Singularity Formation in Incompressible Fluids and Related Models

Thesis by
Jiajie Chen

In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2022
Defended May 12, 2022

Jiajie Chen
ORCID: 0000-0002-0194-1975

ACKNOWLEDGEMENTS

First of all, I would like to express my deepest gratitude to my adviser, Professor Thomas Yizhao Hou, for introducing me to the subject of singularity formation in incompressible fluids, his constant guidance, and generous support. Prof. Hou's steadfast resolution in solving challenging mathematical problems influences me profoundly, and his many viewpoints shape my perspective of research. I am also grateful to him for continuously encouraging me to focus on the hard problems.

I have been fortunate to have many chances to discuss with leading experts in mathematical fluid mechanics. I am indebted to Prof. Vladimir Sverak for generously sharing with me many valuable viewpoints on various problems and many insightful discussions. In particular, I first learned the De Gregorio model studied in this thesis from Prof. Sverak at the AIM Square "Towards a 3D Euler singularity" when I was a first-year graduate student. I have benefited a lot from Prof. Tarek Elgindi for his ideas and perspective on singularity formation. I am also grateful to him for many valuable discussions. I have also learned a lot about singularity formation and some important problems from Prof. Alexander Kiselev via the AIM Square, workshops, and seminars. I am indebted to Prof. Zhifei Zhang for introducing me to mathematical fluid mechanics when I was an undergrad, for valuable discussions, and for his advice on my development. In addition, I am grateful to Dr. In-Jee Jeong, Prof. Hao Jia, Prof. Vlad Vicol, Dr. Dongyi Wei, Prof. Sijue Wu, Dr. Xiaoqian Xu, and Prof. Yao Yao for stimulating discussions and suggestions. I have benefited tremendously from their knowledge.

I want to thank Professors Philip Isett, Houman Ohwadi, and Andrew Stuart for serving on my thesis committee. I enjoyed the beautiful courses on advanced topics by Professors Rupert Frank, Andrew Stuart, and Joel Tropp very much. These courses broadened my academic interests. In addition, I am very grateful to these professors for their suggestions on my development. I am also thankful to be supported by many amazing Caltech staff so that I can fully concentrate on my studies. I want to thank Diana Bohler, Jolene Brink, Sydney Garstang, Maria Lopez, Carmen Sirois at the CMS department, Laura Flower Kim from ISP, and Tess Legaspi from the registrar's office for their tremendous help.

I am grateful to the research group of Prof. Hou. In particular, I learned about the dynamic rescaling formulation and the self-similar singularity from Dr. Pengfei Liu at the beginning of my graduate study. I have benefited greatly from Dr. De Huang for numerous discussions and learned many numerical methods from him.

I have been fortunate enough to have the company of many good friends during my Ph.D. I am grateful to many long-time friends since middle school, in particular Xiaoqi Chen, Zhijie Chen, Jianyang Fu, and Shukun Wu, for their support in many aspects. I thank Zhijie Chen and Jianyang Fu for doing several challenging hiking projects in California and outdoor sports together, which were the stream of inspiration. I have received many guidances on coding and help in many ways from Xiaoqi Chen. I am also grateful to Shukun Wu for his guidance and help during my postdoc application. Many friends at Caltech have greatly enriched my life, and I wish to thank them all. I also thank Donghao Wang for his many suggestions and opinions during my postdoc application.

Last but not least, I would like to thank my parents and my sister for their unconditional love and support. I am very grateful to receive the best possible education from middle school to undergraduate in China, which is impossible without my parents' support. My parents set me a great example of diligence, which I benefit from forever.

ABSTRACT

Whether the three-dimensional (3D) incompressible Euler equations can develop a finite-time singularity from smooth initial data with finite energy is a major open problem in partial differential equations. A few years ago, Tom Hou and Guo Luo obtained strong numerical evidence of a potential finite time singularity of the 3D axisymmetric Euler equations with boundary from smooth initial data. So far, there is no rigorous justification. In this thesis, we develop a framework to study the Hou-Luo blowup scenario and singularity formation in related equations and models. In addition, we analyze the obstacle to singularity formation.

In the first part, we propose a novel framework of analysis based on the dynamic rescaling formulation to study singularity formation. Our strategy is to reformulate the problem of proving finite time blowup into the problem of establishing the nonlinear stability of an approximate self-similar blowup profile using the dynamic rescaling equations. Then we prove finite time blowup of the 2D Boussinesq and the 3D Euler equations with $C^{1,\alpha}$ velocity and boundary. This result provides the first rigorous justification of the Hou-Luo scenario using $C^{1,\alpha}$ velocity.

In the second part, we further develop the framework for smooth data. The method in the first part relies crucially on the low regularity of the data, and there are several essential difficulties to generalize it to study the Hou-Luo scenario with smooth data. We demonstrate that some of the challenges can be overcome by proving the asymptotically self-similar blowup of the Hou-Luo model. Applying this framework, we establish the finite time blowup of the De Gregorio (DG) model on the real line ($\mathbb{R}$) with smooth data. Our result resolves the open problem on the regularity of this model on $\mathbb{R}$ that has been open for quite a long time.

In the third part, we investigate the competition between advection and vortex stretching, an essential difficulty in studying the regularity of the 3D Euler equations. This competition can be modeled by the DG model on S^1. We consider odd initial data with a specific sign property and show that the regularity of the initial data in this class determines the competition between advection and vortex stretching. For any $0 < \alpha < 1$, we construct a finite time blowup solution from some C^α initial data. On the other hand, we prove that the solution exists globally for C^1 initial data. Our results resolve some conjecture on the finite time blowup of this model and imply that singularities developed in the DG model and the generalized Constantin-Lax-Majda model on S^1 can be prevented by stronger advection.

PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Jiajie Chen, Thomas Y. Hou, and De Huang. On the finite time blowup of the De Gregorio model for the 3D Euler equations. *Comm. Pure Appl. Math.*, 74(6):1282–1350, 2021. ISSN 0010-3640. doi: 10.1002/cpa.21991. URL `https://doi.org/10.1002/cpa.21991`.
J.C. participated in the conception of the project, performed the analysis, and participated in the numerical computation, computer-assisted verification, and the writing of the manuscript.

[2] Jiajie Chen. Singularity formation and global well-posedness for the generalized Constantin-Lax-Majda equation with dissipation. *Nonlinearity*, 33(5):2502– 2532, 2020. ISSN 0951-7715. doi: 10.1088/1361-6544/ab74b0. URL `https://doi.org/10.1088/1361-6544/ab74b0`.
J.C. is the sole author of the manuscript.

[3] Jiajie Chen and Thomas Y. Hou. Finite time blowup of 2D Boussinesq and 3D Euler equations with $C^{1,\alpha}$ velocity and boundary. *Comm. Math. Phys.*, 383(3): 1559–1667, 2021. ISSN 0010-3616. doi: 10.1007/s00220-021-04067-1. URL `https://doi.org/10.1007/s00220-021-04067-1`.
J.C. participated in the conception of the project, performed the analysis, and participated in the writing of the manuscript.

[4] Jiajie Chen. On the slightly perturbed De Gregorio model on S^1. *Arch. Ration. Mech. Anal.*, 241(3):1843–1869, 2021. ISSN 0003-9527. doi: 10.1007/s00205- 021-01685-w. URL`https://doi.org/10.1007/s00205-021-01685-w`.
J.C. is the sole author of the manuscript.

[5] Jiajie Chen, Thomas Y. Hou, and De Huang. Asymptotically self-similar blowup of the Hou–Luo model for the 3D Euler equations. *arXiv preprint arXiv:2106.05422,* 2021.
J.C. participated in the conception of the project, performed the analysis, and participated in the numerical computation, computer-assisted verification, and the writing of the manuscript.

[6] Jiajie Chen. On the regularity of the De Gregorio model for the 3D Euler equations. *To appear in J. Eur. Math. Soc.*, *arXiv preprint arXiv:2107.04777,* 2021.
J.C. is the sole author of the manuscript.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

Chapter 1

INTRODUCTION

1.1 The 3D incompressible Euler equations

The three-dimensional (3D) incompressible Euler equations in fluid dynamics describe the motion of ideal incompressible flows. The equations are among the most fundamental nonlinear partial differential equations and have been used to model ocean currents, weather patterns, and other fluids-related phenomena. Despite their wide range of applications, whether the 3D Euler equations can develop a finite-time singularity from smooth initial data remains open, which is generally viewed as one of the major open questions in mathematical fluid mechanics. See the surveys [24, 51, 53, 66, 89] and the references therein.

There are three fundamental difficulties associated with the analysis of the 3D Euler equations: nonlinearity, nonlocality, and the competing effects from advection and vortex stretching. To illustrate these difficulties, we consider the vorticity-stream function formulation [89]:

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \omega \cdot \nabla \mathbf{u}, \tag{1.1}$$

where $\omega = \nabla \times \mathbf{u} : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ is the *vorticity* of the fluid, and $\mathbf{u}(x,t) : \mathbb{R}^3 \times [0, T) \to \mathbb{R}^3$ is the velocity vector related to ω via the *Biot-Savart law*:

$$\mathbf{u} = \nabla \times (-\Delta)^{-1} \omega. \tag{1.2}$$

The above nonlocal relation comes from the divergence-free condition $\nabla \cdot \mathbf{u} = 0$ enforcing the incompressibility of the fluids. Thus, the Euler equations (1.1) are a nonlinear and nonlocal system, making analysis of these equations extremely challenging.

The term $\omega \cdot \nabla \mathbf{u}$ in (1.1) is called the vortex stretching term. Note that $\nabla \mathbf{u}$ is formally of the same order as ω. Under some decay conditions in the far field, using Calderon-Zygmund estimates, one can show that

$$c_p \|\omega\|_{L^p} \le \|\nabla \mathbf{u}\|_{L^p} \le C_p \|\omega\|_{L^p}, \quad p \in (1, \infty)$$

with constants c_p, C_p depending on p. Thus the vortex stretching term scales quadratically as a function of vorticity, suggesting that (1.1) may develop a finite time singularity. In 2D, the vortex stretching term is absent, and one can get an *a-priori* estimate of $||\omega(t)||_\infty$ for all time. Then the global regularity follows from the Beale-Kato-Majda blow-up criterion (2.2) [1]. Thus, the vortex stretching term is the main source of difficulty in obtaining the global regularity of the 3D Euler equations. For the third difficulty of competing effects from advection and vortex stretching, we will discuss it in Section 1.4.

The 3D Euler equations are locally well-posed for sufficiently regular data, and several blow-up criteria have been developed for the Euler equations. Moreover, the equations enjoy several properties, such as symmetry groups and energy conservation. We will review these classical results in Section 2.1.

There have been a number of attempts on the numerical search of a potential singularity of the Euler equations, see e.g., [2, 40, 58, 66, 69, 75]. We refer to a review article [53] for more discussions on potential Euler singularities.

1.2 The Hou-Luo scenario

In [86, 87], Hou-Luo obtained strong numerical evidence that the 3D axisymmetric Euler equations (2.5) in a periodic cylinder develop a potential finite time singularity on the boundary from smooth initial data with finite energy. It is by far the most promising blowup scenario for (1.1) with smooth data. We refer to Section 2.1.3 for the settings of the Hou-Luo scenario.

Review of related works The Hou-Luo scenario has inspired several important subsequent developments, see e.g., [22, 23, 78, 80]. In [78], Kiselev-Sverak proved that the gradient of the vorticity in the 2D Euler can achieve the double exponential growth rate (the fastest possible growth rate) using a setting similar to that of the Hou-Luo scenario. In [80], Kiselev-Ryzhik-Yao-Zlatos established singularity formation of a sequence of vortex patch models with boundary using a similar flow structure. We also mention the works of Zlatos [106] and Kiselev-He [62] on the small-scale creation of the 2D Euler and the SQG equations, where a similar hyperbolic flow structure in $\mathbb{R}^2$ is considered.

A lot of efforts have been devoted to studying the Hou-Luo scenario, in particular via the 2D Boussinesq equations (1.3)

$$\begin{aligned} \omega_t + \mathbf{u}\cdot\nabla\omega &= \theta_x, \\ \theta_t + \mathbf{u}\cdot\nabla\theta &= 0, \quad \mathbf{u} = (u,v)^T = \nabla^{\perp}(-\Delta)^{-1}\omega, \end{aligned} \tag{1.3}$$

where $\mathbf{u}$ is the velocity satisfying the no flow boundary condition $v(x,0)=0$, ω is the vorticity, and θ is the density. Since the potential singularity [86] occurs on the boundary, away from the symmetry axis $r=0$, it is well known that the 3D axisymmetric Euler equations (2.5) are similar to the 2D Boussinesq equations [89]. Note that the question of finite time blowup of the Boussinesq equations from smooth initial data is listed by Yudovich as one of the great problems of mathematical hydrodynamics [105]. Due to the difficulties in the 3D Euler equations discussed in Section 1.1 (see also Section 1.4), only a few methods have been developed to study the singularity formation of the 3D Euler equations and related equations.

One way to make progress in understanding the Hou-Luo scenario is by studying simplified 1D model equations capturing certain features of the full equations. The study of 1D models for hydrodynamical equations has a long history, and one of the earliest works is the Constantin-Lax-Majda model [26] for the effect of vortex stretching. Hou-Luo [86] derived the first 1D model for the Hou-Luo scenario by restricting (1.3) on the boundary under some closure assumption. In [22], Choi-Kiselev-Yao simplified the Biot-Savart law in the HL model and established finite time blowup. In [70], Hou and Liu established the self-similar singularity of the CKY model [22] using the property that the CKY model can be reformulated as a local ODE system. By exploiting the symmetry properties of the solution and some monotonicity property of the velocity kernel, Choi et al. established the singularity formation of the HL model in [23] using a Lyapunov functional argument. There are other 1D models for the Hou-Luo scenario, see e.g., [36, 37, 64], and finite time blowups of these models were established therein.

There are some 2D models for the Hou-Luo scenario by simplifying the nonlocal Biot-Savart law $\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}$ in (1.3), an essential difficulty in the study of (1.1) and (1.3), see e.g., [65, 79]. In [65, 79], the authors studied modified 2D Boussinesq equations with θ_x in (1.3) replaced by θ/x. In these works, the simplified Biot-Savart law has a positive kernel, and the authors proved finite time blowup for smooth initial data using the method of characteristics and a functional argument [79] or barrier functions [65].

Unlike studying simplified model equations, Elgindi-Jeong considered the 2D Boussinesq and the 3D axisymmetric Euler equations in a domain with a corner using $\mathring{C}^{0,\alpha}$ data and settings similar to the Hou-Luo scenario and established finite time blowup [43, 45]. This line of research is further extended in [47] to establish singularity formation of (1.1) under octahedral symmetry with bounded and piecewise-smooth vorticity. In these works, the behavior of the solutions is governed by exact 1D models or ODEs. We refer to the excellent survey [77] for more discussions related to the Hou-Luo scenario.

Singularity formation in the 3D Euler equations Recently, Elgindi [42] proved a remarkable result on singularity formation of the 3D axisymmetric Euler equations without swirl for $C^{1,\alpha}$ velocity with sufficiently small α. In [48], Elgindi-Ghoul-Masmoudi further established the stability of the blowup solutions in [42] and constructed $C^{1,\alpha}$ blowup solutions with finite energy. Note that the result in [42] cannot be generalized to smooth data since it is well known that the 3D axisymmetric Euler equations without swirl have a global smooth solution for smooth initial data [89].

Despite all the previous efforts, rigorous proof of the Hou-Luo blowup scenario or finite time blowup of the Euler equations with smooth data remains open. There seem to be some essential difficulties in generalizing the methods in [22, 23, 65, 78–80] to study the singularity formation of the 3D Euler equations.

One of the main contributions of this thesis is developing a novel framework to study the singularity formation of the 3D Euler equations and related equations based on the dynamic rescaling formulation (or modulation technique), see e.g., [74, 92]. This framework contributes to the publications [13–16, 19, 20]. Many important ideas were first developed in work [19].

1.3 A framework to study singularity formation

Our framework of analysis builds on the scaling symmetry of the equations and the dynamic rescaling formulation. See Section 2.1 for more discussions.

To introduce our framework, we use the Boussinesq equations (1.3). The main idea of our framework is to prove that the vorticity enjoys a decomposition

$$\omega(x,t) \approx \frac{1}{(T-t)}\Omega\Big(\frac{x}{(T-t)^{c_l(t)}}, t)\Big), \tag{1.4}$$

and the profile Ω is bounded from below $||\Omega(\cdot,t)||_\infty \geq c$ uniformly for some $c > 0$ up to the blowup time T. The dynamic rescaling formulation [92] allows us to reformulate the physical equations (1.3) into the following dynamic rescaling equations of the profile by choosing appropriate scaling parameters

$$\omega_t + (c_l\mathbf{x} + \mathbf{u}) \cdot \nabla\omega = c_\omega\omega + \theta_x, \quad \theta_t + (c_l\mathbf{x} + \mathbf{u}) \cdot \nabla\theta = (c_l + 2c_\omega)\theta, \tag{1.5}$$

where $\mathbf{u} = \nabla^\perp(-\Delta)^{-1}\omega, \mathbf{x} = (x, y)$, and the scaling factors c_ω, c_l satisfy some normalization conditions depending on (ω, θ). The potential blowup time T in (1.4) is mapped to $t = \infty$ in (1.5) and the profile Ω in (1.4) becomes the solution ω to (1.5). We refer the reformulation to Section 2.1.4.

The challenge is to prove that the profile $\omega(x, t)$ is bounded from below for all $t > 0$. Due to the nonlinear terms, standard energy estimates only yield a short time estimate of the solution to (1.5). Our idea is that if (1.5) has a stable approximate steady state (ASS) with a small residual error, by establishing its nonlinear stability, we can control the solution for all time. Thus, we reformulate the problem of proving finite time singularity into the problem of establishing the nonlinear stability of an ASS in the dynamic rescaling equations. We will discuss how to construct an ASS in Section 1.3.6.1. In the whole framework, the most challenging step is to establish the linear stability of the ASS.

1.3.1 Linear stability analysis

Given an ASS $(\bar\omega, \bar{\mathbf{u}}, \bar\theta, \bar c_\omega, \bar c_l)$ of (1.5), we linearize (1.5) around $(\bar\omega, \bar{\mathbf{u}}, \bar\theta, \bar c_\omega, \bar c_l)$ to obtain the linearized equations for the perturbations $(\tilde\omega, \tilde\theta, \tilde c_l, \tilde c_\omega)$:

$$\begin{aligned}\partial_t\tilde\omega &= -(\bar c_l\mathbf{x} + \bar{\mathbf{u}}) \cdot \nabla\tilde\omega + \tilde\theta_x + \bar c_\omega\tilde\omega - (\tilde c_l\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla\bar\omega + \tilde c_\omega\bar\omega + N(\omega) + \bar F_\omega,\\ \partial_t\tilde\theta &= -(\bar c_l\mathbf{x} + \bar{\mathbf{u}}) \cdot \nabla\tilde\theta + (\bar c_l + 2\bar c_\omega)\tilde\theta - (\tilde c_l\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla\bar\theta + (\tilde c_l + 2\tilde c_\omega)\bar\theta + N(\theta) + \bar F_\theta,\end{aligned} \tag{1.6}$$

where $\mathbf{x} = (x, y)$, $\tilde{\mathbf{u}} = \nabla^\perp(-\Delta)^{-1}\tilde\omega$ is nonlocal, $N(\omega), N(\theta)$ are the nonlinear terms, and $\bar F_\omega, \bar F_\theta$ are the residual error of the approximate steady state.

Our motivation to establish linear stability in the framework is inspired by [85], where Liu showed that the eigenvalues of the discretized linearized operator of (1.6) have negative real parts bounded from above by roughly -0.3. However, since the linearized operators in (1.6) and other equations, e.g., (2.5) and (1.16), that we study in this thesis are not compact, we *cannot* approximate them by finite rank operators that can be analyzed by computing the eigenvalues of these finite rank operators. Moreover, the operators are non-normal and contain several nonlocal terms involving the velocity $\tilde{\mathbf{u}}$, which are difficult to control. Since $\bar\omega, \bar\theta$ have size $O(1)$, the damping factor is small, and there is no regularizing effect, e.g., viscosity, we would not be able to prove linear stability of the ASS if we overestimate the effect of these nonlocal terms.

For example, one may try to estimate the term $\mathbf{u}\cdot\nabla\bar{\theta}_x$ in (1.6) using a weighted Sobolev estimate

$$||\mathbf{u}\rho_1||_{L^2} \leq C||\omega\rho_2||_{L^2}, \tag{1.7}$$

for some singular weights ρ_1, ρ_2 (see the discussion below (1.8) for the motivation of singular weights). Yet, the constant C is typically unknown and can be large. Such an estimate can lead to the failure of linear stability analysis. This makes it extremely difficult to establish the linear stability for (1.6) and other equations. We remark that the linear stability analysis is *problem-dependent.*

To further establish nonlinear stability, we need to control an energy of the perturbations and understand the damping mechanism.

Some damping mechanisms For several equations, the damping terms in the energy estimates can be derived from the local terms, such as $\bar{c}_l\mathbf{x}\cdot\nabla\tilde{\omega}$, $\bar{c}_l\mathbf{x}\cdot\nabla\tilde{\theta}$, $\bar{c}_\omega\omega$, by performing the estimates in a suitably weighted functional space X with singular weights. In general, the scaling parameters $\bar{c}_l, \bar{c}_\omega$ satisfy $\bar{c}_l > 0, \bar{c}_\omega < 0$. Then $\bar{c}_\omega\omega$ in (1.6) is already a damping term.

To understand the effect of the operator $\bar{c}_l\mathbf{x}\cdot\nabla$, we consider a 1D toy model

$$f_t = -\bar{c}_l x f_x + \mathcal{R}(x), \quad f(x) = O(|x|^l) \text{ near } x = 0,\ l > 0, \tag{1.8}$$

where $\bar{c}_l x f_x$ models $\bar{c}_l\mathbf{x}\cdot\nabla\theta$ or $\bar{c}_l\mathbf{x}\cdot\nabla\omega$ in (1.6) and $\mathcal{R}(x)$ models other terms in (1.6). Performing a weighted L^2 estimate on f with a weight $\rho = x^{-k}, k > 1$ and applying integration by parts, we obtain

$$\frac{1}{2}\frac{d}{dt}\int f^2\rho = \int f_t f\rho = \int(-\bar{c}_l x f_x + \mathcal{R})f x^{-k} = \int -\frac{k-1}{2}\bar{c}_l f^2 x^{-k} + \mathcal{R} f x^{-k}.$$

Since $k > 1$, we see that $-\bar{c}_l x f_x$ contributes to a damping term $\int -\frac{k-1}{2}\bar{c}_l f^2 x^{-k}$ for the energy $\int f^2\rho$. Moreover, for larger k, the damping factor $\frac{k-1}{2}\bar{c}_l$ is larger. If the remaining part $\int \mathcal{R} f x^{-k}$ is smaller, we obtain the linear stability. In order for $\int f^2 x^{-k}$ to be well defined, since $f(x) = O(|x|^l)$ near $x = 0$ (1.8), we require $k < 2l + 1$. Hence, the higher vanishing order of f near $x = 0$, the larger damping factor we can derive by choosing a more singular weight. Yet, the estimate of the remaining term $\int \mathcal{R} f x^{-k}$ can get worse due to the more singular weight. Thus, we need to design the singular weight carefully. Note that the idea of applying singular weights in stability analysis has been used in [73, 83].

An alternative approach to derive the damping term from $\bar{c}_l x f_x$ (and similarly $\bar{c}_l \mathbf{x} \cdot \nabla$ (1.6)) is to perform an energy estimate on $\partial_x^k f$ with suitably large $k > 0$. Yet, it has a disadvantage that it leads to more nonlocal terms that are difficult to control and a worse structure of (1.6) due to the Leibniz rule.

Normalization conditions In the dynamic rescaling equations (1.6), we have the freedom to choose time-dependent scaling parameters $\tilde{c}_l, \tilde{c}_\omega$. See Section 2.1.4 for more discussions. We choose some normalization conditions for the scaling factors $\tilde{c}_l, \tilde{c}_\omega$ to enforce that the perturbations $(\tilde{\omega}, \tilde{\theta})$ vanish at the origin with higher order. This allows us to choose more singular weights and obtain a larger damping factor in the energy estimates. For example, in the blowup analysis of the Boussinesq equations (1.3) in Chapter 2, by choosing appropriate normalization conditions, we can improve the vanishing order of $\tilde{\omega}, \tilde{\nabla}\theta$ near the origin from $O(|x|^\alpha)$ to $O(|x|^{2\alpha})$ for some small $\alpha > 0$.

Moreover, it is important to choose the appropriate normalization conditions for $\tilde{c}_l, \tilde{c}_\omega$ to eliminate potential dynamically unstable modes in (1.6). This is crucial in the analysis of the HL model in Chapter 4.

Control of the nonlocal terms To control the nonlocal terms in (1.6), we use sharp functional inequalities and exploit cancellation among various nonlocal terms. It is crucial to exploit the cancellation in the equations since applying sharp functional inequalities alone can still overestimate the effect of the nonlocal terms. See the discussion around (1.7). Since we only have limited sharp functional inequalities and weighted inequalities that capture the nonlocal cancellation, we need to perform the energy estimate very carefully. We refer to (1.12) for an example of nonlocal cancellation.

Our goal is to apply these sharp estimates to show that the nonlocal terms can be treated as perturbations to the damping terms and establish linear stability estimates of the perturbations in some lower-order functional space X_1.

1.3.2 Nonlinear stability

Once we establish the linear stability estimates in space X_1, we can similarly perform linear stability estimates in higher-order functional spaces. Then we can close the nonlinear estimate using embedding inequalities. The above strategies and analysis culminate in a nonlinear energy estimate for some energy $E(\tilde{\omega}(t), \tilde{\theta}(t))$ of the perturbations

$$\frac{d}{dt} E \le C(\bar{\omega}, \bar{\theta}) E^2 - \lambda E + \varepsilon. \tag{1.9}$$

The crucial damping term $-\lambda E$ with $\lambda > 0$ comes from the linear stability, CE^2 controls the nonlinear terms $N(\omega), N(\theta)$ in (1.6), and ε is the weighted norm of the residual error $\bar{F}_\omega, \bar{F}_\theta$ in (1.6) of the approximate steady state.

The mechanism of nonlinear stability is that for small perturbation E and error ε, CE^2 and ε can be treated as small perturbations to the damping term $-\lambda E$, resulting in stability. Applying a bootstrap argument, we obtain that if the error ε and initial perturbation $E(0)$ are small

$$\varepsilon < \varepsilon^* = \frac{\lambda^2}{4C}, \quad E(0) = E(\tilde{\omega}_0, \tilde{\theta}_0) < E^* = \frac{2\varepsilon}{\lambda}, \tag{1.10}$$

the assumption

$$E(t) = E(\tilde{\omega}(t), \tilde{\theta}(t)) < E^* \tag{1.11}$$

holds true for all time. Thus, we can close the nonlinear estimate (1.9). Estimate (1.10) provides an upper bound ε^* on the required accuracy ε of the approximate steady state.

A significant difference between the above estimates and those in Section 1.3.1 is that we have a small parameter ε. As long as ε is sufficiently small : $4C\varepsilon < \lambda^2$ (1.10), thanks to the damping term $-\lambda E$ from the linear stability, we can afford a large constant $C(\bar{\omega}, \bar{\theta})$ in the nonlinear estimates and close the nonlinear estimates. Thus, the above estimates are much simpler than those in Section 1.3.1. We refer to Section 1.3.6.1 for constructing an approximate steady state $(\bar{\omega}, \bar{\theta})$ to (1.5) with a sufficiently small residual error.

1.3.3 Finite time blowup from finite energy initial data

It is essential to have initial data with finite energy $||\mathbf{u}_0||_{L^2} < +\infty$ for singularity formation in fluids equations. However, the approximate steady state $\bar{\mathbf{u}}$ we construct for several equations has sublinear growth in the far-field and thus has infinite energy $||\bar{\mathbf{u}}||_{L^2} = \infty$. Since the ASS is nonlinearly stable, it is natural to choose perturbation $(\tilde{\omega}, \tilde{\theta})$ small in the energy norm E to truncate the ASS. Note that $\bar{\mathbf{u}}$ is growing. The fundamental idea is to design a weighted energy norm E with suitably decaying weights so that the truncation is small in the energy E.

We illustrate this idea using the blowup analysis of (1.3) with $C^{1,\alpha}$ data discussed in Chapter 2. The ASS of the velocity has sublinear growth $|\bar{\mathbf{u}}| \sim |x|^{1-\alpha}$ for large x and small $\alpha > 0$, and the associated vorticity $\bar{\omega} = \nabla \times \bar{\mathbf{u}}$ has a slow decay $|x|^{-\alpha}$. In the energy estimate of the perturbation $\tilde{\omega}$, we choose a functional space $L^2(\rho)$ with weight $\rho(x)$ that has a radial decay rate $|x|^{\gamma}, \gamma \in (-2, -2+2\alpha)$. Since the weight has a fast decay, we can truncate the ASS in the far-field with perturbation $\tilde{\omega}$ that is small in $L^2(\rho)$. This provides initial data with finite energy $||u_0||_2 < +\infty$. Using the rescaling argument in Section 2.1.4 and the idea in (1.4), we establish the finite time blowup.

1.3.4 Convergence to the steady state

Based on the nonlinear stability estimates (1.11), we can further establish that the solution $(\bar{\omega} + \tilde{\omega}(t), \bar{\theta} + \tilde{\theta}(t))$ to (1.5) converges to the exact steady state (1.5). We first rewrite (1.6) as follows :

$$Z_t = \mathcal{L}Z + N(Z) + \bar{F}_Z,$$

where $Z = (\tilde{\omega}, \tilde{\theta})$, $\mathcal{L}$ denotes the linearized operators, $N(Z) = (N(\tilde{\omega}), N(\tilde{\theta}))$, and $\bar{F}_Z = (\bar{F}_\omega, \bar{F}_\theta)$ in (1.6). The key observation is that the residual error $\bar{F}_Z$ and the operator $\mathcal{L}$ are time-independent. Taking time-derivative, we yield

$$\partial_t Z_t = \partial_t \mathcal{L}Z + \partial_t N(Z) + \partial_t \bar{F}_Z = \mathcal{L}Z_t + \partial_t N(Z).$$

We can estimate $\mathcal{L}Z_t$ using the same linear stability analysis of $\mathcal{L}$ under the norm X_1 in Sections 1.3.1, and estimate the nonlinear part $\partial_t N(Z)$ in the norm X_1 using embedding inequalities. These estimates imply

$$\frac{d}{dt}||Z_t||_{X_1} \leq -\lambda||Z_t||_{X_1} + C_2(\bar{\omega}, \bar{\theta})||Z_t||_{X_1} E(Z).$$

From $E(Z(t)) < \frac{2\varepsilon}{\lambda}$ ((1.10) and (1.11)), if the error ε is small enough, we get that $||Z_t||_{X_1}$ decays exponentially fast and further establish the convergence from $\bar{Z} + Z(t)$ to the steady state Z_∞ of (1.5). The convergence and a rescaling argument (see Section 2.1.4) implies that the blowup is asymptotically self-similar. We refer to Section 3.4.3, Chapter 3 for a concrete example and argument.

1.3.5 $C^{1,\alpha}$ velocity

Using the strategies in Section 1.3 and adopting several methods from Elgindi's important work [42], Hou and I proved the following blowup result.

Theorem 1.1. *For small α, the solution of the Boussinesq equations* (1.3) *in $\mathbb{R}_+^2$ develops a focusing asymptotically self-similar singularity in finite time for some initial data $\omega \in C_c^{\alpha}(\mathbb{R}_+^2), \theta \in C_c^{1,\alpha}(\mathbb{R}_+^2)$ with finite energy $||\mathbf{u}||_{L^2} < +\infty$.*

We also proved similar blowup results for the 3D Euler equations with boundary and $C^{1,\alpha}$ velocity [16]. See Theorems 2.1, 2.2 in Chapter 2 for the precise statements of these theorems. These results provide the first rigorous justification of the Hou-Luo scenario [86] using $C^{1,\alpha}$ data.

For a class of $\mathbf{u}, \theta \in C^{1,\alpha}$ with sufficiently small α, following [42], we derive the leading order terms in the nonlocal velocity (1.5), and show that some nonlocal terms in the dynamic rescaling equations (1.5) become lower order terms. This allows us to further construct an ASS analytically. As discussed in Section 1.3.1, one of the main difficulties is to control the nonlocal terms effectively in the stability analysis of (1.6). One of the key ingredients is to exploit the nonlocal cancellation between ω and $\mathbf{u}$ using the following inequality

$$\iint_D -(u_x - u_x(0))\omega \frac{\sin(2\beta)}{r^{1+k\alpha}} dr d\beta = -C_k \alpha \iint_D \frac{(u_x - u_x(0))^2}{r^{1+k\alpha}} dr d\beta + l.o.t. \tag{1.12}$$

for sufficiently small α, where $k \in [3/2, 4]$ and (r, β) is the polar coordinate in $\mathbb{R}_2^+$ and $D = [0, \infty] \times [0, \pi/2]$. The interaction term on the left appears in the weighted L^2 energy estimate of $(\tilde{\omega}, \tilde{\theta}_x, \tilde{\theta}_y)$ in (1.6). On the right hand side, the main term is a damping term since it has a negative sign. We apply this crucial damping term to control the nonlocal terms $\tilde{\mathbf{u}}$ in (1.6). To exploit the above cancellation, we use the coupling structures in (1.6) and carefully design the weighted energy estimates. With these sharp estimates, we follow the ideas in Sections 1.3.1, 1.3.2 to establish the stability analysis of (1.6). To generalize the analysis of (1.6) to that of the 3D Euler equations, we further develop a localized elliptic estimate and control the support of the solution.

Using the idea in Section 1.3.3, we can obtain finite time blowup of (1.3) and (2.5) from finite energy initial data. Note that the initial data of the singular solution in [42] does not have finite energy. Constructing a singular solution from finite energy initial data was established in a subsequent paper [48]. Our argument to construct finite energy initial data in [16] develops independently from that in [48]. We refer to Chapter 2 for complete proof.

1.3.6 Smooth data

Yet, several important methods in [16], Chapter 2, rely crucially on the small parameter α from the $C^{1,\alpha}$ regularity. For smooth data, the leading order structure for $C^{1,\alpha}$ velocity is not available and a few nonlocal terms that we can neglect using $C^{1,\alpha}$ data are not small. Moreover, we need to address the challenging problem of constructing an ASS.

1.3.6.1 Construction of the approximate steady state

An analytic construction of ASS is applicable if the dynamic rescaling equations are perturbations to simpler equations whose steady state can be derived, see e.g., [16, 42] for the 3D Euler equations with $C^{1,\alpha}$ velocity, and [13–15, 19, 44] for model problems with specific parameters.

In general, it is very difficult to construct an ASS analytically for fluids equations due to the nonlinearity and the nonlocality of the velocity. For the Hou-Luo scenario, constructing a smooth ASS of the Euler equations (2.5) or the Boussinesq equations (1.3) is even more challenging due to the coupled system. It is almost impossible to do it analytically.

An advantage of considering an ASS in our framework over an exact steady state is that we avoid constructing an exact steady state which is much more challenging. An ASS with a small residual error can be obtained by solving the dynamic rescaling equations (1.5) for a long enough time numerically. We can represent the ASS using suitably chosen piecewise polynomial basis functions. The residual error is estimated a posteriori and incorporated in the energy estimate for nonlinear stability as a small error term.

We have first applied the framework introduced in Section 1.3 and the above idea to construct an ASS numerically and establish the finite time blowup of the De Gregorio model (1.15). We defer our discussion to Section 1.4.

1.3.6.2 The Hou-Luo model

The most essential difficulty in applying our framework to study the Hou-Luo scenario with smooth data [86, 87] is establishing the linear stability of (1.6).

Due to the difficulties stated in Section 1.3.1 and the first paragraph in Section 1.3.6, the stability mechanism of (1.6) is far from clear. To understand the stability mechanism, we consider the HL model proposed in [86]

$$\omega_t + u\partial_x \omega = \theta_x, \quad \theta_t + u\partial_x \theta = 0, \quad u_x = H\omega, \tag{1.13}$$

which can be seen as the restriction of the Boussinesq equations (1.3) to the boundary. Here, H is the Hilbert transform.

The analysis of the HL model captures the difficulties of constructing an ASS and the linear stability analysis in the Boussinesq equations on the boundary, where the potential singularity develops. Firstly, Hou-Luo [86] showed numerically that the blowup exponents and profiles of the HL model and the 3D axisymmetric Euler equations on the boundary are quantitatively similar. Secondly, we obtain numerical evidence that the eigenvalues of the discretized linearized operator in the HL model all have negative real parts bounded from above by -0.38, which is similar to that of (1.6) (about -0.3) reported in [85].

In joint work with Hou and Huang [20], we successfully applied the framework and ideas in Section 1.3 and the numerical construction in Section 1.3.6.1 to establish the following result.

Theorem 1.2. *There is a family of initial data* (θ_0, ω_0) *with* $\theta_{0,x}, \omega_0 \in C_c^\infty$, *such that the solution of the HL model* (1.13) *develops a focusing asymptotically self-similar singularity in finite time.*

The more precise and stronger statement of the above theorem is given in Chapter 4. We refer to Chapter 4 for the stability mechanism of the HL model and its implication for (1.6).

1.3.7 Energy estimates with computer assistance

We *do not* require computer assistance in the blowup analysis of the Boussinesq equations and the 3D Euler equations with $C^{1,\alpha}$ velocity in Chapter 2 and the DG model discussed in Chapter 5. For other problems, such as the analysis of the DG model and HL model with smooth data, we need to use computer assistance in the following aspects.

Firstly, as discussed in Section 1.3.6.1, we need to construct the ASS numerically in general. We use numerical analysis with rigorous error control to verify that the residual error of the ASS is small in the energy norm. Secondly, the crucial part of the stability analysis is to use energy estimates to establish linear stability. In the energy estimates, instead of bounding several coefficients by some absolute constants, which leads to overestimates, we keep track of these coefficients. Since these coefficients depend on the ASS constructed numerically, we use numerical computation with rigorous error control to verify several inequalities that involve these coefficients. We remark that we *do not* compute the eigenvalues of the linearized operator $\mathcal{L}$ in the dynamic rescaling equations to establish linear stability since $\mathcal{L}$ is *not compact*.

1.4 Competition between advection and vortex stretching

The Hou-Luo scenario has significantly advanced the understanding of the potential singularity formation of the Euler equations with boundary. In the case without the boundary, the understanding is much poorer. One of the difficulties is the competition between advection and vortex stretching. In the Hou-Luo scenario with $C^{1,\alpha}$ velocity, one of our contributions is to show that the boundary plays a vital role in weakening the advection. See Chapter 2.

This competition in the 3D Euler equations has been studied in [72], where Hou-Jin-Liu considered a family of models of the 3D axisymmetric Euler equations (2.5) by adding a weight ε to advection in the equations of $u^\theta/r, \omega^\theta/r$

$$\begin{aligned} \partial_t \frac{u^\theta}{r} + \varepsilon(u^r\partial_r + u^z\partial_z)\frac{u^\theta}{r} &= -2\frac{u^r u^\theta}{r}, \\ \partial_t \frac{\omega^\theta}{r} + \varepsilon(u^r\partial_r + u^z\partial_z)\frac{\omega^\theta}{r} &= \partial_z(\frac{u^\theta}{r})^2. \end{aligned} \tag{1.14}$$

The model reduces to the 3D Euler equations (2.5) when $\varepsilon = 1$. The authors presented numerical evidence that (1.14) develops finite time singularity for weak advection ($\varepsilon \leq 0.3$), but a similar singularity scenario does not persist for strong advection. These results suggest the following principle:

> *The vortex stretching tends to lead to the growth of the solution, while the advection tends to stabilize the solution.*

Note that the stabilizing effect of advection has also been studied in [67, 68] for the 3D Navier Stokes and Euler equations.

Given the difficulties discussed in Section 1.1, experts have devised simpler models, which capture some of the difficulties and the above effects and can be better understood. De Gregorio [33, 34] introduced an 1D model (DG)

$$\omega_t + u\omega_x = u_x\omega, \quad u_x = H\omega, \tag{1.15}$$

to model the effects of advection and vortex stretching in the 3D Euler equations (1.1), where H is the Hilbert transform. The DG model is a modification of the Constantin-Lax-Majda model (CLM) [101] which does not include the advection $u\omega_x$ in (1.15). By adding a weight a to the advection $u\omega_x$ in (1.15), Okamoto et al. [97] introduced the following one-parameter family of models (gCLM)

$$\omega_t + au\omega_x = u_x\omega, \quad u_x = H\omega. \tag{1.16}$$

The domain of (1.15), (1.16) can be $\mathbb{R}$ or S^1. In these models, $u\omega_x$ and $u_x\omega$ models the advection $\mathbf{u}\cdot\nabla\omega$ and the vortex stretching $\nabla\mathbf{u}\cdot\omega$ in the 3D Euler equations (1.1), respectively. The Biot-Savart law (1.2) $\nabla\mathbf{u} = \nabla\nabla\times(-\Delta)^{-1}\omega$ is modeled by $u_x = H\omega$, which preserves the same scaling. We refer to Section 3.1 for basic properties of these models.

1.4.1 Conjectures and open problems

Despite the significant simplifications from the Euler equations (1.1) to these models, the analysis of (1.15) or (1.16) with $a > 0$ remains very challenging since it captures all the three difficulties in Section 1.1 for the Euler equations, in particular the competition between advection and vortex stretching. There are some open problems and conjectures in the literature for (1.16) and (1.15).

(I) Regularity of the DG model The evidence in [33, 34], numerical simulations [88, 97], and the report in [73] suggest global regularity of (1.15) from smooth initial data. These lead to the conjecture in [44, 88, 97] that the DG model is globally well-posed for smooth initial data. The question of regularity for the DG model is listed as one of the open problems in [57].

(II) Finite time blowup of gCLM model It is conjectured in [44, 88, 98] that the gCLM model (1.16) with $a < 1$ can blow up from some C_c^∞ initial data. A related conjecture was stated earlier in [97].

(III) Finite time blowup of the DG model In [44, 104], it is conjectured that (1.15) develops a finite time singularity from initial data $\omega_0 \in C^\alpha$ or $\omega_0 \in H^s$ for any $\alpha \in (0, 1)$ and $s < \frac{3}{2}$.

The borderline case of these conjectures is the DG model (or the gCLM model with $a = 1$) with $\omega \in C^1$. In such a case, the advection and the vortex stretching are balanced, which can be understood by a Taylor expansion near $x = 0$ for odd u, ω

$$u\omega_x = u_x(0)\omega_x(0)x + l.o.t.,\ u_x\omega = u_x(0)\omega_x(0)x + l.o.t.,\ u\omega_x \approx u_x\omega. \quad (1.17)$$

Formally, below the borderline case, i.e. $a < 1$ or $a = 1$ with C^α data, the advection is weaker than the vortex stretching, and blowup may occur, which is Conjectures (II) and (III).

1.4.2 Finite time blowup

Before our study, the only blowup result of (1.16) with $a > 0$ was established recently by Elgindi-Jeong [44], where they constructed smooth self-similar profiles for sufficiently small a and C^α self-similar profiles for all $a \in \mathbb{R}$ with $|a\alpha|$ sufficiently small using a fixed point argument. Note that the initial data in [44] do not have finite energy. There are other blowup results for (1.16), see e.g., [5, 26, 28, 49]. We refer to Chapter 3 for a detailed review.

Applying the framework in Section 1.3 and the numerical construction in Section 1.3.6.1, Hou, Huang, and I [19] have established the following result.

Theorem 1.3. *There exist some $C_c^\infty(\mathbb{R})$ initial data such that the solution of the DG model on $\mathbb{R}$ develops an expanding and asymptotically self-similar singularity in finite time with compactly supported self-similar profile $\Omega \in H^1(\mathbb{R})$.*

Our result resolves the open problem (I) in the case of $\mathbb{R}$, which has been open for quite a long time. It also resolves Conjecture (III) and can be seen as the endpoint case of Conjecture (II). This blowup result does not generalize to the case of a circle (S^1) due to the expanding nature of the blowup solution.

In [14], applying our framework, we proved singularity formation of (1.16) on S^1 with C^∞ initial data for any $a \in (1-\delta, 1)$ with some $\delta > 0$. It resolves the endpoint case of Conjecture (II) on S^1. On the other hand, for (1.16) with the same initial data and $a \in [1, 1+\delta)$, we showed that such a singular scenario does not persist by proving global regularity of (1.16). In [15], we further established the singularity formation of (1.15) on S^1 with $C^\alpha \cap H^s$ data and any $0 < \alpha < 1, s < \frac{3}{2}$. This result resolves Conjecture (III) on S^1. In these works, we need to carefully quantify that the advection is weaker than the vortex stretching since there is a strong competition between these two effects (1.17). The behavior of the singular solution is similar to

$$\omega_t = u_x\omega - au\omega_x \approx \omega^2 - \beta\omega^2 = (1-\beta)\omega^2, \quad \omega_0 > 0,$$

for $\beta = a$ in the first result, and $\beta = \alpha, a = 1$ in the second result. As long as $\beta < 1$, the blowup can occur. The key novelty of these works is that we give a sharp characterization of two effects and justify the above heuristic using our framework, which is non-trivial due to the nonlocality of the velocity u. The C^α result [15] relates to an important idea in [44] that the advection can be weakened by choosing some C^α data with sufficiently small α. However, we do not require α to be sufficiently small in [15].

1.4.3 Global regularity of the DG model

An important question is whether stronger advection can prevent the above singularity formations. In fact, in contrast to the above blowup results for the DG model and gCLM model with $a < 1$, it is conjectured that the DG model (gCLM with $a = 1$) on S^1 is globally well-posed for smooth initial data. See open problem (I) in Section 1.4.1. Due to the destabilizing effect of vortex stretching, the lack of a-priori conserved quantity, and dissipation in (1.15), this question has been open for quite a long time. There have been only a few results in recent years, see e.g., [73, 101] and more discussions in Chapter 3. In [73], Jia-Stewart-Sverak established global regularity for initial data near the equilibrium $\sin x$ by proving its nonlinear stability.

In [15], we considered initial data ω_0 with period π and in class X: ω_0 is odd and $\omega_0 \leq 0$ (or $\omega_0 \geq 0$) on $[0, \pi/2]$. The singularity formation of the DG model and gCLM model in Section 1.4.2 and [13–15, 19, 44, 49] all develops from initial data with the same sign and symmetry properties as those in X. Thus, to establish the global regularity of the DG model on S^1 with C^∞ initial data, we need to address the important question of whether there is a finite time blowup in this class. We proved the following result:

Theorem 1.4. *Suppose that $\omega_0 \in X \cap C^1$ and $\int_0^{\pi/2} \left| \frac{\omega_{0,x}^2}{\omega_0} \sin(2x) \right| dx < +\infty$. There exists a global solution ω of the DG model with initial data ω_0.*

We remark that $\int_0^{\pi/2} \left| \frac{\omega_{0,x}^2}{\omega_0} \sin(2x) \right| dx < +\infty$ is a mild assumption and holds true for generic data in X. See Chapter 5 for more discussion. This global regularity result and the blowup result from some C^α initial data in Section 1.4.2 characterize the regularity of (1.15) with initial data in X. Moreover, it justifies that stronger advection can prevent singularity formation of (1.15) and (1.16) on S^1 and confirms the principle below (1.14).

Recall the heuristic (1.17). The C^1 case is the borderline case, and we expect that the advection is (almost) stronger than the vortex stretching. To prove global regularity from C^1 data, we need to quantify this heuristic, which is very challenging since (1.15) is nonlocal and there is a transition from finite time blowup for C^α data to global regularity for C^1 data.

Quadratic form In the special case of $\omega_0 \in C^{1,\alpha} \cap X$ with $\omega_{0,x}(0) = 0$, which is the case above the borderline case (1.17), the key step is to establish

$$\frac{d}{dt}\int_0^{\frac{\pi}{2}} \omega \cot^2 x dx = \int_0^{\frac{\pi}{2}} (u_x\omega - u\omega_x) \cot^2 x dx \geq 0. \tag{1.18}$$

The quantity $Q(2,t) = \int_0^{\pi/2} \omega \cot^2 x dx$ distinguishes the C^α case and the case of $\omega \in C^{1,\alpha}$ with $\omega_x(0) = 0$ since $\omega \cot^2 x$ is not integrable for C^α data in general but is integrable in the other case. The above estimate quantifies that the stabilizing effect of advection is stronger than the effect of vortex stretching in some sense. We exploit the convolution structure in the quadratic form in (1.18) and use an idea from Bochner's theorem for a positive-definite function to establish (1.18). It implies that $|Q(2,t)|$ is monotone decreasing in time, which is an important *a-priori* estimate for (1.15).

One-point blow-up criterion Another key ingredient in the proof is a crucial one-point blow-up criterion for (1.15). Based on a novel equation discovered in [83], we prove that for $\omega_0 \in X \cap H^1$ with an additional mild assumption, the solution to (1.15) cannot be extended beyond T if and only if

$$\int_0^T |u_x(0,t)| dt = \infty. \tag{1.19}$$

The DG model satisfies the BKM-type blow-up criterion [1] that $\int_0^T ||\omega(t)||_\infty dt$ or $\int_0^T ||u_x(t)||_\infty dt$ controls the blowup of the solution. For initial data in X, criterion (1.19) improves the latter criterion significantly by replacing $||u_x||_\infty$ by one point $|u_x(0)|$. Based on the *a-priori* estimate of $Q(2,t)$ and (1.19), we establish global well-posedness for $\omega_0 \in C^{1,\alpha} \cap X$ with $\omega_{0,x}(0) = 0$.

The borderline case of $\omega_0 \in C^1$ (1.17) is much more challenging since the advection and vortex stretching are precisely balanced. We will study it in detail in Chapter 5.

1.4.4 Singularity formation with dissipation

A natural question is whether our framework can be applied to study the singularity formation of viscous fluid equations. In [13], we gave a positive answer by generalizing this framework to establish singularity formation of gCLM with a viscous term

$$\omega_t + au\omega_x = u_x\omega + \nu\Delta\omega, \quad u_x = H\omega$$

for a close to $\frac{1}{2}$ and $\nu \geq 0$. The novelty of our approach is first to show that the blowup in the inviscid case is stable and then to show that the viscous term is asymptotically small compared to the nonlinear terms under the blowup scaling.

Summary of the Thesis

The remaining thesis is organized as follows:

Singularity formation in incompressible fluids

In Chapter 2, we first review some basic properties of the 3D Euler equations and introduce the Hou-Luo scenario and the dynamic rescaling formulation. Then we consider the 2D Boussinesq equations and the 3D Euler equations in a setting similar to the Hou-Luo scenario. We follow the framework introduced in Section 1.3 to establish finite time blowup of these equations with boundary and $C^{1,\alpha}$ velocity. The most essential difficulty is establishing the linear stability analysis of the approximate steady state in Section 2.6. We need to perform energy estimates carefully and exploit several nonlocal cancellations in the system. The blowup analysis for $C^{1,\alpha}$ initial data captures certain essential features of the Hou-Luo scenario and some essential difficulties in analyzing such a scenario with smooth data.

Finite time blowup of the De Gregorio model on $\mathbb{R}$

In Chapter 3, we further develop the framework in Section 1.3 for smooth data and establish finite time asymptotically self-similar blowup of the DG model (1.15) on $\mathbb{R}$ from C_c^∞ initial data. Moreover, we prove finite time blowup of the gCLM model (1.16) with small $|a|$ from C_c^∞ initial data and with any $|a|$ from C_c^α initial data with small $|a\alpha|$. We will discuss how to construct the approximate steady state for the DG model numerically and incorporate it in the energy estimate rigorously in Section 3.5.

Asymptotically self-similar singularity of the Hou-Luo Model

In Chapter 4, we study the stability mechanism of the blowup in the Hou-Luo scenario with smooth data using the HL model. Based on our framework with numerical construction of the approximate steady state, we establish finite time asymptotically self-similar singularity of the HL model from a class of smooth initial data with compact support. Compared to the analysis of the DG model, the analysis of the HL model is much more complicated since it is a coupled system. The analysis of the HL model serves as an intermediate step toward the analysis of the full 2D Boussinesq equations with smooth data.

Competition between advection and vortex stretching

In Chapter 5, we investigate the DG model on S^1 to analyze the competition between advection and vortex stretching, an essential difficulty in studying the regularity of the 3D Euler equations. The behaviors of the solutions to the DG model on S^1 are much more complicated than those on $\mathbb{R}$. We consider initial data ω_0 with period π and in class X: ω_0 is odd and $\omega_0 \leq 0$ (or $\omega_0 \geq 0$) on $[0, \pi/2]$. The importance of this class of data has been discussed in Section 1.4.3. We prove global well-posedness for initial data $\omega_0 \in H^1 \cap X$ with $\omega_0(x)x^{-1} \in L^\infty$. On the other hand, for any $\alpha \in (0, 1)$, we construct a finite time blowup solution from a class of initial data with $\omega_0 \in C^\alpha \cap C^\infty(S^1 \backslash \{0\}) \cap X$. Our results imply that singularities developed in the DG model and the gCLM model on S^1 can be prevented by stronger advection.

In Chapter 6, we make some concluding discussions toward justifying the Hou-Luo scenario with smooth data. In addition, we discuss potential applications of our framework to other equations and list some related problems.

Chapter 2

SINGULARITY FORMATION IN INCOMPRESSIBLE FLUIDS

In this chapter, we will first present some preliminaries about 3D incompressible Euler equations, the Hou-Luo scenario, and the dynamic rescaling formulation in Section 2.1. These fundamental results relate to three important ingredients in studying singularity formation.

Firstly, the *a-priori* energy identity of the equations is not strong enough to control the nonlinear terms in the equations, and thus finite time blowup could occur. Secondly, it is useful to consider a specific scenario for potential singularity formation, where the solutions enjoy certain sign and symmetry properties. These properties are crucial to generating a stable finite time blowup for the equations and models we study in this thesis. We will consider the Hou-Luo scenario for the 2D Boussinesq and 3D Euler equations and similar scenarios for 1D models studied in Chapters 3-5. Thirdly, the 3D Euler equations and related equations and models enjoy some scaling symmetries. Based on the scaling symmetries, we can apply the dynamic rescaling formation and construct approximately self-similar blowup solutions. This approach allows us to obtain strong control of the solution up to the blowup time. See also Section 1.3.

We will discuss the main results and ideas in Section 2.2 and prove the main results in the rest of the chapter.

2.1 Preliminaries

2.1.1 The blow-up criterion for 3D Euler equations

The 3D Euler equations (1.1) are locally well-posed for initial data $\mathbf{u}_0 \in C^{1,\alpha}, \alpha > 0$ or $\mathbf{u}_0 \in H^s, s \geq 3$ [89]. The solution satisfies energy conservation

$$||u(t)||_{L^2} = ||u_0||_{L^2}. \tag{2.1}$$

The regularity of the solution is propagated and the celebrated Beale-Kato-Majda (BKM) blow-up criterion [1] states that the unique local solution to

(1.1) blows up at finite time T if and only if

$$\int_0^T ||\omega(\cdot,t)||_\infty dt = \infty. \tag{2.2}$$

The BKM criterion imposes a constraint on the blowup rate of $||\omega||_\infty$. For example, a blowup rate $||\omega||_\infty \sim C(T-t)^\beta$ with $\beta > -1$ is excluded since the associated integral is finite. Note that the *a-priori* estimate (2.1) is much weaker than the one required to control $||\omega||_\infty$. There are other blowup criteria [27, 35] based on geometric aspects of the 3D Euler flows.

2.1.2 Symmetry groups

There are several symmetry groups for the Euler equations, including the Galilean invariance, the rotation symmetry, and the scaling symmetry [89]. The latter two are of fundamental importance in studying the potential singularity formation of (1.1). Suppose that $\mathbf{u}(x,t)$ is a solution to (1.1).

Rotation symmetry. For any orthogonal matrix $Q \in \mathbb{R}^{3\times 3}$ ($Q^T = Q^{-1}$),

$$\mathbf{u}_Q(t,x) \triangleq Q^T\mathbf{u}(Qx,t)$$

is also a solution to (1.1).

Scaling symmetry. The Euler equations (1.1) enjoy the scaling symmetry with a two-parameter symmetry group. For any $\lambda, \tau \in \mathbb{R}$,

$$\mathbf{u}_{\lambda,\tau} \triangleq \frac{\lambda}{\tau}\mathbf{u}\Big(\frac{x}{\lambda}, \frac{t}{\tau}\Big) \tag{2.3}$$

is also a solution to (1.1).

2.1.2.1 Axisymmetric flows

The rotation symmetry implies an special class of solution to (1.1), the 3D axisymmetric flow. Denote by (r,θ,z) the cylindrical coordinate in $\mathbb{R}^3$: $r = (x^2+y^2)^{1/2}$, $\beta = \arctan(y/x)$, and e_r, e_θ, e_z the orthonormal unit vectors

$$e_r = (\cos\theta, \sin\theta, 0)^T, \quad e_\theta = (\sin\theta, -\cos\theta, 0)^T, \quad e_z = (0,0,1)^T.$$

The velocity field $\mathbf{u}$ is axisymmetric if

$$\mathbf{u} = u^r(r,z,t)e_r + u^\theta(r,z,t)e_\theta + u^z(r,z,t)e_z. \tag{2.4}$$

The velocity u^θ is called the swirl velocity. The associated vorticity is given by

$$\omega(r,z,t) = \nabla \times \mathbf{u} = \omega^r e_r + \omega^\theta e_\theta + \omega^z e_z,$$
$$\omega^r = -\partial_z u^r, \quad \omega_z = \frac{1}{r}(ru^\theta)_r, \quad \omega^\theta = \partial_z u^r - \partial_r u^z.$$

The rotation symmetry implies that the axisymmetry is preserved by the flow. Under the axisymmetric assumption, (1.1) reduces to the following system for $(u^\theta, \omega^\theta)$ on (r,z)

$$\begin{aligned} \partial_t(ru^\theta) + u^r(ru^\theta)_r + u^z(ru^\theta)_z &= 0, \\ \partial_t \frac{\omega^\theta}{r} + u^r(\frac{\omega^\theta}{r})_r + u^z(\frac{\omega^\theta}{r})_z &= \frac{1}{r^4}\partial_z((ru^\theta)^2). \end{aligned} \tag{2.5}$$

The specialty of the 3D axisymmetric Euler equation is that it has a strong connection to the 2D Boussinesq equations, and can be approximated by Boussinesq equations away from the symmetry axis $r = 0$ [89].

Thanks to the rotation symmetry, we can further impose a reflection symmetry on the solution to (2.5). In particular, the following symmetry in z is preserved by the dynamic

$$\omega^\theta(r,z) = -\omega^\theta(r,-z), \quad u^\theta(r,z) = u^\theta(r,-z). \tag{2.6}$$

The axisymmetry and the above reflection symmetry provides the most promising candidate for a potential blowup solution of (1.1) [86, 87]. See Section 1.2 for more discussions.

2.1.2.2 Scaling symmetry and self-similar singularity

Due to the scaling symmetry (2.3), it is natural to look for self-similar singularity of (1.1)

$$\omega(x,t) = \frac{1}{(T-t)^{c_\omega}}\Omega\Big(\frac{x-x_0}{(T-t)^{c_l}}\Big), \tag{2.7}$$

where c_ω, c_l are the blowup exponents, $x_0 \in \mathbb{R}^3$, $T > 0$ is the blowup time, and Ω is the blowup profile.

Self-similar singularities develop in many nonlinear PDEs governing physical phenomena. There are examples in the nonlinear Schrödinger equations [74, 81, 92, 93], the nonlinear wave equation [95], and the nonlinear heat equation [94]. See [41] for a survey of self-similar singularity.

There are some efforts in ruling out the self-similar singularity of the 3D Euler equations [8–10]. In [8, 9], the blowup is excluded if the blowup admits the power law $||\omega(t)||_\infty = O((T-t)^{-1})$, and the profile Ω has very fast decay [8] or is small in some specific norm [9]. Nevertheless, possibilities of self-similar singularity are not entirely ruled out. Recently, there has been important progress in constructing (asymptotically) self-similar singularity of (1.1) [16, 42, 86, 87]. We refer to Sections 1.2 for more discussions of these results.

The scaling symmetry is closely related to the dynamic rescaling formulation [81, 92] and the modulation technique [74], which will be discussed in Section 2.1.4.

2.1.3 The Hou-Luo scenario

We give more details about the Hou-Luo scenario [86, 87] introduced in Section 1.2 for the 3D axisymmetric Euler equations (2.5). The solution $(\omega^\theta, u^\theta)$ enjoys the axisymmetry (2.4) and the reflection symmetry in z (2.6). To understand the Hou-Luo scenario, we consider the Boussinesq equations in $\mathbb{R}_+^2$ (1.3).

The solution enjoys the following sign and symmetry properties

$$\begin{aligned} &\omega(x,y), \quad \theta(x,y), \quad \theta_x(x,y) > 0, \quad x > 0, \\ &\omega(x,y) = -\omega(-x,y), \quad \theta(x,y) = \theta(-x,y). \end{aligned} \tag{2.8}$$

The symmetry is the same as (2.6) under the connection $\omega \leftrightarrow \omega^\theta, \theta \leftrightarrow (u^\theta)^2$. The vorticity induces a clockwise flow in the first quadrant. Due to the symmetry in x, we obtain a hyperbolic flow in $\mathbb{R}_+^2$. The potential singularity occurs at the origin $(0,0)$, which is on the boundary and a stagnation point of the flow. Due to the no-flow boundary condition $v(0,y) = 0$, along the boundary, the fluid is compressed at the origin. See Figure 2.1 for an illustration.

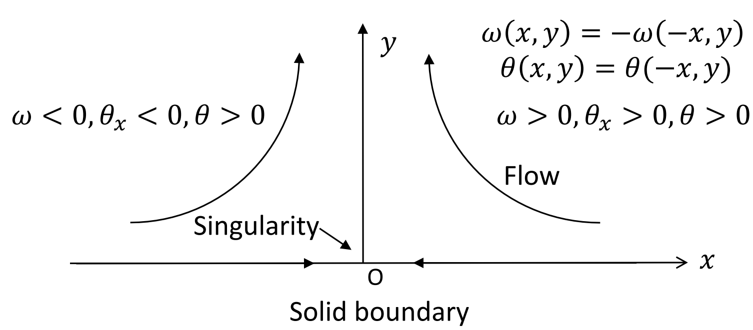

Figure 2.1: Illustration of the Hou-Luo scenario.

2.1.4 Dynamic rescaling formulation

Our framework of analysis is based on the scaling symmetry in Section 2.1.2 and the following dynamic rescaling formulation. Let $\omega(x,t), \theta(x,t), \mathbf{u}(x,t)$ be the solutions of (1.3). It is easy to show that

$$\begin{aligned} &\tilde{\omega}(x,\tau) = C_\omega(\tau)\omega(C_l(\tau)x, t(\tau)), \quad \tilde{\theta}(x,\tau) = C_\theta(\tau)\theta(C_l(\tau)x, t(\tau)), \\ &\tilde{\mathbf{u}}(x,\tau) = C_\omega(\tau)C_l(\tau)^{-1}\mathbf{u}(C_l(\tau)x, t(\tau)), \end{aligned} \tag{2.9}$$

are the solutions to the dynamic rescaling equations

$$\begin{aligned} &\tilde{\omega}_\tau(x,\tau) + (c_l(\tau)\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla\tilde{\omega} = c_\omega(\tau)\tilde{\omega} + \tilde{\theta}_x, \\ &\tilde{\theta}_\tau(x,\tau) + (c_l(\tau)\mathbf{x} + \tilde{\mathbf{u}}) \cdot \nabla\tilde{\theta} = 0, \end{aligned} \tag{2.10}$$

where $\mathbf{u} = (u,v)^T = \nabla^\perp(-\Delta)^{-1}\tilde{\omega}$, $\mathbf{x} = (x,y)^T$,

$$\begin{aligned} &C_\omega(\tau) = \exp\left(\int_0^\tau c_\omega(s)d\tau\right), \quad C_l(\tau) = \exp\left(\int_0^\tau -c_l(s)ds\right), \\ &C_\theta(\tau) = \exp\left(\int_0^\tau c_\theta(s)d\tau\right), \quad t(\tau) = \int_0^\tau C_\omega(\tau)d\tau, \end{aligned} \tag{2.11}$$

and the rescaling parameter $c_l(\tau), c_\theta(\tau), c_\omega(\tau)$ satisfies

$$c_\theta(\tau) = c_l(\tau) + 2c_\omega(\tau). \tag{2.12}$$

Let us explain the above relation. Using (2.9) and (2.11), we have

$$\tilde{\mathbf{u}} \cdot \nabla\tilde{\omega} = C_\omega(\tau)^2\mathbf{u} \cdot \nabla\omega, \quad \tilde{\theta}_x = C_\theta(\tau)C_l(\tau)\theta_x.$$

To obtain (2.10) from (1.3), we require that the scaling factors of $\tilde{\mathbf{u}} \cdot \nabla\tilde{\omega}$ and $\tilde{\theta}_x$ are the same, which implies $C_\omega(\tau)^2 = C_\theta(\tau)C_l(\tau)$. Using this relationship and (2.11), we obtain (2.12).

The Boussinesq equations have the same scaling-invariant property as (2.3) with two parameters. We have the freedom to choose the time-dependent scaling parameters $c_l(\tau)$ and $c_\omega(\tau)$ according to some normalization conditions. After we determine the normalization conditions for $c_l(\tau)$ and $c_\omega(\tau)$, the dynamic rescaling equation (2.10) is completely determined and the solution of (2.10) is equivalent to that of the original equation using the scaling relationships in (2.9)-(2.11), as long as $c_l(\tau)$ and $c_\omega(\tau)$ remain finite.

The dynamic rescaling formulation was introduced in [81, 92] to study the self-similar blowup of the nonlinear Schrödinger equations. This formulation is closely related to the modulation technique, which has been developed by Merle, Raphael, Martel, Zaag and others. It has been a very effective tool to analyze singularity formation for many problems like the nonlinear Schrödinger equation [74, 93], the nonlinear wave equation [95], the nonlinear heat equation [94], the generalized KdV equation [90], and other dispersive problems. Recently, this method has been applied to study singularity formation in the 3D Euler and 2D Boussinesq equations [16, 42, 48], the De Gregorio model [15, 19], the gCLM model [14, 19, 49], and the HL model for 3D Euler equations [20].

If there exists $C > 0$ such that for any $\tau > 0$, $c_\omega(\tau) \leq -C < 0$ and the solution $\tilde{\omega}$ is nontrivial, e.g., $||\tilde{\omega}(\tau, \cdot)||_{L^\infty} \geq c > 0$ for all $\tau > 0$, we then have

$$C_\omega(\tau) \leq e^{-C\tau}, \ t(\infty) \leq \int_0^\infty e^{-C\tau} d\tau = C^{-1} < +\infty \ ,$$

and

$$|\omega(C_l(\tau)x, t(\tau))| = C_\omega(\tau)^{-1} |\tilde{\omega}(x,\tau)| \geq e^{C\tau} |\tilde{\omega}(x,\tau)|$$

blows up at finite time $T = t(\infty)$. This corresponds to the heuristic (1.4) in the Introduction, Chapter 1. If $(\tilde{\omega}(\tau), \tilde{\theta}(\tau), c_l(\tau), c_\omega(\tau), c_\theta(\tau))$ converges to a steady state $(\omega_\infty, \theta_\infty, c_{l,\infty}, c_{\omega,\infty}, c_{\theta,\infty})$ of (2.10) as $\tau \to \infty$, one can verify that

$$\begin{aligned} \omega(x,t) &= \frac{1}{1-t} \omega_\infty \left(\frac{x}{(1-t)^\gamma} \right), \quad \gamma = -\frac{c_{l,\infty}}{c_{\omega,\infty}}, \\ \theta(x,t) &= \frac{1}{(1-t)^{c_{\theta,\infty}/c_{\omega,\infty}}} \theta_\infty \left(\frac{x}{(1-t)^\gamma} \right), \end{aligned} \tag{2.13}$$

is a self-similar solution of (1.3). Due to this connection, we will not distinguish the (approximate) steady state of the dynamic rescaling equations and the (approximate) self-similar profile of the original equations throughout the thesis.

To simplify our presentation, we still use t to denote the rescaled time in the rest of the thesis and drop $\tilde{\cdot}$ in (2.10). A similar formulation and transform apply to other equations, including the 3D Euler equations, the DG model, the gCLM model, and the HL model for 3D Euler equations.

2.1.5 Basic notations

Throughout this thesis, we use the notation $A \lesssim B$ if there is some absolute constant $C > 0$ with $A \leq CB$, and denote $A \asymp B$ if $A \lesssim B$ and $B \lesssim A$. The notation $\bar{\cdot}$ is reserved for the approximate steady states, e.g., $\bar{\Omega}$ denotes the approximate steady state for Ω. We will use C, C_1, C_2 for some absolute constant, which may vary from line to line. We use $K_1, K_2, ...$ and $\mu_1, \mu_2, ...$ to denote some absolute constant which does not vary.

2.2 Main results and ideas

2.2.1 Main results

The main results of this chapter are summarized by the following two theorems. In our first main result, we prove finite time blowup of the Boussinesq equations with $C^{1,\alpha}$ initial data for the velocity field and the density.

Theorem 2.1. *Let ω be the vorticity and θ be the density in the 2D Boussinesq equations described by* (2.16)-(2.18)*. There exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$, the unique local solution of the 2D Boussinesq equations in the upper half plane develops a focusing asymptotically self-similar singularity in finite time for some initial data $\omega \in C_c^\alpha(\mathbb{R}_+^2), \theta \in C_c^{1,\alpha}(\mathbb{R}_+^2)$. In particular, the velocity field is $C^{1,\alpha}$ with finite energy. Moreover, the self-similar profile $(\omega_\infty, \theta_\infty)$ satisfies $\omega_\infty, \nabla\theta_\infty \in C^{\frac{\alpha}{40}}$.*

By asymptotically self-similar, we mean that the solution in the dynamic rescaling equations (see Definition in Section 2.1.4) converges to the self-similar profile in a suitable norm. We will specify the norm in the convergence in Section 2.9.6.3.

In our second result, we prove the finite time singularity formation for the 3D axisymmetric Euler equations with large swirl in a cylinder $D = \{(r, z) : r \leq 1, z \in \mathbb{T}\}$ that is periodic in z (axial direction) with period 2, where r is the radial variable and $\mathbb{T} = \mathbb{R}/(2\mathbb{Z})$.

Theorem 2.2. *Consider the 3D axisymmetric Euler equations in the cylinder $r, z \in [0, 1] \times \mathbb{T}$. Let ω^θ be the angular vorticity and u^θ be the angular velocity. There exists $\alpha_0 > 0$ such that for $0 < \alpha < \alpha_0$, the unique local solution of the 3D axisymmetric Euler equations given by* (2.178)-(2.180) *develops a singularity in finite time for some initial data $\omega^\theta \in C^\alpha(D), u^\theta \in C^{1,\alpha}(D)$ supported away from the axis $r = 0$ with $u^\theta \geq 0$. In particular, the velocity field in each period has finite energy.*

Our analysis shows that the singular solution in Theorem 2.2 in the dynamic rescaling formulation remains very close to an approximate blowup profile in some norm (see Section 2.10) for all time (or up to the blowup time in the original formulation). It is conceivable that it converges to a self-similar blowup profile of 2D Boussinesq at the blowup time so that the blowup solution is asymptotically self-similar. However, we cannot prove this result using the current analysis since the domain D is not invariant under dilation. We leave it to our future work.

In the recent works [43, 45], Elgindi and Jeong proved finite time singularity formation for the 2D Boussinesq and 3D axisymmetric equations in a physical domain with a corner and $\mathring{C}^{0,\alpha}$ data. The domain we study in this chapter does not have a corner. In the case of the 3D Euler equations, our physical domain includes the symmetry axis. In comparison, the domain studied in [43] does not include the symmetry axis.

2.2.2 Main ingredients in our analysis

Our analysis follows the framework developed by Elgindi in [42] and our framework introduced in Section 1.3. We use the Boussinesq equations to illustrate the main ideas in our analysis.

Guided by the framework in Section 1.3, we reformulate the equations using an equivalent dynamic rescaling formulation (see e.g., [81, 92]). We follow [42] to derive the leading order system. In the derivation, we have used the argument in [42] to obtain the leading order approximation of the stream function for small α. Moreover, as observed by Elgindi and Jeong in [44] (see also [42]), the advection terms are relatively small compared with the nonlinear vortex stretching term when we work with C^α solution with small α for vorticity or $\nabla\theta$, which vanishes weakly near the origin, e.g., $|x|^\alpha$. In the 2D Boussinesq equations (2.16)-(2.17), the vortex stretching term for the ω equation is given by θ_x. Within the above C^α class of solution, the transport term $\mathbf{u}\cdot\nabla\omega$ may not be smaller than θ_x. For example, one can choose ω, θ so that $\mathbf{u}\cdot\nabla\omega = O(1)$ and $\theta_x = O(1)$. We further look for solutions of the 2D Boussinesq equation (2.16)-(2.17) by letting $\omega = \alpha\tilde{\omega}$, $\theta = \alpha\tilde{\theta}$ with $\tilde{\omega} = O(1)$ and $\tilde{\theta} = O(1)$ as $\alpha \to 0$. Formally, the nonlinear transport term $\mathbf{u}\cdot\nabla\omega$ becomes relatively small compared with θ_x due to the weakening effect of advection for C^α data and the weak nonlinear effect due to the fact that $\omega = O(\alpha)$ and $\theta_x = O(\alpha)$ for small α at a given time. Thus, we can ignore the contributions from the advection terms for small α when we work with this class of ω and θ. See more discussion in Section 2.3.4. In addition, inspired by our own computation of the Hou-Luo singularity scenario [86, 87], we look for θ that is anisotropic in the sense that θ_y is small compared with θ_x. We will justify that this property is preserved dynamically for our singular solution. As a result, we can decouple the θ_y equation from the leading order equations for ω and θ_x. This gives rise to a leading order coupled system of Riccati type for ω and θ_x, which is similar to the scalar leading order equation obtained in [42]. Inspired by the solution structure of the leading order system in [42], we are able to find a class of closed form solutions of this leading order system.

The most essential part of our analysis is to establish linear stability of the approximate steady state using the dynamic rescaling equations. Following the strategies in Section 1.3.1, we design some singular weights to extract the damping effect from the linearized operator around the approximate steady state. In order for the perturbation from the approximate steady state to be well defined in the weighted norm with a more singular weight, we impose some vanishing conditions on the perturbation at the origin by choosing some normalization conditions. This leads to some nonlocal terms related to the scaling parameter c_ω in the linearized equations, which are not present in [42].

Compared with the scalar linearized equation considered in [42], the linearized equations for the 2D Boussinesq equations lead to a more complicated coupled system and we need to deal with a few more nonlocal terms that are of $O(1)$ as $\alpha \to 0$. Thus we cannot apply the coercivity estimate of the linearized operator in [42], which is one of the key steps in constructing the self-similar solution in [42]. One of the main difficulties in our linear stability analysis is to control the nonlocal terms. If we use a standard energy estimate to handle these nonlocal terms, we will over-estimate their contributions to the linearized equations and would not be able to obtain the desired linear stability result. Since the damping term has a relatively small coefficient, we need to exploit the coupling structure in the system and take into account the cancellation among different nonlocal interaction terms in order to obtain linear stability. For this purpose, we design our singular weights that are adapted to the approximate self-similar profile and contain different powers of R^{-k} to account the interaction in the near field, the intermediate field and the far field. To control the nonlocal scaling parameter c_ω, we will derive a separate ODE for c_ω, which captures the damping effect of c_ω.

We have used the elliptic estimate and several nonlinear estimates from [42] in our nonlinear stability analysis. The presence of swirl (the angular velocity u^θ) or density (θ) introduces additional technical difficulties. Since the approximate steady state for $\nabla\theta$ does not decay in certain direction, we need to design different weighted Sobolev spaces carefully for different derivatives and further develop several nonlinear estimates. To obtain the L^∞ estimate of a directional derivative of θ, which is necessary to close the nonlinear stability analysis, we make use of the hyperbolic flow structure. Once we obtain nonlinear stability, using the ideas in Section 1.3.3, we establish finite time blowup from a class of compactly supported initial data ω_0 and θ_0 with finite energy by truncating the approximate steady state and using a rescaling argument. We further establish convergence of the solution of the dynamic rescaling equations to the self-similar profile using a time-differentiation argument introduced in Section 1.3.4. This argument has also been used in our recent joint work with Hou and Huang in [19] and developed independently in [42].

2.2.3 From the 2D Boussinesq to the 3D Euler equations

For the 3D Euler equations, we consider the domain within one period, i.e. $D_1 = \{(r,z) : r \in [0,1], |z| \leq 1\}$. We will construct a singular solution that is supported near $r = 1, z = 0$ up to blowup time and blows up at $r = 1, z = 0$. Since the support is away from the symmetry axis, we show that the 3D Euler equations are essentially the same as the 2D Boussinesq equations up to some lower order terms. This connection is well known; see e.g., [89]. Then we generalize the proof of Theorem 2.1 to prove Theorem 2.2. To justify this connection rigorously, we need two steps. The first step is to establish the elliptic estimates in the new domain. The second step is to control the support of the solution and show that it remains close to $r = 1, z = 0$ up to the blowup time.

2.2.3.1 Control of the support

The reason that the support of the singular solution remains close to $(r,z) = (1,0)$ is due to the following properties of the singular solution. Firstly, the singular solution is focusing, which is characterized by the rescaling parameters $c_l(\tau) > \frac{1}{2\alpha}$ for all $\tau > 0$. See the definition of c_l in Section 2.1.4. Secondly, the velocity in the dynamic rescaling formulation has sublinear growth in the support of the solution. These properties hold for the singular solution of the

2D Boussinesq equations. We prove that they remain true for the 3D Euler in Section 2.10. Using these properties, we derive an ODE to control the size of the support and show that it remains small up to the blowup time. See more discussion in Section 2.10.3.5. Similar ideas and estimates to control the support have been used in [19] to generalize the singularity formation of the De Gregorio type model from the real line to a circle.

2.2.3.2 The elliptic estimates

The elliptic equation for the stream function $\tilde{\psi}(r,z)$ in D_1 reads

$$\mathcal{L}\tilde{\psi} \triangleq -(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz})\tilde{\psi} + \frac{1}{r^2}\tilde{\psi} = \omega^\theta, \tag{2.14}$$

where ω^θ is the angular vorticity. We impose the periodic boundary condition in z and a no-flow boundary condition on $r = 1 : \tilde{\psi}(1,z) = 0$. See [86, 89]. Since the solution is supported near $r = 1, z = 0$, we will only use $\tilde{\psi}(r,z)$ for (r,z) near $(1,0)$ in our analysis. In this case, $r^{-1} \approx 1$ and the term $-\frac{1}{r}\partial_r\tilde{\psi} + \frac{1}{r^2}\tilde{\psi}$ in $\mathcal{L}\tilde{\psi}$ is of lower order compared with $\partial_{rr}\tilde{\psi} + \partial_{zz}\tilde{\psi}$. In the dynamic rescaling equations, we obtain a small factor $C_l(\tau)$ for the term $-\frac{1}{r}\partial_r\tilde{\psi} + \frac{1}{r^2}\tilde{\psi}$ and treat it as a perturbation in $\mathcal{L}\tilde{\psi}$. Moreover, if we relabel the variables (r,z) as (y,x) in $\mathbb{R}^2$, we formally have $\mathcal{L}\tilde{\psi} \approx -\Delta_{2D}\tilde{\psi}$. In Section 2.10.2, we will justify this connection rigorously and then generalize the elliptic estimates that we obtain for the 2D Boussinesq to the 3D Euler equations.

2.2.4 Connections to the Hou-Luo scenario

Many settings of our problem are similar to those considered in [86, 87]. See more discussions after Lemma 2.4.1. The driving mechanism for the finite time singularity that we consider in this chapter is essentially the same as that for the 3D axisymmetric Euler equations with solid boundary considered in [86, 87]. In both cases, the swirl (the angular velocity u^θ) and the boundary play an essential role in generating a sustainable finite time singularity. It is the strong compression of the angular velocity u^θ toward the symmetry plane $z = 0$ along the axial (z) direction on the boundary $r = 1$ that creates a large gradient in u^θ. Then the nonlinear forcing term $\partial_z(u^\theta)^2$ induces a rapid growth in the angular vorticity ω^θ, ultimately leading to a finite time blowup. Moreover, the singularities that we consider occur at the solid boundary, which are the same as the one reported in [86, 87].

We would like to emphasize that the presence of boundary plays a crucial role in the singularity formation even with $C^{1,\alpha}$ initial data for the velocity and θ. If we remove the boundary, a promising potential blowup scenario for the 2D Boussinesq equation is to have a hyperbolic flow structure near the origin with 4-fold symmetry for θ, i.e. θ is odd in y and even in x. Similar scenario has been used in [106]. Since $\theta(x,y)$ is odd with respect to y and $\theta \in C^{1,\alpha}$, a typical θ is of the form: $\theta(x,y) \approx c_1 \alpha x^{1+\alpha} y + l.o.t., c_1 \neq 0$ near the origin. From our derivation and analysis of the leading order system, it is the nonlinear coupling between ω and θ_x that generates the blow-up mechanism. However, without the boundary, $\theta_x \approx c_1\alpha(1+\alpha)x^\alpha y + l.o.t.$ and it does not vanish to the order $O(|y|^\kappa)$ near $y = 0$ with a small exponent $\kappa > 0$. The advection of θ_x along the y direction is not small compared with the vortex stretching term $-u_x\theta_x$ in the θ_x equation (2.21). Thus, we can no longer neglect the contribution from the y advection term and we cannot derive our leading order system in this case. In fact, the transport of θ_x along the y direction provides a strong destabilizing effect to the singularity formation and would likely destroy the self-similar focusing blowup mechanism [67, 68].

If we approximate the velocity field (u, v) by $(xu_x(0,0,t), yv_y(0,0,t))$ (note that $u_x(0,0,t) + v_y(0,0,t) = 0$) as was done in a toy model introduced in [42], we have the following result. For any $\omega_0, \nabla\theta_0 \in C_c^\alpha(\mathbb{R}^2)$, which is in the local well-posedness class for the 2D Boussinesq equations [11], under the 4-fold symmetry assumption, the solution of the toy model exists globally. The key point is that due to the odd symmetry of θ_0 with respect to y and the assumption that $\theta_0 \in C^{1,\alpha}$, θ_0 must vanish linearly in y, i.e. $|\theta(x,y)| \lesssim |y|$. The proof follows an estimate similar to that presented in [42] and we defer it to Appendix A.0.8.

In the presence of the boundary $(y = 0)$, θ can be nonzero on $y = 0$, which removes the above constraint $|\theta(x,y)| \lesssim |y|$. Then we can further weaken the transport terms in the 2D Boussinesq as discussed in Section 2.2.2. Although the leading order system for the 2D Boussinesq equations and the 3D Euler equations with $C^{1,\alpha}$ initial velocity and the boundary looks qualitatively similar to that for the 3D Euler equations without swirl and without boundary obtained in [42], the physical driving mechanisms of the finite time singularity behind these two blowup scenarios are quite different. In our case, the swirl and the boundary play a crucial role. Our numerical study suggests that even for smooth initial data, θ_x is an order of magnitude larger than θ_y and the effect of advection is relatively weak compared with the vortex stretching term. More importantly, Liu [85] provided a strong evidence that the linearized operator should be stable even for smooth initial data. See Section 1.3.1 for the discussion. The essential step in proving this rigorously is the linear stability analysis, which requires us to estimate the Biot-Savart law without the availability of the leading order structure for $C^{1,\alpha}$ velocity and control a few more nonlocal terms that we can neglect using the $C^{1,\alpha}$ initial data. In some sense, our blowup analysis for $C^{1,\alpha}$ initial data captures certain essential features of the Hou-Luo scenario [86, 87] and some essential difficulties in analyzing such a scenario.

Organization of the Chapter The rest of the chapter is organized as follows. In Sections 2.3-2.5, we provide some basic set-up for our analysis, including the derivation of the leading order system, the dynamic rescaling formulation, the reformulation using the polar coordinates (R, β), and the construction of the approximate self-similar solution. Section 2.6 is devoted to the linear stability analysis of the leading order system. In Section 2.7, we perform higher order estimates of the leading order system as part of the nonlinear stability analysis. Sections 2.8 and 2.9 are devoted to the nonlinear stability analysis of the original system. In Section 2.10, we extend our analysis for the 2D Boussinesq equations to the 3D axisymmetric Euler equations. Some concluding remarks are provided in Chapter 6 and some technical estimates are deferred to Appendix A.

Notations We use $\langle \cdot, \cdot \rangle, || \cdot ||_{L^2}$ to denote the inner product in (R, β) and its L^2 norm

$$\langle f, g \rangle = \int_0^\infty \int_0^{\pi/2} f(R,\beta) g(R,\beta) dR d\beta, \quad ||f||_{L^2} = \sqrt{\langle f, f \rangle}. \tag{2.15}$$

We also simplify $|| \cdot ||_{L^2}$ as $|| \cdot ||_2$. We remark that we use $dRd\beta$ in the definition of the inner product rather than $RdRd\beta$.

2.3 Derivation of the leading order system

In this section, we will derive the leading order system used for our analysis later in the chapter. We first recall that the 2D Boussinesq equations on the upper half space are given by the following system:

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \theta_x, \tag{2.16}$$

$$\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \tag{2.17}$$

where the velocity field $\mathbf{u} = (u, v)^T : \mathbb{R}^2_+ \times [0, T) \to \mathbb{R}^2_+$ is determined via the Biot-Savart law

$$-\Delta \psi = \omega, \quad u = -\psi_y, \quad v = \psi_x, \tag{2.18}$$

with no flow boundary condition

$$\psi(x, 0) = 0 \quad x \in \mathbf{R}$$

and ψ is the stream function. The reader should not confuse the vector field $\mathbf{u}$ with its first component u.

The 2D Boussinesq equations have the following scaling-invariant property. If (ω, θ) is a solution pair to (2.16)-(2.18), then

$$\omega_{\lambda,\tau}(x, t) = \frac{1}{\tau}\omega\left(\frac{x}{\lambda}, \frac{t}{\tau}\right), \quad \theta_{\lambda,\tau}(x, t) = \frac{\lambda}{\tau^2}\theta\left(\frac{x}{\lambda}, \frac{t}{\tau}\right) \tag{2.19}$$

is also a solution pair to (2.16)-(2.18) for any $\lambda, \tau > 0$.

Next, we follow the ideas in Section 2.2.2 to derive the leading order system for the solutions $\omega, \nabla\theta \in C^\alpha$ with small α.

2.3.1 The setup

We look for a solution of (2.16)-(2.18) with the following symmetry

$$\omega(x, y) = -\omega(x, -y), \quad \theta(x, y) = \theta(-x, y)$$

for all $x, y \geq 0$. Accordingly, the stream function ψ (2.18) is odd with respect to x

$$\psi(x, y) = -\psi(-x, y).$$

It is easy to see that the equations (2.16)-(2.18) preserve these symmetries during time evolution. With these symmetries, it suffices to solve (2.16)-(2.18) on $(x, y) \in [0, \infty) \times [0, \infty)$ with the following boundary conditions

$$\psi(x, 0) = \psi(0, y) = 0$$

for the elliptic equation (2.18).

Taking x, y derivative on (2.17), respectively, we obtain

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \theta_x, \tag{2.20}$$

$$\theta_{xt} + \mathbf{u} \cdot \nabla \theta_x = -u_x \theta_x - v_x \theta_y, \tag{2.21}$$

$$\theta_{yt} + \mathbf{u} \cdot \nabla \theta_y = -u_y \theta_x - v_y \theta_y. \tag{2.22}$$

Under the odd symmetry assumption, we have $u(0, y) = 0$. If the initial data $\theta(0, y) = 0$, this property is preserved. Therefore, we can recover θ from θ_x by integration. We will perform a-prior estimate of the above system, which is formally a closed system for $(\omega, \theta_x, \theta_y)$.

2.3.2 Reformulation using polar coordinates

Next, we reformulate (2.20)-(2.22) using the polar coordinates introduced by Elgindi in [42]. We assume that $\alpha < 1/10$. We introduce

$$r = \sqrt{x^2 + y^2}, \quad \beta = \arctan(y/x), \quad R = r^\alpha,$$

Notice that $r\partial_r = \alpha R \partial_R$. We denote

$$\begin{aligned} &\Omega(R, \beta, t) = \omega(x, y, t), \quad \Psi = \frac{1}{r^2}\psi, \quad \eta(R, \beta, t) = (\theta_x)(x, y, t), \\ &\xi(R, \beta, t) = (\theta_y)(x, y, t). \end{aligned} \tag{2.23}$$

We have

$$\begin{aligned} \partial_x &= \cos(\beta)\partial_r - \frac{\sin(\beta)}{r}\partial_\beta = \frac{\cos(\beta)}{r}\alpha R \partial_R - \frac{\sin(\beta)}{r}\partial_\beta, \\ \partial_y &= \sin(\beta)\partial_r + \frac{\cos(\beta)}{r}\partial_\beta = \frac{\sin(\beta)}{r}\alpha R \partial_R + \frac{\cos(\beta)}{r}\partial_\beta. \end{aligned} \tag{2.24}$$

Then using (2.18), we derive

$$\begin{aligned} u &= -(r^2\Psi)_y = -2r\sin\beta\Psi - \alpha r R \sin\beta \partial_R \Psi - r\cos\beta\partial_\beta\Psi, \\ v &= (r^2\Psi)_x = 2r\cos\beta\Psi + \alpha r R\cos\beta\partial_R\Psi - r\sin\beta\partial_\beta\Psi. \end{aligned} \tag{2.25}$$

Using the new variables R, β, we can reformulate the Biot-Savart law (2.18) as

$$-\alpha^2 R^2 \partial_{RR}\Psi - \alpha(4 + \alpha)R\partial_R\Psi - \partial_{\beta\beta}\Psi - 4\Psi = \Omega \tag{2.26}$$

with boundary condition

$$\Psi(R, 0) = \Psi(R, \frac{\pi}{2}) = 0.$$

For the transport term in (2.20)-(2.22), we use (2.24) to derive

$$u\partial_x + v\partial_y \to -(\alpha R\partial_\beta \Psi)\partial_R + (2\Psi + \alpha R\partial_R \Psi)\partial_\beta. \tag{2.27}$$

Recall the notations Ω, η, ξ in (2.23) for $\omega, \theta_x, \theta_y$ in the (R, β) coordinates. Using (2.27), we can rewrite (2.20)-(2.22) in (R, β) coordinates as follows

$$\Omega_t + \Big(-(\alpha R\partial_\beta \Psi)\partial_R + (2\Psi + \alpha R\partial_R \Psi)\partial_\beta \Big)\Omega = \eta, \tag{2.28}$$

$$\eta_t + \Big(-(\alpha R\partial_\beta \Psi)\partial_R + (2\Psi + \alpha R\partial_R \Psi)\partial_\beta \Big)\eta = -u_x\eta - v_x\xi, \tag{2.29}$$

$$\xi_t + \Big(-(\alpha R\partial_\beta \Psi)\partial_R + (2\Psi + \alpha R\partial_R \Psi)\partial_\beta \Big)\xi = -u_y\eta - v_y\xi. \tag{2.30}$$

The formulas of ∇u in (R, β) coordinates are rather lengthy and presented in (2.150).

2.3.3 Leading order approximations of the Biot-Savart law and the velocity

Next, we use an important result of Elgindi in [42] to obtain a leading order approximation of the modified stream function. Using this approximation, we can simplify the transport terms and ∇u, and further derive the the leading order system of (2.28)-(2.30).

Following [42], we decompose the modified stream function Ψ as follows

$$\begin{aligned} \Psi &= \frac{1}{\pi\alpha}\sin(2\beta)L_{12}(\Omega) + \text{ lower order terms}, \\ L_{12}(\Omega) &= \int_R^\infty \int_0^{\pi/2} \frac{\sin(2\beta)\Omega(s,\beta)}{s} ds d\beta. \end{aligned} \tag{2.31}$$

For $\omega \in C^\alpha$ with sufficiently small $\alpha > 0$, the leading order term in Ψ is given by the first term on the right hand side. The lower order terms (l.o.t.) are relatively small compared to the first term and we will control them later using the elliptic estimates. We will perform the L^2 estimate for the solution of (2.26) and one can see that the a-priori estimate blows up as $\alpha \to 0$. For $\alpha = 0$, (2.26) becomes

$$L_0(\Psi) = -\partial_{\beta\beta}\Psi - 4\Psi,$$

with boundary conditions $\Psi(R, 0) = \Psi(R, \pi/2) = 0$, which is self-adjoint and has kernel $\sin(2\beta)$. In this case, to solve $L_0(\Psi) = \Omega$, a necessary and sufficient condition is that Ω is orthogonal to $\sin 2\beta$. Imposing this constraint when we perform the elliptic estimate leads to the leading order term in Ψ (2.31).

Following the same procedure as in [42], we drop the $O(\alpha)$ terms in (2.24), (2.25) and the lower order terms in (2.31) to extract the leading order term of the velocity u, v

$$\begin{aligned} u &= -\frac{2r\cos\beta}{\pi\alpha}L_{12}(\Omega) + l.o.t., \quad v = \frac{2r\sin\beta}{\pi\alpha}L_{12}(\Omega) + l.o.t., \\ u_x &= -v_y = -\frac{2}{\pi\alpha}L_{12}(\Omega) + l.o.t., \quad u_y = l.o.t., \quad v_x = l.o.t. \end{aligned} \tag{2.32}$$

The complete calculation and the formulas of the lower order terms are given in (2.150)-(2.152). Similarly, the leading order term in the transport terms (2.27) is

$$\begin{aligned} &-(\alpha R\partial_\beta\Psi)\partial_R + (2\Psi + \alpha R\partial_R\Psi)\partial_\beta \\ &= -\frac{2}{\pi}\cos(2\beta)L_{12}(\Omega)R\partial_R + \frac{2}{\pi\alpha}\sin(2\beta)L_{12}(\Omega)\partial_\beta + l.o.t. \end{aligned} \tag{2.33}$$

Later on, we will prove that the self-similar blowup is non-linearly stable and we will control the above lower order terms using the elliptic estimates. These terms will be treated as small perturbations and are harmless to the self-similar blowup.

2.3.4 Decoupling and simplifying the system

We will look for solution θ of (2.20)-(2.22) (or equivalently (2.28)-(2.30)) such that $\theta_x \in C^\alpha$, θ_x is odd, and θ_y is relatively small compared to θ_x, i.e. θ is *n*ot isotropic. The anisotropic property of θ will enable us to further simplify (2.28)-(2.30). The reason that we have this property is due to the following key observation. For the purpose of illustration, we construct a function θ that has the same qualitative feature as our solution θ. We first construct θ_x of the form: $\theta_x = \frac{x^\alpha}{1+(x^2+y^2)^{\alpha/2}}$ for $x, y \geq 0$. Then for x, y close to 0, we have

$$\theta \approx \frac{1}{1+\alpha}\cdot\frac{x^{1+\alpha}}{1+(x^2+y^2)^{\alpha/2}}, \quad |\theta_y| \approx \left|\frac{\alpha}{1+\alpha}\cdot\frac{xy}{x^2+y^2}\cdot\frac{x^\alpha(x^2+y^2)^{\alpha/2}}{(1+(x^2+y^2)^{\alpha/2})^2}\right| \lesssim \alpha\theta_x. \tag{2.34}$$

Compared to θ_x, θ_y is relatively small. Equivalently, ξ is small relative to η. Moreover, ξ is weakly coupled with Ω, η in (2.28)-(2.29) since $v_x = l.o.t.$ according to (2.32). Hence, we can decouple ξ from the η equation in (2.29) as follows

$$\eta_t + \Big(-(\alpha R\partial_\beta\Psi)\partial_R + (2\Psi + \alpha R\partial_R\Psi)\partial_\beta\Big)\eta = -u_x\eta + l.o.t.$$

These key observations motivate us to focus on the system (2.28)-(2.29) about Ω, η.

Using the calculations of ∇u (2.32), the transport terms (2.33) and treating ξ (θ_y) as a lower order term, we can simplify (2.28)-(2.30) as follows

$$\Omega_t - \frac{2}{\pi}\cos(2\beta)L_{12}(\Omega)R\partial_R\Omega + \frac{2}{\pi\alpha}\sin(2\beta)L_{12}(\Omega)\partial_\beta\Omega = \eta + l.o.t., \tag{2.35}$$

$$\eta_t - \frac{2}{\pi}\cos(2\beta)L_{12}(\Omega)R\partial_R\eta + \frac{2}{\pi\alpha}\sin(2\beta)L_{12}(\Omega)\partial_\beta\eta = \frac{2}{\pi\alpha}L_{12}(\Omega)\eta + l.o.t., \tag{2.36}$$

where the equations are evaluated at (R, β) with $R = (x^2 + y^2)^{\alpha/2}, \beta = \arctan(y/x)$. Notice that in (2.36), the first transport term looks much smaller than the other transport term and the nonlinear term which contains a $1/\alpha$ factor. Thus we can ignore it in our leading order approximation. For the angular transport term, we use an argument introduced in [42] and look for approximate solutions (Ω, η) of the form

$$\Omega(R, \beta, t) = \alpha\Gamma(\beta)\Omega_*(R, t), \quad \eta(R, \beta, t) = \alpha\Gamma(\beta)\eta_*(R, t), \quad \Gamma(\beta) = (\cos(\beta))^\alpha. \tag{2.37}$$

We have added the factor α in the above form, which is slightly different from [42]. For $\beta \in [0, \pi/2]$, we gain a small factor α from the angular derivative:

$$|\sin(2\beta)\partial_\beta\Gamma(\beta)| = |2\alpha\sin^2(\beta)(\cos(\beta))^\alpha| \leq 2\alpha\Gamma(\beta).$$

Hence, the angular transport term in (2.36) becomes smaller compared to the nonlinear term.

Using (2.37) and the above estimate, formally, we obtain that the transport terms in (2.35) is of order α^2 and η in (2.35) is of order α. Therefore, we drop the transport terms in (2.35). This additional consideration is not required in [42] for 3D asymmetric Euler without swirl.

We remark that in our dynamic rescaling formulation, η is comparable to the nonlinear term $\alpha^{-1}L_{12}(\Omega)\eta$. Therefore, we drop the transport terms and the lower order terms in (2.35),(2.36) to derive a leading order system for (Ω, η)

$$\Omega_t = \eta, \quad \eta_t = \frac{2}{\pi\alpha}L_{12}(\Omega)\eta, \quad L_{12}(\Omega) = \int_R^\infty \int_0^{\pi/2} \frac{\Omega(s, \beta)\sin(2\beta)}{s} ds d\beta. \tag{2.38}$$

It is not difficult to see that if the initial data Ω, η are non-negative and are odd with respect to x, the solutions preserve these properties dynamically. In the first equation, Ω tends to align with η during the evolution. Then the nonlinear term in the second equation is of order η^2, which is the driving force of finite time singularity of the leading order system.

2.4 Self-similar solution of the leading order system

The leading order system (2.38) is crucial in our analysis and it captures the leading behavior of the blowup solution of the Boussinesq equations (2.16)-(2.18). In this section, we construct the self-similar solution of the leading order system (2.38) for (Ω, η). Notice that $L_{12}(\Omega)$ does not depend on the angular component β. Inspired by the solution structure of the leading order system in [42], we look for a self-similar solution in the form

$$\Omega(R, \beta, t) = (T-t)^{c_\omega} \Omega_* \left(\frac{R}{(T-t)^{\alpha \cdot c_l}} \right) \Gamma(\beta),$$
$$\eta(R, \beta, t) = (T-t)^{c_\theta - c_l} \eta_* \left(\frac{R}{(T-t)^{\alpha \cdot c_l}} \right) \Gamma(\beta),$$

where c_ω, c_l, c_θ are the scaling parameters. The reason that we use the scaling factor $(T-t)^{\alpha \cdot c_l}$ in the space variable R is that $R = r^\alpha$ and $\frac{R}{(T-t)^{\alpha \cdot c_l}} = \left(\frac{r}{(T-t)^{c_l}} \right)^\alpha$, where $r = \sqrt{x^2 + y^2}$. Factor $(T-t)^{c_l}$ corresponds to the scaling of the original variables x, y and $(T-t)^{c_\theta}$ is the scaling of θ in (2.20)-(2.22). See (2.19) for the scaling invariance of the Boussinesq equations.

Plugging the self-similar solutions ansatz into (2.38), we obtain

$$\begin{aligned} & -(T-t)^{c_\omega - 1} c_\omega \Omega_*(z) \Gamma(\beta) + (T-t)^{c_\omega - 1} \alpha c_l z \partial_z \Omega_*(z) \Gamma(\beta) \\ = & (T-t)^{c_\theta - c_l} \eta_*(z) \Gamma(\beta), \\ & -(T-t)^{c_\theta - c_l - 1}(c_\theta - c_l) \eta_*(z) \Gamma(\beta) + (T-t)^{c_\theta - c_l - 1} \alpha c_l z \partial_z \eta_*(z) \Gamma(\beta) \\ = & (T-t)^{c_\theta - c_l + c_\omega} \eta_*(z) \Gamma(\beta) \frac{2}{\pi \alpha} \int_z^\infty \frac{\Omega_*(s)}{s} ds \cdot \int_0^{\pi/2} \Gamma(\beta) \sin(2\beta) d\beta, \end{aligned} \tag{2.39}$$

where $z = R \cdot (T-t)^{-\alpha c_l} \geq 0$. From the above equations, we obtain that the scaling parameters $(c_\omega, c_l, c_\theta)$ satisfies

$$c_\omega - 1 = c_\theta - c_l, \quad c_\theta - c_l - 1 = c_\omega + c_\theta - c_l,$$

which implies

$$c_\omega = -1, \quad c_\theta = c_l + 2.$$

Denote

$$c = \frac{2}{\pi} \int_0^\pi \Gamma(\beta) \sin(2\beta) d\beta.$$

Plugging the relations among the scaling parameters into (2.39) and factorizing the temporal variable, we derive

$$\begin{aligned} \alpha c_l z\partial_z \Omega_* \Gamma(\beta) &= -\Omega_* \Gamma(\beta) + \eta_* \Gamma(\beta), \\ \alpha c_l z\partial_z \eta_* \Gamma(\beta) &= -2\eta_* \Gamma(\beta) + \frac{c}{\alpha}\eta_* \Gamma(\beta) \int_z^\infty \frac{\Omega_*(s)}{s} ds. \end{aligned} \tag{2.40}$$

We can factorize the angular part $\Gamma(\beta)$ to further simplify the above equations. Surprisingly, the above equations have explicit solutions of the form

$$\Omega_*(z) = \frac{az}{(b+z)^2}, \quad c_l = \frac{1}{\alpha}$$

(recall that $z \geq 0$). We determine η_* from the first equation in (2.40)

$$\eta_*(z) = \alpha c_l z \partial_z \Omega_* + \Omega_* = z\partial_z \Omega_* + \Omega_* = \frac{2abz}{(b+z)^3}.$$

Then (η_*, Ω_*) solves (2.40) exactly if and only if

$$z\partial_z \eta_* + 2\eta_* - \frac{c}{\alpha}\eta_* \int_z^\infty \frac{\Omega_*(s)}{s} ds = 0$$

which is equivalent to

$$0 = z\left(-\frac{6abz}{(b+z)^4} + \frac{2ab}{(b+z)^3}\right) + \frac{4abz}{(b+z)^3} - \frac{c}{\alpha}\frac{2abz}{(b+z)^3}\frac{a}{b+z} = -\frac{2ab(-3\alpha b + ac)z}{\alpha(b+z)^4}.$$

Hence, we obtain

$$a = \frac{3\alpha b}{c}.$$

Using the above formula, we can derive the solutions (Ω_*, η_*) of (2.38). We remark that there is a free parameter b in the solutions (Ω_*, η_*). After we impose a normalization condition, e.g., the derivative of Ω_* at $z = 0$, we can determine b. For simplicity, we choose $b = 1$ and then a becomes $a = 3\alpha/c$. Consequently, we obtain the following result.

Lemma 2.4.1. *The leading order system* (2.38) *admits a family of self-similar solutions*

$$\Omega(R, \beta, t) = \frac{\alpha}{c}\frac{1}{T-t}\Gamma(\beta)\Omega_*\left(\frac{R}{T-t}\right), \quad \eta(R, \beta, t) = \frac{\alpha}{c}\frac{1}{(T-t)^2}\Gamma(\beta)\eta_*\left(\frac{R}{T-t}\right),$$

for some $T > 0$, where

$$\Omega_*(z) = \frac{3z}{(1+z)^2}, \quad \eta_* = \frac{6z}{(1+z)^3}, \quad c = \frac{2}{\pi}\int_0^{\pi/2} \Gamma(\beta)\sin(2\beta) d\beta \neq 0.$$

We will choose $\Gamma(\beta) = (\cos(\beta))^\alpha$ in the later discussion.

Properties of θ_x, ω The self-similar profile (Ω, η) of the leading order system (2.38) in Lemma 2.4.1 is indeed anisotropic in x, y direction. Moreover, θ_x and ω are positive in the first quadrant. For $\Gamma(\beta) = (\cos(\beta))^\alpha$, the self-similar profile of θ_x in the first quadrant is

$$\theta_x = C\alpha\Gamma(\beta)\frac{R}{(1+R)^3} = C\alpha\frac{|x|^\alpha}{(1+(x^2+y^2)^{\alpha/2})^3},$$

for some constant C. If x^2+y^2 is small, the formal argument (2.34) shows that θ_y is relatively small compared to θ_x. We will estimate it precisely in Lemma A.0.8 in Appendix A.

Hyperbolic flow field The leading order of the flow structure corresponding to the self-similar solution of the leading order system can be obtained using (2.32)

$$L_{12}(\Omega)(R, \beta, t) = \frac{\pi\alpha}{2}\frac{1}{T-t}\frac{3}{1+R/(T-t)} = \frac{\pi\alpha}{2}\frac{3}{(T-t)+R},$$
$$u(x, y, t) = -\frac{3r\cos\beta}{(T-t)+R} + l.o.t., \quad v(x, y, t) = \frac{3r\sin(\beta)}{(T-t)+R} + l.o.t.$$

In the first quadrant, the flow is clockwise since $u < 0, v > 0$. Moreover, the odd symmetry of ω implies that the flow is hyperbolic near the origin. These properties of the solutions are similar to those considered in [86, 87].

2.5 The dynamic rescaling formulation

Applying the dynamic rescaling formulation in Section 2.1.4, we obtain the dynamic rescaling equations of (2.20)-(2.22) with Biot-Savart law (2.18) in (2.10)-(2.12).

2.5.1 Reformulation using the (R, β) coordinates

Taking x, y derivative on the θ equation in (2.10), we obtain a system similar to (2.20)-(2.22).

$$\begin{aligned} \omega_t + (c_l\mathbf{x} + \mathbf{u}) \cdot \nabla\theta_x &= c_\omega\omega + \theta_x, \\ \theta_{xt} + (c_l\mathbf{x} + \mathbf{u}) \cdot \nabla\theta_x &= (c_\theta - c_l - u_x)\theta_x - v_x\theta_y, \\ \theta_{yt} + (c_l\mathbf{x} + \mathbf{u}) \cdot \nabla\theta_y &= (c_\theta - c_l - v_y)\theta_y - u_y\theta_x, \end{aligned} \tag{2.41}$$

where we have dropped $\tilde{\cdot}$ to simplify the notations. We make a change of variable $R = r^\alpha, \beta = \arctan(y/x)$ and introduce

$$\Omega(R, \beta, t) = \omega(x, y, t), \quad \eta(R, \beta, t) = (\theta_x)(x, y, t), \quad \xi(R, \beta, t) = (\theta_y)(x, y, t)$$

in (2.41) as we did in Section 2.3. Notice that the stretching term and the damping term satisfy

$$c_l \mathbf{x}\cdot\nabla\omega(x,y,t) = c_l r\partial_r\omega(r,\beta,t) = \alpha c_l R\partial_R\Omega(R,\beta,t), \quad c_\omega\omega(x,y,t) = c_\omega\Omega(R,\beta,t),$$

and similar relations hold for θ_x, θ_y. The reformulated system (2.41) under (R,β) coordinates reads

$$\begin{aligned} \Omega_t + \alpha c_l R\partial_R\Omega + (\mathbf{u}\cdot\nabla)\Omega &= c_\omega\Omega + \eta \\ \eta_t + \alpha c_l R\partial_R\eta + (\mathbf{u}\cdot\nabla)\eta &= (2c_\omega - u_x)\eta - v_x\xi \\ \xi_t + \alpha c_l R\partial_R\xi + (\mathbf{u}\cdot\nabla)\xi &= (2c_\omega - v_y)\xi - u_y\eta, \end{aligned} \tag{2.42}$$

with the Biot-Savart law in the (R,β) coordinates (2.25) and (2.26), where we have used $c_\theta - c_l = 2c_\omega$ (2.12). For now, we do not expand $u\cdot\nabla$ using (2.27) and u_x, u_y, v_x, v_y due to their complicated expressions. Using the same argument as that in Section 2.3.4, the leading terms in (2.42) are given by

$$\begin{aligned} \Omega_t + \alpha c_l R\partial_R\Omega &= c_\omega\Omega + \eta + l.o.t., \\ \eta_t + \alpha c_l R\partial_R\eta &= (2c_\omega + \frac{2}{\pi\alpha}L_{12}(\Omega))\eta + l.o.t., \\ \xi_t + \alpha c_l R\partial_R\xi &= (2c_\omega - \frac{2}{\pi\alpha}L_{12}(\Omega))\xi + l.o.t., \end{aligned} \tag{2.43}$$

where we have dropped the transport terms and simplified $u_x, u_y, v_x, v_y, u/x, v/y$ using (2.32). We remark that the first two equations in (2.43) are exactly the dynamic rescaling formulation of the leading order system (2.38).

2.5.2 Constructing an approximate steady state

Notice that the system (2.43) captures the leading order terms in the system (2.42) and that the self-similar profile of (2.38) corresponds to the steady state of the first two equations in (2.43) after neglecting the lower order terms. It motivates us to use the self-similar solutions of (2.38) in Lemma 2.4.1 as the building block to construct the approximate steady state of (2.42). Firstly, we construct

$$\begin{aligned} \bar{\Omega}(R,\beta) = \frac{\alpha}{c}\Gamma(\beta)\frac{3R}{(1+R)^2}, \quad \bar{\eta}(R,\beta) = \frac{\alpha}{c}\Gamma(\beta)\frac{6R}{(1+R)^3}, \quad \bar{c}_\omega = -1, \\ \bar{c}_l = \frac{1}{\alpha} + 3, \quad \Gamma(\beta) = (\cos(\beta))^\alpha, \quad c = \frac{2}{\pi}\int_0^{\pi/2}\Gamma(\beta)\sin(2\beta)d\beta. \end{aligned} \tag{2.44}$$

Notice that $(\bar{\Omega}, \bar{\eta})$ is a solution of (2.40) with $c_l = \frac{1}{\alpha}$. We modify $\bar{c}_l$ so that the approximate error vanishes quadratically near $R = 0$, which will be discussed later. The corresponding $\bar{\theta}$ can be obtained by integrating $\bar{\theta}_x$ with condition $\bar{\theta}(0, y) = 0$, which is discussed in Appendix A.0.5, and $\bar{u}, \bar{v}$ are obtained from the Biot-Savart law (2.25), (2.26). We can derive the leading order terms using (2.31) and (2.32)

$$L_{12}(\bar{\Omega}) = \int_R^\infty \int_0^{\pi/2} \frac{\bar{\Omega}(s,\beta)\sin(2\beta)}{s} ds = \frac{\pi}{2}\frac{3\alpha}{1+R}, \quad \bar{\Psi} = \frac{\sin(2\beta)}{2}\frac{3}{1+R} + l.o.t.,$$
$$\bar{u}_x = -\bar{v}_y = -\frac{2}{\pi\alpha} L_{12}(\Omega) + l.o.t. = \frac{3}{1+R} + l.o.t., \quad \bar{u}_y, \ \bar{v}_x = l.o.t. \tag{2.45}$$

We will explain later why we choose the above $\Gamma(\beta)$. Lemma A.0.1 in Appendix A shows that $\Gamma(\beta)$ is essentially equal to the constant 1 in some weighted norm.

We define the error of the approximate steady state below

$$\begin{aligned} \bar{F}_\omega &\triangleq \bar{c}_\omega \bar{\Omega} + \bar{\eta} - \alpha \bar{c}_l R \partial_R \bar{\Omega} - (\bar{\mathbf{u}} \cdot \nabla)\bar{\Omega}, \\ \bar{F}_\eta &\triangleq (2\bar{c}_\omega - \bar{u}_x)\bar{\eta} - \bar{v}_x \bar{\xi} - \alpha \bar{c}_l R \partial_R \bar{\eta} - (\bar{\mathbf{u}} \cdot \nabla)\bar{\eta}, \\ \bar{F}_\xi &\triangleq (2\bar{c}_\omega - \bar{v}_y)\bar{\xi} - \bar{u}_y \bar{\eta} - \alpha \bar{c}_l R \partial_R \bar{\xi} - (\bar{\mathbf{u}} \cdot \nabla)\bar{\xi}. \end{aligned} \tag{2.46}$$

The criteria to choose Γ in (2.44) is that F_ω, F_η, F_ξ vanish quadratically near $R = 0$ since we will perform energy estimates with a singular weight in the later sections. Using the formula (2.27) for $\bar{u} \cdot \nabla$ and (2.44), one can obtain the following expansion of $\bar{F}_\omega$ near $R = 0$

$$\bar{F}_\omega = -3\alpha R \partial_R \bar{\Omega} - (\bar{u} \cdot \nabla)\bar{\Omega} = \frac{9\alpha R}{c}(\alpha\Gamma\cos(2\beta) - \sin(2\beta)\partial_\beta\Gamma - \alpha\Gamma) + O(R^2),$$

where we have used the explicit formula (2.44) in the first equality and the factor 3 comes from $\bar{c}_l = \frac{1}{\alpha} + 3$ in (2.44). In order for $\bar{F}_\omega$ to vanish quadratically near $R = 0$, we have no choice but to set the coefficient in the $O(R)$ term to be zero, which gives

$$\alpha\Gamma\cos(2\beta) - \sin(2\beta)\partial_\beta\Gamma - \alpha\Gamma = 0.$$

To solve the above first order ODE for Γ, we choose the boundary condition $\Gamma(\pi/2) = 0$ and requires $\Gamma(\beta) > 0$ for $\beta \in (0, \pi/2]$. The solution of this ODE is exactly given by the formula of $\Gamma(\beta)$ in (2.44). As we can see, such choice of Γ is unique and is a consequence of the condition that $\bar{F}_\omega = O(R^2)$ near $R = 0$. This condition plays an essential role in our stability analysis for the approximate self-similar profile. With this $\Gamma(\beta)$, we also have $\bar{F}_\eta, \bar{F}_\xi = O(R^2)$ near $R = 0$. We justify these rigorously in Section 2.9.

2.5.3 Normalization conditions

For initial data $\bar{\Omega}+\Omega, \bar{\eta}+\eta, \bar{\xi}+\xi$ of (2.42), we treat Ω, η, ξ as perturbation and choose time-dependent scaling parameters $c_l + \bar{c}_l, c_\omega + \bar{c}_\omega$ as follows

$$c_\omega(t) = -\frac{2}{\pi\alpha} L_{12}(\Omega(t))(0), \quad c_l(t) = -\frac{1-\alpha}{\alpha}\frac{2}{\pi\alpha} L_{12}(\Omega(t))(0) = \frac{1-\alpha}{\alpha} c_\omega(t). \tag{2.47}$$

Here, $c_l(t), c_\omega(t)$ are treated as the perturbation of the scaling parameters $\bar{c}_l, \bar{c}_\omega$. Suppose that $F_\Omega(t), F_\eta(t), F_\xi(t)$ are the time-dependent update in (2.42),i.e.

$$F_\Omega(t) = (c_\omega + \bar{c}_\omega)(\Omega+\bar{\Omega}) + (\eta+\bar{\eta}) - \alpha(c_l+\bar{c}_l)R\partial_R(\Omega+\bar{\Omega}) + ((\mathbf{u}+\bar{\mathbf{u}})\cdot\nabla)(\Omega+\bar{\Omega}),$$

and so on. The reason we choose (2.47) is that we want $F_\Omega(t), F_\eta(t), F_\xi(t)$ vanishes quadratically near $R = 0$ for any perturbation $\Omega(t), \eta(t), \xi(t)$ that vanishes quadratically near $R = 0$, so that we can choose a singular weight to analyze the stability of the approximate steady state. Similar consideration has been used in our previous work with Hou and Huang on the asymptotically self-similar blowup of the Hou-Luo model from smooth initial data [20]. We will provide rigorous estimates for these terms in Section 2.9.

2.6 Linear stability

We present our linear stability analysis in this section. In Section 2.6.1, we linearize the dynamic rescaling formulation in the (R, β) coordinates (2.42) around the approximate steady state $(\bar{\Omega}, \bar{\eta}, \bar{\xi}, \bar{c}_l, \bar{c}_\omega)$. In Section 2.6.2, we outline the steps in the linear stability analysis. In the rest of the Section, we establish the linear stability of the leading terms in the linearized system. Throughout this section, we use $\Omega, \eta, \xi, c_l, c_\omega$ to denote the perturbations around the approximate profile (2.44) and assume that $\Omega \in L^2(\varphi), \eta \in L^2(\varphi), \xi \in L^2(\psi)$ for some singular weights φ, ψ to be determined later.

2.6.1 Linearized system

We linearize (2.42) around $(\bar{\Omega}, \bar{\eta}, \bar{\xi}, \bar{c}_l, \bar{c}_\omega)$ (2.44) and derive the equations for the perturbation Ω, η, ξ as follows

$$\begin{aligned}
\Omega_t + (1+3\alpha)R\partial_R\Omega + (\bar{\mathbf{u}}\cdot\nabla)\Omega &= -\Omega + \eta + c_\omega(\bar{\Omega} - R\partial_R\bar{\Omega}) \\
&\quad + (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\Omega} + \bar{F}_\Omega + N_\omega, \\
\eta_t + (1+3\alpha)R\partial_R\eta + (\bar{\mathbf{u}}\cdot\nabla)\eta &= (-2-\bar{u}_x)\eta - u_x\bar{\eta} + c_\omega(2\bar{\eta} - R\partial_R\bar{\eta}) \\
&\quad + (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\eta} - v_x\bar{\xi} - \bar{v}_x\xi + \bar{F}_\eta + N_\eta, \\
\xi_t + (1+3\alpha)R\partial_R\xi + (\bar{\mathbf{u}}\cdot\nabla)\xi &= (-2-\bar{v}_y)\xi - v_y\bar{\xi} + c_\omega(2\bar{\xi} - R\partial_R\bar{\xi}) \\
&\quad + (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\xi} - u_y\bar{\eta} - \bar{u}_y\eta + \bar{F}_\xi + N_\xi,
\end{aligned} \tag{2.48}$$

where we have used $\bar{c}_l = 1/\alpha + 3, \bar{c}_\omega = -1$ (2.44), $\alpha c_l(t) = c_\omega(t) - \alpha c_\omega(t)$ (2.47) and $-\alpha c_l R\partial_R \bar{g} = -c_\omega R\partial_R \bar{g} + \alpha c_\omega R\partial_R \bar{g}$ for $g = \bar{\Omega}, \bar{\eta}, \bar{\xi}$. The error $\bar{F}_\Omega, \bar{F}_\eta, \bar{F}_\xi$ are defined in (2.46) and the nonlinear terms are defined below

$$\begin{aligned} N_\Omega &= c_\omega \Omega + \eta - \alpha c_l R\partial_R \Omega - (\mathbf{u}\cdot\nabla)\Omega, \\ N_\eta &= (2c_\omega - u_x)\eta - v_x\xi - \alpha c_l R\partial_R\eta - (\mathbf{u}\cdot\nabla)\eta, \\ N_\xi &= (2c_\omega - v_y)\xi - u_y\eta - \alpha c_l R\partial_R\xi - (\mathbf{u}\cdot\nabla)\xi. \end{aligned} \tag{2.49}$$

We focus on the linearized equation of (2.48). From (2.33) and (2.45), we have

$$\begin{aligned} 3\alpha R\partial_R + \bar{u}\cdot\nabla &= 2\bar{\Psi}\partial_\beta + \left\{-\alpha R\partial_\beta\bar{\Psi}\partial_R + \alpha R\partial_R\bar{\Psi}\partial_\beta\right\} \\ &= \frac{3\sin(2\beta)}{1+R}\partial_\beta + l.o.t. \end{aligned} \tag{2.50}$$

We will justify the above decomposition using integration by parts to avoid loss of derivatives. We will also show that

$$(\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\Omega}, \quad (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\eta}, \quad (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\xi} \tag{2.51}$$

in (2.48) are lower order terms. Moreover, we will justify that $\bar{\xi}$ is small and is of order α^2 in Lemma A.0.8 so that we can treat $v_x\bar{\xi}$ as a lower order term in the η equation.

Using (2.32), (2.45), (2.50), (2.51) and then collecting the lower order terms with a small factor α, the error terms $\bar{F}$ and the nonlinear terms N in the remaining term $\mathcal{R}$, we derive the leading order terms in the linearized equations

$$\Omega_t + R\partial_R\Omega + \frac{3\sin(2\beta)}{1+R}\partial_\beta\Omega = -\Omega + \eta + c_\omega(\bar{\Omega} - R\partial_R\bar{\Omega}) + \mathcal{R}_\Omega, \tag{2.52}$$

$$\begin{aligned} \eta_t + R\partial_R\eta + \frac{3\sin(2\beta)}{1+R}\partial_\beta\eta &= (-2 + \frac{3}{1+R})\eta + \frac{2}{\pi\alpha}L_{12}(\Omega)\bar{\eta} \\ &\quad + c_\omega(2\bar{\eta} - R\partial_R\bar{\eta}) + \mathcal{R}_\eta, \end{aligned} \tag{2.53}$$

$$\begin{aligned} \xi_t + R\partial_R\xi + \frac{3\sin(2\beta)}{1+R}\partial_\beta\xi &= (-2 - \frac{3}{1+R})\xi - \frac{2}{\pi\alpha}L_{12}(\Omega)\bar{\xi} \\ &\quad + c_\omega(2\bar{\xi} - R\partial_R\bar{\xi}) + \mathcal{R}_\xi, \end{aligned} \tag{2.54}$$

where the full expansion of $\mathcal{R}$ is given in (2.154) and their estimates are deferred to Section 2.9. In the following subsections, we establish the linear stability for (2.52)-(2.54). The contribution of $\mathcal{R}$ is small. Using this property, we can further establish the nonlinear stability of the approximate profile (2.44) using a bootstrap argument.

We introduce the following notation

$$\tilde{L}_{12}(\Omega)(R) \triangleq L_{12}(\Omega)(R) - L_{12}(\Omega)(0) = -\int_0^R \int_0^{\pi/2} \frac{\Omega(s,\beta)\sin(2\beta)}{s} d\beta dx. \tag{2.55}$$

According to the normalization condition of c_ω (2.47), we can simplify

$$c_\omega + \frac{2}{\pi\alpha} L_{12}(\Omega)(R) = \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)(R). \tag{2.56}$$

Definition 2.6.1. We define the differential operators

$$D_R = R\partial_R, \quad D_\beta = \sin(2\beta)\partial_\beta$$

and the linear operators $\mathcal{L}_i$

$$\begin{aligned}
&\mathcal{L}_1(\Omega,\eta) \triangleq -D_R\Omega - \frac{3}{1+R}D_\beta\Omega - \Omega + \eta + c_\omega(\bar{\Omega} - D_R\bar{\Omega}),\\
&\mathcal{L}_2(\Omega,\eta) \triangleq -D_R\eta - \frac{3}{1+R}D_\beta\eta + (-2 + \frac{3}{1+R})\eta + \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta} + c_\omega(\bar{\eta} - D_R\bar{\eta}),\\
&\mathcal{L}_3(\Omega,\xi) \triangleq -D_R\xi - \frac{3}{1+R}D_\beta\xi + (-2 - \frac{3}{1+R})\xi - \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\xi} + c_\omega(3\bar{\xi} - D_R\bar{\xi}),
\end{aligned} \tag{2.57}$$

where $\tilde{L}_{12}(\Omega)$ is defined in (2.55) and $\bar{\Omega}, \bar{\eta}$ are defined in (2.44). Define the local part of $\mathcal{L}_i$ by eliminating $c_\omega, \tilde{L}_{12}(\Omega)$

$$\begin{aligned}
\mathcal{L}_{10}(\Omega,\eta) &\triangleq -D_R\Omega - \frac{3}{1+R}D_\beta\Omega - \Omega + \eta,\\
\mathcal{L}_{20}(\eta) &\triangleq -D_R\eta - \frac{3}{1+R}D_\beta\eta + (-2 + \frac{3}{1+R})\eta,\\
\mathcal{L}_{30}(\xi) &\triangleq -D_R\xi - \frac{3}{1+R}D_\beta\xi + (-2 - \frac{3}{1+R})\xi.
\end{aligned} \tag{2.58}$$

With the above notations, (2.52)-(2.54) can be reformulated as

$$\Omega_t = \mathcal{L}_1(\Omega,\eta) + \mathcal{R}_\Omega, \quad \eta_t = \mathcal{L}_2(\Omega,\eta) + \mathcal{R}_\eta, \quad \xi_t = \mathcal{L}_3(\xi) + \mathcal{R}_\xi, \tag{2.59}$$

where we have used the following identities to rewrite the $L_{12}(\Omega), c_\omega$ terms in (2.53)-(2.54)

$$\begin{aligned}
\frac{2L_{12}(\Omega)}{\pi\alpha}\bar{\eta} + c_\omega(2\bar{\eta} - D_R\bar{\eta}) &= \frac{2\tilde{L}_{12}(\Omega)}{\pi\alpha}\bar{\eta} + c_\omega(\bar{\eta} - D_R\bar{\eta}),\\
-\frac{2L_{12}(\Omega)}{\pi\alpha}\bar{\xi} + c_\omega(2\bar{\xi} - D_R\bar{\xi}) &= -\frac{2\tilde{L}_{12}(\Omega)}{\pi\alpha}\bar{\eta} + c_\omega(3\bar{\xi} - D_R\bar{\xi}).
\end{aligned}$$

2.6.1.1 Key observations

There are several key observations that play a crucial role in our analysis. Firstly, the leading order terms in the Ω, η equations (2.52)-(2.53) do not couple the ξ term, which is consistent with our derivation for the leading order system (2.38).

Secondly, in the ξ equation, the coupling between Ω and ξ through the nonlocal term $L_{12}(\Omega)$ and c_ω (2.47) is weak due to the fact that $\bar{\xi}$ is much smaller than $\bar{\Omega}, \bar{\eta}$. After removing these nonlocal terms, (2.54) only involves local terms about ξ. By choosing a suitable singular weight, we will show that ξ is linearly stable up to the weak nonlocal term.

Thirdly, all the nonlocal terms in (2.52)-(2.53), e.g., $c_\omega, L_{12}(\Omega)$, have coefficients with small angular derivative. For example, using (2.44), we have

$$c_\omega(\bar{\Omega} - R\partial_R \bar{\Omega}) = c_\omega \cdot \frac{\alpha}{c}\Gamma(\beta)\frac{6R^2}{(1+R)^3}. \tag{2.60}$$

We can apply the weighted angular derivative to gain a small factor α

$$|\sin(2\beta)\partial_\beta \Gamma(\beta)| = |2\alpha \sin^2(\beta)\Gamma(\beta)| \le 2\alpha\Gamma(\beta).$$

A similar observation and estimate have been obtained for a different Γ in [42].

2.6.1.2 The angular transport term

To understand the effect of the angular transport term in (2.52)-(2.54), we choose a weight $\varphi(R,\beta) = A(R)(\sin(\beta))^{-\gamma_1}(\cos(\beta))^{-\gamma_2}$ and then perform the L^2 estimate and use integration by parts to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle \Omega^2, \varphi\rangle &= -\Big\langle \frac{3\sin(2\beta)}{1+R}\partial_\beta\Omega, \Omega\varphi(R,\beta)\Big\rangle + \text{ other terms (o.t.)} \\ &= \Big\langle \frac{3(\sin(2\beta)\varphi)_\beta}{2(1+R)\varphi}, \Omega^2\varphi\Big\rangle + o.t..\end{aligned}$$

It is not difficult to show that

$$\frac{3(\sin(2\beta)\varphi)_\beta}{2(1+R)\varphi}\Big|_{R=0} = 3(1-\gamma_1)\cos^2(\beta) - 3(1-\gamma_2)\sin^2(\beta).$$

Suppose that $\gamma_1, \gamma_2 \le 1$. If β is small, the angular transport term contributes a growing factor $3(1-\gamma_1) > 0$ to the energy norm.

To establish the linear stability, it is natural to first establish the (weighted) L^2 estimate of (2.52)-(2.54). However, the above argument shows that for small

$\beta > 0$ the angular transport term destabilizes the profile of the singularity using the singular weights $A(R)(\sin(\beta))^{-\gamma_1}(\cos(\beta))^{-\gamma_2}$ with $\gamma_1 \leq 1$. A possible approach to address this issue in the estimate is to choose γ_1 close to or larger than 1, i.e. a very singular weight in the β direction is desired. In [42], γ_1 is chosen to be close to 1 so that such growing factor is minimized. For (2.52)-(2.54), due to the presence of the nonlocal term, e.g., $c_\omega(\bar{\Omega} - R\partial_R\bar{\Omega})$, which only vanishes of order $\sin(2\beta)^{\alpha/2}$ near $\beta = 0, \pi/2$, if we use a very singular weight for the angular component β, such nonlocal term will be very difficult to control.

To handle the angular transport term in the L^2 estimate, we observe that $\sin(2\beta)\partial_\beta\bar{\Omega}$ is small since $\bar{\Omega}$ varies slowly in β. We expect that a similar smallness result holds for the perturbation term $\sin(2\beta)\partial_\beta\Omega$ and we will justify it in Section 2.6.4. This observation motivates us *not* to perform integration by parts for the angular transport term in the weighted L^2 estimate.

2.6.2 Outline of the linear stability analysis

We decompose the linear stability analysis of (2.52)-(2.54), or equivalently (2.59) into several steps. Based on the first observation in Section 2.6.1.1, we separate the estimates of the system of Ω, η (2.52)-(2.53) and the equation of ξ (2.54).

In Section 2.6.3, we estimate the local part of the linearized operators $\mathcal{L}_i$ (2.57), i.e. $\mathcal{L}_{i0}$ (2.58). The argument is mainly based on integration by parts.

Instead of first performing the weighted L^2 estimate of the system, we perform the weighted L^2 estimate of the angular derivative in Section 2.6.4. The motivation is that using the third observation in Section 2.6.1.1, we gain a small factor $\alpha^{1/2}$ for the nonlocal terms in the equations of $D_\beta\Omega, D_\beta\eta$. Therefore, we can treat the nonlocal terms as small perturbations and use the estimates of $\mathcal{L}_{i0}$ in Section 2.6.3 to establish the estimates of $D_\beta\Omega, D_\beta\eta$. See also the motivation in Section 2.6.1.2. Once we obtain the estimates of $D_\beta\Omega, D_\beta\eta$, we can treat the angular transport terms in the weighted L^2 estimates of the equations of Ω, η (2.52)-(2.53) as perturbations. This overcomes the difficulty discussed in Section 2.6.1.2.

In Section 2.6.5, we use two models to illustrate the cancellations in (2.52),(2.53), which are crucial for the estimates of $\tilde{L}_{12}(\Omega), c_\omega$. This motivates several technical estimates in Section 2.6.6.

In Section 2.6.6, we establish the weighted L^2 estimates of Ω, η with less singular weights, and obtain the damping terms for $c_\omega, \tilde{L}_{12}(\Omega)$. We design the less singular weights carefully to fully exploit the cancellations discussed in Section 2.6.5. This is the most difficult part in the whole analysis.

After we obtain the damping terms for $c_\omega, \tilde{L}_{12}(\Omega)$, we can treat the nonlocal terms in (2.52)-(2.53) as perturbations. Using the estimates of the local operators $\mathcal{L}_{i0}$ in Section 2.6.3, we further establish weighted L^2 estimates of Ω, η with more singular weights that are introduced in [42] in Section 2.6.7. This enables us to apply several key estimates in [42] in our nonlinear estimates and simplify the whole estimates.

From the second observation in Section 2.6.1.1, we treat the nonlocal terms in the ξ equation (2.54) as small perturbations. We estimate $D_\beta \xi, \xi$ in Section 2.6.8 using the estimate of $\mathcal{L}_{30}$ in Section 2.6.3.

2.6.3 Estimates of $\mathcal{L}_{10}, \mathcal{L}_{20}, \mathcal{L}_{30}$

We first introduce several singular weights that will be used throughout the chapter.

Definition 2.6.2. Define φ_i, ψ_i by

$$\begin{aligned} &\varphi_1 \triangleq \frac{(1+R)^4}{R^4} \sin(2\beta)^{-\sigma}, \quad \varphi_2 \triangleq \frac{(1+R)^4}{R^4} \sin(2\beta)^{-\gamma}, \\ &\psi_1 \triangleq \frac{(1+R)^4}{R^4} (\sin(\beta)\cos(\beta))^{-\sigma}, \quad \psi_2 \triangleq \frac{(1+R)^4}{R^4} \sin(\beta)^{-\sigma}\cos(\beta)^{-\gamma}, \end{aligned} \tag{2.61}$$

where $\sigma = \frac{99}{100}, \gamma = 1 + \frac{\alpha}{10}$.

The weights φ_1, φ_2 have been introduced in [42] for stability analysis.

The weights φ_1 and ψ_1 are essentially the same. We introduce ψ_1 for consistency and the following reasons. Firstly, we will apply the weights φ_i to Ω, η and the weights ψ_i to ξ. In particular, we will construct weighted H^3 norm $\mathcal{H}^3(\varphi)$ for Ω, η and $\mathcal{H}^3(\psi)$ for ξ in (2.129). Secondly, φ_1 and φ_2 have similar forms, and ψ_1 and ψ_2 also have similar forms. It is easy to see that $\varphi_1 \lesssim \varphi_2, \psi_1 \lesssim \psi_2$. We choose ψ_2 less singular than φ_2 for β close to 0 since $\bar{\xi}$ does not decay in R when $R\sin(\beta)^\alpha$ is fixed and β is small. See Lemma A.0.8 regarding the estimate of $\bar{\xi}$.

Recall $\mathcal{L}_{10}, \mathcal{L}_{20}, \mathcal{L}_{30}$ (2.58) in Definition 2.6.1. The following Lemmas will be used repeatedly.

Lemma 2.6.3. *For some $\delta, \delta_1, \delta_2 > 0$, consider the weights*

$$\varphi(R,\beta) = \frac{(1+R)^4}{R^4}(\sin(2\beta))^{-\delta}, \ \psi(R,\beta) = \frac{(1+R)^4}{R^4}(\sin(\beta))^{-\delta_1}(\cos(\beta))^{-\delta_2}. \tag{2.62}$$

Assume $\varphi^{1/2}\Omega, \varphi^{1/2}\eta \in L^2$. We have

$$\langle \mathcal{L}_{10}(\Omega,\eta), \Omega\varphi\rangle + \langle \mathcal{L}_{20}(\eta), \eta\varphi\rangle \leq (-\frac{1}{4} + 3|1-\delta|)(||\Omega\varphi^{1/2}||_2^2 + ||\eta\varphi^{1/2}||_2^2). \tag{2.63}$$

Assume that $\psi^{1/2}\xi \in L^2$. Denote $a \vee b \triangleq \max(a,b)$. Then it holds true that

$$\langle \mathcal{L}_{30}(\xi), \xi\psi\rangle \leq \Big(-\frac{1}{2} + 3(|1-\delta_1| \vee |1-\delta_2|)\Big)||\xi\psi^{1/2}||_2^2. \tag{2.64}$$

We will apply Lemma 2.6.3 to the singular weights in Definition 2.6.2, i.e. $\varphi = \varphi_1$ or φ_2 and $\psi = \psi_1$ or ψ_2. Hence, the exponents we will use are $\delta = \sigma = \frac{99}{100}$ or $\delta = \gamma = 1 + \frac{\alpha}{10}$, $\delta_1 = \sigma$, $\delta_2 = \sigma$ or $\delta_2 = \gamma$. Since these exponents are very close to 1, we have the order $|1-\delta| \approx 0, |1-\delta_1| \vee |1-\delta_2| \approx 0$. The reader can regard the terms $|1-\delta|, |1-\delta_1| \vee |1-\delta_2| \approx 0$.

Remark 2.6.4. The constant $-\frac{1}{4}$ in (2.63) can be improved to $-\frac{1}{2} + \varepsilon$ for any $\varepsilon > 0$ by considering $\lambda_\varepsilon\langle \mathcal{L}_{10}(\Omega,\eta), \Omega\varphi\rangle + \langle \mathcal{L}_{20}(\eta), \eta\varphi\rangle$ for some $\lambda_\varepsilon > 0$, and $-\frac{1}{2}$ in (2.64) can be improved to $-\frac{3}{2}$. Yet, we do not need these sharper estimates.

Proof of Lemma 2.6.3. By definition of φ, ψ, we have

$$\begin{aligned}
\frac{(3\sin(2\beta)\varphi)_\beta}{2(1+R)\varphi} &= \frac{3}{2(1+R)}\frac{(\sin(2\beta)^{1-\delta})_\beta}{\sin(2\beta)^{-\delta}} = \frac{3\cos(2\beta)\cdot(1-\delta)}{1+R} \leq 3|1-\delta|,\\
\frac{(3\sin(2\beta)\psi)_\beta}{2(1+R)\psi} &= \frac{3}{(1+R)}\frac{(\sin(\beta)^{1-\delta_1}\cos(\beta)^{1-\delta_2})_\beta}{\sin(\beta)^{-\delta_1}\cos(\beta)^{-\delta_2}}\\
&= \frac{3}{1+R}((1-\delta_1)\cos^2(\beta) - (1-\delta_2)\sin^2(\beta))\\
&\leq 3\max(|1-\delta_1|, |1-\delta_2|),\\
\frac{(R\varphi)_R}{2\varphi} = \frac{(R\psi)_R}{2\psi} &= \left(\frac{(1+R)^4}{R^3}\right)_R \frac{R^4}{2(1+R)^4} = \frac{2R}{1+R} - \frac{3}{2} = \frac{1}{2} - \frac{2}{1+R}.
\end{aligned} \tag{2.65}$$

Using integration by parts for the transport terms in $\mathcal{L}_{10}$ (2.58), we yield

$$\langle -D_R\Omega, \Omega\varphi\rangle = \Big\langle -R\varphi, \frac{1}{2}\partial_R\Omega^2\Big\rangle = \Big\langle \frac{1}{2}(R\varphi)_R, \Omega^2\Big\rangle = \Big\langle \frac{(R\varphi)_R}{2\varphi}, \Omega^2\varphi\Big\rangle.$$

Similar calculation applies to $-\frac{3}{1+R}D_\beta\Omega$ in $\mathcal{L}_{10}$. Using the above calculations, we get

$$\begin{aligned}&\langle \mathcal{L}_{10}(\Omega,\eta),\Omega\varphi\rangle = \Big\langle \frac{(R\varphi)_R}{2\varphi} + \frac{(3\sin(2\beta)\varphi)_\beta}{2(1+R)\varphi}, \Omega^2\varphi\Big\rangle - \langle \Omega,\Omega\varphi\rangle + \langle\Omega,\eta\varphi\rangle\\ \leq&\Big\langle \frac{1}{2} - \frac{2}{1+R} + 3|1-\delta| - 1,\ \Omega^2\varphi\Big\rangle + \langle \Omega,\eta\varphi\rangle\\ =&\Big\langle -\frac{1}{2} - \frac{2}{1+R} + 3|1-\delta|,\ \Omega^2\varphi\Big\rangle + \langle\Omega,\eta\varphi\rangle.\end{aligned}$$

Similarly, using integration by parts for the transport terms in $\mathcal{L}_{20}$ (2.58) and (2.65), we get

$$\begin{aligned}&\langle \mathcal{L}_{20}(\eta),\eta\varphi\rangle = \Big\langle \frac{(R\varphi)_R}{2\varphi} + \frac{(3\sin(2\beta)\varphi)_\beta}{2(1+R)\varphi}, \eta^2\varphi\Big\rangle + \Big\langle (-2+\frac{3}{1+R}),\eta^2\varphi\Big\rangle\\ \leq&\Big\langle \frac{2R}{1+R} - \frac{3}{2} + 3|1-\delta| + (-2+\frac{3}{1+R}),\eta^2\varphi\Big\rangle = \Big\langle -\frac{1}{2} - \frac{R}{1+R} + 3|1-\delta|,\eta^2\varphi\Big\rangle.\end{aligned} \tag{2.66}$$

We estimate the interaction term between Ω,η. Note that

$$4(\frac{1}{4}+\frac{2}{1+R})(\frac{1}{4}+\frac{R}{1+R}) > \frac{2}{1+R}+\frac{R}{1+R} \geq 1.$$

Using the Cauchy-Schwarz inequality implies

$$\langle\Omega,\eta\varphi\rangle \leq \Big\langle \frac{1}{4}+\frac{2}{1+R},\Omega^2\varphi\Big\rangle + \Big\langle\frac{1}{4}+\frac{R}{1+R},\eta^2\varphi\Big\rangle.$$

Combining the above estimates, we prove

$$\begin{aligned}&\langle \mathcal{L}_{10}(\Omega,\eta),\Omega\varphi\rangle + \langle\mathcal{L}_{20}(\Omega,\eta),\eta\varphi\rangle \leq \Big\langle -\frac{1}{2} - \frac{2}{1+R} + 3|1-\delta|,\ \Omega^2\varphi\Big\rangle\\ &+\Big\langle -\frac{1}{2} - \frac{R}{1+R} + 3|1-\delta|,\eta^2\varphi\Big\rangle + \Big\langle\frac{1}{4}+\frac{2}{1+R},\Omega^2\varphi\Big\rangle + \Big\langle\frac{1}{4}+\frac{R}{1+R},\eta^2\psi\Big\rangle\\ \leq&\left(-\frac{1}{4}+3|1-\delta|\right)(||\Omega\varphi^{1/2}||_2^2 + ||\eta\varphi^{1/2}||_2^2).\end{aligned}$$

Recall $\mathcal{L}_{30}$ in Definition 2.6.1. For (2.64), we use the computations (2.65)-(2.66) to obtain

$$\begin{aligned}&\langle\mathcal{L}_{30}(\xi),\xi\psi\rangle = \Big\langle\frac{(R\psi)_R}{2\psi} + \frac{(3\sin(2\beta)\psi)_\beta}{2(1+R)\psi},\xi^2\psi\Big\rangle + \Big\langle(-2-\frac{3}{1+R}),\xi^2\psi\Big\rangle\\ \leq&\Big\langle\frac{2R}{1+R} - \frac{3}{2} + 3(|1-\delta_1|\vee|1-\delta_2|) + (-2-\frac{3}{1+R}),\xi^2\psi\Big\rangle\\ \leq&\Big(-\frac{1}{2}+3(|1-\delta_1|\vee|1-\delta_2|)\Big)||\xi\psi^{1/2}||_2^2.\end{aligned}$$

□

2.6.4 Weighted L^2 estimate of the angular derivative $D_\beta\Omega, D_\beta\eta$

Definition 2.6.5. Define an energy $E(\beta,1)$ and a remaining term $\mathcal{R}(\beta,1)$ by

$$\begin{aligned} E(\beta,1)(\Omega,\eta) &\triangleq \Big(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2\Big)^{1/2}, \\ \mathcal{R}(\beta,1) &\triangleq \langle D_\beta\mathcal{R}_\Omega, D_\beta\Omega\varphi_2\rangle + \langle D_\beta\mathcal{R}_\eta, D_\beta\eta\varphi_2\rangle. \end{aligned} \tag{2.67}$$

To simplify the notations, we drop Ω,η in $E(\beta,1)$. The main result in this subsection is the following. This proposition enables us to treat the angular transport terms in (2.52)-(2.54) as perturbations. A similar estimate has been established in [42].

Proposition 2.6.6. *Assume that $\varphi_2^{1/2}D_\beta\Omega,\ \varphi_2^{1/2}D_\beta\eta \in L^2$. We have*

$$\begin{aligned} &\langle D_\beta\mathcal{L}_1(\Omega,\eta),(D_\beta\Omega)\varphi_2\rangle + \langle D_\beta\mathcal{L}_2(\Omega,\eta),(D_\beta\eta)\varphi_2\rangle \\ \le& -(\frac{1}{5}-\alpha)(E(\beta,1))^2 + C\alpha(L_{12}^2(\Omega)(0) + ||\tilde{L}_{12}(\Omega)R^{-1}||^2_{L^2(R)}), \end{aligned} \tag{2.68}$$

where $\mathcal{L}_1,\mathcal{L}_2$ are defined in Definition 2.6.1.

We will use the following basic property of $D_\beta = \sin(2\beta)\partial_\beta,\ \Gamma(\beta) = \cos(\beta)^\alpha$ repeatedly

$$\begin{aligned} D_\beta\Gamma(\beta) &= -2\alpha\sin^2(\beta)\cos^\alpha(\beta) = -2\alpha\sin^2(\beta)\Gamma(\beta), \\ |D_\beta\Gamma(\beta)| &\le 2\alpha\sin(\beta)\Gamma(\beta). \end{aligned} \tag{2.69}$$

Proof. Notice that the angular transport term in (2.52)-(2.53) can be written as $\frac{3}{1+R}D_\beta$ and that D_β commutes with the derivatives in (2.52)-(2.53) and $\mathcal{L}_{10},\mathcal{L}_{20}$ (2.58). We have

$$\begin{aligned} D_\beta\mathcal{L}_1(\Omega,\eta) &= D_\beta(\mathcal{L}_{10}(\Omega,\eta) + c_\omega D_\beta(\bar{\Omega} - R\partial_R\bar{\Omega})) \\ &= \mathcal{L}_{10}(D_\beta\Omega, D_\beta\eta) + c_\omega D_\beta(\bar{\Omega} - R\partial_R\bar{\Omega}), \\ D_\beta\mathcal{L}_2(\Omega,\eta) &= D_\beta(\mathcal{L}_{20}(\Omega,\eta) + \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta} + c_\omega(\bar{\Omega} - R\partial_R\bar{\Omega})) \\ &= \mathcal{L}_{10}(D_\beta\Omega, D_\beta\eta) + \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)D_\beta\bar{\eta} + c_\omega D_\beta(\bar{\eta} - R\partial_R\bar{\eta}), \end{aligned} \tag{2.70}$$

where we have used (2.56). Applying Lemma 2.6.3 with $\varphi = \varphi_2$ and $\delta = \gamma = 1 + \frac{\alpha}{10}$, we derive

$$\begin{aligned} &\langle\mathcal{L}_{10}(D_\beta\Omega, D_\beta\eta),(D_\beta\Omega)\varphi_2\rangle + \langle\mathcal{L}_{20}(D_\beta\Omega, D_\beta\eta),(D_\beta\eta)\varphi_2\rangle \\ \le&(-\frac{1}{4} + 3|1-\gamma|)\Big(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2\Big) \\ \le&(-\frac{1}{4} + \alpha)\Big(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2\Big). \end{aligned} \tag{2.71}$$

Recall $c_\omega = -\frac{2}{\pi\alpha} L_{12}(\Omega)(0)$. Using (A.16) in Lemma A.0.6 and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&|\langle c_\omega D_\beta(\bar{\Omega} - R\partial_R \bar{\Omega}), (D_\beta \Omega)\varphi_2\rangle| + |\langle c_\omega D_\beta(\bar{\eta} - R\partial_R\bar{\eta}), (D_\beta \eta)\varphi_2\rangle| \\ \lesssim& \alpha^{1/2}|L_{12}(\Omega)(0)|(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2)^{1/2}.\end{aligned} \tag{2.72}$$

Recall the notation $\tilde{L}_{12}(\Omega)$ (2.55). Applying Lemma A.0.3 and (A.15) in Lemma A.0.6, we derive

$$\Big|\Big|\frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)D_\beta\bar{\eta}\varphi_2^{1/2}\Big|\Big|_2 \lesssim \alpha||\tilde{L}_{12}(\Omega)R^{-1}||^2_{L^2(R)}.$$

Therefore, using the Cauchy-Schwarz inequality, we yield

$$\langle\frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)D_\beta\bar{\eta}, D_\beta(\eta)\varphi_2\rangle \lesssim \alpha^{1/2}||\tilde{L}_{12}(\Omega)R^{-1}||_{L^2(R)}||D_\beta\eta\varphi_2^{1/2}||_2. \tag{2.73}$$

Combining (2.71), (2.72), (2.73) and adding the inner product about two terms in (2.70), we prove

$$\begin{aligned}&\langle D_\beta \mathcal{L}_1(\Omega,\eta), (D_\beta\Omega)\varphi_2\rangle + \langle D_\beta\mathcal{L}_2(\Omega,\eta), (D_\beta\eta)\varphi_2\rangle \\ \leq& -(\frac{1}{4} - \alpha)(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2) + C\alpha^{1/2}\Big|\Big|\tilde{L}_{12}(\Omega)R^{-1}\Big|\Big|_{L^2(R)}||D_\beta\eta\varphi_2^{1/2}||_2 \\ &+ C\alpha^{1/2}|L_{12}(\Omega)(0)|\Big(||D_\beta\Omega\varphi_2^{1/2}||_2^2 + ||D_\beta\eta\varphi_2^{1/2}||_2^2\Big)^{1/2},\end{aligned}$$

where C is some absolute constant. Using the notation $E(\beta, 1)$ (2.67), the Cauchy-Schwarz inequality concludes the proof of Proposition 2.6.6 (notice that $-1/4 < -1/5$). □

2.6.5 Ideas in the estimates of the nonlocal terms

Recall $c_\omega, \tilde{L}_{12}(\Omega)$ from (2.47), (2.55)

$$\begin{aligned}c_\omega &= -\frac{2}{\pi\alpha}L_{12}(\Omega)(0) = -\frac{2}{\pi\alpha}\int_0^\infty\int_0^{\pi/2}\frac{\Omega\sin(2\beta)}{R}dRd\beta, \\ \tilde{L}_{12}(\Omega)(R) &= -\int_0^R\int_0^{\pi/2}\frac{\Omega\sin(2\beta)}{R}dRd\beta.\end{aligned} \tag{2.74}$$

The most difficult part in the linear stability analysis of (2.59),(2.57) (or equivalently (2.52)-(2.54)) lies in the nonlocal terms $\tilde{L}_{12}(\Omega), c_\omega$. Note that the constant in the coercivity estimates of the local part of the linear operators $\mathcal{L}_i$, i.e. $\mathcal{L}_{i0}$, is small. For example, this constant is about $-\frac{1}{4}$ in Lemma 2.6.3. We cannot estimate the nonlocal terms in some weighted Sobolev norm and treat them as small perturbations since these nonlocal terms are $O(1)$ for small α. It is crucial for us to exploit the cancellation among various terms so that we can obtain sharp estimates of these nonlocal terms.

We use two models to study $\tilde{L}_{12}(\Omega)$ and the c_ω term. Similar models have been used in our previous work with Hou and Huang on the asymptotically self-similar blowup of the Hou-Luo model [20].

2.6.5.1 Model 1 for nonlocal interaction

We consider the following coupled system

$$\partial_t \Omega = \eta, \quad \eta_t = \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta} \tag{2.75}$$

to study the cancellation between the nonlocal term $\frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}$ in the η equation and η in the Ω equation in (2.59). The above model is derived by dropping other terms in (2.59). The profile $\bar{\eta}$ satisfies $\bar{\eta}(0,\beta) = 0$ and $\bar{\eta} > 0$ for $R > 0$.

The motivation to exploit nonlocal cancellation is inspired by our previous joint works with Hou and Huang on the De Gregorio model [19] and the Hou-Luo model for smooth initial data [20]. In these works, the nonlocal cancellations between Hf and f, where H is the Hilbert transform, play an important role.

From Lemma C.0.4, we have a similar cancellation between $\tilde{L}_{12}(\Omega)$ and Ω. Roughly speaking, $\tilde{L}_{12}(\Omega)$ behaves like $-\Omega$. We perform $L^2(\rho_1)$ estimate on Ω and $L^2(\rho_2)$ estimate on η for some singular weights ρ_1, ρ_2 to be determined and combine both estimates

$$\frac{1}{2}\frac{d}{dt}(\langle \Omega, \Omega\rho_1\rangle + \langle \eta, \eta\rho_2\rangle) = \langle \Omega, \eta\rho_1\rangle + \langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\rho_2\rangle \triangleq I.$$

Formally, I is the sum of the projections of η onto two opposite directions. To exploit this cancellation using Lemma C.0.4, we choose $\rho_1 = \sin(2\beta)\rho_0, \rho_2 = \lambda\frac{\alpha\pi}{2\bar{\eta}}\rho_0$ with some $\lambda > 0$ and singular weight ρ_0, such as $\rho_0 = R^{-3}, R^{-2}$, to obtain

$$I = \langle \Omega \sin(2\beta), \eta\rho_0\rangle + \langle \lambda\tilde{L}_{12}(\Omega), \eta\rho_0\rangle = \langle \Omega\sin(2\beta) + \lambda\tilde{L}_{12}(\Omega), \eta\rho_0\rangle.$$

For $k \in [\frac{3}{2}, 4]$, applying Young's inequality $ab \le sa^2 + \frac{1}{4s}b^2$ for some $s > 0$, we yield

$$I \le s||(\Omega\sin(2\beta) + \lambda\tilde{L}_{12}(\Omega))R^{-k/2}||_2^2 + (4s)^{-1}||\eta\rho_0 R^{k/2}||_2^2 \triangleq A + B.$$

If $k-1>\frac{\pi}{2}\lambda$, using Lemma C.0.4, we obtain

$$
\begin{aligned}
A &= s||\Omega\sin(2\beta)^{1/2}R^{-k/2}||_2^2 - s((k-1)\lambda - \frac{\pi}{2}\lambda^2)\Big|\Big|\tilde{L}_{12}(\Omega)R^{-k/2}\Big|\Big|_{L^2(R)}^2 \\
&\le s||\Omega\sin(2\beta)^{1/2}R^{-k/2}||_2^2.
\end{aligned}
$$

We remark that even estimating the first term in I, which is $\langle \Omega\sin(2\beta), \eta\rho_0\rangle$ and does not involve the nonlocal term, we get an upper bound $s||\Omega\sin(2\beta)^{1/2}R^{-k/2}||_2^2+B$. The above calculation shows that by designing the weights ρ_1,ρ_2 carefully, we can exploit the nonlocal cancellation and obtain an even better estimate. Moreover, we gain a damping term for $\tilde{L}_{12}(\Omega)$ from A.

We will use similar ideas to estimate the $\tilde{L}_{12}(\Omega)$ term in the linearized equation (2.52)-(2.54).

2.6.5.2 Model 2 for the c_ω term

We consider the following coupled system

$$
\partial_t\Omega = \eta + c_\omega\bar{g}, \quad \partial_t\eta = c_\omega\bar{f}, \tag{2.76}
$$

where $\bar{f}(0,\beta)=0, \bar{g}(0,\beta)=0$, $\bar{f},\bar{g}>0$ for $R>0$ with $\bar{f}R^{-1},\bar{g}R^{-1}\in L^1$. Note that the profiles $\bar{\eta}-R\partial_R\bar{\eta}, \bar{\Omega}-R\partial_R\bar{\Omega}$ satisfy similar properties. This system models the c_ω terms in the Ω,η equations in (2.59) by dropping other terms.

Denote $W=\sin(2\beta)R^{-1}$. Recall c_ω in (2.74). We have

$$
c_\omega = -\frac{2}{\pi\alpha}\langle\Omega, \sin(2\beta)R^{-1}\rangle = -\frac{2}{\pi\alpha}\langle\Omega, W\rangle.
$$

Denote $B=\frac{2}{\pi\alpha}\langle\bar{g},W\rangle$. By definition, $B>0$. We derive an ODE for c_ω using the Ω equation

$$
\partial_t\langle\Omega,W\rangle = c_\omega\langle\bar{g},W\rangle + \langle\eta,W\rangle = -\frac{2}{\pi\alpha}\langle\Omega,W\rangle\langle\bar{g},W\rangle + \langle\eta,W\rangle = -B\langle\Omega,W\rangle + \langle\eta,W\rangle.
$$

Multiplying both sides by $(\frac{2}{\pi\alpha})^2\langle\Omega,W\rangle = -\frac{2}{\pi\alpha}c_\omega$, we get

$$
\frac{1}{2}\frac{d}{dt}c_\omega^2 = -Bc_\omega^2 - \frac{2}{\pi\alpha}c_\omega\langle\eta,W\rangle \triangleq I_1 + I_2. \tag{2.77}
$$

We see that the $c_\omega\bar{g}$ term in the Ω equation in (2.76) provides a damping term for c_ω in this ODE. In the $L^2(\rho_2)$ estimates of η in (2.76), we have

$$
\partial_t\langle\eta,\eta\rho_2\rangle = c_\omega\langle\eta,\bar{f}\rho_2\rangle \triangleq I_3.
$$

Since $\bar{f}\rho_2, W > 0$, we can exploit the cancellation between the integral I_2 in (2.77) and I_3. By combining the estimates of both terms, we can obtain better estimates of I_2 and I_3.

In the estimates of (2.52)-(2.54), we will derive a similar ODE for c_ω, which provides a damping term for c_ω^2. This damping term is crucial for us to control the nonlocal c_ω terms in (2.52)-(2.54). There is a coupling term $-c_\omega\langle\eta, W\rangle$ in this ODE similar to I_2 in (2.77). Using an idea similar to the one stated above, we will combine the estimates of such term and the c_ω term in the η equation in (2.53).

2.6.6 Weighted L^2 estimate of Ω, η with a less singular weight

In this subsection, we prove Proposition 2.6.8 to be introduced on the weighted L^2 estimate of Ω, η with less singular weights.

The proof consists of several steps and we sketch it below. Firstly, we introduce the weights in our weighted estimates and motivate the choices of these weights. In Section 2.6.6.2, we estimate the local part of $\mathcal{L}_1, \mathcal{L}_2$ using mainly integration by parts argument, which is similar to that in Section 2.6.3. In Section 2.6.6.3, we use some ideas and estimates similar to those in Model 1 to estimate the interaction among Ω, η and $\tilde{L}_{12}(\Omega)$. In Section 2.6.6.4, we use a direct calculation to estimate the c_ω term in the Ω equation in (2.59). Due to the special form of the weight φ_0 in (2.78), the main term in this estimate is a damping term for $L_{12}^2(\Omega)(0)$. In Section 2.6.6.5, we use some ideas and estimates similar to those in Model 2 in Section 2.6.5.2 to estimate the c_ω term in the η equation. In Section 2.6.6.6, we estimate the angular transport term in the Ω, η equations in (2.59) and treat it as perturbations. In Sections 2.6.6.7, 2.6.6.8, we summarize these estimates, and establish some inequalities to conclude the proof of Proposition 2.6.8.

Since the amount of damping in the energy estimate is small, we cannot overestimate several terms and need to track the coefficients in the estimates. Thus the estimates involve several explicit calculations, which will be presented in Appendix A.0.2. These calculations,(2.83) and (2.92) can also be verified with the aid of Mathematica. [1] In view of Lemma A.0.1, in the following estimates, the reader can regard $\Gamma(\beta) \approx 1, c \approx \frac{2}{\pi}$.

[1] The Mathematica code for these calculations can be found via the link `https://www.dropbox.com/s/y6vfhxi3pa8okvr/Calpha_calculations.nb?dl=0`.

Definition 2.6.7. To exploit the cancellation of the system, we define the following weights

$$\begin{aligned} \psi_0 &\triangleq \frac{9}{8}\frac{\alpha}{c\bar{\eta}}\left(R^{-3} + \frac{3}{2}\frac{1+R}{R^2}\right) = \frac{3}{16}\left(\frac{(1+R)^3}{R^4} + \frac{3}{2}\frac{(1+R)^4}{R^3}\right)\Gamma(\beta)^{-1}, \\ \varphi_0 &\triangleq \frac{(1+R)^3}{R^3}\sin(2\beta), \quad \rho \triangleq R^{-3} + R^{-2}, \end{aligned} \tag{2.78}$$

where $\bar{\eta}$, $\Gamma(\beta) = \cos^{\alpha}(\beta)$ are given in (2.44).

Compared to φ_2 in (2.61), the above weights are less singular in the R, β components.

2.6.6.1 The forms of the singular weights

There are several considerations to choose the above weights ψ_0, φ_0. Firstly, to obtain the damping terms in the energy estimate similar to that in Lemma 2.6.3, the weights in the R direction can be a linear combinations of R^{-k} with various k [19, 42]. See also Lemma 2.6.3. For R near 0, we need the weight to be singular, e.g., R^{-k_1} for a large k_1. For very large R, we need the weight with slow decay, e.g., R^{-k_2} with small k_2. However, using only these two powers R^{-k_1} and R^{-k_2} are not sufficient. Suppose that we use a weight $\varphi_0 = R^{-k_2} + cR^{-k_1}$ with well chosen k_1, k_2, c. Applying a calculation similar to that in (2.66) in Lemma 2.6.3 to $\langle L_{20}\eta, \eta\varphi_0\rangle$, we can obtain $\langle D, \eta^2\varphi_0\rangle$ for some coefficient $D(R,\beta)$. However, D may not be negative in the whole domain as the one that we obtain in (2.66) or $|D(R,\beta)|$ with $R = O(1)$ may become much smaller than $|D(0,\beta)|$ and $|D(\infty,\beta)|$. In either case, we cannot establish linear stability since the nonlocal terms are not small. Therefore, we need to add several powers R^{-k} in φ_0, ψ_0. The first formula of ψ_0 in (2.78) is more important than the second, and it contains three different powers.

Secondly, we add $\bar{\eta}$ in the denominator in ψ_0 to cancel the variable coefficient in our energy estimates, and design φ_0 with the factor $\sin(2\beta)$. These forms are similar to that in Model 1 in Section 2.6.5.1, where we choose $\rho_1 = \sin(2\beta)\rho_0, \rho_2 = \frac{c}{\bar{\eta}}\rho_0$ for some weight ρ_0. These special forms are important and enable us to combine the estimates among $L_{12}(\Omega), \Omega$ and η. This is the most important motivation in designing ψ_0, φ_0 in (2.78). See Model 1 in Section 2.6.5.1 and estimate (2.82).

Thirdly, we choose φ_0 with the factor $\frac{(1+R)^3}{R^3}$ to derive a damping term for $L_{12}(\Omega)(0)$ from the nonlocal term $c_\omega(\bar{\Omega} - D_R\bar{\Omega})$. See (2.85).

The main result in this section is the following

Proposition 2.6.8. *Define an energy $E(R,0)$ and a remaining term $\mathcal{R}(R,0)$*

$$\begin{aligned} E(R,0) &= (||\Omega\varphi_0^{1/2}||_2^2 + ||\eta\psi_0^{1/2}||_2^2 + \mu_0 L_{12}^2(\Omega)(0))^{1/2}, \\ \mathcal{R}(R,0) &= \langle \mathcal{R}_\Omega, \Omega\varphi_0\rangle + \langle \mathcal{R}_\eta, \eta\psi_0\rangle + \mu_0 L_{12}(\Omega)(0)\langle \mathcal{R}_\Omega, \sin(2\beta)R^{-1}\rangle, \quad \mu_0 = \frac{81}{4\pi c}. \end{aligned} \tag{2.79}$$

Assume that Ω, η satisfies that $E(R,0), E(\beta) < +\infty$. For some absolute constant μ_1, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}((E(R,0)^2 + \mu_1 E(\beta,1)^2)) \leq -(\frac{1}{9} - C\alpha)((E(R,0)^2 + \mu_1 E(\beta,1)^2)) \\ &- (4 - C\alpha)L_{12}^2(\Omega)(0) - (\frac{1}{4} - C\alpha)\left|\left|\tilde{L}_{12}^2\rho^{1/2}\right|\right|_{L^2(R)}^2 + \mathcal{R}(R,0) + \mu_1\mathcal{R}(\beta,1), \end{aligned}$$

where the energy $E(\beta,1)$ and the remaining term $\mathcal{R}(\beta,1)$ are defined in (2.67).

Recall $\mathcal{L}_1, \mathcal{L}_2$ in Definition 2.6.1. Direct calculations with weights φ_0, ψ_0 imply

$$\begin{aligned} \langle \mathcal{L}_1(\Omega,\eta), \Omega\varphi_0\rangle &= -\langle R\partial_R\Omega, \Omega\varphi_0\rangle - \langle \Omega, \Omega\varphi_0\rangle + \langle \eta, \Omega\varphi_0\rangle \\ &\quad + c_\omega\langle \bar{\Omega} - R\partial_R\bar{\Omega}, \Omega\varphi_0\rangle - \left\langle \frac{3}{1+R}D_\beta\Omega, \Omega\varphi_0\right\rangle, \\ \langle \mathcal{L}_2(\Omega,\eta), \eta\psi_0\rangle &= -\langle R\partial_R\eta, \eta\psi_0\rangle + \left\langle (-2 + \frac{3}{1+R})\eta, \eta\psi_0\right\rangle + \left\langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\psi_0\right\rangle \\ &\quad + c_\omega\langle \bar{\eta} - R\partial_R\bar{\eta}, \eta\psi_0\rangle - \left\langle \frac{3}{1+R}D_\beta\eta, \eta\psi_0\right\rangle, \end{aligned} \tag{2.80}$$

where we have used the notation $D_\beta = \sin(2\beta)\partial_\beta$ to simplify the formula. We treat the sum of the first two terms on the right hand side as damping terms.

2.6.6.2 The damping terms

We first handle the first two terms on the right hand side of the $\mathcal{L}_1$ equation in (2.80). Using integration by parts for ∂_R, we derive

$$\begin{aligned} &- \langle R\partial_R\Omega, \Omega\varphi_0\rangle - \langle \Omega, \Omega\varphi_0\rangle = -\langle R\varphi_0, \frac{1}{2}\partial_R\Omega^2\rangle - \langle \Omega, \Omega\varphi_0\rangle = \left\langle \frac{1}{2}(R\varphi_0)_R - \varphi_0, \Omega^2\right\rangle, \\ &- \langle R\partial_R\eta, \eta\psi_0\rangle + \left\langle (-2 + \frac{3}{1+R})\eta, \eta\psi_0\right\rangle = \left\langle \frac{1}{2}(R\psi_0)_R + (-2 + \frac{3}{1+R})\psi_0, \eta^2\right\rangle. \end{aligned}$$

Using the formulas of ψ_0, φ_0 (2.78), we compute the coefficients in the inner products in Appendix A.0.2.1 and obtain

$$
\begin{aligned}
&- \langle R\partial_R \Omega, \Omega\varphi_0\rangle - \langle \Omega, \Omega\varphi_0\rangle = -\Big\langle (2R^{-3} + \frac{9}{2}R^{-2} + 3R^{-1} + \frac{1}{2})\sin(2\beta), \Omega^2\Big\rangle, \\
&- \langle R\partial_R \eta, \eta\psi_0\rangle + \Big\langle (-2 + \frac{3}{1+R})\eta, \eta\psi_0\Big\rangle \\
= &- \Big\langle \frac{3(1+R)^2}{32R^4}(1 + 4R + 3R^2 + 3R^3)\Gamma(\beta)^{-1}, \eta^2\Big\rangle.
\end{aligned} \tag{2.81}
$$

2.6.6.3 Estimate of interaction between Ω and η

We use ideas in Model 1 in Section 2.6.5.1 to combine the estimates of $\langle \Omega, \eta\psi\rangle$ and $\langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\psi_0\rangle$. Using (2.44) and (2.78), we can compute

$$
\begin{aligned}
I \triangleq \Big\langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\psi_0\Big\rangle = \Big\langle \frac{9}{4\pi c}\tilde{L}_{12}(\Omega), \eta\Big(\frac{1}{R^3} + \frac{3}{2}\frac{1+R}{R^2}\Big)\Big\rangle, \\
II \triangleq \langle \Omega, \eta\varphi_0\rangle = \Big\langle \Omega\sin(2\beta), \eta\Big(\frac{1}{R^3} + 3\frac{1+R}{R^2} + 1\Big)\Big\rangle,
\end{aligned}
$$

where c is defined in (2.44) and satisfies $c = \frac{2}{\pi} + O(\alpha)$ (see Lemma A.0.1). We design ψ_0 (2.78) so that the denominator in ψ_0 and the coefficient $\bar{\eta}$ in I cancel.

Applying the Cauchy-Schwarz inequality, we yield

$$
\begin{aligned}
I + II = &\Big\langle \Omega\sin(2\beta) + \frac{9}{4\pi c}\tilde{L}_{12}(\Omega), \eta R^{-3}\Big\rangle \\
&+ \Big\langle \Omega\sin(2\beta) + \frac{9}{8\pi c}\tilde{L}_{12}(\Omega), 3\eta\frac{1+R}{R^2}\Big\rangle + \langle \Omega\sin(2\beta), \eta\rangle \\
&\leq \frac{4}{3}\Big\langle (\Omega\sin(2\beta) + \frac{9}{4\pi c}\tilde{L}_{12}(\Omega))^2, R^{-3}\Big\rangle + \frac{1}{4\cdot 4/3}\langle \eta^2, R^{-3}\rangle \\
&+ 6\Big\langle (\Omega\sin(2\beta) + \frac{9}{8\pi c}\tilde{L}_{12}(\Omega))^2, R^{-2}\Big\rangle + \frac{3^2}{4\cdot 6}\Big\langle \eta^2, \frac{(1+R)^2}{R^2}\Big\rangle \\
&+ \frac{1}{3}\Big\langle \Omega^2, \frac{1+R}{R}\sin(2\beta)^2\Big\rangle + \frac{3}{4}\Big\langle \eta^2, \frac{R}{1+R}\Big\rangle = \sum_{i=1}^{6} J_i.
\end{aligned} \tag{2.82}
$$

We design the special forms ψ_0, φ_0 in (2.78) to obtain the good form $\Omega\sin(2\beta) + C\tilde{L}_{12}(\Omega)$ for some $C > 0$ in $I + II$. Next, we exploit the cancellation between Ω and $\tilde{L}_{12}(\Omega)$ using Lemma C.0.4. We apply Lemma C.0.4 with $k = 2, 3$ to simplify J_1, J_3 defined above:

$$
\begin{aligned}
J_1 + J_3 =& \Big\langle \left(\frac{4}{3}R^{-3} + 6R^{-2}\right)\sin(2\beta)^2, \Omega^2 \Big\rangle \\
& - \frac{4}{3}\left(2\cdot\frac{9}{4\pi c} - \frac{\pi}{2}\frac{9^2}{(4\pi c)^2}\right) ||\tilde{L}_{12}(\Omega)R^{-3/2}||^2_{L^2(R)} \\
& - 6(\frac{9}{8\pi c} - \frac{\pi}{2}\frac{9^2}{(8\pi c)^2})\ ||\tilde{L}_{12}(\Omega), R^{-1}||^2_{L^2(R)} \triangleq M_1 + M_2 + M_3.
\end{aligned}
$$

We further simplify M_2, M_3 defined above. Using Lemma A.0.1, we have $|\pi c - 2| \lesssim \alpha$ and

$$
\begin{aligned}
&-\frac{4}{3}\cdot\frac{9}{4\pi c}(2 - \frac{\pi}{2}\frac{9}{4\pi c}) \leq -\frac{4}{3}\cdot\frac{9}{8}(2 - \frac{\pi}{2}\cdot\frac{9}{8}) + C\alpha < -\frac{1}{4} + C\alpha, \\
&-6\cdot\frac{9}{8\pi c}(1 - \frac{\pi}{2}\frac{9}{8\pi c}) \leq -6\cdot\frac{9}{16}(1 - \frac{\pi}{2}\cdot\frac{9}{16}) + C\alpha < -\frac{1}{4} + C\alpha,
\end{aligned}
\tag{2.83}
$$

for some absolute constant C. It follows that

$$
\begin{aligned}
M_2 + M_3 &\leq (-\frac{1}{4} + C\alpha)(||\tilde{L}_{12}(\Omega), R^{-3/2}||^2_{L^2(R)} + ||\tilde{L}_{12}(\Omega), R^{-1}||^2_{L^2(R)}) \\
&= (-\frac{1}{4} + C\alpha)||\tilde{L}_{12}(\Omega)\rho^{1/2}||^2_{L^2},
\end{aligned}
$$

where we have used the notation ρ defined in (2.78). Therefore, we yield the damping for $\tilde{L}_{12}(\Omega)$.

Remark 2.6.9. The above computations of J_1, J_2, J_3, J_4 are exactly the same as those in Model 1 in Section 2.6.5.1. We choose the constants in the weights (2.78) carefully so that when we apply Lemma C.0.4, the constant $-((k-1)\lambda - \frac{\pi}{2}\lambda^2)$ in (A.7) is negative, i.e. (2.83).

Using (2.82), the above estimate of $M_2 + M_3$ in $J_1 + J_3$ and $\sin(2\beta)^2 \leq \sin(2\beta)$, we prove

$$
\begin{aligned}
&\Big\langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\psi_0 \Big\rangle + \langle \Omega, \eta\varphi_0 \rangle = I + II \\
\leq& \Big\langle \left(\frac{4}{3}R^{-3} + 6R^{-2} + \frac{1+R}{3R}\right)\sin(2\beta), \Omega^2 \Big\rangle \\
+& \Big\langle \frac{3}{16}R^{-3} + \frac{3}{8}\frac{(1+R)^2}{R^2} + \frac{3}{4}\frac{R}{1+R}, \eta^2 \Big\rangle - (\frac{1}{4} - C\alpha)||\tilde{L}_{12}(\Omega)\rho^{1/2}||^2_{L^2(R)}.
\end{aligned}
\tag{2.84}
$$

2.6.6.4 Estimate of the projection c_ω in the Ω equation

We estimate the terms involving c_ω in (2.80) in this subsection. Notice that c_ω defined in (2.47) is the projection of Ω onto some function. Using (2.44) and (2.78), we can calculate

$$\langle \bar{\Omega} - R\partial_R \bar{\Omega}, \Omega \varphi_0 \rangle = \Big\langle \frac{\alpha}{c}\Gamma(\beta)\frac{6R^2}{(1+R)^3}, \Omega \frac{(1+R)^3}{R^3}\sin(2\beta) \Big\rangle = \frac{6\alpha}{c}\Big\langle \frac{\sin(2\beta)\Gamma(\beta)}{R}, \Omega \Big\rangle.$$

We show that the above projection is almost equal to $L_{12}(\Omega)(0)$. Notice that

$$\begin{aligned} &\frac{1}{c}\Big\langle \frac{\sin(2\beta)\Gamma(\beta)}{R}, \Omega \Big\rangle - \frac{\pi}{2}\Big\langle \frac{\sin(2\beta)}{R}, \Omega \Big\rangle \\ =&\frac{1}{c}\Big\langle \frac{\sin(2\beta)(\Gamma(\beta)-1)}{R},\ \Omega \Big\rangle + (\frac{1}{c} - \frac{\pi}{2})\langle \frac{\sin(2\beta)}{R}, \Omega \rangle \triangleq I + II. \end{aligned}$$

Using Lemma A.0.1, (2.78) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |I| &\lesssim \alpha \langle \frac{1}{R}\sin(2\beta)^{1/2}, |\Omega| \rangle \lesssim \alpha \Big|\Big| \Omega \frac{(1+R)^{3/2}}{R^{3/2}} \sin(2\beta)^{1/2} \Big|\Big|_2 \Big|\Big| \frac{1}{R} \cdot \frac{R^{3/2}}{(1+R)^{3/2}} \Big|\Big|_2 \\ &\lesssim \alpha ||\Omega \varphi_0^{1/2}||_2, \\ |II| &\lesssim \alpha \langle \frac{1}{R}\sin(2\beta), |\Omega| \rangle \lesssim \alpha \Big|\Big| \Omega \frac{(1+R)^{3/2}}{R^{3/2}} \sin(2\beta) \Big|\Big|_2 \Big|\Big| \frac{1}{R} \cdot \frac{R^{3/2}}{(1+R)^{3/2}} \sin(2\beta) \Big|\Big|_2 \\ &\lesssim \alpha ||\Omega \varphi_0^{1/2}||_2. \end{aligned}$$

It follows that

$$\Big| \frac{1}{\alpha}\langle \bar{\Omega} - R\partial_R \bar{\Omega}, \Omega \varphi_0 \rangle - 6 \cdot \frac{\pi}{2}\Big\langle \frac{\sin(2\beta)}{R}, \Omega \Big\rangle \Big| \leq 6|I + II| \lesssim \alpha ||\Omega \varphi_0^{1/2}||_2.$$

Recall the definition of c_ω in (2.47). Using the above estimate and then the formula of $L_{12}(\Omega)(0)$ (2.31), we have

$$\begin{aligned} &c_\omega \langle \bar{\Omega} - R\partial_R \bar{\Omega}, \Omega \varphi_0 \rangle = -\frac{2}{\pi} L_{12}(\Omega)(0) \cdot \frac{1}{\alpha} \langle \bar{\Omega} - R\partial_R \bar{\Omega}, \Omega \varphi_0 \rangle \\ \leq& -\frac{2}{\pi} L_{12}(\Omega)(0) \cdot 6 \cdot \frac{\pi}{2}\Big\langle \frac{\sin(2\beta)}{R}, \Omega \Big\rangle + C\alpha |L_{12}(\Omega)(0)| \cdot ||\Omega \varphi_0^{1/2}||_2 \\ =& -6(L_{12}(\Omega)(0))^2 + C\alpha |L_{12}(\Omega)(0)| \cdot ||\Omega \varphi_0^{1/2}||_2 \\ \leq& -(6 - C\alpha)(L_{12}(\Omega)(0))^2 + C\alpha ||\Omega \varphi_0^{1/2}||_2. \end{aligned} \tag{2.85}$$

By choosing φ_0 in (2.78) carefully, we obtain a damping term for $L_{12}(\Omega)(0)^2$ from $c_\omega(\bar{\Omega} - R\partial_R \bar{\Omega})$. This is one of the motivations to choose the special form of φ_0.

2.6.6.5 Estimate of the projection c_ω in the η equation

We use some ideas and estimates similar to those in Model 2 in Section 2.6.5.2 to estimate the c_ω term in the η equation (2.80).

Using $c_\omega = -\frac{2}{\pi\alpha} L_{12}(\Omega)(0)$ (2.47) and expanding the coefficient $(\bar{\eta} - R\partial_R \bar{\eta})\psi_0$ using the formulas (2.44) and (2.78), which is presented in Appendix A.0.2.2, we derive

$$\begin{aligned} c_\omega \langle \bar{\eta} - R\partial_R \bar{\eta}, \eta\psi_0 \rangle &= -\frac{27}{4\pi c} L_{12}(\Omega)(0) \Big\langle \eta, \frac{1}{(1+R)R^2} \Big\rangle - \frac{81}{8\pi c} L_{12}(\Omega)(0) \Big\langle \eta, \frac{1}{R} \Big\rangle \\ &\triangleq A_1 + A_2. \end{aligned} \tag{2.86}$$

An ODE for $L_{12}(\Omega)(0)$ Using the Ω equation in (2.59) and derivation similar to that in Model 2 in Section 2.6.5.2, we derive the following ODE for $L_{12}(\Omega)(0)$ in Appendix A.0.2.3

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\frac{81}{4\pi c} L_{12}^2(\Omega)(0) = \frac{81}{4\pi c}\Big(&- 4L_{12}^2(\Omega)(0) + L_{12}(\Omega)(0)\Big\langle \eta, \frac{\sin(2\beta)}{R} \Big\rangle \\ &- L_{12}(\Omega)(0) \Big\langle \frac{3\sin(2\beta)}{(1+R)R}, D_\beta \Omega \Big\rangle + L_{12}(\Omega)(0) \Big\langle \mathcal{R}_\Omega, \frac{\sin(2\beta)}{R} \Big\rangle \Big). \end{aligned} \tag{2.87}$$

Note that we have multiplied both sides of the ODE for $L_{12}^2(\Omega)(0)$ by the constant $\frac{81}{4\pi c}$ and will include $\frac{81}{4\pi c} L_{12}^2(\Omega)(0)$ in the energy $E(R,0)$ (2.79). The first term on the right hand side provides damping for $L_{12}^2(\Omega)(0)$, which is similar to that in (2.77). It enables us to control the term A_1, A_2 in (2.86). Based on the idea in Model 2 in Section 2.6.5.2 and the fact that the integrands in A_2 and $L_{12}(\Omega)(0)\Big\langle \eta, \frac{\sin(2\beta)}{R} \Big\rangle$ in (2.87) have different signs, we combine the estimate of A_2 in (2.86) and the η term in (2.87) as follows to exploit cancellation

$$A_3 \triangleq A_2 + \frac{81}{4\pi c} L_{12}(\Omega)(0) \Big\langle \eta, \frac{\sin(2\beta)}{R} \Big\rangle = \frac{81}{8\pi c} L_{12}(\Omega)(0) \Big\langle \eta, \frac{1}{R}(-1 + 2\sin(2\beta)) \Big\rangle. \tag{2.88}$$

Next, we estimate A_1 in (2.86) and A_3 by treating them as perturbation. Applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} A_3 &\leq \frac{81}{8\pi c} |L_{12}(\Omega)(0)| \cdot \Big|\Big| \eta \frac{(1+R)^2}{R^{3/2}} \Big|\Big|_2 \Big|\Big| \frac{R^{3/2}}{(1+R)^2} \frac{1}{R} (1 - 2\sin(2\beta)) \Big|\Big|_2, \\ A_1 &\leq \frac{27}{4\pi c} |L_{12}(\Omega)(0)| \cdot \Big|\Big| \eta \frac{(1+R)^{3/2}}{R^2} \Big|\Big|_2 \Big|\Big| \frac{R^2}{(1+R)^{3/2}} \cdot \frac{1}{(1+R)R^2} \Big|\Big|_2. \end{aligned} \tag{2.89}$$

The integrals on R, β in (2.89) equal to $\sqrt{\frac{1}{6}(\frac{3\pi}{2} - 4)}, \sqrt{\frac{\pi}{8}}$, respectively, which are computed in Appendix A.0.2.4. Then we reduce (2.89) to

$$A_3 \leq b_1 |L_{12}(\Omega)(0)| \Big|\Big| \eta \frac{(1+R)^2}{R^{3/2}} \Big|\Big|_2, \quad A_1 \leq b_2 |L_{12}(\Omega)(0)| \Big|\Big| \eta \frac{(1+R)^{3/2}}{R^2} \Big|\Big|_2. \tag{2.90}$$

where b_1, b_2 are given by

$$b_1 \triangleq \frac{81}{8\pi c}\sqrt{\frac{1}{6}(\frac{3\pi}{2}-4)}, \quad b_2 \triangleq \frac{27}{4\pi c}\sqrt{\frac{\pi}{8}}.$$

Using the Young's inequality $ab \leq sa^2 + \frac{1}{4s}b^2$ for any $s > 0$, we get

$$\begin{aligned} A_1 + A_3 \leq & \frac{1}{32}\Big|\Big|\eta\frac{(1+R)^2}{R^{3/2}}\Big|\Big|_2^2 + \frac{9}{128}\Big|\Big|\eta\frac{(1+R)^{3/2}}{R^2}\Big|\Big|_2^2 \\ & + L_{12}^2(\Omega)(0)\Big(\frac{b_1^2}{4\times 1/32} + \frac{b_2^2}{4\times 9/128}\Big). \end{aligned} \tag{2.91}$$

Using Lemma A.0.1 for the estimate of c and a direct calculation yield

$$\begin{aligned} & \frac{b_1^2}{1/8} + \frac{b_2^2}{9/32} - \frac{81}{4\pi c}\cdot 4 = 8\left(\frac{81}{8\pi c}\right)^2 \frac{1}{6}(\frac{3\pi}{2}-4) + \frac{32}{9}\left(\frac{27}{4\pi c}\right)^2\frac{\pi}{8} - \frac{81}{\pi c} \\ \leq & \frac{4}{3}\left(\frac{81}{16}\right)^2(\frac{3\pi}{2}-4) + \frac{4\pi}{9}\left(\frac{27}{8}\right)^2 - \frac{81}{2} + C\alpha < C\alpha. \end{aligned} \tag{2.92}$$

Combining the identities (2.86), (2.88), the damping term of $L_{12}^2(\Omega)(0)$ in (2.87) and the estimate (2.91), we prove

$$\begin{aligned} & c_\omega\langle\bar\eta - R\partial_R\bar\eta, \eta\psi_0\rangle + \frac{81}{4\pi c}L_{12}(\Omega)(0)\Big\langle\eta, \frac{\sin(2\beta)}{R}\Big\rangle - \frac{81}{4\pi c}\cdot 4L_{12}^2(\Omega)(0) \\ = & A_1 + A_2 + \frac{81}{4\pi c}L_{12}(\Omega)(0)\Big\langle\eta, \frac{\sin(2\beta)}{R}\Big\rangle - \frac{81}{4\pi c}\cdot 4L_{12}^2(\Omega)(0) \\ = & A_1 + A_3 - \frac{81}{4\pi c}\cdot 4L_{12}^2(\Omega)(0) \\ = & \langle\eta^2, \frac{1}{32}\frac{(1+R)^4}{R^3} + \frac{9}{128}\frac{(1+R)^3}{R^4}\rangle + L_{12}^2(\Omega)(0)\left(\frac{b_1^2}{1/8} + \frac{b_2^2}{9/32} - \frac{81}{\pi c}\right) \\ \leq & \frac{3}{16}\Big\langle\eta^2, \frac{1}{6}\frac{(1+R)^4}{R^3} + \frac{3}{8}\frac{(1+R)^3}{R^4}\Big\rangle + C\alpha L_{12}^2(\Omega)(0), \end{aligned} \tag{2.93}$$

where we have used (2.92) to derive the last inequality.

2.6.6.6 Estimate of the angular transport term

From the definition of the weights (2.61), (2.78), we have

$$\varphi_0 \lesssim \varphi_2, \quad (1+R)^{-1}\psi_0 \lesssim \psi_2, \quad \Big|\Big|\frac{3\sin(2\beta)}{(1+R)R}\varphi_2^{-1/2}\Big|\Big|_2 \lesssim 1.$$

Therefore, we can estimate the angular transport terms in (2.80), (2.87) as follows

$$\begin{aligned} & -\langle\frac{3D_\beta\Omega}{1+R}, \Omega\varphi_0\rangle \lesssim ||D_\beta\Omega\varphi_2^{1/2}||_2||\Omega\varphi_0^{1/2}||_2, \quad -\langle\frac{3D_\beta\eta}{1+R}, \eta\psi_0\rangle \lesssim ||D_\beta\eta\psi_2^{1/2}||_2||\eta\psi_0^{1/2}||_2, \\ & -\frac{81}{4\pi c}L_{12}(\Omega)(0)\Big\langle\frac{3\sin(2\beta)}{(1+R)R}, D_\beta\Omega\Big\rangle \lesssim |L_{12}(\Omega)(0)|\Big|\Big|D_\beta\Omega\varphi_2^{1/2}||_2, \end{aligned}$$

where we have used $c^{-1} \lesssim 1$ (see Lemma A.0.1). Using the energy notations $E(\beta,1)$ (2.67) and $E(R,0)$ (2.79), we further derive

$$
\begin{aligned}
&-\langle \frac{3D_\beta\Omega}{1+R}, \Omega\varphi_0\rangle - \langle \frac{3D_\beta\eta}{1+R}, \eta\psi_0\rangle - \frac{81}{4\pi c} L_{12}(\Omega)(0)\Big\langle \frac{3\sin(2\beta)}{(1+R)R}, D_\beta\Omega\Big\rangle \\
\le& K_1 E(R,0)E(\beta,1),
\end{aligned} \tag{2.94}
$$

for some absolute constant K_1. We remark that the absolute constants $K_1, K_2, ...$ do not change from line to line.

2.6.6.7 Completing the estimates with a less singular weight

Combining the estimates (2.80)-(2.84), (2.85), (2.87), (2.93), (2.94) and using the notations $E(R,0), \mathcal{R}(R,0)$ (2.79), we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}E(R,0)^2 &= \frac{1}{2}\frac{d}{dt}\left(||\Omega\varphi_0^{1/2}||_2^2 + ||\eta\psi_0^{1/2}||_2^2 + \frac{81}{4\pi c}L_{12}^2(\Omega)(0) \right) \\
&\le \langle \Omega^2, \sin(2\beta)D(\Omega)\rangle + \langle \eta^2, D(\eta)\rangle + (-\frac{1}{4} + C\alpha)||\tilde{L}_{12}(\Omega)\rho^{1/2}||_{L^2(R)}^2 \\
&\quad + L_{12}^2(\Omega)(0)\,(-6 + C\alpha) + C\alpha\langle \Omega^2, \varphi_0\rangle + K_1E(R,0)E(\beta,1) + \mathcal{R}(R,0),
\end{aligned} \tag{2.95}
$$

where $D(\Omega), D(\eta)$ are given by

$$
\begin{aligned}
D(\Omega) &\triangleq -2R^{-3} - \frac{9}{2}R^{-2} - 3R^{-1} - \frac{1}{2} + \frac{4}{3}R^{-3} + 6R^{-2} + \frac{1+R}{3R}, \\
D(\eta) &\triangleq -\frac{3(1+R)^2}{32R^4}(1 + 4R + 3R^2 + 3R^3)\Gamma(\beta)^{-1} \\
&\quad + \left(\frac{3}{16}R^{-3} + \frac{3}{8}\frac{(1+R)^2}{R^2} + \frac{3R}{4(1+R)} \right) + \frac{3}{16}\Big(\frac{1}{6}\frac{(1+R)^4}{R^3} + \frac{3}{8}\frac{(1+R)^3}{R^4} \Big).
\end{aligned} \tag{2.96}
$$

Recall the weights φ_0, ψ_0 in (2.78). In Appendix A.0.2.5, we estimate $D(\Omega), D(\eta)$ and prove

$$
\sin(2\beta)D(\Omega) \le -\frac{1}{6}\varphi_0, \quad D(\eta) \le -\frac{1}{8}\psi_0, \tag{2.97}
$$

which only involves elementary estimates.

For $L_{12}^2(\Omega)(0)$ in (2.95), we use Lemma A.0.1 about c ($c\pi = 2 + O(\alpha)$) to get

$$
-6 + C\alpha \le -\frac{1}{8} \times \frac{81}{8} - 4 + C\alpha \le -\frac{1}{8} \times \frac{81}{4\pi c} - 4 + C\alpha,
$$

which implies

$$
(-6 + C\alpha)L_{12}^2(\Omega)(0) \le -\frac{1}{8} \cdot \frac{81}{4\pi c}L_{12}^2(\Omega)(0) - (4 - C\alpha)L_{12}^2(\Omega)(0), \tag{2.98}
$$

where C is some absolute constant and may vary from line to line. Observe that

$$K_1 E(R,0)E(\beta,1) \leq \frac{1}{100}E(R,0)^2 + 100K_1^2 E^2(\beta,1). \tag{2.99}$$

Recall $E(R,0)$ in (2.79). Finally, substituting the estimates (2.97)-(2.99) in (2.95), we prove

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(R,0)^2 \leq &-(\frac{1}{6} - C\alpha)||\Omega\varphi_0^{1/2}||_2^2 - \frac{1}{8}||\eta\psi_0^{1/2}||_2^2 - \frac{1}{8}\cdot\frac{81}{4\pi c}L_{12}^2(\Omega)(0)\\ &- (4 - C\alpha)L_{12}^2(\Omega)(0) - (\frac{1}{4} - C\alpha)||\tilde{L}_{12}(\Omega)\rho^{1/2}||_{L^2(R)}\\ &+ \frac{1}{100}E(R,0)^2 + 100K_1^2E^2(\beta,1) + \mathcal{R}(R,0)\\ \leq&(-\frac{1}{9} + C\alpha)E^2(R,0) - (\frac{1}{4} - C\alpha)||\tilde{L}_{12}(\Omega)\rho^{1/2}||_{L^2(R)}\\ &- (4 - C\alpha)L_{12}^2(\Omega)(0) + 100K_1^2E^2(\beta,1) + \mathcal{R}(R,0),\end{aligned} \tag{2.100}$$

where we have used $-\frac{1}{6} + C\alpha + \frac{1}{100}, -\frac{1}{8} + \frac{1}{100} < -\frac{1}{9} + C\alpha$ to derive the last inequality.

2.6.6.8 Linear stability with a less singular weight

Using the reformulation (2.59), and the notations $E(\beta,1)$ and $\mathcal{R}(\beta,1)$ defined in (2.67), we have

$$\frac{1}{2}\frac{d}{dt}(E(\beta,1))^2 = \langle D_\beta\mathcal{L}_1(\Omega,\eta),(D_\beta\Omega)\varphi_2\rangle + \langle D_\beta\mathcal{L}_2(\Omega,\eta),(D_\beta\eta)\varphi_2\rangle + \mathcal{R}(\beta,1). \tag{2.101}$$

Now we combine (2.68) and (2.100) to establish the linear stability of (2.52)-(2.53) with the less singular weight (2.78). Firstly, we choose an absolute constant μ_1 such that

$$100K_1^2 < \frac{1}{20}\mu_1,$$

where the absolute constant K_1 is determined in (2.94). From (2.78), we have $R^{-2} \leq \rho$. Hence,

$$||\tilde{L}_{12}(\Omega)R^{-1}||_{L^2(R)}^2 \leq ||\tilde{L}_{12}(\Omega)\rho^{1/2}||_{L^2(R)}^2.$$

Combining Proposition 2.6.6, (2.100), the formulation (2.101), and the above estimates, we establish the estimate for $E(R,0)^2 + \mu_1 E(\beta,1)^2$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}((E(R,0)^2 + \mu_1E(\beta,1)^2)) \leq -(\frac{1}{9} - C\alpha)((E(R,0)^2 + \mu_1E(\beta,1)^2))\\ - (4 - C\alpha)L_{12}^2(\Omega)(0) - (\frac{1}{4} - C\alpha)||\tilde{L}_{12}^2\rho^{1/2}||_{L^2(R)}^2 + \mathcal{R}(R,0) + \mu_1\mathcal{R}(\beta,1).\end{aligned} \tag{2.102}$$

The proof of Proposition 2.6.8 is now complete.

2.6.7 Weighted L^2 estimate of Ω, η with a more singular weight

With the linear stability (2.102) with a less singular weight, we can proceed to perform the weighted L^2 estimate with a more singular weight.

Definition 2.6.10. Define an energy $E(R,1)$ and a remaining term $\mathcal{R}(R,1)$ by

$$\begin{aligned} E(R,1) &\triangleq \left(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2 \right)^{1/2}, \\ \mathcal{R}(R,1) &\triangleq \langle \mathcal{R}_\Omega, \Omega\varphi_1\rangle + \langle \mathcal{R}_\eta, \eta\varphi_1\rangle, \end{aligned} \tag{2.103}$$

where φ_1, ψ_1 are given in Definition 2.6.2.

The main result in this Section is the following.

Proposition 2.6.11. *Assume that $\Omega\varphi_1^{1/2}, \ \eta\varphi_1^{1/2} \in L^2$. We have that*

$$\begin{aligned} &\langle \mathcal{L}_1(\Omega,\eta), \Omega\varphi_1\rangle + \langle \mathcal{L}_2(\Omega,\eta), \eta\varphi_1\rangle \\ \leq& -\frac{1}{6}(E(R,1))^2 + K_3\left(L_{12}^2(\Omega)(0) + \left|\left| \tilde{L}_{12}(\Omega)R^{-1}\right|\right|_{L^2(R)}^2 \right), \end{aligned}$$

where $\mathcal{L}_1, \mathcal{L}_2$ are defined in Definition 2.6.1, $K_3 > 0$ is some fixed absolute constant.

Proof of Proposition 2.6.11. A direct calculation yields

$$\begin{aligned} \langle \mathcal{L}_1(\Omega,\eta), \Omega\varphi_1\rangle &= \langle \mathcal{L}_{10}(\Omega,\eta), \Omega\varphi_1\rangle + c_\omega\langle \bar{\Omega} - R\partial_R\bar{\Omega}, \Omega\varphi_1\rangle, \\ \langle \mathcal{L}_2(\Omega,\eta), \eta\varphi_1\rangle &= \langle \mathcal{L}_{20}(\eta), \eta\varphi_1\rangle + \frac{2}{\pi\alpha}\langle \tilde{L}_{12}(\Omega)\bar{\eta}, \eta\varphi_1\rangle + c_\omega\langle \bar{\eta} - R\partial_R\bar{\eta}, \eta\varphi_1\rangle. \end{aligned} \tag{2.104}$$

Applying Lemma 2.6.3 with $\varphi = \varphi_1$ and $\delta = \sigma = \frac{99}{100}$, we yield

$$\begin{aligned} &\langle \mathcal{L}_{10}(\Omega,\eta), \Omega\varphi_1\rangle + \langle \mathcal{L}_{20}(\eta), \eta\varphi_1\rangle \\ \leq& (-\frac{1}{4} + 3|1-\sigma|)(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2) < -\frac{1}{5}(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2). \end{aligned}$$

Recall $c_\omega = -\frac{2}{\pi\alpha}L_{12}(\Omega)(0)$ (2.47). Using (A.16) in Lemma A.0.6 and the Cauchy-Schwarz inequality, we obtain

$$|c_\omega\langle(\bar{\Omega} - R\partial_R\bar{\Omega}), \Omega\varphi_1\rangle| + |c_\omega\langle(\bar{\eta} - R\partial_R\bar{\eta}), \eta\varphi_1\rangle| \lesssim |L_{12}(\Omega)(0)|(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2)^{1/2}.$$

For $\tilde{L}_{12}(\Omega)$ in (2.104), using the Cauchy-Schwarz inequality, we derive

$$\left\langle \frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\bar{\eta}, \eta\varphi_1 \right\rangle \lesssim \alpha^{-1}||\tilde{L}_{12}(\Omega)\bar{\eta}\varphi_1^{1/2}||_2 ||\eta\varphi_1^{1/2}||_2 \lesssim \left|\left|\tilde{L}_{12}(\Omega)R^{-1}\right|\right|_{L^2(R)} ||\eta\varphi_1^{1/2}||_2,$$

where we have applied Lemma A.0.3 and (A.15) in Lemma A.0.6 in the second inequality.

Using the Cauchy-Schwarz inequality and the energy notation $E(R,1)$ (2.103), we complete the proof of Proposition 2.6.11. $\square$

2.6.8 Weighted L^2 estimate of $D_\beta \xi$ and ξ

The estimates of ξ are simpler since the main terms in the equation of ξ (2.54) do not couple with Ω, η directly. We use the weights ψ_1, ψ_2 in Definition 2.6.2.

Proposition 2.6.12. *Suppose that $\psi_1^{1/2}\xi,\ \psi_2^{1/2} D_\beta \xi \in L^2$. We have*

$$\begin{aligned} \langle \mathcal{L}_3(\Omega,\xi), \xi\psi_1\rangle \leq & (-\frac{1}{3} + C\alpha) ||\xi \psi_1^{1/2}||_2^2 \\ & + C\alpha \left(L_{12}^2(\Omega)(0) + ||\tilde{L}_{12}(\Omega) R^{-1}||^2_{L^2(R)} \right), \quad (2.105) \\ \langle D_\beta \mathcal{L}_3(\Omega,\xi), (D_\beta\xi)\psi_2\rangle \leq & (-\frac{1}{3} + C\alpha) ||D_\beta \xi \psi_2^{1/2}||_2^2 \\ & + C\alpha \left(L_{12}^2(\Omega)(0) + ||\tilde{L}_{12}(\Omega) R^{-1}||^2_{L^2(R)} \right). \quad (2.106) \end{aligned}$$

Proof of Proposition 2.6.12. Since D_β commutes with $\mathcal{L}_3$ (see Definition 2.6.1) and $\tilde{L}_{12}(R)$ does not depend on β, a direct calculation implies

$$\begin{aligned} \langle \mathcal{L}_3(\Omega,\xi), \xi\psi_1\rangle &= \langle \mathcal{L}_{30}(\xi), \xi\psi_1\rangle - \frac{2}{\pi\alpha}\langle \tilde{L}_{12}(\Omega)\bar{\xi}, \xi\psi_1\rangle + c_\omega \langle 3\bar{\xi} - D_R\bar{\xi}, \xi\psi_1\rangle \\ \langle D_\beta\mathcal{L}_3(\Omega,\xi), (D_\beta\xi)\psi_2\rangle &= \langle \mathcal{L}_{30}(D_\beta\xi), (D_\beta\xi)\psi_2\rangle - \frac{2}{\pi\alpha}\langle \tilde{L}_{12}(\Omega) D_\beta\bar{\xi}, (D_\beta\xi)\psi_2\rangle \\ & \quad + c_\omega\langle D_\beta(3\bar{\xi} - D_R\bar{\xi}), \xi\psi_2\rangle. \end{aligned} \quad (2.107)$$

Applying (2.64) in Lemma 2.6.3 with $\psi = \psi_1$ (a constant multiple of ψ does not change the estimate in (2.64)) and with $\psi = \psi_2$ (see Definition 2.6.2), respectively, we derive

$$\begin{aligned} \langle \mathcal{L}_{30}(\xi), \xi\psi_1\rangle &\leq (-\frac{1}{2} + 3|1-\sigma|) ||\xi\psi_1^{1/2}||_2^2 < -\frac{3}{8}||\xi\psi_1^{1/2}||_2^2, \\ \langle \mathcal{L}_{30}(D_\beta\xi), (D_\beta\xi)\psi_2\rangle &\leq (-\frac{1}{2} + 3(|1-\gamma| \vee |1-\sigma|)) ||D_\beta\xi\psi_2^{1/2}||_2^2 \\ &\leq (-\frac{3}{8} + \alpha) ||D_\beta\xi\psi_2^{1/2}||_2^2, \end{aligned} \quad (2.108)$$

where $\gamma = 1 + \frac{\alpha}{10}, \sigma = \frac{99}{100}$. Using the Cauchy-Schwarz inequality, we yield

$$\begin{aligned} \left| -\frac{2}{\pi\alpha}\langle \tilde{L}_{12}(\Omega)\bar{\xi}, \xi\psi_1\rangle \right| &\lesssim \alpha^{-1} ||\tilde{L}_{12}(\Omega)\bar{\xi}\psi_1^{1/2}||_2 ||\xi\psi_1^{1/2}||_2 \\ &\lesssim \alpha ||\tilde{L}_{12}(\Omega)R^{-1}||_2 ||\xi\psi_1^{1/2}||_2, \end{aligned} \quad (2.109)$$

where we have applied Lemma A.0.3 and (A.25) in Lemma A.0.8 to derive the second inequality.

Using the Cauchy-Schwarz inequality, (2.47) and Lemma A.0.8, we obtain

$$\begin{aligned} c_\omega \langle 3\bar{\xi} - D_R\bar{\xi}, \xi\psi_1\rangle &\lesssim \alpha^{-1}|L_{12}(\Omega)(0)| \cdot ||(3\bar{\xi} - D_R\bar{\xi})\psi_1^{1/2}||_2 ||\xi\psi_1^{1/2}||_2 \\ &\lesssim \alpha|L_{12}(\Omega)(0)| \cdot ||\xi\psi_1^{1/2}||_2. \end{aligned} \tag{2.110}$$

Plugging (2.108)-(2.110) in (2.107) and using the Cauchy-Schwarz inequality , we prove (2.105).

The proof of (2.106) is completely similar. We apply estimates similar to those in (2.109)-(2.110) and Lemmas A.0.3, A.0.8 to control the c_ω and $\tilde{L}_{12}(\Omega)$ terms. Combining these estimates, using the second inequality in (2.108) and then the Cauchy-Schwarz inequality prove (2.106). □

2.6.8.1 The weighted L^2 energy

Using the reformulation (2.59), we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2) &= \langle \mathcal{L}_1(\Omega,\eta), \Omega\varphi_1\rangle + \langle \mathcal{L}_2(\Omega,\eta), \eta\varphi_1\rangle \\ &\quad + \langle \mathcal{R}_\Omega, \Omega\varphi_1\rangle + \langle \mathcal{R}_\eta, \eta\varphi_1\rangle, \\ \frac{1}{2}\frac{d}{dt}||\xi\psi_1^{1/2}||_2^2 &= \langle \mathcal{L}_3(\xi), \xi\psi_1\rangle + \langle \mathcal{R}_\xi, \xi\psi_1\rangle, \\ \frac{1}{2}\frac{d}{dt}||D_\beta\xi\psi_2^{1/2}||_2^2 &= \langle D_\beta\mathcal{L}_3(\xi), (D_\beta\xi)\psi_2\rangle + \langle D_\beta\mathcal{R}_\xi, D_\beta\xi\psi_2\rangle. \end{aligned}$$

Recall the energy $E(R,1)$ and the remaining term $\mathcal{R}(R,1)$ in Definition 2.6.10.

$$E(R,1) = (||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\varphi_1^{1/2}||_2^2)^{1/2}, \quad \mathcal{R}(R,1) = \langle \mathcal{R}_\Omega, \Omega\varphi_1\rangle + \langle \mathcal{R}_\eta, \eta\varphi_1\rangle.$$

Combining the above reformulation, Propositions 2.6.8, 2.6.11, 2.6.12 and $R^{-2} \leq \rho$ (2.78), we know that there is some absolute constant μ_2, which is small enough, e.g., $\mu_2 K_3 < \frac{1}{100}$, such that the following estimate holds

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\Big(E(R,0)^2 + \mu_1 E(\beta,1)^2 + \mu_2 E(R,1)^2 + ||\xi\psi_1^{1/2}||_2^2 + ||D_\beta\xi\psi_2^{1/2}||_2^2\Big) \\ \leq &-(\frac{1}{9} - C\alpha)\Big(E(R,0)^2 + \mu_1 E(\beta,1)^2 + \mu_2 E(R,1)^2 + ||\xi\psi_1^{1/2}||_2^2 + ||D_\beta\xi\psi_2^{1/2}||_2^2\Big) \\ &-(3 - C\alpha)L_{12}^2(\Omega)(0) - (\frac{1}{5} - C\alpha)\Big|\Big|\tilde{L}_{12}^2\rho^{1/2}\Big|\Big|_{L^2(R)}^2 + \mathcal{R}_0(\Omega,\eta,\xi), \end{aligned} \tag{2.111}$$

where $\mathcal{R}_0$ is defined below. We define the following weighted L^2 energy and the remaining term $\mathcal{R}_0$ [2]

$$E_0(\Omega,\eta,\xi) \triangleq \Big(E(R,0)^2 + \mu_1 E(\beta,1)^2 + \mu_2 E(R,1)^2 + ||\xi\psi_1^{1/2}||_2^2 + ||D_\beta \xi \psi_2^{1/2}||_2^2 \Big)^{1/2},$$
$$\mathcal{R}_0(\Omega,\eta,\xi) \triangleq \mathcal{R}(R,0) + \mu_1 \mathcal{R}(\beta,1) + \mu_2 \mathcal{R}(R,1) + \langle \mathcal{R}_\xi, \xi\psi_1\rangle + \langle D_\beta \mathcal{R}_\xi, (D_\beta \xi)\psi_2\rangle, \tag{2.112}$$

where $(E(R,0),\mathcal{R}(R,0)),(E(\beta,1),\mathcal{R}(\beta,1)),(E(R,1),\mathcal{R}(R,1))$ are defined in (2.79), (2.67) and (2.103), respectively, and μ_i are some fixed absolute constants.

We do not need the extra damping for $\tilde{L}_{12}(\Omega)\rho^{1/2}$ and $L_{12}(\Omega)(0)$ in (2.111) due to Lemma A.0.4 and the fact that E_0 is stronger than $||\Omega \frac{(1+R)^2}{R^2}||_{L^2}$. Using (A.9), we know that $C\alpha||\tilde{L}_{12}(\Omega)\rho^{1/2}||^2_{L^2(R)}$, $C\alpha|L_{12}(\Omega)(0)|^2$ can be bounded by $C\alpha E_0^2$. Hence, using the notation $E_0, \mathcal{R}_0$, we derive the following result from (2.111).

Corollary 2.6.13. *Let $E_0(\Omega,\eta,\xi),\mathcal{R}_0(\Omega,\eta,\xi)$ be the energy and the remaining term defined in* (2.112)*. Under the assumptions of Propositions 2.6.6, 2.6.11 and 2.6.12, we have*

$$\frac{1}{2}\frac{d}{dt}E_0^2 \leq -(\frac{1}{9} - C\alpha)E_0^2 + \mathcal{R}_0.$$

2.7 Higher order estimates and the energy functional

In this section, based on the weighted L^2 estimates established in Corollary 2.6.13, we proceed to perform the higher order estimates in the spirit of Propositions 2.6.11, 2.6.12 so that we can complete the nonlinear analysis. In subsection 2.7.1, we perform the weighted H^1 estimates of $\mathcal{L}_i$ and illustrate how to apply several lemmas to control different terms in $D_R\mathcal{L}_i$. In Section 2.7.2, we use a similar argument to establish weighted H^2 and H^3 estimates. In these estimates, we treat the nonlocal terms as perturbations and apply Lemma 2.6.3 recursively.

Since $\bar{\xi}(x,y)$ does not decay in the x direction when y is fixed (see the estimates of $\bar{\xi}$ in Lemma A.0.8), we cannot obtain the decay estimate for its perturbation ξ. Hence, in order to obtain the L^∞ control of ξ and its derivatives, which will be used later to estimate the nonlinear terms, we cannot apply a $H^k \hookrightarrow L^\infty$ type Sobolev embedding. We perform the L^∞ estimates of ξ and its

[2] In fact, E_0 contains a L^2 norm of the angular derivative $D_\beta\Omega, D_\beta\eta, D_\beta\xi$.

derivative directly in Section 2.7.3. This difficulty is not present in [42] by removing the swirl. The coefficient of the damping term in (2.54) is given by $I_1 = -2 - \frac{3}{1+R} \leq -2$. This simple inequality is actually related to the flow structure. In fact, I_1 is the leading order term of $-2 - \bar{v}_y$ (see (2.48) and (2.45)), and the positive sign of $\bar{v}_y$ is related to the hyperbolic flow structure $\bar{u} < 0, \bar{v} > 0$ and $\bar{v}(x, 0) = 0$. See more discussions after Lemma 2.4.1. The fact that I_1 is bounded uniformly away from 0 enables us to establish the L^∞ estimate of ξ.

2.7.1 Weighted H^1 estimates

We remark that the weighted H^1 estimate with angular derivatives is already established in Section 2.6.4 about $D_\beta \Omega, D_\beta \eta$ and Section 2.6.8 about $D_\beta \xi$. Recall the weighted differential operator $D_R = R\partial_R$ in Definition 2.6.1. We define an energy and a remaining term

$$
\begin{aligned}
E(R,2)(\Omega,\eta,\xi) &\triangleq \left(||D_R \Omega \varphi_1^{1/2}||_2^2 + ||D_R \eta \varphi_1^{1/2}||_2 + ||D_R \xi \psi_1^{1/2}||_2^2 \right)^{1/2}, \\
\mathcal{R}(R,2)(\Omega,\eta,\xi) &\triangleq \langle D_R \mathcal{R}_\Omega, D_R \Omega \varphi_1 \rangle + \langle D_R \mathcal{R}_\eta, D_R \eta \varphi_1 \rangle + \langle D_R \mathcal{R}_\xi, D_R \xi \psi_1 \rangle,
\end{aligned} \tag{2.113}
$$

where φ_1, ψ_1 are defined in (2.61).

Proposition 2.7.1. *Under the assumption of Corollary 2.6.13 and that $\varphi_1^{1/2} D_R \Omega$, $\varphi_1^{1/2} D_R \eta, \psi_1^{1/2} D_R \xi \in L^2$, for some fixed absolute constant K_4, we have*

$$
\begin{aligned}
&\langle D_R \mathcal{L}_1(\Omega,\eta), (D_R\Omega)\varphi_1 \rangle + \langle D_R \mathcal{L}_2(\Omega,\eta), (D_R\eta)\varphi_1 \rangle + \langle D_R \mathcal{L}_3(\xi), (D_R\xi)\psi_1 \rangle \\
&\leq -\frac{1}{6} E^2(R,2) + K_4 E_0^2,
\end{aligned}
$$

where $E_0, E(R,2)$ are defined in (2.112) *and* (2.113).

Proof. Since D_R commutes with D_R, D_β in $\mathcal{L}_i, \mathcal{L}_{i0}$ (Definition 2.6.1), we have

$$
\begin{aligned}
D_R \mathcal{L}_1(\Omega,\eta) &= \mathcal{L}_{10}(D_R\Omega, D_R\eta) - D_R \frac{3}{1+R} \cdot D_\beta \Omega + c_\omega D_R(\bar{\Omega} - R\partial_R \bar{\Omega}) \\
&= \mathcal{L}_{10}(D_R\Omega, D_R\eta) + \sum_{i=1}^{2} I_i, \\
D_R \mathcal{L}_2(\Omega,\eta) &= \mathcal{L}_{20}(D_R\eta) - D_R \frac{3}{1+R} \cdot D_\beta \eta + D_R(-2 + \frac{3}{1+R}) \cdot \eta \\
&\quad + \frac{2}{\pi\alpha} \tilde{L}_{12}(\Omega) \cdot D_R \bar{\eta} + \frac{2}{\pi\alpha} D_R \tilde{L}_{12}(\Omega) \cdot \bar{\eta} + c_\omega D_R(\bar{\eta} - R\partial_R \bar{\eta}) \\
&= \mathcal{L}_{20}(D_R\eta) + \sum_{i=1}^{5} II_i,
\end{aligned}
$$

$$
\begin{aligned}
D_R\mathcal{L}_3(\Omega,\xi) &= \mathcal{L}_{30}(D_R\xi) - D_R\frac{3}{1+R}\cdot D_\beta\xi + D_R(-2-\frac{3}{1+R})\cdot\xi \\
&\quad -\frac{2}{\pi\alpha}\tilde{L}_{12}(\Omega)\cdot D_R\bar{\xi} - \frac{2}{\pi\alpha}D_R\tilde{L}_{12}(\Omega)\cdot\bar{\xi} + c_\omega D_R(3\bar{\xi} - R\partial_R\bar{\xi}) \\
&= \mathcal{L}_{30}(D_R\xi) + \sum_{i=1}^{5} III_i.
\end{aligned}
$$

Applying (2.63) with $\varphi = \varphi_1$ (see (2.61)), and (2.64) with $\psi = \psi_1$ (see (2.61)) in Lemma 2.6.3, and $3|1-\sigma| < \frac{1}{30}$, we yield

$$
\begin{aligned}
&\langle \mathcal{L}_{10}(D_R\Omega, D_R\eta), (D_R\Omega)\varphi_1\rangle + \langle \mathcal{L}_{20}(D_R\eta), (D_R\eta)\varphi_1\rangle \\
&\qquad\qquad \le -\frac{1}{5}\Big(||D_R\Omega\varphi_1^{1/2}||_2^2 + ||D_R\eta\varphi_1^{1/2}||_2^2\Big), \\
&\langle \mathcal{L}_{20}(D_R\xi), (D_R\xi)\psi_1\rangle \le -\frac{3}{8}||D_R\xi\psi_1^{1/2}||_2^2.
\end{aligned}
$$

Notice that φ_2, ψ_2 (2.61) satisfy $\varphi_1 \le \varphi_2,\ \psi_1 \le \psi_2$. For the terms not involving $\tilde{L}_{12}(\Omega), c_\omega$, we use E_0 defined in (2.112) to control the weighted L^2 norm of $D_\beta\Omega, D_\beta\eta$. It is easy to see that

$$
\begin{aligned}
&||I_1\varphi_1^{1/2}||_{L^2} \lesssim ||D_\beta\Omega\varphi_2^{1/2}||_{L^2} \lesssim E_0, \quad ||II_1\varphi_1^{1/2}||_{L^2} \lesssim ||D_\beta\eta\varphi_2^{1/2}||_{L^2} \lesssim E_0, \\
&||II_2\varphi_1^{1/2}||_{L^2} \lesssim ||\eta\varphi_1^{1/2}||_{L^2} \lesssim E_0, \qquad ||III_1\psi_1^{1/2}||_{L^2} \lesssim ||D_\beta\xi\psi_2^{1/2}||_{L^2} \lesssim E_0, \\
&||III_2\psi_1^{1/2}||_{L^2} \lesssim ||\xi\psi_1^{1/2}||_{L^2} \lesssim E_0.
\end{aligned}
$$

Recall $c_\omega = -\frac{2}{\pi\alpha}L_{12}(\Omega)(0)$. Applying (A.16) in Lemma A.14 to I_2, II_5 and (A.26) in Lemma A.0.8 to III_5, we obtain

$$
\begin{aligned}
&||I_2\varphi_1^{1/2}||_{L^2} \lesssim |L_{12}(\Omega)(0)| \lesssim E_0, \quad ||II_5\varphi_1^{1/2}||_{L^2} \lesssim |L_{12}(\Omega)(0)| \lesssim E_0, \\
&||III_5\psi_1^{1/2}||_{L^2} \lesssim \alpha|L_{12}(\Omega)(0)| \lesssim \alpha E_0.
\end{aligned}
$$

Finally, for the $\tilde{L}_{12}(\Omega)$ terms, we apply Lemma A.0.3. To apply Lemma A.0.3, we need the L^∞ norm of some angular integrals, whose estimates are given in (A.15) in Lemma A.0.6 about $\bar{\Omega}, \bar{\eta}$ and (A.25) in Lemma A.0.8 about $\bar{\xi}$. Using these estimates, we obtain

$$
\begin{aligned}
&||II_3\varphi_1^{1/2}||_{L^2} \lesssim ||\tilde{L}_{12}(\Omega)R^{-1}||_{L^2(R)} \lesssim E_0, \qquad ||II_4\varphi_1^{1/2}||_{L^2} \lesssim ||R^{-1}\Omega||_{L^2} \lesssim E_0, \\
&||III_3\psi_1^{1/2}||_{L^2} \lesssim \alpha||\tilde{L}_{12}(\Omega)R^{-1}||_{L^2(R)} \lesssim \alpha E_0, \quad ||III_4\psi_1^{1/2}||_{L^2} \lesssim \alpha||R^{-1}\Omega||_{L^2} \lesssim \alpha E_0.
\end{aligned}
$$

The result now follows using the Cauchy-Schwarz inequality (notice that $-\frac{1}{5} < -\frac{1}{6},\ \alpha < 1$) and applying the energy notation (2.113). □

Using the reformulation (2.59), we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E^2(R,2) &= \frac{1}{2}\frac{d}{dt}\left(||D_R\Omega\varphi_1^{1/2}||_2^2 + ||D_R\eta\varphi_1^{1/2}||_2^2 + ||D_R\xi\psi_1^{1/2}||_2^2\right)^{1/2} \\ &= \langle D_R\mathcal{L}_1(\Omega,\eta),(D_R\Omega)\varphi_1\rangle + \langle D_R\mathcal{L}_2(\Omega,\eta),(D_R\eta)\varphi_1\rangle \\ &\quad + \langle D_R\mathcal{L}_3(\xi),(D_R\xi)\psi_1\rangle + \mathcal{R}(R,2).\end{aligned}$$

Therefore, it is not difficult to combine the above reformulation, Corollary 2.6.13 and Proposition 2.7.1 to prove the following results.

Corollary 2.7.2. *Suppose that Ω,η,ξ satisfy $E_0(\Omega,\eta,\xi), E(R,2)(\Omega,\eta,\xi) < +\infty$, where $E_0, E(R,2)$ are defined in (2.112) and (2.113), respectively. Then there exists an absolute constant μ_3, such that, the following statement holds true. The weighted H^1 energy E_1 and its associated remaining term $\mathcal{R}_1$ defined by*

$$\begin{aligned}&E_1(\Omega,\eta,\xi) \triangleq \left(E_0^2(\Omega,\eta,\xi) + \mu_3 E^2(R,2)(\Omega,\eta,\xi)\right)^{1/2}, \\ &\mathcal{R}_1(\Omega,\eta,\xi) \triangleq \mathcal{R}_0 + \mu_3\mathcal{R}(R,2),\end{aligned} \tag{2.114}$$

where $\mathcal{R}_0, \mathcal{R}(R,2)$ are defined in (2.112) and (2.113), satisfy

$$\frac{1}{2}\frac{d}{dt}E_1^2 \le (-\frac{1}{10} + C\alpha)E_1^2 + \mathcal{R}_1.$$

2.7.2 Weighted H^2 and H^3 estimates

Recall the weights φ_i, ψ_i in Definition 2.6.2. For $D_R^k\Omega, D_R^k\eta, k = 2,3$, we use weight φ_1; for other second or third derivative terms $D_R^iD_\beta^j\Omega, D_R^iD_\beta^j\eta$, $j \ge 1$ we use weight φ_2. For $D_R^k\xi, k = 2,3$, we use weight ψ_1; for other second or third derivative terms $D_R^iD_\beta^j\xi$, $j \ge 1$, we use weight ψ_3.

In the same spirit of the weighted H^1 estimates established in Sections 2.6.4 and 2.7.1, we perform the weighted H^2 and H^3 estimates. We estimate the second and third derivative terms in the order of $D_\beta^2, D_\beta D_R, D_R^2, D_\beta^3, D_\beta^2 D_R, D_\beta D_R^2, D_R^3$. The motivation to first estimate the angular derivative terms is the same as that in Sections 2.6.1.2 and 2.6.4. This order of energy estimates has been used in [42]. In these estimates, we treat the nonlocal terms as perturbations and apply Lemma 2.6.3 recursively.

Similar to the weighted H^1 energy function E_1 and Corollary 2.7.2, there exist some absolute constants $\mu_{j,k}$, which can be determined in the order $(j,k) = (2,0),(2,1),(2,2),(3,0),(3,1),(3,2),(3,3)$, such that the weighted H^3 energy functional $E_3 \ge 0$ and its associated remaining term $\mathcal{R}_3$ defined below satisfy the estimates stated in Corollary 2.7.3.

$$
\begin{aligned}
E_3^2(\Omega,\eta,\xi) \triangleq & E_1^2 + \sum_{l=2,3}\sum_{0\le k\le l} \mu_{l,k}\Big(||D_R^k D_\beta^{l-k}\Omega\varphi_i^{1/2}||_2^2 \\
& + ||D_R^k D_\beta^{l-k}\eta\varphi_i^{1/2}||_2^2 + ||D_R^k D_\beta^{l-k}\xi\psi_i^{1/2}||_2^2 \Big), \\
\mathcal{R}_3(\Omega,\eta,\xi) \triangleq & \mathcal{R}_1 + \sum_{l=2,3}\sum_{0\le k\le l} \mu_{l,k}\Big(\langle D_R^k D_\beta^{l-k}\mathcal{R}_\Omega, (D_R^k D_\beta^{l-k}\Omega)\varphi_i\rangle \\
& + \langle D_R^k D_\beta^{l-k}\mathcal{R}_\eta, (D_R^k D_\beta^{l-k}\eta)\varphi_i\rangle + \langle D_R^k D_\beta^{l-k}\mathcal{R}_\xi, (D_R^k D_\beta^{l-k}\xi)\psi_i\rangle \Big),
\end{aligned}
\tag{2.115}
$$

where $E_1, \mathcal{R}_1$ are defined in (2.114), $(\varphi_i, \psi_i) = (\varphi_3, \psi_3)$ for $k = 0, 1, 2$ and (φ_1, ψ_1) otherwise.

Corollary 2.7.3. *Suppose $E_3(\Omega, \eta, \xi) < +\infty$. Then the energy E_3 satisfies*

$$
\frac{1}{2}\frac{d}{dt}E_3^2(\Omega,\eta,\xi) \le (-\frac{1}{12} + C\alpha)E_3^2 + \mathcal{R}_3.
$$

We refer the weighted H^2 estimates to the arXiv version of [16]. The weighted H^3 estimates can be generalized in a straightforward manner.

2.7.3 $\mathcal{C}^1$ estimates

We introduce the following weights for the weighted C^1 estimates

$$
\phi_1 = \frac{1+R}{R}, \quad \phi_2 = 1 + (R\sin(2\beta)^\alpha)^{-\frac{1}{40}}, \tag{2.116}
$$

and the following $\mathcal{C}^1$ norm

$$
\begin{aligned}
||f||_{\mathcal{C}^1} &\triangleq ||f||_\infty + ||\phi_1 D_R f||_\infty + ||\phi_2 D_\beta f||_\infty \\
&= ||f||_\infty + ||\frac{1+R}{R}D_R f||_\infty + ||(1 + (R\sin(2\beta)^\alpha)^{-\frac{1}{40}})D_\beta f||_\infty.
\end{aligned}
\tag{2.117}
$$

To close the nonlinear estimates, we need to control the L^∞ norm of Ω, η, ξ and their derivatives. For Ω, η, the weighted H^3 estimates that we have obtained guarantee that $\Omega, \eta \in \mathcal{C}^1$, which will be established precisely in later sections. For ξ, however, since the weight ψ_2 (see Definition 2.6.2) is less singular in β for β close to 0, the weighted H^3 space associated to ξ is not embedded continuously into $\mathcal{C}^1$. Alternatively, we perform $\mathcal{C}^1$ estimates on ξ directly. This difficulty is absent in [42] by removing the swirl.

Firstly, the transport term in the ξ equation in (2.48), including the nonlinear part in N_ξ, is given by

$$
\mathcal{A}(\xi) \triangleq (1+3\alpha)D_R\xi + \alpha c_l D_R\xi + (\bar{\mathbf{u}}\cdot\nabla)\xi + (\mathbf{u}\cdot\nabla)\xi. \tag{2.118}
$$

The main damping term in the ξ equation is $(-2-\bar{v}_y)\xi$. (2.45) shows that $-\bar{v}_y = -\frac{3}{1+R} + l.o.t.$ Therefore, we consider

$$(-2-\bar{v}_y)\xi = (-2-\frac{3}{1+R})\xi + \Xi_1, \quad \Xi_1 \triangleq (\frac{3}{1+R} - \bar{v}_y)\xi. \tag{2.119}$$

We further introduce Ξ_2 to denote the lower order terms in the ξ equation (2.48)

$$\Xi_2 = -v_y\bar{\xi} + c_\omega(2\bar{\xi} - R\partial_R\bar{\xi}) + (\alpha c_\omega R\partial_R - (\mathbf{u}\cdot\nabla))\bar{\xi} - (u_y\bar{\eta} + \bar{u}_y\eta). \tag{2.120}$$

Then the ξ equation in (2.48) can be simplified as

$$\partial_t\xi + A(\xi) = (-2-\frac{3}{1+R})\xi + \Xi_1 + \Xi_2 + \bar{F}_\xi + N_o, \tag{2.121}$$

where we have moved part of the nonlinear term N_ξ defined in (2.49) to the transport term $A(\xi)$ and N_o is given by

$$N_o = (2c_\omega - v_y)\xi - u_y\eta. \tag{2.122}$$

Notice that $-\frac{3}{1+R} \leq 0$. Multiplying ξ on both sides and then performing L^∞ estimate yield

$$\frac{1}{2}\frac{d}{dt}||\xi||_\infty^2 \leq -2||\xi||_\infty^2 + ||\xi||(||\Xi_1||_{L^\infty} + ||\Xi_2||_\infty + ||\bar{F}_\xi||_\infty + ||N_\xi||_\infty), \tag{2.123}$$

where the transport term $A(\xi)$ vanishes.

Before we perform weighted C^1 estimates, we rewrite $\mathcal{A}(\xi)$ defined in (2.118) as follows

$$\mathcal{A}(\xi) = ((1+3\alpha+\alpha c_l)D_R\xi + \frac{3}{1+R}D_\beta\xi) + (((\mathbf{u}+\bar{\mathbf{u}})\cdot\nabla - \frac{3}{1+R}D_\beta)\xi) \triangleq \mathcal{A}_1(\xi) + \mathcal{A}_2(\xi). \tag{2.124}$$

Recall the weights ϕ_1, ϕ_2 in (2.116). Observe that D_β commutes with $\mathcal{A}_1$ and D_R commutes with D_R, D_β. Denote by $[P,Q]$ the commutator $PQ-QP$. A direct calculation shows that

$$\begin{aligned}
&\phi_1 D_R\mathcal{A}\xi - \mathcal{A}(\phi_1 D_R\xi) \\
=&\phi_1 D_R\frac{3}{1+R}\cdot D_\beta\xi - (1+3\alpha+\alpha c_l)D_R\phi_1\cdot D_R\xi + [\phi_1 D_R, \mathcal{A}_2]\xi, \\
=&-\frac{3}{1+R}D_\beta\xi + (1+3\alpha+\alpha c_l)\frac{1}{1+R}\phi_1 D_R\xi + [\phi_1 D_R, \mathcal{A}_2]\xi, \\
&\phi_2 D_\beta\mathcal{A}\xi - \mathcal{A}(\phi_2 D_\beta\xi) = -\mathcal{A}_1(\phi_2 - 1)\cdot D_\beta\xi + [\phi_2 D_\beta, \mathcal{A}_2]\xi,
\end{aligned} \tag{2.125}$$

where we have used $\mathcal{A}_1(1) = 0$ in the last equality. Hence, using (2.121) and the above calculation, we obtain the equation of $\phi_1 D_R \xi$

$$\partial_t(\phi_1 D_R \xi) + \mathcal{A}(\phi_1 D_R \xi) = \frac{3}{1+R} D_\beta \xi - (1 + 3\alpha + \alpha c_l)\frac{1}{1+R}\phi_1 D_R \xi - [\phi_1 D_R, \mathcal{A}_1 \xi]$$
$$+ \phi_1 D_R((-2 - \frac{3}{1+R})\xi) + \phi_1 D_R(\Xi_1 + \Xi_2 + \bar{F}_\xi + N_o).$$

We remark that $-(1+3\alpha)\frac{1}{1+R}\phi_1 D_R \xi$ is a damping term, though we will not use it. Performing L^∞ estimate for $\phi_1 D_\beta \xi$, we obtain the following estimate, which is similar to (2.123)

$$\frac{1}{2}\frac{d}{dt}||\phi_1 D_R \xi||_\infty^2 \leq -(2 - |\alpha c_l|)||\phi_1 D_R \xi||_\infty^2 + 3||\phi_1 D_R \xi||_\infty ||\xi||_\infty$$
$$+ ||\phi_1 D_R \xi||_{L^\infty}(3||D_\beta \xi||_\infty + ||[\phi_1 D_R, \mathcal{A}_2]\xi||_\infty + ||\phi_1 D_R(\Xi_1 + \Xi_2 + \bar{F}_\xi + N_o)||_{L^\infty}), \tag{2.126}$$

where we have used $|\frac{3}{1+R}| \leq 3$ and

$$\phi_1 D_R \xi \cdot \phi_1 D_R(-2 - \frac{3}{1+R})\xi = \phi_1 D_R \xi \cdot ((-2 - \frac{3}{1+R})\phi_1 D_R \xi + \phi_1 \frac{3R\xi}{(1+R)^2})$$
$$\leq -2(\phi_1 D_R \xi)^2 + 3||\phi_1 D_R \xi||_\infty ||\xi||_\infty.$$

Similarly, using (2.121), (2.125) and performing L^∞ estimate on $\phi_2 D_\beta \xi$ yield

$$\frac{1}{2}\frac{d}{dt}||\phi_2 D_\beta \xi||_\infty^2 \leq -2||\phi_2 D_\beta \xi||_\infty^2 + ||\phi_2 D_\beta \xi||_\infty ||\mathcal{A}_1(\phi_2 - 1)\cdot D_\beta \xi||_{L^\infty}$$
$$+ ||\phi_2 D_\beta \xi||_\infty(||[\phi_2 D_\beta, \mathcal{A}_2]\xi||_\infty + ||\phi_2 D_\beta(\Xi_1 + \Xi_2 + \bar{F}_\xi + N_o)||_{L^\infty}), \tag{2.127}$$

where we have used

$$\phi_2 D_\beta \xi \cdot \phi_2 D_\beta(-2 - \frac{3}{1+R})\xi \leq -2(\phi_2 D_\beta \xi)^2.$$

We defer the estimates of the remaining terms in (2.123),(2.126),(2.127) which are small, to Section 2.9.

2.7.4 The energy functional and the $\mathcal{H}^m$ norm

Using all the energy notations (2.67), (2.79), (2.103), (2.112), (2.113),(2.114) and (2.115), we obtain the full expression of E_3 (2.115)

$$E_3^2 = ||\Omega\varphi_0^{1/2}||_2^2 + ||\eta\psi_0^{1/2}||_2^2 + \frac{81}{4\pi c}L_{12}^2(\Omega)(0) + ||D_\beta \xi \psi_2^{1/2}||_2^2$$
$$+ \mu_1\left(||D_\beta \Omega \varphi_2^{1/2}||_2^2 + ||D_\beta \eta \psi_2^{1/2}||_2^2\right) + \mu_2\left(||\Omega\varphi_1^{1/2}||_2^2 + ||\eta\psi_1^{1/2}||_2^2\right) + ||\xi\psi_1^{1/2}||_2^2$$
$$+ \mu_3\left(||D_R \Omega \varphi_1^{1/2}||_2^2 + ||D_R \eta \psi_1^{1/2}||_2^2 + ||D_R \xi \psi_1^{1/2}||_2^2\right)$$
$$+ \sum_{l=2,3}\sum_{0\leq k\leq l}\mu_{l,k}\left(\langle||D_R^k D_\beta^{l-k}\Omega\varphi_i^{1/2}||_2^2 + ||D_R^k D_\beta^{l-k}\eta\varphi_i^{1/2}||_2^2 + ||D_R^k D_\beta^{l-k}\xi\psi_i^{1/2}||_2^2\right), \tag{2.128}$$

where $(\varphi_i, \psi_i) = (\varphi_1, \psi_1)$ for $k = l$, $(\varphi_i, \psi_i) = (\varphi_2, \psi_2)$ for $k \neq l$ and $l = 2, 3$.

Recall φ_i, ψ_i in Definition 2.6.2. We define the $\mathcal{H}^m(\rho)$ norm with $m \geq 0$ as follows

$$||f||_{\mathcal{H}^m(\rho)} \triangleq \sum_{0 \leq k \leq m} ||D_R^k f \rho_1^{1/2}||_{L^2} + \sum_{i+j \leq m-1} ||D_R^i D_\beta^{j+1} f \rho_2^{1/2}||_{L^2}. \tag{2.129}$$

The $\mathcal{H}^0(\varphi)$ norm is the same as $L^2(\varphi_1)$ norm. For the $\mathcal{H}^3(\varphi)$ norm, we use (2.129) with $\rho_i = \varphi_i$; for the $\mathcal{H}^3(\psi)$ norm, we use (2.129) with $\rho_i = \psi_i, i = 1, 2$. We simplify $\mathcal{H}^3(\varphi)$ as $\mathcal{H}^3$. We apply the $\mathcal{H}^3$ norm for Ω, η and the $\mathcal{H}^3(\psi)$ norm for ξ. We use the $\mathcal{H}^m$ norm to establish the elliptic estimate in the next Section. We will only use the $\mathcal{H}^2, \mathcal{H}^2(\psi)$ and $\mathcal{H}^3, \mathcal{H}^3(\psi)$ norms. Remark that the $\mathcal{H}^m$ norm is different from the canonical Sobolev H^m norm.

From the Definition 2.6.2 of φ_i, ψ_i, we have a simple relationship between $\mathcal{H}^m$ and $\mathcal{H}^m(\psi)$.

Lemma 2.7.4. *For $\frac{\gamma-\sigma}{2} \leq \lambda \leq \frac{1}{2}$ and $m \leq 3$, we have*

$$||f||_{\mathcal{H}^m(\psi)} \lesssim ||f||_{\mathcal{H}^m}, \quad ||\sin(\beta)^\lambda f||_{\mathcal{H}^m} \lesssim ||f||_{\mathcal{H}^m(\psi)}. \tag{2.130}$$

The proof follows from simple inequalities $\psi_i \lesssim \varphi_i, \sin(\beta)^\lambda \varphi_i \lesssim \psi_i$, $D_\beta^i \sin(\beta)^\lambda \cdot \varphi_2 = 2\lambda \cos^2(\beta) \sin(\beta)^\lambda \varphi_2 \lesssim \psi_1$ for $i \leq 3$, and expanding the norm.

We also define the corresponding inner products on $\mathcal{H}^3$ and $\mathcal{H}^3(\psi)$, which are equivalent to $\mathcal{H}^3, \mathcal{H}^3(\psi)$

$$\begin{aligned}
\langle f, g \rangle_{\mathcal{H}^3} &\triangleq \mu_1 \langle D_\beta f, D_\beta g \varphi_2 \rangle + \mu_2 \langle f, g\varphi_1 \rangle + \mu_3 \langle D_R f, D_R g \varphi_1 \rangle \\
&\quad + \sum_{k=2,3} \mu_{k,k} ||D_R^k f \varphi_1^{1/2}||_{L^2} + \sum_{j \geq 1, \ 2 \leq i+j \leq 3} \mu_{i+j,i} \langle D_R^i D_\beta^j f, D_R^i D_\beta^j g \varphi_2 \rangle, \\
\langle f, g \rangle_{\mathcal{H}^3(\psi)} &\triangleq \langle D_\beta f, D_\beta g \psi_2 \rangle + \langle f, g\psi_1 \rangle + \mu_3 \langle D_R f, D_R g \psi_1 \rangle \\
&\quad + \sum_{k=2,3} \mu_{k,k} ||D_R^k f \psi_1^{1/2}||_{L^2} + \sum_{j \geq 1, \ 2 \leq i+j \leq 3} \mu_{i+j,i} \langle D_R^i D_\beta^j f, D_R^i D_\beta^j g \psi_2 \rangle.
\end{aligned} \tag{2.131}$$

Clearly, using these notations and (2.112), (2.113), (2.114), (2.115), we have

$$\begin{aligned}
E_3^2 &= \frac{81}{4\pi c} L_{12}^2(\Omega)(0) + \langle \Omega^2, \varphi_0 \rangle + \langle \eta^2, \psi_0 \rangle + \langle \Omega, \Omega \rangle_{\mathcal{H}^3} + \langle \eta, \eta \rangle_{\mathcal{H}^3} + \langle \xi, \xi \rangle_{\mathcal{H}^3(\psi)}, \\
\mathcal{R}_3 &= \langle \mathcal{R}_\Omega, \Omega\varphi_0 \rangle + \langle \mathcal{R}_\eta, \eta\psi_0 \rangle + \frac{81}{4\pi c} L_{12}(\Omega)(0) \langle \mathcal{R}_\Omega, \sin(2\beta) R^{-1} \rangle \\
&\quad + \langle \mathcal{R}_\Omega, \Omega \rangle_{\mathcal{H}^3} + \langle \mathcal{R}_\eta, \eta \rangle_{\mathcal{H}^3} + \langle \mathcal{R}_\xi, \xi \rangle_{\mathcal{H}^3(\psi)}.
\end{aligned} \tag{2.132}$$

We also have the following simple inequality

$$||\Omega||_{\mathcal{H}^3}^2 + ||\eta||_{\mathcal{H}^3}^2 + ||\xi||_{\mathcal{H}^3(\psi)}^2 \lesssim E_3^2(\Omega, \eta, \xi). \tag{2.133}$$

2.8 Elliptic regularity estimates and estimate of nonlinear terms

In this section, we first follow the argument in [42] to establish the $\mathcal{H}^3$ estimates for the elliptic operator and justify that the leading order term of the (modified) stream function can be written as (2.31) in subsection 2.8.1. Then to simplify our nonlinear estimates, we will generalize several estimates derived in [42] in subsection 2.8.2.

The fact that $\bar{\xi}$ (see Lemma A.0.8) and ξ do not decay in certain direction makes the estimates of nonlinear terms complicated since we cannot apply the same weighted Sobolev norm to Ω, η, ξ. More precisely, the $\mathcal{H}^k(\psi)$ norm for ξ is weaker than the $\mathcal{H}^k$ norm for Ω, η (see (2.130)). To compensate this, we use a combination of $\mathcal{C}^1$ norm and $\mathcal{H}^k(\psi)$ norm for ξ. We will establish several estimates for ξ in subsection 2.8.2. Moreover, estimating the $\mathcal{H}^k$ norm of $v_x\xi$ in the η equation (2.48) will be more difficult since ξ is in a weaker Sobolev space. In subsection 2.8.3, we will estimate the nonlinear term $v_x\xi$ in the η equation (2.48). We will also perform a new estimate of the transport term with weighted H^3 data.

Recall that the Biot-Savart law in $\mathbb{R}_+^2$ is given by (2.18), which can be reformulated using the polar coordinates as

$$-\partial_{rr}\psi - \frac{1}{r}\partial_r\psi - \frac{1}{r^2}\partial_{\beta\beta}\psi = \omega,$$

where $r = \sqrt{x^2+y^2}, \beta = \arctan(y/x)$. We introduce $R = r^\alpha$ and $\Psi(R,\beta) = \frac{1}{r^2}\psi(r,\beta), \Omega(R,\beta) = \omega(r,\beta)$. It is easy to verify that the above elliptic equation is equivalent to

$$\mathcal{L}_\alpha(\Psi) \triangleq -\alpha^2 R^2 \partial_{RR}\Psi - \alpha(4+\alpha)R\partial_R\Psi - \partial_{\beta\beta}\Psi - 4\Psi = \Omega. \tag{2.134}$$

The boundary condition of Ψ is given by

$$\Psi(R,0) = \Psi(R,\pi/2) = 0, \quad \lim_{R\to\infty}\Psi(R,\beta) = 0. \tag{2.135}$$

2.8.1 $\mathcal{H}^3$ estimates

Recall that the $\mathcal{H}^m, m \geq 0$ norm defined in Section 2.7.4 is given by

$$||f||_{\mathcal{H}^m} \triangleq \sum_{0\leq k\leq m} ||D_R^k f \frac{(1+R)^2}{R^2\sin(2\beta)^{\sigma/2}}||_{L^2} + \sum_{i+j\leq m-1} ||D_R^i D_\beta^{j+1} f \frac{(1+R)^2}{R^2\sin(2\beta)^{\gamma/2}}||_{L^2}, \tag{2.136}$$

where $\sigma = 99/100, \gamma = 1 + \alpha/10$ and we have used the definition of φ_i in Definition 2.6.2. The $\mathcal{H}^0$ norm is the same as $L^2(\varphi_1)$ norm.

Proposition 2.8.1. *Assume that* $0 < \alpha \leq \frac{1}{4}, 1 < \gamma \leq \frac{5}{4}$, *and* Ω *satisfies* $||\Omega||_{\mathcal{H}^3} < +\infty$ *with*

$$\int_0^{\pi/2} \Omega(R,\beta)\sin(2\beta)d\beta = 0 \tag{2.137}$$

for every R. *The solution of* (2.134) *satisfies*

$$\alpha^2||R^2\partial_{RR}\Psi||_{\mathcal{H}^3} + \alpha||R\partial_{R\beta}\Psi||_{\mathcal{H}^3} + ||\partial_{\beta\beta}\Psi||_{\mathcal{H}^3} \leq C||\Omega||_{\mathcal{H}^3}$$

for some absolute constant C *independent of* α *and* γ.

Remark 2.8.2. We need the orthogonality assumption (2.137) since $\sin(2\beta)$ is in the null space of the self-adjoint operator $L_0(\Psi) = -\partial_{\beta\beta}\Psi - 4\Psi$ with boundary condition $\Psi(0) = \Psi(\pi/2) = 0$, which is the limiting operator in (2.134) as $\alpha \to 0$. See more discussion on the connection between this orthogonality assumption and the elliptic estimate in the arXiv version of [16].

Since the $\mathcal{H}^2$ norm is the same as that in [42] and the $\mathcal{H}^2$ estimates can be easily extended to the $\mathcal{H}^3$ estimates, the complete proof follows from the same argument in [42]. Here, the proof is even simpler since there is no first order angular derivative term in (2.134), i.e. $\partial_\beta(\tan(\beta)\Psi)$, which is one of the major difficulties in obtaining the elliptic estimate in [42].

The singular term In general the vorticity Ω does not satisfy the assumption (2.137) in Proposition 2.8.1. Suppose that Ψ is the solution of (2.134). Consider $\tilde{\Psi} = \Psi + G\sin(2\beta)$. The goal is to construct G so that $\mathcal{L}_\alpha(\tilde{\Psi})$ satisfies (2.137), i.e. $\int_0^{\pi/2}\mathcal{L}_\alpha(\tilde{\Psi})\sin(2\beta)d\beta = 0$. Recall the notation $L_{12}(\Omega)$ in (2.31). Following the argument in [42], in Appendix A.0.3, we derive

$$G = -\frac{1}{\pi\alpha}L_{12}(\Omega)(R) + \bar{G}, \quad \bar{G} \triangleq -\frac{1}{\alpha\pi}R^{-\frac{4}{\alpha}}\int_0^R\int_0^{\pi/2}\Omega(s,\beta)\sin(2\beta)s^{\frac{4}{\alpha}-1}ds. \tag{2.138}$$

Although there is a large factor $1/\alpha$ in $\bar{G}$, it can be proved that $||\bar{G}||_{\mathcal{H}^3}$ can be bounded by $C||\Omega||_{\mathcal{H}^3}$ using a Hardy-type inequality. We refer the reader to [42] and [44] for more details.

Using Proposition 2.8.1 and an argument similar to that in [42], we have the following result, which is similar to Theorem 2 in [42].

Proposition 2.8.3. *Assume that* $\alpha \leq \frac{1}{4}$ *and* $\Omega \in \mathcal{H}^3$. *Let* Ψ *be the solution to* (2.134) *with boundary condition* (2.135). *Then we have*

$$\alpha^2 ||R^2 \partial_{RR} \Psi||_{\mathcal{H}^3} + \alpha ||R\partial_{R\beta}\Psi||_{\mathcal{H}^3} + ||\partial_{\beta\beta}(\Psi - \frac{1}{\alpha\pi}\sin(2\beta)L_{12}(\Omega))||_{\mathcal{H}^3} \leq C||\Omega||_{\mathcal{H}^3}$$

for some absolute constant C *independent of* α, γ *in the definition* (2.136).

Remark 2.8.4. The $\mathcal{H}^3$ norm of $\alpha D_R \partial_\beta \Psi$ is not included in Theorem 2 in [42]. Yet, the estimate of such term can be derived easily from Proposition 2.8.1 and the estimate of G defined in (2.138).

2.8.2 Estimates of nonlinear terms

In this subsection, we generalize several estimates of nonlinear terms derived in [42], which will be used in our nonlinear stability estimate in the next section.

We define the $\mathcal{W}^{l,\infty}$ norm:

$$||f||_{\mathcal{W}^{l,\infty}} \triangleq \sum_{0\leq k+j\leq l, j\neq 0} \Big|\Big| \sin(2\beta)^{-\frac{\alpha}{5}} D_R^k \frac{(\sin(2\beta)\partial_\beta)^j}{\frac{\alpha}{10} + \sin(2\beta)} f \Big|\Big|_{L^\infty} + \sum_{0\leq k\leq l} \Big|\Big| D_R^k f \Big|\Big|_{L^\infty}. \tag{2.139}$$

A similar $\mathcal{W}^{l,\infty}$ has been used in [42] and our $\mathcal{W}^{l,\infty}$ norm is slightly different from that in [42]. We replace the operator $(R+1)^k \partial_R^k$ by $D_R^k = (R\partial_R)^k$. The reason for doing this is that the stronger weight $(R+1)^k$ is not necessary in the derivation of the product rule in [42] related to $\mathcal{W}^{l,\infty}$, and that the differential operator D_R commutes with $\mathcal{L}_\alpha$ in the elliptic equation (2.134), while ∂_R does not. Therefore, the higher order elliptic estimates related to ∂_R can depend on the value of α. We will only use these estimates when α is very small.

Functions in $\mathcal{W}^{7,\infty}$ From Proposition A.0.7 in Appendix A, we know that $\Gamma(\beta), \bar{\Omega}, \bar{\eta} \in \mathcal{W}^{7,\infty}$.

Remark 2.8.5. We do not apply the $\mathcal{W}^{l,\infty}$ norm to $\bar{\xi}$, ξ.

Recall the $\mathcal{C}^1$ norm in (2.117). For the $\mathcal{C}^1$ and $\mathcal{W}^{1,\infty}$ norms, we have a simple result.

Proposition 2.8.6. *For any* $f, g \in \mathcal{C}^1$ *and* $\frac{1+R}{R} p \in \mathcal{W}^{1,\infty}$, *we have*

$$||fg||_{\mathcal{C}^1} \leq ||f||_{\mathcal{C}^1} ||g||_{\mathcal{C}^1}, \quad ||p||_{\mathcal{C}^1} \lesssim ||\frac{1+R}{R} p||_{\mathcal{W}^{1,\infty}}.$$

The $\mathcal{W}^{4,\infty}$ version of the following result has been established in [42], whose generalization to $\mathcal{W}^{l,\infty}$ is straightforward.

Proposition 2.8.7. *Assume that $f, g \in \mathcal{W}^{l,\infty}$. Then we have*

$$||fg||_{\mathcal{W}^{l,\infty}} \lesssim_l ||f||_{\mathcal{W}^{l,\infty}} ||g||_{\mathcal{W}^{l,\infty}}.$$

Recall from (2.45) that $L_{12}(\bar{\Omega}) = \frac{3\pi\alpha}{2} \frac{1}{1+R}$. We define $\bar{\Psi}$ by

$$\mathcal{L}_\alpha(\bar{\Psi}) = -\alpha^2 R^2 \partial_{RR}\bar{\Psi} - \alpha(4+\alpha)R\partial_R\bar{\Psi} - \partial_{\beta\beta}\bar{\Psi} - 4\Psi = \bar{\Omega},$$

where $\mathcal{L}_\alpha$ is the operator in (2.134). We have the following estimates.

Proposition 2.8.8. *For $\alpha \leq \frac{1}{4}$, we have*

$$\begin{aligned} &||\frac{1+R}{R}\partial_{\beta\beta}(\bar{\Psi} - \frac{\sin(2\beta)}{\pi\alpha}L_{12}(\bar{\Omega}))||_{\mathcal{W}^{7,\infty}} \lesssim \alpha, \qquad ||L_{12}(\bar{\Omega})||_{\mathcal{W}^{7,\infty}} \lesssim \alpha, \\ &\alpha||\frac{1+R}{R}D_R^2\bar{\Psi}||_{\mathcal{W}^{5,\infty}} + \alpha||\frac{1+R}{R}\partial_\beta D_R\bar{\Psi}||_{\mathcal{W}^{5,\infty}} \\ &+ ||\frac{1+R}{R}\partial_{\beta\beta}(\bar{\Psi} - \frac{\sin(2\beta)}{\pi\alpha}L_{12}(\bar{\Omega}))||_{\mathcal{W}^{5,\infty}} \lesssim \alpha. \end{aligned}$$

The proof of the first inequality follows from the same argument in [42]. The proof of the second inequality follows from $L_{12}(\bar{\Omega}) = \frac{3\pi\alpha}{2(1+R)}$ in (2.45) and a direct calculation. The third inequality follows from the first two inequalities. We refer more details to the arXiv version of [16].

2.8.2.1 Some embedding Lemmas

A similar version of the following estimate has been established in [42]. We remark that we have modified the weight for the R variable in the $\mathcal{W}^{l,\infty}$ norm. We refer the proof to the arXiv version of [16].

Proposition 2.8.9. *Assume that $\frac{(1+R)^3}{R^2} f \in \mathcal{W}^{3,\infty}$, then we have $f \in \mathcal{H}^3$ and*

$$||f||_{\mathcal{H}^3} \lesssim ||\frac{(1+R)^3}{R^2} f||_{\mathcal{W}^{3,\infty}}.$$

We have the following decay estimate.

Lemma 2.8.10. *Suppose that $\xi \in \mathcal{H}^2(\psi)$, we have*

$$||R^{1/2}\sin(2\beta)^{1/4}\xi||_{L^\infty} \lesssim ||\xi||_{\mathcal{H}^2(\psi)}.$$

The above estimate also holds for $\xi \in \mathcal{H}^2$ since $\mathcal{H}^2$ is stronger than $\mathcal{H}^2(\psi)$ (see Lemma 2.7.4).

Proof. Using a direct calculation yields

$$||\sin(2\beta)^{1/2} R\xi^2||_{L^\infty} \lesssim ||\partial_R \partial_\beta (\sin(2\beta)^{1/2} R\xi^2)||_{L^1} = ||\partial_\beta (\sin(2\beta)^{1/2}(\xi^2 + 2\xi D_R \xi))||_{L^1}$$
$$\lesssim ||\sin(2\beta)^{-1/2}(\xi^2 + 2\xi D_R\xi)||_{L^1} + ||\sin(2\beta)^{1/2}(2\xi\partial_\beta\xi + 2\partial_\beta\xi D_R\xi + 2\xi\partial_\beta D_R\xi)||_{L^1}.$$

Recall the definition of $\mathcal{H}^2(\psi)$ (2.129) and the weights in Definition 2.6.2. Using the Cauchy-Schwarz inequality concludes the proof. ☐

Lemma 2.8.11. *We have*

$$||f||_{L^\infty} \lesssim \alpha^{-1/2} ||f||_{\mathcal{H}^2},$$
$$||f||_{\mathcal{C}^1} = ||f||_{L^\infty} + ||\frac{1+R}{R} D_R f||_{L^\infty} + ||(1 + (R\sin(2\beta)^\alpha)^{-\frac{1}{40}}) D_\beta f||_{L^\infty} \lesssim \alpha^{-1/2} ||f||_{\mathcal{H}^3},$$

provided that the right hand side is bounded.

The first inequality has been established in [42]. Recall the definition of $\mathcal{H}^3$ and its associated weights in (2.136). The second inequality follows from the argument in the proof of Lemma 2.8.10, the Cauchy-Schwarz inequality and

$$||\frac{1}{1+R}\sin(2\beta)^{\gamma/2-1}||_{L^2} \lesssim \alpha^{-1/2}, \quad ||\frac{R^{1-\frac{\alpha}{40}}}{(1+R)^2}\sin(2\beta)^{\gamma/2-1-\frac{\alpha}{40}}||_{L^2} \lesssim \alpha^{-1/2}.$$

2.8.2.2 The product rules

In this subsection, we generalize the estimates of nonlinear terms and the transport terms derived in [42] to the $\mathcal{H}^3(\psi)$ norm.

Denote the sum space $X \triangleq \mathcal{H}^3 \oplus \mathcal{W}^{5,\infty}$ with sum norm

$$||f||_X \triangleq \infty\{||g||_{\mathcal{H}^3} + ||h||_{\mathcal{W}^{5,\infty}} : f = g + h\}. \tag{2.140}$$

We use the following product rules to estimate the nonlinear terms.

Proposition 2.8.12. *For all $f \in X, g \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap \mathcal{C}^1$, we have*

$$\begin{aligned} ||fg||_{\mathcal{H}^3} &\lesssim \alpha^{-1/2} ||f||_X ||g||_{\mathcal{H}^3}, \\ ||f\xi||_{\mathcal{H}^3(\psi)} &\lesssim \alpha^{-1/2} ||f||_X (\alpha^{1/2} ||\xi||_{\mathcal{C}^1} + ||\xi||_{\mathcal{H}^3(\psi)}). \end{aligned} \tag{2.141}$$

The first inequality has been proved in [42]. We will focus on the product rule with $\mathcal{H}^3(\psi)$ norm.

Proof. If $f \in \mathcal{W}^{5,\infty}$, applying the same argument in [42] yields

$$||f\xi||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1/2}||f||_{\mathcal{W}^{5,\infty}}||\xi||_{\mathcal{H}^3(\psi)}.$$

Now, we assume $f \in \mathcal{H}^3$. We consider the third derivative $D^3 = D_R^i D_\beta^j$ terms since other are terms are easier. If $(D^3, \psi_i) = (D_R^3, \psi_1), (D_\beta^3, \psi_2)$, we use a $L^2 \times L^\infty$ interpolation

$$\begin{aligned} ||D^3(f\xi)\psi_i^{1/2}||_2^2 &\lesssim \sum_{k=0,1} ||D^k f D^{3-k}\xi\psi_i^{1/2}||_2^2 + \sum_{k=2,3} ||D^k f D^{3-k}\xi\psi_i^{1/2}||_2^2 \\ &\lesssim ||f||_{\mathcal{C}^1}||\xi||_{\mathcal{H}^3(\psi)} + ||f||_{\mathcal{H}^3(\psi)}||\xi||_{\mathcal{C}^1} \\ &\lesssim \alpha^{-1/2}||f||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)} + ||f||_{\mathcal{H}^3}||\xi||_{\mathcal{C}^1}, \end{aligned}$$

where we have applied Lemma 2.8.11 to $||f||_{\mathcal{C}^1}$ and Lemma 2.7.4 to obtain the last inequality.

If $D^3 = D_R^2 D_\beta$ or $D_\beta^2 D_R$, the corresponding singular weight in the $\mathcal{H}^3(\psi)$ norm is ψ_2. We consider the term $D_R^2\xi D_\beta f \psi_2^{1/2}$ in the L^2 estimate of $D^3(f\xi)\psi_2^{1/2}$, which is a typical and the most difficult term. The previous $L^2 \times L^\infty$ estimate fails since $D_R^2\xi\psi_2^{1/2} \notin L^2(R,\beta)$. Recall the Definition 2.6.2 of ψ_2, φ_2. Denote

$$\begin{aligned} W &= \frac{(1+R)^4}{R^4}, \quad P = \sin(\beta)^{-\sigma}\cos(\beta)^{-\gamma}, \\ Q &= \sin(2\beta)^{-\gamma},\ S = \sin(2\beta)^{-\sigma},\ \lambda = \gamma - \sigma. \end{aligned} \tag{2.142}$$

Clearly, we have $\varphi_2 = WQ, \psi_2 = WP,\ \psi_1 \asymp WS$ and $P \lesssim \sin(\beta)^\lambda Q$. We use a $L^2(R, L^\infty(\beta)) \times L^\infty(R, L^2(\beta))$ estimate [3]

$$\begin{aligned} ||D_R^2\xi D_\beta f(WP)^{1/2}||_2^2 &\le \Big|\Big| || \sin(\beta)^{\lambda/2} D_R^2\xi(R,\cdot)||^2_{L^\infty(\beta)} ||D_\beta f Q^{1/2}(R,\cdot)||^2_{L^2(\beta)} W \Big|\Big|_{L^1(R)} \\ &\triangleq ||A(R)^2 B(R)^2 W||_{L^1(R)}. \end{aligned} \tag{2.143}$$

We further estimate the integrands $A(R), B(R)$. Applying the Poincare inequality yields

$$\begin{aligned} A(R) &\lesssim ||\partial_\beta(\sin(\beta)^{\lambda/2} D_R^2\xi(R,\cdot))||_{L^1(\beta)} + || \sin(\beta)^{\lambda/2} D_R^2\xi(R,\cdot)||_{L^2(\beta)} \\ &\triangleq A_1(R) + A_2(R). \end{aligned}$$

[3]The $L^2(R, L^\infty(\beta)) \times L^\infty(R, L^2(\beta))$ estimate of the mixed derivatives term in the $\mathcal{H}^2$ norm is due to Dongyi Wei. We are grateful to him for telling us this estimate. We apply this idea to derive the estimates in the $\mathcal{H}^3(\psi)$ norm.

Using the Cauchy-Schwarz inequality, we can bound the first term as follows

$$
\begin{aligned}
A_1(R) &\lesssim ||\sin(\beta)^{\lambda/2-1} D_R^2 \xi(R,\cdot)||_{L^1(\beta)} + ||\sin(\beta)^{\lambda/2}\sin(2\beta)^{-1} D_\beta D_R^2 \xi(R,\cdot)||_{L^1(\beta)} \\
&\lesssim ||S^{1/2} D_R^2 \xi(R,\cdot)||_{L^2(\beta)} ||S^{-1/2}\sin(\beta)^{\lambda/2-1}||_{L^2} \\
&\quad + ||P^{1/2} D_\beta D_R^2 \xi(R,\cdot)||_{L^2(\beta)} ||P^{-1/2}\sin(\beta)^{\lambda/2}\sin(2\beta)^{-1}||_{L^2}.
\end{aligned}
$$

Recall P, S, λ defined in (2.142) and $\gamma = 1 + \frac{\alpha}{10}$. A simple calculation yields

$$
\begin{aligned}
||S^{-1/2}\sin(\beta)^{\lambda/2-1}||_{L^2} &\lesssim ||\sin(\beta)^{\gamma/2-1}||_{L^2(\beta)} \lesssim \alpha^{-1/2}, \\
||P^{-1/2}\sin(\beta)^{\lambda/2}\sin(2\beta)^{-1}||_{L^2} &\lesssim ||\sin(\beta)^{\gamma/2-1}\cos(\beta)^{\gamma/2-1}||_{L^2(\beta)} \lesssim \alpha^{-1/2}.
\end{aligned}
$$

Combining the above estimates, we derive

$$
\begin{aligned}
A &\lesssim A_1(R) + A_2(R) \\
&\lesssim \alpha^{-1/2}(||S^{1/2} D_R^2 \xi(R,\cdot)||_{L^2(\beta)} + ||P^{1/2} D_\beta D_R^2 \xi(R,\cdot)||_{L^2(\beta)}) + ||D_R^2 \xi(R,\cdot)||_{L^2(\beta)}.
\end{aligned}
$$

Recall $WS \lesssim \psi_1, WP \lesssim \psi_2$. Consequently, we have

$$
||A^2(R)W||_{L^1(R)} \lesssim \alpha^{-1} ||\xi||^2_{\mathcal{H}^3(\psi)}.
$$

Recall $B(R)$ in (2.143). Since $D_\beta f Q^{1/2} W^{1/2}, D_R D_\beta f Q^{1/2} W^{1/2} \in L^2$, we have $\lim \infty_{R\to 0} B(R) = 0$ and yield

$$
||B^2||_{L^\infty(R)} \leq ||\partial_R B^2||_{L^1(R)} \lesssim ||\partial_R D_\beta f Q^{1/2}||_{L^2} ||D_\beta f Q^{1/2}||_{L^2} \lesssim ||f||^2_{\mathcal{H}^3},
$$

where we have used $\partial_R = R^{-1} D_R, R^{-1} \lesssim W^{1/2}$ and $WQ = \varphi_2$ to obtain the last inequality. Plugging the estimates of A and B in (2.143), we yield the desired estimate on $||D_R^2 \xi D_\beta f \psi_2^{1/2}||_{L^2}$. □

We generalize the $\mathcal{H}^2$ estimate of transport term derived in the earlier arXiv version of [42] as follows.

Proposition 2.8.13. *Assume that $u, \partial_\beta u, D_R u \in \mathcal{H}^3$ and $\Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap \mathcal{C}^1$ we have*

$$
\begin{aligned}
|\langle \Omega, u D_R \Omega \rangle_{\mathcal{H}^3}| &\lesssim \alpha^{-\frac{1}{2}} \left(||u||_{\mathcal{H}^3} + ||\partial_\beta u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}\right) ||\Omega||^2_{\mathcal{H}^3}, \\
|\langle \xi, u D_R \xi \rangle_{\mathcal{H}^3(\psi)}| &\lesssim \alpha^{-\frac{1}{2}} \left(||u||_{\mathcal{H}^3} + ||\partial_\beta u||_{\mathcal{H}^3} + ||D_R u||_{\mathcal{H}^3}\right) (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{\mathcal{C}^1})^2.
\end{aligned}
$$

Moreover, for all $u, D_R u \in X = \mathcal{H}^3 \oplus \mathcal{W}^{5,\infty}$ and $\Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap \mathcal{C}^1$, we have

$$
\begin{aligned}
|\langle \Omega, u D_\beta \Omega \rangle_{\mathcal{H}^3}| &\lesssim \alpha^{-1/2} \left(||u||_X + ||D_R u||_X\right) ||\Omega||^2_{\mathcal{H}^3}, \\
|\langle \xi, u D_\beta \xi \rangle_{\mathcal{H}^3(\psi)}| &\lesssim \alpha^{-1/2} \left(||u||_X + ||D_R u||_X\right) (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{\mathcal{C}^1})^2.
\end{aligned}
$$

The proof follows from the argument in the proof of Proposition 2.8.12 and that in the earlier arXiv version of [42]. Here, the proof is easier since the data is more regular (than $\mathcal{H}^2$), i.e. $\mathcal{H}^3$ or $\mathcal{H}^3(\psi)$, and then the estimate of several nonlinear terms can be done by applying L^∞ estimate on one term. To estimate the mixed derivative terms, e.g., $\langle D_R^2 D_\beta \xi, D_R^2 D_\beta (u D_\beta \xi)\psi_2 \rangle$, we apply the $L^2(R, L^\infty(\beta)) \times L^\infty(R, L^2(\beta))$ argument similar to that in the proof of Proposition 2.8.12.

The following result is a simple $\mathcal{H}^3, \mathcal{H}^3(\psi)$ generalization of another transport estimate in the earlier arXiv version of [42].

Proposition 2.8.14. *Let $\mathcal{H}^3(\rho)$ be either $\mathcal{H}^3$ or $\mathcal{H}^3(\psi)$. For all $g \in \mathcal{H}^3(\rho)$, u with $||D_R^i u||_{L^\infty} < \infty$ for $i \leq 3$ and $||D_R^i D_\beta^j \partial_\beta u||_{L^\infty} < \infty$ for $i + j \leq 2$, we have*

$$|\langle g, u D_R g \rangle_{\mathcal{H}^3(\rho)}| \lesssim \alpha^{-1/2} \Big(\sum_{0 \leq i \leq 3} ||D_R^i u||_{L^\infty} + \sum_{i+j \leq 2} ||D_R^i D_\beta^j \partial_\beta u||_{L^\infty} \Big) ||g||^2_{\mathcal{H}^3(\rho)},$$

The proof follows simply from applying L^∞ estimate on the u term and integration by parts.

2.8.3 A new estimate of the transport term and the estimate of $v_x \xi$

In this subsection, we establish a new estimate of the transport term which is necessary to close the nonlinear estimate and estimate $||v_x \xi||_{\mathcal{H}^3}$ which is not covered by Proposition 2.8.12.

Proposition 2.8.15. *Let Ψ be a solution of* (2.134)*. Suppose that $g, \Omega \in \mathcal{H}^3, \xi \in \mathcal{H}^3(\psi) \cap \mathcal{C}^1$. We have*

$$|\langle g, \frac{1}{\sin(2\beta)} D_R \Psi D_\beta g \rangle_{\mathcal{H}^3}| \lesssim \alpha^{-3/2} ||\Omega||_{\mathcal{H}^3} ||g||^2_{\mathcal{H}^3},$$
$$|\langle \xi, \frac{1}{\sin(2\beta)} D_R \Psi D_\beta \xi \rangle_{\mathcal{H}^3(\psi)}| \lesssim \alpha^{-3/2} ||\Omega||_{\mathcal{H}^3} (||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2} ||\xi||_{\mathcal{C}^1})^2.$$

If one apply Proposition 2.8.13 with $u = \frac{D_R \Psi}{\sin(2\beta)}$, $||D_R u||_{\mathcal{H}^3}$ in the upper bound cannot be bounded by $||\Omega||_{\mathcal{H}^3}$.

Proof. Denote $u = \frac{D_R \Psi}{\sin(2\beta)}$. The estimate of the transport term is similar to that in Proposition 2.8.12 except that we need to perform integration by parts for the terms $\langle D^3 g, u D^3 D_\beta \varphi \rangle$ in the estimate. We focus on a typical and difficult term $\langle D_R^2 D_\beta \xi, D_R^2 u D_\beta^2 \xi \psi_2 \rangle$ to see why we can improve the estimate in Proposition 2.8.13. Other terms can be estimated similarly.

For this term, it suffices to estimate the L^2 norm of $D_R^2 u D_\beta^2 \xi \psi_2^{1/2}$. Recall $\psi_2 = WP$ with W, P defined in (2.142). We have

$$||D_R^2 u D_\beta^2 \xi \psi_2^{1/2}||_2 \leq ||D_R^2 u W^{1/2}||_{L^2(R, L^\infty(\beta))} ||D_\beta^2 \xi P^{1/2}||_{L^\infty(R, L^2(\beta))} \triangleq A \cdot B.$$

The term A can be bounded by $C\alpha^{-1/2}||u||_{\mathcal{H}^3}$, which is further bounded by $C\alpha^{-3/2}||\Omega||_{\mathcal{H}^3}$ using Proposition 2.8.3. The term B is bounded by $C||\xi||_{\mathcal{H}^3(\psi)}$. It is similar to the argument in the proof of Proposition 2.8.12 and we omit the detail. □

Finally, we estimate the nonlinear term $v_x \xi$ in the η equation (2.48).

Proposition 2.8.16. *Let $\Psi, \bar{\Psi}$ be a solution of* (2.134) *with source term $\Omega, \bar{\Omega}$, respectively, and $V_1(\Psi)$ be the operator which is related to v_x and is to be defined in* (2.150). *Assume that $\xi \in \mathcal{H}^3(\psi) \cap \mathcal{C}^1, \Omega \in \mathcal{H}^3$. We have*

$$\begin{aligned} &||V_1(\Psi)\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}(\alpha^{1/2}||\xi||_{\mathcal{C}^1} + ||\xi||_{\mathcal{H}^3(\psi)}), \\ &||V_1(\bar{\Psi})\xi||_{\mathcal{H}^3} \lesssim \alpha^{1/2}||\xi||_{\mathcal{H}^3(\psi)}. \end{aligned} \tag{2.144}$$

The difficulty lies in that $\mathcal{H}^3(\psi)$ is weaker than $\mathcal{H}^3$ (see Lemma 2.7.4). We can not apply Proposition 2.8.12 directly to estimate $v_x \xi$. We need to use a key fact that v_x vanishes on $\beta = 0$.

Proof. We use the formula of $V_1(\Psi)$ (2.152) to be derived

$$\begin{aligned} V_1(\Psi) = {} & \alpha(1 + 2\cos^2\beta) D_R \Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi_* + 2\Psi_* + \sin^2(\beta)\partial_\beta^2 \Psi_* \\ & + \alpha^2 \cos^2(\beta) D_R^2 \Psi \triangleq A(\Psi) + \alpha^2 \cos^2(\beta) D_R^2 \Psi. \end{aligned}$$

where $\Psi_* = \Psi - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\Omega)$. We first consider the second inequality in (2.144). Notice that $V_1(\bar{\Psi})$ vanishes on $\beta = 0$. More precisely, Proposition 2.8.8 implies $\sin(\beta)^{-1/2} V_1(\bar{\Psi}) \in \mathcal{W}^{5,\infty}$. Applying the product rule in $\mathcal{H}^3$ norm in Proposition 2.8.12, Lemma 2.7.4 and then Proposition 2.8.8, we yield

$$||V_1(\bar{\Psi})\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\sin(\beta)^{-1/2} V_1(\bar{\Psi})||_{\mathcal{W}^{5,\infty}} ||\sin(\beta)^{1/2}\xi||_{\mathcal{H}^3} \lesssim \alpha^{1/2}||\xi||_{\mathcal{H}^3(\psi)}.$$

Next, we consider the first inequality in (2.144). From Proposition 2.8.3, we know that $\sin(\beta)^{-1/2} A(\Psi) \in \mathcal{H}^3$. Applying Propositions 2.8.12, 2.8.3 and Lemma 2.7.4, we derive

$$||A(\Psi)\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||A(\Psi)\sin(\beta)^{-1/2}||_{\mathcal{H}^3}||\xi \sin(\beta)^{1/2}||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)}.$$

Finally, we focus on the term $g \triangleq \alpha^2 D_R^2 \Psi$ in $V_1(\Psi)$. We consider the third derivative terms $D^3(D_R^2\Psi \cdot \xi)$ with $D^3 = D_R^i D_\beta^j, i+j=3$ in the $\mathcal{H}^3$ estimate since other terms are easier. If $D^3 = D_R^3$, we need to estimate the L^2 norm of $D_R^3(g\xi)\varphi_1^{1/2}$. Since $\varphi_1 \asymp \psi_1$, the estimate follows from the argument in the proof of Proposition 2.8.12 and we obtain

$$||D_R^3(\alpha^2 D_R^2 \Psi \xi)\varphi_1^{1/2}||_2^2 \lesssim \alpha^{3/2}||\Omega||_{\mathcal{H}^3}(||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2}||\xi||_{\mathcal{C}^1}).$$

Otherwise, we need to estimate the L^2 norm of $D^2 D_\beta(g\cdot\xi)\varphi_2^{1/2}$ with $D^2 = D_R^i D_\beta^j, i+j=2$ (note that D_β commutes with D_R). We rewrite $D_\beta(g\xi)$ as follows

$$\begin{aligned}D_\beta(g\xi) &= \partial_\beta g(\sin(2\beta)\xi) + gD_\beta\xi \\ &= \sin(2\beta)^{3/4}\partial_\beta g(\sin(2\beta)^{1/4}\xi) + \sin(2\beta)^{1/4}(\sin(2\beta)^{-1/2}g)\sin(2\beta)^{1/4}D_\beta\xi.\end{aligned}$$

Notice that $\sin(2\beta)^{1/4}\varphi_2 \lesssim \varphi_1, \psi_1$. Using the idea in the discussion of Lemma 2.7.4 and expanding the $\mathcal{H}^2$ norm, one can verify easily that

$$\begin{aligned}||D^2(D_\beta(g\xi))\varphi_2^{1/2}||_{L^2} \lesssim & ||\sin(2\beta)^{1/2}\partial_\beta g \cdot \sin(2\beta)^{1/4}\xi||_{\mathcal{H}^2} \\ &+ ||\sin(2\beta)^{-1/2} g \sin(2\beta)^{1/4} D_\beta \xi||_{\mathcal{H}^2}.\end{aligned}$$

Applying the $\mathcal{H}^2$ version of the product rule in Proposition 2.8.12 (it is given in [42]), Proposition 2.8.3 to $g = \alpha^2 D_R^2\Psi$, and Lemma 2.7.4, we obtain

$$\begin{aligned}||D^2(D_\beta(g\xi))\varphi_2^{1/2}||_{L^2} &\lesssim \alpha^{-1/2}||\sin(2\beta)^{1/2}\partial_\beta g||_{\mathcal{H}^2}||\sin(2\beta)^{1/4}\xi||_{\mathcal{H}^2} \\ &\quad + \alpha^{-1/2}||\sin(2\beta)^{-1/2}g||_{\mathcal{H}^2}||\sin(2\beta)^{1/4}D_\beta\xi||_{\mathcal{H}^2} \\ &\lesssim \alpha^{3/2}||\Omega||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)}.\end{aligned}$$

Combining the estimates of $A(\Psi)$ and $\alpha^2 D_R^2\Psi$ completes the proof. □

2.9 Nonlinear stability

In this section, we complete the estimates of the remaining terms $\mathcal{R}_3$ in Corollary 2.7.3 and in (2.123),(2.126),(2.127). We will prove the following for the energy E_3 in (2.115) and $E(\xi,\infty)$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E_3^2 \le & -\frac{1}{12}E_3^2 + C\alpha^{1/2}(E_3^2 + \alpha||\xi||_{\mathcal{C}^1}^2) \\ & + C\alpha^{-3/2}(E_3 + \alpha^{1/2}||\xi||_{\mathcal{C}^1})^3 + C\alpha^2 E_3, \qquad (2.145)\end{aligned}$$

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E(\xi,\infty)^2 \le & -E(\xi,\infty)^2 + C||\xi||_{\mathcal{C}^1}(\alpha^{1/2}E_3 + \alpha||\xi||_{\mathcal{C}^1}) \\ & + C||\xi||_{\mathcal{C}^1}(\alpha^{-1}E_3^2 + \alpha^{-1}E_3||\xi||_{\mathcal{C}^1}) + C\alpha^2 E(\xi,\infty), \qquad (2.146)\end{aligned}$$

for any initial perturbation Ω, η, ξ with $E_3(\Omega, \eta, \xi) < +\infty$ and $E(\xi, \infty) < +\infty$, where

$$E(\xi, \infty) \triangleq (||\xi||_\infty^2 + ||\phi_2 D_\beta \xi||_\infty^2 + \mu_4 ||\phi_1 D_R \xi||_\infty^2)^{1/2} \tag{2.147}$$

for some absolute constants μ_4. $E(\xi, \infty)$ is equivalent to $||\xi||_{\mathcal{C}^1}$ (2.117) once we determine the absolute constants μ_4.

The major step is the linear stability that gives the damping term $(-\frac{1}{12} + C\alpha)E_2^3$ and $(-1 + C\alpha)E(\xi, \infty)^2$. We have already established the linear stability in Corollary 2.7.3 and estimates (2.123), (2.126), (2.127). The remaining terms $\mathcal{R}_3$ in Corollary 2.7.3 and in (2.123), (2.126), (2.127) contribute other terms in (2.145)-(2.146). We will further construct an energy $E^2(\Omega, \eta, \xi) \triangleq \alpha E(\xi, \infty)^2 + E_3^2(\Omega, \eta, \xi)$ and these remaining terms are relatively small at the threshold $E = O(\alpha^2)$. Then we can close the nonlinear estimate.

We will first derive several formulas for later use in subsection 2.9.1. Then we estimate the remaining terms mentioned above. In subsection 2.9.2 and 2.9.3, we will apply the product rules obtained in subsection 2.8.2 to estimate the transport terms and nonlinear terms and then complete the estimate (2.145). We will derive the $\mathcal{C}^1$ estimate (2.146) in subsection 2.9.5 and prove finite time blowup in subsection 2.9.6. We remark that estimates similar to the $\mathcal{C}^1$ estimates (2.146) are not required in [42] since there is no swirl.

Notations Throughout this section, χ is the radial cutoff function in Lemma A.0.4. We use $\Psi_*, \bar{\Psi}_*$ to denote the lower order terms in $\Psi, \bar{\Psi}$, i.e.

$$\Psi_* \triangleq \Psi - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\Omega), \quad \bar{\Psi}_* \triangleq \bar{\Psi} - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\bar{\Omega}). \tag{2.148}$$

Ψ_* and Ψ enjoys the elliptic estimate in Proposition 2.8.1 and $\bar{\Psi}, \bar{\Psi}_*$ satisfy Proposition 2.8.8.

2.9.1 Formulas of the velocity and related terms

In this subsection, we derive the formulas of the velocity in terms of the stream function in the (R, β) coordinates to be used later and then collect the remaining terms to be estimated in the nonlinear stability analysis.

Denote

$$\begin{aligned} &u \triangleq U(\Psi), \quad v \triangleq V(\Psi), \quad u_x \triangleq U_1(\Psi), \\ &u_y \triangleq U_2(\Psi), \quad v_x \triangleq V_1(\Psi), \quad v_y \triangleq V_2(\Psi). \end{aligned} \tag{2.149}$$

The formula of U, V in terms of Ψ are given in (2.25). We also collect them below. Using (2.24)-(2.25), $D_R = R\partial_R, r\partial_r = \alpha D_R$ and the incompressible condition $u_x + v_y = 0$, we compute

$$
\begin{aligned}
U(\Psi) &= -2r\sin\beta\Psi - \alpha r \sin\beta D_R\Psi - r\cos\beta\partial_\beta\Psi, \\
V(\Psi) &= 2r\cos\beta\Psi + \alpha r\cos\beta D_R\Psi - r\sin\beta\partial_\beta\Psi, \\
U_1(\Psi) &= -\frac{1}{2}\alpha^2\sin(2\beta)D_R^2\Psi - \frac{\alpha}{2}\sin(2\beta)D_R\Psi + \frac{\sin(2\beta)}{2}\partial_\beta^2\Psi \\
&\quad - \cos(2\beta)\partial_\beta\Psi - \alpha\cos(2\beta)\partial_\beta D_R\Psi, \\
U_2(\Psi) &= \alpha(-1 - 2\sin^2\beta)D_R\Psi - \alpha D_R D_\beta\Psi - D_\beta\Psi - 2\Psi \\
&\quad - \alpha^2\sin^2(\beta)D_R^2\Psi - \cos^2(\beta)\partial_\beta^2\Psi, \\
V_1(\Psi) &= \alpha(1 + 2\cos^2\beta)D_R\Psi - \alpha D_R D_\beta\Psi - D_\beta\Psi + 2\Psi \\
&\quad + \alpha^2\cos^2(\beta)D_R^2\Psi + \sin^2(\beta)\partial_\beta^2\Psi, \\
V_2(\Psi) &= -U_1(\Psi).
\end{aligned}
\tag{2.150}
$$

Recall $\Psi = \frac{\sin(2\beta)}{\pi\alpha}L_{12}(\Omega) + \Psi_*$. For the terms not involving the R-derivative, e.g., Ψ, $\partial_\beta\Psi$, we compute the contributions from the leading order part of Ψ, i.e. $\frac{\sin(2\beta)}{\pi\alpha}L_{12}(\Omega)$, and Ψ_* separately,

$$
\begin{aligned}
U(\Psi) &= -\frac{2r\cos(\beta)}{\pi\alpha}L_{12}(\Omega) - 2r\sin(\beta)\Psi_* - \alpha r\sin\beta D_R\Psi - r\cos\beta\partial_\beta\Psi_* \\
&\triangleq -\frac{2r\cos(\beta)}{\pi\alpha}L_{12}(\Omega) + U(\Psi, \Psi_*), \\
V(\Psi) &= \frac{2r\sin(\beta)}{\pi\alpha}L_{12}(\Omega) + 2r\cos\beta\Psi_* + \alpha r\cos\beta D_R\Psi - r\sin\beta\partial_\beta\Psi_* \\
&\triangleq \frac{2r\sin(\beta)}{\pi\alpha}L_{12}(\Omega) + V(\Psi, \Psi_*), \\
U_1(\Psi) &= -\frac{2}{\pi\alpha}L_{12}(\Omega) - \frac{\alpha^2}{2}\sin(2\beta)D_R^2\Psi - \frac{\alpha}{2}\sin(2\beta)D_R\Psi - \cos(2\beta)\partial_\beta\Psi_* \\
&\quad - \alpha\cos(2\beta)\partial_\beta D_R\Psi + \frac{\sin(2\beta)}{2}\partial_\beta^2\Psi_* \triangleq -\frac{2}{\pi\alpha}L_{12}(\Omega) + U_1(\Psi, \Psi_*), \\
V_2(\Psi) &= -U_1(\Psi) = \frac{2}{\pi\alpha}L_{12}(\Omega) - U_1(\Psi, \Psi_*).
\end{aligned}
\tag{2.151}
$$

The first term in the formulas of U, V, U_1, V_2 is the leading order term. Observe that

$$
\begin{aligned}
-D_\beta\sin(2\beta) - 2\sin(2\beta) - \cos^2(\beta)\partial_\beta^2\sin(2\beta) = 0, \\
-D_\beta\sin(2\beta) + 2\sin(2\beta) + \sin^2(\beta)\partial_\beta^2\sin(2\beta) = 0.
\end{aligned}
$$

For the terms not involving the R-derivative in $U_2(\Psi), V_1(\Psi)$ (2.150), the contributions from $\sin(2\beta)L_{12}(\Omega)$ cancel each other. Hence, we have

$$\begin{aligned} U_2(\Psi) &= \alpha(-1-2\sin^2\beta)D_R\Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi_* \\ &\quad - 2\Psi_* - \alpha^2 \sin^2(\beta) D_R^2 \Psi - \cos^2(\beta)\partial_\beta^2 \Psi_*, \\ V_1(\Psi) &= \alpha(1+2\cos^2\beta)D_R\Psi - \alpha D_R D_\beta \Psi - D_\beta \Psi_* \\ &\quad + 2\Psi_* + \alpha^2 \cos^2(\beta) D_R^2 \Psi + \sin^2(\beta)\partial_\beta^2 \Psi_*. \end{aligned} \tag{2.152}$$

We decompose U, V in (2.151)-(2.152) so that we can apply the elliptic estimate in Propositions 2.8.3, 2.8.8 to $U(\Psi,\Psi_*), V(\Psi,\Psi_*), U_1(\Psi,\Psi_*), V_2(\Psi,\Psi_*), U_2(\Psi), V_1(\Psi)$.

Recall the formula of $\mathbf{u}\cdot\nabla$ in (2.27)

$$u\cdot\nabla = -(\alpha R\partial_\beta\Psi)\partial_R + (2\Psi + \alpha R\partial_R\Psi)\partial_\beta.$$

Since $\Psi = \frac{\sin(2\beta)}{\pi\alpha}L_{12}(\Omega) + \Psi_*, D_\beta = \sin(2\beta)\partial_\beta$, we have

$$\begin{aligned} u\cdot\nabla &= (-\frac{2\cos(2\beta)}{\pi}L_{12}(\Omega) - \alpha\partial_\beta\Psi_*)D_R + (\frac{2}{\pi\alpha}L_{12}(\Omega) + \frac{2\Psi_* + \alpha D_R\Psi}{\sin(2\beta)})D_\beta \\ &\triangleq \frac{2}{\pi\alpha}L_{12}(\Omega)D_\beta + \mathcal{T}(\Omega), \\ \mathcal{T}(\Omega) &\triangleq -\frac{2\cos(2\beta)}{\pi}L_{12}(\Omega)D_R - \alpha\partial_\beta\Psi_* D_R + \frac{2\Psi_* + \alpha D_R\Psi}{\sin(2\beta)}D_\beta. \end{aligned} \tag{2.153}$$

Using (2.45), we have $\frac{2}{\pi\alpha}L_{12}(\bar\Omega) = \frac{3}{1+R}$ and

$$\bar{\mathbf{u}}\cdot\nabla = \frac{3}{1+R}D_\beta + \mathcal{T}(\bar\Omega).$$

Recall the formulations (2.52)-(2.54) and their equivalence (2.59). We use the notations (2.149) to rewrite u_x, u_y and so on, and the above computations to expand the remaining terms $\mathcal{R}$ in (2.52)-(2.54). $\mathcal{R}$ consists of three parts: the lower order terms in the linearized equation (denote as P), the error term $\bar F$

(2.46) and the nonlinear term N (2.49). The formula of P is given below

$$\begin{aligned}
\mathcal{R}_\Omega =& P_\Omega + \bar{F}_\Omega + N_\Omega, \quad \mathcal{R}_\eta = P_\eta + \bar{F}_\eta + N_\eta, \quad \mathcal{R}_\xi = P_\xi + \bar{F}_\xi + N_\xi, \\
P_\Omega =& (-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\Omega + (\alpha c_\omega D_R - (\mathbf{u}\cdot\nabla))\bar{\Omega}, \\
P_\eta =& (-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\eta + (\alpha c_\omega D_R - (\mathbf{u}\cdot\nabla))\bar{\eta} - (U_1(\bar{\Psi}) + \frac{3}{1+R})\eta \\
& - (U_1(\Psi) + \frac{2}{\pi\alpha} L_{12}(\Omega))\bar{\eta} - (V_1(\bar{\Psi})\xi + V_1(\Psi)\bar{\xi}), \\
P_\xi =& (-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\xi + (\alpha c_\omega D_R - (\mathbf{u}\cdot\nabla))\bar{\xi} + (-V_2(\bar{\Psi}) + \frac{3}{1+R})\xi \\
& + (-V_2(\Psi) + \frac{2}{\pi\alpha} L_{12}(\Omega))\bar{\xi} - (U_2(\Psi)\bar{\eta} + U_2(\bar{\Psi})\eta).
\end{aligned} \tag{2.154}$$

We remark that P is the difference between the linear part of (2.48) and (2.52)-(2.54).

Recall $\bar{c}_\omega = -1, \bar{c}_l = \frac{1}{\alpha} + 3$ and $\bar{\Omega}, \bar{\eta}$ in (2.44). Notice that $c_l = \frac{1}{\alpha}, \Omega_* = \frac{3\alpha}{c}\frac{R}{(1+R)^2}, \eta_* = \frac{6\alpha}{c}\frac{R}{(1+R)^3}, \Gamma = \cos(\beta)^\alpha$ is a solution of (2.40) and $\bar{\Omega}, \bar{\eta}$ satisfy $\bar{\Omega} = \Omega_*\Gamma(\beta), \ \bar{\eta} = \eta_*\Gamma(\beta), \ \frac{c}{\alpha}\int_R^\infty \frac{\Omega_*}{s} ds = \frac{3}{1+R}$. Hence, we have

$$D_R\bar{\Omega} = \bar{c}_\omega\bar{\Omega} + \bar{\eta}, \quad D_R\bar{\eta} = 2\bar{c}_\omega\bar{\Omega} + \frac{3}{1+R}\bar{\eta}.$$

Hence, we can simplify $\bar{F}_\Omega, \bar{F}_\eta$ in (2.46) as

$$\begin{aligned}
\bar{F}_\Omega &= (-3\alpha D_R - \bar{\mathbf{u}}\cdot\nabla)\bar{\Omega}, \\
\bar{F}_\eta &= (-\frac{3}{1+R} - U_1(\bar{\Psi}))\bar{\eta} - V_1(\bar{\Psi})\bar{\xi} + (-3\alpha D_R - \bar{\mathbf{u}}\cdot\nabla)\bar{\eta},
\end{aligned} \tag{2.155}$$

where we have used the notations in (2.149) for $\bar{u}_x, \bar{u}_y, \bar{v}_x, \bar{v}_y$.

Recall the definition of the $\mathcal{H}^3, \mathcal{H}^3(\psi)$ inner product in (2.131) and the remaining terms $\mathcal{R}_3$ in (2.115),(2.132). See also the full expression of the weighted H^3 energy E_3 (2.128) related to $\mathcal{R}_3$ Clearly, we have

$$\begin{aligned}
\mathcal{R}_3 = & \langle \mathcal{R}_\Omega, \Omega\varphi_0\rangle + \langle \mathcal{R}_\eta, \eta\psi_0\rangle + \frac{81}{4\pi c} L_{12}(\Omega)(0)\langle \mathcal{R}_\Omega, \sin(2\beta)R^{-1}\rangle \\
& + \langle \mathcal{R}_\Omega, \Omega\rangle_{\mathcal{H}^3} + \langle \mathcal{R}_\eta, \eta\rangle_{\mathcal{H}^3} + \langle \mathcal{R}_\xi, \xi\rangle_{\mathcal{H}^3(\psi)}.
\end{aligned} \tag{2.156}$$

We remark that $\langle\cdot,\cdot\rangle$ in the first three terms is the L^2 inner product defined in (2.15). We assume that $\Omega, \eta \in \mathcal{H}^3, \Omega \in L^2(\varphi), \eta \in L^2(\psi), \ \xi \in \mathcal{H}^3(\psi), \xi \in \mathcal{C}^1$. We will choose initial perturbations Ω, η, ξ in these classes. In subsection 2.9.2, we estimate the transport terms in the last three terms in $\mathcal{R}_3$. In subsection 2.9.3, we estimate the nonlinear terms in the last three terms in $\mathcal{R}_3$. In subsection 2.9.4, we estimate the first three terms in $\mathcal{R}_3$.

2.9.2 Analysis of the transport terms in P, N, F

In this subsection, we estimate the transport terms in P, N and F in $\mathcal{H}^3$ or $\mathcal{H}^3(\psi)$ norm. Our main tools in this and the next few subsections are the product rules, the elliptic estimates obtained in Section 2.8 and Lemma A.0.4 on $L_{12}(\Omega)$.

The reader should pay attention to the subtle cancellation near $R = 0$ in the estimates in subsections 2.9.2.3, 2.9.2.4.

2.9.2.1 Transport terms I : $(-3\alpha D_R - \mathcal{T}(\bar{\Omega}))g$ in P

We estimate

$$
\begin{aligned}
&I_1 = |\langle(-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\Omega, \Omega\rangle_{\mathcal{H}^3}|, \quad I_2 = |\langle(-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\eta, \eta\rangle_{\mathcal{H}^3}|, \\
&I_3 = |\langle(-3\alpha D_R - \mathcal{T}(\bar{\Omega}))\xi, \xi\rangle_{\mathcal{H}^3(\psi)}|.
\end{aligned}
$$

Recall $\mathcal{T}(\bar{\Omega})$ in (2.153)

$$
3\alpha D_R + \mathcal{T}(\bar{\Omega}) = 3\alpha D_R - \frac{2\cos(2\beta)}{\pi} L_{12}(\bar{\Omega}) D_R - \alpha\partial_\beta \bar{\Psi}_* D_R + \frac{1}{\sin(2\beta)}(2\bar{\Psi}_* + \alpha D_R \bar{\Psi}) D_\beta.
$$

Applying Proposition 2.8.8 to estimate the above coefficients, then Proposition 2.8.13 to the D_β transport terms and Proposition 2.8.14 to the D_R transport terms yield

$$
I_1 \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3}^2, \quad I_2 \lesssim \alpha^{1/2} ||\eta||_{\mathcal{H}^3}^2, \quad I_3 \lesssim \alpha^{1/2} ||\xi||_{\mathcal{H}^3(\psi)}^2.
$$

2.9.2.2 Transport term II : $-\alpha c_l R\partial_R g - (\mathbf{u}\cdot\nabla)g$ in N (2.49)

We are going to estimate

$$
\begin{aligned}
&|\langle(-\alpha c_l D_R - (\mathbf{u}\cdot\nabla))\Omega, \Omega\rangle_{\mathcal{H}^2}|, \quad |\langle(-\alpha c_l D_R - (\mathbf{u}\cdot\nabla))\eta, \eta\rangle_{\mathcal{H}^2}|, \\
&|\langle(-\alpha c_l D_R - (\mathbf{u}\cdot\nabla))\xi, \xi\rangle_{\mathcal{H}^2(\psi)}|.
\end{aligned}
$$

Recall $\alpha c_l = -\frac{2(1-\alpha)}{\pi\alpha} L_{12}(\Omega)(0)$ in (2.47) and the computation about $\mathbf{u}\cdot\nabla$ in (2.153)

$$
\begin{aligned}
(-\alpha c_l D_R - (\mathbf{u}\cdot\nabla)) = &(\frac{2(1-\alpha)}{\pi\alpha} L_{12}(\Omega)(0) + \frac{2\cos(2\beta)}{\pi} L_{12}(\Omega) + \alpha\partial_\beta \Psi_*) D_R \\
&- (\frac{2}{\pi\alpha} L_{12}(\Omega) + \frac{2\Psi_*}{\sin(2\beta)} + \frac{\alpha D_R \Psi}{\sin(2\beta)}) D_\beta.
\end{aligned}
$$

For the first two D_R transport terms, we apply Proposition 2.8.14 and Lemma A.0.4 to estimate $||D_R^k L_{12}(\Omega)||_{L^\infty}$ for $k \leq 3$. For the third, fourth $((\frac{2}{\pi\alpha} L_{12}(\Omega)) D_\beta)$

and fifth $(\frac{2\Psi_*}{\sin(2\beta)})D_\beta)$ transport terms, we apply Proposition 2.8.13, Proposition 2.8.3 to $\partial_\beta \Psi_*, \frac{\Psi_*}{\sin(2\beta)}$ and (A.10) in Lemma A.0.4 to $L_{12}(\Omega)$. For the last transport term, we use Proposition 2.8.15. Hence, we derive

$$
\begin{aligned}
&|I_1| \lesssim \alpha^{-3/2}||\Omega||_{\mathcal{H}^3}^3, \quad |I_2| \lesssim \alpha^{-3/2}||\Omega||_{\mathcal{H}^3}||\eta||_{\mathcal{H}^3}^2, \\
&|I_3| \lesssim \alpha^{-3/2}||\Omega||_{\mathcal{H}^3}(||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2}||\xi||_{\mathcal{C}^1})^2.
\end{aligned}
$$

The largest term is $\frac{2}{\pi\alpha}L_{12}(\Omega)D_\beta$, which leads to $\alpha^{-3/2}$ in the upper bound.

2.9.2.3 Transport term III : $(\alpha c_\omega D_R - (\mathbf{u}\cdot\nabla))\bar{g}$ in P

Next, we estimate

$$
||\alpha c_\omega D_R - (u\cdot\nabla))\bar{\Omega}||_{\mathcal{H}^3}, \quad ||\alpha c_\omega D_R - (u\cdot\nabla))\bar{\eta}||_{\mathcal{H}^3}, \quad ||\alpha c_\omega D_R - (u\cdot\nabla))\bar{\xi}||_{\mathcal{H}^3(\psi)}.
$$

Recall that $\mathcal{H}^3$ contains a singular weight $\frac{(1+R)^4}{R^4}$. We use the explicit form $\Gamma(\beta) = \cos(\beta)^\alpha$ and a careful calculation to cancel the singular weight R^{-4} near $R = 0$. Using the formula for c_ω in (2.47) and the computation in (2.153), we have

$$
\begin{aligned}
&(\alpha c_\omega D_R - (u\cdot\nabla))g \\
&= \left(-\frac{2}{\pi}L_{12}(\Omega)(0)D_R + \frac{2\cos(2\beta)}{\pi}L_{12}(\Omega)D_R - \frac{2}{\pi\alpha}L_{12}(\Omega)D_\beta\right)g \\
&\quad + (\alpha\partial_\beta\Psi_* D_R - (\sin(2\beta))^{-1}(2\Psi_* + \alpha D_R\Psi)D_\beta)g \triangleq I(g) + II(g).
\end{aligned}
\tag{2.157}
$$

Denote $Q = L_{12}(\Omega) - \chi L_{12}(\Omega)(0)$. We use $L_{12}(\Omega) = Q + \chi L_{12}(\Omega)(0)$ to rewrite $I(g)$

$$
\begin{aligned}
I &= \frac{2}{\pi}L_{12}(\Omega)(0)(-D_R g + \cos(2\beta)\chi D_R g - \frac{1}{\alpha}\chi D_\beta g) + \frac{2}{\pi}Q(\cos(2\beta)D_R g - \frac{1}{\alpha}D_\beta g) \\
&\triangleq I_1 + I_2.
\end{aligned}
\tag{2.158}
$$

Using (2.69) and the formula of $g = \bar{\Omega}, \bar{\eta}$ in (2.44), we have

$$
D_\beta\Gamma = -2\alpha\sin^2(\beta)\Gamma, \quad D_\beta g = -2\alpha\sin^2(\beta)g.
$$

It follows that

$$
\begin{aligned}
I_1 &= \frac{2}{\pi}L_{12}(\Omega)(0)(-D_R g + \cos(2\beta)\chi D_R g + 2\sin^2(\beta)\chi g) \\
&= \frac{2}{\pi}L_{12}(\Omega)(0)(-(1-\chi)D_R g + 2\sin^2(\beta)\chi(-D_R g + g)).
\end{aligned}
\tag{2.159}
$$

Since the smooth cutoff function χ satisfies $1-\chi(R)=0$ for $R \le 1$. I_1 vanishes quadratically near $R=0$. For $(g, \mathcal{H}^3(\rho)) = (\bar{\Omega}, \mathcal{H}^3), (\bar{\eta}, \mathcal{H}^3)$ or $(\bar{\xi}, \mathcal{H}^3(\psi))$, applying Lemma A.0.6 to $g = \bar{\Omega}, \bar{\eta}$, (A.26) in Lemma A.0.8 to $g = \bar{\xi}$ and using a direct calculation yield

$$
\begin{aligned}
||I_1(g)||_{\mathcal{H}^3(\rho)} &\lesssim |L_{12}(\Omega)(0)|(||(1-\chi)g||_{\mathcal{H}^3(\rho)} + ||D_R g - g||_{\mathcal{H}^3(\rho)}) \\
&\lesssim \alpha |L_{12}(\Omega)(0)| \lesssim \alpha ||\Omega||_{\mathcal{H}^3},
\end{aligned}
$$

where we have used (A.9) in Lemma A.0.4 in the last inequality.

Recall $Q = L_{12}(\Omega) - \chi L_{12}(\Omega)(0)$ and $I_2, II(g)$ in (2.157), (2.158). For $g = \bar{\Omega}, \bar{\eta}$, applying the product estimate in Proposition 2.8.12, we get

$$
\begin{aligned}
||I_2(g||_{\mathcal{H}^3} &\lesssim \alpha^{-1/2} ||Q||_{\mathcal{H}^3} (||D_R g||_{\mathcal{W}^{5,\infty}} + \alpha^{-1} ||D_\beta g||_{\mathcal{W}^{5,\infty}}) \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3}, \\
||II(g)||_{\mathcal{H}^3} &\lesssim \alpha^{-1/2} ||\Omega||_{\mathcal{H}^3} (\alpha ||D_R g||_{\mathcal{W}^{5,\infty}} + ||D_\beta g||_{\mathcal{W}^{5,\infty}}) \lesssim \alpha^{3/2} ||\Omega||_{\mathcal{H}^3},
\end{aligned}
$$

where we have applied Proposition 2.8.3 to Ψ, Lemma A.0.4 to Q and Proposition A.0.7 to $g = \bar{\Omega}, \bar{\eta}$. For $g = \bar{\xi}$, applying Proposition 2.8.12 yields [4]

$$
\begin{aligned}
||I_2(\bar{\xi})||_{\mathcal{H}^3(\psi)} &\lesssim \alpha^{-1/2} ||Q||_{\mathcal{H}^3} (\alpha^{1/2} ||D_R \bar{\xi}||_{\mathcal{C}^1} + ||D_R \bar{\xi}||_{\mathcal{H}^3(\psi)} \\
&\quad + \alpha^{1/2} ||D_\beta \bar{\xi}||_{\mathcal{C}^1} + ||D_\beta \bar{\xi}||_{\mathcal{H}^3(\psi)}) \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3}, \\
||II(\bar{\xi})||_{\mathcal{H}^3(\psi)} &\lesssim \alpha^{-1/2} ||\Omega||_{\mathcal{H}^3} (\alpha^{3/2} ||D_R \bar{\xi}||_{\mathcal{W}\mathcal{C}^1} + \alpha ||D_R \bar{\xi}||_{\mathcal{H}^3(\psi)} \\
&\quad + \alpha^{1/2} ||D_\beta \bar{\xi}||_{\mathcal{C}^1} + ||D_\beta \bar{\xi}||_{\mathcal{H}^3(\psi)}) \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3},
\end{aligned}
$$

where we have used Lemma A.0.8 to estimate the norm of $\bar{\xi}$. Hence, we prove

$$
\begin{aligned}
&||\alpha c_\omega D_R - (u \cdot \nabla))\bar{\Omega}||_{\mathcal{H}^3} + ||\alpha c_\omega D_R - (u \cdot \nabla))\bar{\eta}||_{\mathcal{H}^3} \\
&+ ||\alpha c_\omega D_R - (u \cdot \nabla))\bar{\xi}||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3}.
\end{aligned}
$$

2.9.2.4 Transport term IV: $(-3\alpha D_R - \bar{\mathbf{u}} \cdot \nabla) g$ in $\bar{F}_\Omega, \bar{F}_\eta, \bar{F}_\xi$

We will prove for $(g, \mathcal{H}^3(\rho)) = (\bar{\Omega}, \mathcal{H}^3), (\bar{\eta}, \mathcal{H}^3), (\bar{\xi}, \mathcal{H}^3(\psi))$

$$
||(-3\alpha D_R - \bar{\mathbf{u}} \cdot \nabla) g||_{\mathcal{H}^3(\rho)} \lesssim \alpha^2. \tag{2.160}
$$

[4] The estimate of $I_2(\bar{\xi}), II(\bar{\xi})$ can be improved to $\alpha^{3/2} ||\Omega||_{\mathcal{H}^3}$ but we do not need this extra smallness here.

From (2.45), we have $\frac{2}{\pi}L_{12}(\bar{\Omega})(0) = 3\alpha$. Hence, we can apply the decomposition in (2.157)-(2.158) to $(-3\alpha D_R - \bar{\mathbf{u}}\cdot\nabla)g$ to get

$$\begin{aligned}
&(-3\alpha D_R - \bar{\mathbf{u}}\cdot\nabla)g = I_1(g) + I_2(g) + II(g),\\
&II(g) = (\alpha\partial_\beta\bar{\Psi}_* D_R - (\sin(2\beta))^{-1}(2\bar{\Psi}_* + \alpha D_R\bar{\Psi})D_\beta)g,\\
&I_1(g) = \frac{2}{\pi}L_{12}(\bar{\Omega})(0)(-D_Rg + \cos(2\beta)\chi D_Rg - \frac{1}{\alpha}\chi D_\beta g),\\
&I_2(g) = \frac{2}{\pi}\bar{Q}(\cos(2\beta)D_Rg - \frac{1}{\alpha}D_\beta g),
\end{aligned} \tag{2.161}$$

where $\bar{Q} = L_{12}(\bar{\Omega}) - \chi L_{12}(\bar{\Omega})(0)$. Notice that the computation (2.159) still holds for $g = \bar{\Omega}, \bar{\eta}$

$$I_1(g) = \frac{2}{\pi}L_{12}(\bar{\Omega})(0)(-(1-\chi)D_Rg + 2\sin^2(\beta)\chi(-D_Rg + g).$$

Recall $L_{12}(\bar{\Omega}) = \frac{3\alpha\pi}{2(1+R)}$. Notice that $(1-\chi)D_Rg, D_Rg - g, QD_Rg, QD_\beta g$ vanish quadratically near $R = 0$. Applying Lemma A.0.6 to $g = \bar{\Omega}, \bar{\eta}$ and using a direct calculation yield

$$||I_1(g)||_{\mathcal{H}^3} \lesssim \alpha|L_{12}(\bar{\Omega})(0)| \lesssim \alpha^2, \quad ||I_2(g)||_{\mathcal{H}^3} \lesssim \alpha^2.$$

Since $\bar{\xi}$ already vanishes quadratically near $R = 0$, using Lemma A.0.8 for $\bar{\xi}$ and a direct calculation give

$$||I_1(\bar{\xi})||_{\mathcal{H}^3(\psi)} \lesssim \alpha|L_{12}(\bar{\Omega})(0)| \lesssim \alpha^2, \quad ||I_2(\bar{\xi})||_{\mathcal{H}^3(\psi)} \lesssim \alpha^2.$$

For $II(g)$ with $g = \bar{\Omega}, \bar{\eta}$, we apply Propositions 2.8.9, 2.8.7 and the triangle inequality to yield

$$\begin{aligned}
||II(g)||_{\mathcal{H}^3} &\lesssim ||\frac{(1+R)^3}{R^2}II(g)||_{\mathcal{W}^{3,\infty}} \lesssim ||\frac{1+R}{R}\alpha\partial_\beta\bar{\Psi}_*||_{\mathcal{W}^{5,\infty}}||\frac{(1+R)^2}{R}D_Rg||_{\mathcal{W}^{3,\infty}}\\
&+ ||\frac{1+R}{R}(\sin(2\beta))^{-1}(2\bar{\Psi}_* + \alpha D_R\bar{\Psi})||_{\mathcal{W}^{5,\infty}}||\frac{(1+R)^2}{R}D_\beta g||_{\mathcal{W}^{3,\infty}} \lesssim \alpha^2,
\end{aligned}$$

where we have applied Proposition 2.8.8 to $\bar{\Psi}, \bar{\Psi}_*$ and Proposition A.0.7 to $g = \bar{\Omega}, \bar{\eta}$.

For $II(\bar{\xi})$, we use Propositions 2.8.12, 2.8.8 and Lemma A.0.8 to get

$$\begin{aligned}
&||II(\bar{\xi})||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1/2}||\partial_\beta\bar{\Psi}_*||_{\mathcal{W}^{5,\infty}}(\alpha^{3/2}||D_R\bar{\xi}||_{\mathcal{W}\mathcal{C}^1} + \alpha||D_R\bar{\xi}||_{\mathcal{H}^3(\psi)})\\
&+ \alpha^{-1/2}||(\sin(2\beta))^{-1}(2\bar{\Psi}_* + \alpha D_R\bar{\Psi})||_{\mathcal{W}^{5,\infty}}(\alpha^{1/2}||D_\beta\bar{\xi}||_{\mathcal{C}^1} + ||D_\beta\bar{\xi}||_{\mathcal{H}^3(\psi)}) \lesssim \alpha^{5/2}.
\end{aligned}$$

2.9.3 Nonlinear forcing terms in P, N, F

The estimates in this subsection are obtained by applying the product estimates in subsection 2.8.2 directly.

2.9.3.1 Nonlinear forcing term in P_η, P_ξ

We are going to estimate

$$
\begin{aligned}
I_1 &= || -(U_1(\bar\Psi) + \frac{3}{1+R})\eta - (U_1(\Psi) + \frac{2}{\pi\alpha}L_{12}(\Omega))\bar\eta||_{\mathcal{H}^3}, \\
II_1 &= ||(-V_2(\bar\Psi) + \frac{3}{1+R})\xi + (-V_2(\Psi) + \frac{2}{\pi\alpha}L_{12}(\Omega))\bar\xi||_{\mathcal{H}^3(\psi)}, \\
I_2 &= ||V_1(\bar\Psi)\xi + V_1(\Psi)\bar\xi||_{\mathcal{H}^3}, \quad II_2 = ||U_2(\Psi)\bar\eta + U_2(\bar\Psi)\eta||_{\mathcal{H}^3(\psi)}.
\end{aligned}
$$

From (2.45), $\frac{2}{\pi\alpha}L_{12}(\bar\Omega) = \frac{3}{1+R}$. Recall the formula of U_i, V_j in (2.151)-(2.152). Applying Propositions 2.8.8, 2.8.3, we obtain

$$
\begin{aligned}
&||U_1(\bar\Psi) + \frac{2}{\pi\alpha}L_{12}(\bar\Omega)||_{\mathcal{W}^{5,\infty}} = || - V_2(\bar\Psi) + \frac{2}{\pi\alpha}L_{12}(\bar\Omega)||_{\mathcal{W}^{5,\infty}} \lesssim \alpha, \\
&||U_1(\Psi) + \frac{2}{\pi\alpha}L_{12}(\Omega)||_{\mathcal{H}^3} = || - V_2(\Psi) + \frac{2}{\pi\alpha}L_{12}(\Omega)||_{\mathcal{W}^{5,\infty}} \lesssim ||\Omega||_{\mathcal{H}^3}, \\
&||U_2(\bar\Psi)||_{\mathcal{W}^{5,\infty}} \lesssim \alpha, \quad ||U_2(\Psi)||_{\mathcal{H}^3} \lesssim ||\Omega||_{\mathcal{H}^3}.
\end{aligned}
\tag{2.162}
$$

Applying Proposition 2.8.12, Lemma A.0.7 to $\bar\eta$ and Lemma A.0.8 to $\bar\xi$, we yield

$$
\begin{aligned}
I_1 &\lesssim \alpha^{1/2}||\eta||_{\mathcal{H}^3} + \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}||\bar\eta||_{\mathcal{W}^{5,\infty}} \lesssim \alpha^{1/2}(||\eta||_{\mathcal{H}^3} + ||\Omega||_{\mathcal{H}^3}), \\
II_1 &\lesssim \alpha^{1/2}(\alpha^{1/2}||\xi||_{\mathcal{C}^1} + ||\xi||_{\mathcal{H}^3(\psi)}) + \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}(\alpha^{1/2}||\bar\xi||_{\mathcal{C}^1} + ||\bar\xi||_{\mathcal{H}^3(\psi)}) \\
&\lesssim \alpha^{1/2}(\alpha^{1/2}||\xi||_{\mathcal{C}^1} + ||\xi||_{\mathcal{H}^3(\psi)}) + \alpha^{3/2}||\Omega||_{\mathcal{H}^3},
\end{aligned}
$$

where we have used Lemma A.0.8 in the last inequality. Using Lemma 2.7.4 and Proposition 2.8.12, we derive

$$
\begin{aligned}
II_2 &\lesssim ||U_2(\Psi)\bar\eta + U_2(\bar\Psi)\eta||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}(||\Omega||_{\mathcal{H}^3}||\bar\eta||_{\mathcal{W}^{5,\infty}} + ||U_2(\bar\Psi)||_{\mathcal{W}^{5,\infty}}||\eta||_{\mathcal{H}^3}) \\
&\lesssim \alpha^{1/2}(||\Omega||_{\mathcal{H}^3} + ||\eta||_{\mathcal{H}^3}).
\end{aligned}
$$

For I_2, we use Proposition 2.8.16 and Lemma A.0.8 to obtain

$$
I_2 \lesssim \alpha^{1/2}||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}(\alpha^{1/2}||\bar\xi||_{\mathcal{C}^1} + ||\bar\xi||_{\mathcal{H}^3(\psi)}) \lesssim \alpha^{1/2}||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{3/2}||\Omega||_{\mathcal{H}^3}.
$$

2.9.3.2 Nonlinear forcing term in N (2.49)

Recall the formula of U_1, V_2 in (2.151). We use the following decomposition

$$
-V_2(\Psi) = U_1(\Psi) = (U_1(\Psi) + \frac{2}{\pi\alpha}L_{12}(\Omega)) - \frac{2}{\pi\alpha}L_{12}(\Omega) = I + II.
$$

Applying Proposition 2.8.3 to I and Lemma A.0.4 to II, we obtain

$$
||V_2(\Psi)||_X = ||U_1(\Psi)||_X \lesssim ||I||_{\mathcal{H}^3} + \alpha^{-1}||L_{12}(\Omega)||_X \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}. \tag{2.163}
$$

Applying Propositions 2.8.12, 2.8.3, we get

$$||U_1(\Psi)\eta||_{\mathcal{H}^3} \lesssim \alpha^{-3/2}||\Omega||_{\mathcal{H}^3}||\eta||_{\mathcal{H}^3},$$
$$||(V_2(\Psi)\xi||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-3/2}||\Omega||_{\mathcal{H}^3}(||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2}||\xi||_{\mathcal{C}^1}).$$

Applying Proposition 2.8.16 to $V_1\xi$, Proposition 2.8.12 and Lemma 2.7.4 to $U_2\eta$ yields

$$||-V_1(\Psi)\xi||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}(||\xi||_{\mathcal{H}^3(\psi)} + \alpha^{1/2}||\xi||_{\mathcal{C}^1}),$$
$$||-U_2(\Psi)\eta||_{\mathcal{H}^3(\psi)} \lesssim ||U_2(\Psi)\eta||_{\mathcal{H}^3} \lesssim \alpha^{-1/2}||\Omega||_{\mathcal{H}^3}||\eta||_{\mathcal{H}^3}.$$

Finally, from (2.47), (A.9), the scalar c_ω satisfies $|c_\omega| \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}$. Hence, we obtain

$$||c_\omega\Omega||_{\mathcal{H}^3} \lesssim \alpha^{-1}||\Omega||^2_{\mathcal{H}^3}, \ ||c_\omega\eta||_{\mathcal{H}^3} \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}||\eta||_{\mathcal{H}^3},$$
$$||c_\omega\xi||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}||\xi||_{\mathcal{H}^3(\psi)}.$$

2.9.3.3 Nonlinear forcing terms in F

Recall that we have estimated the transport term $(-3\alpha D_R - \bar{\mathbf{u}}\nabla)g$ in F_Ω, F_η, F_ξ in (2.160). The remaining terms in $\bar{F}_\eta$ and $\bar{F}_\xi$ (see (2.46), (2.155)) are

$$I = (-\frac{3}{1+R} - U_1(\bar{\Psi}))\bar{\eta} - V_1(\bar{\Psi})\bar{\xi}, \quad II = (2\bar{c}_\omega - V_2(\bar{\Psi}))\bar{\xi} - U_2(\bar{\Psi})\bar{\eta} - D_R\bar{\xi}, \tag{2.164}$$

where we have used $-\alpha\bar{c}_l D_R = -D_R - 3\alpha D_R$ since $\bar{c}_l = \frac{1}{\alpha} + 3$ (2.44). From (2.45), we have $\frac{2}{\pi\alpha}L_{12}(\bar{\Omega}) = \frac{3}{1+R}$. Using U_i, V_j in (2.151)-(2.152), $\bar{\eta}$ (2.44) and Proposition 2.8.8, we have

$$||\frac{1+R}{R}(U_1(\bar{\Psi}) + \frac{3}{1+R})||_{\mathcal{W}^{5,\infty}} \lesssim \alpha,$$
$$||\frac{1+R}{R}U_2(\bar{\Psi})||_{\mathcal{W}^{5,\infty}} \lesssim \alpha, \quad ||\frac{(1+R)^2}{R}\bar{\eta}||_{\mathcal{W}^{5,\infty}} \lesssim \alpha.$$

Applying the embedding in Proposition 2.8.9 and then the algebra property of $\mathcal{W}^{3,\infty}$ in Proposition 2.8.7 to $\bar{\eta}$ and the above estimates, we get

$$||(-\frac{3}{1+R} - U_1(\bar{\Psi}))\bar{\eta}||_{\mathcal{H}^3} \lesssim \alpha^2, \quad ||U_2(\bar{\Psi})\bar{\eta}||_{\mathcal{H}^3(\psi)} \lesssim ||U_2(\bar{\Psi})\bar{\eta}||_{\mathcal{H}^3} \lesssim \alpha^2,$$

where we have used (2.130) in the second inequality. Applying the product estimates in Propositions 2.8.12, 2.8.16, Proposition 2.8.8 to $V_2(\bar{\Psi})$ and Lemma A.0.8 to $\bar{\xi}$, we yield

$$||(V_2(\bar{\Psi}) - \frac{3}{1+R})\bar{\xi}||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{-1/2}\cdot\alpha(\alpha^{1/2}||\bar{\xi}||_{\mathcal{C}^1} + ||\bar{\xi}||_{\mathcal{H}^3(\psi)}) \lesssim \alpha^{5/2},$$
$$||V_1(\bar{\Psi})\bar{\xi}||_{\mathcal{H}^3} \lesssim \alpha^{1/2}||\bar{\xi}||_{\mathcal{H}^3(\psi)} \lesssim \alpha^{5/2}.$$

For the remaining part in II, we simply use $\bar{c}_\omega = -1$ and Lemma A.0.8 to get

$$||2\bar{c}_\omega \bar{\xi} - D_R \bar{\xi}||_{\mathcal{H}^3(\psi)} + ||\frac{3}{1+R}\bar{\xi}||_{\mathcal{H}^3(\psi)} \lesssim \alpha^2.$$

Therefore, combining the formula of $\bar{F}$ in (2.46), (2.155), the estimate (2.160) and the above estimates of I, II, we prove

$$||\bar{F}_\Omega||_{\mathcal{H}^3} \lesssim \alpha^2, \quad ||\bar{F}_\eta||_{\mathcal{H}^3} \lesssim \alpha^2, \quad ||\bar{F}_\xi||_{\mathcal{H}^3(\psi)} \lesssim \alpha^2. \tag{2.165}$$

2.9.4 Analysis of the remaining terms in $\mathcal{R}_3$

It remains to estimate

$$\langle \mathcal{R}_\Omega, \Omega\varphi_0 \rangle, \quad \langle \mathcal{R}_\eta, \eta\psi_0 \rangle, \quad \frac{81}{4\pi c} L_{12}(\Omega)(0) \langle \mathcal{R}_\Omega, \sin(2\beta) R^{-1} \rangle, \tag{2.166}$$

in $\mathcal{R}_3$ (2.156). Recall the definition of φ_0, ψ_0 in Definition 2.78 and φ_1 in Definition 2.6.2. Note that $\psi_0(R,\beta)$ grows linearly for large R. Clearly, we have

$$\varphi_0 \lesssim \varphi_1,$$
$$\psi_0 = \frac{9}{32} R\Gamma(\beta)^{-1} + \frac{3}{16}\left(\frac{(1+R)^3}{R^4} + \frac{3}{2}\frac{(1+R)^4}{R^3} - \frac{3}{2}R \right)\Gamma(\beta)^{-1} \triangleq \psi_{0,1} + \psi_{0,2}.$$

Since the weights $\varphi_0, \psi_{0,2}, R^{-1}\sin(2\beta)$ are much weaker than the weights φ_1, the estimates of

$$\langle \mathcal{R}_\Omega, \Omega\varphi_0 \rangle, \quad \langle \mathcal{R}_\eta, \eta\psi_{0,2} \rangle, \quad \frac{81}{4\pi c} L_{12}(\Omega)(0) \langle \mathcal{R}_\Omega, \sin(2\beta) R^{-1} \rangle$$

follows from the same argument as that in the last two sections and a similar bound can be derived. It remains to estimate $\langle R_\eta, \eta R\Gamma(\beta)^{-1} \rangle$. Compared to φ_1, $R\Gamma(\beta)^{-1}$ is much less singular in R and β. We focus on how to control the growing factor R. We use the decay estimate of $\bar{\eta}$ in Lemma A.0.6 and $\bar{\xi}$ in Lemma A.0.8. In particular, for $i + j \leq 7$ we have

$$|D_R^i D_\beta^j \bar{\eta}| \lesssim \alpha(1+R)^{-2}, \quad |D_R^i D_\beta^j \bar{\xi}| \lesssim |\bar{\xi}| \lesssim \alpha^2 (1+R)^{-2} \sin(\beta)^{-2\alpha}. \tag{2.167}$$

Recall the decomposition of $\mathcal{R}_\eta$ in (2.154) and the error $\bar{F}_\eta$ defined in (2.155). We use argument similar to that in the last subsection to estimate $||\bar{F}_\eta (R\Gamma(\beta)^{-1})^{1/2}||_2$. A typical term in $\bar{F}_\eta$ can be estimated as follows

$$\begin{aligned} &\int_0^\infty \int_0^{\pi/2} V_1(\bar{\Psi})^2 \bar{\xi}^2 R\Gamma(\beta)^{-1} dR d\beta \\ \lesssim &\alpha^2 \int_0^\infty \int_0^{\pi/2} \alpha^4 (1+R)^{-4} \sin(\beta)^{-4\alpha} R\Gamma(\beta)^{-1} dR d\beta \lesssim \alpha^6 \lesssim \alpha^4, \end{aligned}$$

where we have applied Proposition 2.8.8 to estimate $V_1(\bar{\Psi})$ and used $\alpha < \frac{1}{8}$ (we will choose α sufficiently small). Similarly, we have $||\bar{F}_\eta (R\Gamma(\beta)^{-1})^{1/2}||_2^2 \lesssim \alpha^4$. Hence, using the Cauchy-Schwarz inequality, we get

$$|\langle \bar{F}_\eta, \eta R\Gamma(\beta)^{-1}\rangle| \lesssim ||\bar{F}_\eta (R\Gamma(\beta)^{-1})^{1/2}||_2 ||\eta (R\Gamma(\beta)^{-1})^{1/2}||_2 \lesssim \alpha^2 ||\eta\psi_0^{1/2}||_2 \lesssim \alpha^2 E_3,$$

where we have used (2.132) to derive the last inequality.

Recall P_η in (2.154), N_η in (2.49) and the formula of $\mathbf{u}\cdot\nabla$ in (2.153). We use integration by parts and then a L^∞ estimate to estimate the transport terms in P_η, N_η. A typical term in these transport terms can be estimated as follows

$$\begin{aligned}|\langle \frac{2}{\pi\alpha} L_{12}(\Omega) D_\beta \eta, \eta R\Gamma^{-1}\rangle| &= |\langle \frac{2}{\pi\alpha} L_{12}(\Omega)\partial_\beta(\sin(2\beta)\Gamma^{-1}), \eta^2 R\rangle| \\ &\lesssim \alpha^{-1} ||L_{12}(\Omega)||_\infty ||\eta (R\Gamma^{-1})^{1/2}||_2^2 \\ &\lesssim \alpha^{-1} ||\Omega\varphi_1^{1/2}||_{L^2} ||\eta\psi_0^{1/2}||_2^2 \lesssim \alpha^{-1} E_3^3,\end{aligned}$$

where we have used $\Gamma(\beta) = \cos(\beta)^\alpha, |\sin(2\beta)\partial_\beta \Gamma(\beta)^{-1}| \lesssim \Gamma(\beta)^{-1}$ in the first inequality, Lemma A.0.4 in the second inequality and (2.132) in the last inequality.

For the nonlinear terms related to η, i.e. $(2c_\omega - U_1(\Psi))\eta$ in N_η (2.49) and $-(U_1(\Psi) + \frac{3}{1+R})\eta$ in P_η (2.154), we also apply a L^∞ estimate. For example, we have

$$\begin{aligned}|\langle (2c_\omega - U_1(\Psi))\eta, \eta R\Gamma(\beta)^{-1}\rangle| &\lesssim ||2c_\omega - U_1(\Psi)||_{L^\infty} ||\eta\psi_0^{1/2}||_2^2 \\ &\lesssim \alpha^{-1} ||\Omega||_{\mathcal{H}^3} ||\eta\psi_0^{1/2}||_2^2 \lesssim \alpha^{-1} E_3^3,\end{aligned}$$

where we have used (2.163) and $|c_\omega| = \frac{2}{\pi\alpha}|L_{12}(\Omega)(0)| \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}$ (see Lemma A.0.4) in the last inequality.

For the terms related to $\bar{\eta}, \bar{\xi}$ in P_η (2.154), i.e. $(U_1(\Psi) + \frac{2}{\pi\alpha} L_{12}(\Omega))\bar{\eta},\ V_1(\Psi)\bar{\xi})$, they can be estimated easily by using the fast decay of $\bar{\xi}, \bar{\eta}$ (2.167).

Finally, for the terms related to ξ, i.e. $V_1(\Psi)\xi$ in N_η (2.49) and $V_1(\bar{\Psi})\xi$ in (2.154), we get

$$\begin{aligned}&|\langle V_1(\bar{\Psi})\xi, \eta R\Gamma^{-1}\rangle| + |\langle V_1(\Psi)\xi, \eta R\Gamma^{-1}\rangle| \\ \lesssim& ||\eta, R^{1/2}\Gamma^{-1/2}||_{L^2} ||\xi R^{1/2}\sin(2\beta)^{1/4}||_{L^\infty} \\ &\cdot (||V_1(\bar{\Psi})\sin(2\beta)^{-1/4}\Gamma^{-1/2}||_{L^2} + ||V_1(\Psi)\sin(2\beta)^{-1/4}\Gamma^{-1/2}||_{L^2}) \\ \lesssim& ||\eta\psi_0^{1/2}||_{L^2} ||\xi||_{\mathcal{H}^3(\psi)} (||V_1(\bar{\Psi})\sin(2\beta)^{-\sigma/2}||_{L^2} + ||V_1(\Psi)\sin(2\beta)^{-\sigma/2}||_{L^2}) \\ \lesssim& E_3^2 (||\bar{\Omega}\sin(2\beta)^{-\sigma/2}||_{L^2} + ||\Omega\sin(2\beta)^{-\sigma/2}||_{L^2}) \lesssim E_3^2(\alpha + E_3),\end{aligned}$$

where we have applied Lemma 2.8.10 in the second inequality, the weighted L^2 (with weight $\sin(2\beta)^{-\sigma}, \sigma = \frac{99}{100}$) version of Proposition 2.8.3 in the third inequality and a direct computation using (2.44) in the last inequality.

Combining the estimates of $\bar{F}_\eta, P_\eta, N_\eta$, we have

$$|\langle \mathcal{R}_\eta, \eta R \Gamma^{-1} \rangle| \lesssim \alpha^{-3/2} E_3^3 + \alpha^{1/2} E_3^2 + \alpha^2 E_3.$$

2.9.4.1 Completing the $\mathcal{H}^3$ and $\mathcal{H}^3(\psi)$ estimates

From (2.133), we can use E_3 to bound $||\Omega||_{\mathcal{H}^3}$, $||\eta||_{\mathcal{H}^3}, ||\xi||_{\mathcal{H}^3(\psi)}$. Combining the estimates in the last few subsections, we prove

$$\begin{aligned} &|\langle \mathcal{R}_\Omega, \Omega \rangle_{\mathcal{H}^3}| + |\langle \mathcal{R}_\eta, \eta \rangle_{\mathcal{H}^3}| + |\langle \mathcal{R}_\xi, \xi \rangle_{\mathcal{H}^3(\psi)}| \\ \lesssim &\alpha^{1/2}(E_3^2 + \alpha ||\xi||_{\mathcal{C}^1}^2) + \alpha^{-3/2}(E_3 + \alpha^{1/2}||\xi||_{\mathcal{C}^1})^3 + \alpha^2 E_3, \end{aligned}$$

where E_3 is defined in (2.115). Combining Corollary 2.7.3 and the above estimates, we prove (2.145).

2.9.5 Remaining terms in the $\mathcal{C}^1$ estimate of ξ

Recall that we perform L^∞ estimates on ξ and its derivatives in subsection 2.7.3. In this subsection, we complete the estimate of the remaining terms in these estimates and derive (2.146). We group together the remaining terms in (2.123), (2.126), (2.127), which remain to be estimated. They can be bounded by

$$\begin{aligned} &||\xi||_{\mathcal{C}^1}(||\Xi_1||_{\mathcal{C}^1} + ||\Xi_2||_{\mathcal{C}^1} + ||\bar{F}_\xi||_{\mathcal{C}^1} + ||N_o||_{\mathcal{C}^1}), \quad ||\xi||_{\mathcal{C}^1}||[\phi_1 D_R, \mathcal{A}_2]\xi||_\infty, \\ &||\xi||_{\mathcal{C}^1}||[\phi_2 D_\beta, \mathcal{A}_2]\xi||_\infty, \quad |\alpha c_l| ||\phi_1 D_R \xi||_{L^\infty}^2, \quad ||\phi_2 D_\beta \xi||_\infty ||\mathcal{A}_1(\phi_2 - 1) \cdot D_\beta \xi||_{L^\infty}. \end{aligned}$$

2.9.5.1 Analysis of Ξ_1, Ξ_2, N_o

Recall Ξ_1, Ξ_2, N_o in (2.119), (2.120),(2.122)

$$\begin{aligned} &\Xi_1 = (\frac{3}{1+R} - V_2(\bar{\Psi}))\xi, \quad N_o = (2c_\omega - V_2(\Psi))\xi - U_2(\Psi)\eta, \\ &\Xi_2 = -V_2(\Psi)\bar{\xi} + c_\omega(2\bar{\xi} - R\partial_R \bar{\xi}) + (\alpha c_\omega R \partial_R - (\mathbf{u} \cdot \nabla))\bar{\xi} - (U_2(\Psi)\bar{\eta} + U_2(\bar{\Psi})\eta), \end{aligned}$$

where we have used $V_2(\Psi) = v_y, U_2(\Psi) = u_y$ (2.149). Recall (2.45), (2.47), (2.148). We have

$$\frac{2}{\pi\alpha} L_{12}(\bar{\Omega}) = \frac{3}{1+R}, \ c_\omega = -\frac{2}{\pi\alpha} L_{12}(\Omega)(0), \ \Psi_* = \Psi - \frac{\sin(2\beta)}{\pi\alpha} L_{12}(\Omega).$$

Then we obtain $V_2(\bar{\Psi}) - \frac{3}{1+R} = -U_1(\bar{\Psi}, \bar{\Psi}_*)$ (see (2.151)) .

For the transport term $(\alpha c_\omega D_R - (\mathbf{u} \cdot \nabla))\bar{\xi}$, we use the decomposition (2.153). Then each term in Ξ_1, Ξ_2, N_o depends only on $L_{12}(\Omega), \Psi, \eta, \xi$ and their approximate steady state, e.g., $V_2(\bar{\Psi})$. To estimate the $\mathcal{C}^1$ norm of the product in Ξ_1, Ξ_2, N_o, using Proposition 2.8.6, we only need to estimate the $\mathcal{C}^1$ norm of each single term.

For the terms depending on Ψ, Ψ_*, e.g., $V_2(\Psi) - \frac{2}{\pi\alpha} L_{12}(\Omega)$ (see (2.151)-(2.152)), we apply Proposition 2.8.3 and Lemma 2.8.11 to obtain the $\mathcal{C}^1$ estimate. For the terms depending on $\bar{\Psi}, \bar{\Psi}_*$, we apply Propositions 2.8.8 and 2.8.6 to estimate the $\mathcal{C}^1$ norm.

For the terms depending on $L_{12}(\Omega)$, we use (A.10) in Lemma A.0.4 to estimate the $\mathcal{C}^1$ norm.

The slightly difficult term is $V_2(\Psi)$. Using the formula of $V_2(\Psi)$ in (2.151), (2.152), Propositions 2.8.3, and Lemmas 2.8.11, A.0.4, we get

$$\begin{aligned} ||V_2(\Psi)||_{\mathcal{C}^1} &\lesssim ||V_2(\Psi) - \frac{2}{\pi\alpha} L_{12}(\Omega)||_{\mathcal{C}^1} + \frac{2}{\pi\alpha}||L_{12}(\Omega)||_{\mathcal{C}^1} \\ &\lesssim (\alpha^{-1/2} + \alpha^{-1})||\Omega||_{\mathcal{H}^3} \lesssim \alpha^{-1}||\Omega||_{\mathcal{H}^3}. \end{aligned} \tag{2.168}$$

Using (A.23)-(A.24) in Lemma A.0.8 and Lemma A.0.6, we have $||\bar{\xi}||_{\mathcal{C}^1} + ||D_R\bar{\xi}||_{\mathcal{C}^1} \lesssim \alpha^2, ||\bar{\eta}||_{\mathcal{C}^1} \lesssim \alpha$. From (2.45), we know $||L_{12}(\bar{\Omega})||_{\mathcal{C}^1} \lesssim \alpha$. Therefore, we get

$$\begin{aligned} &||\Xi_1||_{\mathcal{C}^1} \lesssim \alpha ||\xi||_{\mathcal{C}^1}, \quad ||\Xi_2||_{\mathcal{C}^1} \lesssim \alpha^{1/2}||\Omega||_{\mathcal{H}^3} + \alpha^{1/2}||\eta||_{\mathcal{H}^3}, \\ &||N_o||_{\mathcal{C}^1} \lesssim \alpha^{-1}||\xi||_{\mathcal{C}^1}||\Omega||_{\mathcal{H}^3} + \alpha^{-1}||\Omega||_{\mathcal{H}^3}||\eta||_{\mathcal{H}^3}. \end{aligned}$$

The largest term in Ξ_2 is given by $(U_2(\Psi)\bar{\eta} + U_2(\bar{\Psi})\eta)$, which leads to the above upper bound.

2.9.5.2 Analysis of $\bar{F}_\xi$

Recall $\bar{F}_\xi$ and $\bar{\mathbf{u}} \cdot \nabla$ defined in (2.46) and (2.153)

$$\begin{aligned} &\bar{F}_\xi = (2\bar{c}_\omega - V_2(\bar{\Psi}))\bar{\xi} - U_2(\bar{\Psi})\bar{\eta} - \alpha \bar{c}_l R \partial_R \bar{\xi} - (\bar{\mathbf{u}} \cdot \nabla)\bar{\xi}, \\ &\bar{\mathbf{u}} \cdot \nabla \bar{\xi} = (-\frac{2\cos(2\beta)}{\pi} L_{12}(\Omega) - \alpha \partial_\beta \Psi_*) D_R \bar{\xi} + (\frac{2}{\pi\alpha} L_{12}(\Omega) + \frac{2\Psi_* + \alpha D_R \Psi}{\sin(2\beta)}) D_\beta \bar{\xi}. \end{aligned}$$

For $\bar{\xi}$ terms, we use $||D_R^i D_\beta^j \bar{\xi}||_{\mathcal{C}^1} \lesssim \alpha^2, i + j \leq 2$ from (A.23)-(A.24) in Lemma A.0.8. For other terms, we use $||\bar{\eta}||_{\mathcal{C}^1} \lesssim \alpha$ from Lemma A.0.6 and apply the strategy in the last subsection to estimate the $\mathcal{C}^1$ norm. We get

$$||\bar{F}_\xi||_{\mathcal{C}^1} \lesssim \alpha^2.$$

2.9.5.3 $||[\phi_2 D_\beta, \mathcal{A}_2]\xi||_\infty$, $||[\phi_1 D_R, \mathcal{A}_2]\xi||_\infty$

Recall $\mathcal{A}_2$ defined in (2.124). Using (2.153), we have

$$\begin{aligned}\mathcal{A}_2(\xi) &= \frac{2}{\pi\alpha} L_{12}(\Omega) D_\beta \xi + (\mathcal{T}(\bar{\Omega}) + \mathcal{T}(\Omega))\xi \\ &= \frac{2}{\pi\alpha} L_{12}(\Omega) D_\beta \xi - \frac{2}{\pi}\cos(2\beta)(L_{12}(\Omega) + L_{12}(\bar{\Omega})) D_R \xi \\ &\quad - \alpha(\partial_\beta \Psi_* + \partial_\beta \bar{\Psi}_*) D_R \xi + \frac{2\Psi_* + \alpha D_R \Psi + 2\bar{\Psi}_* + \alpha D_R \bar{\Psi}}{\sin(2\beta)} D_\beta \xi \\ &\triangleq (H_1 D_\beta + H_2 D_R + H_3 D_R + H_4 D_\beta)\xi.\end{aligned}$$

Recall ϕ_1, ϕ_2 defined in (2.116). For $D = D_R, D_\beta$ and $\phi = \phi_1, \phi_2$, a direct computation yields

$$|\phi^{-1} D\phi| \lesssim 1. \tag{2.169}$$

Let $H\tilde{D}$ be a term in the above formula of $\mathcal{A}_2$ and $(D, \phi) = (D_R, \phi_1)$ or (D_β, ϕ_2). Using (2.169) and the $\mathcal{C}^1$ norm defined in (2.117) to control the L^∞ norm of $\phi D H, \phi D\xi, \tilde{D}\xi, H$, we obtain

$$\begin{aligned}|[\phi D, H\tilde{D}]\xi| &= |\phi D H \cdot \tilde{D}\xi - H\tilde{D}\phi \cdot D\xi| \\ &\leq ||H||_{\mathcal{C}^1} ||\xi||_{\mathcal{C}^1} + ||H||_{L^\infty} ||\phi^{-1}\tilde{D}\phi||_{L^\infty} ||\phi D\xi||_{L^\infty} \lesssim ||H||_{\mathcal{C}^1} ||\xi||_{\mathcal{C}^1}.\end{aligned}$$

Applying the strategy in Section 2.9.5.1 to estimate the $\mathcal{C}^1$ norm of $\Psi, \bar{\Psi}, L_{12}(\Omega)$ terms, we get

$$\begin{aligned}&||H_1||_{\mathcal{C}^1} \lesssim \alpha^{-1} ||\Omega||_{\mathcal{H}^3}, \quad ||H_2||_{\mathcal{C}^1} \lesssim ||\Omega||_{\mathcal{H}^3} + \alpha, \\ &||H_3||_{\mathcal{C}^1} \lesssim \alpha^{1/2} ||\Omega||_{\mathcal{H}^3} + \alpha^2, \quad ||H_4||_{\mathcal{C}^1} \lesssim \alpha^{-1/2} ||\Omega||_{\mathcal{H}^3} + \alpha.\end{aligned}$$

The largest term is $\alpha^{-1} L_{12}(\Omega)$ in H_1, which is estimated by (A.10) in Lemma A.0.4 and using $D_\beta L_{12}(\Omega) = 0$.

Combining the above estimates, we conclude that

$$||[D_R, \mathcal{A}_2]\xi||_\infty, \ ||[D_\beta, \mathcal{A}_2]\xi||_\infty \lesssim ||\xi||_{\mathcal{C}^1} (\alpha^{-1} ||\Omega||_{\mathcal{H}^3} + \alpha).$$

2.9.5.4 Analysis of $|\alpha c_l|, ||\mathcal{A}_1(\phi_2 - 1) \cdot D_\beta \xi||_{L^\infty}$

Using (2.47) and (A.9) in Lemma A.0.4, we obtain

$$|\alpha c_l| \leq C\alpha^{-1}|L_{12}(\Omega)(0)| \leq C\alpha^{-1}||\Omega||_{\mathcal{H}^3}.$$

Using the formulas of $\phi_2, \mathcal{A}_1$ in (2.116), (2.124), we get

$$\begin{aligned}
|\phi_2^{-1}\mathcal{A}_1(\phi_2 - 1)| =& |\phi_2^{-1}((1 + 3\alpha + \alpha c_l)D_R + \frac{3}{1+R}D_\beta)(R\sin(2\beta)^\alpha)^{-1/40}| \\
\leq& \phi_2^{-1}(\frac{1}{40}(1 + 3\alpha + \alpha c_l) + C\alpha)(R\sin(2\beta)^\alpha)^{-1/40} \\
\leq& \frac{1}{40}(1 + 3\alpha + C\alpha^{-1}||\Omega||_{\mathcal{H}^3}) + C\alpha,
\end{aligned}$$

where we have used $D_R(R\sin(2\beta)^\alpha)^{-1/40} = -\frac{1}{40}(R\sin(2\beta)^\alpha)^{-1/40}$, $|D_\beta(R\sin(2\beta)^\alpha)^{-1/40}| \lesssim \alpha|(R\sin(2\beta)^\alpha)^{-1/40}|$ in the first inequality. Therefore, we get

$$||\mathcal{A}_1(\phi_2 - 1) \cdot D_\beta\xi||_{L^\infty} \leq (\frac{1}{40} + C\alpha + C\alpha^{-1}||\Omega||_{\mathcal{H}^3})||\phi_2 D_\beta \xi||_{L^\infty}.$$

2.9.5.5 Completing the $\mathcal{C}^1$ estimates

From (2.133), we can use E_3 to further bound $||\Omega||_{\mathcal{H}^3}, ||\eta||_{\mathcal{H}^3}, ||\xi||_{\mathcal{H}^3(\psi)}$. Plugging all the above estimates of the remaining terms in (2.123), (2.126), (2.127), we prove

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}||\xi||_\infty^2 \leq& -2||\xi||_\infty^2 + C\alpha^2||\xi||_\infty \\
&+ C||\xi||_{\mathcal{C}^1}(\alpha^{1/2}E_3 + \alpha||\xi||_{\mathcal{C}^1} + \alpha^{-1}E_3^2 + \alpha^{-1}E_3||\xi||_{\mathcal{C}^1}), \\
\frac{1}{2}\frac{d}{dt}||\phi_2 D_\beta\xi||_\infty^2 \leq& -(2 - \frac{1}{40})||\phi_2 D_\beta \xi||_\infty^2 + C\alpha^2||\phi_2 D_\beta\xi||_\infty \\
&+ C||\xi||_{\mathcal{C}^1}(\alpha^{1/2}E_3 + \alpha||\xi||_{\mathcal{C}^1} + \alpha^{-1}E_3^2 + \alpha^{-1}E_3||\xi||_{\mathcal{C}^1}), \\
\frac{1}{2}\frac{d}{dt}||\phi_1 D_R\xi||_\infty^2 \leq& -2||\phi_1 D_R\xi||_\infty^2 + 3||\phi_1 D_R \xi||_\infty(||\phi_2 D_\beta\xi||_\infty + ||\xi||_\infty) \\
&+ C||\xi||_{\mathcal{C}^1}(\alpha^{1/2}E_3 + \alpha||\xi||_{\mathcal{C}^1} + \alpha^{-1}E_3^2 + \alpha^{-1}E_3||\xi||_{\mathcal{C}^1}) + C\alpha^2||\phi_1 D_R\xi||_\infty.
\end{aligned}$$

Hence, for some absolute constant μ_4, e.g., $\mu_4 = \frac{1}{10}$, the energy defined in (2.147) satisfies (2.146).

2.9.6 Finite time blowup with finite energy velocity field

2.9.6.1 The bootstrap argument

Now, we construct the energy

$$E(\Omega, \eta, \xi) = (E_3(\Omega, \eta, \xi)^2 + \alpha E(\xi, \infty)^2)^{1/2}. \tag{2.170}$$

Adding the estimates (2.145) and $\alpha\times$(2.146), we have

$$\frac{1}{2}\frac{d}{dt}E^2(\Omega,\eta,\xi) \leq -\frac{1}{12}E^2 + K\alpha^{1/2}E^2 + K\alpha^{-3/2}E^3 + K\alpha^2 E, \tag{2.171}$$

for some universal constant K, where we have used the fact that $E(\xi,\infty)$ is equivalent to $||\xi||_{\mathcal{C}^1}$ since μ_4 is an absolute constant. We know that there exists a small absolute constant $\alpha_1 < \frac{1}{1000}$ and K_*, such that, for any $\alpha < \alpha_1$ and $E = K_*\alpha^2$, we have

$$-\frac{1}{12}E^2 + K\alpha^{1/2}E^2 + K\alpha^{-3/2}E^3 + K\alpha^2 E < 0. \tag{2.172}$$

If $E(\Omega(\cdot,0),\eta(\cdot,0),\xi(\cdot,0)) < K_*\alpha^2$, we have

$$E(\Omega(t),\eta(t),\xi(t)) < K_*\alpha^2, \tag{2.173}$$

for all time $t > 0$, where we have used the time-dependent normalization condition (2.47) for $c_\omega(t), c_l(t)$. Applying Lemma A.0.4 to $L_{12}(\Omega)(0)$ and Lemma 2.8.11 to Ω,η, we derive

$$\begin{aligned}
&|c_\omega(t)| = \frac{2}{\pi\alpha}|L_{12}(\Omega)(0)| < C\alpha^{-1}||\Omega||_{\mathcal{H}^3} \leq C\alpha^{-1}E \leq K_9\alpha,\\
&|c_l(t)| = |\frac{1-\alpha}{\alpha}\frac{2}{\pi\alpha}L_{12}(\Omega)(0)| < C\alpha^{-2}E \leq K_9,\\
&||\Omega||_{L^\infty} + ||\eta||_{L^\infty} < CE \leq C\alpha^2 \leq K_9\alpha\min(||\bar{\Omega}||_{L^\infty}, ||\bar{\eta}||_{L^\infty}),\\
&||\xi||_{L^\infty} < C\alpha^{-1/2}E \leq K_9\alpha^{3/2},
\end{aligned}$$

where we have used $||\bar{\Omega}||_{L^\infty}, ||\bar{\eta}||_{L^\infty} \geq C\alpha$ according to (2.44) and Lemma A.0.1 in the last inequality, and $K_9 > 0$ is some absolute constant. We further take

$$\alpha_0 = \min(\alpha_1,\ \frac{3\pi}{4K_*},\ \frac{K_*^2}{4K_{10}^2},\ \frac{1}{16(K_9+1)^4}), \tag{2.174}$$

where K_{10} is the constant defined in Lemma A.0.11. For $\alpha < \alpha_0$, using $\bar{c}_\omega = -1, \bar{c}_l = \frac{1}{\alpha} + 3$ and the formula of $\bar{\Omega},\bar{\eta}$ in (2.44), we further yield

$$\begin{aligned}
&-\frac{3}{2} < c_\omega + \bar{c}_\omega < -\frac{1}{2}, \quad c_l + \bar{c}_l > \frac{1}{2\alpha} + 3,\\
&||\Omega + \bar{\Omega}||_{L^\infty} \asymp ||\bar{\Omega}||_{L^\infty} \asymp \alpha, \quad ||\eta + \bar{\eta}||_{L^\infty} \asymp ||\bar{\eta}||_{L^\infty} \asymp \alpha, \quad ||\xi||_{L^\infty} \leq \frac{1}{2}\alpha^{3/2}.
\end{aligned} \tag{2.175}$$

2.9.6.2 Finite time blowup

For Hölder initial data, the local well-posedness of the solutions follows from the argument in [11] for the 2D Boussinesq equations. The Beale-Kato-Majda type blowup criterion still applies to the Boussinesq equations in the specified domain. The time integral of $||\nabla\theta||_{L^\infty}$ controls the breakdown of the solutions in the 2D Boussinesq equations [11]. We will control this quantity and show that there exists T_0 such that $\int_0^{T_0} ||\nabla\theta(\cdot,s)||_\infty ds = \infty$ in the 2D Boussinesq equations. The solutions remain in the same regularity class as that of the initial data before the blowup time. In particular, the velocity field is in $C^{1,\alpha}$ before the blowup time.

Let $\chi(\cdot) : [0,\infty) \to [0,1]$ be a smooth cutoff function, such that $\chi(R) = 1$ for $R \leq 1$ and $\chi(R) = 0$ for $R \geq 2$. We choose perturbation $\Omega = (\chi(R/\lambda) - 1)\bar{\Omega}, \theta(R,\beta) = (\chi(R/\lambda) - 1)\bar{\theta}$ and $\eta = \theta_x$, $\xi = \theta_y$ can be obtained accordingly, where $\bar{\theta}(x,y)$ is recovered from $\bar{\theta}_x$ by integration (A.20). Obviously, $\Omega, \eta, \xi \equiv 0$ for $R \leq \lambda$. Using Lemma A.0.11 for Ω, η, ξ and $\alpha < \alpha_0$ (see (2.174)), we obtain that these initial perturbations satisfy $E(\Omega(0), \eta(0), \xi(0)) < 2K_{10}\alpha^{5/2} \leq K_*\alpha^2$ for sufficiently large λ. We remark that the initial perturbation is of size $C\alpha^{5/2}$ even for extremely large λ because $\bar{\xi}$ does not decay in the $\mathcal{C}^1$ norm for large R. It is important to add a small weight α in $E(\xi,\infty)$ when we define the final energy in (2.170).

In particular, the initial data $\bar{\Omega} + \Omega = \chi(R/\lambda)\bar{\Omega}$ (recall $\Omega(R,\beta) = \omega(x,y)$), $\bar{\theta} + \theta = \chi(R/\lambda)\bar{\theta}$ have compact support and thus we have finite energy $||u + \bar{u}||_{L^2} < +\infty, ||\theta + \bar{\theta}||_{L^2} < +\infty$. $c_\omega(t), c_l(t)$ are determined by (2.47).

Denote by $\omega_{phy}, \theta_{phy}$ the corresponding solutions in the original Boussinesq equation (2.16)-(2.17), which are related to the rescaled variables ω, θ via the rescaling formula (2.9), (2.11)

$$\begin{aligned}
&\omega_{phy}(x, t(\tau)) = C_\omega(\tau)^{-1}(\omega + \bar{\omega})(C_l(\tau)^{-1}x, \tau), \\
&\theta_{phy}(x, t(\tau)) = C_\theta(\tau)^{-1}(\theta + \bar{\theta})(C_l(\tau)^{-1}x, \tau), \\
&C_\omega(\tau) = \exp\Big(\int_0^\tau c_\omega(s) + \bar{c}_\omega ds\Big), \\
&C_l(\tau) = \exp\Big(-\int_0^\tau c_l(s) + \bar{c}_l ds\Big), \quad t(\tau) = \int_0^\tau C_\omega(\tau) d\tau.
\end{aligned} \tag{2.176}$$

We remark that the scaling parameters in (2.11) become $(c_\omega + \bar{c}_\omega, c_l + \bar{c}_l)$. Denote

$$M(\tau) \triangleq \int_0^{t(\tau)} ||\nabla\theta_{phy}(s)||_{L^\infty} ds.$$

Using a change of variable $s = t(p)$ and $\partial_x(\theta+\bar{\theta}) = (\eta+\bar{\eta}), \partial_y(\theta+\bar{\theta}) = (\xi+\bar{\xi})$, we obtain

$$\begin{aligned} M(\tau) &= \int_0^\tau ||\nabla\theta_{phy}(t(p))||_{L^\infty} C_\omega(p)dp \\ &= \int_0^\tau C_\omega(p)^{-1}(||(\eta+\bar{\eta})(p)||_{L^\infty} + ||(\xi+\bar{\xi})(p)||_{L^\infty})dp, \end{aligned}$$

where we have used the formula (2.176) and $C_\theta^{-1}(p)C_l^{-1}(p) = C_\omega(p)^{-2}$ according to (2.11),(2.12) in the second equality. Using the bootstrap estimates (2.175) and Lemma A.0.8 about $\bar{\xi}$, we obtain

$$M(\tau) \asymp \alpha \int_0^\tau C_\omega(p)^{-1}dp.$$

Using (2.175) and (2.176), we have $e^{-3p/2} < C_\omega(p) < e^{-p/2}$. Therefore, we obtain $M(\tau) < +\infty \quad \forall \tau < +\infty$ and

$$\int_0^\infty M(\tau)d\tau \geq C\alpha \int_0^\infty \int_0^\tau e^{p/2}dpd\tau = \infty, \quad t(\infty) \leq \int_0^\infty e^{-p/2}dp < +\infty.$$

Denote $T^* = t(\infty)$. Applying the BKM type blowup criterion in [11], we obtain that the solutions remain in the same regularity class as that of the initial data before T^* and develop a finite time singularity at T^*. Similarly, by rescaling the time variable, we prove that $||\omega_{phy}||_{L^\infty}$ and $||\nabla\theta_{phy}||_{L^\infty}$ blowup at T^*.

Remark 2.9.1. The crucial nonlinear estimate (2.171) and *a* priori estimate (2.173), i.e. the bootstrap estimate for small perturbation, offer strong control on the perturbation and the exact solution before the blowup time. In particular, it allows us to truncate the far field of the approximate steady state, which leads to a small perturbation only, to obtain initial data with finite energy.

2.9.6.3 Convergence to the self-similar solution

Taking the time derivative of (2.48), using the *a* priori estimate (2.173) for the small perturbation and analysis similar to that in the previous Section, we can further perform $\mathcal{H}^2$ estimates on Ω_t, η_t, $\mathcal{H}^2(\psi)$ and L^∞ estimates on ξ_t. In particular, following the argument in our previous joint work with Hou and Huang [19], we can further obtain that there exists an exact self-similar solution $\Omega_\infty, \eta_\infty \in \mathcal{H}^3, \xi_\infty \in \mathcal{H}^3(\psi) \cap L^\infty$, such that the solution of the dynamic rescaling equation with initial data constructed in Subsection 2.9.6.2 converges to $(\Omega_\infty, \eta_\infty, \xi_\infty)$ exponentially fast. The convergence is in the $\mathcal{H}^2$ norm for the variables Ω, η and both $\mathcal{H}^2(\psi)$ and L^∞ norm for the variable ξ.

Using the a-priori estimate (2.173) and Lemma A.0.8, we have $||\bar{\xi} + \xi(t)||_{\mathcal{C}^1} \leq C\alpha^{3/2}$ for all time in the dynamic rescaling equation. Using Lemma A.0.13, we know that the space $\mathcal{C}^1$ (the weighted C^1 space) can be embedded continuously into the standard Hölder space $C^{\alpha/40}$. Therefore, the $\mathcal{C}^1$ estimate of $\bar{\xi} + \xi$ implies that $\bar{\xi}+\xi(t) \in C^{\alpha/40}$ with uniform Hölder norm. Since $\bar{\xi}+\xi(t)$ converges to ξ_∞ in L^∞, we have $\xi_\infty \in C^{\alpha/40}$. Finally, using the same argument, the fact that $\Omega_\infty, \eta_\infty \in \mathcal{H}^3$ and the embedding $\mathcal{H}^3 \hookrightarrow \mathcal{C}^1$ in Lemma 2.8.11, we conclude $\Omega_\infty, \eta_\infty, \xi_\infty \in C^{\alpha/40}$.

Notice that $c_l + \bar{c}_l > \frac{1}{2\alpha}$ from (2.175). Thus, the self-similar blowup is focusing. This completes the proof of Theorem 2.1.

2.10 Finite time blowup of 3D axisymmetric Euler equations

In this section, we prove Theorem 2.2. We first review the setup of the problem. In Section 2.10.1, we reformulate the 3D Euler equations and discuss the connection between the 3D Euler and 2D Boussinesq; see e.g., [89]. In Section 2.10.2, we establish the elliptic estimates. In Section 2.10.3, we will construct initial data and control the support of the solution under some bootstrap assumptions. With these estimates, the rest of the proof follows essentially the nonlinear stability analysis of the 2D Boussinesq equations and is sketched in the same subsection.

Notations In this section, we use x_1, x_2, x_3 to denote the Cartesian coordinates in $\mathbb{R}^3$, and

$$r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3, \quad \vartheta = \arctan(x_2/x_1) \tag{2.177}$$

to denote the cylindrical coordinates. The reader should not confuse r with the radial variable in the 2D Boussinesq.

Let $\mathbf{u}$ be the axi-symmetric velocity and $\omega = \nabla \times \mathbf{u}$ be the vorticity vector. In the cylindrical coordinates, we have the following representation

$$\mathbf{u}(r,z) = u^r(r,z)\mathbf{e}_r + u^\theta(r,z)\mathbf{e}_\theta + u^z(r,z)\mathbf{e}_z, \ \omega = \omega^r(r,z)\mathbf{e}_r + \omega^\theta(r,z)\mathbf{e}_\theta + \omega^z(r,z)\mathbf{e}_z,$$

where $\mathbf{e}_r, \mathbf{e}_\theta$ and $\mathbf{e}_z$ are the standard orthonormal vectors defining the cylindrical coordinates,

$$\mathbf{e}_r = (\frac{x_1}{r}, \frac{x_2}{r}, 0)^T, \quad \mathbf{e}_\theta = (\frac{x_2}{r}, -\frac{x_1}{r}, 0)^T, \quad \mathbf{e}_z = (0,0,1)^T,$$

and $r = \sqrt{x_1^2 + x_2^2}$ and $z = x_3$.

We study the 3D axisymmetric Euler equations in a cylinder $D = \{(r,z) : r \in [0,1], z \in \mathbb{T}\}, \mathbb{T} = \mathbb{R}/(2\mathbb{Z})$ that is periodic in z. The equations are given below:

$$\begin{aligned} &\partial_t (ru^\theta) + u^r (ru^\theta)_r + u^z (ru^\theta)_z = 0, \\ &\partial_t \frac{\omega^\theta}{r} + u^r (\frac{\omega^\theta}{r})_r + u^z (\frac{\omega^\theta}{r})_z = \frac{1}{r^4} \partial_z ((ru^\theta)^2). \end{aligned} \tag{2.178}$$

The radial and axial components of the velocity can be recovered from the Biot-Savart law

$$-(\partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz})\tilde{\psi} + \frac{1}{r^2}\tilde{\psi} = \omega^\theta, \quad u^r = -\tilde{\psi}_z, \quad u^z = \tilde{\psi}_r + \frac{1}{r}\tilde{\psi} \tag{2.179}$$

with a no-flow boundary condition on the solid boundary $r = 1$

$$\tilde{\psi}(1, z) = 0 \tag{2.180}$$

and a periodic boundary condition in z.

We consider solution ω^θ with odd symmetry in z, which is preserved by the equations dynamically. Then $\tilde{\psi}$ is also odd in z. Moreover, since $\tilde{\psi}$ is 2-periodic in z, we obtain

$$\tilde{\psi}(r, 2k-1) = 0. \quad \text{for all} \quad k \in \mathbb{Z} \tag{2.181}$$

This setup of the problem is essentially the same as that in [86, 87].

Equation (2.179) is equivalent to $-\Delta(\tilde{\psi}\sin(\vartheta)) = \omega \sin(\vartheta)$, where $\vartheta = \arctan(x_2/x_1)$ and Δ is the Laplace operator in $\mathbb{R}^3$. We further assume that $\omega^\theta \in C^\alpha(D)$ with support away from $r = 0$. It follows $\omega^\theta \sin(\vartheta) \in C^\alpha(D)$. Note that the cylinder $D_{k,l} \triangleq \{(r,z) : r \in [0,1], 2k-1 \leq z \leq 2l-1\}$ satisfies the exterior sphere condition. Under the boundary condition (2.180)-(2.181), using Theorems 4.3, 4.6 in [54] we obtain a unique solution $\tilde{\psi}\sin\vartheta \in C^{2,\alpha}(D_{k,l}) \cap C(\bar{D}_{k,l})$ for any $k < l, k, l \in \mathbb{Z}$. This further implies the existence and the uniqueness of solution of (2.179)-(2.181).

Due to the periodicity in z direction, it suffices to consider the equations in the first period $D_1 = \{(r,z) : r \in [0,1], |z| \leq 1\}$. We have the following pointwise estimate on $\tilde{\psi}$, which will be used to estimate $\tilde{\psi}$ away from the supp(ω^θ) in Section 2.10.2.

Lemma 2.10.1. *Let $\tilde{\psi}$ be a solution of* (2.179)-(2.180), *and $\omega^\theta \in C^\alpha(D_1)$ for some $\alpha > 0$ be odd in z with* $\mathrm{supp}(\omega^\theta) \cap D_1 \subset \{(r,z) : (r-1)^2 + z^2 < 1/4\}$. *For $\frac{1}{4} < r \leq 1, |z| \leq 1$, we have*

$$|\tilde{\psi}(r,z)| \lesssim \int_{D_1} |\omega^\theta(r_1,z_1)| \Big(1 + |\log((r-r_1)^2 + (z-z_1)^2)|\Big) r_1 dr_1 dz_1.$$

If the domain of the equation (2.179) is $\mathbb{R}^3$, the estimate is straightforward by using the Green function. For the domain we consider, the Green function would be complicated. The proof is based on comparing $\tilde{\psi}\sin(\vartheta)$ with the solutions of $-\Delta(\psi_\pm \sin(\vartheta)) = f_\pm(r,z)\sin(\vartheta)$ in $\mathbb{R}^3$, where $f_\pm$ are some functions related to ω^θ. We defer the proof to Appendix A.0.7.

If the initial data u^θ of (2.178)-(2.180) is non-negative, u^θ remains non-negative before the blowup, if it exists. Then, u^θ can be uniquely determined by $(u^\theta)^2$. We introduce the following variables

$$\tilde{\theta} \triangleq (ru^\theta)^2, \quad \tilde{\omega} = \omega^\theta / r. \tag{2.182}$$

We reformulate (2.178)-(2.180) as

$$\begin{gathered} \partial_t \tilde{\theta} + u^r \tilde{\theta}_r + u^z \tilde{\theta}_z = 0, \quad \partial_t \tilde{\omega} + u^r \tilde{\omega}_r + u^z \tilde{\omega}_z = \frac{1}{r^4}\tilde{\theta}_z, \\ -(\partial_r^2 + \frac{1}{r}\partial_r + \partial_z^2 - \frac{1}{r^2})\tilde{\psi} = r\tilde{\omega}, \quad \tilde{\psi}(1,z) = 0, \quad u^r = -\tilde{\psi}_z, \quad u^z = \frac{1}{r}\tilde{\psi} + \tilde{\psi}_r. \end{gathered} \tag{2.183}$$

2.10.1 Dynamic rescaling formulation

We introduce new coordinates (x,y) centered at $r = 1, z = 0$ and its related polar coordinates

$$\begin{aligned} & x = C_l(\tau)^{-1} z, \quad y = (1-r)C_l(\tau)^{-1}, \\ & \rho = \sqrt{x^2+y^2}, \quad \beta = \arctan(y/x), \quad R = \rho^\alpha, \end{aligned} \tag{2.184}$$

where $C_l(\tau)$ is defined below (2.187). The reader should not confuse ρ with the notations for the weights, and the relation $R = \rho^\alpha$ with $R = r^\alpha$ in the 2D Boussinesq. By definition, we have

$$z = C_l(\tau)x, \quad r = 1 - C_l(\tau)y = 1 - C_l(\tau)\rho\sin(\beta). \tag{2.185}$$

Consider the following dynamic rescaling formulation centered at $r = 1, z = 0$

$$\begin{aligned}\theta(x,y,\tau) &= C_\theta(\tau)\tilde{\theta}(1-C_l(\tau)y, C_l(\tau)x, t(\tau)),\\ \omega(x,y,\tau) &= C_\omega(\tau)\tilde{\omega}(1-C_l(\tau)y, C_l(\tau)x, t(\tau)),\\ \psi(x,y,\tau) &= C_\omega(\tau)C_l(\tau)^{-2}\tilde{\psi}(1-C_l(\tau)y, C_l(\tau)x, t(\tau)),\end{aligned} \tag{2.186}$$

where $C_l(\tau), C_\theta(\tau), C_\omega(\tau), t(\tau)$ are given by $C_\theta = C_l^{-1}(0)C_\omega^2(0)\exp\Big(\int_0^\tau c_\theta(s)d\tau\Big)$,

$$\begin{aligned}&C_\omega(\tau) = C_\omega(0)\exp\Big(\int_0^\tau c_\omega(s)d\tau\Big),\\ &C_l(\tau) = C_l(0)\exp\Big(\int_0^\tau -c_l(s)ds\Big), \quad t(\tau) = \int_0^\tau C_\omega(\tau)d\tau,\end{aligned} \tag{2.187}$$

and the rescaling parameter $c_l(\tau), c_\theta(\tau), c_\omega(\tau)$ satisfies $c_\theta(\tau) = c_l(\tau) + 2c_\omega(\tau)$. We remark that $C_\theta(\tau)$ is determined by C_l, C_ω via $C_\theta = C_\omega^2 C_l^{-1}$. We have this relation due to the same reason as that of (2.12). We choose $(r,z) = (1,0)$ as the center of the above transform since the singular solution is concentrated near this point. We have $0 \leq y \leq C_l^{-1}, |x| \leq C_l^{-1}$ since $r \in [0,1], |z| \leq 1$. We have a minus sign for ∂_y

$$\partial_y\theta = -C_\theta C_l(\tau)\tilde{\theta}_r, \quad \partial_y\omega = -C_\omega C_l(\tau)\tilde{\omega}_r, \quad \partial_y\psi = -C_\omega C_l(\tau)^{-1}\tilde{\psi}_r.$$

Let $(\tilde{\theta}, \tilde{\omega})$ be a solutions of (2.183). It is easy to show that ω, θ satisfy

$$\theta_t + c_l\mathbf{x}\cdot\nabla\theta + (-u^r)\theta_y + u^z\theta_x = c_\theta\theta, \quad \omega_t + c_l\mathbf{x}\cdot\nabla\omega + (-u^r)\omega_y + u^z\omega_x = c_\omega\omega + \frac{1}{r^4}\theta_x.$$

The Biot-Savart law in (2.183) depends on the rescaling parameter C_l, τ

$$-(\partial_{xx}+\partial_{yy})\psi + \frac{1}{r}C_l\partial_y\psi + \frac{1}{r^2}C_l^2\psi = r\omega, \ u^r(r,x) = -\psi_x, \ u^z(r,x) = \frac{1}{r}C_l\psi - \psi_y,$$

where $r = 1 - C_l(\tau)y$ (2.185). We introduce $u = u^z, v = -u^r$. Then, we can further simplify

$$\begin{aligned}&\theta_t + (c_l\mathbf{x} + \mathbf{u}\cdot\nabla)\theta = c_\theta\theta, \quad \omega_t + (c_l\mathbf{x} + \mathbf{u}\cdot\nabla)\omega = \theta_x + \frac{1-r^4}{r^4}\theta_x,\\ &-(\partial_{xx}+\partial_{yy})\psi + \frac{1}{r}C_l\partial_y\psi + \frac{1}{r^2}C_l^2\psi = r\omega, \quad u(x,y) = -\psi_y + \frac{1}{r}C_l\psi, \quad v = \psi_x,\end{aligned} \tag{2.188}$$

with boundary condition $\psi(x,0) \equiv 0$. If C_l is extremely small, we expect that the above equations are essentially the same as the dynamic rescaling formulation (2.10) of the Boussinesq equations. We look for solutions of (2.188) with the following symmetry

$$\omega(x,y) = -\omega(-x,y), \quad \theta(x,y) = \theta(-x,y).$$

Obviously, the equations preserve these symmetries and thus it suffices to solve (2.188) on $x, y \geq 0$ with boundary condition $\psi(x, 0) = \psi(y, 0) = 0$ for the elliptic equation.

2.10.2 The elliptic estimates

In this Section, we use the ideas in Section 2.2.3.2 to estimate the time-dependent elliptic equation in (2.188). We first estimate ψ away from $\mathrm{supp}(\omega)$. In Section 2.10.2.1, we outline the estimates. In the remaining subsections, we localize the elliptic equation and establish the $\mathcal{H}^3$ elliptic estimates.

Under the polar coordinates (2.184) $\rho = \sqrt{x^2 + y^2}, \beta = \arctan(y/x)$, we reformulate (2.188) as

$$-\partial_{\rho\rho}\psi - \frac{1}{\rho}\partial_\rho\psi - \frac{1}{\rho^2}\partial_{\beta\beta}\psi + \frac{C_l}{r}\sin(\beta)\partial_\rho\psi + \frac{C_l}{r}\frac{\cos(\beta)}{\rho}\partial_\beta\psi + \frac{C_l^2}{r^2}\psi = r\omega. \quad (2.189)$$

Recall $R = \rho^\alpha$ from (2.184). Denote

$$\begin{aligned} &\Psi(R, \beta) = \frac{1}{\rho^2}\psi(\rho, \beta), \quad \Omega(R, \beta) = \omega(\rho, \beta), \\ &\eta(R, \beta) = (\theta_x)(\rho, \beta), \quad \xi(R, \beta) = (\theta_y)(\rho, \beta). \end{aligned} \quad (2.190)$$

Since we rescale the cylinder $D_1 = \{(r, z) : r \leq 1, |z| \leq 1\}$, the domain for (x, y) is

$$\tilde{D}_1 \triangleq \{(x, y) : |x| \leq C_l^{-1}, y \in [0, C_l^{-1}]\}. \quad (2.191)$$

We focus on the sector $\rho \leq C_l^{-1}$, or equivalently $R \leq C_l^{-\alpha}$, and $\beta \in [0, \pi/2]$ due to the symmetry of the solutions. Notice that $\rho\partial_\rho = \alpha R\partial_R = \alpha D_R$. It is easy to verify that (2.189) is equivalent to

$$\begin{aligned} &-\alpha^2 R^2 \partial_{RR}\Psi - \alpha(4 + \alpha)R\partial_R\Psi - \partial_{\beta\beta}\Psi - 4\Psi \\ &+ \frac{C_l\rho}{r}(\sin(\beta)(2 + \alpha D_R)\Psi + \cos(\beta)\partial_\beta\Psi) + \frac{C_l^2\rho^2}{r^2}\Psi = r\Omega. \end{aligned} \quad (2.192)$$

We keep the notation $\rho = R^{1/\alpha}, r = 1 - C_l\rho\sin(\beta)$ to simplify the formulation. The boundary condition of Ψ is given by (in the sector $R \leq C_l^{-\alpha}$)

$$\Psi(R, 0) = \Psi(R, \pi/2) = 0. \quad (2.193)$$

Definition 2.10.2. We define the size of support of (θ, ω) of (2.188)

$$S(\tau) = \mathrm{ess}\infty\{\rho : \theta(x, y, \tau) = 0, \omega(x, y, \tau) = 0 \text{ for } x^2 + y^2 \geq \rho^2\}.$$

Obviously, the support of Ω, η defined in (2.190) is $S(\tau)^\alpha$. After rescaling the spatial variable, the support of $(\tilde{\theta}, \tilde{\omega})$ of (2.183) satisfies

$$\text{supp } \tilde{\theta}(t(\tau)), \ \text{supp } \tilde{\omega}(t(\tau)) \subset \{(r,z) : ((r-1)^2 + z^2)^{1/2} \leq C_l(\tau)S(\tau)\}.$$

We will construct initial data of (2.188) with compact support $S(0) < +\infty$ and use the idea described in Section 2.2.3.1 to prove that $C_l(\tau)S(\tau)$ remains sufficiently small for all $\tau > 0$.

Remark 2.10.3. There are several small parameters $\alpha, C_l(\tau), C_l(\tau)S(\tau)$ in the following estimates. We will choose α to be small. For most estimates, the constants are independent of $C_l(\tau)$. We will choose $C_l(0)$ to be much smaller than α at the final step. This allows us to prove that $C_l(\tau), C_l(\tau)S(\tau), (C_l(\tau)S(\tau))^\alpha$ are very small. One can regard $C_l(\tau) \approx 0$. Recall the relation (2.185) about r. In the support of the solution, we have $r = 1 - C_l\rho\sin(\beta) \approx 1$. We treat the error terms in these approximations as small perturbations.

Recall the L^2 inner product defined in (2.15). Using the estimate in Lemma 2.10.1, we obtain in the following Lemma that the L^2 norm of Ψ away from the support of the solution is small. It will be used later to localize(2.192).

Lemma 2.10.4. *Suppose that the assumptions in Lemma 2.10.1 hold true. Let $S(\tau)$ be the support size of $\omega(\tau), \theta(\tau)$. Assume $C_l(\tau)S(\tau) < \frac{1}{4}$. For any $M > (2S(\tau))^\alpha$, we have*

$$\begin{aligned} &||\Psi \mathbf{1}_{M \leq R \leq (2C_l)^{-\alpha}}||_{L^2} \lesssim C(M) \cdot ||\Omega||_{L^2}, \\ &C(M) \triangleq (1 + |\log(C_l M^{1/\alpha})|) S M^{-1/\alpha} ||\Omega||_{L^2}. \end{aligned}$$

The proof follows from the estimate in Lemma 2.10.1, the Cauchy-Schwarz inequality and a direct calculation, which is standard. We refer it to the arXiv version of [16]. We will choose M so that $C(M)$ is small, e.g., $C(M) \lesssim 1$ or $C(M) \lesssim 3^{-1/\alpha}$. If we use an estimate similar to Proposition 2.8.3 and then restrict it to $M \leq R \leq (2C_l)^{-\alpha}$, the constant in the upper bound is α^{-1}, which is not sufficient for our purpose.

Remark 2.10.5. We restrict the domain of the integral D_I to $R \leq (2C_l)^{-\alpha}$, which is equivalent to $\rho \leq (2C_l)^{-1}$ due to (2.184), so that D_I is in $\tilde{D}_1$ (2.191). We impose $R \geq M > (2S(\tau))^\alpha$ so that D_I is away from the support of the solution. Since $S(\tau), C_l(\tau)$ are the variables defined in (x,y) coordinates, when we pass to (R,β) coordinates, we have a α power for these variables, e.g., $(S(\tau))^\alpha, (C_l(\tau))^\alpha$.

2.10.2.1 Outline of the estimates

In Section 2.10.2.2, we use (2.192) to derive the elliptic equation (2.196) for $\chi\Psi$ with some cutoff function χ. The equation is similar to (2.134) in the 2D Boussinesq and has an extra error term Z_χ. We first establish the L^2 estimate of $\chi_1\Psi$ in the same Section 2.10.2.2. To estimate the terms involving derivatives of χ, e.g., $D_R^2\chi\Psi$, we use Lemma 2.10.4. The L^2 estimate enables us to estimate the error term Z_χ. The advantage of localizing (2.192) is that $\chi\Psi$ can be treated as a solution of the elliptic equation (2.134) in $\mathbb{R}_2^+$. Then, in Section 2.10.2.3, we apply the $\mathcal{H}^k$ version of the key elliptic estimate in Proposition 2.8.3 recursively to $\chi_i\Psi$ with χ_i that has smaller support, and establish the higher order elliptic estimates.

2.10.2.2 Localizing the elliptic equation

We will take advantage of the fact that $C_l(\tau)S(\tau)$ can be extremely small and localize the elliptic equation. Firstly, we assume that $C_l(\tau)S(\tau) < \frac{1}{4}$. Recall the relation (2.185) about r. Then we have $r = 1 - C_l\rho\sin(\beta) \geq \frac{3}{4}, r^{-1} \lesssim 1$.

Let $\chi_1(\cdot) : [0,\infty) \to [0,1]$ be a smooth cutoff function, such that $\chi_1(R) = 1$ for $R \leq 1$, $\chi_1(R) = 0$ for $R \geq 2$ and $(D_R\chi_1)^2 \lesssim \chi_1$. This assumption can be satisfied if $\chi_1 = \chi_0^2$ where χ_0 is another smooth cutoff function. Denote $\chi_\lambda(R) = \chi_1(R/\lambda)$. It is easy to verify that

$$(D_R\chi_\lambda)^2 = (R/\lambda\partial_R\chi_1(R/\lambda))^2 \lesssim \chi_1(R/\lambda) = \chi_\lambda(R), \quad |D_R^k\chi_\lambda| \lesssim \mathbf{1}_{\lambda\leq R\leq 2\lambda}, \tag{2.194}$$

for $k \leq 5$, where we have used $|D_R^2\chi_1| \lesssim \chi_1$ in the first inequality. Denote

$$\Psi_\chi = \Psi\chi_\lambda, \quad \Omega_\chi = \Omega\chi_\lambda.$$

At this moment, we just simplify χ_λ as χ. Note that $R^2\partial_{RR} + R\partial_R = D_R^2$ and

$$\begin{aligned} r\Omega_\chi &= (1 - C_l\rho\sin(\beta))\Omega_\chi = \Omega_\chi - C_l\rho\sin(\beta)\Omega_\chi, \\ \alpha D_R(\chi\Psi) &= \alpha D_R\chi\Psi + \alpha\chi D_R\Psi, \\ \alpha^2 D_R^2(\chi\Psi) &= \alpha^2\chi D_R^2\Psi + 2\alpha D_R\chi\cdot\alpha D_R\Psi + \alpha^2 D_R^2\chi\Psi. \end{aligned} \tag{2.195}$$

Multiplying χ on both sides of (2.192), and using (2.195) and a direct calculation yield

$$-\alpha^2 D_R^2\Psi_\chi - 4\alpha D_R\Psi_\chi - \partial_{\beta\beta}\Psi_\chi - 4\Psi_\chi = \Omega_\chi + Z_\chi, \quad Z_\chi = Z_1 + Z_2 + Z_3, \tag{2.196}$$

with boundary condition (2.193), where Z_1, Z_2 and Z_3 are given below

$$
\begin{aligned}
Z_1 &= -\frac{C_l\rho}{r}(\sin(\beta)(2\Psi_\chi + \alpha D_R\Psi_\chi) + \cos(\beta)\partial_\beta\Psi_\chi) - \frac{C_l^2\rho^2}{r^2}\Psi_\chi, \\
Z_2 &= \frac{C_l\sin(\beta)\rho}{r}\alpha D_R\chi\Psi - (\alpha^2 D_R^2\chi + 4\alpha D_R\chi)\Psi - 2\alpha^2 D_R\chi D_R\Psi, \\
Z_3 &= -C_l\rho\sin(\beta)\Omega_\chi.
\end{aligned}
\tag{2.197}
$$

Recall that $R = \rho^\alpha, r = 1 - C_l y = 1 - C_l\rho\sin(\beta)$ from (2.184), (2.185) and $L_{12}(f)(0)$ from (2.31). Next, we derive $L_{12}(Z_{\chi_\lambda})(0)$. It will be used in Section 2.10.2.3 when we apply Proposition 2.8.3.

Firstly, for sufficiently smooth Ω, Ψ with Ω vanishing at least linear near $R = 0$, we show that $L_{12}(Z_{\chi_\lambda})(0)$ is independent of the cutoff radial λ for $\lambda \geq (S(\tau))^\alpha$. From $\lambda \geq (S(\tau))^\alpha$, we have $\Omega = \Omega \cdot \chi_\lambda = \Omega_{\chi_\lambda}$. For any $\varepsilon > 0$, using integration by parts, we get

$$
\begin{aligned}
\langle \partial_{\beta\beta}\Psi_\chi + 4\Psi_\chi,\ \sin(2\beta)R^{-1}\mathbf{1}_{R\geq\varepsilon}\rangle &= \langle -4\Psi_\chi + 4\Psi_\chi, \sin(2\beta)R^{-1}\mathbf{1}_{R\geq\varepsilon}\rangle = 0, \\
\langle \alpha^2 D_R^2\Psi_\chi + 4\alpha D_R\Psi_\chi,\ \sin(2\beta)R^{-1}\rangle &= \langle \alpha^2\partial_R(D_R\Psi_\chi) + 4\alpha\partial_R\Psi_\chi, \sin(2\beta)\rangle \\
&= -4\alpha\int_0^{\pi/2}\Psi(0,\beta)\sin(2\beta)d\beta.
\end{aligned}
$$

Note that Ψ may not vanish at $R = 0$. Since $\rho = R^{1/\alpha}$ vanishes at $R = 0$, it is easy to see that Z_χ vanishes at $R = 0$. Therefore, integrating both sides of (2.196) with $\sin(2\beta)R^{-1}\mathbf{1}_{R\geq\varepsilon}$, and then using the above computations and taking $\varepsilon \to 0$, for $\lambda \geq (S(\tau)^\alpha)$, we have

$$
L_{12}(Z_{\chi_\lambda})(0) = -L_{12}(\Omega)(0) + 4\alpha\int_0^{\pi/2}\Psi(0,\beta)\sin(2\beta)d\beta. \tag{2.198}
$$

Next, we perform L^2 estimate for Ψ_χ. It will be used later to estimate Z_χ in (2.196).

Lemma 2.10.6. *There exists $\alpha_2 > 0$ such that if $\alpha < \alpha_2, C_l S < 4^{-1/\alpha-1}$, for $\lambda = \frac{1}{4}C_l^{-\alpha}$, the solution of (2.196) satisfies*

$$
\alpha^2||D_R\Psi_{\chi_\lambda}||_{L^2}^2 + \alpha||\Psi_{\chi_\lambda}||_{L^2}^2 + \alpha||\partial_\beta\Psi_{\chi_\lambda}||_{L^2}^2 \lesssim \alpha^{-1}||\Omega||_{L^2}^2.
$$

Firstly, we have $\lambda = \frac{1}{4}C_l^{-\alpha} > S^\alpha$ and $\Omega_\chi = \Omega\chi_\lambda = \Omega$. We impose $C_l S < 4^{-1/\alpha-1}$ so that $\lambda > (2S)^\alpha$ and $C(M) \lesssim C\alpha^{-1}$ in Lemma 2.10.4 with $M = \lambda$. At this step, this bound is good enough for us to treat Z_2 in (2.196)-(2.197) as perturbation. In the following estimates, we treat the small factor C_l in Z_χ approximately equal to zero and $r \approx 1$. See also Remark 2.10.3.

Proof. We simplify χ_λ as χ. Multiplying (2.196) by Ψ_χ and using integration by parts, we get

$$\begin{aligned} I &\triangleq \alpha^2 ||R\partial_R \Psi_\chi||_{L^2}^2 + \frac{4\alpha-\alpha^2}{2} ||\Psi_\chi||_{L^2}^2 + ||\partial_\beta \Psi_\chi||_{L^2}^2 - 4||\Psi_\chi||_{L^2}^2 \\ &= \langle \Omega, \Psi_\chi \rangle + \langle Z_1 + Z_3, \Psi_\chi \rangle + \langle Z_2, \Psi_\chi \rangle. \end{aligned} \tag{2.199}$$

Using the Fourier series expansion with basis $\{\sin(2n\beta)\}_{n\geq 1}$, one can verify that

$$||\partial_\beta \Psi_\chi||_{L^2}^2 \geq 4||\Psi_\chi||_{L^2},$$

which is sharp with equality when $\Psi_\chi = \sin(2\beta)$. Therefore, multiplying the above inequality by $1 - \frac{\alpha}{4}$ and then applying it to the left hand side of (2.199) yields

$$\begin{aligned} I &\geq \alpha^2 ||D_R \Psi_\chi||_{L^2}^2 + \frac{2\alpha - \alpha^2}{2} ||\Psi_\chi||_{L^2}^2 + \frac{\alpha}{4} ||\partial_\beta \Psi_\chi||_{L^2}^2 \\ &\geq \alpha^2 ||D_R \Psi_\chi||_{L^2}^2 + \frac{\alpha}{2} ||\Psi_\chi||_{L^2}^2 + \frac{\alpha}{4} ||\partial_\beta \Psi_\chi||_{L^2}^2, \end{aligned}$$

where we have used $\alpha \leq 1$.

Within the support of $\chi = \chi_\lambda$, we have $R \leq 2\lambda$. By assumption, we have $\lambda = \frac{1}{4} C_l^{-\alpha} > 4^\alpha S^\alpha$. It follows that

$$\begin{aligned} &C_l \rho \mathbf{1}_{R \leq 2\lambda} = C_l R^{\frac{1}{\alpha}} \mathbf{1}_{R\leq 2\lambda} \leq C_l (2\lambda)^{\frac{1}{\alpha}} = 2^{-\frac{1}{\alpha}} \lesssim \alpha^2, \\ &|\log(C_l \lambda^{\frac{1}{\alpha}})| \lesssim \alpha^{-1}, \quad 2S \leq \lambda^{1/\alpha}. \end{aligned} \tag{2.200}$$

The Z_1, Z_3 terms (2.197) contain the small factor $C_l \rho$. Since $r^{-1} \lesssim 1$, we get

$$\begin{aligned} &||Z_1||_{L^2} \lesssim \alpha^2 (||\Psi_\chi||_{L^2} + ||\alpha D_R \Psi_\chi||_{L^2} + ||\partial_\beta \Psi_\chi||_{L^2}) \lesssim \alpha^2 \alpha^{-1/2} I^{1/2} \lesssim \alpha^{3/2} I^{1/2}, \\ &||Z_3||_{L^2} \lesssim \alpha^2 ||\Omega||_{L^2} \lesssim ||\Omega||_{L^2}. \end{aligned}$$

We perform integration by parts for the last term $-2\alpha^2 D_R \chi D_R \Psi$ in Z_2 (2.197)

$$\begin{aligned} -2\alpha^2 \langle D_R \chi D_R \Psi, \Psi \chi \rangle &= -\alpha^2 \langle R \chi D_R \chi, \partial_R \Psi^2 \rangle = \alpha^2 \langle (R\chi D_R \chi)_R, \Psi^2 \rangle \\ &= \alpha^2 \langle (D_R \chi)^2 + \chi D_R^2 \chi + \chi D_R \chi, \Psi^2 \rangle. \end{aligned}$$

Using the above identity, (2.194) for $|D_R^k \chi|$ and (2.200), we obtain

$$|\langle Z_2, \Psi_\chi \rangle| \lesssim (\alpha^2 + \alpha) ||\Psi \mathbf{1}_{\lambda \leq R \leq 2\lambda}||_{L^2}^2 \lesssim \alpha ||\Psi \mathbf{1}_{\lambda \leq R \leq 2\lambda}||_{L^2}^2 \lesssim \alpha ||\Psi \mathbf{1}_{\lambda \leq R \leq (2C_l)^{-\alpha}}||_{L^2}^2,$$

where we have used $2\lambda < (2C_l)^{-\alpha}$ in the last inequality. Since $(2S)^\alpha \leq \lambda$ and $S\lambda^{-1/\alpha} \lesssim 1$ (see (2.200)), we apply Lemma 2.10.4 with $M = \lambda$ and (2.200) to get

$$|\langle Z_2, \Psi_\chi \rangle| \lesssim \alpha (1 + |\log(C_l \lambda^{1/\alpha})|)^2 (S\lambda^{-1/\alpha})^2 ||\Omega||_{L^2}^2 \lesssim \alpha^{-1} ||\Omega||_{L^2}^2.$$

Plugging the estimates of Z_1, Z_2, Z_3 and $||\Psi_\chi||_{L^2} \lesssim \alpha^{-1/2} I^{1/2}$ into (2.199) and then using the Cauchy-Schwarz inequality, we prove

$$I \leq C\alpha^{-1/2} I^{1/2} ||\Omega||_{L^2} + C\alpha \cdot I + C\alpha^{-1} ||\Omega||_{L^2}^2.$$

Now we choose

$$\alpha_2 = \min((2C)^{-1}, 4^{-1}). \tag{2.201}$$

Then for $\alpha < \alpha_2$, we have $C\alpha < \frac{1}{2}$. Solving the above inequality yields $I \lesssim \alpha^{-1} ||\Omega||_{L^2}^2$. □

2.10.2.3 Localized $\mathcal{H}^3$ estimates

Notice that the elliptic equation (2.196) is localized to $R \leq 2\lambda \leq \frac{1}{2} C_l^{-\alpha}$, which is away from the boundary of the rescaled domain $\tilde{D}_1 = [0, C_l^{-1}] \times [-C_l^{-1}, C_l^{-1}]$. Therefore, Ψ_χ can be treated as a solution of (2.196) in the whole space $R \geq 0, \beta \in [0, \pi/2]$ with source term $\Omega_\chi + Z_\chi$. We can apply Proposition 2.8.3 to improve the elliptic estimate in Lemma 2.10.6. In this estimate, we need to further estimate $\Omega_\chi + Z_\chi$ and $L_{12}(\Omega_\chi + Z_\chi)$.

The term in Z_χ (2.196),(2.197) either has a small factor $C_l \rho \approx 0$ (see Remark 2.10.3), or is localized to $\lambda \leq R \leq 2\lambda$ due to the factor $(D_R \chi)^k$, where λ is the parameter in the cutoff function $\chi(R/\lambda)$. To show that the second type of term is small, we use Lemma 2.10.4 and interpolation. Using the smallness of these variables and Lemmas 2.10.4 and 2.10.6, we can treat Z_χ as a small perturbation. Since $\rho = R^{1/\alpha}$ and $D_R \chi = 0$ for $|R| \leq 1$, the singular weight $W = \frac{(1+R)^k}{R^k}, k = 1, 2$ is treated approximately as 1 in the following estimates of terms involving $\rho, D_R \chi$.

Proposition 2.10.7. *Let Ψ be the solution of (2.192) and $W = \frac{(1+R)^k}{R^k}$ for $k = 1$ or 2. If $\alpha < \alpha_2$ (2.201), $C_l S < \alpha \cdot 8^{-1/\alpha - 1}$, for $\lambda = \frac{1}{8} C_l^{-\alpha}$, we have*

$$\begin{aligned} &\alpha^2 ||R^2 \partial_{RR} \Psi_{\chi_\lambda} W||_{L^2} + \alpha ||R \partial_{R\beta} \Psi_{\chi_\lambda} W||_{L^2} \\ &+ ||\partial_{\beta\beta} (\Psi_{\chi_\lambda} - \frac{\sin(2\beta)}{\alpha\pi} (L_{12}(\Omega) + \chi_1 L_{12}(Z_{\chi_\lambda})(0))) W||_{L^2} \lesssim ||\Omega W||_{L^2}, \end{aligned}$$

where Z_χ is defined in (2.196),(2.197) and χ_1 is the cutoff function. Moreover, for $\nu \geq (S(\tau))^\alpha$, $L_{12}(Z_{\chi_\nu})(0)$ does not depend on ν and satisfies

$$|L_{12}(Z_{\chi_\lambda})(0)| = |L_{12}(Z_{\chi_\nu})(0)| \lesssim (4^{-\frac{1}{\alpha}} \alpha^{-1} + \min(\alpha, (8^{1/\alpha} C_l S)^{1/2})) ||\Omega \frac{1+R}{R}||_{L^2}. \tag{2.202}$$

Remark 2.10.8. Let χ_{λ_1} be the cutoff function in Lemma 2.10.6. We choose $\lambda = \frac{1}{8}C_l^{-\alpha}$ so that $\chi_{\lambda_1} \equiv 1$ in $\mathrm{supp}(\chi_\lambda)$. This allows us to apply Lemma 2.10.6 to estimate various terms in $\mathrm{supp}(\chi_\lambda)$. We use $L_{12}(Z_{\chi_\lambda})(0)$ to correct Ψ so that $\Psi_{\chi_\lambda} - \frac{\sin(2\beta)}{\pi\alpha}(L_{12}(\Omega) - \chi_1 L_{12}(Z_{\chi_\lambda})(0))$ vanishes near $R = 0$. Choosing small $C_l S, \alpha$ later, we use (2.202) to show that $L_{12}(Z_{\chi_\lambda})(0)$ is very small.

Proof. **Step 1**. We apply the elliptic estimate in Proposition 2.8.3 in the weighted L^2 case, which can be proved using the same argument in [42], to obtain

$$\begin{aligned} I \triangleq\ & \alpha^2 ||R^2 \partial_{RR} \Psi_{\chi_\lambda} W||_{L^2} + \alpha ||R \partial_{R\beta} \Psi_{\chi_\lambda} W||_{L^2} \\ & + ||\partial_{\beta\beta}(\Psi_{\chi_\lambda} - \frac{\sin(2\beta)}{\alpha\pi}(L_{12}(\Omega_\chi + Z_\chi))W||_{L^2} \lesssim ||(\Omega_\chi + Z_\chi)W||_{L^2}. \end{aligned} \tag{2.203}$$

Under the assumption $C_l S < \alpha 8^{-1/\alpha - 1}$, we have $(2S)^\alpha < \frac{1}{8}C_l^{-\alpha} = \lambda$. Thus, $\Omega_\chi = \chi_\lambda \Omega = \Omega$. Recall $Z_\chi = Z_1 + Z_2 + Z_3$ in (2.197) and $\rho = R^{1/\alpha}$. Within the support of χ, we have

$$C_l \rho W = C_l R^{1/\alpha - 2}(1+R)^2 \lesssim C_l (2\lambda)^{1/\alpha} \leq 4^{-1/\alpha}. \tag{2.204}$$

We can apply Lemma 2.10.6 to estimate the $L^2(W^2)$ norm of Z_1

$$||Z_1 W||_{L^2} \lesssim ||C_l \rho W \chi||_{L^\infty}(||\Psi_\chi||_{L^2} + \alpha ||D_R \Psi_\chi||_{L^2} + ||\partial_\beta \Psi_\chi||_{L^2}) \lesssim 4^{-1/\alpha}\alpha^{-1}||\Omega||_{L^2}. \tag{2.205}$$

Estimate of Z_3 defined in (2.197) is trivial

$$||Z_3 W||_{L^2} \lesssim 4^{-1/\alpha}||\Omega||_{L^2}. \tag{2.206}$$

Recall Z_2 defined in (2.197). Notice that the support of Z_2 lies in $\lambda \leq R \leq 2\lambda$ due to the $D_R \chi$ term. Within this annulus, we get $W \lesssim 1$. Due to the smallness of $C_l \rho$ from (2.204), we have

$$||Z_2 W||_{L^2} \lesssim \alpha ||\Psi \mathbf{1}_{\lambda \leq R \leq 2\lambda}||_{L^2} + \alpha^2 ||D_R \chi D_R \Psi||_{L^2}. \tag{2.207}$$

Using $\lambda = \frac{1}{8}C_l^{-\alpha}$ and $C_l S < \alpha 8^{-1/\alpha - 1}$, we obtain

$$|\log(C_l \lambda^{1/\alpha})| = |\log(8^{-1/\alpha})| \lesssim \alpha^{-1}, \quad S\lambda^{-1/\alpha} = 8^{1/\alpha} C_l S < \alpha. \tag{2.208}$$

Since $\lambda \geq (2S(\tau))^\alpha$, applying Lemma 2.10.4 with $M = \lambda$ and (2.208) to $C(M)$, we get

$$||\Psi \mathbf{1}_{\lambda \leq R \leq 2\lambda}||_{L^2} \lesssim \alpha^{-1} 8^{1/\alpha} C_l S ||\Omega||_{L^2} \lesssim ||\Omega||_{L^2}. \tag{2.209}$$

Applying Lemma 2.10.6 to $D_R\Psi_\chi$, and using (2.207), (2.209), we yield

$$||Z_2 W||_{L^2} \lesssim \alpha ||\Omega||_{L^2} + \alpha^{1/2}||\Omega||_{L^2} \lesssim \alpha^{1/2}||\Omega||_{L^2}. \tag{2.210}$$

Plugging (2.205)-(2.210) into (2.203) and using $4^{-1/\alpha}\alpha^{-1} \lesssim 1$, we prove

$$I \lesssim ||\Omega_\chi W||_{L^2} \lesssim ||\Omega W||_{L^2}, \tag{2.211}$$

Step 2: Smallness of Z_2. We use interpolation and the smallness of $||\Psi \mathbf{1}_{\lambda\leq R\leq 2\lambda}||_{L^2}$ (2.209) to refine the estimate of Z_2 in (2.210). The refinement is used to estimate the term $\alpha^{-1}L_{12}(Z_\chi)$ in I (2.203), and is important to prove (2.202). Using integration by parts, we obtain

$$\begin{aligned} J &\triangleq ||\alpha^2 D_R\chi D_R\Psi||_2^2 = \alpha^4\langle R(D_R\chi)^2 D_R\Psi,\ \partial_R\Psi\rangle \\ &= -\alpha^4\langle \partial_R(R(D_R\chi)^2)D_R\Psi, \Psi\rangle - \alpha^4\langle R(D_R\chi)^2\partial_R D_R\Psi, \Psi\rangle. \end{aligned} \tag{2.212}$$

Using (2.194), we get $|\partial_R(R(D_R\chi)^2)| \lesssim |D_R\chi|\mathbf{1}_{\lambda\leq R\leq 2\lambda}, (D_R\chi)^2 \leq \chi\mathbf{1}_{\lambda\leq R\leq 2\lambda}$. Using the Cauchy-Schwarz inequality, we yield

$$J \lesssim \alpha^2(\alpha^2||D_R\chi D_R\Psi||_2 + \alpha^2||\chi D_R^2\Psi||_{L^2})||\Psi\mathbf{1}_{\lambda\leq R\leq 2\lambda}||_{L^2}. \tag{2.213}$$

We further estimate $\alpha^2||\chi D_R^2\Psi||_{L^2}$. Using $|D_R^2\chi| \lesssim \mathbf{1}_{\lambda\leq R\leq 2\lambda}$ and (2.195), we obtain

$$\alpha^2||\chi_\lambda D_R^2\Psi||_{L^2} \lesssim \alpha^2||D_R^2\Psi_\chi||_{L^2} + \alpha^2||D_R\chi D_R\Psi||_{L^2} + \alpha^2||\Psi\mathbf{1}_{\lambda\leq R\leq 2\lambda}||_{L^2}.$$

By definition, we have $\alpha^2||D_R\chi D_R\Psi||_{L^2} = J^{1/2}$. Using (2.203),(2.211),(2.209), we obtain

$$\alpha^2||\chi_\lambda D_R^2\Psi||_2 \lesssim ||\Omega W||_2 + J^{1/2} + \alpha^2||\Omega||_2 \lesssim ||\Omega W||_2 + J^{1/2}. \tag{2.214}$$

Plugging $\alpha^2||D_R\chi D_R\Psi||_{L^2} = J^{1/2}$, the first inequality in (2.209) and the estimate (2.214) in (2.213), we establish

$$\begin{aligned} J &\lesssim \alpha^2(J^{1/2} + ||\Omega W||_2 + J^{1/2})\alpha^{-1}8^{1/\alpha}C_l S||\Omega||_{L^2} \\ &\lesssim \alpha(J^{1/2} + ||\Omega W||_2)8^{1/\alpha}C_l S||\Omega W||_{L^2}. \end{aligned}$$

The above inequality is a quadratic inequality on $B = J^{1/2}/||\Omega W||_2 :\ B^2 \lesssim A(B+1), A = \alpha 8^{1/\alpha}C_l S \leq \alpha$, which implies $B \lesssim A^{1/2}$. Thus, we prove

$$J^{1/2} \lesssim \alpha^{1/2}(8^{1/\alpha}C_l S)^{1/2}||\Omega W||_{L^2}.$$

Combining the above estimate of J and (2.207)-(2.208), we yield

$$\begin{aligned}||Z_2 W||_{L^2} &\lesssim (8^{1/\alpha} C_l S + \alpha^{\frac{1}{2}} (8^{1/\alpha} C_l S)^{1/2}) ||\Omega W||_{L^2} \\ &\lesssim \min(\alpha, (8^{1/\alpha} C_l S)^{1/2}) ||\Omega W||_{L^2}.\end{aligned} \tag{2.215}$$

Using Lemma A.0.4, (2.205), (2.206) and (2.215), we establish

$$\begin{aligned}&||L_{12}(Z_{\chi_\lambda}) - \chi_1 L_{12}(Z_{\chi_\lambda})(0) W||_{L^2} \lesssim ||Z_\chi W||_{L^2} \\ \lesssim &||(Z_1 + Z_2 + Z_3) W||_{L^2} \lesssim \alpha ||\Omega W||_{L^2},\end{aligned} \tag{2.216}$$

where we have used $4^{-1/\alpha} \alpha^{-1} \lesssim \alpha$. Combining (2.203), (2.211) and (2.216), we complete the proof of the first estimate. We remark that we only need the bound $||Z_2 W||_{L^2} \lesssim \alpha ||\Omega W||_{L^2}$ from (2.215) in this estimate.

Step 3: Estimate of $L_{12}(Z_{\chi_\lambda})(0)$. Note that the previous estimates hold true for the weights $W = W_k \triangleq \frac{(1+R)^k}{R^k}$ with $k = 1$ or 2. Recall $Z_{\chi_\lambda} = Z_1 + Z_2 + Z_3$ (2.196). Using Lemma A.0.4, (2.205), (2.206) and (2.215), we prove

$$\begin{aligned}|L_{12}(Z_{\chi_\lambda})(0)| &\lesssim ||Z_{\chi_\lambda} W_1||_{L^2} \lesssim ||(Z_1 + Z_2 + Z_3) W||_{L^2} \\ &\lesssim (4^{-\frac{1}{\alpha}} \alpha^{-1} + \min(\alpha, (8^{1/\alpha} C_l S)^{1/2})) ||\Omega W_1||_{L^2},\end{aligned}$$

which is (2.202). Using (2.198), we yield that $L_{12}(Z_{\chi_\nu})$ is independent of ν for $\nu \geq S(\tau)^\alpha$. □

Proposition 2.10.9. *Suppose that Ψ is the solution of* (2.192) *and $\Omega \in \mathcal{H}^3$. If $\alpha < \alpha_2$* (2.201)*, $\lambda = 2^{-13} C_l^{-\alpha}$, $C_l S < \alpha \cdot (2^{13})^{-1/\alpha - 1}$, then we have*

$$\begin{aligned}&\alpha^2 ||R^2 \partial_{RR} \Psi_{\chi_\lambda}||_{\mathcal{H}^3} + \alpha ||R \partial_{R\beta} \Psi_{\chi_\lambda}||_{\mathcal{H}^3} \\ &\quad + ||\partial_{\beta\beta} (\Psi_{\chi_\lambda} - \frac{\sin(2\beta)}{\alpha\pi} (L_{12}(\Omega) + \chi_1 L_{12}(Z_{\chi_\lambda})(0)))||_{\mathcal{H}^3} \lesssim ||\Omega||_{\mathcal{H}^3}, \\ &|L_{12}(Z_{\chi_\lambda})(0)| \lesssim 3^{-\frac{1}{\alpha}} ||\Omega \frac{1+R}{R}||_{L^2}.\end{aligned}$$

Moreover, $L_{12}(Z_{\chi_\nu})(0)$ does not depend on ν for $\nu \geq (S(\tau))^\alpha$.

The small factor $3^{-1/\alpha}$ will be used later to absorb α^{-k} for several $k \in Z_+$, i.e. $3^{-1/\alpha} \alpha^{-k} \lesssim_k 1$. The estimate of $L_{12}(Z_{\chi_\lambda})(0)$ follows from (2.202). The proof of the first inequality follows from the idea discussed at the beginning of Section 2.10.2.3 and estimates similar to the Step 1 in the proof of Proposition 2.10.7. Suppose that Ω has size 1. Using the smallness of $C_l, C_l S$ (see Remark 2.10.3), Lemma 2.10.4 and Proposition 2.10.7, formally, we get that $C_l \rho \Omega$ has size 0,

$C_l \rho \mathcal{T} \Psi_\chi$ has size ≈ 0 for $\mathcal{T} = \partial_\beta, D_R$ or Id, $\alpha^2 D_R \chi D_R \Psi$ has size $\alpha^2 \alpha^{-1} = \alpha$ and $D_R^k \chi \Psi$ has size ≈ 0. Hence, $Z_\chi, \frac{1}{\alpha}(L_{12}(Z_\chi) - \chi_1 L_{12}(Z_\chi)(0))$ have size less than $\alpha, 1$, respectively, which enables us to treat them as perturbation. Moreover, the terms Z_1, Z_2 in Z_χ (2.196) have derivatives whose orders are lower than $D_R^2 \Psi_\chi, \partial_\beta^2 \Psi_\chi$. The term $Z_3 = -C_l \rho \sin(\beta) \Omega$ in (2.197) does not involve Ψ and its estimate is trivial. These allow us to use induction to establish higher order estimates.

Denote $\Psi_{\chi_\lambda, *} = \Psi_{\chi_\lambda} - \frac{\sin(2\beta)}{\alpha \pi}(L_{12}(\Omega) + \chi_1 L_{12}(Z_{\chi_\lambda})(0))$. To simplify our discussions, we introduce some notations for different elliptic estimates. Recall $\mathcal{H}^m$ defined in (2.136) and $\mathcal{H}^0 = L^2(\varphi_1)$. For some weight $\widetilde{W}$, differential operator $\mathcal{T} = D_\beta^j D_R^i$ and constant μ, we denote by $\mathcal{P}(\widetilde{W}, \mathcal{T}, \mu, \alpha, C_l, \Psi, \Omega)$ the following elliptic estimate for the solution Ψ of (2.193)

$$
\begin{aligned}
&\alpha^2 ||\mathcal{T} D_R^2 \Psi_{\chi_\lambda} \widetilde{W}^{\frac{1}{2}}||_2 + \alpha ||\mathcal{T} D_R \partial_\beta \Psi_{\chi_\lambda} \widetilde{W}^{\frac{1}{2}}||_2 + ||\mathcal{T} \partial_\beta^2 \Psi_{\chi_\lambda, *} \widetilde{W}^{\frac{1}{2}}||_2 \\
\lesssim & ||D_R^i \Omega||_{\mathcal{H}^j} + ||\Omega||_{\mathcal{H}^j},
\end{aligned}
\tag{2.217}
$$

where $\lambda = \mu C_l^{-\alpha}$ is the parameter for the cutoff function. We put $D_R^i \Omega$ in the upper bound since D_R commutes with the elliptic operator $\mathcal{L}_\alpha$ (2.134), which was observed in [42]. The upper bound controls the D_β^j derivatives of $D_R^i \Omega$. We simplify $\mathcal{P}(\widetilde{W}, \mathcal{T}, \mu, \alpha, C_l, \Psi, \Omega)$ as $\mathcal{P}(\widetilde{W}, \mathcal{T}, \mu)$.

Recall the weights φ_i (2.6.2) and the $\mathcal{H}^3$ norm (2.136). Denote

$$
\begin{aligned}
&\mathcal{P}_0 = \mathcal{P}(\frac{(1+R)^4}{R^4}, Id, 2^{-3}), \ \mathcal{P}_{1+j} = \mathcal{P}(\varphi_1, (D_R)^j, 2^{-4-j]}), \ 0 \leq j \leq 3, \\
&\mathcal{P}_{5+j} = \mathcal{P}(\varphi_2, (D_R)^j D_\beta, 2^{-8-j}), \ 0 \leq j \leq 2, \\
&\mathcal{P}_{8+j} = \mathcal{P}(\varphi_2, (D_R)^j D_\beta^2, 2^{-11-j}), \ j = 0, 1, \quad \mathcal{P}_{10} = \mathcal{P}(\varphi_2, D_\beta^3, 2^{-13}),
\end{aligned}
\tag{2.218}
$$

We establish $\mathcal{P}_i$ in an increasing order by induction. In other words, we first establish (2.217) for $\mathcal{T}$ being D_R derivatives, then for $\mathcal{T}$ including one D_β derivatives, and so on. Estimate $\mathcal{P}_0$ is established in Proposition 2.10.7, and it serves as the base case. This order of estimates has been used in [42]. The support of the cutoff function in $\mathcal{P}_l$ satisfies

$$
\chi^{(l)} \equiv 1 \quad \text{in} \quad \mathrm{supp}(\chi^{(l+1)}) \cup \mathrm{supp}(\Omega), \quad \chi^{(l)} \triangleq \chi_1(R/(2^{-3-l} C_l^{-\alpha})). \tag{2.219}
$$

Hence, to prove $\mathcal{P}_n$, we can apply $\mathcal{P}_l, l \leq n-1$. The $\mathcal{H}^3$ elliptic estimate follows from all $\mathcal{P}_i$.

Proof. We demonstrate the ideas in the induction by mainly proving the $L^2(\varphi_1)$ elliptic estimate $\mathcal{P}_1$. To establish $\mathcal{P}_n, n \geq 1$, in step I, we use the $\mathcal{P}_n$ version of the elliptic estimates in Proposition 2.8.3 with source term $\Omega + Z_\chi$, which can be proved by the argument in [42]. We simplify $\chi^{(n)}$ as χ. In the case of $n = 1$, using the $\mathcal{P}_1$ elliptic estimates (or the $L^2(\varphi_1)$ estimates), we obtain

$$\begin{aligned} &\alpha^2 ||R^2 \partial_{RR} \Psi_{\chi_\lambda} \varphi_1^{1/2}||_{L^2} + \alpha ||R \partial_{R\beta} \Psi_{\chi_\lambda} \varphi_1^{1/2}||_{L^2} \\ &+ ||\partial_{\beta\beta} (\Psi_{\chi_\lambda} - (\pi\alpha)^{-1} \sin(2\beta)(L_{12}(\Omega_\chi + Z_\chi)) \cdot \varphi_1^{1/2}||_{L^2} \lesssim ||(\Omega_\chi + Z_\chi)\varphi_1^{1/2}||_{L^2}. \end{aligned} \tag{2.220}$$

In step II, we apply Lemma A.0.4 to the $L_{12}(\cdot)$ terms and the elliptic estimate we have obtained, i.e. $\mathcal{P}_i, i \leq n-1$, to control the Z_χ terms. In the case of $n = 1$, $\mathcal{P}_i, i \leq n-1$ is $\mathcal{P}_0$, which has been established in Proposition 2.10.7. Our goal is to establish

$$||Z_\chi \varphi_1^{1/2}||_{L^2} \lesssim ||\Omega \varphi_1^{1/2}||_{L^2}, \tag{2.221}$$

$$||\partial_{\beta\beta} \Big(\frac{\sin(2\beta)}{\alpha\pi} (L_{12}(Z_\chi) - \chi_1 L_{12}(Z_\chi)(0)) \Big) \varphi_1^{1/2}||_{L^2} \lesssim ||\Omega \varphi_1^{1/2}||_{L^2}, \tag{2.222}$$

in the case of $n = 1$, and similar estimates in the case of $n > 1$.

Recall $Z_\chi = Z_1 + Z_2 + Z_3$ and (2.197). By triangle inequality, it suffices to establish the above estimates for Z_i separately. Note that Z_3 in (2.197) does not involve Ψ and contains the small factor $C_l \rho$ (see (2.224) below). The above estimates (and similar estimates in the case of $n > 1$) for Z_3 are straightforward by applying Lemma A.0.4 to estimate the $L_{12}(\cdot)$ term. The above estimates (and similar estimates appeared in the proof of $\mathcal{P}_n, n > 1$) for Z_1, Z_2 are established by the following substeps.

Firstly, Z_1, Z_2 defined in (2.197) only contain the first order derivative D_R, ∂_β of Ψ_χ, which are lower order than the leading terms $D_R^2 \Psi_\chi, \partial_\beta^2 \Psi_\chi$ in (2.196). Hence, we can apply the previous elliptic estimates, e.g., $\mathcal{P}_0$ or Proposition 2.10.7 for $n = 1$, to estimate the norm of higher order derivatives of Z_1, Z_2 or the norm of Z_1, Z_2 with more singular weight. To estimate the Ψ terms in Z_1, Z_2 that do not involve D_R derivative, e.g., $\partial_\beta \Psi_\chi, \Psi_\chi$, we decompose Ψ_χ into

$$\begin{aligned} \Psi_{\chi,*} &\triangleq \Psi_\chi - \frac{\sin(2\beta)}{\pi\alpha} (L_{12}(\Omega) + \chi_1 L_{12}(Z_\chi)(0)), \\ \Psi_{\chi,2} &\triangleq \frac{\sin(2\beta)}{\pi\alpha} (L_{12}(\Omega) + \chi_1 L_{12}(Z_\chi)(0)). \end{aligned} \tag{2.223}$$

We apply the elliptic estimates to estimate $\Psi_{\chi,*}$, Lemma A.0.4 and Proposition 2.10.7 for $L_{12}(Z_\chi)(0)$ to estimate $\Psi_{\chi,2}$. Formally, compared to Ω, $\Psi_{\chi,*}$ and $\Psi_{\chi,2}$ have size $1, \alpha^{-1}$, respectively.

Secondly, Z_1 and Z_2 contain small factors. For Z_1 (2.197), since $\lambda = CC_l^{-\alpha}$ for $C \in [2^{-13}, 2^{-4}]$, in $\mathrm{supp}(\chi)$, we get a small factor

$$C_l\rho\chi \leq \mathbf{1}_{R\leq 2\lambda} C_l \rho \leq C_l(2\lambda)^{1/\alpha} \leq 8^{-1/\alpha} \lesssim_k \alpha^k \tag{2.224}$$

for any $k \in Z_+$. For Z_2 defined in (2.197), the first term in Z_2 also contains the small factor $C_l\rho$, the second and the fourth terms contains a small factor α^2 and the third term contains α. These small factors cancel the factor α^{-1} in $\Psi_{\chi,2}$ in (2.223). In the case of $n = 1$, we estimate typical terms, $C_l\rho r^{-1}\partial_\beta \Psi_\chi, \alpha D_R \chi \Psi$, in Z_χ (2.197). Denote

$$W = \frac{(1+R)^2}{R^2}. \tag{2.225}$$

Recall $\varphi_1 = (f(\beta)W)^2, f(\beta) = \sin(2\beta)^{-\sigma/2}, \sigma = \frac{99}{100}$ (2.6.2). Using

$$||g(R,\beta)f(\beta)||_{L^2(\beta)} \lesssim ||g(R,\beta)||_{L^\infty(\beta)} \lesssim ||\partial_\beta g(R,\cdot)||_{L^2(\beta)}, \quad g = \partial_\beta \Psi_\chi,\ \Psi,$$

$\rho = R^{1/\alpha}$, $D_R\chi = 0$ for $|R| \leq 1$, Proposition 2.10.7 and Lemma A.0.4, we get

$$||C_l\rho r^{-1}\partial_\beta\Psi_\chi\varphi_1^{1/2}||_2 \lesssim \alpha^4||\partial_\beta\Psi_\chi f||_2 \lesssim \alpha^4||\partial_{\beta\beta}\Psi_\chi||_2 \lesssim \alpha^3||\Omega W||_2 \lesssim \alpha^3||\Omega\varphi_1^{1/2}||_{L^2},$$
$$||\alpha D_R\chi\Psi\varphi_1^{1/2}||_2 \lesssim \alpha||D_R\chi\Psi f||_2 \lesssim \alpha||D_R\chi\partial_\beta\Psi||_2 \lesssim \alpha\cdot\alpha^{-1}||\Omega W||_2 \lesssim ||\Omega\varphi_1^{1/2}||_{L^2},$$

where we have used (2.219) with $l = 0$ so that we can apply Proposition 2.10.7 to estimate $D_R\chi\partial_\beta\Psi$. Other terms in Z_χ can be estimated similarly. We prove (2.221). Estimates similar to (2.221) in the case of $n > 1$ are proved similarly.

Thirdly, we consider (2.222) and similar estimates appeared in the proof of $\mathcal{P}_n$ with $n > 1$, which are more difficult to prove since they contain α^{-1}. Recall φ_1, φ_2 in (2.78) and W in (2.225). Using Lemma A.0.4 and (A.12) in its proof, for any $p, l \geq 0$ and $q = 1, 2$, we obtain

$$\begin{aligned} &||D_\beta^p D_R^l \partial_\beta^2 \frac{\sin(2\beta)}{\pi\alpha}(L_{12}(Z_1+Z_2) - \chi_1 L_{12}(Z_1+Z_2)(0))\varphi_q^{1/2}||_2 \\ \lesssim&\alpha^{-1}\sum_{i\leq\max(l-1,0)}||D_R^i(Z_1+Z_2)W||_2. \end{aligned} \tag{2.226}$$

We need to further estimate the right hand side. The most difficult term in Z_1, Z_2 (2.196),(2.197) is $\alpha D_R\chi\Psi$, since other terms contain smaller factors

$\alpha^2, C_l\rho$ and their weighted Sobolev norm can be bounded by $\alpha(||D_R^l\Omega||_{\mathcal{H}^p} + ||\Omega||_{\mathcal{H}^p})$ using the same argument as that in Step 2. Formally, $\alpha\Psi$ has size 1 compared to Ω. Exploiting the factor $D_R\chi$, we show that $\alpha D_R\chi\Psi$ has size α.

To estimate $\alpha D_R^i(D_R\chi\Psi)$, we have two types of terms $I_1 \triangleq \alpha D_R^{i+1}\chi\Psi$ and $I_{j,m} \triangleq \alpha D_R^j\chi D_R^m\Psi$ with $j, m \geq 1$ and $j+m = i+1$. Note that $|\log(C_l\lambda^{1/\alpha})| \lesssim \alpha^{-1}$, $S\lambda^{-1/\alpha} \leq 2^{13/\alpha}C_lS \lesssim \alpha$. Applying Lemma 2.10.4 with $M = \lambda$ to $D_R^{i+1}\chi\Psi$, we get

$$||\alpha D_R^{i+1}\chi\Psi W||_2 \lesssim ||\alpha D_R^{i+1}\chi\Psi||_2 \lesssim \alpha||\Omega||_2. \tag{2.227}$$

When $i = 0$, we do not have $I_{j,m}$. Recall $i \leq \max(l-1, 0)$ in the summation in (2.226). Thus, in the case of $n = 1$, combining (2.226) and (2.227) implies (2.222). The same argument applies to the case of $l \leq 1$.

It remains to estimate $I_{j,m}$ with $j, m \geq 1$, in the case of $l \geq 2$. Recall $L_{12}(\cdot)$ from (2.31), $\Psi_{\chi,2}$ from (2.223) and $\chi = \chi^{(n)}$. Since $\mathrm{supp}(D_R^j\chi) = \{\lambda_n \leq R \leq 2\lambda_n\}$ is away from $\mathrm{supp}(\Omega) \cup \mathrm{supp}(\chi_1) \subset \{R \leq S^\alpha\}$ (see (2.219)), we get

$$\begin{aligned}&D_R^j\chi D_R^m\Psi_{\chi^{(m)},2}\\ =&D_R^j\chi \cdot \frac{\sin(2\beta)}{\pi\alpha}\Big(-\int_0^{\pi/2} D_R^{m-1}\Omega(R,\beta)\sin(2\beta)d\beta + D_R^m\chi_1 L_{12}(Z_{\chi^{(m)}})(0)\Big) = 0.\end{aligned}$$

Thus, we can subtract the singular term from $D_R^j\chi D_R^m\Psi$

$$D_R^j\chi D_R^m\Psi = D_R^j\chi D_R^m(\Psi - \Psi_{\chi^{(m)},2}). \tag{2.228}$$

Formally, compared to Ω, $D_R^j\chi D_R^m\Psi$ has size 1. In the summation in (2.226), we have $i \leq l-1$. Since $j+m = i+1$ and $j \geq 1$, we get $m = i+1-j \leq l-1$. By definition (2.218), the weighted $D_\beta^pD_R^l$ elliptic estimates appear in $\mathcal{P}_n$, if $n \geq l$ and $l \leq 3$. Thus, using the elliptic estimate $\mathcal{P}_m$ ($m \leq l-1 \leq n-1$) in the induction hypothesis, i.e. $\mathcal{T} = D_R^m, \tilde{W} = \varphi_1$ in (2.217), we yield

$$\begin{aligned}||\alpha D_R^j\chi D_R^m(\Psi - \Psi_{\chi^{(m)},2})W||_2 &\lesssim \alpha||D_R^m(\Psi_{\chi^{(m)}} - \Psi_{\chi^{(m)},2})W||_2\\ &\lesssim \alpha(||D_R^m\Omega||_{\mathcal{H}^0} + ||\Omega||_{\mathcal{H}^0}),\end{aligned} \tag{2.229}$$

where $\mathcal{H}^0 = L^2(\varphi_1)$. Combining (2.226)-(2.229), we establish the $\mathcal{P}_n$ version of (2.222) for $Z_1 + Z_2$.

Therefore, combining (2.220)-(2.222), we obtain the $L^2(\varphi_1)$ elliptic estimate, i.e. $\mathcal{P}_1$. Repeating this argument, we can obtain the $\mathcal{P}_l, 2 \leq l \leq 10$ and $\mathcal{H}^3$ elliptic estimates.

Note that the assumption on λ, C_l, S, i.e. $C_l S < \alpha \cdot (2^{13})^{-1/\alpha - 1}$, implies $\lambda \geq S^\alpha$ and

$$4^{-1/\alpha}\alpha^{-1} \lesssim 3^{-1/\alpha}, \quad (8^{1/\alpha}C_l S)^{1/2} \leq ((2^{10})^{-1/\alpha})^{1/2} \leq 3^{-1/\alpha}.$$

Since the estimate (2.202) in Proposition 2.10.7 does not depend on λ as long as $\lambda \geq (S(\tau))^\alpha$, using (2.202) and the above calculation, we establish the desired estimate on $L_{12}(Z_{\chi_\lambda})(0)$. □

Remark 2.10.10. The term $D_R^j \chi D_R^m \Psi$ can also be estimated using an argument similar to that in the Step 2 of the proof of Proposition 2.10.7. We find the above approach simpler.

Recall $\bar{\Omega}$ in (2.44). We have a result similar to Proposition 2.8.8.

Proposition 2.10.11. *Let $\bar{\Psi}_0(t)$ be the solution of* (2.192) *with source term $\bar{\Omega}_0 = \bar{\Omega}\chi(R/\nu)$. If $\alpha < \alpha_2$* (2.201)*, $\lambda = 2^{-13}C_l^{-\alpha}$, $C_l S < \alpha(2^{13})^{-1/\alpha-1}$, $2\nu < \lambda$, then we have*

$$\begin{aligned}
&\alpha|| \frac{1+R}{R} D_R^2 \bar{\Psi}_{0,\chi_\lambda} ||_{\mathcal{W}^{5,\infty}} + \alpha|| \frac{1+R}{R} R\partial_{R\beta} \bar{\Psi}_{0,\chi_\lambda} ||_{\mathcal{W}^{5,\infty}} \\
&+ || \frac{1+R}{R} \partial_{\beta\beta} (\bar{\Psi}_{0,\chi_\lambda} - \frac{\sin(2\beta)}{\alpha\pi} (L_{12}(\bar{\Omega}_0) + \chi_1 L_{12}(\bar{Z}_{\chi_\lambda})(0))) ||_{\mathcal{W}^{5,\infty}} \lesssim \alpha, \\
&|L_{12}(\bar{Z}_{\chi_\lambda})(0)| \lesssim 3^{-\frac{1}{\alpha}},
\end{aligned}$$

where $\bar{Z}_{\chi_\lambda}$ associated to $\bar{\Psi}_0$ is defined in (2.196),(2.197)*. Moreover, $L_{12}(\bar{Z}_{\chi_\mu})(0)$ does not depend on μ for $\mu \geq (S(\tau))^\alpha$ and enjoys the above estimate for $L_{12}(\bar{Z}_{\chi_\lambda})$.*

Remark 2.10.12. Although $\bar{\Omega}_0 = \bar{\Omega}\chi_\nu$ is time-independent, the equation (2.179) is not and $\bar{\Psi}_0(t)$ depends on how we rescale the space. The factor 2ν is the support size of $\bar{\Omega}_0$. We impose $\lambda > 2\nu$ so that $\chi_\lambda = 1$ in the support of $\bar{\Omega}_0$.

The proof follows from the argument in the proof of Propositions 2.8.8, 2.10.7 and 2.10.9.

2.10.3 Nonlinear stability

We apply the nonlinear stability analysis of the 2D Boussinesq equations to prove Theorem 2.2. In Section 2.10.3.1, we impose the bootstrap assumption on the support size. In Section 2.10.3.2, we construct the approximate steady state and impose the normalization conditions, which are small perturbations to those in the 2D Boussinesq. In Section 2.10.3.3, we estimate the terms in the 3D Euler (2.188) that are different from the 2D Boussinesq (2.42). These terms contain factors that are much smaller than α and we treat them as perturbations. In Section 2.10.3.4, we generalize the nonlinear stability estimates in the 2D Boussinesq to the 3D Euler. In Section 2.10.3.5, we use the ideas described in Section 2.2.3.1 to control the growth of the support. In Section 2.10.3.6, we prove finite time blowup.

2.10.3.1 Bootstrap assumption on the support size

Recall α_2 defined in (2.201) in Lemma 2.10.6. We first require $\alpha < \alpha_2$. We impose the first bootstrap assumption: for $t \geq 0$, we have

$$C_l(t) \max(S(t), S(0)) < \alpha \cdot (2^{13})^{-\frac{1}{\alpha}-1} \triangleq K(\alpha). \tag{2.230}$$

Under the above Bootstrap assumption, the support of ω, θ in D_1 does not touch the symmetry axis and $z = \pm 1$, and the assumption in Proposition 2.10.9 is satisfied. We will choose $C_l(0)$ at the final step, which guarantees the smallness in (2.230).

2.10.3.2 Approximate steady state and the normalization condition

Since the rescaled domain $\tilde{D}_1$ (2.191) is bounded, we construct approximate steady state with bounded support. We localize $\bar{\Omega}, \bar{\theta}$ defined in (2.44) to construct the approximate steady state for (2.188)

$$\bar{\Omega}_0 \triangleq \chi_\nu \bar{\Omega}, \quad \bar{\theta}_0 \triangleq \chi_\nu \bar{\theta} = \chi_\nu (1 + xJ(\bar{\eta})), \tag{2.231}$$

where $\chi_\nu = \chi_1(R/\nu)$ and we have applied the integral operator $J(f)$ in Lemma A.0.11. We can choose $\chi_1 = \tilde{\chi}_1^2$ for another smooth cutoff function $\tilde{\chi}_1$ such that $\chi_1^{1/2} = \tilde{\chi}_1$ is smooth. We use $\bar{\omega}_0$ to denote $\bar{\Omega}_0$ in the (x, y) coordinates. Clearly, the support size of $\bar{\Omega}_0, \bar{\theta}_0$ is 2ν. Using the computation in (A.44), we have

$$\begin{aligned} \bar{\eta}_0 &= \partial_x(\chi_\nu \bar{\theta}) = \alpha \cos^2(\beta) D_R \chi_\nu \cdot J(\bar{\eta}) + \alpha \cos(\beta) r^{-1} D_R \chi_\nu + \chi_\nu \bar{\eta}, \\ \bar{\xi}_0(R, \beta) &= \partial_y(\chi_\nu \bar{\theta}) = \alpha \sin(\beta) \cos(\beta) D_R \chi_\nu \cdot J(\bar{\eta}) + \alpha \sin(\beta) r^{-1} D_R \chi_\nu + \chi_\nu \bar{\xi}, \end{aligned} \tag{2.232}$$

Let $\bar{\Psi}_0(t)$ be the solution of (2.196) with source term $\bar{\Omega}_0$. Applying Lemma A.0.11 and the analysis in its proof, we know that $\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0$ enjoys the same estimates as that of $\bar{\Omega}, \bar{\eta}, \bar{\xi}$ in Lemmas A.0.6 and A.0.8.

We need to adjust the time-dependent normalization condition for $c_\omega(t), c_l(t)$. Firstly, we choose the time-dependent cutoff radial $\lambda(t) = 2^{-13}(C_l(t))^{-\alpha}$ according to Proposition 2.10.9.

Define $\bar{Z}_{\chi_{\lambda(0)}}(t)$ according to (2.197), or equivalently (2.198), with $\Psi = \bar{\Psi}_0(t)$, $\Omega = \bar{\Omega}_0$ and $\chi = \chi_{\lambda(0)}$. It does not depend on the cutoff radial if $\lambda(0) \geq (2\nu)^\alpha$, where 2ν is the size of support of $\bar{\Omega}_0$. We use the following conditions

$$\begin{aligned} \bar{c}_\omega(t) &= -1 - \frac{2}{\pi\alpha} L_{12}(\bar{\Omega}_0 - \bar{\Omega} + \bar{Z}_{\chi_{\lambda(0)}})(0), \\ \bar{c}_l(t) &= \frac{1}{\alpha} + 3 - \frac{1-\alpha}{\alpha}\frac{2}{\pi\alpha} L_{12}(\bar{\Omega}_0 - \bar{\Omega} + \bar{Z}_{\chi_{\lambda(0)}})(0). \end{aligned} \tag{2.233}$$

We remark that $\bar{c}_\omega(t), \bar{c}_l(t)$ is time-dependent. Without the Z term, the above conditions for $\bar{c}_\omega, \bar{c}_l$ are the same as that in (2.44) with a correction due to the difference between the profiles $(\bar{\Omega}, \bar{\eta})$ in (2.44) and $\bar{\Omega}_0, \bar{\eta}_0$ in (2.231)-(2.232). For this difference, we use (2.47) to correct $\bar{c}_\omega, \bar{c}_l$.

For any perturbation $\Omega(t)$, we use the following conditions for $c_\omega(t), c_l(t)$

$$c_\omega(t) = -\frac{2}{\pi\alpha} L_{12}(\Omega(t) + Z_{\chi_{\lambda(t)}}(t))(0), \quad c_l(t) = \frac{1-\alpha}{\alpha} c_\omega(t). \tag{2.234}$$

Without the Z term, the above conditions for $c_\omega(t), c_l(t)$ are the same as that in (2.47).

We add the Z terms in (2.233), (2.234) since the behavior of Ψ, which is the solution of (2.192), is characterized by $L_{12}(\Omega + Z_\chi)(0)$ for R close to 0 according to the elliptic estimate in Proposition 2.10.9. For the 2D Boussinesq equation, we use $L_{12}(\Omega)(0)$ to determine c_ω, c_l since it also characterizes the behavior of Ψ near $R = 0$ according to Proposition 2.8.3.

We choose the above conditions so that the error of the approximate steady state vanishes quadratically in R near $R = 0$ and that the update of $\Omega(t), \eta(t)$ (ω, θ_x) in equation (2.188) also vanishes quadratically in R near $R = 0$ if the initial perturbation $\Omega(\cdot, 0), \eta(\cdot, 0)$ $(\theta_x(0))$ vanishes quadratically. We also determine $\bar{c}_\omega, \bar{c}_l$ in (2.44) and c_ω, c_l in (2.47) based on this principle.

Remark 2.10.13. We will choose ν to be very large, relative to α^{-1}. Therefore, we treat $\bar{\Omega}_0 \approx \bar{\Omega}, \bar{\theta}_0 \approx \bar{\theta}$. Due to the small factor $3^{-1/\alpha}$ in Propositions 2.10.9, 2.10.11, we treat $L_{12}(Z_\chi)(0)$, $L_{12}(\bar{Z}_\chi)(0) \approx 0$. From Remark 2.10.3 and the bootstrap assumption (2.230), we also have $C_l \approx 0, C_l S \approx 0, r \approx 1$. We treat the error terms in these approximations as perturbation.

2.10.3.3 Estimate of the lower order terms

The equations (2.188) are slightly different from (2.42) for the Boussinesq systems. We show how to estimate their differences. Suppose that $\omega(t), \theta(t)$ are the perturbations and the support size of $\bar{\omega}_0 + \omega(t), \bar{\theta}_0 + \theta(t)$ is $S(t)$.

Assume that the bootstrap assumption (2.230) holds true. For the term $\frac{1-r^4}{r^4}\theta_x$, within the support of ω, θ, we have $\rho \leq S(t), r = 1 - C_l \rho \sin(\beta) \in [3/4, 1]$. We get

$$|\frac{1-r^4}{r^4}| \lesssim 1 - r \leq C_l\rho \leq C_l S(t) < \alpha(2^{13})^{-1/\alpha - 1}, \tag{2.235}$$

which is extremely small compared to α. Since $\rho = R^{1/\alpha}$, the factor $C_l\rho, 1 - r^4$ vanish high order in R near $R = 0$. Hence, $\frac{1-r^4}{r^4}\theta_x$ is a smooth (near $R = 0$) small error term.

For the term $\frac{1}{r}C_l\psi$ in $u = -\psi_y + \frac{1}{r}C_l\psi$ defined in (2.188). Under the (R, β) coordinates, it becomes $\frac{C_l\rho}{r}(\rho\Psi(R,\beta))$. Compared to $-\psi_y = -(\rho^2\Psi)_y$ in (2.25), $\frac{C_l\rho}{r}(\rho\Psi(R,\beta))$ vanishes on $\beta = 0, \pi/2$ and contains a small smooth factor $C_l\rho = C_l R^{1/\alpha}$ within the support of ω, θ.

The last difference is the elliptic estimate between Propositions 2.8.3 and 2.10.9. Notice that in (2.188), we only use $\Psi(R,\beta)$ for (R,β) within the support of ω, θ. We have $\Psi_{\chi_{\lambda(t)}}(R,\beta) = \Psi(R,\beta)$ for $\lambda(t) = 2^{-13}C_l^{-\alpha}, R \leq S(t)$. Finally, $\chi_1 L_{12}(Z_{\chi_{\lambda(t)}})(0)$ in Proposition 2.10.9 only affects the equation near $R = 0$. Since $\frac{1+R}{R}\bar{\Omega} \in L^2$, using the estimate in Proposition 2.10.9, we get

$$\begin{aligned} &|L_{12}(\bar{Z}_{\chi_{\lambda(0)}})(0)| = |L_{12}(\bar{Z}_{\chi_{\lambda(t)}})(0)| \lesssim \alpha 3^{-1/\alpha}, \\ &|L_{12}(Z_{\chi_{\lambda(t)}})(0)| \lesssim 3^{-1/\alpha}||\frac{1+R}{R}\Omega||_{L^2}, \end{aligned} \tag{2.236}$$

where we have used $\lambda(t) \geq (S(t))^\alpha$ due to (2.230) to obtain the first identity, and used (2.44),(2.231) and $||\frac{1+R}{R}\bar{\Omega}_0||_{L^2} \lesssim \alpha$ to obtain the first inequality. The terms in (2.236) are treated as small terms with amplitude close to 0.

Using the argument in Section 2.9, we can estimate these lower order terms in $\mathcal{H}^3, \mathcal{H}^3(\psi)$ or $\mathcal{C}^1$ norm accordingly and obtain a small constant in the estimate bounded by $C(1 + \alpha^{-\kappa})(3^{-1/\alpha} + C_l S)$, where $\kappa, C > 0$ are some absolute constant.

2.10.3.4 Nonlinear stability

Notice that the domain $\tilde{D}_1$ (2.191) of the dynamic rescaling equation is bounded and is different from $\mathbb{R}_2^+$. We cannot apply directly the estimates in Sections

2.6-2.9 because in these estimates, we linearize the equations around $\bar{\Omega}, \bar{\eta}, \bar{\xi}$ which are defined globally.

We consider the system of $\theta_x, \theta_y, \omega$ obtained from (2.188) and then linearize it around the approximate steady state $\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0, \bar{c}_\omega, \bar{c}_l$ constructed in Section 2.10.3.2 to obtain a system similar to (2.52)-(2.54) for the perturbation (Ω, η, ξ) with $\bar{\Omega}, \bar{\eta}, \bar{\xi}, \frac{3}{1+R}(= \frac{2}{\pi\alpha} L_{12}(\bar{\Omega}))$ replaced by $\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0, \frac{2}{\pi\alpha} L_{12}(\bar{\Omega}_0)$. We also put the lower order terms discussed in Section 2.10.3.3 into the remaining terms $\mathcal{R}_\Omega, \mathcal{R}_\eta, \mathcal{R}_\xi$.

According to Lemma A.0.11, we know that $\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0$ converges to $\bar{\Omega}, \bar{\eta}, \bar{\xi}$ in the $\mathcal{H}^3, \mathcal{H}^3(\psi)$ norm as $\nu \to \infty$ (ν is the cutoff radial in (2.231)). Moreover, we can easily generalize the $\mathcal{H}^3, \mathcal{H}^3(\psi)$ convergence to the higher order convergence. We choose the same weights and the same energy norm as that in Section 2.6-2.9. Then for sufficient large ν, due to these convergence results, under the bootstrap assumption (2.230), we can obtain the following $\mathcal{H}^3, \mathcal{H}^3(\psi)$ estimates similar to that in Corollary 2.7.3

$$\frac{1}{2}\frac{d}{dt} E_3^2(\Omega, \eta, \xi) \leq (-\frac{1}{13} + C\alpha) E_3^2 + \mathcal{R}_3,$$

where $E_3, \mathcal{R}_3$ are defined in (2.115). We have a slightly weaker estimate ($\frac{1}{13} < \frac{1}{12}$) due to the small difference between $(\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0)$ and $(\bar{\Omega}, \bar{\eta}, \bar{\xi})$.

Remark 2.10.14. The choice of ν is independent of $C_l(0)$. We will choose initial data with the size of the support $S(0) \geq 2\nu$. Though $S(0) > \nu$ is large, we choose $C_l(0)$ small enough at the final step and verify (2.230).

Recall the equation (2.121) for the 2D Boussinesq equation in the $\mathcal{C}^1$ estimate of ξ. The damping part in (2.121) is $(-2 - \frac{3}{1+R})\xi$. For the 3D Euler equation, it is replaced by $(-2 - \frac{2}{\pi\alpha} L_{12}(\bar{\Omega}_0))\xi$. For sufficient large ν, using the convergence results, we can obtain estimates similar to (2.123), (2.126), (2.127) with slightly larger constants, e.g., $-2, 3$ are replaced by $-2 + \frac{1}{100}, 3 + \frac{1}{100}$.

There exists a large absolute constant ν_0, such that for $\nu > \nu_0$, ν satisfies the above requirements, and we have

$$|\frac{2}{\pi\alpha} L_{12}(\bar{\Omega} - \bar{\Omega}_0)(0)| \leq \frac{1}{100}. \tag{2.237}$$

To estimate the $\mathcal{H}^3$ norm of $\mathcal{R}_\Omega, \mathcal{R}_\eta$ and $\mathcal{H}^3(\psi), \mathcal{C}^1$ norms of $\mathcal{R}_\xi$, we apply the estimates in Section 2.9 and the argument in Section 2.10.3.3. Therefore, for $\nu > \nu_0$, under the bootstrap assumption, we obtain the following nonlinear estimate for compactly supported perturbations $\Omega(t), \eta(t), \xi(t)$ around $(\bar{\Omega}_0, \bar{\eta}_0, \bar{\xi}_0)$, which is similar to (2.171),

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}E^2(\Omega,\eta,\xi) \leq & -\frac{1}{13}E^2 + C(\alpha^{1/2}E^2 + \alpha^{-3/2}E^3 + \alpha^2 E) \\ & + C(\alpha, C_l(t), S(t))(E^2 + E + E^3), \end{aligned} \tag{2.238}$$

where the energy E is defined in (2.170). The last term is from the estimates of the lower order terms discussed in Section 2.10.3.3, e.g., $\frac{1-r^4}{r^4}\theta_x, \frac{1}{r}C_l\psi$, and $C(\alpha, C_l(t), S(t)) = C(1+\alpha^{-\kappa})(3^{-1/\alpha} + C_l(t)S(t)))$, for some universal constant C. Under the bootstrap assumption 2.230, we further obtain

$$C(\alpha, C_l(t), S(t)) \lesssim (1+\alpha^{-\kappa})3^{-1/\alpha} \lesssim \alpha^3. \tag{2.239}$$

Combining (2.238), (2.239), we obtain that there exist α_3 with $0 < \alpha_3 < \alpha_2$ (α_2 is the constant in (2.201) in Lemma 2.10.6) and an absolute constant $\tilde{K} > 0$, such that if $E(\Omega(0), \eta(0), \xi(0)) < \tilde{K}\alpha^2$, under the bootstrap assumption 2.230, we have

$$E(\Omega(t), \eta(t), \xi(t)) < \tilde{K}\alpha^2. \tag{2.240}$$

Recall $c_\omega, c_l, \bar{c}_\omega, \bar{c}_l$ defined in (2.233), (2.234). Using (2.236), (2.237), $|L_{12}(\Omega)(0)| \lesssim ||\Omega||_{\mathcal{H}^3} \lesssim E \lesssim \alpha^2$, we obtain

$$|c_\omega + \bar{c}_\omega + 1| < \frac{1}{100} + C3^{-1/\alpha} + C\alpha, \quad c_l + \bar{c}_l > \frac{1}{\alpha} + 3 - \frac{1}{100\alpha} - C3^{-1/\alpha}\alpha^{-1} - C.$$

We further choose α_4 with $0 < \alpha_4 < \alpha_3$, such that for $\alpha < \alpha_4$,

$$-\frac{3}{2} < c_\omega + \bar{c}_\omega < -\frac{1}{2}, \quad c_l + \bar{c}_l > \frac{3}{4\alpha}. \tag{2.241}$$

2.10.3.5 Growth of the support

Recall Definition 2.10.2 of $S(\tau)$. Finally, we use the idea in Section 2.2.3.1 to estimate the growth of the support $S(\tau)$ of the solutions $\omega + \bar{\omega}_0, \theta + \bar{\theta}_0$. Denote

$$\widehat{\mathbf{u}}(t) = \mathbf{u}(t) + \bar{\mathbf{u}}(t), \ \widehat{\Psi}(t) = \Psi(t) + \bar{\Psi}_0(t), \ \widehat{c_l}(t) = c_l(t) + \bar{c}_l.$$

Applying (2.24)-(2.25) and (2.27) to $\widehat{\Psi}$, we can rewrite the transport term $\widehat{\mathbf{u}} \cdot \nabla$ in (2.188) as

$$\begin{aligned} \widehat{\mathbf{u}} \cdot \nabla &= (-\partial_y \widehat{\psi} + C_l r^{-1} \widehat{\psi})\partial_x + \partial_x \widehat{\psi} \partial_y \\ &= (\frac{\alpha C_l \rho \cos(\beta)}{r} R\widehat{\Psi} - \alpha R \partial_\beta \widehat{\Psi})\partial_R + (2\widehat{\Psi} + \alpha R \partial_R \widehat{\Psi} - \frac{C_l \rho \sin(\beta)}{r}\widehat{\Psi})\partial_\beta, \end{aligned}$$

where $\rho = R^{1/\alpha}, r = 1 - C_l \rho \sin(\beta)$. The above formula is different from (2.27) due to the extra term $C_l r^{-1} \widehat{\psi} \partial_x$. Notice that $\widehat{c}_l \mathbf{x} \cdot \nabla$ becomes $\alpha \widehat{c}_l R \partial_R$ under the (R, β) coordinates. For a point which is inside the support of $\omega + \bar{\omega}, \theta + \bar{\theta}$ and has coordinates $(R(t)), (\beta(t))$, its trajectory under the flow $(\widehat{c}_l \mathbf{x} + \widehat{\mathbf{u}}) \cdot \nabla$ is governed by

$$\begin{aligned} \frac{d}{dt} R(t) = \alpha \widehat{c}_l R(t) + & \frac{\alpha C_l(t) \rho(t) \cos(\beta(t))}{r(t)} R(t) \widehat{\Psi}(R(t), \beta(t)) \\ & - \alpha R(t) \partial_\beta \widehat{\Psi}(R(t), \beta(t)), \end{aligned} \tag{2.242}$$

where the relation between $\widehat{c}_l(t), C_l(t)$ is given in (2.187).

Lemma 2.10.15. *Under the assumption of Propositions 2.10.9, 2.10.11 and that $\Omega \in \mathcal{H}^3$, for $R \leq S(t)$, we have*

$$\begin{aligned} & |(1 + R^{1/3}) \widehat{\Psi}(R, \beta)| + |(1 + R^{1/3}) \partial_\beta \widehat{\Psi}(R, \beta)| \\ \lesssim & \alpha^{-1} ||\Omega||_{\mathcal{H}^2} + 1 \lesssim \alpha^{-1} E(\Omega(t), \eta(t), \xi(t)) + 1. \end{aligned}$$

Recall the weights φ_i in Definition 2.6.2 for the $\mathcal{H}^3$ norm (2.136). Denote by $\tilde{\mathcal{H}}^3$ the modified $\mathcal{H}^3$ space with radial weight $\frac{(1+R)^2}{R^2}$ in the $\mathcal{H}^3$ space replaced by $\frac{1+R}{R}$. The $\tilde{\mathcal{H}}^3$ version of the elliptic estimates in Proposition 2.10.9 can be obtained by the same argument. Since $\bar{\Omega}_0 + \Omega$ is in $\tilde{\mathcal{H}}^3$ space ($\bar{\Omega}_0$ vanishes linearly near $R = 0$), applying the $\tilde{\mathcal{H}}^3$ elliptic estimate to $\widehat{\Psi} - \widehat{\Psi}_2$ and $L_{12}(Z_\chi)(0)$, where $\widehat{\Psi}_2 = \frac{\sin(2\beta)}{\pi\alpha}(L_{12}(\Omega) + \chi_1 L_{12}(Z_\chi)(0))$, and Lemma A.0.4 to $\widehat{\Psi}_2$, we obtain

$$||\partial_{\beta\beta} \widehat{\Psi}||_{\tilde{\mathcal{H}}^3} \lesssim \alpha^{-1} ||\Omega + \bar{\Omega}_0||_{\tilde{\mathcal{H}}^3} \lesssim \alpha^{-1} ||\Omega||_{\mathcal{H}^3} + 1 \lesssim \alpha^{-1} E(\Omega(t), \eta(t), \xi(t)) + 1.$$

Applying the argument in the proof of Lemma 2.8.10, we establish the decay estimate.

Now we assume that the initial data satisfies $E(\Omega(0), \eta(0), \xi(0)) < \tilde{K} \alpha^2$. Under the bootstrap assumption (2.230), we have *a* priori estimates (2.240), (2.241).

Plugging the bootstrap assumption 2.230, (2.240) and Lemma 2.10.15 in (2.242), we derive

$$\frac{d}{dt} R(t) \leq \alpha \widehat{c}_l R(t) + C\alpha(\alpha^{-1} E + 1) R(t)^{2/3} \leq \alpha \widehat{c}_l R(t) + C\alpha R(t)^{2/3},$$

where we have used $C_l(t)\rho(t) \leq C_l(t) S(t) \lesssim 1, r^{-1} \lesssim 1$. From the formula of $C_l(t)$, we know $\frac{d}{dt} C_l(t) = -\widehat{c}_l(t) C_l(t)$. Multiplying $C_l^\alpha(t)$ on both sides, we get

$$\frac{d}{dt} C_l^\alpha R(t) \leq C\alpha C_l^\alpha R^{2/3}(t) = C\alpha (C_l^\alpha R)^{2/3} C_l(t)^{\alpha/3}.$$

From the *a* priori estimate (2.241) and the formula of C_l in (2.187), we know $C_l^{\alpha}(t) \leq C_l^{\alpha}(0)\exp(-\frac{t}{2})$. Then solving this ODE, we yield

$$\begin{aligned}(C_l^{\alpha}R(t))^{1/3} &\leq (C_l(0)^{\alpha}S(0)^{\alpha})^{1/3} + C\alpha\int_0^{\infty} C_l^{\alpha/3}(0)\exp(-\frac{b}{6})db \\ &\leq C_l(0)^{\alpha/3}(S(0)^{\alpha/3} + C\alpha).\end{aligned}$$

Taking the supremum over $(R(t), \beta(t))$ within the support of Ω, θ, we prove

$$C_l(t)S(t) \leq C(\alpha, S(0))C_l(0). \tag{2.243}$$

2.10.3.6 Finite time blowup

For fixed $\alpha < \alpha_4, \nu > \nu_0$, we choose zero initial perturbation $\Omega(0) = 0, \eta(0) = 0, \xi(0) = 0$. Then the initial data is $(\bar{\Omega}_0, \bar{\theta}_0)$ defined in (2.231) which has compact support with support size $S(0) = 2\nu$. We choose initial rescaling $C_l(0)$ such that $C(\alpha, S(0))C_l(0) < K(\alpha)/2$, where $K(\alpha)$ is defined in (2.230). Using the *a* priori estimates (2.240), (2.241) and (2.243), we know that the bootstrap assumption in (2.230) can be continued. Thus these estimates hold true for all time.

Since $-\frac{3}{2} < c_{\omega} + \bar{c}_{\omega} < -\frac{1}{2}$ ((2.241)) and the solutions ω, θ are close to $\bar{\omega}_0, \bar{\theta}_0$ for all time in the dynamic rescaling equation, using the argument in Subsection 2.9.6 and the BKM blowup criterion in [1], we prove that the solutions remain in the same regularity class as that of the initial data before $T^* < +\infty$ and develop a finite time singularity at T^*, where $T^* = t(\infty) = \int_0^{\infty} C_{\omega}(\tau)d\tau < +\infty$.

Since $\bar{\theta}_0 + \theta(t) \geq 0$ and the support of ω, θ is away from the axis, we can recover $u^{\theta} = \tilde{\theta}^{1/2}/r, \omega^{\theta}$ from θ, ω via (2.182), (2.186). From (2.231) and the discussion below (2.231), $\chi_{\nu}^{1/2}(R) = \tilde{\chi}_1(R/\nu)$ is smooth. Since $\bar{\theta}(x, y) \geq 1$ (A.20), it is even in x, and $\bar{\theta} \in C^{1,\alpha}$, we get $\bar{\theta}^{1/2} \in C^{1,\alpha}$. It follows $\bar{\theta}_0^{1/2} = \chi_{\nu}^{1/2}\bar{\theta}^{1/2} = \tilde{\chi}_1(R/\nu)\bar{\theta}^{1/2} \in C^{1,\alpha}$. Using $u_0^{\theta} = \tilde{\theta}_0^{1/2}/r$, the relation (2.186), and $r \geq \frac{1}{2}$ in $\mathrm{supp}(u_0^{\theta})$, we get $u_0^{\theta} \in C^{1,\alpha}$. Due to the regularity on $u_0^{\theta}, \omega_0^{\theta}$ and the fact that in D_1, they are supported near $(r, z) = (1, 0)$, the solutions have finite energy in D_1.

Chapter 3

FINITE TIME BLOWUP OF THE DE GREGORIO MODEL ON $\mathbb{R}$

In this chapter, we study the singularity formation of the De Gregorio (DG) model (3.1) and the generalized Constantin-Lax-Majda model (3.2) model on $\mathbb{R}$. To establish finite time blowup of the DG model on $\mathbb{R}$ from C_c^∞ data, we will develop the important method of constructing the approximate steady state (approximate self-similar profile) with a small residual error numerically and incorporating the residual error as a small and lower order term in the energy estimate. We refer to Sections 1.4 and 1.4.1 for the background and conjectures of these models.

3.1 Preliminaries for the 1D models

Recall the De Gregorio (DG) model

$$\omega_t + u\omega_x = u_x\omega, \quad u_x = H\omega. \tag{3.1}$$

and the gCLM model discussed in Section 1.4

$$\omega_t + au\omega_x = u_x\omega, \quad u_x = H\omega, \tag{3.2}$$

where H is the Hilbert transform defined below and the domain can be $\mathbb{R}$ or S^1 (periodic case)

$$H_{\mathbb{R}}\omega(x) = \frac{1}{\pi}P.V.\int_{\mathbb{R}} \frac{\omega(y)}{x-y}dy, \quad H_{S^1}\omega(x) = \frac{1}{L}P.V.\int_0^L \omega(y)\cot(\frac{(x-y)\pi}{L})dy. \tag{3.3}$$

Here L is the period of the circle. If $a = 1$, (3.2) is the DG model. If $a = 0$, (3.2) becomes the Constantin-Lax-Majda (CLM) model [26]

$$\omega_t = u_x\omega, \quad u_x = H\omega. \tag{3.4}$$

These models admit the same symmetry groups in Section 2.1.2 as the 3D Euler equations: the Galilean invariance, the rotation symmetry, and the scaling symmetry. In 1D, the rotation symmetry reduces to the reflection symmetry: if $\omega(x,t)$ is a solution to (3.2), then $-\omega(-x,t)$ is also a solution to (3.2). As a result, the odd symmetry $\omega(x,t) = -\omega(-x,t)$ is preserved. The local well-posedness of (3.2) in $C^{k,\alpha}$ with any $k \in \mathbb{Z}_+ \cup \{0\}$ and $\alpha \in (0,1)$ can be established by the particle trajectory method [89]. See also [97]. Solutions to these models satisfy the BKM blow-up criterion (2.2). See also [10] and [73] for the local well-posedness theory and the blow-up criterion.

Connections with the SQG equation In [5], Castro-Córdoba observed that a solution $\omega(y,t)$ of the De Gregorio model (3.1) can be extended to a solution of the surface quasi-geostrophic (SQG) equation

$$\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \quad \mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta \tag{3.5}$$

with infinite energy via the connection $\theta(x,y,t) = x\omega(y,t)$.

Under the radial homogeneity ansatz $\theta(t,r,\beta) = r^{2-2\alpha} g(t,\beta)$, Elgindi-Jeong [46] established a connection between a solution θ to the generalized SQG equation and a solution $g(t,\beta)$ to the gCLM model (3.2) with $a > 1$ up to some lower order term in the velocity operator.

See Section 5.1.3 for more discussion on the connections between these models and incompressible fluids.

3.1.1 Existing results

The gCLM model (3.2) has been studied actively in recent years since it can characterize the competition between advection and vortex stretching in different scenarios and has concrete connection to fluids equations [5, 46]. For $a < 0$, the advection would work together with the vortex stretching to produce a singularity. Indeed, Castro and Córdoba [5] proved the finite time blow-up for $a \leq 0$ based on a Lyapunov functional argument. The case of $a = 0$ reduces to the CLM model and finite time singularity and finite time singularity was established in [26].

Smooth self-similar profiles of (3.2) for sufficiently small a and C^α self-similar profiles for all $a \in \mathbb{R}$ with $|a\alpha|$ sufficiently smal were established in [44]. In joint work with Hou and Huang [19], we established finite time asymptotically self-similar blowup of (3.2) with small $|a|$ for C_c^∞ initial data, and with arbitrary large $|a|$ for C^α initial data with small $|a\alpha|$. We will prove these results in this Chapter 3. Similar results were obtained independently by Elgindi-Ghoul-Masmoudi [49] on the stability of the self-similar solutions constructed in [44].

Regarding the global regularity of the De Gregorio model (3.1), Jia-Stewart-Sverak [73] proved the nonlinear stability of a steady state $A\sin(2x)$ of (3.1) with period π using spectral theories. As a result of [73], there exists global solution to (3.1) from smooth initial data close to $A\sin(2x)$. In [83], Lei-Liu-Ren discovered a novel equation (see (5.9)) and a conserved quantity for initial data ω_0 with a fixed sign and established the global regularity of (3.1) for such initial data. We note that for strictly positive or negative initial data ω_0, the CLM model (3.4) does not blow up.

Lushnikov-Silantyev-Siegel [88] provided strong numerical evidence for singularity formation of (3.2) with various a and obtained a critical value $a_c \approx 0.6890665$. For (3.2) on a circle, they discovered a new type of self-similar blowup solutions of the form $\omega(x,t) = \frac{1}{t_c - t} f(x)$ for $a_c < a \leq 0.95$, which is neither focusing nor expanding. The blowup results we established in [14] are inspired by these discoveries.

There are other 1D models for 3D Euler equations and the SQG equation, see e.g., [28] and Section 1.2, and we refer to [44] for an excellent survey.

3.2 Main results

Let Ω, c_l, c_ω be the solution of the self-similar equation of (3.2) given below

$$(c_l x + aU)\Omega_x = (c_\omega + U_x)\Omega, \quad U_x = H\Omega, \tag{3.6}$$

with $c_\omega < 0$ and a self-similar profile $\Omega \neq 0$ in some weighted H^1 space. Then for some given $T > 0$,

$$\omega(x,t) = \frac{1}{(T-t)|c_\omega|}\Omega\left(\frac{x}{(T-t)^\gamma}\right), \quad \gamma = -\frac{c_l}{c_\omega}, \tag{3.7}$$

is a self-similar singular solution of (3.2).

We define some notions about the self-similar singularities to be used in this chapter.

Definition 3.2.1 (Two types of asymptotically self-similar singularities)**.** We say that a singular solution ω of (3.2) is asymptotically self-similar if there exists a solution of (3.6) (Ω, c_l, c_ω) with $\Omega \neq 0$ in some weighted H^1 space and $c_\omega < 0$ such that the following statement holds true. By rescaling ω dynamically, i.e. $C_\omega(t)\omega(C_l(t)x, t)$ for some time dependent scaling factors $C_\omega(t), C_l(t) > 0$, it converges to Ω as $t \to T^-$ in some weighted L^2 norm, where $T > 0$ is the blowup time. In addition, we say that the asymptotically self-similar singularity is of the *expanding* type if the self-similar solution (3.7) associated to (Ω, c_l, c_ω) satisfies $\gamma < 0$ and of the *focusing* type if $\gamma > 0$. We call γ the scaling exponent.

Remark 3.2.2. We will specify in later Sections the weighted L^2 norm in which the dynamically rescaled function of ω converges to the self-similar profile Ω in the following Theorems. We will also specify in later Sections the stronger weighted H^1 norm that the self-similar profile Ω belongs to, so that the Hilbert transform $U_x = H\Omega$ is well defined and (Ω, c_l, c_ω) is a solution of (3.6). In the case of small $|a|$, we refer to Propositions 3.4.1, 3.4.2 and Section 3.4.3 for more precise statements. Similar statements also apply to other cases.

Our first main result is regarding the finite time singularity of the original De Gregorio model.

Theorem 3.1. *There exist some C_c^∞ initial data on $\mathbb{R}$ such that the solution of* (3.2) *with $a = 1$ develops an expanding and asymptotically self-similar singularity in finite time with scaling exponent $\gamma = -1$ and compactly supported self-similar profile $\Omega \in H^1(\mathbb{R})$.*

Although the initial data and the self-similar profile Ω have compact support, due to the expanding nature of the blowup, the support of the solution will become unbounded at the blowup time.

Remark 3.2.3. Surprisingly, the blowup solution in Theorem 3.1 satisfies the property that $||\omega(x,t)/x||_{L^\infty}$ is uniformly bounded up to the blowup time (that is, $\sup_{t\in[0,T)} ||\omega(x,t)/x||_\infty < +\infty$), which can be seen from the special scaling exponent $\gamma = -1$ and the proof of Theorem 3.1.

Remark 3.2.4. The uniform boundedness of $||\omega(t)/x||_{L^\infty}$ over $[0,T)$ implies that $\omega(x,t)$ cannot blowup at any finite x, which is consistent with the expanding nature of the blowup.

The second result is finite time blowup of (3.2) for small $|a|$ with C_c^∞ initial data.

Theorem 3.2. *There exists a positive constant $\delta > 0$ such that for $|a| < \delta$, the solution of* (3.2) *with some C_c^∞ initial data develops a focusing and asymptotically self-similar singularity in finite time with self-similar profile $\Omega \in H^1(\mathbb{R})$.*

The third result is finite time blowup of (3.2) for all a with C_c^α initial data.

Theorem 3.3. *There exists* $C_1 > 0$ *such that for* $0 < \alpha < \min(1/4,\ C_1/|a|)$, *the solution of* (3.2) *with some* C_c^α *initial data develops a focusing and asymptotically self-similar singularity in finite time with self-similar profile* Ω *satisfying* $|x|^{-1/2}\Omega \in L^2$ *and* $|x|^{1/2}\Omega_x \in L^2$.

The blowup results in Theorem 3.2 and Theorem 3.3 also hold for the De Gregorio model on the circle.

Theorem 3.4. *Consider* (3.2) *on the circle. (1) There exists* $C_1 > 0$ *such that if* $|a| < C_1$, *the solution of* (3.2) *develops a singularity in finite time for some* C_c^∞ *initial data. (2) If* $0 < \alpha < \min(1/4,\ C_1/|a|)$, *then the solution of* (3.2) *develops a finite time singularity for some initial data* $\omega_0 \in C^\alpha$ *with compact support.*

Remark 3.2.5. Due to the fact that (3.2) on a circle does not enjoy the perfect spatial scaling symmetry, we do not establish the result on the asymptotically self-similar singularity in the above theorem.

The initial data ω_0 we constructed for the previous theorems satisfied the property that ω_0 is odd and $\omega_0 \leq 0$ for $x > 0$. Theorem 5 in the arXiv version of [19] shows that for large $a > 0$, the Hölder regularity with a small Hölder exponent α for ω_0 in this class is crucial for the focusing asymptotically self-similar blow-up.

We remark that an important observation made by Elgindi and Jeong in [44] is that the advection term can be substantially weakened by choosing C^α data with small α. We use this property in the proof of Theorem 3.3.

The proof of Theorem 3.4 mainly follows from those of Theorems 3.2, 3.3. There are two additional steps. If the asymptotically self-similar blowup of (3.2) on $\mathbb{R}$ from compactly supported initial data is focusing, we can show that the support of the solution at the blow-up time remains finite. Moreover, we show that in the support of ω, the difference between the velocities generated by the Hilbert transform on the real line and on S^1 can be arbitrarily small by choosing initial data with small support. Thus, the blowup mechanism of (3.2) on $\mathbb{R}$ generalizes to (3.2) on the circle. We have applied a similar argument to study finite time blowup of 3D Euler equations in Section 1.1. For these reasons, we refer to Section 5.2 in [19] for the proof of Theorem 3.4.

Organization of this Chapter The rest of the chapter is organized as follows. In Section 3.3, we outline our general strategy that we use to prove nonlinear stability for various cases. In Section 3.4, we study the De Gregorio model with small $|a|$. In Section 3.5, we construct an approximate self-similar profile with a small residual error numerically for the case of $a = 1$ and apply our method of analysis to prove the finite time self-similar blowup for C_c^∞ initial data. In Section 3.6, we study the case with any $a \in \mathbb{R}$ and prove finite time singularity for any $a \in \mathbb{R}$ on $\mathbb{R}$ for some C^α initial data with compact support. Finally, in Section 3.7, we use a Lyapunov functional argument to prove finite time blowup for all $a < 0$ with smooth initial data.

Notations Since the functions that we consider in this chapter, e.g., ω, u, have odd or even symmetry, we just need to consider $\mathbb{R}^+$. The inner product is defined on $\mathbb{R}^+$, i.e.

$$\langle f, g \rangle \triangleq \int_0^\infty fg dx, \quad ||f||_{L^p} \triangleq \left(\int_0^\infty |f|^p dx \right)^{1/p}.$$

In Section 3.5, we further restrict the inner product and the norm to the interval $[0, L]$, e.g $\langle f, g \rangle = \int_0^L fg dx$, since the support of $\omega, \bar{\omega}$ lies in $[-L, L]$.

We will use the basic notations introduced in Section 2.1.5. In addition, we use $\rightarrow$ to denote strong convergence and $\rightharpoonup$ to denote weak convergence in some norm. The upper bar notation is reserved for the approximate profile, e.g., $\bar{\omega}$. The letters e, f, a_1, a_2, a_3 are reserved for some parameters that we will choose in Section 3.5.

3.3 Ideas in establishing nonlinear stability

We will follow the general framework introduced in Section 1.3. We use both analytic and numerical approaches to construct the approximate self-similar profile in various cases. The analytic approach is based on a class of self-similar profiles of the Constantin-Lax-Majda model (CLM) [26], or equivalent (3.2) with $a = 0$, which are derived in [44]. In [44], the exact self-similar profiles of (3.2) with $a \neq 0$ are also constructed in various cases. We remark that our analysis *d*oes not rely on these profiles of (3.2) with $a \neq 0$.

This analytic approach does not apply to study (3.2) with $a = 1$ and smooth data since it cannot be treated as a small perturbation to the CLM model. Instead, we will use the numerical construction in Section 1.3.6.1 to obtain a piecewise smooth approximate steady state $\bar{\omega}$ with a small residual error.

A very essential part of our analysis is to prove linear and nonlinear stability of the approximate steady state of the dynamic rescaling equation. The dynamic rescaling equation of (3.2) is given below

$$\omega_t + (c_l(t)x + au)\omega_x = (c_\omega(t) + u_x)\omega \ , \tag{3.8}$$

where $c_l(t)$ and $c_\omega(t)$ are time-dependent scaling parameters. See (3.13)-(3.15) and subsection 2.1.4 for more discussion on the dynamic rescaling formulation. Let $(\bar{\omega}, \bar{u}, \bar{c}_l, \bar{c_\omega})$ be an approximate steady state of the dynamic rescaling equation. We define the linearized operator $L(\omega)$

$$\begin{aligned} L(\omega) &= -(\bar{c}_l x + a\bar{u})\omega_x + (\bar{c}_\omega + \bar{u}_x)\omega + (u_x + c_\omega)\bar{\omega} - (au + c_l x)\bar{\omega}_x, \\ u_x &= H\omega, \end{aligned} \tag{3.9}$$

where the scaling factors c_l and c_ω, which depend on ω, will be chosen later. Let ω be the perturbation around the approximate steady state $\bar{\omega}$. The stability around $\bar{\omega}$ is reduced to analyzing the nonlinear stability of the dynamic equation

$$\omega_t = L(\omega) + N(\omega) + F \tag{3.10}$$

around $\omega = 0$. The perturbation ω lies in $\mathcal{H}(\Omega)$, a Hilbert space on a domain Ω. Here $F = (\bar{c}_\omega + \bar{u}_x)\bar{\omega} - (\bar{c}_l x + a\bar{u})\bar{\omega}_x$ is the residual error and $N(\omega) = (c_\omega + u_x)\omega - (c_l x + u)\omega_x$ is the non-linear operator. We remark that $L(\omega)$ and $N(\omega)$ are nonlocal operators since $u_x = H(\omega)$ is nonlocal. Due to the presence of the non-linear operator N and the error term F, it is not sufficient to only show that the spectrum of L has negative real parts.

Our approach is to first perform the weighted L^2 estimate with appropriate weight function φ to establish the linear stability (we drop the terms $N(\omega)$ and F to illustrate the main ideas)

$$\frac{1}{2}\frac{d}{dt}\langle \varphi\omega, \omega\rangle = \langle \varphi\omega, L(\omega)\rangle \le -\lambda \langle \varphi\omega, \omega\rangle, \quad \omega \in \mathcal{H}(\Omega) \tag{3.11}$$

for some $\lambda > 0$ and then extend the above estimates to the weighted H^1 estimates. We can use a bootstrap argument to establish the nonlinear stability of (3.10), provided that F is sufficiently small in the energy norm.

We will focus on the linear stability (3.11) to illustrate the main ideas. The linearized equation around some approximate self-similar profile $(\bar{\omega}, \bar{u}, \bar{c}_l, \bar{c}_\omega)$ reads

$$\omega_t = -(\bar{c}_l x + a\bar{u})\omega_x + (\bar{c}_\omega + \bar{u}_x)\omega + (u_x + c_\omega)\bar{\omega} - (au + c_l x)\bar{\omega}_x \ . \tag{3.12}$$

The linear stability of the profile is mainly due to the damping effect from some local terms and cancellation among several nonlocal terms.

3.3.1 Derivation of the damping term

The damping effect of the equation comes from two parts that depend locally on ω: the stretching term $(\bar{c}_l x + a\bar{u})\omega_x$ and the vortex stretching term $(\bar{c}_\omega + \bar{u}_x)\omega$. An important observation of the approximate profile is that $(\bar{c}_\omega + \bar{u}_x)$ is negative for large $|x|$, thus the vortex stretching term $(\bar{c}_\omega + \bar{u}_x)\omega$ is a damping term for large $|x|$. This is the main source of the damping effect for large $|x|$. However, $(\bar{c}_\omega + \bar{u}_x)\omega$ is not a damping term for x near 0 since $\bar{c}_\omega + \bar{u}_x$ is positive.

The profile we constructed satisfies $\bar{c}_l x + a\bar{u} > 0$ for all $x > 0$ within the support of the solution and $\bar{c}_l x + a\bar{u} \approx Cx$ near $x = 0$ for some $C > 0$, which can be seen in later sections. Thus for x close to 0, we can follow the ideas in Section 1.3.1 to derive the damping term in the weighted L^2 estimate with singular weight from the local terms $\bar{c}_l x + a\bar{u} > 0$.

Another subtlety in our analysis is that we do not use a singular weight to derive a damping term from $(\bar{c}_l x + \bar{u})\omega_x$ in all cases with different a. In the case of $a = 1$, we need to estimate the perturbation near the endpoints $x = 0, \pm L$ carefully. We choose a singular weight φ of order $O((x-L)^{-2})$ near $x = L$ in order to obtain a sharp estimate of u. See more discussions in next Section.

3.3.2 Estimates of the nonlocal terms

The linearized equation (3.12) contains several nonlocal terms that are difficult to control. To estimate the vortex stretching term $(u_x + c_\omega)\bar{\omega}$ in (3.12), we take full advantage of the cancellation between u_x and ω, see Lemmas B.0.3 and B.0.4. To control the last term $-(au + c_l x)\bar{\omega}_x$ in (3.12), we have to choose appropriate functional spaces (X, Y) and develop several functional inequalities $||u||_X \leq C_{XY}||\omega||_Y$ with a sharp constant C_{XY}. For example, we need to make use of the isometry property of the Hilbert transform. We remark that an overestimate of the constant C_{XY} could lead to the failure of the linear stability analysis since the effect of the advection term can be overestimated. To implement the above ideas in obtaining the damping term and estimating the nonlocal terms, we need to design the singular weight very carefully. See (3.18) and (3.62) for some singular weights that are used in our analysis.

Singular weights similar to those in Sections 3.4, 3.6 and in the form of linear combination of $|x|^{-k}$ have also been designed independently in [42, 49] for the stability analysis.

Energy estimates with computer assistance In the case of $a = 1$, we need to use computer-assisted analysis and refer to Section 1.3.7 for its role in our analysis. There is another computer-assisted approach to establishing stability by tracking the spectrum of a given operator and quantifying the spectral gap; see, e.g., [7]. The key difference between this approach and our approach is that we *do not* use computation to quantify the spectral gap of the linearized operator L in (3.9), since L is not a compact operator due to the Hilbert transform $u_x = H\omega$. We refer to [55] for an excellent survey of other computer-assisted proofs in PDE.

3.4 Finite time self-similar blowup for small $|a|$

In this section, we will present the proof of Theorem 3.2. We use this example to illustrate the main ideas in our method of analysis by carrying stability analysis around an approximate self-similar profile with a small residual error by using a dynamic rescaling formulation. In this case, we have an analytic expression for the approximate steady state $\bar{\omega}$.

3.4.1 Dynamic rescaling formulation

We will prove Theorem 3.2 by using a dynamic rescaling formulation. Let $\omega(x,t), u(x,t)$ be the solutions of the original equation (3.2). Following the ideas in Section 2.1.4, we obtain that

$$\tilde{\omega}(x,\tau) = C_\omega(\tau)\omega(C_l(\tau)x, t(\tau)), \quad \tilde{u}(x,\tau) = C_\omega(\tau)C_l(\tau)^{-1}u(C_l(\tau)x, t(\tau)) \tag{3.13}$$

are the solutions to the dynamic rescaling equations

$$\tilde{\omega}_\tau(x,\tau) + (c_l(\tau)x + a\tilde{u})\tilde{\omega}_x(x,\tau) = c_\omega(\tau)\tilde{\omega} + \tilde{u}_x\omega, \quad \tilde{u}_x = H\tilde{\omega}, \tag{3.14}$$

where

$$C_\omega(\tau) = \exp\Big(\int_0^\tau c_\omega(s)d\tau\Big),\ C_l(\tau) = \exp\Big(\int_0^\tau -c_l(s)ds\Big),\ t(\tau) = \int_0^\tau C_\omega(\tau)d\tau\ . \tag{3.15}$$

We have the freedom to choose the time-dependent scaling parameters $c_l(\tau)$ and $c_\omega(\tau)$ according to some normalization conditions. Suppose that $\tilde{\omega}(\tau)$ converges to Ω_∞ in some weighted L^2 norm and $c_l(\tau), c_\omega(\tau)$ converge to $c_{l,\infty}, c_{\omega,\infty}$, respectively, as $\tau \to \infty$, with $(\Omega_\infty, c_{l,\infty}, c_{\omega,\infty})$ being a steady state of (3.14) and $\Omega_\infty \neq 0$ in some weighted H^1 space. Since the steady state equation of (3.14) is the same as the self-similar equation (3.6), we can use (3.7) to obtain a self-similar singular solution of (3.2). We refer to Propositions 3.4.1, 3.4.2 and Section 3.4.3 for more details about the convergence and the regularity of Ω_∞ in the case of small $|a|$. Similar statements apply to other cases.

To simplify our presentation, we still use t to denote the rescaled time in the rest of the chapter.

3.4.2 Nonlinear stability of the approximate self-similar profile

Consider the dynamic rescaling equation

$$\omega_t + (c_l x + au)\omega_x = (c_\omega + u_x)\omega, \quad u_x = H\omega. \tag{3.16}$$

For $a = 0$, we have the following analytic steady state obtained in [44]

$$\bar{\omega} = \frac{-x}{b^2 + x^2}, \quad \bar{u}_x = \frac{b}{b^2 + x^2}, \ c_l = 1, \ c_\omega = -1 \ , \tag{3.17}$$

where $b = 1/2$. The above steady state can also be obtained by using the exact formula of the solution of (3.2) with $a = 0$ given in [26] and analyzing the profile for smooth solution near the blowup time.

We will use the strategy and the general ideas outlined in Section 3.3 to establish the linear and nonlinear stability of the approximate self-similar profile.

Choosing an appropriate singular weight function plays a crucial role in the stability analysis. We will use the following weight functions in our L^2 and H^1 estimates:

$$\varphi = -\frac{1}{\bar{\omega} x^3} - \frac{1}{b^2 \bar{\omega} x} = \frac{(b^2 + x^2)^2}{b^2 x^4}, \tag{3.18}$$

$$\psi = x^2 \varphi = -\frac{1}{\bar{\omega} x} - \frac{x}{b^2 \bar{\omega}} = \frac{(b^2 + x^2)^2}{b^2 x^2} \ , \tag{3.19}$$

where $\bar{\omega}$ is defined in (3.17) and $b = 1/2$. Note that $\varphi \asymp x^{-4} + 1$ and $\psi \asymp x^{-2} + x^2$.

Theorem 3.2 is the consequence of the following two propositions.

Proposition 3.4.1. *Let $\bar{\omega}, \varphi, \psi$ be the function and weights defined in* (3.17), (3.18) *and* (3.19). *There exist some absolute constants $a_0, \mu, c > 0$, such that* *if* $|a| < a_0$ *and the initial data* $\bar\omega + \omega_0$ *of* (3.16) *(*ω_0 *is the initial perturbation) satisfies that* ω_0 *is odd,* $\omega_0 \in H^2$*,* $\omega_{0,x}(0) = 0$ *and* $E(0) < c|a|$*, where*

$$E^2(t) \triangleq \langle \omega^2(t), \varphi \rangle + \mu \langle \omega_x^2(t), \psi \rangle,$$

then we have (a) In the dynamic rescaling equation (3.16)*, the perturbation remains small for all time:* $E(t) < c|a|$ *for all* $t > 0$*; (b) The physical equation* (3.2) *with initial data* $\bar\omega + \omega_0$ *develops a singularity in finite time.*

Proposition 3.4.2. *There exists some universal constant* δ *with* $0 < \delta < a_0$ *such that, if* $|a| < \delta$ *and the initial perturbation* ω_0 *satisfies the assumptions in Proposition 3.4.1, then the solution of the dynamic rescaling equation* (3.16)*,* $(\bar\omega + \omega, \bar c_l + c_l, \bar c_\omega + c_\omega)$*, converges to* $(\Omega_\infty, c_{l,\infty}, c_{\omega,\infty})$ *with* $\Omega_\infty - \bar\omega \in L^2(\varphi), \Omega_{\infty,x} - \bar\omega_x \in L^2(\psi)$*,* $c_{l,\infty} > 0, c_{\omega,\infty} < 0$*. Moreover,* $\bar\omega + \omega$ *converges to* Ω_∞ *in* $L^2(\varphi)$ *exponentially fast and* $(\Omega_\infty, c_{l,\infty}, c_{\omega,\infty})$ *is the steady state of* (3.16)*. In particular, the physical equation* (3.2) *with initial data* $\bar\omega + \omega_0$ *develops a focusing and asymptotically self-similar singularity in finite time with self-similar profile* $\Omega_\infty \in H^1(\mathbb{R})$*.*

In Appendix B, we describe some properties of the Hilbert transform. We will use these properties to estimate the velocity.

Proof of Proposition 3.4.1. For any $|a| \le a_0$, where $a_0 > 0$ is to be determined, we consider the following approximate self-similar profile by perturbing c_l in (3.17) :

$$\begin{aligned} &\bar\omega = \frac{-x}{b^2 + x^2}, \quad \bar u_x = H\bar\omega = \frac{b}{b^2 + x^2}, \quad \bar u = \arctan \frac{x}{b} , \\ &\bar c_l = 1 - a\bar u_x(0) = 1 - 2a, \quad \bar c_\omega = -1 , \end{aligned} \tag{3.20}$$

where $b = 1/2$. We consider the equation for perturbation ω, u around the above approximate self-similar profile

$$\omega_t + (\bar c_l x + a\bar u)\omega_x = (\bar c_\omega + \bar u_x)\omega + (u_x + c_\omega)\bar\omega - (au + c_l x)\bar\omega_x + N(\omega) + F(\bar\omega) , \tag{3.21}$$

where N and F are the nonlinear terms and the error, respectively, and are defined below:

$$N(\omega) = (c_\omega + u_x)\omega - (c_l x + au)\omega_x, \quad F(\bar\omega) = -a(\bar u - \bar u_x(0)x)\bar\omega_x . \tag{3.22}$$

We choose the following normalization condition for c_l and c_ω

$$c_l(t) = -au_x(t,0), \quad c_\omega(t) = -u_x(t,0). \tag{3.23}$$

Note that $\bar\omega$ is smooth and odd, the initial data $\omega_0+\bar\omega \in H^2$ and the evolution of (3.16) preserves the odd symmetry of the solution. Standard local well-posedness results imply that $\omega(t,\cdot)+\bar\omega$ remains in H^2 locally in time, so does $\omega(t,\cdot)$. Using the above normalization condition, the original equation (3.16) and the fact that ω, u are odd, we can derive the evolution equation for $\omega_x(t,0)$ as follows

$$\begin{aligned}\frac{d}{dt}(\omega_x(t,0)+\bar\omega_x(0)) &= [(c_\omega+\bar c_\omega+u_x+\bar u_x)(\bar\omega+\omega)]_x\Big|_{x=0} \\ &\quad - [(\bar c_l x + a\bar u + c_l x + au)(\omega_x+\bar\omega_x)]_x\Big|_{x=0} \\ &= [(c_\omega+\bar c_\omega+u_x+\bar u_x)-(\bar c_l+c_l+a\bar u_x+au_x)](\bar\omega_x+\omega_x)\Big|_{x=0} \\ &= [(\bar c_\omega+\bar u_x)-(\bar c_l+a\bar u_x)](\bar\omega_x+\omega_x)\Big|_{x=0} = 0,\end{aligned}$$

where we have used (3.20) and $\bar u_x(0)=2$ to obtain the last equality. It follows

$$\frac{d}{dt}\omega_x(t,0) = \frac{d}{dt}(\omega_x(t,0)+\bar\omega_x(0)) = 0, \tag{3.24}$$

which implies $\omega_x(t,0) \equiv \omega_{0,x}(0)$.

In the following discussion, our goal is to construct an energy functional $E^2(\omega) \triangleq \langle \omega^2, \varphi\rangle + \mu\langle \omega_x^2, \psi\rangle$ for some universal constant μ and show that E satisfies an ODE inequality

$$\frac{1}{2}\frac{d}{dt}E^2(\omega) \le CE^3 - (1/4 - C|a|)E^2 + C|a|E,$$

for some universal constant C. Then we will use a bootstrap argument to establish nonlinear stability.

Linear Stability We use φ defined in (3.18) for the following weighted L^2 estimates. Note that φ is singular and is of order $O(x^{-4})$ near $x=0$. For an initial perturbation $\omega_0 \in H^2$ that is odd and satisfies $\omega_{0,x}(0)=0$, $\omega(t,\cdot)$ preserves these properties locally in time (see (3.24)). We will choose $\omega_0(x)$ that has $O(|x|^{-1})$ decay as $|x| \to \infty$ (same decay as $\bar\omega$). Hence, $\langle \omega^2, \varphi\rangle$ is finite. We perform the weighted L^2 estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle \omega^2,\varphi\rangle &= \langle -(\bar c_l x + a\bar u)\omega_x + (\bar c_\omega + \bar u_x)\omega, \omega\varphi\rangle + \langle (u_x+c_\omega)\bar\omega, \omega\varphi\rangle \\ &\quad - \langle (au+c_l x)\bar\omega_x, \omega\varphi\rangle + \langle N(\omega), \omega\varphi\rangle + \langle F(\bar\omega), \omega\varphi\rangle \\ &\triangleq I + II + III + N_1 + F_1.\end{aligned} \tag{3.25}$$

For I, we use integration by parts to obtain

$$I = \left\langle \frac{1}{2\varphi}((\bar{c}_l x + a\bar{u})\varphi)_x + (\bar{c}_\omega + \bar{u}_x), \omega^2\varphi \right\rangle.$$

Recall $\bar{c}_l = 1 - 2a$ (3.20). Using the explicit formula of profile (3.20) and weight (3.18), we can evaluate the terms in I that do not involve a as follows

$$\frac{1}{2\varphi}(x\varphi)_x + (\bar{c}_\omega + \bar{u}_x) = \frac{b^2x^4}{2(b^2+x^2)^2}\left(\frac{(b^2+x^2)^2}{b^2x^3}\right)_x + \frac{b}{b^2+x^2} - 1$$
$$= \frac{b^2x^4}{2(b^2+x^2)^2}\left(4\frac{x(b^2+x^2)}{b^2x^3} - 3\frac{(b^2+x^2)^2}{b^2x^4}\right) + \frac{b}{b^2+x^2} - 1 = \frac{2x^2+b}{x^2+b^2} - \frac{5}{2} = -\frac{1}{2}, \tag{3.26}$$

where we have used $b = 1/2$. From (3.20) and (3.18), we have

$$\begin{aligned} \left|\left|\frac{1}{2\varphi}[(\bar{c}_l x - x + a\bar{u})\varphi]_x\right|\right|_{L^\infty} =& |a|\left|\left|\frac{1}{2\varphi}((-2x+\bar{u})\varphi)_x\right|\right|_{L^\infty} \\ \le& |a|\left|\left|\frac{-2+\bar{u}_x}{2} + \frac{-2x+\bar{u}}{x}\frac{x\varphi_x}{2\varphi}\right|\right|_{L^\infty} \\ \le& |a|(1+||\bar{u}_x||_\infty)\left(1+\left|\left|\frac{x\varphi_x}{\varphi}\right|\right|_\infty\right) \lesssim |a|. \end{aligned} \tag{3.27}$$

Hence, we can estimate I as follows

$$I = \left\langle \frac{1}{2\varphi}((\bar{c}_l x + a\bar{u})\varphi)_x + (\bar{c}_\omega + \bar{u}_x), \omega^2\varphi \right\rangle \le -\left(\frac{1}{2} - C|a|\right)\langle \omega^2, \varphi\rangle, \tag{3.28}$$

for some absolute constant C. Denote $\tilde{u} \triangleq u(x) - u_x(0)x$. (3.23) implies that

$$c_l x + au = a\tilde{u}, \quad \tilde{u}_x = u_x + c_\omega.$$

Using the definition of II in (3.25),(B.5) and (B.6), we obtain

$$II = -\left\langle (u_x - u_x(0))\omega, \frac{1}{x^3} + \frac{1}{b^2x}\right\rangle = -\frac{\pi}{2b^2}u_x^2(0) \le 0. \tag{3.29}$$

For III, we use the Cauchy–Schwarz inequality to get

$$III = -a\langle \tilde{u}\omega, \bar{\omega}_x\varphi\rangle \le |a|\left|\left|\tilde{u}\sqrt{x^{-6}+x^{-4}}\right|\right|_2\left|\left|\bar{\omega}_x\left(x^{-6}+x^{-4}\right)^{-1/2}\varphi\omega\right|\right|_2. \tag{3.30}$$

For $\tilde{u}$, we use the Hardy inequality (B.8) to obtain

$$\langle \tilde{u}^2, x^{-6}+x^{-4}\rangle \lesssim \langle \tilde{u}_x^2, x^{-4}+x^{-2}\rangle \lesssim \langle \omega^2, x^{-4}+x^{-2}\rangle \lesssim \langle \omega^2, \varphi\rangle. \tag{3.31}$$

Note that (3.20) and (3.18) implies

$$\left|\bar{\omega}_x\left(x^{-6}+x^{-4}\right)^{-1/2}\varphi\right| = \left|\frac{-b^2+x^2}{(b^2+x^2)^2}\cdot\frac{x^3}{(x^2+1)^{1/2}}\cdot\frac{b^2+x^2}{bx^2}\varphi^{1/2}\right| \lesssim \varphi^{1/2}.$$

We get

$$III \le C|a|\langle \omega^2, \varphi\rangle. \tag{3.32}$$

Combining the estimates (3.28), (3.29) and (3.32), we obtain

$$\frac{1}{2}\frac{d}{dt}\langle \omega^2, \varphi\rangle \le -(1/2 - C|a|)\langle \omega^2, \varphi\rangle + N_1 + F_1. \tag{3.33}$$

Weighted H^1 estimate The weighted H^1 estimate is similar to the L^2 estimate. We use the weight ψ defined in (3.19) and perform the weighted H^1 estimates

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi\rangle &= \langle -((\bar{c}_l x + a\bar{u})\omega_x)_x + ((\bar{c}_\omega + \bar{u}_x)\omega)_x, \omega_x \psi\rangle \\ &\quad + \langle ((u_x + c_\omega)\bar{\omega})_x, \omega_x\psi\rangle - \langle ((au + c_l x)\bar{\omega}_x)_x, \omega_x \psi\rangle \\ &\quad + \langle N(\omega)_x, \omega_x\psi\rangle + \langle F(\omega)_x, \omega_x \psi\rangle \\ &\triangleq I + II + III + N_2 + F_2 \ .\end{aligned} \tag{3.34}$$

For I, we obtain by using integration by parts that

$$\begin{aligned} I &= \langle -(\bar{c}_l x + a\bar{u})\omega_{xx} + (-\bar{c}_l - a\bar{u}_x + \bar{c}_\omega + \bar{u}_x)\omega_x + \bar{u}_{xx}\omega, \omega_x \psi\rangle \\ &= \Big\langle \frac{1}{2\psi}((\bar{c}_l x + a\bar{u})\psi)_x + (\bar{c}_\omega - \bar{c}_l + (1-a)\bar{u}_x), \omega_x^2 \psi \Big\rangle - \Big\langle \frac{1}{2}(\bar{u}_{xx}\psi)_x, \omega^2\Big\rangle. \end{aligned}$$

Similar to (3.26), we use formula (3.20), (3.19) to evaluate the terms that do not involve a.

$$\begin{aligned}\frac{1}{2\psi}(x\psi)_x + (\bar{c}_\omega - 1 + \bar{u}_x) &= \frac{b^2 x^2}{2(b^2+x^2)^2}\left(\frac{(b^2+x^2)^2}{b^2 x}\right)_x - 2 + \frac{b}{b^2+x^2} = -\frac{1}{2}, \\ (\bar{u}_{xx}\psi)_x &= \left(-\frac{2bx}{(b^2+x^2)^2}\cdot\frac{(b^2+x^2)^2}{b^2x^2}\right)_x = \frac{2}{bx^2} > 0 \ .\end{aligned}$$

Similar to (3.27), we use (3.20) and (3.19) to show that the remaining terms in I are small. We get

$$\begin{aligned} &\Big|\Big|\frac{1}{2\psi}((\bar{c}_l x - x + a\bar{u})\psi)_x - (\bar{c}_l - 1) - a\bar{u}_x\Big|\Big|_{L^\infty} \\ =&|a|\Big|\Big|\frac{1}{2\psi}((-2x+\bar{u})\psi)_x + 2 - \bar{u}_x\Big|\Big|_{l^\infty} \lesssim |a|,\end{aligned}$$

where we have used $\bar{c}_l - 1 = -2a$. Therefore, we can estimate I as follows

$$I \leq -(\frac{1}{2} - C|a|)\langle \omega_x^2, \psi\rangle, \tag{3.35}$$

where C is some absolute constant. For II, we have

$$\begin{aligned} II &= \langle ((u_x + c_\omega)\bar{\omega})_x, \omega_x\psi\rangle = \langle u_{xx}\bar{\omega}, \omega_x\psi\rangle + \langle (u_x + c_\omega)\bar{\omega}_x, \omega_x\psi\rangle \\ &= -\Big\langle u_{xx}\omega_x, \frac{1}{x} + \frac{x}{b^2}\Big\rangle - \langle \tilde{u}_x, \omega_x\bar{\omega}_x\psi\rangle \triangleq II_1 + II_2 \ ,\end{aligned} \tag{3.36}$$

where $\tilde{u} = u - u_x(0)x, \tilde{u}_x = u_x - u_x(0)$. Note that

$$u_{xx} = H\omega_x, \quad \omega_x(0) = u_{xx}(0) = 0.$$

Applying (B.5) with (u_x, ω) replaced by (u_{xx}, ω_x) and (B.7), we obtain

$$\left\langle u_{xx}\omega_x, \frac{1}{x} \right\rangle = 0, \quad \langle u_{xx}\omega_x, x\rangle = 0. \tag{3.37}$$

It follows that

$$II_1 = -\left\langle u_{xx}\omega_x, \frac{1}{x} \right\rangle - \frac{1}{b^2}\langle u_{xx}\omega_x, x\rangle = 0. \tag{3.38}$$

For II_2 in (3.36), we use an argument similar to (3.30) to obtain

$$|II_2| \lesssim \langle \tilde{u}_x^2, x^{-4} + x^{-2}\rangle^{1/2} \cdot \langle (x^{-4} + x^{-2})^{-1}(\bar{\omega}_x\psi)^2, \omega_x^2\rangle^{1/2}.$$

(3.31) shows that this first term in the RHS is bounded by $\langle \omega^2, \varphi\rangle^{1/2}$. For the second term, we use the definition (3.20) and (3.19) to obtain

$$\left|(x^{-4} + x^{-2})^{-1}(\bar{\omega}_x\psi)^2\right| = \left|\frac{x^4}{x^2+1}\left(\frac{-b^2+x^2}{(b^2+x^2)^2}\right)^2 \frac{(b^2+x^2)^2}{b^2x^2}\right|\psi \lesssim \psi.$$

Hence, we have

$$II_2 \lesssim \langle \omega^2, \varphi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2}. \tag{3.39}$$

For III in (3.34), we note that $c_l x + au = a(u - u_x(0)x)$. Similarly, we have

$$|III| \lesssim |a|\langle \omega^2, \varphi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2}. \tag{3.40}$$

In summary, combining (3.35),(3.36), (3.38), (3.39) and (3.40), we prove that

$$\frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi\rangle \le C\langle \omega^2, \varphi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2} - (\frac{1}{2} - C|a|)\langle \omega_x^2, \psi\rangle + N_2 + F_2, \tag{3.41}$$

where C is some absolute constant.

Estimate of nonlinear and error terms We use the following estimate to control $\|u_x\|_\infty$

$$||u_x||_\infty \le C||u_x||_2^{1/2}||u_{xx}||_2^{1/2} = C||\omega||_2^{1/2}||\omega_x||_2^{1/2} \le C\langle \omega^2, \varphi\rangle^{1/4}\langle \omega_x^2, \psi\rangle^{1/4}.$$

Recall the definition of $N(\omega), F(\bar{\omega})$ in (3.22). For the nonlinear part N_1, N_2, we have

$$\begin{aligned} N_1 &= \langle N(\omega), \omega\varphi\rangle \lesssim (|a|+1)||u_x||_\infty\langle \omega^2, \varphi\rangle \lesssim ||u_x||_\infty\langle \omega^2, \varphi\rangle \ , \\ N_2 &= \langle N(\omega)_x, \omega_x\psi\rangle \lesssim (|a|+1)||u_x||_\infty\langle \omega_x^2, \psi\rangle \lesssim ||u_x||_\infty\langle \omega_x^2, \psi\rangle, \end{aligned} \tag{3.42}$$

where we use that $|a| < 1$ since we only consider small $|a|$ in Theorem 3.2. We note that $F(\bar{\omega})$ (3.22) satisfies $F(\bar{\omega}) = O(x^3)$ near 0 and $F(\bar{\omega}) = O(x^{-1})$ for large x. From (3.18) and (3.19), we have $F(\bar{\omega}) \in L^2(\varphi)$ and $(F(\bar{\omega}))_x \in L^2(\psi)$. Then for the error terms F_1, F_2, we can use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} |F_1| &= |\langle F(\bar{\omega}), \omega\varphi\rangle| \le \langle F^2(\bar{\omega}), \varphi\rangle^{1/2}\langle \omega^2, \varphi\rangle^{1/2} \lesssim |a|\langle \omega^2, \varphi\rangle^{1/2} \ , \\ |F_2| &= |\langle (F(\bar{\omega}))_x, \omega_x\psi\rangle| \le \langle (F(\bar{\omega}))_x^2, \psi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2} \lesssim |a|\langle \omega_x^2, \psi\rangle^{1/2}. \end{aligned} \tag{3.43}$$

Nonlinear Stability Let $\mu < 1$ be some positive parameter to be determined. We consider the following energy norm

$$E^2(t) \triangleq \langle \omega^2, \varphi \rangle + \mu \langle \omega_x^2, \psi \rangle.$$

Using the previous estimates on u_x and the Cauchy–Schwarz inequality, we have

$$\langle \omega^2, \varphi \rangle^{1/2} \langle \omega_x^2, \psi \rangle^{1/2} \leq \mu^{-1/2} E^2, \quad ||u_x||_\infty \leq C \langle \omega^2, \varphi \rangle^{1/4} \langle \omega_x^2, \psi \rangle^{1/4} \leq C \mu^{-1/4} E.$$

Combining (3.33), (3.41), (3.42), (3.43) and the above estimate, we derive

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} E^2(t) &\leq - \left(\frac{1}{2} - C|a| \right) E^2 + C\mu \langle \omega^2, \varphi \rangle^{1/2} \langle \omega_x^2, \psi \rangle^{1/2} + C|a|E + C||u_x||_\infty E^2 \\ &\leq - \left(\frac{1}{2} - C|a| - C\sqrt{\mu} \right) E^2 + C|a|E + C\mu^{-1/4} E^3 \ , \end{aligned}$$

where C is some absolute constant. Now we choose μ such that $C\sqrt{\mu} < 1/4$. Note that μ is also a universal constant. It follows that

$$\frac{1}{2} \frac{d}{dt} E^2(t) \leq - \left(\frac{1}{4} - C_1|a| \right) E^2 + C_1|a|E + C_1 E^3 \ , \tag{3.44}$$

where C_1 is a universal constant. For $c_\omega(t)$ and $c_l(t)$, they satisfy the following estimate

$$|c_\omega(t)| = |u_x(t,0)| \leq C_2 E, \quad |c_l(t)| = |a u_x(0)| \leq C_2 E \ ,$$

for some absolute constant C_2. Hence there exist absolute constants $a_0, c > 0$ with $C_1 a_0 < 1/8$, such that for $|a| < a_0$, if $E(0) < c|a|$, using a bootstrap argument, we obtain

$$E(t) < c|a|, \quad |c_\omega(t)|, |c_l(t)| \leq C_2 E(t) < C_2 c|a|, \tag{3.45}$$

for all $t > 0$. We can further require

$$a_0 < \min\left(\frac{1}{8C_1}, \frac{1}{2C_2 c}\right) \ ,$$

so that we get $|c_\omega(t)|, |c_l(t)| < C_2 c|a| < \frac{1}{2}$, which implies

$$\bar{c}_\omega + c_\omega(t) < -1/2, \quad c_l(t) + \bar{c}_l > 1/2. \tag{3.46}$$

As a result, we can choose small initial perturbation ω_0 which modifies $\bar{\omega}$ in the far field so that we have an initial data $\bar{\omega} + \omega_0$ with compact support. We can also require that $\omega_{0,x}(0) = 0$ and $E(0) < c|a|$. Then the bootstrap result and $\bar{c}_\omega + c_\omega(\tau) < -1/2 < 0$ imply the finite time blowup. We conclude the proof of Proposition 3.4.1. $\square$

Based on the a-priori estimate, we can further obtain the convergence result.

3.4.3 Convergence to the self-similar solution

Proof of Proposition 3.4.2. An important observation is that the approximate self-similar profile is time-independent. Therefore, we take the time derivative in (3.21) to obtain

$$\begin{aligned}\omega_{tt} + (\bar{c}_l x + a\bar{u})\omega_{tx} = (\bar{c}_\omega + \bar{u}_x)\omega_t + (u_{x,t} + c_{\omega,t})\bar{\omega} \\ - (au_t + c_{l,t}x)\bar{\omega}_x + N(\omega)_t \ ,\end{aligned} \tag{3.47}$$

where the error term $F(\bar{\omega})$ vanishes since it depends on the approximate self-similar profile only. Note that the normalization condition also implies

$$\frac{d}{dt}\omega_x(t,0) = 0.$$

Exponential convergence Note that the linearized operator in (3.47) is exactly the same as that in the weighted L^2 estimate (3.21). Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}\langle \omega_t^2, \varphi\rangle \leq -(1/2 - C|a|)\langle \omega_t^2, \varphi\rangle + \langle N(\omega)_t, \omega_t\varphi\rangle. \tag{3.48}$$

The nonlinear part reads

$$\begin{aligned}N(\omega)_t &= (c_{\omega,t} + u_{x,t})\omega + (c_\omega + u_x)\omega_t - (c_{l,t}x + au_t)\omega_x - (c_l x + au)\omega_{x,t} \\ &\triangleq I + II + III + IV \ ,\end{aligned} \tag{3.49}$$

where $c_{\omega,t} = -u_{x,t}(0), c_{l,t} = -au_{x,t}(0)$ according to the (3.23). We are going to show that

$$|\langle N(\omega)_t, \omega_t\varphi\rangle| \lesssim E(t)\langle \omega_t^2, \varphi\rangle. \tag{3.50}$$

From previous estimates, we can control $||\omega||_\infty$, $||u_x||_\infty, ||\frac{u}{x}||_{L^\infty}, |c_\omega|, |c_l|$ by $E(t)$. Using (B.8) with $p = 2, 4$, $x^{-4} + x^{-2} \lesssim \varphi$ (see (3.18)) and the L^2 isometry of the Hilbert transform, we have

$$\begin{aligned}||(u_x - u_x(0))(x^{-4} + x^{-2})^{1/2}||_2 \lesssim ||\omega\varphi^{1/2}||_{L^2} \lesssim E(t), \\ ||(u_{x,t} - u_{x,t}(0))(x^{-4} + x^{-2})^{1/2})||_2 \lesssim ||\omega_t\varphi^{1/2}||_{L^2}.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\Big|\frac{u_t(x)}{x}\Big| &= \frac{1}{\pi}\Big|\int_{y>0}\log\Big|\frac{x+y}{x-y}\Big|\frac{1}{x}\omega_t(y)dy\Big| \\ &\lesssim \langle \omega_t^2, \varphi\rangle^{1/2}\Big\langle \Big(\log\Big|\frac{x+y}{x-y}\Big|\frac{1}{x}\Big)^2, \varphi^{-1}\Big\rangle^{1/2} \lesssim \langle \omega_t^2, \varphi\rangle^{1/2}.\end{aligned} \tag{3.51}$$

Taking $x=0$ in the above estimate, we also yield the bound for $|u_{x,t}(0)|$ and thus that for $|c_{\omega,t}|, |c_{l,t}|$. The tail behavior of φ (3.18) satisfies

$$\varphi = \frac{b^2}{x^4} + \frac{2}{x^2} + \frac{1}{b^2} = O(x^{-2}) + b^{-2}, \quad \varphi - b^{-2} = \frac{b^2}{x^4} + \frac{2}{x^2} < \varphi.$$

Recall $\tilde{u} = u - u_x(0)x$ and (3.23). We can estimate different parts of $N(\omega)_t$ as follows

$$\begin{aligned}
|\langle I, \omega_t \varphi\rangle| &\leq |\langle (c_{\omega,t} + u_{x,t})\omega, \omega_t(\varphi - b^{-2})\rangle| + b^{-2}|\langle (c_{\omega,t} + u_{x,t})\omega, \omega_t\rangle| \\
&\lesssim \langle \tilde{u}_{x,t}^2, (x^{-4} + x^{-2})\rangle^{1/2} ||\omega||_\infty \langle \omega_t^2, \varphi\rangle^{1/2} + b^{-2}|c_{\omega,t}| ||\omega||_2 ||\omega_t \varphi^{1/2}||_2 \\
&\quad + b^{-2}||u_{x,t}||_2 ||\omega||_\infty ||\omega_t||_2 \lesssim E(t)\langle \omega_t^2, \varphi\rangle \ , \\
\langle II + IV, \omega_t \varphi\rangle &= \left\langle c_\omega + u_x + \frac{((c_l x + au)\varphi)_x}{2\varphi}, \omega_t^2 \varphi\rangle \right\rangle \\
&\lesssim ||u_x||_\infty \langle \omega_t^2, \varphi\rangle \lesssim E(t)\langle \omega_t^2, \varphi\rangle \ , \\
\langle III, \omega_t \varphi\rangle &= \left\langle c_{l,t} + a\frac{u_t}{x}, \omega_x x \varphi^{1/2} \omega_t \varphi^{1/2}\right\rangle \\
&\lesssim \left|\left| c_{l,t} + a\frac{u_t}{x}\right|\right|_\infty ||\omega_x \varphi^{1/2} x||_2 ||\omega_t \varphi^{1/2}||_2 \lesssim E(t)\langle \omega_t^2, \varphi\rangle,
\end{aligned}$$

where we have used $|x\varphi_x/\varphi| \lesssim 1$ to estimate $II + IV$ and $||\omega_x \varphi^{1/2} x||_2 = ||\omega_x \psi^{1/2}||_2 \lesssim E(t)$ to obtain the last inequality. In summary, we have proved (3.50). Consequently, by substituting the above estimates and (3.45) into (3.48), we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle \omega_t^2, \varphi\rangle &\leq -(1/2 - C|a|)\langle \omega_t^2, \varphi\rangle + C_3 E(t)\langle \omega_t^2, \varphi\rangle \\
&\leq -(1/2 - C|a|)\langle \omega_t^2, \varphi\rangle + C_3 c|a|\langle \omega_t^2, \varphi\rangle \\
&= -(1/2 - C|a| - C_3 c|a|)\langle \omega_t^2, \varphi\rangle
\end{aligned}$$

for some universal constant C_3. Thus, there exists $0 < \delta < a_0$ such that

$$C\delta + C_3 c\delta < \frac{1}{4}.$$

Hence, if $|a| < \delta$, we obtain

$$\frac{d}{dt}\langle \omega_t^2, \varphi\rangle \leq -(1/2 - C|a| - C_3 c|a|)\langle \omega_t^2, \varphi\rangle \leq -\frac{1}{4}\langle \omega_t^2, \varphi\rangle. \tag{3.52}$$

It follows that $\langle \omega_t^2, \varphi\rangle$ converges to 0 exponentially fast as $t \to \infty$ and that $\omega(t)$ is a Cauchy sequence in $L^2(\varphi)$ as $t \to \infty$. It admits a limit ω_∞ and we have

$$||(\omega(t) - \omega_\infty)\varphi^{1/2}||_2 \leq e^{-t/4}. \tag{3.53}$$

According to the a-priori estimate $\langle \omega_x(t,\cdot)^2, \psi\rangle < E^2(t) < (ca)^2$, there is a subsequence $\omega(t_n)$ of $\omega(t)$, such that $\omega_x(t_n)\psi^{1/2}$ converges weakly in L^2, and the limit must be $\omega_{\infty,x}\psi^{1/2}$. Therefore, we conclude that $\omega_\infty \in L^2(\varphi)$ and $\omega_{\infty,x} \in L^2(\psi)$. Using these convergence results, we obtain

$$c_l(t) = -au_x(t,0) \to -aH\omega_\infty(0), \quad c_\omega = -u_x(t,0) \to -H\omega_\infty(0), \qquad (3.54)$$

as $t \to \infty$. Using the formulas of $\bar\omega$ in (3.17), φ, ψ in (3.18) and the above result, we obtain $\omega_\infty, \bar\omega \in H^1(\mathbb{R})$, which implies $\omega_\infty + \bar\omega \in H^1(\mathbb{R})$.

Convergence to self-similar solution Finally, we verify that $\omega_\infty + \bar\omega$ with some $c_{l,\infty}, c_{\omega,\infty}$ is a steady state of (3.16).

We use $\Omega, U, \kappa_l, \kappa_\omega$ to denote the original solution of (3.16)

$$\Omega = \omega + \bar\omega, \quad U = u + \bar u, \quad \kappa_l = c_l + \bar c_l, \quad \kappa_\omega = c_\omega + \bar c_\omega.$$

In particular, we define $(\Omega_\infty, U_\infty)$ by

$$\Omega_\infty = \omega_\infty + \bar\omega, \quad U_{\infty,x} = H(\Omega_\infty).$$

Notice that

$$\omega_t = \Omega_t = (\kappa_\omega + U_x)\Omega - (\kappa_l x + aU)\Omega_x \triangleq K(t).$$

Due to the exponential convergence (3.52), we have

$$\langle K(t)^2, \varphi\rangle \to 0 \quad \text{as } t \to +\infty. \qquad (3.55)$$

Suppose that $\{\omega(t_n,\cdot)\}_{n\geq 1}$ is a subsequence of $\{\omega(t,\cdot)\}_{t\geq 0}$ such that as $n \to \infty$, $t_n \to \infty$ and $\omega_x(t_n)\psi^{1/2}$ converges weakly to $\omega_{\infty,x}\psi^{1/2}$ in L^2. From (3.53), we obtain that $\{\omega(t_n)\}_{n\geq 1}$ converges strongly to ω_∞ in $L^2(\varphi)$. Therefore, $\Omega(t_n) - \Omega_\infty$ converges strongly to 0 in $L^2(\varphi)$ and $\Omega_x(t_n)\psi^{1/2} - \Omega_{\infty,x}\psi^{1/2}$ converges weakly to 0 in L^2. From (3.54), we obtain that $\kappa_l(t_n), \kappa_\omega(t_n)$ converge to some scaling factors $c_{l,\infty}, c_{\omega,\infty}$, respectively.

Using these convergence results, the relation $\psi = x^2\varphi$ between two weights and the standard convergence argument, we obtain that $K(t_n)\varphi^{1/2} - K(\infty)\varphi^{1/2}$ converges weakly to 0 in L^2, i.e.

$$\begin{aligned}&((\kappa_\omega + U_x)\Omega - (\kappa_l x + aU)\,\Omega_x)\varphi^{1/2}\\ &-((c_{\omega,\infty} + U_{\infty,x})\Omega_\infty - (c_{l,\infty}x + aU_\infty)\Omega_{\infty,x})\varphi^{1/2} \quad \rightharpoonup \quad 0.\end{aligned} \qquad (3.56)$$

We refer to the arXiv version of [19] for the detailed proof of this result.

Note that (3.55) shows that $K(t_n) \to 0$ in $L^2(\varphi)$. We get

$$(c_{\omega,\infty} + U_{\infty,x})\Omega_\infty - (c_{l,\infty}x + aU_\infty)\Omega_{\infty,x} = 0$$

in $L^2(\varphi)$. The *a-priori* estimate (3.46) and the convergence result imply that $c_{l,\infty} > 1/2 > 0,\ c_{\omega,\infty} < -1/2 < 0$. Therefore, the solution $\Omega(t)$ in the dynamic rescaling equation converges to Ω_∞ in $L^2(\varphi)$ and $(\Omega_\infty, c_{l,\infty}, c_{\omega,\infty})$ is a steady state of (3.16), or equivalently, a solution of the self-similar equation (3.6). Using the rescaling relations (3.13) and (3.15), we obtain that the singularity is asymptotically self-similar. Since $\gamma = -\frac{c_{l,\infty}}{c_{\omega,\infty}} > 0$, the asymptotically self-similar singularity is focusing. The regularity $\Omega_\infty \in H^1(\mathbb{R})$ follows from the result below (3.54). □

Remark 3.4.3. An argument similar to that of proving convergence to the self-similar solutions by time-differentiation given above has been developed independently in [42]. There is a difference between two approaches in the sense that an artificial time variable was introduced in [42], while we use the dynamic rescaling time variable.

3.5 Finite time blowup for $a = 1$ with C_c^∞ initial data

In this section, we will prove Theorem 3.1 regarding the finite time self-similar blowup of the original De Gregorio model with $a = 1$. Compared to the De Gregorio model with small $|a|$ analyzed in the previous Section, the case of $a = 1$ is much more challenging since we do not have a small parameter a in the advection term $u\omega_x$. The smallness of $|a|$ has played an important role both in the construction of analytic approximate self-similar profile (3.20) and the stability analysis, where we treat the advection term as a small perturbation. We will use the same method of analysis presented in the previous section except that the approximate steady state is constructed numerically. Since our approximate steady state is constructed numerically, we also present a general strategy how to obtain rigorous error bounds for various terms using Interval arithmetic guided by numerical error analysis, see subsection 3.5.3.

To begin with, we consider (3.2) with $a = 1$. The associated dynamic rescaling equation reads

$$\omega_t + (c_l x + u)\omega_x = (c_\omega + u_x)\omega\ , \quad u_x = H\omega\ . \tag{3.57}$$

For an odd initial datum ω_0 supported in $[-L, L]$, we use the following normalization conditions

$$c_l = -\frac{u(L)}{L}, \quad c_\omega = c_l. \tag{3.58}$$

We fix $L = 10$. With the above conditions, we have $(c_l x + u)\Big|_{x=\pm L} = 0$ and

$$\begin{aligned}\partial_t \omega_x(t,0) &= \partial_x((u_x + c_\omega)\omega - (c_l x + u)\omega_x)\Big|_{x=0} \\ &= (c_\omega + u_x(t,0) - c_l - u_x(t,0))\omega_x(t,0) = 0.\end{aligned} \tag{3.59}$$

Thus $\omega_x(t,0)$ remains constant and $x = \pm L$ is a stationary point of (3.57) and the support of ω will remain in $[-L, L]$, as long as the solution of the dynamic rescaling equation remains smooth.

The reader who is not interested in the numerical computation can skip the following discussion on the numerical computation and go directly to Section 3.5.1.1 and later subsections for the description of the approximate profile and the analysis of linear stability.

3.5.1 Construction of the approximate self-similar profile

We approximate the steady state of (3.57) numerically by using the normalization conditions (3.58). Since ω is supported on $[-L, L]$ and remains odd for all time, we restrict the computation in the finite domain $[0, L]$ and adopt a uniform discretization with grid points $x_i = ih, i = 0, 1, ..., n = 8000, h = L/8000$. In what follows, the subscript i of ω_i^k stands for space discretization, and the superscript k stands for time discretization. We solve (3.57) numerically using the following discretization scheme:

1. Initial guess is chosen as $\omega_i^0 = -\frac{L-x_i}{\pi}\sin(\frac{\pi x_i}{L}), i = 0, 1, ..., n.$

2. The whole function ω^k is obtained from grid point values w_i^k using a standard cubic spline interpolation on $[-L, L]$, with odd extension of w^k on $[-L, 0]$. We approximate $w_{x,i}^k$ at the boundary using a second order extrapolation:

$$w_x^k(-L) = w_x^k(L) = w_{x,n}^k = \frac{3\omega_n^k - 4\omega_{n-1}^k + \omega_{n-2}^k}{2h}.$$

   The resulting ω^k is a piecewise cubic polynomial and $\omega^k \in C^{2,1}$. The derivative point values $w_{x,i}^k$ are evaluated to be $w_x^k(x_i)$.

3. The values of u^k and u_x^k at grid points are obtained using the kernel integrals:
$$u_i^k = \frac{1}{\pi}\int_0^L \omega^k(y)\log\left|\frac{x_i-y}{x_i+y}\right| dy, \quad u_{x,i}^k = \frac{1}{\pi}\int_0^L \frac{2y}{x_i^2-y^2}\omega^k(y)dy.$$
In particular, for each x_i, the contributions to the above integrals from the neighboring intervals $[x_{i-m}, x_{i+m}]$ are integrated explicitly using the piecewise cubic polynomial expressions of ω; the contributions from the intervals $[0,L]\backslash[x_{i-m}, x_{i+m}]$ are approximate by using a piecewise 8-point Legendre-Gauss quadrature, in order to avoid large round-off error. We choose $m = 8$. We compute u_{xx}^k similarly and will use it later.

4. The integration in time is performed by the 4_{th} order Runge-Kutta scheme with adaptive time stepping. The discrete time step size $\Delta t_k = t_{k+1} - t_k$ is given by $\Delta t_k = \frac{1}{2}\frac{h}{\max_i |c_l x_i + u_i^k|}$, respecting the CFL stability condition $|c_l x + u^k|\frac{h}{\Delta t_k} \leq 1$.

5. After each time step, we apply a local smoothing on w_i^k to prevent oscillation:
$$w_i^k \longleftarrow \frac{1}{4}w_{i-1}^k + \frac{1}{2}w_i^k + \frac{1}{4}w_{i+1}^k, \quad i = 1, \ldots, n-1.$$

Our computation stops when the pointwise residual
$$F_{\omega,i}^k = (c_\omega^k + u_{x,i}^k)\omega_i^k - (c_l^k x_i + u_i^k)\omega_{x,i}^k$$
satisfies $\max_i |F_{\omega,i}^k| \leq 10^{-5}$. Then we use $\bar{\omega} = \omega^k$ as our approximate self-similar profile. The corresponding scaling factors are
$$\bar{c}_l = \bar{c}_\omega = -0.6991$$
by rounding up to 4 significant digits.

We remark that we observe second order convergence in space and fourth order convergence in time for the numerical method described above. However, we do not actually need to do convergence study (by refining the discretization) for our scheme, as we can measure the accuracy of our approximate self-similar profile *a* posteriori. The criterion for a good approximate self-similar profile is that it is piecewise smooth and has a small residual error in the energy norm.

All the numerical computations and quantitative verifications are performed by MATLAB (version 2019a) in the double-precision floating-point operation.

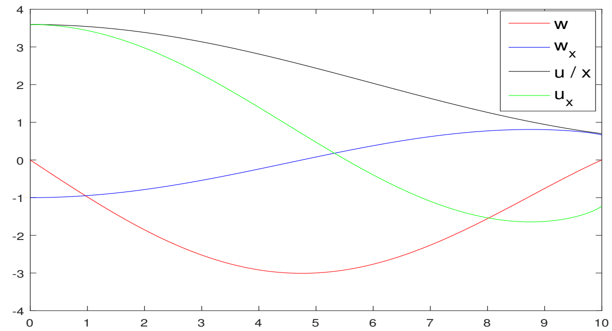

Figure 3.1: Approximate self-similar profile.

The MATLAB codes can be found via the link [17]. To make sure that our computer-assisted proof is rigorous, we adopt the standard method of interval arithmetic (see [96, 100]). In particular, we use the MATLAB toolbox INTLAB (version 11 [99]) for the interval computations. Every single real number p in MatLab is represented by an interval $[p_l, p_r]$ that contains p, where p_l, p_r are precise floating-point numbers of 16 digits. Every computation of real number summation, multiplication or division is performed using the interval arithmetic, and the outcome is hence represented by the resulting interval $[P_l, P_r]$ that strictly contains P. We then obtain a rigorous upper bound on $|P|$ by rounding up $\max\{|P_l|, |P_r|\}$ to 2 significant digits (or 4 when necessary). We remark that, when encountering a non-essential ill-conditioned computation, especially a division, we will replace it by an alternative well-conditioned one. For example, for some function $f(x)$ such that $f(0) = 0, f_x(0) < +\infty$, the evaluation of $\frac{f(x)}{x}$ at $x = 0$ will be replaced by the evaluation of $f_x(0)$.

3.5.1.1 Compact support of the approximate profile

The approximate profile $\bar{\omega}$ we obtain actually has compact support. Below we explain how we obtain a compactly supported approximate self-similar profile. First let us assume that ω is a solution of the steady state equation (or equivalently self-similar equation), i.e. setting $\omega_t = 0$ in (3.57),

$$(c_l x + u)\omega_x = (c_\omega + u_x)\omega, \quad u_x = H\omega.$$

Differentiating both sides and then evaluating the resulting equation at $x = 0$, we obtain

$$(c_l + u_x)\omega_x|_{x=0} = (c_\omega + u_x)\omega_x|_{x=0},$$

which implies $c_l = c_\omega$, provided that $\omega_x(0) \neq 0$. Suppose that we have a finite time self-similar blowup. Then the scaling factor c_ω is negative. See the discussion in Section 2.1.4. It follows that $c_l = c_\omega < 0$. This also holds true for the approximate profile: $\bar{c}_l = \bar{c}_\omega < 0$. Moreover, we have that $\bar{u} > 0$ for $x > 0$ and grows sublinearly for large x. The difference between the signs of $\bar{c}_l x$ and $\bar{u}(x)$ and their different growth rates for large $|x|$ lead to the following change of sign in the approximate profile

$$\bar{c}_l x_0 + \bar{u}(x_0) = 0, \quad \bar{c}_l x + \bar{u}(x) > 0 \text{ for } 0 \leq x < x_0, \quad \bar{c}_l x + \bar{u}(x) < 0 \text{ for } x > x_0,$$

for some $x_0 > 0$. We expect that a similar change of sign occurs in the dynamic variable $c_l x + u$ and the solution of (3.57) will form a shock. When we solve $\bar{\omega}$ numerically, we can fix the point where the sign of $c_l x + u$ changes by imposing (3.58). Moreover, the approximate profile satisfies that $\bar{c}_\omega + \bar{u}_x(x)$ is negative for $x > x_0$ (see Figure 3.1). For $x > x_0$, we expect that the dynamic variable $c_\omega + u_x(x)$ is also negative, which implies that $(c_\omega + u_x(x))\omega$ in (3.57) is a damping term. For $x > x_0$, due to the transport term $(c_l x + u)\omega_x$ with $c_l x + u(x) < 0$ and the damping effect $(c_\omega + u_x(x))\omega$, the solution tends to have compact support. For this reason, in our computation, we have chosen the initial data with compact support and controlled the support of the solution by imposing (3.58). As a result, the approximate profile also has compact support.

3.5.1.2 Regularity of the approximate profile

In the domain $[-L, L]$, since $\bar{\omega}$ is obtained from the cubic spline interpolation, it has the regularity $C^{2,1}[-L, L]$. Moreover, since $\bar{\omega}(x) = 0$ for $|x| \geq L$, $\bar{\omega}$ is a Lipschitz function on the real line. We remark that $\bar{\omega}$ is in $H^1(\mathbb{R})$ but not in $H^2(\mathbb{R})$ since $\bar{\omega}_x$ is discontinuous at $x = \pm L$ (see Figure 3.1). Multiplying $(x^2 - L^2)$, we get a compactly supported and global Lipschitz function $(x^2 - L^2)\bar{\omega}_x$. Hence we can define the Hilbert transform of $((x^2 - L^2)\bar{\omega}_x)_x$ which is in L^p for any $1 \leq p < +\infty$.

Applying (B.4) in Lemma B.0.2, we have

$$\bar{u}_{xx} = H\bar{\omega}_x, \quad \bar{u}_{xxx}(x^2 - L^2) = H(\bar{\omega}_{xx}(x^2 - L^2)).$$

Using the regularity of $\bar{\omega}$, we have that $\bar{u}$ is at least C^3 in $(-L, L)$ and $\bar{u}_{xx}$ grows logarithmically near $x = \pm L$ since $\bar{\omega}_x$ is discontinuous at $x = \pm L$.

3.5.1.3 Regularity of the perturbation

We will choose an odd initial perturbation ω_0 such that $\omega_0 + \bar{\omega} \in C_c^\infty$ and $\omega_{0,x}(0) = 0$. Standard local well-posedness result shows that $\omega + \bar{\omega}$ remains smooth locally in time. Hence, the regularity of ω and $\bar{\omega}$ are the same before blowup. Since the odd symmetry of the solution $\omega + \bar{\omega}$ is preserved and $\bar{\omega}$ is odd, this implies that ω is odd. From this property and $\omega_x(0) = 0$ (see (3.59)), ω is of order $O(x^3)$ near $x = 0$. On the other hand, we have $\omega(\pm L) = 0$ since its support lies in $[-L, L]$. In the following derivation, the boundary terms when we perform integration by parts on ω terms will vanish, which can be justified by these vanishing conditions. We will use this property without explicitly mentioning it.

In [91], the De Gregorio model (3.2) with $a = 1$ was solved numerically on $\mathbb{R}$ for $t \in [0, 1]$. The author demonstrated the growth of the solution numerically and plotted the solutions at several times that have similar profiles, which share some similar structure with our $\bar{\omega}$.

3.5.2 Linear stability of the approximate self-similar profile

Linear stability analysis plays a crucial role in establishing the existence and stability of the self-similar profile. We will establish the linear stability of the approximate self-similar profile in this subsection.

Linearizing (3.57) around $\bar{\omega}, \bar{u}, \bar{c}_l, \bar{c}_\omega$ yields

$$\omega_t + (\bar{c}_l x + \bar{u})\omega_x = (\bar{c}_\omega + \bar{u}_x)\omega + (u_x + c_\omega)\bar{\omega} - (u + c_l x)\bar{\omega}_x + N(\omega) + F(\bar{\omega}) \;, \quad (3.60)$$

where ω, u, c_l, c_ω are the perturbations of the approximate self-similar profile, N and F are the nonlinear terms and the residual error, respectively,

$$N(\omega) = (c_\omega + u_x)\omega - (c_l x + u)\omega_x, \quad F(\bar{\omega}) = (\bar{c}_\omega + \bar{u}_x)\bar{\omega} - (\bar{c}_l x + \bar{u})\bar{\omega}_x. \quad (3.61)$$

Main ideas in our linear stability analysis Compared to (3.21), (3.60) does not contain a small parameter a in the nonlocal term $(u + c_l x)\bar{\omega}_x$, which makes it substantially harder to establish linear stability. There are three key observations in our linear stability estimates. First of all, we observe that the $u_x\bar{\omega}$ term (vortex stretching) is harmless to the linear stability analysis as we have shown in Section 3.4. We construct the weight function carefully to fully exploit the cancellation between u_x and ω (see Lemma B.0.3). Secondly, we observe that there is a competition between the advection term $u\omega_x$ and the vortex stretching term $u_x\omega$. We expect some cancellation between their perturbation $u\bar{\omega}_x$ and $u_x\bar{\omega}$. By exploiting this cancellation, we obtain a sharper estimate of u/x by ω, which improves the corresponding estimate using the Hardy inequality (B.8). Roughly speaking, for x close to 0, the term u/x can be bounded by $\omega/5$ in some appropriate norm; similarly, for x close to L, the term $(u(x)-u(L))/(x-L)$ can be bounded by $\omega/3$ in some appropriate norm. The small constants, $1/5$ and $1/3$, are essential for us to obtain sharp estimates on the non-local term u. If we had used a rough estimate with constant $1/5$ replacing by $1/2$, we would have failed to establish linear stability. Using the first two observations, the estimate of most interactions can be reduced to the estimate of some boundary terms. In order to obtain a sharp stability constant, we express these boundary terms as the projection of ω onto some functions and exploit the cancellation between different projections to obtain the desired linear stability estimate.

Due to the odd symmetry of u,ω, we just need to focus on the positive real line. Denote

$$\langle f,g\rangle \triangleq \int_0^L fg dx, \quad ||f||_p = ||f||_{L^p[0,L]}$$

for any $1 \leq p \leq \infty$. For most integrals we consider, it is the same as the integral from 0 to $+\infty$ since the support of ω lies in $[-L,L]$. Define a singular weight function on $[0,L]$

$$\varphi \triangleq \left(-\frac{1}{x^3} - \frac{e}{x} - \frac{f\cdot 2x}{L^2-x^2}\right)\cdot\left(\chi_1\left(\bar{\omega} - \frac{x\bar{\omega}_x}{5}\right) + \chi_2\left(\bar{\omega} - \frac{(x-L)\bar{\omega}_x}{3}\right)\right)^{-1}, \tag{3.62}$$

where $\chi_1,\chi_2 \geq 0$ are cutoff functions such that $\chi_1+\chi_2=1$ and

$$\chi_1(x) = \begin{cases} 1 & x\in[0,4] \\ 0 & x\in[6,10] \end{cases}, \quad \chi_1(x) = \frac{\exp\left(\frac{1}{x-4}+\frac{1}{x-6}\right)}{1+\exp\left(\frac{1}{x-4}+\frac{1}{x-6}\right)} \ \forall x\in[4,6].$$

Note that the denominator in (3.62) is negative in $(0,L)$ and that $\varphi>0$ is a singular weight and is of order $O(x^{-4})$ near $x=0$, $O((x-L)^{-2})$ near $x=L$.

Performing the weighted L^2 estimate on (3.60) yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle\omega^2,\varphi\rangle &= \Big\langle -(\bar{c}_l x+\bar{u})\omega_x + (\bar{c}_w+\bar{u}_x)\omega, \omega\varphi\Big\rangle + \Big\langle(u_x+c_\omega),\bar{\omega}\omega\varphi\rangle \\ &\quad - \langle(c_l x+u),\bar{\omega}_x\omega\varphi\Big\rangle + \langle N(\omega),\omega\varphi\rangle + \langle F(\bar{\omega}),\omega\varphi\rangle \\ &\triangleq D+I+N_1+F_1.\end{aligned} \tag{3.63}$$

For D, we use integration by parts to obtain

$$D = \left\langle \frac{1}{2\varphi}((\bar{c}_l x + \bar{u})\varphi)_x + (\bar{c}_w + \bar{u}_x), \omega^2\varphi \right\rangle \triangleq \langle D(\bar{\omega}), \omega^2\varphi\rangle. \tag{3.64}$$

From (3.62), we know that $\varphi(x) = O(x^{-4})$ near $x = 0$ and $\varphi(x) = O((x-L)^{-2})$ near $x = L$. Using these asymptotic properties of φ, one can obtain that

$$D(\bar{\omega})(0) = -(\bar{c}_l + \bar{u}_x(0))/2 < 0, \quad D(\bar{\omega})(L) = (\bar{c}_l + \bar{u}_x(L))/2 < 0.$$

We can verify rigorously that $D(\bar{\omega})(x)$ is pointwisely negative on $[0, L)$. In particular, we treat $\langle D(\bar{\omega}), \omega^2\varphi\rangle$ as a damping term. See Section 3.3.1 for the discussions on the derivation of the damping term.

We estimate the interaction near $x = 0$ and $x = L$ differently. First we split the I term into two terms as follows:

$$\begin{aligned} I = &\langle (u_x + c_\omega)\bar{\omega} - (c_l x + u)\bar{\omega}_x, \omega\varphi\chi_1\rangle \\ &+ \langle (u_x + c_\omega)\bar{\omega} - (c_l x + u)\bar{\omega}_x, \omega\varphi\chi_2\rangle \triangleq I_1 + I_2. \end{aligned} \tag{3.65}$$

We use different decompositions of $(u_x + c_\omega)\bar{\omega} - (c_l x + u)\bar{\omega}_x$ for x close to 0 and to L. For x close to 0 (the χ_1 part), we use $c_\omega = c_l$ to obtain

$$\begin{aligned} &(u_x + c_\omega)\bar{\omega} - (c_l x + u)\bar{\omega}_x = (u_x + c_\omega)\left(\bar{\omega} - \frac{\bar{\omega}_x x}{5}\right) + x\bar{\omega}_x\left(\frac{u_x + c_\omega}{5} - \frac{u + c_l x}{x}\right) \\ &= (u_x + c_\omega)\left(\bar{\omega} - \frac{\bar{\omega}_x x}{5}\right) + x\bar{\omega}_x\left(\frac{u_x - u_x(0)}{5} - \frac{u - u_x(0)x}{x} - \frac{4(c_\omega + u_x(0))}{5}\right). \end{aligned}$$

For x close to L (the χ_2 part), using $c_\omega = c_l = -u(L)/L$ (3.58), we have

$$u + c_l x = u - u(L) + c_l(x - L).$$

Therefore, we obtain

$$\begin{aligned} &(u_x + c_\omega)\bar{\omega} - (c_l x + u)\bar{\omega}_x = (u_x + c_\omega)\bar{\omega} - (x - L)\bar{\omega}_x \cdot \frac{u - u(L) + c_l(x - L)}{x - L} \\ &= (u_x + c_\omega)\Big(\bar{\omega} - \frac{\bar{\omega}_x(x - L)}{3}\Big) + (x - L)\bar{\omega}_x\Big(\frac{u_x + c_\omega}{3} - \frac{u - u(L) + c_l(x - L)}{x - L}\Big) \\ &= (u_x + c_\omega)\Big(\bar{\omega} - \frac{\bar{\omega}_x(x - L)}{3}\Big) - \frac{2}{3}(x - L)\bar{\omega}_x(c_\omega + u_x(L)) \\ &\quad + (x - L)\bar{\omega}_x\Big(\frac{u_x - u_x(L)}{3} - \frac{u - u(L) - u_x(L)(x - L)}{x - L}\Big). \end{aligned}$$

Using (3.65) and the above decompositions near $x = 0$, we get

$$\begin{aligned} I_1 = &\left\langle \left(\frac{1}{5}\frac{u_x - u_x(0)}{x^2} - \frac{u - u_x(0)x}{x^3}\right),\ x^3\bar{\omega}_x\omega\varphi\chi_1 \right\rangle \\ &+ \left\langle (c_\omega + u_x), \left(\bar{\omega} - \frac{1}{5}\bar{\omega}_x x\right)\omega\chi_1\varphi \right\rangle - \frac{4}{5}(c_\omega + u_x(0))\langle x\bar{\omega}_x, \omega\chi_1\varphi\rangle \\ \triangleq &\ I_{11} + I_{12} + I_{13}. \end{aligned} \tag{3.66}$$

Similarly, near $x = L$, we have

$$I_2 = \left\langle \left(\frac{1}{3} \frac{u_x - u_x(L)}{x - L} - \frac{u - u(L) - u_x(L)(x-L)}{(x-L)^2} \right),\ (x-L)^2 \bar{\omega}_x \omega \varphi \chi_2 \right\rangle$$
$$+ \left\langle (c_\omega + u_x), \left(\bar{\omega} - \frac{1}{3} \bar{\omega}_x (x - L) \right) \omega \varphi \chi_2 \right\rangle - \frac{2}{3}(c_\omega + u_x(L)) \langle (x-L)\bar{\omega}_x, \omega\varphi\chi_2 \rangle$$
$$\triangleq I_{21} + I_{22} + I_{23}. \tag{3.67}$$

3.5.2.1 The first part: the interior interaction

To handle the first term on the right hand side of (3.66) and (3.67), i.e. I_{11}, I_{21}, we use the Cauchy–Schwarz inequality to obtain

$$\begin{aligned} I_{11} &\leq \left|\left| \left(\frac{1}{5} \frac{u_x - u_x(0)}{x^2} - \frac{u - u_x(0)x}{x^3} \right) \right|\right|_2 ||x^3 \bar{\omega}_x \omega \varphi \chi_1||_2, \\ I_{21} &\leq \left|\left| \frac{1}{3} \frac{u_x - u_x(L)}{x - L} - \frac{u - u(L) - u_x(L)(x-L)}{(x-L)^2} \right|\right|_2 ||(x-L)^2 \bar{\omega}_x \omega \varphi \chi_2||_2. \end{aligned} \tag{3.68}$$

Using integration by parts yields

$$\begin{aligned} &\left|\left| \left(\frac{1}{5} \frac{u_x - u_x(0)}{x^2} - \frac{u - u_x(0)x}{x^3} \right) \right|\right|_2^2 \\ = &\frac{1}{25} \left|\left| \frac{u_x - u_x(0)}{x^2} \right|\right|_2^2 - \frac{2}{5} \int_0^L \frac{(u_x - u_x(0)) \cdot (u - u_x(0)x)}{x^5} dx + \left|\left| \frac{u - u_x(0)x}{x^3} \right|\right|_2^2 \\ = &\frac{1}{25} \left|\left| \frac{u_x - u_x(0)}{x^2} \right|\right|_2^2 - \frac{1}{5} \frac{(u - u_x(0)x)^2}{x^5} \Big|_0^L \\ &\quad - \frac{1}{5} \cdot 5 \int_0^L \frac{(u - u_x(0)x)^2}{x^6} dx + \left|\left| \frac{u - u_x(0)x}{x^3} \right|\right|_2^2 \\ = &\frac{1}{25} \left|\left| \frac{u_x - u_x(0)}{x^2} \right|\right|_2^2 - \frac{1}{5L^5} (u(L) - u_x(0)L)^2 \\ = &\frac{1}{25} \left|\left| \frac{u_x - u_x(0)}{x^2} \right|\right|_2^2 - \frac{1}{5L^3} (c_\omega + u_x(0))^2 \\ \leq &\frac{1}{25} \left|\left| \frac{\omega}{x^2} \right|\right|_2^2 - \frac{1}{5L^3} (c_\omega + u_x(0))^2, \end{aligned} \tag{3.69}$$

where we have used $c_\omega = c_l = -u(L)/L$ in the second to the last line. To obtain the last inequality, we have used estimate (B.8) with $p = 4$, the facts that the integral in $|| \cdot ||_2$ is from 0 to L and that ω is supported in $[-L, L]$. Denote $v \triangleq u - u(L) - u_x(L)(x - L)$. Obviously, we have

$$v(L) = v_x(L) = 0,\ v(0) = -u(L) + u_x(L)L = L(c_\omega + u_x(L)).$$

Using the above formula and integration by parts, we obtain

$$\begin{aligned}
&\Big|\Big|\frac{1}{3}\frac{u_x-u_x(L)}{x-L}-\frac{u-u(L)-u_x(L)(x-L)}{(x-L)^2}\Big|\Big|_2^2=\Big|\Big|\frac{1}{3}\frac{v_x}{x-L}-\frac{v}{(x-L)^2}\Big|\Big|_2^2\\
&=\frac{1}{9}\Big|\Big|\frac{v_x}{x-L}\Big|\Big|_2^2-\frac{2}{3}\int_0^L\frac{vv_x}{(x-L)^3}dx+\Big|\Big|\frac{v}{(x-L)^2}\Big|\Big|_2^2\\
&=\frac{1}{9}\Big|\Big|\frac{v_x}{x-L}\Big|\Big|_2^2-\frac{1}{3}\frac{v^2}{(x-L)^3}\Big|_0^L-\frac{1}{3}\cdot 3\int_0^L\frac{v^2}{(x-L)^4}dx+\Big|\Big|\frac{v}{(x-L)^2}\Big|\Big|_2^2\\
&=\frac{1}{9}\Big|\Big|\frac{v_x}{x-L}\Big|\Big|_2^2+\frac{1}{3}\frac{v(0)^2}{(0-L)^3}=\frac{1}{9}\Big|\Big|\frac{u_x-u_x(L)}{x-L}\Big|\Big|_2^2-\frac{1}{3L}(c_\omega+u_x(L))^2.
\end{aligned}\tag{3.70}$$

Using a formula similar to (B.1) yields

$$(u_x-u_x(L))(x-L)^{-1}=H\left(\omega(x-L)^{-1}\right).$$

We further obtain the following by using the L^2 isometry of the Hilbert transform

$$\int_0^L\frac{(u_x-u_x(L))^2}{(x-L)^2}dx=\int_{\mathbb{R}}\frac{\omega^2}{(x-L)^2}dx-\int_{x\notin[0,L]}\frac{(u_x-u_x(L))^2}{(x-L)^2}dx. \tag{3.71}$$

Note that the Cauchy–Schwarz inequality implies

$$\begin{aligned}
&\int_{x\notin[0,L]}\frac{(u_x-u_x(L))^2}{(x-L)^2}dx\geq\int_{-L}^0\frac{(u_x-u_x(L))^2}{(x-L)^2}dx\\
&\geq\left(\int_{-L}^0(u_x-u_x(L))dx\right)^2\left(\int_{-L}^0(x-L)^2dx\right)^{-1}\\
&=(u(0)-u(-L)-u_x(L)L)^2\left(\frac{7}{3}L^3\right)^{-1}=\frac{3}{7}\frac{(c_\omega+u_x(L))^2L^2}{L^3}=\frac{3}{7}\frac{(c_\omega+u_x(L))^2}{L}.
\end{aligned}$$

Combining (3.70), (3.71) and the above inequality, we get

$$\begin{aligned}
&\Big|\Big|\frac{1}{3}\frac{u_x-u_x(L)}{x-L}-\frac{u-u(L)-u_x(L)(x-L)}{(x-L)^2}\Big|\Big|_2^2\\
&=\frac{1}{9}\int_{\mathbb{R}}\frac{\omega^2}{(x-L)^2}dx-\frac{1}{9}\int_{x\notin[0,L]}\frac{(u_x-u_x(L))^2}{(x-L)^2}dx-\frac{1}{3L}(c_\omega+u_x(L))^2\\
&\leq\frac{1}{9}\int_{\mathbb{R}}\frac{\omega^2}{(x-L)^2}dx-\left(\frac{1}{3L}+\frac{1}{21L}\right)(c_\omega+u_x(L))^2.
\end{aligned}\tag{3.72}$$

Combining (3.68) , (3.69) and (3.72) and using the elementary inequality $xy \leq \lambda x^2 + \frac{1}{4\lambda}y^2$, we obtain the estimate for I_{11}, I_{21},

$$\begin{aligned}
I_{11} + I_{21} \leq 25a_1 \Big|\Big| \left(\frac{1}{5}\frac{u_x - u_x(0)}{x^2} - \frac{u - u_x(0)x}{x^3} \right) \Big|\Big|_2^2 + \frac{1}{100a_1} ||x^3 \bar{\omega}_x \omega \varphi \chi_1||_2^2 \\
+ 9a_2 \Big|\Big| \frac{1}{3}\frac{u_x - u_x(L)}{x - L} - \frac{u - u(L) - u_x(L)(x-L)}{(x-L)^2} \Big|\Big|_2^2 \\
+ \frac{1}{36a_2} ||(x-L)^2 \bar{\omega}_x \omega \varphi \chi_2||_2^2 \\
\leq a_1 \Big|\Big| \frac{\omega}{x^2} \Big|\Big|_2^2 + \frac{1}{100a_1} ||x^3 \bar{\omega}_x \omega \varphi \chi_1||_2^2 + a_2 \int_{\mathbb{R}} \frac{\omega^2}{(x-L)^2} dx \\
+ \frac{1}{36a_2} ||(x-L)^2 \bar{\omega}_x \omega \varphi \chi_2||_2^2 - a_2 \left(\frac{3}{L} + \frac{3}{7L} \right) (c_\omega + u_x(L))^2,
\end{aligned} \tag{3.73}$$

where $a_1, a_2 > 0$ are some parameters to be chosen later.

3.5.2.2 The second part

Combining I_{12}, I_{22} in (3.66), (3.67), respectively, and using the definition of φ (3.62), we obtain

$$\begin{aligned}
I_{12} + I_{22} = & \Big\langle (c_\omega + u_x), \Big\{ \left(\bar{\omega} - \frac{1}{5}\bar{\omega}_x x \right) \chi_1 + \left(\bar{\omega} - \frac{1}{3}\bar{\omega}_x (x-L) \right) \chi_2 \Big\} \omega \varphi \Big\rangle \\
= & \Big\langle (c_\omega + u_x)\omega, \left(-\frac{1}{x^3} - \frac{e}{x} - \frac{f \cdot 2x}{L^2 - x^2} \right) \Big\rangle \\
= & (c_\omega + u_x(0)) \Big\langle \omega, -\frac{1}{x^3} - \frac{e}{x} \Big\rangle + \Big\langle (u_x - u_x(0))\omega, -\frac{1}{x^3} - \frac{e}{x} \Big\rangle \\
& + \Big\langle (c_\omega + u_x)\omega, -\frac{f \cdot 2x}{L^2 - x^2} \Big\rangle,
\end{aligned} \tag{3.74}$$

where e and f are constants in the definition of φ (3.62). Since $\omega \in C^{2,1}$ and $\omega(0) = \omega_x(0) = \omega_{xx}(0) = 0$, we have $\omega \cdot x^{-3} \in L^1$ and the above integrals are well-defined. Using (B.5) and (B.6), we obtain

$$\begin{aligned}
\Big\langle (u_x - u_x(0))\omega, \frac{1}{x^3} \Big\rangle = \frac{1}{2} \int_{\mathbf{R}} \frac{(u_x - u_x(0))\omega}{x^3} dx = 0, \\
\Big\langle (u_x - u_x(0))\omega, \frac{1}{x} \Big\rangle = \frac{1}{2} \int_{\mathbf{R}} \frac{(u_x - u_x(0))\omega}{x} dx = \frac{\pi}{4} u_x^2(0).
\end{aligned} \tag{3.75}$$

Note that $(c_\omega + u_x)\omega$ is odd. The Tricomi identity Lemma B.0.1 implies

$$\begin{aligned}
& \Big\langle (c_\omega + u_x)\omega, -\frac{2x}{L^2 - x^2} \Big\rangle = -\int_{\mathbf{R}^+} (c_\omega + u_x)\omega \left(\frac{1}{L-x} - \frac{1}{L+x} \right) dx \\
= & -\int_{\mathbf{R}} \frac{(c_\omega + u_x)\omega}{L - x} dx = -\pi H((c_\omega + u_x)\omega)(L) \\
= & -\pi c_\omega H\omega(L) - \pi H(u_x \omega)(L) = -\pi c_\omega u_x(L) - \frac{\pi}{2}(u_x^2(L) - \omega^2(L)) \\
= & -\pi c_\omega u_x(L) - \frac{\pi}{2} u_x^2(L).
\end{aligned} \tag{3.76}$$

Combining (3.74), (3.75) and (3.76), we obtain

$$\begin{aligned} I_{12}+I_{22} =&(c_\omega+u_x(0))\Big\langle \omega, \Big(-\frac{1}{x^3}-\frac{e}{x}\Big)\Big\rangle - \frac{\pi e}{4}u_x^2(0) \\ &- f\pi c_\omega u_x(L) - \frac{f\pi}{2}u_x^2(L). \end{aligned} \tag{3.77}$$

3.5.2.3 The remaining part: the boundary interaction

Let $a_3 \triangleq a_2(\frac{3}{L}+\frac{3}{7L})$. The negative term that appears in the last term of (3.73) can be written as

$$-a_2(\frac{3}{L}+\frac{3}{7L})(c_\omega+u_x(L))^2 = -a_3(c_\omega+u_x(L))^2. \tag{3.78}$$

Combining (3.78), (3.77), I_{13}, I_{23} in (3.66) and (3.67), we obtain

$$\begin{aligned} & I_{12}+I_{22}+I_{13}+I_{23}-a_3(c_\omega+u_x(L))^2 \\ =\ & (c_\omega+u_x(0))\Big\langle \omega, \Big(-\frac{1}{x^3}-\frac{e}{x}\Big)\Big\rangle - \frac{e\pi}{4}u_x^2(0) - f\pi c_\omega u_x(L) - \frac{f\pi}{2}u_x^2(L) \\ & -\frac{4}{5}(c_\omega+u_x(0))\langle \omega, x\bar{\omega}_x\chi_1\varphi\rangle - \frac{2}{3}(c_\omega+u_x(L))\langle\omega,(x-L)\bar{\omega}_x\chi_2\varphi\rangle \\ & - a_3(c_\omega+u_x(L))^2 \\ =\ & u_x(0)\Big(\Big\langle\omega, \Big(-\frac{1}{x^3}-\frac{e}{x}\Big) - \frac{4}{5}x\bar{\omega}_x\chi_1\varphi\Big\rangle - \frac{e\pi}{4}u_x(0)\Big) \\ & + c_\omega\Big(\Big\langle\omega, \Big(-\frac{1}{x^3}-\frac{e}{x}\Big) - \frac{4}{5}x\bar{\omega}_x\chi_1\varphi - \frac{2}{3}(x-L)\bar{\omega}_x\chi_2\varphi\Big\rangle - f\pi u_x(L) - a_3c_\omega\Big) \\ & + u_x(L)\Big(\Big\langle\omega, -\frac{2}{3}(x-L)\bar{\omega}_x\chi_2\varphi\Big\rangle - \frac{f\pi}{2}u_x(L) - 2a_3c_\omega - a_3u_x(L)\Big). \end{aligned} \tag{3.79}$$

Note that

$$\begin{aligned} u_x(0) &= -\frac{2}{\pi}\int_0^L \frac{\omega}{x}dx, \quad u_x(L) = \frac{1}{\pi}\int_0^L \frac{2x}{L^2-x^2}\omega dx, \\ c_\omega &= -\frac{u(L)}{L} = \frac{1}{L\pi}\int_0^L \log\Big(\frac{L+x}{L-x}\Big)\omega(x)dx. \end{aligned}$$

All the integrals in (3.79) and $c_\omega, u_x(0), u_x(L)$ are the projection of ω onto some explicit functions. We use the cancellation of these functions to obtain a sharp estimate of the right hand side of (3.79). Denote

$$
\begin{aligned}
g_{c_\omega} &\triangleq \frac{1}{L\pi}\log\left(\frac{L+x}{L-x}\right), \quad g_{u_x(0)} \triangleq -\frac{2}{\pi x}, \quad g_{u_x(L)} \triangleq \frac{2x}{\pi(L^2-x^2)}, \\
g_1 &\triangleq \left(-\frac{1}{x^3}-\frac{e}{x}\right) - \frac{4}{5}x\bar{\omega}_x\chi_1\varphi - \frac{e\pi}{4}g_{u_x(0)}, \\
g_2 &\triangleq \left(-\frac{1}{x^3}-\frac{e}{x}\right) - \frac{4}{5}x\bar{\omega}_x\chi_1\varphi - \frac{2}{3}(x-L)\bar{\omega}_x\chi_2\varphi - f\pi g_{u_x(L)} - a_3 g_{c_\omega}, \\
g_3 &\triangleq -\frac{2}{3}(x-L)\bar{\omega}_x\chi_2\varphi - \left(\frac{f\pi}{2}+a_3\right)g_{u_x(L)} - 2a_3 g_{c_\omega}.
\end{aligned}
\tag{3.80}
$$

With these notations, we can rewrite (3.79) as follows

$$
\begin{aligned}
&u_x(0)\langle\omega, g_1\rangle + c_\omega\langle\omega, g_2\rangle + u_x(L)\langle\omega, g_3\rangle \\
=&\langle\omega, g_{u_x(0)}\rangle\langle\omega, g_1\rangle + \langle\omega, g_{c_\omega}\rangle\langle\omega, g_2\rangle + \langle\omega, g_{u_x(L)}\rangle\langle\omega, g_3\rangle.
\end{aligned}
\tag{3.81}
$$

For some function $R \in C([0,L]), R > 0$ to be chosen, we introduce

$$
\begin{aligned}
&y \triangleq (R\varphi)^{1/2}\omega, \quad f_1 \triangleq (R\varphi)^{-1/2}g_{u_x(0)}, \quad f_2 \triangleq (R\varphi)^{-1/2}g_1, \\
&f_3 \triangleq (R\varphi)^{-1/2}g_{c_\omega}, \quad f_4 \triangleq (R\varphi)^{-1/2}g_2, \\
&f_5 \triangleq (R\varphi)^{-1/2}g_{u_x(L)}, \quad f_6 \triangleq (R\varphi)^{-1/2}g_3.
\end{aligned}
\tag{3.82}
$$

Our goal is to find the best constant of the following inequality for any $\omega \in L^2(\varphi)$

$$
\langle f_1, y\rangle\langle f_2, y\rangle + \langle f_3, y\rangle\langle f_4, y\rangle + \langle f_5, y\rangle\langle f_6, y\rangle \le C_{opt}||y||_2^2, \tag{3.83}
$$

which is equivalent to

$$
\langle\omega, g_{u_x(0)}\rangle\langle\omega, g_1\rangle + \langle\omega, g_{c_\omega}\rangle\langle\omega, g_2\rangle + \langle\omega, g_{u_x(L)}\rangle\langle\omega, g_3\rangle \le C_{opt}\langle R, \omega^2\varphi\rangle,
$$

so that we can estimate (3.81) by $\langle R, \omega^2\varphi\rangle$ with a sharp constant. From the definition of functions g, f, we have that $g_3 \in \text{span}(g_{c_\omega}, g_{u_x(0)}, g_{u_x(L)}, g_1, g_2)$ and

$$
f_6 \in \text{span}(f_1, f_2, ..., f_5) \triangleq V, \quad \text{dim}V = 5. \tag{3.84}
$$

Without loss of generality, we assume $y \in V$ since $||Py||_2 \le ||y||_2$ and $\langle y, f_i\rangle = \langle Py, f_i\rangle$, where P is the orthogonal projector onto V. Suppose that $\{e_i\}_{i=1}^5$ is an orthonormal basis (ONB) in V with respect to the L^2 inner product on $[0, L]$. It can be obtained via the Gram-Schmidt procedure. Then we have $z = \sum_{i=1}^5\langle z, e_i\rangle e_i$ for any $z \in V$. We consider the linear map $T : V \to \mathbb{R}^5$ defined by $(Tz)_i = \langle z, e_i\rangle, \ \forall z \ \in V$. It is obvious that T is a linear isometry from $(V, \langle \cdot, \cdot \rangle_{L^2})$ to $\mathbb{R}^5$ with the Euclidean inner product, i.e. $||Tz||_{l^2} = ||z||_{L^2}$. Denote $v = Ty, v_i = Tf_i \in \mathbb{R}^5$. Using the linear isometry, i.e. $\langle f_i, y \rangle = v^T v_i$ and $||y||_2^2 = v^T v$, we can reduce (3.83) to

$$\sum_{1 \le i \le 3} (v^T v_{2i-1})(v_{2i}^T v) = v^T (\sum_{1 \le i \le 3} v_{2i-1} v_{2i}^T) v \le C_{opt} v^T v.$$

Denote $M \triangleq \sum_{1 \le i \le 3} v_{2i-1} v_{2i}^T \in \mathbb{R}^{5 \times 5}$. Then the above inequality becomes $v^T M v \le C_{opt} v^T v$. Using the fact that $v^T M v = v^T M^T v$, we can symmetrize it to obtain

$$v^T \frac{M + M^T}{2} v \le C_{opt} v^T v.$$

Since $(M^T + M)/2$ is symmetric, the optimal constant C_{opt} is the maximal eigenvalue of $(M + M^T)/2$, i.e.

$$C_{opt} = \lambda_{\max} \left(\frac{M + M^T}{2} \right) = \lambda_{\max} (\frac{1}{2} \sum_{1 \le i \le 3} (v_{2i-1} v_{2i}^T + v_{2i} v_{2i-1}^T)). \tag{3.85}$$

We remark that maximal eigenvalue $\lambda_{\max}$ is independent of the choice of the ONB of V. For other ONB, the resulting $\lambda_{\max}$ will be $\lambda_{\max}(Q(M + M^T)Q^T/2)$ for some orthonormal matrix $Q \in \mathbb{R}^{5 \times 5}$, which is the same as (3.85). Using (3.79), (3.81), (3.83) and (3.85), we have proved

$$\begin{aligned} & I_{12} + I_{22} + I_{13} + I_{23} - a_3 (c_\omega + u_x(L))^2 \\ \le & \lambda_{\max} (\frac{1}{2} \sum_{1 \le i \le 3} (v_{2i-1} v_{2i}^T + v_{2i} v_{2i-1}^T)) \langle R, \omega^2 \varphi \rangle, \end{aligned} \tag{3.86}$$

where $v_i \in \mathbb{R}^5$ is the coefficient of f_i (see (3.82)) expanded under an ONB $\{e_i\}_{i=1}^5$ of $V = \text{span}(f_1, f_2, ..., f_5)$, i.e. the j-th component of v_i satisfies $v_{ij} = \langle f_i, e_j \rangle$. We will choose R so that $\lambda_{\max} < 1$ and then the left hand side can be controlled by $\langle R, \omega^2 \varphi \rangle$.

3.5.2.4 Summary of the estimates

In summary, we collect all the estimates of $I_{ij}, i = 1, 2, j = 1, 2, 3$, (3.66), (3.67), (3.73) and (3.86) to conclude

$$\begin{aligned}
&\langle (u_x + c_\omega)\bar{\omega} - (c_l x + u), \bar{\omega}_x, \omega\varphi\rangle = I = I_1 + I_2 = \sum_{i=1,2,j=1,2,3} I_{ij} \\
\leq & a_1 \Big|\Big|\frac{\omega}{x^2}\Big|\Big|_2^2 + \frac{1}{100a_1}||x^3\bar{\omega}_x\omega\varphi\chi_1||_2^2 + a_2\int_{\mathbb{R}} \frac{\omega^2}{(x-L)^2}dx \\
&+ \frac{1}{36a_2}||(x-L)^2\bar{\omega}_x\omega\varphi\chi_2||_2^2 + \lambda_{\max}(\frac{1}{2}\sum_{1\leq i\leq 3}(v_{2i-1}v_{2i}^T + v_{2i}v_{2i-1}^T))\langle R, \omega^2\varphi\rangle \\
\triangleq & \langle A(\bar{\omega}), \omega^2\varphi\rangle + \lambda_{\max}(\frac{1}{2}\sum_{1\leq i\leq 3}(v_{2i-1}v_{2i}^T + v_{2i}v_{2i-1}^T))\langle R, \omega^2\varphi\rangle,
\end{aligned} \tag{3.87}$$

where $A(\bar{\omega})$ is the sum of the four terms in the first inequality and is given by

$$A(\bar{\omega}) = \Big(\frac{a_1}{x^4} + \frac{a_2}{(x-L)^2} + \frac{a_2}{(x+L)^2}\Big)\varphi^{-1} + \frac{(x^3\bar{\omega}_x\chi_1)^2\varphi}{100a_1} + \frac{((x-L)^2\bar{\omega}_x\chi_2)^2\varphi}{36a_2}. \tag{3.88}$$

Optimizing the parameters To optimize the estimate, we choose

$$\begin{aligned}
e = 0.005, \quad f = 0.004, \quad a_1 = \frac{1}{6}, \\
a_2 = 1.4f = 0.0056, \quad a_3 = \frac{a_2}{L}(3 + \frac{3}{7}) = 0.00192.
\end{aligned} \tag{3.89}$$

After specifying these parameters, the coefficient of the damping term $D(\bar{\omega})$ (see (3.63)) and the coefficient of the estimate of the interior interaction $A(\bar{\omega})$ are completely determined. Then we choose

$$R(\bar{\omega}) = -D(\bar{\omega}) - A(\bar{\omega}) - 0.3 \tag{3.90}$$

in (3.82). The numerical values of $D(\bar{\omega})$, $A(\bar{\omega})$ and $R(\bar{\omega})$ on the grid points are plotted in the first subfigure in Figure 3.2. We can verify rigorously (see the discussion below) that $R(\bar{\omega}) = -D(\bar{\omega}) - A(\bar{\omega}) - 0.3 > 0$. In particular, the coefficient of the damping term satisfies $D(\bar{\omega}) < -0.3 - A(\bar{\omega})$ and is negative pointwisely. The corresponding f_i in (3.83) are determined. The optimal constant in (3.86) can be computed :

$$C_{opt} = \lambda_{\max}(\frac{1}{2}\sum_{1\leq i\leq 3}(v_{2i-1}v_{2i}^T + v_{2i}v_{2i-1}^T)) < 1. \tag{3.91}$$

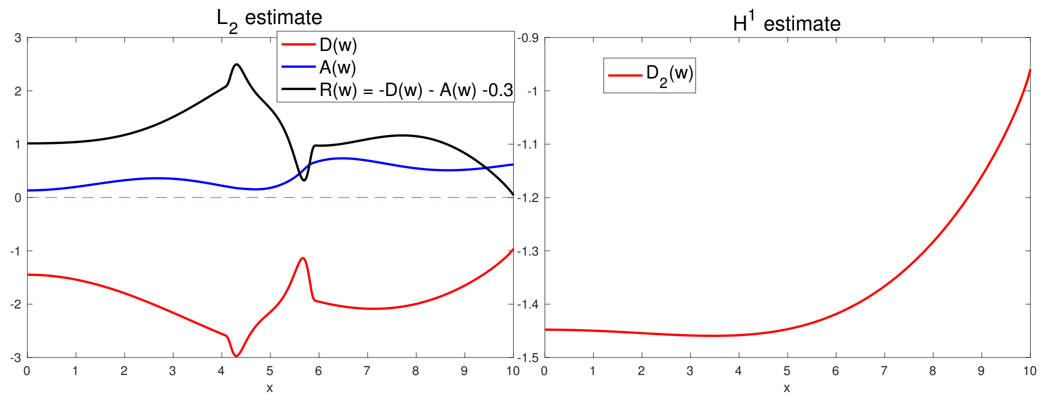

Figure 3.2: Illustration of the linear stability estimates. Left: Coefficients of the damping term $D(\bar{\omega})$ in the L^2 estimate, the estimate of the interior interaction $A(\bar{\omega})$ and the remaining terms $R(\bar{\omega})$. Right: Coefficient of the damping term $D_2(\bar{\omega})$ in the H^1 estimate.

Combining $\langle D(\bar{\omega}), \omega^2\varphi\rangle$ in (3.63), (3.87) and (3.91), we obtain the linear estimate

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle\omega^2,\varphi\rangle &= \langle D(\bar{\omega}),\omega^2\varphi\rangle + I + N_1 + F_1 \\ &\leq \langle D(\bar{\omega}),\omega^2\varphi\rangle + \langle A(\bar{\omega}),\omega^2\varphi\rangle + \langle R(\bar{\omega}),\omega^2\varphi\rangle + N_1 + F_1 \\ &= -0.3\langle\omega^2,\varphi\rangle + N_1 + F_1.\end{aligned} \tag{3.92}$$

For those who are not interested in the rigorous verification of the numerical values, they can skip the following discussion and jump to Section 3.5.4 for the weighted H^1 estimate.

3.5.3 Rigorous verification of the numerical values

We will use the following strategy to verify $R(\bar{\omega}) > 0$ (3.90), $C_{opt} < 1$ (3.91) and $D_2(\bar{\omega}) < -0.95$ (3.98) to be discussed later. These quantities appear in the weighted Sobolev estimates and are determined by the profile.

(a) **Obtaining an explicit approximate self-similar profile.** As described in section 3.5.1, our approximate self-similar profile $\bar{\omega}$ is expressed in terms of a piece-wise cubic polynomial over the grid points $x_i = \frac{iL}{n}, i = 0, \cdots, n$. The function values, $\bar{\omega}(x_i), \bar{\omega}_x(x_i)$, which are used to construct the cubic Hermite spline, are computed accurately up to double-precision, and will be represented in the computations using the interval arithmetic with exact floating-point bounding intervals. All the following computer-assisted estimates are based on the rigorous interval arithmetic.

(b) **Accurate point values of $\bar{u}, \bar{u}_x, \bar{u}_{xx}$.** We have described how to compute the value of $\bar{u}_x(x)$ (or $\bar{u}(x), \bar{u}_{xx}(x)$) from certain integrals involving $\bar{\omega}$ on $[-L, L]$ in paragraph (3) in Section 3.5.1. For any $x \in [0, L]$, the integral contribution to $\bar{u}_x(x)$ from mesh intervals within $m = 8$ mesh points distance from x is computed exactly using analytic integration. In the outer domain that is $8h$ distance away from x, the integrand $\bar{\omega}(y)/(x-y)$ is not singular and we use a composite 8-point Legendre-Gauss quadrature. There are two types of errors in this computation. The first type of error is the round-off error in the computation. The second type of error is due to the composite Gaussian quadrature that we use to approximate the integral in the outer domain. Notice that in each interval $[ih, (i+1)h]$ away from x, $\bar{\omega}$ is a cubic polynomial and the integrand $\bar{\omega}(y)/(x-y)$ is smooth. We can estimate high order derivatives of the integrand rigorously in these intervals. With the estimates of the derivatives, we can further establish error estimates of the Gaussian quadrature. In particular, we prove the following error estimates of the composite Gaussian quadrature in the computation of $\bar{u}_x, \bar{u}, \bar{u}_{xx}$ in the Supplementary material [18, Section 7]

$$Error_{GQ}(u_x) < 2\cdot 10^{-17}, \quad Error_{GQ}(u) < 2\cdot 10^{-19}, \quad Error_{GQ}(u_{xx}) < 5\cdot 10^{-18}. \tag{3.93}$$

These two types of errors will be taken into account in the interval representations of $\bar{u}_x$. That is, each $\bar{u}_x(x)$ will be represented by $[\lfloor \bar{u}_x(x)-\epsilon \rfloor_f, \lceil \bar{u}_x(x)+\epsilon \rceil_f]$ in any computation using the interval arithmetic, where $\lfloor \cdot \rfloor_f$ and $\lceil \cdot \rceil_f$ stand for the rounding down and rounding up to the nearest floating-point value, respectively. We remark that we will need the values of $\bar{u}_x(x)$ at finitely many points only. The same arguments apply to $\bar{u}(x)$ and $\bar{u}_{xx}(x)$ as well.

(c) **Rigorous estimates of integrals.** In many of our discussions, we need to rigorously estimate the integral of some function $g(x)$ on $[0, L]$. In particular, we want to obtain c_1, c_2 such that $c_1 \leq \int_0^L g(x)dx \leq c_2$. A straightforward way to do so is by constructing two sequences of values $g^{up} = \{g_i^{up}\}_{i=1}^n, g^{low} = \{g_i^{low}\}_{i=1}^n$ such that

$$g_i^{up} \geq \max_{x \in [x_{i-1}, x_i]} g(x) \quad \text{and} \quad g_i^{low} \leq \min_{x \in [x_{i-1}, x_i]} g(x).$$

Then we can bound

$$h \cdot \sum_{i=1}^n g_i^{low} \leq \int_0^L g(x)dx \leq h \cdot \sum_{i=1}^n g_i^{up}.$$

In most cases, we will construct g^{up} and g^{low} from the grid point values of g and an estimate of its first derivative. Let $g^{\max} = \{g_i^{\max}\}_{i=1}^N$ denote the sequence such that $g_i^{\max} = \max\{|g_i^{up}|, |g_i^{low}|\}$. Then if we already have $g_x^{\max}$, we can construct g^{up} and g^{low} as

$$g_i^{up} = g(x_i) + h \cdot (g_x^{\max})_i \quad \text{and} \quad g_i^{low} = g(x_i) - h \cdot (g_x^{\max})_i.$$

We can use this method to construct the piecewise upper bounds and lower bounds for many functions we need. For example, our approximate steady state $\bar{\omega}$ is constructed to be piecewise cubic polynomial using the standard cubic spline interpolation. Since $\bar{\omega}_{xxx}$ is piecewise constant, we have $\bar{\omega}_{xxx}^{up}$ and $\bar{\omega}_{xxx}^{low}$ for free from the grid point values of $\bar{\omega}$. Then we can construct $\bar{\omega}_{xx}^{up/low}, \bar{\omega}_x^{up/low}$ and $\bar{\omega}^{up/low}$ recursively.

Note that for some explicit functions, we can construct the associated sequences of their piecewise upper bounds and lower bounds more explicitly. For example, for a monotone function g, g^{up} and g^{low} are just the grid point values.

Moreover, we can construct the piecewise upper bounds and lower bounds for more complicated functions. For instance, if we have $f_a^{up/low}$ and $f_b^{up/low}$ for two functions, then we can construct $g^{up/low}$ for $g = f_a f_b$ using standard interval arithmetic. In this way, we can estimate the integral of all the functions we need in our computer-aided arguments.

Sometimes we need to handle the ratio between two functions, which may introduce a removable singularity. For example, in the construction of $D(\bar{\omega})^{up}$ and $D(\bar{\omega})^{low}$ for $D(\bar{\omega})$ in (3.64), it involves $\frac{x\varphi_x}{\varphi}, \frac{\bar{u}\varphi_x}{\varphi}$ and φ is a singular weight of order x^{-4} near $x = 0$. Directly applying interval arithmetic to the ratio near a removable singularity can lead to large errors. We hence need to treat this issue carefully. For example, let us explain how to reasonably construct $g^{up/low}$ for a $g(x) = f(x)/x$ such that $f(x)$ has continuous first derivatives and $f(0) = 0$. Suppose that we already have $f^{up/low}$ and $f_x^{up/low}$. Then for some small number $\varepsilon > 0$, we let

$$g_i^{up} = \max\left\{\frac{f_i^{up}}{x_{i-1}}, \frac{f_i^{low}}{x_{i-1}}, \frac{f_i^{up}}{x_i}, \frac{f_i^{low}}{x_i}\right\} \quad \text{for each index } i \text{ such that } x_{i-1} \geq \varepsilon.$$

Otherwise, for $x \in [0, \varepsilon)$, we have

$$g(x) = \frac{f(x)}{x} = f_x(\xi(x)) \quad \text{for some } \xi(x) \in [0, x) \subset [0, \varepsilon).$$

Then we choose $g_i^{up} = \max_{x\in[0,\varepsilon]} f_x^{up}$ for every index i such that $x_i \le \varepsilon$. The parameter ε needs to be chosen carefully. On the one hand, ε should be small enough so that the bound $f_x(\xi(x)) \le \max_{\tilde{x}\in[L-\varepsilon,L]} |f_x(\tilde{x})|$ is sharp for $x \in [L-\varepsilon, L]$. On the other, the ratio ε/h must be large enough so that $f_i^{up}/x_{i-1}, f_i^{low}/x_{i-1}, f_i^{up}/x_i$ and f_i^{low}/x_i are close to each other for $x_{i-1} \ge \varepsilon$. Other types of removable singularities can be handled in a similar way.

See more detailed discussions in the Supplementary Material [18, Section 1.3].

(d) **Estimates of some (weighted) norms of $\bar{\omega}, \bar{u}$.** Once we have used the preceding method to obtain $\bar{\omega}_{xxx}^{up/low}, \bar{\omega}_{xx}^{up/low}, \bar{\omega}_x^{up/low}$ and $\bar{\omega}^{up/low}$ from the grid point values of $\bar{\omega}$, we can further estimate some (weighted) norms of $\bar{\omega}$, e.g., $||\bar{\omega}_x||_{L^\infty}$, $||\bar{\omega}_{xx}||_{L^\infty}$, rigorously. Moreover, from the discussion of the regularity of $\bar{u}, \bar{\omega}$ in Section (3.5.1.2), the norms of $\bar{u}$, such as $||\bar{u}_x||_\infty$ and $||\bar{u}_{xx}||_{L^2}$, can be bounded by some norms of $\bar{\omega}$. See more detailed discussions in the Supplementary Material [18, Section 1.3].

(e) **Rigorous and accurate estimates of certain integrals.** Our rigorous estimate for integrals in the preceding part (c) is only first order accurate. Yet this method is not accurate enough if the target integral is supposed to be a very small number. When we need to obtain a more accurate estimate of the integral of some function P, we use the composite Trapezoidal rule

$$\int_{ah}^{bh} P(x)dx = \sum_{a\le i<b} (P(x_i) + P(x_{i+1}))h/2 + \text{error}(P).$$

The composite Trapezoidal rule uses the values of P on the grid points only, which can be obtained up to the round off error. The numerical integral error, error(P), can be bounded by the L^1 norm of its second order derivative, i.e. $C||P''||_{L^1}h^2$ for some absolute constant C. We use this approach to obtain integral estimates of some functions involving the residual $F(\bar{\omega})$. For each function P that we integrate, we prove in the Supplementary Material [18, Section 3] that $||P''||_{L^1}$ can be bounded by some (weighted) norms of $\bar{u}, \bar{\omega}$, e.g., $||\bar{\omega}||_{L^\infty}$, $||\bar{u}_x||_{L^\infty}$ and $||\frac{\bar{\omega}_{xx}}{x}||_{L^2}$. Since these norms can be estimated by the method discussed previously, we can establish rigorous error bound for the integral.

(f) **Rigorous estimates of C_{opt}.** Denote by M_s the matrix in (3.85)

$$M_s \triangleq \frac{1}{2}\sum_{i=1}^{3}(v_{2i-1}v_{2i}^T + v_{2i}v_{2i-1}^T) = \frac{1}{2}V_1V_2^T,$$

where $V_1 \triangleq [v_1, v_2, v_3, v_4, v_5, v_6] \in \mathbb{R}^{6\times 6}$ and $V_2 \triangleq [v_2, v_1, v_4, v_3, v_6, v_5] \in \mathbb{R}^{6\times 6}$, and $\{v_i\}_{i=1}^6$ are defined as in Section 3.5.2.3. Note that M_s is symmetric, but not necessarily positive semidefinite. The optimal constant C_{opt} is then the maximal eigenvalue of M_s. To rigorously estimate C_{opt}, we first bound it by the Schatten p-norm of M_s:

$$C_{opt} \le \|M_s\|_p \triangleq \mathrm{Tr}[|M_s|^p]^{1/p} \quad \text{for all } p \ge 1. \tag{3.94}$$

Here $|M_s| = \sqrt{M_s^T M_s} = \sqrt{M_s^2}$. In particular, if p is an even number, we have $|M_s|^p = M_s^p$. Therefore, we have

$$\mathrm{Tr}[|M_s|^p] = 2^{-p} \cdot \mathrm{Tr}[(V_1 V_2^T)^p] = 2^{-p} \cdot \mathrm{Tr}[(V_2^T V_1)^p] \triangleq 2^{-p} \cdot \mathrm{Tr}[X^p]$$

where $X = V_2^T V_1$. Note that each entry of X is the inner product between some v_i and v_j, $i, j = 1, \dots, 6$. Recall from (3.84) and its following paragraph that $v_i = Tf_i, i = 1, 2, ..., 6$ and that $T : V \to \mathbb{R}^5$ is a linear isometry. We have

$$\langle f_i, f_j \rangle = \langle Tf_i, Tf_j \rangle = v_i^T v_j.$$

Therefore, to compute the entries of X, we only need to compute the pairwise inner products between $f_1, \dots, f_6$ (we do not need to compute the coordinate vectors v_i explicitly). This is done by interval arithmetic based on the discussion in the preceding part (c). Therefore each entry X_{ij} of X is represented by a pair of numbers that we can bound from above and below. Once we have the estimate of X, we can compute an upper bound of $\mathrm{Tr}[X^p]$ stably and rigorously by interval arithmetic, which then gives a bound on C_{opt} via (3.94). In particular, we choose $p = 4$ in our computation, and we can rigorously verify that $C_{opt} < 1$.

3.5.4 Weighted H^1 estimate

We choose

$$\psi = -\frac{1}{\bar{\omega}}\left(\frac{1}{x} - \frac{x}{L^2}\right), \quad x \in [0, L], \tag{3.95}$$

as the weight for the weighted H^1 estimate. Note that the weight ψ is non-negative for $0 \le x \le L$, and is of order x^{-2} near $x = 0$ and $O(1)$ near $x = L$. We can perform the weighted H^1 estimate as follows

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi\rangle &= \langle -((\bar{c}_l x + \bar{u})\omega_x)_x + ((\bar{c}_\omega + \bar{u}_x)\omega)_x, \omega_x\psi\rangle \\ &\quad + \langle ((u_x + c_\omega)\bar{\omega})_x, \omega_x\psi\rangle - \langle ((u + c_l x)\bar{\omega}_x)_x, \omega_x\psi\rangle \\ &\quad + \langle N(\omega)_x, \omega_x\psi\rangle + \langle F(\bar{\omega})_x, \omega_x\psi\rangle \\ &\triangleq I + II + III + N_2 + F_2. \end{aligned} \tag{3.96}$$

For I, we use $\bar{c}_l = \bar{c}_\omega$ and integration by parts to get

$$\begin{aligned} I &= \Big\langle -(\bar{c}_l x + \bar{u})\omega_{xx} + \bar{u}_{xx}\omega, \omega_x \psi \Big\rangle \\ &= \Big\langle \frac{1}{2\psi}((\bar{c}_l x + \bar{u})\psi)_x, \omega_x^2 \psi \Big\rangle + \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle \\ &\triangleq \langle D_2(\bar{\omega}), \omega_x^2\psi\rangle + \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle. \end{aligned} \tag{3.97}$$

The first term in I is a damping term. We plot the numerical values of $D_2(\bar{\omega})$ on the grid points in Figure 3.2. We can verify rigorously that it is bounded from above by -0.95. Thus we have

$$\begin{aligned} I &= \langle D_2(\bar{\omega}), \omega_x^2\psi\rangle + \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle \\ &\le -0.95\langle \omega_x^2, \psi\rangle + \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle \triangleq -0.95\langle \omega_x^2, \psi\rangle + I_2, \end{aligned} \tag{3.98}$$

where $I_2 = \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle$. For II, III, we note that

$$\begin{aligned} II + III &= \langle u_{xx}\bar{\omega} + (u_x + c_\omega)\bar{\omega}_x - (u_x + c_l)\bar{\omega}_x - (c_l x + u)\bar{\omega}_{xx}, \omega_x\psi\rangle \\ &= \langle u_{xx}\bar{\omega}, \omega_x\psi\rangle - \langle (c_l x + u)\bar{\omega}_{xx}, \omega_x\psi\rangle \triangleq II_1 + II_2. \end{aligned}$$

Using the definition of ψ, we get

$$II_1 = \langle u_{xx}\bar{\omega}, \omega_x\psi\rangle = \Big\langle u_{xx}\omega_x, -\frac{1}{x} + \frac{x}{L^2}\Big\rangle.$$

Since $\omega_x(0) = 0$ by the normalization condition and $u_{xx}(0) = 0$ by the odd symmetry, we can use the same cancellation as we did in (3.37) to get

$$\Big\langle u_{xx}\omega_x, -\frac{1}{x}\Big\rangle = 0, \quad \langle u_{xx}\omega_x, x\rangle = 0.$$

Therefore II_1 vanishes and we get

$$II + III = II_2 = -\langle (c_l x + u)\bar{\omega}_{xx}, \omega_x\psi\rangle, \tag{3.99}$$

which is a cross term. In fact, after performing integration by parts, it becomes interaction among some lower order terms, i.e. of the order lower than ω_x (e.g., u, u_x, ω).

Remark 3.5.1. So far, we have established all the delicate estimates of the linearized operator that exploit cancellations of various nonlocal terms. We have obtained the linear stability at the L^2 level and the linear stability estimates for the terms of the same order as ω_x, e.g., u_{xx}, in the weighted H^1 estimates after performing integration by parts. The remaining estimates *do not* require specific structure of the equation. Suppose that we have a sequence of approximate steady states ω_{h_i} with h_i converging to 0 that enjoy similar estimates and have approximation error $\langle F(\omega_{h_i})^2, \varphi\rangle + \langle F(\omega_{h_i})_x^2, \psi\rangle$ of order h_i^{β} for some constant $\beta > 0$ independent of h_i, where $F(\omega_{h_i})$ is defined similarly as that in (3.61). Then we can apply the above stability analysis to the profile ω_h and the argument in Sections 3.4.2,3.4.3 to finish the remaining steps of the proof by choosing a sufficiently small $h = h_n$. Here h plays a role similar to the small parameter a in these sections. An important observation is that h_n and the required approximation error to close the whole argument can be estimated effectively. Once we have determined h_n, we can construct the approximate steady state ω_{h_n} numerically and verify whether ω_{h_n} enjoys similar estimates and has the desired approximation error a posteriori.

In the following discussion, we first give some rough bounds and show that the remaining terms can be bounded by the weighted L^2 or H^1 norm of ω with constants depending continuously on $\bar{\omega}$. This property implies that similar bounds will also hold true if we replace the approximate steady state $\bar{\omega}$ by another profile $\hat{\omega}$, if $\bar{\omega} - \hat{\omega}$ is sufficiently small in some energy norm. We will provide other steps in the computer-assisted part of this chapter later in this section.

The remaining linear terms in the weighted H^1 estimate are $I_2 = \langle \bar{u}_{xx}\omega, \omega_x\psi\rangle$ in (3.98) and II_2 in (3.99). Denote $\rho = x^{-2} + (x-L)^{-2}$. Note that $u + c_l x|_{x=0,L} = 0, c_\omega = c_l$. Applying integration by parts to the integral $||u_x + c_l - \frac{1}{2}(u + c_l x)/x||_2^2, ||u_x + c_l - \frac{1}{2}(u+c_l x)/(x-L)||_2^2$ and using an argument similar to those in (3.69), (3.70), we get

$$||(u + c_l x)\rho^{1/2}||_2^2 = \int_0^L (u + c_l x)^2 (\frac{1}{x^2} + \frac{1}{(x-L)^2})dx \le 8\int_0^L (u_x + c_l)^2 dx.$$

Using the L^2 isometry of the Hilbert transform, the identity $\int_0^L u_x dx = u(L) = -L \cdot c_l$ and expanding the square, we further obtain

$$\begin{aligned} ||(u + c_l x)\rho^{1/2}||_2^2 &\le 8||\omega||_2^2 + 8(2c_l \cdot u(L) + Lc_l^2) \\ &\le 8||\omega||_2^2 \le 8||\varphi^{-1}||_{L^\infty}\langle \omega^2, \varphi\rangle. \end{aligned}$$

Applying the Cauchy–Schwarz inequality, we can estimate I_2, II_2 as follows

$$\begin{aligned} |I_2| &= |\langle \bar{u}_{xx}\omega, \omega_x\psi\rangle| \le ||\bar{u}_{xx}\psi^{1/2}\varphi^{-1/2}||_{L^\infty[0,L]}\langle \omega^2, \varphi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2}, \\ |II_2| &= |\langle (c_l x + u)\bar{\omega}_{xx}, \omega_x\psi|\rangle \le ||\rho^{-1/2}\bar{\omega}_{xx}\psi^{1/2}||_{L^\infty[0,L]}\langle (c_l x + u)^2, \rho\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2}. \end{aligned}$$

Hence, combining the above estimates, we yield

$$|I_2| + |II_2| \le C_1(\bar{\omega})\langle \omega^2, \varphi\rangle^{1/2}\langle \omega_x^2, \psi\rangle^{1/2}, \tag{3.100}$$

where

$$C_1(\bar{\omega}) \triangleq ||\bar{u}_{xx}\psi^{1/2}\varphi^{-1/2}||_{L^\infty[0,L]} + \sqrt{8}||\rho^{-1/2}\bar{\omega}_{xx}\psi^{1/2}||_{L^\infty[0,L]}||\varphi^{-1}||_{L^\infty}^{1/2} \tag{3.101}$$

and $\rho = x^{-2} + (x-L)^{-2}$. From the definitions of φ, ψ (3.62), (3.95), the quantities appeared in $C_1(\bar{\omega})$ satisfy that

$$\begin{aligned}
&\varphi^{-1} = O((x^{-4} + (x-L)^{-2})^{-1}),\\
&|\bar{u}_{xx}\psi^{1/2}\varphi^{-1/2}| = O(|\bar{u}_{xx}(x^{-1} + (L-x)^{-1})^{-1}|),\\
&|(x^{-2} + (x-L)^{-2})^{-1/2}\bar{\omega}_{xx}\psi^{1/2}| = O(|(1 + (x-L)^{-2})^{-1/2}\bar{\omega}_{xx}|).
\end{aligned}$$

In particular, these quantities are bounded for any $x \in [0, L]$ and thus $C_1(\bar{\omega})$ is finite.

Therefore, combining (3.96), (3.98), (3.99) and (3.100), we prove for any $\varepsilon > 0$,

$$\frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi\rangle \le -0.95\langle \omega_x^2, \psi\rangle + \varepsilon\langle \omega_x^2, \psi\rangle + (4\varepsilon)^{-1}C_1(\bar{\omega})^2\langle \omega^2, \varphi\rangle + N_2 + F_2, \tag{3.102}$$

From (3.92) and (3.102), we can choose $\varepsilon, \mu > 0$ and construct the energy $E(t)^2 = \langle \omega^2, \varphi\rangle + \mu\langle \omega_x^2, \psi\rangle$ such that

$$\frac{d}{dt}E(t)^2 \le -C(\mu, \varepsilon)E(t)^2 + N_1 + F_1 + \mu(N_2 + F_2), \tag{3.103}$$

where $C(\mu, \varepsilon) > 0$ depends on μ, ε. For example, one can choose $\varepsilon = 0.65$, $\mu = 0.4\varepsilon C_1(\bar{\omega})^{-2}$ to obtain $C(\mu, \varepsilon) = 0.2$. We have now completed the weighted L^2 and H^1 estimates at the linear level.

3.5.4.1 Nonlinear stability

Recall that N, F are defined in (3.61), N_1, F_1 in (3.63), and N_2, F_2 in (3.96). Since $c_l = c_\omega$, a direct calculation yields $\partial_x N(\omega) = u_{xx}\omega - (c_l x + u)\omega_{xx}$.

Using integration by parts similar to that in (3.64) and (3.97), we obtain

$$\begin{aligned}
N_1 + \mu N_2 = &\Big\langle \frac{1}{2\varphi}((c_l x + u)\varphi)_x + (c_\omega + u_x), \omega^2\varphi \Big\rangle \\
&+ \mu\Big\langle \frac{1}{2\psi}((c_l x + u)\psi)_x, \omega_x^2\psi \Big\rangle + \mu\langle u_{xx}\omega, \omega_x\psi\rangle.
\end{aligned}$$

Recall $E(t) = (\langle \omega^2, \varphi\rangle + \mu \langle \omega_x^2, \psi\rangle)^{1/2}$. We can estimate u_x, ω, u_{xx} as follows

$$
\begin{aligned}
||u_x||_{L^\infty} &\leq 2||u_x||_{L^2(\mathbb{R}^+)}^{1/2} ||u_{xx}||_{L^2(\mathbb{R}^+)}^{1/2} \leq 2||\omega||_2^{1/2} ||\omega_x||_2^{1/2} \\
&\leq 2\mu^{-1/4} ||\varphi^{-1}||_{L^\infty}^{1/4} ||\psi^{-1}||_{L^\infty}^{1/4} E(t), \\
||\omega||_{L^\infty} &\leq ||\omega_x||_{L^1} \leq \langle \omega_x^2, \psi\rangle^{1/2} ||\psi^{-1}||_{L^1[0,L]}^{1/2} \leq \mu^{-1/2} ||\psi^{-1}||_{L^1[0,L]}^{1/2} E(t), \\
||u_{xx} x^{-1}||_2 &\leq ||\omega_x x^{-1}||_2 \leq \mu^{-1/2} ||\psi^{-1/2} x^{-1}||_{L^\infty} E(t),
\end{aligned}
$$

where we have used (B.3), $\omega_x(0) = 0$ and the L^2 isometry of the Hilbert transform to obtain the last estimate. Recall $c_l = c_\omega = -u(L)/L$ (3.58). We have $c_l x + u|_{x=0,L} = 0, |c_l| = |c_\omega| \leq ||u_x||_{L^\infty}$ and

$$
\begin{aligned}
|c_l x + u| &\leq \min(|x|, |L-x|) \cdot ||c_\omega + u_x||_{L^\infty[0,L]} \\
&\leq 2\min(|x|, |L-x|) ||u_x||_\infty.
\end{aligned}
$$

For any $x \in [0, L]$, using the Leibniz rule, we derive

$$
\begin{aligned}
&\Big| \frac{((c_l x + u)\varphi)_x}{\varphi} \Big| + \Big| \frac{((c_l x + u)\psi)_x}{\psi} \Big| \\
\leq & 2(2 + \Big|\Big| (|x| \wedge (L-x)) (\frac{|\varphi_x|}{\varphi} + \frac{|\psi_x|}{\psi}) \Big|\Big|_{L^\infty}) ||u_x||_{L^\infty} \triangleq C_2(\bar{\omega}) ||u_x||_{L^\infty}.
\end{aligned} \tag{3.104}
$$

Combining the above estimates, we prove

$$
\begin{aligned}
N_1 + \mu N_2 &\leq (C_2(\bar{\omega}) + 2) ||u_x||_{L^\infty} (\langle \omega^2, \varphi\rangle + \mu \langle \omega_x^2, \psi\rangle) \\
&\quad + \mu ||x\psi^{1/2}||_{L^\infty} ||u_{xx} x^{-1}||_2 ||\omega||_{L^\infty} \langle \omega_x^2, \psi\rangle^{1/2} \\
&\leq C_3(\bar{\omega}, \mu) E(t)^3,
\end{aligned} \tag{3.105}
$$

where

$$
\begin{aligned}
C_3(\bar{\omega}, \mu) &= 2\mu^{-1/4} (C_2(\bar{\omega}) + 2) ||\varphi^{-1}||_{L^\infty}^{1/4} ||\psi^{-1}||_{L^\infty}^{1/4} \\
&\quad + \mu^{-1/2} ||x\psi^{1/2}||_{L^\infty} ||x^{-1}\psi^{-1/2}||_{L^\infty} ||\psi^{-1}||_{L^1[0,L]}^{1/2}.
\end{aligned}
$$

We remark that the above L^∞ norms are taken over $[0, L]$. From the definition of φ, ψ, it is not difficult to verify that $C_3(\bar{\omega}, \mu) < +\infty$.

To estimate the error term, we use the Cauchy–Schwarz inequality

$$
\begin{aligned}
F_1 + \mu F_2 &= \langle F(\bar{\omega}), \omega\varphi\rangle + \mu \langle F(\bar{\omega})_x, \omega_x \psi\rangle \\
&\leq (\langle F(\bar{\omega})^2, \varphi\rangle + \mu \langle F(\bar{\omega})_x^2, \psi\rangle)^{1/2} E(t) \triangleq \text{error}(\bar{\omega}) E(t),
\end{aligned} \tag{3.106}
$$

Guideline for the remaining computer assisted steps Recall the definition of φ, ψ in (3.62) and (3.95). From the weighted L^2 and H^1 estimates, and the estimates of the nonlinear terms, we see that the coefficients and constants, e.g., $D(\bar{\omega})$ in (3.64), $C_1(\bar{\omega})$ in (3.101) and $C_3(\bar{\omega},\mu)$ in (3.105), depend continuously on $\bar{\omega}$. Hence, for two different approximate steady states $\omega_{h_1},\omega_{h_2}$ computed using different mesh $h_2 < h_1$, if $\omega_{h_1}-\omega_{h_2}$ is small in some norm, e.g., some weighted L^2 or H^1 norm, we expect that all of these estimates hold true for these two profiles with very similar coefficients and constants. At the same time, the residual error of the profile computed using the finer mesh error(ω_{h_2}) can be much smaller than that of the coarse mesh error(ω_{h_1}). In particular, if the numerical solution ω_h exhibits convergence in a suitable norm as we refine the mesh size h, then we can obtain a sequence of approximate steady states that enjoy similar estimates with decreasing residual error(ω_{h}). See also the Remark 3.5.1. From our numerical computation, we did observe such convergence of ω_h computed using several meshes with decreasing mesh size h. Using the estimates that we have established, we can obtain nonlinear estimate for each profile $\bar{\omega}$ similar to (3.44)

$$\frac{1}{2}\frac{d}{dt}E^2(t) \leq -K_1(\bar{\omega})E^2(t) + K_2(\bar{\omega})E^3(t) + \mathrm{error}(\bar{\omega})E(t),$$

where $E(t)^2 = \langle \omega^2, \varphi\rangle + \mu(\bar{\omega})\langle \omega_x^2, \psi\rangle$ and the positive constants $K_1(\bar{\omega})$, $K_2(\bar{\omega})$, $\mu(\bar{\omega})$ depend continuously on $\bar{\omega}$. From this inequality, we can estimate the size of error($\bar{\omega}$) that is required to close the bootstrap argument. A sufficient condition is that there exists $y>0$ such that $-K_1(\bar{\omega})y^2 + K_2(\bar{\omega})y^3 + \mathrm{error}(\bar{\omega})y < 0$, which is equivalent to

$$4\cdot \mathrm{error}(\bar{\omega})\cdot K_2(\bar{\omega}) < K_1(\bar{\omega})^2. \tag{3.107}$$

Hence, we obtain a good estimate on error($\bar{\omega}$) that is required to close the whole estimate.

In practice, we first compute an approximate steady state $\bar{\omega}_h$ using a relatively coarse mesh, e.g., mesh size $h = L/1000$ or $L/2000$ (correspond to 1000 or 2000 grid points). Then we can perform all the weighted L^2, H^1 estimates and determine the weights φ,ψ, the decomposition in the estimates and all the parameters in (3.89) to obtain the linear stability, and perform the nonlinear estimates. After we obtain these estimates, we can determine an upper bound of error($\bar{\omega}$) using (3.107) and choose a finer mesh with mesh size h_2 to construct a profile $\bar{\omega}_{h_2}$ with a residual error less than this upper bound. After we extend all the corresponding estimates to the profile $\bar{\omega}_{h_2}$, we found that the corresponding constants and coefficients in the estimates are almost the same as those that we have obtained using $\bar{\omega}_h$ constructed by a coarser mesh. Therefore, we can perform analysis on $\bar{\omega}_{h_2}$ and close the whole argument.

In the Supplementary material [18, Sections 2,4], we will provide much sharper estimates of the cross terms (3.100), (3.102) and the nonlinear terms (3.105). These sharper estimates provide an estimate of the upper bound of error($\bar{\omega}$) in (3.107) that is not too small. This enables us to choose a modest mesh to construct an approximate profile with a residual error less than this upper bound. In particular, we choose $h = 2.5 \cdot 10^{-5}$ and the computational cost of $\bar{\omega}_h$ is affordable even for a personal laptop computer. The rigorous estimate for the residual error of this profile in the energy norm is established in the Supplementary material [18, Section 3]. More specifically, we can prove the following estimate, which improves the estimate given by (3.102) significantly.

Lemma 3.5.2. *The weighted H^1 estimate satisfies*

$$\frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi\rangle = I + II_2 + N_2 + F_2 \leq -0.25\langle \omega_x^2, \psi\rangle + 7.5\langle \omega^2, \varphi\rangle + N_2 + F_2,$$

where I, II_2 combine the damping and the cross terms and are defined in (3.98), (3.99), *respectively.*

These refinements are not necessary if one can construct an approximate profile with a much smaller residual error using a more powerful computer with probably $10 - 100$ times more grid points. With these refined estimates and the rigorous estimate of the residual error of $\bar{\omega}_h$, we choose $\mu = 0.02$ and bootstrap assumption $E(t) = \langle \omega^2, \varphi\rangle + \mu\langle \omega_x^2, \psi\rangle < 5 \cdot 10^{-4}$ to complete the final bootstrap argument. We refer the reader to the Supplementary material [18, Section 5] for the detailed estimates in the bootstrap argument.

The remaining steps are the same as those in the proof of Theorem 3.2. Recall the weights φ (3.62) and ψ (3.95) in the weighted L^2 and H^1 estimates and the regularity of the approximate profile $\bar{\omega}$ in Section 3.5.1.2. Note that φ is of order $O(x^{-4})$ near $x = 0$ and $O((x-L)^{-2})$ near $x = L$, and ψ is of order $O(x^{-2})$ near $x = 0$ and $O(1)$ near $x = L$. We can choose a small and odd initial perturbation ω supported in $[-L, L]$ with vanishing $\omega_x(0) = 0$ such that ω restricted to $[0, L]$ satisfies $\omega \in L^2(\varphi), \omega_x \in L^2(\psi)$ and $\omega + \bar{\omega} \in C_c^{\infty}$. The bootstrap result implies that for all time $t > 0$, the solution $\omega(t) + \bar{\omega}$, $c_l + \bar{c}_l = c_\omega(t) + \bar{c}_\omega$ remain close to $\bar{\omega}, \bar{c}_\omega (\bar{c}_\omega < -0.69)$, respectively. Moreover, in the Supplementary Material [18, Section 6], we have established the following estimate

$$\frac{1}{2}\frac{d}{dt}\langle \omega_t^2, \varphi\rangle \le -0.15\langle \omega_t^2, \varphi\rangle.$$

Using this estimate and a convergence argument similar to that in Section 3.4.3, we prove that the solution eventually converges to the self-similar profile ω_∞ with scaling factors $c_{l,\infty} = c_{\omega,\infty} < 0$. Since $\gamma = -\frac{c_{l,\infty}}{c_{\omega,\infty}} = -1 < 0$, the asymptotically self-similar singularity is expanding. Thus we obtain an expanding and asymptotically self-similar blowup of the original De Gregorio model with scaling exponent $\gamma = -1$ in finite time.

3.6 Finite time blowup for C^α initial data

In [44], Elgindi and Jeong obtained the C^α self-similar solution ω_α of the Constantin-Lax-Majda equation

$$c_l x\omega_x = (c_\omega + u_x)\omega$$

for all $\alpha \in (0, 1]$, which reads

$$\begin{aligned} w_\alpha &= -\frac{2\sin\left(\frac{\alpha\pi}{2}\right)\operatorname{sgn}(x)|x|^\alpha}{1 + 2\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha + |x|^{2\alpha}}, \\ u_{\alpha,x} &= \frac{2(1+\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha)}{1 + 2\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha + |x|^{2\alpha}}, \quad c_l = \frac{1}{\alpha}, \quad c_\omega = -1, \end{aligned} \tag{3.108}$$

where c_l, c_ω are the scaling parameters.

In this section, we will use the above solutions to construct approximate self-similar solutions analytically and use the same method of analysis presented in Section 3.4 to prove finite time asymptotically self-similar singularity for C^α initial data with small α on both the real line and on the circle. We will focus on solution of (3.2) with odd symmetry that is preserved during the evolution. In particular, we will construct odd approximate steady state and analyze the stability of odd perturbation around the approximate steady state.

3.6.1 Finite time blowup on $\mathbb{R}$ with C_c^α initial data

In this section, we prove Theorem 3.3. Throughout the proof, we impose $|a\alpha| < 1$ and $\alpha < \frac{1}{4}$. We choose the following weights in the stability analysis

$$\varphi_\alpha = -\frac{1}{\operatorname{sgn}(x)\omega_\alpha}\frac{1 + 2\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha + |x|^{2\alpha}}{|x|^{1+2\alpha}}, \quad \psi_\alpha = \frac{1}{\alpha^2}\varphi_\alpha x^2. \tag{3.109}$$

We choose these weights so that the estimates of $\langle \omega^2, \varphi_\alpha\rangle$ and $\langle \omega_x^2, \psi_\alpha\rangle$ are comparable in the energy estimates.

3.6.1.1 Normalization conditions and approximate steady state

The self-similar equation of DG model with parameter a reads

$$(c_l x + au)\omega_x = (c_\omega + u_x)\omega \ . \tag{3.110}$$

For any $a > 0, \alpha \in (0,1)$, we construct C^α approximate self-similar profile of (3.110) below

$$\omega_\alpha, \quad u_\alpha, \quad \bar{c}_{l,\alpha} = \frac{1}{\alpha} - au_{\alpha,x}(0) = \frac{1}{\alpha} - 2a, \quad \bar{c}_\omega = -1. \tag{3.111}$$

The only difference between the above solution and the C^α self similar solutions of CLM (3.108) is the c_l term. The above solution satisfies (3.110) up to an error

$$F_\alpha(\omega_\alpha) = -(\bar{c}_l x - \frac{1}{\alpha}x + au_\alpha)\omega_{\alpha,x} = -a(u_\alpha - u_{\alpha,x}(0)x)\omega_{\alpha,x}. \tag{3.112}$$

Linearizing the dynamic rescaling equation (3.8) around the approximate self-similar profile in (3.111), we obtain the following equation for the perturbation ω, u, c_l, c_ω:

$$\begin{aligned}\omega_t + (\bar{c}_{l,\alpha}x + au_\alpha)\omega_x =& (\bar{c}_\omega + u_{\alpha,x})\omega + (u_x + c_\omega)\omega_\alpha \\ & - (au + c_l x)\omega_{\alpha,x} + N(\omega) + F_\alpha(\omega_\alpha) \ ,\end{aligned} \tag{3.113}$$

where the error term $F_\alpha(\omega_\alpha)$ is given in (3.112) and the nonlinear part is given by

$$N(\omega) = (c_\omega + u_x)\omega - (c_l x + au)\omega_x.$$

We choose the following normalization conditions for $c_l(t), c_\omega(t)$

$$c_l(t) = -au_x(t,0), \quad c_\omega(t) = -u_x(t,0). \tag{3.114}$$

Using (3.111) and $u_{\alpha,x}(0) = 2$, we can rewrite the above conditions as

$$c_l(t) + \bar{c}_l = \frac{1}{\alpha} - a(u_x(t,0) + u_{\alpha,x}(0)), \quad c_\omega + \bar{c}_\omega = 1 - (u_x(t,0) + u_{\alpha,x}(0)). \tag{3.115}$$

3.6.1.2 Estimate of the velocity and the self-similar solution

We introduce the notation

$$\tilde{u} \triangleq u - u_x(0)x, \quad \tilde{u}_x = u_x - u_x(0) \ , \tag{3.116}$$

and use the weighs defined in (3.109) to perform the L^2, H^1 estimates.

We first state some useful properties of the C^α approximate self-similar solution that we will use in our stability analysis.

Lemma 3.6.1. *For $\alpha \in (0,1]$, we have the following estimates for the self-similar solutions defined in (3.108). (a) Uniform estimates on the damping effect*

$$\begin{aligned} &\frac{1}{2\varphi_\alpha}\left(\frac{1}{\alpha}x\varphi_\alpha\right)_x + (\bar{c}_\omega + u_{\alpha,x}) = -1/2 \, , \\ &\frac{1}{2\psi_\alpha}\left(\frac{1}{\alpha}x\psi_\alpha\right)_x + (\bar{c}_\omega + u_{\alpha,x}) - \frac{1}{\alpha} = -1/2 \, , \\ &\frac{(u_{\alpha,xx}\psi_\alpha)_x}{2\psi_\alpha}x^2 = \frac{4\alpha^2|x|^\alpha(|x|^\alpha + \cos\left(\frac{\alpha\pi}{2}\right))}{(1+2\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha + |x|^{2\alpha})^2} \geq 0 \, . \end{aligned} \tag{3.117}$$

(b) Vorticity and velocity estimates:

$$\left|\left|\frac{xw_{\alpha,x}}{w_\alpha}\right|\right|_\infty \lesssim \alpha, \quad \left|\left|\frac{x^2 w_{\alpha,xx}}{w_\alpha}\right|\right|_\infty \lesssim \alpha, \quad \left|\left|\frac{x^2\omega_{\alpha,xx} + x\omega_{\alpha,x}}{\omega_\alpha}\right|\right|_\infty \lesssim \alpha^2, \tag{3.118}$$

$$\left|\frac{u_\alpha}{x} - u_{\alpha,x}(0)\right| \lesssim |x|^\alpha \wedge 1, \quad \left|\frac{u_\alpha}{x} - u_{\alpha,x}\right| \lesssim \alpha(|x|^\alpha \wedge 1) \, . \tag{3.119}$$

(c) Asymptotic estimates of $\varphi_\alpha, \psi_\alpha$:

$$\begin{aligned} &\varphi_\alpha \asymp \frac{1}{\alpha}(|x|^{-1-3\alpha} + |x|^{-1+\alpha}) \, , \\ &\psi_\alpha = \frac{1}{\alpha^2}x^2\varphi_\alpha \asymp \frac{1}{\alpha^3}(|x|^{1-3\alpha} + |x|^{1+\alpha}) \, , \\ &\left|\left|\frac{x\psi_{\alpha,x}}{\psi_\alpha} - 1\right|\right|_\infty \lesssim \alpha, \quad \left|\left|\frac{x\varphi_{\alpha,x}}{\varphi_\alpha} + 1\right|\right|_\infty \lesssim \alpha \, , \end{aligned} \tag{3.120}$$

where $A \asymp B$ means that $A \leq CB$ and $B \leq CA$ for some universal constant C.

(d) The smallness of the weighted L^2 and H^1 errors:

$$\langle F_\alpha(\omega_\alpha)^2, \varphi_\alpha\rangle \lesssim a^2\alpha^2, \quad \langle (F_\alpha(\omega_\alpha))_x^2, \psi_\alpha\rangle \lesssim a^2\alpha^2, \tag{3.121}$$

$$\langle (|x|^\alpha \wedge 1)^2\omega_{\alpha,x}^2, \psi_\alpha\rangle \lesssim 1. \tag{3.122}$$

These estimates can be established by using the explicit formulas of ω_α, u_α, $\bar{c}_{l,\alpha}$, $\bar{c}_\omega$, φ_α, ψ_α, $F_\alpha(\omega_\alpha)$ given in (3.108), (3.109), (3.111) and (3.112), which are elementary. Therefore, we will not present the estimates here and refer the reader to the arXiv version of [19] for the details.

Remark 3.6.2. We will use (3.117) to derive the damping terms in the weighted L^2 and H^1 estimates. Using (3.118), we gain a small factor α from the derivatives of ω_α. This enables us to show that the perturbation term $u\omega_{\alpha,x}$ is small. Estimates (3.120) shows that $x\psi_{\alpha,x}/\psi_\alpha$, $x\varphi_{\alpha,x}/\varphi_\alpha$ are close to 1 and -1, respectively, which allows us to estimate $\varphi_{\alpha,x}, \psi_{\alpha,x}$ effectively.

Lemma 3.6.3 (L^∞ estimate)**.**

$$||u_x||_\infty \lesssim \langle \omega^2, \varphi_\alpha \rangle^{1/4} \langle \omega_x^2, \psi_\alpha \rangle^{1/4}, \tag{3.123}$$

$$\left| \tilde{u}_x - \frac{\tilde{u}}{x} \right| \lesssim \alpha \langle \omega_x^2, \psi_\alpha \rangle^{1/2} |x^\alpha| \wedge 1 \lesssim \alpha \langle \omega_x^2, \psi_\alpha \rangle^{1/2}, \tag{3.124}$$

$$|\omega(x)| \lesssim \alpha \langle \omega_x^2, \psi_\alpha \rangle^{1/2} |x^\alpha| \wedge 1, \tag{3.125}$$

where $\tilde{u} = u - u_x(0)x$*.*

The proofs of these estimates are standard so we only sketch the main ideas and refer to the arXiv version of [19] for the details. The weights $\psi_\alpha, \varphi_\alpha$ can be simplified by applying (3.120). Estimate (3.123) follows from the interpolation between the weighted L^2 norm of u_x and u_{xx} and by using the weighted estimates of the Hilbert transform in Lemma B.0.4. To prove (3.124), we can first rewrite $\tilde{u}_x - \frac{\tilde{u}}{x}$ as an integral of ω_x with some kernel. Then the estimate can be established by the Cauchy–Schwarz inequality and estimating the integrals of some explicit functions. Estimate (3.125) is proved by estimating $\omega(x)$ by the L^1 norm of ω_x and the Cauchy–Schwarz inequality.

Estimate (3.124) shows that we can gain a small factor α from $\tilde{u}_x - \frac{\tilde{u}}{x} = u_x - u/x$.

We use a strategy similar to that in the proof of Theorem 3.2 to prove Theorem 3.3. The key step is establishing linear stability by taking advantage of the following:

(a) the stretching effect $\bar{c}_{l,\alpha} x \omega_x$ and the damping term $(\bar{c}_\omega + u_{x,\alpha})\omega$;

(b) the cancellation (B.11), (B.5) involving the vortex stretching term $u_x \omega_\alpha$;

(c) the smallness of the advection term $a u \omega_{\alpha,x}$ (see (3.118)) by choosing $|a\alpha|$ to be sufficiently small.

To control the velocity u, we need to use Lemma B.0.4 in Appendix B, which states some nice properties of the Hilbert transform for a Hölder continuous function.

3.6.1.3 Linear estimate

We first perform the weighted L^2 estimate with respect to (3.113). We proceed as follows

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\langle\omega^2,\varphi_\alpha\rangle &= \langle-(\bar{c}_{l,\alpha}x+au_\alpha)\omega_x+(\bar{c}_\omega+u_{\alpha,x})\omega,\omega\varphi_\alpha\rangle \\ &\quad+\langle(u_x+c_\omega)\omega_\alpha,\omega\varphi_\alpha\rangle-\langle(au+c_lx)\omega_{\alpha,x},\omega\varphi_\alpha\rangle \\ &\quad+\langle N(\omega),\omega\varphi_\alpha\rangle+\langle F_\alpha(\omega_\alpha),\omega\varphi_\alpha\rangle \\ &\triangleq I+II+III+N+F.\end{aligned} \tag{3.126}$$

For I, we use integration by parts, (3.117) and $\bar{c}_{l,\alpha}=\frac{1}{\alpha}-au_{\alpha,x}(0)$ to get

$$\begin{aligned}I &= \Big\langle\frac{1}{2\varphi_\alpha}((\bar{c}_{l,\alpha}x+au_\alpha)\varphi_\alpha)_x+(\bar{c}_\omega+u_{\alpha,x}),\omega^2\varphi_\alpha\Big\rangle \\ &= -\frac{1}{2}\langle\omega^2,\varphi_\alpha\rangle+a\Big\langle\frac{1}{2\varphi_\alpha}((u_\alpha-u_{\alpha,x}(0)x)\varphi_\alpha)_x,\omega^2\varphi_\alpha\Big\rangle.\end{aligned} \tag{3.127}$$

For the second term, we use (3.119) and (3.120) to yield

$$\begin{aligned}&\Big|\frac{1}{2\varphi_\alpha}((u_\alpha-u_{\alpha,x}(0)x)\varphi_\alpha)_x\Big|=\Big|\frac{1}{2}(u_{\alpha,x}-u_{\alpha,x}(0))+\frac{u_\alpha-u_{\alpha,x}(0)x}{x}\frac{x\varphi_{\alpha,x}}{2\varphi_\alpha}\Big| \\ &=\Big|\frac{1}{2}(u_{\alpha,x}-\frac{u_\alpha}{x})+\frac{u_\alpha-u_{\alpha,x}(0)x}{x}\left(\frac{x\varphi_{\alpha,x}}{2\varphi_\alpha}+\frac{1}{2}\right)\Big|\lesssim\alpha+1\cdot\alpha\lesssim\alpha\,.\end{aligned}$$

Combining (3.127) with the above estimate, we derive

$$I\leq-\frac{1}{2}\langle\omega^2,\varphi_\alpha\rangle+C|a|\alpha\langle\omega^2,\varphi_\alpha\rangle=-\left(\frac{1}{2}-C|a|\alpha\right)\langle\omega^2,\varphi_\alpha\rangle, \tag{3.128}$$

where $C>0$ is some universal constant.

Recall the definitions of φ_α in (3.109), $c_l=-au_x(0),c_\omega=-u_x(0)$ in (3.114) and $\tilde{u},\tilde{u}_x$ in (3.116). We have $c_lx+au=a\tilde{u},c_\omega+u_x=\tilde{u}_x$. For II, we use the cancellation (B.11) and (B.5) to get

$$\begin{aligned}II &= \langle\tilde{u}_x\omega_\alpha,\omega\varphi_\alpha\rangle \\ &= -\Big\langle\tilde{u}_x\omega\cdot\mathrm{sgn}(x),|x|^{-1-2\alpha}+2\cos\left(\frac{\alpha\pi}{2}\right)||x|^{-1-\alpha}+|x|^{-1}\Big\rangle \\ &\leq-\langle\tilde{u}_x\omega\cdot\mathrm{sgn}(x),|x|^{-1}\rangle=-\frac{\pi}{2}u_x^2(0)\leq 0.\end{aligned} \tag{3.129}$$

For III, we have

$$\begin{aligned}|III|=|\langle(au+c_lx)\omega_{\alpha,x},\omega\varphi_\alpha\rangle|&=\Big|a\Big\langle\frac{\tilde{u}}{x}\frac{\omega_{\alpha,x}x}{\omega_\alpha},\omega\frac{1+2\cos\left(\frac{\alpha\pi}{2}\right)|x|^\alpha+|x|^{2\alpha}}{|x|^{1+2\alpha}}\Big\rangle\Big| \\ &\lesssim|a|\Big\langle\Big|\frac{\tilde{u}}{x}\Big|\Big|\frac{\omega_{\alpha,x}x}{\omega_\alpha}\Big|,|\omega|(|x|^{-1-2\alpha}+|x|^{-1})\Big\rangle.\end{aligned}$$

Using the estimate for ω_α (3.118) and the Hardy inequality (B.12), we obtain

$$\begin{aligned} |III| &\lesssim |a|\alpha\Big\langle \Big|\frac{\tilde{u}}{x}\Big|, |\omega|(|x|^{-1-2\alpha}+|x|^{-1})\Big\rangle \\ &\lesssim |a|\alpha\langle \tilde{u}^2, |x|^{-3-3\alpha}\rangle^{1/2}\langle \omega^2, |x|^{-1-\alpha}\rangle^{1/2} \\ &\quad + |a|\alpha\langle \tilde{u}^2, |x|^{-3-\alpha}\rangle^{1/2}\langle \omega^2, |x|^{-1+\alpha}\rangle^{1/2} \\ &\lesssim |a|\alpha\alpha^{-1}\langle \omega^2, |x|^{-1-3\alpha}\rangle^{1/2}\langle \omega^2, |x|^{-1-\alpha}\rangle^{1/2} \\ &\quad + |a|\alpha\alpha^{-1}\langle \omega^2, |x|^{-1-\alpha}\rangle^{1/2}\langle \omega^2, |x|^{-1+\alpha}\rangle^{1/2} \\ &\lesssim |a|\alpha\langle \omega^2, \varphi_\alpha\rangle, \end{aligned} \tag{3.130}$$

where we have used (3.120) to obtain the last inequality.

Plugging (3.128), (3.129) and (3.130) in (3.126), we establish

$$\frac{1}{2}\frac{d}{dt}\langle \omega^2, \varphi_\alpha\rangle \leq -\left(\frac{1}{2} - C|a|\alpha\right)\langle \omega^2, \varphi_\alpha\rangle + \langle N(\omega), \omega\varphi_\alpha\rangle + \langle F_\alpha(\omega_\alpha), \omega\varphi_\alpha\rangle. \tag{3.131}$$

3.6.1.4 Weighted H^1 estimate

Recall the definition of the weight ψ_α in (3.109). We now perform the weighted H^1 estimate with respect to (3.113)

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi_\alpha\rangle &= \langle -((\bar{c}_{l,\alpha}x + au_\alpha)\omega_x)_x + ((\bar{c}_\omega + u_{\alpha,x})\omega)_x,\ \omega_x\psi_\alpha\rangle \\ &\quad + \langle ((u_x + c_\omega)\omega_\alpha)_x,\ \omega_x\psi_\alpha\rangle - \langle ((au + c_l x)\omega_{\alpha,x})_x,\ \omega_x\psi_\alpha\rangle \\ &\quad + \langle N(\omega)_x . \omega_x\psi_\alpha\rangle + \langle F_\alpha(\omega_\alpha)_x,\ \omega_x\psi_\alpha\rangle \\ &\triangleq I + II + III + N_2 + F_2. \end{aligned}$$

The estimate of each term in I, II, III is very similar to that in the weighted L^2 estimates in last section and the weighted H^1 estimates in Section 3.4.2 so we only sketch the estimates. Note that I only involves the local terms. We can first apply integration by parts and then use the second and the third identities in (3.117) to obtain the damping term similar to (3.128). For II, we have

$$II = -\langle u_{xx}\omega_\alpha, \omega_x\psi_\alpha\rangle + \langle \tilde{u}_x\omega_{\alpha,x}, \omega_x\psi_\alpha\rangle \triangleq II_1 + II_2,$$

where $\tilde{u} = u + c_\omega x = u - u_x(0)x$ (see (3.116)). To estimate II_1, we use the nonlocal cancellation (B.11), (B.5) with (u_x, ω) replaced by $(xu_{xx}, x\omega_x)$ to obtain an estimate similar to (3.129), which has a favorable sign. For II_2 and III, they involve the derivative of ω_α, which gives a small factor α. We can use Lemmas 3.6.1, 3.6.3 to estimate the profiles and the weights, and use

Lemma B.0.4 to estimate $\tilde{u}$ and $\tilde{u}_x$. We present the estimate of a typical term below. Consider the following decomposition for III

$$
\begin{aligned}
III &= -\langle ((au + c_l x)\omega_{\alpha,x})_x, \omega_x \psi_\alpha \rangle = -a\langle \tilde{u}_x \omega_{\alpha,x} + \tilde{u}\omega_{\alpha,xx}, \omega_x \psi_\alpha \rangle \\
&= -a\Big\langle (\tilde{u}_x - \frac{\tilde{u}}{x})\omega_{\alpha,x}, \omega_x \psi_\alpha \Big\rangle - a\Big\langle \frac{\tilde{u}}{x}(\omega_{\alpha,x} + x\omega_{\alpha,xx}), \omega_x \psi_\alpha \Big\rangle \triangleq III_1 + III_2.
\end{aligned}
$$

The advantage of the above decomposition of is that we gain a small factor α by applying (3.124) to $\tilde{u}_x - \frac{\tilde{u}}{x}$ and the third estimate in (3.118) to $(\omega_{\alpha,x} + x\omega_{\alpha,xx})$. Using (3.124), the Cauchy–Schwarz inequality and (3.122), we get

$$
\begin{aligned}
III_1 &\leq |a|\alpha \langle \omega_x^2, \psi_\alpha \rangle^{1/2} \cdot \langle (|x|^\alpha \wedge 1)|\omega_{\alpha,x}|, |\omega_x|\psi_\alpha \rangle \\
&\lesssim |a|\alpha \langle \omega_x^2, \psi_\alpha \rangle \cdot \langle (|x|^\alpha \wedge 1)^2 \omega_{\alpha,x}^2, \psi_\alpha \rangle^{1/2} \lesssim |a|\alpha \langle \omega_x^2, \psi_\alpha \rangle.
\end{aligned}
$$

Similarly, other terms in II_2, III_2 can be bounded by $|a|\alpha\langle \omega_x^2, \psi_\alpha \rangle$ or the interpolation between $\langle \omega_x^2, \psi_\alpha \rangle$ and $\langle \omega^2, \varphi_\alpha \rangle$. We refer to the arXiv version of [19] for the detailed estimates. In particular, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle \omega_x^2, \psi_\alpha \rangle \leq -\left(\frac{1}{2} - C|a|\alpha\right)\langle \omega_x^2, \psi_\alpha \rangle + C\langle \omega^2, \varphi_\alpha \rangle^{1/2} \langle \omega_x^2, \psi_\alpha \rangle^{1/2} \\
+ \langle N(\omega)_x, \omega_x \psi_\alpha \rangle + \langle F_\alpha(\omega_\alpha)_x,\ \omega_x \psi_\alpha \rangle,
\end{aligned}
\tag{3.132}
$$

for some universal constant C.

In the following two subsections, we aim to control the nonlinear and error terms

$$
\langle N(\omega), \omega\varphi_\alpha \rangle,\ \langle F_\alpha(\omega_\alpha), \omega\varphi_\alpha \rangle,\ \langle N(\omega)_x, \omega_x\psi_\alpha \rangle, \langle F_\alpha(\omega_\alpha)_x,\ \omega_x\psi_\alpha \rangle
$$

in (3.131) and (3.132).

3.6.1.5 Estimates of nonlinear terms

Recall from (3.114) and (3.116) that

$$
c_l x + au = a(u - u_x(0)x) = a\tilde{u}, \quad c_\omega + u_x = u_x - u_x(0) = \tilde{u}_x.
$$

For the nonlinear terms in (3.131) and (3.132), we use integration by parts to obtain

$$
\begin{aligned}
\langle N(\omega), \omega\varphi_\alpha\rangle &= \langle (c_\omega + u_x)\omega - (c_l x + au)\omega_x, \omega\varphi_\alpha\rangle = \Big\langle \tilde{u}_x + \frac{(a\tilde{u}\varphi_\alpha)_x}{2\varphi_\alpha}, \omega^2\varphi_\alpha\Big\rangle \\
&= \langle \tilde{u}_x, \omega^2\varphi_\alpha\rangle + \frac{a}{2}\Big\langle \left(\tilde{u}_x + \frac{\tilde{u}}{x}\frac{x\varphi_{\alpha,x}}{\varphi_\alpha}\right), \omega^2\varphi_\alpha\Big\rangle \triangleq I_1 + I_2, \\
\langle N(\omega)_x, \omega_x\psi_\alpha\rangle &= \langle ((c_\omega + u_x)\omega - (c_l x + au)\omega_x)_x, \omega_x\psi_\alpha\rangle \\
&= \langle u_{xx}\omega + \tilde{u}_x\omega_x, \omega_x\psi_\alpha\rangle - a\Big\langle \tilde{u}_x\omega_x + \tilde{u}\omega_{xx}, \omega_x\psi_\alpha\Big\rangle \\
&= \langle \tilde{u}_x\omega_x, \omega_x\psi_\alpha\rangle + \langle u_{xx}\omega, \omega_x\psi_\alpha\rangle + a\Big\langle -\tilde{u}_x + \frac{(\tilde{u}\psi_\alpha)_x}{2\psi_\alpha}, \omega_x^2\psi_\alpha\Big\rangle \\
&\triangleq II_1 + II_2 + II_3.
\end{aligned}
$$

For each term I_i, II_j, we use Lemma 3.6.3 to control the L^∞ norm of $\omega, \tilde{u}/x, \tilde{u}_x$ or $\tilde{u}_x - \tilde{u}/x$, and use $\langle \omega^2, \varphi_\alpha\rangle$, $\langle \omega_x^2, \psi_\alpha\rangle$ to control other terms. We present the estimate of II_3 that has a large coefficient a and is more complicated. Other terms can be estimated similarly. For II_3, we notice that

$$
-\tilde{u}_x + \frac{(\tilde{u}\psi_\alpha)_x}{2\psi_\alpha} = -\frac{1}{2}\tilde{u}_x + \frac{1}{2}\frac{\tilde{u}}{x}\frac{\psi_{\alpha,x}x}{\psi_\alpha} = -\frac{1}{2}\left(\tilde{u}_x - \frac{\tilde{u}}{x}\right) + \frac{1}{2}\frac{\tilde{u}}{x}\left(\frac{\psi_{\alpha,x}x}{\psi_\alpha} - 1\right).
$$

Then we use the L^∞ estimate (3.124) to control $\tilde{u}_x - \tilde{u}/x$, (3.123) to control $\tilde{u}/x = u/x - u_x(0)$ and (3.120) to estimate the terms involving ψ_α. This gives

$$
\begin{aligned}
II_3 &= \frac{a}{2}\Big\langle -\left(\tilde{u}_x - \frac{\tilde{u}}{x}\right) + \frac{\tilde{u}}{x}\left(\frac{\psi_{\alpha,x}x}{\psi_\alpha} - 1\right), \omega_x^2\psi_\alpha\Big\rangle \\
&\lesssim |a|\left(\Big|\Big|\tilde{u}_x - \frac{\tilde{u}}{x}\Big|\Big|_{L^\infty} + ||u_x||_\infty\Big|\Big|\frac{\psi_{\alpha,x}x}{\psi_\alpha} - 1\Big|\Big|_{L^\infty}\right)\langle \omega_x^2, \psi_\alpha\rangle \\
&\lesssim (|a|\alpha\langle \omega_x^2, \psi_\alpha\rangle^{1/2} + |a|\alpha\langle \omega^2, \varphi_\alpha\rangle^{1/4}\langle \omega_x^2, \psi_\alpha\rangle^{1/4})\langle \omega_x^2, \psi_\alpha\rangle \\
&\lesssim (\langle \omega^2, \varphi_\alpha\rangle + \langle \omega_x^2, \psi_\alpha\rangle)^{3/2},
\end{aligned}
\tag{3.133}
$$

where we have used $|a\alpha| < 1$. Similarly, we have

$$
I_1, I_2, II_1, II_2 \lesssim (\langle \omega^2, \varphi_\alpha\rangle + \langle \omega_x^2, \psi_\alpha\rangle)^{3/2}. \tag{3.134}
$$

Combining (3.133) and (3.134), we obtain the following estimates for the nonlinear terms

$$
\begin{aligned}
\langle N(\omega), \omega\varphi_\alpha\rangle &= I_1 + I_2 \lesssim (\langle \omega^2, \varphi_\alpha\rangle + \langle \omega_x^2, \psi_\alpha\rangle)^{3/2}, \\
\langle N(\omega)_x, \omega_x\psi_\alpha\rangle &= II_1 + II_2 + II_3 \lesssim (\langle \omega^2, \varphi_\alpha\rangle + \langle \omega_x^2, \psi_\alpha\rangle)^{3/2}.
\end{aligned}
\tag{3.135}
$$

3.6.1.6 Estimates of the error terms

Recall the error terms in the weighted L^2, H^1 estimates in (3.131) and (3.132) are given by

$$\langle F_\alpha(\omega_\alpha), \omega\varphi_\alpha\rangle, \quad \langle (F_\alpha(\omega_\alpha))_x, \omega_x\psi_\alpha\rangle.$$

Using the Cauchy–Schwarz inequality and the error estimate (3.121), we obtain

$$\begin{aligned}\langle F_\alpha(\omega_\alpha), \omega\varphi_\alpha\rangle &\leq \langle F_\alpha(\omega_\alpha)^2, \varphi_\alpha\rangle^{1/2}\langle \omega^2, \varphi_\alpha\rangle^{1/2} \lesssim |a|\alpha\langle \omega^2, \varphi_\alpha\rangle^{1/2}, \\ \langle (F_\alpha(\omega_\alpha))_x, \omega_x\psi_\alpha\rangle &\leq \langle (F_\alpha(\omega_\alpha))_x^2, \psi_\alpha\rangle^{1/2}\langle \omega_x^2, \psi_\alpha\rangle^{1/2} \lesssim |a|\alpha\langle \omega_x^2, \psi_\alpha\rangle^{1/2}.\end{aligned} \tag{3.136}$$

3.6.1.7 Nonlinear stability and convergence to self-similar solution

Now, we combine the weighted L^2, H^1 estimates (3.131), (3.132), the estimates of nonlinear terms (3.135) and error terms (3.136). Using these estimates and an argument similar to that in the analysis of nonlinear stability in Section 3.4.2 , we can choose an absolute constant $0 < \mu$ such that the following energy

$$E^2(t) \triangleq \langle \omega^2, \varphi_\alpha\rangle + \mu\langle \omega_x^2, \psi_\alpha\rangle$$

satisfies the differential inequality

$$\frac{1}{2}\frac{d}{dt}E^2(t) \leq -\left(\frac{3}{8} - C|a|\alpha\right)E^2(t) + C|a|\alpha E(t) + CE^3(t), \tag{3.137}$$

where $C > 0$ is an absolute constant. From (3.120), we have

$$\begin{aligned}|u_x(0)| &\lesssim \int |\omega_x(y)||\log(y)|dy \lesssim (\int \omega_x^2\psi_\alpha)^{1/2}(\int \psi_\alpha^{-1}|\log y|^2)^{1/2} \\ &\lesssim E(t)\left(\alpha^3\int |\log y|^2(|y|^{1+\alpha} + |y|^{1-3\alpha})^{-1}dy\right)^{1/2} \lesssim E(t)(\alpha^3\alpha^{-1})^{1/2} \lesssim \alpha E(t).\end{aligned}$$

The normalization condition (3.114) implies

$$|c_\omega(t)| = |u_x(t,0)| \leq C_3\alpha E(t), \quad |c_l(t)| = |au_x(0)| \leq C_3|a|\alpha E(t), \tag{3.138}$$

for some absolute constant $C_3 > 0$.

The remaining steps are essentially the same as those in the proof of Theorem 3.2 in Sections 3.4.2, 3.4.3 for the De Gregorio model in the case of small $|a|$ so we omit the details here. We refer to the arXiv version of [19] for the details and conclude the proof of Theorem 3.3.

3.7 Finite time blowup for negative a with C^∞ initial data

For the sake of completeness, we state the finite time blowup result of (3.2) for negative a with smooth initial data.

Theorem 3.5. *Let $\omega \in C^\infty(S^1)$ be an odd function such that $u_x(0) = H\omega(0) > 0$. Then (3.2) with $a < 0$ develops a singularity in finite time.*

The real line case has been proved by Castro and Córdoba [5]. We consider the case of S^1 with π periodic solution. The corresponding Hilbert transform is given by

$$u_x = H\omega = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \omega(y)\cot(x-y)dy.$$

Proof. Taking the Hilbert transform on (3.2) yields

$$(u_x)_t = \frac{1}{2}(u_x^2 - \omega^2) - aH(u\omega_x).$$

Note that $\omega(0) = 0$. Choosing $x = 0$ gives

$$\frac{d}{dt}u_x(t,0) = \frac{1}{2}u_x(t,0)^2 - aH(u\omega_x)(t,0). \tag{3.139}$$

Next we show that $H(u\omega_x)(t,0) \le 0$. Since ω is odd, π-periodic, and smooth locally in time , it admits a decomposition

$$\omega(t,x) = \sum_{n\ge1} a_n(t)\sin(2nx), \quad \omega_x = \sum_{n\ge1} 2na_n(t)\cos(2nx),$$

for some $a_n(t)$ decays sufficiently fast as $n \to +\infty$. It is easy to show that

$$u(t,x) = -\sum_{n\ge1}\frac{a_n}{2n}\sin(2nx).$$

Next, we compute $u/\sin(x), \omega_x\cos x$. Using telescoping, we get

$$\frac{\sin(2nx)}{\sin(x)} = \sum_{1\le k\le n} 2\cos((2k-1)x),$$
$$\cos(2nx)\cos x = \frac{\cos(2n-1)x + \cos(2n+1)x}{2}.$$

It follows that

$$\frac{u}{\sin x} = -\sum_{n\ge1}\frac{a_n}{2n}\sum_{1\le k\le n}2\cos((2k-1)x) = -\sum_{k\ge1}\cos((2k-1)x)\sum_{n\ge k}\frac{a_n}{n},$$
$$\omega_x\cos x = \sum_{n\ge1}2na_n\frac{\cos(2n-1)x+\cos(2n+1)x}{2}$$
$$= \sum_{n\ge1}\cos((2n-1)x)(na_n + (n-1)a_{n-1}),$$

where $a_0 = 0$, and we have used summation by parts to get the last two identities, which are valid since a_n decays sufficiently fast. Using the orthogonality of $\{\cos((2n-1)x)\}_{n\geq 1}$ on $L^2(-\pi/2, \pi/2)$, we derive

$$H(u\omega_x)(t,0) = -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \frac{u}{\sin x}\omega_x \cos(x)dx = \frac{1}{2}\sum_{k\geq 1}(\sum_{n\geq k}\frac{a_n}{n})(ka_k + (k-1)a_{k-1}). \tag{3.140}$$

Denote $S_k \triangleq \sum_{n\geq k} \frac{a_n}{n}$ for $k \geq 1$ and $S_0 = 0$. Since a_n decays sufficiently fast, so does S_n. We then have $a_k = k(S_k - S_{k+1})$ and

$$ka_k + (k-1)a_{k-1} = k^2(S_k - S_{k+1}) + (k-1)^2(S_{k-1} - S_k).$$

We can reduce $H(u\omega_x)(t,0)$ to

$$\begin{aligned} H(u\omega_x)(t,0) &= \frac{1}{2}\sum_{k\geq 1} S_k(k^2(S_k - S_{k+1}) + (k-1)^2(S_{k-1} - S_k)) \\ &= \frac{1}{2}\sum_{k\geq 1} S_k^2(2k-1) - \frac{1}{2}\sum_{k\geq 1} S_k S_{k+1} k^2 + \frac{1}{2}\sum_{k\geq 1} S_k S_{k-1}(k-1)^2 \\ &= \frac{1}{2}\sum_{k\geq 1} S_k^2(2k-1) \geq 0. \end{aligned}$$

Consequently, for $a < 0$, (3.139) implies

$$\frac{d}{dt}u_x(t,0) \geq \frac{1}{2}u_x^2(t,0).$$

Since $u_x(0,0) > 0$, it follows that the solution must develop a finite time singularity. ∎

Chapter 4

ASYMPTOTICALLY SELF-SIMILAR SINGULARITY OF THE HOU-LUO MODEL

In this chapter, we study the asymptotically self-similar singularity of Hou-Luo model [86] introduced in Section 1.3.6.2

$$\begin{aligned} \omega_t + u\omega_x &= \theta_x, \\ \theta_t + u\theta_x &= 0, \quad u_x = H\omega. \end{aligned} \tag{4.1}$$

Since the HL model can be seen as the restriction of the 2D Boussinesq equations on the boundary and it captures several features of the 2D Boussinesq equations (see Section 1.3.6.2), our analysis of (4.1) serves as an intermediate step toward the analysis of the full 2D Boussinesq equations. The analysis of (4.1) is much more complicated than that of the DG model in Chapter 3 since (4.1) is a more complicated coupled system, while the DG model (3.1) is a scalar equation.

4.1 Main results

The main result of this chapter is stated by the informal theorem below. The more precise and stronger statement will be given by Theorem 4.2 in Section 2.10.1.

Theorem 4.1. *There is a family of initial data (θ_0, ω_0) with $\theta_{0,x}, \omega_0 \in C_c^\infty$, such that the solution of the HL model (4.1) will develop a focusing asymptotically self-similar singularity in finite time. The self-similar blowup profile $(\theta_\infty, \omega_\infty)$ is unique within a small energy ball and its associated scaling exponents $c_{l,\infty}, c_{\omega,\infty}$ satisfy $|\lambda - 2.99870| \leq 6\cdot 10^{-5}$ with $\lambda = c_{l,\infty}|c_{\omega,\infty}|^{-1}$. Moreover, the C^γ norm of θ is uniformly bounded up to the blowup time T, and the C^β norm of θ blows up at T for any $\beta \in (\gamma, 1]$ with $\gamma = \frac{\lambda-2}{\lambda}$.*

Using the self-similar profile $(\theta_\infty, \omega_\infty)$, we can construct the self-similar blowup solution

$$\omega_*(x,t) = \frac{1}{(1-t)|c_{\omega,\infty}|}\omega_\infty(\frac{x}{(1-t)^\lambda}), \quad \theta_*(x,t) = \frac{1}{(1-t)^{2-\lambda}|c_{\omega,\infty}|}\theta_\infty(\frac{x}{(1-t)^\lambda}), \tag{4.2}$$

that blows up at $T = 1$. The blowup exponent $\lambda \approx 2.99870$ in the HL model is surprisingly close to the blowup exponent $\lambda \approx 2.9215$ of the 3D Euler equations considered by Luo-Hou [86, 87]. An important property that characterizes the stable nature of the blowup in the HL model is that $\bar{c}_l x + \bar{u} \geq 0.49x, \bar{c}_l = 3, \bar{u} < 0$ for any $x \geq 0$, here $\bar{u}, \bar{c}_l$ are the velocity and the scaling exponents of an approximate self-similar profile. We use this property to extract the main damping effect from the linearized operator in the near field using some carefully designed singular weights.

As we will show later, $\bar{c}_l x + \bar{u}$ is the velocity field for the linearized equation in the dynamic rescaling formulation. The inequality $\bar{c}_l x + \bar{u} \geq 0.49x, x \geq 0$ implies that the perturbation is transported from the near field to the far field and then damped by the damping term $\bar{c}_\omega \omega$ in the ω equation and by $2\bar{c}_\omega \theta_x$ in the θ_x equation. This is the main physical mechanism that generates the dynamic stability of the self-similar blowup in the HL model. We believe that this also captures the dynamic stability of the blowup scenario considered by Luo-Hou along the boundary [86, 87], whose numerical evidence has been reported in [85].

There are four important components of our analysis for the HL model. The first one is to construct the approximate steady state with sufficiently small residual error by decomposing it into a semi-analytic part that captures the far field behavior of the solution and a numerically computed part that has compact support. See more discussion in Section 4.4. The second one is that we extract the damping effect from the local terms in the linearized equations by using carefully designed singular weights. The third one is that the contributions from the advection terms are relatively weak compared with those coming from the vortex stretching terms. As a result, we can treat those terms coming from advection as perturbation to those from vortex stretching. The last one is to apply some sharp functional inequalities to control the nonlocal terms and take into account cancellation among various nonlocal terms. This enables us to show that the contributions from the nonlocal terms are relatively small compared with those from the local terms and can be controlled by the damping terms. We refer to Section 2.10.1 for more detailed discussion of the main ingredients in our stability analysis.

We believe that the analysis of the 2D Boussinesq equations and 3D Euler equations with smooth initial data and boundary would benefit from the four important components mentioned above. The stability analysis of the HL model is established based on some weighted L^2 space. For the 2D Boussinesq equations and 3D Euler equations, a wider class of functional spaces, e.g., weighted L^p or weighted C^α spaces, can be explored to derive larger damping effect from the linearized equations and to further establish stability analysis.

There is an interesting implication of our blowup results for the self-similar solution (ω_*, θ_*) defined in (4.2). In Section 4.6.1, we show that the profile satisfies $\lim_{x\to\infty} \theta_\infty(x)|x|^{-\gamma} = C$ for some $C > 0$ (see (4.2)). Thus, we have $\lim_{t\to 1} \theta_*(x,t) \to C|x|^\gamma$ for any $x \neq 0$. Since $0 < \gamma < 1$, the self-similar solution forms a cusp singularity at $x = 0$ as $t \to 1$. Moreover, from Theorem 4.1, for a class of initial data θ_0, the C^γ norm of the singular solution θ is uniformly bounded up to the blowup time. Note that from Theorem 4.1, we have $|\gamma - 0.33304| < 2\cdot 10^{-5}$, thus $\gamma \approx \frac{1}{3}$ and $\lim_{t\to 1} \theta_*(x,t) = C|x|^\gamma \approx C|x|^{1/3}$. Similarly, we can generalize the method of analysis to prove $\lim_{t\to 1} \omega_*(x,t) = C_2|x|^{(\gamma-1)/2} \approx C_2|x|^{-1/3}$. Interestingly, the limiting behavior is closely related to a family of explicit solutions of (4.1) discovered by Hoang and Radosz in [64]

$$\omega(x,t) = k|x|^{-1/3}\mathrm{sgn}(x), \quad \theta(x,t) = c_1 k^2 |x|^{1/3} + c_2 k^3 t, \tag{4.3}$$

where $c_1, c_2 > 0$ are suitable constants and $k > 0$ is arbitrary. We remark that from Theorem 4.1, the $C^{1/3}$ norm of θ from a class of smooth initial data that we consider blows up at the singularity time since $\frac{1}{3} > \gamma$, while the non-smooth θ in (4.3) remains in $C^{1/3}$ for all time.

The cusp formation and the Hölder regularity on θ are related to the $C^{1/2}$ conjecture by Silvestre and Vicol in [103] and the cusp formation on the Cordoba-Cordoba-Fontelos (CCF) model [19, 28, 76, 84], which is the $\theta-$equation in (4.1) coupled with $u = H\theta$. The cusp formation of a closely related model was established in [63], and the $C^{1/2}$ conjecture was studied in [44, 49] for a class of $C^{1,\alpha}$ initial data with small α. Using the same method for the HL model, we have obtained an approximate self-similar profile for the CCF model with residual $O(10^{-8})$ and $\gamma = 0.5414465$, which is accurate up to six digits. This blowup exponent γ is qualitatively similar to that obtained in [88] for the generalized Constantin-Lax-Majda model (gCLM) (see[97]) with $a = -1$. In a follow-up work, we will generalize our method of analysis to study the cusp formation of the CCF model, and rigorously prove that $\theta \in C^\gamma$ up to the singularity time with $\gamma > 1/2$. Moreover, the C^β norm of θ will blow up at the singularity time for any $\beta > \gamma$.

The rest of the chapter is organized as follows. In Section 2.10.1, we outline some main ingredients in our stability analysis by using the dynamic rescaling formulation. Section 4.3 is devoted to linear stability analysis. In Section 4.4, we discuss some technical difficulty in obtaining an approximate steady state with a residual error of order 10^{-10}. In Section 4.5, we perform nonlinear stability analysis and establish the finite time blowup result. In Section 4.6, we estimate the Hölder regularity of the singular solution. In Section 4.7, we give a formal derivation to demonstrate that both the HL model and the 2D Boussinesq equations with $C^{1,\alpha}$ initial data for velocity and θ and with boundary have the same leading system for small α. We make some concluding remarks in Chapter 6. Some technical estimates and derivations are deferred to Appendix C.

4.2 Outline of the main ingredients in the stability analysis

In this section, we will outline the main ingredients in our stability analysis by using the dynamic rescaling formulation for the HL model. We will follow the general framework introduced in Section 1.3. The most essential part of our analysis lies in the linear stability. We need to use a number of techniques to extract the damping effect from the linearized operator around the approximate steady state of the dynamic rescaling equations and obtain sharp estimates of various nonlocal terms. Since the damping coefficient we obtain is relatively small (about 0.03), we need to construct an approximate steady state with a very small residual error of order 10^{-10}. This is extremely challenging since the solution is supported on the whole real line with a slowly decaying tail in the far field. We use analytic estimates and numerical analysis with rigorous error control to verify that the residual error is small in the energy norm. See detailed discussions in Section 4.4 and Section 10 of the Supplementary Material [21].

Passing from linear stability to nonlinear stability is relatively easier since the perturbation is quite small due to the small residual error. Yet we need to verify various inequalities involving the approximate steady state using the interval arithmetic [55, 96, 100] and numerical analysis with computer assistance. The most essential part of the linear stability analysis can be established based on the grid point values of the approximate steady state constructed on a relatively coarse grid, which does not involve the lengthy rigorous verification. See more discussion in Section 4.3.13. The reader who is not interested in the rigorous verification can skip the lengthy verification process presented in the Supplementary Material [21].

4.2.1 Dynamic rescaling formulation

An essential tool in our analysis is the dynamic rescaling formulation. Let $\omega_{phy}(x,t), \theta_{phy}(x,t)$ be the solutions of the physical equations (4.1). Following the ideas in Section 2.1.4, we obtain that

$$\omega(x,\tau) = C_\omega(\tau)\omega_{phy}(C_l(\tau)x, t(\tau)), \quad \theta(x,\tau) = C_\theta(\tau)\theta_{phy}(C_l(\tau)x, t(\tau))$$

are the solutions to the dynamic rescaling equations

$$\omega_\tau + (c_l x + u)\omega_x = c_\omega \omega + \theta_x, \quad \theta_\tau + (c_l x + u)\theta_x = c_\theta \theta, \quad u_x = H\omega, \tag{4.4}$$

where

$$\begin{aligned} t(\tau) &= \int_0^\tau C_\omega(s)ds, \quad C_\omega(\tau) = \exp\left(\int_0^\tau c_\omega(s)ds\right), \\ C_l(\tau) &= \exp\left(\int_0^\tau -c_l(s)ds\right), \quad C_\theta(\tau) = \exp\left(\int_0^\tau c_\theta(s)ds\right). \end{aligned}$$

In order for the dynamic rescaling formulation to be equivalent to the original HL model, we must enforce a relationship among the three scaling parameters, c_l, c_ω and c_θ, i.e. $c_\theta = c_l + 2c_\omega$.

To simplify our presentation, we still use t to denote the rescaled time in (4.4). Taking the x derivative on the θ equation in (4.4) yields

$$\begin{aligned} \omega_t + (c_l x + u)\omega_x &= c_\omega \omega + \theta_x, \\ (\theta_x)_t + (c_l x + u)\theta_{xx} &= (c_\theta - c_l - u_x)\theta_x = (2c_\omega - u_x)\theta_x, \quad u_x = H\omega, \end{aligned} \tag{4.5}$$

where $c_\theta = c_l + 2c_\omega$. We still have two degrees of freedom in choosing c_l, c_ω to uniquely determine the dynamic rescaled solution. We impose the following normalization conditions on c_ω, c_l

$$c_l = 2\frac{\theta_{xx}(0)}{\omega_x(0)}, \quad c_\omega = \frac{1}{2}c_l + u_x(0). \tag{4.6}$$

These two normalization conditions play the role of forcing

$$\theta_{xx}(t,0) = \theta_{xx}(0,0), \quad \omega_x(t,0) = \omega_x(0,0) \tag{4.7}$$

for all time. Our study shows that enforcing $\theta_{xx}(t,0)$ to be independent of time is essential for stability by eliminating a dynamically unstable mode in the dynamic rescaling formulation.

4.2.2 Main result

Throughout this chapter, we will consider solution of (4.4) with odd ω, θ_x and $\theta(t,0)=0$. Under this setting, it is not difficult to show that the odd symmetries of θ_x, ω, u and the condition $\theta(t,0)=0$ are preserved by the equations.

Due to the symmetry, we restrict the inner product and L^2 norm to $\mathbb{R}_+$

$$\langle f, g\rangle \triangleq \int_0^\infty fg dx, \quad ||f||_2^2 = \int_0^\infty f^2 dx. \tag{4.8}$$

Let ψ, φ be the singular weights defined in (4.25), and λ_i be the parameter given in (C.25). We use the following energy in our energy estimates

$$\begin{aligned} E^2(f,g) \triangleq & ||f\psi^{1/2}||_2^2 + \lambda_1 ||g\psi^{1/2}||_2^2 + \lambda_2 \frac{\pi}{2}(Hg(0))^2 + \lambda_3 \langle f, x^{-1}\rangle^2 \\ & + \lambda_4 (||D_x f \psi^{1/2}||_2^2 + \lambda_1 ||D_x g \varphi^{1/2}||_2^2), \end{aligned} \tag{4.9}$$

where $Hg(0) = -\frac{1}{\pi}\int_{\mathbb{R}} g x^{-1} dx$ is related to c_ω in (4.6). Our main result is the following.

Theorem 4.2. *Let $(\bar\theta, \bar\omega, \bar c_l, \bar c_\omega)$ be the approximate self-similar profile constructed in Section 4.4, and $E_* = 2.5\cdot 10^{-5}$. For odd initial data $\theta_{0,x}, \omega_0$ of (4.4) with $\theta_0(0) = 0$ and a small perturbation to $(\bar\theta_x, \bar\omega)$, $E(\theta_{0,x} - \bar\theta_x, \omega_0 - \bar\omega) \leq E_*$, we have (a) $E(\theta_x - \bar\theta_x, \omega - \bar\omega) \leq E_*$ for all time.*

(b) The solution $(\theta, \omega, c_l, c_\omega)$ converges to a steady state of (4.4) $(\theta_\infty, \omega_\infty, c_{l,\infty}, c_{\omega,\infty})$

$$||(\theta_x(t) - \theta_{\infty,x})\psi^{1/2}||_2 + ||(\omega(t)-\omega_\infty)\varphi^{1/2}||_2 + ||c_l(t) - c_{l,\infty}||_2 + ||c_\omega(t) - c_{\omega,\infty}||_2 \leq Ce^{-\kappa_2 t}$$

exponentially fast, for some $\kappa_2 > 0, C > 0$. Moreover, $(\theta_\infty, \omega_\infty, c_{l,\infty}, c_{\omega,\infty})$ enjoys the regularity $E(\theta_{x,\infty} - \bar\theta_x, \omega_\infty - \bar\omega) \leq E_*$, and is the unique steady state in the class $E(\theta_x - \bar\theta_x, \omega - \bar\omega) \leq E_*$ with normalization conditions (4.6) and $\theta(0) = 0$, and odd assumption on θ_x, ω.

(c) Let $\gamma = \frac{c_{\theta,\infty}}{c_{l,\infty}} = \frac{c_{l,\infty} + 2c_{\omega,\infty}}{c_{l,\infty}}$. We have $|\frac{c_{\omega,\infty}}{c_{l,\infty}} - 2.99870| \leq 6\cdot 10^{-5}$. Moreover, the solution enjoys the Hölder estimates $\theta_\infty \in C^\gamma$ and $\sup_{t\geq 0} ||\theta||_{C^\gamma} \lesssim 1$.

(d) For the physical equations (4.1) with the above initial data, the solution blows up in finite time T with the following blowup estimates for any $\gamma < \beta \leq 1$

$$||\theta_{phy}(t)||_{C^\beta} \gtrsim (T-t)^{-\delta}, \quad \delta = \frac{2(\beta-\gamma)}{1-\gamma} > 0.$$

If in addition $\theta_{0,x}|x|^{1-\gamma} \in L^{\infty}$, the C^{γ} norm is uniformly bounded up to the blowup time: $\sup_{t\in[0,T)} ||\theta_{phy}(t)||_{C^{\gamma}} \lesssim 1$.

The assumption $\theta_{0,x}|x|^{1-\gamma} \in L^{\infty}$ in (d) is to ensure the decay $|\theta_{0,x}| \leq C|x|^{\gamma-1}$, which is consistent with $\theta_0 \in C^{\gamma}$. In fact, if $\theta_0 \in C^{\gamma}$, we get $|\theta_0(x)| \lesssim 1 + |x|^{\gamma}$. Then, formally, $\theta_{0,x}$ has a decay rate $|x|^{\gamma-1}$.

4.2.3 Main ingredients in our stability analysis

The key step to prove Theorem 4.2 is the stability analysis. We will outline several important ingredients to establish it in this subsection.

4.2.3.1 The stability of the linearized operator

The most essential part of our analysis is the linear stability of the linearized operator around the approximate steady state $(\bar{\theta}, \bar{\omega}, \bar{c}_l, \bar{c}_{\omega})$. To simplify our notation, we still use ω, u, θ, c_l, and c_{ω} to denote the perturbation. The linearized system for the perturbation is given below by neglecting the nonlinear and error terms:

$$\begin{aligned} \partial_t \theta_x + (\bar{c}_l x + \bar{u})\theta_{xx} &= (2\bar{c}_{\omega} - \bar{u}_x)\theta_x + (2c_{\omega} - u_x)\bar{\theta}_x - u\bar{\theta}_{xx}, \\ \omega_t + (\bar{c}_l x + \bar{u})\omega_x &= \bar{c}_{\omega}\omega + \theta_x + c_{\omega}\bar{\omega} - u\bar{\omega}_x, \quad c_{\omega} = u_x(t,0), \quad c_l = 0. \end{aligned} \tag{4.10}$$

The condition $c_{\omega} = u_x(t,0), c_l = 0$ is a consequence of the normalization conditions (4.7). There are two groups of terms in the above system, one representing the local terms and the other representing the nonlocal terms. Among the nonlocal terms, we can further group them into three subgroups, one from the vortex stretching term, one from the advection term, and the remaining from the rescaling factor c_{ω}.

As in our previous works [16, 19], we design some singular weights to extract the damping effect from the local terms. As we mentioned before, we will use $\bar{c}_l x + \bar{u} \geq 0.49x$ to extract an $O(1)$ damping effect. Since the damping coefficient that we can extract from the local terms is relatively small and the linearized operator is not a normal operator, we typically expect to have a transient growth for a standard energy norm of the solution to (4.10). This will present considerable difficulty for us to obtain nonlinear stability since the approximate steady state also introduces a residual error. To overcome this difficulty, we need to design a weighted energy norm carefully so that the energy of the solution to the linearized equations decreases monotonically in time. We remark that weighted energy estimates with singular weights have also been used in [13, 14, 42, 73] for nonlinear stability analysis.

4.2.3.2 Control of nonlocal terms

The most challenging part of the linear stability analysis is how to control several nonlocal terms that are of $O(1)$. It is essential to obtain sharp estimates of these nonlocal terms by applying sharp weighted functional inequalities, e.g., Lemma C.0.8, and taking into account the cancellation among different nonlocal terms and the structure of the coupled system. We have also used the L^2 isometry property and several other properties of the Hilbert transform in an essential way. We remark that some of these properties of the Hilbert transform have been used in the previous works, see e.g., [5, 13, 19, 29, 44]. Based on our observation that the blowup is driven by vortex stretching and the advection is relatively weak compared with vortex stretching, we will treat the nonlocal terms that are generated by the advection terms, e.g., $u\bar{\theta}_{xx}$ in (4.10), as perturbation to the linearized vortex stretching terms, e.g., $u_x\bar{\theta}_x$ in (4.10). We will use the following five strategies in our analysis.

(1) The decomposition of the velocity field. We first denote $\tilde{u} \triangleq u - u_x(0)x$ and choose a constant $c = 1/(2p-1)$ where p is related to the order of the singular weight $|x|^{-p}$ being used. We further decompose $\tilde{u}$ into a main term and a remainder term as follows:

$$\tilde{u} = cx\tilde{u}_x + (\tilde{u} - cx\tilde{u}_x) \triangleq \tilde{u}_M + \tilde{u}_R, \tag{4.11}$$

where $\tilde{u}_M = cx\tilde{u}_x$ and $\tilde{u}_R = (\tilde{u} - cx\tilde{u}_x)$. The contribution from the remainder term $\tilde{u}_R$ is smaller than $x\tilde{u}_x$ due to an identity (see Appendix C.1.1)

$$||(\tilde{u} - \frac{1}{2p-1}\tilde{u}_x x)x^{-p}||_2^2 = \frac{1}{(2p-1)^2}\int_{\mathbb{R}_+} \frac{\tilde{u}_x^2}{x^{2p-2}} dx. \tag{4.12}$$

We can choose $p = 3$ in the near field, which enables us to gain a small factor of $1/5$ in estimating the $\tilde{u}_R$ term in terms of the weighted norm of $\tilde{u}_x$.

(2) Exploiting the nonlocal cancellation between $\tilde{u}_x$ and ω. For the main term $\tilde{u}_M = cx\tilde{u}_x$ and the vortex stretching term $-u_x\bar{\theta}_x$, we use an orthogonality between $\tilde{u}_x$ and ω

$$\langle \tilde{u}_x, \omega x^{-3}\rangle = \langle H\omega - H\omega(0), \omega x^{-3}\rangle \ = \ 0 \tag{4.13}$$

(see Lemma C.0.4). We will use similar orthogonal properties to exploit the cancellation between $-\tilde{u}_x\bar{\theta}_x$ in the θ_x equation and θ_x in the ω equation in (4.10) by performing the weighted L^2 estimates for θ_x and ω together. To illustrate this idea, we consider the following model:

Model 1 for nonlocal interaction

$$\partial_t \theta_x = -(u_x - u_x(0))\bar{\theta}_x, \quad \omega_t = \theta_x.$$

The above system is derived by dropping other terms in (4.10). The profile $\bar{\theta}_x$ satisfies $\bar{\theta}_x(0) = 0$ and $\bar{\theta}_x > 0$ for $x > 0$.

By performing $L^2(\rho_1)$ estimate on θ_x and $L^2(\rho_2)$ estimate on ω, we get

$$\frac{1}{2}\frac{d}{dt}(\langle \theta_x, \theta_x \rho_1\rangle + \langle \omega, \omega\rho_2\rangle) = -\langle (u_x - u_x(0))\bar{\theta}_x\rho_1, \theta_x\rangle + \langle \omega\rho_2, \theta_x\rangle \triangleq I. \quad (4.14)$$

From (4.13), we know that $(u_x - u_x(0))x^{-2}$ and ωx^{-1} are orthogonal. Formally, I is the sum of the projections of θ_x onto two directions that are orthogonal. To exploit this orthogonality, we choose $\rho_1 = (\mu x \bar{\theta}_x)^{-1}\rho_2$ with any $\mu > 0$. We can rewrite I as follows

$$I = \langle -(u_x - u_x(0))x^{-2}, \theta_x\bar{\theta}_x\rho_1 x^2\rangle + \langle \mu\omega x^{-1}, \theta_x\bar{\theta}_x\rho_1 x^2\rangle \triangleq \langle A + B, \theta_x\bar{\theta}_x\rho_1 x^2\rangle,$$

where $A = -(u_x - u_x(0))x^{-2}$ and $B = \mu\omega x^{-1}$. Applying the Cauchy-Schwarz inequality yields

$$I \le ||A + B||_2 ||\theta_x\bar{\theta}_x\rho_1 x^2||_2.$$

The equality can be achieved if $\theta_x\bar{\theta}_x\rho_1 x^2 = c(A + B)$ for some c. Expanding $||A + B||_2$ and using Lemma C.0.4 with $f = \omega$ and $g = u$, we get

$$||A + B||_2^2 = ||A||_2^2 + ||B||_2^2 + 2\langle A, B\rangle = ||A||_2^2 + ||B||_2^2, \quad (4.15)$$

which is sharper than the trivial estimate $||A + B||_2 \le ||A||_2 + ||B||_2$. The $||A||_2^2$ term can be further bounded by $||\omega x^{-1}||_2^2$ using the L^2 isometry of the Hilbert transform in Lemma C.0.2. The $||B||_2^2$ term can be bounded by the weighted L^2 norm of ω directly.

(3) Additional damping effect from c_ω. Another nonlocal term in (4.10) is $c_\omega = u_x(t, 0) = H(\omega)(t, 0)$. Physically, the role of c_ω is to rescale the amplitude of the blowup profile ω in the original physical variable so that the magnitude

of the dynamic rescaled profile remains $O(1)$ for all time. Thus, we expect that the dynamic rescaling parameter c_ω should also offer some stabilizing effect to the blowup profile and the linearized system (4.10). Indeed, by deriving an ODE for c_ω, we can extract an additional damping term, which will be used to control other nonlocal terms associated with c_ω. To illustrate this idea, we consider the following model:

Model 2 for the c_ω term

$$\partial_t \theta_x = c_\omega \bar{f}, \quad \partial_t \omega = \theta_x + c_\omega \bar{g}, \tag{4.16}$$

where $\bar{f}, \bar{g}$ are odd and $\bar{f}, \bar{g} > 0$ for $x > 0$ with $\bar{f} x^{-1}, \bar{g} x^{-1} \in L^1$. Note that the profile satisfies that $\bar{\theta}_x - x\bar{\theta}_{xx}, \bar{\omega} - x\bar{\omega}_x$ are odd and positive for $x > 0$. This system models the c_ω terms in (4.10) with coupling θ_x in ω equation by dropping other terms. Recall

$$c_\omega = -\frac{1}{\pi} \int_{\mathbb{R}} \frac{\omega}{x} dx = -\frac{2}{\pi} \langle \omega, x^{-1} \rangle.$$

Obviously, it can be bounded by some weighted L^2 norm of ω using the Cauchy-Schwarz inequality. Yet, the constant in this estimate is large. Denote $A = \langle \bar{f}, x^{-1} \rangle, B = \langle \bar{g}, x^{-1} \rangle$. By definition, $A, B > 0$. We derive an ODE for c_ω using the ω equation

$$\partial_t \langle \omega, x^{-1} \rangle = c_\omega \langle \bar{g}, x^{-1} \rangle + \langle \theta_x, x^{-1} \rangle = -\frac{2}{\pi} B \langle \omega, x^{-1} \rangle + \langle \theta_x, x^{-1} \rangle.$$

We see that the c_ω term in the ω equation in (4.16) has a damping effect, which is not captured by the weighted L^2 estimates. To handle the coupled term, we also derive an ODE for $\langle \theta_x, x^{-1} \rangle$ using the θ_x equation

$$\partial_t \langle \theta_x, x^{-1} \rangle = c_\omega \langle \bar{f}, x^{-1} \rangle = -\frac{2}{\pi} A \langle \omega, x^{-1} \rangle.$$

Multiplying both sides of these ODEs by $\langle \omega, x^{-1} \rangle$ or $\langle \theta_x, x^{-1} \rangle$, we yield

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \langle \omega, x^{-1} \rangle^2 &= -\frac{2}{\pi} B \langle \omega, x^{-1} \rangle^2 + \langle \theta_x, x^{-1} \rangle \langle \omega, x^{-1} \rangle, \\ \frac{1}{2} \partial_t \langle \theta_x, x^{-1} \rangle^2 &= -\frac{2}{\pi} A \langle \theta_x, x^{-1} \rangle \langle \omega, x^{-1} \rangle. \end{aligned} \tag{4.17}$$

The $\langle \theta_x, x^{-1} \rangle \langle \omega, x^{-1} \rangle$ terms in the above ODEs have cancellation. This implies that the c_ω term in the θ_x equation and θ_x term in the ω equation have cancellation, which is not captured by the weighted L^2 estimate. We will derive similar ODEs in the analysis of (4.10) and obtain damping term similar to $-\frac{2}{\pi}B\langle\omega, x^{-1}\rangle^2$ in the above ODEs, which enables us to control the c_ω terms in (4.10) effectively.

(4) Estimating the **u** term in (4.10). To estimate the u terms in (4.10) effectively, we have two approaches. The first approach is to exploit the cancellation between u and ω similar to that in Model 1. See Lemma C.0.4. The second approach is to decompose $\tilde{u}$ into the main term $\tilde{u}_M = cx\tilde{u}_x$ and an error term $\tilde{u}_R$ as (4.34). For $\tilde{u}_M$, we employ the estimates on u_x discussed previously. The error term $\tilde{u}_R$ enjoys better estimate (4.12) and is treated as a perturbation.

(5) Obtaining sharp estimates for other interaction terms. To obtain sharper estimates for a number of quadratic interaction terms, we introduce a number of parameters in various intermediate steps and optimize these parameters later by solving a constrained optimization problem. In the ODE for c_ω and the weighted L^2 estimates, we need to control a number of quadratic interaction terms, e.g., $\langle\omega, x^{-1}\rangle\cdot\langle\theta_x, x^{-1}\rangle$. We treat these interaction terms as the products of projection of θ_x and ω onto some low dimensional subspaces and reduce them to some quadratic forms in a finite dimensional space. This connection enables us to reduce the problem of obtaining sharp estimates of these terms to computing the largest eigenvalue $\lambda_{\max}$ of a matrix. We then compute $\lambda_{\max}$ as part of the constrained optimization problem to determine these parameters and obtain a sharper upper bound in the energy estimate.

4.3 Linear stability

In this section, we establish the linear stability of (4.23) in some weighted L^2 spaces.

4.3.1 Linearized operators around approximate steady state

The approximate steady state of (4.5) $(\bar{\theta}_x, \bar{\omega})$ we construct are odd with scaling factors

$$\bar{c}_l = 3, \quad |\bar{c}_\omega + 1.00043212| < 10^{-8}, \quad \bar{c}_\omega \approx -1.$$

It has regularity $\bar{\omega}, \bar{\theta}_x \in C^3$ and decay rates $\partial_x^i \bar{\omega} \sim x^{\alpha - i}, \partial_x^i \bar{\theta}_x \sim x^{2\alpha - i}, i = 0, 1, 2$ with α slightly smaller than $-1/3$. One can find plots of $(\bar{\omega}, \bar{\theta}_x)$ (with particular rescaling) in Figure 4.2 in Section 4.5.5. See detailed discussion in Section 4.4. Note that we do not require a C^∞ approximate steady state in our analysis, since the C^3 approximate steady state is regular enough for us to perform weighted H^1 estimates and establish its nonlinear stability.

Linearizing around the approximate steady state $(\bar{\theta}_x, \bar{\omega})$, we obtain the equations for the perturbation

$$\begin{aligned} \partial_t \theta_x + (\bar{c}_l x + \bar{u})\theta_{xx} &= (2\bar{c}_\omega - \bar{u}_x)\theta_x + (2c_\omega - u_x)\bar{\theta}_x - u\bar{\theta}_{xx} + F_\theta + N(\theta), \\ \omega_t + (\bar{c}_l x + \bar{u})\omega_x &= \bar{c}_\omega \omega + \theta_x + c_\omega \bar{\omega} - u\bar{\omega}_x + F_\omega + N(\omega), \end{aligned} \tag{4.18}$$

where the error terms F_θ, F_ω and the nonlinear terms $N(\theta), N(\omega)$ read

$$\begin{aligned} F_\theta &= (2\bar{c}_\omega - \bar{u}_x)\bar{\theta}_x - (\bar{c}_l x + \bar{u}) \cdot \bar{\theta}_{xx}, \quad F_\omega = \bar{\theta}_x + \bar{c}_\omega \bar{\omega} - (\bar{c}_l x + \bar{u}) \cdot \bar{\omega}_x, \\ N(\theta) &= (2c_\omega - u_x)\theta_x - u\theta_{xx}, \quad N(\omega) = c_\omega \omega - u\omega_x. \end{aligned} \tag{4.19}$$

We consider odd initial perturbation $\omega_0, \theta_{0,x}$ with $\omega_{0,x}(0) = 0, \theta_{0,xx}(0) = 0$. Note that the normalization conditions (4.6),(4.7) implies

$$c_\omega = u_x(0), \quad c_l = 0, \quad \theta_{xx}(t,0) = \theta_{0,xx}(0) = 0, \quad \omega_x(t,0) = \omega_{0x}(0) = 0, \tag{4.20}$$

for the perturbation. Since ω, θ_x are odd, these normalization conditions imply that near $x = 0$, $\omega = O(x^3), \theta_x = O(x^3)$ for sufficient smooth solution. This important property enables us to use a more singular weight in our stability analysis to extract a larger damping coefficient.

We rewrite the c_ω and u terms as follows

$$\begin{aligned} (2c_\omega - u_x)\bar{\theta}_x - u\bar{\theta}_{xx} &= -(u_x - u_x(0))\bar{\theta}_x - (u - u_x(0)x)\bar{\theta}_{xx} + c_\omega(\bar{\theta}_x - x\bar{\theta}_{xx}), \\ c_\omega \bar{\omega} - u\bar{\omega}_x &= -(u - u_x(0)x)\bar{\omega}_x + c_\omega(\bar{\omega} - x\bar{\omega}_x). \end{aligned} \tag{4.21}$$

Denote $\Lambda = (-\Delta)^{1/2}$. From $\partial_x u = H\omega$ and $\Lambda = \partial_x H$, we have $u(x) = -\Lambda^{-1}\omega(x) = \frac{1}{\pi}\int \log|x-y|\omega(y)dy$. Using this notation, we get $u - u_x(0)x = -\Lambda^{-1}\omega - H\omega(0)x$. We introduce the following linearized operators

$$\begin{aligned} \mathcal{L}_{\theta 1}(f,g) &= -(\bar{c}_l x + \bar{u})f_x + (2\bar{c}_\omega - \bar{u}_x)f \\ &\quad - (Hg - Hg(0))\bar{\theta}_x - (-\Lambda^{-1}g - Hg(0)x)\bar{\theta}_{xx}, \\ \mathcal{L}_{\omega 1}(f,g) &= -(\bar{c}_l x + \bar{u})g_x + \bar{c}_\omega g + f - (-\Lambda^{-1}g - Hg(0)x)\bar{\omega}_x, \\ \mathcal{L}_\theta(f,g) &= \mathcal{L}_{\theta 1}(f,g) + Hg(0)(\bar{\theta}_x - x\bar{\theta}_{xx}), \\ \mathcal{L}_\omega(f,g) &= \mathcal{L}_{\omega 1}(f,g) + Hg(0)(\bar{\omega} - x\bar{\omega}_x). \end{aligned} \tag{4.22}$$

Using these operators, we can rewrite (4.18) as follows

$$\begin{aligned} \partial_t \theta_x &= \mathcal{L}_{\theta 1}(\theta_x, \omega) + c_\omega(\bar{\theta}_x - x\bar{\theta}_{xx}) + F_\theta + N(\theta), \\ \partial_t \omega &= \mathcal{L}_{\omega 1}(\theta_x, \omega) + c_\omega(\bar{\omega} - x\bar{\omega}_x) + F_\omega + N(\omega). \end{aligned} \tag{4.23}$$

Clearly, $\mathcal{L}_\theta, \mathcal{L}_\omega$ are the linearized operators associated to (4.18). The motivation of introducing $\mathcal{L}_{\theta 1}, \mathcal{L}_{\omega 1}$ is that the estimates of these operators will be used importantly in both the weighted L^2 and weighted H^1 estimates.

4.3.2 Singular weights

For some $e_1, e_2, e_3 > 0$ determined by the profile $(\bar{\omega}, \bar{\theta})$, we introduce

$$\xi_1 = e_1 x^{-2/3} - (\bar{\theta}_x + \frac{1}{5} x\bar{\theta}_{xx}), \quad \xi_2 = e_2 x^{-2/3} - (\bar{\theta}_x + \frac{3}{7} x\bar{\theta}_{xx}), \quad \xi_3 = -\frac{e_3}{3} x^{-4/3} - \bar{\omega}_x. \tag{4.24}$$

Following the guideline of the construction of the singular weight in [19], we design different parts of the singular weight that have different decays as follows

$$\begin{aligned} &\psi_n = \frac{1}{\bar{\theta}_x + \frac{1}{5} x\bar{\theta}_{xx} + \chi\xi_1}(\alpha_1 x^{-4} + \alpha_2 x^{-3}), \quad \psi_f = \frac{1}{\bar{\theta}_x + \frac{3}{7} x\bar{\theta}_{xx} + \chi\xi_2}\alpha_3 x^{-4/3}, \\ &\varphi_s = \alpha_4 x^{-4}, \quad \varphi_n = \alpha_5(\alpha_1 x^{-3} + \alpha_2 x^{-2}), \quad \varphi_f = \alpha_6 x^{-2/3}, \end{aligned} \tag{4.25}$$

where the parameters are positive and chosen in (C.24), and the cutoff function χ defined in Appendix C.1.2 is supported in $|x| \geq \rho_2$ for $\rho_2 > 10^8$. The subscripts f, n, s are short for *far, near, singular*. We use the following weights in the weighted Sobolev estimate

$$\psi = \psi_n + \psi_f, \quad \varphi = \varphi_s + \varphi_n + \varphi_f. \tag{4.26}$$

We introduce χ, ξ_1, ξ_2 and add them in the definition of ψ_n, ψ_f for the following purposes. Firstly, recall from the beginning of Section 4.3.1 that $\bar{\theta}_x + cx\bar{\theta}_{xx}$ with $c = \frac{1}{5}$ or $c = \frac{3}{7}$ has decay $x^{2\alpha}$ which is close to $x^{-2/3}$. In particular, for sufficient large x, it can be well approximated by $ex^{-2/3}$ for some constant e. The parameters e_1, e_2 in (4.24) are determined in this way. Secondly, in the far field, where $\chi(x) = 1$, the weights ψ_n, ψ_f reduce to $c_1 x^{-7/3}, c_2 x^{-2/3}$ for some c_1, c_2, respectively. These explicit powers are much simpler than the weights in the near field and have forms similar to those in φ. They will be useful for the analytic estimates (see Section 4.3.6) and simplify the computer-assisted verification of the estimates in the far field. We introduce ξ_3 similar to ξ_1, ξ_2 and it will be used later.

Remark 4.3.1. Since χ is supported in $|x| > 10^8$ and the profile $(\bar{\omega}, \bar{\theta}_x)$ decays for large $|x|$, we gain a small factor in the estimates of the terms involving χ. Thus, the upper bound in these estimates are very small. The reader can safely skip the technicalities due to the χ terms.

4.3.2.1 The form of the singular weights

We add $\bar{\theta}_x, \bar{\theta}_{xx}$ terms in the denominators in ψ_n, ψ_f to cancel the variable coefficients in our energy estimates. In Model 1 in Section 4.2.3.2, we have chosen $\rho_1 = (\mu x \bar{\theta}_x)^{-1} \rho_2$ so that we can combine the estimates of two interactions in (4.14). Here, we design ψ_n, φ_n with a similar relation $\psi_n = \frac{1}{f} x^{-1} \varphi_n, f = \bar{\theta}_x + \frac{1}{5} x \bar{\theta}_{xx} + \chi \xi_1$ for the same purpose. Similar consideration applies to ψ_f, φ_f. See also estimates (4.35), (4.40). This idea has been used in [16, 19] for stability analysis.

The profile satisfies $\bar{\theta}_x + \frac{1}{5} x \bar{\theta}_{xx}, \bar{\theta}_x + \frac{3}{7} x \bar{\theta}_{xx} > 0$ for $x > 0$. The weight ψ is of order x^{-5} for x close to 0, while it is of order $x^{-2/3}$ for large x. We choose φ of order x^{-4} near 0 so that we can apply the sharp weighted estimates in Lemma C.0.8 to control u_x and u.

We will use the following notations repeatedly

$$\tilde{u} \triangleq u - u_x(0)x, \quad \tilde{u}_x = u_x - u_x(0). \tag{4.27}$$

4.3.3 Weighted L^2 estimates

Performing weighted L^2 estimates on (4.23) with weights ψ, φ, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\langle \theta_x, \theta_x \psi \rangle &= \langle \mathcal{L}_{\theta 1} \theta_x, \theta_x \psi \rangle + c_\omega \langle \bar{\theta}_x - x \bar{\theta}_{xx}, \theta_x \psi \rangle + \langle N(\theta), \theta_x \psi \rangle + \langle F_\theta, \theta_x \psi \rangle \\
&= \Big(- \langle (\bar{c}_l x + \bar{u}) \theta_{xx}, \theta_x \psi \rangle + \langle (2\bar{c}_\omega - \bar{u}_x) \theta_x, \theta_x \psi \rangle \Big) \\
&\quad + \Big(- \langle (u_x - u_x(0)) \bar{\theta}_x + (u - u_x(0)x) \bar{\theta}_{xx}, \theta_x \psi \rangle + c_\omega \langle \bar{\theta}_x - x \bar{\theta}_{xx}, \theta_x \psi \rangle \Big) \\
&\quad + \langle N(\theta), \theta_x \psi \rangle + \langle F_\theta, \theta_x \psi \rangle \triangleq D_1 + Q_1 + N_1 + F_1, \\
\frac{1}{2}\frac{d}{dt}\langle \omega, \omega \varphi \rangle &= \langle \mathcal{L}_{\omega 1} \omega, \omega \varphi \rangle + c_\omega \langle \bar{\omega}_x - x \bar{\omega}_{xx}, \omega_x \varphi \rangle + \langle N(\omega), \omega \varphi \rangle + \langle F_\omega, \omega \varphi \rangle \\
&= \Big(- \langle (\bar{c}_l x + \bar{u}) \omega_x, \omega \varphi \rangle + \langle \bar{c}_\omega \omega, \omega \varphi \rangle \Big) + \Big(\langle \theta_x, \omega \varphi \rangle - \langle (u - u_x(0)x) \bar{\omega}_x, \omega \varphi \rangle \\
&\quad + c_\omega \langle \bar{\omega} - x \bar{\omega}_x, \omega \varphi \rangle \Big) + \langle N(\omega), \omega \varphi \rangle + \langle F_\omega, \omega \varphi \rangle \triangleq D_2 + Q_2 + N_2 + F_2.
\end{aligned} \tag{4.28}$$

Our goal in the remaining part of this section is to establish an estimate similar to

$$D_1 + \lambda_1 D_2 + Q_1 + \lambda_1 Q_2 \leq -c(||\theta_x \psi^{1/2}||_2^2 + \lambda_1 ||\omega \varphi^{1/2}||_2^2), \tag{4.29}$$

for some $\lambda_1 > 0$ with $c > 0$ as large as possible. This implies the linear stability of (4.23) with $N_i, F_i = 0$ in the energy norm $||\theta_x \psi^{1/2}||_2^2 + \lambda_1 ||\omega \varphi^{1/2}||_2^2$. The actual estimate is slightly more complicated and we will add $c_\omega^2, \langle \theta_x, x^{-1} \rangle^2$ to the energy. We ignore the term c_ω and $\langle \theta_x, x^{-1} \rangle^2$ for now to illustrate the main ideas. See (4.74).

The D_1, D_2 terms only involve the local terms about θ_x, ω and we treat them as damping terms. The Q_i term denotes the quadratic terms other than D_i in the weighted L^2 estimates; the N_i and F_i terms represent the nonlinear terms and error terms in (4.23).

For D_1, D_2, performing integration by parts on the transport term, we obtain

$$D_1 = \langle D_\theta, \theta_x^2 \psi \rangle, \quad D_2 = \langle D_\omega, \omega^2 \varphi \rangle, \tag{4.30}$$

where D_θ, D_ω are given by

$$D_\theta = \frac{1}{2\psi}((\bar{c}_l x + \bar{u})\psi)_x + 2\bar{c}_\omega - \bar{u}_x, \quad D_\omega = \frac{1}{2\varphi}((\bar{c}_l x + \bar{u})\varphi)_x + \bar{c}_\omega.$$

We will verify that $D_\theta, D_\omega \leq -c < 0$ for some constant $c > 0$ in (C.30), Appendix C.3. The weight ψ in (4.26) involves three parameters $\alpha_1, \alpha_2, \alpha_3$. We choose the approximate values of α_i with $\alpha_i > 0$ so that $D_\theta \leq -c$ with c as large as possible and varies slowly. This enables us to obtain a large damping coefficient. After we choose $\alpha_1, \alpha_2, \alpha_3$, we choose positive α_4, α_5 and α_6 in the weight φ in (4.26) so that $D_\omega \leq -c_1$ with c_1 as large as possible and varies slowly. The final values are given in (C.24). See also Figure 4.1 for plots of the grid point values of D_θ, D_ω.

Using the notations in (4.27), we can rewrite $Q_1 + \lambda_1 Q_2$ as follows

$$\begin{aligned} Q_1 + \lambda_1 Q_2 = & -\langle \tilde{u}_x \bar{\theta}_x + \tilde{u} \bar{\theta}_{xx}, \theta_x \psi \rangle + \lambda_1 \langle \omega, \theta_x \varphi \rangle - \lambda_1 \langle \tilde{u} \bar{\omega}_x, \omega \varphi \rangle \\ & + c_\omega \langle (\bar{\theta}_x - x \bar{\theta}_{xx}), \theta_x \psi \rangle + \lambda_1 c_\omega \langle (\bar{\omega} - x \bar{\omega}_x), \omega \varphi \rangle. \end{aligned} \tag{4.31}$$

The terms in $Q_1 + \lambda_1 Q_2$ are the interactions among u, ω, θ_x and do not have a favorable sign. Our goal is to prove that they are perturbation to the damping terms D_1, D_2 and establish (4.29). This is challenging since the coefficients of the quadratic terms in $Q_1 + \lambda_1 Q_2$ and in D_i are comparable.

4.3.3.1 Decompositions on Q_i

Recall different parts of the weights in (4.25). They provide a natural decomposition of the global interaction among u, ω, θ_x into the near field and the far field interaction. We have a straightforward partition of unity

$$\psi_n \psi^{-1} + \psi_f \psi^{-1} = 1, \quad \varphi_n \varphi^{-1} + \varphi_f \varphi^{-1} + \varphi_s \varphi^{-1} = 1. \tag{4.32}$$

According to different singular orders and decay rates of the weights in (4.25), $\psi_f \psi^{-1}, \varphi_f \varphi^{-1}$ are mainly supported in the far field, $\psi_n \varphi^{-1}$ in the near field, $\varphi_n \varphi^{-1}$ near $|x| \approx 1$, and $\varphi_s \varphi^{-1}$ near 0. Next, we decompose the interaction using these weights. Using $\psi = \psi_f + \psi_n$, we get

$$-\langle \tilde{u}_x \bar{\theta}_x, \theta_x \psi \rangle = -\langle \tilde{u}_x (\bar{\theta}_x + \chi \xi_1), \theta_x \psi_n \rangle - \langle \tilde{u}_x (\bar{\theta}_x + \chi \xi_2), \theta_x \psi_f \rangle + \langle \tilde{u}_x \chi (\xi_1 \psi_n + \xi_2 \psi_f), \theta_x \rangle.$$

We decompose the first two terms on the right hand side of (4.31) as follows

$$\begin{aligned}
& - \langle \tilde{u}_x \bar{\theta}_x + \tilde{u} \bar{\theta}_{xx}, \theta_x \psi \rangle + \lambda_1 \langle \omega, \theta_x \varphi \rangle \\
=& \Big(- \langle \tilde{u}_x (\bar{\theta}_x + \chi \xi_2) + \tilde{u} \bar{\theta}_{xx}, \theta_x \psi_f \rangle + \lambda_1 \langle \theta_x, \omega \varphi_f \rangle \Big) \\
& + \Big(- \langle \tilde{u}_x (\bar{\theta}_x + \chi \xi_2) + \tilde{u} \bar{\theta}_{xx}, \theta_x \psi_n \rangle + \lambda_1 \langle \theta_x, \omega \varphi_n \rangle \Big) + \lambda_1 \langle \theta_x, \omega \varphi_s \rangle \\
& + \langle \tilde{u}_x \chi (\xi_1 \psi_n + \xi_2 \psi_f), \theta_x \rangle \triangleq I_f + I_n + I_s + I_{r1}.
\end{aligned} \tag{4.33}$$

The subscripts f, n, s, r are short for *far, near, singular, remainder*. Denote $I_{u\omega} = -\lambda_1 \langle \tilde{u} \bar{\omega}_x, \omega \varphi \rangle$ in (4.31). The main terms in (4.31) are I_f, I_n and I_s. From the above discussion on (4.32), the interactions in I_n, I_f, I_s are mainly supported in different regions. Since u depends on ω linearly, $I_{u\omega}$ can be seen as the interaction between ω and itself. This type of interaction is different from I_n, I_f, I_s. Since $c_\omega = u_x(0) = -\frac{1}{\pi} \int_{\mathbb{R}} \omega dx$, the terms $c_\omega \langle (\bar{\theta}_x - x \bar{\theta}_{xx}), \theta_x \psi \rangle, \lambda_1 c_\omega \langle (\bar{\omega} - x \bar{\omega}_x), \omega \varphi \rangle$ in (4.31) are the projections of ω, θ_x onto some rank-1 space. The estimate of the c_ω terms is smaller than that of $I_n, I_f, I_s, I_{u\omega}$. The term I_{r1} is very small compared to other terms and will be estimated directly.

We will exploit the structure of the interactions in (4.31) using the above important decompositions.

4.3.4 Outline of the estimates

In order to establish the weighted L^2 estimates similar to (4.29), we first develop sharp estimates on each term in the above decomposition. In these estimates, we introduce several parameters, when we apply the Cauchy-Schwarz or Young's inequality. These parameters are important in our estimates. Since the coefficients in the damping term D_1, D_2 (4.30) are relative small, we can treat the interaction term as perturbation to the damping term using the energy estimates, only for certain range of parameters. See more discussion in Remark 4.3.3. Thus, the upper bound in these estimates depend on several parameters. Then, using these estimates, we reduce the estimate similar to (4.29) to some inequality constraints on the parameters with explicit coefficients. See (4.57) for an example. Finally, to obtain an overall sharp energy estimate, e.g., (4.29) with $c > 0$ as large as possible, we determine these parameters guided by solving a constrained optimization problem.

In our energy estimates, to obtain the sharp weighted estimates of xu_x, u with singular weight x^{-2p} by applying Lemma C.0.8, we can only use a few exponents $p = 3, 2, \frac{5}{3}$. Thus, we need to perform the energy estimates very carefully. The linear combinations of different powers in Lemma C.0.8, e.g., $\alpha x^{-4} + \beta x^{-2}$, plays a role similar to that of interpolating different singular weights, e.g., x^{-4}, x^{-2}. It enables us to obtain sharp weighted estimates with singular weight x^{-2q} and intermediate exponent q. In our weighted estimates of u_x, u, we choose some weights with a few parameters, see e.g., (4.50). Moreover, to generalize the cancellations and estimates in the Model 1 in Section 4.2.3.2 to the more complicated linearized system (4.10), we also need to perform the energy estimates carefully so that we can apply the cancellation in Lemma C.0.4.

4.3.5 Estimates of the interaction in the near field I_n

We use ideas in Model 1 in Section 4.2.3.2 to estimate the main term introduced below and ideas in Section 4.2.3.2 to estimate u.

Firstly, we choose $c = \frac{1}{5}$ in the decomposition 4.11 $\tilde{u} = \frac{1}{5}x\tilde{u}_x + \tilde{u} - \frac{1}{5}x\tilde{u}_x$, and decompose $\tilde{u}_x(\bar{\theta}_x + \chi\xi_1) + \tilde{u}\bar{\theta}_{xx}$ into the main term $\mathcal{M}$ and the remainder $\mathcal{R}$ as follows

$$\tilde{u}_x(\bar{\theta}_x + \chi\xi_1) + \tilde{u}\bar{\theta}_{xx} = \tilde{u}_x(\bar{\theta}_x + \frac{1}{5}\bar{\theta}_{xx}x + \chi\xi_1) + (\tilde{u} - \frac{1}{5}\tilde{u}_x x)\bar{\theta}_{xx} \triangleq \mathcal{M} + \mathcal{R}. \quad (4.34)$$

This term also appears in I_f and we will use another decomposition in Section 4.3.6.

Recall I_n in (4.33). Using the above decomposition, we yield

$$\begin{aligned} I_n &= -\langle \tilde{u}_x(\bar{\theta}_x + \chi\xi_1) + \tilde{u}\bar{\theta}_{xx}, \theta_x\psi_n\rangle + \lambda_1\langle \omega, \theta_x\varphi_n\rangle \\ &= \Big(- \langle \mathcal{M}, \theta_x\psi_n\rangle + \lambda_1\langle \omega, \theta_x\varphi_n\rangle \Big) + \langle -\mathcal{R}, \theta_x\psi_n\rangle \triangleq I_{\mathcal{M}} + I_{\mathcal{R}}. \end{aligned}$$

The estimates of $I_{\mathcal{M}}$ are similar to that in Model 1 in Section 4.2.3.2. Recall the formulas of ψ_n, φ_n in (4.25). Using Young's inequality $ab \leq t_2a^2 + \frac{1}{4t_2}b^2$ for $t_2 > 0$, we obtain

$$\begin{aligned} I_{\mathcal{M}} &= -\langle \tilde{u}_x, \theta_x(\alpha_2x^{-3} + \alpha_1x^{-4})\rangle + \langle \omega, \theta_x\lambda_1\alpha_5(\alpha_2x^{-2} + \alpha_1x^{-3})\rangle \\ &= \langle -\tilde{u}_xx^{-2} + \lambda_1\alpha_5\omega x^{-1}, \theta_x(\alpha_2x^{-1} + \alpha_1x^{-2})\rangle \\ &\leq t_2|| - \tilde{u}_xx^{-2} + \lambda_1\alpha_5\omega x^{-1}||_2^2 + \frac{1}{4t_2}||\theta_x(\alpha_2x^{-1} + \alpha_1x^{-2})||_2^2. \end{aligned} \tag{4.35}$$

Remark 4.3.2. We design the special form ψ_n in (4.25) so that the denominator in ψ_n and the coefficient $\bar{\theta}_x + \frac{1}{5}\bar{\theta}_{xx}x + \chi\xi_1$ in $\mathcal{M}$ cancel each other. This allows us to obtain a desirable term of the form $J \triangleq -\tilde{u}_xx^{-2} + \lambda_1\alpha_5\omega x^{-1}$. The term $t_2||J||_2^2$ in (4.35) is a quadratic form in ω, where we can exploit the cancellation between $\tilde{u}_x$ and ω to obtain a sharp estimate. See Model 1 for the motivation.

Using the weighted estimate in Lemma C.0.8 and Lemma C.0.4 with $f = \omega$ and $g = u$, we get

$$\begin{aligned} & t_2|| - \tilde{u}_xx^{-2} + \lambda_1\alpha_5\omega x^{-1}||_2^2 \\ =& t_2\Big(||\tilde{u}_xx^{-2}||_2^2 - 2\lambda_1\alpha_5\langle \tilde{u}_x, \omega x^{-3}\rangle + (\lambda_1\alpha_5)^2||\omega x^{-1}||_2^2 \Big) \\ =& t_2\Big(||\omega x^{-2}||_2^2 + (\lambda_1\alpha_5)^2||\omega x^{-1}||_2^2 \Big) = t_2\Big\langle \omega^2, x^{-4} + (\lambda_1\alpha_5)^2x^{-2} \Big\rangle. \end{aligned} \tag{4.36}$$

The cancellation is exactly the same as (4.15) in Model 1. For $I_{\mathcal{R}}$, using Young's inequality $ab \leq t_{22}a^2 + \frac{1}{4t_{22}}b^2$, (4.12) with $p = 3$ and the weighted estimate in Lemma C.0.8, we obtain

$$\begin{aligned} I_{\mathcal{R}} &= \langle (\tilde{u} - \frac{1}{5}\tilde{u}_xx)\bar{\theta}_{xx}, \theta_x\psi_n\rangle \leq t_{22}||(\tilde{u} - \frac{1}{5}\tilde{u}_xx)x^{-3}||_2^2 + \frac{1}{4t_{22}}||x^3\bar{\theta}_{xx}\psi_n\theta_x||_2^2 \\ &= \frac{t_{22}}{25}||\tilde{u}_xx^{-2}||_2^2 + \frac{1}{4t_{22}}||x^3\bar{\theta}_{xx}\psi_n\theta_x||_2^2 = \frac{t_{22}}{25}||\omega x^{-2}||_2^2 + \frac{1}{4t_{22}}||x^3\bar{\theta}_{xx}\psi_n\theta_x||_2^2. \end{aligned} \tag{4.37}$$

The remainder $I_{\mathcal{R}}$ is much smaller than $I_{\mathcal{M}}$ since we get a small factor $\frac{1}{2p-1} = \frac{1}{5}$ from (4.12). Combining the above estimates, we establish the estimate for $I_n = I_{\mathcal{M}} + I_{\mathcal{R}}$

$$\begin{aligned} I_n \leq & \left\langle \omega^2, t_2 x^{-4} + \frac{t_{22}}{25} x^{-4} + t_2 (\lambda_1 \alpha_5)^2 x^{-2} \right\rangle \\ & + \left\langle \theta_x^2, \frac{1}{4t_2} (\alpha_2 x^{-1} + \alpha_1 x^{-2})^2 + \frac{1}{4t_{22}} (x^3 \bar{\theta}_{xx} \psi_n)^2 \right\rangle. \end{aligned} \tag{4.38}$$

Remark 4.3.3. If we neglect other terms in (4.31) except I_n, a necessary condition for (4.29) is

$$I_n + D_1 + \lambda_1 D_2 \leq -c(||\theta_x \psi^{1/2}||_2 + \lambda_1 ||\omega \varphi^{1/2}||_2^2) \tag{4.39}$$

with $c > 0$, where D_1, D_2 are the damping terms in (4.30). We cannot determine the ratio λ_1 between two norms and t_i in Young's inequality without using the profile $(\bar{\theta}, \bar{\omega})$. For example, if we use equal weights $\lambda_1 = 1, t_2 = t_{22} = \frac{1}{2}$, we cannot apply estimate (4.38) to establish (4.39) even with $c = 0$. Therefore, we introduce several parameters, especially when we apply Young's inequality. At this step, we do not fix λ_1, t_{ij} such that the subproblem (4.39) holds with $c > 0$ as large as possible. In fact, such parameters may not be ideal for (4.29) since the final energy estimate involves other terms in (4.29),(4.31) to be estimated later on. Instead, we identify the ranges $\lambda_1 \in [0.31, 0.33], t_2 \in [5, 5.8], t_{22} \in [13, 14]$, such that a weaker version of (4.39) with $c = 0.01$ holds with the estimate (4.38) on I_n. See Appendix C.3.1 for rigorous verification. Similarly, we will obtain the ranges of other parameters t_i introduced in later estimates. We will determine the values of λ_1, t_{ij} in these ranges by combining the estimates of I_f, I_n and other terms in (4.31).

The estimates (4.35), (4.36) on the main term is crucial. If we estimate two inner products separately without using the cancellation between $\tilde{u}_x, \omega$ in Lemma C.0.4 with $f = \omega$ and $g = u$, we would fail to establish (4.39) even with $c = 0$ since the damping term D_i is relatively small.

Remark 4.3.4. Several key ideas in the above estimates will be used repeatedly later. Firstly, we will perform decompositions on $\tilde{u}$ into the main term and the remainder similar to (4.34). Secondly, we will use Lemmas C.0.4, C.0.5 to estimate the inner product between $\tilde{u}$ and ω similar to (4.36). Thirdly, we will use Lemma C.0.8 to estimate weighted norms of $\tilde{u}_x, \tilde{u}$ similar to (4.36).

4.3.6 Estimates of the interaction in the far field I_f

We use ideas and estimates similar to that of I_n to estimate I_f. The main difference is that to estimate the inner product between $\tilde{u}_x$ and ω, instead of using Lemma C.0.4, we will use Lemma C.0.5. See estimates (4.35) and (4.40).

Firstly, we choose $c = \frac{3}{7}$ in (4.11) and decompose $\tilde{u}_x(\bar{\theta}_x + \chi\xi_2) + \tilde{u}\bar{\theta}_{xx}$ into the main term $\mathcal{M}$ and the remainder $\mathcal{R}$ as follows

$$\tilde{u}_x(\bar{\theta}_x + \chi\xi_2) + \tilde{u}\bar{\theta}_{xx} = \tilde{u}_x(\bar{\theta}_x + \frac{3}{7}x\bar{\theta}_{xx} + \chi\xi_2) + (\tilde{u} - \frac{3}{7}x\tilde{u}_x)\bar{\theta}_{xx} \triangleq \mathcal{M} + \mathcal{R}.$$

We choose $c = \frac{3}{7}$, which is different from that in (4.11), since we will apply (4.12) with a different power p later. Recall I_f in (4.33). The above formula implies

$$\begin{aligned} I_f &= -\langle \tilde{u}_x(\bar{\theta}_x + \chi\xi_2) + \tilde{u}\bar{\theta}_{xx}, \theta_x\psi_f\rangle + \lambda_1\langle\theta_x, \omega\varphi_f\rangle \\ &= \Big(- \langle \mathcal{M}, \theta_x\psi_f\rangle + \lambda_1\langle\omega, \theta_x\varphi_f\rangle\Big) + \langle -\mathcal{R}, \theta_x\psi_f\rangle \triangleq I_{\mathcal{M}} + I_{\mathcal{R}}. \end{aligned}$$

Recall the weights ψ_f, φ_f in (4.25). Using Young's inequality $a\cdot b \leq t_1 a^2 + \frac{1}{4t_1}b^2$ for some $t_1 > 0$ to be determined, we obtain

$$\begin{aligned} I_{\mathcal{M}} &= \langle -\alpha_3\tilde{u}_x x^{-4/3} + \lambda_1\alpha_6\omega x^{-2/3}, \theta_x\rangle = \langle -\alpha_3\tilde{u}_x x^{-1} + \lambda_1\alpha_6\omega x^{-1/3}, \theta_x x^{-1/3}\rangle \\ &\leq t_1|| - \alpha_3\tilde{u}_x x^{-1} + \lambda_1\alpha_6\omega x^{-1/3}||_2^2 + \frac{1}{4t_1}||\theta_x x^{-1/3}||_2^2 \triangleq I_{\mathcal{M},1} + I_{\mathcal{M},2}. \end{aligned} \tag{4.40}$$

We design the special form ψ_f in (4.25) to obtain a desirable term of the form $-\alpha_3\tilde{u}_x x^{-1} + \lambda_1\alpha_6\omega x^{-1/3}$. See also Remark 4.3.2. We further estimate $I_{\mathcal{M},1}$. Applying Lemma C.0.8 and Lemma C.0.5, we derive

$$\begin{aligned} I_{\mathcal{M},1} =& t_1(||\alpha_3\tilde{u}_x x^{-1}||_2^2 - 2\alpha_3\lambda_1\alpha_6\langle\tilde{u}_x, \omega x^{-4/3}\rangle + (\lambda_1\alpha_6)^2||\omega x^{-1/3}||_2^2) \\ =& t_1(\alpha_3^2||\omega x^{-1}||_2^2 - \frac{2\alpha_3\lambda_1\alpha_6}{2\sqrt{3}}(||\tilde{u}_x x^{-2/3}||_2^2 - ||\omega x^{-2/3}||_2^2) + (\lambda_1\alpha_6)^2||\omega x^{-1/3}||_2^2) \\ =& t_1\langle\omega^2, \alpha_3^2 x^{-2} + \frac{\alpha_3\lambda_1\alpha_6}{\sqrt{3}}x^{-4/3} + (\lambda_1\alpha_6)^2 x^{-2/3}\rangle - \frac{t_1\alpha_3\lambda_1\alpha_6}{\sqrt{3}}||\tilde{u}_x x^{-2/3}||_2^2. \end{aligned} \tag{4.41}$$

Remark 4.3.5. The negative sign in $-t_1 2\alpha_3\lambda_1\alpha_6\langle\tilde{u}_x, \omega x^{-4/3}\rangle$ in (4.41) is crucial. Firstly, we can bound the positive term $\frac{\alpha_3\lambda_1\alpha_6 t_1}{\sqrt{3}}||\omega x^{-2/3}||_2$ derived from the identity in Lemma C.0.5 directly without an overestimate. Secondly, $-\frac{t_1\alpha_3\lambda_1\alpha_6}{\sqrt{3}}||\tilde{u}_x x^{-2/3}||_2^2$ from the same identity provides a good quantity that allows us to control the weighted norm of $\tilde{u}, \tilde{u}_x$ with a slowly decaying weight using Lemma C.0.8.

We introduce D_u to denote the parameter in (4.41)

$$D_u = \frac{t_1 \alpha_3 \lambda_1 \alpha_6}{\sqrt{3}}. \tag{4.42}$$

We use Young's inequality $ab \leq t_{12} a^2 + \frac{1}{4t_{12}} b^2$ for some $t_{12} > 0$ and (4.12) with $p = \frac{5}{3}$ to estimate $I_{\mathcal{R}}$ directly

$$\begin{aligned} I_{\mathcal{R}} &= -\langle (\tilde{u} - \frac{3}{7} x \tilde{u}_x), \bar{\theta}_{xx} \psi_f \theta_x \rangle \leq t_{12} ||(\tilde{u} - \frac{3}{7} \tilde{u}_x x) x^{-5/3}||_2^2 + \frac{1}{4t_{12}} ||\theta_x \psi_f \bar{\theta}_{xx} x^{5/3}||_2^2 \\ &= t_{12} \cdot \frac{9}{49} ||\tilde{u}_x x^{-2/3}||_2^2 + \frac{1}{4t_{12}} ||\theta_x \psi_f \bar{\theta}_{xx} x^{5/3}||_2^2. \end{aligned} \tag{4.43}$$

The remainder $I_{\mathcal{R}}$ is smaller since we get a factor $\frac{1}{2p-1} = \frac{3}{7}$ from (4.12).

Combining the above estimates, we obtain the estimate of $I_f = I_{\mathcal{M},1} + I_{\mathcal{M},2} + I_{\mathcal{R}}$

$$\begin{aligned} I_f \leq & t_1 \Big\langle \omega^2, \alpha_3^2 x^{-2} + \frac{\alpha_3 \lambda_1 \alpha_6}{\sqrt{3}} x^{-4/3} + (\lambda_1 \alpha_6)^2 x^{-2/3} \Big\rangle \\ & + \Big\langle \theta_x^2, \frac{1}{4t_1} x^{-2/3} + \frac{1}{4t_{12}} (\psi_f \bar{\theta}_{xx} x^{5/3})^2 \Big\rangle - \Big(D_u - \frac{9}{49} t_{12} \Big) ||\tilde{u}_x x^{-2/3}||_2^2 \ . \end{aligned} \tag{4.44}$$

Similar to the discussion in Remark 4.3.3, in order for $I_f + D_1 + D_2 \leq -c(||\theta_x \psi^{1/2}||_2 + \lambda_1 ||\omega \varphi^{1/2}||_2^2)$ with $c = 0.01$, we can choose $t_1 \in [1.2, 1.4], t_{12} \in [0.55, 0.65]$. See Appendix C.3.1 for the verification.

4.3.7 Estimates of the interaction with the most singular weight I_s

Recall I_s in (4.33) and $\psi_s = \alpha_4 x^{-4}$ in (4.25). Using Young's inequality $ab \leq t_4 a^2 + \frac{1}{4t_4} b^2$ for $t_4 > 0$, we yield

$$I_s = \lambda_1 \langle \omega, \theta_x \varphi_s \rangle = \lambda_1 \alpha_4 \langle \omega, \theta_x x^{-4} \rangle \leq t_4 \langle \omega^2, x^{-3} \rangle + \frac{(\lambda_1 \alpha_4)^2}{4t_4} \langle \theta_x^2, x^{-5} \rangle. \tag{4.45}$$

In order for $I_s + D_1 + D_2 \leq -c(||\theta_x \psi^{1/2}||_2 + \lambda_1 ||\omega \varphi^{1/2}||_2^2)$ with $c = 0.01$, we can choose $t_4 \in [3, 5]$. See Appendix C.3.1 for the verification. We do not combine estimates of I_s with the estimates for the interaction between $\tilde{u}$ and θ_x in Section 4.3.5 since the weight x^{-4} is too singular. In fact, to apply estimate similar to that in (4.14) in Model 1, the weight for θ_x near 0 is 2 order more singular than that of ω. In this case, it is of order x^{-6} near 0 and more singular than ψ.

4.3.8 Estimates of the interaction u and ω

Firstly, we rewrite $-\lambda_1\langle \tilde{u}, \bar{\omega}_x \omega\varphi\rangle$ in (4.31) by decomposing $\bar{\omega}_x$ into the main term $\bar{\omega}_x + \chi\xi_3$ and the error term ξ_3

$$I_{u\omega} \triangleq -\lambda_1\langle \tilde{u}, \bar{\omega}_x \omega\varphi\rangle = -\lambda_1\langle \tilde{u}, (\bar{\omega}_x + \chi\xi_3)\omega\varphi\rangle + \lambda_1\langle \tilde{u}, \chi\xi_3\omega\varphi\rangle \triangleq J + I_{r2}. \quad (4.46)$$

We will estimate I_{r2} in Section 4.3.9 and show that it is very small. See also Remark 4.3.1.

Difficulty The main difficulty in establishing a sharp estimate for J is the slow decay of the coefficient $(\bar{\omega}_x + \chi\xi_3)\varphi$. A straightforward estimate similar to (4.43) yields $|J| \leq \lambda_1 ||\tilde{u}x^{-p}||_2 ||\bar{\omega}_x \omega\rho x^p||_2$. In view of the weighted estimate in Lemma C.0.8, we have effective estimates of $(u - u_x(0)x)x^{-p}$ for exponent $p = 3, 2$ or $\frac{5}{3}$. In order to further control $||\bar{\omega}_x\omega\varphi x^p||_2$ by the weighted L^2 norm $||\omega\varphi^{1/2}||_2$, we cannot choose $p = 3$ or $p = 2$ due to the fast growth of x^p for large $|x|$. On the other hand, if we choose $p = \frac{5}{3}$, the resulting constant $\frac{36}{49}$ in Lemma C.0.8 is much larger than the constant $\frac{4}{25}, \frac{4}{9}$ corresponding to $p = 3, p = 2$.

To overcome this difficulty, we exploit the cancellations between u and ω in both the near field and the far field, which is similar to that in the estimate of I_n, I_f. We decompose the coefficient $(\bar{\omega}_x + \chi\xi_3)\varphi$ in J into the main terms $\mathcal{M}_i$ and the remainder $\mathcal{K}_{u\omega}$

$$\begin{aligned}(\bar{\omega}_x + \chi\xi_3)\varphi &= -\frac{1}{3}e_3\alpha_6 x^{-2} + \tau_1 x^{-4} + \Big((\bar{\omega}_x + \chi\xi_3)\varphi + \frac{1}{3}e_3\alpha_6 x^{-2} - \tau_1 x^{-4}\Big) \\ &\triangleq \mathcal{M}_1 + \mathcal{M}_2 + \mathcal{K}_{u\omega},\end{aligned} \quad (4.47)$$

where e_3, α_6 are defined in (4.24) and (4.25) and $\tau_1 > 0$ is some parameter.

Let us motivate the above decomposition. From the definitions of ξ_3, φ in (4.24)-(4.26) and the discussion therein, we have $\bar{\omega}_x + \chi\xi_3 \approx -\frac{1}{3}e_3 x^{-4/3}, \varphi \approx \alpha_6 x^{-2/3}$ for some $e_3, \alpha_6 > 0$ and large $|x|$. Thus, $(\bar{\omega}_x + \chi\xi_3)\varphi$ can be approximated by $-\frac{1}{3}e_3\alpha_6 x^{-2}$ for large $|x|$. Since $\varphi \approx \alpha_4 x^{-4}$ and $\bar{\omega}_x \approx \bar{\omega}_x(0) > 0$ near 0, $(\bar{\omega}_x + \chi\xi_3)\varphi$ is approximated by $\tau_1 x^{-4}$ for some $\tau_1 > 0$ in the near field.

Using the above formula, we can decompose J as follows

$$\begin{aligned}J &= -\lambda_1\langle \tilde{u}, (\bar{\omega}_x + \chi\xi_3)\omega\varphi\rangle = -\lambda_1\langle \tilde{u}, \mathcal{M}_1\omega\rangle - \lambda_1\langle \tilde{u}, \mathcal{M}_2\omega\rangle - \lambda_1\langle \tilde{u}, \mathcal{K}_{u\omega}\omega\rangle \\ &\triangleq I_{\mathcal{M}1} + I_{\mathcal{M}2} + I_{\mathcal{R}}.\end{aligned}$$

To estimate the main terms, we use cancellations in Lemma C.0.4. Using $\tilde{u} = u - u_x(0)x$ defined in (4.27), $-u_x(0)\langle x, \omega x^{-2}\rangle = \frac{\pi}{2}u_x^2(0)$ and Lemma C.0.4 with $f = \omega$ and $g = u$, we get

$$I_{\mathcal{M}_1} = \frac{\lambda_1 e_3 \alpha_6}{3}\langle \tilde{u}, \omega x^{-2}\rangle = \frac{\lambda_1 e_3\alpha_6}{3}\frac{\pi}{4}u_x(0)^2 - \frac{\lambda_1 e_3 \alpha_6}{3}\langle \Lambda \frac{u}{x}, \frac{u}{x}\rangle,$$

where $\Lambda = (-\partial_x^2)^{1/2}$. We denote by $A(u)$ the right hand side of the above equation

$$A(u) \triangleq \frac{\lambda_1 e_3\alpha_6}{3}\frac{\pi}{4}u_x(0)^2 - \frac{\lambda_1 e_3 \alpha_6}{3}\langle \Lambda \frac{u}{x}, \frac{u}{x}\rangle. \tag{4.48}$$

Since $e_3\alpha_6\lambda_1 > 0$ and $\langle \Lambda\frac{u}{x}, \frac{u}{x}\rangle \geq 0$, the second term in $A(u)$ is a good term and we will use it in the weighted H^1 estimate.

Although $I_{\mathcal{M}2}$ is a quadratic form on ω, it does not have a good sign similar to the identities in Lemma C.0.4. Yet, we can approximate $\tilde{u}$ by $\tilde{u}_x$ using (4.11) and then use the cancellation between $\tilde{u}_x$ and ω. Choosing $c = \frac{1}{5}$ in 4.11 and using the cancellation in Lemma C.0.4 with $f = \omega$ and $g = u$, we obtain

$$\begin{aligned} I_{\mathcal{M}2} &= -\lambda_1\tau_1\langle \tilde{u}, x^{-4}\omega\rangle = -\lambda_1\tau_1\langle \tilde{u} - \frac{1}{5}\tilde{u}_x x, \omega x^{-4}\rangle - \lambda_1\tau_1\langle \frac{\tilde{u}_x}{5}, \omega x^{-3}\rangle \\ &= -\lambda_1\tau_1\langle \tilde{u} - \frac{1}{5}\tilde{u}_x x, \omega x^{-4}\rangle. \end{aligned}$$

The form $\tilde{u} - \frac{1}{5}\tilde{u}_x x$ allows us to gain a small factor $\frac{1}{5}$ using (4.12) with $p = 3$. Using Young's inequality $ab \leq ca^2 + \frac{1}{4c}b^2$, (4.12) with $p = 3$ and Lemma C.0.8, we obtain

$$\begin{aligned} I_{\mathcal{M}2} &\leq \lambda_1\tau_1\Big(t_{34}||(\tilde{u} - \frac{1}{5}\tilde{u}_x)x^{-3}||_2^2 + \frac{1}{4t_{34}}||\omega x^{-1}||_2^2\Big) \\ &= \lambda_1\tau_1\Big(\frac{t_{34}}{25}||\tilde{u}_x x^{-2}||_2^2 + \frac{1}{4t_{34}}||\omega x^{-1}||_2^2\Big) \\ &= \lambda_1\tau_1(\frac{t_{34}}{25}||\omega x^{-2}||_2^2 + \frac{1}{4t_{34}}||\omega x^{-1}||_2^2) = \lambda_1\tau_1\Big\langle \omega^2, \frac{t_{34}}{25}x^{-4} + \frac{1}{4t_{34}}x^{-2}\Big\rangle, \end{aligned} \tag{4.49}$$

for some $t_{34} > 0$. For $t_{31}, t_{32} > 0$ to be defined, we denote

$$S_{u1} = t_{31}x^{-6} + t_{32}x^{-4} + 2\cdot 10^{-5}x^{-10/3}. \tag{4.50}$$

We estimate $I_{\mathcal{R}}$ directly using Young's inequality and the weighted estimate in Lemma C.0.8

$$\begin{aligned} I_{\mathcal{R}} &= -\lambda_1\langle \tilde{u}, \mathcal{K}_{u\omega}\omega\rangle \leq \lambda_1(||\tilde{u}S_{u1}^{1/2}||_2^2 + \frac{1}{4}||S_{u1}^{-1/2}\mathcal{K}_{u\omega}\omega||_2^2) \\ &\leq \lambda_1\Big\langle \omega^2, \frac{4t_{31}}{25}x^{-4} + \frac{4t_{32}}{9}x^{-2} + \frac{1}{4}\mathcal{K}_{u\omega}^2 S_{u1}^{-1}\Big\rangle + \frac{36\lambda_1}{49}\cdot 2\cdot 10^{-5}||\tilde{u}_x x^{-2/3}||_2^2, \end{aligned} \tag{4.51}$$

where $\mathcal{K}_{u\omega}$ in defined (4.47).

Remark 4.3.6. From (4.24),(4.25) and (4.26), we have asymptotically $\mathcal{K}_{u\omega} \sim Cx^{-4}$ for x close to 0. The slowly decaying part in $\mathcal{K}_{u\omega}$ is given by $f = (\bar{\omega}_x + \chi\xi_3)\alpha_6 x^{-2/3} + \frac{1}{3}e_3\alpha_6 x^{-2} = (1-\chi)(\bar{\omega}_x + \frac{1}{3}e_3 x^{-4/3})\alpha_6 x^{-2/3}$. In the support of $1-\chi$, since $-\frac{1}{3}e_3 x^{-4/3}$ approximates $\bar{\omega}_x$, f can be approximated by cx^{-2} with some very small constant c. We add x^{-6}, $2 \cdot 10^{-5} x^{-10/3}$ in S_{u1} so that $\mathcal{K}_{u\omega}^2 S_{u1}^{-1}$ can be bounded by φ. We also add the power x^{-4} in S_{u1} to obtain a sharper estimate. See also Remark 4.3.3 for the discussion on the parameters.

Combining the above estimates on $I_{\mathcal{M}i}, I_{\mathcal{R}}$, we prove

$$
\begin{aligned}
I_{u\omega} = J + I_{r2} \leq \lambda_1 \Big\langle \omega^2, \frac{4t_{31}}{25}x^{-4} + \frac{4t_{32}}{9}x^{-2} + \frac{1}{4}\mathcal{K}_{u\omega}^2 S_{u1}^{-1} + \tau_1(\frac{t_{34}}{25}x^{-4} + \frac{1}{4t_{34}}x^{-2})\Big\rangle \\
+ A(u) + \frac{72\lambda_1}{49} \cdot 10^{-5}||\tilde{u}_x x^{-2/3}||_2^2 + I_{r2}.
\end{aligned}
\tag{4.52}
$$

The term I_{r2} was not estimated and we keep it on both sides. We can determine the ranges of parameters $t_{31}, t_{32}, t_{34}, \tau_1$ so that $J + D_1 + D_2 \leq -0.01(||\theta_x \psi^{1/2}||_2 + \lambda_1||\omega\varphi^{1/2}||_2^2)$.

4.3.9 Estimates of the I_{r1}, I_{r2}

Recall I_{r1}, I_{r2} in (4.33) and (4.46). Since χ is supported in the far field $|x| \geq \rho_2 > 10^8$ and the profile $(\bar{\omega}, \bar{\theta}_x)$ decays, we can get a small factor in the estimate of these terms. We establish the following estimate in Appendix C.1.2

$$
|I_{r1}| + |I_{r2}| \leq \langle G_\theta, \theta_x^2 \rangle + \langle G_\omega, \omega^2 \rangle + G_c c_\omega^2, \tag{4.53}
$$

where G_θ, G_ω, G_c are given by

$$
\begin{aligned}
G_\theta &= 10^{10} \cdot \frac{(2+\sqrt{3})^2}{4}\chi^2(\xi_1\psi_n + \xi_2\psi_f)^2, \quad G_c = \frac{\lambda_1^2||x\xi_3\chi^{1/2}\varphi^{1/2}||_2^2}{4} \cdot 10^2, \\
G_\omega &= 10^{-10}x^{-4/3} + 10^{-5}x^{-2/3} + \frac{10^5}{4}(\frac{6\lambda_1(2+\sqrt{3})}{5})^2(x^{4/3}\chi\xi_3\varphi)^2 + 10^{-2}\chi\varphi.
\end{aligned}
\tag{4.54}
$$

These functions are very small compared to the weights φ, ψ (4.25)-(4.26). We focus on a typical term G_θ to illustrate the smallness. From (4.24)-(4.25), for large $|x|$, ξ_i, ψ_f, ψ have decay rate $x^{-2/3}$, ψ_n has a decay rate at least x^{-2}. For $\partial_x^i \bar{\theta}_x$, we recall from the beginning of Section 4.3.1 that it has decay rate $x^{2\alpha - i}$ with α slightly smaller than $-\frac{1}{3}$. Thus $|\chi(\xi_1\psi_n + \xi_2\psi_f)|^2/\psi$ has decay rate x^{-2}. Since χ is supported in $|x| \geq \rho_2 > 10^8$, we get a small factor

$x^{-2}\mathbf{1}_{|x|>\rho_2} < 10^{-16}$, which absorbs the large constant 10^{10} in G_θ. Therefore, G_θ is very small compared to ψ.

4.3.10 Summary: estimates of $\mathcal{L}_{\theta 1}, \mathcal{L}_{\omega 1}$

Recall $\mathcal{L}_{\theta 1}, \mathcal{L}_{\omega 1}$ in (4.28), the quadratic terms in (4.28), (4.31). Combining (4.30),(4.44),(4.38), (4.45), (4.52) and (4.53), we obtain the estimate on $\mathcal{L}_{\theta 1}, \mathcal{L}_{\omega 1}$

$$\begin{aligned}\langle \mathcal{L}_{\theta 1}\theta_x, \theta_x\psi\rangle + \lambda_1\langle \mathcal{L}_{\omega 1}\omega, \omega\varphi\rangle \leq \langle D_\theta + A_\theta\psi^{-1}, \theta_x^2\psi\rangle + \langle \lambda_1 D_\omega + A_\omega\varphi^{-1}, \omega^2\varphi\rangle \\ - \Big(D_u - \frac{9}{49}t_{12} - \frac{72\lambda_1}{49}\cdot 10^{-5}\Big)||\tilde{u}_x x^{-2/3}||_2^2 + A(u) + G_c c_\omega^2,\end{aligned} \tag{4.55}$$

where $A(u)$ is defined in (4.48), D_u, t_{12} are given in (C.24), and the A_θ, A_ω terms are the sum of the integrals of ω^2, θ_x^2 in the upper bounds in (4.44),(4.38), (4.45), (4.53) given by

$$\begin{aligned}A_\theta &\triangleq \Big(\frac{1}{4t_1}x^{-2/3} + \frac{1}{4t_{12}}(\psi_f\bar{\theta}_{xx}x^{5/3})^2\Big) \\ &\quad + \Big(\frac{1}{4t_2}(\alpha_2 x^{-1} + \alpha_1 x^{-2})^2 + \frac{1}{4t_{22}}(x^3\bar{\theta}_{xx}\psi_n)^2\Big) + \frac{(\lambda_1\alpha_4)^2}{4t_4}x^{-5} + G_\theta, \\ A_\omega &\triangleq t_1\Big(\alpha_3^2 x^{-2} + \frac{\alpha_3\lambda_1\alpha_6}{\sqrt{3}}x^{-4/3} + (\lambda_1\alpha_6)^2 x^{-2/3}\Big) \\ &\quad + \Big(t_2 x^{-4} + \frac{t_{22}}{25}x^{-4} + t_2(\lambda_1\alpha_5)^2 x^{-2}\Big) + t_4 x^{-3} \\ &\quad + \lambda_1\Big(\frac{4t_{31}}{25}x^{-4} + \frac{4t_{32}}{9}x^{-2} + \frac{1}{4}\mathcal{K}_{u\omega}^2 S_{u1}^{-1} + \tau_1(\frac{t_{34}}{25}x^{-4} + \frac{1}{4t_{34}}x^{-2})\Big) + G_\omega.\end{aligned} \tag{4.56}$$

In the previous estimates, we have obtained the ranges of t_{ij} such that $I + D_1 + D_2 \leq -0.01(||\theta_x\psi^{1/2}||_2^2 + \lambda_1||\omega\varphi^{1/2}||_2^2)$ for several terms I in (4.31), e.g., $I = I_f, I_n$. We further determine the approximate values of λ_1, t_{ij} so that

$$D_\theta + A_\theta\psi^{-1} \leq -c, \quad \lambda_1 D_\omega + A_\omega\varphi^{-1} \leq -\lambda_1 c \tag{4.57}$$

with $c > 0$ as large as possible. The functions in (4.57) depend on the parameters and other explicit functions. The above task is equivalent to solving a constrainted optimization problem by maximizing c, subject to the constraints (4.57) and $\lambda_1, t_{ij} > 0$ within an interval that we have determined.

Estimates (4.55), (4.57) imply the linear stability estimate (4.29) up to the c_ω terms in (4.31), $A(u)$ (4.48) and $G_c c_\omega^2$. In Section 4.3.11, we further control

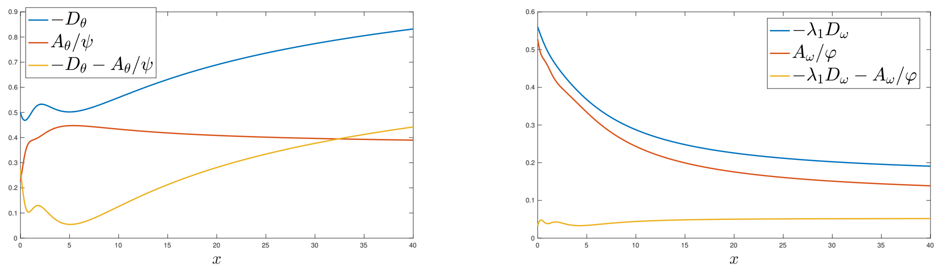

Figure 4.1: Illustration of linear stability estimates. Left : Grid point values of $-D_\theta$, $A_\theta\psi^{-1}$ and $-D_\theta - A_\theta\psi^{-1}$ for $x \in [0, 40]$. Right: Those of $-\lambda_1 D_\omega$, $A_\omega\phi^{-1}$ and $-\lambda_1 D_\omega - A_\omega\phi^{-1}$ (with $\lambda_1 = 0.32$).

these c_ω terms. The estimate of these c_ω terms are small. We will perturb λ_1, t_{ij} around their approximate values and finalize the choices of λ_1, t_{ij}. The final values of these parameters are given in (C.24), (C.25).

The main reasons that we can establish (4.57) are the followings. Firstly, we exploit several cancellations using Lemma C.0.8 and apply sharp weighted estimates in Lemma C.0.8 to estimate the nonlocal terms. Secondly, we have $I + D_1 + D_2 \leq -c(||\theta_x\psi^{1/2}||_2^2 + \lambda_1||\omega\varphi^{1/2}||_2^2)$ for I being the main terms in (4.31), i.e. $I = I_f, I_n$ or I_s. Thirdly, the interactions in I_f, I_n, I_s are mainly supported in different regions. See the discussion after (4.33). Finally, the main term in $I_{u\omega}$ is estimated using several cancellations.

To illustrate that the inequalities in (4.57) can actually be achieved, we plot in Figure 4.1 the grid point values of the functions $-D_\theta - A_\theta\psi^{-1}$ and $-\lambda_1 D_\omega - A_\omega\varphi^{-1}$ for $x \in [0, 40]$ with the parameters λ_1, t_{ij} given in (C.24), (C.25). It is shown that their grid point values are all positive and uniformly bounded away from 0. In fact, the minimum of the grid point values of $-D_\theta - A_\theta\psi^{-1}$ is above 0.032 and that of $-\lambda_1 D_\omega - A_\omega\phi^{-1}$ is above 0.054.

Estimate (4.55) on $\mathcal{L}_{\theta 1}, \mathcal{L}_{\omega 1}$ is important and we will use it in Section 4.5 to establish the weighted H^1 estimates.

4.3.11 Estimate of the c_ω term

We use the idea in Model 2 in Section 4.2.3.2 to obtain the damping term for c_ω by deriving the ODEs for c_ω^2 and $\langle \theta_x, x^{-1}\rangle^2$. We introduce some notations

$$d_\theta \triangleq \langle \theta_x, x^{-1}\rangle, \quad \bar{d}_\theta \triangleq \langle \bar{\theta}_x, x^{-1}\rangle, \quad \bar{u}_{\theta,x} \triangleq H\bar{\theta}_x, \quad u_\Delta = \tilde{u} - \frac{1}{5}\tilde{u}_x x. \tag{4.58}$$

4.3.11.1 Derivation of the ODEs

Recall $c_\omega = u_x(0) = -\frac{2}{\pi}\int_0^{+\infty} \frac{\omega}{x} dx$ from (4.20). Using a derivation similar to that in Model 2 in Section 4.2.3.2, , we derive the following ODE in Appendix C.1.3

$$\frac{1}{2}\frac{d}{dt}\frac{\pi}{2}c_\omega^2 = \frac{\pi}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega^2 + c_\omega \int_0^\infty \frac{\bar{u}\omega_x + u\bar{\omega}_x}{x} dx - c_\omega d_\theta - c_\omega \int_0^\infty \frac{F_\omega + N(\omega)}{x} dx. \tag{4.59}$$

The ODE for d_θ^2 (C.8) is derived similarly in Appendix C.1.3.1. There is a cancellation between these two ODEs, which is captured by Model 2 in Section 4.2.3.2. To exploit this cancellation, we combine two ODEs and derive the following ODE in Appendix C.1.3 with $\lambda_2, \lambda_3 > 0$ to be chosen

$$\frac{1}{2}\frac{d}{dt}(\frac{\lambda_2\pi}{2}c_\omega^2 + \lambda_3 d_\theta^2) = \frac{\pi\lambda_2}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega^2 + 2\bar{c}_\omega\lambda_3 d_\theta^2 + \mathcal{T}_0 + \mathcal{R}_{ODE}, \tag{4.60}$$

where $\mathcal{T}_0$ is the sum of the quadratic terms that do not have a fixed sign

$$\begin{aligned}\mathcal{T}_0 =& -(\lambda_2 - \lambda_3\bar{d}_\theta)c_\omega d_\theta + \lambda_2 c_\omega\langle\omega, f_2\rangle - \lambda_3 d_\theta\langle\theta_x, f_6\rangle + \lambda_3 d_\theta\langle\omega, f_4\rangle \\ &+ \lambda_2 c_\omega\langle u_\Delta x^{-1}, f_8\rangle - \lambda_3 d_\theta\langle u_\Delta x^{-1}, f_9\rangle,\end{aligned} \tag{4.61}$$

u_Δ is defined in (4.58), f_i defined in (C.5) are some functions depending on the profile $(\bar{\omega}, \bar{\theta})$, and $\mathcal{R}_{ODE}$ is the sum of the remaining terms in the ODEs given by

$$\mathcal{R}_{ODE} \triangleq -\lambda_2 c_\omega\langle F_\omega + N(\omega), x^{-1}\rangle + \lambda_3 d_\theta\langle F_\theta + N(\theta), x^{-1}\rangle. \tag{4.62}$$

Since the approximate steady state satisfies $\bar{c}_\omega < 0, \bar{u}_x(0) < 0$, $\frac{\pi\lambda_2}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega^2$ and $2\bar{c}_\omega\lambda_3 d_\theta^2$ in (4.60) are damping terms.

4.3.11.2 Derivation of the $\mathcal{T}_0$ term

Let us explain how we obtain (4.60). The ODEs of c_ω^2, d_θ^2 ((4.59) and (C.8)) involves the integrals of the nonlocal terms u, u_x in the form of $\langle\tilde{u}, f\rangle$ or $\langle\tilde{u}_x, g\rangle$ for some functions f, g. To estimate these terms effectively, we use the antisymmetry property of the Hilbert transform in Lemma C.0.3 to transform these terms into integrals of ω. We first consider $\langle\tilde{u}_x, g\rangle$ and $\langle u_x, g\rangle$. Using $u_x = H\omega$, $\frac{u_x - u_x(0)}{x} = H(\frac{\omega}{x})$ and Lemma C.0.3, we get

$$\langle u_x, g\rangle = \langle H\omega, g\rangle = -\langle\omega, Hg\rangle, \quad \langle\tilde{u}_x, g\rangle = \Big\langle H\left(\frac{\omega}{x}\right), xg\Big\rangle = -\Big\langle\frac{\omega}{x}, H(xg)\Big\rangle. \tag{4.63}$$

For $\langle \tilde{u}, f\rangle$, we first approximate f by p_x for some function p and then perform a decomposition $\tilde{u} = cx\tilde{u}_x + (\tilde{u} - cx\tilde{u}_x)$. We obtain

$$\langle \tilde{u}, f\rangle = \langle \tilde{u}, p_x\rangle + \langle \tilde{u}, f - p_x\rangle = \langle \tilde{u}, p_x\rangle + \langle cx\tilde{u}_x, f - p_x\rangle + \langle \tilde{u} - cx\tilde{u}_x, f - p_x\rangle \triangleq I_1 + I_2 + I_3.$$

The last term enjoys much better estimate than $\langle \tilde{u}, f\rangle$ due to (4.12) and the fact that $f - p_x$ is much smaller than f. For I_1, I_2, using integration by parts, we get

$$I_1 + I_2 = \langle \tilde{u}_x, -p + cx(f - p_x)\rangle.$$

Using (4.63), we can further rewrite the above term as an integral of ω.

In addition, we introduce the function f_i to simplify the integrals of ω, θ_x. These derivations lead to the $\mathcal{T}_0$ term. We refer the details to Appendix C.1.3.

We remain to estimate the c_ω terms in (4.28) in the weighted L^2 estimates and f_3, f_7 that are defined in (C.5). We combine $\mathcal{T}_0$ and these c_ω terms, and define

$$\mathcal{T} \triangleq \mathcal{T}_0 + c_\omega\langle \bar{\theta}_x - x\bar{\theta}_{xx},\ \theta_x\psi\rangle + \lambda_1 c_\omega\langle \bar{\omega} - x\bar{\omega}_x, \omega\varphi\rangle = \mathcal{T}_0 + c_\omega\langle \omega, f_3\rangle + c_\omega\langle \theta_x, f_7\rangle. \tag{4.64}$$

In the weighted L^2 estimates, it remains to estimate $\mathcal{T}$. Though each term in $\mathcal{T}$ can be estimated by the weighted L^2 norms of ω, θ_x and $c_\omega^2, \langle \theta_x, x^{-1}\rangle^2$ using the Cauchy-Schwarz inequality, these straightforward estimates do not lead to sharp estimates since these Cauchy-Schwarz inequalities do not achieve (or are close to) equalities for the same functions. We use the optimal-constant argument in [19] to obtain a sharp estimate on $\mathcal{T}$.

4.3.11.3 Sharp estimates on $\mathcal{T}$

For positive $T_1, T_2, T_3 \in C(\mathbb{R}_+)$ and positive parameter $s_1, s_2 > 0$ to be determined, we consider the following inequality with sharp constant C_{opt}

$$\mathcal{T} \leq C_{opt}(||\omega T_1^{1/2}||_2^2 + ||\theta_x T_2^{1/2}||_2^2 + ||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 + s_1 c_\omega^2 + s_2 d_\theta^2), \tag{4.65}$$

where u_Δ is defined in (4.58). We define several functions

$$\begin{aligned} &X = \omega T_1^{1/2}, \quad Y = \theta_x T_2^{1/2}, \quad Z = u_\Delta x^{-1} T_3^{1/2}, \\ &g_1 = -\frac{2}{\pi} x^{-1} T_1^{-1/2},\ g_2 = f_2 T_1^{-1/2},\ g_3 = f_3 T_1^{-1/2},\ g_4 = f_4 T_1^{-1/2}, \\ &g_5 = x^{-1} T_2^{-1/2},\ g_6 = f_6 T_2^{-1/2},\ g_7 = f_7 T_2^{-1/2},\ g_8 = f_8 T_3^{-1/2},\ g_9 = f_9 T_3^{-1/2}. \end{aligned} \tag{4.66}$$

Notice that each term in (4.61) and (4.64) can be seen as the projection of X, Y, Z onto some function g_i. For example, c_ω, d_θ can be written as follows

$$c_\omega = u_x(0) = -\frac{2}{\pi}\int_0^\infty \frac{\omega}{x}dx = \langle X, g_1\rangle, \quad d_\theta = \int_0^\infty \frac{\theta_x}{x}dx = \langle Y, g_5\rangle.$$

Using the definition of $\mathcal{T}$ in (4.61), (4.64) and the functions in (4.66), we rewrite (4.65) as

$$\begin{aligned}\mathcal{T} =&\langle X, g_1\rangle\langle X, g_3\rangle + \langle X, g_1\rangle\langle Y, g_7\rangle - (\lambda_2 - \lambda_3\bar{d}_\theta)\langle X, g_1\rangle\langle Y, g_5\rangle + \lambda_2\langle X, g_1\rangle\langle X, g_2\rangle \\ &- \lambda_3\langle Y, g_5\rangle\langle Y, g_6\rangle + \lambda_3\langle Y, g_5\rangle\langle X, g_4\rangle + \lambda_2\langle X, g_1\rangle\langle Z, g_8\rangle - \lambda_3\langle Y, g_5\rangle\langle Z, g_9\rangle \\ \le& C_{opt}(||X||_2^2 + ||Y||_2^2 + ||Z||_2^2 + s_1\langle X, g_1\rangle^2 + s_2\langle Y, g_5\rangle^2).\end{aligned} \tag{4.67}$$

We project X, Y, Z onto the following finite dimensional spaces

$$\begin{aligned}&X \in \text{span}\{g_1, g_2, g_3, g_4\} \triangleq \Sigma_1, \quad Y \in \text{span}\{g_5, g_6, g_7\} \triangleq \Sigma_2, \\ &Z \in \text{span}\{g_8, g_9\} \triangleq \Sigma_3,\end{aligned} \tag{4.68}$$

which only makes the right hand side of (4.67) smaller. Then (4.67) completely reduces to an optimization problem on the finite dimensional space. Using the optimal-constant argument in [19], we obtain

$$C_{opt} = \lambda_{\max}(D^{-1/2}M_s D^{-1/2}),$$

where D, M_s defined in (C.17) are symmetric matrices with entries determined by the inner products among g_i. In particular, C_{opt} can be computed rigorously and we present the details in Appendix C.1.5.

4.3.12 Summary of the estimates

Recall the c_ω terms in (4.28), the operators in (4.22). Combining (4.55), (4.61) and (4.64), we yield

$$\begin{aligned}&\langle \mathcal{L}_\theta\theta_x, \theta_x\psi\rangle + \lambda_1\langle \mathcal{L}_\omega\omega, \omega\varphi\rangle + \mathcal{T}_0 = \langle \mathcal{L}_{\theta 1}\theta_x, \theta_x\psi\rangle + \lambda_1\langle \mathcal{L}_{\omega 1}\omega, \omega\varphi\rangle + \mathcal{T} \\ \le&\langle D_\theta + A_\theta\psi^{-1}, \theta_x^2\psi\rangle + \langle \lambda_1 D_\omega + A_\omega\varphi^{-1}, \omega^2\varphi\rangle \\ -&\Big(D_u - \frac{9}{49}t_{12} - \frac{72\lambda_1}{49}\cdot 10^{-5}\Big)||\tilde{u}_x x^{-2/3}||_2^2 + \mathcal{T} + A(u) + G_c c_\omega^2 \triangleq J.\end{aligned} \tag{4.69}$$

We use the remaining damping of $\omega, \theta_x, \tilde{u}_x$ and the argument in Section 4.3.11.3 to control $\mathcal{T}$. In (C.19), Appendix C.1.6, we define $T_i > 0, s_i > 0$ that are used to compute the upper bound of $C_{opt} < 1$ in (4.65). These functions and scalars are essentially determined by four parameters $\lambda_2, \lambda_3, \kappa, t_{61} > 0$. Using the estimate (4.65), we obtain

$$\mathcal{T} \leq ||\omega T_1^{1/2}||_2^2 + ||\theta_x T_2^{1/2}||_2^2 + ||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 + s_1 c_\omega^2 + s_2 d_\theta^2. \tag{4.70}$$

The u_Δ term can be further bounded by $||\tilde{u}_x x^{-2/3}||_2$ and $||\omega x^{-2}||_2$ similar to (4.51), which is established in (C.22) in Appendix C.1.6. Plugging (C.22) and (4.70) in (4.69), we obtain

$$J \leq -\kappa ||\theta_x \psi^{1/2}||_2^2 - \kappa\lambda_1 ||\omega\varphi^{1/2}||_2^2 + (s_1 + G_c) c_\omega^2 + s_2 d_\theta^2 - 10^{-6} ||\tilde{u}_x x^{-2/3}||_2^2 + A(u), \tag{4.71}$$

for $\kappa > 0$ determined in Appendix C.2. The details are elementary and presented in Appendix C.1.6. For $\lambda_2, \lambda_3 > 0$ given in (C.25), we define the weighted L^2 energy

$$E_1^2(\theta_x, \omega) = ||\theta_x \psi^{1/2}||_2^2 + \lambda_1 ||\omega \psi^{1/2}||_2^2 + \lambda_2 \frac{\pi}{2} \cdot \frac{4}{\pi^2} \langle \omega, x^{-1} \rangle^2 + \lambda_3 \langle \theta_x, x^{-1} \rangle^2. \tag{4.72}$$

Note that $\frac{2}{\pi} \langle \omega, x^{-1} \rangle = -u_x(0) = -c_\omega$ (4.20). Recall the relations of different operators in (4.22). Combining the equations (4.28), (4.60) and using the estimates (4.69) and (4.71), we establish

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt} E_1^2(\theta, \omega) = & \langle \mathcal{L}_\theta \theta_x, \theta_x \psi \rangle + \lambda_1 \langle \mathcal{L}_\omega \omega, \omega \varphi \rangle + \mathcal{T}_0 + \frac{\pi \lambda_2}{2} (\bar{c}_\omega + \bar{u}_x(0)) c_\omega^2 \\
& + 2\bar{c}_\omega \lambda_3 d_\theta^2 + \mathcal{R}_{L^2} \\
\leq & -\kappa ||\theta_x \psi^{1/2}||_2^2 - \kappa\lambda_1 ||\omega \varphi^{1/2}||_2^2 - 10^{-6} ||\tilde{u}_x x^{-2/3}||_2^2 + A(u) \\
& + \Big(\frac{\pi \lambda_2}{2} (\bar{c}_\omega + \bar{u}_x(0)) + s_1 + G_c \Big) c_\omega^2 + (2\bar{c}_\omega \lambda_3 + s_2) d_\theta^2 + \mathcal{R}_{L^2},
\end{aligned}$$

where $\mathcal{R}_{L^2}$ is given by

$$\mathcal{R}_{L^2} \triangleq N_1 + F_1 + \lambda_1 N_2 + \lambda_1 F_2 + \mathcal{R}_{ODE} \tag{4.73}$$

and N_i, F_i are defined in (4.28) and $\mathcal{R}_{ODE}$ in (4.62). Recall $A(u)$ in (4.48), $c_\omega = u_x(0)$. Using the definitions of s_i in (C.19), we get

$$\frac{\pi \lambda_2}{2} (\bar{c}_\omega + \bar{u}_x(0)) + s_1 + G_c + \frac{\pi \lambda_1 e_3 \alpha_6}{12} = -r_{c_\omega}, \quad s_2 + 2\bar{c}_\omega \lambda_3 = -\kappa \lambda_3,$$

for $r_{c_\omega}, \kappa > 0$ determined in Appendix C.2. Hence, we obtain

$$\begin{aligned}
& A(u) + \Big(\frac{\pi \lambda_2}{2} (\bar{c}_\omega + \bar{u}_x(0)) + s_1 + G_c \Big) c_\omega^2 + (2\bar{c}_\omega \lambda_3 + s_2) d_\theta^2 \\
= & - r_{c_\omega} c_\omega^2 - \kappa \lambda_3 d_\theta^2 - \frac{\lambda_1 e_3 \alpha_6}{3} \langle \Lambda \frac{u}{x}, \frac{u}{x} \rangle.
\end{aligned}$$

Therefore, we obtain

$$\frac{1}{2}\frac{d}{dt}E_1^2(\theta,\omega) \leq -\kappa||\theta_x\psi^{1/2}||_2^2 - \kappa\lambda_1||\omega\varphi^{1/2}||_2^2 - 10^{-6}||\tilde{u}_x x^{-2/3}||_2^2 - \frac{\lambda_1 e_3\alpha_6}{3}\langle\Lambda\frac{u}{x},\frac{u}{x}\rangle$$
$$- r_{c_\omega}c_\omega^2 - \kappa\lambda_3 d_\theta^2 + \mathcal{R}_{L^2} \triangleq Q + \mathcal{R}_{L^2}. \tag{4.74}$$

These parameters satisfy $r_{c_\omega} \geq \frac{\pi}{2}\kappa\lambda_2$. Thus (4.74) implies

$$\frac{1}{2}\frac{d}{dt}E_1^2(\theta,\omega) \leq -\kappa E_1^2(\theta,\omega) + \mathcal{R}_{L^2}, \tag{4.75}$$

and we establish the linear stability. See also (4.29). Compared to (4.75), (4.74) contains extra damping terms $-||\tilde{u}_x x^{-2/3}||_2^2$, $-\langle\Lambda\frac{u}{x},\frac{u}{x}\rangle$ and $-(r_{c_\omega} - \frac{\pi}{2}\kappa\lambda_2)c_\omega^2$. We choose $r_{c_\omega} > \kappa\frac{\pi}{2}\lambda_2$ and keep these terms in (4.74) mainly to obtain sharper constants in our later weighted H^1 estimates.

4.3.13 From linear stability to nonlinear stability with rigorous verification

In this subsection, we describe some main ideas how to go from linear stability to nonlinear stability with computer-assisted proof.

(1) As we discuss at the beginning of Section 4.2, the most challenging and essential part in the proof is the weighted L^2 linear stability analysis established in Section 4.3, since there is *no* small parameter and the linearized equations (4.10) are complicated.

(2) The weighted L^2 linear stability estimates can be seen as a-priori estimates on the perturbation, and we proceed to perform higher order energy estimates in a similar manner and establish the nonlinear energy estimate for some energy $E(t)$ of the perturbation

$$\frac{d}{dt}E^2 \leq CE^3 - \lambda E^2 + \varepsilon E. \tag{4.76}$$

Here, $-\lambda E^2$ with $\lambda > 0$ comes from the linear stability, CE^3 with some constant $C(\bar{\omega},\bar{\theta}) > 0$ controls the nonlinear terms, and ε is the weighted norm of the residual error of the approximate steady state. See more details in Section 4.5. To close the bootstrap argument $E(t) < E^*$ with some threshold $E^* > 0$, a sufficient condition is that $\varepsilon < \varepsilon^* = \lambda^2/(4C)$, which provides an upper bound on the required accuracy of the approximate steady state.

The essential parts of the estimates in (1), (2) are established based on the grid point values of $(\bar{\omega},\bar{\theta})$ constructed using a moderate fine mesh. These parts do not involve the lengthy rigorous verification in the Supplementary Material [21]. These estimates already provide a strong evidence of nonlinear stability.

A significant difference from this step and step (1) is that we have a small parameter ε. As long as ε is sufficiently small, thanks to the damping term $-\lambda E^2$ established in step (1), we can afford a large constant $C(\bar{\omega}, \bar{\theta})$ in the estimate of the nonlinear terms and close the nonlinear estimates. We can complete all the nonlinear estimates in this step.

(3) We follow the general approach in [19] to construct an approximate steady state with residual error below a required level ε^*. To achieve the desired accuracy, the construction is typically done by solving (4.5) for a sufficiently long time using a fine mesh. The difficulty of the construction depends on the target accuracy ε^*, and we refer to Section 4.4 for more discussion on the new difficulty and the construction of the approximate steady state for the HL model. Here, the mesh size plays a role similar to a small parameter that we can use. In practice, the profile $(\bar{\omega}_1, \bar{\theta}_1)$ constructed using a moderate fine mesh Ω_1 is close to the one (ω_2, θ_2) constructed using a finer mesh Ω_2 with higher accuracy. As a result, the constants $C(\bar{\omega}, \bar{\theta})$ and λ that we estimate in (4.76) using different approximate steady states (ω_i, θ_i) are nearly the same. This refinement procedure allows us to obtain an approximate steady state, based on which we close the nonlinear estimates (4.76). We refer more discussion of this philosophy to [19].

(4) Finally, we follow the standard procedure to perform rigorous verification on the estimates to pass from the grid point value to its continuous counterpart. Estimates that require rigorous verification with computer assistance are recorded in Appendix C.3. In the verification step, we can evaluate the approximate steady state on a much finer mesh Ω_3 with many more grid points so that they almost capture the whole behavior of the solution. Then, we use the regularity of the solution to pass from finite grid points to the whole real line. In this procedure, the mesh size in Ω_3 plays a role similar to a small parameter that we can exploit. In practice, to perform the rigorous verification, we evaluate the solution computed in a mesh with about 5000 grid points using a much denser mesh with more than $5 \cdot 10^5$ grid points.

In summary, in steps (2)-(4), we can take advantage of a small parameter which can be either the small error or the small mesh size, while there is no small parameter in step (1). Though these three steps could be technical, they are relatively standard from the viewpoint of analysis.

We remark that the approach of computer-assisted proof has played an important role in the analysis of many PDE problems, especially in computing explicit tight bounds of complicated (singular) integrals [6, 32, 56] or bounding the norms of linear operators [4, 50]. We refer to [55] for an excellent survey on computer-assisted proofs in establishing rigorous analysis for PDEs, which also explains the use of interval arithmetics that guarantees rigorous computer-assisted verifications. Examples of highly nontrivial results established by the use of interval arithmetics can be found in, for example, [52, 60, 82, 102]. Our approach to establish stability analysis with computer assistance is different from existing computer-assisted approach, e.g., [7], where the stability is established by numerically tracking the spectrum of a given operator and quantifying the spectral gap. See Sections 1.3.7 and the last paragraph in 3.3.1 for more discussions and explanations.

4.4 On the approximate steady state

The proof of the main Theorem 4.2 heavily relies on an approximate steady state solution $(\bar{\theta}, \bar{\omega}, \bar{c}_l, \bar{c}_\omega)$ to the dynamic rescaling equations (4.5), which is smooth enough, e.g., $\bar{\omega}, \bar{\theta}_x \in C^3$. Moreover, as discussed in Section 4.3.13, the residual error of the approximate steady state must be small enough in order to close the nonlinear estimates. In particular, the residual error ε requirement depends on the stability gap λ via the inequality $\varepsilon < \lambda^2/(4C)$.

For comparison, we refer the reader to our previous work on proving the finite-time, approximate self-similar blowup of the 1D De Gregorio model via a similar computer-aided strategy [19], where the corresponding approximate steady state is constructed numerically on a compact domain $[-10, 10]$. The stability gap that the authors proved in that work is relatively large (around 0.3), and thus the point-wise error requirement on the residual can be relaxed to 10^{-6}.

For the HL model, however, the stability gap $\lambda \approx \kappa = 0.03$ (see (C.25)) that we can prove in the linear stability analysis (4.75) is much smaller, which leads to a much stronger requirement on the residual error. More precisely, we need to bound the residual in a weighted norm by 5.5×10^{-7} with weights (4.26) that are singular of order $x^{-k}, k \geq 4$ near 0 and decay slowly for large x. This effectively requires the point-wise values of the residual to be as small as 10^{-10}. To achieve this goal, it is not sufficient to simply follow the method in [19], mainly due to the following reasons:

1. The steady state solution to (4.5) is supported on the whole real line and has a slowly decaying tail in the far field (see below). If we approximate the steady state on a finite domain $[-L, L]$, we need to use an unreasonably large L (roughly $L \geq 10^{30}$) for the tail part beyond $[-L, L]$ to be considered as a negligible error, since truncating the tail leads to an error of order $L^{c_\omega/c_l} \approx L^{-1/3}$. It is then impractical to achieve a uniformly small residual by only using mesh-based algorithms such as spline interpolations.

2. Numerically computing the Hilbert transform of a function supported on the whole real line $\mathbb{R}$ is sensitively subject to round-off errors. For example, when computing u from an odd function ω via the Hilbert transform, we need to evaluate the convolution kernel $\log(|y-x|/|y+x|)$, which will be mistaken as 0 by a computer program using double-precision if $|x/y| < 10^{-16}$. Such round-off errors, when accumulated over the whole mesh, are unacceptable in our case since we have a very high accuracy requirement for the computation of the approximate steady state solution.

To design a practical method of obtaining a sufficiently accurate construction, we must have some a priori knowledge on the behavior of a steady state $(\omega_\infty, \theta_\infty, c_{l,\infty}, c_{\omega,\infty})$. Assume that the velocity u_∞ grows (if it grows) only sub-linearly in the far field, i.e. $u_\infty(x)/x, u_{\infty,x}(x) \to 0$ as $x \to \infty$. Substituting this ansatz into the steady state equation of θ_x in (4.5) yields

$$\frac{\theta_{\infty,xx}}{\theta_{\infty,x}} \sim \frac{2c_{\omega,\infty}}{c_{l,\infty}} \cdot x^{-1}, \quad \text{which implies} \quad \theta_{\infty,x} \sim x^{2c_{\omega,\infty}/c_{l,\infty}}.$$

Furthermore, using these results to the steady state equation of ω in (4.5) yields

$$\frac{\omega_{\infty,x}}{\omega_\infty} \sim \frac{c_{\omega,\infty}}{c_{l,\infty}} \cdot x^{-1}, \quad \text{which implies} \quad \omega_\infty \sim x^{c_{\omega,\infty}/c_{l,\infty}}.$$

From our preliminary numerical simulation, we have $c_{\omega,\infty}/c_{l,\infty}$ close to $-1/3$. This straightforward argument implies that ω_∞ and $\theta_{\infty,x}$ should behave asymptotically like $x^{c_{\omega,\infty}/c_{l,\infty}}$, $x^{2c_{\omega,\infty}/c_{l,\infty}}$ as $x \to +\infty$, respectively, which in turn justifies the sub-linear growth of u_∞.

Guided by these observations, we will construct our approximate steady state as the combination of two parts:

$$\bar{\omega} = \omega_b + \omega_p, \quad \bar{\theta} = \theta_b + \theta_p. \tag{4.77}$$

We will call (ω_b, θ_b) the explicit part and (ω_p, θ_p) the perturbation part. The explicit part (ω_b, θ_b) is constructed analytically to approximate the asymptotic tail behavior of the steady state for $x \geq L$, and satisfies $\omega_b, \theta_{b,x} \in C^5$ and $\omega_b \sim x^{\alpha}, \theta_{x,b} \sim x^{2\alpha}$ with $\alpha \approx \bar{c}_\omega / \bar{c}_l < -\frac{1}{3}$. The construction of ω_b and its Hilbert transform relies on the following crucial identity

$$H(\mathrm{sgn}(x)|x|^{-a}) = -\cot\frac{\pi a}{2} \cdot |x|^{-a}, \quad a \in (0,1), \tag{4.78}$$

which is proved in the proof of Lemma C.0.5 in Appendix C. It indicates that the leading order behavior of Hf for large x is given by $-\cot\frac{\pi a}{2} \cdot |x|^{-a}$, if f is odd with a decay rate $|x|^{-a}$. By perturbing $\mathrm{sgn}(x)|x|^{-a}$ and (4.78), we construct $\omega_b \in C^5$ and obtain the leading order behavior of $H\omega_b$ for large x. This is one of the main reasons that we can compute the Hilbert transform of a function with slow decay accurately and overcome large round-off error in its computation. The perturbation part (ω_p, θ_p) is constructed numerically using a quintic spline interpolation and methods similar to those in [19] in the domain $[-L, L]$ for some reasonably large L (around 10^{16}). By our construction, they satisfy that $\omega_p, \theta_{p,x} \in C_c^3$. Since achieving a small residual error is critical to our proof, a large portion of the Supplementary Material [21] is devoted to the construction (Section 10) and error estimates of the approximate steady state (Section 11-15) with the above decomposition, especially the ω_b part.

4.4.1 Connection to the approximate steady state of the 2D Boussinesq in $\mathbb{R}_+^2$

To generalize the current framework to the 2D Boussinesq equations, an important step is to construct an approximate steady state with a sufficiently residual error. The construction of the approximate steady state of the HL model provides important guidelines on this. The steady state equations of the dynamic rescaling formulation of the 2D Boussinesq, see e.g., [16], read

$$(c_l x + \mathbf{u}) \cdot \nabla\omega = c_\omega \omega + \theta_x,$$
$$(c_l x + \mathbf{u}) \cdot \nabla\theta = (c_l + 2c_\omega)\theta, \quad \mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\omega.$$

Denote $r = |x|$. Assume that the velocity u grows sub-linearly in the far field: $\frac{u(x)}{r} \to 0$ as $r \to \infty$ and the scaling factors satisfy $c_l > 0, c_\omega < 0$. Note that

$x \cdot \nabla = r\partial_r$. Passing to the polar coordinate $(r, \beta), r = |x|, \beta = \arctan \frac{x_2}{x_1}$ and dropping the lower order terms, we yield

$$c_l r\partial_r \omega(r, \beta) = c_\omega \omega + \theta_x + l.o.t., \quad c_l r\partial_r \theta(r, \beta) = (2c_\omega + c_l)\theta + l.o.t.$$

Using an argument similar to the above argument for the HL model, we obtain

$$\omega(r, \beta) \sim p(\beta) r^\alpha, \quad \theta(r, \beta) \sim q(\beta) r^{1+2\alpha}, \quad \alpha = \frac{c_\omega}{c_l} < 0.$$

We remark that θ_x has a decay rate $r^{2\alpha}$ faster than that of ω. The computation in [87] suggests that $\alpha \approx -\frac{1}{3}$. Thus, the profile (if it exists) for the 2D Boussinesq does not have a fast decay, and we also encounter the difficulties similar to (1) and (2). In particular, the 2D analog of difficulty (2) is to obtain the stream function $\psi = (-\Delta)^{-1}\omega$ accurately in $\mathbb{R}_+^2$. To design a practical method that overcomes these difficulties, it is important to perform a decomposition similar to (4.77), where $\omega_p, \nabla\theta_p$ have compact support and ω_b, θ_b capture the tail behavior of the steady state. For the 2D Boussinesq, ω_b, θ_b become semi-analytic since the angular part $p(\beta), q(\beta)$ cannot be determined a-priori. To overcome the difficulty of solving the stream function in the far field, we seek a generalization of (4.78). We consider the ansatz $\psi = r^{2+\alpha} f(\beta)$ and solve

$$-\Delta(r^{2+\alpha} f(\beta)) = r^\alpha p(\beta)$$

with boundary condition $f(0) = f(\pi/2) = 0$ due to the Dirichlet boundary condition and the odd symmetry for the solution ω. In the polar coordinate, the above equation is equivalent to

$$(-\partial_\beta^2 - (2+\alpha)^2) f(\beta) = p(\beta), \quad f(0) = f(\pi/2) = 0.$$

Solving the above equation is significantly simpler than solving $-\Delta\psi = \omega$ in $\mathbb{R}_+^2$ since it is one-dimensional and in a compact domain. The above two formulas are a generalization of (4.78) that connects the leading order far field behavior of ω with that of the velocity. We believe that the above decomposition is crucial to construct the approximate steady state with sufficiently small residual error for the 2D Boussinesq equations. The supplementary material on the analysis of the decomposition (4.77) for the HL model can be seen as a preparation for the more complicated decomposition in the 2D Boussineq equations.

4.5 Nonlinear stability and finite time blowup

In this section, we further establish nonlinear stability analysis of (4.23).

4.5.1 Weighted H^1 estimate

In order to obtain nonlinear stability, we first establish the weighted H^1 estimate similar to

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||D_x\theta_x\psi^{1/2}||_2^2+\lambda_1||D_x\omega\varphi^{1/2}||_2^2)\\ \leq&-c(||D_x\theta_x\psi^{1/2}||_2^2+\lambda_1||D_x\omega\varphi^{1/2}||_2^2)+CE_1^2(\theta,\omega)+\mathcal{R}_{H^1}\end{aligned} \tag{4.79}$$

for some $c, C > 0$, where $D_x = x\partial_x$, E_1 is defined in (4.72) and $\mathcal{R}_{H^1}$ are the error terms and nonlinear terms in the weighted H^1 estimate to be introduced.

In the work of Elgindi-Ghoul-Masmoudi [49], they made a good observation that the weighted H^1 estimates of the equation studied in [49] can be established by performing weighted L^2 estimates of $x\partial_x f$ with the same weight as that in the weighted L^2 estimate, since the commutator between the linearized operator and $x\partial_x$ is of lower order. Inspired by this observation, we perform weighted L^2 estimates on $x\partial_x\theta_x$ and $x\partial_x\omega$. However, one important difference between our problem and that considered in [49] is that the commutator between the linearized operator in (4.23) and $x\partial_x$ is not of lower order.

Denote $D_x = x\partial_x$. Similar weighted derivatives have been used in [16, 42, 49] for stability analysis. We derive the equations for $D_x\theta_x, D_x\omega$. Taking D_x on both side of (4.23), we get

$$\begin{aligned}\partial_t D_x\theta_x &= \mathcal{L}_{\theta 1}(D_x\theta_x, D_x\omega) + c_\omega D_x(\bar{\theta}_x - x\bar{\theta}_{xx}) + [D_x, \mathcal{L}_{\theta 1}](\theta_x,\omega) + D_xF_\theta + D_xN(\theta),\\ \partial_t D_x\omega &= \mathcal{L}_{\omega 1}(D_x\theta_x, D_x\omega) + c_\omega D_x(\bar{\omega} - x\bar{\omega}_x) + [D_x, \mathcal{L}_{\omega 1}](\theta_x,\omega) + D_xF_\theta + D_xN(\theta),\end{aligned} \tag{4.80}$$

where $[D_x, \mathcal{L}](f,g) \triangleq D_x\mathcal{L}(f,g) - \mathcal{L}(D_xf, D_xg)$. In Appendix C.1.4, we obtain the following formulas for the commutators

$$\begin{aligned}[D_x, \mathcal{L}_{\theta 1}](\theta_x,\omega) &= -(\bar{u}_x - \frac{\bar{u}}{x})D_x\theta_x - D_x\tilde{u}_x\theta_x - D_x\tilde{u}\bar{\theta}_{xx} - \tilde{u}(\bar{\theta}_{xx} + D_x\bar{\theta}_{xx}),\\ [D_x, \mathcal{L}_{\omega 1}](\theta_x,\omega) &= -(\bar{u}_x - \frac{\bar{u}}{x})D_x\omega - \tilde{u}(\bar{\omega}_x + D_x\bar{\omega}_x),\end{aligned} \tag{4.81}$$

where $\tilde{u}, \tilde{u}_x$ are defined in (4.27).

Performing the weighted H^1 estimates, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\langle D_x\theta_x, D_x\theta_x\psi\rangle + \lambda_1\langle D_x\omega, D_x\omega\varphi\rangle\Big)\\
=&\Big(\langle \mathcal{L}_{\theta 1}(D_x\theta, D_x\omega), D_x\theta_x\psi\rangle + \lambda_1\langle \mathcal{L}_{\omega 1}(D_x\theta, D_x\omega), D_x\omega\varphi\rangle\Big)\\
&+\Big(\langle [D_x, \mathcal{L}_{\theta 1}](\theta_x, \omega), D_x\theta_x\psi\rangle + \lambda_1\langle [D_x, \mathcal{L}_{\omega 1}](\theta_x, \omega), D_x\omega\varphi\rangle\Big)\\
&+\Big(\langle c_\omega D_x(\bar\theta_x - x\bar\theta_{xx}), D_x\theta_x\psi\rangle + \lambda_1\langle c_\omega D_x(\bar\omega - x\bar\omega_x), D_x\omega\varphi\rangle\Big) + \mathcal{R}_{H^1}\\
&\triangleq Q_1 + Q_2 + Q_3 + \mathcal{R}_{H^1},
\end{aligned} \tag{4.82}$$

where $\mathcal{R}_{H^1}$ is the remaining term in the weighted H^1 estimate

$$\begin{aligned}
\mathcal{R}_{H^1} = &\langle D_x N(\theta), D_x\theta_x\psi\rangle + \lambda_1\langle D_x N(\omega), D_x\omega\varphi\rangle\\
&+ \langle D_x F_\theta, D_x\theta_x\psi\rangle + \lambda_1\langle D_x F_\omega, D_x\omega\varphi\rangle.
\end{aligned} \tag{4.83}$$

4.5.1.1 Estimate of Q_1

Applying the estimate of $\mathcal{L}_{\theta,1}, \mathcal{L}_{\omega 1}$ in (4.55) to $(D_x\theta_x, D_x\omega)$, we obtain

$$\begin{aligned}
Q_1 \leq &\langle D_\theta + A_\theta\psi^{-1}, (D_x\theta_x)^2\psi\rangle + \langle \lambda_1 D_\omega + A_\omega\varphi^{-1}, (D_x\omega)^2\varphi\rangle\\
&+ A(-\Lambda^{-1}(D_x\omega)) + G_c\cdot(HD_x\omega(0))^2,
\end{aligned} \tag{4.84}$$

where G_c is defined in (4.54), and we have dropped the term related to $||\tilde u_x x^{-2/3}||_2^2$ in (4.55) since $D_u - \frac{9}{49}t_{12} - \frac{72\lambda_1}{49}\cdot 10^{-5} > 0$. In addition, we have replaced $u = -\Lambda^{-1}\omega$ in $A(u)$ in (4.55) by $-\Lambda^{-1}(D_x\omega)$ and replaced $c_\omega = H\omega(0)$ by $HD_x\omega(0)$. Recall the definition of $A(u)$ in (4.48). Since $\Lambda = H\partial_x$ and $H\circ H = -Id$, we yield

$$\partial_x(-\Lambda^{-1}D_x\omega)(0) = HD_x\omega(0) = -\frac{1}{\pi}\int_{\mathbb{R}}\omega_x dx = 0,$$

which implies

$$G_c\cdot(H(D_x\omega)(0))^2 = 0, \quad A(-\Lambda^{-1}D_x\omega) \leq 0. \tag{4.85}$$

We treat Q_1 as the damping terms in the weighted H^1 estimate since from (C.31), we have

$$D_\theta + A_\theta\psi^{-1} \leq -\kappa, \quad \lambda_1 D_\omega + A_\omega\varphi^{-1} \leq -\lambda_1\kappa, \quad \kappa > 0. \tag{4.86}$$

4.5.1.2 Estimate of Q_2

Recall the commutators in (4.81). The profile satisfies $\bar{u}_x - \frac{\bar{u}}{x} > 0$ and thus $-(\bar{u}_x - \frac{\bar{u}}{x})f$ with $f = D_x\theta_x, D_x\omega$ is a damping term in the $D_x\theta_x$ or $D_x\omega$ equation. We do not estimate these terms.

For the term $D_x\bar{u}_x\theta_x$ in (4.81), using integration by parts, we get

$$-\langle D_x\bar{u}_x\theta_x, D_x\theta_x\psi\rangle = -\langle x^2\bar{u}_{xx}\psi, \frac{1}{2}\partial_x(\theta_x)^2\rangle = \frac{1}{2}\langle (x^2\bar{u}_{xx}\psi)_x, \theta_x^2\psi\rangle.$$

The approximate steady state satisfies the following inequality

$$(x^2\bar{u}_{xx}\psi)_x \leq 0.02\psi, \tag{4.87}$$

which will be verified rigorously by the methods in the Supplementary Material [21]. We record it in (C.34), Appendix C.3. Using (4.87), we obtain

$$-\langle D_x\bar{u}_x\theta_x, D_x\theta_x\psi\rangle \leq \varepsilon_1||\theta_x\psi^{1/2}||_2^2, \quad \varepsilon_1 = 0.01. \tag{4.88}$$

The nonlocal terms in (4.81) are of lower order than $D_x\omega$ and we estimate then directly. We introduce some weights

$$S_{u2} = t_{71}x^{-6} + t_{72}x^{-4} + 2\cdot 10^{-6}x^{-10/3}, \quad S_{u3} = t_{81}x^{-6} + t_{82}x^{-4} + 2\cdot 10^{-6}x^{-10/3}, \tag{4.89}$$

for some parameters $t_{ij} > 0$ to be determined. Using Young's inequality, we get

$$\begin{aligned} &|\langle D_x\tilde{u}\bar{\theta}_{xx}, D_x\theta_x\psi\rangle| + |\langle \tilde{u}(\bar{\theta}_{xx} + D_x\bar{\theta}_{xx}), D_x\theta_x\psi\rangle| \\ \leq& ||D_x\tilde{u}S_{u2}^{1/2}||_2^2 + \frac{1}{4}||S_{u2}^{-1/2}\bar{\theta}_{xx}D_x\theta_x\psi||_2^2 \\ &+ ||\tilde{u}S_{u3}^{1/2}||_2^2 + \frac{1}{4}||S_{u3}^{-1/2}(\bar{\theta}_{xx} + D_x\bar{\theta}_{xx})D_x\theta_x\psi||_2^2. \end{aligned} \tag{4.90}$$

We introduce the weights S_{u2}, S_{u3} for a reason similar to that of S_{u1} in Remark 4.3.6. Recall $D_x\tilde{u} = \tilde{u}_x$ and $\tilde{u}, \tilde{u}_x$ in (4.27). Using the weighted estimates in Lemma C.0.8 yields

$$\begin{aligned} &||D_x\tilde{u}S_{u2}^{1/2}||_2^2 + ||\tilde{u}S_{u3}^{1/2}||_2^2 \\ \leq&\Big\langle \omega^2, (t_{71} + \frac{4t_{81}}{25})x^{-4} + (t_{72} + \frac{4t_{82}}{9})x^{-2}\Big\rangle + (1 + \frac{36}{49})\cdot 2\cdot 10^{-6}||\tilde{u}_x x^{-2/3}||_2^2. \end{aligned} \tag{4.91}$$

In (4.91), we do not estimate $||\tilde{u}_x x^{-2/3}||_2^2$ in $||D_x\tilde{u}S_{u2}^{1/2}||_2$ and keep it on both sides.

Remark 4.5.1. We will choose large enough parameters t_{ij} in S_{u2}, S_{u3} (4.89) so that the weighted L^2 norm of $D_x\theta_x$ terms in (4.90) are relative small compared to the damping term of $D_x\theta_x$ in the weighted H^1 estimate (4.82), e.g., Q_1 in (4.84). See also (4.86). The weighted L^2 norm of ω and $||\tilde{u}_x x^{-2/3}||_2$ in (4.91) will be bounded using the damping terms in the weighted L^2 estimate (4.74). The same argument applies to controlling the weighted L^2 norm of $D_x\omega$ term in (4.96).

Next, we estimate the $\tilde{u}((\bar{\omega}_x + D_x\bar{\omega})$ term in (4.81). The idea is similar to that in Section 4.3.8. We perform the following decomposition

$$\begin{aligned} & -\lambda_1\langle \tilde{u}(\bar{\omega}_x + D_x\bar{\omega}_x), D_x\omega\varphi\rangle \\ =& -\lambda_1\langle \tilde{u}(\bar{\omega}_x + D_x\bar{\omega}_x - \frac{1}{3}\chi\xi_3), D_x\omega\varphi\rangle - \frac{1}{3}\lambda_1\langle \tilde{u}\chi\xi_3, D_x\omega\varphi\rangle \triangleq J + I_{r3}. \end{aligned} \tag{4.92}$$

The estimate of I_{r3} is similar to (4.53) and we obtain the following estimate in Appendix C.1.2

$$|I_{r3}| \leq \langle G_{\omega 2}, \omega^2\rangle + \langle G_{\omega 3}, (D_x\omega)^2\rangle + G_{c2}c_\omega^2, \tag{4.93}$$

where $G_{\omega 2}, G_{\omega 3}$ and G_{c2} are given by

$$\begin{aligned} & G_{\omega 2} = \frac{1}{4\cdot 10^6}(\frac{2\lambda_1(2+\sqrt{3})}{5})^2 x^{-2/3}, \quad G_{c2} = \frac{\lambda_1^2||x\xi_3\chi^{1/2}\varphi^{1/2}||_2^2}{36}\cdot 10^3, \\ & G_{\omega 3} = 10^6(x^{4/3}\chi\xi_3\varphi)^2 + 10^{-3}\chi\varphi. \end{aligned} \tag{4.94}$$

These functions are small due to the same reason that we describe in Section 4.3.9.

For J, we perform a decomposition

$$J = -\lambda_1\Big\langle \tilde{u}, \Big((\bar{\omega}_x + D_x\bar{\omega}_x - \frac{1}{3}\chi\xi_3)\varphi - \frac{e_3\alpha_6}{9}x^{-2}\Big)D_x\omega\Big\rangle - \frac{\lambda_1 e_3\alpha_6}{9}\langle \tilde{u}, D_x\omega x^{-2}\rangle \triangleq I_1 + I_2.$$

Note that $\tilde{u} = u - u_x(0)x$ and $\int_0^\infty xD_x\omega x^{-2}dx = \int_0^\infty \omega_x dx = 0$. Using Lemma C.0.4 with $f = \omega$ and $g = u$, we get

$$\begin{aligned} I_2 &= -\frac{\lambda_1 e_3\alpha_6}{9}\Big(\langle u, D_x\omega x^{-2}\rangle - u_x(0)\int_0^\infty xD_x\omega x^{-2}dx\Big) \\ &= -\frac{\lambda_1 e_3\alpha_6}{9}\langle u, \omega_x x^{-1}\rangle = \frac{\lambda_1 e_3\alpha_6}{9}\langle \Lambda\frac{u}{x}, \frac{u}{x}\rangle, \end{aligned}$$

which can be controlled using the damping term in (4.74). Denote

$$S_{u4} = t_{91}x^{-6} + t_{92}x^{-4} + 5\cdot 10^{-4}x^{-10/3}, \quad \mathcal{K}_{u\omega 2} = (\bar{\omega}_x + D_x\bar{\omega}_x - \frac{1}{3}\chi\xi_3)\varphi - \frac{e_3\alpha_6}{9}x^{-2}. \tag{4.95}$$

For I_1, using Young's inequality and the weighted estimate in Lemma C.0.8, we get

$$\begin{aligned}
&|I_1| \leq \lambda_1 \langle S_{u4}, \tilde{u}^2 \rangle + \frac{\lambda_1}{4} \langle \mathcal{K}_{u\omega 2}^2 S_{u4}^{-1}, (D_x\omega)^2 \rangle \\
\leq & \lambda_1 \langle \omega^2, \frac{4t_{91}}{25} x^{-4} + \frac{4t_{92}}{9} x^{-2} \rangle + \frac{36\lambda_1}{49} \cdot 5 \cdot 10^{-4} ||\tilde{u}_x x^{-2/3}||_2^2 + \frac{\lambda_1}{4} \langle \mathcal{K}_{u\omega 2}^2 S_{u4}^{-1}, (D_x\omega)^2 \rangle.
\end{aligned} \tag{4.96}$$

We introduce the weight S_{u4} for a reason similar to that of S_{u1} in Remark 4.3.6. The $||\tilde{u}_x x^{-2/3}||_2^2$ term is further controlled by the corresponding damping term in (4.74).

Combining the above estimates on the commutators in (4.81), we obtain

$$\begin{aligned}
Q_2 \leq & \langle -(\bar{u}_x - \frac{\bar{u}}{x}) + B_\theta \psi^{-1}, (D_x\theta_x)^2 \psi \rangle + \langle -\lambda_1 (\bar{u}_x - \frac{\bar{u}}{x}) + B_\omega \varphi^{-1}, (D_x\omega)^2 \varphi \rangle \\
& + \varepsilon_1 ||\theta_x \psi^{1/2}||_2^2 + \langle A_{\omega 2}, \omega^2 \rangle + \frac{\lambda_1 e_3 \alpha_6}{9} \langle \Lambda \frac{u}{x}, \frac{u}{x} \rangle \\
& + \Big((1 + \frac{36}{49}) \cdot 2 \cdot 10^{-6} + \frac{36\lambda_1}{49} \cdot 5 \cdot 10^{-4} \Big) ||\tilde{u}_x x^{-2/3}||_2^2 + G_{c2} c_\omega^2,
\end{aligned} \tag{4.97}$$

where G_{c2} is defined in (4.94). The term $(\bar{u}_x - \frac{\bar{u}}{x})$ comes from the commutators (4.81) and we do not estimate them in Q_2 in (4.82). The terms $B_\theta, B_\omega, A_{\omega 2}$ are the sum of the coefficients in the integrals of $(D_x\theta_x)^2, (D_w\omega)^2, \omega^2$ in the above estimates

$$\begin{aligned}
B_\theta &\triangleq \frac{1}{4} S_{u2}^{-1} (\bar{\theta}_{xx} \psi)^2 + \frac{1}{4} S_{u3}^{-1} (\bar{\theta}_{xx} + D_x \bar{\theta}_{xx})^2 \psi^2, \quad B_\omega \triangleq \frac{\lambda_1}{4} \mathcal{K}_{u\omega 2}^2 S_{u4}^{-1} + G_{\omega 3}, \\
A_{\omega 2} &\triangleq (t_{71} + \frac{4t_{81}}{25}) x^{-4} + (t_{72} + \frac{4t_{82}}{9}) x^{-2} + \lambda_1 (\frac{4t_{91}}{25} x^{-4} + \frac{4t_{92}}{9} x^{-2}) + G_{\omega 2}.
\end{aligned} \tag{4.98}$$

4.5.1.3 Estimate of Q_3

Recall the c_ω terms in (4.80). Denote by K_1, K_2 the following L^2 norms

$$K_1 \triangleq ||\partial_x (x^3 \bar{\theta}_{xxx} \psi) \psi^{-1/2}||_2, \quad K_2 \triangleq ||\partial_x (x^3 \bar{\omega}_{xx} \varphi) \varphi^{-1/2}||_2. \tag{4.99}$$

Using integration by parts and the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
& |c_\omega \langle D_x (\bar{\theta}_x - x\bar{\theta}_{xx}), D_x \theta_x \psi \rangle| \\
= & |c_\omega \langle -x^2 \bar{\theta}_{xxx} \cdot (x\psi), \ \partial_x \theta_x \rangle| = |c_\omega \langle \partial_x (x^3 \bar{\theta}_{xxx} \psi), \theta_x \rangle| \\
\leq & |c_\omega| \cdot ||\partial_x (x^3 \bar{\theta}_{xxx} \psi) \psi^{-1/2}||_2 ||\theta_x \psi^{1/2}||_2 = K_1 |c_\omega| \cdot ||\theta_x \psi^{1/2}||_2,
\end{aligned}$$

where we have used $x\partial_x(f - x\partial_x f) = -x^2 f_{xx}, f = \bar{\theta}_x$ in the first equality. Similarly, we have

$$\begin{aligned}\lambda_1 |c_\omega \langle D_x(\bar{\omega} - x\bar{\omega}_x), D_x \omega\varphi\rangle| &\leq \lambda_1 |c_\omega| \cdot ||\partial_x(x^3 \bar{\omega}_{xx}\varphi)\varphi^{-1/2}||_2 ||\omega\varphi^{1/2}||_2 \\ &= \lambda_1 K_2 |c_\omega| \cdot ||\omega\varphi^{1/2}||_2.\end{aligned}$$

Using Young's inequality, we obtain

$$\begin{aligned}Q_3 \leq & K_1 |c_\omega| \cdot ||\theta_x \psi^{1/2}||_2 + \lambda_1 K_2 |c_\omega| \cdot ||\omega\varphi^{1/2}||_2 \\ \leq & \gamma_1 ||\theta_x \psi^{1/2}||_2^2 + \gamma_2 ||\omega\varphi^{1/2}||_2^2 + c_\omega^2 \Big(\frac{K_1^2}{4\gamma_1} + \frac{(\lambda_1 K_2)^2}{4\gamma_2} \Big),\end{aligned} \tag{4.100}$$

where $\gamma_1, \gamma_2 > 0$ are chosen in (C.26).

4.5.1.4 Summary of the estimates

We determine the parameters t_{ij} in the estimates in Sections 4.5.1.1-4.5.1.3 and choose κ_2 so that

$$\begin{aligned}& D_{\theta_2} + B_\theta \psi^{-1} \leq -\kappa_2, \quad D_{\omega 2} + B_\omega \varphi^{-1} \leq -\kappa_2 \lambda_1, \\ & D_{\theta 2} \triangleq D_\theta + A_\theta \psi^{-1} - (\bar{u}_x - \frac{\bar{u}}{x}), \quad D_{\omega 2} \triangleq \lambda_1 D_\omega + A_\omega \varphi^{-1} - \lambda_1 (\bar{u}_x - \frac{\bar{u}}{x}).\end{aligned} \tag{4.101}$$

The terms $D_{\theta 2}, D_{\omega 2}$ are the coefficients of the damping terms in the weighted H^1 estimate (4.82) and are already determined in the weighted L^2 estimates. The terms $B_\theta \psi^{-1}, B_\omega \varphi^{-1}$ defined in (4.98) are the coefficients in the weighted L^2 norm of $D_x \theta_x, D_x \omega$ in (4.90), (4.96). The motivation of (4.101) is that we use the damping terms to control the weighted L^2 norms of $D_x \theta_x, D_x \omega$ in the estimates of Q_i. The idea is the same as that in Remark 4.5.1.

We first choose $\kappa_2 < \kappa = 0.03$ in Appendix C.2, where κ is related to (4.75). This choice is motivated by our estimate (4.106). The dependences of $A_{\omega 2}, B_\theta, B_\omega$ on t_{ij} are given in (4.98), (4.89), (4.95). Inequalities in (4.101) can be seen as constraints on t_{ij}. We choose t_{ij} subject to the constraints (4.101) such that $||A_{\omega 2}\varphi^{-1}||_\infty$ is as small as possible. This enables us to obtain sharper constant a_{H^1} in the weighted H^1 estimate (4.103). After t_{ij} are determined, we verify (4.101) and

$$||A_{\omega 2}\varphi^{-1}||_\infty \leq a_{H^1}, \tag{4.102}$$

using the methods in the Supplementary Material [21], and record them in (C.35), Appendix C.3, where a_{H^1} is given in (C.26).

Combining (4.84), (4.85), (4.88), (4.97), (4.100) and (4.101), we prove

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(||(D_x\theta_x)\psi^{1/2}||_2^2 + \lambda_1||(D_x\omega)\varphi^{1/2}||_2^2)\\ \le& -\kappa_2||(D_x\theta_x)\psi^{1/2}||_2^2 - \kappa_2\lambda_1||(D_x\omega)\varphi^{1/2}||_2^2 + (\varepsilon_1+\gamma_1)||\theta_x\psi^{1/2}||_2^2\\ &+ (a_{H^1}+\gamma_2)||\omega\varphi^{1/2}||_2^2 + \Big(G_{c2} + \frac{K_1^2}{4\gamma_1} + \frac{(\lambda_1K_2)^2}{4\gamma_2}\Big)c_\omega^2 + \frac{\lambda_1 e_3\alpha_6}{9}\langle\Lambda\frac{u}{x},\frac{u}{x}\rangle\\ &+ \Big((1+\frac{36}{49})\cdot 2\cdot 10^{-6} + \frac{36\lambda_1}{49}\cdot 5\cdot 10^{-4}\Big)||\tilde{u}_x x^{-2/3}||_2^2 + \mathcal{R}_{H^1}.\end{aligned} \tag{4.103}$$

Recall the weighted L^2 energy E_1 in (4.72). For some $\lambda_4 > 0$, we construct the energy

$$\begin{aligned}E^2(\theta_x,\omega) &= E_1^2(\theta_x,\omega) + \lambda_4(||D_x\theta_x\psi^{1/2}||_2^2 + \lambda_1||D_x\omega\varphi^{1/2}||_2^2)\\ &= ||\theta_x\psi^{1/2}||_2^2 + \lambda_1||\omega\psi^{1/2}||_2^2 + \lambda_2\frac{\pi}{2}c_\omega^2\\ &\quad + \lambda_3 d_\theta^2 + \lambda_4(||D_x\theta_x\psi^{1/2}||_2^2 + \lambda_1||D_x\omega\varphi^{1/2}||_2^2).\end{aligned} \tag{4.104}$$

Note that $c_\omega, ||\theta_x\psi^{1/2}||_2, ||\omega\varphi^{1/2}||_2$ in (4.103) can be bounded by the energy E_1 in (4.72). The terms $\langle\Lambda\frac{u}{x},\frac{u}{x}\rangle$ and $||\tilde{u}_x x^{-2/3}||_2^2$ can be bounded by their damping terms in (4.74). To motivate later estimates and the choice of several parameters, we neglect these two terms. Then (4.103) implies (4.79) with $c = \kappa_2$ and some $C > 0$. Combining (4.75) and (4.79), we get

$$\frac{1}{2}\frac{d}{dt}E^2(\theta_x,\omega) \le -(\kappa-\lambda_4 C)E_1^2 - \kappa_2(||D_x\theta_x\psi^{1/2}||_2^2 + \lambda_1||D_x\omega\varphi^{1/2}||_2^2) + \mathcal{R}_{L^2} + \lambda_4\mathcal{R}_{H^1}, \tag{4.105}$$

where $\kappa = 0.03$. We first choose $\kappa_2 < \kappa$ and then λ_4 small enough, such that

$$\kappa - \lambda_4 C \ge \kappa_2. \tag{4.106}$$

Then we obtain the linear stability of (4.23) in the energy norm E.

4.5.2 Nonlinear stability

Combining (4.74) and (4.103), we derive

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}E^2(\theta_x,\omega) \le -\kappa||\theta_x\psi^{1/2}||_2^2 - \kappa\lambda_1||\omega\varphi^{1/2}||_2^2 - r_{c_\omega}c_\omega^2 - \kappa\lambda_3 d_\theta^2\\ &- \lambda_4\kappa_2||(D_x\theta_x)\psi^{1/2}||_2^2 - \lambda_4\kappa_2\lambda_1||(D_x\omega)\varphi^{1/2}||_2^2 + \lambda_4(\varepsilon_1+\gamma_1)||\theta_x\psi^{1/2}||_2^2\\ &+ \lambda_4(a_{H^1}+\gamma_2)||\omega\varphi^{1/2}||_2^2 + \lambda_4\Big(G_{c2} + \frac{K_1^2}{4\gamma_1} + \frac{(\lambda_1K_2)^2}{4\gamma_2}\Big)c_\omega^2\\ &+ (\frac{\lambda_1 e_3\alpha_6}{9}\lambda_4 - \frac{\lambda_1 e_3\alpha_6}{3})\langle\Lambda\frac{u}{x},\frac{u}{x}\rangle + \mathcal{R}_{L^2} + \lambda_4\mathcal{R}_{H^1}\\ &+ \Big(\Big((1+\frac{36}{49})\cdot 2\cdot 10^{-6} + \frac{36\lambda_1}{49}\cdot 5\cdot 10^{-4}\Big)\lambda_4 - 10^{-6}\Big)||\tilde{u}_x x^{-2/3}||_2^2.\end{aligned}$$

Since $\kappa_2 < \kappa$, we choose small $\lambda_4 > 0$ in Appendix C.2 so that

$$\lambda_4 \cdot \frac{\lambda_1 e_3 \alpha_6}{9} < \frac{\lambda_1 e_3 \alpha_6}{3}, \quad \Big((1+\frac{36}{49}) \cdot 2 \cdot 10^{-6} + \frac{36\lambda_1}{49} \cdot 5 \cdot 10^{-4}\Big)\lambda_4 < 10^{-6},$$
$$r_{c_\omega} - \lambda_4\Big(\frac{K_1^2}{4\gamma_1} + \frac{(\lambda_1 K_2)^2}{4\gamma_2}\Big) - \lambda_4 G_{c2} > \kappa_2 \cdot \frac{\pi\lambda_2}{2},$$
$$\kappa - \lambda_4\gamma_1 - \lambda_4\varepsilon_1 \geq \kappa_2, \quad \kappa\lambda_1 - \lambda_4\gamma_2 - \lambda_4 a_{H^1} \geq \kappa_2\lambda_1, \quad \kappa\lambda_3 \geq \kappa_2\lambda_3, \tag{4.107}$$

where K_1, K_2 are defined in (4.99). The above inequalities will be verified rigorously by the methods in the Supplementary Material [21]. Note that $r_{c_\omega} > \frac{\pi}{2}\lambda_2\kappa$ and $\kappa_2 < \kappa$. The above conditions are essentially the same as (4.106). We keep the damping term $\langle \Lambda \frac{u}{x}, \frac{u}{x}\rangle$ and $||\tilde{u}_x x^{-2/3}||_2^2$ in (4.74) to control the corresponding terms in (4.103). Plugging the above estimates and (4.107) into the differential inequality, we yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E^2(\theta_x,\omega) \leq & -\kappa_2||\theta_x\psi^{1/2}||_2^2 - \kappa_2\lambda_1||\omega\varphi^{1/2}||_2^2 - \kappa_2\frac{\pi\lambda_2}{2}c_\omega^2 - \kappa_2\lambda_3 d_\theta^2 \\ & - \lambda_4\kappa_2||(D_x\theta_x)\psi^{1/2}||_2^2 - \lambda_4\kappa_2\lambda_1||(D_x\omega)\varphi^{1/2}||_2^2 + \mathcal{R}_{L^2} + \lambda_4\mathcal{R}_{H^1} \\ \leq & -\kappa_2 E^2(\theta_x,\omega) + \mathcal{R}_{L^2} + \lambda_4\mathcal{R}_{H^1} \triangleq -\kappa_2 E^2(\theta_x,\omega) + \mathcal{R},\end{aligned} \tag{4.108}$$

where $\mathcal{R} = \mathcal{R}_{L^2} + \lambda_4\mathcal{R}_{H^1}$ and $\kappa_2 = 0.024$ is given in (C.26).

4.5.2.1 Outline of the estimates of the nonlinear and error terms

Recall the definitions of $\mathcal{R}_{L^2}$ and $\mathcal{R}_{H^1}$ in (4.73) and (4.83). The nonlinear terms in $\mathcal{R}_{L^2}, \mathcal{R}_{H^1}$, e.g., $\langle D_x N(\theta), D_x\theta_x\psi\rangle$, depend cubically on θ_x, ω. In the Supplementary Material [21], we use the energy $E(\theta_x,\omega)$ and interpolation to control $||u_x||_\infty$ and $||\theta_x||_\infty$. Using these L^∞ estimates, we further estimate the nonlinear terms in $\mathcal{R}$. For example, a typical nonlinear term in $\mathcal{R}$ can be estimated as follows

$$\begin{aligned}|\langle u\theta_{xx}, \theta_x\psi\rangle| &= \frac{1}{2}|\langle u\psi, \partial_x(\theta_x)^2\rangle| = \frac{1}{2}|\langle (u\psi)_x, \theta_x^2\rangle| = \frac{1}{2}|\langle u_x\psi + u\psi_x, \theta_x^2\rangle| \\ &\leq \frac{1}{2}(||u_x||_{L^\infty} + ||\frac{u}{x}||_\infty||\frac{x\psi_x}{\psi}||_{L^\infty})||\theta_x\psi^{1/2}||_2^2 \\ &\leq \frac{1}{2}||u_x||_{L^\infty}(1 + ||\frac{x\psi_x}{\psi}||_{L^\infty})||\theta_x\psi^{1/2}||_2^2,\end{aligned}$$

where we have used $|\frac{u}{x}| \leq ||u_x||_\infty$ in the last inequality since $u(0) = 0$. The above upper bound can be further bounded by $E^3(\theta_x,\omega)$.

The error terms in $\mathcal{R}_{L^2}, \mathcal{R}_{H^1}$, e.g., $F_1 = \langle F_\theta, \theta_x \psi \rangle$, depend linearly on θ_x, ω. We estimate these terms using the Cauchy-Schwarz inequality. A typical term F_1 can be estimated as follows

$$|F_1| \leq ||F_\theta \psi^{1/2}||_2 ||\theta_x \psi^{1/2}||_2.$$

The error term $||F_\theta \psi^{1/2}||_2$ is small and $||\theta_x \psi^{1/2}||_2$ can be further bounded by $E(\theta_x, \omega)$.

In the Supplementary Material [21], we work out the constants in these estimates and establish the following estimates

$$\mathcal{R} = \mathcal{R}_{L^2} + \lambda_4 \mathcal{R}_{L^2} \leq 36E^3 + \varepsilon E, \quad \varepsilon = 5.5 \cdot 10^{-7}.$$

4.5.2.2 Nonlinear stability and finite time blowup

Plugging the above estimate on $\mathcal{R}$ in (4.108), we establish the nonlinear estimate

$$\frac{1}{2}\frac{d}{dt} E^2(\theta_x, \omega) \leq -\kappa_2 E(\theta_x, \omega)^2 + 36E(\theta_x, \omega)^3 + \varepsilon E(\theta_x, \omega),$$

where $\kappa_2 = 0.024$ is given in (C.26). We choose the threshold $E_* = 2.5 \cdot 10^{-5}$ in the Bootstrap argument. Since

$$-\kappa_2 E_*^2 + 36E_*^3 + \varepsilon E_* < 0,$$

the above differential inequality implies that if $E(0) < E_*$, the bootstrap assumption

$$E(\theta_x(t), \omega(t)) < E_* \tag{4.109}$$

holds for all $t > 0$. Consequently, we can choose odd initial perturbations θ_x, ω which satisfy $\omega_x(0) = \theta_{xx}(0) = 0, E(\theta_x, \omega) < E_*$ and modify the far field of $\bar{\theta}, \bar{\omega}$ so that $\bar{\omega} + \omega, \theta_x + \bar{\theta}_x \in C_c^\infty$. The bootstrap result implies that for all time $t > 0$, the solution $\omega(t) + \bar{\omega}, \theta_x(t) + \bar{\theta}_x, c_l(t) + \bar{c}_l, c_\omega(t) + \bar{c}_\omega$ remain close to $\bar{\omega}, \bar{\theta}_x, \bar{c}_l, \bar{c}_\omega$, respectively. Using the rescaling argument in Section 4.2, we obtain finite time blowup of the HL model.

4.5.3 Convergence to the steady state

We use the time-differentiation argument in [19] to establish convergence. The initial perturbations (θ_x, ω) satisfy the properties in the previous Section. Since the linearized operators and the error terms in (4.23) are time-independent, differentiating (4.23) in t, we get

$$\partial_t (\theta_x)_t = \mathcal{L}_\theta((\theta_x)_t, \omega_t) + \partial_t N(\theta), \quad \partial_t (\omega)_t = \mathcal{L}_\omega((\theta_x)_t, \omega_t) + \partial_t N(\omega).$$

Applying the estimates of $\mathcal{L}_\theta, \mathcal{L}_\omega$ in Section 4.3 and (4.75) to $(\theta_x)_t, \omega_t$, we obtain

$$\frac{1}{2}\frac{d}{dt}E_1((\theta_x)_t,\omega_t)^2 \leq -\kappa E_1((\theta_x)_t,\omega_t)^2 + \mathcal{R}_2,$$

where the energy notation E_1 is defined in (4.72) and $\mathcal{R}_2$ is given by

$$\begin{aligned} E_1((\theta_x)_t,\omega_t) &= ||(\theta_x)_t\psi^{1/2}||_2^2 + \lambda_1||\omega_t\psi^{1/2}||_2^2 + \lambda_2\frac{\pi}{2}(\partial_t c_\omega)^2 + \lambda_3(\partial_t d_\theta)^2, \\ \mathcal{R}_2 &= \langle \partial_t N(\theta), (\theta_x)_t\psi\rangle + \lambda_1\langle\partial_t N(\omega), \omega_t\varphi\rangle \\ &\quad - \lambda_2\partial_t c_\omega\langle\partial_t N(\omega), x^{-1}\rangle + \lambda_3\partial_t d_\theta\langle\partial_t N(\theta), x^{-1}\rangle. \end{aligned}$$

The term $\partial_t c_\omega$ in the above estimates is from

$$H\omega_t(0) = \partial_t H\omega(0) = \partial_t c_\omega.$$

Similarly, we obtain the term $\partial_t d_\theta$. Using the a-priori estimate $E(\theta_x,\omega) < E_*$ in (4.109) and the energy $E_1((\theta_x)_t,\omega_t)$, we can further estimate $\mathcal{R}_2$. In the Supplementary Material [21], we prove

$$\frac{1}{2}\frac{d}{dt}E_1((\theta_x)_t,\omega_t)^2 \leq -0.02E_1((\theta_x)_t,\omega_t)^2. \tag{4.110}$$

Using this estimate and the argument in [19], we prove that the solution $\omega + \bar{\omega}, \theta_x + \bar{\theta}_x$ converge to the steady state $\omega_\infty, \theta_{\infty,x}$ in $L^2(\varphi), L^2(\psi)$ and $c_l(t), c_\omega(t)$ converge to $c_{l,\infty}, c_{\omega,\infty}$ exponentially fast. Moreover, the steady state admits regularity $(D_x)^i(\omega_\infty - \bar{\omega}) \in L^2(\varphi), (D_x)^i(\theta_{x,\infty} - \bar{\theta}_x) \in L^2(\psi)$ for $i = 0, 1$. We obtain θ_∞ from $\theta_{\infty,x}$ by imposing $\theta_\infty(0) = 0$ and integration.

Recall the energy E in (4.104). Since

$$\lambda_2\pi/2 > 3 > 1.5^2, \quad E > (\lambda_2\pi/2)^{1/2}|c_\omega| \geq 1.5|c_\omega|$$

(see (C.25)), using the convergence result, the a-priori estimate (4.109) and (4.20), we obtain

$$E(\theta_{x,\infty} - \bar{\theta}_x, \omega_\infty - \bar{\omega}) \leq E_*, \quad c_{l,\infty} = \bar{c}_l = 3, \quad |c_{\omega,\infty} - \bar{c}_\omega| \leq \frac{2}{3}E_* = \frac{5}{3}\cdot 10^{-5}. \tag{4.111}$$

Recall $\bar{c}_\omega < -1.0004$ from the beginning of Section 4.3.1. Thus, $c_{\omega,\infty} < -1$ and we conclude that the blowup is focusing and asymptotically self-similar with blowup scaling $\lambda = \frac{c_{l,\infty}}{-c_{\omega,\infty}}$ satisfying

$$|\lambda - \bar{\lambda}| \leq |\frac{c_{l,\infty}}{c_{\omega,\infty}} - \frac{\bar{c}_l}{\bar{c}_\omega}| + |\frac{\bar{c}_l}{\bar{c}_\omega} + \bar{\lambda}| < 3|\bar{c}_\omega - c_{\omega,\infty}| + 10^{-5} < 6\cdot 10^{-5}, \quad \bar{\lambda} = 2.99870,$$

where $\bar{\lambda}$ is the determined by the first 6 digits of $-\bar{c}_l/\bar{c}_\omega$.

4.5.4 Uniqueness of the self-similar profiles

Suppose that (ω_1, θ_1) and (ω_2, θ_2) are two initial perturbations which are small in the energy norm $E(\omega_i, \theta_{i,x}) < E_*$. The associated solution $(\omega_i, \theta_{i,x})$ solves (4.23)

$$\partial_t \theta_{i,x} = \mathcal{L}_\theta(\theta_{i,x}, \omega_i) + F_\theta + N(\theta_i), \quad \partial_t \omega_i = \mathcal{L}_\omega(\theta_{i,x}, \omega_i) + F_\omega + N(\omega_i).$$

Denote

$$\delta\omega \triangleq \omega_1 - \omega_2, \quad \delta\theta \triangleq \theta_1 - \theta_2, \quad \delta N_\theta = N(\theta_1) - N(\theta_2), \quad \delta N_\omega = N(\omega_1) - N(\omega_2). \tag{4.112}$$

A key observation is that the forcing terms F_θ, F_ω do not depend on (ω_i, θ_i). Thus, we derive

$$\partial_t \delta\theta_x = \mathcal{L}_\theta(\delta\theta_x, \delta\omega) + \delta N_\theta, \quad \partial_t \delta\omega = \mathcal{L}_\omega(\delta\theta_x, \delta\omega) + \delta N_\omega.$$

Applying the estimates of $\mathcal{L}_\theta, \mathcal{L}_\omega$ in Section 4.3 and (4.75), we get

$$\frac{1}{2}\frac{d}{dt} E_1(\delta\theta_x, \delta\omega)^2 \leq -\kappa E_1(\delta\theta_x, \delta\omega)^2 + \mathcal{R}_3$$

where the energy notation E_1 is defined in (4.72) and $\mathcal{R}_3$ is given by

$$\mathcal{R}_3 = \langle \delta N_\theta, \delta\theta_x \psi\rangle + \lambda_1 \langle \delta N_\omega, \delta\omega\varphi\rangle - \lambda_2 c_\omega(\delta\omega) \cdot \langle \delta N_\omega, x^{-1}\rangle + \lambda_3 d_\theta(\delta\theta_x) \cdot \langle \delta N_\theta, x^{-1}\rangle.$$

The above formulations are very similar to that in Section 4.5.3. Formally, the difference operator δ is similar to the time differentiation ∂_t. In the Supplementary Material [21], we show that $(\delta\theta_x, \delta\omega)$ enjoys the same estimates as that of $(\partial_t\theta_x, \omega_t)$ in (4.110)

$$\frac{1}{2}\frac{d}{dt} E_1(\delta\theta_x, \delta\omega)^2 \leq -0.02 E_1(\delta\theta_x, \delta\omega)^2. \tag{4.113}$$

As a result, $E_1(\delta\theta_x, \delta\omega)$ converges to 0 exponentially fast and the two solutions $(\omega_i + \bar{\omega}_i, \theta_i + \bar{\theta}_i), i = 1, 2$ converge to steady states $(\omega_{i,\infty}, \theta_{i,\infty})$ with the same ω_∞ and $\theta_{\infty,x}$. Since $\theta_{i,\infty}(0) = 0$, two steady states are the same.

4.5.5 Numerical evidence of stronger uniqueness

The above discussion argues that the steady state is unique at least within a small energy norm ball. However, our numerical computation suggests that the steady state of the dynamical rescaling equations (4.5),(4.6) is unique (up to rescaling) for a much larger class of smooth initial data ω, θ with $\theta(0) = 0$ that satisfy the following conditions:

1. odd symmetry: $\omega(x)$ and $\theta_x(x)$ are odd functions of x;

2. non-degeneracy condition: $\omega_x(0) > 0$ and $\theta_{xx}(0) > 0$;

3. sign condition: $\omega(x), \theta_x(x) > 0$ for $x > 0$.

In fact, these conditions are consistent with the initial data considered by Luo-Hou in [86, 87] restricted on the boundary. They are preserved by the equations as long as the solution exists. Moreover, this class of initial data leads to finite time blowup of the HL model [23].

Here we present the convergence study for the dynamic rescaling equations for four sets of initial data that belong to the function class described above. The four initial data of ω are given by $\omega^{(i)}(x) = a_i f_i(b_i x), i = 1, 2, 3, 4$, where

$$f_1(x) = \frac{x}{1+x^2}, \quad f_2(x) = \frac{xe^{-(x/10)^2}}{1+x^2}, \quad f_3(x) = \frac{x}{1+x^4}, \quad f_4 = \frac{x(1-x^2)^2}{(1+x^2)^3},$$

and the parameters a_i, b_i are chosen to normalize the initial data such that they satisfy the same normalization conditions:

$$\omega_x^{(i)}(0) = 1 \quad \text{and} \quad u_x^{(i)}(0) = -2.5, \quad i = 1, 2, 3, 4.$$

The initial data of θ_x are chosen correspondingly as

$$\theta_x^{(i)} = (c_l x + u^{(i)})\omega_x^{(i)} - c_\omega \omega^{(i)}, \quad i = 1, 2, 3, 4,$$

so that the initial residual of the ω equation is everywhere 0. The initial value of the scaling parameters are set to be $c_l = 3$ and $c_\omega = -1$, respecting our preliminary numerical result that $\bar{c_l}/\bar{c_\omega} \approx -3$. Note that all these initial data of ω, θ_x are far away from the approximate steady (with proper rescaling) with $O(1)$ distance in the the energy norm that is used in our analysis. In particular, we have $\omega^{(1)}(x) = O(x^{-1})$, $\omega^{(2)}(x) = O(x^{-1}e^{-(x/10)^2})$, $\omega^{(3)}(x) = O(x^{-3})$ for $x \to +\infty$, while the approximate steady should satisfy $\bar{\omega}(x) = O(x^{\bar{c}_\omega/\bar{c}_l})$ where $\bar{c}_\omega/\bar{c}_l$ is approximately $-1/3$ according to our numerical results. Moreover, $\omega^{(4)}(x)$ has two peaks, while $\bar{\omega}(x)$ only has one. Figure 4.2(a) plots the four initial data of ω for $x \in [0, 40]$.

With each set of these initial data, we numerically solve the dynamic rescaling equations (4.5) subject to the normalization conditions (4.6) using the algorithm described in Section 10 of the Supplementary Material [21] (by modifying

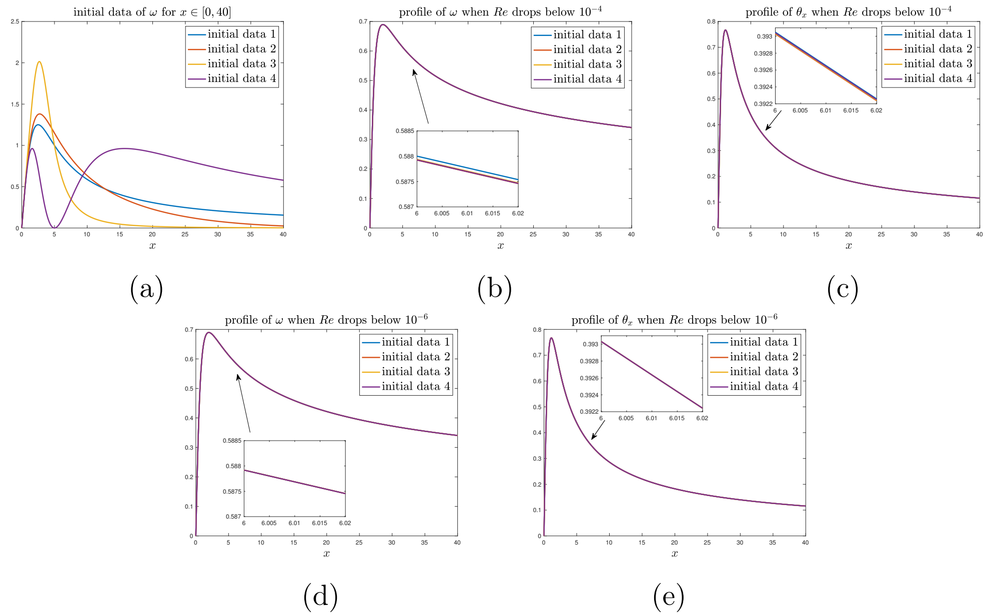

Figure 4.2: Profiles of ω, θ_x with different initial data. (A) Four different initial data of ω; (B)(C) Profiles of ω and θ_x when Re drops below 10^{-4} the first time. (D)(E) Profiles of ω and θ_x when Re drops below 10^{-6} the first time.

the initial values of the part ω_p and $(\theta_x)_p$). We verify the uniqueness of the steady state by comparing the profiles of ω at the first time the maximum grid-point residual $Re := \max_i\{|F_\omega(x_i)|, |F_{\theta_x}(x_i)|\}$ drops below some small number ϵ. Here the residuals F_ω and F_{θ_x} are defined as

$$F_\omega = -(c_l x + u)\omega_x + c_\omega + v, \quad F_{\theta_x} = -(c_l x + u)\theta_{xx} + (2c_\omega - u_x)\theta_x. \tag{4.114}$$

Figure 4.2 (b) and (c) plot the solutions of ω when $Re \leq 10^{-4}$ and when $Re \leq 10^{-6}$, respectively. We can see that the profiles of ω from different initial data are barely distinguishable when the residual is smaller than 10^{-4}; they become even closer to each other when the residual is even smaller. This implies that the solutions in the four cases of computation should converge to the same steady state.

4.6 Hölder regularity of the blowup solution

To estimate the C^γ norm with $\gamma = \frac{c_{\theta,\infty}}{c_{l,\infty}}$ of the solution θ, we will use the following estimate

$$\begin{aligned}\frac{|f(y) - f(x)|}{|x-y|^\gamma} &= |x-y|^{-\gamma}|\int_x^y f_x(z)dz| \lesssim |x-y|^{-\gamma}\int_x^y z^{\gamma-1}dz \cdot ||f_x x^{1-\gamma}||_\infty \\ &\lesssim |x-y|^{-\gamma}(y^\gamma - x^\gamma)\cdot ||f_x x^{1-\gamma}||_\infty \lesssim ||f_x x^{1-\gamma}||_\infty\end{aligned} \tag{4.115}$$

for any $0 \le x < y$. The difficulty lies in the decay estimate of θ_x since the previous a-priori estimates only imply that θ_x decays with rate slower than $x^{\gamma-1}$. The decay rate $x^{\gamma-1}$ is sharp since it is exactly the decay rate of the self-similar profile $\theta_{\infty,x}$, which will be established in Section 4.6.1. In Section 4.6.2, we establish the decay estimates of the perturbation. In Section 4.6.3, we estimate the Hölder norm of the solution.

Notations In this Section, we use the notation $A \lesssim B$ if there exists some finite constant $C > 0$, such that $A \le CB$. The constant C *can* depend on the norms of the approximate steady state $(\bar\theta, \bar\omega)$ and the self-similar profile $(\theta_\infty, \omega_\infty)$ constructed in Section 4.5.3, e.g., $||\theta_x||_\infty, ||\bar\theta_x||_\infty$, as long as these norms are finite. These constants do not play an important role in characterizing several exponents and thus we do not need to track them.

4.6.1 Decay estimates of the self-similar profile

Recall that we have constructed the self-similar profile $(\theta_\infty, \omega_\infty)$ in Section 4.5.3. Using the estimate (4.111), we obtain

$$\begin{aligned} &|u_\infty(x)| \lesssim |x|^{5/6}, \quad |c_{l,\infty}x + u_\infty(x)| \ge 0.3|x|, \\ &u_{\infty,x} \in L^\infty, \quad \theta_\infty(1) \ne 0, \quad \theta_{x,\infty} \in L^\infty, \end{aligned} \tag{4.116}$$

whose proofs are referred to Section 10 in the Supplementary Material [21]. Recall that the profile $(\theta_\infty, \omega_\infty)$ solves

$$(c_{l,\infty}x + u_\infty)\theta_{\infty,x} = c_{\theta,\infty}\theta_\infty, \quad u_{\infty,x} = H\omega_\infty. \tag{4.117}$$

Solving the ODE on θ_∞, we obtain

$$\begin{aligned} &\theta_\infty(x) = \theta_\infty(1)\exp(J(x)), \\ &J(x) \triangleq \int_1^x \frac{c_{\theta,\infty}}{c_{l,\infty}y + u_\infty(y)}dy, \quad \theta_{\infty,x} = \frac{c_{\theta,\infty}\theta_\infty(x)}{c_{l,\infty}x + u_\infty(x)}. \end{aligned} \tag{4.118}$$

Denote $\gamma = \frac{c_{\theta,\infty}}{c_{l,\infty}}$. Using the estimates on u_∞ in (4.116), we obtain $|J(x) - \gamma\log(x)| \lesssim 1$. Thus, for some constant $C_1 > 0$ depending on the profile, we get

$$\lim_{x\to\infty} \theta_\infty(x)x^{-\gamma} = C_1\theta_\infty(1) \ne 0.$$

Plugging the above limit and (4.116) in the formula of $\theta_{\infty,x}$ in (4.118), we yield

$$\lim_{x\to\infty} \theta_{x,\infty}x^{1-\gamma} = \lim_{x\to\infty} \frac{c_{\theta,\infty}x}{c_{l,\infty}x + u_\infty(x)} \cdot \theta_\infty(x)x^{-\gamma} = C_1\gamma\theta_\infty(1). \tag{4.119}$$

Combining the above estimate and $\theta_{\infty,x} \in L^\infty$ from (4.116), we prove

$$||\theta_{\infty,x} x^{1-\gamma}||_\infty \lesssim 1. \tag{4.120}$$

Differentiating (4.117) and using $c_{\theta,\infty} = c_{l,\infty} + 2c_{\omega,\infty}$, we get

$$(c_{l,\infty} x + u_\infty)\theta_{\infty,xx} = (c_{\theta,\infty} - c_{l,\infty} - u_{\infty,x})\theta_{\infty,x} = (2c_{\omega,\infty} - u_{\infty,x})\theta_{\infty,x}. \tag{4.121}$$

Using (4.116), we further obtain

$$\Big|\frac{x\theta_{\infty,xx}}{\theta_{\infty,x}}\Big| = \Big|\frac{(2c_{\omega,\infty} - u_{\infty,x})x}{c_{l,\infty} x + u_\infty}\Big| \lesssim \Big|\frac{x}{c_{l,\infty} x + u_\infty}\Big| \lesssim 1. \tag{4.122}$$

4.6.2 Decay estimates of the perturbation

Note that we have constructed $(\theta_\infty, \omega_\infty)$ in Section 4.5.3 with estimate (4.111). We treat them as known functions. Similar to (4.18), (4.19), linearizing the θ_x equation around the self-similar profile, we get

$$\partial_t \theta_x + (c_{l,\infty} x + u_\infty + u)\theta_{xx} = (2c_{\omega,\infty} - u_{\infty,x})\theta_x + (2c_\omega - u_x)\theta_{\infty,x} - u\theta_{\infty,xx} + (2c_\omega - u_x)\theta_x,$$

with normalization conditions

$$c_\omega = u_x(0), \quad c_l = 0, \quad c_\theta = c_l + 2c_\omega. \tag{4.123}$$

Here, the nonlinear terms are given by $u\theta_{xx}, (2c_\omega - u_x)\theta_x$, and the error term is 0 since we linearize the equation around the exact steady state. To obtain the decay estimates of θ_x with a decay rate $O(|x|^{\gamma-1})$, we choose ρ with a growth rate $O(|x|^{1-\gamma})$ and perform L^∞ estimate on $\theta_x\rho$, which will imply $|\theta_x| \leq |\rho^{-1}| \lesssim |x|^{\gamma-1}$ for large x. We derive the equation for $\theta_x\rho$ as follows

$$\begin{aligned} \partial_t(\theta_x\rho) + (c_{l,\infty} x + u_\infty + u)(\theta_x\rho)_x &= I(\rho)\theta_x\rho + J, \\ I(\rho) &\triangleq 2c_{\omega,\infty} - u_{\infty,x} + (c_{l,\infty} x + u_\infty)\rho_x\rho^{-1}, \\ J &\triangleq (2c_\omega - u_x)\theta_{\infty,x}\rho - u\theta_{\infty,xx}\rho + u\theta_x\rho_x + (2c_\omega - u_x)\theta_x\rho. \end{aligned} \tag{4.124}$$

For a typical function ρ with a growth rate $O(|x|^{\gamma-1})$, e.g., $\rho = \mathrm{sgn}(x)|x|^{\gamma-1}$, since u_∞ has sublinear growth (4.116), for large $x > 0$, we get

$$I(\rho) = 2c_{\omega,\infty} + c_{l,\infty} x (x^{1-\gamma})_x x^{\gamma-1} + l.o.t. = 2c_{\omega,\infty} + c_{l,\infty}(1-\gamma) + l.o.t. = l.o.t.,$$

where we have $c_{l,\infty}(1-\gamma) = c_{l,\infty} - c_{\theta,\infty} = -2c_{\omega,\infty}$ to obtain the last equality. Thus, we expect that $I(\rho)$ is not uniformly negative, i.e. $I(\rho) \leq -c$ for some

$c > 0$, and we do not obtain a damping term in the L^∞ estimate of $\theta_x \rho$, which is different from the weighted L^2 and H^1 estimates in Sections 4.3, 4.5. In some sense, the decay rate $O(|x|^{\gamma-1})$ is critical. An ideal choice of ρ with the desired growth rate is $\theta_{\infty,x}^{-1}$, since we have (4.119) and $I(\rho)$ term in (4.124) vanishes:

$$\begin{aligned} I(\rho) &= 2c_{\omega,\infty} - u_{\infty,x} - \frac{(c_{l,\infty} x + u_\infty)\theta_{\infty,xx}}{\theta_{\infty,x}} \\ &= \frac{(2c_{\omega,\infty} - u_{\infty,x})\theta_{\infty,x} - (c_{l,\infty} x + u_\infty)\theta_{\infty,xx}}{\theta_{\infty,x}} = 0, \end{aligned}$$

where we have used (4.121) to obtain the last equality.

Recall $c_\omega = u_x(t,0)$. Using $\rho = \theta_{\infty,x}^{-1}$, $|\frac{x\theta_{\infty,xx}}{\theta_{\infty,x}}| \lesssim 1$ in (4.122) and $|\frac{u}{x}| \lesssim ||u_x||_\infty$, we get

$$\begin{aligned} |J| &= \left| (2c_\omega - u_x)\frac{\theta_{\infty,x}}{\theta_{\infty,x}} - u\frac{\theta_{\infty,xx}}{\theta_{\infty,x}} - u\theta_x \frac{\theta_{\infty,xx}}{\theta_{\infty,x}^2} + (2c_\omega - u_x)\theta_x \rho \right| \\ &\lesssim ||u_x||_\infty (1 + ||\theta_x \rho||_\infty). \end{aligned}$$

For $\theta_x(\cdot, 0)\rho \in L^\infty$, performing L^∞ estimates in (4.124), we yield

$$\frac{d}{dt}||\theta_x \rho||_\infty \lesssim ||u_x||_\infty (1 + ||\theta_x \rho||_\infty). \tag{4.125}$$

Next, we control $||u_x||_\infty$. Recall the energy E in (4.104) and the a-priori estimates in (4.109),(4.111)

$$E(\theta_{x,\infty} + \theta_x - \bar{\theta}_x, \omega_\infty + \omega - \bar{\omega}) \leq E_*, \quad E(\theta_{x,\infty} - \bar{\theta}_x, \omega_\infty - \bar{\omega}) \leq E_*.$$

Using the triangle inequality, for any $t \geq 0$, we get

$$||\theta_x \psi^{1/2}||_2 + ||D_x \theta_x \psi^{1/2}||_2 + ||\omega \varphi^{1/2}||_2 + ||D_x \omega \varphi^{1/2}||_2 + |c_\omega(\omega)| + |d_\theta(\theta_x)| \lesssim 1. \tag{4.126}$$

Denote $\kappa_3 = 0.02$. Applying (4.113) to two solutions $(\theta_\infty, \omega_\infty)$ and $(\theta_\infty + \theta, \omega_\infty + \omega)$, we get

$$\begin{aligned} ||\theta_x(t)\psi^{1/2}||_2 + ||\omega(t)\varphi^{1/2}||_2 + |c_\omega(t)| &\lesssim E_1(\theta_x(t), \omega(t)) \\ &\leq e^{-\kappa_3 t} E_1(\theta_x(0), \omega(0)) \lesssim e^{-\kappa_3 t}, \end{aligned} \tag{4.127}$$

where we have used (4.126) to obtain the last inequality. Since $H(D_x \omega)(0) = 0$, using Lemma C.0.1, we get $H(D_x \omega) = D_x H\omega = x u_{xx}$. From (4.25) and (4.26), we have $x^{-4/3} + x^{-2/3} \lesssim \varphi$. Applying Lemma C.0.6 to $f = D_x\omega$ and $f = \omega$ (note that $H(D_x\omega)(0) = 0$), we obtain

$$||u_x||_\infty^2 \lesssim \int_{\mathbb{R}} |u_{xx}u_x|dx = \int_{\mathbb{R}} |H(D_x\omega) \cdot H\omega x^{-1}|dx \lesssim ||H(D_x\omega)x^{-2/3}||_2||H\omega x^{-1/3}||_2$$
$$\lesssim ||D_x\omega x^{-2/3}||_2||\omega x^{-1/3}||_2 \lesssim ||D_x\omega\varphi^{1/2}||_2||\omega\varphi^{1/2}||_2 \lesssim e^{-\kappa_3 t/2}. \quad (4.128)$$

Plugging the above estimate in (4.125), we yield

$$\frac{d}{dt}||\theta_x\rho||_\infty \lesssim e^{-\kappa_3 t/4}(1 + ||\theta_x\rho||_\infty).$$

Since $\kappa_3 > 0$, solving the differential inequality and using $|x^{1-\gamma}| \lesssim |\theta_{x,\infty}^{-1}|$ from (4.120), we prove

$$\sup_{t\geq 0} ||\theta_x(t)\rho||_\infty \lesssim 1, \quad \sup_{t\geq 0} ||\theta_x(t)x^{1-\gamma}||_\infty \lesssim \sup_{t\geq 0} ||\theta_x(t)\theta_{x,\infty}^{-1}||_\infty \lesssim 1.$$

Since θ is even, using (4.115), (4.120) and the above estimate, we prove

$$\sup_{t\geq 0} ||\theta_\infty + \theta(t)||_{C^\gamma} \lesssim 1. \quad (4.129)$$

Remark 4.6.1. Since we do not have a damping term in the L^∞ estimate (4.125), the exponential convergence estimates in (4.127), (4.128) play a crucial role in obtaining (4.129).

4.6.3 Hölder regularity

Denote $\hat{\theta} = \theta_\infty + \theta$ and by θ_{phy} the solution with initial data $\hat{\theta}(0, \cdot)$ in the physical space. Recall the rescaling relation and the normalization conditions (4.123)

$$\begin{aligned} C_\omega(\tau) &= \exp(\int_0^\tau c_\omega(s) + c_{\omega,\infty}ds), \quad t(\tau) = \int_0^\tau C_\omega(s)ds, \\ C_\theta(\tau) &= \exp(\int_0^\tau c_\theta(s) + c_{\theta,\infty}ds), \quad C_l(\tau) = \exp(-\int_0^\tau (c_l(s) + c_{l,\infty})ds), \\ \hat{\theta}(x,\tau) &= C_\theta(\tau)\theta_{phy}(C_l(\tau)x, t(\tau)), \quad c_\theta = 2c_\omega, \quad c_l = 0. \end{aligned} \quad (4.130)$$

From assumptions $\hat{\theta}_x(0)|x|^{1-\gamma} \in L^\infty$ in (d) in Theorem 4.2, $E(\hat{\theta}_x(0)-\bar{\theta}_x, \hat{\omega}(0)-\bar{\omega}) \lesssim 1$, and estimates (4.119) and $E(\theta_{\infty,x} - \bar{\theta}_x, \omega_\infty - \bar{\omega}) \lesssim 1$, it is not difficult to obtain that $\theta_x\rho \in L^\infty$. Thus, $\theta, \hat{\theta}$ enjoys the energy estimates in Section 4.6.2. Using (4.127), (4.129), (4.130) and $\gamma c_{l,\infty} = c_{\theta,\infty}$, we prove

$$\sup_{\tau\geq 0} ||\theta_{phy}(t(\tau))||_{C^\gamma} = \sup_{\tau\geq 0} ||\hat{\theta}(\tau)||_{C^\gamma} C_\theta^{-1}C_l^{-\gamma} = \sup_{\tau\geq 0} ||\hat{\theta}(\tau)||_{C^\gamma} \exp(\int_0^\tau -2c_\omega d\tau) \lesssim 1.$$

4.6.3.1 Blowup in higher Hölder norm

We show that for any $\beta > \gamma$, the C^β norm of the solution blows up. Since $1 \lesssim \psi(x)$ for $x \in [0,1]$, using (4.127) and Cauchy-Schwarz inequality, we get

$$|\theta(1) - \theta(0)| = |\int_0^1 \theta_x(y)dy| \lesssim \Big(\int_0^1 \theta_x(y)^2 dy\Big)^{1/2} \lesssim ||\theta_x \psi^{1/2}||_2 \lesssim e^{-\kappa_3 \tau}. \tag{4.131}$$

Recall the formulas in (4.130). Denote $T = t(\infty)$. Since $|c_\omega(\tau)|$ decays exponentially (4.127) and $c_{\omega,\infty} < -\frac{1}{2}$, we obtain

$$\begin{aligned} &C_\omega(\tau) \gtrsim e^{c_{\omega,\infty}\tau}, \quad C_\theta(\tau)^{-1} \gtrsim e^{-c_{\theta,\infty}\tau}, \quad C_l(\tau)^{-1} \gtrsim e^{c_{l,\infty}\tau}, \\ &T - t(\tau) = \int_\tau^\infty C_\omega(s)ds \gtrsim \int_\tau^\infty e^{c_{\omega,\infty}s}ds \gtrsim e^{c_{\omega,\infty}\tau}. \end{aligned}$$

Recall $\gamma c_{l,\infty} = c_{\theta,\infty} = c_{l,\infty} + 2c_{\omega,\infty}$. Denote $\delta = -\frac{\beta c_{l,\infty} - c_{\theta,\infty}}{c_{\omega,\infty}} = \frac{2(\beta-\gamma)}{1-\gamma} > 0$. We have

$$S \triangleq \liminf_{\tau\to\infty} ||\theta_{phy}(x,\tau)||_{C^\beta}(T - t(\tau))^\delta \gtrsim \liminf_{\tau\to\infty} ||\hat\theta(x,\tau)||_{C^\beta} C_\theta^{-1} C_l^{-\beta} \exp(\delta c_{\omega,\infty}\tau).$$

Note that $\theta_\infty(0) = 0$. Using (4.131), we have $||\hat\theta(\tau)||_{C^\beta} \geq |\hat\theta(\tau,1) - \hat\theta(\tau,0)| \geq |\theta_\infty(1)| - C\exp(-\kappa_3\tau)$. Using this estimate, $\delta = -\frac{\beta c_{l,\infty} - c_{\theta,\infty}}{c_{\omega,\infty}}$ and (4.116), we establish

$$S \gtrsim \liminf_{\tau\to\infty} |\theta_\infty(1)| \exp((-c_{\theta,\infty} + \beta c_{l,\infty} + \delta c_{\omega,\infty})\tau) \gtrsim |\theta_\infty(1)| > 0.$$

We conclude the proof of result (d) in Theorem 4.2.

Remark 4.6.2. The exponential convergence in (4.127) is crucial for us to obtain the unique Hölder exponent γ that characterizes the regularity of the singular solution and the sharp blowup rate. It enables us to essentially treat the perturbation as 0.

4.7 Connection between the HL model and the Boussinesq equations

In this section, we discuss the connection between the leading order system of the HL model and that of the 2D Boussinesq equations in $\mathbb{R}_2^+$ with low regularity initial data.

4.7.1 The leading order system for the 2D Boussinesq equations

The 2D Boussinesq equations in $\mathbb{R}_2^+$ read

$$\begin{aligned} \omega_t + \mathbf{u} \cdot \nabla \omega &= \theta_x, \\ \theta_t + \mathbf{u} \cdot \nabla \theta &= 0, \end{aligned} \tag{4.132}$$

where the velocity field $\mathbf{u} = (u, v)^T : \mathbb{R}_+^2 \times [0, T) \to \mathbb{R}_+^2$ is determined via the Biot-Savart law

$$-\Delta \psi = \omega, \quad u = -\psi_y, \quad v = \psi_x,$$

with no flow boundary condition $\psi(x, 0) = 0 \quad x \in \mathbb{R}$.

Consider the polar coordinate (r, β) in $\mathbb{R}_2^+ : r = (x^2 + y^2)^{1/2}, \beta = \arctan(y/x)$. For $\alpha > 0$, denote

$$R = r^\alpha, \quad \Omega(R, \beta) = \omega(x, y), \quad \eta(R, \beta) = \theta_x(x, y), \quad \xi(R, \beta) = \theta_y(x, y).$$

In [16], the following leading order system of (4.132) is derived based on the framework developed in [42] under the assumption that $\omega, \nabla\theta$ are in some Hölder space C^α with sufficient small α

$$\Omega_t = \eta, \quad \eta_t = \frac{2}{\pi\alpha} L_{12}(\Omega)\eta, \quad L_{12}(\Omega) = \int_R^\infty \int_0^{\pi/2} \frac{\Omega(s, \beta)\sin(2\beta)}{s} ds d\beta. \tag{4.133}$$

An important observation made in [16] is that for certain class of C^α data, θ is anisotropic in the sense that $|\theta_y| \lesssim \alpha|\theta_x|$. Moreover, this property is preserved dynamically. Therefore, the θ_y variable does not appear in the leading order system. Define the following operators

$$Pf(R) = \int_0^{\pi/2} f(R, \beta)\sin(2\beta)d\beta, \quad Sf(R) = \frac{2}{\pi\alpha}\int_R^\infty f(S)\frac{dS}{S}. \tag{4.134}$$

By definition, we have

$$\frac{2}{\pi\alpha} L_{12}(\Omega) = \frac{2}{\pi\alpha}\int_R^\infty P\Omega(s)\frac{ds}{s} = S(P\Omega). \tag{4.135}$$

Since $L_{12}(\Omega)$ does not depend on β, we apply the operator P to both sides of (4.133) to obtain

$$\partial_P \Omega = P\eta, \quad \partial_t P\eta = \frac{2}{\pi\alpha} L_{12}(\Omega) P\eta = S(P\Omega) \cdot P\eta. \tag{4.136}$$

The above system is an 1D coupled system on $P\Omega, P\eta$. Once $P\Omega, P\eta$ are determined, we can obtain an explicit solution of (4.133).

4.7.2 The leading order system for the HL model

We use the observation made in [44] that the advection can be substantially weakened by choosing C^α data with sufficiently small α. Suppose that $\omega, \theta_x \in C^\alpha$ with small α. Then the advection terms in the system of (ω, θ_x) in the HL model become lower order terms

$$\omega_t = \theta_x + l.o.t., \quad (\theta_x)_t = -u_x\theta_x + l.o.t., \quad u_x = H\omega. \tag{4.137}$$

The above system is already very similar to (4.133) by taking $\Omega = \omega, \eta = \theta_x$. We further perform a simplification for the Hilbert transform. We impose extra assumptions that ω, θ_x are odd, which are preserved dynamically. Due to these symmetries, it suffices to consider the HL model on $\mathbb{R}_+$. For $x > 0$, symmetrizing the kernel, we get

$$\begin{aligned} H\omega(x) &= \frac{1}{\pi}\int_{\mathbb{R}_+} \omega(y)(\frac{1}{x-y} - \frac{1}{x+y})dy \\ &= \frac{1}{\pi}\int_{\mathbb{R}_+} \omega(y)\frac{2y}{x^2-y^2}dy = \frac{1}{\pi}\int_{\mathbb{R}_+} \omega(y)\frac{2}{(x/y)^2-1}\frac{dy}{y}. \end{aligned}$$

We learn the following formal derivation of the leading order part of general singular integral operator from Dr. Elgindi. [1] Denote

$$X = x^\alpha, \quad Y = y^\alpha, \quad \Omega(X) = \omega(x), \quad \eta(X) = \theta_x(x). \tag{4.138}$$

Using the above change of variables and $\frac{dy}{y} = \frac{1}{\alpha}\frac{dY}{Y}$, we get

$$H\omega(x) = \frac{1}{\alpha\pi}\int_{\mathbb{R}_+} \omega(Y^{1/\alpha})\frac{2}{(\frac{X}{Y})^{1/\alpha}-1}\frac{dY}{Y} = \frac{1}{\alpha\pi}\int_{\mathbb{R}_+} \Omega(Y)K_\alpha(X,Y)\frac{dY}{Y},$$

where $K_\alpha(X,Y) = \frac{2}{(\frac{X}{Y})^{1/\alpha}-1}$. Next, we consider the leading order part of $K_\alpha(X,Y)$ as $\alpha \to 0^+$. Note that

$$\lim_{\alpha\to 0^+}(\frac{X}{Y})^{1/\alpha} = 0, \text{ for } X < Y, \quad \lim_{\alpha\to 0^+}(\frac{X}{Y})^{1/\alpha} = \infty, \text{ for } X > Y.$$

Hence, for $X \neq Y$ and $X, Y > 0$, we get

$$\lim_{\alpha\to 0^+} K_\alpha(X,Y) = -2\cdot \mathbf{1}_{Y>X}.$$

[1] Similar derivation was presented in the One World PDE Seminar "Singularity formation in incompressible fluids" by Dr. Elgindi. `https://www.youtube.com/watch?v=29zUjm7xFlI&feature=youtu.be`

Therefore, formally, we get

$$H\omega(x) = -\frac{2}{\alpha\pi}\int_X^\infty \omega(Y)\frac{dY}{Y} + l.o.t. = -S\Omega(X) + l.o.t., \tag{4.139}$$

where the operator S is defined in (4.134). Now, plugging the above formula in (4.137), dropping the lower order terms in (4.137) and applying the notations (4.138), we derive another leading order system for the HL model

$$\partial_t\Omega(X) = \eta(X), \quad \partial_t\eta(X) = S\Omega(X)\cdot\eta(X). \tag{4.140}$$

The above system is exactly the same as that in (4.136). We remark that the lower order term in the simplification (4.139) needs to be estimated rigorously. In general, the system (4.137) is more complicated than (4.140) since the Hilbert transform is nonlocal and is a singular operator, while we can obtain a local relation between Sf and f by taking derivative $\partial_X(Sf)(X) = -\frac{2}{\pi\alpha}\frac{f(X)}{X}$. Note that $\mathbf{1}_{X<Y} = \mathbf{1}_{x<y}$. Undoing the change of variables in (4.138), we get

$$S\Omega(X) = \frac{2}{\pi\alpha}\int_{\mathbb{R}_+}\mathbf{1}_{x<y}\Omega(Y)\frac{dY}{Y} = \frac{2}{\alpha\pi}\int_{\mathbb{R}_+}\mathbf{1}_{x<y}\omega(y)\cdot\alpha\frac{dy}{y} = \frac{2}{\pi}\int_x^\infty \omega(y)\frac{dy}{y}. \tag{4.141}$$

The operator on the right hand side is closely related to the Choi-Kiselev-Yao (CKY) simplification of the Hilbert transform [22]. Therefore, the leading order system (4.140) can be seen as the CKY's simplification of (4.137) without the lower order terms.

Chapter 5

COMPETITION BETWEEN ADVECTION AND VORTEX STRETCHING

In this chapter, we investigate the DG model on S^1 to study the competition between advection and vortex stretching, an essential difficulty in studying the regularity of 3D Euler equations. In Chapter 3, we have established the finite time blowup of the De Gregorio (DG) model on $\mathbb{R}$ with smooth data. Yet, the blowup mechanism does not generalize to the case of S^1 due to the expanding nature of the blowup solution on $\mathbb{R}$. The dynamic of the solution in the case of S^1 is much more complicated, and we will show that the solutions exhibit dichotomous behaviors from initial data with different regularity. We refer to Sections 1.4, 1.4.1, and 3.1 for the background and conjectures of the DG model and existing results.

We focus on odd initial data ω_0 with period π in class X (see (5.1)): $\omega_0(x) \geq 0$ or $\omega_0(x) \leq 0$ for all $x \in [0, \frac{\pi}{2}]$. These properties are preserved dynamically. The class of initial data in X seems to provide the most promising scenario for a potential blowup solution of (3.1) on S^1 up to now for the following reasons. Firstly, the initial data considered in Chapter 3 [19] that lead to finite time blowup of (3.1) on $\mathbb{R}$ satisfies the same sign and symmetry properties as those in X. Secondly, for the gCLM model [97] (3.2) with $a > 0$, which is closely related to (3.1), singularity formation [13, 14, 19, 44, 49] all develops from initial data with the same sign and symmetry properties as those in X. In particular, in [14], we established that the gCLM model on S^1 with a slightly less than 1, which can be seen as a slight perturbation to (3.1), develops finite time singularity from some smooth initial data in X. Thirdly, this scenario can be seen as a 1D analog of the hyperbolic blowup scenario for the 3D Euler equations reported by Hou-Luo [86, 87]. See also [16, 78, 80]. In fact, the restriction of the (angular) vorticity in [16, 78, 80, 86, 87] to the boundary has the same sign and symmetry properties as those in X. Thus, to establish global regularity of (3.1) for general smooth initial data, which relates to Conjecture I in Section 1.4.3, we need to address the important question of whether there is a finite time blowup in this class. We note that the initial data considered in [73] is close to the steady state $A\sin(2x)$ of (3.1). Thus it belongs to or is close to that in X.

Note that the CLM model (3.4) can only blow up in finite time at the zeros of ω [26]. Since the vortex stretching is the driving force for a potential blowup of (3.1), it is likely that a potential singularity of (3.1) with general data is also located at the zeros of ω. For a zero x_0 of ω, across which ω changes sign, the leading order term of ω near x_0 is $\partial_x^k\omega(x_0)(x-x_0)^k$ for some odd $k \in Z_+$. It has the same sign and symmetry properties as those in X. Thus, our analysis of (3.1) with $\omega \in X$ can provide valuable insights on the local analysis of these potential singularities. For a zero x_0 of ω, across which ω does not change sign, the local analysis could benefit from [83].

5.1 Main results

Throughout this chapter, we consider initial data ω_0 in the following class X

$$X \triangleq \Big\{ f : f \text{ is odd }, \pi-\text{periodic and } f(x) \leq 0, x \in [0, \frac{\pi}{2}] \Big\}, \tag{5.1}$$

unless we specify otherwise. We assume $\omega_0 \leq 0$ on $[0, \frac{\pi}{2}]$ without loss of generality. For the case of $\omega_0 \geq 0$ on $[0, \frac{\pi}{2}]$, we can consider a new variable $\omega_{new}(x) \triangleq \omega(x+\frac{\pi}{2})$ and then reduce it to the previous case. It is not difficult to show that the solution $\omega(t)$ remains in X.

Our first main result is a one-point blowup criterion. A similar blowup criterion has been obtained in our previous work [13] for the DG model and the gCLM model with dissipation.

Theorem 5.1. *Suppose that $\omega_0 \in X \cap H^1$ and $\int_0^{\pi/2} \left|\frac{\omega_{0,x}^2}{\omega_0}\sin(2x)\right| dx < +\infty$. The unique local in time solution of (3.1) cannot be extended beyond $T > 0$ if and only if*

$$\int_0^T u_x(0,t)dt = \infty. \tag{5.2}$$

For $\omega \in X \cap H^1$, we have $u_x(0,t) \geq 0$. Suppose that ω vanishes to the order $|x|^\beta, \beta > 0$ near $x = 0$. Then $\frac{\omega_x^2}{\omega}\sin(2x)$ is of order $|x|^{2(\beta-1)-\beta+1} = |x|^{\beta-1}$ near $x = 0$, which is locally integrable. A similar conclusion holds for the local integrability near $x = \frac{\pi}{2}$. For $\omega \in C^{1,\alpha} \cap X$, the sign condition in X implies that ω degenerates at its zeros in $S^1 \backslash \{0, \pi/2\}$ with an order $\beta > 1$, if it exists, and thus $\frac{\omega_x^2}{\omega}\sin(2x)$ is still locally integrable. In particular, for $\omega_0 \in C^\infty \cap X$ with a finite number of zeros and a finite order of degeneracy, the assumption

$\int_0^{\pi/2} \left|\frac{\omega_{0,x}^2}{\omega_0}\sin(2x)\right| dx < +\infty$ holds automatically. Based on Theorem 5.1, we obtain the following global well-posedness result.

Theorem 5.2. *Suppose that $\omega_0 \in X \cap H^1$, $\omega_0(x)x^{-1} \in L^\infty$, and $A(\omega_0) = \int_0^{\pi/2} \left|\frac{\omega_{0,x}^2}{\omega_0}\sin(2x)\right| dx < +\infty$. There exists a global solution ω of (3.1) with initial data ω_0. In particular, (a) for $\omega_0 \in X \cap C^{1,\alpha}$ with $\alpha \in (0,1)$ and $A(\omega_0) < +\infty$, there exists a global solution from ω_0; (b) for $\omega_0 \in X \cap C^1$ with $A(\omega_0) < +\infty$, the unique local solution $\omega \in \cap_{\alpha<1} C^\alpha$ from ω_0 exists globally. If the initial data further satisfies $\omega_0 \in C^{1,\alpha}$ with $\alpha \in (0,1)$ and $\omega_{0,x}(0) = 0$, we have*

$$||\omega(t)||_{L^1} + |u_x(0,t)| \le K(\omega_0)e^{CQ(2)t},$$
$$||\omega(t)||_{L^\infty} \le K(\omega_0)\exp(2\exp(K(\omega_0)\exp(CQ(2)t))),$$

where $Q(2) = \int_0^{\pi/2} |\omega_0| \cot^2 y dy$ and $K(\omega_0)$ is some constant depending on $H\omega_0(0), H\omega_0(\frac{\pi}{2})$, $||\omega_0||_{L^1}, Q(2), A(\omega_0)$.

In the general case, the a-priori estimates are much weaker. See Lemma 5.5.4 and Remark 5.5.5 for more discussions. Since $H^s \hookrightarrow C^{1,\alpha}$ for $s > \alpha + \frac{3}{2}$, Theorem 5.2 implies the global well-posedness (GWP) in $H^s \cap X$ with $s > \frac{3}{2}$. The condition $\omega_0(x)x^{-1} \in L^\infty$ in Theorem 5.2 is necessary since we can obtain a finite time blowup for ω_0 that is less regular near $x = 0$.

Theorem 5.3. *For any $0 < \alpha < 1, s < \frac{3}{2}$, there exists $\omega_0 \in X \cap C^\alpha \cap H^s \cap C^\infty(S^1 \backslash \{0\})$ with $\int_0^{\pi/2} \left|\frac{\omega_{0,x}^2}{\omega_0}\sin(2x)\right| dx < +\infty$, such that the solution of (3.1) with initial data ω_0 develops a singularity in finite time. In particular, we have $\int_0^T u_x(0,t)dt = \infty$.*

One can establish the local well-posedness of (3.1) in $C^{k,\alpha}$ with any $k \in \mathbb{Z}_+ \cup \{0\}$ and $\alpha \in (0,1)$ using the particle trajectory method [89]. From the ill-posedness result for the incompressible Euler equations in [3], it is conceivable that (3.1) is ill-posed in C^1. For C^1 initial data, there is a unique local solution in $\cap_{\alpha<1} C^\alpha$. Thus, in view of the above Theorems, in the class $\omega \in X$, the blowup criterion in Theorem 5.1 and the regularity results in Theorems 5.2 and 5.3 are sharp.

Theorem 5.2 verifies the conjecture on the GWP of (3.1) on S^1 and rules out potential blowup of (3.1) from initial data in $C^\infty \cap X$. It also addresses the conjecture made in [44] in the case of S^1 that the strong solution to (3.1) is global for C^1 initial data in class X. Note that the smooth initial data that lead to singularity formation of the gCLM model (3.2) on S^1 [13, 14, 19] or the CLM model [26] can be chosen in the class in Theorem 5.2. Thus, Theorem 5.2 implies that the advection in (3.1) can prevent singularity formation in the CLM model or the gCLM model for such initial data. The global regularity results in Theorem 5.2 can be generalized to the DG model (3.1) with an external force $f\omega$ linear in ω, where $f \in C^\infty$ is a given even function. Theorem 5.3 resolves the conjecture made in [44, 104] that (3.1) develops a finite time singularity from initial data $\omega_0 \in C^\alpha$ or $\omega_0 \in H^s$ for any $\alpha \in (0,1)$ and $s < \frac{3}{2}$ in the case of S^1. The case of $\mathbb{R}$ has been resolved in [19] with $\omega_0 \in C_c^\infty$.

In [44], Elgindi-Jeong made an important observation that the advection can be substantially weakened by choosing C^α data with sufficiently small α, and constructed C^α self-similar blowup solution of (3.1) on $\mathbb{R}$ with small α. For (3.1) on S^1, a finite time blowup from C_c^α data with small α was obtained in [19]. In Theorem 5.3, the Hölder exponent α can be arbitrary close to 1. As we will see in the proof, it suffices to weaken the advection slightly. Theorem 5.3 is inspired by our previous work [14], where we constructed a finite time blowup solution for the gCLM model (3.2) with a slightly less than 1 and smooth initial data.

5.1.1 Connection with the CLM model

The CLM model (3.4) can be solved explicitly [26]

$$\begin{aligned} \omega(x,t) &= \frac{4\omega_0(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}, \\ H\omega(x,t) &= \frac{2H\omega_0(x)(2 - tH\omega_0(x)) - 2t\omega_0^2(x)}{(2 - tH\omega_0(x))^2 + t^2\omega_0^2(x)}. \end{aligned} \tag{5.3}$$

We consider the solution of (3.4) with period π . From (5.3), the solution can blow up at x in finite time if and only if $\omega_0(x) = 0$ and $H\omega_0(x) > 0$. Consider odd ω_0 with $\omega_0 < 0$ on $(0, \frac{\pi}{2})$. Since $H\omega_0(0) > 0$ and $H\omega_0(\frac{\pi}{2}) < 0$, the only point x with $\omega_0(x) = 0$ and $H\omega_0(x) > 0$ is $x = 0$. Within this class of initial data, from Theorem 5.1, $u_x(0,t)$ controls the blowup in both the CLM model and the De Gregorio model. On the other hand, the CLM model blows up in finite time for smooth initial data, while from Theorems 5.2, 5.3, the advection term in the De Gregorio model can prevent singularity formation if the initial data is smooth enough.

5.1.2 Competition between advection and vortex stretching

The competition between advection and vortex stretching and its relation with the vanishing order of $\omega \in X$ near $x = 0$ can be illustrated by a simple Taylor expansion. Suppose that near $x = 0$, $\omega = -x^a + l.o.t.$ for $a > 0$ and $u = cx + l.o.t.$ for some $c > 0$, where $l.o.t.$ denotes the lower order terms. We impose the latter assumption on u since $u = -(-\partial_{xx})^{-1/2}\omega$ is odd and at least C^1 with $u_x(0) > 0$ for nontrivial $\omega \in X$. The leading order term of $u\omega_x$ and $u_x\omega$ near $x = 0$ are given by

$$u\omega_x = -acx^a + l.o.t., \quad u_x\omega = -cx^a + l.o.t.$$

This simple calculation suggests that $a - 1$ characterizes the relative strength between the advection $|u\omega_x|$ and the vortex stretching $|u_x\omega|$ near $x = 0$. The advection is weaker than, comparable to, and stronger than the vortex stretching if $a < 1$, $a = 1$, and $a > 1$, respectively. Considering the stabilizing effect of advection [14, 67, 97] and the destabilizing effect of vortex stretching [26], one would expect that there exists singularity formation in the case of $a < 1$ and global well-posedness in the case of $a \geq 1$. Theorems 5.2 and 5.3 confirm this formal analysis. In the case of $a = 1$, e.g., $\omega_0 \in C^{1,\alpha}$ with $\omega_{0,x}(0) \neq 0$ in Theorem 5.2, the effects of two terms balance, making it very challenging to establish the GWP result in Theorem 5.2. To prove these results, we need to quantitatively characterize the competition in three different cases and precisely control the effects of advection and vortex stretching. See more discussions in Section 5.2.

5.1.3 Connections with incompressible fluids

5.1.3.1 The effect of advection

Theorem 5.2 provides some valuable insights on potential singularity formation in incompressible fluids. We consider the 2D Boussinesq equations

$$\omega_t + \mathbf{u} \cdot \nabla \omega = \theta_x, \quad \theta_t + \mathbf{u} \cdot \nabla \theta = 0, \tag{5.4}$$

where ω is the vorticity, θ is the density, and $\mathbf{u}$ is the velocity field determined by $\nabla^{\perp}(-\Delta)^{-1}\omega$.

In the whole space, a promising potential blowup scenario is the hyperbolic-flow scenario with θ_x, ω being odd in both x, y, and positive θ_x, ω in the first quadrant. The induced flow is clockwise in the first quadrant near the origin.

A similar scenario has been used in [62, 106]. In this scenario, the flow in the y-direction in the first quadrant moves away from the origin. To understand the effect of $y-$advection, we derive a model on θ_x, which is the driving force for the growth in (5.4). Taking $x-$derivative on (5.4) and using the incompressible condition $u_{2,y} = -u_{1,x}$, we yield

$$\partial_t \theta_x + \mathbf{u} \cdot \nabla \theta_x = -u_{1,x}\theta_x - u_{2,x}\theta_y = u_{2,y}\theta_x - u_{2,x}\theta_y. \tag{5.5}$$

Dropping θ_y term and the advection in x direction and simplifying $\omega = \theta_x$, we further derive

$$\partial_t \theta_x + u_2 \partial_y \theta_x = u_{2,y}\theta_x, \tag{5.6}$$

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\theta_x, \quad u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x. \tag{5.7}$$

See more motivations for these simplifications in Appendix D.0.2. Note that the $\theta-$equation in (5.4) with (5.7) reduces to the incompressible porous media equation [30, 31]. Equation (5.6) captures the competition between the vortex stretching $u_{2,y}\theta_x$ and the y-advection $u_2\partial_y\theta_x$ in (5.5). This model relates to (3.1) via the connections $\theta_x \to -\omega, \partial_{xy}(-\Delta)^{-1} \to -H$. Moreover, the solutions of the two models enjoy similar sign and symmetry properties. See more discussions in Appendix D.0.2. The connection between $\partial_{xy}(-\Delta)^{-1}$ and H can be justified under some assumptions [20, 23, 71], though it may not be consistent with the current setting.

Valuable insight from Theorem 5.2 and the connection between the above model and (3.1) is that if $\theta_x(x, y)$ vanishes near $y = 0$ to order $|y|^a$ with $a \geq 1$, the advection may be strong enough to destroy potential singularity formation. In the hyperbolic flow scenario, due to the odd symmetry in y, a typical θ near the origin is of the form $\theta(x, y) \approx c_1 x^{1+\alpha} y + l.o.t.$ for $\theta \in C^{1,\alpha}$ and $\theta(x, y) \approx c_1 x^2 y + l.o.t.$ for $\theta \in C^{\infty}$. In both cases, θ_x vanishes linearly in y, and thus the effect of $y-$advection can be an obstacle to singularity formation. Such effect can be overcome by imposing a solid boundary on $y = 0$ and singularity formation with $C^{1,\alpha}$ velocity has been established in [16]. For smooth data, the importance of boundary has been studied in [86, 87]. In the absence of a boundary, new mechanisms to overcome the advection or a new scenario may be required to obtain singularity formation of (5.4) in $\mathbb{R}^2$.

5.1.3.2 Connections with the SQG equation

In [5], Castro-Córdoba observed that a solution $\omega(y,t)$ of the De Gregorio model (3.1) can be extended to a solution of the SQG equation

$$\theta_t + \mathbf{u}\cdot\nabla\theta = 0, \quad \mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta \tag{5.8}$$

with infinite energy via the connection $\theta(x,y,t) = x\omega(y,t)$. We can perform derivations for (5.8) similar to those in (5.4)-(5.7). Under this connection, the terms dropped in the derivations are $e$xactly 0, and the SQG equation in the hyperbolic-flow scenario [62] reduces $e$xactly to the DG model (3.1) with a solution in class X. Hence, our analysis of (3.1) provides valuable insight into the effect of advection in (5.8) in such a scenario. Moreover, from Theorem 5.2, we obtain a new class of globally smooth non-trivial solutions to (5.8) with infinite energy. Note that a globally smooth solution to (5.8) with finite energy has been constructed in [7]. See also [59]. Singularity formation of (5.8) from smooth initial data with infinite energy follows from [19].

Under the radial homogeneity ansatz $\theta(t,r,\beta) = r^{2-2\alpha}g(t,\beta)$, Elgindi-Jeong [46] established a connection between a solution θ to the generalized SQG equation and a solution $g(t,\beta)$ to the gCLM model (3.2) with $a > 1$ up to some lower order term in the velocity operator. Our analysis of the global regularity of (3.1) sheds useful light on the analysis of (3.2) with $a > 1$ and constructing globally non-trivial solutions to the generalized SQG equation using the connection in [46]. In particular, our argument to analyze $u_x(0)$ and a singular integral, which is defined in (5.12) and characterizes the competition between advection and vortex stretching in (3.1), can be generalized to the gCLM model with $a > 1$. See more discussions in Chapter 6.

Organization of the Chapter The rest of the chapter is organized as follows. In Section 5.2, we discuss the main ideas in the proofs of the main theorems. In Section 5.3, we establish the one-point blowup criterion. In Section 5.4, we discuss the stabilizing effect of the advection in (3.1) and study the positive-definiteness of several quadratic forms, which are the building blocks for the GWP results in Theorem 5.2. In Section 5.5, we prove Theorem 5.2. In Section 5.6, we construct finite time blowup of (3.1) with $C^\alpha \cap H^s$ data. We make some concluding remarks on the potential generalization of the results in Chapter 6. Some technical Lemmas and derivations are deferred to Appendix D.

5.2 Main ideas and the outline of the proofs

In this section, we discuss the main ideas and outline the proofs of the main theorems.

5.2.1 Difference between the De Gregorio on $\mathbb{R}$ and on S^1

Note that the initial condition considered in [19] that leads to finite time blowup of (3.1) on $\mathbb{R}$ has the same sign and symmetry properties as those in X. To establish the well-posedness results in Theorems 5.1 and 5.2, we need to understand the mechanism on S^1 that prevents singularity formation similar to [19].

For (3.1) on S^1 with $\omega \in X$, we have two special points $x = 0, x = \pi/2$, which correspond to $x = 0, x = \infty$ in the case of $\mathbb{R}$. One of the key differences between two cases is captured by the evolution of $||\omega||_{L^1}$

$$\frac{d}{dt}\Big(- \int_0^{\pi/2} \omega(x) dx \Big) = \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) \cot(x+y) dx dy,$$

which is derived in (5.22),(5.23). Since $\omega \le 0$ on $[0, \pi/2]$, $- \int_0^{\pi/2} \omega(x) dx$ is the same as $||\omega||_{L^1}$.

For $x + y \le \frac{\pi}{2}$, the interaction on the right hand side has a positive sign due to $\cot(x+y) \ge 0$, which leads to the growth of $||\omega||_{L^1}$. On the other hand, for $x + y \ge \frac{\pi}{2}$, the interaction has a negative sign, which contributes to the decrease of $||\omega||_{L^1}$. The former and the latter interaction can be seen as the interaction near 0 and $\pi/2$, respectively. The latter plays a crucial role in our proof as a damping term. For comparison, a similar ODE can be derived for (3.1) on $\mathbb{R}$ with $\cot(x+y)$ replaced by $\frac{1}{x+y}$. The interaction is always positive and can contribute to the unbounded growth of the singular solution in [19] in the far field. Yet, for (3.1) on S^1, similar growth near $x = \pi/2$ is prevented due to the above damping term.

Moreover, for (3.1) on S^1 with $\omega \in X$, we have $-u \in X$ and thus $u_x(0) > 0$ and $u_x(\frac{\pi}{2}) < 0$ for nontrivial ω. The sign of $u_x(\frac{\pi}{2})$ suggests that near $x = \frac{\pi}{2}$, the vortex stretching $u_x \omega$ in (3.1) depletes the growth of the solution. Using these observations, we show that the nonlinear terms near $x = \frac{\pi}{2}$ are harmless. Thus, the main difficulty is the analysis of (3.1) near $x = 0$.

5.2.2 The one-point blowup criterion

In [83], an important equation was discovered

$$\frac{1}{2}\partial_t((\sqrt{\omega})')^2 = -\frac{1}{2}u(((\sqrt{\omega})')^2)' - \frac{1}{2}H\omega((\sqrt{\omega})')^2 + \frac{1}{4}(H\omega)'\omega', \tag{5.9}$$

which implies

$$\frac{1}{2}\partial_t \frac{\omega_x^2}{\omega} = -\frac{1}{2}\Big(u\frac{\omega_x^2}{\omega}\Big)_x + \omega_x H\omega_x. \tag{5.10}$$

Identity (5.10) can also be obtained from the equation of ω_x and ω^{-1} using (3.1).

To prove Theorem 5.1, one of the key steps is the estimate of a new quantity $\int_0^{\pi/2} \frac{\omega_x^2}{\omega}\sin(2x)dx$. The vanishing property of $\sin(2x)$ near $x = 0, \frac{\pi}{2}$ cancels the singularity caused by $\frac{1}{\omega}$ for $\omega \in X$. Since $\omega(t)$ remains in X (5.1) and $\omega \leq 0$ on $[0, \frac{\pi}{2}]$, $\frac{\omega_x^2}{\omega}\sin(2x)$ has a fixed sign. To control the nonlinear terms in the energy estimate, we will exploit the conservation form $\Big(u\frac{\omega_x^2}{\omega}\Big)_x$, use an important cancellation on a quadratic form of ω_x and a crucial extrapolation inequality on u. Using some estimates in [13, 19], we derive a-priori estimates on $u_x(0), ||\omega||_{L^1}, \int_0^{\pi/2}\frac{\omega_x^2}{\omega}\sin(2x)dx$, which controls $\omega(x)$ away from $x = \frac{\pi}{2}$ by interpolation. By exploiting the damping mechanisms near $x = \pi/2$ discussed in Section 5.2.1, we further show that $u_x(\frac{\pi}{2}, t)$ cannot blow up before the blowup of $u_x(0,t)$. With these estimates, we obtain an a-priori estimate on $||\omega||_{L^\infty}$ in terms of $\int_0^t u_x(0,s)ds$, and establish the one-point blowup criterion by applying the Beale-Kato-Majda type blowup criterion [1, 73]. See also [97].

5.2.3 Global well-posedness

To prove Theorem 5.2 using Theorem 5.1, we need to further control $u_x(0)$. In the special case of $\omega_0 \in C^{1,\alpha}$ with $\omega_{0,x}(0) = 0$, the key step is to establish

$$\frac{d}{dt}\int_0^{\frac{\pi}{2}} \omega \cot^2 x dx = \int_0^{\frac{\pi}{2}} (u_x\omega - u\omega_x)\cot^2 x dx \geq 0. \tag{5.11}$$

The quantity $\int_0^{\pi/2}\omega\cot^2 x dx$ is well-defined for $\omega \in C^{1,\alpha}$ with $\omega_x(0) = 0$ and $\alpha > 0$. The above inequality quantifies that the stabilizing effect of advection is stronger than the effect of vortex stretching in some sense for ω in this case. We will exploit the convolution structure in the quadratic form in (5.11) and use an idea from Bochner's theorem for a positive-definite function to establish (5.11). We remark that an inequality similar to (5.11) has been established in the arXiv version of [19], where a more singular function $\cot^\beta x$

with $\beta \geq 2.2$ is used. The inequality (5.11) is stronger than that in [19] since $\int_0^{\pi/2} \omega(\cot x)^\beta dx$ is not well-defined for $\omega \in C^{1,\alpha}$ with $\alpha \in (0, \beta - 2)$ and $\omega_x(0) = 0$. Since $\omega \leq 0$ on $[0, \pi/2]$, (5.11) implies an a-priori estimate of $\int_0^{\pi/2} |\omega \cot^2 x| dx$, based on which we can further control $||\omega||_{L^1}, u_x(0)$ and establish the global well-posedness.

In the general case, ω_0 can vanish only linearly near $x = 0$. The proof is much more challenging since $\int_0^{\pi/2} |\omega \cot^2 x| dx$ is not well-defined, and there is no similar coercive conserved quantity. Note that in this case, for ω_0 close to $A \sin 2x$ in the C^2 norm, the solution $\omega(x,t)$ converges to $A \sin 2x$ as $t \to \infty$ [73]. As pointed out in [73], this imposes strong constraints on possible conserved quantities. Thus, it is not expected that there is any good conserved quantity similar to some weighted norm of ω.

To illustrate our main ideas, we consider $\omega_0 \in C^{1,\alpha} \cap X$ with $\omega_{0,x} \neq 0$. In this case, the only conserved quantities seems to be $\omega_x(x,t) \equiv \omega_{0,x}(x)$ for $x = 0, \frac{\pi}{2}$. Surprisingly, the one-point conservation law $\omega_x(0,t) \equiv \omega_{0,x}(0)$ allows us to control $Q(\beta, t)$ defined below for $\beta < 2$. We remark that we do not have monotonicity of $Q(\beta,t)$ in t similar to (5.11) when $\beta < 2$. A crucial observation is the following leading order structure

$$Q(\beta, t) \triangleq \int_0^{\pi/2} -\omega(y,t)(\cot y)^\beta dy = \frac{-\omega_x(0)}{2-\beta} + \mathcal{R}(\beta, t), \quad |\mathcal{R}(\beta,t)| \lesssim_\alpha ||\omega||_{C^{1,\alpha}}, \tag{5.12}$$

for any $\beta < 2$. As long as $\omega(t)$ remains in $C^{1,\alpha}$, we can choose β sufficiently close to 2, such that $(2-\beta)Q(\beta,t)$ is comparable to $-\omega_x(0)$, which is time-independent. Using this observation, an ODE of $Q(\beta,t)$ similar to (5.11) but with a nonlinear forcing term and an additional extrapolation-type estimate, we can control $Q(\beta(t), t)$ with $\beta(t)$ sufficiently close to 2. In the case of the less regular initial data $\omega_0 \in X \cap H^1$ with $\omega_0 x^{-1} \in L^\infty$, we will establish an estimate similar to (5.12). This enables us to further control $u_x(0)$ and establish the global well-posedness.

5.2.4 Finite time blowup

To prove Theorem 5.3, we follow the framework introduced in Section 1.3. See also the work of Chen-Hou-Huang [19]. We also adopt an idea developed in our previous work [14] that a singular solution of the gCLM model (3.2) can be constructed by perturbing the equilibrium $\sin(2x)$ of (3.1). We first construct a C^α approximate self-similar profile of (3.1) $\omega_\alpha = C \cdot \mathrm{sgn}(x) |\sin 2x|^\alpha$ with $\alpha < 1$

sufficiently close to 1. Our key observation is that for $\alpha < 1$, the advection $u\omega_x$ is slightly weaker than the vortex stretching $u_x\omega$. See the discussions in the paragraph before Section 5.1.1 and in Section 5.1.2. Then we establish the nonlinear stability of the profile ω_α in the dynamic rescaling formulation of (3.1) based on the coercivity estimates of a linearized operator established in [83] and several weighted estimates. Using the nonlinear stability results and the argument in [14, 19], we further establish finite time blowup.

The finite time singularity of (3.1) on $\mathbb{R}$ from C_c^∞ initial data established in [19] has expanding support, and the vorticity blows up at ∞. The singularities of the gCLM model (3.2) with weak advection constructed in [13, 19, 44, 49] are focusing, and the blowups occur at the origin. Due to the relatively strong advection and the compactness of a circle, the C^α singular solution of (3.1) on S^1 we construct is neither expanding nor focusing, which is similar to that in [14]. Moreover, the solution blows up in most places at the blowup time. Compared to the analysis of the gCLM model in [14], the blowup analysis of (3.1) with C^α data is more complicated due to the less regular profile and its estimates in the nonlinear stability analysis with singular weights.

5.3 One-point blowup criterion

In this section, we establish the one-point blowup criterion in Theorem 5.1.

Recall the class X defined in (5.1) and the Hilbert transform on a circle with period π

$$u_x = H\omega = \frac{1}{\pi} P.V. \int_{-\pi/2}^{\pi/2} \omega(y)\cot(x-y)dy, \quad u = -\frac{1}{\pi}\int_{-\pi/2}^{\pi/2} \omega(y) \log\Big|\frac{\sin(x+y)}{\sin(x-y)}\Big| dy. \tag{5.13}$$

For (3.1) with initial data $\omega_0 \in X$, it is not difficult to obtain that $\omega(\cdot,t), -u(\cdot,t)$ remain in X.

5.3.1 Energy estimate

To perform energy estimate using (5.10), we multiply both sides of (5.10) with $-\sin(2x) \in X$ so that $-\frac{\omega_x^2}{\omega}\sin(2x) \geq 0$. Integrating them over S^1, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{S^1} -\frac{\omega_x^2}{\omega}\sin(2x)dx = \frac{1}{2}\int_{S^1}\Big(u\frac{\omega_x^2}{\omega}\Big)_x \sin(2x)dx - \int_{S^1}\omega_x H\omega_x \sin(2x)dx \triangleq I + II. \tag{5.14}$$

We introduce the following functionals

$$\begin{aligned} A(\omega) &\triangleq \int_{S^1} -\frac{\omega_x^2}{\omega} \sin(2x) dx, E(\omega) = A(\omega) + u_x(0) + ||\omega||_{L^1}, \\ U(t) &\triangleq \int_0^t u_x(0,s) ds. \end{aligned} \tag{5.15}$$

We choose the special function $\sin 2x$ due to the crucial cancellation in Lemma D.0.3

$$II = \int_{S^1} \omega_x H\omega_x \sin(2x) dx = 0. \tag{5.16}$$

For I, using integration by parts, we obtain

$$\begin{aligned} I &= -\frac{1}{2} \int_{S^1} u \frac{\omega_x^2}{\omega} (\sin(2x))_x dx = -\int_{S^1} \frac{u\cos(2x)}{\sin(2x)} \frac{\omega_x^2}{\omega} \sin(2x) dx \\ &= -2 \int_0^{\pi/2} \frac{u\cos(2x)}{\sin(2x)} \frac{\omega_x^2}{\omega} \sin(2x) dx. \end{aligned}$$

A crucial observation is that by taking advantage of the conservation form $(u\frac{\omega_x^2}{\omega})_x$ and performing estimate on (5.10) with an explicit function, the coefficient $\frac{u\cos(2x)}{\sin(2x)}$ in the nonlinear term I for x away from $x = 0, \frac{\pi}{2}$ is of lower order than u_x, ω. We further estimate I from above. Since $\omega, -u \in X$, we derive $-\frac{\omega_x^2}{\omega}\sin(2x) \geq 0$, $\frac{u}{\sin(2x)} \geq 0$, and $\cos(2x) \leq 0$ on $[\frac{\pi}{4}, \frac{\pi}{2}]$. It follows

$$I \leq -2 \int_0^{\pi/4} \frac{u\cos(2x)}{\sin(2x)} \frac{\omega_x^2}{\omega} \sin(2x) dx \lesssim \Big|\Big| \frac{u}{\sin x} \Big|\Big|_{L^\infty[0,\frac{\pi}{4}]} A(\omega), \tag{5.17}$$

where $A(\omega)$ is defined in (5.15). The fact that the nonlinear term in $[\pi/4, \pi/2]$ is harmless is related to the discussion in Section 5.2.1. To control $\frac{u}{\sin x}$, we use the following extrapolation.

Lemma 5.3.1. *Suppse that $\omega \in X$ satisfies $A(\omega) < +\infty, u_x(0) < +\infty$ and $\omega \in L^1$. We have*

$$\Big|\Big| \frac{u}{\sin x} \Big|\Big|_{L^\infty[0,\frac{\pi}{4}]} \lesssim (u_x(0) + ||\omega||_{L^1} + 1) \log(||\omega||_{L^\infty[0,\frac{\pi}{3}]} + 2), \tag{5.18}$$

$$\begin{aligned} || \, |\cos x|^{1/2} \omega ||_{L^\infty} &\lesssim (A(\omega)(u_x(0) + ||\omega||_{L^1}))^{1/2}, \\ ||\sin x \cdot \omega||_{L^\infty} &\lesssim (A(\omega)|u_x(\pi/2)|)^{1/2}. \end{aligned} \tag{5.19}$$

We remark that $||\omega||_{L^\infty[0,\frac{\pi}{3}]}$ can be further bounded by $|| \, |\cos x|^{1/2}\omega||_{L^\infty}$.

Proof. Denote

$$K(x,y)=\frac{\sin y}{\sin x}\log\Big|\frac{\sin(x+y)}{\sin(x-y)}\Big|=\frac{\sin y}{\sin x}\log\Big|\frac{\tan x+\tan y}{\tan x-\tan y}\Big|,$$
$$f(x)=x\log\Big|\frac{x+1}{x-1}\Big|.$$

From (5.13), we get

$$\frac{u}{\sin x}=-\frac{1}{\pi}\int_0^{\pi/2}\omega(y)\frac{1}{\sin x}\log\Big|\frac{\sin(x+y)}{\sin(x-y)}\Big|dy=-\frac{1}{\pi}\int_0^{\pi/2}\frac{\omega(y)}{\sin y}K(x,y)dy. \tag{5.20}$$

For $\varepsilon<\frac{1}{10}$ to be determined, we decompose (5.20) as follows

$$\begin{aligned}\Big|\frac{u}{\sin x}\Big| &\lesssim \int_0^{\pi/2}\mathbf{1}_{|y/x-1|>\varepsilon}\Big|\frac{\omega(y)}{\sin y}K(x,y)\Big|dy\\ &\quad+\int_0^{\pi/2}\mathbf{1}_{|y/x-1|\le\varepsilon}\Big|\frac{\omega(y)}{\sin x}\Big|\log\Big|\frac{\sin(x+y)}{\sin(x-y)}\Big|dy\triangleq I+II.\end{aligned}$$

Denote $z=\frac{\tan y}{\tan x}$. For $|y/x-1|>\varepsilon, x,y\in[0,\pi/2]$, we have

$$|z-1|=\Big|\frac{\tan y-\tan x}{\tan x}\Big|=\Big|\frac{\sin(x-y)}{\cos x\cdot\cos y\cdot\tan x}\Big|=\Big|\frac{\sin(x-y)}{\cos y\cdot\sin x}\Big|\gtrsim\frac{|x-y|}{x}\gtrsim\varepsilon.$$

For $x\in[0,\frac{\pi}{4}]$ and $y\in[0,\frac{\pi}{2}]$, using $\sin x\asymp\tan x,\sin y\le\tan y$ and the above estimate, we get

$$K(x,y)\lesssim\frac{\tan y}{\tan x}\log\Big|\frac{\tan x+\tan y}{\tan x-\tan y}\Big|=z\log\Big|\frac{z+1}{z-1}\Big|=f(z)\lesssim\log\varepsilon^{-1},$$

where we have used $f(z)\lesssim 1$ for $z>2$ and $z<\frac{1}{2}$ to obtain the last inequality. It follows

$$\begin{aligned}I&\lesssim\log\varepsilon^{-1}\int_0^{\pi/2}\frac{|\omega|}{\sin y}dy\lesssim\log\varepsilon^{-1}\int_0^{\pi/2}(-\omega(y))(\cot y+1)dy\\&\lesssim\log\varepsilon^{-1}(u_x(0)+||\omega||_1).\end{aligned}$$

For II, since $|\frac{y}{x}-1|\le\varepsilon<\frac{1}{10}$ and $x\in[0,\pi/4]$, we yield $y\in[0,\frac{\pi}{3}]$. Since $\sin z\asymp z$ on $[0,3\pi/4]$, we get $\Big|\frac{\sin(x+y)}{\sin(x-y)}\Big|\lesssim\Big|\frac{x+y}{x-y}\Big|$. Using these estimates, we derive

$$\begin{aligned}II&\lesssim||\omega||_{L^\infty(0,\frac{\pi}{3})}\int_{|y/x-1|\le\varepsilon}(1+\log\Big|\frac{y+x}{y-x}\Big|)\frac{1}{x}dy\\&=||\omega||_{L^\infty(0,\frac{\pi}{3})}\int_{1-\varepsilon}^{1+\varepsilon}(1+\log\Big|\frac{1+z}{1-z}\Big|)dz.\end{aligned}$$

Using a change of variable $s = z - 1 \in [-\varepsilon, \varepsilon]$, we further obtain

$$II \lesssim ||\omega||_{L^\infty(0,\frac{\pi}{3})} \int_{|s|\leq \varepsilon} \log |s|^{-1} ds \lesssim \varepsilon \log \varepsilon^{-1} ||\omega||_{L^\infty(0,\frac{\pi}{3})}.$$

Choosing $\varepsilon = (||\omega||_{L^\infty(0,\frac{\pi}{3})} + 10)^{-1} < \frac{1}{10}$, we prove

$$\begin{aligned} ||u(\sin x)^{-1}||_{L^\infty[0,\pi/4]} &\lesssim (u_x(0) + ||\omega||_1 + 1) \log \varepsilon^{-1} \\ &\lesssim (u_x(0) + ||\omega||_1 + 1) \log(||\omega||_{L^\infty(0,\frac{\pi}{3})} + 2), \end{aligned}$$

which is exactly (5.18).

For $x \in [0, \frac{\pi}{2}]$, using the Cauchy-Schwarz inequality, we prove

$$\begin{aligned} |\omega(x)(\cos x)^{1/2}| &\leq (\cos x)^{1/2} \int_0^x |\omega_x(y)| dy \leq \int_0^x |\omega_x(y)| (\cos y)^{1/2} dy \\ &\lesssim \Big(\int_0^{\pi/2} \frac{\omega_x^2}{|\omega|} \sin(2x) dx \int_0^{\pi/2} |\omega| (\cot x + 1) dx \Big)^{1/2} \\ &\lesssim (A(\omega)(u_x(0) + ||\omega||_{L^1}))^{1/2}, \end{aligned}$$

which is the first inequality in (5.19). The proof of the second inequality in (5.19) is similar. □

5.3.1.1 Estimates of $||\omega||_{L^1}, u_x(0)$

To close the energy estimate using Lemma 5.3.1, we further estimate $||\omega||_{L^1}, u_x(0)$ in terms of $U(t)$. Similar estimates have been established in [13] and the arXiv version of [19]. Integrating (3.1) over $[0, \frac{\pi}{2}]$ and using integration by parts, we yield

$$\frac{d}{dt} \int_0^{\pi/2} -\omega dx = \int_0^{\pi/2} -u_x \omega + u \omega_x dx = -2 \int_0^{\pi/2} u_x \omega dx \triangleq III. \tag{5.21}$$

Since ω is odd, symmetrizing the kernel in (5.13), we obtain

$$\begin{aligned} III &= -\frac{2}{\pi} \int_0^{\pi/2} \omega(x) \int_0^{\pi/2} \omega(y) \Big(\cot(x-y) - \cot(x+y) \Big) dy dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) \cot(x+y) dx dy \\ &= \frac{4}{\pi} \int_0^{\pi/2} (-\omega(x)) \Big(- \int_0^x \omega(y) \cot(x+y) dy \Big). \end{aligned} \tag{5.22}$$

Since $-\omega(x) \geq 0$ on $[0, \frac{\pi}{2}]$ and $\cot z$ is decreasing on $[0, \pi]$, we get

$$-\int_0^x \omega(y)\cot(x+y)dy \leq -\int_0^x \omega(y)\cot y dy \leq -\int_0^{\pi/2} \omega(y)\cot y dy \lesssim u_x(0). \tag{5.23}$$

It follows

$$III \lesssim u_x(0)\int_0^{\pi/2}(-\omega(y))dy, \quad \frac{d}{dt}\int_0^{\pi/2} -\omega(y)dy = III \lesssim u_x(0)\int_0^{\pi/2} -\omega(y)dy.$$

Using Gronwall's inequality, we establish

$$||\omega(t)||_{L^1} \leq ||\omega_0||_{L^1}\exp(C\int_0^t u_x(0,s)ds) \lesssim ||\omega_0||_{L^1}\exp(CU(t)).$$

Taking the Hilbert transform on both side of (3.1) and applying Lemma D.0.1, we derive

$$\begin{aligned}\frac{d}{dt}u_x(0) &= H(u_x\omega - u\omega_x)(0) = 2H(u_x\omega)(0) - H(\partial_x(u\omega))(0) \\ &= u_x^2(0) - \omega^2(0) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\cot y(u\omega)_x(y)dy \\ &= u_x^2(0) + \frac{1}{\pi}\int_{-\pi/2}^{\pi/2}\frac{1}{\sin^2 y}u\omega dy.\end{aligned} \tag{5.24}$$

Note that $u\omega \leq 0$ for all x and $u_x(0) \geq 0$ for $\omega \in X$. It follows

$$\frac{d}{dt}u_x(0) \leq u_x^2(0).$$

Using Gronwall's inequality, we obtain

$$0 \leq u_x(0,t) \leq u_x(0,0)\exp(U(t)) = H\omega_0(0)\exp(U(t)).$$

Plugging the above estimates, (5.16), (5.17) and Lemma 5.3.1 in (5.14), we obtain

$$\begin{aligned}\frac{d}{dt}A(\omega) \lesssim\ & C(||\omega_0||_{L^1}, H\omega_0(0))\exp(CU(t)) \\ & \cdot A(\omega)\log\Big((A(\omega)(u_x(0) + ||\omega||_{L^1}))^{1/2} + 2\Big),\end{aligned}$$

where $C(||\omega_0||_{L^1}, H\omega_0(0))$ is some constant only depending on $||\omega_0||_{L^1}, H\omega_0(0)$. Recall the energy $E(\omega)$ in (5.15). Combining the above estimates, we establish

$$\frac{d}{dt}E(\omega) \lesssim C(||\omega_0||_{L^1}, H\omega_0(0))\exp(CU(t)) \cdot E\log(E+2).$$

Solving the differential inequality, we prove

$$E(\omega) \leq (E(\omega_0)+2)\exp(\exp(C(||\omega_0||_{L^1}, H\omega_0(0))\int_0^t \exp(CU(s))ds)). \tag{5.25}$$

5.3.2 Estimate near $x = \frac{\pi}{2}$

In view of Lemma 5.3.1, we have control of $||\omega||_{L^\infty[0,a]}$ using $A(\omega), u_x(0)$ and $||\omega||_{L^1}$ only away from $x = \frac{\pi}{2}$, i.e. $a < \frac{\pi}{2}$, due to the vanishing weight $(\cos x)^{1/2}$. We further estimate $u_x(\pi/2, t)$ so that we can apply Lemma 5.3.1 to control $||\omega||_\infty$. This will enable us to apply the BKM type blowup criterion for (3.1) to establish Theorem 5.1.

Using a derivation similar to that in (5.24), we obtain

$$\frac{d}{dt}u_x(\frac{\pi}{2}) = u_x^2(\frac{\pi}{2}) + \frac{1}{\pi}\int_0^\pi \frac{1}{\cos^2(y)}u\omega dy \triangleq I + II. \tag{5.26}$$

A crucial observation is that for $\omega \in X$, $u_x(\frac{\pi}{2}) = \frac{1}{\pi}\int_0^\pi \omega(y)\tan(y)dy$ is negative. Thus the vortex stretching term $u_x^2(\frac{\pi}{2})$ depletes the growth of $u_x(\frac{\pi}{2})$, which is the main mechanism that $u_x(\frac{\pi}{2})$ does not blowup as long as $U(t)$ is bounded. See also Section 5.2.1. On the other hand, since $u\omega \leq 0$, the advection term $\frac{1}{\pi}\int_0^\pi \frac{1}{\cos^2(y)}u\omega dy$ is negative and contributes to the growth of $u_x(\frac{\pi}{2})$. Our goal is to show that the growing effect is weaker. The main difficulty is the singular functions $(\cos y)^{-2}, \tan y$ near $y = \frac{\pi}{2}$ in I and II since we can control ω away from $y = \frac{\pi}{2}$.

For II, we decompose it as follows

$$II = \frac{1}{\pi}\int_0^\pi \tan^2(y)u\omega dy + \frac{1}{\pi}\int_0^\pi u\omega dy = II_1 + II_2.$$

Since II_2 does not involve a singular function, the estimate of II_2 is simple. Using (5.13), we get

$$\begin{aligned}|u(x)| &\lesssim \int_0^\pi |\omega(y)||\cos y|^{1/2}|\cos y|^{-1/2}|\log|\sin(x-y)||dy \\ &\lesssim ||\,|\cos x|^{1/2}\omega||_\infty|||\cos x|^{-1/2}||_{L^{4/3}}||\log x||_{L^4} \lesssim ||\,|\cos x|^{1/2}\omega||_\infty.\end{aligned}$$

It follows

$$|II_2| \leq ||u||_{L^\infty}||\omega||_{L^1} \lesssim ||\,|\cos x|^{1/2}\omega||_\infty ||\omega||_{L^1}. \tag{5.27}$$

For I and II_1, our goal is to establish

$$I + II_1 \geq \frac{1}{4}u_x^2(\frac{\pi}{2}) - C|u_x(\frac{\pi}{2})| \cdot ||\omega||_{L^\infty}. \tag{5.28}$$

We will further use Lemma 5.3.1 and $\varepsilon-$Young's inequality to estimate $|u_x(\frac{\pi}{2})| \cdot ||\omega||_{L^\infty}$ and close the estimate of $u_x(\frac{\pi}{2})$ in (5.26). Note that near $y = \frac{\pi}{2}$, we have $(\cos y)^{-1}, \tan y = \frac{1}{\pi/2-y} + O(|\pi/2 - y|)$. For simplicity, we consider the coordinate near $\frac{\pi}{2}$ and introduce

$$f = \omega(x + \frac{\pi}{2}), \quad g = u(x + \frac{\pi}{2}), \quad s(x,y) = \frac{\tan y}{\tan x}. \tag{5.29}$$

Remark 5.3.2. Since $\tan z = z + O(z^3), \sin z = z + O(z^3)$ near $z = 0$, in the following derivations, we essentially treat $\tan z, \sin z$ similar to z.

Clearly, $g_x = Hf$, g and f are odd and $f \geq 0, g \leq 0$ on $(0, \frac{\pi}{2})$. Using (5.13), (5.29), $(\tan(x + \pi/2))^2 = (\tan x)^{-2}$ and symmetrizing the integrals in I, II_1, we get

$$\begin{aligned}
I &= (H\omega(\frac{\pi}{2}))^2 = (Hf(0))^2 = \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} f(x)f(y)\cot x \cot y dx dy \\
&= \frac{4}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{f(x)f(y)}{\tan x \cdot \tan y} dx dy, \\
II_1 &= \frac{1}{\pi}\int_0^{\pi} \frac{fg}{\tan^2 x} dx = \frac{2}{\pi}\int_0^{\pi/2} \frac{fg}{\tan^2 x} dx \\
&= -\frac{2}{\pi^2}\int_0^{\pi/2} \frac{f(x)}{\tan^2 x}\int_0^{\pi/2} f(y) \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy \\
&= -\frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} f(x)f(y)(\frac{1}{\tan^2 x} + \frac{1}{\tan^2 y}) \log\left|\frac{\sin(x+y)}{\sin(x-y)}\right| dy.
\end{aligned}$$

Recall s from (5.29). Note that

$$\left|\frac{\sin(x+y)}{\sin(x-y)}\right| = \left|\frac{\tan x + \tan y}{\tan x - \tan y}\right| = \left|\frac{s+1}{1-s}\right|, \quad \frac{1}{\tan x} = s\frac{1}{\tan y}. \tag{5.30}$$

We further obtain

$$I + II_1 = \frac{1}{\pi^2}\int_0^{\pi/2}\int_0^{\pi/2} \frac{f(x)f(y)}{\tan^2 y}\Big(4s - (1+s^2)\log\left|\frac{s+1}{1-s}\right|\Big) dx dy.$$

Note that $f(x)f(y) \geq 0$ for $x, y \in [0, \frac{\pi}{2}]$. The competition between I, II_1 is characterized by the interaction kernel $K(s) = 4s - (1+s^2)\log\left|\frac{s+1}{1-s}\right|, s \in [0, \infty)$. An important observation is that for large s or small s, $K(s) \approx 2s$. In particular, it is easy to obtain

$$\begin{aligned}
&K(s) = s^2 K(s^{-1}), \\
&K(s) \geq s - (1+s^2)\log\left|\frac{1+s}{1-s}\right| \mathbf{1}_{a \leq s \leq a^{-1}} \geq s - C\log\left|\frac{1+s}{1-s}\right| \mathbf{1}_{a \leq s \leq a^{-1}}
\end{aligned}$$

for some absolute constant $0 < a < 1$ and $C > 0$. It follows

$$I + II_1 \geq \frac{1}{\pi^2} \int_0^{\pi/2} \int_0^{\pi/2} \frac{f(x)f(y)}{\tan^2 y} \Big(s - C \log \Big| \frac{s+1}{1-s} \Big| \mathbf{1}_{a \leq s \leq a^{-1}} \Big) dx dy.$$

Repeating the above derivations, we get

$$I + II_1 \geq \frac{1}{4} (Hf(0))^2 - C \int_0^{\pi/2} \frac{f(y)}{\tan^2 y} \int_0^{\pi/2} \log \Big| \frac{s+1}{1-s} \Big| \mathbf{1}_{a \leq s \leq a^{-1}} f(x) dx dy. \tag{5.31}$$

Next, we show that

$$|J(y)| \lesssim ||f||_{L^\infty}, \quad J(y) \triangleq \frac{1}{\tan y} \int_0^{\pi/2} \log \Big| \frac{s+1}{1-s} \Big| \mathbf{1}_{a \leq s \leq a^{-1}} f(x) dx.$$

We consider a change of variable $z = \tan x$. The restriction $s \in [a, a^{-1}]$ implies $z \in [a \tan y, a^{-1} \tan y]$. Using $dx = \frac{1}{1+z^2} dz$ and (5.30), we yield

$$\begin{aligned} J(y) &\lesssim \frac{||f||_\infty}{\tan y} \int_{a \tan y}^{a^{-1} \tan y} \log \Big| \frac{z + \tan y}{z - \tan y} \Big| \frac{1}{1+z^2} dz \\ &\lesssim \frac{||f||_\infty}{\tan y} \int_{a \tan y}^{a^{-1} \tan y} \log \Big| \frac{z + \tan y}{z - \tan y} \Big| dz \lesssim ||f||_{L^\infty} \int_a^{a^{-1}} \log \Big| \frac{\tau + 1}{\tau - 1} \Big| d\tau \lesssim ||f||_{L^\infty}, \end{aligned}$$

where we have used another change of variable $z = \tau \tan y$ to obtain the third estimate.

Recall $f = \omega(x + \frac{\pi}{2})$ from (5.29). Plugging the above estimates in (5.31), we establish

$$I + II_1 \geq \frac{1}{4}(Hf(0))^2 - C \int_0^{\pi/2} \frac{|f(y)|}{\tan y} dy ||f||_{L^\infty} = \frac{1}{4}(Hf(0))^2 - C|Hf(0)| \cdot ||f||_{L^\infty},$$

where we have used the facts that f is odd and that f has a fixed sign on $[0, \frac{\pi}{2}]$ to obtain the equality. We prove (5.28).

5.3.2.1 Estimate of $u_x(\frac{\pi}{2})$

Combining the estimates (5.26)-(5.28), we obtain

$$\frac{d}{dt} u_x(\frac{\pi}{2}) \geq \frac{1}{4} u_x^2(\frac{\pi}{2}) - C|u_x(\frac{\pi}{2})| \cdot ||\omega||_{L^\infty} - || \, |\cos x|^{1/2} \omega||_\infty ||\omega||_{L^1} \triangleq J.$$

Recall the energies in (5.15). Using Lemma 5.3.1, we derive

$$||\omega||_{L^\infty} \lesssim |u_x(\frac{\pi}{2})|^{1/2} (E(\omega))^{1/2} + E(\omega), \quad || \, |\cos x|^{1/2} \omega||_\infty ||\omega||_{L^1} \lesssim E^2(\omega).$$

Using $\varepsilon-$Young's inequality, we yield

$$
\begin{aligned}
J &\geq \frac{1}{4}u_x^2(\frac{\pi}{2}) - C|u_x(\frac{\pi}{2})|\Big(|u_x(\frac{\pi}{2})|^{1/2}(E(\omega))^{1/2} + E(\omega)\Big) - CE^2(\omega) \\
&\geq \frac{1}{8}u_x^2(\frac{\pi}{2}) - CE^2(\omega).
\end{aligned}
$$

Since $u_x(\frac{\pi}{2}) \leq 0$, we derive

$$
\frac{d}{dt}|u_x(\frac{\pi}{2})| \leq -\frac{1}{8}u_x^2(\frac{\pi}{2}) + CE^2(\omega).
$$

Using the estimate (5.25), we prove

$$
\begin{aligned}
&|u_x(t,\frac{\pi}{2})| \leq |H\omega_0(\frac{\pi}{2})| + C\int_0^t E^2(\omega(s))ds \\
&\leq |H\omega_0(\frac{\pi}{2})| + C(E(\omega_0)+2)^2 \exp(2\exp(C(||\omega_0||_{L^1}, H\omega_0)\int_0^t \exp(CU(s))ds)).
\end{aligned} \tag{5.32}
$$

5.3.2.2 The blowup criterion

Using (5.25), (5.32) and Lemma 5.3.1, we prove

$$
||\omega||_{L^\infty} \leq K_1(\omega_0)\exp(2\exp(K_1(\omega_0)\int_0^t \exp(CU(s))ds)), \tag{5.33}
$$

where C is some absolute constant, and the constant $K_1(\omega_0)$ depends on $H\omega_0(0), H\omega_0(\frac{\pi}{2}), ||\omega_0||_{L^1}$ and $A(\omega_0)$. Applying the BKM-type blowup criterion, we conclude the proof of Theorem 5.1.

5.4 Stabilizing effect of the advection and several quadratic forms

In order to apply Theorem 5.1 to establish the well-posedness result, we need to control $u_x(0)$. Yet, $u_x(0)$ itself does not enjoy a good estimate. Recall the ODE of $u_x(0)$ from (5.24).

$$
\frac{d}{dt}u_x(0) = u_x^2(0) + \frac{2}{\pi}\int_0^{\pi/2}\frac{u\omega}{\sin^2 y}dy.
$$

Since $u_x(0) \geq 0$ for $\omega \in X$, the quadratic nonlinearity $u_x^2(0)$ makes it very difficult to obtain a long time estimate on $u_x(0)$. Since $u_x(0) = \frac{-2}{\pi}\int_0^{\pi/2}\omega(y)\cot y dy$ can be viewed as a weighted integral of ω with a singular weight near 0, it motivates us to estimate other weighted integral that controls $u_x(0)$.

For $\beta \in (1,3)$, we introduce

$$Q(\beta,t) \triangleq -\int_0^{\pi/2} \omega(y,t)(\cot y)^{\beta} dy, \quad B(\beta,t) \triangleq \int_0^{\frac{\pi}{2}} (u_x\omega - u\omega_x)\cot^{\beta} x dx. \tag{5.34}$$

For $\omega \in X \cap H^1$, $Q(\beta,t), B(\beta,t)$ are well-defined if ω vanishes near $x=0$ at order $|x|^{\gamma}$ with $\gamma > \beta - 1$. For $\omega \in X$, since $\omega \leq 0$ on $[0,\pi/2]$, we have $Q(\beta,t) \geq 0$. The boundedness of $Q(\beta,t)$ implies that ω cannot be too large near 0, and it allows us to control the weighted integral of ω near 0. In Section 5.5, we will combine it and $||\omega||_{L^1}$ to further control $u_x(0)$.

Remark 5.4.1. The special singular function $(\cot y)^{\beta}$ and functional $Q(\beta,t)$ are motivated by the homogeneous function $|y|^{-\beta}$ and $\int_{\mathbb{R}^+} \omega/y^{\beta} dy$, which were used to analyze the gCLM model on the real line in the arXiv version of [19].

Using (3.1), we obtain the ODE of $Q(\beta,t)$

$$\frac{d}{dt}Q(\beta,t) = -B(\beta,t). \tag{5.35}$$

We should further estimate $B(\beta,t)$. The key Lemma to prove Theorem 5.2 is the following. To simplify the notation, we will drop "t" in some places.

Lemma 5.4.2. *Suppose that $\omega \in C^{\alpha}$ is odd with $\alpha \in (0,1)$ and $\omega(x)x^{-1} \in L^{\infty}$. There exists some absolute constant $\beta_0 \in (1,2)$, such that for $\beta \in [\beta_0,2)$, we have*

$$B(\beta) \geq -(2-\beta)\Big(u_x(0)Q(\beta) + \frac{1}{\pi}\iint_{[0,\pi/2]^2} \omega(x)\omega(y)(\cot y)^{\beta-1}\frac{s(s^{\beta-1}-1)}{s^2-1} dxdy\Big), \tag{5.36}$$

where $s(x,y) = \frac{\cot x}{\cot y}$. If in addition $\omega \in C^{1,\alpha}$ with $\alpha \in (0,1)$ and $\omega_x(0) = 0$, for $\beta = 2$, we have

$$B(2) \geq 0.$$

Note that in Lemma 5.4.2, we do not impose the sign condition: $\omega \leq 0$ (or ≥ 0) on $[0,\pi/2]$. Thus, it is likely that Lemma 5.4.2 can be generalized to study (3.1) with a larger class of data.

Lemma 5.4.2 quantifies the stabilizing effect of the advection, and reflects that the advection is stronger or almost stronger than the vortex stretching for ω vanishes at least linearly near $x=0$, which has been discussed heuristically in Section 5.1.2. In fact, if $\omega \in C^{1,\alpha}$ with $\omega_x(0) = 0$, using (5.35) and Lemma

5.4.2, we obtain that $Q(2,t)$ is bounded uniformly in t and thus ω can not be too large near 0. In the general case, ω can vanish only linearly near $x = 0$. Then $Q(2,t)$ is not well-defined since $\omega(\cot y)^2$ is not integrable. In this case, we apply (5.36). Though $Q(\beta,t)$ may not be bounded uniformly in t, the critical small factor $2-\beta$ indicates that $Q(\beta,t)$ cannot grow too fast.

5.4.1 Symmetrization and derivation of the kernel

To prove Lemma 5.4.2, we first symmetrize the quadratic form $B(\beta)$ and derive its associated interaction kernel. The symmetrization idea has been used in [23] to analyze some quadratic forms in the Hou-Luo model. Denote

$$s = \frac{\tan y}{\tan x} = \frac{\cot x}{\cot y}. \tag{5.37}$$

Since ω is odd, applying (5.13) and following the symmetrization argument in the arXiv version of [19], we derive (5.38) in Appendix D.0.3 if ω vanishes near $x = 0$ at order $|x|^{\gamma}$ with $\gamma > \beta - 1$

$$B(\beta) = \frac{1}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)P_\beta(x,y)dxdy, \tag{5.38}$$

where

$$\begin{aligned} P_\beta(x,y) &= (\cot y)^{\beta-1}\left(\frac{\beta}{2}(s^{\beta-1}+1)\log\left|\frac{s+1}{s-1}\right| - (s^{\beta-1}-1)\frac{2s}{s^2-1}\right) \\ &\quad + (\cot y)^{\beta+1}\left(\frac{\beta}{2}(s^{\beta+1}+1)\log\left|\frac{s+1}{s-1}\right| - (s^{\beta+1}-1)\frac{2s}{s^2-1}\right) \\ &\triangleq (\cot y)^{\beta-1}P_{2,\beta}(s) + (\cot y)^{\beta+1}P_{1,\beta}(s). \end{aligned} \tag{5.39}$$

Similar derivations and kernels were obtained in the arXiv version of [19]. The logarithm terms come from the advection $u\omega_x$ and are positive. Other terms $-(s^{\tau}-1)\frac{2s}{s^2-1}, \tau = \beta-1, \beta+1$ are from the vortex stretching $u_x\omega$ and are negative. Thus, the kernel P_β captures the competition between two terms. The main term in P_β is $(\cot y)^{\beta+1}P_{1,\beta}(s)$ since $(\cot y)^{\beta+1}$ is more singular. For s near 1, $P_\beta(s)$ is positive due to the singularity in $\log\left|\frac{1+s}{s-1}\right|$. It is not difficult to see that

$$\lim_{s\to\infty} P_{1,\beta}(s) = (\beta-2)s^{\beta}, \quad \lim_{s\to\infty} P_{2,\beta}(s) = (\beta-2)s^{\beta-2}. \tag{5.40}$$

Formally, as β increases, the kernel $P_\beta(x,y)$ becomes more positive-definite. Recall the ODE of $Q(\beta)$ from (5.34), (5.35). The higher vanishing order of ω

near 0, the larger β we can choose with $Q(\beta)$ being well-defined, and it is more likely that $Q(\beta,t)$ is decreasing and bounded uniformly in t. Therefore, the higher vanishing order of ω near 0 reflects the stronger effect of the advection, which potentially depletes the growing effect of the vortex stretching. The asymptotics (5.40) suggests that to obtain the positive definiteness of P_β, β should be at least 2. Indeed, such result is proved in the arXiv version of [19] for $\beta = 2.2$ under the sign condition $\omega \in X$ (5.1) by showing that $P_{i,\beta}(s) \geq 0$ pointwisely. However, the method in [19] can not be applied to the critical case $\beta = 2$ since numerical result shows that $P_{1,2}(s) < 0$ for $s \leq 0.5$ or $s \geq 2$.

For $\beta < 2$, it is not expected that P_β is positive-definite and the gap is of order $2-\beta$ quantified in Lemma 5.4.2. We study the modified kernel and its associated quadratic form

$$
\begin{aligned}
K_{1,\beta}(s) &= P_{1,\beta}(s) + (2-\beta)(s+s^\beta), \quad K_{2,\beta}(s) = P_{2,\beta}(s) + (2-\beta)\frac{(s^{\beta-1}-1)s}{s^2-1}, \\
K_\beta &= (\cot y)^{\beta+1} K_{1,\beta} + (\cot y)^{\beta-1} K_{2,\beta}, \\
\tilde{B}(\beta) &= \int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y) K_\beta(x,y)\,dxdy,
\end{aligned}
\tag{5.41}
$$

where P_i, s are defined in (5.39), (5.37). Using (5.38), (5.39), (5.41), and the following identities

$$
\begin{aligned}
&(s+s^\beta)(\cot y)^{\beta+1} = \cot x (\cot y)^\beta + (\cot x)^\beta \cot y, \\
&\frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \omega(x)\omega(y)(s+s^\beta)(\cot y)^{\beta+1} dxdy \\
&\quad = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \omega \cot y\, dy \int_0^{\frac{\pi}{2}} \omega (\cot y)^\beta dy = u_x(0) Q(\beta),
\end{aligned}
\tag{5.42}
$$

we derive

$$
\begin{aligned}
\frac{\tilde{B}(\beta)}{\pi} &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \omega(x)\omega(y)\Big\{ P_\beta(x,y) \\
&\quad + (2-\beta)\Big((s+s^\beta)(\cot y)^{\beta+1} + \frac{(s^{\beta-1}-1)s}{s^2-1}(\cot y)^{\beta-1}\Big)\Big\} dxdy \\
&= B(\beta) + (2-\beta)\Big(u_x(0) Q(\beta) \\
&\quad + \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \omega(x)\omega(y)\frac{(s^{\beta-1}-1)s}{s^2-1}(\cot y)^{\beta-1} dxdy\Big).
\end{aligned}
\tag{5.43}
$$

Hence, Lemma 5.4.2 is equivalent to $\tilde{B}(\beta) \geq 0$, or the positive definiteness of K_β for $\beta \in [\beta_0, 2]$.

Our key observation is that $s(x,y) = \frac{\cot x}{\cot y}$ can be written as $p(u-v)$, for some function p and variables u, v, and K_β can be written as a convolution kernel after a change of variable. This allows us to follow the idea in Bochner's theorem for a positive-definite function to leverage the positive part of $K_\beta(s)$ and establish that K_β is positive-definite.

In the following derivation, we restrict β to $\beta \in [1.9, 2]$. The reader can think of the special case $\beta = 2$, since we will choose β to be sufficiently close to 2.

5.4.1.1 Reformulation of $K_{1,\beta}$

We introduce

$$
\begin{aligned}
&F_1(x) \triangleq \omega(x)(\cot x)^{\frac{\beta+1}{2}}, \\
&\tilde{K}_{1,\beta}(s) \triangleq s^{-\frac{\beta+1}{2}} K_{1,\beta} = \frac{\beta}{2}(s^{\frac{\beta+1}{2}} + s^{-\frac{\beta+1}{2}}) \log \left| \frac{s+1}{s-1} \right| - \frac{s^{\frac{\beta+1}{2}} - s^{-\frac{\beta+1}{2}}}{s^2-1} 2s \\
&\qquad + (2-\beta)(s^{\frac{\beta-1}{2}} + s^{\frac{1-\beta}{2}}).
\end{aligned} \tag{5.44}
$$

Recall $s \cot y = \cot x$ from (5.37). Using $s^{\frac{\beta+1}{2}} (\cot y)^{\beta+1} = (\cot y \cot x)^{\frac{\beta+1}{2}}$, we derive

$$
(\cot y)^{\beta+1} K_{1,\beta}(s) = (\cot y)^{\beta+1} s^{\frac{\beta+1}{2}} s^{-\frac{\beta+1}{2}} K_{1,\beta}(s) = (\cot y \cot x)^{\frac{\beta+1}{2}} \tilde{K}_{1,\beta}(s).
$$

Hence, we can rewrite the quadratic form associated with $K_{1,\beta}$ in $\tilde{B}(\beta)$ (5.41) as follows

$$
\begin{aligned}
B_1(\beta) &\triangleq \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y)(\cot y)^{\beta+1} K_{1,\beta}(s) dx dy \\
&= \int_0^{\pi/2} \int_0^{\pi/2} F_1(x) F_1(y) \tilde{K}_{1,\beta}(s) dx dy.
\end{aligned} \tag{5.45}
$$

For $x, y \in [0, \pi/2]$, we consider a change of variable

$$
x = \arctan e^r, \quad y = \arctan e^t, \quad F_2(z) = \frac{e^z F_1(\arctan e^z)}{1+e^{2z}}, \quad W_{1,\beta}(z) = \tilde{K}_{1,\beta}(e^z). \tag{5.46}
$$

The variables $r = \log \tan x$ maps $(0, \frac{\pi}{2})$ to $\mathbb{R}$. Using $\frac{dx}{dr} = \frac{e^r}{1+e^{2r}}$ and $s = \frac{\tan y}{\tan x} = e^{t-r}$, we obtain

$$
\begin{aligned}
B_1 &= \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{F_1(\arctan e^r) F_1(\arctan e^t)}{(1+e^{2r})(1+e^{2t})} \tilde{K}_{1,\beta}(e^{t-r}) e^r e^t dt dr \\
&= \int_{\mathbb{R}} \int_{\mathbb{R}} F_2(r) F_2(t) W_{1,\beta}(t-r) dt dr.
\end{aligned}
$$

Recall F_1 in (5.44). Since $\cot(\arctan e^r) = e^{-r}$, we can rewrite F_2 in terms of ω

$$F_2(r) = \frac{e^r \omega(\arctan e^r)(\cot(\arctan e^r))^{\frac{\beta+1}{2}}}{1+e^{2r}} = \frac{e^{-\frac{\beta-1}{2}r}\omega(\arctan e^r)}{1+e^{2r}}.$$

Next, we discuss the integrability of $W_{1,\beta}$ and F_2. Since $\omega(x)x^{-1} \in L^\infty$, $\arctan x \lesssim \min(x,1)$ and $\beta \in [1.9, 2]$, we get

$$|F_2(r)| \lesssim e^{-\frac{\beta-1}{2}r} \min(1, e^r) \lesssim \min(e^{r/4}, e^{-r/4}).$$

Recall the definition of $\tilde{K}_{1,\beta}$ in (5.44). Clearly, $|\tilde{K}_{1,\beta}(s)|^p, |W_{1,\beta}(z)|^p$ are locally integrable for any $p > 0$. Using (5.40), $\left| \log \left| \frac{s+1}{s-1} \right| - \frac{2}{s} \right| \lesssim s^{-3}$ for $s > 2$ and a direct estimate, we obtain

$$\tilde{K}_{1,\beta}(s) = \tilde{K}_{1,\beta}(s^{-1}), \quad |\tilde{K}_{1,\beta}(s)| \lesssim s^{-\frac{\beta-1}{2}} \lesssim s^{-1/4} \text{ for } s > 2.$$

Note that for large s, the leading exponents $s^{\frac{\beta-1}{2}}$ appeared in each term of $\tilde{K}_{1,\beta}$ are canceled. As a result, we yield

$$W_{1,\beta}(z) = \tilde{K}_{1,\beta}(e^z) = \tilde{K}_{1,\beta}(e^{-z}) = W_{1,\beta}(-z), \quad |W_{1,\beta}(z)| \lesssim e^{-|z|/4} \text{ for } |z| > 1. \tag{5.47}$$

Denote by $\hat{f} = \int_{\mathbb{R}} \exp(-ix\xi) f(x) dx$ the Fourier transform of f. Using the Plancherel theorem, for some absolute constant $C_1 > 0$, we get

$$B_1(\beta) = C_1 \int_{\mathbb{R}} |\hat{F}_2(\xi)|^2 \widehat{W}_{1,\beta}(\xi) d\xi. \tag{5.48}$$

5.4.1.2 Reformulation of $K_{2,\beta}$

Similarly, we reformulate the kernel $K_{2,\beta}$ and its associated quadratic form in $\tilde{B}(\beta)$ in (5.41) as follows

$$\begin{aligned} B_2(\beta) &\triangleq \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y)(\cot y)^{\beta-1} K_{2,\beta}(s) dx dy \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} F_4(r) F_4(s) W_{2,\beta}(t-r) dt dr = C_1 \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{F}_4(\xi)|^2 \widehat{W}_{2,\beta}(\xi) d\xi \end{aligned} \tag{5.49}$$

for some absolute constant $C_1 > 0$, where

$$\begin{aligned} F_4(r) &= \frac{e^{\frac{3-\beta}{2}r}\omega(\arctan e^r)}{1+e^{2r}}, \quad W_{2,\beta}(z) = \tilde{K}_{2,\beta}(e^z), \\ \tilde{K}_{2,\beta}(s) &= \frac{\beta}{2}\Big((s^{\frac{\beta-1}{2}} + s^{-\frac{\beta-1}{2}}) \log\left|\frac{s+1}{s-1}\right| - (s^{\frac{\beta-1}{2}} - s^{-\frac{\beta-1}{2}}) \frac{2s}{s^2-1} \Big). \end{aligned} \tag{5.50}$$

The variable F_4 corresponds to $\omega(x)(\cot x)^{\frac{\beta-1}{2}}$ after a change of variable. For $\tilde{K}_{2,\beta}, W_{2,\beta}, F_4$ with $s > 2, |z| > 1$, we have

$$\begin{aligned} &|F_4(r)| \lesssim \min(e^{\frac{r}{4}}, e^{-\frac{r}{4}}), \qquad W_{2,\beta}(z) = W_{2,\beta}(-z), \quad \tilde{K}_{2,\beta}(s) = \tilde{K}_{2,\beta}(s^{-1}), \\ &|W_{2,\beta}(z)| \lesssim e^{-|z|/4}, \qquad |\tilde{K}_{2,\beta}(s)| \lesssim s^{-1/4}. \end{aligned}$$

5.4.2 Positivity of $W_{j,\beta}$

Recall the formulas of $B_j(\beta)$ (5.48), (5.49). To show that $B_j(\beta) \geq 0$, we only need to prove $\widehat{W}_{j,\beta}(\xi) \geq 0$ for any ξ. Since $W_{j,\beta}$ is even, it is equivalent to show that

$$G_{j,\beta}(\xi) \triangleq \frac{1}{2}\hat{W}_{j,\beta}(\xi) = \frac{1}{2}\int_{\mathbb{R}} W_{j,\beta}(x)e^{-ix\xi}dx = \int_{\mathbb{R}_+} W_{j,\beta}(x)\cos(x\xi)dx \geq 0 \tag{5.51}$$

for any ξ. Since $G_{j,\beta}(\xi), \widehat{W}_{j,\beta}(\xi)$ are even, we can further restrict to $\xi \geq 0$. We first study the positivity of $G_{1,\beta}$, which is much more difficult than that of $G_{2,\beta}$.

5.4.2.1 Positivity of $G_{1,\beta}$

Since we are interested in the case where β close to 2, using continuity, we can essentially reduce proving $G_{1,\beta} \geq 0$ to the special case $\beta = 2$.

Lemma 5.4.3. *Let $W = W_{1,2}, G = G_{1,2}$. Suppose that there exists $x_0 > 0, M > 0$, such that*

$$G(\xi) > 0, \quad \xi \in [0, M], \tag{5.52}$$

$$W''(x) > 0, \quad x \in [0, x_0], \tag{5.53}$$

$$-W'(x_0) - \frac{1}{M}\Big(|W''(x_0)| + \int_{x_0}^{\infty} |W'''(x)|dx\Big) > 0. \tag{5.54}$$

Then there exists $\beta_0 \in (1, 2)$, such that for any $\beta \in [\beta_0, 2]$ and ξ, we have $G_{1,\beta}(\xi) \geq 0$.

Using continuity of $W_{1,\beta}$ in β and the smallness of $2 - \beta$, we will show that (5.52)-(5.54) hold for $W_{1,\beta}, G_{1,\beta}$. The proof of this part is standard and is deferred to Appendix D.0.4.

Next, we prove that (5.53), (5.54) implies $G_{1,2}(\xi) \geq 0$ on $[M, \infty]$, which along with (5.52) prove $G_{1,2}(\xi) \geq 0$. The same argument applies to $G_{1,\beta}$. We simplify $W_{1,2}, G_{1,2}$ defined in (5.44), (5.46), (5.51) as W, G.

Large ξ We will choose M to be relatively large. This allows us to exploit the oscillation in the integral $G(\xi)$ (5.51) for $\xi \geq M$. From the definition of $W(x)$ in (5.44) and (5.46), we know that $W(x)$ is smooth away from $x = 0$ and $W(x)$ is singular of order $\log|x|$ near $x = 0$. Using integration by parts twice, we yield

$$\begin{aligned} G(\xi) &= \xi^{-1} \int_{\mathbb{R}_+} W(x) \partial_x \sin(x\xi) dx = -\xi^{-1} \int_{\mathbb{R}_+} W'(x) \sin(x\xi) dx \\ &= -\xi^{-2} \int_{\mathbb{R}_+} W'(x) \partial_x (1 - \cos(x\xi)) dx = \xi^{-2} \int_{\mathbb{R}_+} W''(x)(1 - \cos(x\xi)) dx, \end{aligned} \tag{5.55}$$

where the boundary term vanishes due to $W(x)\sin(x\xi) = O(x \log x)$ and $W'(x)(1 - \cos x\xi) = O(\frac{1}{x} x^2) = O(x)$ and the fast decay (5.47). The advantage of the above formula is that we obtain a nonnegative coefficient $1 - \cos(x\xi)$. For some $x_0 > 0$, we define

$$G_1(\xi) \triangleq \int_0^{x_0} W''(x)(1 - \cos(x\xi)) dx, \quad G_2(\xi) \triangleq \int_{x_0}^{\infty} W''(x)(1 - \cos(x\xi)) dx. \tag{5.56}$$

It suffices to verify $G_1(\xi) \geq 0$ and $G_2(\xi) \geq 0$. Thanks to (5.53) and $1 - \cos(\xi x) \geq 0$, we obtain $G_1(\xi) \geq 0$. For $G_2(\xi)$, the main term is associated with 1 since $\cos(x\xi)$ oscillates. In fact, using integration by parts, we yield

$$\begin{aligned} G_2(\xi) &= -W'(x_0) - \int_{x_0}^{\infty} W''(x) \cos(x\xi) dx \\ &= -W'(x_0) - \xi^{-1} \int_{x_0}^{\infty} W''(x) \partial_x \sin(x\xi) dx \\ &= -W'(x_0) + W''(x_0) \frac{\sin(x_0\xi)}{\xi} + \int_{x_0}^{\infty} W'''(x) \frac{\sin(x\xi)}{\xi} dx \\ &\geq -W'(x_0) - \frac{1}{M} \Big(|W''(x_0)| + \int_{x_0}^{\infty} |W'''(x)| dx \Big), \end{aligned}$$

where we have used $\xi \geq M$ in the last inequality. We choose $x_0 > 0$ and decompose the integral into two domains $x \leq x_0$ and $x > x_0$ in (5.56) since W''' in the above derivation is not integrable near $x = 0$. Using the assumption (5.54), we obtain $G_2(\xi) \geq 0$.

5.4.2.2 Verification of the conditions in Lemma 5.4.3

We discuss how to verify (5.52)-(5.54) below.

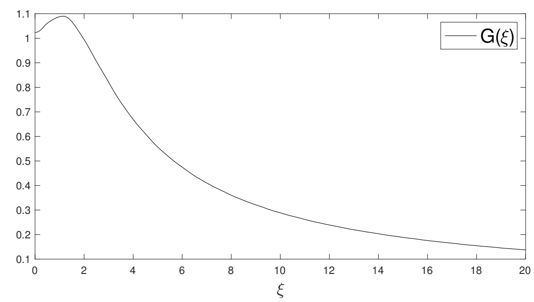

Figure 5.1: Rigorous lower bound of $G(\xi)$ for $\xi = ih, h = 0.05, 0 \leq i \leq 400$, $G(ih) > 0$.

Firstly, $G(\xi)$ is smooth in ξ and the Lipschitz constant satisfies

$$|\partial_\xi G| \leq \int_{\mathbb{R}_+} |W(x)| x dx \triangleq b_1. \tag{5.57}$$

The constant b_1 will be estimated rigorously. For small $\xi \in [0, M]$, we compute a lower bound of the integral $G(\xi)$ rigorously for the discrete points $\xi = ih, i = 0, 1, 2.., n$, $M = nh$, and verify $G(ih) > 0$. For $\xi \in [ih, (i+1)h]$, we use

$$G(\xi) \geq \min(G(ih), G((i+1)h)) - \frac{h}{2} b_1 > 0 \tag{5.58}$$

and verify the second inequality to obtain $G(\xi) > 0$. This enables us to establish (5.52).

For (5.53) and (5.54), let us first motivate why they hold true for some x_0, M. Using (5.44) and (5.46) yields the asymptotic behavior of $W(x)$ for x near 0

$$W(x) \approx -C \log |e^x - 1| \approx -C \log x, \quad W'(x) \approx -\frac{C}{x} < 0, \quad W''(x) \approx \frac{C}{x^2} > 0,$$

for some constant $C > 0$. See also (D.3) for a detailed derivation. Since W''' is integrable away from 0, (5.53), (5.54) hold true for small x_0 and large M.

In practice, we choose $x_0 = \log \frac{5}{3}$ and $M = 20$ in Lemma 5.4.3. Note that $W_{1,2}$ is an explicit function. We prove (5.53) for $x_0 = \log \frac{5}{3}$ in Appendix D.0.4. We discuss how to compute the integrals in (5.58) and (5.54) and verify these conditions, which are independent of ξ, rigorously in Appendix D.0.6. This allows us to establish the conditions in Lemma 5.4.3. The rigorous lower bound of $G(\xi)$ for $\xi = ih \in [0, M]$ is plotted in Figure 5.1, and $G(\xi)$ is strictly positive.

5.4.2.3 Positivity of $G_{2,\beta}$

Recall $W_{2,\beta}, G_{2,\beta}$ defined in (5.50) and (5.51). For $G_{2,\beta}$, it is easier to establish its positivity than that of $G_{1,\beta}$. From the argument in Section 5.4.2.1 and (5.55), a sufficient condition for $G_{2,\beta}(\xi) \geq 0$ is the convexity of $W_{2,\beta}$. We have the following result.

Lemma 5.4.4. *For any $\beta \in (1,2]$, we have $W_{2,\beta}''(x) \geq 0$ for $x \geq 0$. As a result, $G_{2,\beta}(\xi) \geq 0$ for any ξ and $\beta \in (1,2]$.*

The proof is based on estimating $W_{2,\beta}''$ directly using its explicit formula and elementary inequalities, which is not difficult and deferred to Appendix D.0.4.

5.4.2.4 Proof of Lemma 5.4.2

Combining Lemma 5.4.3 and Lemma 5.4.4, we establish that there exists $\beta_0 \in (1,2)$, such that for $\beta \in [\beta_0, 2]$ and any ξ, $\widehat{W}_{j,\beta}(\xi) = 2G_{1,\beta}(\xi) \geq 0, j = 1,2$. From (5.48), (5.49), we prove $B_j(\beta) \geq 0$. Recall the definitions of $\tilde{B}(\beta), B_1(\beta), B_2(\beta)$ from (5.41), (5.45), and (5.49). We obtain $\tilde{B}(\beta) = B_1(\beta) + B_2(\beta) \geq 0$.

Note that to obtain the equivalence between the forms of $B(\beta)$ in (5.34) and (5.38), we require that ω vanishes near $x = 0$ at order $|x|^\gamma$ with $\gamma > \beta - 1$. Using the relation (5.43) between $\tilde{B}(\beta)$ and $B(\beta)$, we prove (5.36) in Lemma 5.4.2 for $\beta \in [\beta_0, 2)$ and odd $\omega \in C^\alpha$ with $\omega x^{-1} \in L^\infty$. If in addition $\omega \in C^{1,\alpha}$ and $\omega_x(0) = 0$, we obtain that the vanishing order of ω near $x = 0$ is larger than 1 and choose $\beta = 2$ to establish $B(2) = \tilde{B}(2) \geq 0$. We conclude the proof of Lemma 5.4.2.

5.5 Global well-posedness

In this section, we use the crucial Lemma 5.4.2 to control $u_x(0,t)$ and then establish the global well-posedness result in Theorem 5.2 using the one-point blowup criterion in Theorem 5.1. We impose the assumptions $\omega_0 \in H^1 \cap X, \omega_0(x)x^{-1} \in L^\infty$, and $A(\omega_0) < +\infty$ stated in Theorem 5.2.

Recall $Q(\beta)$ defined in (5.34). To apply Theorem 5.1, from Hölder's inequality

$$|u_x(0)| \lesssim \int_0^{\pi/2} |\omega(y)| \cot y dy \lesssim Q(\beta)^{1/\beta} ||\omega||_{L^1}^{1-1/\beta}, \tag{5.59}$$

we only need to control $||\omega||_{L^1}$ and $Q(\beta)$. In (5.22),(5.23), we derive the evolution of $||\omega||_{L^1}$

$$\frac{d}{dt} - \int_0^{\pi/2} \omega(x)dx = \frac{2}{\pi}\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\cot(x+y)dxdy. \tag{5.60}$$

Recall the discussion of the interaction on the right hand side in Section 5.2.1. For $x+y \geq \frac{\pi}{2}$, the interaction has a negative sign and it will play a crucial role as a damping term.

5.5.1 Special case: $\omega_0 \in C^{1,\alpha}, \omega_{0,x}(0) = 0$

For initial data ω_0 with $\omega_{0,x}(0) = 0$, $\omega_x(0,t) = 0$ is preserved and $Q(2,t) = -\int_0^{\pi/2} \omega(y)\cot^2 y dy$ is well-defined. Using (5.35) and Lemma 5.4.2, we obtain

$$\frac{d}{dt}Q(2,t) = -B(2,t) \leq 0.$$

Since $\omega \leq 0$ on $(0, \frac{\pi}{2})$, we derive $Q(2,t) \geq 0$ and

$$\int_0^{\pi/2} |\omega|\cot^2 y dy = Q(2,t) \leq Q(2,0) = \int_0^{\pi/2} |\omega_0|\cot^2 y dy < +\infty.$$

Next, we estimate $||\omega||_{L^1}$. We first establish an estimate similar to (5.23)

$$-\int_0^{x} \omega(y)\cot(x+y)dy \leq -\int_0^{\pi/2} \omega(y)\cot^2 y dy = Q(2,t) \tag{5.61}$$

for $x \in [0, \frac{\pi}{2}]$. Since $\cot z \leq 0$ for $z \geq \frac{\pi}{2}$, $\cot y \leq 1$ on $[0, \frac{\pi}{4}]$, and $\cot y$ is decreasing on $[0,\pi]$, for $0 \leq y \leq x \leq \frac{\pi}{2}$, we get

$$\mathbf{1}_{y\leq x}\cot(x+y) \leq \mathbf{1}_{y\leq \frac{\pi}{4}}\mathbf{1}_{y\leq x}\cot(x+y) \leq \mathbf{1}_{y\leq \frac{\pi}{4}}\mathbf{1}_{y\leq x}\cot y \leq \cot y^2,$$

where we have used $x+y \geq \frac{\pi}{2}, \cot(x+y) \leq 0$ if $\frac{\pi}{4} \leq y \leq x$ in the first inequality. Since $\omega \leq 0$ on $[0, \frac{\pi}{2}]$, we prove (5.61). Plugging (5.61) in the estimates (5.21)-(5.22), we derive

$$\frac{d}{dt}\int_0^{\pi/2} -\omega dx \lesssim Q(2,t)\int_0^{\pi/2} -\omega dx \leq Q(2,0)\int_0^{\pi/2} -\omega dx.$$

Using the above estimate and the interpolation (5.59) with $\beta = 2$, we obtain

$$||\omega||_{L^1} \leq ||\omega_0||_{L^1} e^{CQ(2,0)t},$$
$$|u_x(0)| \lesssim (Q(2,t)||\omega||_{L^1})^{1/2} \lesssim (Q(2,0)||\omega_0||_{L^1})^{1/2} e^{CQ(2,0)t},$$

for some constant $C > 0$. Applying the same argument as that in Sections 5.3.1 and 5.3.2 with $U(t)$ replacing by $CQ(2,0)t$, we establish

$$||\omega||_{L^\infty} \leq K(\omega_0)\exp(2\exp(K(\omega_0)\exp(CQ(2,0)t))),$$

where we have used $\int_0^t \exp(CQ(2,0)s)ds \lesssim K(\omega_0)e^{CQ(2,0)t}$ and $K(\omega_0)$ is some constant depending on $H\omega_0(0), H\omega_0(\frac{\pi}{2}), ||\omega_0||_{L^1}, Q(2,0)$ and $A(\omega_0)$. We prove the result in Theorem 5.2 for the case of $\omega_0 \in C^{1,\alpha}$ with $\omega_{0,x}(0) = 0$.

We remark that the above a-priori estimates can be generalized to initial data ω_0 with lower regularity, e.g., $\omega_0/|x|^{1+\alpha} \in L^\infty$ for some $\alpha > 0$ and $\omega_0 \in X \cap H^1$.

5.5.2 General case

Recall from Section 5.2.3 the difficulties and ideas in the general case where ω_0 can vanish only linearly near $x = 0$. In this case, the monotone quantity $Q(2,t)$ in the previous case is not well-defined and not applicable. We will exploit a relation similar to the conservation law $\omega_x(0,t) = \omega_{0,x}(0)$ and control $Q(\beta,t)$ for β sufficiently close to 2.

5.5.2.1 Estimate of ωx^{-1}

For the less regular initial data $\omega_0 \in H^1$ with $\omega_0 x^{-1} \in L^\infty$, $\omega_x(0,t)$ is not well-defined. Instead of using the conservation law $\omega_x(0,t) = \omega_{0,x}(0)$, we show that $\omega(x,t)x^{-1}$ cannot grow too fast for x near 0. Consider the flow map

$$\frac{d}{dt}\Phi(x,t) = u(\Phi(x,t),t), \quad \Phi(x,0) = x. \tag{5.62}$$

We focus on $x \in [0, \frac{\pi}{2}]$. Since $u(x,t) \geq 0$, $u(0,t) = 0$, and $u(\frac{\pi}{2},t) = 0$, we get

$$\frac{d}{dt}\Phi(x,t) \geq 0, \quad 0 \leq \Phi(x,t_1) \leq \Phi(x,t_2), \tag{5.63}$$

for $t_1 \leq t_2$. Using (3.1), we derive the equation of ω/x

$$\partial_t \frac{\omega}{x} + u\partial_x(\frac{\omega}{x}) = (u_x - \frac{u}{x})\frac{\omega}{x}.$$

Fix $\gamma \in (0, \frac{1}{2})$. Using the embedding $H^1 \hookrightarrow C^\gamma$, we have $\omega, u_x \in C^\gamma$. Since $u_x(x) - \frac{u(x)}{x} = 0$ at $x = 0$ and $\omega \leq 0$ on $[0,\pi/2]$, for $x \in [0,\pi/2]$, we yield

$$\begin{aligned}\frac{d}{dt}\Big(-\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\Big) &= (u_x(\Phi(x,t),t) - \frac{u(\Phi(x,t),t)}{\Phi(x,t)})\Big(-\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\Big) \\ &\lesssim |\Phi(x,t)|^\gamma ||\omega||_{H^1}\Big|\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\Big|.\end{aligned}$$

Denote

$$m \triangleq ||\omega_0 x^{-1}||_{L^\infty}.$$

Using Gronwall's inequality and (5.63), we derive

$$\begin{aligned}\Big|\frac{\omega(\Phi(x,t),t)}{\Phi(x,t)}\Big| &\leq \exp(C\int_0^t |\Phi(x,s)|^\gamma ||\omega(s)||_{H^1} ds)||\frac{\omega_0}{x}||_{L^\infty} \\ &\leq m\exp(C|\Phi(x,t)|^\gamma \int_0^t ||\omega(s)||_{H^1} ds).\end{aligned}$$

Since $\Phi(\cdot,t)$ is a bijection from $[0,\pi/2]$ to $[0,\pi/2]$ and x is arbitrary, we yield

$$|\frac{\omega(x,t)}{x}| \leq m\exp(C|x|^\gamma \int_0^t ||\omega(s)||_{H^1} ds) \leq m(1+C|x|^\gamma \exp(C\int_0^t ||\omega(s)||_{H^1} ds), \tag{5.64}$$

where we have used $|x| \leq \pi/2, e^{Ax} \leq 1 + Ax \cdot e^{Ax} \leq 1 + Cxe^{CA}$ for some absolute constant C in the last inequality. The above estimate shows that $\limsup_{x\to 0} |\omega(x,t)/x|$ is bounded uniformly in t, which is an analog of $\omega_x(0,t) = \omega_{0,x}(0)$. Moreover, we obtain that $\omega(x,t)x^{-1} \in L^\infty$.

5.5.2.2 Weighted L^1 estimates

From the local well-posedness result and (5.64), we have $\omega(t) \in X \cap H^1$ and $\omega(x,t)x^{-1} \in L^\infty$, and $\omega(t)$ satisfies the assumptions in Lemma 5.4.2. A key step to control $Q(\beta,t)$ is establishing the following weighted L^1 estimates.

Lemma 5.5.1. *Let β_0 be the parameter in Lemma 5.4.2. For $\beta \in [\beta_0, 2)$, we have*

$$\frac{d}{dt}Q(\beta,t) \leq C(2-\beta)Q^2(\beta,t) + C(2-\beta)D(t), \quad \frac{d}{dt}||\omega||_{L^1} \leq CQ^2(\beta,t) - C_2 D(t), \tag{5.65}$$

for some absolute constant $C, C_2 > 0$, where $D(t) \geq 0$ is a damping term

$$D(t) = -\int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\cot(x+y)\mathbf{1}_{x+y>\pi/2} dxdy. \tag{5.66}$$

As a result, for some absolute constant $\lambda > 0$, we have

$$\frac{d}{dt}(Q(\beta,t) + \lambda(2-\beta)||\omega||_{L^1}) \lesssim (2-\beta)Q^2(\beta,t). \tag{5.67}$$

At first glance, the estimate (5.67) looks terrible due to the quadratic nonlinearity $Q^2(\beta,t)$. Yet, we have a crucial small factor $2-\beta$, which can compensate the nonlinearity. The boundedness of ωx^{-1} for x near 0 (5.64) implies the following leading order structure of $Q(\beta, t)$

$$\begin{aligned} Q(\beta,t) &= -\int_0^{\pi/2} \omega(x,t)(\cot x)^{\beta} dx \leq m \int_0^1 x \cdot x^{-\beta} dx + \mathcal{R}(\beta,t) \\ &\leq \frac{m}{2-\beta} + \mathcal{R}(\beta,t), \end{aligned}$$

where the remainder $\mathcal{R}(\beta, t)$ is of order lower than $(2-\beta)^{-1}$. For β sufficiently close to 2, we get $(2-\beta)Q(\beta,t) \lesssim m$, which is time-independent. Formally, the nonlinearity in (5.67) becomes linear. In Section 5.5.2.3, we will apply (5.67) and this key observation to prove Theorem 5.2.

The first estimate in (5.65) is highly nontrivial since the forcing term $u_x(0)Q(\beta)$ (see (5.71)) cannot be controlled by $Q^2(\beta)$. The idea behind Lemma 5.5.1 is that for the forcing terms $B(\beta,t)$ in (5.35) and (5.36) and that in (5.60), we use the more singular integral $Q(\beta,t)$ to control them near $x = 0$, and the magic damping term $D(t)$ from (5.60) to control them near $x = \pi/2$. To prove Lemma 5.5.1, we need several inequalities, whose proofs are deferred to Appendix D.0.5.

Lemma 5.5.2. *Denote $a \wedge b = \min(a,b)$. For $x, y \in [0, \pi/2], \beta \in [3/2, 2]$, we have*

$$\cot(x+y) \leq \mathbf{1}_{x+y\geq \pi/2} \cot(x+y) + (\cot x \cot y)^{\beta}, \tag{5.68}$$

$$\cot y (\cot x)^{\beta-2} \wedge \cot x (\cot y)^{\beta-2} \lesssim (\cot x \cot y)^{\beta} + \mathbf{1}_{x+y\geq\pi/2} \cot(\pi - x - y), \tag{5.69}$$

$$\cot y \mathbf{1}_{y\geq \pi/3} \lesssim (\cot x \cot y)^{\beta} + \mathbf{1}_{x+y\geq\pi/2} \cot(\pi - x - y). \tag{5.70}$$

Proof of Lemma 5.5.1. Using $\omega(x)\omega(y) \geq 0$ for $x, y \in [0, \pi/2]^2$ and (5.68), we obtain

$$\begin{aligned} &\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \omega(x)\omega(y) \cot(x+y) dx dy \\ \leq &- D(t) + \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y)(\cot x \cot y)^{\beta} dx dy \leq -D(t) + Q^2(\beta,t), \end{aligned}$$

where $D(t)$ is defined in (5.66). Using the above estimate and (5.60), we prove the second estimate in (5.65). Recall the ODE of $Q(\beta,t)$ (5.35). Applying

Lemma 5.4.2, for $\beta \in [\beta_0, 2)$, we get

$$\begin{aligned}\frac{d}{dt}Q(\beta,t) \le (2-\beta)\Big(u_x(0)Q(\beta,t) \\ + \frac{1}{\pi}\iint_{[0,\pi/2]^2} \omega(x)\omega(y)(\cot y)^{\beta-1}\frac{s(s^{\beta-1}-1)}{s^2-1}dxdy\Big) \triangleq (2-\beta)(I_1+I_2),\end{aligned} \tag{5.71}$$

where $s = \frac{\cot x}{\cot y}$. Next, we estimate $f(s) = \frac{s(s^{\beta-1}-1)}{s^2-1}$. Note that $\beta \in (3/2, 2)$. For $s \ge 0$, the following estimate is straightforward

$$0 \le f(s) \lesssim \mathbf{1}_{s<1/2}s + \mathbf{1}_{1/2\le s\le 2} + \mathbf{1}_{s\ge 2}s^{\beta-2} \lesssim s \wedge s^{\beta-2}.$$

Since $s = \frac{\cot x}{\cot y}$, using the above estimate and (5.69), we yield

$$\begin{aligned} f(s)(\cot y)^{\beta-1} &\lesssim (s \wedge s^{\beta-2}) \cdot (\cot y)^{\beta-1} = \cot y(\cot x)^{\beta-2} \wedge \cot x(\cot y)^{\beta-2} \\ &\lesssim (\cot x \cot y)^{\beta} + \mathbf{1}_{x+y\ge\pi/2}\cot(\pi - x - y) \\ &= (\cot x\cot y)^{\beta} - \mathbf{1}_{x+y\ge\pi/2}\cot(x+y). \end{aligned}$$

Using $\omega(x)\omega(y) \ge 0$ for $x, y \in [0, \pi/2]$, the above estimate and (5.66), we derive

$$\begin{aligned} 0 \le I_2 &\lesssim \int_0^{\pi/2}\int_0^{\pi/2} \omega(x)\omega(y)\Big((\cot x\cot y)^{\beta} - \mathbf{1}_{x+y\ge\pi/2}\cot(x+y)\Big)dxdy \\ &= Q^2(\beta,t) + D(t). \end{aligned}$$

For I_1, we cannot establish the desired estimate by comparing the kernel similar to the above since

$$\cot y(\cot x)^{\beta} \lesssim (\cot x\cot y)^{\beta} - \mathbf{1}_{x+y\ge\pi/2}\cot(x+y)$$

does not holds for x close to 0 and y close to $\pi/2$. In fact, for $\pi/2 - y = t^{\beta}, x = t$, with t sufficiently small, the left hand side is of $O(1)$, while the right hand side is $o(t)$. The main difficulty lies in that $(\cot y)^{\beta}$ is too weak to control $\cot y$ for y close to $\pi/2$.

A key observation is that we can further impose the restriction $Q(\beta,t) \le u_x(0) \lesssim ||\omega||_{L^1}$. In fact, if $u_x(0) \le Q(\beta,t)$, we obtain the trivial estimate

$$I_1 = u_x(0)Q(\beta,t) \le Q^2(\beta,t).$$

In the other case $Q(\beta,t) \le u_x(0)$, thanks to the interpolation (5.59), we derive

$$u_x(0) \lesssim Q(\beta,t)^{1/\beta}||\omega||_{L^1}^{1-1/\beta} \le (u_x(0))^{1/\beta}||\omega||_{L^1}^{1-1/\beta},$$

which implies $u_x(0) \lesssim ||\omega||_{L^1}$. Now, we decompose $I_1 = u_x(0)Q(\beta,t)$ as follows

$$
\begin{aligned}
I_1 &\lesssim \int_0^{\pi/2} |\omega| \cot y dy Q(\beta,t) = \int_0^{\pi/3} |\omega| \cot y dy Q(\beta,t) + \int_{\pi/3}^{\pi/2} |\omega| \cot y dy Q(\beta,t) \\
&\triangleq J_1 + J_2.
\end{aligned}
$$

For J_1, since $\cot y \lesssim (\cot y)^\beta$ for $y \leq \pi/3$, we get $J_1 \lesssim Q^2(\beta,t)$. For J_2, using $Q(\beta,t) \leq u_x(0) \lesssim ||\omega||_{L^1}$, we yield

$$
J_2 \lesssim \int_{\pi/3}^{\pi/2} |\omega(y)| \cot y ||\omega||_{L^1} \lesssim \int_{\pi/3}^{\pi/2} \omega(y) \cot y dy \int_0^{\pi/2} \omega(x) dx,
$$

where we have used $\omega(x) \leq 0$ on $[0,\pi/2]$ to obtain the last inequality. Applying (5.70) and $\cot(\pi - x - y) = -\cot(x+y)$, we obtain

$$
\begin{aligned}
J_2 &\lesssim \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y)\Big((\cot x \cot y)^\beta - \mathbf{1}_{x+y\geq\pi/2} \cot(x+y)\Big) dx dy \\
&= Q^2(\beta,t) + D(t).
\end{aligned}
$$

Combining the above estimates on J_1, J_2, in the other case $Q(\beta,t) \leq u_x(0)$, we prove

$$
I_1 \lesssim J_1 + J_2 \lesssim Q^2(\beta,t) + D(t).
$$

Combining the above estimates on I_1, I_2, we establish the first inequality in (5.65). Estimate (5.67) follows directly from (5.65) by choosing $\lambda > 0$ with $C_2\lambda \geq 2C$, e.g., $\lambda = \frac{2C}{C_2}$. □

Remark 5.5.3. We cannot apply (5.59) to estimate $u_x(0)$ in I_1 directly, since such estimate only offers

$$
\frac{d}{dt}(Q(\beta,t) + \mu||\omega||_{L^1}) \lesssim (2-\beta)^\gamma (Q(\beta,t) + \mu||\omega||_{L^1})^2
$$

with power $\gamma < 1$ for any well chosen μ, which is not sufficient for our purpose. Compared to (5.67), the above estimate loses a small factor $(2-\beta)^{1-\gamma}$, which is due to the fact that we do not have a good estimate on $||\omega||_{L^1}$, while for $Q(\beta,t)$ we have the crucial small factor $2-\beta$. We only add minimal amount of $||\omega||_{L^1}$ in the energy in (5.67) due to a similar reason.

5.5.2.3 A bootstrap estimate

Now, we are in a position to establish the global well-posedness result in Theorem 5.2 in the general case. It follows from a bootstrap lemma.

Lemma 5.5.4. *Suppose that ω_0 satisfies the assumptions in Theorem 5.2. Denote $m = ||\omega_0 x^{-1}||_{L^\infty}$. There exists some absolute constant c, such that for $\delta = \frac{c}{m}$, if $\int_0^T u_x(0,s)ds < +\infty$, we have $\int_0^{T+\delta} u_x(0,s)ds < +\infty$.*

Proof. Without loss of generality, we assume $m > 0$. Recall $Q(\beta, t)$ from (5.34). Denote

$$H(\beta, t) = Q(\beta, t) + \lambda(2-\beta)||\omega||_{L^1}.$$

In view of Theorem 5.1 and (5.59), for $\omega_0 \in H^1 \cap X$, the solution $\omega(x,t)$ remains in H^1 if $H(\beta, t) < +\infty$ for some $\beta < 2$. Thus, it suffices to control H. Using Lemma 5.5.1, we have

$$\frac{d}{dt} H(\beta, t) \leq \mu(2-\beta) H^2(\beta, t) \tag{5.72}$$

for some absolute constant $\mu > 0$ and any $\beta \in [\beta_0, 2)$. Since $\int_0^T u_x(0,s)ds < 0$, using Theorem 5.1, we obtain $\sup_{t \leq T} ||\omega(t)||_{H^1} < +\infty, ||\omega(T)||_{L^1} < +\infty$. Using (5.64), we obtain

$$\begin{aligned} Q(\beta, T) = \int_0^{\pi/2} |\omega| (\cot y)^\beta dy &\leq \int_0^1 |\omega| y^\beta dy + C \int_0^{\pi/2} |\omega| dy \\ &\leq m \int_0^1 \Big(y^{1-\beta} + C y^{\gamma+1-\beta} \exp(CT \sup_{t \leq T} ||\omega(t)||_{H^1}) \Big) dy + C||\omega(T)||_{L^1} \\ &\leq \frac{m}{2-\beta} + Cm \exp(CT \sup_{t \leq T} ||\omega(t)||_{H^1}) + C||\omega(T)||_{L^1}, \end{aligned} \tag{5.73}$$

where C is some absolute constant and we have used $|(\cot x)^\beta - x^{-\beta}| \lesssim |\cot x - x^{-1}| x^{-\beta+1} \lesssim x^{-\beta+2} \lesssim 1$ in the first inequality. Thus, there exists β_1 slightly less than 2 , such that

$$\begin{aligned} H(\beta_1, T) &= Q(\beta_1, T) + \lambda(2-\beta_1)||\omega(T)||_{L^1} \\ &\leq \frac{m}{2-\beta_1} + Cm \exp(CT \sup_{t \leq T} ||\omega(t)||_{H^1}) + C||\omega(T)||_{L^1} \leq \frac{2m}{2-\beta_1}. \end{aligned}$$

Solving the ODE (5.72) with $\beta = \beta_1$ on $t \geq T$, we yield

$$\frac{d}{dt} H^{-1}(\beta_1, t) \geq -\mu(2-\beta_1),$$

which along with the estimate on $H(\beta_1, T)$ imply

$$H^{-1}(\beta_1, T+\tau) \geq H^{-1}(\beta_1, T) - \mu(2-\beta_1)\tau \geq \frac{2-\beta_1}{2m} - \mu(2-\beta_1)\tau.$$

Note that μ is absolute. We choose $\delta = \frac{1}{4m\mu}$. Then, for $t \in [T, T+\delta]$, we yield

$$H^{-1}(\beta_1, t) \geq \frac{2-\beta_1}{2m} - \frac{2-\beta_1}{4m} = \frac{2-\beta_1}{4m}, \quad H(\beta_1, t) \leq \frac{4m}{2-\beta_1}. \tag{5.74}$$

Applying (5.59), we obtain $u_x(0,t) \lesssim \frac{m}{(2-\beta_1)^2}$ on $[T, T+\delta]$. We conclude the proof. $\square$

Remark 5.5.5. Denote $V(t) = \int_0^t (u_x(0,s)+1)ds$. We can obtain an a-priori estimate for $V(t)$ by tracking the bounds in the above proof. Using standard energy estimates and (5.33), we obtain

$$\begin{aligned} &Cm\exp(Ct\sup_{s\leq t}||\omega(s)||_{H^1}) + C||\omega(t)||_{L^1} \leq g(V(t), C_1), \\ &g(x,c) \triangleq c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(c\cdot\exp(cx)))))), \end{aligned}$$

for some constant $C_1 > 1$ depending only on the initial data. Note that the estimate of $||\omega||_{L^\infty}$ (5.33) is triple exponential growth, and then the estimate of $||\omega||_{H^1}$ is a quintuple one due to extrapolation in bounding $||u_x||_{L^\infty}$. These estimates further lead to the above sextuple exponential growth. For any $T \geq 0$, choosing β_1 with $2-\beta_1 = c\cdot\frac{m}{g(V(T),C_1)}$ for some absolute constant c and using (5.59), (5.74), we yield

$$V(T+\delta) \leq g(V(T), C_2),$$

for some constant $C_2 > 0$ depending only on ω_0. Since δ and C_2 are independent of T, iterating the above estimate yields an a-priori estimate for $V(t)$ with any $t \geq 0$.

Remark 5.5.6. The above estimate is consistent with the heuristic in the paragraph below (5.67) that the nonlinearity $(2-\beta)Q^2$ in (5.67) or $(2-\beta)H^2$ is essentially linear. In fact, for $t \in [T, T+\delta]$, (5.74) implies $(2-\beta_1)Q(\beta_1,t) \leq (2-\beta_1)H(\beta_1,t) \leq 4m$. Formally, $Q(\beta,t)$ grows exponentially in t for β close to 2, which we can barely afford, while in the previous case, $Q(2,t)$ is bounded uniformly. This argument is similar in spirit to extrapolation, e.g., the BKM blowup criterion [1].

5.6 Finite time blowup for $C^\alpha \cap H^s$ data

In this section, we prove Theorem 5.3 on finite time blowup for (3.1) with $C^\alpha \cap H^s$ data for any $\alpha \in (0,1)$ and $s \in (1/2, 3/2)$. We will use ideas outlined in Section 5.2.

Since we will adopt several estimates established in [14, 83], for consistency, throughout this section, we assume that the solution ω is 2π periodic. This modification also simplifies our notations. Theorem 5.3 can be established by applying the same argument to $\omega_{\pi}(x) \triangleq \omega_{2\pi}(2x)$. As a result, the Hilbert transform and the set X (5.1) becomes

$$
\begin{aligned}
Hf &\triangleq \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} \cot \frac{x-y}{2} f(y) dy, \\
X &\triangleq \Big\{ f : f \text{ is odd }, 2\pi - \text{periodic and } f(x) \leq 0, x \in [0, \pi] \Big\}.
\end{aligned}
$$

5.6.1 Slightly weakening the effect of advection

Recall the discussion on the competition between advection and vortex stretching in Section 5.1.2. To characterize that the advection is relatively weak for $\omega \in C^{\alpha} \cap X$ with $\omega \approx -Cx^{\alpha}$ near $x = 0$, we study (3.1) using the dynamic rescaling formulation

$$
\omega_t + u\omega_x = (c_{\omega} + u_x)\omega, \quad u_x = H\omega \tag{5.75}
$$

derived in (5.77)-(5.79) with the normalization condition

$$
c_{\omega}(t) = (\alpha - 1) u_x(0, t), \tag{5.76}
$$

where c_{ω} is a rescaling factor. If $u_x(0, t)$ is bounded away from $0 : u_x(0, t) \geq C > 0$ for all t, the competition between advection and the vortex stretching is encoded in the sign of c_{ω} since $\mathrm{sign}(c_{\omega}) = \mathrm{sign}(\alpha - 1)$, which can determine the long time behavior of the solution. See the discussion below (5.79). We remark that the idea and condition (5.76) are similar to those in [14], which play a crucial role in establishing singularity formation for the gCLM model.

5.6.2 Dynamic rescaling formulation

We follow the method in [14, 19] to construct finite time blowup solution using the dynamic rescaling formulation of (3.1). Let $\omega(x, t), u(x, t)$ be the solutions of equation (3.1). Following the ideas in Section 2.1.4, we obtain that

$$
\tilde{\omega}(x, \tau) = C_{\omega}(\tau) \omega(x, t(\tau)), \quad \tilde{u}(x, \tau) = C_{\omega}(\tau) u(x, t(\tau)) \tag{5.77}
$$

are the solutions to the dynamic rescaling equations

$$
\tilde{\omega}_{\tau} + \tilde{u}\tilde{\omega}_x = c_{\omega}\tilde{\omega} + \tilde{u}_x \tilde{\omega}, \quad \tilde{u}_x = H\tilde{\omega}, \tag{5.78}
$$

where

$$
C_{\omega}(\tau) = \exp\left(\int_0^{\tau} c_{\omega}(s) ds \right), \quad t(\tau) = \int_0^{\tau} C_{\omega}(s) ds. \tag{5.79}
$$

Remark 5.6.1. In (5.77)-(5.78), we do not rescale the spatial variable. This is different from the dynamic rescaling equation in Section 2.1.4 which contains a factor $C_l(\tau)$ in (5.77) and a stretching term $c_l(\tau)x\omega_x$ in (5.78). Here, we simply choose $c_l(\tau) \equiv 0$ and $C_l(\tau) \equiv 1$.

We will impose some normalization condition on the time-dependent scaling parameter $c_\omega(\tau)$, and establish that $-C_1 \leq c_\omega(\tau) \leq -C < 0$ for all $\tau > 0$ and some $C_1, C > 0$. To simplify our presentation, we still use t to denote the rescaled time in the rest of this section, unless specified, and drop $\tilde{\cdot}$ in (5.78). Then (5.78) reduces to (5.75).

5.6.3 Construction of the C^α approximate steady state

Based on the discussion in Sections 5.1.2 and 5.6.1, we first construct an approximate steady state $(\omega_\alpha, c_{\omega,\alpha})$ of (5.75) with $\omega_\alpha \in C^\alpha$ and $\omega_\alpha \approx -Cx^\alpha$ near $x = 0$. Following the idea in [14], we perform the construction by perturbing the equilibrium $\sin(x)$ of (3.1). A natural choice of ω_α is

$$\omega_\alpha = -\mathrm{sgn}(x)|\sin(x)|^\alpha c_\alpha, \quad c_\alpha = (\frac{1}{\pi}\int_0^\pi (\sin x)^\alpha \cot\frac{x}{2} dx)^{-1}. \tag{5.80}$$

We choose the above c_α to normalize $H\omega_\alpha(0) = 1$. Let u_α be the associated velocity with $u_{\alpha,x} = H\omega_\alpha$. We choose $c_{\omega,\alpha}$ according to (5.76)

$$c_{\omega,\alpha} = (\alpha - 1)u_{\alpha,x}(0) = \alpha - 1. \tag{5.81}$$

Denote

$$\omega_1 = -\sin x, \quad u_1 = \sin x, \quad \eta_\alpha = \omega_\alpha - \omega_1. \tag{5.82}$$

For α close to 1, we expect that $(\omega_\alpha, u_\alpha)$ are close to (ω_1, u_1).

Lemma 5.6.2. *Let $\kappa_1 = \frac{3}{4}, \kappa_2 = \frac{7}{8}$. For $\kappa_2 < \frac{9}{10} < \alpha < 1$ and $x \in [-\pi, \pi]$, we have*

$$|\partial_x^i \eta_\alpha| \lesssim (1-\alpha)|\sin x|^{\kappa_2 - i}, \quad i = 1,2,3, \tag{5.83}$$

$$|H\eta_\alpha| \lesssim (1-\alpha)|x|^{\kappa_1}, \quad |\partial_x H\eta_\alpha| \lesssim (1-\alpha)|\sin x|^{\kappa_1 - 1}, \tag{5.84}$$

$$|(\alpha-1)\omega_\alpha - \sin x(\omega_{\alpha,xx} - \omega_{1,xx})| \lesssim ((1-\alpha) \wedge |x|^2)|\sin x|^{\alpha-1}. \tag{5.85}$$

For x near 0, the above estimates on ω_α are similar to those for $\omega_\alpha = -x^\alpha$ and $\omega_1 = -x$. The reader can think of κ_1, κ_2 close to 1, and that α is even closer to 1.

Proof. Due to symmetry, it suffices to consider $x \geq 0$.

Firstly, using Lemma D.0.4 and $1 \lesssim \alpha - \kappa_2$, we obtain

$$|(\sin x)^{\alpha} - \sin x| = (\sin x)^{\kappa_2} (\sin x)^{\alpha-\kappa_2} (1 - (\sin x)^{1-\alpha}) \lesssim (1-\alpha)(\sin x)^{\kappa_2}. \quad (5.86)$$

Recall c_α defined in (5.80). Using the above estimate, we obtain

$$\frac{1}{\pi} \int_0^{\pi} |(\sin x)^{\alpha} - \sin x| \cot \frac{x}{2} dx \lesssim 1 - \alpha, \quad |c_\alpha - 1| \lesssim 1 - \alpha. \quad (5.87)$$

Next, we establish the estimate of ω_α defined in (5.80). A direct calculation yields

$$\omega_{\alpha,x} = -c_\alpha \alpha (\sin x)^{\alpha-1} \cos x, \quad \omega_{\alpha,xx} = -c_\alpha \alpha(\alpha-1)(\sin x)^{\alpha-2} \cos^2 x + \alpha c_\alpha (\sin x)^{\alpha}. \quad (5.88)$$

We consider a typical case $i = 3$ in (5.83), and the case $i = 1$ or 2 can be proved similarly. Recall $\omega_1, u_1, \eta_\alpha$ from (5.82). Using (5.86), (5.87) and $\kappa_2 < \alpha$, we get

$$\begin{aligned} |\eta_{\alpha,xx}| = |\omega_{\alpha,xx} - \sin x| &\lesssim |\alpha c_\alpha (\sin x)^{\alpha} - \sin x| + (1-\alpha)(\sin x)^{\alpha-2} \\ &\lesssim |(\sin x)^{\alpha} - \sin x| + (1-\alpha)(\sin x)^{\alpha-2} \lesssim (1-\alpha)(\sin x)^{\kappa_2 - 2}. \end{aligned}$$

For (5.85), the first bound $(1-\alpha)|\sin x|^{\alpha-1}$ follows directly from (5.83). Using (5.88), $|\omega_{1,xx}| = \sin x$ and a direct calculation, we yield

$$\begin{aligned} &|(\alpha-1)\omega_\alpha - \sin x(\omega_{\alpha,xx} - \omega_{1,xx})| \\ \leq &|c_\alpha(\alpha-1)\alpha(\sin x)^{\alpha-1}(\cos x - \cos^2 x)| + C(\sin x)^{\alpha+1} + \sin x|\omega_{1,xx}| \\ \lesssim &(\sin x)^{a-1}|x|^2 + (\sin x)^{a+1} \lesssim (\sin x)^{a-1}|x|^2, \end{aligned}$$

where we have used $|1 - \cos x| \lesssim x^2$.

Next, we prove (5.84). Denote $D_x = \sin x \partial_x$. Using (5.83) and $\kappa_2 = \frac{7}{8}$ close to 1, we have

$$\begin{aligned} &||\partial_x \eta_\alpha||_{L^4} \lesssim (1-\alpha) || \, |\sin x|^{\kappa_2 - 1} ||_{L^4} \lesssim 1 - \alpha, \\ &||\partial_x (D_x \eta_\alpha)||_{L^4} \lesssim || \, |\partial_x \eta_\alpha| + |\sin x \partial_x^2 \eta_\alpha| \, ||_{L^4} \lesssim (1-\alpha) |\sin x|^{\kappa_2 - 1} ||_{L^4} \lesssim 1 - \alpha. \end{aligned} \quad (5.89)$$

Recall from (5.80) that $u_{\alpha,x}(0) = H\omega_\alpha(0) = 1 = u_{1,x}(0)$. It implies $H\eta_\alpha(0) = 0$. Since the Hilbert transform is L^4 bounded, using Hölder's inequality and (5.89), we yield

$$\begin{aligned}|H\eta_\alpha(x)| = |\int_0^x \partial_x H\eta_\alpha(y)dy| \leq ||\partial_x H\eta_\alpha||_{L^4}\Big(\int_0^x 1dy\Big)^{3/4} \lesssim ||\partial_x H\eta_\alpha||_{L^4} x^{3/4} \\ = x^{3/4}||H\partial_x\eta_\alpha||_{L^4} \lesssim x^{3/4}||\partial_x\eta_\alpha||_{L^4} \lesssim (1-\alpha)x^{3/4}.\end{aligned}$$

Since $D_x H\eta_\alpha$ vanishes on $x = 0, \pi$, using an estimate similar to the above, we yield

$$|D_x H\eta_\alpha(x)| \lesssim ||\partial_x(D_x H\eta_\alpha(x))||_{L^4}(|x|^{3/4}\wedge|\pi-x|^{3/4}) \lesssim ||\partial_x(D_x H\eta_\alpha(x))||_{L^4}|\sin x|^{3/4}.$$

Applying Lemma D.0.2 ($n = 2$), we yield

$$\partial_x(D_x H\eta_\alpha) = \partial_x(H(D_x\eta_\alpha) - H(D_x\eta_\alpha)(0)) = \partial_x(H(D_x\eta_\alpha) = H(\partial_x D_x\eta_\alpha).$$

Applying (5.89) and the fact that H is L^4 bounded, we establish

$$|D_x H\eta_\alpha(x)| \lesssim ||H(\partial_x D_x\eta_\alpha)||_{L^4}|\sin x|^{3/4} \lesssim ||\partial_x D_x\eta_\alpha||_{L^4}|\sin x|^{3/4} \lesssim (1-\alpha)|\sin x|^{3/4},$$

which implies the second inequality in (5.84). □

The above L^4 estimate on $H\eta_\alpha$ can be replaced by L^p estimates with larger p, which offers more vanishing order of $H\eta_\alpha$ near $x = 0$. Here, the power $|x|^{3/4}$ is sufficient for our later weighted energy estimates.

5.6.4 Nonlinear stability of the approximate steady state

In this section, we follow [14, 19] to perform stability analysis around $(\omega_\alpha, c_{\omega,\alpha})$ constructed in (5.80), (5.81) and establish the finite time blowup results. We first introduce some weighted norms and spaces.

Definition 5.6.3. Define the singular weight $\rho = (\sin\frac{x}{2})^{-2}$, the standard inner product $\langle\cdot,\cdot\rangle$ on S^1, the weighted norms $||\cdot||_{\mathcal{H}}$ and the Hilbert spaces $\mathcal{H}$ as follows

$$\langle f, g\rangle = \int_0^{2\pi} fg dx, \quad ||f||_{\mathcal{H}}^2 \triangleq \frac{1}{4\pi}\int_{-\pi}^{\pi}\frac{|f_x|^2}{\sin^2\frac{x}{2}}dx, \quad \mathcal{H} \triangleq \{f|f(0) = 0, ||f||_{\mathcal{H}} < +\infty\} \tag{5.90}$$

with inner products $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ induced by the $\mathcal{H}$ norm.

The $\mathcal{H}$ norm was introduced in [83] for the stability analysis of the De Gregorio model. By definition, we have

$$\langle f, g\rangle_{\mathcal{H}} = (4\pi)^{-1}\langle f_x, g_x\rho\rangle. \tag{5.91}$$

5.6.4.1 Linearized equation

Linearizing (5.75) around $\omega_\alpha, c_{\omega,\alpha}$, we obtain the equation for the perturbation ω, c_ω (Note that $(\omega + \omega_\alpha, c_\omega + c_{\omega,\alpha})$ is the solution of (5.75))

$$\begin{aligned} \omega_t &= -u_\alpha \omega_x + u_{\alpha,x}\omega + u_x \omega_\alpha - u\omega_{\alpha,x} + c_{\omega,\alpha}\omega + c_\omega \omega_\alpha + N(\omega) + F(\omega_\alpha) \\ &\triangleq \mathcal{L}_\alpha \omega + N(\omega) + F(\omega_\alpha), \end{aligned} \tag{5.92}$$

where the nonlinear term $N(\omega)$ and error term $F(\omega_\alpha)$ are given by

$$N(\omega) = (c_\omega + u_x)\omega - u\omega_x, \quad F(\omega_\alpha) = (c_{\omega,\alpha} + u_{\alpha,x})\omega_\alpha - u_\alpha \omega_{\alpha,x}. \tag{5.93}$$

We choose the normalization condition on c_ω according to (5.76)

$$c_\omega = (\alpha - 1)u_x(0). \tag{5.94}$$

Under the conditions (5.76), (5.94), it is easy to obtain that the slope of ω/x^α is fixed, i.e.

$$\lim_{x\to 0} \frac{\omega(x,t) + \omega_\alpha(x)}{x^\alpha} = \lim_{x\to 0} \frac{\omega(x,0) + \omega_\alpha(x)}{x^\alpha}, \quad \lim_{x\to 0} \frac{\omega(x,t)}{x^\alpha} = \lim_{x\to 0} \frac{\omega(x,0)}{x^\alpha}.$$

In particular, if the initial perturbation $\omega_0(x)$ vanishes near $x = 0$ with order higher than x^α, e.g., $x^{2\alpha}$, the perturbation $\omega(x,t)$ will also vanish near $x = 0$ with higher order. This allows us to perform energy estimates on ω with a singular weight near $x = 0$.

We treat the linearized operator $\mathcal{L}_\alpha$ as a perturbation to $\mathcal{L}_1$

$$\mathcal{L}_1 \omega = -u_1 \omega_x + u_{1,x}\omega + u_x \omega_1 - u\omega_{1,x} = -\sin x \omega_x + \cos x \omega - u_x \sin x + u \cos x,$$

where we have used the explicit formulas (5.82), and perform the following decomposition

$$\begin{aligned} \mathcal{L}_\alpha \omega &= \mathcal{L}_1 \omega - (u_\alpha - u_1)\omega_x + (u_{\alpha,x} - u_{1,x})\omega + u_x(\omega_\alpha - \omega_1) \\ &\quad - u(\omega_{\alpha,x} - \omega_{1,x}) + c_{\omega,\alpha}\omega + c_\omega \omega_\alpha \\ &= \mathcal{L}_1 \omega - u(\eta_\alpha)\omega_x + H\eta_\alpha \cdot \omega + u_x \eta_\alpha - u\eta_{\alpha,x} + c_{\omega,\alpha}\omega + c_\omega \omega_\alpha \triangleq \mathcal{L}_1 \omega + \mathcal{R}_\alpha \omega, \end{aligned} \tag{5.95}$$

where $u(\eta_\alpha)$ denotes the odd velocity u with $u_x = H\eta_\alpha$. In fact, we have $u(\eta_\alpha) = -(-\partial_{xx})^{-1/2}\eta_\alpha$.

The operator $\mathcal{L}_1$ enjoys an important coercive estimate established in [83]. The following slight modification of the result in [83] is from [14].

Lemma 5.6.4. *Suppose that* $f, g \in \mathcal{H}$ *and* $\int_{S^1} f dx = 0$. *Denote* $e_0(x) = \cos x - 1$ *and*

$$f_e = \langle f, e_0 \rangle_{\mathcal{H}}, \quad \langle f, g \rangle_Y \triangleq \langle f - f_e e_0, g - g_e e_0 \rangle_{\mathcal{H}}.$$

We have : (a) Equivalence of norms : $(\mathcal{H}/\mathbb{R} \cdot e_0, \langle \cdot, \cdot \rangle_Y)$ *is a Hilbert space and the induced norm* $|| \cdot ||_Y$ *satisfies* $\frac{1}{2}||f||_{\mathcal{H}} \leq ||f||_Y \leq ||f||_{\mathcal{H}}$.

(b) Orthogonality : $||e_0||_{\mathcal{H}} = 1$ *and*

$$\langle f - f_e e_0, e_0 \rangle_{\mathcal{H}} = 0, \quad ||f||^2_{\mathcal{H}} = f_e^2 + ||f||_Y^2.$$

(c) Coercivity : $\langle \mathcal{L}_1 f, f \rangle_Y \leq -\frac{3}{8}||f||_Y^2$.

Using (5.91) and the above result (b), we can represent $\langle \cdot, \cdot \rangle_Y$ as follows

$$\langle f, g \rangle_Y = \langle f - f_e e_0, g \rangle_{\mathcal{H}} = (4\pi)^{-1} \langle f_x + f_e \sin x, g_x \rho \rangle, \tag{5.96}$$

where we have used $\partial_x e_0 = -\sin x$.

5.6.4.2 Weighted H^1 estimates

We consider odd perturbation ω, which satisfies $\int_{S^1} \omega dx = 0$. Recall the linearized equation (5.92) and the decomposition (5.95). Performing energy estimate on $\langle \omega, \omega \rangle_Y$ yields

$$\frac{1}{2}\frac{d}{dt}\langle \omega, \omega \rangle_Y = \langle \mathcal{L}_1 \omega, \omega \rangle_Y + \langle \mathcal{R}_\alpha \omega, \omega \rangle_Y + \langle N(\omega), \omega \rangle_Y + \langle F(\omega_\alpha), \omega \rangle_Y. \tag{5.97}$$

The estimate of the first term $\langle \mathcal{L}_1 \omega, \omega \rangle_Y$ follows from Lemma 5.6.4

$$\langle \mathcal{L}_1 \omega, \omega \rangle_Y \leq -\frac{3}{8}||\omega||_Y^2. \tag{5.98}$$

For the remainder $\mathcal{R}_\alpha$ in (5.95), a direct calculation yields

$$\begin{aligned} \partial_x \mathcal{R}_\alpha \omega &= -u(\eta_\alpha)\omega_{xx} + \partial_x H \eta_\alpha \cdot \omega + u_{xx}\eta_\alpha - u\eta_{\alpha,xx} + c_{\omega,\alpha}\omega_x + c_\omega \omega_{a,x} \\ &\triangleq -u(\eta_\alpha)\omega_{xx} + \mathcal{R}_{\alpha,2}\omega. \end{aligned}$$

Applying (5.96), we derive

$$\begin{aligned} &\langle \mathcal{R}_\alpha \omega, \omega \rangle_Y = (4\pi)^{-1} \langle \partial_x \mathcal{R}_\alpha \omega, (\omega_x + \omega_e \sin x)\rho \rangle \\ =&(4\pi)^{-1} \langle -u(\eta_\alpha)\omega_{xx}, (\omega_x + \omega_e \sin x)\rho \rangle + (4\pi)^{-1} \langle \mathcal{R}_{\alpha,2}, (\omega_x + \omega_e \sin x)\rho \rangle \triangleq I + II. \end{aligned} \tag{5.99}$$

Recall $\rho = (\sin \frac{x}{2})^{-2}$. Since $\sin \frac{x}{2} \asymp x$, we can essentially treat ρ as x^{-2}. For II, it suffices to estimate $||\mathcal{R}_{\alpha,2}\omega\rho^{1/2}||_2$. Since $c_\omega = (\alpha-1)u_x(0)$ (5.94), we decompose $\mathcal{R}_{\alpha,2}$ as follows

$$\begin{aligned}\mathcal{R}_{\alpha,2} = \partial_x H\eta_\alpha \cdot \omega + u_{xx}\eta_\alpha - (u - u_x(0)\sin x)\eta_{\alpha,xx} \\ + u_x(0)((\alpha-1)\omega_{\alpha,x} - \sin x \cdot \eta_{\alpha,xx}) + c_{\omega,\alpha}\omega_x.\end{aligned} \tag{5.100}$$

Next, we estimate the $L^2(\rho)$ norm of each term. The main difficulty is the estimate of the nonlocal term, e.g., $||u_{xx}\eta_\alpha\rho^{1/2}||_2$, due to the singular weight ρ near $x=0$ and that the profiles $\omega_\alpha, \eta_\alpha$ are not smooth near $x = 0, \pi$. Since $\eta_\alpha\rho^{1/2} \notin L^\infty$ (see (5.80),(5.82)), we need to perform a weighted estimate on u_{xx}. It is based on the lemma below, which shows that the Hilbert transform commutes with $\frac{1}{x}$ up to some lower order terms.

Lemma 5.6.5. *Suppose that $\frac{f}{x} \in L^2([-\pi,\pi])$. We have*

$$\Big|\frac{Hf - Hf(0)}{x} - H(\frac{f}{x})\Big| \lesssim \int_{-\pi}^{\pi} \Big|\frac{f(y)}{y}\Big| dy.$$

The proof is deferred to Appendix D.0.1. Since u, ω are odd, we get $u_{xx}(0) = 0$. Applying the above Lemma with $f = u_x$ and using the fact that H is L^2 bounded, we yield

$$||\frac{u_{xx}}{x}||_{L^2} = ||\frac{H\omega_x}{x}||_{L^2} \lesssim ||H(\frac{\omega_x}{x})||_{L^2} + ||\frac{\omega_x}{x}||_{L^1} \lesssim ||\frac{\omega_x}{x}||_{L^2} \lesssim ||\omega||_{\mathcal{H}}. \tag{5.101}$$

Applying (5.83) in Lemma 5.6.2, we obtain

$$||u_{xx}\eta_\alpha\rho^{1/2}||_{L^2} \lesssim ||u_{xx}x^{-1}||_{L^2}||\eta_\alpha||_{L^\infty} \lesssim (1-\alpha)||\omega||_{\mathcal{H}}.$$

Denote $\tilde{u} = u - u_x(0)\sin x$. Next we estimate $||\tilde{u}\eta_{\alpha,xx}\rho^{1/2}||_2$. From (5.80) and (5.83), $\eta_{\alpha,xx}$ is similar to $|\sin x|^{\alpha-2}$, which is singular both at $x = 0, \pi$. To overcome the singularities from $\eta_{\alpha,xx}$ and $\rho^{1/2}$, we estimate $\tilde{u}(\sin x)^{-1}x^{-1}$. For $|x| \geq \frac{\pi}{2}$, since $\tilde{u}(\pi) = 0$ and $|\sin x| \lesssim |\pi - |x||^{-1}$, we yield

$$|\tilde{u}(\sin x)^{-1}x^{-1}| \lesssim |\tilde{u}|\pi - |x||^{-1}| \lesssim ||\partial_x\tilde{u}||_\infty \lesssim ||u_{xx}||_2 \lesssim ||\omega||_{\mathcal{H}}.$$

For $|x| \leq \frac{\pi}{2}$, since $\tilde{u}(0) = \partial_x\tilde{u}(0) = 0$, using integration by parts, we obtain

$$\begin{aligned}|\tilde{u}(\sin x)^{-1}x^{-1}| &\lesssim \frac{|\tilde{u}|}{x^2} = \frac{1}{x^2}\Big|\int_0^x \partial_{yy}\tilde{u}(y)\cdot(x-y)dy\Big| \\ &\lesssim \frac{1}{x^2}||\partial_{yy}\tilde{u}\cdot y^{-1}||_2\Big(\int_0^x y^2(x-y)^2dy\Big)^{1/2}.\end{aligned}$$

Since $\partial_{yy}\tilde{u}(y) = \partial_{yy}u + u_x(0)\sin y$, using (5.101), we derive

$$|\tilde{u}(\sin x)^{-1}x^{-1}| \lesssim x^{-2}(||u_{xx}x^{-1}||_2 + |u_x(0)|)x^{5/2} \lesssim ||\omega||_{\mathcal{H}}.$$

Since $\rho^{1/2} = (\sin\frac{x}{2})^{-1} \asymp x^{-1}$, applying the above estimate and (5.83), we obtain

$$\begin{aligned}||(u - u_x(0)\sin x)\eta_{\alpha,xx}\rho^{1/2}||_2 &\lesssim ||\tilde{u}(\sin x)^{-1}\rho^{1/2}||_\infty||\eta_{\alpha,xx}\sin x||_2 \\ &\lesssim (1-\alpha)||\omega||_{\mathcal{H}}||\ |\sin x|^{\kappa_2 - 1}||_2 \lesssim (1-\alpha)||\omega||_{\mathcal{H}}.\end{aligned}$$

The estimates of other terms in (5.100) and I in (5.99) are relatively simple. Since ω vanishes at $x = 0, \pi$, using the Hardy-type inequality in Lemma D.0.5, we yield

$$\begin{aligned}||\omega(\sin x)^{-1}\rho^{1/2}||_2 &\lesssim ||\omega x^{-2}||_2 + ||\omega|\pi - |x||^{-1}||_2 \\ &\lesssim ||\omega_x x^{-1}||_2 + ||\omega_x||_2 \lesssim ||\omega_x x^{-1}||_2 \lesssim ||\omega||_{\mathcal{H}}.\end{aligned}$$

Applying the above estimates and (5.84) in Lemma 5.6.2, we obtain

$$||\partial_x H\eta_\alpha \cdot \omega\rho^{1/2}||_2 \lesssim ||\omega(\sin x)^{-1}\rho^{1/2}||_2||\sin(x)\partial_x H\eta_\alpha||_{L^\infty} \lesssim (1-\alpha)||\omega||_{\mathcal{H}}.$$

Applying (5.85) in Lemma 5.6.2 and $(1-\alpha)\wedge x^2 \lesssim (1-\alpha)^{1/2}|x|$, we yield

$$\begin{aligned}||u_x(0)((\alpha-1)\omega_{\alpha,x} - \sin x\cdot\eta_{\alpha,xx})\rho^{1/2}||_2 &\lesssim ||u_x||_{L^\infty}||(1-\alpha)\wedge|x|^2)|\sin x|^{\alpha-1}x^{-1}||_2 \\ &\lesssim ||u_{xx}||_2(1-\alpha)^{1/2}||\ |\sin x|^{\alpha-1}||_2 \\ &\lesssim (1-\alpha)^{1/2}||\omega_x||_2 \lesssim (1-\alpha)^{1/2}||\omega||_{\mathcal{H}}.\end{aligned}$$

Recall $c_{\omega,\alpha} = (\alpha - 1)$ from (5.81). The estimate of the last term in (5.100) is trivial

$$||c_{\omega,\alpha}\omega_x\rho^{1/2}||_2 \lesssim (1-\alpha)||\omega||_{\mathcal{H}}.$$

Combining the above $L^2(\rho)$ estimates of each term in (5.100), we establish

$$\begin{aligned}|II| &\lesssim ||\mathcal{R}_{\alpha,2}\rho^{1/2}||_2||(\omega_x + \omega_e\sin x)\rho^{1/2}||_2 \\ &\lesssim (1-\alpha)^{\frac{1}{2}}||\omega||_{\mathcal{H}}(||\omega||_{\mathcal{H}} + |\omega_e|) \lesssim (1-\alpha)^{\frac{1}{2}}||\omega||^2_{\mathcal{H}},\end{aligned} \tag{5.102}$$

where we have applied $|\omega_e| \lesssim ||\omega||_{\mathcal{H}}$ from Lemma 5.6.4 in the last inequality.

Next, we estimate the term I from (5.99). Applying integration by parts, we yield

$$\begin{aligned}I_1 &\triangleq \langle -u(\eta_\alpha)\omega_{xx},\ \omega_x\rho\rangle = \langle -u(\eta_\alpha)\rho,\ \frac{1}{2}\partial_x(\omega_x)^2\rangle = \frac{1}{2}\langle\partial_x(u(\eta_\alpha)\rho)\rho^{-1},\ \omega_x^2\rho\rangle, \\ I_2 &\triangleq \langle -u(\eta_\alpha)\omega_{xx},\ \omega_e\sin x\cdot\rho\rangle = \omega_e\langle\partial_x(u(\eta_\alpha)\rho\cdot\sin x),\ \omega_x\rangle.\end{aligned}$$

Since $\rho = (\sin\frac{x}{2})^{-2}, |\partial_x \rho| \lesssim \rho |x|^{-1}$, and $\partial_x u(\eta_\alpha) = H\eta_\alpha$, applying (5.84), we derive

$$
\begin{aligned}
|\partial_x (u(\eta_\alpha)\rho)| &\lesssim (|\partial_x u(\eta_\alpha)| + \Big|\frac{u(\eta_\alpha)}{x}\Big|)\rho \lesssim ||\partial_x u(\eta_\alpha)||_\infty \rho \lesssim (1-\alpha)\rho, \\
|\partial_x (u(\eta_\alpha)\rho \cdot \sin x)| &\lesssim |u(\eta_\alpha)\rho| + |x\partial_x (u(\eta_\alpha)\rho)| \\
&\lesssim ||\partial_x u(\eta_\alpha)||_\infty |x\rho| + (1-\alpha)|x|\rho \lesssim (1-\alpha)|x|\rho.
\end{aligned}
$$

Using the above estimate and the result (b) in Lemma 5.6.4, we establish

$$
\begin{aligned}
|I_1| &\lesssim ||\partial_x (u(\eta_\alpha)\rho)\rho^{-1}||_\infty ||\omega_x \rho^{1/2}||_2^2 \lesssim (1-\alpha)||\omega||_{\mathcal{H}}^2, \\
|I_2| &\lesssim (1-\alpha)|\omega_e| \cdot ||\omega_x x\rho||_{L^1} \lesssim (1-\alpha)||\omega||_{\mathcal{H}} ||\omega_x \rho^{1/2}||_{L^1} \lesssim (1-\alpha)||\omega||_{\mathcal{H}}^2.
\end{aligned}
\tag{5.103}
$$

Plugging the estimates (5.102) and (5.103) in (5.99) and then applying Lemma 5.6.4, we obtain

$$
|\langle \mathcal{R}_\alpha \omega, \omega \rangle_Y| \lesssim (1-\alpha)^{1/2} ||\omega||_{\mathcal{H}}^2 \lesssim (1-\alpha)^{1/2}||\omega||_Y^2. \tag{5.104}
$$

5.6.4.3 Estimates of nonlinear and error terms

Recall the nonlinear term $N(\omega)$ and error term $F(\omega_\alpha)$ from (5.93). Since $N(\omega)$ is similar to that in [14, 83] and the perturbation ω lies in the same space $\mathcal{H}$, the estimate of $N(\omega)$ is almost identical to that in [14, 83]. In particular, we yield

$$
|\langle N(\omega), \omega\rangle_Y| \lesssim ||\omega||_{\mathcal{H}}^3 \lesssim ||\omega||_Y^3 \tag{5.105}
$$

and refer the detailed estimates to [14, 83].

In the following derivation, we use the implicit notation $O(f)$ to denote some term g that satisfies $|g| \lesssim f$. It can vary from line to line. Due to symmetry, we focus on $x \in [0, \pi]$.

For the error term $F(\omega_\alpha)$, we first compute $\partial_x F(\omega_\alpha)$

$$
\partial_x F(\omega_\alpha) = u_{\alpha,xx}\omega_\alpha - u_\alpha \omega_{\alpha,xx} + c_{\omega,\alpha}\omega_{\alpha,x}. \tag{5.106}
$$

Recall $u_1 = \sin x, \omega_1 = -\sin x, \eta_\alpha = \omega_\alpha - \omega_1$ from (5.82), and $u_{\alpha,x} - u_{1,x} = H\eta_\alpha$. Applying Lemma 5.6.2 and $|\omega_\alpha| \lesssim |\sin x|^\alpha$ (5.80), we yield

$$
\begin{aligned}
u_{\alpha,xx}\omega_\alpha &= (u_{1,xx} + \partial_x H\eta_{\alpha,x})\omega_\alpha = u_{1,xx}\omega_\alpha + O((1-\alpha)|\sin x|^{\kappa_1 - 1 + \alpha}) \\
&= u_{1,xx}\omega_1 - \sin x \cdot \eta_\alpha + O((1-\alpha)|\sin x|^{\kappa_1 - 1 + \alpha}) \\
&= (\sin x)^2 + O((1-\alpha)|\sin x|^{\kappa_1 - 1 + \alpha}).
\end{aligned}
\tag{5.107}
$$

We decompose the second term in (5.106) as follows

$$\begin{aligned} u_\alpha \omega_{\alpha,xx} &= u_\alpha \eta_{\alpha,xx} + u_\alpha \omega_{1,xx} \\ &= (u_\alpha - \sin x)\eta_{\alpha,xx} + \sin x \cdot \eta_{\alpha,xx} + u_\alpha \omega_{1,xx} \triangleq I_1 + I_2 + I_3. \end{aligned} \tag{5.108}$$

Using (5.84), we yield

$$\begin{gathered} |u_{\alpha,xx}| \lesssim |u_{1,xx}| + |\partial_x H \eta_\alpha| \lesssim |\sin x|^{\kappa_1 - 1}, \\ |u_\alpha - \sin x| \lesssim (||u_x(\eta_\alpha)||_\infty + 1)|\sin x| \lesssim |\sin x|. \end{gathered}$$

Recall $u_{\alpha,x}(0) = 1$ from (5.80). For $0 \le x \le \frac{\pi}{2}$, the above estimate implies

$$\begin{aligned} |u_\alpha - \sin x| &\le |u_\alpha - x| + C|x|^3 = \Big| \int_0^x (u_{\alpha,x}(x) - u_{\alpha,x}(0)) dx \Big| + C|x|^3 \\ &= \Big| \int_0^x u_{\alpha,xx}(y) \cdot (x-y) dy \Big| + C|x|^3 \lesssim \int_0^x y^{\kappa_1 - 1}(x-y) dy + C|x|^3 \lesssim |x|^{\kappa_1 + 1}. \end{aligned}$$

Therefore, we yield

$$|u_\alpha - \sin x| \lesssim \mathbf{1}_{x \le \pi/2} |x|^{\kappa_1 + 1} + \mathbf{1}_{x > \pi/2} |\sin x| \lesssim |\sin x| \cdot |x|^{\kappa_1},$$

which along with (5.83) imply the estimate of I_1 in (5.108)

$$|I_1| \lesssim (1-\alpha)|\sin x|^{\kappa_2 - 1}|x|^{\kappa_1}.$$

For I_3 in (5.108), applying (5.84) and $u_1 = \sin x, \omega_1 = -\sin x$, we get

$$\begin{aligned} I_3 &= u_1 \omega_{1,xx} + (u_\alpha - u_1)\omega_{1,xx} = (\sin x)^2 + O(|\sin x|^2 ||u_{\alpha,x}||_\infty) \\ &= (\sin x)^2 + O((1-\alpha)|\sin x|^2). \end{aligned}$$

Recall $c_{\omega,\alpha} = \alpha - 1$ from (5.81). We combine I_2 in (5.108) and $c_{\omega,\alpha}\omega_{\alpha,x}$ in (5.106) and then apply (5.85) to obtain

$$\begin{aligned} |c_{\omega,\alpha}\omega_{\alpha,x} - I_2| &= |(\alpha-1)\omega_{\alpha,x} - \sin x \cdot \eta_{\alpha,xx}| \lesssim ((1-\alpha) \wedge |x|^2)|\sin x|^{\alpha-1} \\ &\lesssim (1-\alpha)^{1/2}|x| \cdot |\sin x|^{\alpha-1}. \end{aligned}$$

Plugging the above estimates on I_i and $c_{\omega,\alpha}$ in (5.108), we establish

$$u_\alpha \omega_{\alpha,xx} - c_{\omega,\alpha}\omega_{\alpha,x} = I_1 + I_3 + (I_2 - c_{\omega,\alpha}\omega_{\alpha,x}) = (\sin x)^2 + O((1-\alpha)^{1/2}|x|^{\kappa_1}|\sin x|^{\kappa_2 - 1}), \tag{5.109}$$

where we have used $|\sin x| \le |\sin x|^{\kappa_2 - 1}, |\sin x| \lesssim |x| \lesssim 1$ and $\kappa_2 < \alpha$ to combine the estimates of I_i in the last estimate.

Recall $\kappa_1 = \frac{3}{4}, \kappa_2 = \frac{7}{8}$ from Lemma 5.6.2. Combining (5.106), (5.107) and (5.109), we establish

$$\begin{aligned}\partial_x F(\omega_\alpha) &= (\sin x)^2 \cdot (1-1) + O((1-\alpha)|\sin x|^{\kappa_1 - 1 + \alpha}) + O(1-\alpha)^{1/2}|x|^{\kappa_1}|\sin x|^{\kappa_2 - 1} \\ &= (1-\alpha)^{1/2}|\sin x|^{\kappa_2 - 1}|x|^{\kappa_1},\end{aligned}$$

where we have used $|\sin x|^{\kappa_1+\alpha-\kappa_2} \lesssim |\sin x|^{\kappa_1} \lesssim |x|^{\kappa_1}$ to obtain the last estimate. Using the above estimate and Lemma 5.6.4, we prove

$$\begin{aligned}|\langle F(\omega_\alpha), \omega\rangle_Y| &\lesssim ||F(\omega_\alpha)||_Y ||\omega||_Y \lesssim ||\partial_x F(\omega_\alpha)\rho^{1/2}||_2 ||\omega||_Y \\ &\lesssim (1-\alpha)^{1/2} ||\,|\sin x|^{\kappa_2 - 1}|x|^{\kappa_1 - 1}||_2 ||\omega||_Y \lesssim (1-\alpha)^{1/2}||\omega||_Y.\end{aligned} \tag{5.110}$$

The integral is bounded since $2\kappa_2 - 2 = -\frac{1}{4} > -1, 2\kappa_2 + 2\kappa_1 - 4 = -\frac{3}{4} > -1$.

5.6.4.4 Nonlinear stability and finite time blowup

Combining (5.98), (5.104), (5.105), and (5.110), we establish the following nonlinear estimate for some absolute constant $C > 0$

$$\frac{1}{2}\frac{d}{dt}||\omega||_Y^2 \le -(\frac{3}{8} - C|1-\alpha|^{1/2})||\omega||_Y^2 + C|1-\alpha|^{1/2}||\omega||_Y + C||\omega||_Y^3.$$

Therefore, there exist absolute constants $\alpha_0 < 1$ sufficiently close to 1 and $\mu > 0$, such that for any $\alpha \in (\alpha_0, 1)$, if the initial perturbation satisfies $||\omega_0||_Y < \mu|1-\alpha|^{1/2}$, then

$$\begin{aligned}&||\omega(t)||_Y < \mu|1-\alpha|^{1/2}, \\ &c_{\omega,\alpha} + c_\omega(t) = (\alpha - 1)(1 + u_x(0)) \le (\alpha-1)(1 - C|\alpha - 1|^{1/2}) \le \frac{1}{2}(\alpha - 1)\end{aligned}$$

holds true for all $t > 0$. Since the weight $\rho = O(1)$ near $x = \pi$ and $(\partial_x \omega_\alpha)^2\rho$ is integrable near $x = \pi$, we can choose initial perturbation ω_0 such that $||\omega_0||_Y < \mu|1-\alpha|^{1/2}$, $\omega_0 \in C^2((-\pi/3, \pi/3))$ and $\omega_0 + \omega_\alpha \in C^\alpha \cap C^\infty(S^1 \backslash \{0\})$. For example, ω_0 can be $-\omega_\alpha$ near $x = \pi$, $\omega_0 = 0$ near $x = 0$ and smooth in the intermediate region. A simple Lemma D.0.6 shows that $\omega_0 + \omega_\alpha \in H^s$ for any $s < \alpha + \frac{1}{2}$, and a direct calculation gives $\int_0^\pi |\sin x \cdot f_x^2/f| dx < +\infty$ where $f = \omega_0 + \omega_\alpha$. Using the rescaling argument in Section 5.6.2, we establish finite time blowup of (3.1) from $\omega_0 + \omega_\alpha$.

The condition $\int_0^T u_{phy,x}(0,t)dt = \infty$ in Theorem 5.3, where u_{phy} is the velocity in (3.1), follows from Theorem 5.1 or a calculation using the above a-priori estimates on the perturbation and the rescaling relations (5.77)-(5.79). Due to the inclusion $C^\alpha \subset C^{\alpha_1}, H^s \subset H^{s_1}$ for $0 < \alpha_1 < \alpha, s_1 < s$, we conclude the proof of Theorem 5.3.

Chapter 6

CONCLUDING DISCUSSIONS AND RELATED PROBLEMS

Toward the Hou-Luo scenario with smooth data

In this thesis, we have centered around the important question of singularity formation for the 3D incompressible Euler equations. Our ultimate goal is to establish singularity formation of the 3D Euler equations with smooth data and boundary.

In Chapter 2, following the framework in Section 1.3 and adopting several important methods from [42], we have proved finite time blowup of the 2D Boussinesq and the 3D axisymmetric Euler equations with solid boundary and $C^{1,\alpha}$ velocity that has finite energy. These results provide the first rigorous justification of the Hou-Luo scenario using $C^{1,\alpha}$ data.

We remark that the analysis presented in Chapter 2 relies heavily on the small parameter α from the $C^{1,\alpha}$ regularity of the velocity. By choosing a sufficiently small α, several important nonlocal terms in the equations can be made arbitrarily small. As a result, the equations can be characterized by a much simpler leading order system. For smooth initial data considered in [86, 87], it is almost impossible to obtain an analytic expression of an approximate steady state with a small residual error for the dynamic rescaling equations.

In Chapters 3, 4, we further develop the framework for smooth data and establish finite time asymptotically self-similar blowup of the De Gregorio model on $\mathbb{R}$ and the Hou-Luo model from C_c^∞ initial data. An important novelty is to construct the approximate steady state with a small residual error by solving the dynamic rescaling equations numerically for a sufficiently long time. The residual error is incorporated in the energy estimates as a small and lower order term. This approach can be generalized to construct the approximate steady state for the 2D Boussinesq equations in the Hou-Luo scenario.

The framework of analysis established in this thesis provides a promising approach to studying the singularity formation of the 2D Boussinesq equations and the 3D axisymmetric Euler equations with smooth initial data and boundary. The key step is to establish the linear stability analysis of the approximate steady state in (1.6). We can follow the general strategy developed in this thesis, in particular Section 1.3 and Chapter 4, by (1) extracting the damping effect from the local terms, (2) treating the advection terms as the perturbation to vortex stretching, and (3) controlling the nonlocal terms by developing sharp functional inequalities on the Biot-Savart law and exploiting cancellation among them to control the nonlocal terms by using the damping effects from the local terms. See also Section 1.3. Compared with the HL model, we will encounter some additional difficulties associated with the advection away from the boundary, and need to estimate more complicated Biot-Savart law in the 2D Boussinesq and the 3D Euler equations. In our future work, we will explore a more effective functional space, e.g., weighted L^p or weighted C^α space, to establish the stability analysis. Such space offers the advantage of weakening the effect of the advection in the stability analysis and extracting larger damping effect from the local terms in the linearized equations. Moreover, it still allows us to estimate the Biot-Savart law effectively.

Singularity formation in related equations

Our framework is useful to study other fluids equations and models. We discuss some equations and problems below.

Incompressible porous medium equation The 2D incompressible porous medium (IPM) equation

$$\theta_t + \mathbf{u} \cdot \nabla \theta = 0, \quad \mathbf{u} = \nabla^{\perp} (-\Delta)^{-1} \partial_x \theta, \tag{6.1}$$

describes the evolution of density transported by incompressible flows. Equation (6.1) has the same level of regularity as the surface quasi-geostrophic (SQG) equation (3.5), which has a Biot-Savart law $\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta$. Whether the IPM equation blows up in finite time remains an open question.

Simplifying the ω equation in the 2D Boussinesq equations (1.3) by $\omega = \theta_x$, we obtain

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta = 0, \quad \omega = \theta_x, \quad \mathbf{u} = \nabla^{\perp} (-\Delta)^{-1} \omega,$$

which is exactly (6.1). See also the discussion in Section 5.1.3. We consider a scenario similar to the Hou-Luo scenario: θ is even in x, $\theta_x(x, y) \geq 0$ for $x > 0$. Note that θ_x is the driving force for the growth of ω in (1.3), and it leads to a strong alignment between θ_x and ω. It is reasonable to expect that this simplification does not change the essential mechanism in generating a potential singularity in the presence of a boundary. In particular, we expect that the method developed in this thesis to study the blowup of the 2D Boussinesq equations can be generalized to study the potential singularity formation of the IPM equation with smooth data and boundary.

In the above scenarios with boundary, the solutions to the 3D axisymmetric Euler equations, the 2D Boussinesq equations, and the IPM equation all enjoy the following parities. The density θ or swirl $(u^\theta)^2$ is even, and the velocity u^z for the 3D Euler or u_1 for Boussinesq and IPM equations is odd with respect to the symmetry axis: $z = 0$ for the 3D Euler or $x = 0$ for the 2D Boussinesq and the IPM equations. These parities are important to generate the potential singularity. See [38] for the discussions of parity in the singularity formation of some 1D models.

These examples of potential finite time blowup for the 3D axisymmetric Euler equations, the 2D Boussinesq equations, and the IPM equations with boundary suggest the following guideline of blowup, which has some flavor of *universality*.

> *In the presence of boundary, singularity formation may develop from smooth initial data if the Biot-Savart law has a certain parity and the solution satisfies suitable sign and symmetry properties.*

Singularity formation without boundary In the case without the boundary, the understanding of the potential singularity of incompressible fluids with smooth data is much poorer. In Chapter 5, we have proved that stronger advection can prevent singularity formation in the DG model on S^1. Thus, one of the guidelines to study finite time blowup without boundary is to look for a scenario where the advection is relatively weak. Meanwhile, it is important to understand how the vortex stretching generates the blowup without the boundary.

For these purposes, one can study the modified 3D axisymmetric Euler equations (1.14) [72] that add a weight ε to the advection. The authors in [72] showed numerically that (1.14) develops a stable self-similar singularity for weak advection. Thus, a natural step is the following.

Problem 1. *Prove that there exists* $\varepsilon_0 > 0$*, such that the modified 3D axisymmetric Euler equations* (1.14) *with any* $0 \le \varepsilon \le \varepsilon_0$ *develop a finite time singularity from some* C_c^∞ *initial data.*

A closely related problem is establishing the singularity formation of Constantin-Lax-Majda type models in 2D

$$\omega_t = \mathcal{K}(\omega)\omega, \quad x \in \mathbb{R}^2, \tag{6.2}$$

where $\mathcal{K}(\omega)$ is some zero order Calderon-Zygmund operator in 2D. Singularity formation of (6.2) is listed as an open problem in [57]. See also [25]. We focus on three specific operators

$$\mathcal{K}(\omega) = \partial_{xy}(-\Delta)^{-1}\omega, \quad \mathcal{K}(\omega) = \partial_x^2(-\Delta)^{-1}\omega, \quad \mathcal{K}(\omega) = \partial_x(-\Delta)^{-1/2}\omega. \tag{6.3}$$

These operators arise naturally in 2D. In fact, we consider the equation of θ_x in (6.1)

$$\partial_t\theta_x + \mathbf{u}\cdot\nabla\theta_x = -u_x\theta_x - v_x\theta_y. \tag{6.4}$$

For the vortex stretching term, we have $-u_x\theta_x = -\partial_{xy}(-\Delta)^{-1}\theta_x \cdot \theta_x$, which relates to the first operator in (6.3). If we remove the advection term and $-v_x\theta_y$ from (6.4), we get (6.2).

The second operator in (6.3) relates to $v_x = \partial_{xx}(-\Delta)^{-1}\omega$ in the 2D Boussinesq equations (1.3). The third operator relates to the SQG (3.5), where the Biot-Savart law is given by $\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1/2}\theta$. The question is the following.

Problem 2. *Prove finite time blowup of* (6.2) *with* $\mathcal{K}(\omega)$ *being one of the operators in* (6.3) *from some* C_c^∞ *initial data.*

Unlike the 1D CLM model (3.4), (6.2) cannot be solved explicitly for general initial data. We believe that our framework provides a promising approach to studying Problems 1, 2.

Based on the solutions to these problems, a potential approach to construct finite time blowup of the full equations is by a continuation argument. One adds the advection back to the equations and then gradually increases the strength of the advection. For strong advection, the singular solution would be unstable. In this case, it would be helpful to consider the equations of the self-similar blowup profile, which is time-independent, rather than the dynamic equations.

General vorticity model

We have shown in this thesis and previous discussions that the competition between the advection and vortex stretching is an essential difficulty for singularity formation in incompressible fluids and can lead to complicated dynamics. In Chapter 5, we construct a finite time blowup solution of the De Gregorio model (3.1) on S^1 from $C^\alpha \cap X$ initial data for any $0 < \alpha < 1$, The blowup result from initial data ω_0 in class X is sharp as we have proved global regularity for $\omega_0 \in C^1 \cap X$. From these results, we better understand the situation where the advection is weaker. In particular, the weakening of the advection is a *local* phenomenon since the initial data $\omega_0 \in C^\infty(S^1 \backslash \{0\}) \cap X$ of the singular solution is only C^α at the origin. This phenomenon is quite surprising since the advection and the vortex stretching are *nonlocal* effects. A natural question is the following.

Problem 3. *How universal are this phenomenon and weakening mechanism? Do they depend on the specific Biot-Savart law?*

In the DG model, the velocity u is determined by the Hilbert transform $u_x = H\omega$. To study the above problem, we can consider the general vorticity model proposed in [38]

$$\omega_t + u\omega_x = u_x \omega, \quad u_x = \mathcal{K}(\omega), \quad x \in S^1, \tag{6.5}$$

where $\mathcal{K}$ is a non-trivial Fourier integral operator with bounded symbol m satisfying $|m(k)| \leq \frac{C}{1+|k|}$. The motivation for this bound on $m(k)$ is that the velocity u should be at least one order more regular than ω to avoid the loss of derivative. In particular, we have $||u_x||_{H^s} \lesssim_s ||\omega||_{H^s}$.

It is conjectured in [38] that (6.5) can blow up from H^s data with $s < \frac{3}{2}$. In the case of Hilbert transform $\mathcal{K} = H$, (6.5) reduces to the DG model (3.1), and we have resolved the conjecture in Chapter 5. The construction of finite time blowup in Chapter 5 is based on perturbing the smooth steady state $\sin(2x)$ of the DG model. We first developed this argument in [14]. For even symbol m, i.e. $\mathcal{K}(\omega)$ is odd if ω is odd, we believe that this argument can be applied to study singularity formation of (6.5) and answer Problem 3 in the context of the 1D vorticity model. Such an answer will allow us to understand the essential feature of the weakening mechanism and then generalize it to full fluids equations.

Global regularity of (6.5) can be investigated using the argument in Chapter 5 with additional assumptions on the operator $\mathcal{K}$.

Global regularity of the De Gregorio model on S^1

In Chapter 5, we have established the global regularity of the DG model on S^1 from initial data $\omega_0 \in H^1 \cap X$ with $\omega_0(x)x^{-1} \in L^\infty$, based on a one-point blowup criterion, where X denotes the class of odd solution ω with $\omega(x,t) \leq 0$ for $x \in [0, \pi/2]$. This result rules out the potential blowup of the DG model on S^1 from smooth initial data in X that provide the most promising blowup candidate up to now.

Our analysis provides valuable insights on the global regularity of (3.1) with more general data, which relates to the Open Problem I in Section 1.4.1. A potential direction is to generalize the one-point blowup criterion to a finite-points version. For simplicity, we assume that the number of zeros of $\omega(x,t)$ is finite, and the zeros are $x_i(t), i = 1, 2, .., n$ with $\partial_x \omega(x_i(t), t) \neq 0$. It is shown in [73] that the number n and $\partial_x \omega(x_i(t), t), i = 1, 2., , .n$ are conserved. Denote $N_{\pm}(t) \triangleq \{x : \omega(x,t) = 0, \ \mathrm{sgn}(\omega_x(x,t)) = \pm 1\}$. A natural generalization of Theorem 5.1 is that the solution of (3.1) cannot be extended beyond T if and only if

$$\int_0^T \sum_{x \in N_-(t)} |u_x(x,t)| dt = \infty. \tag{6.6}$$

A weaker version is that $\sum_{i=1}^n |u_x(x_i(t), t)|$ controls the breakdown of the solution. These blowup criteria are consistent with that of the CLM model. See the discussion in Section 5.1.1. We believe that these criteria are important for establishing global regularity from general smooth initial data.

Passing from (6.6) to the global regularity, one may estimate functionals and quadratic forms similar to those in Section 5.4 in a suitable moving frame. We remark that our proof of Lemma 5.4.2 does not require the assumption of the sign of ω. Thus, it is conceivable that the argument can be adapted to study other scenarios.

BIBLIOGRAPHY

[1] James T. Beale, Tosio Kato, and Andrew J. Majda. Remarks on the breakdown of smooth solutions for the 3-D Euler equations. *Comm. Math. Phys.*, 94(1):61–66, 1984.

[2] Oluş N. Boratav and Richard B. Pelz. Direct numerical simulation of transition to turbulence from a high-symmetry initial condition. *Phys. Fluids (1994-present)*, 6(8):2757–2784, 1994.

[3] Jean Bourgain and Dong Li. Strong illposedness of the incompressible Euler equation in integer C^m spaces. *Geom. Funct. Anal.*, 25(1):1–86, 2015.

[4] Roberto Castelli, Marcio Gameiro, and Jean-Philippe Lessard. Rigorous numerics for ill-posed PDEs: Periodic orbits in the Boussinesq equation. *Arch. Ration. Mech. Anal.*, 228(1):129–157, 2018.

[5] Angel Castro and Diego Córdoba. Infinite energy solutions of the surface quasi-geostrophic equation. *Adv. Math.*, 225(4):1820–1829, 2010.

[6] Angel Castro, Diego Córdoba, Javier Gómez-Serrano, and Alberto Martín Zamora. Remarks on geometric properties of SQG sharp fronts and α-patches. *Discrete Contin. Dyn. Syst.*, 34(12):5045–5059, 2014.

[7] Angel Castro, Diego Córdoba, and Javier Gómez-Serrano. Global smooth solutions for the inviscid SQG equation. *Mem. Amer. Math. Soc.*, 266(1292):v+89, 2020.

[8] Dongho Chae. Nonexistence of self-similar singularities for the 3D incompressible Euler equations. *Comm. Math. Phys.*, 273(1):203–215, 2007.

[9] Dongho Chae. Nonexistence of asymptotically self-similar singularities in the Euler and the Navier–Stokes equations. *Math. Ann.*, 338(2):435–449, 2007.

[10] Dongho Chae. On the generalized self-similar singularities for the Euler and the Navier-Stokes equations. *J. Funct. Anal.*, 258(9):2865–2883, 2010.

[11] Dongho Chae, Sung-Ki Kim, and Hee-Seok Nam. Local existence and blow-up criterion of Hölder continuous solutions of the Boussinesq equations. *Nagoya Mathematical Journal*, 155:55–80, 1999.

[12] Jiajie Chen. Codes for verifications in the paper "On the regularity conjecture of the De Gregorio model for the 3D Euler equations". `https://www.dropbox.com/sh/8qu5otbukb9patp/AAAW9B32nexiB1r4iMWfTFYva?dl=0`.

[13] Jiajie Chen. Singularity formation and global well-posedness for the generalized Constantin-Lax-Majda equation with dissipation. *Nonlinearity*, 33(5):2502–2532, 2020.

[14] Jiajie Chen. On the slightly perturbed De Gregorio model on S^1. *Arch. Ration. Mech. Anal.*, 241(3):1843–1869, 2021.

[15] Jiajie Chen. On the regularity of the De gregorio model for the 3D Euler equations. *To appear in J. Eur. Math. Soc., arXiv preprint arXiv:2107.04777*, 2021.

[16] Jiajie Chen and Thomas Y. Hou. Finite time blowup of 2D Boussinesq and 3D Euler equations with $C^{1,\alpha}$ velocity and boundary. *Comm. Math. Phys.*, 383(3):1559–1667, 2021.

[17] Jiajie Chen, Thomas Y. Hou, and De Huang. Matlab codes for computer-aided proofs in the paper "Asymptotically self-similar blowup of the Hou–Luo model for the 3D Euler equations". `https://www.dropbox.com/sh/qjs6p6d9n3uiq8r/AABCDI-rZeVuTmBxGQuLJbUva?dl=0`.

[18] Jiajie Chen, Thomas Y. Hou, and De Huang. Supplementary materials for the paper "On the finite time blowup of the De Gregorio model for the 3D Euler equations". *arXiv:1905.06387*, 2019.

[19] Jiajie Chen, Thomas Y. Hou, and De Huang. On the finite time blowup of the De Gregorio model for the 3D Euler equations. *Comm. Pure Appl. Math.*, 74(6):1282–1350, 2021.

[20] Jiajie Chen, Thomas Y. Hou, and De Huang. Asymptotically self-similar blowup of the Hou–Luo model for the 3D Euler equations. *arXiv preprint arXiv:2106.05422*, 2021.

[21] Jiajie Chen, Thomas Y. Hou, and De Huang. Supplementary materials for the paper for the paper "Asymptotically self-similar blowup of the Hou–Luo model for the 3D Euler equations". *arXiv preprint arXiv:2106.05422*, 2021.

[22] Kyudong Choi, Alexander Kiselev, and Yao Yao. Finite time blow up for a 1D model of 2D Boussinesq system. *Comm. Math. Phys.*, 334(3): 1667–1679, 2015.

[23] Kyudong Choi, Thomas Y. Hou, Alexander Kiselev, Guo Luo, Vladimir Sverak, and Yao Yao. On the finite-time blowup of a one-dimensional model for the three-dimensional axisymmetric Euler equations. *Comm. Pure Appl. Math.*, 70(11):2218–2243, 2017.

[24] Peter Constantin. On the Euler equations of incompressible fluids. *Bull. Amer. Math. Soc. (N.S.)*, 44(4):603–621, 2007.

[25] Peter Constantin and Weiran Sun. Remarks on Oldroyd-B and related complex fluid models. *Commun. Math. Sci.*, 10(1):33–73, 2012.

[26] Peter Constantin, Peter D. Lax, and Andrew J. Majda. A simple one-dimensional model for the three-dimensional vorticity equation. *Comm. Pure Appl. Math.*, 38(6):715–724, 1985.

[27] Peter Constantin, Charles Fefferman, and Andrew J. Majda. Geometric constraints on potentially singular solutions for the 3-D Euler equations. *Comm. Partial Differential Equations*, 21(3-4):559–571, 1996.

[28] Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos. Formation of singularities for a transport equation with nonlocal velocity. *Ann. of Math. (2)*, 162(3):1377–1389, 2005.

[29] Antonio Córdoba, Diego Córdoba, and Marco A. Fontelos. Integral inequalities for the Hilbert transform applied to a nonlocal transport equation. *J. Math. Pures Appl. (9)*, 86(6):529–540, 2006.

[30] Diego Córdoba, Francisco Gancedo, and Rafael Orive. Analytical behavior of two-dimensional incompressible flow in porous media. *J. Math. Phys.*, 48(6):065206, 2007.

[31] Diego Cordoba, Daniel Faraco, and Francisco Gancedo. Lack of uniqueness for weak solutions of the incompressible porous media equation. *Arch. Ration. Mech. Anal.*, 200(3):725–746, 2011.

[32] Diego Córdoba, Javier Gómez-Serrano, and Andrej Zlatoš. A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable. *Anal. PDE*, 10(2):367–378, 2017.

[33] Salvatore De Gregorio. On a one-dimensional model for the three-dimensional vorticity equation. *J. Statist. Phys.*, 59(5-6):1251–1263, 1990.

[34] Salvatore De Gregorio. A partial differential equation arising in a 1D model for the 3D vorticity equation. *Math. Methods Appl. Sci.*, 19(15): 1233–1255, 1996.

[35] Jian Deng, Thomas Y. Hou, and Xinwei Yu. Geometric properties and nonblowup of 3D incompressible Euler flow. *Comm. Partial Differential Equations*, 30(1-3):225–243, 2005.

[36] Tam Do, Vu Hoang, Maria Radosz, and Xiaoqian Xu. One-dimensional model equations for hyperbolic fluid flow. *Nonlinear Anal.*, 140:1–11, 2016.

[37] Tam Do, Alexander Kiselev, and Xiaoqian Xu. Stability of blowup for a 1D model of axisymmetric 3D Euler equation. *J. Nonlinear Sci.*, 28(6): 2127–2152, 2018.

[38] Theodore D. Drivas and Tarek M. Elgindi. Singularity formation in the incompressible Euler equation in finite and infinite time. *arXiv preprint arXiv:2203.17221*, 2022.

[39] Javier Duoandikoetxea and Javier Duoandikoetxea Zuazo. *Fourier analysis*, volume 29. American Mathematical Soc., 2001.

[40] Weinan E and Chi-Wang Shu. Small-scale structures in Boussinesq convection. *Phys. Fluids*, 6(1):49–58, 1994.

[41] Jens Eggers and Marco A. Fontelos. The role of self-similarity in singularities of partial differential equations. *Nonlinearity*, 22(1):R1–R44, 2009.

[42] Tarek M. Elgindi. Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. *Ann. of Math. (2)*, 194(3): 647–727, 2021.

[43] Tarek M. Elgindi and In-Jee Jeong. Finite-time singularity formation for strong solutions to the axi-symmetric 3D Euler equations. *Ann. PDE*, 5 (2):Paper No. 16, 51, 2019.

[44] Tarek M. Elgindi and In-Jee Jeong. On the effects of advection and vortex stretching. *Arch. Ration. Mech. Anal.*, 235(3):1763–1817, 2020.

[45] Tarek M. Elgindi and In-Jee Jeong. Finite-time singularity formation for strong solutions to the Boussinesq system. *Ann. PDE*, 6(1):Paper No. 5, 50, 2020.

[46] Tarek M. Elgindi and In-Jee Jeong. Symmetries and critical phenomena in fluids. *Comm. Pure Appl. Math.*, 73(2):257–316, 2020.

[47] Tarek M. Elgindi and In-Jee Jeong. The incompressible Euler equations under octahedral symmetry: singularity formation in a fundamental domain. *Adv. Math.*, 393:108091, 2021.

[48] Tarek M. Elgindi, Tej-Eddine Ghoul, and Nader Masmoudi. On the stability of self-similar blow-up for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$. *arXiv preprint arXiv:1910.14071*, 2019.

[49] Tarek M. Elgindi, Tej-eddine Ghoul, and Nader Masmoudi. Stable self-similar blow-up for a family of nonlocal transport equations. *Anal. PDE*, 14(3):891–908, 2021.

[50] Alberto Enciso, Javier Gómez-Serrano, and Bruno Vergara. Convexity of Whitham's highest cusped wave. *arXiv preprint arXiv:1810.10935*, 2018.

[51] Charles L. Fefferman. Existence and smoothness of the Navier-Stokes equation. *The millennium prize problems*, pages 57–67, 2006.

[52] David Gabai, G. Robert Meyerhoff, and Nathaniel Thurston. Homotopy hyperbolic 3-manifolds are hyperbolic. *Ann. of Math. (2)*, 157(2):335–431, 2003.

[53] John D. Gibbon. The three-dimensional Euler equations: Where do we stand? *Phys. D*, 237(14-17):1894–1904, 2008.

[54] David Gilbarg and Neil S. Trudinger. *Elliptic partial differential equations of second order*. Springer, 2015.

[55] Javier Gómez-Serrano. Computer-assisted proofs in PDE: A survey. *SeMA J.*, 76(3):459–484, 2019.

[56] Javier Gómez-Serrano and Rafael Granero-Belinchón. On turning waves for the inhomogeneous Muskat problem: a computer-assisted proof. *Nonlinearity*, 27(6):1471, 2014.

[57] Loukas Grafakos, Malabika Pramanik, Andreas Seeger, Betsy Stovall, et al. Some problems in harmonic analysis. *arXiv preprint arXiv:1701.06637*, 2017.

[58] Rainer Grauer and Thomas Sideris. Numerical computation of 3D incompressible ideal fluids with swirl. *Phys. Rev. Lett.*, 67(25):3511, 1991.

[59] Philippe Gravejat and Didier Smets. Smooth travelling-wave solutions to the inviscid surface quasi-geostrophic equation. *Int. Math. Res. Not. IMRN*, 2019(6):1744–1757, 2019.

[60] Thomas C. Hales. A proof of the Kepler conjecture. *Ann. of Math. (2)*, 162(3):1065–1185, 2005.

[61] Godfrey H. Hardy, John E. Littlewood, and George Pólya. *Inequalities*. Cambridge University Press, Cambridge, 1988.

[62] Siming He and Alexander Kiselev. Small-scale creation for solutions of the SQG equation. *Duke Math. J.*, 1(1):1–15, 2021.

[63] Vu Hoang and Maria Radosz. Cusp formation for a nonlocal evolution equation. *Arch. Ration. Mech. Anal.*, 224(3):1021–1036, 2017.

[64] Vu Hoang and Maria Radosz. Singular solutions for nonlocal systems of evolution equations with vorticity stretching. *SIAM J. Math. Anal.*, 52 (2):2158–2178, 2020.

[65] Vu Hoang, Betul Orcan-Ekmekci, Maria Radosz, and Hang Yang. Blowup with vorticity control for a 2D model of the Boussinesq equations. *J. Differential Equations*, 264(12):7328–7356, 2018.

[66] Thomas Y. Hou. Blow-up or no blow-up? A unified computational and analytic approach to 3D incompressible Euler and Navier-Stokes equations. *Acta Numer.*, 18:277–346, 2009.

[67] Thomas Y. Hou and Zhen Lei. On the stabilizing effect of convection in three-dimensional incompressible flows. *Comm. Pure Appl. Math.*, 62 (4):501–564, 2009.

[68] Thomas Y. Hou and Congming Li. Dynamic stability of the three-dimensional axisymmetric Navier-Stokes equations with swirl. *Comm. Pure Appl. Math.*, 61(5):661–697, 2008.

[69] Thomas Y. Hou and Ruo Li. Dynamic depletion of vortex stretching and non-blowup of the 3-D incompressible Euler equations. *J. Nonlinear Sci.*, 16(6):639–664, 2006.

[70] Thomas Y. Hou and Pengfei Liu. Self-similar singularity of a 1D model for the 3D axisymmetric Euler equations. *Res. Math. Sci.*, 2:Art. 5, 26, 2015.

[71] Thomas Y. Hou and Guo Luo. On the finite-time blowup of a 1D model for the 3D incompressible Euler equations. *arXiv preprint arXiv:1311.2613*, 2013.

[72] Thomas Y. Hou, Tianling Jin, and Pengfei Liu. Potential singularity for a family of models of the axisymmetric incompressible flow. *J. Nonlinear Sci.*, 28(6):2217–2247, 2018.

[73] Hao Jia, Samuel Stewart, and Vladimir Sverak. On the De Gregorio modification of the Constantin–Lax–Majda model. *Arch. Ration. Mech. Anal.*, 231(2):1269–1304, 2019.

[74] Carlos E. Kenig and Frank Merle. Global well-posedness, scattering and blow-up for the energy-critical, focusing, non-linear Schrödinger equation in the radial case. *Invent. Math.*, 166(3):645–675, 2006.

[75] Robert M. Kerr. Evidence for a singularity of the three-dimensional, incompressible Euler equations. *Phys. Fluids A*, 5(7):1725–1746, 1993.

[76] Alexander Kiselev. Regularity and blow up for active scalars. *Math. Model. Nat. Phenom.*, 5(04):225–255, 2010.

[77] Alexander Kiselev. Small scales and singularity formation in fluid dynamics. In *Proceedings of the International Congress of Mathematicians*, volume 3, 2018.

[78] Alexander Kiselev and Vladimir Sverak. Small scale creation for solutions of the incompressible two-dimensional Euler equation. *Ann. of Math. (2)*, 180(3):1205–1220, 2014.

[79] Alexander Kiselev and Changhui Tan. Finite time blow up in the hyperbolic boussinesq system. *Adv. Math.*, 325:34–55, 2018.

[80] Alexander Kiselev, Lenya Ryzhik, Yao Yao, and Andrej Zlatoš. Finite time singularity for the modified SQG patch equation. *Ann. of Math. (2)*, 184(3):909–948, 2016.

[81] Michael J. Landman, George C. Papanicolaou, Catherine Sulem, and Pierre-Louis Sulem. Rate of blowup for solutions of the nonlinear Schrödinger equation at critical dimension. *Phys. Rev. A (3)*, 38(8): 3837–3843, 1988.

[82] Oscar E. Lanford. A computer-assisted proof of the Feigenbaum conjectures. In *Universality in Chaos*, pages 245–252. Routledge, 2017.

[83] Zhen Lei, Jie Liu, and Xiao Ren. On the Constantin-Lax-Majda model with convection. *Comm. Math. Phys.*, 375(1):765–783, 2020.

[84] Dong Li and Jose Rodrigo. Blow-up of solutions for a 1D transport equation with nonlocal velocity and supercritical dissipation. *Adv. Math.*, 217(6):2563–2568, 2008.

[85] Pengfei Liu. *Spatial Profiles in the Singular Solutions of the 3D Euler Equations and Simplified Models*. PhD thesis, California Institute of Technology, 2017. `https://resolver.caltech.edu/CaltechTHESIS:09092016-000915850`.

[86] Guo Luo and Thomas Y. Hou. Toward the finite-time blowup of the 3D axisymmetric Euler equations: A numerical investigation. *Multiscale Model. Simul.*, 12(4):1722–1776, 2014.

[87] Guo Luo and Thomas Y. Hou. Potentially singular solutions of the 3D axisymmetric Euler equations. *Proc. Natl. Acad. Sci.*, 111(36):12968–12973, 2014.

[88] Pavel M. Lushnikov, Denis A. Silantyev, and Michael Siegel. Collapse versus blow-up and global existence in the generalized Constantin-Lax-Majda equation. *J. Nonlinear Sci.*, 31(5):Paper No. 82, 56, 2021.

[89] Andrew J. Majda and Andrea L. Bertozzi. *Vorticity and incompressible flow*, volume 27. Cambridge University Press, 2002.

[90] Yvan Martel, Frank Merle, and Pierre Raphaël. Blow up for the critical generalized Korteweg–de Vries equation. I: Dynamics near the soliton. *Acta Math.*, 212(1):59–140, 2014.

[91] Angel Castro Martınez. Nonlinear and nonlocal models in fluid mechanics. 2010. `https://www.icmat.es/Thesis/ACastro.pdf`.

[92] David W. McLaughlin, George C. Papanicolaou, Catherine Sulem, and Pierre-Louis Sulem. Focusing singularity of the cubic Schrödinger equation. *Phys. Rev. A*, 34(2):1200, 1986.

[93] Frank Merle and Pierre Raphael. The blow-up dynamic and upper bound on the blow-up rate for critical nonlinear Schrödinger equation. *Ann. of Math. (2)*, 161(1):157–222, 2005.

[94] Frank Merle and Hatem Zaag. Stability of the blow-up profile for equations of the type $u_t = \delta u + |u|^{p-1}u$. *Duke Math. J*, 86(1):143–195, 1997.

[95] Frank Merle and Hatem Zaag. On the stability of the notion of non-characteristic point and blow-up profile for semilinear wave equations. *Comm. Math. Phys.*, 333(3):1529–1562, 2015.

[96] Ramon E. Moore, R. Baker Kearfott, and Michael J. Cloud. *Introduction to interval analysis*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2009.

[97] Hisashi Okamoto, Takashi Sakajo, and Marcus Wunsch. On a generalization of the Constantin-Lax-Majda equation. *Nonlinearity*, 21(10): 2447–2461, 2008.

[98] Hisashi Okamoto, Takashi Sakajo, and Marcus Wunsch. Steady-states and traveling-wave solutions of the generalized constantin–lax–majda equation. *Discrete Contin. Dyn. Syst.*, 34(8):3155, 2014.

[99] Siegfried M. Rump. INTLAB - INTerval LABoratory. In Tibor Csendes, editor, *Developments in Reliable Computing*, pages 77–104. Kluwer Academic Publishers, Dordrecht, 1999. `http://www.ti3.tuhh.de/rump/`.

[100] Siegfried M. Rump. Verification methods: rigorous results using floating-point arithmetic. *Acta Numer.*, 19:287–449, 2010.

[101] Steven Schochet. Explicit solutions of the viscous model vorticity equation. *Comm. Pure Appl. Math.*, 39(4):531–537, 1986.

[102] Richard Evan Schwartz. Obtuse triangular billiards II: One hundred degrees worth of periodic trajectories. *Experiment. Math.*, 18(2):137–171, 2009.

[103] Luis Silvestre and Vlad Vicol. On a transport equation with nonlocal drift. *Trans. Amer. Math. Soc.*, 368(9):6159–6188, 2016.

[104] Vladimir Sverak. On certain models in the PDE theory of fluid flows. *Journées Équations aux dérivées partielles*, pages 1–26, 2017.

[105] Victor I. Yudovich. Eleven great problems of mathematical hydrodynamics. *Mosc. Math. J.*, 3(2):711–737, 746, 2003.

[106] Andrej Zlatoš. Exponential growth of the vorticity gradient for the Euler equation on the torus. *Adv. Math.*, 268:396–403, 2015.

Appendix A

APPENDIX TO CHAPTER 2

In Appendix A.0.1, we estimate $\Gamma(\beta)$ and the constant c appeared in the approximate profile (2.44). In Appendix A.0.2, we perform the derivations and establish several inequalities in the linear stability analysis in Section 2.6.6. In Appendix A.0.3, we derive the singular term (2.138) in the elliptic estimates. In Appendix A.0.4, we will establish several estimates of $L_{12}(\Omega)$ that are used frequently in the nonlinear stability analysis. Notice that we only have the formula of $\bar{\eta} = \bar{\theta}_x$ in (2.44). We need to recover $\bar{\theta}, \bar{\xi} = \bar{\theta}_y$ from $\bar{\eta}$ via integration. Yet, we do not have a simple formula to perform integration. Alternatively, we derive useful estimates for $\bar{\xi}$ in Appendix A.0.5. Some estimates of $\bar{\Omega}, \bar{\eta}$ are also obtained there. In Appendix A.0.6, we show that the truncation of the approximate steady state would contribute only to a small perturbation under the norm we use, and we prove Lemma 2.10.1. In Appendix A.0.7, we prove Lemma 2.10.1. In Appendix A.0.8, we study the toy model introduced in [42].

A.0.1 Estimates of $\Gamma(\beta)$ and the constant c

Lemma A.0.1. *For $x \in [0,1]$, the following estimate holds uniformly for $\lambda \geq 1/10$,*

$$(1 - x^{\kappa})x^{\lambda} \leq \frac{\kappa}{\lambda}. \tag{A.1}$$

Consequently, for $\beta \in [0, \pi/2]$, $2 \geq \lambda \geq 1/10$, we have

$$|(\Gamma(\beta) - 1)(\sin(2\beta))^{\lambda}| \lesssim |(\cos^{\alpha}(\beta) - 1)(\cos(\beta))^{\lambda}| \lesssim \alpha,$$

and

$$\Big| c - \frac{2}{\pi} \Big| = \Big| \frac{2}{\pi} \int_0^{\pi/2} (\Gamma(\beta) - 1)\sin(2\beta) d\beta \Big| \leq 2\alpha.$$

Proof. Using change of a variable $t = x^{\kappa}$, it suffices to show that for $t \in [0,1], (1-t)t^{\lambda/\kappa} \leq \frac{\kappa}{\lambda}$. Notice that $\lambda \geq 1/10$ and $t \leq 1$. Using Young's inequality, we derive

$$\begin{aligned}(1-t)t^{\lambda/\kappa} &= \frac{\kappa}{\lambda} \cdot (\frac{\lambda}{\kappa}(1-t))t^{\lambda/\kappa} \\ &\leq \frac{\kappa}{\lambda}\Big(\frac{\frac{\lambda}{\kappa}(1-t) + \frac{\lambda}{\kappa}t}{1 + \frac{\lambda}{\kappa}} \Big)^{1+\lambda/\kappa} = \frac{\kappa}{\lambda}\Big(\frac{\lambda}{\lambda+\kappa}\Big)^{1+\lambda/\kappa} \leq \frac{\kappa}{\lambda},\end{aligned}$$

which implies (A.1). The remaining inequalities in the Lemma follows directly from (A.1). □

A.0.2 Computations in the linear stability analysis

We perform the derivations and establish several inequalities in the linear stability analysis in Section 2.6.6.

The calculations and estimates presented below can also be verified using *M*athematica [1] since we have simple and explicit formulas.

A.0.2.1 Derivations of (2.81)

Recall the formulas of ψ_0, φ_0 in (2.78). A direct calculation yields

$$\begin{aligned}\frac{1}{2}(R\varphi_0)_R - \varphi_0 =& \Big(\frac{1}{2}\Big(R\cdot\frac{(1+R)^3}{R^3}\Big)_R - \frac{(1+R)^3}{R^3}\Big)\sin(2\beta)\\ =& \Big(\frac{1}{2}\Big(-2R^{-3}-3R^{-2}+1\Big) - \frac{(1+R)^3}{R^3}\Big)\sin(2\beta)\\ =& -\Big(2R^{-3}+\frac{9}{2}R^{-2}+3R^{-1}+\frac{1}{2}\Big)\sin(2\beta).\end{aligned}$$

Denote $\psi_0 = A(R)\Gamma(\beta)^{-1}$. For the coefficient in the η integral in (2.81), we have

$$\begin{aligned}\frac{1}{2}(R\psi_0)_R + (-2+\frac{3}{1+R})\psi_0 &= \Big(\frac{1}{2}(RA(R))_R + (-2+\frac{3}{1+R})A(R)\Big)\Gamma(\beta)^{-1}\\ &\triangleq (I+II)\Gamma(\beta)^{-1}.\end{aligned}$$

[1] The Mathematica code for these calculations can be found via the link `https://www.dropbox.com/s/y6vfhxi3pa8okvr/Calpha_calculations.nb?dl=0`.

Note that $A(R) = \frac{3}{16}\left(\frac{(1+R)^3}{R^4} + \frac{3}{2}\frac{(1+R)^4}{R^3}\right)$ (2.78). A direct calculation implies

$$
\begin{aligned}
I &= \frac{3}{32}\left(\frac{(1+R)^3}{R^3} + \frac{3}{2}\frac{(1+R)^4}{R^2}\right)_R \\
&= \frac{3}{32}\left(3\frac{(1+R)^2}{R^3} - 3\frac{(1+R)^3}{R^4} + 6\frac{(1+R)^3}{R^2} - 3\frac{(1+R)^4}{R^3}\right) \\
&= \frac{3}{32}\left(\frac{(1+R)^2}{R^4}(3R - 3(1+R) + 6(1+R)R^2 - 3(1+R)^2 R)\right) \\
&= \frac{3(1+R)^2}{32R^4}(-3 - 3R + 3R^3), \\
II &= \left(-2 + \frac{3}{1+R}\right)\frac{3}{32}\left(2\frac{(1+R)^3}{R^4} + 3\frac{(1+R)^4}{R^3}\right) \\
&= \frac{3(1+R)^2}{32R^4}(-2 - 2R + 3)(2 + 3R(1+R)), \\
I + II &= \frac{3(1+R)^2}{32R^4}(-3 - 3R + 3R^3 + (1-2R)(2 + 3R + 3R^2)) \\
&= \frac{3(1+R)^2}{32R^4}(-1 - 4R - 3R^2 - 3R^3).
\end{aligned}
$$

The above calculations imply (2.81).

A.0.2.2 Derivations of (2.86)

From (2.44), we know

$$
\frac{\bar{\eta} - R\partial_R\bar{\eta}}{\bar{\eta}} = \frac{(1+R)^3}{6R}\Big(\frac{6R}{(1+R)^3} - R\cdot\frac{6}{(1+R)^3} + R\cdot\frac{18R}{(1+R)^4}\Big) = \frac{3R}{1+R}.
$$

Using the above identity, (2.78) and $c_\omega = -\frac{2}{\pi\alpha}L_{12}(\Omega)(0)$ (2.47), we can compute

$$
\begin{aligned}
(\bar{\eta} - R\partial_R\bar{\eta})\psi_0 c_\omega =& \frac{\bar{\eta} - R\partial_R\bar{\eta}}{\bar{\eta}}\frac{9}{8}\frac{\alpha}{c}\Big(R^{-3} + \frac{3}{2}\frac{1+R}{R^2}\Big)c_\omega \\
=& \frac{27\alpha}{8c}\frac{R}{1+R}\Big(R^{-3} + \frac{3}{2}\frac{1+R}{R^2}\Big)c_\omega \\
=& \Big(\frac{27\alpha}{8c}\frac{1}{(1+R)R^2} + \frac{81\alpha}{16c}\frac{1}{R}\Big)\cdot\frac{-2}{\pi\alpha}L_{12}(\Omega)(0) \\
=& \Big(-\frac{27}{4\pi c}\frac{1}{(1+R)R^2} - \frac{81}{8\pi c}\frac{1}{R}\Big)L_{12}(\Omega)(0),
\end{aligned}
$$

which implies (2.86).

A.0.2.3 Derivation of the ODE (2.87) for $L_{12}(\Omega)(0)$

Multiplying $\sin(2\beta)/R$ on both sides of (2.52) and then integrating (2.52), we derive

$$\frac{d}{dt}L_{12}(\Omega)(0) = -\Big\langle R\partial_R\Omega, \frac{\sin(2\beta)}{R}\Big\rangle - L_{12}(\Omega)(0) + c_\omega\Big\langle\bar{\Omega} - R\partial_R\bar{\Omega}, \frac{\sin(2\beta)}{R}\Big\rangle + \Big\langle\eta, \frac{\sin(2\beta)}{R}\Big\rangle - \Big\langle\frac{3}{1+R}D_\beta\Omega, \frac{\sin(2\beta)}{R}\Big\rangle + \Big\langle\mathcal{R}_\Omega, \frac{\sin(2\beta)}{R}\Big\rangle.$$

The first term vanishes by an integration by parts argument. Using (2.44) and (2.47), we can compute the third term

$$\begin{aligned}&c_\omega\Big\langle\bar{\Omega} - R\partial_R\bar{\Omega}, \frac{\sin(2\beta)}{R}\Big\rangle\\ =&\frac{\alpha}{c}c_\omega\int_0^\infty\int_0^{\pi/2}\Gamma(\beta)\frac{6R^2}{(1+R)^3}\cdot\frac{\sin(2\beta)}{R}d\beta dR = \frac{\pi\alpha}{2}c_\omega\int_0^\infty\frac{6R}{(1+R)^3}dR\\ =&3\pi\alpha c_\omega\left(-(1+R)^{-1}+\frac{1}{2}(1+R)^{-2}\right)\Big|_0^\infty = \frac{3\pi\alpha}{2}c_\omega = -3L_{12}(\Omega)(0).\end{aligned}$$

It follows that

$$\frac{d}{dt}L_{12}(\Omega)(0) = -4L_{12}(\Omega)(0) + \Big\langle\eta, \frac{\sin(2\beta)}{R}\Big\rangle - \Big\langle\frac{3\sin(2\beta)}{(1+R)R}, D_\beta\Omega\Big\rangle + \Big\langle\mathcal{R}_\Omega, \frac{\sin(2\beta)}{R}\Big\rangle.$$

Multiplying $\frac{81}{4\pi c}L_{12}(\Omega)(0)$ to the both sides, we derive (2.87).

A.0.2.4 Computations of the integrals in (2.89)

A simple calculation implies that for any $k > 2$

$$\begin{aligned}&\int_0^\infty(1+R)^{-k}dR = \frac{1}{k-1},\\ &\int_0^\infty\frac{R}{(1+R)^k}dR = \int_0^\infty\frac{1}{(1+R)^{k-1}} - \frac{1}{(1+R)^k}dR = \frac{1}{(k-1)(k-2)}.\end{aligned}\tag{A.2}$$

For the integral in β, we get

$$\begin{aligned}\int_0^{\pi/2}(1-2\sin(2\beta))^2d\beta &= \frac{\pi}{2} - 4\int_0^{\pi/2}\sin(2\beta)d\beta + 4\int_0^{\pi/2}(\sin(2\beta))^2d\beta\\ &= \frac{\pi}{2} - 4 + 4\cdot\frac{\pi}{4} = \frac{3\pi}{2} - 4.\end{aligned}$$

Using (A.2) with $k = 4$ and the above calculation, we can compute

$$\begin{aligned}\Big|\Big|\frac{R^{3/2}}{(1+R)^2}\frac{1}{R}(1-2\sin(2\beta))\Big|\Big|_2^2 &= \int_0^\infty\frac{R}{(1+R)^4}dR\cdot\int_0^{\pi/2}(1-2\sin(2\beta))^2d\beta\\ &= \frac{1}{6}(\frac{3\pi}{2} - 4).\end{aligned}$$

Using (A.2), we can calculate

$$\Big|\Big|\frac{R^2}{(1+R)^{3/2}}\cdot\frac{1}{(1+R)R^2}\Big|\Big|_2^2=\int_0^{\pi/2}1d\beta\cdot\int_0^{\infty}(1+R)^{-5}dR=\frac{\pi}{8}.$$

A.0.2.5 Estimates of $D(\Omega), D(\eta)$ and the proof of (2.97)

We introduce

$$\begin{aligned}D_1(\eta)&\triangleq-\frac{3(1+R)^2}{32R^4}(1+4R+3R^2+3R^3),\\D_2(\eta)&\triangleq\left(\frac{3}{16}R^{-3}+\frac{3}{8}\frac{(1+R)^2}{R^2}+\frac{3R}{4(1+R)}\right)+\frac{3}{16}\left(\frac{1}{6}\frac{(1+R)^4}{R^3}+\frac{3}{8}\frac{(1+R)^3}{R^4}\right).\end{aligned}$$

Recall $D(\Omega), D(\eta)$ in (2.96) and the weights φ_0,ψ_0 defined in (2.78). By definition, $D(\eta)=D_1(\eta)\Gamma(\beta)^{-1}+D_2(\eta)$. Thus, (2.97) is equivalent to

$$\sin(2\beta)D(\Omega)\le-\frac{1}{6}\varphi_0,\quad D_1(\eta)\Gamma(\beta)^{-1}+D_2(\eta)\le-\frac{1}{8}\psi_0.\tag{A.3}$$

To prove the first inequality, it suffices to prove

$$D(\Omega)=-2R^{-3}-\frac{9}{2}R^{-2}-3R^{-1}-\frac{1}{2}+\frac{4}{3}R^{-3}+6R^{-2}+\frac{1+R}{3R}\le-\frac{(1+R)^3}{6R^3},$$

which is equivalent to proving

$$(-2+\frac{4}{3}+\frac{1}{6})R^{-3}+(-\frac{9}{2}+6+\frac{1}{2})R^{-2}+(-3+\frac{1}{3}+\frac{1}{2})R^{-1}+(-\frac{1}{2}+\frac{1}{3}+\frac{1}{6})\le0.$$

It is further equivalent to

$$-\frac{1}{2}R^{-3}+2R^{-2}-\frac{13}{6}R^{-1}\le0,$$

which is valid since $2\sqrt{\frac{1}{2}\times\frac{13}{6}}>2$. Hence, we prove the first inequality in (A.3).

For the second inequality in (A.3), firstly, we use $\Gamma(\beta)D_2(\eta)\le D_2(\eta)$ ($\Gamma(\beta)=\cos^{\alpha}(\beta)$ (2.44)) to obtain

$$\begin{aligned}D_3(\eta)\triangleq&D_1(\eta)+D_2(\eta)\Gamma(\beta)\le D_1(\eta)+D_2(\eta)\\=&\frac{3}{16}\Big\{-\frac{(1+R)^2}{2R^4}(1+4R+3R^2+3R^3)\\&+R^{-3}+2\frac{(1+R)^2}{R^2}+\frac{4R}{1+R}+\frac{1}{6}\frac{(1+R)^4}{R^3}+\frac{3}{8}\frac{(1+R)^3}{R^4}\Big\}.\end{aligned}\tag{A.4}$$

Recall the definition of ψ_0 in (2.78). Multiplying both sides of the second inequality in (A.3) by $\Gamma(\beta)$, we obtain that the inequality is equivalent to

$$D_3(\eta) \leq \frac{3}{16}\left(-\frac{1}{8}\frac{(1+R)^3}{R^4} - \frac{3}{16}\frac{(1+R)^4}{R^3}\right). \tag{A.5}$$

We split the negative term in the upper bound of $D_3(\eta)$ in (A.4) as follows

$$\begin{aligned}
&-\frac{(1+R)^2}{2R^4}(1+4R+3R^2+3R^3)\\
&= -\frac{(1+R)^2}{2R^4}\left\{(1+R)+(3R^2)+R(1+R)^2+R(2-2R+2R^2)\right\}\\
&= -\frac{(1+R)^3}{2R^4} - \frac{3}{2}\frac{(1+R)^2}{R^2} - \frac{(1+R)^4}{2R^3} - \frac{(1+R)^2(1-R+R^2)}{R^3}.
\end{aligned}$$

It follows that

$$\begin{aligned}
D_3(\eta) \leq \frac{3}{16}\Big\{&\frac{(1+R)^3}{R^4}\left(-\frac{1}{2}+\frac{3}{8}\right) + \frac{(1+R)^4}{R^3}\left(-\frac{1}{2}+\frac{1}{6}\right)\\
&+\frac{1}{2}\frac{(1+R)^2}{R^2} - \frac{(1+R)^2(1-R+R^2)}{R^3} + \frac{1}{R^3} + \frac{4R}{1+R}\Big\}\\
= \frac{3}{16}\Big\{&-\frac{1}{8}\frac{(1+R)^3}{R^4} - \frac{1}{3}\frac{(1+R)^4}{R^3}\\
&+\frac{1}{2}\frac{(1+R)^2}{R^2} - \frac{(1+R)(1+R^3)}{R^3} + \frac{1}{R^3} + \frac{4R}{1+R}\Big\}.
\end{aligned}$$

Observe that

$$\begin{aligned}
-\frac{1}{3}\frac{(1+R)^4}{R^3} + \frac{1}{2}\frac{(1+R)^2}{R^2} &= -\frac{3}{16}\frac{(1+R)^4}{R^3} + \left(-\frac{7}{48}\frac{(1+R)^4}{R^3} + \frac{1}{2}\frac{(1+R)^2}{R^2}\right)\\
&\leq -\frac{3}{16}\frac{(1+R)^4}{R^3},\\
-\frac{(1+R)(1+R^3)}{R^3} + \frac{1}{R^3} + \frac{4R}{1+R} &= -\frac{1}{R^2} - (1+R) + \frac{4R}{1+R}\\
&= -\frac{1}{R^2} - \frac{(R-1)^2}{(1+R)} \leq 0,
\end{aligned}$$

where we have used $\frac{7}{48}\frac{(1+R)^2}{R} \geq \frac{7}{48} \times 4 \geq 1/2$ to derive the first inequality. Therefore, we prove (A.5), which further implies the second inequality in (A.3).

A.0.3 Derivation of the singular term (2.138) in the elliptic estimates

Suppose that Ψ is the solution of (2.134). Consider $\tilde{\Psi} = \Psi + G\sin(2\beta)$. Notice that if $\alpha = 0$, $\sin(2\beta)$ is the kernel of the operator $\mathcal{L}_\alpha$ in (2.134) (it is self-adjoint if $\alpha = 0$). We have

$$\mathcal{L}_\alpha(\tilde{\Psi}) = \Omega + \mathcal{L}_\alpha(G\sin(2\beta)) = \Omega - (\alpha^2 R^2 \partial_{RR} G + \alpha(\alpha+4)R\partial_R G)\sin(2\beta).$$

We look for $G(R)$ that satisfies $G(R) \to 0$ as $R \to +\infty$ and $\mathcal{L}_\alpha(\tilde{\Psi})$ is orthogonal to $\sin(2\beta)$:

$$0 = \int_0^{\pi/2} \sin(2\beta)(\Omega - (\alpha^2 R^2 \partial_{RR} G + \alpha(\alpha+4) R \partial_R G) \sin(2\beta)) d\beta$$

for every R, which implies

$$\alpha^2 R^2 \partial_{RR} G + \alpha(\alpha+4) R \partial_R G = \frac{4}{\pi} \Omega_*, \tag{A.6}$$

where $\Omega_*(R) = \int_0^{\pi/2} \Omega(R,\beta) \sin(2\beta) d\beta$ and we have used $\int_0^{\pi/2} \sin^2(2\beta) d\beta = \frac{\pi}{4}$. The above ODE is first order with respect to $\partial_R G$ and can be solved explicitly. Multiplying the integrating factor $\frac{1}{\alpha^2} R^{-2+\frac{4+\alpha}{\alpha}}$ to both sides and then integrating from 0 to R yield

$$R^{\frac{4+\alpha}{\alpha}} \partial_R G = \frac{4}{\alpha^2 \pi} \int_0^R \Omega_*(t) t^{\frac{4}{\alpha}-1} dt.$$

Imposing the vanishing condition $G(R) \to 0$ as $R \to +\infty$, we yield

$$G = -\frac{4}{\alpha^2 \pi} \int_R^\infty s^{-\frac{4+\alpha}{\alpha}} \int_0^s \Omega_*(t) t^{\frac{4}{\alpha}-1} dt ds.$$

Using integration by parts, we further derive

$$\begin{aligned} G &= \frac{1}{\alpha\pi} \int_R^\infty \partial_s (s^{-\frac{4}{\alpha}}) \int_0^s \Omega_*(t) t^{\frac{4}{\alpha}-1} dt ds \\ &= -\frac{1}{\alpha\pi} \int_R^\infty \frac{\Omega_*(s)}{s} ds - \frac{1}{\alpha\pi} R^{-\frac{4}{\alpha}} \int_0^R \Omega_*(s) s^{\frac{4}{\alpha}-1} ds. \end{aligned}$$

Using the above formula and the notation $L_{12}(\Omega)$ (2.31), we derive (2.138).

A.0.4 Estimates of $L_{12}(\Omega)$

Recall $\tilde{L}_{12}(\Omega) = L_{12}(\Omega) - L_{12}(\Omega)(0)$. We have the following important cancellation between $\tilde{L}_{12}(\Omega)$ and Ω.

Lemma A.0.2. *For $k \in [3/2, 4]$ and any $\lambda > 0$, we have*

$$\begin{aligned} \langle \sin(2\beta)\Omega \tilde{L}_{12}(\Omega), R^{-k} \rangle &= -\frac{k-1}{2} \left|\left| \tilde{L}_{12}(\Omega) R^{-k/2} \right|\right|_{L^2(R)}^2, \\ \langle (\sin(2\beta)\Omega + \lambda \tilde{L}_{12}(\Omega))^2, R^{-k} \rangle &= \langle R^{-k} (\sin(2\beta))^2, \Omega^2 \rangle \\ &\quad -((k-1)\lambda - \frac{\pi}{2}\lambda^2) \left|\left| \tilde{L}_{12}(\Omega) R^{-k/2} \right|\right|_{L^2(R)}^2. \end{aligned} \tag{A.7}$$

Proof. From the definition of $\tilde{L}_{12}(\omega)(R)$ in (2.55), we know that it does not depend on β and

$$\int_0^{\pi/2} \Omega(s,\beta)\sin(2\beta)d\beta = -(\partial_R \tilde{L}_{12}(R))R.$$

Using integration by parts, we obtain

$$\begin{aligned}\langle \sin(2\beta)\Omega\tilde{L}_{12}(\Omega), R^{-k}\rangle &= \int_0^\infty (-(\partial_R\tilde{L}_{12}(R))R)\tilde{L}_{12}(\Omega)R^{-k}dR \\ &= -\frac{k-1}{2}\int_0^\infty \tilde{L}_{12}(\Omega)^2 R^{-k}dR,\end{aligned}$$

which is the first identity in (A.7). The second identity in (A.7) follows from $\langle \tilde{L}_{12}^2(\Omega), R^{-k}\rangle = \frac{\pi}{2}||\tilde{L}_{12}(\Omega)R^{-k/2}||^2_{L^2(R)}$ and the first identity. ☐

To estimate $\tilde{L}_{12}(\Omega)g$ in $\mathcal{L}_i$, we use the following simple Lemma.

Lemma A.0.3. *Let g be some function depending on $\bar{\Omega}, \bar{\eta}, \bar{\xi}$ and φ be some weights. We have*

$$\begin{aligned}\langle \tilde{L}_{12}^2(\Omega)g^2, \varphi\rangle &\lesssim ||R^{-1}\tilde{L}_{12}(\Omega)||^2_{L^2(R)}\Big|\Big|\int_0^{\pi/2} R^2 g^2(R,\beta)\varphi(R,\beta)d\beta\Big|\Big|_{L^\infty(R)}, \\ \langle (D_R^k\tilde{L}_{12}(\Omega))^2 g^2, \varphi\rangle &\lesssim ||R^{-1}D_R^{k-1}\Omega||^2_{L^2}\Big|\Big|\int_0^{\pi/2} R^2 g^2(R,\beta)\varphi(R,\beta)d\beta\Big|\Big|_{L^\infty(R)},\end{aligned} \tag{A.8}$$

for $k \geq 1$, provided that the upper bound is well-defined, where $D_R = R\partial_R$.

Proof. The first inequality follows directly from that $\tilde{L}_{12}(\Omega)$ does not dependent on β. Recall the definition of $\tilde{L}_{12}(\Omega)$ in (2.55) and $D_R = R\partial_R$. Notice that for $k \geq 1$, we have

$$D_R^k\tilde{L}_{12}(\Omega) = -\int_0^{\pi/2} D_R^{k-1}\Omega(R,\beta)\sin(2\beta)d\beta.$$

Using the Cauchy-Schwarz inequality, we prove

$$\begin{aligned}\langle (D_R^k\tilde{L}_{12}(\Omega))^2 g^2, \varphi\rangle &= \int_0^\infty \left((\int_0^{\pi/2} D_R^{k-1}\Omega(R,\beta)\sin(2\beta)d\beta)^2 \int_0^{\pi/2} g^2\varphi d\beta \right) dR \\ &\lesssim \int_0^\infty (\int_0^{\pi/2} (D_R^{k-1}\Omega)^2 d\beta)(\int_0^{\pi/2} g^2\varphi d\beta)dR \\ &\leq ||R^{-1}D_R^{k-1}\Omega||^2_{L^2}\Big|\Big|\int_0^{\pi/2} R^2 g^2\varphi(R,\beta)d\beta\Big|\Big|_{L^\infty(R)}.\end{aligned}$$

☐

Lemma A.0.4. *Let $\chi(\cdot) : [0,\infty) \to [0,1]$ be a smooth cutoff function, such that $\chi(R) = 1$ for $R \leq 1$ and $\chi(R) = 0$ for $R \geq 2$. For $k = 1, 2$, we have*

$$
\begin{aligned}
&||L_{12}(\Omega)||_{L^\infty} \lesssim ||\frac{1+R}{R}\Omega||_{L^2}, \quad ||\tilde{L}_{12}(\Omega)(R^{-2}+R^{-3})^{1/2}||^2_{L^2(R)} \lesssim ||\Omega\frac{(1+R)^2}{R^2}||^2_{L^2}, \\
&||L_{12}(\Omega)||_2 \lesssim ||\Omega||_2, \quad ||\frac{(1+R)^k}{R^k}(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)||_{L^2(R)} \lesssim ||\frac{(1+R)^k}{R^k}\Omega||_{L^2}.
\end{aligned}
\tag{A.9}
$$

provided that the right hand side is bounded. Moreover, if $\Omega \in \mathcal{H}^3$, then for $0 \leq k \leq 3, 0 \leq l \leq 2$, we have

$$
\begin{aligned}
&||L_{12}(\Omega) - L_{12}(\Omega)(0)\chi||_{\mathcal{H}^3} + ||D_R(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)||_{\mathcal{H}^3} \lesssim ||\Omega||_{\mathcal{H}^3}, \\
&||D_R^k L_{12}(\Omega)||_\infty + ||D_R^k(L_{12}(\Omega) - \chi L_{12}(\Omega)(0))||_\infty \lesssim ||\Omega||_{\mathcal{H}^3}, \\
&||(1+R)\partial_R D_R^l L_{12}(\Omega)||_\infty + ||(1+R)\partial_R D_R^l(L_{12}(\Omega) - \chi L_{12}(\Omega)(0))||_\infty \lesssim ||\Omega||_{\mathcal{H}^3}, \\
&||L_{12}(\Omega)||_X + ||D_R L_{12}(\Omega)||_X \lesssim ||\Omega||_{\mathcal{H}^3},
\end{aligned}
\tag{A.10}
$$

where $X \triangleq \mathcal{H}^3 \oplus \mathcal{W}^{5,\infty}$ is defined in (2.140).

Remark A.0.5. We subtract $\chi L_{12}(\Omega)(0)$ near $R = 0$ since $L_{12}(\Omega)$ does not vanishes at $R = 0$.

Proof. Recall $L_{12}(\Omega)$ in (2.31) and $\tilde{L}_{12}(\Omega)$ in (2.55). Using the Cauchy-Schwarz and the Hardy inequality, we get

$$
\begin{aligned}
&||L_{12}(\Omega)||_{L^\infty} \lesssim \langle|\Omega|, \frac{1}{R}\rangle \lesssim ||\frac{1+R}{R}\Omega||_{L^2}||\frac{1}{1+R}||_{L^2(R)} \lesssim ||\frac{1+R}{R}\Omega||_{L^2}, \\
&||\frac{1}{R^l}\tilde{L}_{12}(\Omega)||_{L^2(R)} \lesssim \int_0^\infty \frac{1}{R^{2l}}\tilde{L}^2_{12}(\Omega)dR \lesssim \int_0^\infty \frac{1}{R^{2l-2}}(\partial_R \tilde{L}_{12}(\Omega))^2 dR \lesssim \langle\Omega^2, R^{-2l}\rangle,
\end{aligned}
\tag{A.11}
$$

for $l = 1, \frac{3}{2}, 2$, which implies the first two inequalities in (A.9). For $k = 1, 2$, observe that

$$
\begin{aligned}
&||\frac{(1+R)^k}{R^k}(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)||_{L^2(R)} \\
\lesssim &||\frac{(1+R)^k}{R^k}\tilde{L}_{12}(\Omega)\chi||_{L^2(R)} + ||\frac{(1+R)^k}{R^k}L_{12}(\Omega)(1-\chi)||_{L^2(R)} \\
\lesssim &||\frac{1}{R^k}\tilde{L}_{12}(\Omega)||_{L^2(R)} + ||L_{12}(\Omega)||_{L^2(R)} \lesssim ||\Omega\frac{(1+R)^k}{R^k}||_{L^2} + ||L_{12}(\Omega)||_{L^2(R)},
\end{aligned}
$$

where we have used (A.11) in the last inequality. Denote $\Omega_* = \int_0^{\pi/2}\Omega d\beta$. From (2.31), we know

$$
L_{12}(\Omega)(R) = \int_R^\infty \frac{\Omega_*(S)}{S}dS = \int_0^\infty K(R,S)\Omega_*(S)dS, \quad K(R,S) = \frac{1}{S}\mathbf{1}_{R\leq S}.
$$

The L^2 boundedness of L_{12} is standard. Notice that K is homogeneous of degree -1, i.e. $K(\lambda R, \lambda S) = \lambda^{-1} K(R,S)$ for $\lambda > 0$. Using change of a variable $S = Rz$, we get

$$L_{12}(\Omega)(R) = \int_0^\infty \frac{1}{R} K(1,z)\Omega_*(Rz) R dz = \int_0^\infty K(1,z)\Omega_*(Rz)dz.$$

Then, the Minkowski inequality implies

$$\begin{aligned} ||L_{12}(\Omega)||_{L^2} &\leq \int_0^\infty K(1,z)||\Omega_*(Rz)||_{L^2(R)}dz \lesssim \int_0^\infty K(1,z)z^{-1/2}||\Omega||_{L^2}dz \\ &= ||\Omega||_{L^2}\int_{z\geq 1} z^{-3/2}dz \lesssim ||\Omega||_{L^2}. \end{aligned}$$

We complete the proof of (A.9). Notice that $D_R L_{12}(\Omega) = -\Omega_*, ||D_R^k \chi||_{L^2} \lesssim 1$ for $1 \leq k \leq 4$ and $D_\beta L_{12}(\Omega) = 0, D_\beta \chi = 0$. Using that $\sin(2\beta)^{-\sigma}$ in the weight $\varphi_1 = \sin(2\beta)^{-\sigma}\frac{(1+R)^4}{R^4}$ is integrable in the β direction and (A.9), we yield

$$\begin{aligned} &||(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)\varphi_1^{1/2}||_{L^2} + ||D_R^k(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)\varphi_1^{1/2}||_{L^2} \\ \lesssim &||(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)\frac{(1+R)^2}{R^2}||_{L^2} + ||D_R^k(L_{12}(\Omega) - L_{12}(\Omega)(0)\chi)\frac{(1+R)^2}{R^2}||_{L^2} \\ \lesssim &||\Omega\frac{(1+R)^2}{R^2}||_{L^2} + ||D_R^{k-1}\Omega_*\frac{(1+R)^2}{R^2}||_{L^2} + |L_{12}(\Omega)(0)|||D_R^k\chi\frac{(1+R)^2}{R^2}||_{L^2} \\ \lesssim &||\Omega\frac{(1+R)^2}{R^2}||_{L^2} + ||D_R^{k-1}\Omega\frac{(1+R)^2}{R^2}||_{L^2} \lesssim ||\Omega||_{\mathcal{H}^3}, \end{aligned} \tag{A.12}$$

which implies the first estimate in (A.10). From the definition of $L_{12}(\Omega)$ in (2.31), we have $D_R L_{12}(\Omega) = L_{12}(D_R\Omega)$. Notice that $|D_R^k\chi(R)| \lesssim 1$. Using (A.9), we prove for $k \leq 3$

$$||D_R^k L_{12}(\Omega)||_{L^\infty} + |L_{12}(\Omega)(0)| \cdot ||D_R^k\chi||_{L^\infty} \lesssim ||\Omega||_{\mathcal{H}^3},$$

which implies the second estimate in (A.10). Similarly, since $\partial_R D_R^l L_{12}(\Omega) = \partial_R L_{12}(D_R^l\Omega) = -R^{-1}D_R^l\Omega_*(R)$, where $\Omega_*(R) = \int_0^{\pi/2}\Omega(R,\beta)d\beta$, and that $l \leq 2$, we have

$$\begin{aligned} ||\partial_R D_R^l L_{12}(\Omega)||_{L^\infty} &= ||R^{-1}D_R^l\Omega_*||_{L^\infty(R)} \\ &\lesssim ||R^{-1}D_R^l\Omega_*||_{L^2(R)}^{1/2}||\partial_R(R^{-1}D_R^l\Omega_*)||_{L^2(R)}^{1/2} \lesssim ||\Omega||_{\mathcal{H}^3}, \end{aligned}$$

which along with the second estimate in (A.10) and $|\partial_R D_R^l \chi L_{12}(\Omega)(0)| \lesssim |L_{12}(\Omega)(0)| \lesssim ||\Omega||_{\mathcal{H}^3}$ completes the proof of the third estimate in (A.10).

Since $\chi L_{12}(\Omega)(0)$ does not depend on β, we apply the first two estimates in (A.10) to yield

$$\begin{aligned}||D_R^i L_{12}(\Omega)||_X &\leq ||D_R^i (L_{12}(\Omega) - \chi L_{12}(\Omega)(0))||_{\mathcal{H}^3} + ||D_R^i \chi L_{12}(\Omega)(0)||_{\mathcal{W}^{5,\infty}} \\ &\lesssim ||\Omega||_{\mathcal{H}^3} + |L_{12}(\Omega)(0)| \lesssim ||\Omega||_{\mathcal{H}^3}\end{aligned}$$

for $i = 0, 1$. We complete the proof of (A.10). □

A.0.5 Estimate of the approximate self-similar solution

In Appendix A.0.5.1, we estimate some norm of $\bar{\Omega}, \bar{\eta}$ using the explicit formulas. For $\bar{\xi}$, it is given by an integration of $\bar{\eta}$ that does not have an explicit formula. We estimates $\bar{\xi}$, its derivatives and some norm in subsection A.0.5.2.

A.0.5.1 Estimate of $\bar{\Omega}, \bar{\eta}$

Recall the formula of $\bar{\Omega}, \bar{\eta}$ in (2.44). A simple calculation yields

$$\bar{\Omega} = \frac{\alpha}{c}\frac{3R\Gamma(\beta)}{(1+R)^2}, \ \bar{\eta} = \frac{\alpha}{c}\frac{6R\Gamma(\beta)}{(1+R)^3}, \ \bar{\Omega} - D_R\bar{\Omega} = \frac{\alpha}{c}\frac{6R^2\Gamma(\beta)}{(1+R)^3}, \ \bar{\eta} - D_R\bar{\eta} = \frac{\alpha}{c}\frac{18R^2\Gamma(\beta)}{(1+R)^4}. \tag{A.13}$$

Without specification, in later sections, we assume that $R \geq 0, \beta \in [0, \pi/2]$.

Lemma A.0.6. *The following results apply to any $k \leq 3, 0 \leq i + j \leq 3, j \neq 1$.*
(a) For $f = \bar{\Omega}, \bar{\eta}, \bar{\Omega} - D_R\bar{\Omega}, \bar{\eta} - D_R\bar{\eta}$, we have

$$|D_R^k f| \lesssim f, \quad |D_R^i D_\beta^j f| \lesssim \alpha \sin(\beta) f. \tag{A.14}$$

(b) Let φ_i be the weights defined in (2.61)*. For $g = \bar{\Omega}, \bar{\eta}$, we have*

$$\int_0^{\pi/2} R^2 (D_R^k g)^2 \varphi_1 d\beta \lesssim \alpha^2, \quad \int_0^{\pi/2} R^2 (D_R^i D_\beta^j g)^2 \varphi_2 d\beta \lesssim \alpha^3, \tag{A.15}$$

uniformly in R and

$$\langle (D_R^k (g - D_R g))^2, \varphi_1 \rangle \lesssim \alpha^2, \quad \langle (D_R^i D_\beta^j (g - D_R g))^2, \varphi_2 \rangle \lesssim \alpha^3. \tag{A.16}$$

Proof. Recall $D_\beta = \sin(2\beta)\partial_\beta, D_R = R\partial_R$. Using $\Gamma(\beta) = \cos(\beta)^\alpha$, (2.69) and a direct calculation gives

$$\begin{aligned}&|D_\beta^j \Gamma(\beta)| \lesssim \alpha \sin(\beta)\Gamma(\beta), \\ &|D_R^i \frac{R}{(1+R)^m}| \lesssim \frac{R}{(1+R)^m}, \quad |D_R^i \frac{R^2}{(1+R)^m}| \lesssim \frac{R^2}{(1+R)^m}.\end{aligned} \tag{A.17}$$

for $1 \leq j \leq 5$, $0 \leq i \leq 5$ and $m = 2, 3, 4$. Combining these estimates and the formulas in (A.13) implies (A.14). As a result, we have the following pointwise estimates for $g = \bar{\Omega}$ or $\bar{\eta}$

$$
\begin{aligned}
&|D_R^k g| \lesssim g \lesssim \alpha \Gamma(\beta) \frac{R}{(1+R)^2}, \quad |D_R^i D_\beta^j g| \lesssim \alpha \sin(\beta) g \lesssim \alpha^2 \sin(\beta)\Gamma(\beta)\frac{R}{(1+R)^2}, \\
&|D_R^k(g - D_R g)| \lesssim g - D_R g \lesssim \alpha \frac{R^2\Gamma(\beta)}{(1+R)^3}, \\
&|D_R^i D_\beta^j (g - D_R g)| \lesssim \alpha \sin(\beta)(g - D_R g) \lesssim \alpha^2 \sin(\beta) \frac{R^2 \Gamma(\beta)}{(1+R)^3},
\end{aligned}
$$

for $k \leq 3$, $i + j \leq 3, j \neq 0$, where we have used $c \approx \frac{2}{\pi}$ in Lemma A.0.1. Recall φ_i in Definition 2.6.2.

$$
\varphi_1 \triangleq (1+R)^4 R^{-4} \sin(2\beta)^{-\sigma}, \quad \varphi_2 \triangleq (1+R)^4 R^{-4} \sin(2\beta)^{-\gamma}.
$$

Notice that for $\sigma = \frac{99}{100}, \gamma = 1 + \frac{\alpha}{10}$, we have

$$
\begin{aligned}
&\int_0^{\pi/2} \Gamma(\beta)^2 \sin(2\beta)^{-\sigma} d\beta \lesssim 1, \\
&\int_0^{\pi/2} \alpha^2 \sin(\beta)^2 \Gamma(\beta)^2 \sin(2\beta)^{-\gamma} d\beta \lesssim \alpha^2 \int_0^{\pi/2} \cos(\beta)^{2\alpha - 1 - \alpha/10} d\beta \lesssim \alpha.
\end{aligned}
$$

Combining the pointwise estimates, the estimates of the angular integral and a simple calculation then gives (A.15), (A.16). □

Recall the $\mathcal{W}^{l,\infty}$ norm in (2.139). We have

Proposition A.0.7. *It holds true that* $\Gamma(\beta), \bar{\Omega}, \bar{\eta} \in \mathcal{W}^{7,\infty}$ *with*

$$
\begin{aligned}
&||\Gamma(\beta)||_{\mathcal{W}^{7,\infty}} \lesssim 1, \quad ||\frac{(1+R)^2}{R}\bar{\Omega}||_{\mathcal{W}^{7,\infty}} + ||\frac{(1+R)^2}{R}\bar{\eta}||_{\mathcal{W}^{7,\infty}} \lesssim \alpha, \\
&||D_\beta \bar{\Omega}||_{\mathcal{W}^{7,\infty}} + ||D_\beta \bar{\eta}||_{\mathcal{W}^{7,\infty}} \lesssim \alpha^2.
\end{aligned}
$$

Proof. It follows from the calculation A.17 and $\sin(\beta)\Gamma(\beta)\sin(2\beta)^{-\alpha/5} \lesssim 1$. □

A.0.5.2 Estimates of $\bar{\xi}$

Recall that the approximate self-similar profile $\bar{\eta}$ (2.44) is given by

$$
(\bar{\theta}_x)(x, y) = \bar{\eta}(R, \theta) = \frac{\alpha}{c} \frac{6R}{(1+R)^3} \cos^\alpha(\beta) = \frac{6\alpha}{c} \frac{x^\alpha}{(1 + (x^2+y^2)^{\alpha/2})^3}. \tag{A.18}
$$

We also use $\bar{\eta}(x,y)$ to denote the above expression. Throughout this section, we use the following notation

$$R = (x^2+y^2)^{\alpha/2}, \quad \beta = \arctan(y/x), \quad S = (z^2+y^2)^{\alpha/2}, \quad \tau = \arctan(y/z), \tag{A.19}$$

where z will be used in the integral. $\bar{\theta}(x,y)$, $\bar{\xi}(R,\theta) = \bar{\theta}_y(x,y)$ can be obtained from $\bar{\eta}(x,y)$ (or $\bar{\theta}_x$) as follows

$$\bar{\theta} = 1 + \int_0^x \bar{\eta}(z,y)dz, \quad \bar{\xi} = \bar{\theta}_y = \int_0^x \bar{\eta}_y(z,y)dz. \tag{A.20}$$

We can choose $\bar{\theta}(0,y) \equiv c$ for other constant $c > 0$, and it does not change $\nabla\bar{\theta}$. Observe that

$$\begin{aligned} \bar{\eta}_y(z,y) &= -\frac{6\alpha}{c} \cdot \frac{3\alpha y}{y^2+z^2} \frac{(z^2+y^2)^{\alpha/2} z^\alpha}{(1+(z^2+y^2)^{\alpha/2})^4} \\ &= -\frac{1}{z}\frac{3\alpha yz}{y^2+z^2}\frac{(z^2+y^2)^{\alpha/2}}{1+(z^2+y^2)^{\alpha/2}}\bar{\eta}(z,y) = -\frac{1}{z}\frac{3\alpha\sin(2\tau)S}{2(1+S)}\bar{\eta}, \end{aligned} \tag{A.21}$$

where we have used the notation S, τ defined in (A.19). Hence, we get

$$\bar{\xi} = \int_0^x -\frac{6\alpha}{c} \cdot \frac{3\alpha y}{y^2+z^2} \frac{(z^2+y^2)^{\alpha/2} z^\alpha}{(1+(z^2+y^2)^{\alpha/2})^4} dz = \int_0^x \frac{1}{z}\left(-\frac{3\alpha\sin(2\tau)S}{2(1+S)}\bar{\eta}\right) dz. \tag{A.22}$$

These integrals cannot be calculated explicitly for general α. We have the following estimates for $\bar{\xi}$.

Lemma A.0.8. *Assume that* $0 \le \alpha \le \frac{1}{1000}$. *For* $R \ge 0, \beta \in [0,\pi/2]$ *and* $0 \le i+j \le 5$, *we have*

$$|D_R^i D_\beta^j \bar{\xi}| \lesssim -\bar{\xi}, \quad |D_R^i D_\beta^j (3\bar{\xi} - R\partial_R\bar{\xi})| \lesssim -\bar{\xi}, \tag{A.23}$$

$$\begin{aligned} |\bar{\xi}| &\lesssim \frac{\alpha^2(x^2+y^2)^{\alpha/2}}{(1+(x^2+y^2)^{\alpha/2})}\frac{y^\alpha}{(1+y^\alpha)^3}\min\left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right) \\ &\lesssim \frac{\alpha^2 R^2}{1+R}\left(\mathbf{1}_{\beta<\pi/4}\frac{\sin^\alpha(\beta)}{(1+R\sin^\alpha(\beta))^3} + \mathbf{1}_{\beta\ge\pi/4}\frac{\cos^{\alpha+1}(\beta)}{(1+R)^3}\right), \\ -\bar{\xi} &\lesssim \alpha^2\cos(\beta), \quad ||\bar{\xi}||_{\mathcal{C}^1} \lesssim ||\frac{1+R}{R}(1+(R\sin(2\beta)^\alpha)^{-\frac{1}{40}})\bar{\xi}||_{L^\infty} \lesssim \alpha^2, \end{aligned} \tag{A.24}$$

where $||\cdot||_{\mathcal{C}^1}$ *is defined in* (2.117). *Let* ψ_1, ψ_2 *be the weights defined in* (2.61). *We have*

$$\int_0^{\pi/2} R^2 (D_R^i D_\beta^j \bar{\xi})^2 \psi_k d\beta \lesssim \alpha^4 \tag{A.25}$$

uniformly in R, *and*

$$\langle (D_R^i D_\beta^j (3\bar{\xi} - R\partial_R\bar{\xi}))^2, \psi_k\rangle \lesssim \alpha^4, \quad \langle (D_R^i D_\beta^j \bar{\xi})^2, \psi_k\rangle \lesssim \langle \bar{\xi}^2, \psi_k\rangle \lesssim \alpha^4, \tag{A.26}$$

where $(D_R^i D_\beta^j, \psi_k)$ *represents* (D_R^i, ψ_1) *for* $0 \leq i \leq 5$, *and* $(D_R^i D_\beta^j, \psi_2)$ *for* $i + j \leq 5, j \geq 1$.

Remark A.0.9. Using (A.22), we have $-\bar{\xi} \geq 0$ for $R \geq 0, \beta \in [0, \pi/2]$.

We have several commutator estimates which enable us to exchange the derivative and integration in (A.22) so that we can estimate $D_R^i D_\beta^j \bar{\xi}$ easily.

Recall the relation between ∂_x, ∂_y and $\partial_R, \partial_\beta$ in (2.24). We have the following relation

$$D_R = R\partial_R = \frac{1}{\alpha}(x\partial_x + y\partial_y), \quad D_\beta = \sin(2\beta)\partial_\beta = 2y\partial_y - 2\alpha \sin^2(\beta) D_R. \tag{A.27}$$

The first relation holds because $R = r^\alpha, R\partial_R = \frac{1}{\alpha} r\partial_r$, and the second relation is obtained by multiplying $\partial_y = \frac{\sin(\beta)}{r}\alpha D_R + \frac{\cos(\beta)}{r}\partial_\beta$ by y and then using $y/r = \sin(\beta), x/r = \cos(\beta)$.

Lemma A.0.10. *Suppose that* $f(0, y) = 0$ *for any* y. *Denote*

$$I(f)(x, y) = \int_0^x \frac{1}{z} f(z, y) dz. \tag{A.28}$$

We have

$$D_R I(f)(x, y) = I(D_S f)(x, y), \tag{A.29}$$

$$D_\beta I(f)(x, y) - I(D_\tau f)(x, y) = -2\alpha \sin^2(\beta) \cdot I(D_S f) + 2\alpha I(\sin^2(\tau) D_S f), \tag{A.30}$$

where R, β, S, τ *are defined in* (A.19), *provided that* f *is sufficiently smooth.*

Proof. Notice that $y\partial_y$ commutes with the z integral. From (A.27), it suffices to prove

$$x\partial_x I(f)(x, y) = I(z\partial_z f).$$

A direct calculation yields

$$x\partial_x I(f)(x, y) = x\partial_x (\int_0^x \frac{1}{z} f(z, y) dz) = f(x, y),$$
$$I(z\partial_z f)(x, y) = \int_0^x \frac{1}{z} \cdot z\partial_z f(z, y) dz = f(x, y).$$

It follows (A.29). Using the fact that both $y\partial_y$ and $R\partial_R$ commute with the z integral and the formula of D_β (A.27) twice, we derive

$$\begin{aligned} D_\beta I(f)(x, y) &= (2y\partial_y - 2\alpha \sin^2(\beta) D_R) I(f) = I(2y\partial_y f) - 2\alpha \sin^2(\beta) I(D_S f) \\ &= I(D_\tau f + 2\alpha \sin^2(\tau) D_S f) - 2\alpha \sin^2(\beta) I(D_S f) \\ &= I(D_\tau f) + 2\alpha I(\sin^2(\tau) D_S f) - 2\alpha \sin^2(\beta) I(D_S f). \end{aligned}$$

Identity (A.30) follows by rearranging the above identity. □

Next, we prove Lemma A.0.8.

Proof of Lemma A.0.8. Step 1. Recall $D_R = R\partial_R, D_\beta = \sin(2\beta)\partial_\beta$. First, we show that

$$|D_R^i D_\beta^j \bar{\xi}| \lesssim \alpha \int_0^x \frac{1}{z} \sin(2\tau) \frac{S}{1+S} \bar{\eta}(z,y) dz \asymp -\bar{\xi} \tag{A.31}$$

for $0 \leq i+j \leq 5$. Using $\Gamma(\beta) = \cos(\beta)^\alpha$,(2.69) and a direct calculation yields

$$\left| D_R^i \frac{R^2}{(1+R)^4} \right| \lesssim \frac{R^2}{(1+R)^4}, \quad |D_\beta^i \Gamma(\beta)| \lesssim \alpha \sin(\beta)\Gamma(\beta), \quad |D_\beta^i \sin(2\beta)| \lesssim \sin(2\beta) \tag{A.32}$$

for $i \leq 5$. Denote

$$f(S,\tau) = \frac{3\alpha}{2} \sin(2\tau) \frac{S}{1+S} \bar{\eta} = \frac{9\alpha^2}{c} \sin(2\tau)\Gamma(\tau) \frac{S^2}{(1+S)^4}. \tag{A.33}$$

We remark that $f = -z\bar{\eta}_y(z,y)$ according to (A.21). Obviously, $f(S,\tau) \geq 0$. Using the above estimates, we get

$$|D_S^i D_\tau^j f| \lesssim f \tag{A.34}$$

for $i+j \leq 5$. Notice that (A.22) implies $\bar{\xi} = -I(f)$ and that $I(\cdot)$ (A.28) is a positive linear operator for $x \geq 0$. We further derive

$$|I(D_S^i D_\tau^j f|)| \leq I(|D_S^i D_\tau^j f|) \lesssim I(f) \tag{A.35}$$

for $i+j \leq 5$. Using (A.29) and the above estimates, we yield

$$|D_R^i \bar{\xi}| = |D_R^i I(f)| = |I(D_S^i f)| \lesssim I(f).$$

For other derivatives $D_R^i D_\beta^j$ with $j \geq 1, i+j \leq 5$, we estimate $D_\beta^2 \bar{\xi}$, which is representative. Using (A.30), we have

$$\begin{aligned}
D_\beta^2 \bar{\xi} =& D_\beta^2 I(f) = D_\beta \left(I(D_\tau f) - 2\alpha \sin^2(\beta) \cdot I(D_S f) + 2\alpha I(D_S f \sin^2(\tau)) \right) \\
=& I(D_\tau^2 f) - 2\alpha \sin^2(\beta) \cdot I(D_S D_\tau(f)) + 2\alpha I(\sin^2(\tau) D_S D_\tau f) \\
&+ D_\beta \left(-2\alpha \sin^2(\beta) \cdot I(D_S f) \right) + D_\beta \left(2\alpha I(D_S f \sin^2(\tau)) \right) \\
=& J_1 + J_2 + J_3 + J_4 + J_5.
\end{aligned}$$

For J_1, J_2, J_3, we simply use $\sin^2(\beta), \sin^2(\tau) \leq 1$ and (A.35) to obtain

$$I_1, J_2, J_3 \lesssim I(|D_R^i D_\tau^j f|) \lesssim I(f) \tag{A.36}$$

for $(i,j) = (0,2),(1,1),(1,1)$, respectively. For J_4, if D_β acts on $\sin^2(\beta)$, we obtain $\alpha D_\beta(\sin^2(\beta)) \cdot I(D_S f)$, which can be bounded as before using (A.35). For the remaining parts in J_4 and J_5, D_β acts on $I(\cdot)$, and we can use (A.30) again to obtain several terms. Each term can be bounded using (A.35) and an argument similar to (A.36). The estimates of other derivatives $D_R^i D_\beta^j$ can be done similarly. We omit these estimates. Since the right hand side of (A.31) is $\frac{2}{3}I(f) = -\frac{2}{3}\bar{\xi} \asymp -\bar{\xi}$, the above estimates imply (A.31).

Step 2. The estimate (A.31) can be generalized to $i+j \le 6$ easily. Hence, we get

$$|D_R^i D_\beta^j (3\bar{\xi} - R\partial_R \bar{\xi})| \lesssim |D_R^i D_\beta^j \bar{\xi}| + |D_R^{i+1} D_\beta^j \bar{\xi}| \lesssim -\bar{\xi},$$

for any $i+j \le 5$, which proves (A.23).

Step 3: Pointwise estimate. In this step, we prove (A.24). From (A.22), we know that the first inequality in (A.24) is equivalent to

$$\int_0^x \frac{y}{y^2+z^2} \frac{z^\alpha (y^2+z^2)^{\alpha/2}}{(1+(y^2+z^2)^{\alpha/2})^4} dz \lesssim \frac{(x^2+y^2)^{\alpha/2}}{(1+(x^2+y^2)^{\alpha/2})} \frac{y^\alpha}{(1+y^\alpha)^3} \min\left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right).$$

For $z \in [0,x]$, we have $z^2+y^2 \le x^2+y^2$. Since $\frac{t}{1+t}$ is increasing with respect to $t \ge 0$, we yield

$$\frac{(y^2+z^2)^{\alpha/2}}{1+(y^2+z^2)^{\alpha/2}} \lesssim \frac{(y^2+x^2)^{\alpha/2}}{1+(y^2+x^2)^{\alpha/2}}.$$

Therefore, it suffices to prove

$$J(x,y) \triangleq \int_0^x \frac{y}{y^2+z^2} \frac{z^\alpha}{(1+(y^2+z^2)^{\alpha/2})^3} dz \lesssim \frac{y^\alpha}{(1+y^\alpha)^3} \min\left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right). \tag{A.37}$$

Case 1 : $x \le 1+y$ Observe that

$$J \le \frac{1}{(1+y^\alpha)^3} \int_0^x \frac{y z^\alpha}{y^2+z^2} dz = \frac{y^\alpha}{(1+y^\alpha)^3} \int_0^{\frac{x}{y}} \frac{t^\alpha}{1+t^2} dt,$$

where we have used change of a variable $z = yt$ to derive the identity. Since $\alpha \le 1/10$, we get

$$\int_0^{\frac{x}{y}} \frac{t^\alpha}{1+t^2} dt \le \int_0^\infty \frac{t^\alpha}{1+t^2} dt \lesssim 1, \quad \int_0^{\frac{x}{y}} \frac{t^\alpha}{1+t^2} dt \le \int_0^{\frac{x}{y}} t^\alpha dt \lesssim \frac{x^{1+\alpha}}{y^{1+\alpha}}.$$

Combining the above estimates, we prove (A.37) for $x \le 1+y$.

Case 2 : $x > 1+y$ Firstly, we have

$$J(x,y) = \int_0^{1+y} \frac{y}{y^2+z^2}\frac{z^\alpha}{(1+(y^2+z^2)^{\alpha/2})^3}dz + \int_{1+y}^{x} \frac{y}{y^2+z^2}\frac{z^\alpha}{(1+(y^2+z^2)^{\alpha/2})^3}dz$$
$$\triangleq J_1 + J_2.$$

We apply the result in Case 1 to estimate J_1

$$J(1+y,y) \lesssim \frac{y^\alpha}{(1+y^\alpha)^3}\min\left(1, \frac{(1+y)^{1+\alpha}}{y^{1+\alpha}}\right) \lesssim \frac{y^\alpha}{(1+y^\alpha)^3}.$$

For J_2, we have

$$J_2 \leq \int_{1+y}^{x} \frac{y}{y^2+z^2}\frac{z^\alpha}{z^{3\alpha}}dz = y^{-2\alpha}\int_{\frac{1+y}{y}}^{\frac{x}{y}} \frac{t^{-2\alpha}}{1+t^2}dt \lesssim y^{-2\alpha}\int_{\frac{1+y}{y}}^{\infty} t^{-2\alpha-2}dt$$
$$\lesssim y^{-2\alpha}\left(\frac{1+y}{y}\right)^{-1-2\alpha} = \frac{y}{(1+y)^{1+2\alpha}} = \frac{y^\alpha}{(1+y)^{3\alpha}}\frac{y^{1-\alpha}}{(1+y)^{1-\alpha}} \lesssim \frac{y^\alpha}{(1+y^\alpha)^3},$$

where we have used change of a variable $z = yt$ to derive the first identity. Noting that $x \geq y$ in this case. We conclude

$$J(x,y) = J_1 + J_2 \lesssim \frac{y^\alpha}{(1+y^\alpha)^3} \leq \frac{y^\alpha}{(1+y^\alpha)^3}\min\left(1, \frac{x^{1+\alpha}}{y^{1+\alpha}}\right).$$

Combining the above two cases, we prove (A.37), which implies the first inequality in (A.24).

Finally, we prove the second inequality in (A.24). Using the notation (A.19), we have

$$R = (x^2+y^2)^{\alpha/2}, \quad y^\alpha = R\sin^\alpha(\beta),$$
$$\frac{(x^2+y^2)^{\alpha/2}}{1+(x^2+y^2)^{\alpha/2}} = \frac{R}{1+R}, \quad \frac{y^\alpha}{(1+y^\alpha)^3} = \frac{R\sin^\alpha(\beta)}{(1+R\sin^\alpha(\beta))^3}.$$

For $x \leq y$, we have $\beta \geq \pi/4,\ 1 \lesssim \sin(\beta),\ x^2+y^2 \lesssim y^2$. Hence,

$$\frac{y^\alpha}{(1+y^\alpha)^3}\frac{x^{1+\alpha}}{y^{1+\alpha}} \lesssim \frac{y^\alpha}{(1+(x^2+y^2)^{\alpha/2})^3}\frac{x^{1+\alpha}}{y^{1+\alpha}} = \frac{R\sin^\alpha(\beta)}{(1+R)^3}\cdot\frac{\cos^{1+\alpha}(\beta)}{\sin^{1+\alpha}(\beta)} \lesssim \frac{R\cos^{1+\alpha}(\beta)}{(1+R)^3}.$$

Combining the above identity and the estimate, we prove the second inequality in (A.24). The last inequality in (A.24) follows directly from (A.23) and the first two inequalities in (A.24).

Step 4: Estimates of the integral Now, we are in a position to prove (A.25) and (A.26). We are going to prove

$$\int_0^{\pi/2} \bar{\xi}^2(R,\beta)\psi_k d\beta \lesssim \frac{\alpha^4}{(1+R)^2}. \tag{A.38}$$

Clearly, (A.25) and (A.26) follow from the above estimate and (A.23).

Notice that ψ_i defined in (2.61) satisfies

$$\psi_1, \psi_2 \lesssim \frac{(1+R)^4}{R^4} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma}, \tag{A.39}$$

where $\gamma = 1 + \frac{\alpha}{10}, \sigma = \frac{99}{100}$. Using (A.24), $1 + R\sin^\alpha(\beta) \geq (1+R)\sin^\alpha(\beta)$, we yield

$$\begin{aligned}
&(1+R)^2 \int_0^{\pi/2} |\bar{\xi}|^2 \psi_k d\beta \\
\lesssim&(1+R)^2 \frac{\alpha^4 R^4}{(1+R)^2} \cdot \Big\{ \int_0^{\pi/4} \frac{\sin^{2\alpha}(\beta)}{((1+R)\sin^\alpha(\beta))^6} \psi_k d\beta + \int_{\pi/4}^{\pi/2} \frac{\cos^{2\alpha+2}}{(1+R)^6} \psi_k d\beta \Big\} \\
\lesssim& \frac{\alpha^4 R^4}{(1+R)^6} \frac{(1+R)^4}{R^4} \Big\{ \int_0^{\pi/4} \sin(\beta)^{-4\alpha} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma} \\
&+ \int_{\pi/4}^{\pi/2} \cos(\beta)^{2+2\alpha} \sin(\beta)^{-\sigma} \cos(\beta)^{-\gamma} d\beta \Big\} \\
\lesssim& \alpha^4 \Big(\int_0^{\pi/4} \sin(\beta)^{-\sigma-4\alpha} d\beta + \int_{\pi/4}^{\pi/2} \cos(\beta)^{2+2\alpha-\gamma} d\beta \Big) \lesssim \alpha^4,
\end{aligned}$$

where we have used $\alpha \leq \frac{1}{1000}$, $4\alpha + \sigma < \frac{199}{200}$, $2 + 2\alpha - \gamma \geq 1$, to derive the last inequality which does not depend on α for $\alpha \leq \frac{1}{1000}$. It follows (A.38). □

A.0.6 Other Lemmas

We use the following Lemma to construct small perturbation.

Lemma A.0.11. *Let $\chi(\cdot) : [0, \infty) \to [0, 1]$ be a smooth cutoff function, such that $\chi(R) = 1$ for $R \leq 1$ and $\chi(R) = 0$ for $R \geq 2$. Denote*

$$\chi_\lambda(R) = \chi(R/\lambda), \quad \bar{\Omega}_\lambda = \chi_\lambda \bar{\Omega}, \quad \bar{\eta}_\lambda = \partial_x(\chi_\lambda \bar{\theta}), \quad \bar{\xi}_\lambda = \partial_y(\chi_\lambda \bar{\theta}), \tag{A.40}$$

where $\bar{\theta}$ is obtained in (A.20). *We have*

$$\begin{aligned}
&\lim_{\lambda \to +\infty} ||\bar{\Omega}_\lambda - \bar{\Omega}||_{\mathcal{H}^3} + ||(1+R)(\bar{\eta}_\lambda - \bar{\eta})||_{\mathcal{H}^3} + ||\bar{\xi}_\lambda - \bar{\xi}||_{\mathcal{H}^3(\psi)} = 0, \\
&\limsup_{\lambda \to +\infty} ||\bar{\xi}_\lambda - \bar{\xi}||_{\mathcal{C}^1} \leq K_{10}\alpha^2,
\end{aligned} \tag{A.41}$$

where $K_{10} > 0$ is some absolute constant. In particular, we also have

$$\lim_{\lambda \to +\infty} L_{12}^2(\bar{\Omega}_\lambda - \bar{\Omega})(0) + \langle (\bar{\Omega}_\lambda - \bar{\Omega})^2, \varphi_0 \rangle + \langle (\bar{\eta}_\lambda - \bar{\eta})^2, \psi_0 \rangle = 0. \tag{A.42}$$

We need a Lemma similar to Lemma A.0.10.

Lemma A.0.12. *Suppose that $f(0,y)=0$ for any y. Denote $J(f)(x,y) = \frac{1}{z}\int_0^x f(z,y)dz$. We have*

$$D_R J(f)(x,y) = J(D_S f)(x,y),$$
$$D_\beta J(f)(x,y) - J(D_\tau f)(x,y) = -2\alpha \sin^2(\beta)\cdot J(D_S f) + 2\alpha J(\sin^2(\tau) D_S f),$$

where R,β,S,τ are defined in (A.19), *provided that f is sufficiently smooth.*

The first identity follows from a direct calculation and the proof of the second is similar to that in Lemma A.0.10. We omit the proof.

Proof of Lemma A.0.11. Step 1: Estimate of $J(\bar\eta)$. Using (A.20) and the operator J in Lemma A.0.12, we get $\bar\theta = 1 + xJ(\bar\eta)$. We have the following estimate for $J(\bar\eta)$

$$|D_R^i D_\beta^j J(\bar\eta)| \lesssim J(\bar\eta) = \frac{1}{x}\int_0^x \bar\eta(z,y)dz \lesssim \bar\eta, \tag{A.43}$$

for $0 \le i+j \le 5$. The proof of the first inequality follows from Lemma A.0.12 and the argument in the proof of (A.31). The proof of the second inequality is similar to that of (A.37) by considering $x \le 1+y$ and $x > 1+y$. We omit the proof.

Step 2: Estimate of $\bar\eta_\lambda - \bar\eta, \bar\xi_\lambda - \bar\xi$. Recall $\bar\eta_\lambda = \partial_x(\chi_\lambda\bar\theta)$, $\bar\xi_\lambda = \partial_y(\chi_\lambda\bar\theta)$ and the formula of ∂_x, ∂_y (2.24). A direct calculation yields

$$\begin{aligned}
\bar\eta_\lambda(R,\beta) - \bar\eta &= \alpha\frac{\cos(\beta)}{r}D_R\chi_\lambda\cdot\bar\theta + (\chi_\lambda - 1)\bar\eta \\
&= \alpha\cos^2(\beta)D_R\chi_\lambda\cdot J(\bar\eta) + \alpha\frac{\cos(\beta)}{r}D_R\chi_\lambda + (\chi_\lambda-1)\bar\eta, \\
\bar\xi_\lambda(R,\beta) - \bar\xi &= \alpha\frac{\sin(\beta)}{r}D_R\chi_\lambda\cdot\bar\theta + (\chi_\lambda - 1)\bar\xi \\
&= \alpha\sin(\beta)\cos(\beta)D_R\chi_\lambda\cdot J(\bar\eta) + \alpha\frac{\sin(\beta)}{r}D_R\chi_\lambda + (\chi_\lambda-1)\bar\xi,
\end{aligned} \tag{A.44}$$

where we have used $\partial_x\bar\theta = \bar\eta, \partial_y\bar\theta = \bar\xi,\ \bar\theta = 1 + xJ(\bar\eta) = 1 + r\cos(\beta)J(\bar\eta)$. From (A.40), we have

$$D_\beta\chi_\lambda = 0, \quad |D_R\chi_\lambda| = (R/\lambda)|\chi'(R/\lambda)| \lesssim 1.$$

Similarly, we have

$$|D_R^k\chi_\lambda| \lesssim 1, \tag{A.45}$$

for $k = 1,2,3,4$. For $G_\lambda(R,\beta) = \alpha\frac{g(\beta)}{r} D_R\chi_\lambda$ with $g(\beta) = \sin(\beta)$ or $\cos(\beta)$ and $i + j \leq 3$, since $R \geq \lambda$ in $\mathrm{supp}(D_R^k\chi_\lambda)$ and $r = R^{-1/\alpha}$, we get

$$|D_\beta^i D_R^j G_\lambda| \lesssim_\alpha R^{-1/\alpha}\mathbf{1}_{R\geq\lambda}, \quad |D_\beta^{i+1} D_R^j G_\lambda| \lesssim_\alpha \sin(2\beta) R^{-1/\alpha}\mathbf{1}_{R\geq\lambda}.$$

Recall the $\mathcal{C}^1$ norm (2.117) and $\mathcal{H}^3$ (2.136). Using the fast decay of G_λ in R and the smoothness in β, we get

$$\lim_{\lambda\to\infty} ||(1+R)G_\lambda||_{\mathcal{H}^3} = 0, \quad \lim_{\lambda\to\infty} ||G_\lambda||_{\mathcal{C}^1} = 0. \tag{A.46}$$

Notice that $\partial_R\chi_\lambda$, $(\chi_\lambda - 1) = 0$ for $R \leq \lambda$. From the formula of $\bar\eta$ and (A.26) in Lemma A.0.8, we know $(\chi_1 - 1)(1+R)\bar\eta \in \mathcal{H}^3$ ($\bar\eta$ decays R^{-2} for large R) and $\bar\xi \in \mathcal{H}^3(\psi)$. Using the estimates of $J(\bar\eta)$ in (A.43), we also have $(\chi_1 - 1)J(\bar\eta) \in \mathcal{H}^3 \subset \mathcal{H}^3(\psi)$. Therefore, applying (A.44), (A.45) to χ_λ, (A.46), the fact that the $\mathcal{H}^3$ norm is stronger than the $\mathcal{H}^3(\psi)$ norm (2.129), and the Dominated Convergence Theorem yields

$$\lim_{\lambda\to\infty} ||(1+R)(\bar\eta_\lambda - \bar\eta)||_{\mathcal{H}^3} = 0, \quad \lim_{\lambda\to\infty} ||\bar\xi_\lambda - \bar\xi||_{\mathcal{H}^3(\psi)} = 0.$$

Similarly, we have

$$\lim_{\lambda\to\infty} ||\bar\Omega_\lambda - \bar\Omega||_{\mathcal{H}^3} = 0.$$

Using (A.43), (A.45) and the fact that $\bar\eta$ decays for large R (see (2.44)), we have

$$\limsup_{\lambda\to\infty} ||\sin(\beta)\cos(\beta) D_R\chi_\lambda \cdot J(\bar\eta)||_{\mathcal{C}^1} = 0.$$

Using (A.23)-(A.24) in Lemma A.0.8, and (A.44)-(A.46), we conclude

$$||(\chi_\lambda - 1)\bar\xi||_{\mathcal{C}^2} \lesssim \alpha^2, \quad \overline{\lim}_{\lambda\to+\infty} ||\bar\xi_\lambda - \bar\xi||_{\mathcal{C}^1} \lesssim \alpha^2.$$

We complete the proof of (A.41).

Recall that the $\mathcal{H}^3$ norm is stronger than $L^2(\varphi_1)$. Using Lemma A.0.4 for $L_{12}(\Omega)(0)$, the fact that $\varphi_0 \lesssim \varphi_1, \psi_0 \lesssim (1+R)\varphi_1$ (see Definition 2.6.2, 2.6.7) and the limit obtained in (A.41), we prove (A.42). □

Let $C^{\frac{\alpha}{40}}$ be the standard Hölder space. Recall the $\mathcal{C}^1$ norm defined in (2.117). We have the following embedding.

Lemma A.0.13. *Suppose that $f \in \mathcal{C}^1(R,\beta)$ and $f(R,\pi/2) = 0$ for $R \geq 0$. We have*

$$||f||_{C^{\frac{\alpha}{40}}} \leq C_\alpha ||f||_{\mathcal{C}^1}$$

for some constant C_α depending on α only.

Proof. Recall the relation between the Cartesian coordinates (x, y) and the polar coordinates $(r, \beta), (R, \beta)$. Since f vanishes on the axis $\beta = \frac{\pi}{2}$. It suffices to prove that f is Hölder in $\mathbb{R}^2_{++}$. Let $(R_1, \beta_1), (R_2, \beta_2)$ be arbitrary two different points in $\mathbb{R}^2_{++}$, i.e. $R_1, R_2 \geq 0, \beta_1, \beta_2 \in [0, \pi/2]$, and $r_1 = R_1^{1/\alpha}, r_2 = R_2^{1/\alpha}$. Without loss of generality, we assume $R_1 \leq R_2$, $\beta_1 \leq \beta_2$ and $||f||_{\mathcal{C}^1} = 1$. From (2.117), we have $|f| \leq 1, |\partial_R f| \leq \frac{1}{1+R}$, $|\partial_\beta f| \leq R^{1/40} \sin(2\beta)^{\alpha/40-1}$. Using

$$\sin(2\beta)^{\alpha/40-1} \lesssim (\sin(\beta)^{\alpha/40-1} + \cos(\beta)^{\alpha/40-1}) \lesssim (\beta^{\alpha/40-1} + (\pi/2 - \beta)^{\alpha/40-1})$$

and the estimates of the derivatives, we obtain

$$\begin{aligned}
&|f(R_1, \beta_1) - f(R_1, \beta_2)| \\
\leq &\int_{\beta_1}^{\beta_2} |\partial_\beta f(R_1, \beta)| d\beta \leq C R_1^{\frac{1}{40}} \int_{\beta_1}^{\beta_2} \left(\beta^{\frac{\alpha}{40}-1} + (\frac{\pi}{2} - \beta)^{\frac{\alpha}{40}-1}\right) d\beta \\
\leq &C_\alpha R_1^{\frac{1}{40}} (\beta_2^{\frac{\alpha}{40}} - \beta_1^{\frac{\alpha}{40}} + (\frac{\pi}{2} - \beta_1)^{\frac{\alpha}{40}} - (\frac{\pi}{2} - \beta_2)^{\frac{\alpha}{40}}) \leq C_\alpha R_1^{\frac{1}{40}} |\beta_2 - \beta_1|^{\frac{\alpha}{40}}, \\
&|f(R_1, \beta_2) - f(R_2, \beta_2)| \\
\leq &\int_{R_1}^{R_2} |\partial_R f(R, \beta_2)| dR \leq \int_{R_1}^{R_2} \frac{1}{1+R} dR = \log \frac{1+R_2}{1+R_1} \lesssim (R_2 - R_1)^{1/40},
\end{aligned}$$

where we have used $\log \frac{1+R_2}{1+R_1} \leq \log(1 + R_2 - R_1)$ and $\log(1+x) \lesssim x^{1/40}$ for $x \geq 0$ in the last inequality. The distance d between two points is

$$\begin{aligned}
d^2 &= (r_1 \cos(\beta_1) - r_2 \cos(\beta_2))^2 + (r_1 \sin(\beta_1) - r_2 \sin(\beta_2))^2 \\
&= (r_1 - r_2)^2 + 2 r_1 r_2 (1 - \cos(\beta_1 - \beta_2)) \\
&= |R_1^{1/\alpha} - R_2^{1/\alpha}|^2 + 4 R_1^{1/\alpha} R_2^{1/\alpha} \sin(\frac{1}{2}(\beta_1 - \beta_2))^2 \\
&\geq C_\alpha (|R_1 - R_2|^{2/\alpha} + R_1^{2/\alpha} |\beta_1 - \beta_2|^2),
\end{aligned}$$

where we have used $R_1 \leq R_2$ in the last inequality. Using triangle inequality and the above estimates, we prove $|f(R_1, \beta_1) - f(R_2, \beta_2)| \lesssim C_\alpha d^{\frac{\alpha}{40}}$. □

A.0.7 Proof of Lemma 2.10.1

Proof of Lemma 2.10.1. We simplify ω^θ as ω and denote by $\vartheta = \arctan(x_2/x_1)$ the angular variable. Recall the cylinder $D_1 = \{(r, z) : r \in [0, 1], |z| \leq 1\}$. We extend $\omega \mathbf{1}_{(r,z) \in D_1}$ to $\mathbb{R}^3$ as follows :

$$\omega_e(r, z) = \omega(r, z) \text{ for } (r, z) \in D_1, \quad \omega_e(r, z) = 0 \text{ for } (r, z) \notin D_1. \tag{A.47}$$

Note that ω_e is only supported in D_1, which is different from ω. Denote

$$\omega_\pm = \max(\pm\omega_e, 0), \quad \mathcal{L} = -\partial_{rr} - \frac{1}{r}\partial_r - \partial_{zz} + \frac{1}{r^2}, \quad \Delta = \partial_{rr} + \frac{1}{r}\partial_r + \partial_{zz} + \frac{1}{r^2}\partial_{\vartheta\vartheta},$$

$$\psi_\pm(r,z) = \frac{1}{4\pi}\int_0^\infty \int_{-\infty}^\infty \int_0^{2\pi} \frac{\sin(\vartheta)\omega_\pm(r_1,z_1)}{\left((z-z_1)^2 + r^2 + r_1^2 - 2\sin(\vartheta) r r_1\right)^{1/2}} r_1 dr_1 dz_1 d\vartheta, \tag{A.48}$$

where Δ is the Laplace operator in $\mathbb{R}^3$ in cylindrical coordinates. Clearly, $\psi_\pm$ solve the Poisson equation in $\mathbb{R}^3$: $-\Delta(\sin(\vartheta)\psi_\pm(r,z)) = \omega_\pm(r,z)\sin(\vartheta)$, which can be verified easily using the Green function of $-\Delta$. Since $\omega_\pm \geq 0$, using the above formula and $\frac{\sin(\vartheta)}{\left((z-z_1)^2+r^2+r_1^2-2\sin(\vartheta)rr_1\right)^{1/2}} - \frac{\sin(\vartheta)}{\left((z-z_1)^2+r^2+r_1^2+2\sin(\vartheta)rr_1\right)^{1/2}} \geq 0$ for $\vartheta \in [0,\pi]$, we get $\psi_\pm \geq 0$.

Let $\tilde{\psi}$ be a solution of (2.179)-(2.180). By definition of $\mathcal{L}$, we have

$$-\Delta(\tilde{\psi}\sin(\vartheta)) = \sin(\vartheta)\mathcal{L}\tilde{\psi} = \omega\sin(\vartheta).$$

Consider the domain $D_1^+ = \{(r,z,\vartheta) : r \in [0,1], |z| \leq 1, \vartheta \in [0,\pi]\}$, which is a half of the cylinder D_1. Next, we compare $\tilde{\psi}\sin(\vartheta)$ and $\psi_+\sin(\vartheta)$ in D_1^+ using the maximal principle for the Laplace operator Δ.

Recall from (A.47) that $\omega_e = \omega$ in $D_1^+ \subset D_1$. For $(r,z,\vartheta) \in D_1^+$, we have $\sin(\vartheta) \geq 0$ and

$$-\Delta((\tilde{\psi} - \psi_+)\sin(\vartheta)) = (\omega - \omega_+)\sin(\vartheta) \leq 0. \tag{A.49}$$

On the boundary of ∂D_1^+, we have $\vartheta \in \{0,\pi\}$, $r = 1$ or $z \in \{-1,1\}$. The boundary related to $\vartheta \in \{0,\pi\}$ is $\{(r,z,\vartheta) : r \in [0,1], |z| \leq 1, \vartheta = 0,\pi\}$, or equivalently $\{(x,y,z) : |x| \leq 1, y = 0, |z| \leq 1\}$ in the Cartesian coordinates. It contains the symmetry axis $r = 0$. Recall that $\tilde{\psi}$ is odd and 2-periodic in z. We obtain (2.181) $\tilde{\psi}(r,\pm 1) = 0$. Recall the boundary condition (2.180) $\tilde{\psi}(1,z) = 0$ and the fact that ψ^+ is nonnegative. We have

$$(\tilde{\psi} - \psi_+)\sin(\vartheta) = 0 \quad \text{for} \quad \vartheta \in \{0,\pi\},$$
$$(\tilde{\psi} - \psi_+)\sin(\vartheta) \leq 0 \quad \text{for} \quad r = 1 \ \text{ or } \ z \in \{-1,1\},$$

where we have used $\sin(\vartheta) \geq 0$ in D_1^+. Applying the maximal principle to (A.49) in the bounded domain D_1^+, we yield $(\tilde{\psi}(r,z) - \psi_+(r,z))\sin(\vartheta) \leq 0$ in D_1^+, which further implies $\tilde{\psi}(r,z) \leq \psi_+(r,z)$ for $r \leq 1, |z| \leq 1$. Similarly, we have $\tilde{\psi} + \psi_- \geq 0$. Hence $|\tilde{\psi}| \leq \psi_+ + \psi_-$.

Recall from (A.47),(A.48) that $\mathrm{supp}(\omega_\pm) \subset \mathrm{supp}(\omega) \cap D_1$ and the assumption $\mathrm{supp}(\omega) \cap D_1 \subset \{(r,z) : (r-1)^2 + z^2 < 1/4\}$ in Lemma 2.10.1. Thus, for $r > \frac{1}{4}$, (r_1, z_1) in the support of $\omega_\pm$ and $|\vartheta| \leq \pi$, we have $r_1 > \frac{1}{2}$ and

$$
\begin{aligned}
(z-z_1)^2 + r^2 + r_1^2 - 2\cos(\vartheta) r r_1 &= (z-z_1)^2 + (r-r_1)^2 + 4\sin^2(\vartheta/2) r r_1 \\
&\asymp (((z-z_1)^2 + (r-r_1)^2)^{1/2} + |\vartheta|)^2.
\end{aligned}
$$

We have similar estimate with $\cos(\vartheta)$ replaced by $\sin(\vartheta)$. Using this estimate and integrating the ϑ variable in the integral about $\psi_\pm$ in (A.48), we complete the proof. □

Remark A.0.14. The above proof can also be established in the Cartesian coordinates, which is essentially the same up to change of variables.

A.0.8 A toy model for the 2D Boussinesq equations

We consider the toy model introduced in [42]

$$
\begin{aligned}
&\omega_t - (x_1\lambda(t), -x_2\lambda(t)) \cdot \nabla\omega = \partial_1\theta, \\
&\theta_t - (x_1\lambda(t), -x_2\lambda(t)) \cdot \nabla\theta = 0, \quad \lambda(t) = \int_{\mathbb{R}^2} \frac{y_1 y_2}{|y|^4}\omega(y,t)dy,
\end{aligned}
$$

where $\partial_1\theta = \partial_{x_1}\theta$. This model can be derived from the 2D Boussinesq equations by approximating the velocity (u,v) by $u_{x_1}(0,0,t)\cdot(x_1, -x_2)$ and rescaling the solution by a constant. Assume that ω is odd in x_1 and x_2, and θ is even in x_1 and odd in x_2. We show that for initial data $\omega_0, \nabla\theta_0 \in C_c^\alpha(\mathbb{R}_2)$, the solution exists globally. We follow the argument in [42]. Without loss of generality, we assume $\mathrm{supp}(\partial_1\theta_0) \subset [-1,1]^2$. Using the derivation in [42], we get

$$
\begin{aligned}
&\omega(x_1,x_2,t) = (\partial_1\theta_0)(\mu(t)x_1, \frac{x_2}{\mu(t)}) \int_0^t \mu(s)ds, \quad \mu(t) \triangleq \exp(\int_0^t \lambda(s)ds), \\
&\frac{\dot{\mu}}{\mu} = 4\int_0^t \mu(s)ds J(t), \quad J(t) \triangleq \int_0^\infty \int_0^\infty \frac{y_1 y_2}{|y|^4} (\partial_1\theta_0)(\mu(t)y_1, \frac{y_2}{\mu(t)}) dy_1 dy_2.
\end{aligned}
\tag{A.50}
$$

Next, we estimate $J(t)$. Denote $\tilde{\theta}(x_1,x_2)=\theta_0(x_1,x_2)-\theta_0(0,x_2)$. Clearly, we have $\partial_1\tilde{\theta}=\partial_1\theta$. We simplify $\mu(t)$ as μ. Since $(\partial_1\tilde{\theta}_0)(\mu y_1,\frac{y_2}{\mu})=\mu^{-1}\partial_1(\tilde{\theta}_0(\mu y_1,\frac{y_2}{\mu}))$, $\mathrm{supp}(\partial_1\tilde{\theta}_0)=\mathrm{supp}(\partial_1\theta_0)\subset[-1,1]^2$, using integration by parts and $\partial_1\frac{y_1y_2}{|y|^4}=\frac{y_2(y_2^2-3y_1^2)}{|y|^6}$, we yield

$$
\begin{aligned}
&J=\mu^{-1}\int_0^{\mu^{-1}}\int_0^\infty\frac{y_1y_2}{|y|^4}\partial_1\Big(\tilde{\theta}_0(\mu(t)y_1,\frac{y_2}{\mu(t)})\Big)dy_1dy_2\\
=&\mu^{-1}\int_0^\infty\frac{\mu^{-1}y_2}{(\mu^{-2}+y_2^2)^2}\tilde{\theta}_0(1,\frac{y_2}{\mu})dy_2-\mu^{-1}\int_0^{\mu^{-1}}\int_0^\infty\frac{y_2(y_2^2-3y_1^2)}{|y|^6}\tilde{\theta}_0(\mu y_1,\frac{y_2}{\mu})dy_1dy_2\\
\triangleq&J_1+J_2.
\end{aligned}
$$

Since $\tilde{\theta}_0\in C^{1,\alpha}$, $\tilde{\theta}_0(0,x_2)=0$ and $\tilde{\theta}_0(x_1,0)=0$, we have $|\tilde{\theta}_0(x_1,x_2)|\lesssim|x_1|^\alpha|x_2|$. It follows

$$
\begin{aligned}
&|J_1|\lesssim\mu^{-1}\int_0^\infty\frac{\mu^{-1}y_2}{(\mu^{-2}+y_2^2)^2}\frac{y_2}{\mu}dy_2=\mu^{-2}\int_0^\infty\frac{z^2}{(1+z^2)^2}dz\lesssim\mu^{-2},\\
&|J_2|\lesssim\mu^{-2}\int_0^{\mu^{-1}}(\mu y_1)^\alpha\int_0^\infty\Big|\frac{y_2^2(y_2^2-3y_1^2)}{|y|^6}\Big|dy_2dy_1\lesssim\mu^{-2}\int_0^{\mu^{-1}}(\mu y_1)^\alpha y_1^{-1}dy_1\lesssim_\alpha\mu^{-2}.
\end{aligned}
$$

Plugging the above estimates in (A.50), we obtain

$$
\Big|\frac{\dot{\mu}}{\mu}\Big|\lesssim_\alpha\mu^{-2}\int_0^t\mu(s)ds.
$$

Thus, μ remains bounded for all time. Formula (A.50) implies that the solution exists globally.

APPENDIX TO CHAPTER 3

Throughout this section, without specification, we assume that ω is smooth and decays sufficiently fast. The general case can be obtained easily by approximation.

The following identity is well known whose proof can be found in e.g., [39, 44].

Lemma B.0.1 (The Tricomi identity)**.** *We have*

$$H(\omega H\omega) = \frac{1}{2}((H\omega)^2 - \omega^2).$$

The Hilbert transform has a nice property that it almost commutes with the power x^{-1}, x.

Lemma B.0.2. *Suppose that $u_x = H\omega$. Then we have*

$$\frac{u_x - u_x(0)}{x} = H\left(\frac{\omega}{x}\right), \ \text{or equivalently} \ \ (H\omega)(x) = (H\omega)(0) + xH\left(\frac{\omega}{x}\right). \tag{B.1}$$

Similarly, we have

$$u_{xx} = H\omega_x, \quad xu_{xx} = H(x\omega_x). \tag{B.2}$$

Suppose that in addition ω is odd. Then we further have

$$x^2 u_{xx} = H(x^2\omega_x), \quad xu_x = H(x\omega), \quad \frac{u_{xx}}{x} = H\left(\frac{\omega_x - \omega_x(0)}{x}\right). \tag{B.3}$$

If ω is odd and a piecewise cubic polynomial supported on $[-L, L]$ with $\omega(L) = \omega(-L) = 0$ (ω', ω'' may not be continuous at $x = \pm L$), then we have

$$u_{xxx}(x^2 - L^2) = H(\omega_{xx}(x^2 - L^2)). \tag{B.4}$$

Proof. The identity (B.1) is very well known. We have

$$\frac{u_x - u_x(0)}{x} = \frac{1}{\pi x} P.V. \int \omega(y)\left(\frac{1}{x-y} + \frac{1}{y}\right) dy = \frac{1}{\pi} P.V. \int \frac{\omega(y)}{(x-y)y} dy = H\left(\frac{\omega}{y}\right)(x).$$

For (B.2), note that

$$H\omega_x = u_{xx}, \quad H(x\omega_x)(0) = -\frac{1}{\pi}\int \omega_x dx = 0.$$

From (B.1), we get

$$H(x\omega_x)(x) = H(x\omega_x)(0) + x(H\omega_x)(x) = xu_{xx}(x).$$

For (B.3), if ω is odd, then we obtain

$$H(x^2\omega_x)(0) = -\frac{1}{\pi}\int x\omega_x dx = \frac{1}{\pi}\int \omega dx = 0.$$

Applying (B.1) again yields

$$H(x^2\omega_x) = H(x^2\omega_x)(0) + xH(x\omega_x) = xH(x\omega_x) = x^2 u_{xx}.$$

For the second identity, since ω is odd, we can apply a similar argument to yield $H(x\omega)(0) = -\frac{1}{\pi}\int \omega dx = 0$ and

$$H(x\omega)(x) = H(x\omega)(0) + xH\omega = xH\omega = xu_x.$$

For the third identity in (B.3), first of all, we have

$$\omega_x = -Hu_{xx}.$$

If ω is odd, then u, u_{xx} are also odd. $\frac{\omega_x - \omega_x(0)}{x}$ and $\frac{u_{xx}}{x}$ are L^2 for ω smooth with suitable decay at infinity. Using an argument similar to that in the proof of (B.1) implies

$$\frac{\omega_x - \omega_x(0)}{x} = -H\left(\frac{u_{xx}}{x}\right).$$

Applying the Hilbert transform on both sides proves the third identity.

Next, we consider (B.4). From the assumption of ω, we know $\omega \in H^1(\mathbf{R})$. We can apply (B.3) to yield

$$x^2 u_{xx} = H(x^2\omega_x), \quad L^2 u_{xx} = L^2 H(\omega_x),$$

which implies $(x^2 - L^2)u_{xx} = H(\omega_x(x^2 - L^2))$. Since ω is a piecewise cubic polynomial on $[-L, L]$ and is continuous globally, we further have that $\omega_x(x^2 - L^2)$ is globally Lipschitz and it is in $H^1(\mathbf{R})$. By the L^2 isometry of the Hilbert transform, we get $u_{xx}(x^2 - L^2) \in H^1(\mathbf{R})$. Using the fact that the derivative commutes with the Hilbert transform, we yield

$$\partial_x H(\omega_x(x^2 - L^2)) = H(\partial_x(\omega_x(x^2 - L^2))),$$

which implies

$$u_{xxx}(x^2 - L^2) + 2u_{xx}x = H(\omega_{xx}(x^2 - L^2) + 2x\omega_x).$$

Using the linearity of the Hilbert transform and $u_{xx}x = H(x\omega_x)$(B.2), we conclude the proof of (B.4). □

The cancellation in the following Lemma is crucial in our linear stability analysis.

Lemma B.0.3. *Suppose $u_x = H\omega$. (a) We have*

$$\int_{\mathbf{R}} \frac{(u_x - u_x(0))\omega}{x} dx = \frac{\pi}{2}(u_x^2(0) + \omega^2(0)) \geq 0. \tag{B.5}$$

Furthermore, if ω is odd (so is u_{xx} due to the symmetry of Hilbert transform), we have

$$\int_{\mathbf{R}} \frac{(u_x - u_x(0))\omega}{x^3} dx = \frac{\pi}{2}(\omega_x^2(0) - u_{xx}^2(0)) = \frac{\pi}{2}\omega_x^2(0) \geq 0. \tag{B.6}$$

In particular, the right hand side of (B.5) *vanishes if $u_x(0) = \omega(x) = 0$.*

(b) We have

$$\int_{\mathbf{R}} u_{xx}\omega_x x dx = 0. \tag{B.7}$$

(c) The Hardy inequality: Suppose that ω is odd and $\omega_x(0) = 0$. For $p = 2, 4$, we have

$$\int \frac{(u - u_x(0)x)^2}{|x|^{p+2}} dx \leq \left(\frac{2}{p+1}\right)^2 \int \frac{(u_x - u_x(0))^2}{|x|^p} dx = \left(\frac{2}{p+1}\right)^2 \int \frac{\omega^2}{|x|^p} dx. \tag{B.8}$$

Proof of (B.5). Note that $u_x = H\omega,\ u_x(0) = -\frac{1}{\pi}\int \frac{\omega}{x} dx$. Using Lemma B.0.1, we get

$$\begin{aligned}\int \frac{(u_x - u_x(0))\omega}{x} dx &= \int \frac{\omega \cdot H\omega}{x} dx - u_x(0) \int \frac{\omega}{x} dx \\ &= -\pi H(\omega \cdot H\omega)(0) + \pi u_x(0) \cdot u_x(0) \\ &= \frac{\pi}{2}(\omega^2(0) - u_x^2(0)) + \pi u_x^2(0) = \frac{\pi}{2}(\omega^2(0) + u_x^2(0)).\end{aligned}$$

If $\omega(0) = 0$, the above estimates are reduced to $\frac{\pi}{2}u_x^2(0)$. □

Proof of (B.6). If ω is odd and smooth, then ω/x is even and smooth and $H(\omega/x)$ is odd. Using (B.1) and Lemma B.0.1, we have

$$\begin{aligned}\int \frac{(u_x - u_x(0))\omega}{x^3} dx &= \int \frac{1}{x}\frac{\omega}{x} H\left(\frac{\omega}{x}\right) dx = -\pi H\left(\frac{\omega}{x} H\left(\frac{\omega}{x}\right)\right)(0) \\ &= \frac{\pi}{2}\left\{\left(\frac{\omega}{x}(0)\right)^2 - H\left(\frac{\omega}{x}\right)(0)^2\right\} = \frac{\pi}{2}(\omega_x^2(0) - u_{xx}^2(0)).\end{aligned}$$

If $u_{xx}(0) = 0$, the above equality is reduced to $\frac{\pi}{2}\omega_x^2(0)$. □

Proof of (B.7). Applying (B.5) with (u_x, ω) replaced by (u_{xx}, ω_x) yields

$$\begin{aligned}\langle u_{xx}\omega_x, x\rangle &= \int \frac{(x\omega_x)H(x\omega_x)}{x} dx = \int \frac{(x\omega_x)(H(x\omega_x) - H(x\omega_x)(0))}{x} dx \\ &= \frac{\pi}{2}((x\omega_x)^2(0) + (xu_{xx})^2(0)) = 0,\end{aligned}$$

where we have used $(xu_{xx})(0) = (x\omega_x)(0) = 0$ to obtain the last equality. □

Proof of (B.8). The first inequality in (B.8) is the standard Hardy inequality [61]. Since ω is odd and $\omega_x(0) = 0$, $\omega/x, \omega/x^2 \in L^2(\mathbf{R})$. From (B.0.3), we have

$$\frac{u_x - u_x(0)}{x} = H\left(\frac{\omega}{x}\right), \quad H\left(\frac{\omega}{x^2}\right) = \frac{1}{x}\left(H\left(\frac{\omega}{x}\right) - H\left(\frac{\omega}{x}\right)(0)\right).$$

Since ω is odd, we obtain $H(\frac{\omega}{x}) = 0$. Hence, we can simplify the second equality as follows

$$H\left(\frac{\omega}{x^2}\right) = \frac{1}{x} H\left(\frac{\omega}{x}\right) = \frac{1}{x}\frac{u_x - u_x(0)}{x} = \frac{u_x - u_x(0)}{x^2}.$$

Applying the L^2 isometry property of the Hilbert transform H to $H(\frac{\omega}{x}), H(\frac{\omega}{x^2})$, we establish the equality in (B.8). □

The following Lemma is an analogy of Lemma B.0.3 for Hölder continuous functions. (B.9),(B.10) and (B.11) are from Córdoba & Córdoba [29].

Lemma B.0.4 (Weighted estimate for C^α functions)**.** *Suppose that* $u_x = H\omega$ *and* ω *is odd in* (B.9), (B.11) *and* (B.12). *(a) For* $\beta \in (0, 2)$, *we have*

$$\begin{aligned}\int \frac{(u_x - u_x(0))^2}{|x|^{1+\beta}} dx &\leq \frac{1}{\tan^2 \frac{\beta\pi}{4} \wedge \cot^2 \frac{\beta\pi}{4}} \int \frac{w^2}{|x|^{1+\beta}} dx \\ &\lesssim \frac{1}{(\beta \wedge (2-\beta))^2} \int \frac{w^2}{|x|^{1+\beta}} dx, \end{aligned} \tag{B.9}$$

$$\begin{aligned}\int \frac{u_x^2}{|x|^{1-\beta}} dx &\leq \frac{1}{\tan^2 \frac{\beta\pi}{4} \wedge \cot^2 \frac{\beta\pi}{4}} \int \frac{w^2}{|x|^{1-\beta}} dx \\ &\lesssim \frac{1}{(\beta \wedge (2-\beta))^2} \int \frac{w^2}{|x|^{1-\beta}} dx, \end{aligned} \tag{B.10}$$

provided that the right hand side is finite, where $a \wedge b = \min(a, b)$*. Note that we do not need to assume that* ω *is odd in* (B.10).

(b) For $\beta \in (0, 2)$*, we have*

$$\int \frac{(u_x - u_x(0))\omega}{\mathrm{sgn}(x)|x|^{1+\beta}} dx \geq 0. \tag{B.11}$$

(c) 1D Hardy inequality [61]: For $\beta \in (0, 1)$*, we have*

$$\int \frac{(u - u_x(0)x)^2}{|x|^{3+\beta}} dx \leq \left(\frac{2}{\beta + 2}\right)^2 \int \frac{(u_x - u_x(0))^2}{|x|^{\beta+1}} dx \lesssim \frac{1}{\beta^2} \int \frac{\omega^2}{|x|^{\beta+1}}. \tag{B.12}$$

The first inequality in (B.12) is the Hardy inequality [61] and the second inequality in (B.12) follows from (B.9).

Appendix C

APPENDIX TO CHAPTER 4

Throughout this section, we assume that ω is smooth and decays sufficiently fast. The general case can be obtained by approximation. The properties of the Hilbert transform in Lemmas C.0.1-C.0.3 are well known, see e.g., [13, 19, 39].

Lemma C.0.1. *Assume that ω is odd. We have*

$$H\omega(x) - H\omega(0) = xH(\frac{\omega}{x}).$$

Lemma C.0.2. *Assume that ω is odd and $\omega_x(0) = 0$. For $p = 1, 2$, we have*

$$(u_x - u_x(0))x^{-p} = H(\omega x^{-p}). \tag{C.1}$$

Consequently, the L^2 isometry property of the Hilbert transform implies

$$||(u_x - u_x(0))x^{-p}||_2^2 = ||\omega x^{-p}||_2^2.$$

Recall the inner product $\langle f, g\rangle = \int_0^\infty fgdx$ (see (4.8))and $\Lambda = (-D)^{1/2} = H\partial_x$.

Lemma C.0.3. *For $f \in L^p, g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$ and $1 < p < \infty$, we have*

$$\langle Hf, g\rangle = -\langle f, Hg\rangle. \tag{C.2}$$

Lemma C.0.4. *Denote $\Lambda = (-\partial_x^2)^{1/2}$. Assume that f is odd and $g_x = Hf, g(0) = 0$. We have*

$$\langle Hf - Hf(0), fx^{-3}\rangle = 0,$$
$$\langle g, f_x x^{-1}\rangle = -\Big\langle \Lambda\frac{g}{x}, \frac{g}{x}\Big\rangle, \quad \langle g, fx^{-2}\rangle = -\Big\langle \Lambda\frac{g}{x}, \frac{g}{x}\Big\rangle - \frac{\pi}{4}g_x(0)^2.$$

Identities similar to those in Lemma C.0.4 have been used in [5, 13, 19, 49]. We refer the proof of Lemma C.0.4 to the arXiv version of this paper [20].

Lemma C.0.5. *Assume that $\omega \in L^2(|x|^{-4/3} + |x|^{-2/3})$ is odd and $u_x = H\omega$. We have*

$$\int_{\mathbb{R}} \frac{(u_x(x) - u_x(0))^2}{|x|^{4/3}} = \int_{\mathbb{R}} \Big(\frac{w^2}{|x|^{4/3}} + 2\sqrt{3}\cdot \mathrm{sgn}(x)\frac{\omega(u_x(x) - u_x(0))}{|x|^{4/3}}\Big)dx.$$

It seems that the identity (4.78) $H(|x|^{-\alpha}) = \tan\left(\frac{\alpha\pi}{2}\right) sgn(x)|x|^{-\alpha}$, which will be used in the proof of Lemma C.0.5, is difficult to locate in the literature. We thus give a proof.

Proof. Firstly, we compute $H(|x|^{-\alpha})$. For $\alpha \in (0,1)$, we have $H(|x|^{-\alpha}) = C_\alpha sgn(x)|x|^{-\alpha}$, for some constant C_α. We determine C_α by applying Lemma C.0.3 to

$$f = |x|^{-\alpha}, \quad Hf = C_\alpha sgn(x)|x|^{-\alpha}, \quad g = -\frac{x}{1+x^2}, \quad Hg = \frac{1}{1+x^2},$$

which implies

$$C_\alpha \int_0^\infty \frac{x^{1-\alpha}}{1+x^2} dx = \int_0^\infty \frac{1}{x^\alpha(1+x^2)} dx.$$

The integrals can be evaluated using the Beta function $\mathrm{B}(x,y)$ and $\mathrm{B}(\beta, 1-\beta) = \frac{\pi}{\sin(\beta\pi)}$ for $\beta \in (0,1)$. In particular, we get

$$C_\alpha = \frac{\mathrm{B}(\frac{\alpha+1}{2}, \frac{1-\alpha}{2})}{\mathrm{B}(\frac{2-\alpha}{2}, \frac{\alpha}{2})} = \frac{\pi/\sin((\alpha+1)\pi/2)}{\pi/\sin((2-\alpha)\pi/2)} = \tan\left(\frac{\alpha\pi}{2}\right).$$

Choosing $\alpha = 1/3$, we get

$$H(|x|^{-1/3}) = \frac{1}{\sqrt{3}} sgn(x)|x|^{-1/3}, \quad H(sgn(x)|x|^{-1/3}) = -\sqrt{3}|x|^{-1/3}. \tag{C.3}$$

Recall that ω is odd. We assume that ω is in the Schwartz space. Applying the Cotlar identity, see e.g., [19, 39],

$$(HF)^2 = F^2 + 2H(F \cdot HF),$$

we yield

$$\begin{aligned} I \triangleq \int_{\mathbb{R}} \frac{(u_x(x) - u_x(0))^2}{|x|^{4/3}} &= \int_{\mathbb{R}} |x|^{2/3} \left(H\left(\frac{\omega}{x}\right)\right)^2 dx \\ &= \int_{\mathbb{R}} \left\{ |x|^{2/3} \left(\frac{\omega}{x}\right)^2 + 2|x|^{2/3} H\left(\frac{\omega}{x} H\left(\frac{\omega}{x}\right)\right) \right\} dx. \end{aligned}$$

Since the Hilbert transform is antisymmetric (Lemma C.0.3), we get $H(\omega H(\frac{\omega}{x})) = -\frac{1}{\pi}\int_{\mathbb{R}} \frac{\omega}{x} H(\frac{\omega}{x}) dx = 0$. Using Lemma C.0.1, we obtain

$$|x|^{2/3} H\Big(\frac{\omega}{x} H(\frac{\omega}{x})\Big) = |x|^{2/3} \frac{1}{x} H\Big(\omega H(\frac{\omega}{x})\Big) = \mathrm{sgn}(x)|x|^{-1/3} H\Big(\omega H(\frac{\omega}{x})\Big).$$

Thus, applying Lemma C.0.3, then (C.3) and $H(\frac{\omega}{x}) = \frac{u_x - u_x(0)}{x}$ in Lemma C.0.1, we prove

$$\begin{aligned} I &= \int_{\mathbb{R}} \Big\{ \frac{\omega^2}{|x|^{4/3}} - 2H\left(sgn(x)|x|^{-1/3}\right)\omega H\left(\frac{\omega}{x}\right)\Big\} dx \\ &= \int_{\mathbb{R}} \Big\{ \frac{\omega^2}{|x|^{4/3}} + 2\sqrt{3}|x|^{-1/3}\omega H\left(\frac{\omega}{x}\right)\Big\} dx \\ &= \int_{\mathbb{R}} \Big\{ \frac{\omega^2}{|x|^{4/3}} + 2\sqrt{3}|x|^{-1/3}\omega \frac{u_x - u_x(0)}{x}\Big\} dx \\ &= \int_{\mathbb{R}} \Big(\frac{\omega^2}{|x|^{4/3}} + 2\sqrt{3} sgn(x) \frac{\omega(u_x(x) - u_x(0))}{|x|^{4/3}}\Big) dx. \end{aligned}$$

To prove the Lemma for general odd $\omega \in L^2(|x|^{-4/3} + |x|^{-2/3})$, or equivalently $\frac{\omega}{x} \in L^2(|x|^{2/3} + |x|^{4/3})$, we approximate $\frac{\omega}{x}$ by the Schwartz function and use the fact that $|x|^{2/3}$ is an A_2 weight [39]. $\square$

The weighted estimates in Lemma C.0.6 were established in [29].

Lemma C.0.6. *For $f \in L^2(x^{-4/3} + x^{-2/3})$, we have*

$$\begin{aligned} ||(Hf - Hf(0))x^{-2/3}||_2 &\le \cot\frac{\pi}{12}||fx^{-2/3}||_2 = (2+\sqrt{3})||fx^{-2/3}||_2, \\ ||Hfx^{-1/3}||_2 &\le \cot\frac{\pi}{12}||fx^{-1/3}||_2 = (2+\sqrt{3})||fx^{-1/3}||_2. \end{aligned}$$

The estimate in the following Lemma is the Hardy inequality [61].

Lemma C.0.7. *Assume that u is odd. Then for $p > \frac{3}{2}$, we have*

$$\int_0^{+\infty} \frac{(u(x) - u_x(0)x)^2}{x^{2p}} \le \frac{4}{(2p-1)^2} \int_0^{+\infty} \frac{(u_x(x) - u_x(0))^2}{x^{2p-2}}.$$

Lemma C.0.8. *Assume that ω is odd and $\omega \in L^2(x^{-4} + x^{-2/3})$. Let $u_x = H\omega$. For any $\alpha, \beta, \gamma \ge 0$, we have*

$$\begin{aligned} &||(u_x - u_x(0))(\alpha x^{-4} + \beta x^{-2})^{1/2}||_2^2 = ||\omega(\alpha x^{-4} + \beta x^{-2})^{1/2}||_2^2, \\ &\Big|\Big|(u - u_x(0)x)\Big(\frac{\alpha}{x^6} + \frac{\beta}{x^4} + \frac{\gamma}{x^{10/3}}\Big)^{1/2}\Big|\Big|_2^2 \\ \le &\Big|\Big|\omega\Big(\frac{4\alpha}{25x^4} + \frac{4\beta}{9x^2}\Big)^{1/2}\Big|\Big|_2^2 + \frac{36\gamma}{49}||(u_x - u_x(0))x^{-2/3}||_2^2. \end{aligned}$$

The first identity follows from Lemma C.0.2. Applying Lemma C.0.7 with $p = 3, 2, \frac{5}{3}$ and then Lemma C.0.2 to the power x^{-4}, x^{-2} yield the second inequality. The constants $\frac{4}{25}, \frac{4}{9}, \frac{36}{49}$ are determined by $\frac{4}{(2p-1)^2}$ with $p = 3, 2, \frac{5}{3}$.

C.1 Derivations and estimates in the linear stability analysis

C.1.1 Derivation of (4.12)

For $p \in [1,3]$, using integration by parts yields

$$||(\tilde{u} - \frac{1}{2p-1}\tilde{u}_x x)x^{-p}||_2^2 = \int_{\mathbb{R}_+} \Big(\frac{1}{(2p-1)^2}\frac{\tilde{u}_x^2}{x^{2p-2}} - \frac{2}{2p-1}\frac{\tilde{u}\tilde{u}_x}{x^{2p-1}} + \frac{\tilde{u}^2}{x^{2p}}\Big)dx$$
$$= \int_{\mathbb{R}_+} \Big(\frac{1}{(2p-1)^2}\frac{\tilde{u}_x^2}{x^{2p-2}} + \frac{1}{2p-1}(\partial_x x^{-(2p-1)})\tilde{u}^2 + \frac{\tilde{u}^2}{x^{2p}}\Big)dx = \frac{1}{(2p-1)^2}\int_{\mathbb{R}_+} \frac{\tilde{u}_x^2}{x^{2p-2}}dx.$$

C.1.2 Estimate of I_{r1}, I_{r2}, I_{r3}

We construct the cutoff function χ in (4.25) as follows

$$\chi(x) = \frac{2}{\pi}\arctan((\frac{x-l_1}{l_2})^3)\mathbf{1}_{x\geq l_1}\ , \quad l_1 = 5\cdot 10^8, \quad l_2 = 10 l_1.$$

Recall I_{r1}, I_{r2} in (4.33), (4.46), and (4.92)

$$I_{r1} = \langle \tilde{u}_x \chi(\xi_1\psi_n + \xi_2\psi_f), \theta_x\rangle, \quad I_{r2} = \lambda_1\langle \tilde{u}, \chi\xi_3\omega\varphi\rangle, \quad I_{r3} = -\frac{1}{3}\lambda_1\langle \tilde{u}\chi\xi_3, D_x\omega\varphi\rangle. \tag{C.4}$$

Recall from the beginning of Section 4.3.1 that $\bar{\omega}, \bar{\theta}_x, \bar{\omega}_x, \bar{\theta}_{xx}$ have decay rates $x^{\alpha}, x^{2\alpha}, x^{\alpha-1}, x^{2\alpha-1}$, respectively, with α slightly smaller than $-\frac{1}{3}$. Using the formulas of ξ_i in (4.24) and $\varphi_f, \varphi_n, \psi$ in (4.25), (4.26), we obtain the decay rates $\chi(\xi_1\psi_n + \xi_2\psi_f) \sim C_1 x^{-4/3}$, $\chi\xi_3\varphi \sim C_2 x^{-2}$ for sufficiently large x, where C_1, C_2 are some constants.

Recall $\tilde{u}_x = u_x - u_x(0)$. Using the Cauchy-Schwarz inequality and Lemmas C.0.6, we obtain

$$\begin{aligned}|I_{r1}| &\leq ||\tilde{u}_x x^{-2/3}||_2||\chi(\xi_1\psi_n + \xi_2\psi_f)\theta_x||_2 \\ &\leq (2+\sqrt{3})||\omega x^{-2/3}||_2||\chi(\xi_1\psi_n + \xi_2\psi_f)\theta_x||_2.\end{aligned}$$

For I_{r2}, we first decompose it as follows using $\tilde{u} = u - u_x(0)x$

$$I_{r2} = \lambda_1\langle u, \chi\xi_3\omega\varphi\rangle - u_x(0)\lambda_1\langle x, \chi\xi_3\omega\varphi\rangle \triangleq J_1 + J_2.$$

Using the Cauchy-Schwarz inequality, Lemma C.0.7 with $p = \frac{4}{3}$ and Lemma C.0.6, we get

$$\begin{aligned}|J_1| &\leq \lambda_1||ux^{-4/3}||_2||x^{4/3}\chi\xi_3\omega\varphi||_2 \leq \frac{6\lambda_1}{5}||u_x x^{-1/3}||_2||x^{4/3}\chi\xi_3\omega\varphi||_2 \\ &\leq \frac{6\lambda_1(2+\sqrt{3})}{5}||\omega x^{-1/3}||_2||x^{4/3}\chi\xi_3\omega\varphi||_2.\end{aligned}$$

Recall $c_\omega = u_x(0)$. For J_2, using Cauchy-Schwarz inequality, we yield

$$|J_2| \leq \lambda_1 |c_\omega| \cdot ||\chi^{1/2}\omega\varphi^{1/2}||_2 ||x\xi_3\chi^{1/2}\varphi^{1/2}||_2.$$

In the above estimates of I_{r1}, if we further bound $||\chi(\xi_1\psi_n + \xi_2\psi_f)\theta_x||_2$ by the weighted L^2 norm $||\theta_x\psi^{1/2}||_2$, we obtain a small factor $\rho_2^{-1/3}$ since χ is supported in $|x| \geq \rho_2$ and the profile has decay. See also the above discussion on the decay rates. Similarly, we get a small factor in the estimates of J_1, J_2 from $||x^{4/3}\chi\xi_3\omega\varphi||_2, ||x^{4/3}\chi\xi_3\omega\varphi||_2$, respectively.

Using Young's inequality $ab \leq ta^2 + \frac{1}{4t}b^2$, we obtain

$$\begin{aligned}|I_{r1}| + |I_{r2}| \leq\ & t_{51}||\omega x^{-2/3}||_2^2 + \frac{(2+\sqrt{3})^2}{4t_{51}}||\chi(\xi_1\psi_n + \xi_2\psi_f)\theta_x||_2^2 + t_{52}||\omega x^{-1/3}||_2^2 \\ &+ \frac{1}{4t_{52}}(\frac{6\lambda_1(2+\sqrt{3})}{5})^2||x^{4/3}\chi\xi_3\omega\varphi||_2^2 \\ &+ t_{53}||\chi^{1/2}\omega\varphi^{1/2}||_2^2 + \frac{\lambda_1^2||x\xi_3\chi^{1/2}\varphi^{1/2}||_2^2}{4t_{53}}c_\omega^2,\end{aligned}$$

where $t_{51} = 10^{-10}, t_{52} = 10^{-5}, t_{53} = 10^{-2}$. We choose these weights t_{5i} so that the terms $ta^2, \frac{1}{4t}b^2$ in Young's inequality are comparable. It follows the estimate (4.53).

Note that replacing ω in I_{r2} in (C.4) by $-\frac{1}{3}D_x\omega$, we obtain I_{r3}. Therefore, applying the same estimate as that of I_{r2} to I_{r3}, we yield

$$\begin{aligned}|I_{r3}| \leq\ & \frac{2\lambda_1(2+\sqrt{3})}{5}||\omega x^{-1/3}||_2||x^{4/3}\chi\xi_3 D_x\omega\varphi||_2 \\ &+ \frac{\lambda_1}{3}|c_\omega| \cdot ||\chi^{1/2}D_x\omega\varphi^{1/2}||_2||x\xi_3\chi^{1/2}\varphi^{1/2}||_2.\end{aligned}$$

Using Young's inequality $ab \leq ta^2 + \frac{1}{4t}b^2$, we establish

$$\begin{aligned}|I_{r3}| \leq\ & t_{94}||x^{4/3}\chi\xi_3 D_x\omega\varphi||_2^2 + \frac{1}{4t_{94}}(\frac{2\lambda_1(2+\sqrt{3})}{5})^2||\omega x^{-1/3}||_2^2 \\ &+ t_{95}||\chi^{1/2}D_x\omega\varphi^{1/2}||_2^2 + \frac{\lambda_1^2||x\xi_3\chi^{1/2}\varphi^{1/2}||_2^2}{36t_{95}}c_\omega^2.\end{aligned}$$

where $t_{94} = 10^6, t_{95} = 10^{-3}$. We choose these weights t_{94}, t_{95} so that the terms $ta^2, \frac{1}{4t}b^2$ in Young's inequality are comparable. It follows (4.93).

C.1.3 Derivations of the ODE (4.60) in Section 4.3.11

We use the following functions in the derivations

$$
\begin{aligned}
&f_2 \triangleq \frac{1}{4}\frac{\bar{u}_x}{x} - \frac{1}{5}(\frac{3}{4}\bar{u}_{xx} + \frac{1}{4}\frac{\bar{u}_x}{x}) - \frac{\bar{u}_x}{x} + \frac{\bar{u}}{x^2}, \quad f_3 \triangleq \lambda_1(\bar{\omega} - x\bar{\omega}_x)\varphi, \\
&f_4 \triangleq \frac{3}{5}\frac{\bar{u}_{\theta,x}}{x} + \frac{1}{5}\left(\frac{3}{5}\bar{u}_{\theta,xx} + \frac{2}{5}\frac{\bar{u}_{\theta,x}}{x}\right), \quad f_6 \triangleq \frac{\bar{u}}{x^2}, \\
&f_7 \triangleq (\bar{\theta}_x - x\bar{\theta}_{xx})\psi, \quad f_8 \triangleq \frac{3}{4}\bar{\omega}_x + \frac{1}{4}\frac{\bar{\omega}}{x}, \quad f_9 \triangleq \frac{3}{5}\bar{\theta}_{xx} + \frac{2}{5}\frac{\bar{\theta}_x}{x}.
\end{aligned} \tag{C.5}
$$

C.1.3.1 Derivations of the ODE for c_ω^2, d_θ^2 and (4.60)

Recall $c_\omega = u_x(0) = -\frac{2}{\pi}\int_0^{+\infty} \frac{\omega}{x} dx$ from (4.20). Multiplying the equation of ω in (4.18) by $-\frac{1}{x}$ and then taking the integral from $0, +\infty$ yield

$$
\begin{aligned}
\frac{d}{dt}\frac{\pi}{2}c_\omega = &\frac{d}{dt}\int_0^{+\infty} \frac{\omega}{-x}dx = \int_0^\infty \frac{(\bar{c}_l x + \bar{u})\omega_x + u\bar{\omega}_x}{x}dx - \int_0^\infty \frac{\theta_x}{x}dx \\
&+ \int_0^\infty \frac{\bar{c}_\omega \omega + c_\omega \bar{\omega}}{-x}dx - \int_0^\infty \frac{F_\omega + N(\omega)}{x}dx \\
= &\int_0^\infty \frac{\bar{u}\omega_x + u\bar{\omega}_x}{x}dx - d_\theta + \frac{\pi}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega - \int_0^\infty \frac{F_\omega + N(\omega)}{x}dx,
\end{aligned}
$$

where we have used the notation d_θ in (4.58) and $\int_0^\infty \frac{f}{-x} = \frac{\pi}{2}Hf(0)$ with $f = \omega, \bar{\omega}$ in the last identity. Multiplying c_ω on both sides, we yield

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\frac{\pi}{2}c_\omega^2 = \frac{\pi}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega^2 + c_\omega \int_0^\infty \frac{\bar{u}\omega_x + u\bar{\omega}_x}{x}dx \\
- c_\omega d_\theta - c_\omega \int_0^\infty \frac{F_\omega + N(\omega)}{x}dx.
\end{aligned} \tag{C.6}
$$

which is exactly (4.59).

We derive the ODE for d_θ using the θ equation in (4.18). Since $\int_{\mathbb{R}_+} \frac{\bar{c}_l x \theta_{xx}}{x} dx = 0$, we get

$$
\begin{aligned}
\frac{d}{dt}d_\theta = 2\bar{c}_\omega \int_{\mathbb{R}_+} \frac{\theta_x}{x} + 2c_\omega \int_{\mathbb{R}_+} \frac{\bar{\theta}_x}{x} - \int_0^\infty \frac{\bar{u}_x\theta_x + \bar{u}\theta_{xx}}{x}dx - \int_0^\infty \frac{u\bar{\theta}_{xx} + u_x\bar{\theta}_x}{x}dx \\
+ \int_0^\infty \frac{F_\theta + N(\theta)}{x}dx \triangleq I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned} \tag{C.7}
$$

We use the notation $\langle \cdot, \cdot \rangle$ in (4.8) to simplify the integral. For I_3, using integration by parts, we obtain

$$
I_3 = -\langle (\bar{u}\theta_x)_x, x^{-1} \rangle = \langle \bar{u}\theta_x, \partial_x x^{-1} \rangle = -\langle \bar{u}\theta_x, x^{-2} \rangle.
$$

Similarly, for I_4, we get

$$I_4 = -\langle u\bar{\theta}_x, x^{-2}\rangle.$$

Recall $c_\omega = u_x(0)$. We rewrite the above term using the decomposition $u = \tilde{u} + u_x(0)x$ (4.27)

$$I_4 = -\langle(\tilde{u} + u_x(0)x)\bar{\theta}_x, x^{-2}\rangle = -\langle\tilde{u}\bar{\theta}_x, x^{-2}\rangle - c_\omega\bar{d}_\theta.$$

where we have used the notation $\bar{d}_\theta$ defined in (4.58). Using (4.58), we can simplify I_1, I_2 as

$$I_1 = 2\bar{c}_\omega d_\theta, \quad I_2 = 2c_\omega\bar{d}_\theta.$$

The $c_\omega d_\theta$ term in I_2 and I_4 are canceled partially. Using these computations and multiplying both sides of (C.7) by d_θ yields

$$\frac{1}{2}\frac{d}{dt}d_\theta^2 = 2\bar{c}_\omega d_\theta^2 + c_\omega\bar{d}_\theta d_\theta - d_\theta\int_0^\infty \frac{\bar{u}\theta_x}{x^2}dx - d_\theta\int_0^\infty \frac{\tilde{u}\bar{\theta}_x}{x^2}dx + d_\theta\int_0^\infty \frac{F_\theta + N(\theta)}{x}dx. \tag{C.8}$$

Since $\bar{d}_\theta > 0$, the term $c_\omega d_\theta$ in (C.8) and (C.6) have cancellation.

The quadratic parts on the right hand sides in (C.6), (C.8) involve the following terms remained to estimate

$$J_1 = \langle\bar{u}, \omega_x x^{-1}\rangle,\ J_2 = \langle u, \bar{\omega}_x x^{-1}\rangle,\ J_3 = \langle\bar{u}, \theta_x x^{-2}\rangle,\ J_4 = \langle\tilde{u}, \bar{\theta}_x x^{-2}\rangle. \tag{C.9}$$

We use the idea in Section 4.3.11.2 to rewrite the integrals of u as the integrals of ω and of $\tilde{u} - \frac{1}{5}\tilde{u}_x x = u_\Delta$ (see (4.58)). We use the functions f_i defined (C.5) to simplify the integrals of θ_x, ω. In Appendix C.1.3.2, we rewrite J_i as follows

$$J_1 + J_2 = \langle\omega, f_2\rangle + \langle u_\Delta x^{-1}, f_5\rangle, \quad J_3 = \langle\theta_x, f_6\rangle, \quad J_4 = \langle u_\Delta x^{-1}, f_9\rangle - \langle\omega, f_4\rangle. \tag{C.10}$$

For some parameters $\lambda_2, \lambda_3 > 0$ to be determined, combining (C.6) and (C.8), we yield

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\left(\frac{\lambda_2\pi}{2}c_\omega^2 + \lambda_3 d_\theta^2\right) = &\frac{\pi\lambda_2}{2}(\bar{c}_\omega + \bar{u}_x(0))c_\omega^2 + \lambda_2 c_\omega(J_1 + J_2) - \lambda_2 c_\omega d_\theta \\ &- \lambda_2 c_\omega\langle F_\omega + N(\omega), x^{-1}\rangle + 2\bar{c}_\omega\lambda_3 d_\theta^2 + \lambda_3 c_\omega\bar{d}_\theta d_\theta \\ &- \lambda_3 d_\theta J_3 - \lambda_3 d_\theta J_4 + \lambda_3 d_\theta\langle F_\theta + N(\theta), x^{-1}\rangle.\end{aligned}$$

Plugging (C.10) in the above ODE, we derive (4.60).

C.1.3.2 Derivations of (C.10) in the ODEs

Recall the integrals J_i from (C.9). We use the idea in Section 4.3.11.2 to derive the formulas in (C.10).

Recall $\tilde{u} = u - u_x(0)x$ from (4.27). Firstly, we consider J_2. Since $\int_0^\infty \bar{\omega}_x dx = 0$, we have

$$J_2 = \langle u - u_x(0)x, \bar{\omega}_x x^{-1} \rangle = \langle \tilde{u}, \bar{\omega}_x x^{-1} \rangle.$$

We approximate the far field of $\bar{\omega}_x x^{-1}$ by $\frac{1}{4}(\frac{\bar{\omega}}{x})_x$ and derive

$$J_2 = \Big\langle \tilde{u}, \frac{\bar{\omega}_x}{x} - \frac{1}{4}(\frac{\bar{\omega}}{x})_x \Big\rangle + \frac{1}{4}\Big\langle \tilde{u}, (\frac{\bar{\omega}}{x})_x \Big\rangle \triangleq J_{21} + J_{22}.$$

Applying integration by parts, (C.1) and (C.2) yields

$$J_{22} = -\frac{1}{4}\langle \tilde{u}_x, \bar{\omega} x^{-1} \rangle = -\frac{1}{4}\Big\langle H\left(\frac{\omega}{x}\right), \bar{\omega} \Big\rangle = \frac{1}{4}\Big\langle \frac{\omega}{x}, H\bar{\omega} \Big\rangle = \frac{1}{4}\Big\langle \frac{\omega}{x}, \bar{u}_x \Big\rangle.$$

In J_{21}, the coefficient

$$\frac{\bar{\omega}_x}{x} - \frac{1}{4}(\frac{\bar{\omega}}{x})_x = \frac{3}{4}\frac{\bar{\omega}_x}{x} + \frac{1}{4}\frac{\bar{\omega}}{x^2}$$

decays much faster than $\bar{\omega}_x x^{-1}$ for large x. We approximate $\tilde{u}$ by $\frac{1}{5}\tilde{u}_x x$

$$J_{21} = \Big\langle \tilde{u}, \frac{3}{4}\frac{\bar{\omega}_x}{x} + \frac{1}{4}\frac{\bar{\omega}}{x^2} \Big\rangle = \Big\langle \tilde{u} - \frac{1}{5}\tilde{u}_x x, \frac{3}{4}\frac{\bar{\omega}_x}{x} + \frac{1}{4}\frac{\bar{\omega}}{x^2} \Big\rangle + \frac{1}{5}\Big\langle \tilde{u}_x x, \frac{3}{4}\frac{\bar{\omega}_x}{x} + \frac{1}{4}\frac{\bar{\omega}}{x^2} \Big\rangle \triangleq I_1 + I_2.$$

Using a direct computation and then applying (C.1) and (C.2), we get

$$\begin{aligned} I_2 &= \frac{1}{5}(\frac{3}{4}\langle \tilde{u}_x, \bar{\omega}_x \rangle + \frac{1}{4}\langle \frac{\tilde{u}_x}{x}, \bar{\omega} \rangle) = \frac{1}{5}(\frac{3}{4}\langle u_x, \bar{\omega}_x \rangle + \frac{1}{4}\langle H\left(\frac{\omega}{x}\right), \bar{\omega} \rangle) \\ &= \frac{1}{5}(-\frac{3}{4}\langle \omega, H\bar{\omega}_x \rangle - \frac{1}{4}\langle \frac{\omega}{x}, H\bar{\omega} \rangle) = -\frac{1}{5}\langle \omega, \frac{3}{4}\bar{u}_{xx} + \frac{1}{4}\frac{\bar{u}_x}{x} \rangle, \end{aligned}$$

where we have used $\int_0^\infty u_x(0)\bar{\omega}_x dx = 0$ in the second identity. Using the notation and function in (4.58), (C.5), we can simplify I_1 as

$$I_1 = \langle u_\Delta x^{-1}, f_8 \rangle.$$

Combining the above calculations on J_{22}, I_1, I_2, we obtain

$$J_2 = I_1 + I_2 + J_{22} = \Big\langle \omega, \frac{1}{4}\frac{\bar{u}_x}{x} - \frac{1}{5}(\frac{3}{4}\bar{u}_{xx} + \frac{1}{4}\frac{\bar{u}_x}{x}) \Big\rangle + \Big\langle \frac{u_\Delta}{x}, f_8 \Big\rangle.$$

For J_1 in (C.9), using integration by parts, we obtain

$$J_1 = \langle \bar{u} x^{-1}, \omega_x \rangle = -\langle \partial_x(\bar{u} x^{-1}), \omega \rangle = \langle -\frac{\bar{u}_x}{x} + \frac{\bar{u}}{x^2}, \omega \rangle.$$

We can simplify $J_1 + J_2$ using the function f_2 in (C.5)

$$J_1 + J_2 = \langle \omega, f_2 \rangle + \langle u_\Delta x^{-1}, f_5 \rangle. \tag{C.11}$$

For J_3, using f_6 defined in (C.5), we get

$$J_3 = \langle \theta_x, \bar{u} x^{-2} \rangle = \langle \theta_x, f_6 \rangle. \tag{C.12}$$

For J_4 in (C.9), we use a similar computation to obtain

$$\begin{aligned} J_4 &= \langle \tilde{u}, \bar{\theta}_x x^{-2} \rangle = \Big\langle \tilde{u}, \frac{\bar{\theta}_x}{x^2} + \frac{3}{5} (\frac{\bar{\theta}_x}{x})_x \Big\rangle - \frac{3}{5} \Big\langle \tilde{u}, (\frac{\bar{\theta}_x}{x})_x \Big\rangle \\ &= \langle \tilde{u}, \frac{3}{5} \frac{\bar{\theta}_{xx}}{x} + \frac{2}{5} \frac{\bar{\theta}_x}{x^2} \rangle - \frac{3}{5} \Big\langle \tilde{u}, (\frac{\bar{\theta}_x}{x})_x \Big\rangle \\ &= \Big\langle \tilde{u} - \frac{1}{5} \tilde{u}_x x, \frac{3}{5} \frac{\bar{\theta}_{xx}}{x} + \frac{2}{5} \frac{\bar{\theta}_x}{x^2} \Big\rangle + \frac{1}{5} \Big\langle \tilde{u}_x x, \frac{3}{5} \frac{\bar{\theta}_{xx}}{x} + \frac{2}{5} \frac{\bar{\theta}_x}{x^2} \Big\rangle - \frac{3}{5} \Big\langle \tilde{u}, (\frac{\bar{\theta}_x}{x})_x \Big\rangle \\ &\triangleq J_{41} + J_{42} + J_{43}. \end{aligned}$$

For J_{41}, using the notations in (4.58) and (C.5), we obtain

$$J_{41} = \langle u_\Delta x^{-1}, f_9 \rangle.$$

For J_{42}, J_{43}, using Lemmas C.0.2 and C.0.3, we derive

$$\begin{aligned} J_{42} &= \frac{1}{5} \Big\langle \tilde{u}_x, \frac{3}{5} \bar{\theta}_{xx} + \frac{2}{5} \frac{\bar{\theta}_x}{x} \Big\rangle = \frac{1}{5} \left(\frac{3}{5} \Big\langle H\omega - H\omega(0), \bar{\theta}_{xx} \Big\rangle + \frac{2}{5} \Big\langle H \left(\frac{\omega}{x} \right), \bar{\theta}_x \Big\rangle \right) \\ &= -\frac{1}{5} \left(\frac{3}{5} \langle \omega, H \bar{\theta}_{xx} \rangle + \frac{2}{5} \langle \frac{\omega}{x}, H \bar{\theta}_x \rangle \right), \\ J_{43} &= \frac{3}{5} \Big\langle \tilde{u}_x, \frac{\bar{\theta}_x}{x} \Big\rangle = \frac{3}{5} \Big\langle H(\frac{\omega}{x}), \bar{\theta}_x \Big\rangle = -\frac{3}{5} \Big\langle \frac{\omega}{x}, H \bar{\theta}_x \Big\rangle. \end{aligned}$$

Combining the above computations and using the notations $\bar{u}_{\theta,x}, f_4$ defined in (4.58), (C.5), we yield

$$\begin{aligned} J_4 = J_{41} + J_{42} + J_{43} &= \langle u_\Delta x^{-1}, f_9 \rangle - \Big\langle \omega, \frac{3}{5} \frac{\bar{u}_{\theta,x}}{x} + \frac{1}{5} \left(\frac{3}{5} \bar{u}_{\theta,xx} + \frac{2}{5} \frac{\bar{u}_{\theta,x}}{x} \right) \Big\rangle \\ &= \langle u_\Delta x^{-1}, f_9 \rangle - \langle \omega, f_4 \rangle. \end{aligned}$$

The formulas in (C.11), (C.12) and the above formula imply (C.10).

C.1.4 Derivations of the commutators in (4.81)

Recall $D_x = x\partial_x$ and the operators in (4.22). We choose $f = \theta_x, g = \omega$ in (4.22). We use the notation $u_x = H\omega$. Then $u = -\Lambda^{-1}\omega$.

Firstly, we compute the commutator related to the transport term. Using $(\bar{c}_l x + \bar{u})\partial_x = (\bar{c}_l + \frac{\bar{u}}{x})D_x$, for $p = \omega$ or θ_x, we yield

$$
\begin{aligned}
&-[D_x, (\bar{c}_l x + \bar{u})\partial_x]p = -[D_x, (\bar{c}_l + \frac{\bar{u}}{x})D_x]p \\
=&- D_x((\bar{c}_l + \frac{\bar{u}}{x})D_x p) + (\bar{c}_l + \frac{\bar{u}}{x})D_x(D_x p) = -D_x(\bar{c}_l + \frac{\bar{u}}{x})D_x p = -(\bar{u}_x - \frac{\bar{u}}{x})D_x p.
\end{aligned} \tag{C.13}
$$

Next, we compute the velocity corresponding to $D_x\omega$. Using Lemma C.0.1, we get

$$
H(D_x\omega) - H(D_x\omega)(0) = xH(\omega_x) = x\partial_x H\omega = xu_{xx}.
$$

Note that $H(D_x\omega)(0) = -\frac{1}{\pi}\int_{\mathbb{R}} \omega_x dx = 0$. We obtain $D_x u_x = xu_{xx} = H(D_x\omega)$. From

$$
(xu_x - u)_x = xu_{xx} = H(D_x\omega), \quad (xu_x - u)(0) = 0,
$$

we obtain that $xu_x - x$ is the velocity corresponding to $D_x\omega$. Therefore, we have

$$
\begin{aligned}
&H\omega = u_x, \quad -\Lambda^{-1}\omega = u, \quad H(D_x\omega)(0) = 0, \\
&H(D_x\omega) = xu_{xx}, \quad -\Lambda^{-1}(D_x\omega) = xu_x - u.
\end{aligned}
$$

Using these formulas, for $q = \bar{\omega}_x$ or $\bar{\theta}_{xx}$ we obtain

$$
\begin{aligned}
&D_x\Big(- (-\Lambda^{-1}\omega - H\omega(0)x)q\Big) - \Big(- (-\Lambda^{-1}D_x\omega - HD_x\omega(0)x)q\Big) \\
=&D_x(-(u - u_x(0)x)q) + (xu_x - u)q \\
=&- (u - u_x(0)x)D_x q + (-(xu_x - u_x(0)x))q + (xu_x - u)q \\
=&- (u - u_x(0)x)(D_x q + q).
\end{aligned} \tag{C.14}
$$

Similarly, we have

$$
\begin{aligned}
&D_x\Big(- (H\omega - H\omega(0)x)q\Big) - (-(HD_x\omega - HD_x\omega(0))q) \\
=&D_x(-(u_x - u_x(0))q) + xu_{xx}q \\
=&- D_x u_x q - (u_x - u_x(0))D_x q + xu_{xx}q = -(u_x - u_x(0))D_x q.
\end{aligned} \tag{C.15}
$$

Since $\bar{c}_\omega\omega, \theta_x$ in $\mathcal{L}_{\omega 1}$ (4.22) vanish in the commutator, applying (C.13) with $p = \omega$ and (C.14) with $q = \bar{\omega}_x$ yields the formula for $[D_x, \mathcal{L}_{\omega 1}]$ in (4.81). Note that

$$
D_x((2\bar{c}_\omega - \bar{u}_x)\theta_x) - (2\bar{c}_\omega - \bar{u}_x)D_x\theta_x = -D_x\bar{u}_x\theta_x.
$$

Combining this computation, (C.13) with $p = \theta_x$, (C.14) with $q = \bar{\theta}_{xx}$ and (C.15) with $q = \bar{\theta}_x$, we derive the formula for $[D_x, \mathcal{L}_{\theta 1}]$ in (4.81).

C.1.5 Derivation and computing C_{opt} in Section 4.3.11.3

Recall the inequality (4.67), the functions in (4.66), and the spaces Σ_i in (4.68). We use the argument similar to that in [19] to derive and compute C_{opt}.

In Section 4.3.11.3, we have reduced (4.67) to an optimization problem on the finite dimensional space $\Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3$ with $X \in \Sigma_1, Y \in \Sigma_2, Z \in \Sigma_3$. Here, we have a direct sum of spaces since there is no inner product among X, Y, Z. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be an orthonormal basis (ONB) of Σ_1 with $\mathbf{e}_1 = \frac{g_1}{||g_1||_2^2}$; $\{\mathbf{e}_5, \mathbf{e}_6, \mathbf{e}_7\}$ be that of Σ_2 with $\mathbf{e}_2 = \frac{g_5}{||g_5||_2^2}$; $\{\mathbf{e}_8, \mathbf{e}_9\}$ be that of Σ_3. Then $\{\mathbf{e}_i\}_{i=1}^9$ is an ONB of $\Sigma \triangleq \Sigma_1 \oplus \Sigma_2 \oplus \Sigma_3$.

Let $v_i \in \mathbb{R}^9$ be the coordinate of g_i in Σ under the basis $\{\mathbf{e}_i\}_{i=1}^9$ and $p = (x, y, z) \in \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^2$ be that of $X + Y + Z$. The vectors v_i and p are column vectors. By abusing notation, we also use $\langle \cdot, \cdot \rangle$ to denote the Euclidean inner product in $\mathbb{R}^9$. With these convections, each summand on the left hand side of (4.67) is a quadratic form in p. For example, we have

$$\langle X, g_1 \rangle \langle Y, g_7 \rangle = \langle p, v_1 \rangle \langle p, v_7 \rangle = (p^T v_1)(v_7^T p) = p^T (v_1 v_7^T) p.$$

Hence, (4.67) is equivalent to

$$p^T M p \le C_{opt} p^T D p, \tag{C.16}$$

where M and D are given by

$$\begin{aligned} M &= v_1 v_3^T + v_1 v_7^T - (\lambda_2 - \lambda_3 \bar{d}_\theta) v_1 v_5^T + \lambda_2 v_1 v_2^T \\ &\quad - \lambda_3 v_5 v_6^T + \lambda_3 v_5 v_4^T + \lambda_2 v_1 v_8^T - \lambda_3 v_5 v_9^T, \\ D &= Id + s_1 v_1 v_1^T + s_2 v_5 v_5^T. \end{aligned} \tag{C.17}$$

By definition of $\mathbf{e}_1, \mathbf{e}_5$, i.e. $\mathbf{e}_1 = \frac{g_1}{||g_1||_2}, \mathbf{e}_5 = \frac{g_5}{||g_5||_2}$, we have $v_1 = ||g_1||_2 E_1, v_5 = ||g_5||_2 E_5$, where $E_i \in \mathbb{R}^9$ is the standard basis of $\mathbb{R}^9$, i.e. the i-th coordinate of E_i is 1 and 0 otherwise. Therefore, D is a diagonal matrix

$$D = \mathrm{diag}(1 + s_1 ||g_1||_2^2,\ 1,\ 1,\ 1,\ 1 + s_2 ||g_5||_2^2,\ 1,\ 1,\ 1,\ 1) \in \mathbb{R}^{9\times 9}.$$

Symmetrizing the left hand side of (C.16) and using a change of variable $q = D^{1/2} p$, we obtain

$$C_{opt} = \lambda_{\max}(D^{-1/2} M_s D^{-1/2}), \quad M_s = \frac{1}{2}(M + M^T).$$

Firstly, M can be written as

$$M = V_1 V_2^T, \quad V_2 = (v_3, v_7, v_5, v_2, v_6, v_4, v_8, v_9),$$
$$V_1 = (v_1, v_1, -(\lambda_2 - \bar{d}_\theta \lambda_3) v_1, \lambda_2 v_1, -\lambda_3 v_5, \lambda_3 v_5, \lambda_2 v_1, -\lambda_3 v_5).$$

Then $M_s = \frac{1}{2}(V_1 V_2^T + V_2 V_1^T) = \frac{1}{2} U_1 U_2^T$ with $U_1 = [V_1, V_2], U_2 = [V_2, V_1] \in \mathbb{R}^{9\times 16}$. Using the argument in [19], for any even integer $p \geq 2$, we obtain

$$\begin{aligned} C_{opt} &\leq (\mathrm{Tr}|D^{-1/2} M_s D^{-1/2}|^p)^{1/p} = 2^{-1}(\mathrm{Tr}(D^{-1/2} U_1 U_2^T D^{-1/2})^p)^{1/p} \\ &= 2^{-1}(\mathrm{Tr}(U_2^T D^{-1} U_1)^p)^{1/p}. \end{aligned} \tag{C.18}$$

We will explain how to rigorously estimate the bound above in the Supplementary Material [21].

C.1.6 Estimate of $\mathcal{T}$ in Section 4.3.12

For $\lambda_2, \lambda_3, t_{61}, \kappa, r_{c_\omega} > 0$ chosen in (C.25), Appendix C.2 and t_{62} determined by these parameters, we define T_i and s_i

$$\begin{aligned} &T_1 = (-\lambda_1 D_\omega - A_\omega \varphi^{-1} - \lambda_1 \kappa)\varphi - t_{61} x^{-4}, \quad T_2 = (-D_\theta - A_\theta \psi^{-1} - \kappa)\psi, \\ &T_3 = 25 t_{61} x^{-4} + t_{62} x^{-4/3}, \quad s_1 = -\frac{\pi}{2}\lambda_2(\bar{c}_\omega + \bar{u}_x(0)) - r_{c_\omega} - \frac{\pi \lambda_1 e_3 \alpha_6}{12} - G_c, \\ &s_2 = -2\bar{c}_\omega \lambda_3 - \kappa \lambda_3, \end{aligned} \tag{C.19}$$

We will verify that $T_i > 0, s_i > 0$ later. The parameter r_{c_ω} is essentially determined by κ. See Appendix C.2.2 for the procedure to determine these parameters. Plugging the above T_i and s_i in (4.65), we can compute the upper bound of C_{opt} in (4.65) using (C.18) with $p = 36$

$$C_{opt} \leq 2^{-1}(\mathrm{Tr}(U_2^T D^{-1} U_1)^p)^{1/p} < 0.9930 < 1, \tag{C.20}$$

which is verified in (C.33), Appendix C.3. Thus from (4.65), we obtain

$$\mathcal{T} \leq ||\omega T_1^{1/2}||_2^2 + ||\theta_x T_2^{1/2}||_2^2 + ||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 + s_1 c_\omega^2 + s_2 d_\theta^2,$$

which is exactly (4.70). By definition of T_1, T_2, we have

$$\langle (D_\theta + A_\theta \psi^{-1})\psi, \theta_x^2 \rangle + \langle T_2, \theta_x^2 \rangle = -\kappa \langle \theta_x^2, \psi \rangle,$$
$$\langle (\lambda_1 D_\omega + A_\omega \varphi^{-1})\varphi, \omega^2 \rangle + \langle T_1, \omega^2 \rangle = -\kappa \lambda_1 \langle \omega^2, \varphi \rangle - t_{61} \langle \omega^2, x^{-4} \rangle.$$

Hence, plugging the above estimate on $\mathcal{T}$ in (4.69), we yield

$$\begin{aligned} J = &-\kappa ||\theta_x \psi^{1/2}||_2^2 - \kappa \lambda_1 ||\omega \varphi^{1/2}||_2^2 - t_{61} ||\omega x^{-2}||_2^2 + s_1 c_\omega^2 + s_2 d_\theta^2 \\ &+ ||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 - \Big(D_u - \frac{9}{49} t_{12} - \frac{72\lambda_1}{49} \cdot 10^{-5}\Big) ||\tilde{u}_x x^{-2/3}||_2^2 + A(u) + G_c c_\omega^2. \end{aligned} \tag{C.21}$$

It remains to estimate the u_Δ term. Recall u_Δ in (4.58) and T_3 in (C.19). A direct calculation yields

$$\begin{aligned} ||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 &= \int_0^\infty (\tilde{u} - \frac{1}{5}\tilde{u}_x x)^2 \cdot 25 t_{61} x^{-6} dx + \int_0^\infty (\tilde{u} - \frac{1}{5}\tilde{u}_x x)^2 \cdot t_{62} x^{-10/3} dx \\ &\triangleq I_1 + I_2. \end{aligned}$$

Using (4.12) with $p = 3$ and Lemma C.0.2, we get

$$I_1 = t_{61} ||\tilde{u}_x x^{-2}||_2^2 = t_{61} ||\omega x^{-2}||_2^2.$$

For I_2, using integration by parts and Lemma C.0.8 about $\tilde{u}$ with $\alpha = \beta = 0$, we get

$$\begin{aligned} I_2 &= t_{62} \int_0^\infty \frac{1}{25} \frac{\tilde{u}_x^2}{x^{4/3}} - \frac{2}{5} \frac{\tilde{u}\tilde{u}_x}{x^{7/3}} + \frac{\tilde{u}^2}{x^{10/3}} dx = t_{62} \int_0^\infty \frac{1}{25} \frac{\tilde{u}_x^2}{x^{4/3}} + \frac{1}{5} \tilde{u}^2 \partial_x x^{-7/3} + \frac{\tilde{u}^2}{x^{10/3}} dx \\ &= t_{62} \int_0^\infty \frac{1}{25} \frac{\tilde{u}_x^2}{x^{4/3}} + (1 - \frac{7}{15}) \frac{\tilde{u}^2}{x^{10/3}} dx \leq t_{62} \int_0^\infty \frac{\tilde{u}_x^2}{x^{4/3}} \left(\frac{1}{25} + \frac{8}{15} \cdot \frac{36}{49} \right) dx. \end{aligned}$$

Combining the estimates of I_1, I_2 yields

$$||\frac{u_\Delta}{x} T_3^{1/2}||_2^2 \leq t_{61} ||\omega x^{-2}||_2^2 + \Big(\frac{1}{25} + \frac{8}{15} \cdot \frac{36}{49} \Big) t_{62} ||\tilde{u}_x x^{-2/3}||_2^2, \tag{C.22}$$

We define t_{62} in Appendix C.2 so that the terms $||\tilde{u}_x x^{-2/3}||_2^2$ in (C.22) and (C.21) are almost canceled. We establish (4.71), i.e.

$$J \leq -\kappa ||\theta_x \psi^{1/2}||_2^2 - \kappa \lambda_1 ||\omega \varphi^{1/2}||_2^2 + (s_1 + G_c) c_\omega^2 + s_2 d_\theta^2 - 10^{-6} ||\tilde{u}_x x^{-2/3}||_2^2 + A(u).$$

C.2 Parameters in the estimates

C.2.1 Parameters

Parameters e_1, e_2, e_3 introduced in (4.24) are determined by the approximate self-similar profiles

$$e_1 = 1.5349, \quad e_2 = 1.2650, \quad e_3 = 1.3729. \tag{C.23}$$

We choose the following parameters for the weights ψ, φ (4.25),(4.26)

$$\alpha_1 = 5.3, \quad \alpha_2 = 3.3, \quad \alpha_3 = 0.68, \quad \alpha_4 = 12.1, \quad \alpha_5 = 2.1, \quad \alpha_6 = 0.77, \tag{C.24}$$

and the following parameters in the linear stability analysis in Section 4.3

$$\lambda_1 = 0.32, \quad t_1 = 1.29, \quad t_{12} = \frac{49}{9} \cdot 0.9 D_u, \quad t_2 = 5.5, \quad t_{22} = 13.5, \quad t_{31} = 3.2,$$
$$t_{32} = 0.5, \quad t_{34} = 2.9, \quad \tau_1 = 4.7, \quad t_4 = 3.8, \quad \lambda_2 = 2.15, \quad \lambda_3 = 0.135,$$
$$t_{61} = 0.16, \quad \kappa = 0.03, \quad r_{c_\omega} = 0.15. \tag{C.25}$$

Parameter λ_1 is introduced in (4.29), (4.31); t_2, t_{22} are introduced in the estimates of I_n (4.35), (4.37); t_1, t_{12} are introduced in the estimate of I_f (4.40), (4.42); t_4 is introduced in the estimate of I_s in (4.45); $(t_{31}, t_{32}), t_{34}, \tau_1$ are introduced in the estimate of $I_{u\omega}$ in (4.50), (4.49) and (4.47), respectively; $\lambda_2, \lambda_3, t_{61}, \kappa, r_{c_\omega}$ are introduced in (C.19) to estimate $\mathcal{T}$ in (4.70).

The parameter D_u introduced in (4.42), t_{62} in (C.19) are determined by the above parameters

$$D_u = \frac{t_1 \alpha_3 \lambda_1 \alpha_6}{\sqrt{3}}, \quad t_{62} = (D_u - \frac{9}{49} t_{12} - \frac{72\lambda_1}{49} \cdot 10^{-5} - 10^{-6})(\frac{1}{25} + \frac{8}{15} \cdot \frac{36}{49})^{-1}.$$

After we complete the weighted L^2 estimate, we choose the following parameters in the weighted H^1 estimates and nonlinear stability estimates

$$\kappa_2 = 0.024, \quad t_{71} = 2.8, \quad t_{72} = 2, \quad t_{81} = 5, \quad t_{82} = 0.7, \quad t_{91} = 1, \quad t_{92} = 1.2,$$
$$\gamma_1 = 0.98, \quad \gamma_2 = 0.07, \quad \lambda_4 = 0.005, \quad E_* = 2.5 \cdot 10^{-5}, \quad a_{H^1} = 0.31. \tag{C.26}$$

Parameters t_{7i}, t_{8i}, t_{9i} are introduced in the estimates of Q_2 (4.89), (4.95); κ_2 in (4.101); γ_1, γ_2 in (4.100); λ_4 in (4.107). Parameter a_{H1} is determined by the above parameters via $A_{\omega 2}$ (4.98) and (4.102)

$$a_{H^1} = 0.31.$$

C.2.2 Choosing parameters in $\mathcal{T}$ and determining κ

We first choose $r_{c_\omega} = \kappa \frac{\pi}{2} \lambda_2$ with small $\kappa = 0.001$. The remaining unknown parameters in the linear stability analysis are $\lambda_2, \lambda_3, t_{61} > 0$. Once $\lambda_2, \lambda_3, t_{61}$ are chosen, the functions T_i and scalars s_i in (C.19) are determined and then we can compute C_{opt} in (4.65) using the argument in Section (4.64) and Appendix C.1.5. We optimize $\lambda_2, \lambda_3, t_{61} > 0$ subject to the constraints $T_i > 0, s_i > 0$, such that $C_{opt} < 0.98$ and C_{opt} is as small as possible. Then we obtain the approximate values for $\lambda_2, \lambda_3, t_{61}$.

Our goal is to obtain κ as large as possible. The estimate of C_{opt} depends on all the parameters in (C.24)-(C.25). We gradually increase κ until $C_{opt} < 0.98$ is violated. We further refine all the parameters in (C.24)-(C.25) one by one and by modifying them around their approximate values to obtain smaller C_{opt}. Then we increase κ again. Repeating this process several times, we obtain larger κ and $\kappa = 0.03$. Finally, we increase r_{c_ω} until $C_{opt} < 0.98$ is violated. This allows us to obtain a damping term for c_ω^2 with a larger coefficient in the weighted L^2 estimate (4.74), Using this procedure, we determine the parameters in (C.24), (C.25) and further establish (4.74).

In our process of determining the parameters, we actually first use the grid point values of the functions and only need to track the constraints, e.g., $T_i > 0$, on the grid points instead of every $x \in \mathbb{R}$. After we determine all parameters, we verify the constraints rigorously by using computer-assisted analysis and establish the desired bound $C_{opt} < 0.993 < 1$ (C.20).

C.3 Rigorous verification

This section is a collection of inequalities that will be rigorously verified with the help of computer programs. The methods of computer-assisted verification are introduced and discussed in detail in the Supplementary Material [21]. All the numerical computations and quantitative verifications are performed in MATLAB (version 2020a) in double-precision floating-point operations. The MATLAB codes can be found via the link [17].

C.3.1 Ranges of the parameters

Denote by

$$
\begin{aligned}
G_1(\lambda_1, t_2, t_{22}) &\triangleq t_2 x^{-4} + \frac{t_{22}}{25} x^{-4} + t_2(\lambda_1 \alpha_5)^2 x^{-2}, \\
G_2(t_2, t_{22}) &\triangleq \frac{1}{4t_2}(\alpha_2 x^{-1} + \alpha_1 x^{-2})^2 + \frac{1}{4t_{22}}(x^3 \bar{\theta}_{xx} \psi_n)^2
\end{aligned}
$$

the coefficients in (4.38). Applying estimate (4.38) on I_n, we establish (4.39) with $c = 0.01$ if

$$
\frac{1}{\lambda_1} G_1(\lambda_1, t_2, t_{22}) \varphi^{-1} + D_\omega \leq -c, \quad G_2(t_2, t_{22}) \psi^{-1} + D_\theta \leq -c,
$$

where D_ω, D_θ defined in (4.30) are the coefficients in D_1, D_2. To verify the above estimate for $\lambda_1 \in [\lambda_{1l}, \lambda_{1u}] = [0.31, 0.33]$, $t_2 \in [t_{2l}, t_{2u}] = [5.0, 5.8], t_{22} \in [t_{22l}, t_{22u}] = [13, 14]$, since G_1, G_2 are monotone in λ_1, t_2, t_{22}, it suffices to verify

$$\frac{1}{\lambda_{1l}} G_1(\lambda_{1u}, t_{2u}, t_{22u})\varphi^{-1} + D_\omega \leq -c, \quad G_2(t_{2l}, t_{22l})\psi^{-1} + D_\theta \leq -c. \tag{C.27}$$

Similarly, in order for $I_f + D_1 + D_2 \leq -0.01(||\theta_x \psi^{1/2}||_2 + \lambda_1 ||\omega\varphi^{1/2}||_2^2)$ with estimate 4.44 on I_f and $\lambda_1 \in [\lambda_{1l}, \lambda_{1u}] = [0.31, 0.33]$, $t_1 \in [t_{1l}, t_{1u}] = [1.2, 1.4]$, $t_{12} \in [t_{12l}, t_{12u}] = [0.55, 0.65]$, it suffices to verify

$$\frac{1}{\lambda_{1l}} G_3(\lambda_{1u}, t_{1u}, t_{12u})\varphi^{-1} + D_\omega \leq -c, \quad G_4(t_{1l}, t_{12l})\psi^{-1} + D_\theta \leq -c, \tag{C.28}$$

where

$$\begin{aligned} G_3(\lambda_1, t_1, t_{12}) &= t_1 \Big(\alpha_3^2 x^{-2} + \frac{\alpha_3 \lambda_1 \alpha_6}{\sqrt{3}} x^{-4/3} + (\lambda_1 \alpha_6)^2 x^{-2/3} \Big), \\ G_4(t_1, t_{12}) &= \frac{1}{4t_1} x^{-2/3} + \frac{1}{4t_{12}} (\psi_f \bar{\theta}_{xx} x^{5/3})^2. \end{aligned}$$

In order for $I_s + D_1 + D_2 \leq -0.01(||\theta_x \psi^{1/2}||_2 + \lambda_1 ||\omega\varphi^{1/2}||_2^2)$ with estimate (4.45) on I_s and $\lambda_1 \in [\lambda_{1l}, \lambda_{1u}] = [0.31, 0.33]$, $t_4 \in [t_{4l}, t_{4u}] = [3.5, 4.0]$, it suffices to verify

$$\frac{1}{\lambda_{1l}} G_5(t_{4u}) + D_\omega \leq -c, \quad G_6(\lambda_{1u}, t_{4l}) + D_\theta \leq -c, \tag{C.29}$$

where

$$G_5(t_4) = t_4 x^{-3} \varphi^{-1}, \quad G_6(\lambda_1, t_4) = \frac{(\lambda_1 \alpha_4)^2}{4t_4} x^{-5} \psi^{-1}.$$

Remark C.3.1. We do not actually use the above estimates. Yet, they provide a useful guideline to determine the parameters t_{ij} in the estimates.

C.3.2 Inequalities on the approximate steady state

To establish the nonlinear estimates in Sections 4.3 and 4.5, we have used several inequalities on the approximate steady state and the parameters defined in Appendix C.2. These inequalities are summarized below.

In (4.30), we derive the damping terms in the weighted L^2 estimate with coefficients D_θ, D_ω. These coefficients are negative uniformly. That is, for some $c > 0$, we have

$$D_\theta, D_\omega \leq -c < 0. \tag{C.30}$$

Recall that we choose the weights T_i and s_i defined in (C.19) and apply the argument in Section 4.3.11.3 to obtain the sharp estimate of the $\mathcal{T}$ term defined in (4.64). This estimate requires that the weights are nonnegative, i.e.

$$\begin{aligned} T_1 &= (-\lambda_1 D_\omega - A_\omega \varphi^{-1} - \lambda_1 \kappa)\varphi - t_{61} x^{-4} > 0, \\ T_2 &= (-D_\theta - A_\theta \psi^{-1} - \kappa)\psi > 0, \\ T_3 &= 25 t_{61} x^{-4} + t_{62} x^{-4/3} > 0. \end{aligned} \tag{C.31}$$

and

$$\begin{aligned} s_1 &= -\frac{\pi}{2}\lambda_2(\bar{c}_\omega + \bar{u}_x(0)) - r_{c_\omega} - \frac{\pi \lambda_1 e_3 \alpha_6}{12} - G_c > 0, \\ s_2 &= -2\bar{c}_\omega \lambda_3 - \kappa \lambda_3 > 0. \end{aligned} \tag{C.32}$$

Using the above T_i, s_i and the argument in Section C.1.5, we establish the following estimate for the constant C_{opt} in (4.65)

$$C_{opt} \leq 2^{-1}(\mathrm{Tr}(U_2^T D^{-1} U_1)^p)^{1/p} < 0.9930 < 1. \tag{C.33}$$

The fact that $C_{opt} < 1$ implies (4.70).

In the weighted H^1 estimates, we have used

$$(x^2 \bar{u}_{xx} \psi)_x \leq 0.02\psi \tag{C.34}$$

in (4.87) to establish (4.88). We have also used

$$\begin{aligned} D_\theta + A_\theta \psi^{-1} - (\bar{u}_x - \frac{\bar{u}}{x}) + B_\theta \psi^{-1} &\leq -\kappa_2, \\ \lambda_1 D_\omega + A_\omega \varphi^{-1} - \lambda_1(\bar{u}_x - \frac{\bar{u}}{x}) + B_\omega \varphi^{-1} &\leq -\kappa_2 \lambda_1. \end{aligned} \tag{C.35}$$

and

$$||A_{\omega 2}\varphi^{-1}||_\infty \leq a_{H^1}, \tag{C.36}$$

originated from (4.87) and (4.102) to establish (4.103).

Appendix D

APPENDIX TO CHAPTER 5

D.0.1 Properties of the Hilbert transform and functional inequalities

The following Cotlar's identity for the Hilbert transform is well known, see e.g., [19, 39, 44].

Lemma D.0.1. *For $f \in C^\infty(S^1)$, we have*

$$H(fHf) = \frac{1}{2}((Hf)^2 - f^2).$$

We have the following commutator identity from Lemma 2.6 in [14].

Lemma D.0.2. *For $f \in H^1(S^1)$ with period $n\pi$, we have*

$$H(\sin(\frac{2x}{n})f_x) - \sin(\frac{2x}{n})Hf_x = -\frac{2}{n^2\pi}\int f\sin(2y)dy = H(\sin(\frac{2x}{n})f_x)(0).$$

The case $n = 2$ is proved in [14]. The general case follows by a rescaling argument.

We use the following important Lemma to establish the energy estimate in Section 5.3.

Lemma D.0.3. *Suppose that $\omega \in H^1$ is π-periodic and odd. We have $\int_{S^1}\omega_x H\omega_x \cdot \sin(2x)dx = 0$.*

Proof. We prove the identity for smooth function $\omega \in C^\infty$, and the general case $\omega \in H^1$ can be obtained by approximation. Applying Lemma D.0.2 with $f = \omega$ and $n = 1$ yields

$$\begin{aligned} S \triangleq \int_{S^1}\omega_x H\omega_x \cdot \sin(2x)dx &= \int_{S^1}\omega_x\Big(H(\sin(2x)\omega_x) - H(\sin(2x)\omega_x)(0)\Big)dx \\ &= \int_{S^1}\omega_x H(\sin(2x)\omega_x)dx. \end{aligned}$$

Denote $f = \sin(2x)\omega_x$. Using $\frac{1}{\sin(2x)} = \frac{1}{2}(\tan x + \cot x) = \frac{1}{2}(\cot(\frac{\pi}{2} - x) + \cot(x))$, (5.13) and Lemma D.0.1, we obtain

$$S = \frac{1}{2}\int_{S^1} (\cot(\frac{\pi}{2} - x) + \cot(x)) f \cdot Hf dx = \frac{\pi}{2}\Big(H(fHf)(\frac{\pi}{2}) - H(fHf)(0)\Big)$$
$$= \frac{\pi}{4}\Big((Hf)^2(\frac{\pi}{2}) - f^2(\frac{\pi}{2}) - (Hf)^2(0) - f^2(0))\Big).$$

Since $\omega \in C^\infty$ and it is odd, we get $f(0) = f(\frac{\pi}{2}) = 0$. Note that

$$Hf(\frac{\pi}{2}) - Hf(0) = \frac{1}{\pi}\int_{S^1} \Big(\cot(\frac{\pi}{2} - x) + \cot x\Big) \sin(2x)\omega_x dx$$
$$= \frac{1}{\pi}\int_{S^1} \frac{2}{\sin(2x)} \sin(2x)\omega_x dx = 0.$$

We obtain $S = 0$ and establish the desired result. □

We use the following Lemma from [16] to estimate the profile in Section 5.6.

Lemma D.0.4. *For $x \in [0, 1]$, $\alpha, \lambda > 0$, we have*

$$(1 - x^\alpha)x^\lambda \leq \frac{\alpha}{\lambda}.$$

We refer the proof to that of Lemma A.0.1.

We have the following Hardy-type inequality [61] in bounded domain.

Lemma D.0.5. *For $p > 1$ and $L > 0$, suppose that $fx^{-p/2}, f_x x^{-p/2+1} \in L^2([0, L])$. We have*

$$\int_0^L \frac{f^2}{x^p} dx \lesssim_p \int_0^L \frac{f_x^2}{x^{p-2}} dx.$$

It can be proved by applying an integration by parts argument. A proof can be founded in the Supplementary material of [20].

Next, we prove the commutator-type Lemma 5.6.5.

Proof of Lemma 5.6.5. A direct calculation yields

$$S \triangleq \frac{1}{x}(Hf - Hf(0)) - H(\frac{f}{x})$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} (\frac{1}{x}\cot\frac{x-y}{2} + \frac{1}{x}\cot\frac{y}{2} - \frac{1}{y}\cot\frac{x-y}{2}) f(y) dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{x}(y\cot\frac{y}{2} - (x-y)\cot\frac{x-y}{2})\frac{f(y)}{y} dy$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{x}(g(y) - g(y-x))\frac{f(y)}{y} dy,$$

where $g(z) = z\cot\frac{z}{2}$ and it satisfies $g(z) = g(-z)$. Since g is Lipschitz on $[-3\pi/2, 3\pi/2]$

$$|g'(z)| = \Big|\cot\frac{z}{2} - \frac{z}{2(\sin\frac{z}{2})^2}\Big| = \frac{|\sin z - z|}{2(\sin\frac{z}{2})^2} \lesssim \frac{z^3}{z^2} \lesssim 1,$$

applying $|g(y) - g(y-x)| \lesssim |x|$, we prove the desired result. □

D.0.2 Derivation of a model for 2D Boussinesq equations

We derive the model (5.6)-(5.7), and discuss its connections with (3.1). Recall the Boussinesq equations (5.4) and (5.5)

$$\partial_t \theta_x + \mathbf{u}\cdot\nabla\theta_x = -u_{1,x}\theta_x - u_{2,x}\theta_y = u_{2,y}\theta_x - u_{2,x}\theta_y.$$

Inspired by the anisotropic property of θ in [16], i.e. $|\theta_y| << |\theta_x|$ near the origin, we drop the θ_y term. To study the y-advection, we further drop the x-advection. Then we obtain (5.6)

$$\partial_t\theta_x + u_2\partial_y\theta_x = u_{2,y}\theta_x.$$

Since θ_x is the forcing term in the ω equation in (5.4), it leads to a strong alignment between θ_x and ω. Thus, we simplify the ω-equation in (5.4) by $\omega = \theta_x$, which leads to the following Biot-Savart law in (5.7)

$$\mathbf{u} = \nabla^{\perp}(-\Delta)^{-1}\theta_x, \quad u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x.$$

This model relates to (3.1) via the connections $\theta_x \to -\omega, \partial_{xy}(-\Delta)^{-1} \to -H$. The velocities of the two models u_2 and u are related via $u_{2,y} = \partial_{xy}(-\Delta)^{-1}\theta_x \approx -H(-\omega) = H\omega = u_x$. Moreover, the solutions of the two models enjoy similar sign and symmetry properties. Suppose that θ_x satisfies the sign and symmetry properties in the hyperbolic-flow scenario. The induced flow $u_2(x,y)$ is odd in y with $u_2(x,y) > 0$ in the first quadrant near $(0,0)$. The odd symmetries of θ_x, u_2 in y are the same as those of ω, u in (3.1) for class X (5.1). Moreover, for fixed $x > 0$, $-\theta_x(x,\cdot)$ and ω satisfy similar sign conditions, and $u_2(x,\cdot)$ and u satisfy similar sign conditions near the origin.

D.0.3 Derivation of (5.38)-(5.39)

Recall the formulas of u_x, u in (5.13) and the quadratic form in (5.34). Using integration by parts, we obtain

$$
\begin{aligned}
B(\beta) &= \int_0^{\pi/2} (2u_x\omega - (u\omega)_x) \cot^\beta x dx \\
&= 2\int_0^{\pi/2} u_x \omega \cot^\beta x dx - \beta \int_0^{\pi/2} u\omega \cot^{\beta-1} x \frac{1}{\sin^2 x} dx \triangleq I + II.
\end{aligned}
$$

The boundary terms $u\omega \cot^\beta x \Big|_0^{\pi/2}$ in the integration by parts vanish since $u(\pi/2) = 0$ and $u(x) = O(x), \omega(x) = O(x^\gamma)$ with $\gamma > \beta - 1$ near $x = 0$ by the assumption in Lemma 5.4.2.

Since ω is odd, using (5.13) and symmetrizing the kernel, we yield

$$
\begin{aligned}
I &= \frac{2}{\pi} \int_0^{\pi/2} \omega(x) \cot^\beta x \int_0^{\pi/2} \omega(y)(\cot(x-y) - \cot(x+y)) dy \\
&= \frac{1}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} \omega(x)\omega(y) P_1(x,y) dx dy,
\end{aligned}
$$

where

$$
P_1(x,y) = \cot^\beta x(\cot(x-y) - \cot(x+y)) + \cot^\beta y(\cot(y-x) - \cot(x+y)).
$$

Recall $s = \frac{\cot x}{\cot y}$ in (5.37). We get $\cot x = s \cot y$. We expand $\cot(x-y), \cot(x+y)$ as follows

$$
\begin{aligned}
\cot(x-y) &= \frac{\cot x \cot y + 1}{\cot y - \cot x} = \frac{s\cot^2 y + 1}{\cot y \cdot (1-s)}, \\
\cot(x+y) &= \frac{\cot x \cot y - 1}{\cot y + \cot x} = \frac{s\cot^2 y - 1}{\cot y \cdot (1+s)}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\cot(x-y) - \cot(x+y) &= \cot y (\frac{s}{1-s} - \frac{s}{1+s}) + \frac{1}{\cot y}(\frac{1}{1-s} + \frac{1}{1+s}) \\
&= \cot y \frac{2s^2}{1-s^2} + \frac{1}{\cot y} \frac{2}{1-s^2}, \\
\cot(y-x) - \cot(x+y) &= \cot y (-\frac{s}{1-s} - \frac{s}{1+s}) + \frac{1}{\cot y}(-\frac{1}{1-s} + \frac{1}{1+s}) \\
&= -\cot y \frac{2s}{1-s^2} - \frac{1}{\cot y} \frac{2s}{1-s^2}.
\end{aligned}
$$

Using the above formulas and $\cot^\beta x = s^\beta \cot^\beta y$, we yield

$$P_1 = \cot^\beta y \cdot s^\beta (\cot y \frac{2s^2}{1-s^2} + \frac{1}{\cot y} \frac{2}{1-s^2}) + \cot^\beta y(-\cot y \frac{2s}{1-s^2} - \frac{1}{\cot y} \frac{2s}{1-s^2})$$
$$= \cot^{\beta+1} y (s^{\beta+1} - 1) \frac{2s}{1-s^2} + \cot^{\beta-1} y (s^{\beta-1} - 1) \frac{2s}{1-s^2}.$$

We remark that $P_1 \leq 0$ since $\frac{1-s^\tau}{1-s^2} \geq 0$ for $s > 0, \tau > 0$.

For II, using (5.13), we get

$$II = \frac{\beta}{\pi} \int_0^{\pi/2} \frac{\omega(x)}{\sin^2 x} \cot^{\beta-1} x \int_0^{\pi/2} \omega(y) \log \Big| \frac{\sin(x+y)}{\sin(x-y)} \Big| dy$$
$$= \frac{1}{\pi} \int_0^{\pi/2} \int_0^{\pi/2} P_2(x,y) \omega(x) \omega(y) dx dy,$$

where
$$P_2 = \frac{\beta}{2} \Big(\frac{\cot^{\beta-1} x}{\sin^2 x} + \frac{\cot^{\beta-1} y}{\sin^2 y} \Big) \log \Big| \frac{\sin(x+y)}{\sin(x-y)} \Big|.$$

Note that
$$\frac{\cot^{\beta-1} z}{\sin^2 z} = \cot^{\beta-1} z + \cot^{\beta+1} z, \quad \Big| \frac{\sin(x+y)}{\sin(x-y)} \Big| = \Big| \frac{\cot x + \cot y}{\cot x - \cot y} \Big| = \Big| \frac{1+s}{1-s} \Big|.$$

We derive
$$P_2(x,y) = \frac{\beta}{2} \Big(\cot^{\beta+1} y (1 + s^{\beta+1}) \log \Big| \frac{1+s}{1-s} \Big| + \cot^{\beta-1} y (1 + s^{\beta-1}) \log \Big| \frac{1+s}{1-s} \Big| \Big).$$

We remark that P_2 is positive. Combining the formulas of P_1, P_2, we derive (5.38)-(5.39).

D.0.4 Positive definiteness of the kernel

In this subsection, we prove Lemmas 5.4.3 and Lemma 5.4.4, which are related to the positive definiteness of the kernel $K_{i,\beta}$. We establish (5.53) for $x_0 = \log \frac{5}{3}$ in Appendix D.0.4.1.

Proof of Lemma 5.4.3. We show that there exists $\beta_0 \in (1,2)$, such that conditions (5.52)-(5.54) hold for $W = W_{1,\beta}, G = G_{1,\beta}$ with $\beta \in [\beta_0, 2]$. Then using the same argument as that in Section 5.4.2.1, we obtain $G_{1,\beta}(\xi) \geq 0$ for all ξ and $\beta \in [\beta_0, 2]$.

Firstly, we impose $\beta \in [1.9, 2]$. Recall $G_{j,\beta}$ defined in (5.51)

$$G_{j,\beta}(\xi) = \int_0^\infty W_{j,\beta}(x) \cos(x\xi) dx, \tag{D.1}$$

and $W_{1,\beta}$ in (5.44), (5.46). Clearly, $W_{1,\beta}(x)$ converges to $W_{1,2}(x)$ as $\beta \to 2$ almost everywhere. Moreover, from the formula of $W_{1,\beta}$ and the decay estimate (5.47), we have

$$|W_{1,\beta}(z)| \lesssim \mathbf{1}_{|z|>1} e^{-|z|/4} + \mathbf{1}_{|z|\leq 1}(1 + |\log |z||), \tag{D.2}$$

where the term $\log |z|$ is due to the logarithm singularity $\log |s-1| = \log |e^z - 1|$ in (5.44). Thus, using dominated convergence theorem, we yield

$$\lim_{\beta \to 2^-} G_{1,\beta}(\xi) = G_{1,2}(\xi).$$

Using (D.1) and (D.2), we obtain that $G_{1,\beta}(\xi)$ is equi-continuous

$$|\partial_\xi G_{j,\beta}(\xi)| \leq \int_0^\infty |W_{j,\beta}(x)||x| dx \lesssim 1.$$

Thus, we obtain that $G_{1,\beta}(\xi)$ converges to $G_{1,2}(\xi)$ uniformly for $\xi \in [0, M]$, where M is the parameter in Lemma 5.4.3.

For x near 0, from (5.44) and (5.46), we have

$$W_{1,\beta}(x) = -\frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x}) \log |e^x - 1| + S_\beta(x),$$

where $S_\beta(x)$ is smooth near $x = 0$. Thus a direct calculation yields

$$\begin{aligned} \partial_{xx} W_{1,\beta}(x) &\geq -\frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x}) \partial_{xx} \log |e^x - 1| - \frac{C}{|x|} \\ &\geq \frac{\beta}{2}(e^{\frac{\beta+1}{2}x} + e^{-\frac{\beta+1}{2}x}) \frac{e^x}{(e^x-1)^2} - \frac{C}{|x|} \geq \frac{\beta}{x^2} - \frac{C}{|x|} \end{aligned} \tag{D.3}$$

for some absolute constant $C > 0$ and $|x| < \frac{1}{2}$. Therefore, there exists $\delta > 0$, such that

$$\partial_{xx} W_{1,\beta}(x) > 0, \quad x \in [0, \delta]. \tag{D.4}$$

Note that $W_{1,\beta}(x) = \tilde{K}_{1,\beta}(e^x)$ (5.44) is smooth for $(\beta, x) \in [1.9, 2] \times [\delta, x_0]$, where x_0 is the parameter in Lemma 5.4.3. We get that $\partial_{xx} W_{1,\beta}(x)$ converges to $\partial_{xx} W_{1,2}(x)$ uniformly for $x \in [\delta, x_0]$ as $\beta \to 2$, and that $\partial_x W_{1,\beta}(x_0) \to \partial_x W_{1,2}(x_0)$ as $\beta \to 2$.

Next, we consider the integral on W''' in (5.54). We need the decay estimate of $W'''_{1,\beta}$. For $r = e^{x_0} > 1$ and $s \geq r > 1$, performing Taylor expansion on $\log |\frac{s+1}{s-1}|$ and $\frac{1}{s^2-1}$, we obtain that the kernel $\tilde{K}_{1,\beta}$ (5.44) enjoys the expansion

$$\tilde{K}_{1,\beta} = \sum_{i \geq 1} a_i(\beta) s^{-\alpha_i(\beta)}, \quad |a_i(\beta)| \lesssim 1, \quad \max(\frac{\beta-1}{2}, \frac{i-2}{10}) \leq \alpha_i(\beta) \leq 10(i+1). \tag{D.5}$$

with $\alpha_i(\beta)$ increasing. Since the expansions for $\log|\frac{s+1}{s-1}|$ and $\frac{1}{s^2-1}$ converge uniformly for $s \geq r > 1$, the above expansion also converges uniformly. Thus, we can exchange the summation and derivatives when we compute $\partial_x^k \tilde{K}_{1,\beta}$. We are interested in the leading order term in the above expansion. It decays at least $s^{-(\beta-1)/2}$ since other terms in $\tilde{K}_{1,\beta}$ that decay more slowly, such as $s^{\frac{\beta-1}{2}}$, are canceled. Using $W_{1,\beta}(x) = \tilde{K}_{1,\beta}(e^x)$ and (D.5), for $x \geq x_0 > 0$, we yield

$$\begin{aligned} |\partial_x^3 W_{1,\beta}(x)| &= \Big|\partial_x^3 \sum_{i\geq 1} a_i(\beta) e^{-\alpha_i(\beta)x}\Big| = \Big|\sum_{i\geq 1} a_i(\beta)(-\alpha_i(\beta))^3 e^{-\alpha_i(\beta)x}\Big| \\ &\lesssim e^{-\frac{\beta-1}{2}x} \lesssim e^{-x/4}, \end{aligned}$$

where the implicit constant can depend on x_0. Note that $\partial_x^3 W_{1,\beta}(x) \to \partial_x^3 W_{1,2}(x)$ for any $x \geq x_0 > 0$ as $\beta \to 2$. Using dominated convergence theorem, we yield

$$\lim_{\beta\to 2^-} \int_{x_0}^{\infty} |\partial_x^3 W_{1,\beta}(x)| dx = \int_{x_0}^{\infty} |\partial_x^3 W_{1,2}(x)| dx. \tag{D.6}$$

Note that the conditions (5.52)-(5.54) hold with strictly inequality for $W = W_{1,2}, G = G_{1,2}$. From the uniform convergences $G_{1,\beta}(\xi) \to G_{1,2}(\xi)$ on $[0, M]$, $\partial_x^2 W_{1,\beta}(x) \to \partial_x^2 W_{1,2}(x)$ on $[\delta, x_0]$, $\partial_x W_{1,\beta}(x_0) \to \partial_x W_{1,2}(x_0)$ as $\beta \to 2$, (D.4) and (D.6), we conclude that there exists $\beta_0 \in (1, 2)$, such that (5.52)-(5.54) hold for $W = W_{1,\beta}, G = G_{1,\beta}$ with $\beta \in [\beta_0, 2]$. □

D.0.4.1 Convexity of $W_{i,\beta}$

We first establish (5.53) for $x_0 = \log \frac{5}{3}$ and then prove Lemma 5.4.4.

Since $W_{i,\beta}$ is given explicitly in (5.44), (5.46) and (5.50),, to simplify the derivations, we have used *Mathematica*. All the symbolic derivations and simplification steps are given in Mathematica (version 12) [12]. We only provide the steps that require estimates.

Suppose that $W(x) = K(e^x)$ and denote $s = e^x$. Using the chain rule, we yield

$$\begin{aligned} \partial_{xx} W_{i,\beta}(x) &= \partial_{xx} \tilde{K}_{i,\beta}(e^x) = e^{2x}(\partial^2 \tilde{K}_{i,\beta})(e^x) + e^x(\partial \tilde{K}_{i,\beta})(e^x) \\ &= s^2 \partial^2 \tilde{K}_{i,\beta}(s) + s\partial \tilde{K}_{i,\beta}(s) \triangleq I_i(s, \beta). \end{aligned} \tag{D.7}$$

To establish (5.53): $\partial_{xx} W_{1,2}(x) > 0$ for $x \in [0, x_0], x_0 = \log \frac{5}{3}$, it suffices to prove $I_1(s, 2) > 0$ for $s \in [1, 5/3]$. For $i = 1, \beta = 2$, using symbolic calculation, we yield

$$I_1(s, 2) = \frac{P_1 + P_2}{4s^{3/2}(1+s)^3}, \quad P_2 = 9(1+s)^4(1-s+s^2)\log\Big|\frac{s+1}{s-1}\Big|.$$

We do not write down the expression of P_1 since it is an intermediate term and is not used directly. We provide its formula in Mathematica [12]. Using $\log(1+z) \le z$ for $z > -1$, we yield

$$\log\left|\frac{1+s}{1-s}\right| = -\log\left|\frac{1-s}{1+s}\right| \ge -\left(-\frac{2}{1+s}\right) = \frac{2}{1+s}. \tag{D.8}$$

Using the above inequality and simplifying the expression, we yield

$$I_1(s,2) \ge \frac{1}{4s^{3/2}(1+s)^3}\left(P_1 + 9(1+s)^4(1-s+s^2)\frac{2}{s+1}\right) = \frac{P_3}{4s^{3/2}(1+s)^3},$$
$$P_3 = -\frac{2(-9+9s+27s^2-18s^3-59s^4+9s^5+9s^6)}{(s-1)^2}.$$

Since $s \in [1, \frac{5}{3}]$, using $s^i \le s^j, i \le j$ and $9s + 9s^2 \le 15 + 25 < 41$, we obtain

$$\begin{aligned}&-9+9s+27s^2-18s^3-59s^4+9s^5+9s^6\\ <&(9s+27s^2-18s^3-18s^4) + s^4(9s+9s^2-41) < 0,\end{aligned}$$

which implies $P_3 > 0$ on $[1, \frac{5}{3}]$. It follows $I_1(s,2) > 0$ on $[1, 5/3]$ and (5.53) with $x_0 = \log\frac{5}{3}$.

Next, we prove Lemma 5.4.4.

Proof. Recall $W_{2,\beta}(x) = \tilde{K}_{2,\beta}(e^x)$ and their formulas from (5.50). Denote $s = e^x$. Using (D.7), it suffices to prove that $I_2(s,\beta) \ge 0$ for all $s = e^x \ge 1$. Using symbolic calculation, we have

$$I_2(s,\beta) = \frac{\beta}{2}s^{-a}\left(I_{2,1}(s,\beta) + a^2(1+s^{2a})\log\frac{1+s}{s-1}\right), \quad a = \frac{\beta-1}{2},$$

where $I_{2,1}(s,\beta)$ is an intermediate term and its formula is given in Mathematica [12]. Since $\beta > 0$, using (D.8), we yield

$$I_2(s,\beta) \ge \frac{\beta}{2}s^{-a}\left(I_{2,1}(s,\beta) + a^2(1+s^{2a})\frac{2}{1+s}\right) \triangleq \frac{\beta}{2}s^{-a}I_{2,2}(s,\beta).$$

Next, we show that $I_{2,2}(s,\beta) \ge 0$. Simplifying the expression, we obtain

$$I_{2,2}(s,\beta) = \frac{P_1+P_2+P_3}{(s^2-1)^3}, \quad P_1 = -2a^2(s^2-1)^2(1-2s+s^{2a}),$$
$$P_2 = 8as(s^2-1)(s^2+s^{2a}), \quad P_3 = 4s(3s^2+s^4-s^{2a}-3s^{2+2a}).$$

Since $a = \frac{\beta-1}{2} \in [0, \frac{1}{2}]$ and $s \geq 1$, we get $2s - 1 - s^{2a} \geq 2s - 1 - s = s - 1 \geq 0$. Thus, we obtain $P_1, P_2 \geq 0$. Using $s^{2a} \leq s$ again, we derive

$$\begin{aligned} P_3 &\geq 4s(3s^2 + s^4 - s - 3s^3) = 4s^2(s^3 - 1 + 3s - 3s^2) \\ &= 4s^2(s-1)(s^2 + s + 1 - 3s) = 4s^2(s-1)^3 \geq 0. \end{aligned}$$

Combining the above estimates of P_i, we establish $I_2(s, \beta) \geq 0$ for $s \geq 1, \beta > 1$, which further implies $\partial_{xx} W_{2,\beta} \geq 0$ for $x \geq 0$. $\square$

D.0.5 Proof of other Lemmas

Proof of Lemma 5.5.2. Recall that $x, y \in [0, \pi/2]$ and $\beta \in [3/2, 2]$. In the following estimates, the reader can think of the special case $\beta = 2$.

For $x + y \leq \frac{\pi}{2}$, since $y \leq \frac{\pi}{2} - x$ and $\cot z$ is decreasing on $[0, \pi]$, we have

$$\cot x \cot y \geq \cot x \cot(\pi/2 - x) = 1. \tag{D.9}$$

Since $\min(x, y) \leq \frac{1}{2}(x + y) \leq \frac{\pi}{4}$, we obtain $\max(\cot x, \cot y) \geq 1$ and

$$(\cot x \cot y)^\beta \geq \cot x \cot y \geq \min(\cot x, \cot y) \geq \cot(x + y).$$

The case $x + y \geq \frac{\pi}{2}$ is trivial, and we prove (5.68) in Lemma 5.5.2. Next, we consider (5.69)

$$I \triangleq \cot y(\cot x)^{\beta-2} \wedge \cot x(\cot y)^{\beta-2} \lesssim (\cot x \cot y)^\beta + \mathbf{1}_{x+y \geq \pi/2} \cot(\pi - x - y) \triangleq J.$$

Note that $\mathbf{1}_{x+y\geq\pi/2} \cot(\pi - x - y)$ is nonnegative. Without loss of generality, we assume $x \leq y$. Since $\beta \leq 2$ and $\cot x \geq \cot y$, we get

$$I = \cot y(\cot x)^{\beta-2}.$$

Case 1: $x + y \leq \pi/2$ Since $x \leq y$ and $x \leq \frac{1}{2}(x + y) \leq \frac{\pi}{4}$, using (D.9), $\cot x \geq 1$, $\cot x \geq \cot y$ and $\beta \in [1, 2]$, we yield

$$J \geq (\cot x \cot y)^\beta \geq (\cot x \cot y)^{\beta-1} \geq (\cot y)^{\beta-1} \geq \cot y(\cot x)^{\beta-2} = I.$$

Case 2: $x + y > \frac{\pi}{2}$ In this case, J contains the term $\cot(\pi - x - y) \geq 0$.

Case 2.a: $x > \frac{\pi}{3}$. Since $y \geq x \geq \frac{\pi}{3}$, we have $\cot y \leq \cot x, \cot x \lesssim 1$ and $\cot(\pi - x - y) \geq \cot \frac{\pi}{3} \gtrsim 1$. It follows

$$I \leq \cot x(\cot x)^{\beta-2} = (\cot x)^{\beta-1} \lesssim 1 \lesssim \cot(\pi - x - y) \lesssim J.$$

Case 2.b: $x \le \frac{\pi}{3}$ and $\pi - x - y \le y$. Since $1 \lesssim \cot x$ and $\cot z$ is decreasing on $[0, \pi]$, we yield

$$I \lesssim \cot y \le \cot(\pi - x - y).$$

Case 2.c: $x \le \frac{\pi}{3}$ and $\pi - x - y \ge y$. Since $y \ge \frac{1}{2}(x+y) \ge \frac{\pi}{4}, x \le \frac{\pi}{3}$, we have

$$\cot x \gtrsim x^{-1}, \quad \cot y \gtrsim \cos y \gtrsim \pi/2 - y.$$

Note that $\pi - x - y \ge y$ implies $\pi/2 - y \ge x/2$. We yield

$$\cot x \cot y \gtrsim \frac{\pi/2 - y}{x} \gtrsim 1,$$

which along with $1 \lesssim \cot x, \cot y \le \cot x, \beta \in [1, 2]$ imply

$$I \le \cot y (\cot y)^{\beta - 2} = (\cot y)^{\beta - 1} \lesssim (\cot x \cot y)^{\beta - 1} \lesssim (\cot x \cot y)^{\beta} \lesssim J.$$

We conclude the proof of (5.69).

Next, we prove (5.70)

$$II \triangleq \cot y \mathbf{1}_{y \ge \pi/3} \lesssim (\cot x \cot y)^{\beta} + \mathbf{1}_{x+y \ge \pi/2} \cot(\pi - x - y) = J.$$

We focus on $y \ge \pi/3$. We consider three cases: (a) $x+y \le \pi/2$, (b) $x+y > \pi/2$ and $\pi - x - y \le y$, (c) $x + y > \frac{\pi}{2}, \pi - x - y \ge y$. In the first case, from (D.9), we have $J \ge 1 \gtrsim II$. In the second case, since $\cot z$ is decreasing, we get

$$J \ge \cot(\pi - x - y) \ge \cot(y) \ge II.$$

In the third case, since $x \le \pi - 2y \le \pi - 2\pi/3 \le \pi/3$, $y \ge \frac{\pi}{3}$ and $\pi/2 - y \ge x/2$, using the same argument as that in the above Case 2.c, we yield

$$\cot x \cot y \gtrsim 1, \quad J \ge (\cot x \cot y)^{\beta} \gtrsim 1 \gtrsim II.$$

So far, we conclude the proof of (5.70) and Lemma 5.5.2. □

The initial data constructed in Section 5.6.4.4 enjoys the following regularity in Sobolev space.

Lemma D.0.6. *Suppose that ω_0 satisfies $\omega_0 + \omega_\alpha \in C^\infty(S^1 \backslash \{0\})$ and $\omega_0 \in C^2(-\pi/3, \pi/3)$, then $\omega_0 + \omega_\alpha \in H^s$ for any $s < \alpha + \frac{1}{2}$.*

Proof. Let χ be a smooth even cutoff function on S^1 (2π periodic) with $\chi(x) = 1$ for $|x| \leq \frac{\pi}{8}$ and $\chi(x) = 0$ for $|x| \geq \frac{\pi}{4}$. We decompose $\omega_0 + \omega_\alpha$ as follows

$$\omega_0 + \omega_\alpha = \chi\omega_\alpha + \chi\omega_0 + (1-\chi)(\omega_0 + \omega_\alpha) = I + II + III.$$

Clearly, $II, III \in C^2 \subset H^{s_1}$ for any $s_1 \leq 2$. Denote $f_\alpha = \chi\omega_\alpha$. Since f_α is odd, it enjoys an expansion $\omega_\alpha(x) = \sum_{k\geq 1} a_k \sin(kx)$. Next, we estimate a_k. Using integration by parts, we yield

$$\begin{aligned} a_k &= C\int_0^\pi f_\alpha \sin(kx)dx = \frac{C}{k}\int_0^\pi f'_\alpha \cos kx dx \\ &= \frac{C}{k}\int_0^\pi (\mathbf{1}_{x\leq 1/k} + \mathbf{1}_{1/k\leq x\leq \pi/4}) f'_\alpha \cos kx dx \triangleq J_1 + J_2, \end{aligned}$$

where the restriction $\mathbf{1}_{x\leq \pi/4}$ is due to the fact that χ is supported in $|x| \leq \pi/4$. Recall the formula of ω_α from (5.80). A direct calculation yields

$$|J_1| \lesssim_\alpha k^{-1}\int_0^{1/k} |f'_\alpha| dx \lesssim \int_0^{1/k} |x|^{\alpha-1} dx \lesssim_\alpha k^{-1-\alpha}.$$

For J_2, using $\cos kx = \partial_x \frac{\sin kx}{k}, |\partial_x^i \omega_\alpha(x)| \lesssim |x|^{\alpha-i}$ and integration by parts again, we derive

$$\begin{aligned} |J_2| &\lesssim_\alpha k^{-1}\Big(\Big|\frac{\sin(k\cdot k^{-1})}{k} f'_\alpha(\frac{1}{k})\Big| + \frac{1}{k}\int_{1/k}^{\pi/4} \Big| f''_\alpha \sin kx \Big| dx\Big) \\ &\lesssim_\alpha k^{-1}(\frac{1}{k}(\frac{1}{k})^{\alpha-1} + \frac{1}{k}\int_{1/k}^{\pi/4} |x|^{\alpha-2} dx) \\ &\lesssim_\alpha k^{-1}(k^{-\alpha} + k^{-1}(k^{-1})^{\alpha-1}) \lesssim_\alpha k^{-\alpha-1}. \end{aligned}$$

Therefore, for $s < \alpha + \frac{1}{2}$, we establish

$$\sum_{k\geq 1} |a_k|^2 k^{2s} \leq \sum_{k\geq 1} k^{-2-2\alpha+2s} < +\infty,$$

which implies $\omega_\alpha \chi = f_\alpha \in H^s$. We conclude the proof. □

D.0.6 Rigorous verification

To establish Lemma 5.4.2, we need to verify conditions (5.52), (5.54) in Lemma 5.4.3. Note that condition (5.53) has been verified in Appendix D.0.4.1.

Since the kernel $W_{1,2}$ is explicit (5.44),(5.46), to simplify the derivations, we have used *Mathematica*. All the symbolic derivations and simplification steps are given in Mathematica (version 12). We only provide the steps that require estimates. All the numerical computations and quantitative verifications are performed in MATLAB (version 2019a) in double-precision floating-point operations. The Mathematica and MATLAB codes can be found via the link [12]. We will also use interval arithmetic [96, 100] and refer the discussions to Appendix D.0.6.4.

To obtain (5.52), using the approach in Section 5.4.2.2, we only need to verify (5.58). Conditions (5.58) and (5.54) involve a finite number of integrals and the Lipschitz constant b_1 in (5.57). Since these conditions are not tight, we use the following simple method to verify them.

To estimate the integral of f on $[A, \infty)$ with $A \geq 0$, we first choose B sufficiently large and partition $[A, B]$ into $A = y_0 < y_1 < ... < y_N = B$. We will estimate the decay of f in the far field in Appendix D.0.6.3, and treat the integral in $[B, \infty)$ as a small error. For each small interval $I = [y_i, y_{i+1}]$, we use a trivial first order method to estimate the integral

$$|I| \min_{x \in I} f(x) \leq \int_I f(x)dx \leq |I| \max_{x \in I} f(x), \quad |I| = y_{i+1} - y_i. \tag{D.10}$$

Denote by $f^u(I), f^l(I)$ the upper and lower bounds for f in I. To use (D.10), we estimate $f^l(I), f^u(I)$ for each interval $I = [y_i, y_{i+1}]$. For simplicity, we drop the dependence on I.

We simplify $W_{1,2}$ defined in (5.44),(5.46) as W. All the integrands involved in (5.58), (5.54), (5.57) are $W(x)\cos(x\xi)$ for $\xi = ih, i = 0, 1, .., \frac{M}{h}, |W(x)x|, |W'''(x)|$. To obtain the piecewise upper and lower bounds for these integrands, using basic interval arithmetics, see e.g., [55]

$$\begin{aligned} &(fg)^u = \max(f^u g^u, f^l g^u, f^u g^l, f^l g^l), \quad (fg)^l = \min(f^u g^u, f^l g^u, f^u g^l, f^l g^l), \\ &|f|^u = \max(|f^l|, |f^u|), \quad (f-g)^l = f^l - g^u, \quad (f-g)^u = f^u - g^l, \end{aligned} \tag{D.11}$$

we only need to obtain the bounds for $\cos(x\xi), W, |Wx|, W'''$. Those for x are trivial.

D.0.6.1 Upper and lower bounds for W, Wx, W'''

We simplify $\tilde{K}_{1,2}$ in (5.44) as $\tilde{K}$. Denote $s = e^x$. Using the chain rule and $W(x) = \tilde{K}(e^x) = \tilde{K}(s)$, we get

$$\begin{aligned} \partial_x^3 W(x) = \partial_x^3 \tilde{K}(e^x) &= e^{3x}(\partial^3 \tilde{K})(e^x) + 3e^{2x}(\partial^2 \tilde{K})(e^x) + e^x(\partial \tilde{K})(e^x) \\ &= s^3 \partial^3 \tilde{K}(s) + 3s^2 \partial^2 \tilde{K}(s) + s\partial \tilde{K}(s) \triangleq D^3 \tilde{K}(s). \end{aligned}$$

Since e^x is increasing, the bounds for W on $[x_l, x_u]$ and those for $\tilde{K}$ on $[e^{x_l}, e^{x_u}]$ enjoys

$$\begin{aligned} f^l = g^l(e^{x_l}, e^{x_u}), \quad f^u = g^u(e^{x_l}, e^{x_u}), \\ (f, g) = (W, \tilde{K}), \quad (\partial_x^3 W, D^3 \tilde{K}), \quad (W(x)x, \tilde{K}(s)\log s). \end{aligned} \tag{D.12}$$

Thus it suffices to get bounds for $\tilde{K}, \tilde{K}\log(s), D^3\tilde{K}$. Recall $\tilde{K}$ from (5.44) with $\beta = 2$

$$\begin{aligned} \tilde{K}(s) &= (s^{\frac{3}{2}} + s^{-\frac{3}{2}}) \log \left|\frac{s+1}{s-1}\right| - \frac{s^{\frac{3}{2}} - s^{-\frac{3}{2}}}{s^2 - 1} 2s \\ &= (s^{\frac{3}{2}} + s^{-\frac{3}{2}}) \log \left|\frac{s+1}{s-1}\right| - 2s^{-\frac{1}{2}} \frac{s^2 + s + 1}{s + 1}. \end{aligned} \tag{D.13}$$

In the interval $s \in [s_l, s_u]$ with $1 \leq s_l < s_u$, using monotonicity, e.g., $s^{3/2} \in [s_l^{3/2}, s_u^{3/2}]$, the fact that $\log \left|\frac{s+1}{s-1}\right|$ is decreasing and (D.11), we get the upper and lower bounds for $\tilde{K}$

$$\begin{aligned} \tilde{K}^l(s_l, s_u) &= (s_l^{3/2} + s_u^{-3/2}) \log \left|\frac{s_u+1}{s_u-1}\right| - 2s_l^{-1/2} \frac{s_u^2 + s_u + 1}{s_l + 1}, \\ \tilde{K}^u(s_l, s_u) &= (s_u^{3/2} + s_l^{-3/2}) \log \left|\frac{s_l+1}{s_l-1}\right| - 2s_u^{-1/2} \frac{s_l^2 + s_l + 1}{s_u + 1}. \end{aligned} \tag{D.14}$$

Next, we consider $\tilde{K}\log s$. For $s \in [s_l, s_u]$ with $s_l \geq 1$, since $\log s \geq 0$, we get

$$\tilde{K}(s)\log(s) \leq \tilde{K}^u \log(s) \leq \max(\tilde{K}^u \log s_l, \tilde{K}^u \log s_u).$$

Similarly, we obtain the lower bound for $\tilde{K}\log s$. Yet, near $s = 1$, the upper bound blows up due to $\log|s_l - 1|$ in $\tilde{K}^u$. Note that $\log s \leq s - 1$. Using (D.8), for $s \geq 1$, we get

$$\partial_s((s-1)\log\left|\frac{s+1}{s-1}\right|) = (\frac{1}{s+1} - \frac{1}{s-1})(s-1) + \log\frac{s+1}{s-1} = -\frac{2}{s+1} + \log\frac{s+1}{s-1} \geq 0.$$

Thus, $\log\left|\frac{s+1}{s-1}\right|(s-1)$ is increasing on $[s_l, s_u]$ and

$$\log\left|\frac{s+1}{s-1}\right| \log s \leq \log\left|\frac{s+1}{s-1}\right| \cdot (s-1) \leq \log\left|\frac{s_u+1}{s_u-1}\right| \cdot (s_u - 1).$$

We obtain the following improvement for the upper bound of $\tilde{K}(s)\log s$ on $[s_l, s_u]$

$$\tilde{K}(s)\log(s) \leq (s_u^{3/2} + s_l^{-3/2}) \log\left|\frac{s_u+1}{s_u-1}\right| \cdot (s_u - 1) - 2s_u^{-1/2} \frac{s_l^2 + s_l + 1}{s_u + 1} \cdot \log(s_l) \tag{D.15}$$

For $D^3\tilde{K}(s)$, firstly, using symbolic computation, we yield

$$\begin{aligned}
D^3\tilde{K}(s) &= \frac{P_{42}(s) - P_{41}(s) + P_5(s)}{P_6(s)}, \quad P_{42}(s) = 180s^3 + 180s^7, \\
P_{41}(s) &= 54s + 54s^2 + 266s^4 + 124s^5 + 266s^6 + 54s^8 + 54s^9, \\
P_5(s) &= 27(s^2-1)^4(1+s+s^2)\log\left|\frac{s+1}{s-1}\right|, \quad P_6(s) = 8(s-1)^3 s^{3/2}(1+s)^4.
\end{aligned} \tag{D.16}$$

Since $1 \le s_l < s_u$ and P_{41}, P_{42}, P_6 are increasing, we get $P_m^u = P_m(s_u), P_m^l = P_m(s_l)$ for index $m = 41, 42$ or $m = 6$. The bounds for P_5 are also trivial

$$P_5^l = 27(s_l^2-1)^4(1+s_l+s_l^2)\log\left|\frac{s_u+1}{s_u-1}\right|, \; P_5^u = 27(s_u^2-1)^4(1+s_u+s_u^2)\log\left|\frac{s_l+1}{s_l-1}\right|.$$

Using the bounds for P_{41}, P_{42}, P_5, P_6 and (D.11), we can further derive the bounds for $D^3\tilde{K}$.

D.0.6.2 Upper and lower bounds for $\cos(x\xi)$

For $f \in C^2([a,b])$ and $x \in [a,b]$, the basic linear interpolation implies $f(x) = \frac{x-a}{b-a}f(b) + \frac{b-x}{b-a}f(a) + \frac{1}{2}f''(x_1)(x-a)(x-b)$ for some $x_1 \in [a,b]$ and

$$\begin{aligned}
&\min(f(a), f(b)) - \frac{(b-a)^2}{8}||f''||_{L^\infty[a,b]} \\
\le& f(x) \le \max(f(a), f(b)) + \frac{(b-a)^2}{8}||f''||_{L^\infty[a,b]}.
\end{aligned}$$

Applying the above estimate to $f(x) = \cos(x\xi)$ and $|f''(x)| \le \xi^2$, we derive the upper and lower bounds for $\cos(x\xi)$ on $[a,b]$.

To verify (5.58), it suffices to get a lower bound for $G(\xi)$ with $\xi = jh$. Applying (D.12), (D.14), the above estimate for $\cos(x\xi)$ and (D.10), we yield

$$\int_{y_i}^{y_{i+1}} \cos(x\xi)W(x)dx \ge (y_{i+1} - y_i) \cdot I^l, \quad I(x) \triangleq \cos(x\xi)W(x).$$

The term I^l can be obtained using (D.11). For y_i close to 0, we should avoid using (D.11) to derive I^l since it involves $W^u(x_l, x_u) = \tilde{K}^u(e^{x_l}, e^{x_u})$ (D.14), which blows up near $x = 0$. For $x\xi \le \pi/2$, since $\cos(x\xi) \ge 0$, we derive I^l using

$$\cos(x\xi)W(x) \ge \cos(x\xi)W^l \ge \min((\cos(\cdot\xi))^l W^l, (\cos(\cdot\xi))^u W^l).$$

For large ξ, the above estimate is not sharp due to large oscillation in $\cos(x\xi)$. Denote $m = \frac{W_l + W_u}{2}$, $h_0 = b - a$. We consider an improved estimate

$$\begin{aligned}
&\int_a^b \cos(x\xi) W(x) dx = \int_a^b \cos(x\xi)(W(x) - m) dx + m \int_a^b \cos(x\xi) dx \\
\geq & m \frac{\sin(x\xi)}{\xi}\Big|_a^b - h_0 |\cos(x\xi)|^u |W - m|^u \\
\geq & \frac{W_l + W_u}{2} \frac{\sin(b\xi) - \sin(a\xi)}{\xi} - h_0 |\cos(x\xi)|^u \frac{W_u - W_l}{2},
\end{aligned}$$

where we have used $W - m \in [W_l - m, W_u - m] = [-\frac{W_u - W_l}{2}, \frac{W_u - W_l}{2}]$.

Using the above estimates, we obtain the lower bound of the integral in $G(\xi)$ (5.51) in a finite domain. The integrals in (5.57) and (5.54) in a finite domain are estimated similarly.

D.0.6.3 Decay estimates of $W, \partial_x^3 W$

It remains to estimate the integrals in (5.58), (5.51), (5.57) and (5.54) in the far field. For $s > 1$, using Taylor expansion, we yield

$$\log \Big| \frac{s+1}{s-1} \Big| = \sum_{k \geq 1} \frac{2}{2k-1} s^{-(2k-1)}, \quad \Big| \log \Big| \frac{s+1}{s-1} \Big| - \frac{2}{s} \Big| \leq \frac{2}{3} \sum_{k \geq 2} s^{-(2k-1)} = \frac{2}{3} \frac{s^{-3}}{1 - s^{-2}}. \tag{D.17}$$

Using the above estimate and (D.13), we obtain

$$\begin{aligned}
|\tilde{K}| &\leq \Big| s^{3/2} \cdot \frac{2}{s} - 2 s^{-\frac{1}{2}} \frac{s^2 + s + 1}{1 + s} \Big| + s^{3/2} \cdot \frac{2}{3} \frac{s^{-3}}{1 - s^{-2}} + s^{-3/2} \log \Big| \frac{s+1}{s-1} \Big| \\
&\triangleq I_1 + I_2 + I_3.
\end{aligned}$$

Note that $I_1 = \frac{2 s^{-1/2}}{s+1} \leq 2 s^{-3/2}$. We derive

$$|\tilde{K}| \leq s^{-3/2} \Big(2 + \frac{2}{3} \frac{1}{1 - s^{-2}} + \log \Big| \frac{s+1}{s-1} \Big| \Big) \triangleq s^{-3/2} \tilde{K}_{tail}(s). \tag{D.18}$$

Next, we estimate $D^3 \tilde{K}$ (D.16). Using (D.17), we decompose P_5 in (D.16) as follows

$$\begin{aligned}
&|P_5 - P_{5,M}| \leq P_{5,err}, \\
&P_{5,M} = 27 (s^2 - 1)^4 (1 + s + s^2) \frac{2}{s}, \quad P_{5,err} = 27 (s^2 - 1)^4 (1 + s + s^2) \frac{2}{3} \frac{s^{-3}}{1 - s^{-2}}.
\end{aligned}$$

Recall P_{41}, P_{42}, P_6 from (D.16). Denote $P_7 = P_{42} - P_{41} + P_{5,M}$. We estimate (D.16) as follows

$$|D^3\tilde{K}| \leq \frac{|P_{42} - P_{41} + P_{5,M}| + P_{5,err}}{P_6} \leq \frac{|P_7|}{P_6} + \frac{P_{5,err}}{P_6}. \tag{D.19}$$

By definition, P_7 is a sum of a polynomial of s and s^{-1}. Simplifying the expression of P_7 (see details in [12]) and using the triangle inequality, we yield

$$|P_7| \leq P_8 = 54 + 54s^{-1} + 216s + 270s^2 + 288s^3 + 58s^4 + 16s^5 + 482s^6 + 18s^7$$
$$\triangleq s^7 P_{8,tail}(s),$$

where $P_{8,tail} \triangleq P_8(s)s^{-7}$ is decreasing in s. For P_6 (D.16) and the error term $P_{5,err}$, we have

$$P_6 = 8(-1+s)^3 s^{3/2}(1+s)^4 \geq s^{7+3/2} \cdot 8(1-s^{-1})^3 \triangleq s^{7+3/2} P_{6,tail}(s),$$
$$\frac{P_{5,err}}{P_6} = \frac{9(1+s+s^2)}{4s^{5/2}(1+s)} \leq s^{-5/2}\frac{9(1+s)}{4} \leq s^{-3/2}\frac{9}{4}(1+s^{-1}) \triangleq s^{-3/2}E_{tail}(s).$$

Plugging the above estimates in (D.16), (D.19), we obtain

$$|D^3\tilde{K}(s)| \leq \frac{|P_7|}{P_6} + \frac{P_{5,err}}{P_6} \leq \frac{P_8}{P_6} + \frac{P_{5,err}}{P_6} \leq s^{-\frac{3}{2}}\left(\frac{P_{8,tail}}{P_{6,tail}} + E_{tail}\right) \triangleq s^{-\frac{3}{2}}\tilde{K}_{tail,2}. \tag{D.20}$$

Clearly, $\tilde{K}_{tail}(s)$ is decreasing. Since $P_{8,tail}, E_{tail}$ are decreasing and $P_{6,tail}$ is increasing, $\tilde{K}_{tail,2}$ is decreasing. Using $W(x) = \tilde{K}(e^x)$, we estimate the integrals in $G(\xi)$ (5.51) and (5.57) in the far field as follows

$$\Big|\int_B^\infty W(x)\cos(x\xi)dx\Big| \leq \tilde{K}_{tail}(e^B)\int_B^\infty e^{-3x/2}dx = \tilde{K}_{tail}(e^B)\frac{2}{3}e^{-3B/2},$$
$$\int_B^\infty |W(x)x|dx \leq \tilde{K}_{tail}(e^B)\int_B^\infty e^{-3x/2}x dx = \tilde{K}_{tail}(e^B)\left(\frac{2B}{3} + \frac{4}{9}\right)e^{-3B/2}, \tag{D.21}$$

and treat them as error. Similarly, we estimate the integral in (5.54) in the far field.

So far, we conclude the estimates of all the integrals in (5.58), (5.51), (5.57) and (5.54).

D.0.6.4 Interval arithmetic

To implement the above estimates and verify (5.58), (5.54) rigorously, we adopt the standard method of interval arithmetic [96, 100]. In particular, we use the

MATLAB toolbox INTLAB (version 11 [99]) for the interval computations. Every single real number p involved in the above estimates is represented by an interval $[p_l, p_r]$ that contains p, where $[p_l, p_r]$ are some floating-point numbers. We refer to [19, 20, 55] for related discussion.