in \mathbb{N},\ 0 \leq b < d.$$

The set $\mathcal{M}(\ell)$ decomposes as a disjoint union of $\sigma_1(\ell) = \sum_{d|\ell} d$ right Γ_{-1}-cosets, with a set of representatives given by the matrices

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \quad \text{with } d \in \mathbb{N},\ 0 \leq b < d,\ a = \ell/d.$$

Given $\alpha \in \mathcal{M}$ there are $\gamma_1, \gamma_2 \in \Gamma_{-1}$ and $a, d \in \mathbb{N}$ with $a|d$ such that

$$\gamma_1 \alpha \gamma_2 = \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}.$$

The $\mathbf{GL}_2(\mathbb{Z})$ case

Here we consider also the case where $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ and $\mathcal{M} = M_2(\mathbb{Z}) \cap \mathrm{GL}_2(\mathbb{Q})$ is the subsemigroup of $\mathrm{GL}_2(\mathbb{Q})$. In this case the explicit description of $\mathbb{Z}\Xi = \mathcal{H}(\Gamma, \mathcal{M})$ is similar to the previous case (see Chapter 5 of [Kri90]), and $\mathcal{H}(\Gamma, \mathcal{M})$ is generated by the Hecke operators

$$T(\ell) = \sum_{\alpha \in \Gamma \backslash \mathcal{M}(\ell)/\Gamma} \Gamma\alpha\Gamma,$$

where here $\mathcal{M}(\ell) = \{\alpha \in M_2(\mathbb{Z}) \,|\, |\det \alpha| = \ell\}$. The Hecke algebra splits into primary components as in the previous case. We refer the reader to [Kri90] for more details.

The case with congruence subgroups

In the case where we also consider a choice of a non-trivial congruence subgroup $G \subset \Gamma$, with a Hecke algebra $\mathbb{Z}\Xi = \mathcal{H}(\Gamma, \mathcal{M})$ as in the previous cases (see Section 2.7 of [Miy06]), we can identify the $\mathbb{Z}$-module $\mathbb{Z}\mathcal{P}$ with

$$\mathbb{Z}\mathcal{P} = \mathbb{Z}[\Gamma \backslash \mathcal{M}]^G,$$

with the identification induced by the injective homomorphism of $\mathbb{Z}$-modules

$$\phi : \mathbb{Z}\mathcal{P} \to \mathbb{Z}[\Gamma\backslash\mathcal{M}], \quad \phi(\Gamma\alpha G) = \sum_i \Gamma\alpha_i,$$

for $\Gamma\alpha G = \sqcup_i \Gamma\alpha_i$ a decomposition into right-cosets. We can then write the action of $\mathbb{Z}\Xi$ on $\mathbb{Z}\mathcal{P}$ described above in (3.6), (3.7) in the form

$$\Gamma\beta\Gamma \cdot \xi = \sum_\alpha a_\alpha \; \Gamma\alpha\beta_j,$$

where

$$\xi = \sum_\alpha a_\alpha \; \Gamma\alpha$$

is a G-invariant element in $\mathbb{Z}[\Gamma\backslash\mathcal{M}]$ and $\Gamma\beta\Gamma = \sqcup_j \Gamma\beta_j$. The action is independent of the choice of representatives (Lemma 2.7.3 of [Miy06]).

Thus, for general elements $h \in \mathbb{Z}\Xi$ and $\xi \in \mathbb{Z}\mathcal{P}$ with $h = \sum_\beta b_\beta \, \Gamma\beta\Gamma$ and $\xi = \sum_\alpha a_\alpha \Gamma\alpha G$, we reformulate (3.6), (3.7) as

$$h \star \xi = \sum_{\alpha,\beta,\gamma} b_\beta \, a_\alpha \, c^\gamma_{\alpha,\beta} \Gamma\gamma G,$$

see (2.7.3) of [Miy06].

The bulk algebra

We now proceed to the construction of a "bulk algebra" (namely, the algebra associated to the bulk space $\mathbb{H}$), which includes the choice of a finite index subgroup $G \subset \Gamma_N$.

Definition 3.1.3. Let $G \subset \Gamma_N$ be a finite index subgroup and let $\mathcal{P}_N = \cup_\alpha \mathbb{P}_{N,\alpha}$ with $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$ for $\alpha \in \mathrm{GL}_2(\mathbb{Q})$. The involutive algebra $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ is given by complex valued functions on the space

$$\mathcal{U}^\pm_{G,\mathcal{P}_N} = \{(g,\rho,z,\xi) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H}^\pm \times \mathcal{P}_N \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\}, \tag{3.8}$$

that are invariant under the action of $\Gamma_N \times \Gamma_N$ by

$$(g,\rho,z,s) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2 z, \gamma_2 s).$$

Moreover, functions in $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ have finite support in $\Gamma_N\backslash\mathrm{GL}_2(\mathbb{Q})$ and in $\mathcal{P}_N$, compact support in $z \in \mathbb{H}^\pm$, and they depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through the projection onto some finite level $p_n : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/n\mathbb{Z})$. The convolution product of $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ is given by

$$(f_1 \star f_2)(g,\rho,z,s) = \sum_{h \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}, h\rho, h(z), hs) f_2(h,\rho,z,s), \tag{3.9}$$

where we are using the notation (3.6), (3.7) for the action of Hecke operators on functions of $\mathcal{P}_N$ and $\mathcal{S}_{\rho,N}$ is the collection of Γ_N-cosets that have some representative element in the set $\{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\}$. In particular, when $N = 1$, $\mathcal{S}_\rho = \mathrm{GL}_2(\mathbb{Z}) \backslash \{g \in \mathrm{GL}_2(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\}$.

The involution is $f^*(g, \rho, z, s) = \overline{f(g^{-1}, g\rho, g(z), gs)}$. The algebra $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ is endowed with a time evolution given by

$$\sigma_t(f)(g, \rho, z, s) = |\det(g)|^{it} f(g, \rho, z, s).$$

We focus here on the algebra $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ and we construct Hilbert space representations analogous to the ones considered for the original GL_2-system.

Consider then the Hilbert space $\mathcal{H}_{\rho,N} = \ell^2(\mathcal{S}_{\rho,N})$, and the representations $\pi_{(\rho,z,s)} : \mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N} \to \mathcal{B}(\mathcal{H}_{\rho,N})$

$$\pi_{(\rho,z,s)}(f)\xi(g) = \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, h(z), hs)\xi(h).$$

We can complete the algebra $\mathcal{A}^c_{\Gamma_N,G,\mathcal{P}_N}$ to a C^*-algebra $\mathcal{A}_{\Gamma_N,G,\mathcal{P}_N}$ in the norm $\|f\| = \sup_{(\rho,z,s)} \|\pi_{(\rho,z,s)}(f)\|_{\mathcal{B}(\mathcal{H}_{\rho,N})}$. The time evolution is implemented in the representation $\pi_{(\rho,z,s)}$ by the Hamiltonian $H\xi(g) = \log|\det(g)|\,\xi(g)$.

3.1.2 The arithmetic algebra

We proceed exactly as in the case of the GL_2-system of [CM06b] to construct an arithmetic algebra associated to $\mathcal{A}_{\Gamma_N,G,\mathcal{P}_N}$. As in [CM06b] this will not be a subalgebra but an algebra of unbounded multipliers.

The arithmetic algebra $\mathcal{A}^{ar}_{\Gamma_N,G,\mathcal{P}_N}$ is the algebra over $\mathbb{Q}$ obtained as follows. We consider functions on $\mathcal{U}^{\pm}_{G,\mathcal{P}_N}$ of (3.8) that are invariant under the action of $\Gamma_N \times \Gamma_N$ by $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2 z)$ and that are finitely supported in $g \in G\backslash\mathrm{GL}_2(\mathbb{Q})$ and in $\mathcal{P}_N$, that depend on ρ through some finite level projection $p_n(\rho) \in M_2(\mathbb{Z}/n\mathbb{Z})$ and that are holomorphic in the variable $z \in \mathbb{H}$ and satisfy the growth condition that $|f(g, \rho, z, s)|$ is bounded by a polynomial in $\max\{1, \Im(z)^{-1}\}$ when $\Im(z) \to \infty$. The resulting algebra $\mathcal{A}^{ar}_{\Gamma_N,G,\mathcal{P}_N}$ acts, via the convolution product (3.9), as unbounded multipliers on the algebra $\mathcal{A}_{\Gamma_N,G,\mathcal{P}_N}$. This construction and its properties are completely analogous to the original case of the GL_2-system described in Section 2.3.2 and we refer the reader to [CM06b], [CM08] for details.

The invariance property ensures that these are modular functions for G (written as Γ_N-invariant functions on $\mathbb{H} \times \mathbb{P}_N$ rather than as G-invariant functions on $\mathbb{H}$). These functions are endowed with the same convolution product (3.9).

3.2 Boundary GL_2-system

We now consider how to extend this setting to incorporate the boundary $\mathbb{P}^1(\mathbb{R})$ of the upper half-plane $\mathbb{H}$ and $\mathbb{Q}$-pseudolattices generalizing the $\mathbb{Q}$-lattices of [CM06b]. A brief discussion of the boundary compactification of the GL_2-system was given in §7.9 of [CM08], but the construction of a suitable quantum statistical mechanical system associated to the boundary was never worked out in detail.

We replace the full $\mathbb{P}^1(\mathbb{R})$ boundary of $\mathbb{H}^{\pm}$ with the smaller interval $[0,1]$. In the $N = 1$ case this choice is natural as this interval meets every orbit of the $\mathrm{GL}_2(\mathbb{Z})$ action and the equivalence relation given by this action can be described equivalently through the shift T of the continued fraction expansion. This action can be implemented via the semigroup of reduced matrices in the form of a crossed product algebra. Inspired by this case, we adopt the same setting for the whole family of algebras parameterized by the nontrivial integer N, with corresponding continued fraction algorithms on the interval $[0,1]$ and associated semigroups. We analyze Hilbert space representations, time evolution, Hamiltonian, partition function, and KMS states.

3.2.1 Continued fraction algorithms

We consider the countable family of N-continued fraction expansions given by

$$[a_0; a_1, a_2, a_3, ...]_N = a_0 + \cfrac{N}{a_1 + \frac{N}{a_2 + \frac{N}{a_3 + ...}}} \tag{3.10}$$

with $a_i \geq N$ when $N \geq 1$ and $a_i \geq |N| + 1$ when $N \leq -1$. We denote the set of allowed digits of the N-continued fraction expansion by Φ_N,

$$\Phi_N = \begin{cases} \mathbb{N}_{\geq N} & \text{when } N \geq 1 \\ \mathbb{N}_{\geq |N|+1} & \text{when } N \leq -1 \end{cases}. \tag{3.11}$$

where we write $\mathbb{N}_{\geq N} := \{n \in \mathbb{N} \,|\, n \geq N\}$.

For each N-continued fraction expansion, we introduce an algebra associated to the boundary $\mathbb{P}^1(\mathbb{R})$ with the action of a certain subsemigroup of $\mathrm{GL}_2(\mathbb{Q})$, called the semigroup of reduced matrices, depending on the choice of N. In the case that $N = 1$ this semigroup of reduced matrices is contained in $\mathrm{GL}_2(\mathbb{Z})$ and in the case that $N = -1$ it is contained in $\mathrm{PSL}_2(\mathbb{Z})$. In the $N = \pm 1$ cases, the associated algebra can be interpreted as a boundary algebra of the GL_2-system. While we have no similar direct geometric interpretation when $|N| > 1$, considering the whole family of systems leads to some interesting observations about the structure of the KMS states.

3.2.2 Boundary dynamics and coset spaces

The N-continued fraction expansion of a real number x can be retrieved via the shift operator $T_N : [0,1] \to [0,1]$ given by

$$T_N(x) = \frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \quad x \neq 0; \qquad T_N(0) = 0. \tag{3.12}$$

For $x \in [0,1)$, one has that $a_0 = 0$ and $a_i = \left\lfloor \frac{N}{T_N^{i-1}(x)} \right\rfloor$ in the case that $N \geq 1$, and $a_0 = 1$ and $a_i = -\left\lfloor \frac{N}{T_N^{i-1}(1-x)} \right\rfloor$ in the case that $N \leq -1$.

Note that for $N \geq 1$, $\lfloor \frac{N}{x} \rfloor = n$ if and only if $\frac{N}{n+1} < x \leq \frac{N}{n}$, so that we can write explicitly

$$T_N(x) = \begin{cases} 0 & \text{if } x = 0 \\ \frac{1}{x} - n & \text{if } \frac{N}{n+1} < x \leq \frac{N}{n} \end{cases} \qquad \text{for } n \in \mathbb{N}.$$

A similar formula holds in the $N \leq -1$ case. In either case, the shift map T_N has discontinuities at rational numbers, and is otherwise continuous and decreasing on each branch.

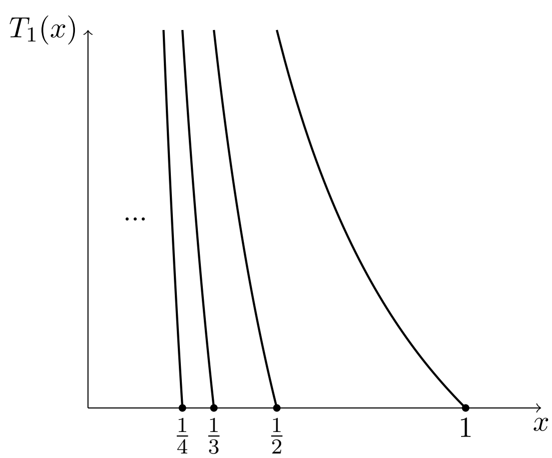

FIGURE 3.1: GRAPH OF THE SHIFT MAP $T_1(x)$

We extend T_N to a map on $[0,1] \times \mathbb{P}$ by

$$T_N : (x,s) \mapsto \left(\frac{N}{x} - \left\lfloor \frac{N}{x} \right\rfloor, \begin{pmatrix} -\lfloor N/x \rfloor & N \\ 1 & 0 \end{pmatrix} \cdot s \right). \tag{3.13}$$

We remark that in the geometrically meaningful case of $N = 1$, the set $[0,1] \times \mathbb{P}$ meets every orbit of the action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$, acting on $\mathbb{P}^1(\mathbb{R})$ by fractional linear transformations and on $\mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$ by the left-action of $\mathrm{GL}_2(\mathbb{Z})$ on itself. Moreover, two points (x, s) and (y, t) in $[0, 1] \times \mathbb{P}$ are in the same $\mathrm{GL}_2(\mathbb{Z})$-orbit if and only if there are integers $n, m \in \mathbb{N}$ such that $T_1^n(x, s) = T_1^m(y, t)$.

Lemma 3.2.1. *The action of the shift map* (3.13) *on* $[0, 1] \times \mathbb{P}$ *extends to an action on* $[0, 1] \times \mathbb{P}_{N,\alpha}$*, with* $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$*, for any given* $\alpha \in GL_2(\mathbb{Q})$*, hence to an action on* $[0, 1] \times \mathcal{P}_N$ *with* $\mathcal{P}_N = \cup_\alpha \mathbb{P}_{N,\alpha}$.

Proof. The action of T_N on $(x, s) \in [0, 1] \times \mathbb{P}$ is implemented by the action of the matrix

$$\begin{pmatrix} -\lfloor N/x \rfloor & N \\ 1 & 0 \end{pmatrix} \in \Gamma_N.$$

The same matrix acts by left multiplication on $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$, hence it determines a map $T_N : [0, 1] \times \mathbb{P}_{N,\alpha} \to [0, 1] \times \mathbb{P}_{N,\alpha}$. □

3.2.3 Disconnection algebra

We recall here from [Spi93] (see also [MM08]) the construction of the disconnection algebra of $\mathbb{P}^1(\mathbb{R})$ along $\mathbb{P}^1(\mathbb{Q})$ and its restriction to $[0, 1]$.

Given a subset $B \subset \mathbb{P}^1(\mathbb{R})$ one considers the abelian C^*-algebra $\mathcal{A}_B$ generated by the algebra $C(\mathbb{P}^1(\mathbb{R}))$ and the characteristic functions of the positively oriented intervals with endpoints in B. If the set U is dense in $\mathbb{P}^1(\mathbb{R})$ then the algebra obtained in this way can be identified with the norm closure of the $*$-algebra generated by these characteristic functions. By the Gelfand–Naimark correspondence, the C^*-algebra $\mathcal{A}_B$ is the algebra of continuous functions on a compact Hausdorff topological space, $\mathcal{A}_B \simeq C(\mathcal{D}_B)$. We refer to this space $\mathcal{D}_B$ as the *disconnection of* $\mathbb{P}^1(\mathbb{R})$ *along* B. The space $\mathcal{D}_B$ is totally disconnected if and only if B is dense in $\mathbb{P}^1(\mathbb{R})$.

In particular, the disconnection $\mathcal{D}_{\mathbb{P}^1(\mathbb{Q})}$ of $\mathbb{P}^1(\mathbb{R})$ along $\mathbb{P}^1(\mathbb{Q})$ can be identified with the ends of the tree of $\mathrm{PSL}_2(\mathbb{Z})$ embedded in the hyperbolic plane $\mathbb{H}$ (see the discussion in §5 of [MM08]).

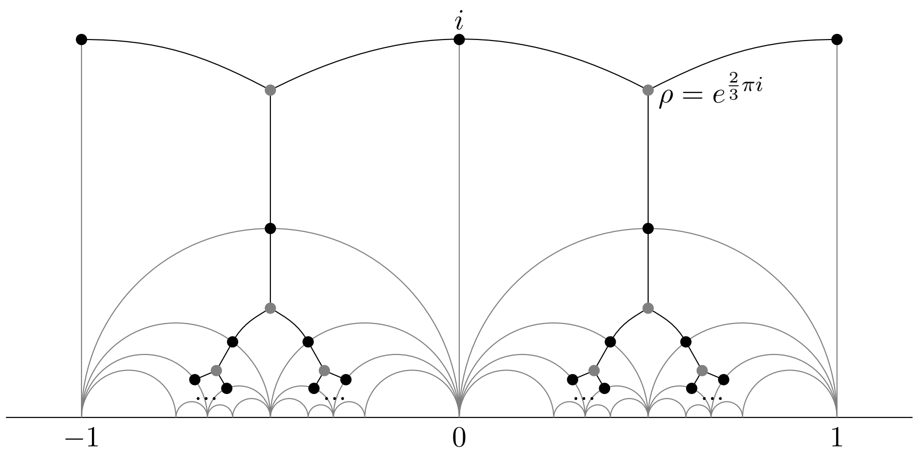

FIGURE 3.2: THE TREE OF $\mathrm{PSL}_2(\mathbb{Z})$ EMBEDDED IN THE HYPERBOLIC PLANE $\mathbb{H}$

In our setting, since the Gauss map of the continued fraction algorithms we are considering has discontinuities, which occur at rational points, we need to work with an algebra of continuous functions over a disconnection of the interval $[0,1]$ at the rationals. The algebra $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$ of the disconnection $\mathcal{D}_{[0,1]\cap\mathbb{Q}}$ of $[0,1]$ along the rational points $[0,1]\cap\mathbb{Q}$ is the image of $C(\mathcal{D}_{\mathbb{P}^1(\mathbb{Q})})$ under the projection given by the characteristic function $\chi_{[0,1]}$ of the interval, which is a continuous function in $C(\mathcal{D}_{\mathbb{P}^1(\mathbb{Q})})$ by construction.

Lemma 3.2.2. *The action*

$$f \mapsto \chi_{X_{N,k}} \cdot f \circ g_{N,k}^{-1} \quad \text{and} \quad \tilde{f} \mapsto f \circ g_{N,k}\,, \tag{3.14}$$

with

$$g_{N,k} = \begin{pmatrix} 0 & N \\ 1 & k \end{pmatrix} \quad \text{and} \quad g_{N,k}^{-1} = \begin{pmatrix} -\frac{k}{N} & 1 \\ \frac{1}{N} & 0 \end{pmatrix}, \tag{3.15}$$

is well defined on $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$.

Proof. This is immediate from (3.11), (3.12), (3.13), but since some readers appeared to be confused about it, we spell it out in full. Indeed, for $x \in [0,1]$, we have

$$g_{N,k}(x) = \frac{N}{x+k} \in [0,1]$$

since $k \geq N$ by (3.11), so $f \circ g_{N,k}$ is still a function in $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$, while for $x \in X_{N,k}$ we have

$$g_{N,k}^{-1}(x) = \frac{-kx+N}{x} \in [0,1]\,,$$

because $X_{N,k}$ is the set of those x for which $k = \lfloor \frac{N}{x} \rfloor$, so that $k \leq N/x \leq k+1$. Thus, even though $f \circ g_{N,k}^{-1}$ is not necessarily in $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$ the product $\chi_{X_{N,k}} \cdot f \circ g_{N,k}^{-1}$ is in $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$. $\square$

Disconnection algebra and coset spaces

We incorporate the coset spaces in the construction of the disconnection algebra in the following way.

Lemma 3.2.3. *Let $\mathbb{C}\mathcal{P}_N$ denote the ring of finitely supported complex valued functions on $\mathcal{P}_N$ and let $\mathcal{B}_N = C(\mathcal{D}_{[0,1]\cap\mathbb{Q}}, \mathbb{C}\mathcal{P}_N)$ denote the algebra of continuous functions from the disconnection of $[0,1]$ at the rationals to $\mathbb{C}\mathcal{P}_N$. Consider the action of the semigroup $\mathbb{Z}_+$ on $\mathcal{B}_N$ determined by the action of T_N on $[0,1]\times\mathcal{P}_N$ of Lemma 3.2.1 and the action on $\mathcal{B}_N$ by Hecke operators acting on $\mathbb{C}\mathcal{P}_N$. These two actions commute.*

Proof. We write functions $f \in \mathcal{B}_N$ in the form $\sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha$ where δ_α is the characteristic function of $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$ and $s_\alpha \in \mathbb{P}_{N,\alpha}$. The action of $\mathbb{Z}_+$ is given by

$$T_N^n : \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha \mapsto \sum_\alpha f_\alpha(T_N^n(x, s_\alpha))\delta_\alpha,$$

with $T_N(x, s_\alpha)$ as in (3.13), while the action of a Hecke operator T_β is given by

$$T_\beta : \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha \mapsto \sum_\gamma (\sum_\alpha c^\gamma_{\beta,\alpha} f_\alpha(x, s_\alpha))\delta_\gamma, \tag{3.16}$$

with $c^\gamma_{\beta\alpha}$ defined as in (3.5), modified appropriately for the choice of N. It is then clear that these two actions commute. □

Lemma 3.2.4. *Let $X_{N,k} \subset [0,1]$ be the subset of points $x \in [0,1]$ with N-continued fraction expansion starting with the digit $k \in \Phi_N$. Let $\mathcal{B}_N$ denote the algebra of continuous complex valued functions on $\mathcal{D}_{[0,1]\cap\mathbb{Q}} \times \mathcal{P}_N$. Let $\tau_N(f) = f \circ T_N$ denote the action of the shift $T_N : [0,1] \to [0,1]$ of the N-continued fraction expansion on $f \in \mathcal{B}_N$.*

Let $\mathcal{B}_N$ act as multiplication operators on $L^2([0,1], d\mu_N)$ with $d\mu_N$ the T_N-invariant measures on $[0,1]$,

$$d\mu_N(x) = \begin{cases} \left(\log \frac{N+1}{N}\right)^{-1} (N+x)^{-1}\, dx & \text{if } N \in \mathbb{Z}\backslash\{0,-1\} \\ (1-x)^{-1}\, dx & \text{if } N = -1 \end{cases}.$$

With the notation (3.15), consider the operators

$$S_{N,k}\xi(x) = \chi_{X_{N,k}}(x) \cdot \xi(g^{-1}_{N,k} x) \qquad \text{and} \qquad \tilde{S}_{N,k}\xi(x) = \xi(g_{N,k}\, x), \tag{3.17}$$

for $\xi \in L^2([0,1], d\mu_N)$, with $\chi_{X_{N,k}}$ the characteristic function of the subset $X_{N,k} \subset [0,1]$.

These satisfy $\tilde{S}_{N,k} = S^*_{N,k}$ *with* $S^*_{N,k} S_{N,k} = 1$ *and* $\sum_k S_{N,k} S^*_{N,k} = 1$. *They also satisfy the relation*

$$\sum_k S_{N,k}\, f\, S^*_{N,k} = f \circ T_N. \tag{3.18}$$

Proof. The shift map of the continued fraction expansion, given by $T_N(x) = N/x - [N/x]$, acts on $x \in X_{N,k}$ as $x \mapsto g^{-1}_{N,k} x$ with the matrix $g^{-1}_{N,k}$ acting by fractional linear transformations. The operators $S_{N,k}$ defined as in (3.17) are not isometries on $L^2([0,1], dx)$ with respect to the Lebesgue measure dx. However, if we consider the T_N-invariant probability measures $d\mu_N$, then we have $d\mu_N \circ g^{-1}_{N,k}|_{X_{N,k}} = d\mu|_{X_{N,k}}$ for all $k \in \mathbb{N}$, hence

$$\begin{aligned}\langle S_{N,k}\xi_1, S_{N,k}\xi_2\rangle &= \int_{X_{N,k}} \bar{\xi}_1 \circ g^{-1}_{N,k}\, \xi_2 \circ g^{-1}_{N,k}\, d\mu_N \\ &= \int_{X_{N,k}} \bar{\xi}_1 \circ g^{-1}_{N,k}\, \xi_2 \circ g^{-1}_{N,k}\, d\mu_N \circ g^{-1}_{N,k} = \int_{[0,1]} \bar{\xi}_1\, \xi_2\, d\mu_N = \langle \xi_1, \xi_2\rangle.\end{aligned}$$

We have $\tilde{S}_{N,k}\, S_{N,k}\xi(x) = \xi(x)\chi_{X_{N,k}}(g_{N,k}x) = \xi(x)$. Moreover, $\tilde{S}_{N,k} = S^*_{N,k}$ in this inner product since we have

$$\begin{aligned}\langle \xi_1, S_{N,k}\xi_2\rangle &= \int_{X_{N,k}} \bar{\xi}_1\, \xi_2 \circ g^{-1}_{N,k}\, d\mu_N \\ &= \int_{X_{N,k}} \bar{\xi}_1\, \xi_2 \circ g^{-1}_{N,k}\, d\mu_N \circ g^{-1}_{N,k} = \int_{[0,1]} \bar{\xi}_1 \circ g_{N,k}\, \xi_2\, d\mu_N = \langle \tilde{S}_{N,k}\xi_1, \xi_2\rangle.\end{aligned}$$

Using Lemma 3.2.2, we also have $\sum_k S_{N,k}\, f\, S^*_{N,k}\xi(x) = \sum_k f(g^{-1}_{N,k}x)\chi_{X_{N,k}}(x)\xi(x) = f(T_N(x))$. Thus we obtain $\sum_k S_{N,k}\, f\, S^*_{N,k} = f \circ T_N$, which in particular also implies $\sum_k S_{N,k}\, S^*_{N,k} = 1$. ∎

3.2.4 Semigroups

Consider the set of matrices in $\mathrm{GL}_2(\mathbb{Q})$

$$\mathrm{Red}_{N,n} := \begin{cases} \left\{ \begin{pmatrix} 0 & N \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & k_n \end{pmatrix} | k_i \in \mathbb{Z}_{\geq N} \right\} & \text{if } N \geq 1 \\ \\ \left\{ \begin{pmatrix} 0 & N \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & k_n \end{pmatrix} | k_i \in \mathbb{Z}_{\geq |N|+1} \right\} & \text{if } N \leq -1 \end{cases} \tag{3.19}$$

Note that $\mathrm{Red}_{N,n} \subset \Gamma_N$ and in particular when $N = 1$, $\mathrm{Red}_{1,n} \subset \mathrm{GL}_2(\mathbb{Z})$, and when $N = -1$, $\mathrm{Red}_{-1,n} \subset \mathrm{SL}_2(\mathbb{Z})$.

The semigroups of reduced matrices are defined as

$$\mathrm{Red}_N := \cup_{n\geq 1} \mathrm{Red}_{N,n}. \tag{3.20}$$

An equivalent description of the Red_1 semigroup is given by ([LZ97])

$$\mathrm{Red}_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}) \,|\, 0 \leq a \leq b,\, 0 \leq c \leq d \right\}.$$

Lemma 3.2.5. *Assigning to a matrix*

$$\gamma = \begin{pmatrix} 0 & N \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & N \\ 1 & n_k \end{pmatrix} \tag{3.21}$$

in Red_N the product $n_1 \cdots n_k \in \mathbb{N}$ is a well-defined semigroup homomorphism.

Proof. We only need to check that the representation of a matrix γ in Red_N as a product (3.21) is unique so that the map is well defined. It is then by construction a semigroup homomorphism.

First we consider the $N = 1$ case. The group $\mathrm{GL}_2(\mathbb{Z})$ has generators

$$\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \qquad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

with relations $(\sigma^{-1}\rho)^2 = (\sigma^{-2}\rho^2)^6 = 1$. The semigroup Red_1 can be equivalently described as the subsemigroup of the semigroup generated by σ and ρ made of all the words in σ, ρ that end in ρ, so elements are products of matrices of the form $\sigma^{n-1}\rho = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$. We have

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & n_{\ell(\gamma)} \end{pmatrix}$$

where $\ell(\gamma)$ is the number of ρ's in the word in σ and ρ representing γ. The semigroup generated by σ and ρ is a free semigroup, as the only relations in $\mathrm{GL}_2(\mathbb{Z})$ between these generators involve the inverse σ^{-1}. If an element $\gamma \in \mathrm{Red}_1$ had two different representations (3.21), for two different ordered sets $\{n_1, \ldots, n_k\}$ and $\{m_1, \ldots, m_l\}$ then we would have a relation

$$\sigma^{n_1-1}\rho\sigma^{n_2-1}\rho \cdots \sigma^{n_k-1}\rho = \sigma^{m_1-1}\rho\sigma^{m_2-1}\rho \cdots \sigma^{m_l-1}\rho$$

involving the generators σ and ρ but not their inverses, which would contradict the fact that σ and ρ generate a free semigroup.

Next we consider the case $N \in \mathbb{Z}\backslash\{-1,0,1\}$. We observe that we can decompose elements of Red_N in terms of ρ, σ and $\eta_N = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix}$, a diagonal matrix depending on N since

$$\begin{pmatrix} 0 & N \\ 1 & n \end{pmatrix} = \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} = \eta_N \sigma^{n-1} \rho.$$

If an element $\gamma \in \mathrm{Red}_N$ for $|N| > 1$ had two different representations (3.21), for two different ordered sets $\{n_1, \ldots, n_k\}$ and $\{m_1, \ldots, m_l\}$ then we would have a relation

$$\eta_N \sigma^{n_1-1} \rho \eta_N \sigma^{n_2-1} \rho \cdots \eta_N \sigma^{n_k-1} \rho = \eta_N \sigma^{m_1-1} \rho \eta_N \sigma^{m_2-1} \rho \cdots \eta_N \sigma^{m_l-1} \rho.$$

As before, there are no relations between ρ and σ. There cannot be a relation involving η_N and ρ and σ. If we had $\mathrm{word}(\eta_N, \rho, \sigma) = 1$ then the determinant of the left-hand side would be $\pm N^r$ where r is the number of times η_N appears in the word, while the determinant of the right-hand side would be 1. Since we are in the case $|N| > 1$, this is a contradiction.

Finally we consider the $N = -1$ case. $\mathrm{PSL}_2(\mathbb{Z})$ can be written as a free product of cyclic groups

$$\mathrm{PSL}_2(\mathbb{Z}) \simeq C_2 \star C_3$$

with generators

$$B = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix} \tag{3.22}$$

of degree 2 and 3 respectively ($B^2 = 1$ and $C^3 = 1$). We can write a matrix in Red_{-1}

$$\gamma = \begin{pmatrix} 0 & -1 \\ 1 & n_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ 1 & n_{\ell(\gamma)} \end{pmatrix}$$

in terms of these generators by noting that in $\mathrm{PSL}_2(\mathbb{Z})$,

$$\begin{pmatrix} 0 & -1 \\ 1 & n \end{pmatrix} = B(CB^{-1})^n$$

and hence

$$\begin{aligned} \gamma &= B(CB^{-1})^{n_1} \cdots B(CB^{-1})^{n_{\ell(\gamma)}} \\ &= B(CB^{-1})^{n_1-1} C^2 B^{-1} (CB^{-1})^{n_2-1} C^2 B^{-1} \ldots C^2 B^{-1} (CB^{-1})^{n_{\ell(\gamma)}-1}. \end{aligned} \tag{3.23}$$

Since each $n_i \geq 2$, this is a reduced sequence of words in C_2 and C_3. Every element in a free product can be written uniquely as a reduced sequence of words. Furthermore, each element of the cyclic groups C_2 and C_3 can be written uniquely as B^k or C^k where k is required to be either positive or negative. The form (3.23) is unique. If an element $\gamma \in \mathrm{Red}_{-1}$ had two different expressions of the form (3.21), it would contradict this uniqueness.

□

Lemma 3.2.6. *Let $\mathcal{B}_N$ denote the algebra of continuous complex valued functions on $\mathcal{D}_{[0,1]\cap\mathbb{Q}} \times \mathcal{P}_N$ that are finitely supported in $\mathcal{P}_N$. The transformations $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ for $\gamma \in Red_N$ define a semigroup action of Red_N on $\mathcal{B}_N$. This action commutes with the action of Hecke operators.*

Proof. We check that $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ is a well-defined semigroup action of Red_N on $\mathcal{B}_N$. For γ of the form (3.21) we have $\alpha_\gamma = \alpha_{g_1} \cdots \alpha_{g_n}$ with the factors $g_i = g_{N,k_i}$ as in (3.15), since for two matrices γ, γ' in Red_N related by $\gamma' = g_{N,k}\gamma$ for some $g_{N,k}$ as in (3.15) we have $\chi_{X_k} \cdot \chi_{X_{\gamma'}} \circ g_{N,k}^{-1} = \chi_{X_\gamma}$.

The commutation with the action of Hecke operators can be checked as in the case of the shift T_N in Lemma 3.2.3. We write elements of the algebra in the form $\sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha$ where δ_α is the characteristic function of $\mathbb{P}_{N,\alpha} = \Gamma_N \alpha G/G$ and $s_\alpha \in \mathbb{P}_{N,\alpha}$, with the action of Hecke operators as in (3.16). The action of $\gamma \in \mathrm{Red}_N$ on the other hand is given by $\alpha_\gamma \sum_\alpha f_\alpha(x, s_\alpha)\delta_\alpha = \sum_\alpha \chi_{X_\gamma}(x)\, f(\gamma^{-1}(x, s_\alpha))\delta_\alpha$. These actions commute, as in the case of Lemma 3.2.3. □

3.2.5 A boundary algebra

We now introduce an algebra associated to the boundary of the bulk-system. In order to explain the reason behind our construction, consider first again the bulk space, namely the upper-half-plane $\mathbb{H}$ or $\mathbb{H} \times \mathcal{P}$ in the case where we fix a choice of a finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$.

In the algebra of the $\Gamma = \mathrm{GL}_2$-system on the bulk space, we consider functions $f(g, \rho, z)$ that are invariant under the action of $\Gamma \times \Gamma$ mapping

$$(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho, \gamma_2 z)$$

(and similarly for the $\mathbb{H} \times \mathcal{P}$ case). This same prescription cannot be used to define a boundary algebra, since the action of $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ (or $\mathrm{SL}_2(\mathbb{Z})$) on the boundary $\mathbb{P}^1(\mathbb{R}) = \partial\mathbb{H}$ has dense orbits, hence requiring this $\Gamma \times \Gamma$-invariance would force continuous functions to be constant.

One possible way around this problem would be to replace invariance under the $\Gamma\times\Gamma$-action (in fact, invariance under the second copy of Γ, as that is the one acting on the

z variable in the bulk, hence on the boundary variable in $\mathbb{P}^1(\mathbb{R})$) by taking an algebra given by a crossed product with Γ. A similar kind of boundary algebra was considered in Section 4 of [MM02]. Using a crossed product with Γ would imply dealing with a boundary algebra that contains a copy of $C^*(\Gamma)$. Invertible ρ's would determine, as in the GL_2-system, representations on the Hilbert space $\mathcal{H} = \ell^2(M_2^+(\mathbb{Z}))$ and in such representation the algebra $C^*(\Gamma)$ generates a type II_1 factor in $\mathcal{H}$. This affects the construction of KMS states for this algebra. Gibbs-type states with respect to the trace Tr_Γ can be evaluated on elements in the commutant of this factor, as discussed in Section 7 of [CM06b]. However, here we do not make this choice in the construction of the boundary algebra, and we leave this to separate future work. This is tied up to the question mentioned in the introduction, of developing a good theory of isogeny for noncommutative tori.

The point of view we follow here on constructing a boundary algebra is based instead on a different observation, namely on the fact that the orbits of the action of $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$ can be equivalently described as the orbits of a discrete dynamical system T acting on the interval $[0, 1]$. Thus, we will replace the crossed product by G with a semigroup crossed product that implements this equivalence relation as part of the algebra. The reason why we prefer this approach to the crossed product by G is because the dynamical system T used here is the same generalized shift of the continuous fractions expansion used in [MM02] to construct limiting modular symbols, and one of our main goals in this paper is obtaining a boundary algebra that is especially suited to relate to limiting modular symbols, hence this viewpoint is more natural here.

Moreover, as already discussed, this viewpoint allows us to see our boundary algebra as one case ($N = \pm 1$ for $\Gamma = \mathrm{GL}_2(\mathbb{Z})$ and $\mathrm{SL}_2(\mathbb{Z})$, respectively) of a countable family of algebras labelled by an integer N, associated to a family of different continued fraction algorithms. Considering this whole family of algebras will help us illustrate some interesting phenomena in the structure of KMS states, even though only the $N = \pm 1$ cases have a direct interpretation as boundary algebras of the respective bulk system and related to the geometry of modular curves.

Thus, in the following we first restrict the boundary variable $\theta \in \mathcal{D}_{\mathbb{P}^1(\mathbb{Q})}$ to the interval $[0, 1]$, that is, to the disconnection $\mathcal{D}_{[0,1]\cap\mathbb{Q}}$, because of the prior observation that the interval $[0, 1]$ meets every $\mathrm{GL}_2(\mathbb{Z})$-orbit. Then we implement the action of the shift operator T in the form of a semigroup crossed product algebra. This corresponds to taking the quotient by the action of T (hence by the action of $\mathrm{GL}_2(\mathbb{Z})$) in a noncommutative way, by considering a crossed product algebra instead of an algebra of functions constant along the orbits. This will be a semigroup crossed product with respect to the semigroup Red_N discussed above, and in a form that will implement the action of the shift operator T as in Lemma 3.2.4. We will work with the algebra of continuous functions on the disconnection $\mathcal{D}_{[0,1]\cap\mathbb{Q}}$. In Corollary 3.2.11 we will further extend this disconnection space by including additional T-invariant subspaces. The reason for this further extension will become clear when we consider such boundary functions that are obtained as integration on certain configurations of geodesics in the bulk space (see Lemma 3.3.1).

Note that if we write, as before, Ξ for the set of cosets $\Gamma\alpha\Gamma$ and $\mathcal{P}$ for the set of cosets $\Gamma\alpha G$, we can identify the sets $\mathcal{P} \simeq \Xi \times \mathbb{P}$, with the finite coset space $\mathbb{P} = \Gamma/G$. It is convenient to use this identification, so that, when we consider the shift operator T (in the case $N = 1$) acting on $[0,1] \times \mathcal{P}$, this can be viewed as the action of T on $[0,1] \times \mathbb{P}$ as in [MM02], with T acting trivially on Ξ.

Definition 3.2.7. Let $\mathcal{A}^c_{\partial,N}$ denote the associative algebra of continuous complex valued functions on

$$\mathcal{U}_{\partial,G,N} = \{(g,\rho,s) \in \mathrm{GL}_2(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathcal{P}_N \,|\, g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{3.24}$$

that are invariant with respect to the action of $\Gamma_N \times \Gamma_N$ by $(g,\rho) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 \rho)$ and are finitely supported in $\mathcal{P}_N$ and in $\Gamma_N\backslash\mathrm{GL}_2(\mathbb{Q})$ with the dependence on ρ through a finite level projection $p_n(\rho) \in M_2(\mathbb{Z}/n\mathbb{Z})$, endowed with the convolution product

$$(f_1 \star f_2)(g,\rho,s) = \sum_{h \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}, h\rho, hs) f_2(h,\rho,s) \tag{3.25}$$

and with the involution $f^*(g,\rho,s) = \overline{f(g^{-1}, g\rho, gs)}$. Let $\pi_{\rho,s} : \mathcal{A}^c_{\partial,N} \to \mathcal{B}(\mathcal{H}_{\rho,N})$ be the representation $\pi_{\rho,s}(f)\xi(g) = \sum_h f(gh^{-1}, h\rho, hs)\xi(h)$ for $h \in \mathcal{S}_{\rho,N}$. Let $\mathcal{A}_{\partial,N}$ denote the C^*-algebra completion of $\mathcal{A}^c_{\partial,N}$ with respect to $\|f\| = \sup_{(\rho,s)} \|\pi_{\rho,s}(f)\|_{\mathcal{H}_{\rho,N}}$. Let $\mathcal{B}_{\partial,N} = C(\mathcal{D}_{[0,1]\cap\mathbb{Q}}, \mathcal{A}_{\partial,N})$ be the algebra of continuous functions from $\mathcal{D}_{[0,1]\cap\mathbb{Q}}$ to $\mathcal{A}_{\partial,N}$, with pointwise product

$$(f_1 \star f_2)(g,\rho,x,s) = \sum_h f_1(gh^{-1}, h\rho, x, hs) f_2(h,\rho,x,s)$$

and involution $f^*(g,\rho,x,s) = \overline{f(g^{-1}, g\rho, x, gs)}$.

Definition 3.2.8. Let $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ be the involutive associative algebra generated by $\mathcal{B}_{\partial,N}$ and by isometries $S_{N,k}$, with $k \in \Phi_N$. It has relations $S^*_{N,k} S_{N,k} = 1$ and $\sum_k S_{N,k} S^*_{N,k} = 1$ and relations of the form

$$S_{N,k}\, f = \chi_{X_{N,k}} \cdot f \circ g^{-1}_{N,k} \cdot S_{N,k} \quad \text{and} \quad S^*_{N,k}\, f = f \circ g_{N,k} \cdot S^*_{N,k}, \tag{3.26}$$

where $\chi_{X_{N,k}}$ is the characteristic function of the subset $X_{N,k} \subset [0,1]$ of points with N-continued fraction expansion starting with k. The matrices $g_{N,k}, g^{-1}_{N,k}$ in $\mathrm{GL}_2(\mathbb{Q})$

are as in (3.15), with $f \circ g^{\pm}_{N,k}(g,\rho,x,s) = f(g,\rho,g^{\pm}_{N,k}(x,s))$. They also satisfy the relation

$$\sum_{k\in\mathbb{N}} S_k f S_k^* = f \circ T_N \tag{3.27}$$

for all $f \in B_{\partial,N}$. Here, for $f = f(g,\rho,x,s)$, we have

$$(f \circ T_N)(g,\rho,x,s) = f(g,\rho,T_N(x,s)),$$

with the action of T_N on $[0,1] \times \mathcal{P}_N$ as in Lemma 3.2.1. The involution on $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ is given by the involution on $\mathcal{B}_{\partial,N}$ and by $S_{N,k} \mapsto S^*_{N,k}$.

In fact, the relation (3.27) follows from the relations (3.26) as in Lemma 3.2.4, but we write it explicitly as it is the relation that implements the dynamical system T_N. Note that we are implicitly using in the construction of the algebra the fact that the semigroup action of Red_N and the action of Hecke operators (that is built into the convolution product of $\mathcal{B}_{\partial,N}$) commute as in Lemma 3.2.6.

3.2.6 Semigroup crossed product

Several examples of semigroup crossed product algebras have been considered in relation to quantum statistical mechanical system, especially in various generalizations of the Bost–Connes system. However, there is no completely standard definition of semigroup crossed product algebra in the literature. For our purposes here, the following setting suffices.

Definition 3.2.9. Let $\mathcal{A}$ be a C^*-algebra, and let $\mathcal{S}$ be a countable semigroup together with a semigroup homomorphism $\beta : \mathcal{S}^{op} \to \mathrm{End}(\mathcal{A})$. For $\ell \in \mathcal{S}$, let $\beta_\ell(1) = e_\ell$ be an idempotent in $\mathcal{A}$ and let α_ℓ denote a partial inverse of β_ℓ on $e_\ell \mathcal{A} e_\ell$. The (algebraic) semigroup crossed product algebra $\mathcal{A} \rtimes \mathcal{S}$ is the involutive $\mathbb{C}$-algebra generated by $\mathcal{A}$ and elements S_ℓ, S_ℓ^*, for all $\ell \in \mathcal{S}$ with the relations

$$S_\ell S_{\ell'} = S_{\ell\ell'}, \quad S_\ell^* S_\ell = 1, \quad S_\ell S_\ell^* = e_\ell, \quad \sum_\ell S_\ell S_\ell^* = 1,$$

$$S_\ell \, X \, S_\ell^* = \alpha_\ell(X), \quad S_\ell^* \, X \, S_\ell = \beta_\ell(X).$$

If $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation as bounded operators on a Hilbert space, and the S_ℓ act as isometries on $\mathcal{H}$, compatibly with the relations above, then semigroup crossed product C^*-algebra (which will also be denoted by $\mathcal{A} \rtimes \mathcal{S}$) is the C^*-completion in $\mathcal{B}(\mathcal{H})$ of the above algebraic crossed product.

Lemma 3.2.10. *The algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ can be identified with the semigroup crossed product $\mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$ of the algebra $\mathcal{B}_{\partial,N}$ of Definition 3.2.7 and the semigroup of* *reducible matrices, with respect to the action* $\alpha : Red_N \to Aut(B_{\partial,N})$ *by* $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$*, where for* γ *of the form* (3.21)*, the set* $X_\gamma \subset [0,1]$ *is the cylinder set consisting of points with* N*-continued fraction expansion starting with the sequence* $n_1, \ldots, n_k$.

Proof. We see as in Lemma 3.2.6 that $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$ defines a semigroup action of Red_N on $\mathcal{B}_{\partial,N}$. The semigroup crossed product algebra is generated by $B_{\partial,N}$ and by isometries μ_γ for $\gamma \in \mathrm{Red}_N$ satisfying $\mu_\gamma \mu_{\gamma'} = \mu_{\gamma\gamma'}$ for all $\gamma, \gamma' \in \mathrm{Red}_N$, $\mu_\gamma^* \mu_\gamma = 1$ for all $\gamma \in \mathrm{Red}_N$ and $\mu_\gamma \, f \, \mu_\gamma^* = \alpha_\gamma(f)$. It suffices to consider isometries $\mu_{g_{N,k}} =: S_{N,k}$ associated to the elements $g_{N,k} \in \mathrm{Red}_N$ as in (3.15) with $\mu_\gamma = S_{n_1} \cdots S_{n_k}$ for $\gamma \in \mathrm{Red}_N$ as in (3.21). We then see that the generators and relations of the algebras $B_{\partial,N} \rtimes \mathrm{Red}_N$ agree with those of the algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ of Definition 3.2.8. □

We consider the following variant of the boundary algebra introduced above, which will be useful for the application discussed in the following section, see in particular Lemma 3.3.1.

Corollary 3.2.11. *Let* $E = \{E_\alpha\}$ *be a collection of subsets* $E_\alpha \subset [0,1]$ *that are invariant under the action of the shift* T_N *of the* N*-continued fraction expansion. We denote by* $\mathcal{D}_E$ *the disconnection space dual to the abelian* C^**-algebra* $C(\mathcal{D}_E)$ *generated by* $C(\mathcal{D}_{[0,1]\cap\mathbb{Q}})$ *and by the characteristic functions* χ_{E_α}*. This then determines an algebra* $\mathcal{A}_{\partial,G,\mathcal{P}_N,E}$ *given by* $B_{\partial,N,E} \rtimes Red_N$ *where* $B_{\partial,N,E} = C(\mathcal{D}_E, \mathcal{A}_{\partial,N})$ *is the algebra of continuous functions from the disconnection space* $\mathcal{D}_E$ *to* $\mathcal{A}_{\partial,N}$ *as in Definition 3.2.7.*

Proof. If the sets E_α are T_N-invariant then the algebra $B_{\partial,N,E}$ is invariant under the action of the semigroup Red_N by $\alpha_\gamma(f) = \chi_{X_\gamma} \cdot f \circ \gamma^{-1}$, since for $\gamma = g_{N,k_1} \cdots g_{N,k_n}$, the matrix γ^{-1} acts on X_γ as the shift T_N^n. Thus, we can form the semigroup crossed product algebra $B_{\partial,N,E} \rtimes \mathrm{Red}_N$ as in Lemma 3.2.10. □

3.2.7 Representations and time evolution

Let $\mathcal{H}_{\rho,N}$ be the same Hilbert space considered above, $\mathcal{H}_{\rho,N} = \ell^2(\mathcal{S}_{\rho,N})$ with $\mathcal{S}_{\rho,N}$ the collection of Γ_N-cosets that have some representative element in the set $\{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\}$. We will consider the case of an invertible $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$. This choice is made to guarantee non-negative spectrum of the Hamiltonian of Proposition 3.2.13 and is also geometrically motivated by the $\mathrm{GL}_2(\mathbb{Z})$ (i.e. $N = 1$) setting as discussed in Section 3.2.5. When $N = 1$ and $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$ is invertible we can write

$$\mathcal{S}_{\rho,1} = \mathrm{GL}_2(\mathbb{Z}) \backslash \{g \in \mathrm{GL}_2(\mathbb{Q}) | g\rho \in M_2(\hat{\mathbb{Z}})\} = \mathrm{GL}_2(\mathbb{Z}) \backslash M_2^\times(\mathbb{Z}),$$

with $M_2^\times(\mathbb{Z}) = \{M \in M_2(\mathbb{Z}) \,|\, \det(M) \neq 0\}$.

Let $\mathcal{W}_N = \cup_n \mathcal{W}_{N,n}$ denote the set of all finite sequences $k_1, \ldots, k_n$ with $k_i \in \Phi_N$, including an element $\emptyset$ corresponding to the empty sequence. Consider the Hilbert spaces $\ell^2(\mathcal{W}_N)$ and $\tilde{\mathcal{H}}_{\rho,N} = \ell^2(\mathcal{W}_N) \otimes \mathcal{H}_{\rho,N}$.

Lemma 3.2.12. *The algebra $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes Red_N$ acts on the Hilbert space $\tilde{\mathcal{H}}_{\rho,N}$ through the representations*

$$\pi_{\rho,x,s}(f)\,(\xi(g) \otimes \epsilon_{k_1,\ldots,k_n}) = \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_\gamma(x, hs))\,\xi(h) \otimes \epsilon_{k_1,\ldots,k_n} \tag{3.28}$$

for $f \in B_{\partial,N}$ and with $g_\gamma = g_{N,k_1} \cdots g_{N,k_n}$, with g_{N,k_i} as in (3.15), and

$$\begin{aligned} &\pi_{\rho,x,s}(S_{N,k})\,(\xi \otimes \epsilon_{k_1,\ldots,k_n}) = \xi \otimes \epsilon_{k,k_1,\ldots,k_n}, \\ &\pi_{\rho,x,s}(S^*_{N,k})(\xi \otimes \epsilon_{k_1,\ldots,k_n}) = \begin{cases} \xi \otimes \epsilon_{k_2,\ldots,k_n} & k_1 = k \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \tag{3.29}$$

In what follows we sometimes write $\pi_{\rho,x,s}(S_{N,k})$ as $S_{N,k}$ because the mapping of these operators does not depend on the choice of (ρ, x, s).

Proof. We check that (3.28) gives a representation of the subalgebra $B_{\partial,N}$ and that the operators $\pi_{\rho,x,s}(f)$, $S_{N,k}$, $S^*_{N,k}$ of (3.28) and (3.29) satisfy the relations $S^*_{N,k}S_{N,k} = 1$, $\sum_k S_{N,k}S^*_{N,k} = 1$, $S_{N,k}\pi_{\rho,x,s}(f) = \pi_{\rho,x,s}(\chi_{X_{N,k}}\, f \circ g^{-1}_{N,k})S_{N,k}$ and $S^*_{N,k}\pi_{\rho,x,s}(f) = \pi_{\rho,x,s}(f \circ g_{N,k})S^*_{N,k}$. For the first property it suffices to see that $\pi_{\rho,x,s}(f_1 \star f_2) = \pi_{\rho,x,s}(f_1) \circ \pi_{\rho,x,s}(f_2)$. We have

$$\pi_{\rho,x,s}(f_1 \star f_2)\,(\xi(g) \otimes \epsilon_{k_1,\ldots,k_n}) = \sum_{h \in \mathcal{S}_{\rho,N}} (f_1 \star f_2)(gh^{-1}, h\rho, g_\gamma(x, hs))\,\xi(h) \otimes \epsilon_{k_1,\ldots,k_n} =$$

$$\sum_{h \in \mathcal{S}_{\rho,N}} \sum_{\ell \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}\ell^{-1}, \ell h\rho, g_\gamma(x, \ell hs)) f_2(\ell, h\rho, g_\gamma(x, hs))\xi(h) \otimes \epsilon_{k_1,\ldots,k_n},$$

where we used Lemma 3.2.6. This is then equal to

$$\sum_{\ell \in \mathcal{S}_{\rho,N}} f_1(gh^{-1}\ell^{-1}, \ell h\rho, g_\gamma(x, \ell hs))(\pi_{\rho,x,s}(f_2)\xi)(\ell) \otimes \epsilon_{k_1,\ldots,k_n} = \pi_{\rho,x,s}(f_1)\pi_{\rho,x,s}(f_2)\xi \otimes \epsilon_{k_1,\ldots,k_n}.$$

The relations $S^*_{N,k}S_{N,k} = 1$ and $\sum_k S_{N,k}S^*_{N,k} = 1$ follow directly from (3.29). For relations between the $S_{N,k}, S^*_{N,k}$ and the $\pi_{\rho,x,s}(f)$, we have

$$S_{N,k}\pi_{\rho,x,s}(f)\,\xi \otimes \epsilon_{k_1,\ldots,k_n} = S_{N,k} \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g_k^{-1} g_k g_\gamma(x, hs))\,\xi(h) \otimes \epsilon_{k_1,\ldots,k_n}$$

$$= \sum_{h \in \mathcal{S}_{\rho,N}} f(gh^{-1}, h\rho, g^{-1}_{N,k} g_{N,k} g_\gamma(x, hs))\,\chi_{X_{N,k}}(g_{N,k}g_\gamma x)\,\xi(h) \otimes \epsilon_{k,k_1,\ldots,k_n},$$

for $g_\gamma = g_{N,k_1}\cdots g_{N,k_n}$, with $\chi_{X_{N,k}}(g_{N,k}g_\gamma x) = 1$, so we get

$$\pi_{\rho,x,s}(\chi_{X_{N,k}}\cdot f\circ g_{N,k}^{-1})\, S_{N,k}\,\xi\otimes\epsilon_{k,k_1,\ldots,k_n}.$$

The second relation is similar: we have

$$S^*_{N,k}\pi_{\rho,x,s}(f)\,\xi\otimes\epsilon_{k_1,\ldots,k_n} = S^*_{N,k}\sum_{h\in\mathcal{S}_{\rho,N}} f(gh^{-1},h\rho,g_k^{-1}g_k g_\gamma(x,hs))\,\xi(h)\otimes\epsilon_{k_1,\ldots,k_n}$$

$$= \sum_{h\in\mathcal{S}_{\rho,N}} f(gh^{-1},h\rho,g_{N,k}g_{\gamma'}(x,hs))\, S^*_{N,k}\xi(h)\otimes\epsilon_{k_1,\ldots,k_n},$$

with $g_{\gamma'} = g_{N,k_2}\cdots g_{N,k_n}$, so that we obtain

$$\pi_{\rho,x,s}(f\circ g_{N,k})\, S^*_{N,k}\,\xi\otimes\epsilon_{k_1,\ldots,k_n}.$$

Thus, (3.28) and (3.29) determine a representation of $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N}\rtimes\mathrm{Red}_N$ by bounded operators on the Hilbert space $\tilde{\mathcal{H}}_{\rho,N}$. □

Proposition 3.2.13. *The transformations* $\sigma_{N,t}(f)(g,\rho,x,s) = |\det(g)|^{it} f(g,\rho,x,s)$ *and* $\sigma_{N,t}(S_{N,k}) = k^{it}S_{N,k}$ *define a time evolution* $\sigma_N : \mathbb{R}\to Aut(\mathcal{A}_{\partial,G,\mathcal{P}_N})$. *In the representations of Lemma 3.2.12 on* $\tilde{\mathcal{H}}_{\rho,N}$ *with* $\rho\in GL_2(\hat{\mathbb{Z}})$ *the time evolution is implemented by the Hamiltonian*

$$H_N\,\xi(g)\otimes\epsilon_{k_1,\ldots,k_n} = \log(|\det(g)|\cdot k_1\cdots k_n)\,\xi(g)\otimes\epsilon_{k_1,\ldots,k_n}, \tag{3.30}$$

with partition function

$$Z_N(\beta) = Tr(e^{-\beta H_N}) = \begin{cases} \dfrac{\zeta(\beta)\zeta(\beta-1)\prod_{p\ prime\,:\,p|N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)}{1+\sum_{n=1}^{N-1} n^{-\beta}-\zeta(\beta)} & \text{if } N>1 \\ \dfrac{\zeta(\beta)\zeta(\beta-1)\prod_{p\ prime\,:\,p|N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)}{1+\sum_{n=1}^{|N|} n^{-\beta}-\zeta(\beta)} & \text{if } N\leq -1 \end{cases} \tag{3.31}$$

with $\zeta(\beta)$ *the Riemann zeta function. In the* $N=1$ *case there is no partition function.*

Proof. We have

$$\sigma_{N,t}(f_1\star f_2)(g,\rho,x,s) = \sigma_{N,t}(\sum_h f_1(gh^{-1},h\rho,x,hs)f_2(h,\rho,x,s)) =$$

$$\sum_h |\det(gh)^{-1}|^{it}|\det(h)|^{it} f_1(gh^{-1},h\rho,x,hs)f_2(h,\rho,x,s) = \sigma_{N,t}(f_1)\star\sigma_t(f_2).$$

We also have $\sigma_{N,t}(S^*_{N,k}) = k^{-it} S^*_{N,k}$ and we see that the action of $\sigma_{N,t}$ is compatible with the relations in $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ and defines a 1-parameter family of algebra homomorphisms. By direct comparison between (3.30) and (3.28) and (3.29) we also see that

$$\pi_{\rho,x,s}(\sigma_{N,t}(f)) = e^{itH}\, \pi_{\rho,x,s}(f)\, e^{-itH} \quad \text{and} \quad \sigma_{N,t}(S_{N,k}) = e^{itH}\, S_{N,k} e^{-itH}\,.$$

We have

$$Z_N(\beta) = \sum_{g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta} \cdot \sum_{k=k_1\cdots k_n : k_i \in \Phi_N} k^{-\beta}$$

where Φ_N is the set of possible digits in the N-continued fraction expansion.

For the first sum, we begin by considering the $N = 1$ case. We now have that $\mathcal{S}_\rho = \mathrm{GL}_2(\mathbb{Z}) \backslash M_2^\times(\mathbb{Z})$, where $M_2^\times(\mathbb{Z}) = \{M \in M_2(\mathbb{Z}) \mid \det(M) \neq 0\}$. Thus, we are counting $\{M \in M_2^\times(\mathbb{Z}) \mid |\det(M)| = n\}$ modulo $\mathrm{GL}_2(\mathbb{Z})$. Up to a change of basis in $\mathrm{GL}_2(\mathbb{Z})$ we can always write a sublattice of $\mathbb{Z}^2$ in the form

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mathbb{Z}^2$$

with $a, d \geq 1$ and $0 \leq b < d$, [Ser77]. Thus, we are equivalently counting such matrices with determinant n. This counting is given by $\sigma(n) = \sum_{d|n} d$ so the first sum is $\sum_{n \geq 1} \sigma(n)\, n^{-\beta} = \zeta(\beta)\zeta(\beta-1)$ as in the original GL_2-system, and converges on $\beta \in (2, \infty)$.

In the general case, we again consider $\rho \in \mathrm{GL}_2(\hat{\mathbb{Z}})$, and we now have that $\mathcal{S}_{\rho,N}$ is the set of matrices in $M_2^\times(\mathbb{Z})$ with determinant not divisible by any prime factor of N, up to the equivalence relation defined by $\mathrm{GL}_2(\mathbb{Z})$. The first sum is then given by

$$\begin{aligned} \sum_{g \in \mathcal{S}_{\rho,N}} |\det(g)|^{-\beta} &= \sum_{n \geq 1 : (N,n)=1} \sigma(n) n^{-\beta} \\ &= \left(\sum_{n \geq 1 : (N,n)=1} n^{-\beta} \right) \left(\sum_{n \geq 1 : (N,n)=1} n^{-(\beta-1)} \right) \\ &= \zeta(\beta)\zeta(\beta-1) \prod_{p\ prime\ :\ p|N} \left(1 - p^{-\beta}\right) \left(1 - p^{-(\beta-1)}\right) \end{aligned}$$

where the counting $\sigma(n) = \sum_{d|n} d$ is identical to the $N = 1$ case. Again, this series converges on $\beta \in (2, \infty)$.

To compute the second sum, let $P_{N,n}$ denote the total number of ordered factorizations of n into positive integer factors in Φ_N. In the $N \geq 1$ case, the sum we are considering is

$$\begin{aligned}
\sum_{n\geq 1} P_{N,n}\, n^{-\beta} &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} n^{-\beta} \sum_{n=n_1\cdots n_k : n_i\geq N} 1 \\
&= \sum_{k\geq 1}\prod_{i=1}^{k}\Big(\sum_{n_i\geq N} n_i^{-\beta}\Big) \\
&= \begin{cases} \sum_{k=1}^{\infty}(\zeta(\beta))^k = \dfrac{1}{1-\zeta(\beta)} & \text{if } N=1 \\ \sum_{k=1}^{\infty}\big(\zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta}\big)^k = \dfrac{1}{1+\sum_{n=1}^{N-1} n^{-\beta} - \zeta(\beta)} & \text{if } N>1. \end{cases}
\end{aligned}$$

In the $N = 1$ case, note that the series $\sum_{k=1}^{\infty}(\zeta(\beta))^k$ converges when $|\zeta(\beta)| < 1$. However, when $\beta > 1$, $\zeta(\beta) > 1$ and the series does not converge there. The first series $\sum_{n\geq 1}\sigma_1(n)\, n^{-\beta} = \zeta(\beta)\zeta(\beta-1)$ converges for $\beta > 2$. Since the second series does not converge anywhere in the region $(2,\infty)$, there is no partition function.

In the $N > 1$ case, the relevant series converges when

$$|\zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta}| = |\zeta(\beta) - (1+\xi(\beta))| < 1$$

where $\xi(\beta) = 0$ when $N = 2$ and $\xi(\beta) = \sum_{n=2}^{N-1} n^{-\beta}$ when $N > 2$. In the range $\beta \in (1,\infty)$ the ζ-function is decreasing to 1 and it crosses the value $\zeta(\beta) = 2$ at a point $\beta_{2,c} \sim 1.728647$. When $N = 2$, the series converges on $(\beta_{2,c},\infty)$. When $N > 2$ we consider the function $\zeta(\beta) - \xi(\beta)$ where $\xi(\beta)$ consists of a finite sum of terms of the form $n^{-\beta}$ each of which are continuous, decreasing to 0 as $\beta \to \infty$ and have some finite value at $\beta = 1$. Since $\lim_{\beta\to\infty}\zeta(\beta) - \xi(\beta) = 1$ and $\lim_{\beta\to 1^+}\zeta(\beta) - \xi(\beta) = \infty$, there will be some point $\beta_{N,c} > 1$ at which $\zeta(\beta_{N,c}) - \xi(\beta_{N,c}) = 2$. The corresponding series then converges on $(\beta_{N,c},\infty)$. Since each $n^{-\beta}$ term is decreasing, we also know that $\beta_{N+1,c} < \beta_{N,c}$.

Similarly, in the $N \leq -1$ case we have

$$\begin{aligned}
\sum_{n\geq 1} P_{N,n}\, n^{-\beta} &= \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} n^{-\beta} \sum_{n=n_1\cdots n_k : n_i\geq |N|+1} 1 \\
&= \sum_{k\geq 1}\prod_{i=1}^{k}\Big(\sum_{n_i\geq |N|+1} n_i^{-\beta}\Big) \\
&= \sum_{k=1}^{\infty}\big(\zeta(\beta) - \sum_{n=1}^{|N|} n^{-\beta}\big)^k = \frac{1}{1+\sum_{n=1}^{|N|} n^{-\beta} - \zeta(\beta)}.
\end{aligned}$$

As before, this series converges on $(\beta_{N,c},\infty)$ where for $N \leq -1$, $\beta_{N,c} = \beta_{1-N,c}$. In particular, $\beta_{-1,c} = \beta_{2,c} \sim 1.728647$. □

Note that in the proof above we have shown that $\beta_{N,c}$ is decreasing in N for positive N, and therefore attains its maximum value at $\beta_{2,c} = \beta_{-1,c} \sim 1.728647$. We also know that $\beta_{N,c} > 1$ for all N.

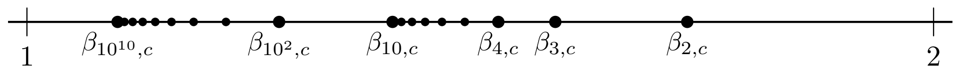

FIGURE 3.3: N-DEPENDENCE OF BOUNDARY-GL_2 CRITICAL TEMPERATURE ($\beta_{N,c}$)

Lemma 3.2.14. *For $N \geq 2$ and $N \leq -1$, the partition function $Z_N(\beta)$ of Proposition 3.2.13 is defined by an absolutely convergent series*

$$Z_N(\beta) = Tr(e^{-\beta H_N}) = \sum_{\lambda \in Spec(H_N)} e^{-\beta\lambda}$$

for $\beta > 2$. Its analytic continuation (3.31) *has poles at $\beta \in \{1, \beta_{N,c}, 2\}$, for a point $1 < \beta_{N,c} < 2$. In the geometrically relevant case of $N = -1$, $\beta_{-1,c} \sim 1.728647$.*

Proof. As argued in the proof of 3.2.13, the denominator of $Z_N(\beta)$ has a single zero at a point $1 < \beta_{N,c} < 2$. The Riemann zeta function $\zeta(\beta)$ has a pole at $\beta = 1$. Therefore, the sum $\sum_{n \geq 1\,:\,(N,n)=1} \sigma_1(n) n^{-\beta}$ is convergent for $\beta > 2$ and its analytic continuation $\zeta(\beta)\zeta(\beta-1)\prod_{p\ prime\,:\,p|N}\left(1-p^{-\beta}\right)\left(1-p^{-(\beta-1)}\right)$ has poles at $\beta = 2$ and $\beta = 1$. □

3.2.8 KMS states.

We classify the KMS states for the family of dynamical systems $(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$. Since we have $\mathcal{A}_{\partial,G,\mathcal{P}_N} = \mathcal{B}_{\partial,N} \rtimes \mathrm{Red}_N$, we consider separately the KMS states for the modified GL_2 part $\mathcal{B}_{\partial,N}$ of the system, and the part of the system generated by the isometries $S_{N,k}$ in the semigroup Red_N, which is a Cuntz-Krieger-Toeplitz type algebra. The KMS states of the Cuntz-Krieger-Toeplitz type algebras have been studied by [EL03], and we draw on their main results. We show that in the $N = 1$ case, corresponding to the standard GL_2-system, there are no KMS states at any temperature, though we may still define ground states. In all other cases, the system has two critical temperatures at $\beta = \beta_{N,c} < 2$ and $\beta = 2$. When $\beta < \beta_{N,c}$ there are no KMS_β states. When $\beta_{N,c} < \beta < 2$, the structure of the KMS_β states will be identical to the structure on the modified GL_2 part of the system alone, though we have not computed this explicitly. When $\beta > 2$, the KMS_β states are given by Gibbs states, whose limit as $\beta \to \infty$ gives the ground states.

Lemma 3.2.15. *The subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ generated by the family $\{S_{N,k}\}_{k \in \Phi_N}$ of isometries is a Cuntz-Krieger-Toeplitz algebra, denoted by $\mathcal{O}_A$ in the setting of [EL03]* *with the infinite matrix* $A = \{A(x,y)\}_{x,y\in\Phi_N}$ *given by* $A(x,y) = 1$ *for all* $x, y \in \Phi_N$. *In other words, it satisfies the following properties on its initial and final projections* $q_k = S^*_{N,k}S_{N,k}$ *and* $p_k = S_{N,k}S^*_{N,k}$. *For all* $k, j \in \Phi_N$*:*

1. $q_k q_j = q_j q_k$,
2. $S^*_{N,k}S_{N,j} = 0$ *if* $j \neq k$,
3. $q_k S_{N,j} = S_{N,j}$,
4. *and* $\prod_{k\in X} q_k \prod_{j\in Y}(1 - q_j) = 0$ *for* X, Y *finite subsets of* Φ_N.

Proof. Conditions (1), (3), and (4) follow from the fact that $S^*_{N,k}S_{N,k} = 1$ (Lemma 3.2.4). Condition (2) is easily verified. If $k \neq j$ then

$$\begin{aligned} S^*_{N,k}S_{N,j}\xi(x) &= S^*_{N,k}(\chi_{X_{N,j}}(x)\xi(g^{-1}_{N,j}x)) \\ &= \chi_{X_{N,j}}(g_{N,k}x)\xi(g_{N,k}g^{-1}_{N,j}x) = 0. \end{aligned}$$

□

Proposition 3.2.16. *The KMS*$_\beta$ *states of the dynamical system* $(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$ *can be characterized as follows. When* $N = 1$*, there are no KMS*$_\beta$ *states for any temperature* β*. When* $N \leq -1$ *or* $N > 1$*, the system has a critical temperature* $\beta_{N,c} \in (1, 2)$*. We then have the following.*

1. *When* $\beta < \beta_{N,c}$ *there are no* β*-KMS states.*
2. *When* $\beta > \beta_{N,c}$*, there is one* β*-KMS state for every* β*-KMS state of the modified* GL_2*-system corresponding to* $\mathcal{B}_{\partial,N}$.
3. *When* $\beta > 2$*, the* β*-KMS states restrict to the* Red_N *part of the system as the unique* β*-KMS state Cuntz-Krieger-Toeplitz algebra and restrict to a* β*-KMS state on the* $\mathcal{B}_{\partial,N}$ *part of the system. In particular, one obtains extremal KMS-states corresponding to the Gibbs states of* $\mathcal{B}_{\partial,N}$*, parameterized by* (ρ, x, s) *with* $\rho \in M_2(\hat{\mathbb{Z}})$ *invertible.*

Proof. If there is a KMS$_\beta$ state on$(\mathcal{A}_{\partial,G,\mathcal{P}_N}, \sigma_{N,t})$, it must restrict to a KMS$_\beta$ state on the subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}_N}$ generated by the family of isometries $\{S_{N,k}\}_{k\in\Phi_N}$. We will first characterize the KMS$_\beta$ states of this subalgebra, which we have established in Lemma 3.2.15 is a Cuntz-Krieger-Toeplitz algebra. We also note that since Φ_N is a countable set and in our case the matrix A is simply a matrix with every entry set to 1, Standing Hypothesis 8.1 (i), and (ii) of [EL03] are satisfied. The dynamics $\sigma_{N,t}$ on the subalgebra take the form $\sigma_{N,t}(S_{N,k}) = k^{it}S_{N,k}$ for each $k \in \Phi_N$, and since

Φ_N is a set of real numbers in the interval $(1, \infty)$ the rest of Standing Hypothesis 8.1 of [EL03] is also satisfied.

We now draw on the main results of [EL03]. Corollary 9.7 states that there is a critical temperature $\dot{\beta}_{N,c}$ above which there is a single KMS$_\beta$ state at each temperature β. Theorem 14.5 states that there is a second critical temperature $\ddot{\beta}_{N,c}$ below which there are no KMS$_\beta$ states at all. These critical temperatures are defined as follows. Let Ω_N be the set of words in Φ_N and $\Omega_{N,j,k}$ be the set of words in Φ_N beginning with j and ending with k. Then $\dot{\beta}_{N,c}$ and $\ddot{\beta}_{N,c}$ are the abscissas of convergence of the series

$$Z(\beta) = \sum_{\mu \in \Omega_N} (\mu)^{-\beta} \quad \text{and} \quad Z_{jk}(\beta) = \sum_{\mu \in \Omega_{N,j,k}} (\mu)^{-\beta}$$

respectively. Note that the abscissa of convergence of the second series is independent of the choice of j and k. We will now calculate these critical temperatures.

The partition function $Z(\beta)$ has already been calculated in the second part of Proposition 3.2.13 and is given by

$$Z(\beta) = \begin{cases} \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta))^k & \text{if } N = 1 \\ \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{N-1} n^{-\beta})^k & \text{if } N > 1 \\ \displaystyle\sum_{k=1}^{\infty} (\zeta(\beta) - \sum_{n=1}^{|N|} n^{-\beta})^k & \text{if } N \leq -1. \end{cases}$$

Modifying this calculation slightly we find that

$$\begin{aligned} Z_{jk}(\beta) &= (jk)^{-\beta} + \sum_{n=1}^{\infty} \sum_{\mu \in \Omega_N : |\mu| = n} j^{-\beta} \mu^{-\beta} k^{-\beta} \\ &= (jk)^{-\beta} \left(1 + \sum_{n=1}^{\infty} \prod_{i=1}^{n} \sum_{k_i \in \Phi_N} k_i^{-\beta} \right) = (jk)^{-\beta} Z(\beta). \end{aligned}$$

Clearly $Z(\beta)$ and $Z_{jk}(\beta)$ have the same abscissa of convergence, $\ddot{\beta}_{N,c} = \dot{\beta}_{N,c}$. In fact in the $N \neq 1$ case, this abscissa of convergence is $\beta_{N,c}$ of 3.2.13. When $N = 1$, neither series converges for any value of β, $\dot{\beta}_{N,c} = \ddot{\beta}_{N,c} = \infty$. Hence there are no KMS$_\beta$ states for any finite inverse temperature β.

Now we focus our attention on the subalgebra corresponding to $\mathcal{B}_{\partial,N}$ in the $N \neq 1$ case. In the range $\beta > 2$, $e^{-\beta H_N}$ is trace class, by Lemma 3.2.14. We therefore have Gibbs states of the form, for $X \in \mathcal{A}_{\partial,G,\mathcal{P}_N}$

$$\varphi_{\beta,N}(X) = \frac{\mathrm{Tr}(\pi_{\rho,x,s}(X)e^{-\beta H_N})}{\mathrm{Tr}(e^{-\beta H_N})}$$
$$= Z_N(\beta)^{-1} \sum_{k_i\in\Phi_N, g\in\mathcal{S}_{\rho,N}} |\det(g)|^{-\beta}(k_1\cdots k_n)^{-\beta}\,\langle \delta_g\otimes\epsilon_{k_1,\dots,k_n}, \pi_{\rho,x,s}(X)\delta_g\otimes\epsilon_{k_1,\dots,k_n}\rangle.$$

depending on our choice of representation $\pi_{\rho,x,s}$.

We now restrict to the subalgebra $\mathcal{B}_{\partial,N}$. The Gibbs states above give, for $f\in\mathcal{B}_{\partial,N}$

$$\varphi_{\beta,\rho,x,s}(f) = \frac{\mathrm{Tr}(\pi_{\rho,x,s}(f)e^{-\beta H_N})}{\mathrm{Tr}(e^{-\beta H_N})}$$
$$= Z(\beta)^{-1} \sum_{k_i\in\Phi_N, g\in\mathcal{S}_{\rho,N}} |\det(g)|^{-\beta}(k_1\cdots k_n)^{-\beta} f(1, g\rho, g_\gamma(x, gs))$$

where $\gamma\in\mathrm{Red}_N$ is determined by $k_1,...,k_n$. These are parameterized by the choice of $\rho\in M_2(\hat{\mathbb{Z}})$ invertible. □

Remark 3.2.1. The standard GL_2-system (when $\Gamma=\mathrm{SL}_2(\mathbb{Z})$) has been studied in [LLN07]. Their analysis of the behavior when $\beta\in(1,2)$ is not directly applicable in our case, because for $N=1$ ($\Gamma=\mathrm{SL}_2(\mathbb{Z})$), the Red_N part of the system has no KMS states at any temperature β. However, it would be interesting to see whether a similar analysis can be applied when $\Gamma=\Gamma_N$ for some $N>1$ or $N\le -1$.

As in [CM06b] we consider the ground states at zero temperature as the weak limit of the Gibbs states for $\beta\to\infty$

$$\varphi_{\infty,\rho,x,s}(f) = \lim_{\beta\to\infty}\varphi_{\beta,\rho,x,s}(f).$$

Corollary 3.2.17. *When $N\neq 1$, the ground states are given by*

$$\varphi_{\infty,\rho,x,s}(f) = f(1,\rho,x,s).$$

Proof. Observe that whenever $N\neq 1$, $\lim_{\beta\to\infty} Z_N(\beta)=1$. Furthermore, the only $g\in\mathcal{S}_{\rho,N}$ with $|\det(g)|=1$ is the identity element, and hence the only terms for which $\lim_{\beta\to\infty}|\det(g)|^{-\beta}$ does not vanish are those for which $g=1$. A word in Φ_N satisfies $k_1...k_n\ge|N|^n$ if $N>1$ and $k_1...k_n\ge(|N|+1)^n$ if $N\le 1$. Hence $\lim_{\beta\to\infty}|k_1...k_n|^{-\beta}$ vanishes unless $k_1...k_n$ is the empty word. We have that the ground states are

$$\varphi_{\infty,\rho,x,s}(f) = \langle\delta_1\otimes\epsilon_\emptyset, \pi_{\rho,x,s}(f)\,\delta_1\otimes\epsilon_\emptyset\rangle = f(1,\rho,x,s), \tag{3.32}$$

the evaluation of the function f at the point $g=1$ and (ρ,x,s) that determines the representation $\pi_{\rho,x,s}$. □

Although in the $N = 1$ case there are no low-temperature KMS states, and hence the weak limit does not exist, we can still define the projection onto the kernel of the Hamiltonian as in 3.32. This satisfies the weak KMS condition in the sense that function

$$F(t) = \varphi_{\infty,\rho,x,s}(f\sigma_t(f')) = f(1,\rho,x,s)f'(1,\rho,x,s)$$

has a bounded holomorphic extension to the upper half plane.

3.3 Averaging on geodesics and boundary values

In this section we construct boundary values of the observables of the GL_2-system of §3.1. We use the theory of limiting modular symbols of [MM02]. We show that the resulting boundary values localize nontrivially at the quadratic irrationalities and at the level sets of the multifractal decomposition considered in [KS07b]. We discuss in particular the case of quadratic irrationalities.

3.3.1 Geodesics between cusps

Let $C_{\alpha,\epsilon,s}$ with $\alpha \in \mathbb{P}^1(\mathbb{Q}) \setminus \{0\}$, $\epsilon \in \{\pm\}$, and $s \in \mathbb{P}$ denote the geodesic in $\mathbb{H}^\epsilon \times \mathbb{P}$ with endpoints at the cusps $(0, s)$ and (α, s) in $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}$.

Let $p_k(\alpha), q_k(\alpha)$ be the successive numerators and denominators of the $\mathrm{GL}_2(\mathbb{Z})$-continued fraction expansion of $\alpha \in \mathbb{Q}$ with $p_n(\alpha)/q_n(\alpha) = \alpha$ and let

$$g_k(\alpha) := \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}).$$

We denote by $C^k_{\alpha,\epsilon,s}$ the geodesic in $\mathbb{H}^\epsilon \times \mathbb{P}$ with endpoints at the cusps

$$\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)} = g_k^{-1}(\alpha) \cdot 0 \quad \text{and} \quad \frac{p_k(\alpha)}{q_k(\alpha)} = g_k^{-1}(\alpha) \cdot \infty,$$

where $g \cdot z$ for $g \in \mathrm{GL}_2(\mathbb{Z})$ and $z \in \mathbb{H}^\pm$ is the action by fractional linear transformations.

For C a geodesic in $\mathbb{H}^\pm$ let ds_C denote the geodesic length element. In the case of $C_{\infty,\epsilon,s}$ we have $ds_{C_{\infty,\epsilon,s}}(z) = dz/z$.

We use the notation $\{g \cdot 0, g \cdot \alpha\}_G$ to denote the homology class determined by the image in the quotient $X_G = G\backslash\mathbb{H} = \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H}^\pm \times \mathbb{P})$ of the geodesic $C_{\alpha,\epsilon,s}$, for g a representative of $s \in \mathbb{P}$. Similarly we write $\{\alpha, \beta\}_G$ for homology classes determined by the images in the quotient X_G of geodesics in $\mathbb{H}^\pm \times \mathbb{P}$ with endpoints α, β at cusps in $\mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}$.

3.3.2 Limiting modular symbols

We recall briefly the construction of limiting modular symbols from [MM02]. We consider here some finite index subgroup $G \subset \Gamma$ of $\Gamma = \mathrm{PGL}_2(\mathbb{Z})$. We denote by $\mathbb{P} = \Gamma/G$ the finite coset space of this subgroup. We also write the quotient modular curve as $X_G = G\backslash\mathbb{H} = \Gamma\backslash(\mathbb{H} \times \mathbb{P})$.

Recall that the classical modular symbols $\{\alpha, \beta\}_G$, with $\alpha, \beta \in \mathbb{P}^1(\mathbb{Q})$ are defined as the homology classes in $H_1(X_G, \mathbb{R})$ defined as functionals that integrate lifts to $\mathbb{H}$ of differentials on X_G along the geodesic arc in $\mathbb{H}$ connecting α and β (see [Man72]). They satisfy additivity and invariance: for all $\alpha, \beta, \gamma \in \mathbb{P}^1(\mathbb{Q})$

$$\{\alpha, \beta\}_G + \{\beta, \gamma\}_G = \{\alpha, \gamma\}_G \quad \text{and} \quad \{g\alpha, g\beta\}_G = \{\alpha, \beta\}_G \ \ \forall g \in G.$$

Thus, it suffices to consider the modular symbols of the form $\{0, \alpha\}_G$ with $\alpha \in \mathbb{Q}$. These satisfy the relation

$$\{0, \alpha\}_G = -\sum_{k=1}^{n} \{\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)}, \frac{p_k(\alpha)}{q_k(\alpha)}\}_G = -\sum_{k=1}^{n} \{g_k^{-1}(\alpha) \cdot 0, g_k^{-1}(\alpha) \cdot \infty\}_G,$$

where $p_k(\alpha), q_k(\alpha)$ are the successive numerators and denominators of the $\mathrm{GL}_2(\mathbb{Z})$-continued fraction expansion of $\alpha \in \mathbb{Q}$ with $p_n(\alpha)/q_n(\alpha) = \alpha$ and

$$g_k(\alpha) = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix}.$$

(There is an analogous formula for the $\mathrm{SL}_2(\mathbb{Z})$-continued fraction.)

Limiting modular symbols were introduced in [MM02], to account for the noncommutative compactification of the modular curves X_G by the boundary $\mathbb{P}^1(\mathbb{R})$ with the G action. One considers the infinite geodesics $L_\theta = \{\infty, \theta\}$ given by the vertical lines $L_\theta = \{z \in \mathbb{H} \,|\, \Re(z) = \theta\}$ oriented from the point at infinity to the point θ on the real line. Upon choosing an arbitrary base point $x \in L_\beta$ let $x(s)$ denote the point on L_β at an arc-length distance s from x in the orientation direction. One considers the homology class $\{x, x(s)\}_G \in H_1(X_G, \mathbb{R})$ determined by the geodesic arc between x and $x(s)$. The limiting modular symbol is defined as the limit (when it exists)

$$\{\{\star, \theta\}\}_G := \lim_{s \to \infty} \frac{1}{s} \{x, x(s)\}_G \in H_1(X_G, \mathbb{R}). \tag{3.33}$$

It was shown in [MM02] that the limit (3.33) exists on a full measure set and can be computed by an ergodic average. More generally, it was shown in [Mar03] that there is a multifractal decomposition of the real line by level sets of the Lyapunov exponent of the shift of the continued fraction expansion plus an exceptional set where the limit does not exist. On the level sets of the Lyapunov exponent the limit is again given by an average of modular symbols associated to the successive terms of the continued fraction expansion. More precisely, as in the previous section, let $T : [0,1] \to [0,1]$ denote the shift map of the continued fraction expansion,

$$T(x) = \frac{1}{x} - \left[\frac{1}{x}\right],$$

extended to a map $T : [0,1] \times \mathbb{P} \to [0,1] \times \mathbb{P}$ with $\mathbb{P} = \Gamma/G$. The Lyapunov exponent of the shift map is given by the limit (when it exists)

$$\lambda(x) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(x)| = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(x), \tag{3.34}$$

where $q_n(x)$ are the successive denominators of the continued fraction expansion of $x \in [0,1]$. There is a decomposition $[0,1] = \cup_\lambda \mathcal{L}_\lambda \cup \mathcal{L}'$ where $\mathcal{L}_\lambda = \{x \in [0,1] \,|\, \lambda(x) = \lambda\}$ and $\mathcal{L}'$ the set on which the limit (3.34) does not exist. For all $\theta \in \mathcal{L}_\lambda$ the limiting modular symbol (3.33) is then given by

$$\{\{\star, \theta\}\}_G = \lim_{n\to\infty} \frac{1}{\lambda n} \sum_{k=1}^{n} \{g_k^{-1}(\theta)\cdot 0, g_k^{-1}(\theta)\cdot\infty\}_G = \lim_{n\to\infty} \frac{1}{\lambda n} \sum_{k=1}^{n} \{\frac{p_{k-1}(\theta)}{q_{k-1}(\theta)}, \frac{p_k(\theta)}{q_k(\theta)}\}_G. \tag{3.35}$$

The results of [MM02] and [Mar03] show that the limiting modular symbol (3.35) vanishes almost everywhere, with respect to the Hausdorff measure of $\mathcal{L}_\lambda$. However, non-vanishing values of the limiting modular symbols are obtained, for example, for all the quadratic irrationalities.

In the case of quadratic irrationalities, one obtains two equivalent descriptions of the limiting modular symbol, one that corresponds to integration on the closed geodesic $C_\theta = \Gamma_\theta \backslash S_\theta$ with S_θ the infinite geodesic with endpoints the Galois conjugate pair θ, θ' and the other in terms of averaged integration on the modular symbols associated to the (eventually periodic) continued fraction expansion. We obtain the identification of homology classes in $H_1(X_G, \mathbb{R})$

$$\{\{\star, \theta\}\}_G = \frac{\sum_{k=1}^{\ell} \{\frac{p_{k-1}(\theta)}{q_{k-1}(\theta)}, \frac{p_k(\theta)}{q_k(\theta)}\}_G}{\lambda(\theta)\ell} = \frac{\{0, g\cdot 0\}_G}{\ell(g)} \in H_1(X_G, \mathbb{R}). \tag{3.36}$$

In the first expression ℓ is the minimal positive integer for which $T^\ell(\theta) = \theta$ and the limit defining the Lyapunov exponent $\lambda(\theta)$ exists for quadratic irrationalities. In the second expression $g \in \Gamma$ is the hyperbolic generator of Γ_θ with fixed points θ, θ', with eigenvalues $\Lambda_g^{\pm}$ and $\{0, g\cdot 0\}_G$ denotes the homology class in $H_1(X_G, \mathbb{R})$ of the closed geodesic C_θ and $\ell(g) = \log \Lambda_g^- = 2 \log \epsilon$ is the length of C_θ.

A more complete analysis of the values of the limiting modular symbols was then carried out in [KS07b], where it was shown that, in fact, the limiting modular symbol is non-vanishing on a multifractal stratification of Cantor sets of positive Hausdorff dimension. We will recall more precisely this result in §3.3.3 below.

The construction recalled above of limiting modular symbols determine non-trivial real homology classes in the quotient X_G associated to geodesics in $\mathbb{H}$ with endpoints in one of the multifractal level sets of [KS07b]. These homology classes pair with 1-forms on X_G, and in particular with weight 2 cusp forms for the finite index subgroup G.

Let $\mathcal{M}_{G,k}$ the $\mathbb{C}$-vector space of modular forms of weight k for the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ and let $\mathcal{S}_{G,k}$ be the subspace of cusp forms. Let $X_G = \mathrm{GL}_2(\mathbb{Z})\backslash(\mathbb{H}^\pm \times \mathbb{P})$ be the associated modular curve. We denote by

$$\langle \cdot, \cdot \rangle : \mathcal{S}_{G,2} \times H_1(X_G, \mathbb{R}) \to \mathbb{C} \tag{3.37}$$

the perfect pairing between cusp forms of weight 2 and modular symbols, which we equivalently write as integration

$$\langle \psi, \{\alpha, \beta\}_G \rangle = \int_{\{\alpha,\beta\}_G} \psi(z)\, dz. \tag{3.38}$$

3.3.3 Boundary values

We consider now a linear map, constructed using cusp forms and limiting modular symbols, that assigns to an observable of the bulk GL_2-system a boundary value.

Let $\mathcal{L} \subset [0,1]$ denote the subset of points such that the Lyapunov exponent (3.34) of the shift of the continued fraction exists. The set $\mathcal{L}$ is stratified by level sets $\mathcal{L}_\lambda = \{x \in [0,1] \,|\, \lambda(x) = \lambda\}$, with the Lyapunov spectrum given by the Hausdorff dimension function $\delta(\lambda) = \dim_H(\mathcal{L}_\lambda)$. Recall also that, for a continuous function ϕ on a T-invariant subset $E \subset [0,1]$, the Birkhoff spectrum (see [FF00]) is the function

$$f_E(\alpha) := \dim_H \mathcal{L}_{\phi,E,\alpha}, \tag{3.39}$$

where $\mathcal{L}_{\phi,E,\alpha}$ are the level sets of the Birkhoff average by

$$\mathcal{L}_{\phi,E,\alpha} := \{x \in E \,|\, \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \phi \circ T^k(x) = \alpha\}. \tag{3.40}$$

In particular, $\mathcal{L}_\lambda = \mathcal{L}_{\phi,\lambda}$ for $\phi(x) = \log |T'(x)|$. Lyapunov and Birkhoff spectra for the shift of the continued fraction expansion are analyzed in [PW99], [Fan+09].

In a similar way, one can consider the multifractal spectrum associated to the level sets of the limiting modular symbol, as analyzed in [KS07b]. Let $f_1, \dots, f_g$ be a basis of the complex vectors space $\mathcal{S}_{G,2}$ of cusp forms of weight 2 for the finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$. Let $\Re(f_i)$, $\Im(f_i)$ be the corresponding basis as a real $2g$-dimensional vector space. Under the pairing (3.37), (3.38), which identifies $\mathcal{S}_{G,2}$ with the dual of $H_1(X_G, \mathbb{R})$, we can define as in [KS07b] the level sets E_α, for $\alpha \in \mathbb{R}^{2g}$, of the limiting modular symbol as

$$E_\alpha := \{(x,s) \in [0,1] \times \mathbb{P} \,|\, \langle f_i, \{\{\star, x\}\}_G\rangle = \alpha \in \mathbb{R}^{2g}\}. \tag{3.41}$$

Equivalently, we write this as

$$E_\alpha = \{(x,s) \in [0,1]\times\mathbb{P} \,|\, (\lim_{n\to\infty} \frac{1}{\lambda(x)n} \int_{\{g_k^{-1}(x)\cdot 0, g_k^{-1}(x)\cdot\infty\}_G} f_i(z)\, dz)_{i=1\dots,g} = \alpha \in \mathbb{R}^{2g}\}.$$

For $(x,s) \in E_\alpha$ we have

$$\lim_{n\to\infty} \frac{1}{\lambda(x)n} \{g_k^{-1}(x)\cdot 0, g_k^{-1}(x)\cdot\infty\}_G = h_\alpha \in H_1(X_G, \mathbb{R}),$$

where the homology class h_α is uniquely determined by the property that $\langle f_i, h_\alpha\rangle = \int_{h_\alpha} f_i(z)dz = \alpha_i$.

The main result of [KS07b] shows that for a given finite index subgroup $G \subset \mathrm{GL}_2(\mathbb{Z})$ with X_G of genus $g \geq 1$, there is a strictly convex and differentiable function $\beta_G : \mathbb{R}^{2g} \to \mathbb{R}$ such that, for all $\alpha \in \nabla\beta_G(\mathbb{R}^{2g}) \subset \mathbb{R}^{2g}$

$$\dim_H(E_\alpha) = \hat{\beta}_G(\alpha), \tag{3.42}$$

where $\hat{\beta}_G(\alpha) = \inf_{v\in\mathbb{R}^{2g}}(\beta_G(v) - \langle\alpha, v\rangle)$ is the Legendre transform, and for all $(x,s) \in E_\alpha$

$$\lim_{n\to\infty} \frac{1}{\lambda(x)n} \{g_k^{-1}(x)\cdot g\cdot 0, g_k^{-1}(x)\cdot g\cdot\infty\}_G = h_\alpha(x,s) \tag{3.43}$$

with g a representative of the class $s \in \mathbb{P} = \mathrm{GL}_2(\mathbb{Z})/G$.

Let $E = \{E_\alpha\}$ be the collection of the level sets E_α of the limiting modular symbol. We let $\mathcal{A}_{\partial,G,\mathcal{P},h_\alpha} = B_{\partial,E} \rtimes$ Red be the algebra associated to the collection E of invariant sets, as in Corollary 3.2.11. As an immediate consequence of the results (3.42), (3.43) of [KS07b] we have the following.

Lemma 3.3.1. *The choice of a cusp form $\psi \in \mathcal{S}_{G,2}$ determines a bounded linear operator $\mathcal{I}_{\psi,\alpha}$ from $\mathcal{A}_{GL_2(\mathbb{Z}),G,\mathcal{P}}$ to B_{∂,E_α} with for $(x,s) \in E_\alpha$*

$$\begin{aligned}\mathcal{I}_{\psi,\alpha}(f)(g,\rho,x,s) &= \int_{\{\star,x\}_G} f(g,\rho,z,s)\,\psi(z)\,dz \\ &= \lim_{n\to\infty} \frac{1}{\lambda(x)\,n} \sum_{k=1}^{n} \int_{\{g_k^{-1}(x)\cdot 0, g_k^{-1}(x)\cdot\infty\}_G} f(g,\rho,z,s)\,\psi(z)\,dz \\ &= \int_{h_\alpha(x)} f(g,\rho,z,s)\,\psi(z)\,dz.\end{aligned} \tag{3.44}$$

Here we pair the form $\omega(z) = f(z)\psi(z)dz$ with the limiting modular symbol $h_\alpha(x,s)$ of (3.43). We use the notation $\omega(z) = \omega_{\rho,s}(z)$ to highlight the dependence on the variables (ρ, s) that come from choosing an element f in the arithmetic algebra $\mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$.

Note that $\mathcal{I}_{\psi,\alpha}$ is only a linear operator and *not* an algebra homomorphism. We obtain a subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}}$ as follows.

Definition 3.3.2. Let $\mathcal{A}_{\mathcal{I},G,\mathcal{P}}$ denote the subalgebra of $\mathcal{A}_{\partial,G,\mathcal{P}} = B_\partial \rtimes$ Red generated by all the images $\mathcal{I}_{\psi,\alpha}(f)$ for $f \in \mathcal{A}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$, for $\psi \in \mathcal{S}_{G,2}$, and for $\alpha \in \nabla\beta_G(\mathbb{R}^{2g})$, and by the S_k, S_k^* with the relations as in Definition 3.2.8. The arithmetic algebra $\mathcal{A}^{ar}_{\mathcal{I},G,\mathcal{P}}$ is obtained in the same way as the algebra generated by the images $\mathcal{I}_{\psi,\alpha}(f)$ with f in the arithmetic algebra $\mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ of §3.1.2, for all $\psi \in \mathcal{S}_{G,2}$ and $\alpha \in \nabla\beta_G(\mathbb{R}^{2g})$, and by the S_k, S_k^* as above.

3.3.4 Evaluation of ground states on boundary values

When we evaluate zero-temperature KMS states on the elements $\mathcal{I}_{\psi,\alpha}(f)$, for an element $f \in \mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$, we obtain the pairing of a cusp form with a limiting modular symbol,

$$\varphi_{\infty,\rho,x,s}(\mathcal{I}_{\psi,\alpha}(f)) = \langle \omega_{\rho,s}, h_\alpha(x) \rangle, \tag{3.45}$$

where $\omega_{\rho,s}(z) = f(1,\rho,z,s)\psi(z)dz$ is a cusp form in $\mathcal{S}_{G,2}$ for all (ρ, s). Since elements $f \in \mathcal{A}^{ar}_{\mathrm{GL}_2(\mathbb{Z}),G,\mathcal{P}}$ depend on the variable $\rho \in M_2(\hat{\mathbb{Z}})$ through some finite projection $\pi_N(\rho) \in \mathbb{Z}/N\mathbb{Z}$, we can write $\omega_{\rho,s}(z)$ as a finite collection $\{\omega_{i,s}(z)\}_{i\in\mathbb{Z}/N\mathbb{Z}}$.

To illustrate the properties of the values of ground states on arithmetic boundary elements, we consider here the particular case where $G = \Gamma_0(N)$ and a state $\varphi_{\infty,\rho,x,s}$ with $s \in \mathbb{P}$. We also choose f and ψ so that the resulting $\omega_{i,s}$ are cusp forms for $\Gamma_0(N)$ that are Hecke eigenforms for all the Hecke operators $T(m)$.

Recall (see [Ser77]) that the Hecke operators $T(m)$ given by

$$T_m = \sum_{\gamma\,:\,\det(\gamma)=m} \Gamma_0(N)\gamma\Gamma_0(N)$$

satisfying $T_nT_m = T_mT_n$ for $(m,n) = 1$ and $T_{p^n}T_p = T_{p^{n+1}} + pT_{p^{n-1}}R_p$ with R_λ the scaling operator that acts on a modular form of weight $2k$ as multiplication by λ^{-2k}. The Hecke operators T_m and the scaling operators R_λ generate a commutative algebra, and the action of T_m on a modular form of weight $2k$ is given by

$$T_m\, f(z) = n^{2k-1} \sum_{a\geq 1, ad=n, 0\leq b<d} d^{-2k}\, f(\frac{az+b}{d}).$$

Proposition 3.3.3. *Let $G = \Gamma_0(N)$ and let ω_{i,s_0}, with $i = 0, \dots, N-1$ and $s_0 = \Gamma_0(N)g_0 \in \mathbb{P}$, be Hecke eigencuspforms of weight 2. For $s \in \mathbb{P}$ with $s = \Gamma_0(N)g_0\gamma$, with $\gamma \in GL_2(\mathbb{Z})$, let $\omega_{i,s} = \omega_{i,s_0\gamma} := \omega_{i,s_0} \circ \gamma^{-1}$. Consider the pairing*

$$\xi_\omega(s) = \langle \omega_{\rho,s}, h_\alpha(x) \rangle = \int_{h_\alpha(x)} \omega_{i,s}$$

with the limiting modular symbol $h_\alpha(x)$, as in (3.44) and (3.45). For $(m,N) = 1$ we have the relations

$$a_{i,m}\, \xi_\omega(s) = \sum_{M \in \bar{A}_m} u_m\, \xi_\omega(s\, M)$$

where $a_{i,m}$ are the Hecke eigenvalues and $A_m = \{M \in M_2(\mathbb{Z}) : \det(M) = m\}$, with $\bar{A}_m = A_m/\{\pm 1\}$ and $\sum_M u_M\, M \in \mathbb{Z}\bar{A}_m$ is the Manin–Heilbronn lift of the Hecke operator T_m.

Proof. The condition that $\omega_{i,s\gamma} = \omega_{i,s} \circ \gamma^{-1}$ implies that $\langle \omega_{i,s}, h_\alpha(x) \rangle = \langle \omega_{i,s_0}, h_\alpha(x,s) \rangle$. The following facts are known from [Man72], [Mer91]. Let

$$A_{m,N} = \{M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \,:\, \det(M) = m,\ N|c\}.$$

Let R be a set of representatives for the classes $\Gamma_0(N)\backslash A_{m,N}$ The Hecke operators act on the modular symbols by $T_m\{\alpha,\beta\} = \sum_{\lambda \in R}\{\lambda\alpha, \lambda\beta\}$. For $(m,N) = 1$ there is a bijection between the cosets $\Gamma_0(N)\backslash A_{m,N}$ and $A_m/\mathrm{SL}_2(\mathbb{Z})$. For $s \in \mathbb{P}$ consider the assignment $\xi_\omega : s \mapsto \xi_\omega(s) = \langle \omega_{i,s}, h_\alpha(x,s) \rangle$, where $\omega_{i,s}$ is a Hecke eigencuspform. It is shown in [Man72], [Mer91] that there is a lift Θ_m of the action of the Hecke operators $T_m \circ \xi = \xi \circ \Theta_m$ (the Manin–Heilbronn lift), which is given by $\Theta_m = \sum_{\gamma \in A_m/\mathrm{SL}_2(\mathbb{Z})} \Upsilon_\gamma$, where Υ_γ is a formal chain of level m connecting ∞ to 0 and of class γ. This means that $\Upsilon_\gamma = \sum_{k=0}^{n-1} \gamma_k$ in $\mathbb{Z}A_m$, for some $n \in \mathbb{N}$ where

$$\gamma_k = \begin{pmatrix} u_k & u_{k+1} \\ v_k & v_{k+1} \end{pmatrix}$$

with $u_0/v_0 = \infty$ and $u_n/v_n = 0$ and where γ_k agrees with γ in $A_m/\mathrm{SL}_2(\mathbb{Z})$. An argument in [Mer91] based on the continued fraction expansion and modular symbols shows that it is always possible to construct such formal chains with $\gamma_k = \gamma g_k$ with $g_k \in \mathrm{SL}_2(\mathbb{Z})$ and that the resulting Θ_m is indeed a lift of the Hecke operators. (The length n of the chain of the Manin–Heilbronn lift is also computed, see §3.2 of [Mer91].) Using the notation of Theorem 4 of [Mer91], we write the Manin–Heilbronn lift as $\Theta_m = \sum_{M \in \bar{A}_m} u_M\, M$ as an element of $\mathbb{Z}A_m$. Each element $M \in \bar{A}_m$ maps $s \mapsto s\, M$ in $\mathcal{P}$, hence one obtains a map $\Theta_m : \mathbb{Z}\mathcal{P} \to \mathbb{Z}\mathcal{P}$. In particular, as shown in Theorem 2 of [Mer91], for $s = \Gamma_0(N)g$ in $\mathbb{P}$ one has $\Theta_m(s) = \sum_{\gamma \in R} \sum_{k=0}^{n-1} \phi(g\gamma\gamma_k)$

where $\phi : A_m \to \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$ is the map that assigns

$$A_m \ni \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \Gamma_0(N) \begin{pmatrix} w & t \\ u & v \end{pmatrix} \in \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$$

with $(c : d) = (u : v)$ in $\mathbb{P}^1(\mathbb{Z}/N\mathbb{Z}) \simeq \mathbb{P} = \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{Z})$. Thus, as in Theorem 2 of [Mer91] we then have $\Theta_m(s) = \sum_{\gamma \in R} \sum_{k=0}^{n-1} \phi(g\gamma) g_k = \sum_{\gamma \in R} \phi(g\gamma)\gamma^{-1}$, seen here as an element in $\mathbb{Z}\mathcal{P}$. Thus, in the pairing of limiting modular symbols and boundary elements we find

$$T_m \xi_\omega(s) = T_m \int_{h_\alpha(x,s)} \omega_{i,s_0} = \int_{h_\alpha(x,s)} T_m \omega_{i,s_0} = a_{i,m} \int_{h_\alpha(x,s)} \omega_{i,s_0},$$

where $a_{i,m}$ are the Hecke eigenvalues of the eigenform ω_{i,s_0}, with $a_{i,1} = 1$. On the other hand, using the Manin–Heilbronn lift we have

$$T_m \xi_\omega(s) = \xi_\omega(\Theta_m(s)) = \sum_{\gamma \in R} \int_{h_\alpha(x,s_\gamma)} \omega_{i,s_0}$$

with $s_\gamma \in \mathcal{P}$ given by $s_\gamma = \phi(g\gamma)\gamma^{-1}$ as above. We write the latter expression in the form

$$\sum_{M \in \bar{A}_m} u_M \int_{h_\alpha(x,s\,M)} \omega_{i,s_0}$$

for consistency with the notation of Theorem 4 of [Mer91]. □

Proposition 3.3.4. *Under the same hypothesis as Proposition 3.3.3, let* $L_{\omega_{i,s_0}}(\sigma) = \sum_m a_{i,m}\, m^{-\sigma}$ *be the L-series associated to the cusp form* $\omega_{i,s_0} = \sum_m a_{i,m}\, q^m$. *For* x *a quadratic irrationality the evaluation* (3.45) *of* KMS_∞ *states satisfies*

$$\langle \omega_{i,s}, h(x) \rangle = \frac{1}{\lambda(x) n} \sum_{k=1}^{n} \langle \omega_{i,s_k}, \{0, \infty\} \rangle = \frac{1}{\lambda(x) n} \sum_{k=1}^{n} L_{\omega_{i,s_k}}(1), \tag{3.46}$$

where $s_k = \Gamma_0(N) g g_k^{-1}(x)$ *for* $s = \Gamma_0(N) g$.

Proof. The special value $L_{\omega_{i,s_0}}(1)$ of the L-function gives the pairing with the modular symbol $\langle \omega_{i,s_0}, \{0, \infty\} \rangle$. Similarly, for $s \in \mathbb{P}$ with $s = \Gamma_0(N) g$, the special value gives

$$L_{\omega_{i,s}}(1) = \langle \omega_{i,s}, \{0, \infty\} \rangle = \langle \omega_{i,s_0}, \{g \cdot 0, g \cdot \infty\} \rangle.$$

In the case of a quadratic irrationality the limiting modular symbol satisfies

$$h(x) = \frac{1}{\lambda(x) n} \sum_{k=1}^{n} \{g_k^{-1}(x) \cdot 0, g_k^{-1}(x) \cdot \infty\}_G,$$

where n is the length of the period of the continued fraction expansion of x and $\lambda(x)$ is the Lyapunov exponent. For $s_k = \Gamma_0(N) g g_k^{-1}(x)$ with $s = \Gamma_0(N) g$, we have $\langle \omega_{i,s_k}, \{0, \infty\} \rangle = \langle \omega_{i,s}, \{g_k^{-1}(x) \cdot 0, g_k^{-1}(x) \cdot \infty\} \rangle$ hence one obtains (3.46). □

As shown in Theorem 3.3 of [Man72], the special value $L_{\omega_{i,s_0}}(1)$ satisfies

$$(\sum_{d|m} d - a_{i,m})L_{\omega_{i,s_0}}(1) = \sum_{d|m,\, b \mod d} \int_{\{0,b/d\}_G} \omega_{i,s_0},$$

since one has

$$\int_0^\infty T_m \omega_{i,s_0} = a_{i,m} \int_0^\infty \omega_{i,s_0} = \sum_{d|m} \sum_{b=0}^{d-1} \int_{\{b/d,0\}_G + \{0,\infty\}_G} \omega_{i,s_0}.$$

For a normalized Hecke eigencuspform $f = \sum_n a_n q^n$, let $L_f(s) = \sum_n a_n n^{-s}$ be the associated L-function and $\Lambda_f(s) = (2\pi)^{-s}\Gamma(s)L_f(s)$ the completed L-function, the Mellin transform $\Lambda_f(s) = \int_0^\infty f(iz)z^{s-1}dz$.

The relation between special values of L-functions and periods of Hecke eigenforms generalizes for higher weights, and it was shown in [Man73] that ratios of these special values of the same parity are algebraic (in the field generated over $\mathbb{Q}$ by the Hecke eigenvalues). For a normalized Hecke eigencuspform $f = \sum_n a_n q^n$ of weight k the coefficients of the period polynomial $r_f(z)$ are expressible in terms of special values of the L-function,

$$r_f(z) = -i \sum_{j=0}^{k-2} \binom{k-2}{j} (iz)^j \Lambda_f(j+1).$$

Manin's Periods Theorem shows that, for $\mathbb{K}_f$ the field of algebraic numbers generated over $\mathbb{Q}$ by the Fourier coefficients, there are $\omega_\pm(f) \in \mathbb{R}$ such that for all $1 \leq s \leq k-1$ with s even $\Lambda_f(s)/\omega_+(f) \in \mathbb{K}_f$, respectively $\Lambda_f(s)/\omega_-(f) \in \mathbb{K}_f$ for s odd.

Shokurov gave a geometric argument based on Kuga varieties and a higher-weight generalization of modular symbols, [Sho81a]. It is expected that the limiting modular symbols of [MM02], as well as the quantum statistical mechanics of the GL_2-system and its boundary described here, will generalize to the case of Kuga varieties, with the relations between periods of Hecke eigencuspforms described in [Man73] arising in the evaluation of zero-temperature KMS states of these systems. The first steps of this project are discussed in the next chapter.

Chapter 4

KUGA VARIETIES AND HIGHER-WEIGHT LIMITING MODULAR SYMBOLS

The first step in extending the work of the previous chapter to a higher-weight setting is to define the limiting modular symbols for weight greater than 2. We begin with the body of work by Shokurov ([Sho81a]) in which the modular symbol for higher weights is defined, based on the Kuga modular varieties and their projections onto the modular curves. We then define a limiting modular symbol by means of a limiting procedure analogously to the standard weight-2 case. We show that the limiting modular symbol can be written as an ergodic average involving the continued fraction expansion, and in particular it converges almost everywhere. To do this we use similar techniques to those of [KS07b]. The idea is to move to a coding space setting where each geodesic in $\mathbb{H}$ is coded using its type changes as it traverses the Farey tessellation. The advantage of this approach is that it allows us to write the limiting modular symbol as an ergodic average everywhere, without having to exclude an exceptional set where the Lyapunov exponent does not converge.

4.1 Shokurov modular symbols of higher weight

We briefly recall the definition of the standard modular symbol . Let $G \subset \mathrm{SL}_2(\mathbb{Z})$ be a modular group, $X_G = G\backslash\mathbb{H}$ the modular curve, and $\Pi = G\backslash\mathbb{P}_1(\mathbb{Q})$ the cusps of the modular curve. Fixing two points $\alpha, \beta \in \mathbb{H} \cup \mathbb{P}_1(\mathbb{Q})$, we define the modular symbol $\{\alpha, \beta\}_G \in H_1(X_G, \mathbb{R})$ by

$$\int_{\{\alpha,\beta\}_G} \omega = \int_\alpha^\beta \phi^*(\omega) \tag{4.1}$$

where the integral on the right-hand side is taken along the geodesic arc connecting α and β and $\phi : \mathbb{H} \to X_G$. Modular symbols have the additivity property

$$\{\alpha, \beta\}_G + \{\beta, \gamma\}_G = \{\alpha, \gamma\}_G.$$

Because of this additivity property, it is sufficient to consider modular symbols of the form $\{0, \alpha\}$ with $\alpha \in \mathbb{Q}$

$$\{0, \alpha\} = -\sum_{k=1}^{N} \{g_k(0), g_k(i\infty)\}_G$$

where

$$g_k = \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix}$$

with p_k/q_k is the kth continued fraction approximant of α and $p_N/q_N = \alpha$. Finally, for any $g \in G$ we have that

$$g\{\alpha, \beta\}_G = \{g(\alpha), g(\beta)\}_G = \{\alpha, \beta\}_G.$$

Following [Sho81a], we define the modular symbols of weight greater than 2. From a pair (G, w) where G is a modular group and w is a weight, one can construct a nonsingular projective variety B_G^w over the complex numbers called a Kuga modular variety. [Sho76] This variety is related to a elliptic surface B_G over the modular curve $\overline{X_G}$. There is a natural projection from B_G^w onto the modular curve $\Phi^w : B_G^w \to \overline{X_G}$.

4.1.1 Kuga modular variety

The Kuga modular variety is constructed using as a starting point the modular elliptic surface, which is an elliptic surface B_G over the modular curve $\Phi : B_G \to \overline{X}_G$. It has the important property that the functional invariant is given by J_G, where J_G is a meromorphic function on $\overline{X}_G$ given by the composition of the morphism

$$\overline{X}_G \to \overline{X}_{\mathrm{SL}_2(\mathbb{Z})}$$

induced by the subgroup structure $G \subset \mathrm{SL}_2(\mathbb{Z})$ with the absolute invariant function

$$j : \overline{X}_{\mathrm{SL}_2(\mathbb{Z})} \to \mathbb{C}$$

extending the standard j-invariant. Such an elliptic modular surface is canonically defined in the case that $-1 \notin G$ by [Shi72]. In the absence of this condition, a non-canonical construction with the desired property is given in [Sho76].

The Kuga modular variety is obtained by taking the Kuga variety, which can be constructed from any non-singular projective surface over a modular curve, of the modular elliptic surface. We give a very brief sketch of this construction. For details, please see [Sho76].

Let Δ' be the set of non-singular points of $\overline{X}_G$, and $\mathcal{U}'$ be its universal cover. There is an action of

$$\mathcal{G}^w = \pi_1(\Delta') \times \mathbb{Z}^w \times \mathbb{Z}^w$$

on $\mathcal{U}' \times \mathbb{C}^w$ given by

$$(\beta, n, m)(u, \xi) = (\beta u, f_\beta(u)(\xi + z(u)n + m))$$

where z is a multivalued function on Δ' defined by

$$j(z(u)) = J_G(u)$$

and a choice of branch gives a function $z : \mathcal{U}' \to \mathbb{H}$, and

$$f_\beta(u) = (cz(u) + d)^{-1}$$

where c, d are given by the entries of the matrix $S(\beta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $S : \pi_1(\Delta') \to \mathrm{SL}_2(\mathbb{Z})$ is a certain representation of the fundamental group.

We define

$$\overline{B}_G^w|_{\Delta'} = \mathcal{G}^w \backslash (\mathcal{U}' \times \mathbb{C}^w).$$

By compactifying and resolving singularities, we then obtain a non-singular projective variety B_G^w, with a canonical projection $\Phi^w : B_G^w \to \overline{X}_G$.

4.1.2 Shokurov symbols

To define the modular symbol of weight $w+2$, we first define $\{\alpha, n, m\}_G$, a boundary modular symbol of weight $w+2$, by a mapping

$$\begin{aligned} \{,,\}_G : \tilde{\mathbb{Q}} \times \mathbb{Z}^w \times \mathbb{Z}^w &\to H_0(\Pi, (R_1\Phi_*\mathbb{Q})^w) \\ (\alpha, n, m) &\mapsto \{\alpha, n, m\}_G \end{aligned}$$

where $\tilde{\mathbb{Q}} = \mathbb{Q} \cup \{i\infty\}$, $(R_1\Phi_*\mathbb{Q})^w$ is a symmetric tensor power of the sheaf $R_1\Phi_*\mathbb{Q} = \mathbb{G} \otimes_{\mathbb{Q}} \mathbb{Q}$, where $\mathbb{G}$ is the homological invariant of B_G.

Remark 4.1.1. In general the sheaf $R_j\Phi_*^w\mathbb{Q}$ is defined by taking the sheaf of local coefficients

$$\cup_{v \in \Delta'} H_j(B_v^w, \mathbb{Q})$$

and extending it over $\overline{X_G}$. We will only need to use the case $R_1\Phi_{*1}\mathbb{Q} = \mathbb{G} \otimes_{\mathbb{Q}} \mathbb{Q}$, which can be interpreted as a rational homological invariant.

This mapping is described carefully in Section 1.1 of [Sho81a], but we summarize the construction here. Let $\alpha \in \tilde{\mathbb{Q}}$, and $n, m \in \mathbb{Z}^w$. Let $p_0 \in \Pi$ be the cusp corresponding to α. There is a decomposition

$$H_0(\Pi, (R_1\Phi_*\mathbb{Q})^w) = \bigoplus_{p \in \Pi} H_0(p, (R_1\Phi_*\mathbb{Q})^w).$$

The modular symbol $\{\alpha, n, m\}$ is trivial on $H_0(p, (R_1\Phi_*\mathbb{Q})^w)$ when $p \neq p_0$, and so will be defined by an element in $H_0(p_0, (R_1\Phi_*\mathbb{Q})^w)$. Let $E \subset \overline{X}_G$ be a small disc around p_0. Let U_α be a neighborhood of α in $\mathbb{H}' = \mathbb{H} \setminus \mathrm{SL}_2(\mathbb{Z})\{e^{\frac{2\pi i}{3}}\}$ that covers E, and let $\tilde{\Gamma}_\alpha : U_\alpha \to E$ be the covering. Choose a point $z_E \in U_\alpha$ and let

$v_E = \tilde{\Gamma}_\alpha(z_E)$. View z_E as a 0-cell in a cellular decomposition of the disc E. Now we let $\{z_E, n, m\}_G^E \subset H_0(E, (R_1\Phi_*\mathbb{Q})^w)$ the cohomology class of the cycle

$$\sum_{j=1}^{w} (n_j e_1 + m_j e_2) v_E$$

where $\{e_1, e_2\}$ is a certain basis which we will not describe in detail here. There is a projective system of spaces $H_0(E, (R_1\Phi_*\mathbb{Q})^w))$ by morphisms

$$H_0(E', (R_1\Phi_*\mathbb{Q})^w)) \to H_0(E, (R_1\Phi_*\mathbb{Q})^w))$$

where $E' \subset E \subset \overline{X_G}$ are nested small discs. Finally, we set

$$\{\alpha, n, m\}_G = \lim_{\substack{\leftarrow \\ E}} \{z_e, n, m\}_G^E.$$

It requires some argument to see that this definition makes sense, but we do not include it here as we will not need to work with this definition directly.

The modular symbol, $\{\alpha, \beta, n, m\}_G$, is then defined ([Sho81a] Lemma 1.2) via the unique mapping

$$\begin{aligned} \tilde{Q} \times \tilde{Q} \times \mathbb{Z}^w \times \mathbb{Z}^w &\to H_1(\overline{X_G}, \Pi, (R_1\Phi_*\mathbb{Q})^w) \\ (\alpha, \beta, n, m) &\mapsto \{\alpha, \beta, n, m\}_G \end{aligned}$$

such that

1. $\partial\{\alpha, \beta, n, m\}_G = \{\beta, n, m\}_G - \{\alpha, n, m\}_G$ where ∂ is the boundary mapping of the pair $(\overline{X_G}, \Pi)$.

2. For any cusp forms $\Psi_1, \Psi_2 \in S_{w+2}(G)$

   $$\langle \{\alpha, \beta, n, m\}_G, (\Psi_1, \overline{\Psi_2}) \rangle = \int_\alpha^\beta \Psi_1 \Pi_{j=1}^w (n_j z + m_j) dz + \int_\alpha^\beta \overline{\Psi_2} \Pi_{j=1}^w (n_j \overline{z} + m_j) d\overline{z}$$

   where $n = (n_1, ..., n_w)$, $m = (m_1, ...m_w)$ and $\langle , \rangle$ is the canonical pairing described in [Sho81b]:

   $$\langle , \rangle : H_1(\overline{X_G}, Y, (R_1\Phi_*\mathbb{Q})^w) \times S_{w+2}(G) \oplus \overline{S_{w+2}(G)} \to \mathbb{C}$$

   where $Y \subset \overline{X_G}$.

Importantly, the pairing $\langle , \rangle$ is non-degenerate on $H_1(\overline{X_G}, (R_1\Phi_*\mathbb{Q})^w) \times S_{w+2}(G) \oplus \overline{S_{w+2}(G)}$ [Sho81b].

The modular symbols of higher weight have a similar additivity property to the weight-2 case:

$$\{\alpha, \beta, n, m\}_G + \{\beta, \gamma, n, m\}_G = \{\alpha, \gamma, n, m\}_G$$

and they transform by elements $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{Z})$ as

$$g|\{\alpha, \beta, n, m\}_G = \{g(\alpha), g(\beta), g \cdot (n, m)\}_G = \{g(\alpha), g(\beta), dn - cm, -bn + am\}_G.$$

Note that this does not give an action directly on $H_1(\overline{X_G}, \Pi, (R_1\Phi_*\mathbb{Q})^w)$, but rather on representations of homology classes as modular symbols. For $g \in G$, we have

$$g|\{\alpha, \beta, n, m\}_G = \{g(\alpha), g(\beta), dn - cm, -bn + am\}_G = \{\alpha, \beta, n, m\}_G. \tag{4.2}$$

Again, due to the additivity property, it is sufficient to consider modular symbols of the form, for $\alpha \in \mathbb{Q}$

$$\{0, \alpha, n, m\}_G = -\sum_{k=1}^{N} \{g_k(0), g_k(i\infty), n, m\}_G.$$

4.1.3 Limiting modular symbols

The paper [MM02] introduced a generalization of the modular symbols to the whole boundary $\mathbb{P}^1(\mathbb{R})$ by considering an infinite geodesic γ_β in $\mathbb{H}$ with one with one end at $\beta \in \mathbb{R} \setminus \mathbb{Q}$ and the other end at $\alpha \in \mathbb{R}$. Let $x_0 \in \mathbb{H}$ be a fixed point on γ_β and $y(\tau)$ a point along γ_β with an arc length distance of τ away from x_0 towards β. The *limiting modular symbol* is defined as the following limit, whenever it exists:

$$\{\{*, \beta\}\}_G = \lim_{\tau \to \infty} \frac{1}{\tau} \{x_0, y(\tau)\}_G \in H_1(X_G, \mathbb{R}) \tag{4.3}$$

where $\{x_0, y(\tau)\}_G$ is the homology class determined by the geodesic arc between x_0 and $y(\tau)$ in $\mathbb{H}$. The limit is independent of the choice of x_0 and of γ_β (§2 of [MM02]).

4.1.4 Shift map and the Lyapunov spectrum

To study the weight-2 limiting modular symbols, we consider a modular curve of the form $X_G = \mathrm{PGL}_2(\mathbb{Z}) \backslash (\mathbb{H} \times \mathbb{P})$ where $\mathbb{P} = \mathrm{PGL}_2(\mathbb{Z})/G$ and the associated shift map

$$\begin{aligned} T : [0,1] \times \mathbb{P} &\to [0,1] \times \mathbb{P} \\ (\beta, t) &\mapsto \left(\frac{1}{\beta} - \left[\frac{1}{\beta} \right], \begin{pmatrix} -[1/\beta] & 1 \\ 1 & 0 \end{pmatrix} t \right). \end{aligned} \tag{4.4}$$

Defining a map $\phi : \mathbb{P} \to H_1(\overline{X_G}, \Pi, \mathbb{R})$ by

$$\phi(s) = \{g(0), g(i\infty)\}_G$$

where $g \in \mathrm{PSL}_2(\mathbb{Z})$ is a representative of the coset $s \in \mathbb{P}$, we see that g_k acts on points $(\beta, t) \in [0,1] \times \mathbb{P}$ as the k^{th} power of the shift operator T. Precisely,

$$\phi(T^k(\beta,t)) = \{g_k(\beta)(0), g_k(\beta)(i\infty)\}_G = -\left\{\frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)}\right\}_G$$

where, as before,

$$g_k(\beta) = \begin{pmatrix} p_{k-1}(\beta) & p_k(\beta) \\ q_{k-1}(\beta) & q_k(\beta) \end{pmatrix}$$

acts by Mobius transformations.

It is shown in [Mar03] that the limiting modular symbol can be computed on certain level sets as a Birkhoff average. The level sets are given by the Lyapunov spectrum of the shift map on the unit interval

$$\begin{aligned} T : [0,1] &\to [0,1] \\ \beta &\mapsto \frac{1}{\beta} - \left[\frac{1}{\beta}\right]. \end{aligned} \tag{4.5}$$

Recall that the Lyapunov exponent of a map $T : [0,1] \to [0,1]$ is given by the T-invariant function

$$\lambda(\beta) = \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(\beta)|.$$

In the particular case of T defined by equation 4.5, the Lyapunov exponent is

$$\lambda(\beta) = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(\beta). \tag{4.6}$$

A theorem of Lévy [Lév29] shows that $\lambda(\beta) = \frac{\pi^2}{6\log 2}$ for almost all β. We can decompose the unit interval into level sets of 4.6, $L_c = \{\beta \in [0,1] : \lambda(\beta) = c\}$

$$[0,1] = \cup_{c\in\mathbb{R}} L_c \cup \{\beta \in [0,1] : \lambda(\beta) \text{ does not exist}\}.$$

Then, we have the following result about the limiting modular symbols.

Proposition 4.1.1 ([Mar03] Theorem 2.1)**.** *For a fixed $c \in \mathbb{R}$ and for $\beta \in L_c$, the limiting modular symbol 4.3 is computed by*

$$\lim_{n\to\infty} \frac{1}{cn} \sum_{k=1}^{n} \phi \circ T^k(\beta, t_0) \tag{4.7}$$

where T is the shift operator defined in 4.4 and t_0 is a base point.

It is easy to check that the shift of the continued fraction expansion is measure-preserving with respect to the Gauss measure

$$d\mu = (\log 2)^{-1} \frac{dx}{1+x}$$

and so the limiting modular symbol exists almost everywhere. However, it is also known that there is an exceptional set of measure 0 and Hausdorff dimension 1 where $\lambda(\beta)$ does not exist ([PW99] Theorem 3). On the exceptional set, the limiting modular symbol cannot be written as the limit 4.7. Finally, in the special case that β is a quadratic irrationality (and hence has a periodic continued fraction expansion) it is shown ([Mar03] Lemma 2.2) that the limiting modular symbol is given by

$$\{\{*, \beta\}\}_G = \frac{\sum_{k=1}^{n} \{g_k^{-1}(\beta) \cdot g(0), g_k^{-1}(\beta) \cdot g(i\infty)\}_G}{\lambda(\beta) n}$$

where n is the period of the continued fraction expansion. In this case it is also known that the limit $\lambda(\beta)$ converges to a positive finite number, so that in particular the limiting modular symbol does not vanish.

To extend this picture to the higher weight setting, we now define $\phi : \mathbb{P} \times \mathbb{Z} \times \mathbb{Z} \to H_1(\overline{X_G}, \Pi, (R_1\Phi_*\mathbb{Q})^w)$ by

$$\phi(s, n, m) = g|\{0, i\infty, n, m\}_G = \{g(0), g(\infty), dn - cm, -bn + am\}_G$$

where $g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$ and g is a representative of the coset $s \in \mathbb{P}$. The action of the shift operator on the higher-weight modular symbols is now described by the relation

$$\begin{aligned} \phi(T^k(\beta, t), n, m) &= \{g_k(0), g_k(i\infty), g_k^{-1} \cdot (n, m)\}_G \\ &= -\left\{ \frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)}, \begin{pmatrix} 0 & -1 \\ -1 & -a_k \end{pmatrix} \cdots \begin{pmatrix} 0 & -1 \\ -1 & -a_1 \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \right\}_G \end{aligned} \tag{4.8}$$

where $\beta = [a_1, ..., a_N]$ is the continued fraction expansion of β. Note that, again, this action is not on $H_1(\overline{X_G}, \Pi, (R_1\Phi_*\mathbb{Q})^w)$, but on representations of the homology classes as modular symbols.

Instead of proceeding with this setting, however, we will move to a related setting where we code each geodesic in the hyperbolic plane using cells of the Farey tessellation. It was introduced by Kessenbómer and Stratmann in [KS07b] in order to obtain a more complete description of the standard modular symbols and their level set structure.

4.2 Twisted continued fraction coding and shift space

Following [KS07b] we define a code space related to the dynamical system given by the shift map in the previous section. We recall that an oriented geodesic in $\mathbb{H}$

can be coded by a sequence of "type changes". Consider the Farey tessellation of $\mathbb{H}$ formed by $\mathrm{PSL}_2(\mathbb{Z})$-translates of the triangle with vertices at 0,1, and $i\infty$. As we travel along a geodesic in the positive direction, each tile is intersected in such a way that one vertex of the triangle is on one side, and two vertices of the triangle are on the other. If the single vertex is on the left, we say the visit to the tile is of type L, and if the single vertex is on the right, we say it is of type R. Let $l = (l_+, l_-)$ be the oriented geodesic with start point l_+ and end point l_- and consider the set

$$\mathcal{L} = \{l = (l_-, l_+)| 0 < |l_+| \leq 1 \leq |l_-|, l_- l_+ < 0, \text{and } l_-, l_+ \in \mathbb{R} \setminus \mathbb{Q}\}.$$

Each $l \in \mathcal{L}$ is coded by the types of its visits

$$...L^{n_{-2}} R^{n_{-1}} y_l L^{n_1} R^{n_2} ... \text{ if } l_- \geq 1$$
$$...R^{n_{-2}} L^{n_{-1}} y_l R^{n_1} L^{n_2} ... \text{ if } l_- \leq -1$$

where y_l is the point where l intersects the imaginary axis.

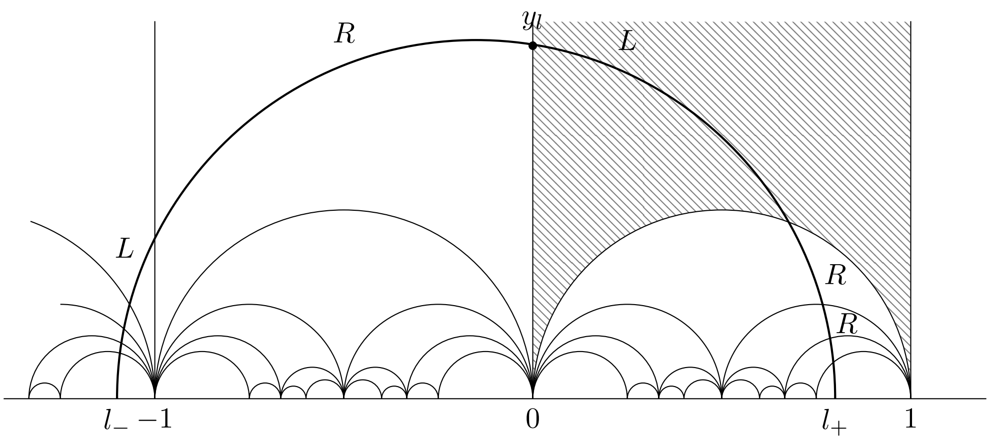

FIGURE 4.1: FAREY TESSELATION AND CODING OF A GEODESIC

This coding is related to the continued fraction expansion of the endpoints l_+ and l_- by

$$l_- = [n_{-1}, n_{-2}, ...]^{-1} \text{ and } l_+ = -[n_1, n_2, ...] \text{ if } l_- \geq 1,$$
$$l_- = -[n_{-1}, n_{-2}, ...]^{-1} \text{ and } l_+ = [n_1, n_2, ...] \text{ if } l_- \leq -1.$$

We now consider the generators S and T of $\mathrm{PSL}_2(\mathbb{Z})$ given by

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

which can also be thought of as their actions on $\mathbb{H}$ as $S : z \mapsto -1/z$ and $T : z \mapsto z+1$ and define the map $\tilde{\mathcal{P}} : \mathcal{L} \to \mathcal{L}$

$$\tilde{\mathcal{P}}(l) = \begin{cases} ST^{-n_1}(l) = (-[n_2, n_3, ...]^{-1}, [n_1, n_{-1}, ...]) & \text{if } l = ([n_1, n_2, ...]^{-1}, -[n_{-1}, n_{-2}, ...]) \\ ST^{n_1}(l) = ([n_2, n_3, ...]^{-1}, -[n_1, n_{-1}, ...],) & \text{if } l = (-[n_1, n_2, ...]^{-1}, [n_{-1}, n_{-2}, ...]) \end{cases}.$$

Let $\mathcal{P}$ be the restriction of $\tilde{\mathcal{P}}$ to the first coordinate. Then the map

$$\begin{aligned}\mathcal{G} : [-1,1] &\to [-1,1]\\ x &\mapsto S\mathcal{P}S(x)\end{aligned}$$

is called the twisted Gauss map. It is related to the shift map T by

$$\mathcal{G}(x) = -\mathrm{sign}(x)T(|x|).$$

We define the shift space to be

$$\Sigma_* = \left\{(x_1, x_2, ...) \in (\mathbb{Z}^\times)^N | x_i x_{i+1} < 0 \forall i \in \mathbb{N}\right\}$$

with the shift map $\sigma_*(x_1, x_2, ...) = (x_2, x_3, ...)$. The map

$$\begin{aligned}\rho : \Sigma_* &\to \mathcal{I}\\ (x_1, x_2, ...) &\mapsto -\mathrm{sign}(x_1)[|x_1|, |x_2|, ...]\end{aligned}$$

where $\mathcal{I} = [-1,1] \cap (\mathbb{R} \diagdown \mathbb{Q})$, is a bijection with the property $\rho \circ \sigma_* = \mathcal{G} \circ \rho$.

We also wish to consider a generalization of this setup where G is a modular subgroup of $\mathrm{PSL}_2(\mathbb{Z})$. Let E_G be a set of fixed representative elements of the left cosets in $G\backslash\mathrm{PSL}_2(\mathbb{Z})$. We now consider the set of oriented geodesics given by

$$\mathcal{L}_G = \bigcup_{e \in E_G} e(\mathcal{L})$$

and the space

$$\overline{\Sigma}_G = \bigcup_{e \in E_G} e(\mathcal{I}) \times \{e\}$$

with the topology inherited from $\mathbb{R}$. The G-twisted Gauss map is

$$\begin{aligned}\mathcal{G}_G : \overline{\Sigma}_G &\to \overline{\Sigma}_G\\ (x, e) &\mapsto \left(eS\mathcal{P}Se^{-1}(x), e\right)\end{aligned}$$

for $x \in e(\mathcal{I})$. It is shown in [KS07b] that a certain proper shift space Σ_G is isomorphic to $\overline{\Sigma}_G$. This shift space is

$$\Sigma_G = \{((x_1, e_1), (x_2, e_2), ...) \in (\mathbb{Z}^\times \times E_G)^N | (x_1, x_2, ...) \in \Sigma_*, \text{ and } e_{k+1} = \tau_{x_k}(e_k) \forall k \in \mathbb{N}\}$$

where $\tau_{x_k} : E_G \to E_G$ is defined by

$$\tau_{x_k}(e_k) \equiv_G e_k S T^{x_k}$$

equipped with the shift map

$$\begin{aligned}\sigma : \Sigma &\to \Sigma\\ ((x_1, e_1), (x_2, e_2), ...) &\mapsto ((x_2, e_2), (x_3, e_3), ...)\end{aligned}$$

and metric

$$d(((x_k, e_k))_k, ((x'_k, e'_k)_k)) = \sum_{i-=1}^{\infty} \frac{1}{2^i} \left(1 - \delta_{(x_i,e_i),(x'_i,e'_i)}\right).$$

The isomorphism $\overline{\Sigma}_G \to \Sigma_G$ is given by

$$(e(\pm[n_1, n_2, ...]), e) \mapsto ((\mp n_1, e), (\mp n_2, \tau_{\mp n_1}(e), (\mp n_3, \tau_{\mp n_2}(\tau_{\mp n_1}(e))), ...).$$

Formulating the limiting modular symbols in terms of the shift space rather than directly in terms of points in $\mathbb{R}$ is useful because the shift space (Σ_G, σ) is known to be finitely irreducible (Prop 3.1 [KS07b]). This means that there is a finite set $W \subset \Sigma_G^*$, where Σ_G^* is the set of finite admissible words in the alphabet $\mathbb{Z}^\times \times E_G$, such that for any $a, b \in \mathbb{Z}^\times \times E_G$ there exists $w \in W$ such that $awb \in \Sigma_G^*$.

It is also shown in [KS07b] that, as elements in $((x_i, e_i))_i \in \Sigma_G$ satisfy $e_{k+1} = \tau_{x_k}(e_k)$, there is a relation in terms of the continued fraction expansion of $x = -\text{sign}(x_1)[|x_1|, |x_2|, ...] = [\tilde{x_1}, \tilde{x_2}, ...]$

$$e_{k+1} \equiv_G e_1 ST^{\tilde{x_1}} ... ST^{\tilde{x_k}} = e_1 \overline{g}_k(x)$$

where

$$\overline{g}_k(x) = \begin{pmatrix} -\text{sign}(x_1) p_{k-1}(|x|) & (-1)^k p_k(|x|) \\ q_{k-1}(|x|) & (-1)^{k+1} \text{sign}(x_1) q_k(|x|) \end{pmatrix}. \tag{4.9}$$

Similar to equation 4.8, we have the relation describing the action of $\overline{g}_k(x)$ on higher-weight modular symbols

$$\begin{aligned} &\overline{g}_k(x)|\{0, i\infty, n, m\}_G \\ &= -\left\{ -\text{sign}(x_1) \frac{p_{k-1}(|x|)}{q_{k-1}(|x|)}, -\text{sign}(x_1) \frac{p_k(|x|)}{q_k(|x|)}, \begin{pmatrix} 0 & 1 \\ -1 & -|x_k| \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & -|x_1| \end{pmatrix} \begin{pmatrix} n \\ m \end{pmatrix} \right\}_G. \end{aligned} \tag{4.10}$$

4.3 Limiting modular symbol for the shift space

We now define a corresponding modular symbol on the shift space. Let $\overline{X}_G = (\mathbb{H} \cup P^1(\mathbb{Q}))/G$. For an element of Σ_G, the associated limiting modular symbol on the shift space is

$$\begin{aligned} \tilde{l}_G : \Sigma_G &\to H_1(\overline{X}_G, \mathbb{R}) \\ ((x_k, e_k))_k &\mapsto \lim_{t \to \infty} \frac{1}{t} \{i, e_1(x + i\exp(-t))\}_G \end{aligned} \tag{4.11}$$

where we set $x = -\text{sign}(x_1)[|x_1|, |x_2|, ...] \in \mathcal{I}$. The modular symbol on the right-hand side, $\{i, e_1(x + i\exp(-t))\}_G \in H_1(X_G, \mathbb{R})$, is the standard modular symbol defined by Equation 4.1.

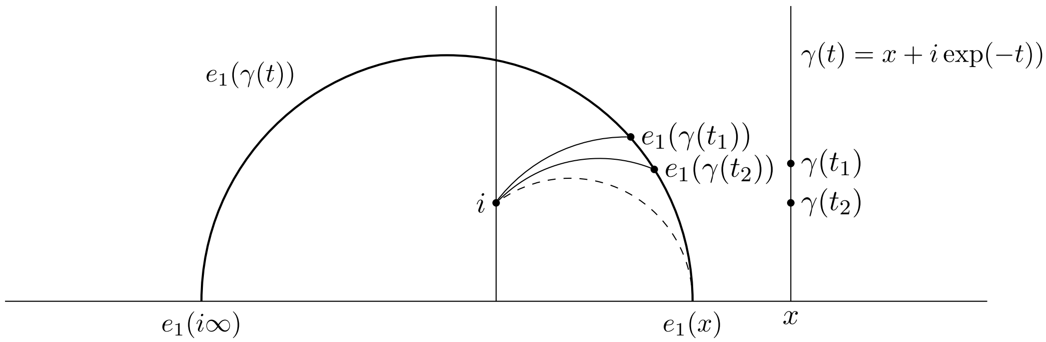

FIGURE 4.2: DEFINITION OF THE LIMITING MODULAR SYMBOL FOR THE SHIFT SPACE

It is known that this limit can be equivalently written by approximating the point $e_1(x)$ by its continued fraction expansion (Proposition 4.2 [KS07b])

$$\tilde{l}_G(((x_k, e_k))_k) = \lim_{n\to\infty} \frac{1}{2\log q_n(|x|)} \sum_{i=1}^{n} \{e_k(i\infty), e_k(0)\}_G. \tag{4.12}$$

We generalize this picture to the higher-weight setting by putting

$$\begin{aligned} \tilde{l}_{G,n,m} &: \Sigma_G \to H_1(\overline{X_G}, \Pi, (R_1\Phi_*\mathbb{Q})^w) \\ ((x_k, e_k))_k &\mapsto \lim_{t\to\infty} \frac{1}{t}\{i, e_1(x + i\exp(-t)), n, m\}_G. \end{aligned} \tag{4.13}$$

We proceed by obtaining a similar result to equation 4.12, but now with modular symbols of higher weight. Importantly, the result holds everywhere on Σ_G.

Theorem 4.3.1. *For $((x_k, e_k))_k \in \Sigma_G$ we have*

$$\tilde{l}_{G,N,M}\left(((x_k, e_k))_k\right) = \lim_{n\to\infty} \frac{1}{2\log q_n(|x|)} \sum_{k=1}^{\infty} \{e_k(i\infty), e_k(0), \tilde{g}_{k-1}^{-1}(x) \cdot (N, M)\}_G$$

where $\tilde{g}_{k-1}(x) = e_1\overline{g_{k-1}}(x)e_k^{-1}$ and we set $x = -sign(x_1)[|x_1|, |x_2|, ...]$.

Proof. The proof follows the strategy outlined in [KS07b], but here we track the additional (n, m)-coordinate data of the higher-weight modular symbol. The general strategy is as follows. We begin by showing that

$$L_{G,N,M}\left(((x_k, e_k))_k\right) := \lim_{n\to\infty} \frac{1}{2\log q_n(|x|)} \sum_{k=1}^{\infty} \{e_k(i\infty), e_k(0), \tilde{g}_{k-1}^{-1}(N, M)\}_G. \tag{4.14}$$

exists if and only if there is a sequence $(t_n)_{n\in\mathbb{N}}$ tending to infinity such that

$$\lim_{n\to\infty} \frac{1}{t_n}\{i, e_1(x + ie^{-t_n}), N, M\}_G \tag{4.15}$$

exists, and that if either limit exists they coincide. Then, we will show that the limit 4.15 does not depend on the particular sequence $(t_n)_{n\in\mathbb{N}}$ chosen.

The main idea is to write the geodesic passing through $e_1(i\infty)$ and $e_1(x)$, in terms of geodesics related to the continued fraction approximants of x. Define a sequence of points in $P^1(\mathbb{Q})$ by

$$\begin{aligned} \xi_1 &= e_1(i\infty) \\ \xi_n &= e_1\left(-\text{sign}(x_1)\frac{p_{n-2}(|x|)}{q_{n-2}(|x|)}\right) \qquad n \geq 2 \end{aligned} \tag{4.16}$$

and let ω_n be the oriented geodesic in $\mathbb{H} \cup P^1(\mathbb{Q})$ which starts at ξ_n and ends at ξ_{n+1}.

Next, let $l(x)$ be the oriented vertical geodesic running from $i\infty$ to x and let $e_1(l(x))$ be its image. The image $e_1(l(x))$ is a geodesic starting at ξ_1 and ending at $e_1(x)$. Define a sequence of points along $e_1(l(x))$ by

$$y_n = \omega_n \cup e_1(l(x)). \tag{4.17}$$

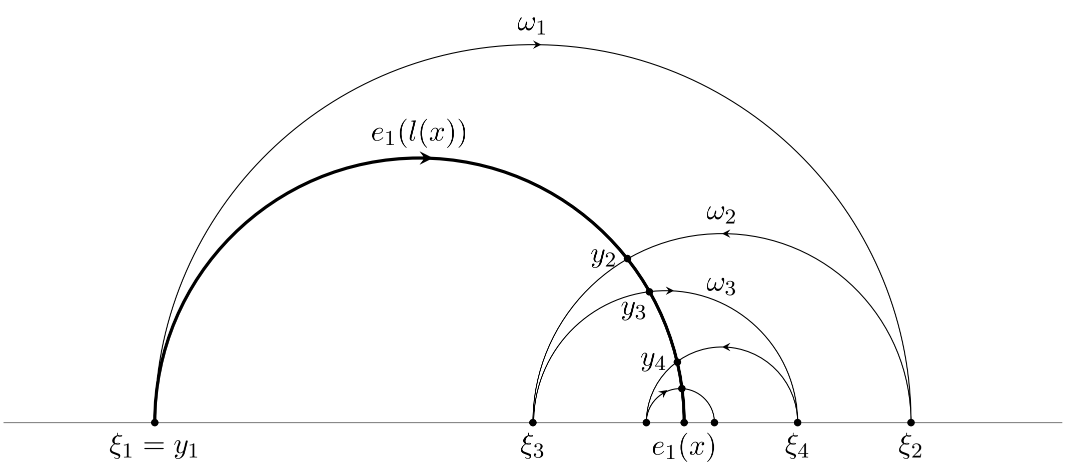

FIGURE 4.3: APPROXIMATING $e_1(l(x))$ BY CONTINUED FRACTIONS

Note that the oriented geodesic path from y_n to y_{n+1} is homologous to the geodesic path running from y_n to ξ_{n+1} to y_{n+1}. Therefore, we have that

$$\{y_n, y_{n+1}, N, M\}_G = \{y_n, \xi_{n+1}, N, M\}_G + \{\xi_{n+1}, y_{n+1}, N, M\}_G$$

for all $n \in \mathbb{N}$.

Recall that we have

$$e_{n+1} \equiv_G e_1\overline{g}_n(x)$$

where $\overline{g}_n$ for $n \geq 2$ is defined in 4.9 and $\overline{g_1} = \text{id}$. Therefore, there exists some $\tilde{g}_n(x) \in G$ such that

$$\tilde{g}_n(x)e_{n+1} = e_1\overline{g}_n(x).$$

By property 4.2 of the modular symbols, and by directly acting g_n on the points 0 and $i\infty$ by fractional linear transformations, we get

$$\begin{aligned}\{e_n(i\infty), e_n(0), \tilde{g}_{n-1}^{-1}(x)\cdot(N,M)\}_G &= \{\tilde{g}_{n-1}(x)e_n(i\infty), \tilde{g}_{n-1}(x)e_n(0), N, M\}_G \\ &= \{e_1\overline{g}_{n-1}(x)(i\infty), e_1\tilde{g}_{n-1}(x)(0), N, M\}_G \\ &= \{\xi_n, \xi_{n+1}, N, M\}_G.\end{aligned}$$

Using homologous paths and additivity of the modular symbols we find

$$\begin{aligned}&\{i, y_{n+1}, N, M\}_G = \{i, y_2, N, M\}_G + \{y_2, y_{n+1}, N, M\}_G \\ &= \{i, y_2, N, M\}_G + \sum_{k=2}^{n}\{y_k, y_{k+1}, N, M\}_G \\ &= \{i, y_2, N, M\}_G + \sum_{k=2}^{n}\left(\{y_k, \xi_{k+1}, N, M\}_G + \{\xi_{k+1}, y_{k+1}, N, M\}_G\right) \\ &= \{i, y_2, N, M\}_G - \{\xi_2, y_2, N, M\}_G - \{y_{n+1}, \xi_{n+1}, N, M\}_G + \sum_{k=2}^{n+1}\{\xi_k, \xi_{k+1}, N, M\}_G \\ &= \{i, \xi_1, N, M\}_G - \{y_{n+1}, \xi_{n+1}, N, M\}_G + \sum_{k=1}^{n+1}\{\xi_k, \xi_{k+1}, N, M\}_G \\ &= \{i, \xi_1, N, M\}_G - \{y_{n+1}, \xi_{n+1}, N, M\}_G + \sum_{k=1}^{n+1}\{e_k(i\infty), e_k(0), \tilde{g}_{k-1}^{-1}\cdot(N,M)\}_G\end{aligned}$$

Let the sequence t_n be defined by the equation

$$e_1(x + ie^{-t_n}) := y_n.$$

An argument involving hyperbolic geometry gives an estimate $e^{-t_n} \sim (q_n(|x|))^2$, for sufficiently large n (see Lemma 3.3 of [KS07a]).

With this we can complete the first part of the proof, concluding the equivalence of the limits:

$$\begin{aligned}L_{G,N,M}\left(((x_k, e_k))_k\right) &= \lim_{n\to\infty}\frac{1}{2\log q_n(|x|)}\sum_{k=1}^{\infty}\{e_k(i\infty), e_k(0), \tilde{g}_{k-1}^{-1}\cdot(N,M)\}_G \\ &= \lim_{n\to\infty}\frac{1}{t_n}(\{i, y_{n+1}, N, M\}_G + \{y_{n+1}, \xi_{n+1}, N, M\}_G - \{i, \xi_1, N, M\}_G) \\ &= \lim_{n\to\infty}\frac{1}{t_n}(\{i, y_n, N, M\}_G + \{y_n, \xi_{n+1}, N, M\}_G - \{i, \xi_1, N, M\}_G) \\ &= \lim_{n\to\infty}\frac{1}{t_n}\{i, y_n, N, M\}_G \\ &= \lim_{n\to\infty}\frac{1}{t_n}\{i, e_1(x + ie^{-t_n}), N, M\}_G\end{aligned}$$

The second step of the proof is to show that the limit $\lim_{n\to\infty}\frac{1}{t_n}\{i, e_1(x+ie^{-t_n}), N, M\}_G$ is independent of the choice of sequence (t_n) tending to infinity. Recall that we have a non-degenerate pairing between the higher-weight modular symbols and the space of cusp forms $S_{w+2}(G) \oplus \overline{S_{w+2}(G)}$. Suppose that $L_{G,N,M}\left(((x_k, e_k))_k\right)$ exists. For $\Phi = (\Phi_1, \Phi_2) \in S_{w+2}(G) \oplus \overline{S_{w+2}(G)}$ arbitrary and $t > 0$, let

$$\alpha_\Phi = \langle L_{G,N,M}\left(((x_k, e_k))_k\right), \Phi\rangle$$

and let

$$n_t = \sup\{n \in \mathbb{N} : 2\log q_n(|x|) \le t\}.$$

Our aim is to show that for all $\Phi \in S_{w+2}(G) \oplus \overline{S_{w+2}}(G)$,

$$\limsup_{t\to\infty} \left| \frac{\langle \{i, e_1(x+ie^{-t}), N, M\}_G, \Phi\rangle}{t} - \frac{\langle \sum_{k=1}^{n_t} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi\rangle}{2\log q_{n_t}(|x|)} \right| = 0,$$

which will allow us to conclude that $\tilde{l}_{G,N,M}\left(((x_k, e_k))_k\right)$ exists and is equal to $L_{G,N,M}\left(((x_k, e_k))_k\right)$. We obtain a bound following exactly the same strategy as [KS07b], but we repeat it here for completeness.

$$\limsup_{t\to\infty} \left| \frac{\langle \{i, e_1(x+ie^{-t}), N, M\}_G, \Phi\rangle}{t} - \frac{\left\langle \sum_{k=1}^{n_t} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}{2\log q_{n_t}(|x|)} \right|$$

$$= \limsup_{t\to\infty} \left| \frac{2\log q_{n_t}(|x|)\langle \{i, e_1(x+ie^{-t}), N, M\}_G, \Phi\rangle}{2t\log q_{n_t}(|x|)} - \frac{t\left\langle \sum_{k=1}^{n_t} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}{2t\log q_{n_t}(|x|)} \right|$$

$$\le \limsup_{t\to\infty} \left| \frac{1}{t}\langle \{i, e_1(x+ie^{-t}), N, M\}_G - \sum_{k=1}^{n_t} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi\rangle \right|$$
$$+ \limsup_{t\to\infty} \left| \frac{2\log q_{n_t}(|x|) - t}{t} \right| \left| \frac{\left\langle \sum_{k=1}^{n_t} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}{2\log q_{n_t}(|x|)} \right|$$

$$\le \limsup_{t\to\infty} \frac{const.}{t} + \limsup_{n\to\infty} \frac{\log|x_{n+1}|}{\log q_n(|x|)} |\alpha_\Phi|$$
$$= |\alpha_\Phi| \limsup_{n\to\infty} \frac{\log|x_{n+1}|}{\log q_n(|x|)}$$

where in the last bound we are using the recursion relation $q_n(|x|) = |x_{n+1}|q_{n-1}(|x|) + q_{n-2}(|x|)$.

In the case that $\alpha_\Phi = 0$ for all $\Phi \in S_{w+2}(G) \oplus \overline{S_{w+2}(G)}$, the result follows immediately. We will assume wlog that there is some Φ such that $\alpha_\Phi > 0$. In this case, we will show that $\limsup_{n\to\infty} \frac{\log|x_{n+1}|}{\log q_n(|x|)} = 0$. To do this, we first observe that

$$\begin{aligned}
\alpha_\Phi &= \lim_{n\to\infty} \frac{\left\langle \sum_{k=1}^{n+1} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}{2\log q_{n+1}(|x|)} \\
&= \lim_{n\to\infty} \frac{\left\langle \sum_{k=1}^{n} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G + \{e_{n+1}(i\infty), e_{n+1}(0), \tilde{g}_{n+1}^{-1}(N,M)\}_G, \Phi \right\rangle}{2\log q_n(|x|) + 2\log|x_{n+1}|} \\
&= \lim_{n\to\infty} \frac{\left\langle \sum_{k=1}^{n} \{e_k(i\infty, e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle \left(1 + \frac{\langle \{e_{n+1}(i\infty), e_{n+1}(0), \tilde{g}_{n+1}^{-1}(N,M)\}_G, \Phi\rangle}{\left\langle \sum_{k=1}^{n} \{e_k(i\infty, e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}\right)}{2\log q_n(|x|)\left(1 + \frac{\log|x_{n+1}|}{\log q_n(|x|)}\right)} \\
&= \alpha_\Phi \lim_{n\to\infty} \frac{1 + \frac{\langle \{e_{n+1}(i\infty), e_{n+1}(0), \tilde{g}_{n+1}^{-1}(N,M)\}_G, \Phi\rangle}{\left\langle \sum_{k=1}^{n} \{e_k(i\infty), e_k(0), \tilde{g}_k^{-1}(N,M)\}_G, \Phi \right\rangle}}{1 + \frac{\log|x_{n+1}|}{\log q_n(|x|)}}.
\end{aligned}$$

Suppose for contradiction that $\limsup_{n\to\infty} \frac{\log|x_{n+1}|}{\log q_n(|x|)} > 0$. Then there is a subsequence $(n_k)_k$ such that $\lim_{k\to\infty} \frac{\log|x_{n_k+1}|}{\log q_{n_k}(|x|)} > 0$, and hence $\lim_{k\to\infty} |x_{n_k+1}| = \infty$. Since we have assumed that $\alpha_\Phi > 0$, we get

$$\begin{aligned}
1 &= \lim_{k\to\infty} \frac{1 + \frac{\langle \{e_{n_k+1}(i\infty), e_{n_k+1}(0), \tilde{g}_{n_k+1}^{-1}(N,M)\}_G, \Phi\rangle}{\left\langle \sum_{j=1}^{n_k} \{e_j(i\infty), e_j(0), \tilde{g}_j^{-1}(N,M)\}_G, \Phi \right\rangle}}{1 + \frac{\log|x_{n_k+1}|}{\log q_{n_k}(|x|)}} \\
&= \lim_{k\to\infty} \frac{\log q_{n_k}(|x|)}{\left\langle \sum_{j=1}^{n_k} \{e_j(i\infty), e_j(0), \tilde{g}_j^{-1}(N,M)\}_G, \Phi \right\rangle} \frac{\langle \{e_{n_k+1}(i\infty), e_{n_k+1}(0), \tilde{g}_{n_k+1}^{-1}(N,M)\}_G, \Phi\rangle}{\log|x_{n_k+1}|} \\
&= \frac{1}{\alpha_\Phi}(0) = 0
\end{aligned}$$

This is a contradiction, so we conclude that $\limsup_{n\to\infty} \frac{\log|x_{n+1}|}{\log q_n(|x|)} = 0$. □

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