Descriptive set theory and dynamics of countable groups

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ABSTRACT

This thesis comprises four papers.

1. We show that for any Polish group G and any countable normal subgroup $\Gamma \triangleleft G$, the coset equivalence relation G/Γ is a hyperfinite Borel equivalence relation. In particular, the outer automorphism group of any countable group is hyperfinite.

2. Given a countable Borel equivalence relation E and a countable group G, we study the problem of when a Borel action of G on X/E can be lifted to a Borel action of G on X.

3. Let Γ be a countable group. A classical theorem of Thorisson states that if X is a standard Borel Γ-space and μ and ν are Borel probability measures on X which agree on every Γ-invariant subset, then μ and ν are equidecomposable, i.e., there are Borel measures $(\mu_\gamma)_{\gamma \in \Gamma}$ on X such that $\mu = \sum_\gamma \mu_\gamma$ and $\nu = \sum_\gamma \gamma\mu_\gamma$. We establish a generalization of this result to cardinal algebras.

4. Let R be a ring equipped with a proper norm. We show that under suitable conditions on R, there is a natural basis under continuous linear injection for the set of Polish R-modules which are not countably generated. When R is a division ring, this basis can be taken to be a singleton.

PUBLISHED CONTENT AND CONTRIBUTIONS

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Chapter 1

INTRODUCTION

1.1 Descriptive set theory

Descriptive set theory is the study of "definable" subsets and functions in Polish spaces, for various notions of definability. Most commonly, we take "definable" to mean Borel, where a **Borel set** is an element of the σ-algebra generated by the open subsets of a Polish space, and where a **Borel function** $X \to Y$ is one under which the preimage of every open set is Borel. Other common notions of "definable" include **analytic** and **Baire-measurable**.

By restricting one's study to definable sets, it is possible to prove more structural theorems than in the general setting. For instance, the continuum hypothesis holds for Borel sets, that is to say, every Borel subset of $\mathbb{R}$ is either countable or of size continuum. This is in stark contrast to arbitrary subsets of $\mathbb{R}$, where the existence of a subset $A \subseteq \mathbb{R}$ with $\aleph_0 < |A| < |\mathbb{R}|$ is independent of ZFC. Another benefit is the absence of pathologies obtained from the Axiom of Choice and other non-constructive arguments, such as the existence of a basis for $\mathbb{R}$ as a $\mathbb{Q}$-vector space, which is possible due to the Axiom of Choice but not in a definable way, in that there is no Borel basis.

1.2 Borel equivalence relations

Over the past forty years, descriptive set theory has seen a wide variety of connections with areas outside of logic, such as ergodic theory, operator algebras, geometric group theory, and more recently, computer science. One important concept which has emerged in these applications is that of a **Borel equivalence relation**, that is, an equivalence relation E on a standard Borel space X such that E is a Borel subset of X^2. Many classification problems in mathematics arise as Borel equivalence relations, such as the classification of finite rank torsion-free abelian groups up to isomorphism, or the classification of unitary operators on the infinite-dimensional Hilbert space up to conjugacy.

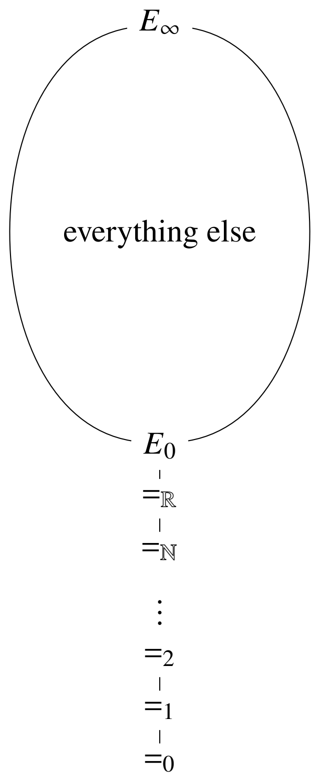
The most important notion in Borel equivalence relations is **Borel reduction**, which lets one talk about the relative hardness of two problems, analogous to polynomial time reduction in complexity theory. Given Borel equivalence relations E and F on

X and Y respectively, a Borel reduction is a Borel map $f : X \to Y$ such that $x \mathrel{E} x'$ iff $f(x) \mathrel{F} f(x')$. Informally, it says that if one can classify up to F-equivalence then one is also able to classify up to E-equivalence by applying f. If there is a Borel reduction from E to F, one says that E is **Borel reducible** to F, denoted $E \leq_B F$. The "simplest" class of Borel equivalence relations are the **smooth** equivalence relations, which are those Borel equivalence relations E which Borel reduce to $=_{\mathbb{R}}$, the equality relation on $\mathbb{R}$. In other words, there are concrete invariants that exactly classify the elements of X up to E-equivalence. For instance, Ornstein's isomorphism theorem says that Bernoulli shifts are classified up to isomorphism by their Kolmogorov-Sinai entropy, and thus isomorphism of Bernoulli shifts is a smooth equivalence relation, with the Borel reduction being the map which sends a shift to its entropy.

1.3 Countable Borel equivalence relations

A large part of my research is focused on **countable Borel equivalence relations (CBER)**, which are Borel equivalence relations with every class countable. An important source of examples arises as follows: given a countable group Γ with a Borel action on a standard Borel space X, the **orbit equivalence relation** E^X_Γ is defined by $x \mathrel{E^X_\Gamma} x'$ iff $\exists \gamma [x' = \gamma \cdot x]$. This is a CBER, and in fact, a theorem of Feldman and Moore shows that every CBER on X is the orbit equivalence relation of some Borel action of a countable group on X. In this way, the theory of CBERs is very intimately connected with the study of countable groups and their Borel and measurable aspects. Amenability plays an important role in the study of hyperfiniteness (defined below), and the use of property (T) groups allows us to invoke such powerful theorems as Popa's cocycle superrigidity theorems.

The Borel reducibility preorder $\leq_B$ on CBERs looks like the following:



where the relations in this diagram are defined as follows (starting from the bottom):

- $=_X$ is the equality relation on the space X.
- E_0 is the **eventual equality** relation on 2^ω defined by

$$x \; E_0 \; y \iff \exists n \forall m \geq n \, [x_m = y_m].$$

- E_∞ is the orbit equivalence relation induced by the shift action of F_2 on 2^{F_2}. This is a **universal** CBER, that is, for every CBER F, we have $F \leq_B E_\infty$.

1.4 Hyperfinite Borel equivalence relations

A relatively low class (in terms of Borel reduction) of CBERs is that of the **hyperfinite** CBERs. A CBER E on a standard Borel space X is hyperfinite if it satisfies any of the following equivalent conditions:

1. There is an increasing sequence $(E_n)_n$ of finite Borel equivalence relations on X such that $E = \bigcup_n E_n$ (an equivalence relation is **finite** if every class is finite);

2. There is a Borel $\mathbb{Z}$-action on X such that $E = E_{\mathbb{Z}}^X$;

3. $E \leq_B E_0$,

From the image above and the third characterization of hyperfiniteness, we see that the hyperfinite CBERs are exactly the next level of complexity after the smooth CBERs, which justifies the view that the hyperfinite CBERs have relatively low complexity.

Other examples of hyperfinite CBERs arise from boundary actions of countable groups. For instance, we have shown in [HSS20] that if Γ is a cubulated hyperbolic group, then action of Γ on its Gromov boundary $\partial\Gamma$ is hyperfinite, generalizing the classical result of Dougherty-Jackson-Kechris [DJK94, Corollary 8.2] that the tail equivalence relation E_t on 2^ω is hyperfinite. The same result for arbitrary (finitely generated) hyperbolic groups has been obtained by Marquis and Sabok in [MS20].

In a completely different direction, we have shown recently that hyperfinite CBERs arise from the cosets of a countable normal subgroup:

Theorem 1.4.1. *[FS22b] If G is a Polish group and Γ is a countable normal subgroup, then G/Γ is hyperfinite (where we write G/Γ to mean E_Γ^G).*

In particular, if Γ is a countable group, then $\mathrm{Out}(\Gamma)$ *is hyperfinite.*

This was a surprising result since generally, a nontrivial construction taking a countable group Γ to a CBER will reflect some of the complexity of Γ. For instance, if a countable group Γ is non-amenable, then the shift action of Γ on 2^Γ is not hyperfinite, and it is an important open question as to whether this characterizes (non)-amenability.

1.5 Lifts of Borel actions

Let E be a countable Borel equivalence relation on X. A **Borel permutation** of X/E is a bijection $f : X/E \to X/E$ such that the set $\{(x, x') \in X^2 : f([x]_E) = [x']_E\}$ is Borel. Let $\mathrm{Sym}_B(X/E)$ denote the group of Borel permutations of X/E. We are concerned with the problem of lifting Borel permutations to Borel isomorphisms on the space X. More precisely, a **Borel automorphism** of E is a Borel isomorphism $T : X \to X$ such that $x \mathrel{E} x'$ iff $f(x) \mathrel{E} f(x')$. Let $\mathrm{Aut}_B(E)$ denote the group of Borel automorphisms of E. Every $T \in \mathrm{Aut}_B(E)$ induces a Borel permutation of X/E sending $[x]_E$ to $[T(x)]_E$, and a Borel permutation of X/E induced by a Borel automorphism of E is called **outer**. The **outer automorphism group** of E, denoted $\mathrm{Out}_B(E)$, is the subgroup of $f \in \mathrm{Sym}_B(X/E)$ which are outer.

For a countable group Γ, an action $\Gamma \curvearrowright X/E$ by Borel permutations is called a **Borel action**, which is equivalently a homomorphism $\Gamma \to \mathrm{Sym}_B(X/E)$. We say a Borel action **lifts** if it factors through the map $\mathrm{Aut}_B(E) \to \mathrm{Sym}_B(X/E)$ described above. We show that for compressible CBERs, every action lifts:

Theorem 1.5.1. *[FKS22, Theorem 3.5] Let Γ be a countable group and let E be a compressible CBER. Then every Borel action $\Gamma \curvearrowright X/E$ lifts.*

If E is not compressible, there can be elements of $\mathrm{Sym}_B(X/E)$ which are not outer, and thus there are Borel actions on X/E which do not lift. For this reason, it is interesting to restrict the setting to that of outer actions. An **outer action** of a countable group Γ is a Borel action $\Gamma \curvearrowright X/E$ by outer permutations, in other words, a homomorphism $\Gamma \to \mathrm{Out}_B(E)$. A common situation where this arises is the following: if $\Gamma \curvearrowright X$ is a Borel action of a countable group on a standard Borel space, and $N \triangleleft \Gamma$ is a normal subgroup, then the action $\Gamma \curvearrowright X$ descends to an outer action $\Gamma \to \mathrm{Out}_B(E_N^X)$.

Let $\mathcal{G}$ be the class of countable groups for which every outer action $\Gamma \to \mathrm{Out}_B(E)$ lifts for every CBER E. We have shown that $\mathcal{G}$ contains a wide variety of groups:

Theorem 1.5.2. *[FKS22, Section 7]*

(1) $\mathcal{G}$ contains all amenable groups.

(2) $\mathcal{G}$ contains all amalgamated products of finite groups.

(3) $\mathcal{G}$ is closed under subgroups.

(4) $\mathcal{G}$ is closed under free products.

The first point generalizes a result of Feldman, Sutherland, and Zimmer in the measurable setting, see [FSZ89, Theorem 3.4]. An interesting feature of the proof of the second point is that it uses Tarski's theory of cardinal algebras (see [Tar49]), which is starting to see applications in the study of countable Borel equivalence relations, for example in [Shi21], where we apply cardinal algebras to the study of invariant measures (see also [KM16], [Che21]).

We also have an upper bound on this class of groups.

Theorem 1.5.3. *[FKS22, Proposition 4.11] Every group Γ in $\mathcal{G}$ is treeable, that is, there is a free pmp action of Γ on a standard probability space which is treeable (a Borel equivalence relation E on X is **treeable** if there is a Borel forest T on X whose connected components are exactly the E-classes).*

There are many examples of groups which are known to not be treeable, for instance, every property (T) group, and every product $\Gamma \times \Delta$, where Γ is infinite and Δ is non-amenable.

1.6 Dichotomies for Polish modules

Many of the cornerstone results in descriptive set theory are dichotomy theorems, which state that either an object satisfies some countability condition, or otherwise, there is a canonical obstruction to uncountability. The fundamental example is Cantor's perfect set theorem, which states that a Polish space is either countable, or it contains a copy of the Cantor space. A more modern example is the G_0-dichotomy of Kechris, Solecki, and Todorčević, which states that a Borel graph G either has countable Borel chromatic number, or there is a Borel homomorphism from a certain graph G_0 with uncountable Borel chromatic number to G.

We have shown a family of dichotomy theorems for vector spaces, and more generally, modules over certain nice classes of rings. If a Polish $\mathbb{Q}$-vector space V is uncountable, then it has dimension $2^{\aleph_0}$, so it is uniquely determined up to isomorphism. However, the existence of a basis uses the Axiom of Choice, and indeed, it turns out that many of these Polish $\mathbb{Q}$-vector spaces are not Borel isomorphic. For Polish $\mathbb{Q}$-vector spaces V and W, write $V \sqsubseteq W$ if there is an injective Borel homomorphism from V to W. We construct a Polish $\mathbb{Q}$-vector space $\ell^1(\mathbb{Q})$, and show that it is the obstruction to countability:

Theorem 1.6.1. *[FS22a] Let F be a countable field, and let V be a Polish $\mathbb{Q}$-vector space. Then exactly one of the following holds:*

1. *V is countable.*
2. *$\ell^1(\mathbb{Q}) \sqsubseteq V$.*

Chapter 2

QUOTIENTS BY COUNTABLE NORMAL SUBGROUPS ARE HYPERFINITE

[FS22] Joshua Frisch and Forte Shinko. "Quotients by countable subgroups are hyperfinite". In: *arXiv:1909.08716, to appear in Groups Geom. Dyn.* (2022).

2.1 Introduction

The purpose of this chapter is to study the complexity of quotient groups G/Γ from the point of view of descriptive set theory. In particular, we focus on the case where G is Polish and Γ is a countable normal subgroup. If Γ is a countable group, then the automorphism group of Γ has a natural Polish group structure, and thus the outer automorphism group of Γ is an example, as is any countable subgroup of an abelian group.

A major recent program is the study of complexity of "definable" equivalence relations. Results in this area are often interpreted to be statements about the difficulty of classification of various natural mathematical objects. A particular focus of the theory of definable equivalence relations, and one where much progress has recently been made, is the study of Borel equivalence relations for which every class is countable, the so-called countable Borel equivalence relations. There is a natural preorder on Borel equivalence relations, called Borel reduction, where E reducing to F is interpreted as E being "easier" than F. The theory of countable Borel equivalence relations has been applied in numerous areas of mathematics. For example, the classification of finitely generated groups [TV99], of subshifts [Cle09], and the arithmetic equivalence of subsets of $\mathbb{N}$ [MSS16] are all equally difficult. In fact, they are equivalent to the universal countable Borel equivalence relation E_∞, which is the hardest countable Borel equivalence relation. On the other hand, many other classification problems are easier. For example, classification of torsion-free finite rank abelian groups is substantially below E_∞ [Tho03; Tho09].

Countable Borel equivalence relations can be characterized as those equivalence relations arising from continuous actions of countable groups on Polish spaces, and thus have very strong interplay with dynamics and group theory. By a foundational result of Slaman-Steel and Weiss [SS88; Wei84], the equivalence relations which arise from a continuous (or more generally, Borel) action of $\mathbb{Z}$ are exactly the **hyperfinite** equivalence relations, which are those which can be written as an increasing union of finite Borel equivalence relations. More generally, it has been shown that every Borel action of a countable abelian group [GJ15], and even of a countable locally nilpotent group [SS13], is hyperfinite. It is an open question whether this holds for all countable amenable groups. By a theorem of Harrington-Kechris-Louveau [HKL90], the hyperfinite equivalence relations only occupy the first two levels of the hierarchy of countable Borel equivalence relations on uncountable Polish spaces under Borel reduction, and thus are considered to have low Borel complexity.

In general, if G is a Polish group and $\Gamma \leq G$ is a countable subgroup, then G/Γ can be rather complicated; we will give a non-hyperfinite example in Section 2.2. However, perhaps surprisingly, if Γ is a *normal* subgroup of G, then the coset equivalence relation G/Γ must have low Borel complexity:

Theorem 2.1.1. *Let G be a Polish group and let Γ be a countable normal subgroup of G. Then G/Γ is hyperfinite.*

Notably, in contrast to the aforementioned results, we require no hypotheses on the algebraic structure of the acting group. The proof proceeds by showing that the equivalence relation is generated by a Borel action of a countable abelian group, which is sufficient by the aforementioned theorem of Gao and Jackson. It is worth noting that it is comparatively easy (albeit still novel) to show that the equivalence relation is an increasing union of hyperfinite equivalence relations (in particular, hyperfinite with respect to any Borel measure), but it is a very important open problem as to whether this is equivalent to being hyperfinite.

We obtain as a consequence the following result about outer automorphism groups:

Corollary 2.1.2. *Let Γ be a countable group. Then* $\mathrm{Out}(\Gamma)$ *is hyperfinite.*

If G is a compact group with a countable normal subgroup $\Gamma \triangleleft G$, then we also show that the algebraic structure of Γ is severely restricted:

Theorem 2.1.3. *Let G be a compact group and let Γ be a countable normal subgroup of G. Then Γ is locally virtually abelian, i.e., every finitely generated subgroup of Γ is virtually abelian.*

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2.2 Preliminaries and examples

Descriptive set theory

A **Polish space** is a second countable, completely metrizable topological space. A **Borel equivalence relation** on a Polish space X is an equivalence relation E which is Borel as a subset of $X \times X$. A Borel equivalence relation is **countable** (resp., **finite**) if every class is countable (resp., finite). A countable Borel equivalence relation E on X is **smooth** if there is a Borel function $f : X \to \mathbb{R}$ such that xEx' if and only if $f(x) = f(x')$. A Borel equivalence relation E is **hyperfinite** (resp., **hypersmooth**) if $E = \bigcup_n E_n$, where each $E_n \subseteq E_{n+1}$ (as a subset of $X \times X$) and each E_n is a finite (resp., smooth) Borel equivalence relation. Given a Borel action of a countable group Γ on a Polish space X, we denote by E_Γ^X the **orbit equivalence relation** of $\Gamma \curvearrowright X$, the Borel equivalence relation whose classes are the orbits of the action. We will say that $\Gamma \curvearrowright X$ is hyperfinite (resp., smooth, hypersmooth) if its orbit equivalence relation E_Γ^X is hyperfinite (resp., smooth, hypersmooth).

A **Polish group** is a topological group whose topology is Polish. If G is a Polish group and $H \leq G$ is a closed subgroup, then the quotient topology on the coset space G/H is Polish (see [BK96, p. 1.2.3]).

Countable subgroups of Polish groups

Let G be a Polish group and let $\Gamma \leq G$ be a countable subgroup. When clear from context, we will abuse notation and identify G/Γ with the coset equivalence relation induced by $\Gamma \curvearrowright G$ (technically, G/Γ is induced by the right action $G \curvearrowleft \Gamma$, but this is isomorphic to the left action $\Gamma \curvearrowright G$ via inversion). For example, we will say that G/Γ is hyperfinite if $\Gamma \curvearrowright G$ is hyperfinite.

Note that since the action $\Gamma \curvearrowright G$ is free, G/Γ cannot be universal among countable Borel equivalence relations (see [Tho09, p. 3.10]).

Example 2.2.1. We give below some examples of G/Γ and the associated Borel complexity, for various G and Γ:

1. $\mathbb{R}/\mathbb{Z}$ is smooth, since $\mathbb{Z} \leq \mathbb{R}$ is a discrete subgroup (see [Kan08, 7.2.1(iv)]).

2. $\mathbb{R}/\mathbb{Q}$ is not smooth, since $\mathbb{Q} \leq \mathbb{R}$ is a dense subgroup (see [Gao09, p. 6.1.10]). Similarly, the commensurability relation $\mathbb{R}^+/\mathbb{Q}^+$ is not smooth. Note that both are hyperfinite, since they arise from Borel actions of countable abelian groups (see [GJ15, p. 8.2]).

3. Let $F_2 \leq \mathrm{SO}_3(\mathbb{R})$ be a free subgroup on two generators. Then $\mathrm{SO}_3(\mathbb{R})/F_2$ is not hyperfinite, since the free action $F_2 \curvearrowright \mathrm{SO}_3(\mathbb{R})$ preserves the Haar measure (see [Gao09, p. 7.4.8]).

4. If Γ is a countable group, then $\mathrm{Inn}(\Gamma)$ is a countable subgroup of $\mathrm{Aut}(\Gamma)$, which is a Polish group under the pointwise convergence topology, and we can consider the quotient $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. For example, when $\Gamma = S_{\mathrm{fin}}$ (the group of finitely supported permutations on $\mathbb{N}$), we have $\mathrm{Out}(S_{\mathrm{fin}}) \cong S_\infty/S_{\mathrm{fin}}$, which is hyperfinite and non-smooth.

In the first, second, and fourth examples, Γ is a normal subgroup of G.

2.3 Proofs

For any group G and any subset $S \subseteq G$, let $C_G(S)$ denote the centralizer of S in G:

$$C_G(S) := \{g \in G : \forall s \in S(gs = sg)\}.$$

Note that if G is a topological group, then $C_G(S)$ a closed subgroup of G.

Proposition 2.3.1. *Let G be a Baire group (i.e., a topological group for which the Baire category theorem holds), and let Γ be a finitely generated subgroup of G, each of whose elements has countable conjugacy class in G. Then $C_G(\Gamma)$ is open in G.*

Proof. Let $\Gamma = \langle \gamma_0, \dots, \gamma_n \rangle$. Since each γ_i has countable conjugacy class in G, we have $[G : C_G(\gamma_i)] \leq \aleph_0$, so by the Baire category theorem, $C_G(\gamma_i)$ is nonmeager. Thus by Pettis's lemma [Kec95, p. 9.9], $C_G(\gamma_i)$ is an open subgroup of G, and thus $C_G(\Gamma)$ is also open, since $C_G(\Gamma) = \bigcap_{i \leq n} C_G(\gamma_i)$. □

As a consequence, under the hypotheses of Proposition 2.3.1, if $Z(\Gamma)$ is additionally assumed to be finite, then Γ is necessarily a discrete subgroup of G.

When G is a compact group, Proposition 2.3.1 implies the following algebraic restriction on Γ:

Theorem 2.3.2. *Let G be a compact group and let Γ be a countable normal subgroup of G. Then Γ is locally virtually abelian, i.e., every finitely generated subgroup of Γ is virtually abelian.*

Proof. Let Δ be a finitely generated subgroup of Γ. Then by Proposition 2.3.1, $C_G(\Delta)$ is an open subgroup of G, so since G is compact, the index of $C_G(\Delta)$ in G is finite. Thus since $Z(\Delta) = \Delta \cap C_G(\Delta)$, the index of $Z(\Delta)$ in Δ is finite. □

We now prove the main theorem.

Theorem 2.3.3. *Let G be a Polish group and let Γ be a countable normal subgroup of G. Then G/Γ is hyperfinite.*

In the special case where Γ is an increasing union of finitely generated subgroups with finite center, this follows from the paragraph following the proof of Proposition 2.3.1. So we are concerned with examples where Γ is not of this form, such as the quotient of $\Delta * \Lambda$, where $\Delta \cong \Lambda \cong F_\infty$, subject to the relation $[[\delta, \lambda], \eta]$ for every $\delta \in \Delta$, $\lambda \in \Lambda$, and $\eta \in \Delta * \Lambda$.

Proof. Let $\Gamma = (\gamma_k)_{k<\omega}$ and denote $\Gamma_k := \langle \gamma_0, \dots, \gamma_k \rangle$. Let $C_k := C_G(\Gamma_k) = C_G(\gamma_0, \dots, \gamma_k)$ and let $Z_k := Z(\Gamma_k) = C_k \cap \Gamma_k$ be the center of Γ_k. By Proposition 2.3.1, C_k is an open subgroup of G.

Let $A := \langle Z_k \rangle_{k<\omega}$, the subgroup of G generated by the Z_k for all $k < \omega$. Then A is an abelian subgroup of G, since each Z_k is abelian, and since Z_k commutes with Z_l (pointwise) for any $k < l$.

The principal fact we use about A is the following:

Lemma 2.3.4. $\Gamma \curvearrowright G/\bar{A}$ *is hyperfinite.*

Proof. Since $\bar{A}$ is a closed subgroup of G, the coset space $G/\bar{A}$ is a standard Borel space, and thus $\Gamma \curvearrowright G/\bar{A}$ induces a Borel equivalence relation.

Every hypersmooth countable Borel equivalence relation is hyperfinite (see [DJK94, p. 5.1]), so it suffices to show that $\Gamma \curvearrowright G/\bar{A}$ is hypersmooth. Since Γ is the increasing union of $(\Gamma_n)_n$, it suffices to show for every n that $\Gamma_n \curvearrowright G/\bar{A}$ is smooth. In fact, we will show for every n that every orbit of $\Gamma_n \curvearrowright G/\bar{A}$ is discrete, which implies smoothness (enumerate a basis, then for each orbit, find the first basic open set isolating an element of the orbit, and select that element).

We need to show for every n and every $g \in G$ that $g\bar{A}$ is isolated in its Γ_n-orbit, or equivalently that $\bar{A}$ is isolated in its $g^{-1}\Gamma_n g$-orbit. By normality of Γ, there is some m for which $g^{-1}\Gamma_n g \subseteq \Gamma_m$, and thus it suffices show for every n that $\bar{A}$ is isolated in its Γ_n-orbit. We claim that $C_n\bar{A}$ isolates $\bar{A}$ in its Γ_n-orbit, or equivalently that $\Gamma_n\bar{A} \cap C_n\bar{A} = \bar{A}$, which is sufficient since C_n is open. We will show the equivalent statement that $\Gamma_n \cap C_n\bar{A} \subseteq \bar{A}$.

Since C_n is an open subgroup of G, it follows that C_nA is closed (since its complement is a union of cosets of an open subgroup). Also note that

$$A = \langle Z_k \rangle_{k>n} \langle Z_l \rangle_{l \leq n} \subseteq C_n(\Gamma_n \cap A).$$

Thus $C_n\bar{A} = C_n(\Gamma_n \cap A)$, since

$$C_n\bar{A} \subseteq \overline{C_nA} = C_nA \subseteq C_n(\Gamma_n \cap A) \subseteq C_n\bar{A}.$$

So we have

$$\Gamma_n \cap C_n\bar{A} = \Gamma_n \cap C_n(\Gamma_n \cap A) = Z_n(\Gamma_n \cap A) \subseteq A \subseteq \bar{A},$$

where the second equation holds since if $c \in C_n$, $\gamma \in \Gamma_n$ and $c\gamma \in \Gamma_n$, then $c \in \Gamma_n\gamma^{-1} = \Gamma_n$, and thus $c \in C_n \cap \Gamma_n = Z_n$. □

We now use this lemma to show that E_Γ^G is induced by the action of a countable abelian group. This is sufficient since by a theorem of Gao and Jackson, every orbit equivalence relation of a countable abelian group is hyperfinite ([GJ15, p. 8.2]).

Since $\Gamma \curvearrowright G/\bar{A}$ is hyperfinite, its orbit equivalence relation is generated by a Borel automorphism T of $G/\bar{A}$ (see [DJK94, p. 5.1]). For each left $\bar{A}$-coset C, let $\gamma_{(C)} \in \Gamma$ be minimal such that $T(C) = \gamma_{(C)}C$, and let $U : G \to G$ be the Borel automorphism defined by $U(g) = \gamma_{(g\bar{A})}g$ (the inverse is defined by $g \mapsto (\gamma_{(T^{-1}(g\bar{A}))})^{-1}g$). This induces a Borel action $\mathbb{Z} \curvearrowright G$, denoted $(n, g) \mapsto n \cdot g$, such that

(i) $g\bar{A}$ and $h\bar{A}$ are in the same Γ-orbit iff for some n, $(n \cdot g)\bar{A} = h\bar{A}$,

(ii) $E_\mathbb{Z}^G \subseteq E_\Gamma^G$,

(iii) and $\mathbb{Z} \curvearrowright G$ commutes with the right multiplication action $G \curvearrowleft (\bar{A} \cap \Gamma)$.

So there is a Borel action of $\mathbb{Z} \times (\bar{A} \cap \Gamma)$ on G such that $E_{\mathbb{Z}\times(\bar{A}\cap\Gamma)}^G \subseteq E_\Gamma^G$. We claim that in fact, $E_{\mathbb{Z}\times(\bar{A}\cap\Gamma)}^G = E_\Gamma^G$. Suppose that $g, h \in G$ are in the same Γ-coset (note that we don't need to specify left/right since Γ is normal). Then $g\bar{A}$ and $h\bar{A}$ are in the same Γ-orbit, so there is some $n \in \mathbb{Z}$ with $(n \cdot g)\bar{A} = h\bar{A}$. Since $E^G_{\mathbb{Z}} \subseteq E^G_{\Gamma}$, we have that $n \cdot g$ is in the same Γ-coset as g, and thus in the same Γ-coset as h. Since $n \cdot g$ and h are in the same left $\bar{A}$-coset, and also in the same (left) Γ-coset, they are in the same left $\bar{A} \cap \Gamma$-coset. Thus $E^G_{\mathbb{Z}\times(\bar{A}\cap\Gamma)} = E^G_{\Gamma}$. Since A is abelian, $\bar{A}$ is also abelian, and thus $\mathbb{Z} \times (\bar{A} \cap \Gamma)$ is a countable abelian group. So E^G_{Γ} is generated by the action of a countable abelian group, and is therefore hyperfinite. □

We can extend this result to a slightly more general class of subgroups:

Corollary 2.3.5. *Let G be a Polish group and let $\Gamma \leq G$ be a countable subgroup of G each of whose elements has countable conjugacy class in G. Then G/Γ is hyperfinite.*

Proof. Since every element of Γ has countable conjugacy class in G, the subgroup $\Delta := \left\langle g\Gamma g^{-1} \right\rangle_{g\in G}$ is a countable normal subgroup of G, and thus by Theorem 2.3.3, E^G_{Δ} is hyperfinite. Since $E^G_{\Gamma} \subseteq E^G_{\Delta}$, we have that G/Γ is also hyperfinite (since hyperfiniteness is closed under subequivalence relations). □

Corollary 2.3.6. *Let Γ be a countable group. Then* $\mathrm{Out}(\Gamma)$ *is hyperfinite.*

Proof. This follows from Theorem 2.3.3, since $\mathrm{Inn}(\Gamma) \triangleleft \mathrm{Aut}(\Gamma)$. □

We end with some open questions:

Question 2.3.7. Let G be a Polish group and let Γ be a countable subgroup. What are the possible Borel complexities of G/Γ? In particular, are they cofinal among orbit equivalence relations arising from free actions?

Definition 2.3.8. For a Polish group G, define the subgroup $Z_\omega(G)$ as follows:

$$Z_\omega(G) := \{g \in G : g \text{ has countable conjugacy class}\}.$$

In general, $Z_\omega(G)$ is a characteristic subgroup of G, analogous to the FC-center, and $Z_\omega(G)$ is $\mathbf{\Pi}^1_1$ by Mazurkiewicz-Sierpiński (see [Kec95, p. 29.19]).

Question 2.3.9. Is there a Polish group G such that $Z_\omega(G)$ is $\mathbf{\Pi}^1_1$-complete?

Chapter 3

LIFTS OF BOREL ACTIONS ON QUOTIENT SPACES

[FKS22] Joshua Frisch, Alexander Kechris, and Forte Shinko. "Lifts of Borel actions on quotient spaces". In: *arXiv:2011.01395, to appear in Israel J. Math* (2022).

3.1 Introduction

Automorphisms of equivalence relations

A **countable Borel equivalence relation (CBER)** is an equivalence relation E on a standard Borel space X such that E is Borel when considered as a subset of X^2. Let $\pi_E : X \to X/E$ denote the quotient map.

Let E be a CBER on X. The **automorphism group** of E, denoted $\mathrm{Aut}_B(E)$ (or $N_B[E]$), is the group of Borel automorphisms of E, that is, Borel automorphisms $T : X \to X$ such that $x \mathrel{E} y \iff T(x) \mathrel{E} T(y)$, under composition. The **inner automorphism group** of E (or the **full group** of E), denoted $\mathrm{Inn}_B(E)$ (or $[E]_B$), is the normal subgroup of $\mathrm{Aut}_B(E)$ consisting of the $T \in \mathrm{Aut}_B(E)$ such that $x \mathrel{E} T(x)$. The normalizer of $\mathrm{Inn}_B(E)$ in the group of Borel automorphisms of X is $\mathrm{Aut}_B(E)$. By a result of Miller and Rosendal [MR07, Proposition 2.1], if E is aperiodic, then the natural map $\mathrm{Aut}_B(E) \to \mathrm{Aut}(\mathrm{Inn}_B(E))$ is an isomorphism. The **outer automorphism group** of E, denoted $\mathrm{Out}_B(E)$, is the quotient group $\mathrm{Aut}_B(E)/\mathrm{Inn}_B(E)$.

Let E and F be CBERs on X and Y respectively. A function $f : X/E \to Y/F$ is **Borel** if the set $\{(x, y) \in X \times Y : f([x]_E) = [y]_F\}$ is Borel, or equivalently by the Lusin-Novikov theorem [Kec95, Theorem 18.10], if there exists a Borel map $T : X \to Y$ such that $f([x]_E) = [T(x)]_F$. The **Borel symmetric group** of X/E, denoted $\mathrm{Sym}_B(X/E)$, is the set of Borel permutations of X/E under composition. There is a natural map $\mathrm{Aut}_B(E) \to \mathrm{Sym}_B(X/E)$, defined by sending $T \in \mathrm{Aut}_B(E)$ to the permutation $[x]_E \mapsto [T(x)]_E$. This morphism has kernel $\mathrm{Inn}_B(E)$, so there is a factorization

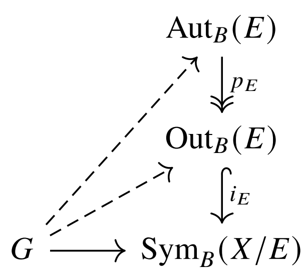
$$\mathrm{Aut}_B(E) \overset{p_E}{\twoheadrightarrow} \mathrm{Out}_B(E) \overset{i_E}{\hookrightarrow} \mathrm{Sym}_B(X/E) \ .$$

A Borel permutation of X/E in the image of this morphism is called an **outer permutation**. In other words, $f \in \mathrm{Sym}_B(X/E)$ is outer if there is $T \in \mathrm{Aut}_B(E)$

such that $f([x]_E) = [T(x)]_E$.

Lifts of Borel actions on quotient spaces

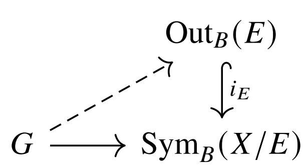
Let E be a CBER on X and let G be a countable group. We write $G \curvearrowright_B (X, E)$ to denote an action of G on X by Borel automorphisms of E, which is equivalent to a morphism $G \to \mathrm{Aut}_B(E)$. An action $G \curvearrowright_B (X, E)$ is **class-bijective** if π_E is class-bijective, that is, the restriction of π_E to every G-orbit is an injection, i.e., $g \cdot x \mathrel{E} x \implies g \cdot x = x$. A **Borel action** of G on X/E, denoted $G \curvearrowright_B X/E$, is an action of G on X/E by Borel permutations, which is equivalent to a morphism $G \to \mathrm{Sym}_B(X/E)$. An action $G \curvearrowright_B X/E$ is **outer** if G acts by outer permutations, or equivalently, if the morphism $G \to \mathrm{Sym}_B(X/E)$ factors through i_E. Every action $G \curvearrowright_B (X, E)$ induces an action $G \curvearrowright_B X/E$ by composing with $i_E \circ p_E$, and π_E is G-equivariant with respect to these actions. We initiate in this chapter the study of the reverse problem: when does a Borel action $G \curvearrowright_B X/E$ have a **lift** to an action $G \curvearrowright_B (X, E)$? In other words, we are interested in the lifting problem



which we will break up into steps by going through $\mathrm{Out}_B(E)$.

Main results

We give in Section 3.3 examples of CBERs E that show that even the first step of the lifting problem



does not always have a positive solution, i.e., that there are Borel actions $G \curvearrowright_B X/E$ which are not outer. In all these examples, E admits an invariant Borel probability measure (i.e, it is generated by a Borel action of a countable group that has an invariant Borel probability measure). On the other hand, we show in Theorem 3.3.5 that the full lifting problem has a positive solution, in a strong sense, when the CBER E admits no such invariant measure or equivalently (by Nadkarni's Theorem) that it is compressible (i.e., there is a Borel injection that sends every equivalence class to a proper subset of itself).

Theorem 3.1.1. *Let E be a compressible CBER. Then every Borel action $G \curvearrowright_B X/E$ has a class-bijective lift $G \curvearrowright_B (X, E)$.*

This theorem follows from a result (see Theorem 3.3.6) about links (see Definition 3.3.3) of pairs $E \subseteq F$ of compressible CBERs that was also proved (by a different method) independently by Ben Miller. Our proof uses some ideas coming from [FSZ89].

We do not know if there are non-compressible E that satisfy Theorem 3.1.1. Using this result and a variant of [KM04, Corollary 13.3], we show, in Corollary 3.3.11, that the full lifting problem has a positive solution generically for an arbitrary **aperiodic** (i.e., having all its classes infinite) CBER E.

Below if $G \curvearrowright_B X/E$, we let $E^{\vee G} \supseteq E$ be the CBER defined as follows:

$$x \; E^{\vee G} \; y \iff \exists g \in G(g \cdot [x]_E = [y]_E).$$

Corollary 3.1.2. *Let E be an aperiodic CBER on a Polish space X. Then for any Borel action $G \curvearrowright_B X/E$, there is a comeager $E^{\vee G}$-invariant Borel subset $Y \subseteq X$ such that $G \curvearrowright_B Y/E$ has a class-bijective lift.*

In Sections 3.4-3.6, we study the lifting problem for outer actions. A lift of an outer action is a solution to the following lifting problem:

$$\begin{array}{ccc} & & \mathrm{Aut}_B(E) \\ & \nearrow & \Big\downarrow p_E \\ G & \longrightarrow & \mathrm{Out}_B(E) \end{array} .$$

Below we use the following terminology. If a group G acts on a set X, we denote by E_G^X the induced equivalence relation whose classes are the G-orbits. An action of group G on a set X is **free** if for any $g \neq 1$ and $x \in X$, $g \cdot x \neq x$. If the set X carries a measure and the action is measure-preserving, we only require that this holds for almost all x. A Borel action of a countable group G on a standard Borel space X is **pmp** if it has an invariant Borel probability measure. A countable group G is **treeable** if it admits a free, pmp Borel action on a standard Borel space X such that the induced CBER E_G^X is treeable, i.e., its classes are the connected components of an acyclic Borel graph on X. For example, all amenable and free groups are treeable but all property (T) groups and all products of an infinite group with a non-amenable group are not treeable.

We now have the following results (see Corollary 3.6.14, Corollary 3.5.12 for (1), and Corollary 3.5.10, Theorem 3.6.13 for (2)). Below a CBER is **smooth** if it admits a Borel set meeting every class in exactly one point.

Theorem 3.1.3.

(1) Every outer action of any abelian group, and in fact any group for which the conjugacy equivalence relation on its space of subgroups is smooth, and any locally finite group has a class-bijective lift.

(2) Every outer action of any amenable group and any amalgamated free product of finite groups has a lift.

The proof of Theorem 3.1.3, (2) for the case of amenable groups makes use of the quasi-tiling machinery developed in the work of Ornstein and Weiss [OW80], [OW87] and also uses some ideas from [FSZ89]. Also the proof of Theorem 3.1.3, (2) for the case of amalgamated free products of finite groups also uses some ideas from [Tse13]. We do not know if the conclusion of (2) can be restrengthened to having a class-bijective lift.

On the other hand we have an upper bound for groups that have this lifting property (see Proposition 3.4.11). The proof of the next result is motivated by [CJ85] and [FSZ89].

Proposition 3.1.4. *If every outer action of a countable group G lifts, then G is treeable.*

We do not know a characterization of the class of countable groups all of whose outer actions have a lift or a class-bijective lift. Section 3.7 contains a summary of what we know about the classes of groups all of whose outer actions have a lift (resp., a class-bijective lift).

Acknowledgements

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3.2 Preliminaries

Countable Borel equivalence relations

We review here some basic notions and results that we will use in the sequel. A general reference is the survey paper [Kec22]. Given a CBER E on X, we denote for each $A \subseteq X$ by $[A]_E = \{x \in X : \exists y \in A\,(x \mathrel{E} y)\}$ the **E-saturation** of A. In particular if $x \in X$, $[\{x\}]_E = [x]_E$ is the equivalence class of E. Dually the **E-hull** of A is the set $\{x \in X : [x]_E \subseteq A\}$. Finally we let $E \restriction A = E \cap A^2$ be the restriction of E to A. A set $A \subseteq X$ is **E-invariant** if $A = [A]_E$. For each set S, we denote by Δ_S the equality relation on S and we also let $I_S = S^2$.

For CBERs E, F on X, Y resp., we denote by $E \oplus F$ the **direct sum** of E, F. Formally this is the equivalence relation on the direct sum $X \sqcup Y$ of X, Y which agrees with E on X and with F on Y. Similarly we define the direct sum $\bigoplus_n E_n$ for a sequence (E_n) of CBERs. The **product** of E, F is the equivalence relation on $X \times Y$ given by $(x, y) \mathrel{E \times F} (x', y') \iff (x \mathrel{E} x')\ \&\ (y \mathrel{F} y')$.

If E, F are CBERs on X and $E \subseteq F$ (as sets of ordered pairs), then E is a **subequivalence relation** of F and F is an **extension** of E. If every F-class contains only finitely many E-classes, we say that F has **finite index** over E, and if for some N every F-class contains at most N E-classes, we say that F has **bounded index** over E. If every F-class contains exactly N E-classes we write $[F : E] = N$. Finally, $E \vee F$ is the smallest equivalence relation containing E and F.

A **complete section** of a CBER E on X is a set $S \subseteq X$ that meets every E-class. A **transversal** of E is a subset $T \subseteq X$ that meets every E-class in exactly one point. If a Borel transversal exists, we say that E is **smooth**. A CBER E is **finite** if every E-class is finite and it is **hyperfinite** if $E = \bigcup_n E_n$, where $E_n \subseteq E_{n+1}$ and E_n is finite, for each n. A canonical non-smooth hyperfinite CBER is E_0 on $2^{\mathbb{N}}$ defined by $x \mathrel{E_0} y \iff \exists m\, \forall n \geq m\, (x_n = y_n)$. We say that a CBER E is **aperiodic** if every E-class is infinite. For any CBER E there is a unique decomposition $X = A \sqcup B$ into E-invariant Borel sets such that $E \restriction A$ is finite and $E \restriction B$ is aperiodic. These are, resp., the finite and infinite parts of E. A CBER E on X is **treeable** if there is an acyclic Borel graph $\Gamma \subseteq X^2$ whose connected components are exactly the E-classes. Every hyperfinite CBER is treeable.

A CBER E on X is **compressible** if there is a Borel injection $T\colon X \to X$ such that $T([x]_E) \subsetneq [x]_E$, for each x. A Borel set $A \subseteq X$ is (E-)compressible if $E \upharpoonright A$ is compressible. In that case $[A]_E$ is compressible as well and there is a Borel injection $T\colon X \to X$ such that $T(x)\ E\ x$, for every x, and $T([A]_E) = A$; see [Kec22, Proposition 3.26]. Recall also from [Kec22, Proposition 3.23] that E is compressible iff $E \cong_B E \times I_{\mathbb{N}}$ (where for two CBERs F_1, F_2 on X_1, X_2, resp., $F_1 \cong_B F_2$ means that they are **Borel isomorphic**, i.e., there is a Borel bijection $T\colon X_1 \to X_2$ that takes F_1 to F_2), and also E is compressible iff it contains a smooth, aperiodic subequivalence relation.

Given CBERs E, F on X, Y, resp., we say that E is **Borel reducible** to F, in symbols $E \leq_B F$, if there is a Borel map $T\colon X \to Y$ such that $x\ E\ x' \iff T(x)\ F\ T(x')$. Such a T is called a **reduction** of E to F. Moreover E, F are **Borel bireducible**, in symbols $E \sim_B F$, if $(E \leq_B F)$ & $(F \leq_B E)$. We have that $E \sim_B F$ iff there is a Borel bijection $T\colon X/E \to Y/F$; see [Kec22, Theorem 3.32].

Given a countable group G and a Borel action of G on X, denote by E_G^X the CBER induced by this action, i.e., the equivalence relation whose classes are exactly the orbits of this action. The Feldman-Moore Theorem (see, e.g., [Kec22, Theorem 3.3]) asserts that for every CBER E on X there is a countable group G and a Borel action of G on X such that $E = E_G^X$.

By a **partial subequivalence relation** of a CBER E on X, we mean an equivalence relation F on a subset $A \subseteq X$ such that $F \subseteq E$. A Borel finite partial subequivalence relation is abbreviated as **fsr**.

Let X now be a standard Borel space and denote by $[X]^{<\infty}$ the standard Borel space of finite subsets of X. If E is a CBER on X, we denote by $[E]^{<\infty}$ the subset of $[X]^{<\infty}$ consisting of all finite sets that are contained in a single E-class. Then $[E]^{<\infty}$ is Borel. For each set $\Phi \subseteq [E]^{<\infty}$, an fsr F of E defined on the set $A \subseteq X$ is **Φ-maximal**, if every F-class is in Φ and every finite set S disjoint from A is not in Φ. We now have the following result; see [KM04, Lemma 7.3]: If E is a CBER and $\Phi \subseteq [E]^{\infty}$ is Borel, then there is a Borel Φ-maximal fsr of E. The **intersection graph** of E is the graph on $[E]^{<\infty}$, where S, T are connected by an edge iff there are distinct and have nonempty intersection. The proof of [KM04, Lemma 7.3] uses the fact that this graph has a countable Borel coloring, i.e., a Borel map $c\colon [E]^{<\infty} \to \mathbb{N}$, which is a coloring of this graph.

For each CBER E on X, denote by INV_E the standard Borel space of invariant Borel probability measures on X, i.e., the Borel probability measures on X for which there is a Borel, measure-preserving action of a countable group G on X with $E_G^X = E$. We also let EINV_E be the Borel subset of INV_E consisting of all ergodic measures in INV_E. Nadkarni's Theorem (see [Kec22, Theorem 5.6]) states that E is compressible iff INV_E is empty. The Ergodic Decomposition Theorem of Farrell and Varadarajan (see [Kec22, Theorem 5.12]) asserts that if $\mathrm{INV}_E \neq \emptyset$, then there is a Borel surjection $\pi : X \twoheadrightarrow \mathrm{EINV}_E$ such that

(i) π is E-invariant;

(ii) If $X_e = \pi^{-1}(\{e\})$, for $e \in \mathrm{EINV}_E$, then $e(X_e) = 1$ and e is the unique E-invariant probability measure concentrating on X_e;

(iii) If $\mu \in \mathrm{INV}_E$, then $\mu = \int \pi(x)\, \mathrm{d}\mu(x) = \int e\, \mathrm{d}\pi_*\mu(e)$.

Moreover this map is unique in the following sense: If π, π' satisfy (i)-(iii), then the set $\{x : \pi(x) \neq \pi'(x)\}$ is compressible.

The sets X_e are the **ergodic components** of E.

We say that E is **uniquely ergodic** (resp., **finitely ergodic, countably ergodic**) if EINV_E is a singleton (resp., finite, countable).

The Classification Theorem for hyperfinite CBERs (see [Kec22, Theorem 8.4]) states that for aperiodic, non-smooth, hyperfinite E, F, we have that $E \cong_B F$ iff EINV_E and EINV_F have the same cardinality.

Cardinal algebras

A **cardinal algebra** is a tuple $(A, 0, +, \sum)$, where $(A, 0, +)$ is a commutative monoid, and $\sum : A^{\mathbb{N}} \to A$ is an infinitary operation satisfying the following axioms:

(i) $\sum_i a_i = a_0 + \sum_i a_{i+1}$;

(ii) $\sum_i (a_i + b_i) = \sum_i a_i + \sum_i b_i$;

(iii) The **refinement axiom**: If $a + b = \sum_i c_i$, then there are $(a_i)_i$ and $(b_i)_i$ such that $a = \sum_i a_i$, $b = \sum_i b_i$, and $a_i + b_i = c_i$;

(iv) The **remainder axiom**: If $(a_i)_i$ and $(b_i)_i$ satisfy $a_i = b_i + a_{i+1}$, then there is some c such that $a_i = c + \sum_j b_{i+j}$.

We will need two consequences of these axioms. For $0 \le n \le \infty$, let na denote the sum of n copies of a (in particular, let ∞a denote $\sum_i a$).

(1) For any a, b,

$$a = a + b \implies a = a + \infty b.$$

To see this, use the remainder axiom with $a_i = a$ and $b_i = b$. This gives some c such that $a = c + \infty b$. Then

$$a + \infty b = c + \infty b + \infty b = c + \infty b = a.$$

(2) The **cancellation law**: For any a, b and $0 < n < \infty$,

$$na = nb \implies a = b;$$

see [Tar49, Theorem 2.34].

We will need the following cardinal algebras:

(1) The collection of all CBERs up to Borel isomorphism is a cardinal algebra under direct sum; see [KM16, p. 3.C].

(2) Let E be a CBER on X. We say that $A, B \subseteq X$ are E**-equidecomposable**, denoted $A \sim_E B$, if there is some Borel bijection $T\colon A \to B$ whose graph is contained in E. This is an equivalence relation, and we denote the class of A by $\widetilde{A}$. Let $\mathcal{K}(E)$ denote the set of E-equidecomposability classes.

Assume now that E is compressible. Then for any countable sequence $\widetilde{A_0}, \widetilde{A_1}, \ldots$, we can assume that the A_n are pairwise disjoint, and we can define the infinitary operation as follows:

$$\sum_n \widetilde{A_n} := \widetilde{\bigcup_n A_n}.$$

(We define + analogously, and we define 0 to be the class of the empty set.) Then $\mathcal{K}(E)$ with these operations is a cardinal algebra; see [Che21, Proposition 4.1].

There is an action $\mathrm{Aut}_B(E) \curvearrowright \mathcal{K}(E)$ (i.e., a group action preserving $(0, +, \sum)$) defined by

$$T \cdot \widetilde{A} = \widetilde{T(A)},$$

and this descends to an action $\mathrm{Out}_B(E) \curvearrowright \mathcal{K}(E)$.

Actions on probability spaces

Let (X, μ) be a standard probability space, i.e., a standard Borel space with a non-atomic Borel probability measure. Let $\mathrm{Aut}_\mu(X)$ denote the group of Borel automorphisms $T : X \to X$ such that $T_*\mu = \mu$, where T and T' are identified if they agree on a conull set.

Let E be a **pmp** CBER on X, i.e., a CBER which is generated by a measure-preserving action of a countable group. Then $\mathrm{Aut}_\mu(E)$ denotes the set of $T \in \mathrm{Aut}_\mu(X)$ such that $x \mathrel{E} y \iff T(x) \mathrel{E} T(y)$, for all x, y in a conull subset of X. Let $\mathrm{Inn}_\mu(E)$ denote the normal subgroup of $T \in \mathrm{Aut}_\mu(E)$ such that $x \mathrel{E} T(x)$ for almost every $x \in X$. Then $\mathrm{Out}_\mu(E)$ denotes the quotient $\mathrm{Aut}_\mu(E)/\mathrm{Inn}_\mu(E)$.

All of the proofs below in the Borel setting go through mutatis mutandis in the pmp setting.

3.3 Borel actions on quotient spaces

Outer and non-outer actions

Not every Borel action $G \curvearrowright_B X/E$ is outer. For example, let $2^{\mathbb{N}} = A \sqcup B$, where A and B are complete Borel sections for E_0 with $\mu(A) \neq \mu(B)$, where μ is Lebesgue measure. Let $E = (E_0 \upharpoonright A) \oplus (E_0 \upharpoonright B)$. Then the involution on X/E sending $[x]_{E_0} \cap A$ to $[x]_{E_0} \cap B$ is not outer, since otherwise we would have $\mu(A) = \mu(B)$.

Note that the following are equivalent:

(1) Every Borel action on X/E is outer;

(2) i_E is a bijection.

This condition is quite strong:

Proposition 3.3.1. *Let G be a countable group and let E be a CBER. Suppose that every action $G \curvearrowright_B X/E$ is outer.*

(1) Whenever $E \cong_B \bigoplus_{g \in G} E_g$, with the E_g pairwise Borel bireducible, then the E_g are pairwise Borel isomorphic.

(2) If G is nontrivial and $E \cong_B E \oplus (E \times I_{\mathbb{N}})$, then E is compressible.

Proof. For (1), suppose E_g lives on X_g, and let F be a CBER on Y such that $F \sim_B E_g$ for every $g \in G$, and for each $g \in G$, fix a Borel bijection $f_g : Y/F \to X_g/E_g$.

Define $G \curvearrowright_B X/E$ for $[x]_E \in X_g/E_g$ by $h \cdot [x]_E = f_{hg}(f_g^{-1}([x]_E))$. By assumption, this action is induced by some $G \to \mathrm{Out}_B(E)$, which induces isomorphisms between the E_g.

For (2), since $E \cong_B E \oplus (E \times I_{\mathbb{N}})$, by working in the cardinal algebra of (Borel isomorphism classes of) CBERs, we have $E \cong_B E \oplus \bigoplus_{g \in G \setminus \{1\}} (E \times I_{\mathbb{N}})$. So by (1), we have $E \cong_B E \times I_{\mathbb{N}}$. □

So if E is non-compressible and satisfies $E \cong_B E \oplus (E \times I_{\mathbb{N}})$, then every nontrivial countable group admits a non-outer action on X/E. There are many such examples:

Example 3.3.2.

(1) (Miller) We have $E_0 \cong E_0 \oplus (E_0 \times I_{\mathbb{N}})$, since they are both uniquely ergodic and hyperfinite. More generally $E \cong_B E \oplus (E \times I_{\mathbb{N}})$, for any aperiodic hyperfinite CBER E.

(2) A countable group G is **dynamically compressible** if every aperiodic orbit equivalence relation of G is Borel reducible to a compressible orbit equivalence relation of G. Examples include amenable groups, and groups containing a non-abelian free group. If G is dynamically compressible, then $E^{\mathrm{ap}}(G, \mathbb{R}) \cong_B E^{\mathrm{ap}}(G, \mathbb{R}) \oplus (E^{\mathrm{ap}}(G, \mathbb{R}) \times I_{\mathbb{N}})$, where $E^{\mathrm{ap}}(G, \mathbb{R})$ denotes the aperiodic part of the shift action of G on $\mathbb{R}^G$; see [Fri+21, 5(B)].

Lifts of compressible CBERs

Every action $G \curvearrowright_B X/E$ induces a CBER $E^{\vee G} \supseteq E$ defined as follows:

$$x \, E^{\vee G} \, y \iff \exists g \in G (g \cdot [x]_E = [y]_E).$$

Every action $G \curvearrowright_B (X, E)$ induces an action $G \curvearrowright_B X/E$, and we write $E^{\vee G}$ for the CBER induced by the latter. Note that $E^{\vee G} = E \vee E_G^X$. If G is a subgroup of $\mathrm{Aut}_B(E)$ or $\mathrm{Out}_B(E)$, we write $E^{\vee G}$ for the CBER given by the (outer) action induced by the inclusion map, and if $T \in \mathrm{Aut}_B(E)$, we write $E^{\vee T}$ for $E^{\vee \langle T \rangle}$.

In [RM21], it is shown that there is a countable basis of pairs $E \subseteq F$ of CBERs such that there is no Borel action $G \curvearrowright_B X/E$ with $F = E^{\vee G}$ (see Section 3.8 for a precise statement).

Given $f \in \mathrm{Sym}_B(X/E)$, a **lift** of f is a map $T \in \mathrm{Aut}_B(E)$ such that $[T(x)]_E = f([x]_E)$ for every $x \in X$. Given an action $G \curvearrowright_B X/E$, a **lift** of $g \in G$ is a lift of its image in $\mathrm{Sym}_B(X/E)$.

The following notion is from [Tse13]:

Definition 3.3.3. Let $E \subseteq F$ be CBERs. An (E, F)**–link** is a CBER $L \subseteq F$ such that for every F-class C, every $E \restriction C$-class meets every $L \restriction C$-class exactly once.

The connection to lifts is the following:

Proposition 3.3.4. *Let $G \curvearrowright_B X/E$. Then the following are equivalent:*

(1) There is an $(E, E^{\vee G})$–link.

(2) There is a class-bijective lift $G \curvearrowright_B (X, E)$.

Proof. (2) $\implies$ (1) E^X_G is a link.

(1) $\implies$ (2) Let $g \cdot x$ be the unique element in $[x]_L \cap (g \cdot [x]_E)$. □

Proposition 3.3.1 perhaps suggests that if E is compressible, then every Borel action on X/E is outer. It turns out that something much stronger is true:

Theorem 3.3.5. *Let E be a compressible CBER. Then every Borel action on X/E has a class-bijective lift.*

By Proposition 3.3.4, it suffices to prove the following, independently established using a different method by Ben Miller (see comments following Corollary 3.3.8 below for his approach):

Theorem 3.3.6. *Let $E \subseteq F$ be compressible CBERs. Then there is a smooth (E, F)–link.*

We will repeatedly use the following, where we identify a positive integer N with $\{0, 1, \ldots, N-1\}$.

Lemma 3.3.7. *Let $E \subseteq F$ be compressible CBERs and let $N \in \{1, 2, \ldots, \mathbb{N}\}$. Then (E, F) is Borel isomorphic to $(E \times I_N, F \times I_N)$, in symbols $(E, F) \cong_B (E \times I_N, F \times I_N)$, i.e., there is a Borel isomorphism that takes E to $E \times I_{\mathbb{N}}$ and F to $F \times I_{\mathbb{N}}$.*

Proof. Since E is compressible, $E \cong_B E \times I_{\mathbb{N}}$. So (E, F) is Borel isomorphic to $(E \times I_{\mathbb{N}}, R)$, for some R, which then must be of the form $F' \times I_{\mathbb{N}}$. Thus $(E, F) \cong_B (E \times I_{\mathbb{N}}, F' \times I_{\mathbb{N}})$, and therefore $(E \times I_N, F \times I_N) \cong_B (E \times I_{\mathbb{N}} \times I_N, F' \times I_{\mathbb{N}} \times I_N) \cong_B (E \times I_{\mathbb{N}}, F' \times I_{\mathbb{N}}) \cong_B (E, F)$, since $I_{\mathbb{N}} \cong_B I_{\mathbb{N}} \times I_N$. □

Proof of Theorem 3.3.6. We can assume that every F-class contains exactly N E-classes, where $N \in \{1, 2, \dots, \mathbb{N}\}$. Below, $i < N$ means $i \in N$.

Fix a Borel action of a countable group Γ generating F.

Fix a **choice sequence** for (E, F), that is, a sequence $(f_i)_{i<N}$ of Borel maps $X \to X$ such that for every $x \in X$, the function $i \mapsto [f_i(x)]_E$ is a bijection from N to $[x]_F/E$. For instance, define f_i inductively by setting $f_0(x) = x$ and $f_i(x) = \gamma \cdot x$, where γ is least (in some enumeration of G) such that $\gamma \cdot x$ is not E-related to any $f_j(x)$ for $j < i$.

We can assume that each f_i is injective. By Lemma 3.3.7, it suffices to define an injective choice sequence for $(E \times I_{\mathbb{N}}, F \times I_{\mathbb{N}})$. Fix a pairing function $\langle -, - \rangle : \mathbb{N} \times \Gamma \to \mathbb{N}$. Then we take the choice sequence for $(E \times I_{\mathbb{N}}, F \times I_{\mathbb{N}})$ defined by $(x, n) \mapsto (f_i(x), \langle n, \gamma \rangle)$, where f_i is a choice sequence for (E, F) and γ is least such that $\gamma \cdot x = f_i(x)$.

We can further assume that each $\operatorname{im} f_i$ is a complete E-section. To see this, endow N with some group operation $\star$, and take the choice sequence for $(E \times I_N, F \times I_N)$ defined by $(x, k) \mapsto (f_{i \star k}(x), k)$, where (f_i) is a choice sequence for (E, F) with each f_i injective.

Moreover, we can assume that each $\operatorname{im} f_i$ is E-compressible. To see this, take the choice sequence for $(E \times I_{\mathbb{N}}, F \times I_{\mathbb{N}})$ defined by $(x, n) \mapsto (f_i(x), n)$, where (f_i) is a choice sequence for (E, F), with each f_i injective and $\operatorname{im} f_i$ a complete E-section.

Finally, we can assume that each f_i is bijective. To see this, since $\operatorname{im} f_i$ is an E-compressible complete section for E, there is some Borel injection T_i such that $T(x)\ E\ x$ for every x, and $T_i(X) = \operatorname{im} f_i$. Then $(T_i^{-1} \circ f_i)$ is a choice sequence for (E, F) with each $T_i^{-1} \circ f_i$ bijective.

Now we can define a smooth $(E \times I_N, F \times I_N)$–link L as follows:

$$(x, i)\ L\ (y, j) \iff f_i^{-1}(x) = f_j^{-1}(y),$$

and we are done again by Lemma 3.3.7. □

Corollary 3.3.8. *Let E be an aperiodic CBER satisfying $E \cong_B E \oplus (E \times I_{\mathbb{N}})$ (for instance, any aperiodic hyperfinite CBER). Then the following are equivalent:*

(1) Every Borel action on X/E has a class-bijective lift.

(2) Every Borel action on X/E has a lift.

(3) Every Borel action on X/E is outer.

(4) There is a nontrivial countable group G such that every action $G \curvearrowright_B X/E$ is outer.

(5) E is compressible.

Proof. (1) $\implies$ (2) Immediate.

(2) $\implies$ (3) Immediate.

(3) $\implies$ (4) Immediate.

(4) $\implies$ (5) Follows from Proposition 3.3.1.

(5) $\implies$ (1) Follows from Theorem 3.3.5. ☐

Concerning Theorem 3.3.6, Ben Miller derives this from the following more general result whose proof uses Proposition 4.1 and 4.2 from [Mil18].

Theorem 3.3.9 (Miller)**.** *Let E and F be compressible CBERs on X and Y respectively, and let $f: X/E \to Y/F$ be Borel. Then the following are equivalent:*

1. *f is smooth-to-one, i.e., for every $y \in Y$, the restriction of E to $\{x \in X : f([x]_E) = [y]_F\}$ is smooth.*
2. *There is a Borel function $T: X \to Y$ such that for every $x \in X$, the restriction $T \upharpoonright [x]_E$ is a bijection from $[x]_E$ to $f([x]_E)$.*

However, one only needs the special case where f is countable-to-one. Applying this to the case where $E \subseteq F$ and $f([x]_E) = [x]_F$, we find a Borel map $T: X \to X$ such that $T \upharpoonright [x]_E$ is a bijection from $[x]_E$ to $[x]_F$. Then we can define the link L by $x \mathrel{L} y \iff T(x) = T(y)$.

To show generic lifting, we need a strengthening of generic compressibility, whose proof is a simple modification of the proof of [KM04, Corollary 13.3]. A more general version appears in [Mil17, Theorem 11.1]. We include a proof for the reader's convenience.

Theorem 3.3.10. *Let $E \subseteq F$ be aperiodic CBERs on a Polish space X. Then there is a comeager F-invariant, E-compressible Borel subset of X.*

Proof. Fix a Borel coloring $c\colon [E]^{<\infty} \to \mathbb{N}$ of the intersection graph. Write $X = \bigsqcup_{n\in\mathbb{N}} A_n$, where each A_n is a Borel set meeting every E-class infinitely often; for instance, write $X = \bigsqcup_{(n,m)\in\mathbb{N}^2} B_{n,m}$, where each $B_{n,m}$ is a complete E-section (see [CM17, p. 1.2.6]), and take $A_n = \bigcup_m B_{n,m}$. Let $\mathbb{N}^{<\mathbb{N}}$ denote the set of finite strings in $\mathbb{N}$. For $s \in \mathbb{N}^{<\mathbb{N}}$, let $\operatorname{len}(s)$ denote the length of s. For $s, t \in \mathbb{N}^{<\mathbb{N}}$, we write $s \preceq t$ to mean that s is a prefix of t. We define fsr's $\{E_s\}_{s\in\mathbb{N}^{<\mathbb{N}}}$ of E such that

(i) if $s \preceq t$, then $E_s \subseteq E_t$,

(ii) A_0 is a transversal for E_s,

(iii) every E_s-class is contained in $\bigsqcup_{k\leq \operatorname{len}(s)} A_k$.

We proceed by induction on the length of s. Let $E_\varnothing$ be the equality relation on A_0. Now for each $a \in A_0$, let $[a]_{E_{s\hat{}i}}$ be the unique set, if it exists, of the form $[a]_{E_s} \sqcup S$, where $S \in [E]^{<\infty}$ is contained in $A_{\operatorname{len}(s)+1}$ and $c([a]_{E_s} \sqcup S) = i$, and otherwise set $[a]_{E_{s\hat{}i}} = [a]_{E_s}$. This defines an fsr E_s with the desired properties.

For every $\alpha \in \mathbb{N}^{\mathbb{N}}$, let $E_\alpha = \bigcup_n E_{\alpha\restriction n}$. We claim that for every $a \in A_0$, we have

$$\forall^* \alpha\ ([a]_{E_\alpha} \text{ is infinite}),$$

where $\forall^* \alpha\, \Phi(\alpha)$ means that the set $\{\alpha \in \mathbb{N}^{\mathbb{N}} : \Phi(\alpha)\}$ is comeager (see [Kec95, 8.J]). It suffices to show that for every n, we have

$$\forall^* \alpha\ (|[a]_{E_\alpha}| > n).$$

Since the set $\{\alpha \in \mathbb{N}^{\mathbb{N}} : |[a]_{E_\alpha}| > n\}$ is open, it suffices to show that it is dense. Fix some $s \in \mathbb{N}^{<\mathbb{N}}$. Let $S \in [E]^{<\infty}$ be a subset of $A_{\operatorname{len}(s)+1}$ with $|S| > n$. Then if $c([a]_{E_s} \sqcup S) = i$, then for every $\alpha \succ s\hat{}i$, we have $|[a]_{E_\alpha}| \geq |[a]_{E_{s\hat{}i}}| > n$, so we are done.

Thus for every $x \in X$, we have

$$\forall a \in A_0 \cap [x]_F\ \forall^* \alpha\ ([a]_{E_\alpha} \text{ is infinite}),$$

or equivalently

$$\forall^* \alpha\ \forall a \in A_0 \cap [x]_F\ ([a]_{E_\alpha} \text{ is infinite}),$$

so by the Kuratowksi-Ulam theorem [Kec95, 8.K], we have

$$\forall^* \alpha\ \forall^* x\ \forall a \in A_0 \cap [x]_F\ ([a]_{E_\alpha} \text{ is infinite}),$$

so in particular, there is some $\alpha \in \mathbb{N}^{\mathbb{N}}$ such that the F-invariant set

$$C := \{x \in X : \forall a \in A_0 \cap [x]_F \, ([a]_{E_\alpha} \text{ is infinite})\}$$

is comeager. Note that C is E-compressible, since $\mathrm{dom}(E_\alpha) \cap C$ is an $(E \upharpoonright C)$-compressible, complete $(E \upharpoonright C)$-section, so we are done. □

Corollary 3.3.11. *Let E be an aperiodic CBER on a Polish space X. Then for any Borel action $G \curvearrowright_B X/E$, there is a comeager $E^{\vee G}$-invariant Borel subset $Y \subseteq X$ such that $G \curvearrowright_B Y/E$ has a class-bijective lift.*

Proof. Apply Theorem 3.3.10 with $F = E^{\vee G}$. Then the result follows from Theorem 3.3.5. □

In conclusion, let us say that an aperiodic CBER E is **outer** if every $G \curvearrowright_B X/E$ is outer, or equivalently i_E is a bijection. We have seen that every compressible CBER is outer, while there are non-outer CBER. However we have the following problems:

Problem 3.3.12.

(1) Are there outer, non-compressible CBER?

(2) Characterize the outer CBERs.

Concerning the first part of this problem, we note the following possible approach to finding such an example:

Assume that there is a a free, pmp action of a countable group G on a standard probability space (X, μ) with the following properties:

(i) G is co-Hopfian (i.e., injective morphisms of G into itself are surjective) and G has no non-trivial finite normal subgroups (e.g., $\mathrm{SL}_3(\mathbb{Z})$);

(ii) The action is totally ergodic (i.e., every infinite subgroup acts ergodically) and satisfies cocycle superrigidity (i.e., every cocycle of the action to a countable group is cohomologous to a homomorphism);

(iii) $\mathrm{Out}_\mu(E_G^X)$ is trivial.

There are many examples that satisfy (ii) and others that satisfy (iii) but it does not seem to be known whether there are examples that satisfy both. Assuming that such an action exists, one can see that the first part of the above problem has a positive answer.

By going to a G-invariant Borel set, we can assume that μ is the unique invariant measure for this action. Then if $Z \subseteq X$ is Borel and G-invariant of measure 1, we have that $Y = X \setminus Z$ is compressible. Put $E = E_G^X$. Let now $f \in \mathrm{Sym}_B(X/E)$ and let $T : X \to X$ be Borel such that $f([x]_E) = [T(x)]_E$. Then T is a reduction of E to E and so it gives rise to a cocycle α of this action into G, which is therefore cohomologous to a homomorphism $\varphi : G \to G$. Thus we can find another Borel map S with $S(x)\ E\ T(x)$ and $S(g \cdot x) = \varphi(g) \cdot S(x)$, a.e. Let $N = \ker(\varphi)$. If it is not trivial, it must be infinite. Then for $g \in N$, $S(g \cdot x) = S(x)$, a.e., so by the ergodicity of the N-action, S is constant, a.e., which is a contradiction. So N is trivial and thus φ is injective, therefore an automorphism. It follows that S is in $\mathrm{Aut}_\mu(E)$ and thus in $\mathrm{Inn}_\mu(E)$. Therefore there is an E-invariant Borel set $Z \subseteq X$ of measure 1 with $f \upharpoonright (Z/E)$ the identity. Then $f \upharpoonright (Z/E)$ can be lifted to the identity of Z. Moreover $Y = X \setminus Z$ is compressible, so, by Theorem 3.3.5 $f \upharpoonright (Y/E)$ can be lifted to some Borel automorphism of $E \upharpoonright Y$. Thus f is an outer permutation.

Concerning the second part of the problem, note that by Corollary 3.3.8, an aperiodic hyperfinite CBER is outer iff it is compressible.

The following problem about the algebraic structure of these groups is also open:

Problem 3.3.13. When is $\mathrm{Out}_B(E)$ a normal subgroup of $\mathrm{Sym}_B(X/E)$?

3.4 Outer actions

A lift of an outer action is a solution to the following lifting problem:

$$\begin{array}{ccc} & & \mathrm{Aut}_B(E) \\ & \nearrow & \downarrow p_E \\ G & \longrightarrow & \mathrm{Out}_B(E) \end{array} .$$

Many outer actions arise from the following construction:

Example 3.4.1. Given a Borel action $G \curvearrowright X$ of a countable group G and a normal subgroup $N \triangleleft G$, there is a morphism $G \to \mathrm{Out}_B(E_N^X)$ defined by

$$g \cdot [x]_{E_N^X} = [g \cdot x]_{E_N^X},$$

and this descends to a morphism $G/N \to \mathrm{Out}_B(E_N^X)$.

Normal subequivalence relations

The concept of normality is central to the study of outer actions:

Definition 3.4.2. Let $E \subseteq F$ be CBERs. We say that E is **normal** in F, denoted $E \triangleleft F$, if any of the following equivalent conditions hold:

(1) There is an action $G \curvearrowright_B (X, E)$ of a countable group G such that $F = E^{\vee G}$.

(2) There is a morphism $G \to \mathrm{Out}_B(E)$ from a countable group G such that $F = E^{\vee G}$.

(3) There is a countable subgroup $G \leq \mathrm{Aut}_B(E)$ such that $F = E^{\vee G}$.

(4) There is a countable subgroup $G \leq \mathrm{Out}_B(E)$ such that $F = E^{\vee G}$.

To see the equivalence, note that (3) $\implies$ (1) $\implies$ (2) is immediate, (2) $\implies$ (4) holds by taking the image of G in $\mathrm{Out}_B(E)$, and (4) $\implies$ (3) holds by fixing a lift $T_g \in \mathrm{Aut}_B(E)$ of each $g \in G$ and taking the subgroup of $\mathrm{Aut}_B(E)$ generated by the T_g.

For CBERs $E \subseteq F$, it is possible that E is not normal in F, but that there is still a Borel action $G \curvearrowright_B X/E$ such that $F = E^{\vee G}$, as witnessed by the example at the beginning of Section 3.3. For more discussion concerning the weaker notion, see Section 3.8.

Proposition 3.4.3. *Let $E \triangleleft F$ be CBERs on X.*

(1) If F' is a CBER with $E \subseteq F' \subseteq F$, then $E \triangleleft F'$.

(2) For any E-invariant subset $Y \subseteq X$, we have $E \upharpoonright Y \triangleleft F \upharpoonright Y$.

Proof. Note that (2) follows immediately from (1) by taking $F' = (F \upharpoonright Y) \oplus (F \upharpoonright (X \setminus Y))$, so it suffices to prove (1).

We first assume that $F = E^{\vee T}$ for some $T \in \mathrm{Aut}_B(E)$. We will show that $F' = E^{\vee T'}$ for some $T' \in \mathrm{Aut}_B(E)$.

For each $x \in X$, let $\leq_x$ be the preorder on $[x]_{F'}/E$ defined by $[y]_E \leq_x [z]_E$ iff there exists some $n \geq 0$ such that $T^n(y) \mathrel{E} z$. If $\leq_x$ is isomorphic to $\mathbb{Z}$ or not antisymmetric, then set $T'(x) = T^n(x)$, where $n > 0$ is least such that $T^n(x) \mathrel{F'} x$. Otherwise, there is a unique isomorphism from $\leq_x$ to either the negative integers $(\{\cdots, -3, -2, -1\}, \leq)$

or to an initial segment of $(\mathbb{N}, \leq)$. So by fixing a transitive $\mathbb{Z}$-action on each of these linear orders, we obtain a transitive $\mathbb{Z}$-action on $[x]_{F'}/E$, and we set $T'(x) = T^n(x)$, where n is unique such that $T^n(x) \in 1 \cdot [x]_E$.

Now suppose that $F = E^{\vee G}$ for some $G \leq \mathrm{Aut}_B(E)$. By above, for each $T \in G$, we can fix some $T' \in \mathrm{Aut}_B(E)$ such that $E^{\vee T'} = F' \cap E^{\vee T}$. Then $F' = E^{\vee H}$, where $H = \langle T' \rangle_{T \in G}$. □

We next make some remarks about smooth links. Let $E \triangleleft F$ be CBERs. Suppose that E is aperiodic and $[F : E] = \infty$, since the finite parts have smooth links via the forthcoming Theorem 3.5.1 and Proposition 3.4.6. If E is compressible, then there is a smooth link by Theorem 3.3.6. On the other hand, if there is a smooth link L, then F must be compressible, since it contains the aperiodic smooth L.

Thus the existence of a link does not imply the existence of a smooth link. For instance, fix a free pmp Borel action $\mathbb{Z}^2 \curvearrowright X$, and consider $E = E^X_{\mathbb{Z}\times\{0\}}$ and $F = E^X_{\mathbb{Z}^2}$. Then there is a link given by the action of $\{0\}\times\mathbb{Z}$, but there is no smooth link, since F is not compressible. If X is the circle and the $\mathbb{Z}^2$-action is by two linearly independent irrational rotations, then E and F are both uniquely ergodic, and by taking copies of these, one can obtain an example with any number of ergodic measures.

If $E \triangleleft F$ with E finitely ergodic, then F is not compressible, since if $\mathrm{EINV}_E = (e_i)_{i<n}$, then $\frac{1}{n}(e_0 + \cdots + e_{n-1}) \in \mathrm{EINV}_F$. Thus there is no smooth link. If EINV_E is infinite, it is still possible for a smooth link to exist. For instance, consider $E = E_0 \times \Delta_{\mathbb{N}}$ and $F = E_0 \times I_{\mathbb{N}}$. In general, the following is open:

Problem 3.4.4. Let $E \triangleleft F$ be CBERs with F is compressible. Is there a smooth (E, F)–link?

Another open question, related to Theorem 3.3.6, is as follows:

Problem 3.4.5. Let $E \triangleleft F \triangleleft F'$ be compressible CBERs. Can every (E, F)–link be extended to an (E, F')–link?

If this were true, then assuming the Continuum Hypothesis, for any compressible CBER E, the epimorphism $p_E\colon \mathrm{Aut}_B(E) \twoheadrightarrow \mathrm{Out}_B(E)$ would split, i.e., there would exist a morphism $s\colon \mathrm{Out}_B(E) \to \mathrm{Aut}_B(E)$ with $p_E \circ s$ equal to the identity. To see this, write $\mathrm{Out}_B(E)$ as an increasing union $\bigcup_{\alpha<\omega_1} G_\alpha$ of countable subgroups. It suffices to obtain class-bijective lifts $G_\alpha \to \mathrm{Aut}_B(E)$ such that if $\alpha < \beta$, then the G_β

lift extends the G_α lift. For λ limit, take the union of the corresponding links for the G_α with $\alpha < \lambda$, and for $\beta = \alpha + 1$ a successor, use a positive answer to Problem 3.4.5.

Basic results

Proposition 3.4.6. *Let E be a smooth CBER.*

(1) *If F is a CBER with $E \triangleleft F$, then there is an (E, F)–link.*

(2) *Every outer action on X/E has a class-bijective lift.*

Proof. By Proposition 3.3.4, it suffices to show (1).

By normality, any two E-classes contained in the same F-class have the same cardinality, so by partitioning the space into F-invariant Borel sets, we can assume that there is some $n \in \{1, 2, \cdots, \mathbb{N}\}$ such that every E-class has cardinality n. Then there is a partition $X = \bigsqcup_{k<n} S_k$ such that each S_k is a transversal for E. Thus the CBER L defined by

$$x \mathrel{L} y \iff (x \mathrel{F} y) \;\&\; (\exists k < n \, [x, y \in S_k])$$

is an (E, F)–link. □

It is clear that if G is a free group, then every outer action of G has a lift. There are also some basic closure properties for the class of groups for which every outer action admits a (class-bijective) lift.

Proposition 3.4.7. *Let $H \leq G$. If every outer action of G has a (class-bijective) lift, then the same holds for H.*

Proof. Let E be a CBER, and fix a morphism $H \to \mathrm{Out}_B(E)$. Let $F = \bigoplus_{G/H} E$. Then there is a morphism $G \to \mathrm{Out}_B(F)$, induced by the action of G on G/H, so we get a lift $G \to \mathrm{Aut}_B(F)$. Restricting to H and E gives the desired lift. □

Proposition 3.4.8. *Let $G \twoheadrightarrow H$ be an epimorphism. If every outer action of G has a class-bijective lift, then the same holds for H.*

Proof. Fix a morphism $H \to \mathrm{Out}_B(E)$. This gives a morphism $G \to \mathrm{Out}_B(E)$. Since by surjectivity $E^{\vee G} = E^{\vee H}$, we are done by Proposition 3.3.4. □

At this point, it is good to show that not every outer action has a lift.

Definition 3.4.9. A countable group G is **treeable** if it admits a free pmp Borel action whose induced equivalence relation is treeable.

Example 3.4.10. There are many examples of groups which are not treeable (see [KM04, p. 30], [Kec22, p. 10.8]):

- Infinite property (T) groups.
- $G \times H$, where G is infinite and H is non-amenable.
- More generally, lattices in products of locally compact Polish groups $G \times H$, where G is non-compact and H is non-amenable.

The proof of the next result is motivated by [CJ85, Theorem 5] and the remark following the proof of [FSZ89, Theorem 3.4].

Proposition 3.4.11. *Suppose that every outer action of G lifts. Then G is treeable.*

Proof. We can assume that $G = F_\infty / N$ for some $N \triangleleft F_\infty$, where F_∞ is the free group on infinitely many generators. Fix a free pmp Borel action $F_\infty \curvearrowright_B (X, \mu)$ (for instance, the Bernoulli shift on 2^{F_∞}), and consider the induced free outer action $G \to \mathrm{Out}_B(E_N^X)$ (see Example 3.4.1). By assumption, there is a lift $G \to \mathrm{Aut}_B(E_N^X)$, which is also a free action. Then E_G^X is treeable and preserves μ, since $E_{F_\infty}^X$ satisfies these properties and contains E_G^X. □

Note that we have no control over the treeable CBER in the proof of Proposition 3.4.11. In particular, the following is open:

Problem 3.4.12. Does every outer action on X/E_0 lift?

3.5 Outer actions of finite groups

The following is a strengthening of [Tse13, Proposition 7.1]:

Theorem 3.5.1. *Let $E \triangleleft F$ be a finite index extension of CBERs. Then there is an (E, F)–link.*

Proof. Let Φ be the set of elements of $[F]^{<\infty}$ which are a transversal for $E \upharpoonright C$ for some F-class C. By [KM04, Lemma 7.3], there is a Φ-maximal fsr R. Let $Y = (\mathrm{dom}(R))_E$ be the E-hull of $\mathrm{dom}(R)$.

Let $G \leq \mathrm{Aut}_B(E)$ be a countable subgroup such that $F = E^{\vee G}$. For every $x \in X \setminus Y$, let $g_x \in G$ be least (in some enumeration of G) such that $g_x \cdot x \in Y$; this exists by Φ-maximality of R. Then the equivalence relation generated by $R \upharpoonright Y$ and $\{(x, g_x \cdot x) : x \in X \setminus Y\}$ is an (E, F)–link. □

Corollary 3.5.2. *Every outer action of a finite group has a class-bijective lift.*

Proof. Follows from Proposition 3.3.4 and Theorem 3.5.1. □

The following is a special case of Corollary 3.6.14, whose proof is much harder.

Corollary 3.5.3. *Every outer action of $\mathbb{Z}$ has a class-bijective lift.*

Proof. On the finite $\mathbb{Z}$-orbits, apply Corollary 3.5.2. On the infinite $\mathbb{Z}$-orbits of X/E, just lift uniquely. □

We next introduce lifts of morphisms:

Definition 3.5.4. Let $H \to G$ be a morphism of countable groups. Then $H \to G$ has the **class-bijective lifting property** if for any CBER E and any diagram of the form

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Aut}_B(E) \\ \big\downarrow & & \big\downarrow p_E \\ G & \longrightarrow & \mathrm{Out}_B(E) \end{array}$$

with $H \to \mathrm{Aut}_B(E)$ class-bijective, there is a class-bijective lift $G \to \mathrm{Aut}_B(E)$.

Proposition 3.5.5. *Let H be a countable group, let $(G_n)_n$ be a countable family of countable groups, let $H \to G_n$ be morphisms, and let G be the amalgamated free product of the G_n over H. If every outer action of H has a class-bijective lift, and each $H \to G_n$ has the class-bijective lifting property, then every outer action of G lifts.*

Proof. Let E be a CBER, and fix $G \to \mathrm{Out}_B(E)$. By assumption, there is a class-bijective lift of $H \to \mathrm{Out}_B(E)$. Then for each n, there is a class-bijective lift $G_n \to \mathrm{Aut}_B(E)$ such that the following diagram commutes:

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Aut}_B(E) \\ \big\downarrow & \nearrow & \big\downarrow p_E \\ G_n & \longrightarrow & \mathrm{Out}_B(E) \end{array}$$

Thus by the universal property of amalgamated products, there is a lift $G \to \mathrm{Aut}_B(E)$. $\square$

Theorem 3.5.6. *Let G be a countable group and let $N \triangleleft G$ be a finite normal subgroup such that every outer action of $H = G/N$ has a class-bijective lift.*

(1) The inclusion $N \hookrightarrow G$ has the class-bijective lifting property.

(2) Every outer action of G has a class-bijective lift.

Proof. (1) implies (2) by Corollary 3.5.2, so it suffices to show (1).

Let E be a CBER on X, and suppose we have

$$\begin{array}{ccc} N & \longrightarrow & \mathrm{Aut}_B(E) \\ \big\downarrow & & \big\downarrow p_E \\ G & \longrightarrow & \mathrm{Out}_B(E) \end{array}$$

with $N \to \mathrm{Aut}_B(E)$ class-bijective, and let $F = E^{\vee N}$. Note that $L = E^X_N$ is an (E, F)–link. There is an induced outer action $H \to \mathrm{Out}_B(F)$. We can assume that $[F : E] = n < \infty$. Let S be a transversal for L, and fix a Borel action $\mathbb{Z}/n\mathbb{Z} \curvearrowright X$ generating L.

Define an injection $\mathrm{Aut}_B(F \upharpoonright S) \hookrightarrow \mathrm{Aut}_B(F)$ as follows: given $T \in \mathrm{Aut}_B(F \upharpoonright S)$, let $T' \in \mathrm{Aut}_B(F)$ be the unique morphism satisfying $T'(k \cdot x) = k \cdot T(x)$ for every $x \in S$ and $k \in \mathbb{Z}/n\mathbb{Z}$. This descends to an injection $\mathrm{Out}_B(F \upharpoonright S) \hookrightarrow \mathrm{Out}_B(F)$ satisfying the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Out}_B(F \upharpoonright S) & \hookrightarrow & \mathrm{Out}_B(F) \\ \big\downarrow i_{F \upharpoonright S} & & \big\downarrow i_F \\ \mathrm{Sym}_B(F \upharpoonright S) & \xrightarrow{\cong} & \mathrm{Sym}_B(F) \end{array}$$

We claim that this injection is a bijection. To see this, let $T \in \mathrm{Aut}_B(F)$. Since $X = \bigsqcup_{k \in \mathbb{Z}/n\mathbb{Z}} k \cdot S$, we have $n\widetilde{S} = \widetilde{X}$ in the cardinal algebra $\mathcal{K}(F \times I_{\mathbb{N}})$. Thus $n\widetilde{T(S)} = \widetilde{T(X)} = \widetilde{X}$, so by the cancellation law, we have $\widetilde{S} = \widetilde{T(S)}$, i.e., there is some $T' \in \mathrm{Inn}_B(F)$ with $T'(T(S)) = S$. Then $(T'T) \upharpoonright S \in \mathrm{Aut}_B(F \upharpoonright S)$ is the desired map.

Thus we obtain an outer action $H \to \mathrm{Out}_B(F \upharpoonright S)$ and by assumption, there is an $(F \upharpoonright S, E^{\vee G} \upharpoonright S)$–link L'. Then the equivalence relation generated by L and L' is an (E, F')–link. $\square$

We will prove next a generalization of Corollary 3.5.2 to morphisms. For that, we need the following result.

Proposition 3.5.7. *Let $E \subseteq F$ be a bounded index extension of CBERs. Then the following are equivalent:*

(1) $E \triangleleft F$.

(2) There is a finite subgroup $G \leq \mathrm{Out}_B(E)$ such that $F = E^{\vee G}$.

Proof. (2) $\implies$ (1) Immediate.

(1) $\implies$ (2) Let $H = (h_n)_n \leq \mathrm{Aut}_B(E)$ be a countable subgroup such that $F = E^{\vee H}$. We define inductively a sequence $(g_n)_n \subseteq \mathrm{Inn}_B(F) \cap \mathrm{Aut}_B(E)$ as follows: for every F-class C, if there is i such that $p_{E\restriction C}(h_i \restriction C) \neq p_{E\restriction C}(g_j \restriction C)$ for all $j < n$, then for the least i with this property, set $g_n \restriction C = h_i \restriction C$; otherwise set $g_n \restriction C = \mathrm{id} \restriction C$.

Note that the sequence $(g_n)_n$ is eventually equal to id_X, since E is of bounded index in F. Thus the group $\tilde{G} = \langle g_n \rangle_{n<\infty} \leq \mathrm{Inn}_B(F) \cap \mathrm{Aut}_B(E)$ is finitely generated. Note also that $F = E^{\vee \tilde{G}}$. Now the image of $\mathrm{Inn}_B(F) \cap \mathrm{Aut}_B(E)$ in $\mathrm{Out}_B(E)$ is locally finite, since it is a subgroup of $(S_n)^{X/F}$ for some finite symmetric group S_n. So the image G of $\tilde{G}$ in $\mathrm{Out}_B(E)$ is finite, and we are done. □

We have a generalization of Theorem 3.5.1:

Theorem 3.5.8. *Let $E \subseteq F \subseteq F'$ be CBERs such that E has finite index in F' and $E \triangleleft F'$. Then every (E, F)–link is contained in an (E, F')–link.*

Proof. By partitioning the underlying standard Borel space X, we can assume that there is some $n < \infty$ such that every F'-class contains at most n F-classes. We proceed by induction on n. The case $n = 1$ is trivial.

Let L be an (E, F)–link and let S be a transversal for L. Let Φ be the set of $A \in [F' \restriction S]^{<\infty}$ which are a transversal for $F \restriction C$ for some F'-class C. By [KM04, Lemma 7.3], there is a Φ-maximal fsr R. Let $Y \subseteq X$ be the set of $x \in X$ such that $[x]_F \subseteq [\mathrm{dom}(R)]_L$ and let $Z = X \setminus Y$. We can assume that no F'-class is contained in Y, since the equivalence relation generated by R and L is an (E, F')–link on such a class. By Φ-maximality of R, no F'-class is contained in Z either. By (2) of Proposition 3.4.3, we have $E \restriction Y \triangleleft F' \restriction Y$, so by the induction hypothesis, there is an $(E \upharpoonright Y, F' \upharpoonright Y)$–link L_Y containing $L \upharpoonright Y$. Similarly, there is an $(E \upharpoonright Z, F' \upharpoonright Z)$–link L_Z containing $L \upharpoonright Z$.

Let S_Y and S_Z be transversals for L_Y and L_Z respectively. It suffices to show that there is some $T \in \mathrm{Inn}_B(F')$ such that $T(S_Y) = S_Z$, since then the smallest equivalence relation containing L_Y and L_Z and $\{(x, T(x)) : x \in S_Y\}$ is an (E, F')–link. In other words, we need to show that $\widetilde{S_Y} = \widetilde{S_Z}$ in the cardinal algebra $\mathcal{K}(F' \times I_{\mathbb{N}})$. By Proposition 3.5.7, there is a finite subgroup $G \leq \mathrm{Out}_B(E)$ such that $F' = E^{\vee G}$. By partitioning X, we can assume that $[F' \upharpoonright Y : E \upharpoonright Y] = n_Y$ and $[F' \upharpoonright Z : E \upharpoonright Z] = n_Z$ for some $n_Y, n_Z < \infty$. Then $\widetilde{Y} = n_Y \widetilde{S_Y}$ and $\widetilde{Z} = n_Z \widetilde{S_Z}$. Let $k = \frac{|G|}{n_Y + n_Z}$. Then for every $x \in X$, we have

$$|\{g \in G : [x]_E \subseteq g \cdot Y\}| = \sum_{[y]_E \subseteq Y} |\{g \in G : [x]_E = g \cdot [y]_E\}| = k n_Y,$$

and thus $|G|\widetilde{Y} = k n_Y \widetilde{X}$. Similarly, $|G|\widetilde{Z} = k n_Z \widetilde{X}$. Thus

$$|G| n_Y n_Z \widetilde{S_Y} = |G| n_Z \widetilde{Y} = k n_Y n_Z \widetilde{X} = |G| n_Y \widetilde{Z} = |G| n_Y n_Z \widetilde{S_Z},$$

which yields $\widetilde{S_Y} = \widetilde{S_Z}$ by the cancellation law. □

Corollary 3.5.9. *Every morphism of finite groups has the class-bijective lifting property.*

Proof. Suppose we have

$$\begin{array}{ccc} H & \longrightarrow & \mathrm{Aut}_B(E) \\ \downarrow & & \downarrow p_E \\ G & \longrightarrow & \mathrm{Out}_B(E) \end{array}$$

with H and G finite, and $H \to \mathrm{Aut}_B(E)$ class-bijective. Then E_H is an $(E, E^{\vee H})$–link, so by Theorem 3.5.8, there is an $(E, E^{\vee G})$–link L_G containing E_H. This lets us define an action of G by setting $g \cdot x$ to be the unique element in both $[x]_{L_G}$ and $g \cdot [x]_E$. □

Corollary 3.5.10. *Every outer action of an amalgamated free product of finite groups has a lift.*

Proof. Let H be a finite group, let $(G_n)_{n<\infty}$ be finite groups, let $H \to G_n$ be morphisms, and let G be the amalgamated free product of the G_n over H. By Corollary 3.5.2, every outer action of H has a class-bijective lift. By Corollary 3.5.9, the morphisms $H \to G_n$ have the class-bijective lifting property. Thus by Proposition 3.5.5, every outer action of G lifts. □

Given CBERs $E \subseteq F$, we say that F/E is **hyperfinite** if there is an increasing sequence $(F_n)_n$ of finite index extensions of E such that $F = \bigcup_n F_n$.

Corollary 3.5.11. *Let $E \triangleleft F$ be CBERs with F/E hyperfinite. Then there is an (E, F)–link.*

Proof. Apply Theorem 3.5.8 countably many times. □

Corollary 3.5.12. *Every outer action of a locally finite group has a class-bijective lift.*

Proof. Immediate from Corollary 3.5.11. □

3.6 Outer actions of amenable groups

Our goal in this section is to show that every outer action of an amenable group lifts. We will prove in 3.6 some special cases of this result, using (as a black box) [FSZ89, Theorem 3.4] (stated in Theorem 3.6.1 below). The general case, which is based on some ideas from the proof of Theorem 3.6.1 in combination with Theorem 3.3.5 will be proved in 3.6.

Special cases

We will use the following result from the pmp setting:

Theorem 3.6.1 ([FSZ89, Theorem 3.4])**.** *Let G be an amenable group and let E be a pmp ergodic CBER. Then any morphism $G \to \mathrm{Out}_\mu(E)$ has a lift.*

Remark 3.6.2. In [FSZ89] this result is stated for free outer actions, i.e., outer actions $\varphi\colon G \to \mathrm{Out}_\mu(E)$ that have the following additional property: if $g \in G$ is not the identity and $T_g \in \mathrm{Aut}_\mu(E)$ maps by the canonical projection to $\varphi(g)$, then $T_g(x) \notin [x]_E$, a.e. Using the ergodicity of E, this is equivalent to the kernel of φ being trivial. Thus for an arbitrary outer action $\varphi\colon G \to \mathrm{Out}_\mu(E)$, if H is the kernel of φ, this gives a free outer action of G/H, which by the special case lifts to an action of G/H which composed with the projection of G to G/H gives a lifting of φ.

Remark 3.6.3. Note that (the measurable version of) Corollary 3.5.10 gives examples of non-amenable groups that satisfy Theorem 3.6.1.

Now Theorem 3.6.1 together with Theorem 3.3.5 implies the following Borel result:

Theorem 3.6.4. *Let G be an amenable group and let E be a uniquely ergodic CBER. Then every morphism $G \to \mathrm{Out}_B(E)$ lifts.*

Proof. Let μ be the ergodic invariant measure for E. Note that any element of $\mathrm{Aut}_B(E)$ preserves μ by unique ergodicity. Thus by Theorem 3.6.1, there is a lift $G \to \mathrm{Aut}_\mu(E)$, so there is a conull E-invariant Borel set $Y \subseteq X$ such that $G \to \mathrm{Out}_B(E \upharpoonright Y)$ lifts to $\mathrm{Aut}_B(E \upharpoonright Y)$. But since the complement is compressible, we are done here by Theorem 3.3.5. □

In fact the following stronger result holds.

Theorem 3.6.5. *Let G be an amenable group and let E be a countably ergodic CBER. Then every morphism $G \to \mathrm{Out}_B(E)$ lifts.*

Proof. Note that G acts on the ergodic components modulo compressible sets, which we can ignore by Theorem 3.3.5. We can assume that this action is transitive. Fix an ergodic component Y, and let $H = \{g \in G : g \cdot Y = Y\}$. By the uniquely ergodic case, there is a lift $H \to \mathrm{Aut}_B(E \upharpoonright Y)$. Let $S \subseteq G$ be a transversal for the left cosets of H in G, with $1 \in S$. For every $s \in S$, choose a lift $T_s \in \mathrm{Aut}_B(E)$, with $T_1 = \mathrm{id}_X$. Now fix $g \in G$ and $s \in S$. We define the action of g on sY. We have $gsY = tY$ for some $t \in S$, so we have $t^{-1}gs \in H$. Thus we can define

$$g \cdot (T_s y) := T_t((t^{-1}gs) \cdot y).$$

□

E-null sets

Let E be an aperiodic CBER on X, so that every $\mu \in \mathrm{EINV}_E$ is non-atomic. A Borel subset $A \subseteq X$ is **E-null** if either of the following equivalent conditions holds:

(1) $\mu(A) = 0$ for every $\mu \in \mathrm{EINV}_E$.

(2) $E \upharpoonright [A]_E$ is compressible.

An **E-conull** set is the complement of an E-null set.

Let $\mathrm{NULL}_E \subseteq \mathcal{B}(X)$ be the σ-ideal of E-null Borel sets, and let ALG_E be the quotient σ-algebra $\mathcal{B}(X)/\mathrm{NULL}_E$. A Borel map $T : X \to X$ is **NULL_E-preserving** if the preimage under T of every E-null set is E-null. Let $\mathrm{End}_{\mathrm{NULL}_E}(E)$ be the monoid of NULL_E-preserving Borel maps $X \to X$ such that $x \mathrel{E} y \implies \varphi(x) \mathrel{E} \varphi(y)$ for all x, y in an E-conull set, where two such maps are identified if they agree on an E-conull set. Let $\mathrm{Aut}_{\mathrm{NULL}_E}(E)$ be the group of invertible elements of $\mathrm{End}_{\mathrm{NULL}_E}(E)$. There is a natural action of $\mathrm{Aut}_{\mathrm{NULL}_E}(E)$ on ALG_E. Denote by $\mathrm{Inn}_{\mathrm{NULL}_E}(E)$ the normal subgroup of $\mathrm{Aut}_{\mathrm{NULL}_E}(E)$ of φ such that $\varphi(x) \mathrel{E} x$ for an E-conull set of x, and denote by $\mathrm{Out}_{\mathrm{NULL}_E}(E)$ the quotient group $\mathrm{Aut}_{\mathrm{NULL}_E}(E)/\mathrm{Inn}_{\mathrm{NULL}_E}(E)$.

Lifts of elements of $\mathrm{Out}_{\mathrm{NULL}_E}(E)$ are defined analogously as in the case of $\mathrm{Out}_B(E)$, as well as lifts of morphisms $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$. Let $G \to \mathrm{Aut}_{\mathrm{NULL}_E}(E)$ be a morphism. Let $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$. There is an action on X/E given by

$$g \cdot [x]_E = [T(x)]_E,$$

where T is a lift of g, which is well-defined for an E-conull set of x. Then $\mathrm{Stab}_G([x]_E)$ is well-defined for an E-conull set of x. We say that this is a **free action** if $\mathrm{Stab}_G([x]_E) = 1$ for an E-conull set of x. A morphism $G \to \mathrm{Aut}_{\mathrm{NULL}_E}(E)$ is **class-bijective** if for every $g \in G$, there is an E-conull set of x such that $\mathrm{Stab}_G(x) = \mathrm{Stab}_G([x]_E)$ (note that $\mathrm{Stab}_G(x)$ is also well-defined for an E-conull set of x). Links are defined as before, except that everything only needs to hold on an E-conull set.

Given $g \in \mathrm{Out}_{\mathrm{NULL}_E}(E)$, a **partial lift** ψ of g is the restriction of a lift ϕ of g to some $A \in \mathrm{ALG}_E$. In this case, we write $\psi\colon A \to B$, where $B = \phi(A)$.

There is a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mathrm{Inn}_B(E) & \longrightarrow & \mathrm{Aut}_B(E) & \longrightarrow & \mathrm{Out}_B(E) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathrm{Inn}_{\mathrm{NULL}_E}(E) & \longrightarrow & \mathrm{Aut}_{\mathrm{NULL}_E}(E) & \longrightarrow & \mathrm{Out}_{\mathrm{NULL}_E}(E) & \longrightarrow & 1 \end{array}.$$

In particular, any morphism $G \to \mathrm{Out}_B(E)$ induces a morphism $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$.

Proposition 3.6.6. *Let E be an aperiodic CBER on X, let G be a countable group and fix a morphism $G \to \mathrm{Out}_B(E)$. Then the following are equivalent:*

(1) $G \to \mathrm{Out}_B(E)$ lifts.

(2) $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$ lifts.

Proof. (1) $\implies$ (2) Immediate.

(2) $\implies$ (1) Denote the lift by $\varphi : G \to \mathrm{Aut}_{\mathrm{NULL}_E}(E)$, and denote by $\varphi_g \in \mathrm{Aut}_{\mathrm{NULL}_E}(E)$ the image of g under φ. For each $g \in G$, pick a representative $T_g : X \to X$ of φ_g. There is an E-conull subset $Y \subseteq X$ such that

(i) $x \; E \; y \iff T_g(x) \; E \; T_g(y)$ for every $g \in G$ and $x, y \in Y$,

(ii) $T_1(x) = x$ for every $x \in Y$,

(iii) $T_g(T_h(x)) = T_{gh}(x)$ for every $g, h \in G$ and $x \in Y$,

(iv) $[T_g(x)]_E = g \cdot [x]_E$ for every $g \in G$ and $x \in Y$.

By taking the $E^{\vee G}$-hull, we can assume that Y is $E^{\vee G}$-invariant. Then the T_g define a lift of $G \to \mathrm{Out}_B(E \upharpoonright Y)$. On $X \setminus Y$, we have that E is compressible, so we are done by Theorem 3.3.5. □

Every $\mu \in \mathrm{EINV}_E$ is a well-defined measure on ALG_E, and there is an action $\mathrm{Aut}_{\mathrm{NULL}_E}(E) \curvearrowright \mathrm{EINV}_E$ given by

$$(\varphi \cdot \mu)(A) = \mu(\varphi^{-1}(A)),$$

which descends to an action of $\mathrm{Out}_{\mathrm{NULL}_E}(E)$.

Proposition 3.6.7. *Let E be an aperiodic CBER, let $g \in \mathrm{Out}_{\mathrm{NULL}_E}(E)$, and let $A, B \in \mathrm{ALG}_E$. Then the following are equivalent:*

(1) $\mu(A) = (g \cdot \mu)(B)$ for every $\mu \in \mathrm{EINV}_E$.

(2) There is a partial lift $\varphi : A \to B$ of g.

(3) There is a lift φ of g with $\varphi(A) = B$.

Proof. (2) $\iff$ (3) By definition.

(3) $\implies$ (1) Immediate.

(1) $\implies$ (3) Let ψ be a lift of g. Then $\mu(A) = (g \cdot \mu)(B) = \mu(\psi^{-1}(B))$, so by replacing B with $\psi^{-1}(B)$, we can assume that $g = 1$. Then the result follows from [KM04, Lemma 7.10] and the remark following it. □

A family $(\varphi_n)_n$ of partial maps is **disjoint** if the family $(\mathrm{dom}\, \varphi_n)_n$ is disjoint and the family $(\mathrm{cod}\, \varphi_n)_n$ is disjoint.

Proposition 3.6.8. *Let E be an aperiodic CBER, fix a morphism $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$, and let $g \in G$. If $(\varphi_n)_n$ are disjoint partial lifts of g, then $\bigsqcup_n \varphi_n$ is a partial lift of g.*

Proof. Suppose $\varphi_n \colon A_n \to B_n$. Let $A = X \setminus \bigsqcup_n A_n$ and let $B = X \setminus \bigsqcup_n B_n$. By Proposition 3.6.7, for any $\mu \in \mathrm{EINV}_E$, we have $\mu(A_n) = (g \cdot \mu)(B_n)$, and thus $\mu(A) = (g \cdot \mu)(B)$. So again by Proposition 3.6.7, there is a partial lift $\varphi \colon A \to B$ of g. Then $\varphi \sqcup \bigsqcup_n \varphi_n$ is a lift of g, and thus the restriction φ_n is a partial lift of g. □

For $A \in \mathrm{ALG}_E$, we write $\mu_E(A) = r$ if for every $\mu \in \mathrm{EINV}_E$, we have $\mu(A) = r$. Recall that for any standard probability space (X, μ), if $A \subseteq X$ and $r \leq \mu(A)$, then there is some $B \subseteq A$ with $\mu(A) = r$, and this B can be found uniformly in μ. By applying this to each E-ergodic component, we obtain the following:

Proposition 3.6.9. *Let E be an aperiodic CBER, let $A \in \mathrm{ALG}_E$, and let $r \in [0, 1]$. If $r \leq \mu_E(A)$, then there is some $B \subseteq A$ such that $\mu_E(B) = r$.*

Quasi-tilings

Let G be a group. Let $\mathrm{Fin}(G)$ denote the set of finite subsets of G, and let $\mathrm{Fin}_1(G)$ denote the set of $A \in \mathrm{Fin}(G)$ containing 1. Given $A, B \in \mathrm{Fin}(G)$, we say that B **λ-covers** A if $|A \cap B| \geq \lambda |A|$.

Let $\mathcal{A}$ be a family in $\mathrm{Fin}(G)$, i.e., a subset of $\mathrm{Fin}(G)$. We say that $\mathcal{A}$ is **ε-disjoint** if there is a disjoint family $\{D_A\}_{A \in \mathcal{A}}$ such that each D_A is a subset of A which $(1 - \varepsilon)$-covers A. Note that if $\mathcal{A}$ is ε-disjoint, then

$$(1 - \varepsilon) \sum_{A \in \mathcal{A}} |A| \leq \left| \bigcup_{A \in \mathcal{A}} A \right|.$$

Given $A \in \mathrm{Fin}(G)$, we say that $\mathcal{A}$ **λ-covers** A if $\bigcup_{B \in \mathcal{A}} B$ λ-covers A.

Let $\mathcal{A}$ be a family in $\mathrm{Fin}_1(G)$ and let $A \in \mathrm{Fin}(G)$. An **$\mathcal{A}$–quasi-tiling** of A is a tuple $C = (C_B)_{B \in \mathcal{A}}$ of subsets of A such that $Bc \subseteq A$ for every $c \in C_B$, and the family $\{BC_B\}_{B \in \mathcal{A}}$ is disjoint. If $1 \in A$, we additionally demand that $1 \in C_B$ for some $B \in \mathcal{A}$. If $\mathcal{A} = \{B\}$ is a singleton, we will write "C is a B-quasi-tiling" as shorthand to mean that (C) is a $\{B\}$-quasi-tiling. We say that C is **ε-disjoint** if for each $B \in \mathcal{A}$, the family $\{Bc\}_{c \in C_B}$ is ε-disjoint. We say that C **λ-covers** A if $\{BC_B\}_{B \in \mathcal{A}}$ λ-covers A. We say that C is an **$(\mathcal{A}, \varepsilon)$-quasi-tiling** of A if it is ε-disjoint and $(1 - \varepsilon)$-covers A.

Given $A \in \mathrm{Fin}(G)$ and $B \in \mathrm{Fin}_1(G)$, let $T(A, B)$ denote the set $\{a \in A : Ba \subseteq A\}$. We say that A is **(B, ε)-invariant** if $T(A, B)$ $(1 - \varepsilon)$-covers A. Note that if A is (B, ε)-invariant, then $|BA| \leq (1 + \varepsilon|B|)|A|$.

Lemma 3.6.10. *Let G be group, let $\delta, \varepsilon > 0$, let $B \in \mathrm{Fin}_1(G)$, and let $A \in \mathrm{Fin}(G)$ be (B, δ)-invariant. Then any maximal ε-disjoint family $\{Bc\}_{c \in C}$ of right translates of B contained in A $\varepsilon(1 - \delta)$-covers A.*

Proof. If $g \in T(A, B)$, then by maximality, we have $|Bg \cap BC| \geq \varepsilon|B|$. Thus

$$\varepsilon(1 - \delta)|A| \leq \varepsilon|T(A, B)| \leq \sum_{g \in T(A,B)} \frac{|Bg \cap BC|}{|B|} \leq \sum_{g \in G} \frac{|Bg \cap BC|}{|B|} = |BC|,$$

where the last equality holds since every element of BC is contained in exactly $|B|$-many right translates of B. □

Let $\mathcal{A}$ be a finite family in $\mathrm{Fin}_1(G)$ and let $\mathbf{p} = (p_B)_{B \in \mathcal{A}}$ be a probability distribution on $\mathcal{A}$. Given an $\mathcal{A}$-quasi-tiling $C = (C_B)_{B \in \mathcal{A}}$ of $A \in \mathrm{Fin}(G)$, we say that C **satisfies $\mathbf{p}$** if $|B||C_B| \leq p_B|A|$ for every $B \in \mathcal{A}$. Given $\varepsilon > 0$, we say that the pair $(\mathcal{A}, \varepsilon)$ **satisfies $\mathbf{p}$** if there is some $\delta > 0$ such that for every $A \in \mathrm{Fin}_1(G)$ larger than $\frac{1}{\delta}$ which is (B, δ)-invariant and contains B for every $B \in \mathcal{A}$, there is an $(\mathcal{A}, \varepsilon)$-quasi-tiling of A satisfying $\mathbf{p}$.

Lemma 3.6.11. *Let G be a group. For every $\varepsilon > 0$, there is a finite probability distribution $\mathbf{p} = (p_i)_{i<k}$ and constants $\eta_i > 0$ for $i < k - 1$ such that if $\mathcal{A} = (B_i)_{i<k}$ is a descending chain in $\mathrm{Fin}_1(G)$ where each B_i for $i < k - 1$ is $(B_{i+1}^{-1}, \frac{\eta_i}{|B_{i+1}|})$-invariant, then $(\mathcal{A}, \varepsilon)$ satisfies $\mathbf{p}$.*

Proof. By scaling, it suffices to find a subprobability distribution. Choose k such that $2\varepsilon \geq (1 - \varepsilon)^k$, define $p_i = \varepsilon(1 - \varepsilon)^i$, and for $i < k - 1$, choose η_i such that

$$\eta_i \leq \frac{1 - 2\varepsilon}{2 \cdot 3^{k-i}}.$$

Let $\mathcal{A} = (B_i)_{i<k}$ be a descending chain in $\mathrm{Fin}_1(G)$ where each B_i is $(B_{i+1}^{-1}, \frac{\eta_i}{|B_{i+1}|})$-invariant, and let $\delta > 0$ be sufficiently small, depending on $(\mathcal{A}, \varepsilon)$, to be specified in the course of the proof. Suppose we have some $A \in \mathrm{Fin}_1(G)$ which is larger than $\frac{1}{\delta}$ and (B, δ)-invariant for every $B \in \mathcal{A}$.

We define a descending sequence $(A_i)_{i<k}$ of subsets of A and 2ε-disjoint B_i-quasi-tilings C_i of A_i such that

(i) $A_0 = A$.

(ii) $A_{i+1} = A_i \setminus B_i C_i$,

(iii) A_i is $(B_i, \frac{1}{3^{k-i}})$-invariant,

(iv)
$$\varepsilon(1-\varepsilon)^{i+2-2^{-i}} \le \frac{|B_i C_i|}{|A|} \le \varepsilon(1-\varepsilon)^{i-2+2^{-i}},$$

(v)
$$(1-\varepsilon)^{i+2-2^{-i+1}} \le \frac{|A_i|}{|A|} \le (1-\varepsilon)^{i-2+2^{-i+1}}.$$

We proceed by induction, starting with $A_0 = A$, defining C_i from A_i, and defining A_{i+1} from C_i via (ii). Note that A_0 satisfies (iii) if we require $\delta \le \frac{1}{3^k}$.

Suppose that A_i has been defined. We will define C_i. Let $\tilde{C_i}$ be a maximal 2ε-disjoint B_i-quasi-tiling of A_i. Since $2\varepsilon\left(1 - \frac{1}{3^{k-i}}\right) > \varepsilon$, by Lemma 3.6.10, $\tilde{C_i}$ is an ε-cover of A_i. Then by removing elements from $\tilde{C_i}$, we obtain a B_i-quasi-tiling $C_i \subseteq \tilde{C_i}$ of A_i such that
$$\varepsilon(1-\varepsilon)^{2^{-i}} \le \frac{|B_i C_i|}{|A_i|} \le \varepsilon(1-\varepsilon)^{-2^{-i}}$$
and
$$(1-\varepsilon)^{1+2^{-i}} \le \frac{|A_{i+1}|}{|A_i|} \le (1-\varepsilon)^{1-2^{-i}},$$
as long as A_i is sufficiently large such that $\frac{|B_i|}{|A_i|}$ is smaller than the length of the interval around ε given by
$$\left[\varepsilon(1-\varepsilon)^{2^{-i}}, \varepsilon(1-\varepsilon)^{-2^{-i}}\right] \cap \left[1-(1-\varepsilon)^{1-2^{-i}}, 1-(1-\varepsilon)^{1+2^{-i}}\right],$$
which occurs for sufficiently large A by (v). Then since $\frac{|B_i C_i|}{|A|} = \frac{|B_i C_i|}{|A_i|}\frac{|A_i|}{|A|}$, we get that (iv) holds. Similarly, (v) holds for A_{i+1}.

It remains to check (iii). Note that
$$T(A_{i+1}, B_{i+1}) = T(A_i, B_{i+1}) \setminus B_{i+1}^{-1} B_i C_i.$$

Since
$$\frac{|A_{i+1}|}{|A_i|} \ge (1-\varepsilon)^{1+2^{-i}} \ge (1-\varepsilon)^2 \ge \frac{1}{2},$$
where we assume that ε is small enough to satisfy the last inequality, the cardinality of $T(A_i, B_{i+1})$ is at least
$$\left(1 - \frac{1}{3^{k-i}}\right)|A_i| \ge |A_i| - \frac{2}{3^{k-i}}|A_{i+1}|.$$

Now B_iC_i is $\left(B_{i+1}^{-1}, \frac{\eta_i}{|B_{i+1}|(1-2\varepsilon)}\right)$-invariant, since

$$\begin{aligned}|\{g \in B_iC_i : B_{i+1}^{-1}g \not\subseteq B_iC_i\}| &\le \sum_{c\in C_i} |\{g \in B_ic : B_{i+1}^{-1}g \not\subseteq B_iC_i\}| \\ &\le \sum_{c\in C_i} |\{g \in B_ic : B_{i+1}^{-1}g \not\subseteq B_ic\}| \\ &\le \sum_{c\in C_i} \frac{\eta_i}{|B_{i+1}|}|B_i| \\ &= \frac{\eta_i}{|B_{i+1}|}|B_i||C_i| \\ &\le \frac{\eta_i}{|B_{i+1}|}\frac{|B_iC_i|}{1-2\varepsilon}.\end{aligned}$$

Since

$$\frac{|A_{i+1}|}{|B_iC_i|} \ge \frac{|A_{i+1}|}{|A_i|} \ge \frac{1}{2} \ge \frac{\eta_i}{1-2\varepsilon}3^{k-i},$$

we have

$$|B_{i+1}^{-1}B_iC_i| \le \left(1 + \frac{\eta_i}{1-2\varepsilon}\right)|B_iC_i| \le |B_iC_i| + \frac{1}{3^{k-i}}|A_{i+1}|.$$

Putting these together, we get

$$|T(A_{i+1}, B_{i+1})| \ge \left(1 - \frac{3}{3^{k-i}}\right)|A_{i+1}|,$$

so (iii) holds. This concludes the construction.

Now

$$\frac{|B_iC_i|}{|A|} \ge \varepsilon(1-\varepsilon)^{i+2-2^{-i}} > \varepsilon(1-\varepsilon)^{i+2} > \varepsilon(1-2\varepsilon)^2(1-\varepsilon)^i,$$

so for each $i < k$, there is a B_i-quasi-tiling $C_i' \subseteq C_i$ of A_i such that

$$\varepsilon(1-2\varepsilon)^2(1-\varepsilon)^i \le \frac{|B_iC_i'|}{|A|} \le \varepsilon(1-2\varepsilon)(1-\varepsilon)^i,$$

as long as A is large enough such that $\frac{|B_i|}{|A|}$ is smaller than the length of the interval

$$\left[\varepsilon(1-2\varepsilon)^2(1-\varepsilon)^i, \varepsilon(1-2\varepsilon)(1-\varepsilon)^i\right].$$

Then $(C_i')_{i<k}$ is a 2ε-disjoint $\mathcal{A}$-quasi-tiling of A which $(1-2\varepsilon)^3$-covers A. We also have

$$\frac{|B_i||C_i'|}{|A|} \le \frac{1}{1-2\varepsilon}\frac{|B_iC_i'|}{|A|} \le \varepsilon(1-\varepsilon)^i = p_i.$$

So we are done by replacing ε in the above argument by any $\bar{\varepsilon}$ such that ε is greater than $2\bar{\varepsilon}$ and $1-(1-2\bar{\varepsilon})^3$. □

A countable group G is **amenable** if for every $B \in \mathrm{Fin}(G)$ and every $\varepsilon > 0$, there is some $A \in \mathrm{Fin}(G)$ which is (B, ε)-invariant. Note that we can assume that A contains B.

Proposition 3.6.12. *Let G be an amenable group and let $(\varepsilon_n)_{n<\infty}$ be a sequence of positive reals. Then there exist for each $n < \infty$, a finite family $\mathcal{A}_n$ in $\mathrm{Fin}_1(G)$ and a probability distribution $\mathbf{p}^n$ on $\mathcal{A}_n$ such that*

(i) $\mathcal{A}_0 = \{\{1\}\}$,

(ii) if $B \in \mathcal{A}_n$ and $A \in \mathcal{A}_{n+1}$, then A is (B, ε_n)-invariant and contains B,

(iii) every $A \in \mathcal{A}_{n+1}$ has an $(\mathcal{A}_n, \varepsilon_n)$-quasi-tiling satisfying $\mathbf{p}^n$,

(iv) $G = \bigcup_n \bigcup_{B \in \mathcal{A}_n} B$.

Proof. Fix an enumeration $(g_n)_n$ of G. We inductively define $\mathcal{A}_n$ and $\mathbf{p}^n$ satisfying the given conditions such that additionally, $(\mathcal{A}_n, \varepsilon_n)$ satisfies $\mathbf{p}^n$. For $n = 0$, take $\mathcal{A}_0 = \{\{1\}\}$, and let $\mathbf{p}^0$ be the unique probability distribution on $\mathcal{A}_0$. Then $(\mathcal{A}_0, \varepsilon_0)$ satisfies $\mathbf{p}^0$. Now suppose that $\mathcal{A}_n$ and $\mathbf{p}^n$ have been defined. Apply Lemma 3.6.11 to ε_{n+1} to obtain a probability distribution $\mathbf{p}^n = (p_i)_{i<k_n}$ and constants $(\eta_i^n)_{i<k_n-1}$. We turn to defining $\mathcal{A}_{n+1} = (B_i^{n+1})_{i<k_{n+1}}$. First we define $B_{k_{n+1}-1}^{n+1}$, by choosing any $B_{k_{n+1}-1}^{n+1} \in \mathrm{Fin}_1(G)$ which contains B and is (B, ε_n)-invariant for every $B \in \mathcal{A}_n$, and contains g_n. and which has an $(\mathcal{A}_n, \varepsilon_n)$-quasi-tiling satisfying $\mathbf{p}^n$ (which is possible since $(\mathcal{A}_n, \varepsilon_n)$ satisifies $\mathbf{p}^n$). Now for any $i < k_{n+1} - 1$, we define B_i^{n+1} from B_{i+1}^{n+1}, by choosing any $B_i^{n+1} \in \mathrm{Fin}_1(G)$ containing B_{i+1}^{n+1} which is $\left((B_{i+1}^{n+1})^{-1}, \frac{\eta_i^n}{|B_{i+1}^{n+1}|}\right)$-invariant, (B, ε_n)-invariant for every $B \in \mathcal{A}_n$, and which has an $(\mathcal{A}_n, \varepsilon_n)$-quasi-tiling satisfying $\mathbf{p}^n$. Then $\mathcal{A}_{n+1}$ satisfies the given conditions and additionally, $(\mathcal{A}_{n+1}, \varepsilon_{n+1})$ satisfies $\mathbf{p}^{n+1}$. □

General case

Theorem 3.6.13. *Every outer action of an amenable group lifts.*

Proof. Let G be an amenable group, and let E be a CBER on X. By Proposition 3.4.6, we can assume that E is aperiodic. By Proposition 3.6.6, it suffices to show that every morphism $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$ lifts to $\mathrm{Aut}_{\mathrm{NULL}_E}(E)$. For the rest of the proof, when we refer to a subset of X, we will mean its equivalence class in ALG_E.

Fix a sequence $(\varepsilon_n)_{n<\infty}$ of positive reals less than 1 such that

$$\sum_n (1 - (1 - \varepsilon_n)(1 - 3\varepsilon_n)) < \infty.$$

Apply Proposition 3.6.12 to $(\varepsilon_n)_n$ to obtain for each $n < \infty$, a finite family $\mathcal{A}_n$ in $\mathrm{Fin}_1(G)$ and a probability distribution $\mathbf{p}^n = (p^n_A)_{A\in\mathcal{A}_n}$ on $\mathcal{A}_n$. For ease of notation, we will write p_A instead of p^n_A.

For each $n < \infty$, we construct a disjoint family $(X_A)_{A\in\mathcal{A}_n} \subseteq \mathrm{ALG}_E$, and partial lifts $\varphi^n_g \in \mathrm{Aut}_{\mathrm{NULL}_E}(E)$ of some $g \in G$ such that

(i) $\varphi^n_1 = \mathrm{id}_X$,

(ii) for $A \in \mathcal{A}_n$, we have $|A|\mu_E(X_A) = p_A$,

(iii) the family $\{\varphi^n_g(X_A) : A \in \mathcal{A}_n, g \in A\}$ is disjoint,

(iv) for $A \in \mathcal{A}_n$, if $g, h, gh \in A$, then φ^n_{gh} and $\varphi^n_g\varphi^n_h$ agree on X_A.

We proceed by induction on n. For $n = 0$, take $X_{\{1\}} = X$ and $\varphi^0_1 = \mathrm{id}_X$. Now suppose that the construction holds for n. We will repeatedly use Proposition 3.6.7, Proposition 3.6.8, and Proposition 3.6.9 to obtain the partial lifts φ^{n+1}_g. For each $A \in \mathcal{A}_{n+1}$, fix an $(\mathcal{A}_n, \varepsilon_n)$-quasi-tiling $(C^A_B)_{B\in\mathcal{A}_n}$ of A. By ε_n-disjointness, for each $B \in \mathcal{A}_n$ there is a disjoint family $\{D^A_{B,c}c\}_{c\in C^A_B}$ where each $D^A_{B,c}$ is a subset of B which $(1-\varepsilon_n)$-covers B. For each $A \in \mathcal{A}_{n+1}$, choose $X_A \subseteq X_B$ where $1 \in C^A_B$, such that $|A|\mu_E(X_A) = p_A$; we can do this since

$$\frac{p_A}{|A|} \leq \frac{|C^A_B|}{|A|} \leq \frac{p_B}{|B|} = \mu_E(B).$$

For each $A \in \mathcal{A}_{n+1}$, each $B \in \mathcal{A}_n$, and each $c \in C^A_B$, define φ^{n+1}_c on X_A so that for every $B \in \mathcal{A}_n$, the family $\{\varphi^{n+1}_c(X_A) : A \in \mathcal{A}_{n+1}, c \in C^A_B\}$ is disjoint and contained in X_B (see Figure 3.1); we can do this since for each $A \in \mathcal{A}_{n+1}$, we have

$$\sum_{c\in C^A_B} \mu_E(X_A) = |C^A_B|\frac{p_A}{|A|} \leq p_A\frac{p_B}{|B|} = p_A\mu_E(X_B).$$

Now for each $A \in \mathcal{A}_{n+1}$, each $B \in \mathcal{A}_n$, each $c \in C^A_B$, and each $h \in D^A_{B,c}$, define φ^{n+1}_{hc} on X_A by setting it equal to $\varphi^n_h\varphi^{n+1}_c$. Then for each $A \in \mathcal{A}_{n+1}$ and each $g \in A$, define φ^{n+1}_g on X_A if it hasn't been already defined, such that the family $\{\varphi^{n+1}_g(X_A) : A \in \mathcal{A}_{n+1}, g \in A\}$ partitions X; this is possible since

$$\sum_{A\in\mathcal{A}_{n+1}}\sum_{g\in A}\mu_E(X_A) = \sum_{A\in\mathcal{A}_{n+1}} |A|\mu_E(X_A) = \sum_{A\in\mathcal{A}_{n+1}} p_A = 1.$$

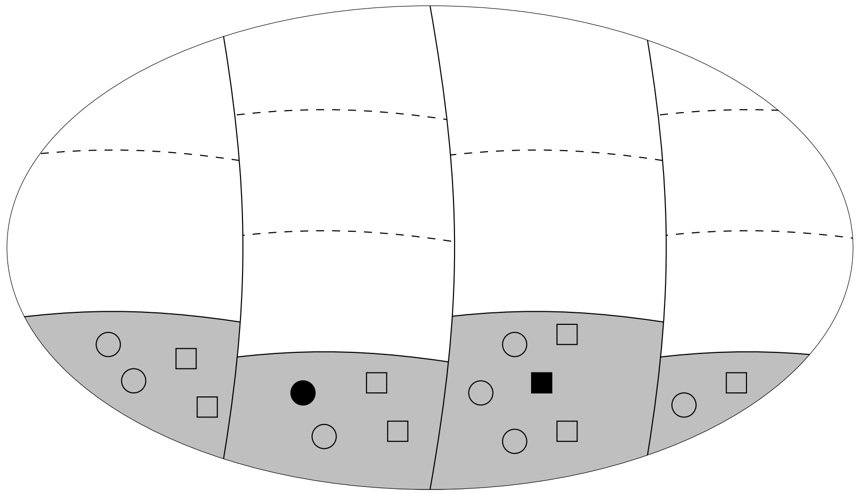

Figure 3.1: The shaded regions are X_B for $B \in \mathcal{A}_n$, and the regions above each X_B are its translates $\varphi_b^n(X_B)$ for $b \in B$. The black disk is some X_A, the other disks are its translates $\varphi_c^{n+1}(X_A)$, and analogously for the squares for some other $A' \in \mathcal{A}_{n+1}$.

Finally, for each $A \in \mathcal{A}_{n+1}$ and $g, h, gh \in A$, define φ_g^{n+1} on $\varphi_h^{n+1}(X_A)$ by setting it to be equal to $\varphi_{gh}^{n+1}(\varphi_h^{n+1})^{-1}$. This concludes the construction.

We claim that for every $g \in G$, the pointwise limit $\varphi_g := \lim_n \varphi_g^n$ exists and is a total function. Let n be large enough such that there is some $C \in \mathcal{A}_{n-1}$ with $g \in C$. Now for any $A \in \mathcal{A}_{n+1}$, $B \in \mathcal{A}_n$, $c \in C_B^A$, and $h \in D_{B,c}^A$ with $gh \in D_{B,c}^A$, we have on X_A,

$$\varphi_g^n \varphi_{hc}^{n+1} = \varphi_g^n \varphi_h^n \varphi_c^{n+1} = \varphi_{gh}^n \varphi_c^{n+1} = \varphi_{ghc}^{n+1} = \varphi_g^{n+1} \varphi_{hc}^{n+1},$$

so φ_g^n and φ_g^{n+1} agree on $\varphi_{hc}^{n+1}(X_A)$. We have

$$|B \setminus g^{-1} D_{B,c}^A| \leq |B \setminus g^{-1} B| + |g^{-1} B \setminus g^{-1} D_{B,c}^A| < 2\varepsilon_n |B|.$$

So φ_g^n and φ_g^{n+1} agree on a set of μ_E-measure at least

$$\begin{aligned} \sum_{A \in \mathcal{A}_{n+1}} \sum_{B \in \mathcal{A}_n} \sum_{c \in C_B^A} \sum_{\substack{h \in D_{B,c}^A \\ gh \in D_{B,c}^A}} \mu_E(\varphi_{hc}^{n+1}(X_A)) &\geq \sum_{A \in \mathcal{A}_{n+1}} \sum_{B \in \mathcal{A}_n} |C_B^A|(1 - 3\varepsilon_n)|B| \frac{p_A}{|A|} \\ &\geq \sum_{A \in \mathcal{A}_{n+1}} (1 - \varepsilon_n)(1 - 3\varepsilon_n) p_A \\ &\geq (1 - \varepsilon_n)(1 - 3\varepsilon_n). \end{aligned}$$

So we are done by the Borel-Cantelli lemma.

Now we claim that $g \mapsto \varphi_g$ is an action. Let $g, h \in G$. Choose n large enough such that there is some $C \in \mathcal{A}_{n-1}$ with $g, h, gh \in C$. Now for any $B \in \mathcal{A}_n$ and $k \in B$ with

$hk, ghk \in B$, we have on X_B,

$$\varphi^n_{gh}\varphi^n_k = \varphi^n_{ghk} = \varphi^n_g\varphi^n_{hk} = \varphi^n_g\varphi^n_h\varphi^n_k,$$

so φ^n_{gh} and $\varphi^n_g\varphi^n_h$ agree on $\varphi^n_k(X_B)$. We have $|B \setminus h^{-1}B| \leq \varepsilon_n|B|$ and $|B \setminus (gh)^{-1}B| \leq \varepsilon_n|B|$. So φ^n_{gh} and $\varphi^n_g\varphi^n_h$ agree on a set of μ_E-measure at least

$$\begin{aligned}\sum_{B\in\mathcal{A}_n}\sum_{\substack{k\in B\\ hk,ghk\in B}} \mu_E(\varphi^n_k(X_B)) &\geq \sum_{B\in\mathcal{A}_n}(1-2\varepsilon_n)|B|\mu_E(\varphi^n_r(X_B))\\ &\geq \sum_{B\in A_n}(1-2\varepsilon_n)p_B\\ &\geq (1-2\varepsilon_n).\end{aligned}$$

So we are done by the Borel-Cantelli lemma. □

We can obtain class-bijective lifts for some amenable groups, including abelian groups and amenable groups with countably many subgroups.

Corollary 3.6.14. *Let G be an amenable group whose conjugacy equivalence relation on its space of subgroups is smooth. Then every outer action of G has a class-bijective lift.*

Proof. For this proof, we will work modulo E-null sets. Fix a morphism $G \to \mathrm{Out}_{\mathrm{NULL}_E}(E)$. Let $(X_e)_{e\in\mathrm{EINV}_E}$ be the ergodic decomposition of E. Let C be a transversal for the conjugacy equivalence relation on the space of subgroups, and for each subgroup $H \leq G$, fix some $g_H \in G$ such that $g_H H g_H^{-1} \in C$. The action $\mathrm{Out}_{\mathrm{NULL}_E}(E) \curvearrowright \mathrm{EINV}_E$ induces an action $G \curvearrowright \mathrm{EINV}_E$. If $e \in \mathrm{EINV}_E$ has stabilizer $H \in C$ under this action, then if N_H is the kernel of $H \to \mathrm{Out}_{\mathrm{NULL}_E}(E \upharpoonright X_e)$, we have $\mathrm{Stab}_H(x) = N_H$ by ergodicity, and thus $H/N_H \to \mathrm{Out}_{\mathrm{NULL}_E}(E \upharpoonright X_e)$ is a free action. Thus by applying Theorem 3.6.13 to X_e, there is a class-bijective lift $H/N_H \to \mathrm{Aut}_{\mathrm{NULL}_E}(E \upharpoonright X_e)$, and this gives a class-bijective lift $H \to \mathrm{Aut}_{\mathrm{NULL}_E}(E \upharpoonright X_e)$, and thus a link. So for each $H \in C$, if we let X_H be the union of the ergodic components with stabilizer H, then there is an $(E \upharpoonright X_H, E^{\vee H} \upharpoonright X_H)$–link L_H. Now for an arbitrary subgroup $H \leq G$, fix a lift ψ_H of g_H. Then the smallest equivalence relation containing L_H and $\{(x, \psi_H(x)) : x \in X_e \text{ with } \mathrm{Stab}(e) = H\}$ for every H is an $(E, E^{\vee G})$–link. □

Remark 3.6.15. There are locally finite groups for which the conjugacy equivalence relation on the space of subgroups is not smooth. Take, for example, a finite group

H with a non-normal subgroup H' and let C be the conjugacy class of H'. Let $G = \bigoplus_n H$ be the infinite direct sum of copies of H. Consider the set X of subgroups of G of the form $\bigoplus_n H_n$, where $H_n \in C$. Then E_0 is Borel reducible to the conjugacy equivalence relation on X, which is therefore non-smooth.

For general amenable groups, the problem is still open:

Problem 3.6.16. Let G be an amenable group. Does every $G \to \mathrm{Out}_B(E)$ have a class-bijective lift?

We remark that in Problem 3.6.16 it suffices to consider hyperfinite E. To see this, note that by Theorem 3.6.13, there is a lift $G \to \mathrm{Aut}_B(E)$. Then it suffices to find an $(E \cap E^X_G, E^X_G)$–link. So by replacing E with $E \cap E^X_G$, we can assume that E is amenable, in the sense of [Kec22, p. 9.1], and this is hyperfinite on an E-conull set, see [Kec22, p. 9.4].

3.7 Summary of lifting results for outer actions

Let $\mathcal{G}$ be the class of groups for which every outer action has a lift. Then:

- $\mathcal{G}$ contains all amenable groups (Theorem 3.6.13).
- $\mathcal{G}$ contains all amalgamated products of finite groups (Corollary 3.5.10).
- $\mathcal{G}$ is closed under subgroups (Proposition 3.4.7).
- $\mathcal{G}$ is closed under free products.
- Every group in $\mathcal{G}$ is treeable (Proposition 3.4.11).

Let $\mathcal{G}_{\mathrm{cb}}$ be the class of groups for which every outer action has a class-bijective lift. Then

- $\mathcal{G}_{\mathrm{cb}}$ contains all locally finite groups (Corollary 3.5.12).
- $\mathcal{G}_{\mathrm{cb}}$ contains all amenable groups whose conjugacy equivalence relation on the space of subgroups is smooth (Corollary 3.6.14).
- $\mathcal{G}_{\mathrm{cb}}$ is closed under subgroups (Proposition 3.4.7).
- $\mathcal{G}_{\mathrm{cb}}$ is closed under quotients (Proposition 3.4.8).

- $\mathcal{G}_{\mathrm{cb}}$ is closed under extensions by a finite normal subgroup (Theorem 3.5.6).

Problem 3.7.1. Characterize the classes $\mathcal{G}$ and $\mathcal{G}_{\mathrm{cb}}$.

3.8 Additional topics

Algebraic properties of automorphism groups

There are several results concerning the algebraic properties of $\mathrm{Inn}_B(E)$ (see [Mil04], [Mer93], [MR07]), and similarly for $\mathrm{Inn}_\mu(E)$ in the pmp case (see [Kec10, §§3-4] and the references therein). In particular, it is known that for aperiodic E, the group $\mathrm{Inn}_B(E)$ is generated by involutions and similarly for $\mathrm{Inn}_\mu(E)$. However, not much seems to be known about the groups $\mathrm{Aut}_B(E), \mathrm{Aut}_\mu(E), \mathrm{Out}_B(E)$, including the question about generation by involutions. There are pmp, ergodic E for which $\mathrm{Aut}_\mu(E)$ is generated by involutions, for example E_0 (see [Kec10, p.46]) and pmp ergodic E that have trivial $\mathrm{Out}_\mu(E)$ (for the existence of such, see [Gef96]). Since E_0 is uniquely ergodic, the question of whether $\mathrm{Aut}_B(E_0)$ is generated by involutions would have a positive answer if $\mathrm{Aut}_B(E)$ is generated by involutions for any hyperfinite compressible E. So it seems natural to consider first the question of generation by involutions of $\mathrm{Aut}_B(E)$, where E is a compressible CBER.

In the case of $\mathrm{Sym}_B(X/E)$, Miller has shown that if $T \in \mathrm{Sym}_B(X/E)$ with $E^{\vee T}$ hyperfinite, then T is a product of three involutions.

Conjugacy of outer actions

A result of Bezuglyi-Golodets [BG87], in combination with Theorem 3.6.1, shows that any two morphisms $\varphi_1, \varphi_2 : G \to \mathrm{Out}_\mu(E_0)$ are conjugate (i.e., there is $\theta \in \mathrm{Out}_\mu(E_0)$ such that $\varphi_1(g) = \theta\varphi_2(g)\theta^{-1}$) iff $\ker(\varphi_1) = \ker(\varphi_2)$. Using Theorem 3.6.4, one can see that the analogous result would hold for morphisms of amenable groups into $\mathrm{Out}_B(E_0)$ if it holds for morphisms of amenable groups into $\mathrm{Out}_B(E)$ for E compressible hyperfinite, which again leads to the question of whether an analog of the Bezuglyi-Golodets theorem holds for morphisms of amenable groups into $\mathrm{Out}_B(E)$, when E is any compressible CBER.

Embeddings of quotients

For a countable group G, let $F_0(G)$ be the CBER on $G^{\mathbb{N}}$ defined by

$$(g_0, g_1, g_2, \ldots)\ F_0(G)\ (h_0, h_1, h_2, \ldots) \iff \exists m\, \forall k > m\, [g_0 \cdots g_k = h_0 \cdots h_k].$$

There is an action $G \to \mathrm{Aut}_B(F_0(G))$ defined by

$$g \cdot (g_0, g_1, g_2, \ldots) = (g \cdot g_0, g_1, g_2, \ldots),$$

inducing an action $G \curvearrowright_B G^{\mathbb{N}}/F_0(G)$. Given CBERs $E \subseteq F$ on X, we say that F/E is **ergodic** if there is no Borel partition $X = A_0 \sqcup A_1$ with each A_i an E-invariant complete F-section.

Let E be a CBER on a Polish space X, and let $G \curvearrowright_B X/E$ be a free action. Then $E^{\vee G}/E$ is ergodic iff there is a G-equivariant Borel injection $G^{\mathbb{N}}/F_0(G) \hookrightarrow X/E$ induced by a continuous embedding $G^{\mathbb{N}} \hookrightarrow X$ (see [Mil04, Theorem 7.2]). If $E^{\vee G}$ is hyperfinite, then there is a G-equivariant Borel injection $X/E \hookrightarrow G^{\mathbb{N}}/F_0(G)$ (see [Mil04, Theorem 8.1]).

Given a pair $E \subseteq F$ of CBERs, we say that F/E is **generated by a Borel action** if there is some Borel action $G \curvearrowright_B X/E$ such that $F = E^{\vee G}$. By [Pin07, Theorem 3], this is equivalent to the existence of a sequence of Borel functions $f_n \colon X/E \to X/E$ such that $x \mathrel{F} y \iff \exists n\, [f_n([x]_E) = [y]_E]$. By [RM21, Theorem 5], there is a countable set of obstructions for being generated by a Borel action. Namely, there is a sequence of pairs $E_n \subseteq F_n$ of CBERs on $2^{\mathbb{N}}$ where F_n/E_n is not generated by a Borel action, such that if $E \subseteq F$ are CBERs on X where F/E is not generated by a Borel action, then there is some n for which there is a continuous embedding $2^{\mathbb{N}} \hookrightarrow X$ which simultaneously reduces E_n to E and F_n to F.

Chapter 4

EQUIDECOMPOSITION IN CARDINAL ALGEBRAS

[Shi21] Forte Shinko. “Equidecomposition in cardinal algebras”. In: *Fund. Math.* 253.2 (2021), pp. 197–204. ISSN: 0016-2736. DOI: 10.4064/fm922-6-2020. URL: https://doi.org/10.4064/fm922-6-2020.

4.1 Introduction

In this chapter, Γ will always denote a countable discrete group. Let X be a standard Borel Γ-space. A classical theorem of Thorisson [Tho96] in probability theory states that if x and x' are random variables on X, then the distributions of x and x' agree on the Γ-invariant subsets of X iff there is a shift-coupling of x and x', i.e., a random variable γ on Γ such that γx and x' are equal in distribution. This characterization in terms of shift-coupling has been applied to various areas of probabilty theory including random rooted graphs [Khe18], Brownian motion [PT15], and point processes [HS13].

This theorem can be reformulated measure-theoretically as follows. Let μ and ν be Borel probability measures on X. Then μ and ν agree on every Γ-invariant set iff either of the following hold:

1. There is a Borel probability measure λ on $\Gamma \times X$ such that $s_*\lambda = \mu$ and $t_*\lambda = \nu$, where $s, t : \Gamma \times X \to X$ are the maps $s(\gamma, x) = x$ and $t(\gamma, x) = \gamma x$, and $s_*\lambda$ is the pushforward measure defined by $s_*\lambda(A) = \lambda(s^{-1}(A))$ (and similarly for $t_*\lambda$).

2. There is a Borel probability measure λ on the orbit equivalence relation $E_G^X := \{(x, y) \in X^2 : \exists\gamma[x = \gamma y]\}$ such that $s_*\lambda = \mu$ and $t_*\lambda = \nu$, where $s, t : E \to X$ are the maps $s(x, y) = x$ and $t(x, y) = y$ (see [Khe18, Theorem 1’]).

By setting μ_γ to be the measure on X defined by $\mu_\gamma(A) := \mu(\{\gamma\} \times A)$, we see that μ and ν agree on every Γ-invariant set iff they are **equidecomposable**, i.e., there are Borel measures $(\mu_\gamma)_{\gamma\in\Gamma}$ on X such that $\mu = \sum_\gamma \mu_\gamma$ and $\nu = \sum_\gamma \gamma\mu_\gamma$, where $\gamma\lambda$ is the pushforward measure $\gamma\lambda(A) = \lambda(\gamma^{-1}A)$. In this chapter, we show that this statement is an instance of a more general result about groups acting on generalized cardinal algebras, a concept introduced by Tarski in [Tar49], leading to a purely algebraic proof of the statement.

Remark 4.1.1. The original result in [Tho96] is stated for actions of locally compact groups, but it is not clear how to formulate an analogous theorem in the setting of cardinal algebras.

Acknowledgments

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4.2 Preliminaries

A **generalized cardinal algebra (GCA)** is a set A equipped with a partial binary operation $+$, a constant 0, and a partial ω-ary operation $\sum$ subject to the following axioms, where we use the notation $\sum_n a_n = \sum(a_n)_n$:

1. If $\sum_n a_n$ is defined, then
$$\sum_n a_n = a_0 + \sum_{n\geq 1} a_n.$$
2. If $\sum_n (a_n + b_n)$ is defined, then
$$\sum_n (a_n + b_n) = \sum_n a_n + \sum_n b_n.$$
3. For any $a \in A$, we have $a + 0 = 0 + a = a$.
4. (Refinement axiom) If $a + b = \sum_n c_n$, then there are $(a_n)_n$ and $(b_n)_n$ such that
$$a = \sum_n a_n, \qquad b = \sum_n b_n, \qquad c_n = a_n + b_n.$$
5. (Remainder axiom) If $(a_n)_n$ and $(b_n)_n$ are such that $a_n = b_n + a_{n+1}$, then there is $c \in A$ such that for each n,
$$a_n = c + \sum_{i\geq n} b_i.$$

These axioms imply in particular that $\sum$ is commutative: if $\sum_n a_n$ is defined and π is a permutation of $\mathbb{N}$, then $\sum_n a_n = \sum_n a_{\pi(n)}$ (see [Tar49, p. 1.38]).

A **cardinal algebra (CA)** is a GCA whose operations $+$ and $\sum$ are total. Cardinal algebras were introduced by Tarski in [Tar49] to axiomatize properties of ZF cardinal arithmetic, such as the cancellation law $n \cdot \kappa = n \cdot \lambda \implies \kappa = \lambda$. More recently, they have been used in [KM16] in the study of countable Borel equivalence relations.

Some examples of GCAs and CAs are as follows.

- [Tar49, p. 14.1] $\mathbb{N}$ and $\mathbb{R}^+$ are GCAs under addition, where $\mathbb{R}^+$ is the set of non-negative real numbers.
- [Sho90, p. 2.1] If X is a measurable space, then the set of measures on X is a CA under addition.
- [Tar49, p. 15.10] Every σ-complete, σ-distributive lattice is a CA under join. In particular, for any set X, the power set $\mathcal{P}(X)$ is a CA under union.
- [Tar49, p. 17.2] The class of cardinals is a CA under addition (although strictly speaking, we require a CA to be a set).
- [Tar49, p. 15.24] Every σ-complete Boolean algebra is a GCA under join of mutually disjoint elements.
  - If X is a measurable space, then the collection $\mathcal{B}(X)$ of measurable sets is a GCA under disjoint union.
  - If (X, μ) is a measure space, then the measure algebra $\mathrm{MALG}(X, \mu)$ is a GCA under disjoint union.

Every GCA is endowed with a relation $\leq$ given by

$$a \leq b \iff \exists c[a + c = b].$$

This is a partial order with least element 0 (see the paragraph following [Tar49, p. 5.18]). Some examples of this partial order are as follows.

- In $\mathbb{N}$ and $\mathbb{R}^+$, $\leq$ coincides with the usual order.
- In the CA of measures on X, $\mu \leq \nu$ iff $\mu(S) \leq \nu(S)$ for every measurable $S \subset X$.

- In the CA induced by a σ-complete, σ-distributive lattice, $\leq$ is the partial order induced by the lattice, i.e., $a \leq b$ iff $a = a \wedge b$.
- For the class of cardinals, $\kappa \leq \lambda$ iff there is an injection $\kappa \hookrightarrow \lambda$, and the fact that this is a partial order is the Cantor-Schroeder-Bernstein theorem.

We say that $a \in A$ is the **meet** (resp. **join**) of a family $(a_i)_{i \in I}$, denoted $\bigwedge a_i$ (resp. $\bigvee a_i$), if it is the meet (resp. join) with respect to $\leq$. We write $a \perp b$ if $a \wedge b = 0$.

A **homomorphism** from a GCA A to a GCA B is a function $\phi : A \to B$ satisfying the following:

1. $\phi(a + b) = \phi(a) + \phi(b)$ whenever $a + b$ is defined.
2. $\phi(\sum_n a_n) = \sum_n \phi(a_n)$ whenever $\sum_n a_n$ is defined.
3. $\phi(0) = 0$.

An **action** of a countable group Γ on a GCA A is a group action $\Gamma \times A \to A$, denoted $(\gamma, a) \mapsto \gamma a$, such that for every $\gamma \in \Gamma$, the map $A \to A$ defined by $a \mapsto \gamma a$ is a homomorphism.

A **Γ-GCA** is a GCA A equipped with an action of a countable group Γ. An element a in A is **Γ-invariant** if $\gamma a = a$ for every $\gamma \in \Gamma$. We say that a and b in A are **equidecomposable** if there exist $(a_\gamma)_{\gamma \in \Gamma}$ in A such that $a = \sum_\gamma a_\gamma$ and $b = \sum_\gamma \gamma a_\gamma$.

The main theorem is as follows, where a GCA A is **cancellative** if for every $a, b \in A$, if $a + b = a$, then $b = 0$.

Theorem 4.2.1. *Let A be a cancellative Γ-GCA with binary meets, and let $\sim$ be an equivalence relation on A such that the following hold:*

1. *Equidecomposable elements are $\sim$-related.*
2. *If $a \sim b$ and $a + c \sim b + d$, then $c \sim d$.*
3. *If $a \sim b$ and $a \perp \gamma b$ for every $\gamma \in \Gamma$, then $a = 0$ (this implies $b = 0$ by symmetry).*

Then $a \sim b$ iff a and b are equidecomposable.

We will show in Section 4.4 that this implies Thorisson's theorem.

4.3 Proof of main theorem

We turn to the proof of the main theorem.

Proof of Theorem 4.2.1. Fix an enumeration $(\gamma_n)_n$ of Γ. Suppose $a \sim b$. We define sequences (a_n) and (b_n) recursively as follows. Let $a_0 = a$ and $b_0 = b$. For the inductive step, choose a_{n+1} and b_{n+1} such that

$$a_n = a_{n+1} + a_n \wedge (\gamma_n b_n)$$
$$b_n = b_{n+1} + (\gamma_n^{-1} a_n) \wedge b_n.$$

By the Remainder axiom, there are some a_∞ and b_∞ such that for any n, we have

$$a_n = a_\infty + \sum_{i \geq n} a_i \wedge (\gamma_i b_i)$$
$$b_n = b_\infty + \sum_{i \geq n} (\gamma_i^{-1} a_i) \wedge b_i.$$

In particular, we have

$$a = a_\infty + \sum_n a_n \wedge (\gamma_n b_n)$$
$$b = b_\infty + \sum_n (\gamma_n^{-1} a_n) \wedge b_n.$$

Thus to show that a and b are equidecomposable, it suffices to show that $a_\infty = b_\infty = 0$.

Now $a_\infty \sim b_\infty$ by the second condition, since $a \sim b$ and $\sum_n a_n \wedge (\gamma_n b_n) \sim \sum_n (\gamma_n^{-1} a_n) \wedge b_n$ (by equidecomposability). Now for any n, we have $b_n \geq b_\infty + (\gamma_n^{-1} a_n) \wedge b_n$, and thus

$$\gamma_n b_n \geq \gamma_n b_\infty + a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n).$$

We also have

$$a_n \geq a_\infty + a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n).$$

Thus

$$a_n \wedge (\gamma_n b_n) \geq a_\infty \wedge (\gamma_n b_\infty) + a_n \wedge (\gamma_n b_n).$$

Since A is cancellative, we have $0 \geq a_\infty \wedge (\gamma_n b_\infty)$, i.e., $a_\infty \wedge (\gamma_n b_\infty) = 0$. Thus $a_\infty \perp \gamma b_\infty$ for every $\gamma \in \Gamma$, and so by our hypothesis, we have $a_\infty = 0$ and $b_\infty = 0$. $\square$

4.4 Applications

By a **finite measure** on a GCA A, we mean a homomorphism from A to $\mathbb{R}^+$.

Corollary 4.4.1. *Let A be a Γ-GCA with countable joins and let μ and ν be finite measures on A. Then μ and ν agree on every Γ-invariant element of A iff they are equidecomposable.*

We recover Thorisson's theorem by setting $A = \mathcal{B}(X)$ (under disjoint union).

Corollary 4.4.2 (Thorisson, [Tho96, Theorem 1])**.** *Let X be a standard Borel Γ-space and let μ and ν be finite Borel measures on X. Then μ and ν agree on every Γ-invariant subset of X iff they are equidecomposable.*

To prove Corollary 4.4.1, we need to define some more notions.

A **closure** of a GCA A is a CA $\overline{A}$ containing A such that the following hold:

1. If a and $(a_n)_n$ are in A, then $a = \sum a_n$ in A iff $a = \sum_n a_n$ in $\overline{A}$.
2. A generates $\overline{A}$, i.e., for every $b \in A$, there exist $(a_n)_n$ in A such that $b = \sum a_n$.

Proposition 4.4.3 ([Tar49, p. 7.8])**.** *Every GCA has a closure.*

Some examples of closures are as follows.

- $\overline{\mathbb{N}}$ is the set of extended natural numbers $\{0, 1, 2, \ldots, \infty\}$.
- $\overline{\mathbb{R}^+}$ is the extended real line $[0, \infty]$.

The following is easy to verify.

Proposition 4.4.4. *If A is a GCA with closure $\overline{A}$ and B is a CA, then every homomorphism $A \to B$ extends uniquely to a homomorphism $\overline{A} \to B$.*

Remark 4.4.5. This shows that the closure is left adjoint to the forgetful functor from the category of CAs to the category of GCAs, so in particular, the closure is unique up to isomorphism.

Let $\mathrm{Hom}(A, B)$ denote the set of all homomorphisms from A to B.

Proposition 4.4.6. *Let A be a GCA. Then $\mathrm{Hom}(A, \mathbb{R}^+)$ is a cancellative GCA with binary meets (under pointwise addition).*

Proof. By [Sho90, p. 2.1], $\operatorname{Hom}(\overline{A}, \overline{\mathbb{R}^+})$ is a CA with binary meets, so $\operatorname{Hom}(A, \overline{\mathbb{R}^+})$ is also CA with binary meets, since it is isomorphic to $\operatorname{Hom}(\overline{A}, \overline{\mathbb{R}^+})$ by Proposition 4.4.4. Thus since $\operatorname{Hom}(A, \mathbb{R}^+)$ is closed $\leq$-downwards in $\operatorname{Hom}(A, \overline{\mathbb{R}^+})$, it is a GCA by [Tar49, 9.18(i)], and it has binary meets. The cancellativity of $\operatorname{Hom}(A, \mathbb{R}^+)$ follows immediately from cancellativity of $\mathbb{R}^+$. □

We can now prove Corollary 4.4.1:

Proof of Corollary 4.4.1. $\operatorname{Hom}(A, \mathbb{R}^+)$ is a cancellative GCA with binary meets, and it has a Γ-action given by $(\gamma\mu)(a) := \mu(\gamma^{-1}a)$. Define the equivalence relation $\sim$ on $\operatorname{Hom}(A, \mathbb{R}^+)$ by setting $\mu \sim \nu$ iff $\mu(a) = \nu(a)$ for every Γ-invariant $a \in A$. It suffices to check the conditions in Theorem 4.2.1. Conditions 1 and 2 are clear. For condition 3, suppose that $\mu \sim \nu$ and $\mu \perp \gamma\nu$ for every $\gamma \in \Gamma$, and fix $a \in A$. We must show that $\mu(a) = 0$. By [Tar49, p. 3.12], we have $\mu \perp \sum \gamma\nu$, and thus by [Sho90, p. 1.14] (which is stated for CAs, but whose proof works without modification for GCAs), we can write $a = b + c$ with $\mu(b) = 0$ and $(\sum \gamma\nu)(c) = 0$. Identifying ν with its extension $\overline{A} \to \overline{\mathbb{R}^+}$, we have $\nu(\sum \gamma c) = 0$. Thus $\nu(\bigvee \gamma c) = 0$, so since $\mu \sim \nu$, we have $\mu(\bigvee \gamma c) = 0$. Thus $\mu(c) = 0$, and thus $\mu(a) = \mu(b) + \mu(c) = 0$. □

We also obtain a criterion for equidecomposability of subsets of a probability space. A **probability measure preserving (pmp)** Γ-action on a standard probabability space (X, μ) is an action of Γ on (X, μ) by measure-preserving Borel automophisms.

Corollary 4.4.7. *Let (X, μ) be a standard probability space with a pmp Γ-action and let $A, B \in \operatorname{MALG}(X, \mu)$. Then A and B agree on every Γ-invariant measure $\ll \mu$ iff they are equidecomposable.* [1]

This generalizes a well-known result (for instance, see [KM04, p. 7.10]) which says that if μ is ergodic, then A and B are equidecomposable iff $\mu(A) = \mu(B)$ (note that in this case, μ is the only Γ-invariant measure $\ll \mu$).

Corollary 4.4.7 will be obtained via a more general result about projections in von Neumann algebras; see Corollary 4.4.8 below.

We recall some notions from the theory of operator algebras; see [Bla06] for a standard reference. A **von Neumann algebra** is a weakly closed $*$-subalgebra M

[1] Ruiyuan (Ronnie) Chen has pointed out that this also follows from the Becker-Kechris comparability lemma [BK96, p. 4.5.1].

of $B(H)$ containing the identity. An element $x \in M$ is **positive** if $x = yy^*$ for some $y \in M$, and the set of positive elements is denoted M_+. There is a partial order on M defined by setting $x \leq y$ iff $y - x$ is positive. An element $p \in M$ is a **projection** if $p = p^* = p^2$, and the set of projections, denoted $P(M)$, is a complete lattice. Two projections p and q are **Murray-von Neumann equivalent**, written $p \sim_{\mathrm{MvN}} q$, if there is some $u \in M$ such that $p = uu^*$ and $q = u^*u$. Then $P(M)/{\sim_{\mathrm{MvN}}}$ is a complete lattice. A projection p is **finite** if for any projection p', if $p \sim_{\mathrm{MvN}} p' \leq p$, then $p = p'$. A von Neumann algebra M is **finite** if 1_M is a finite projection. A **trace** on M is a map $\tau : M_+ \to \overline{\mathbb{R}^+}$ such that $\tau(mm^*) = \tau(m^*m)$, and a trace is **finite** if its image is contained in $\mathbb{R}^+$. A trace is **faithful** if $\tau(m) = 0$ implies $m = 0$, and a trace is **normal** if it is weakly continuous.

If M is a von Neumann algebra, then $P(M)/{\sim_{\mathrm{MvN}}}$ is a GCA under join of orthogonal projections [Fil65], and if M is finite, then this GCA is cancellative. A **Γ-action** on a von Neumann algebra M is an action of Γ on M by weakly continuous $(+, 0, \cdot, 1, *)$-homomorphisms. Every von Neumann algebra M with a Γ-action gives rise to a Γ-GCA, and a trace τ on M is said to be **Γ-invariant** if $\tau(\gamma m) = \tau(m)$ for every $m \in M$ and $\gamma \in \Gamma$.

Corollary 4.4.8. *Let M be a finite von Neumann algebra with a Γ-action which admits a faithful normal finite Γ-invariant trace, and let $[p], [q] \in P(M)/{\sim_{\mathrm{MvN}}}$. Then $[p]$ and $[q]$ agree on every finite Γ-invariant trace on M iff they are equidecomposable.*

Proof. Let $A = P(M)/{\sim_{\mathrm{MvN}}}$, which is a cancellative Γ-GCA with binary meets. Now define the equivalence relation on A by setting $[p] \sim [q]$ if $[p]$ and $[q]$ agree on every Γ-invariant trace on M. It suffices to check the conditions in Theorem 4.2.1. Conditions 1 and 2 are clear. For condition 3, suppose that $[p] \sim [q]$ and $[p] \perp \gamma[q]$ for every $\gamma \in \Gamma$, and fix a faithful normal finite Γ-invariant trace τ on M. Then setting $\overline{p} = \bigvee \gamma p$, the map $m \mapsto \tau(\overline{p} m \overline{p})$ is a finite Γ-invariant trace on M. Since $\tau(\overline{p} q \overline{p}) = 0$ and $p \sim q$, we have $\tau(p) = \tau(\overline{p} p \overline{p}) = 0$. Thus $p = 0$. □

Corollary 4.4.7 follows by applying this to $L^\infty(X, \mu)$.

Proof of Corollary 4.4.7. Let $M = L^\infty(X, \mu)$. This is a finite Γ-von Neumann algebra and μ induces a faithful normal finite Γ-invariant trace on M. Now $P(M)/{\sim_{\mathrm{MvN}}}$ is isomorphic to $\mathrm{MALG}(X, \mu)$ as a lattice (with Γ-action), so they give rise to isomorphic Γ-GCAs, and thus we are done by Corollary 4.4.8. □

Chapter 5

A DICHOTOMY FOR POLISH MODULES

[FS22] Joshua Frisch and Forte Shinko. "A dichotomy for Polish modules". In: *arXiv:2009.05855, to appear in Israel J. Math* (2022).

5.1 Introduction

The Axiom of Choice allows us to construct many abstract algebraic homomorphisms between topological algebraic systems which are incredibly non-constructive. A longstanding theme in descriptive set theory is to study to what extent we can, and to what extent we provably cannot, construct such homomorphisms in a "definable" way. Here the notion of definability is context-dependent but often includes continuous, Borel, or projective maps.

A classical example of such an abstract construction, which provably cannot be constructed with "nice" sets is the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. It is well-known that such a basis cannot be Borel, or more generally, analytic. Similar phenomena show up when constructing Hamel bases for topological vector spaces, or constructing an isomorphism of the additive groups of $\mathbb{R}$ and $\mathbb{C}$.

A more recent theme in descriptive set theory is that such undefinability criteria can often be leveraged in order to gain, and hopefully utilize, additional structure. For example, Silver's theorem [Sil80] and the Glimm-Effros dichotomy [HKL90] interpret the non-reducibility of Borel equivalence relations not as a pathology but rather as the first step in the burgeoning theory of invariant descriptive set theory (see [Gao09] for background). Similarly, work starting with [KST99] studies and exploits the difference between abstract chromatic numbers and more reasonably definable (for example, continuous or Borel) chromatic numbers. A key feature in many of these theories (and all of the above examples) is the existence of dichotomy theorems, which state that either an object is simple, or there is a canonical obstruction contained inside of it. This is usually stated in terms of preorders, saying that there is a natural basis for the preorder of objects which are not simple (recall that a **basis** for a preorder P is a subset $B \subseteq P$ such that for every $p \in P$, there is some $b \in B$ with $b \leq p$).

In this chapter, we apply a descriptive set-theoretic approach to vector spaces and more generally, modules, over a locally compact Polish ring[1]. For a Polish ring R, a **Polish R-module** is a topological left R-module whose underlying topology is Polish. Given Polish R-modules M and N, we say that M **embeds** into N, denoted $M \sqsubseteq^R N$, if there is a continuous linear injection from M into N. One particularly nice aspect of Polish modules is that the notion of "definable" reduction is much simpler than in the general case. By Pettis's lemma, any Baire-measurable homomorphism between Polish modules is in fact automatically continuous (see [Kec95, p. 9.10]). Thus there is no loss of generality in considering continuous homomorphisms rather than a priori more general Borel homomorphisms.

Our main results give a dichotomy for Polish modules being countably generated. More precisely, we give a countable basis under $\sqsubseteq^R$ for Polish modules which are not countably generated. While these results are stated in a substantial level of generality (they are true for all left-Noetherian countable rings and many Polish division rings), we feel that the most interesting cases are over some of the most concrete rings. For example, over $\mathbb{Q}$, we show the existence of a unique (up to bi-embeddability) minimal uncountable Polish vector space $\ell^1(\mathbb{Q})$. We further show that nothing bi-embeddable with $\ell^1(\mathbb{Q})$ is locally compact, and thus that every uncountable-dimensional locally compact Polish vector space (for example, $\mathbb{R}$) is strictly more complicated than $\ell^1(\mathbb{Q})$.

Another case of particular interest is the case of $\mathbb{Z}$-modules, that is, abelian groups. We show that there is a countable basis of minimal uncountable abelian Polish groups (one for each prime number and one for characteristic 0). Furthermore, there exists a maximal abelian Polish group by [Shk99], as well as many natural but incomparable elements (for example, $\mathbb{Q}_p$ and $\mathbb{R}$ are incomparable under $\sqsubseteq^{\mathbb{Q}}$ as are $\mathbb{Q}_p$ and $\mathbb{Q}_r$ for $p \neq r$).

Our dichotomy theorems will hold for rings equipped with a proper norm. A **(complete, proper) norm** on an abelian group A is a function $\|\cdot\|\colon A \to [0, \infty)$ such that the map $(a, b) \mapsto \|a - b\|$ is a (complete, proper) metric on A (recall that a metric is proper if every closed ball is compact). A **norm** on a ring R is a norm $|\cdot|$ on $(R, +)$ such that $|rs| \leq |r||s|$ for every $r, s \in R$. A **proper normed ring** is a ring equipped with a proper norm. Every countable ring admits a proper norm (see Section 5.3). Given a proper normed ring R, the R-module $\ell^1(R)$ is defined as

[1] All rings will be assumed to be unital.

follows:

$$\ell^1(R) = \left\{ (r_k)_k \in R^{\mathbb{N}} : \sum_k \frac{|r_k|}{k!} < \infty \right\}$$

(here, $\frac{1}{k!}$ can be replaced with any summable sequence). Then $\|(r_k)_k\| := \sum_k \frac{|r_k|}{k!}$ is a complete separable norm on $(\ell^1(R), +)$, turning $\ell^1(R)$ into a Polish R-module.

The following theorems will be obtained as special cases of results in Section 5.5.

A **division ring** is a ring R such that every nonzero $r \in R$ has a two-sided inverse.

Theorem 5.1.1. *Let R be a proper normed division ring and let M be a Polish R-vector space. Then exactly one of the following holds:*

(1) $\dim_R(M)$ *is countable.*

(2) $\ell^1(R) \sqsubseteq^R M$.

This seems to be new, even when R is a finite field, in which case $\ell^1(R) = R^{\mathbb{N}}$. This also implies a special case of [Mil12, Theorem 24], which says that if $\dim_R(M)$ is uncountable, then there is a linearly independent perfect set (see Corollary 5.5.2).

An analogous statement holds for a large class of discrete rings. A ring is **left-Noetherian** if every increasing sequence of left ideals stabilizes.

Theorem 5.1.2. *Let R be a left-Noetherian discrete proper normed ring and let M be a Polish R-module. Then exactly one of the following holds:*

(1) M is countable.

(2) $\ell^1(S) \sqsubseteq^R M$ for some nonzero quotient S of R.

Note that this basis is countable since a countable left-Noetherian ring only has countably many left ideals.

For abelian Polish groups, we obtain an irreducible basis (see Theorem 5.4.3):

Theorem 5.1.3. *Let A be an uncountable abelian Polish group. Then one of the following holds:*

1. $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.

2. $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ *for some prime p.*

Related statements have been shown by Solecki, see [Sol99, Proposition 1.3, Theorem 1.7].

The theorems in Section 5.5 will be shown for a substantially broader class of modules. In order to contextualize this, we remark that considering even very basic module homomorphisms (for example, the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ as $\mathbb{Q}$-vector spaces) naturally leads us to consider the broader class of quotients of Polish modules by sufficiently definable submodules. Such quotient modules are in general not Polish (they are not necessarily even standard Borel) but are still important objects of descriptive set-theoretic interest. They play a crucial role in [BLP20] in the form of "groups with a Polish cover", and they also form some of the most classical examples of countable Borel equivalence relations (for example, the commensurability relation on the positive reals naturally comes equipped with an abelian group structure). The embedding order on quotient modules will be defined analogously to the homomorphism reductions for Polish groups studied in [Ber14; Ber18].

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5.2 Polish modules

Most Polish modules which cannot be written as direct sums, even over a field. This will follow from a more general statement about Polish groups.

Given a Polish group G, a family $(H_x)_{x\in\mathbb{R}}$ of subgroups of G is

(i) **analytic** if the set $\{(g,x) \in G \times \mathbb{R} : g \in H_x\}$ is analytic;

(ii) **independent** if for every finite $F \subseteq \mathbb{R}$ and every $x \in \mathbb{R} \setminus F$, we have $H_x \cap \langle H_y \rangle_{y\in F} = 1$;

(iii) **generating** if $(H_x)_{x\in\mathbb{R}}$ generates G.

In particular, if G is the direct sum or the free product of $(H_x)_{x\in\mathbb{R}}$, then $(H_x)_x$ is an independent generating family.

Proposition 5.2.1. *Let G be a Polish group, and let $(H_x)_{x\in\mathbb{R}}$ be an analytic independent generating family of subgroups of G. Then there are only countably many $x \in \mathbb{R}$ with H_x nontrivial, and only finitely many $x \in \mathbb{R}$ with H_x uncountable.*

Proof. Let A_n be the set of $g \in G$ which can be written in the form $h_0 h_1 \cdots h_{n-1}$ with each h_i in some H_x. Then A_n is analytic, and thus Baire-measurable. Since $G = \bigcup_n A_n$, there is some A_n which is non-meager. By Pettis's lemma, we can replace n with $2n$ and assume that A_n has non-empty interior. Thus G can be covered by countably many right translates $(A_n g_k)_k$ of A_n.

Let $X \subseteq \mathbb{R}$ be the set of $x \in \mathbb{R}$ with H_x nontrivial, and suppose that X is uncountable. For each $x \in X$, fix some nontrivial $h_x \in H_x$. Fix an equivalence relation E on X with every class of cardinality $n + 1$. Then there must be two E-classes $(x_i)_{i\leq n}$ and $(y_i)_{i\leq n}$ such that $h_{x_0} h_{x_1} \cdots h_{x_n}$ and $h_{y_0} h_{y_1} \cdots h_{y_n}$ are in the same $A_n g_k$. But then

$$h_{x_0} h_{x_1} \cdots h_{x_n} (h_{y_0} h_{y_1} \cdots h_{y_n})^{-1} \in A_n g_k (A_n g_k)^{-1} = A_{2n},$$

which is a contradiction by independence. Thus X is countable.

Now $G = \bigcup_F \langle H_x \rangle_{x\in F}$, where the union is taken over all finite $F \subseteq X$, so since X is countable, there is some F for which $H_F := \langle H_x \rangle_{x\in F}$ is non-meager, and thus open, since H_F is analytic. Then G/H_F is countable, so if $x \notin F$, then H_x is countable by independence. □

In particular, this implies an unpublished result of Ben Miller showing that an uncountable-dimensional Polish vector space does not have an analytic basis.

If $M \sqsubseteq^R N$ and $N \sqsubseteq^R M$, then we say that M and N are **bi-embeddable**. Note that if M and N are R-modules, and S is a subring of R, then $M \sqsubseteq^R N$ implies $M \sqsubseteq^S N$. In particular, if M and N are $\sqsubseteq^S$-incomparable, then they are $\sqsubseteq^R$-incomparable. In general, the preorder $\sqsubseteq^R$ can contain incomparable elements. For example, $\mathbb{R}$ is $\sqsubseteq^{\mathbb{Z}}$-incomparable with the p-adic rationals $\mathbb{Q}_p$, for any prime p. To see this, we have $\mathbb{R} \not\sqsubseteq^{\mathbb{Z}} \mathbb{Q}_p$ since $\mathbb{R}$ is connected, but $\mathbb{Q}_p$ is totally disconnected. On the other hand, $\mathbb{Q}_p \not\sqsubseteq^{\mathbb{Z}} \mathbb{R}$ since $\mathbb{Q}_p$ has a nontrivial compact subgroup, but $\mathbb{R}$ does not. So $\mathbb{R}$ and $\mathbb{Q}_p$ are $\sqsubseteq^{\mathbb{Z}}$-incomparable, and thus also $\sqsubseteq^{\mathbb{Q}}$-incomparable.

For certain rings, no locally compact module embeds into $R^{\mathbb{N}}$, and thus a minimum for $\sqsubseteq^R$ cannot be locally compact:

Proposition 5.2.2. *Let R be a Polish ring with no nontrivial compact subgroups, and let M be a locally compact Polish R-module. If $M \sqsubseteq^R R^{\mathbb{N}}$, then M is countably generated.*

Proof. Fix a continuous linear injection $f\colon M \hookrightarrow R^{\mathbb{N}}$. Since R has no nontrivial compact subgroups, the same holds for $R^{\mathbb{N}}$, and thus for M. Fix a complete norm $\|\cdot\|$ compatible with $(M,+)$. Let $\pi_n\colon R^{\mathbb{N}} \to R^n$ denote the projection to the first n coordinates, and let $M_n = \ker(\pi_n \circ f)$, which is a closed submodule of M. Fix ε such that the closed ε-ball around $0 \in M$ is compact, and let $C = \{m \in M : \frac{\varepsilon}{2} \leq \|m\| \leq \varepsilon\}$. Then $C \cap \bigcap_n M_n = \varnothing$, so since C is compact, there is some n such that $C \cap M_n = \varnothing$. We claim that M_n is discrete. To see this, suppose that the $\frac{\varepsilon}{2}$-ball around $0 \in M$ contained some nonzero $m \in M_n$. Then the subgroup generated by m is not compact, so there is a minimal $k \in \mathbb{N}$ with $\|km\| \geq \frac{\varepsilon}{2}$, and hence $km \in C$, which is not possible. Thus M_n is countable, so if we pick preimages $(m_i)_{i<n}$ in M of the standard basis of R^n, then M is generated by $M_n \cup (m_i)_{i<n}$, and thus countably generated. □

We do not know anything about the preorder $\sqsubseteq^R$ restricted to locally compact modules, including the existence of a minimum or maximum element.

If M_0 and M_1 are Polish R-modules with Baire-measurable submodules N_0 and N_1 respectively, we write $M_0/N_0 \sqsubseteq^R M_1/N_1$ if there is a continuous linear map $M_0 \to M_1$ which descends to an injection $M_0/N_0 \hookrightarrow M_1/N_1$. This map is a Borel reduction of $E^{M_0}_{N_0}$ to $E^{M_1}_{N_1}$, where $E^{M_i}_{N_i}$ is the coset equivalence relation of N_i in M_i (see [Gao09] for background on Borel reductions). In particular, we have $\mathbb{R}/\mathbb{Q} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}$, since $E^{\mathbb{R}}_{\mathbb{Q}}$ is not smooth. We also have $\mathbb{R} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$, since any nontrivial continuous linear map $\mathbb{R} \to \mathbb{R}$ is surjective, and thus $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are $\sqsubseteq^{\mathbb{Q}}$-incomparable.

5.3 Proper normed rings

Every proper normed ring is locally compact and Polish. There are many examples of proper normed rings:

- The usual norms on $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$ are proper.
- The p-adic norm on $\mathbb{Q}_p$ is proper.
- Every countable ring R admits a proper norm as follows. Let $w\colon R \to \mathbb{N}$ be a finite-to-one function such that $w(0) = 0$, $w(r) \geq 2$ if $r \neq 0$, and $w(r) = w(-r)$. We extend w to every term t in the language $(+,\cdot) \cup R$ by

$w(r+s) = w(r)+w(s)$ and $w(r \cdot s) = w(r)w(s)$. Then let $|r|$ be the minimum of $w(t)$ over all terms t representing r.

- Let R be a proper normed ring. If $S \leq R$ is a closed subring, then there is a proper norm on S obtained by restricting the norm on R. If $I \triangleleft R$ is a closed two-sided ideal, then there is a proper norm on R/I given by $|r + I| = \min_{s \in r+I} |s|$.

In general, we do not know if every locally compact Polish ring admits a compatible proper norm.

Given a closed two-sided ideal $I \triangleleft R$, there is a natural quotient map $\ell^1(R) \twoheadrightarrow \ell^1(R/I)$ with kernel $\ell^1(I) := \ell^1(R) \cap I^{\mathbb{N}}$.

If R is finite proper normed ring, then $\ell^1(R) = R^{\mathbb{N}}$, which in particular is homeomorphic to Cantor space. For infinite discrete rings, there is also a unique homeomorphism type. Recall that **complete Erdős space** is the space of square-summable sequences of irrational numbers with the ℓ^2-norm topology.

Proposition 5.3.1. *Let R be an infinite discrete proper normed ring. Then $\ell^1(R)$ is homeomorphic to complete Erdős space.*

To show this, we will use a characterization due to Dijkstra and van Mill [DM09, Theorem 1.1]. A topological space is **zero-dimensional** if it is nonempty and it has a basis of clopen sets.

Theorem 5.3.2 (Dijkstra-van Mill)**.** *Let X be a separable metrizable space. Then X is homeomorphic to complete Erdős space iff there is a zero-dimensional metrizable topology τ on X coarser than the original topology such that every point in X has a neighborhood basis (for the original topology) consisting of closed nowhere dense Polish subspaces of (X, τ).*

Proof of Proposition 5.3.1. We check the condition from Theorem 5.3.2. Let τ be the product topology on $R^{\mathbb{N}}$, which is zero-dimensional and metrizable. It is enough to show that every closed ball is a closed nowhere dense Polish subspace of $(\ell^1(R), \tau)$. By translation, it suffices to consider balls of the form $B = \{m \in \ell^1(R) : \|m\| \leq \varepsilon\}$. Note that B is closed in $R^{\mathbb{N}}$. Thus (B, τ) is Polish, and B is closed in $(\ell^1(R), \tau)$. It remains to show that the complement of B is dense in $(\ell^1(R), \tau)$. Let U be a nonempty open subset of $(\ell^1(R), \tau)$. We can assume that there is a finite sequence

$(r_k)_{k<n}$ in R such that U is the set of sequences in $\ell^1(R)$ starting with $(r_k)_{k<n}$. Since R is infinite and the norm is proper, there is some $r \in R$ with $|r| > n!\varepsilon$. Then $(r_0, \ldots, r_{n-1}, r, 0, 0, 0, \ldots) \in U \setminus B$. □

5.4 Special cases

For a general Polish ring R, we do not know much about the preorder $\sqsubseteq^R$, including the following:

Problem 5.4.1. Is there a maximum Polish R-module under $\sqsubseteq^R$?

This is known for some particular rings, which we mention below.

Principal ideal domains

Recall that a **principal ideal domain (PID)** is an integral domain in which every ideal is generated by a single element. There is an irreducible basis for uncountable Polish modules over a PID:

Theorem 5.4.2. *Let R be a proper normed discrete PID and let M be a Polish R-module. Then exactly one of the following holds:*

1. *M is countable.*
2. *There a prime ideal $\mathfrak{p} \lhd R$ such that $\ell^1(R/\mathfrak{p}) \sqsubseteq^R M$.*

Moreover, the $\ell^1(R/\mathfrak{p})$ are $\sqsubseteq^R$-incomparable for different $\mathfrak{p}$.

Proof. Suppose that M is not countable. By Theorem 5.1.2, there is some proper ideal $I \lhd R$ such that $\ell^1(R/I) \sqsubseteq^R M$. Then since R is a PID, there is some prime ideal $\mathfrak{p} \lhd R$ and some nonzero $s \in R$ such that $I = \mathfrak{p}s$. Then the linear injection $R/\mathfrak{p} \hookrightarrow R/I$ defined by $r \mapsto rs$ induces a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/I)$.

It remains to show that if $\mathfrak{p}$ and $\mathfrak{q}$ are prime ideals with $\ell^1(R/\mathfrak{p}) \sqsubseteq^R \ell^1(R/\mathfrak{q})$, then $\mathfrak{p} = \mathfrak{q}$. Fix a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/\mathfrak{q})$. Since $R/\mathfrak{p}$ is an integral domain, the annihilator of any nonzero element of $\ell^1(R/\mathfrak{p})$ is $\mathfrak{p}$, and similarly for $\mathfrak{q}$. Then for any nonzero $x \in \ell^1(R/\mathfrak{p})$, its image in $\ell^1(R/\mathfrak{q})$ must have the same annihilator since the map is injective, and thus $\mathfrak{p} = \mathfrak{q}$. □

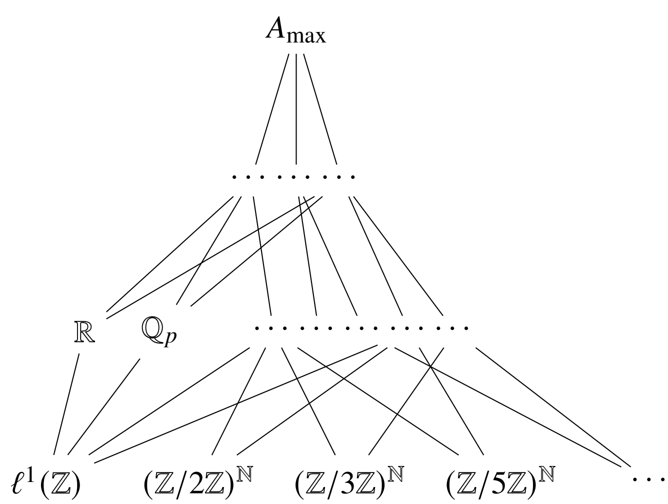
Abelian groups

Applying Theorem 5.4.2 with $R = \mathbb{Z}$ gives an irreducible basis for uncountable abelian groups:

Theorem 5.4.3. *Let A be an uncountable abelian Polish group. Then one of the following holds:*

1. $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.
2. $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ *for some prime p.*

By [Shk99], there is a $\sqsubseteq^{\mathbb{Z}}$-maximum abelian Polish group $A_{\max}$. So the preorder $\sqsubseteq^{\mathbb{Z}}$ on uncountable abelian Polish groups looks like the following:



$\mathbb{Q}$-vector spaces

Fix a proper norm on $\mathbb{Q}$. By Proposition 5.2.2, a $\sqsubseteq^{\mathbb{Q}}$-minimum uncountable Polish $\mathbb{Q}$-vector space cannot be locally compact. By Theorem 5.1.2, we have $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} \mathbb{R}$, where the strictness is due to $\ell^1(\mathbb{Q})$ being totally disconnected. However, it is open as to whether there is an intermediate vector space:

Problem 5.4.4. Is there a Polish $\mathbb{Q}$-vector space V such that $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} V \sqsubset^{\mathbb{Q}} \mathbb{R}$?

Real vector spaces

We consider the order $\sqsubseteq^{\mathbb{R}}$ on uncountable-dimensional Polish $\mathbb{R}$-vector spaces. By Theorem 5.1.1, there is a minimum element $\ell^1(\mathbb{R})$, which is bi-embeddable with the usual space ℓ^1 of absolutely summable sequences. By Proposition 5.2.2, any uncountable-dimensional locally compact Polish $\mathbb{R}$-vector space must be strictly above ℓ^1. By [Kal77], there is a maximum Polish $\mathbb{R}$-vector space $V_{\max}$.

5.5 Proof of the main theorems

Every abelian Polish group A has a compatible complete norm defined by $\|a\| = d(a, 0)$, where d is an invariant metric on A (see [BK96, pp. 1.1.1, 1.2.2]). If $B \subseteq A$ is a Baire-measurable subgroup, then by Pettis's lemma, B is either open or meager (see [Kec95, p. 9.11]).

Setting $N = 0$ in the following theorem recovers Theorem 5.1.1.

Theorem 5.5.1. *Let R be a proper normed division ring, let M be a Polish R-vector space, and let $N \subseteq M$ be an F_σ vector subspace. Then exactly one of the following holds:*

(1) $\dim_R(M/N)$ *is countable.*

(2) $\ell^1(R) \sqsubseteq^R M/N$.

In most natural examples, N is F_σ, such as for $\ell^1(\mathbb{N}) \subseteq \ell^2(\mathbb{N})$. It would be interesting to prove this for more general subspaces.

Proof. Suppose that the dimension of M/N is uncountable. Then N is not open, so N is meager, i.e., we have $N = \bigcup_k F_k$ for some increasing sequence $(F_k)_k$ of closed nowhere dense sets. Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. For every k, we define $\varepsilon_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R. We proceed by induction on k. Choose $\varepsilon_k > 0$ such that

(i) $\varepsilon_k < \frac{1}{2}\varepsilon_i$ for every $i < k$,

(ii) for every $(r_i)_{i<k}$ such that $\sum_{i<k} \frac{|r_i|}{i!} \leq k$ and there is some $l < k$ with $r_l = 1$ and $r_i = 0$ for $i < l$, the open ε_k-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

(i) $m_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$,

(ii) $\|rm_k\| < \frac{1}{2}\varepsilon_k$ whenever $\frac{|r|}{k!} \leq k$.

We verify that this is possible. When choosing ε_k, to satisfy the second condition, note that the set of considered $(r_i)_{i<k}$ is compact, so the set of $\sum_{i<k} r_i m_i$ is also compact, and it is disjoint from N (and hence F_k) by the choice of $(m_i)_{i<k}$. Thus such an ε_k must exist. When choosing m_k, note that the first condition holds for a comeager set of m_k, since $N + Rm_0 + Rm_1 \cdots + Rm_{k-1}$ is analytic, and it is not open, since otherwise M/N would have countable dimension. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \leq k$ is compact. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Let $n > l$ be sufficiently large such that $\sum_k \frac{|r_k|}{k!} \leq n$ and $0 \in F_n$. Then

$$\varepsilon_n \leq \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i} m_{n+i}\| < \frac{1}{2}\varepsilon_{n+i}$, and thus $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$ by inductively using $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$. Thus

$$\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}} \left\| \sum_{k<n} r_k m_k \right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\| \sum_k r_k m_k \right\| < 2 \left\| \sum_{k<n} r_k m_k \right\|.$$

It remains to show that the induced map $\ell^1(R) \to M/N$ is an injection. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Suppose that $n > l$ is sufficiently large such that $\sum_k \frac{|r_k|}{k!} \leq n$. Since $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$, we have $\sum_{i \geq 0} \|r_{n+i} m_{n+i}\| < \varepsilon_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. □

We recover [Mil12, Theorem 24] for proper normed division rings:

Corollary 5.5.2 (Miller)**.** *Let R be a proper normed division ring, and let M be a Polish R-module. If $\dim_R(M)$ is uncountable, then there is a linearly independent perfect subset of M.*

Proof. By Theorem 5.1.1, we can assume that $M = \ell^1(R)$. Fix an enumeration $(q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. For every $x \in \mathbb{R}$, define $\chi_x \in \ell^1(R)$ by

$$(\chi_x)_n = \begin{cases} 1 & q_n < x \\ 0 & \text{otherwise} \end{cases}.$$

Then $(\chi_x)_{x\in\mathbb{R}}$ is an uncountable linearly independent Borel subset of $\ell^1(R)$, so we are done by taking any perfect subset of this. □

There is an analogous generalization of Theorem 5.1.2.

Theorem 5.5.3. *Let R be a left-Noetherian discrete proper normed ring, let M be a Polish R-module, and let $N \subseteq M$ be an F_σ submodule. Then exactly one of the following holds:*

(1) M/N is countable.

(2) $\ell^1(R)/\ell^1(I) \sqsubseteq^R M/N$ for some proper[2] two-sided ideal $I \lhd R$. In particular, there is a linear injection $\ell^1(R/I) \hookrightarrow M/N$.

Proof. Suppose that M/N is not countable. Then N is not open, and thus meager. Let $(U_k)_k$ be a descending neighborhood basis of $0 \in M$, and let $I_k = \{r \in R : rU_k \subseteq N\}$. Then $(I_k)_k$ is an increasing sequence of ideals, so since R is left-Noetherian, this sequence stabilizes at some $I = I_n$. Note that I is a proper ideal, since otherwise $U_n \subseteq N$, a contradiction to N being meager. Note also that I is a two-sided ideal, since if $r \in R$, then there is some $k > n$ with $rU_k \subseteq U_n$, and thus $IrU_k \subseteq IU_n \subseteq N$, and thus $Ir \subseteq I$. By replacing M with the submodule generated by U_n (which is analytic non-meager, and therefore open), we can assume that for every nonempty open $V \subseteq M$, we have $\{r \in R : rV \subseteq N\} = I$. Then for every $r \notin I$, the subgroup $\{m \in M : rm \in N\}$ is not open, and therefore meager. Thus more generally, if $m' \in M$, then $\{m \in M : rm \in N + m'\}$ is meager.

[2] By proper, we mean a proper subset (no relation to proper norms).

Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. Let $(F_k)_k$ be an increasing sequence of closed nowhere dense sets with $N = \bigcup_k F_k$. For every k, we define $\varepsilon_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R/I. We proceed by induction on k. Choose $\varepsilon_k > 0$ such that

(i) $\varepsilon_k < \frac{1}{2}\varepsilon_i$ for every $i < k$,

(ii) for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i$ nonzero and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, we have $\varepsilon_k \leq \|\sum_{i<k} r_i m_i\|$,

(iii) for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i \notin N$ and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, the open ε_k-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

(i) $rm_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$ for every $r \notin I$,

(ii) $\|rm_k\| < \frac{1}{2}\varepsilon_k$ whenever $\frac{|r|}{k!} \leq k$.

We verify that this is possible. When choosing ε_k, for the second and third condition, there is only a finite set of $\sum_{i<k} r_i m_i$ to consider, and for the third condition, this set is disjoint from N, and hence from F_k. Thus such an ε_k must exist. When choosing m_k, for the first condition, for a fixed $r \notin I$ and $m' \in Rm_0 + \cdots + Rm_{k-1}$, we have shown earlier that $\{rm \notin N + m'\}$ is comeager, so by quantifying over the countably many r and m', the set of m_k satisfying the first condition is comeager. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \leq k$ is finite. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$. We can assume that there is some n such that $\sum_{k<n} r_k m_k$ is nonzero and $\sum_{k<n} \frac{|r_k|}{k!} \leq n$. Then

$$\varepsilon_n \leq \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i}m_{n+i}\| < \frac{1}{2}\varepsilon_{n+i}$, and thus $\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$ by inductively using $\varepsilon_{k+1} < \frac{1}{2}\varepsilon_k$. Thus

$$\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\left\|\sum_{k<n} r_k m_k\right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\|\sum_k r_k m_k\right\| < 2\left\|\sum_{k<n} r_k m_k\right\|.$$

It remains to show that the kernel of the induced map $\ell^1(R) \to M/N$ is $\ell^1(I)$. The kernel clearly contains $\ell^1(I)$, since $IM \subseteq N$. Now let $(r_k)_k \in \ell^1(R) \setminus \ell^1(I)$. Since the image of $(r_k)_k$ in M/N is linearly independent over R/I, if n is sufficiently large, then $\sum_{k<n} r_k m_k \notin N$ and $\sum_k \frac{|r_k|}{k!} \leq n$. Since $\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}\varepsilon_n$, we have $\sum_{i\geq 0} \|r_{n+i}m_{n+i}\| < \varepsilon_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. □

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