MANY-VALUED LOGICS

Thesis

by

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INTRODUCTION

This thesis is not intended to add any substance to the philosophical controversies being waged about "many-valued logics". Its purpose is to develope, for the first time as far as the writer knows, the interesting mathematical theory which lies behind these logics. It is an effort to provide a sane method of attack on discovering the properties of operations on a finite number of elements to a finite number of elements.

It is likely that the greatest practical application in the future of the study of these logics will lie in the discovery of relations between operations on a finite number of elements and the principles upon which they are based. Consider the case when n is 2, the Boolean Algebra, and all of the binary operations on two elements such that the result of the operation is again one of two elements. Study of all possible combinations shows that there are 16 operations definable that satisfy these conditions. The relationships between these operations and their properties can be fairly easily established by trial and error methods because of the smallness of their number. Now take n as 3. All of the possible binary operations on 3 elements to 3 elements are 16,183 in number. In case we wish to select a particular binary operation, the methods of trial and error are impractical. This problem was faced when the problem of finding a single binary operation which would generate each of the other binary operations was suggested. The analogous case for n is 2 was solved by inpection of the 16 operations. The number of binary operations on n elements to n elements is n^{n^2}.

Lewis and Langford have proven propositions in 3-valued logic. To do this they made a table in which they placed the results of applying each of 3 truth-values to the proposition. If the proposition had the truth-value "certainly true" in each case, then the proposition was said to be assertable. It is obvious that for the case of a general n we can not make out a table of this sort. Besides, this would involve the proving of a proposition for each value of n. Because of these difficulties we abandon this method and develope a more general one. Then we prove propositions which will hold for any finite integral value of n. These proofs hold as well for n = 1,000,000 as n = 2.

Summarizing briefly the results of the following chapters, we have:

(1) Generalized many results of the Boolean Algebra, obtaining a single binary operations which will generate all of the remaining operations of the logic; generalized the Boolean expansion.

(2) Obtained more completely the properties of five types of implication, one of which was defined by Lukasiewicz.

(3) Discovered which of the important propositions of Whitehead and Russell carry over to L_n for five different types of implication.

(4) Developed two types of arithmetic of L_n.

In closing, I wish to express my appreciation to Professor E.T. Bell for his direction of this thesis and his many suggestions relating to this work.

CHAPTER ONE

THE DEVELOPMENT OF THE ALGEBRA OF N-VALUED LOGIC

INTRODUCTION. In 1920 Lukasiewicz[1] defined in terms of a matrix a "three-valued logic". A year later Post[2] generalized two-valued truth systems, giving an m-valued system. This system was defined in terms of two operators which were generalizations of the negation and disjunction of two-valued logic. Lukasiewicz[1] gave a short characterization of an n-valued system in 1922. This was followed by a paper[3] in 1930 defining implication and negation for an n-valued system. Lewis[4] and Langford extended the results concerning the three-valued logic given by Lukasiewicz and Tarski in their papers by using the truth-tables in terms of which Post had defined his n-valued system.

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1. See J. Lukasiewicz and A.Tarski,"Untersuchungen über den Aussagenkalkül", C.R. Soc.d.Sc. et d. Let. de Varsovie, XXIII (1930), Classe III, p. 32, footnote 5; p. 39, footnote 17. This paper is referred to as LT in subsequent references.
2. E.L. Post,"Introduction to a General Theory of Propositions", Amer. Jour. of Math., XLIII (1921), pp. 163-185. See particularly pp. 180-185. Referred to as P.
3. Lukasiewicz,"Philosophische Bemerkungen zu mehrwertigen Systemen des Aussagenkalküls", C.R. Soc.d.Sc. et d. Let. de Varsovie,XXIII (1930), Classe III, pp. 51 ff. Called L.
4. Lewis and Langford: Symbolic Logic, Century Co., 1932. See in particular Chapter VII. Henceforth called LL.

In this chapter we define the implication and negative of Lukasiewicz in terms of the negative and disjunctive of Post. These in turn are defined in terms of a single operator[5], $p|q$. The operators of Post are introduced since they will allow us to generate the matrix of any order function on n truth-values. This statement may not be made about the implication and negation of Lukasiewicz since they are not symbolically complete[6]. To the above operators we add three others, equivalence and two products, pq and pxq. With these relations we develop an extension of the algebra on two truth-values to n truth-values. A large portion of the properties of Chapter Two of Symbolic Logic by Lewis and Langford have been generalized. By generalization we mean that in the case when n is two the generalized property becomes the Boolean property of which it is the generalization. In the next chapter this algebra is applied to the propositions listed by Whitehead and Russell as the most important in divisions 2,3,4, and 5 of the Principia Mathematica.

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5. Webb,"Definition of Post's Generalized Negative and Maximum in Terms of one Binary Operation", Amer. Jour. of Math., LVIII (1936), pp. 193-194. Post was familiar with a result of this nature. See P p. 183, §15. His operator was defined by $p|q . \equiv . \sim_m p . \vee_m . \sim_m q$. The definition in the paper cited above was $p|q . \equiv . \sim_m (p \vee_m q)$.

6. For this statement I am indebted to Mr. J.C.C. McKinsey of the University of California. For the definition of "symbolically complete" see LL page 231.

The notation used follows that of Whitehead and Russell. Np was introduced instead of $\sim p$ to avoid confusion with Post's negation. It is more convenient to have the subscript of the truth-values, t_i, range $i = 0,1,\ldots,n-1$ than in the traditional manner since this allows the use of congruences.

NOTATION AND DEFINITIONS. Let L_n be a logic of n, where n is a positive integer, truth-values t_0, t_1, ..., t_{n-1}. The t_i are marks such that to each of the t_i any one of the n truth-values of the system may be assigned. One interpretation that may be given to them is that t_i is less likely to be true than t_j if $i<j$, t_i is as likely to be true as t_j if $i = j$, and t_i is more likely to be true than t_j if $i>j$. Then t_{n-1} is taken to be certainly true and t_0 certainly false. During the remainder of this chapter t_{n-1} may be interpreted as being certainly true since we accept a proposition as being assertable when we can show that it has the truth-value t_{n-1} for all possible truth-values that the component elementary propositions may assume. Since these demonstrations depend upon the subscripts and not upon the truth-values correlated with the subscript, we can correlate any truth-value with t_{n-1} and obtain a series of propositions having the truth-value correlated to t_{n-1} for all possible truth-values of its component propositions.

Let L_n be the logic based on the implication and negation of Lukasiewicz[7] and in the case of L_3, as modified by Lewis and

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7. See LT and L.

Langford[8]. P_n represents the logic of Post[9]. p, q, r, x, y, and z are elementary propositions in L_n, L_n and P_n. $p \in L_n$ signifies that p is in L_n, etc.

In place of the matrices of Lewis and Langford, we adopt arithmetical methods of showing what values the matrix possesses. We denote this arithmetic, which includes the ordinary operations of $+$, $-$, $\cdot$, $=$, $>$, $<$, defined as usual, and the integers a, b, ..., e, h, i, j, k by A. To associate the truth-values $t_i \in L_n$ with $i \in A$, we use the following two symbols:

If p has the truth-value t_i, $[p]' = i \quad (i = 0,1,\ldots,n-1)$. Df

$[f(p,q,\ldots,r;a,b,\ldots,e)]$ indicates Df

1. $f(p,q,\ldots,r;a,b,\ldots,e)$ is considered as a convenient method of writing $f([p]',[q]',\ldots,[r]';a,b,\ldots,e)$ where $p,q,\ldots,r \in L_n$; $a,b,\ldots,e \in A$.

2. $f([p]',[q]',\ldots,[r]';a,b,\ldots,e)$ may be any rational polynomial with arguments and operations in A.

3. If for $[p]' = i$, $[q]' = j$,..., $[r]' = k$

$f([p]',[q]',\ldots,[r]';a,b,\ldots,e) = c$, then $[f(p,q,\ldots,r;a,b,\ldots,e)] = d$ where $c \equiv d \bmod n$, $0 \leqq d < n$.

If we enclose a system of brackets in another set of brackets, we shall consider the expression to mean that we shall operate with the inner brackets before considering the outer set of brackets; e.g., by $[[t_i]'+j]$ we mean $[i+j]$.

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8. See LL Chapter VII.

9. See P.

The chief difference between $[\ ]$ and $[\ ]'$ is that all operations indicated in $[\ ]'$ are in L_n while all operations indicated in $[\ ]$ are in A. An example is $[p \supset q]' = [n-1+q-p]$ if $[p] \succ [q]$. This statement might be written as follows: If p has the truth-value t_i and q the truth-value t_j, where $i \succ j$, then the truth-value of $t_i \supset t_j$ is t_k where $k = n-1+j-i$.

It is convenient to define:

If $[p] = i$, $[q] = j$, then Df

$[\max(p,q)] = j$ where $i \leqq j$

$= i$ where $i > j$.

If $[p] = i$, $[q] = j$, then Df

$[\min(p,q)] = j$ where $i \geqq j$

$= i$ where $i < j$.

It is evident from the properties of congruences that $[[a] + b]$ may be written as $[a + b]$. Accordingly, we shall consider $[a + \max(p,q)]$ to mean $[[a] + [\max(p,q)]]$, etc.

Dots are used here as in the two-valued logic for punctuation.

We shall define all operations of L_n in terms of $p|q$. The truth-table for $p|q$ is given by:

1.01 $[p|q]' = [1 + \max(p,q)]$ Df

Other operations in L_n to be used are defined as follows:

1.02 $p^0 .=. p$, $p^{i+1} .=. p^i | p^i$ $(i = 0,1,\ldots,n-3)$ Df

1.03 $p \vee q .=. (p|q)^{n-1}$ Df

1.04 $Np .=. p^{[2n-2p-1]}$ Df

1.05 $pq .=. N(Np . \vee . Nq)$ Df

We shall at times find it more convenient to use $p.q$ than pq. In such instances $p.q$ will be considered as merely another way of writing pq.

1.06 $p \supset q .=. (pq)^{[n-1-p]}$ Df

1.07 $p \equiv q .=: p \supset q . q \supset p$ Df

1.08 $p \times q .=. (p^{n-1} \vee q^{n-1})^{1}$ Df

THEOREMS READILY DEDUCIBLE. From the preceding definitions we can readily show the following theorems concerning their properties.

1.1 $[p^h]' = [p+h]$

Proof: $[p^h]' = [p^{h-1} | p^{h-1}]' = [1 + \max([p^{h-1}]', [p^{h-1}]')]$ (1.02,1.01)

$= [1+[p^{h-1}]']$

Continuing this process

$$[p^h]' = [1+[p^{h-1}]'] = [2+[p^{h-2}]'] = \dots = [h+[p^0]']$$

Or $[p^h]' = [h+p]$.

1.2 $[p \vee q]' = [\max(p,q)]$

Proof: $[p \vee q]' = [(p|q)^{n-1}]' = [1+\max(p,q)+n-1] = [\max(p,q)]$
(1.03,1.1)

1.3 $[Np]' = [n-1-p]$

Proof: $[Np]' = [p+2n-2p-1] = [n-1-p]$ (1.04,1.1)

1.4 $[pq]' = [\min(p,q)]$

Proof: $[pq] = [n-1-\max(n-1-p,n-1-q)]$ (1.05,1.3,1.2)

If $[p] < [q]$, $[n-1-p] > [n-1-q]$ and $[pq]' = [p]$

If $[p] > [q]$, $[n-1-p] < [n-1-q]$ and $[pq]' = [q]$

If $[p] = [q]$, $[n-1-p] = [n-1-q]$ and $[pq]' = [p] = [q]$

Hence $[pq]' = [\min(p,q)]$

1.5 $[p \supset q]' = n-1$ if $[p] \leqq [q]$

$= [n-1+q-p]$ if $[p] > [q]$

Proof: $[p \supset q]' = [\min(p,q)+n-1-p]$ (1.06,1.1)

If $[p] \leqq [q]$, $[p \supset q]' = [p+n-1-p] = n-1$

If $[p] > [q]$, $[p \supset q]' = [n-1+q-p]$

1.6 $[p \equiv q]' = n-1$ if and only if $[p] = [q]$

$= [n-1+q-p]$ if $[p] > [q]$

$= [n-1+p-q]$ if $[p] < [q]$

Proof: (1.07,1.4,1.5)

From this result we readily see that 1.07 implies that two propositions may be asserted as equivalent when and only when they have the same truth-table. Using 1.6 and the preceding theorems we immediately get the following relations of equivalence between the operations in L_n with those of L_n and P_n. The difference in notation of truth-values must be considered. t_m in L_n becomes $\frac{m}{n-1}$ in L_n and t_{n-m} in P_n. Thus $[n-1+q-p]$ in L_n becomes $[1+q-p]$ in L_n and $[1+q-p]$ in P_n.

1.7[10] $p \supset q . \equiv . pCq$; $p \supset q . \in L_n$, $pCq . \in L_n$.

1.8[11] $Np . \equiv . (Np)$; $Np . \in L_n$, $(Np) \in L_n$.

1.9[11] $p \equiv q . \equiv . pEq$, $p \equiv q . E . pEq$; $p \equiv q . \in L_3$, $pEq . \in L_3$.

1.10 $p \supset q . \supset q : \equiv . p \vee q$

Proof: $[p \supset q . \supset q]' = [n-1+q-(n-1)] = [q]$ if $[p] \leqq [q]$

Since $[n-1+q-p] > [q]$ if $[p] \neq n-1$

$= [q]$ if $[p] = n-1$, we have

10. See LL p. 213 footnote or L p. 72.

11. See LL p. 214. pEq is undefined for L_n.

$[p \supset q . \supset q]' = [n-1+q-(n-1)-q+p] = [p]$ if $[p]>[q]$ and $[p] \not> n-1$

$[p \supset q . \supset q]' = n-1$ if $[p] = n-1$

Hence, $[p \supset q . \supset q]' = [\max(p,q)] = [p \vee q]'$

1.11[12] $p \vee q . \equiv . pOq$; $pVq . \in L_3$, $pOq . \in L_3$.

1.12[12] $pq . \equiv . pAq$; $pq . \in L_3$, $pAq . \in L_3$.

1.13[13] $p^1 . \equiv . \sim_n p$, $p^i . \equiv . \sim_n^i p$ $(i = 2,3,\ldots,n-1)$; p^1, $p^i . \in L_n$;

$\sim_n p$, $\sim_n^i p . \in P_n$.

1.14[13] $pVq . \equiv . pV_n q$; $pVq . \in L_n$, $pV_n q . \in P_n$.

1.15 $[p \times q]' = 0$ if $[pq]' = 0$

$= [\max(p,q)]$ if $[pq]' \neq 0$

Proof: $[p \times q]' = [1 + \max([n-1+p], [n-1+q])]$ (1.08,1.1,1.2)

. If $[pq]' = 0$, $[p \times q]' = [1+n-1] = 0$

If $[pq]' \neq 0$, $[\max([n-1+p], [n-1+q])] = [n-1+\max(p,q)]$

Or $[p \times q]' = [1+n-1 \max(p,q)] = [\max(p,q)]$ if $[pq]' \neq 0$

1.16 $pVq . \equiv . qVp$

1.17 $(pVq)Vr . \equiv . pV(qVr)$

Proof: $[(pVq)Vr]' = [\max\{\max(p,q),r\}] = [\max\{p,\max(q,r)\}]$

$= [pV(qVr)]'$

1.18 $pq . \equiv . qp$

1.19 $(pq)r . \equiv . p(qr)$

1.20 $p \times q . \equiv . q \times p$

1.21 $p \times (q \times r) . \equiv . (p \times q) \times r$

Proof: If $[(pq)r]' = 0$ then 1.21 reduces to 1.18, if $[(pq)r]' \neq 0$ then 1.21 reduces to 1.17

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12. See LL p. 214.

13. See P p. 180.

1.22 $p(qVr).\equiv:pq.V.pr$

Proof: By 1.16 without loss of generality we can take $[q] \geqq [r]$.

If $[p] \geqq [q] \geqq [r]$, then $[p(qVr)]' = [q]$ and $[pq.V.pr]' = [q]$

If $[q] > [p] \geqq [r]$, then $[p(qVr)]' = [p]$ and $[pq.V.pr]' = [p]$

If $[q] \geqq [r] > [p]$, then $[p(qVr)]' = [p]$ and $[pq.V.pr]' = [p]$.

1.23 $pV(qr).\equiv.(pVq)(pVr)$

Proof: Similar to that of 1.22.

1.24 $pV(qVr).\equiv:pVq.V.pVr$

Proof: $[\max\{p,\max(q,r)\}] = [\max\{\max(p,q),\max(p,r)\}]$.

1.25 $p\times(qVr).\equiv:p\times q.V.p\times r$

Proof: If $[p] = 0$, $[p\times(qVr)]' = 0$ and $[p\times q.V.p\times r]' = 0$

If $[q] = 0$, $[qVr]' = [r]$ and $[p\times q.V.p\times r]' = [p\times r]'$

Hence $[p\times(qVr)]' = [p\times q.V.p\times r]'$ if $[q] = 0$.

Similarly for $[r] = 0$.

If $[p] \neq 0$, $[q] \neq 0$, $[r] \neq 0$, then we can replace $\times$ by V and 1.25 becomes 1.24.

1.26 $p.\equiv:p.V.pq$

Proof: $[p.V.pq]' = [\max\{p,\min(p,q)\}]$

If $[p] \leqq [q]$ then $[p.V.pq]' = [p]$

If $[p] > [q]$ then $[p.V.pq]' = [p]$

1.09 $pVqVr.=.pV(qVr)$, $pqr.=.p(qr)$, $p\times q\times r.=.p\times(q\times r)$ Df

1.010 $X_i.=.x^0 x^1 \ldots x^{i-1} x^{i+1} \ldots x^{n-1}$ Df

1.011 $\sum_{i=0}^{n-1} x_i.=.x_0 V x_1 V \ldots V x_{n-1}$ similarly for $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\cdots\sum_{k=0}^{n-1} x_{i,j,\ldots,k}$ Df

where x_i and $x_{i,j,\ldots,k}$ $(i,j,\ldots,k = 0,1,\ldots,n-1)$ are any elementary propositions.

1.27 $[X_i]' = 1$ if $[x] = n-i$

$= 0$ if $[x] \neq n-i$

Proof: If $[x] = n-j$ where $j \neq i$ then $[x^j]' = [n-j+j] = 0$. But x^j occurs in X_i, hence $[X_i]' = 0$ if $[x] \neq n-i$ (1.4, 1.09, 1.010, 1.18). If $[x] = n-i$, $[x^i]' = 0$, $[x^{[i+1]}]' \neq 1$ and $[X_i]' = 1$, since t_0 does not occur in X_i.

1.28 $[X_i \times Y_j \times \ldots \times Z_k]' = 1$ if $[x] = n-i, [y] = n-j, \ldots, [z] = n-k$

$= 0$ otherwise.

Proof: (1.27, 1.15).

1.29 $X_i X_j . \equiv . t_0$ $\qquad i \neq j$

Proof: (1.4, 1.27).

1.30 $X_i \times X_j \equiv . t_0$ $\qquad i \neq j$

Proof: (1.15, 1.27).

1.31 $\sum_{i=0}^{n-1} X_i . \equiv . t_1$

Proof: If $[x] = n-i$, $[X_i]' = 1$, $[X_0 V X_1 V \ldots V X_{i-1} V X_{i+1} V \ldots V X_{n-1} = 0$ and $\left[\sum_{j=0}^{n-1} X_j\right]' = 1$ $(i = 0, 1, \ldots, n-1)$. (1.2, 1.27).

1.32 $\left\{\sum_{i=0}^{n-1} X_i\right\} \times \left\{\sum_{j=0}^{n-1} Y_j\right\} \times \ldots \times \left\{\sum_{k=0}^{n-1} Z_k\right\} . \equiv . t_1$

Proof: (1.31).

1.33 $\left\{\sum_{i=0}^{n-1} X_i\right\} \times \left\{\sum_{j=0}^{n-1} Y_j\right\} \times \ldots \times \left\{\sum_{k=0}^{n-1} Z_k\right\} . \equiv . \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\cdots\sum_{k=0}^{n-1}\{X_i \times Y_j \times \ldots \times Z_k\}$

Proof: (1.28, 1.32, 1.2, 1.17).

1.012 By $F(x, y, \ldots, z)$ we represent any function of $x, y, \ldots, z$. Df

Or to be more explicit, if $[x] = i, [y] = j, \ldots, [z] = k$, then $F(x, y, \ldots, z)$ has a definite value h. $F(t_i, t_j, \ldots, t_k)$ has the truth-value that $F(x, y, \ldots, z)$ has when $x, y, \ldots, z$ are replaced by $t_i, t_j, \ldots, t_k$ respectively.

1.013 If $F(x, y, \ldots, z) . \equiv . \sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\cdots\sum_{k=0}^{n-1}\{A_{i,j,\ldots,k} \times (X_{n-i} \times Y_{n-j} \times \ldots \times Z_{n-k})\}$ then $\sum_{i=0}^{n-1}\sum_{j=0}^{n-1}\cdots\sum_{k=0}^{n-1}\{A_{i,j,\ldots,k} \times (X_{n-i} \times Y_{n-j} \times \ldots \times Z_{n-k})\}$, where $A_{i,j,\ldots,k}$ represents some definite truth-value t_h which depends upon $F(x, y, \ldots, z)$, is said to be the normal form for $F(x, y, \ldots, z)$.

1.34 $F(x) .\equiv. \sum_{i=0}^{n-1} F(t_i) \times X_{n-i}$

Proof: $[F(t_i) \times X_{n-i}]' = [F(t_i)]'$ if $[x] = i$

$= 0$ otherwise (1.15, 1.27)

Hence $\left[\sum_{i=0}^{n-1} F(t_i) \times X_{n-i}\right]' = [F(t_i)]'$ if $[x] = i$.

1.35 $F(x,y,\ldots,z) .\equiv. \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \cdots \sum_{k=0}^{n-1} \{F(t_i,t_j,\ldots,t_k) \times (X_{n-i} Y_{n-j} \ldots Z_{n-k})\}$

Proof: Generalization of that of 1.34.

The above is a proof that $F(x,y,\ldots,z)$ may always be expressed in normal form. $A_{i,j,\ldots,k}$ is $F(t_i,t_j,\ldots,t_k)$.

1.36 $x^i X_i .\equiv. t_o$, $x^i \times X_i .\equiv. t_o$

Proof: If $[x] = n-i$, $[x^i]' = 0$. Hence $[x^i X_i]' = 0$ and $[x^i \times X_i]' = 0$ for $[x] = i$ $(i = 0,1,\ldots,n-1)$. (1.4, 1.27).

X_1 becomes $\sim x$ in L_2. This allows us to consider 1.36 as a generalization of $x(\sim x) = 0$ in L_2. 1.35 can be used as a proof that any function $F(x,y,\ldots,z)$ can be generated from $p|q$. In such a proof we would substitute $(t_o)^k$, where

$$k = [F(t_i,t_j,\ldots,t_h)]'$$

for $F(t_i,t_j,\ldots,t_h)$ as the coefficient of $X_{n-i} Y_{n-j} \ldots Z_{n-h}$. Then, since we have obtained t_o in terms of $p|q$, we have obtained $F(x,y,\ldots,z)$ in terms of $p|q$. This may also be considered as a means of determining an expression for any single valued function on a finite number of elements to a finite number of elements. We also see that in L_2 1.35 becomes the Boolean expansion.

CHAPTER TWO

IMPLICATION

In L_2 we find that $p \supset p$ and $p \supset q . q \supset r : \supset : p \supset r$ hold. However in L_n the implication of Lukasiewicz fails to possess the latter property, that of transitivity. There are a great number of possible choices of matrices defining implication relations[15]. If by pIq we mean "p implies q" and if we take $[pIq]' = n-1$ if and only if $[p] \leqq [q]$ it is interesting to discover the necessary and sufficient conditions that must be imposed upon pIq before the proposition pIq.qIr:I:pIr holds in L_n. Using our earlier interpretation for truth-values we see that the condition $[pIq]' = n-1$ if and only if $[p] \leqq [q]$ involves the principle "a proposition implies any which is equally or more probable; and is implied by any which is equally or less probable."[16]

An investigation of the possibilities of pIq defined in the above manner leads us to the following theorem:

6.1 If $[pIq]' = n-1$ when and only when $[p] \leqq [q]$ then in order for $[pIq.qIr:I:pIr]' = n-1$ where p,q,r may assume any truth-value t_i, t_j, t_k it is necessary and sufficient that

1. If $[p] > [q] > [r]$
   then $[pIq]' \geqq [pIr]'$
   and $[pIr]' \leqq [qIr]'$
2. And if p,q,r have particular truth-values, say t_i, t_j, and t_k respectively, where $i > j > k$ then either

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15. See in particular LL pp. 229 and 230.

16. LL p. 230.

$[qIr]' \geq [pIr]'$

and $[pIq]' = [pIr]'$

or $[qIr]' = [pIr]'$

and $[pIq]' \geq [pIr]'$

but we cannot have both of the relations

$[qIr]' > [pIr]'$

and $[pIq]' > [pIr]'$ holding simultaneously.

Proof: 1. If $[p] \leq [r]$

then $[pIr]' = n-1$, making $[pIq.qIr:I:pIr]' = n-1$.

2. $[p] > [r]$

a. $[r] \geq [q]$.

Then $[pIq.qIr:I:pIr]'$ becomes $[pIq.I.pIr]'$ since $[qIr]' = n-1$.

Therefore, if $[p] > [r] \geq [q]$ then

$[pIq]' \leq [pIr]'$ in order for the proposition pIq.qIr:I:pIr to hold since by hypothesis $[pIq]' = n-1$ if and only if $[p] \leq [q]$.

b. $[q] \geq [p] > [r]$.

$[pIq.qIr:I:pIr]'$ becomes $[qIr.I.pIr]'$ since $[pIq]' = n-1$.

Hence, as above, if $[q] \geq [p] > [r]$ then

$[qIr]' \leq [pIr]'$.

c. $[p] > [q] > [r]$.

In pIq.qIr:I:pIr from <u>a</u> and <u>b</u> we see that the conditions

$[qIr]' \geq [pIr]'$

$[pIq]' \geq [pIr]'$ must hold.

If these conditions hold

$$[pIq.qIr]' \geq [pIr]'$$

However, it is necessary that the equality sign hold in the above statement, otherwise by hypothesis $[pIq.qIr:I:pIr]' \neq n-1$. Then, inorder for the equality sign to hold, for a particular set of truth-values such that $[p] > [q] > [r]$, it is necessary and sufficient that either

$$[qIr]' = [pIr]'$$
$$[pIq]' \geqq [pIr]'$$

or

$$[qIr]' \geqq [pIr]'$$
$$[pIq]' = [pIr]'.$$

The above conditions are evidently sufficient.

$p \supset q$ does not satisfy the conditions for transitivity when $n > 2$ since if $[p] = 2$, $[q] = 1$, $[r] = 0$

$[p \supset q]' = [n-1+0-2] = n-3$, $[p \supset q.q \supset r]' = n-2$.

Hence $[p \supset q.q \supset r: \supset :p \supset r]' = [n-1+n-3-(n-2)] = n-2$.

In other words, for these particular values of p,q,r

$$[p \supset q]' > [p \supset r]'$$
$$[q \supset r]' > [p \supset r]'.$$

There are many operations which satisfy 6.1. In particular we shall study three of these operations.

Let us make the definitions:

6.01 $pI_1q.=.p \supset q$ Df

6.02 $pI_2q.=.N\{pI_1q.I_1.(pI_1q)^1\}$ Df

6.03 $pI_3q.=:pI_2q.Vq$ Df

6.04 $pI_4q.=:pI_2q.V.Np$ Df

6.05 $pI_5q .=. pI_2q . V . q . V . Np$ Df

From these definitions by means of the theorems of the preceding chapter, we determine their truth-tables.

6.2 $[pI_1q]' = n-1$ if $[p] \leqq [q]$

$= [n-1+q-p]$ if $[p] > [q]$.

Proof: (1.5).

6.3 $[pI_2q]' = n-1$ if $[p] \leqq [q]$

$= 0$ if $[p] > [q]$

Proof: If $[p] \leqq [q]$ then $[pI_1q]' = n-1$ and $[pI_1q . I_1 . (pI_1q)^1]' = [t_{n-1} I_1 t_0]' = 0$.

Hence $[pI_2q]' = [N\{pI_1q . I_1 (pI_1q)^1\}]' = n-1$ if $[p] \leqq [q]$.

If $[p] > [q]$ then say $[pI_1q]' = i$ where $i \neq n-1$,

then $[pI_2q]' = [N\{t_i I_1 t_{i+1}\}]' = [n-1-(n-1)] = 0$.

6.4 $[pI_3q]' = n-1$ if $[p] \leqq [q]$

$= [q]$ if $[p] > [q]$

Proof: $[pI_2q]' = n-1$ if $[p] \leqq [q]$, hence $[pI_2q . V . q]' = n-1$.

$[pI_2q]' = 0$ if $[p] > [q]$, hence $[pI_2q . V . q]' = [q]$ in this case.

6.5 $[pI_4q]' = n-1$ if $[p] \leqq [q]$

$= [n-1-p]$ if $[p] > [q]$

Proof: Same type as in 6.4.

6.6 $[pI_5q]' = n-1$ if $[p] \leqq [q]$

$= [Np . V . q]'$ if $[p] > [q]$

Proof: (6.3, 1.2)

By theorems 6.3, 6.4, 6.5 it is evident that pI_2q, pI_3q, and pI_4q satisfy the necessary and sufficient conditions of 6.1 making each of these types of implication transitive. If we check the conditions for pI_5q in 6.1 for the particular values of $[p] = n-1$, $[q] = n-2$, $[r] = 0$, we find when $n > 2$ that

$$[pI_5q]' > [pI_5r]'$$

$[qI_5r]' \quad [pI_5r]'$ contrary to conditions 6.1 part 2.

Thus, we can say that pI_5q is not transitive.

It is interesting to determine which of the propositions listed as the most important by Whitehead and Russell in divisions 2,3,4, and 5 of Principia Mathematica hold in L_n. We list these important propositions in the Principia Mathematica below:

2.02 q.I.pIq

2.03 pINq.I.qINp

2.15 NpIq.I.NqIp

2.16 pIq.I.NqINp

2.17 NqINp.I.pIq

2.04 p.I.qIr:I:q.I.pIr

2.05 qIr.I:pIq.I.pIr

2.06 pIq.I:qIr.I.pIr

2.08 pIp

2.21 Np.I.pIq

3.2 p.I:q.I.pq

3.26 pq.I.p

3.27 pq.I.q

3.3 pq.I.r:I:p.I.qIr

3.31 p.I.qIr:I:pq.I.r

3.33 pIq.qIr:I:pIr

3.43 pIq.pIr:I:p.I.qr

3.35 p.pIq.I.q

3.45 pIq.I:pr.I.qr

3.47 pIr.qIs.I:pq.I.rs

4.01 pEq.=.pIq.qIp Df

4.1 pIq.E.NqINp

4.11 pEq.E.NpENq

4.13 p.E.N(Np)

4.2 pEp

4.21 pEq.E.qEp

4.22 pEq.qEr:I:pEr

4.24 p.E.pp

4.25 p.E.pVp

4.3 pq.E.qp

4.31 pVq.E.qVp

4.32 (pq)r.E.p(qr)

4.33 (pVq)Vr.E.pV(qVr)

4.4 p.qVr:E:pq.V.pr

4.41 p.V.qr:E:pVq.pVr

4.71 pIq.E:p.E.pq

4.73 q.I:p.E.pq

5.1 pq.I.pEq

5.32 p.I.qEr:E:pq.E.pr

5.6 p.Nq.I.r:E:p.I.qVr

Utilizing the results of theorems 6.2, 6.3, 6.4, 6.5, and 6.6 we obtain the following table of results. A stands for assertable and N for not assertable. Thus, in the table we find A opposite 2.02 and under pI_1q. That means if we replace I by I_1 in 2.02 above, the proposition is assertable in L_n. The rest of the table reads in the same manner.

	pI_1q	pI_2q	pI_3q	pI_4q	pI_5q
2.02	A	N	A	N	A
2.03	A	A	N	N	A
2.15	A	A	N	N	A
2.16	A	A	N	N	A
2.17	A	A	N	N	A
2.04	A	N	A	N	A
2.05	A	A	A	N	A
2.06	A	A	A	N	A
2.08	A	A	A	A	A
2.21	A	N	N	A	A
3.2	A	N	A	N	A
3.26	A	A	A	A	A
3.27	A	A	A	A	A
3.3	A	N	A	N	A
3.31	N	A	A	N	N
3.33	N	A	A	A	N
3.35	N	A	A	N	N
3.43	A	A	A	A	A
3.45	A	A	A	A	A
3.47	A	A	A	A	A
4.1	A	A	N	N	A
4.11	A	A	N	N	A
4.13	A	A	A	A	A
4.2	A	A	A	A	A
4.21	A	A	A	A	A
4.22	N	A	A	A	N

	pI_1q	pI_2q	pI_3q	pI_4q	pI_5q
4.24	A	A	A	A	A
4.25	A	A	A	A	A
4.3	A	A	A	A	A
4.31	A	A	A	A	A
4.32	A	A	A	A	A
4.33	A	A	A	A	A
4.4	A	A	A	A	A
4.41	A	A	A	A	A
4.71	A	A	A	A	A
4.73	A	N	A	N	A
5.1	A	N	A	N	A
5.32	N	N	A	N	N
5.6	N	N	N	N	N
Total of A's	33	30	31	21	33

The above table is verified on succeeding pages.

The five propositions which Lukasiewicz[14] states as bein a sufficient condition that the system L_n may be put on a postulational basis hold in L_n for pI_1q. They follow:

$p.I_1.qI_1p$, $pI_1q.I_1:qI_1r.I_1.pI_1r$, $pI_1q.I_1q:I_1:qI_1p.I_1p$,
$pI_1q.I_1.qI_1p:I_1;qI_1p$, $NpI_1Nq.I_1.qI_1p$.

By utilizing definition 4.01 $pE_iq.=.pI_iq.qI_ip$ $(i = 0,1,\ldots,5)$ we obtain the following theorems concerning the truth-tables of pE_iq:

6.7 $[pE_2q]' = n-1$ if $[p] = [q]$
$= 0$ otherwise.

14. See LT p. 41 (following Theorem 26).

6.8 $[pE_3q]' = n-1$ if $[p] = [q]$
$= [pq]'$ otherwise

6.9 $[pE_4q]' = n-1$ if $[p] = [q]$
$= [Np.Nq]'$ otherwise

6.10 $[pE_5q]' = n-1$ if $[p] = [q]$
$= [(Np.V.q)(Nq.V.p)]'$ otherwise

VERIFICATION OF THE RESULTS OF THE TABLE.

Below we have proven the results of the table. We first list the proposition number and then have five sub-numbers under this heading. The first sub-number refers to the verification of the result for pI_1q, the second for pI_2q, etc.

2.02 q.I.pIq

True for pIq when $[p] \leq [q]$.

1. $[p] > [q]$.

   $[pI_1q]' = [n-1+q-p]$

   But $[q] \leq [n-1+q-p]$ since $[n-1-p] \geq 0$.

   Hence the proposition holds for all $[p]$, $[q]$.

2. Let $[p] = 2$, $[q] = 1$, then

   $[q.I_2.pI_2q]' = 0$, so does not hold when $n > 2$.

3. If $[p] > [q]$ then $[pI_3q]' = [q]$, therefore theorem holds.

4. Let $[q] = n-2$, $[p] = n-1$, then theorem does not hold for $n > 2$.

5. $[p] > [q]$.

   $[q] \geq [Np]'$ evidently holds.

   Similarly for $[q] < [Np]'$, when $[q.I_5.pI_5q]'$ becomes $[q.I_5.Np]'$

2.03 pINq.I.qINp

True if $[n-1-p] \geq [q]$.

1. $[n-1-p] < [q]$.

Then $[pI_1Nq]' = [n-1+n-1-q-p]$ since $[p] > [n-1-q]$ and also $[qI_1Np]' = [n-1+n-1-q-p]$ making the proposition hold.

2. Holds since $[pI_2Nq]' = 0 = [qI_2Np]$ when $[n-1-p] < [q]$.

3. Proposition does not hold when $n > 2$ for $[p] = n-1$, $[q] = n-2$.

4. Proposition does not hold when $n > 2$ for $[p] = n-2$, $[q] = n-1$.

5. When $[Np]' < [q]$, then $[Nq]' < [p]$ and $[qI_5Np]' = [Nq.V.Np]'$, $[pI_5Nq]' = [Np.V.Nq]'$. Theorem holds.

2.15 NpIq.I.NqIp.

True if $[Nq] \leq [p]$.

Proof follows same lines as in 2.03.

2.16 pIq.I.NqINp.

Proof same type as in 2.03.

2.17 NqINp.I.pIq.

Proof same type as in 2.03.

2.04 p.I.qIr:I:q.I.pIr.

True if $[p] \leq [r]$.

1. $[p] > [r]$.

True if $[q] \leq [n-1+r-p]$

Take $[q] > [n-1+r-p]$, then $[q] > [r]$ and $[n-1-q+r] < [p]$.

Accordingly, $[q.I_1.pI_1r]' = [n-1+n-1+r-p-q]$

and $[p.I_1.qI_1r]' = [n-1+n-1+r-p-q]$

and the theorem holds for this case.

2. Let $[p] = 2$, $[q] = 1$, and $[r] = 1$. For this case the theorem does not hold for $n > 2$.

3. $[p] > [r]$.

Holds if $[q] \leq [r]$.

Let $[q] \geq [p] > [r]$, then $[q.I_3.pI_3r]' = [r] = [p.I_3.qI_3r]'$.

We get the same result when $[p] > [q] > [r]$, making the theorem hold.

4. Does not hold for $n > 2$ when $[p] = n-2$, $[q] = n-1$, $[r] = n-3$.

5. $[p] > [r]$.

Then $[pI_5r]' = [Np.V.r]'$.

Holds if $[q] \leqq [Np.V.r]'$

Take $[q] > [Np.V.r]'$, then $[q] > [r]$, $[q] > [Np]'$, and $[p] > [Nq]'$.

Hence $[q.I_5.pI_5r]' = [Nq.V.Np.V.r]'$

and $[p.I_5.qI_5r]' = [Nq.V.Np.V.r]'$ making theorem hold.

2.05 qIr.I:pIq.I.pIr.

True if $[p] \leqq [r]$.

1. $[p] > [r]$.

$[q] \geqq [p] > [r]$, $[pI_1q]' = n-1$ and $[pI_1r]' = [n-1+r-p]$ making

$[pI_1q.I_1.pI_1r]' = [n-1+r-p]$

$[qI_1r]' = [n-1+r-q]$, but $[n-1+r-q] \leqq [n-1+r-p]$ making theorem hold for this case.

$[p] > [q] > [r]$,

$[pI_1q.I_1.pI_1r]' = [n-1+r-q] = [qI_1r]'$, so theorem holds.

$[p] > [r] \geqq [q]$, $[pI_1r]' \geqq [pI_1q]'$ so theorem holds.

2. Holds.

3. $[p] > [r]$, $[pI_3r]' = [r]$

$[q] \leqq [r]$, holds since then $[qI_3r]' = [r]$.

$[q] \geq [p] > [r]$, $[pI_3q.I_3.pI_3r]' = [r] = [qI_3r]'$.

$[p] > [q] > [r]$, $[pI_3q.I_3.pI_3r]' = [r] = [qI_3r]'$.

4. False if $[p] = 1$, $[q] = 1$, $[r] = 0$ and $n > 2$.

5. $[p] > [r]$, then $[pI_5r]' = [Np.V.r]'$.

$[q] \leqq [r]$ holds since $[pI_5q]' = [Np.V.q]'$ and $[Np.V.q]' \leqq [Np.V.r]'$.

$[p] > [q] > [r]$, $[Np.V.q]' \geqq [Np.V.r]'$.

Theorem holds if equality sign is true.

Let then $[Np.V.q]' > [Np.V.r]'$, or $[q] > [Np.V.r]'$.

Then $[pI_5q.I_5.pI_5r]' = [Nq.V.Np.V.r]'$

But $[qI_5r]' = [Nq.V.r]'$, and $[Nq.V.r]' \leqq [Nq.V.Np.V.r]'$ so the theorem holds.

$[q] \geqq [p] > [r]$, $[pI_5q.I_5.pI_5r]' = [Np.V.r]'$ and $[qI_5r]' = [Nq.V.r]'$

$[Nq]' \leqq [Np]'$, therefore $[Nq.V.r]' \leqq [Np.V.r]'$ and the theorem holds.

3.06 pIq.I:qIr.I.pIr

True for $[p] \leqq [r]$.

1. $[q] > [p] > [r]$, $[pI_1r]' = [n-1+r-p]$, $[qI_1r]' = [n-1+r-q]$

since $[n-1+r-p] \geqq [n-1+r-q]$, $[qI_1.I_1.pI_1r]' = n-1$.

$[p] > [q] > [r]$

$$[qI_1r.I_1.pI_1r]' = [n-1+n-1+r-p-(n-1)-r+q]$$
$$= [n-1+q-p]$$

But $[pI_1q]' = [n-1+q-p]$ so theorem holds.

$[p] > [r] \geqq [q]$, $[pI_1q]' = [n-1+q-p]$

and $[qI_1r.I_1.pI_1r]' = [n-1+r-p]$

But $[n-1+r-p] \leqq [n-1+q-p]$ so theorem holds.

2. Evident.

3. $[p] > [r]$, $[pI_3r]' = [r]$.

$[q] \leqq [r]$, then $[qI_3r.I_3.pI_3r]' = [r]$ and $[pI_3q]' = [q] \leqq [r]$.

$[q] > [r]$, then $[qI_3r.I_3.pI_3r]' = n-1$.

4. Not assertable for $n > 2$ if $[p] = 1$, $[q] = 0$, $[r] = 0$.

5. $[p] > [r]$, $[pI_5r]' = [Np.V.r]'$.

$[q] \leqq [r]$, $[qI_5r]' = n-1$, $[qI_5r.I_5.pI_5r]' = [Np.V.r]'$.

$[pI_5q]' = [Np.V.q]' \leqq [Np.V.r]$, so theorem holds for this case.

$[q] \geqq [p] > [r]$, $[qI_5r]' = [Nq.V.r]'$, $[pI_5r]' = [Np.V.r]'$,

$[qI_5r]' \leqq [pI_5r]'$ making this case hold.

$[p] > [q] > [r]$, $[qI_5r]' = [Nq.V.r]'$, $[pI_5r]' = [Np.V.r]'$.

Hence theorem holds if equality is true in

$[qI_5r]' \geqq [pI_5r]'$.

If $[qI_5r]' > [pI_5r]'$, $[qI_5r.I_5.pI_5r]' = [q.V.Np.V.r]'$

but $[pI_5q]' = [Nq.V.q]'$, so theorem holds for all cases.

2.08 pIp

Evident.

2.21 Np.I.pIq

True for $[p] \leqq [q]$.

1. $[p] > [q]$, $[pI_1q]' = [n-1+q-p] \geqq [Np]$.
2. Not assertable for $[p] = 1$, $[q] = 0$ when $n > 2$.
3. Not assertable for $[p] = 1$, $[q] = 0$ when $n > 2$.
4. True since when $[p] > [q]$, $[pI_4q]' = [Np]'$.
5. $[p] > [q]$, $[pI_5q]' = [Np.V.q]' \geqq [Np]'$.

3.2 p.I:q.I.pq

True when $[p] \geqq [q]$.

1. $[p] < [q]$, then $[q.I_1.pq]' = n-1$.
2. Not assertable for $[p] = 1$, $[q] = 2$ when $n > 2$.
3. $[p] < [q]$, $[pI_3:q.I_3.pq]' = [pI_3p]' = n-1$.
4. Not assertable for $[p] = 1$, $[q] = 2$ when $n > 2$.
5. $[p] < [q]$, $[p.I_5:qI_5p]' = [p.I_5:Nq.V.p]' = n-1$.

3.26 pq.I.p

True if $[p] \leqq [q]$.

If $[p] > [q]$ then we have qIp which for these values is assertable for all five types of implication.

3.27 pq.I.q

Same as 3.26.

3.3 pq.I.r:I:p.I.qIr

True for $[q] > [r]$.

1. $[q] > [r]$, $[qI_1r]' = [n-1+r-q]$

   $[p] \leq [n-1+r-q]$ true

   $[p] > [n-1+r-q]$, then $[p.I_1.qI_1r]' = [n-1+n-1+r-q-p]$

   Since $[p] > [r]$ $[pq.I_1.r]' = [n-1+r-min(p,q)]'$

   Therefore $[pq.I_1.r]' \leqq [p.I_1.qI_1r]'$ and theorem holds.

2. Not assertable for n > 2 when $[p] = 1$, $[q] = 2$, $[r] = 1$.

3. $[q] > [r]$, $[qI_3r]' = [r]$

   Holds if $[p] \leqq [r]$.

   $[p] > [r]$, $[p.I_3.qI_3r]' = [r]$, $[pq.I_3.r]' = [r]$, so theorem holds.

4. $[q] > [r]$, $[qI_4r]' = [Nq]'$

   $[p] \leqq [Nq]'$ holds.

   $[p] > [Nq]'$, $[p.I_4.qI_4p]' = [Np]'$

   For n > 2 the theorem is not assertable for $[p] = 2$, $[q] = 1$, $[r] = 0$.

5. $[q] > [r]$, $[qI_5r]' = [Nq.V.r]'$

   $[p] \leqq [Nq.V.r]'$ holds.

   $[p] > [Nq.V.r]'$, so $[pq.I_5.r]' = [Np.V.Nq.V.r]'$, making proposition hold.

3.31 p.I.qIr:I:pq.I.r

True if $[pq]' \leq [r]$.

1. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 1$, $[r] = 0$.
2. If $[p] > [r]$, $[q] > [r]$ then $[p.I_2.qI_2r]' = 0$, and $[pq.I_2.r]' = 0$, making the theorem hold.
3. Holds since for $[p] > [r]$, $[q] > [r]$, $[pq.I_3.r]' = [r]$ and $[p.I_3.qI_3r]' = [r]$.
4. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 1$, $[r] = 0$.
5. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 1$, $[r] = 0$.

3.33 pIq.qIr:I:pIr

For proofs see earilier part of the chapter.

1. Not assertable.
2. Assertable.
3. Assertable.
4. Assertable.
5. Not assertable.

3.35 p.pIq:I.q

True when $[p] \leq [q]$

1. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 0$.
2. Holds since when $[p] > [q]$, $[pI_2q]' = 0$.
3. $[p] > [q]$, $[p.pI_3q:I_3:q]' = [pq.I_3.q]' = n-1$.
4. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 0$.
5. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 0$.

3.43 pIq.pIr:I:p.I.qr

True when $[p] \leq [qr]'$

1. $[q] \geqq [r] > [p]$, then $[pI_1q]' = [n-1+p-q]$, $[pI_1r]' = [n-1+p-r]$

   or $[pI_1q]' \leqq [pI_1r]'$

   $[p.I_1.qr]'$, so theorem holds for this case.

   $[r] > [q] > [p]$, same as above since q and r are symmetrical.

2. $[q] > [p]$ or $[r] > [p]$, then $[pI_2q.pI_2r]' = 0 = [p.I_2.qr]'$

3. $[p] > [qr]'$, $[p.I_3.qr]' = [qr]'$, and either $[pI_3q]' = q$ or $[pI_3r]' = [r]$, so $[pI_3q.pI_3r]' = [qr]'$ making the theorem hold.

4. $[p] > [qr]'$, $[p.I_4.qr]' = [Np]' = [pI_4q.pI_4r]'$.

5. $[p] > [qr]'$ and $[q] \leqq [r]$. Then $[p.I_5.qr]' = [Np.V.qr]' = [Np.V.q]'$

   $[pI_5q]' = [Np.V.q]'$, hence $[pI_5q.pI_5r]' = [Np.V.q]'$.

   The case is the same for $[q] > [r]$.

3.45 pIq.I:pr.I.qr

True when $[p] \leqq [q]$, $[r] \geqq [pVq]'$

1. $[p] > [r] > [q]$, $[pr.I_1.qr]' = [rI_1q]' = [n-1+q-r]$

   $[pI_1q]' = [n-1+q-p] < [n-1+q-r]$.

2. $[p] > [r] > [q]$, $[pr.I_2.qr]' = 0 = [pI_2q]'$.

3. $[p] > [r] > [q]$, $[pr.I_3.qr]' = [q] = [pI_3q]'$.

4. $[p] > [r] > [q]$, $[pr.I_4.qr]' = [Nr]'$, $[pI_4q]' = [Np]'$

   But $[Np]' < [Nr]'$.

5. $[p] > [r] > [q]$, $[pr.I_5.qr]' = [Nr.V.q]'$

   $[pI_5q]' = [Np.V.q]' \leqq [Nr.V.q]'$

3.47 pIr.qIs:I:pq.I.rs

True if $[pq]' \leqq [rs]'$.

1. $[pq]' > [rs]'$, and take $[p] \leqq [q]$.

   $[s] \leqq [r]$, $[s] < [p]$, then $[pq.I_1.rs]' = [n-1+s-p]$

$[qI_1s]' = [n-1+s-q] \leqq [n-1+s-p]$

Hence $[pI_1r.qI_1s]' \leqq [pq.I_1.rs]'$.

$[r]>[s]$, $[r]<[p]$, then $[pq.I_1.rs]' = [n-1+r-p]$

and $[pI_1r]' = [n-1+r-p]$, so $[pI_1.qI_1s]' \leqq [pq.I_1.rs]'$

Since the proposition is symmetric in p and q the treatment for $[p]>[q]$ is the same as in the above case, making the proposition hold.

2. If $[pq]'>[rs]'$ then either $[pI_2r]' = 0$ or $[qI_2s]' = 0$ making the proposition hold.

3. $[pq]'>[rs]'$, $[s]\leqq[r]$, then $[pq.I_3.rs]' = [s]$,

$[pI_3r.qI_3s]' = [s]$.

The treatment is the same for $[s]>[r]$.

4. $[pq]'>[rs]'$, $[s]\leqq[r]$, then $[pq.I_4.rs]' = [Np.V.Nq]'$

and $[qI_4s]' = [Nq]'$, making $[pI_4r.qI_4s]' \leqq [Np.V.Nq]'$.

The treatment is the same for $[s]>[r]$.

5. $[pq]'>[rs]'$, $[s]\leqq[r]$, then $[pq.I_5.rs]' = [Np.V.Nq.V.s]'$

but $[qI_5s]' = [Nq.V.s]' \leqq [Np.V.Nq.V.s]'$, making the theorem hold for this case.

The treatment for $[s]>[r]$ is of the same type.

4.1 pIq.E:Nq.I.Np

True for $[p]\leqq[q]$, when $[Nq]'\leqq[Np]'$.

1. $[p]>[q]$, making $[Nq]'>[Np]'$, $[pI_1q]' = [n-1+q-p] = [Nq.I_1.Np]'$.

2. Evident.

3. Not assertable for $n>2$ when $[p] = 1$, $[q] = 0$.

4. Not assertable for $n>2$ when $[p] = 1$, $[q] = 0$.

5. $[p]>[q]$, making $[Nq]'>[Np]'$, $[pI_5q]' = [Np.V.q]' = [Nq.I_5.Np]'$.

4.11 pEq:E:Np.E.Nq

True when $[p] = [q]$.

1. $[p]>[q]$, then $[pE_1q]' = [n-1+q-p]$, and $[Np.E_1.Nq]' = [n-1+q-p]$.
$[p]<[q]$, then $[pE_1q]' = [n-1+p-q]$, and $[Np.E_1.Nq]' = [n-1+p-q]$.
2. $[p]\neq[q]$, then $[pE_2q]' = 0$ and $[Np.E_2.Nq]' = 0$.
3. Not assertable for $n>2$ when $[p] = 0$, and $[q] = 1$.
4. Not assertable for $n>2$ when $[p] = 0$, and $[q] = 1$.
5. $[p]>[q]$, then $[pE_5q]' = [Np.V.q]'$, and $[Np.E_5.Nq]' = [Np.V.q]'$.
$[p]<[q]$, then $[pE_5q]' = [Nq.V.p]'$, and $[Np.E_5.Nq]' = [Nq.V.p]'$.

4.13 p.E.N(Np)

True since $[N(Np)]' = [n-1-(n-1)+p] = [p]$.

4.2 pEp

Obviously true.

4.21 pEq.E.qEp

By inspection of theorems 1.6, 6.7, 6.8, 6.9, and 6.10 it is evident that $[pE_iq]' = [qE_ip]'$ $(i = 1,2,3,4,5)$.

4.22 pEq.qEr:I:pEr

True when $[p] = [q]$, or $[p] = [r]$, or $[q] = [r]$.

1. Not assertable for $n>2$ when $[p] = 1$, $[q] = 2$, $[r] = 0$.
2. If $[p]\neq[r]$ then either $[p]\neq[q]$, or $[q]\neq[r]$, in which case $[pE_2r]' = 0 = [pE_2q.qE_2r]'$ making the theorem hold.
3. $[p]\neq[q]$, $[q]\neq[r]$, $[p]\neq[r]$, then $[pE_3q.qE_3r]' = [pqr]'$ and $[pE_3r]' = [pr]'$, thus making $[pE_3q.qE_3r]' \leqq [pE_3r]'$.
4. $[p]\neq[q]$, $[q]\neq[r]$, $[p]\neq[r]$, then $[pE_4q.qE_4r]' = [Np.Nq.Nr]'$ and $[pE_4r]' = [Np.Nr]'$, thus making $[pE_4q.qE_4r]' \leqq [pE_4r]'$.
5. Not assertable for $n>2$ when $[p] = 0$, $[q] = 1$, $[r] = n-1$.

If we prove that a proposition pE_iq $(i = 1,2,3,4,5)$ is assertable for some particular i then pE_jq, where the truth-values of p and q do not change as j changes, is assertable for all j (j = 1,2,...,5), since $[pE_kq]' = n-1$ (k = 1,2,...,5) if and only if $[p] = [q]$. We have utilized this fact in propositions 4.24,4.25,4.3,4.31,4.32,4.33,4.4,4.41 giving in each case the proof for pE_1q, or where the proof for pE_1q may be found.

4.24 p.E.pp

$[pp]' = [p]$.

4.25 p.E.pVp

$[pVp]' = [p]$.

4.3 pq.E.qp

(1.18).

4.31 pVq. E.qVp

(1.16).

4.32 (pq)r.E.p(qr)

(1.19).

4.33 (pVq)Vr.E.pV(qVr)

(1.17).

4.4 p.qVr:E:pq.V.pr

(1.22).

4.41 p.V.qr:E:pVq.pVr

(1.23).

4.71 pIq.E:p.E.pq

Obviously true when $[p] \leq [q]$.

1. $[p] > [q]$, then $[pI_1q]' = [n-1+q-p]$, and $[p.E_1.pq]' = [n-1+q-p]$
2. $[p] > [q]$, then $[pI_2q]' = 0$, and $[p.E_2.pq]' = 0$.
3. $[p] > [q]$, then $[pI_3q]' = [q]$, and $[p.E_3.pq]' = [q]$.
4. $[p] > [q]$, then $[pI_4q]' = [Np]'$, and $[p.E_4.pq]' = [Np]'$.
5. $[p] > [q]$, then $[pI_5q]' = [Np.V.q]'$, and $[p.E_5.pq]' = [Np.V. q]'$.

4.73 q.I:p.E.pq

True when $[p] \leqq [q]$.

1. $[p] > [q]$, then $[p.E_1.pq]' = [n-1+q-p]$. But $[q] \leqq [n-1+q-p]$.

2. Not assertable for $n > 2$ when $[p] = 2$, $[q] = 1$.

3. $[p] > [q]$, then $[p.E_3.pq]' = [q]$.

4. Not assertable for $n > 2$ when $[p] = n-1$, $[q] = 1$.

5. $[p] > [q]$, then $[p.E_5.pq]' = [Np.V.q]'$. But q Np.V.q.

5.1 pq.I.pEq

True when $[p] = [q]$.

1. $[p] > [q]$, $[pE_1q]' = [n-1+q-p]$, but $[q] \leqq [n-1+q-p]$.

   $[p] < [q]$, $[pE_1q]' = [n-1+p-q]$, but $[p] \leqq [n-1+p-q]$.

2. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 2$.

3. $[p] \neq [q]$, then $[pE_3q]' = [pq]'$.

4. Not assertable for $n > 2$ when $[p] = n-1$, $[q] = n-2$.

5. $[p] < [q]$, $[pE_5q]' = [p.V.Nq]'$, and $[pq] \leqq [p.V.Nq]'$.

   $[p] > [q]$, similar to $[p] < [q]$.

5.32 p.I.qEr:E:pq.E.pr

True if $[q] = [r]$.

1. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 1$, $[r] = 0$.

2. Not assertable for $n > 2$ when $[p] = 1$, $[q] = 2$, $[r] = 1$.

3. $[p] \geqq [q] > [r]$, then $[p.I_3.qE_3r]' = [r]$

   and $[pq.E_3.pr]' = [qE_3r]' = [r]$.

   $[q] > [p] > [r]$, then $[p.I_3.qE_3r]' = [r]$

   and $[pq.E_3.pr]' = [pE_3r]' = [r]$.

   Similarly we can show that $[p] \geqq [r] > [q]$, and $[r] > [p] > [q]$ hold.

   $[p] \leqq [qr]'$, then proposition becomes $[p.I_3.qE_3r:E_3:pE_3p]'$

This obviously holds, making the proposition hold for p,q,r ranging over all possible truth-values.

4. Not assertable for $n > 2$ when $[p] = n-1, [q] = 1, [r] = 0$.

5. Not assertable for $n > 2$ when $[p] = 1, [q] = 1, [r] = 0$.

5.6 p.Nq:I.r.:E:p.I.qVr

1. Not assertable for $n > 2$ when $[p] = 1, [q] = 1, [r] = 0$.
2. Not assertable for $n > 2$ when $[p] = 1, [q] = 1, [r] = 0$.
3. Not assertable for $n > 2$ when $[p] = n-1, [q] = 1, [r] = 0$.
4. Not assertable for $n > 2$ when $[p] = n-1, [q] = 1, [r] = 0$.
5. Not assertable for $n > 2$ when $[p] = n-1, [q] = n-2, [r] = 1$.

Considering now the five propositions of Lukasiewicz:

$p:I_1:qI_1p$, $pI_1q.I_1:qI_1r.I_1.pI_1r$, $pI_1q.I_1q:I_1:qI_1p.I_1p$,

$pI_1q.I_1.qI_1p:I_1:qI_1p$, $Np.I_1.Nq:I_1.qI_1p$.

The first, second and fifth of these propositions we have proven already. They are respectively, 2.02, 2.06 and 2.17. Considering the remaining two we find that they both hold.

The proofs follow:

$pI_1q.I_1q:I_1:qI_1p.I_1p$

$[p] \geqq [q]$, $[qI_1p.I_1.p]' = [n-1+p-(n-1)] = [p]$

and $[pI_1q]' = [n-1+q-p] \geqq [q]$ so that $[pI_1q.I_1q]' = [p]$.

$[p] < [q]$, $[pI_1q.I_1q]' = [q] = [qI_1p.I_1p]'$.

Thus the proposition holds.

$pI_1q.I_1.qI_1p:I_1:qI_1p$

$[q] \leqq [p]$, obviously true.

$[q] > [p]$, $[qI_1p]' = [n-1+p-q] = [pI_1q.I_1.qI_1p]'$.

CHAPTER THREE

THE ARITHMETIC OF N-VALUED LOGIC

INTRODUCTION. This chapter is based largely upon efforts to extend the results given in the papers "Arithmetic of Logic" by Dr. E.T. Bell[15] and "On Bell's Arithmetic of Boolean Algebra" by W.A. Hurwitz[16]. Most of the operations that we shall use are generalizations of those given by Bell and Hurwitz. However, in a few cases the obvious generalization is found to be defective, and we have to seek more obscure generalizations, or even at times, operations that are not generalizations of the corresponding operations in the above mentioned papers.

This chapter is meant to be merely a short exploration of the possibilities of this topic. Many continuations of some of the items of this section will probably be evident. However, lack of time and space prevent a more thorough treatment of this subject. Later the writer intends to more thoroughly explore the generalizations of the ideas in the papers of Bell and Hurwitz, as well as attempting to adapt the results in the papers of Bernstein[17], Stone[18], and Von Neumann[19] to a logic of n-values.

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15. Trans. Amer. Math. Soc., 29 (1927), pp. 597-611. Henceforth referred to as B.
16. Trans. Amer. Math. Soc., 30 (1928), pp. 420-424. Called H.
17. Bernstein, "On Finite Boolean Algebras", Amer. Jour. of Math., LVII (1935), pp. 733-742.
18. Stone, Proc. of Nat. Acad. of Sciences, 20, pp. 197-202.
19. J. Von Neumann and M.H. Stone, Fundamenta Mathematicae, XXV (1935), pp. 353-378.

We shall say that we have developed the arithmetic of n-valued logic if we can find operations in L_n which satisfy the abstraction of the postulates covering the corresponding operations in the rational arithmetic. For these postulate systems we shall consult the papers of Bell and Hurwitz. They are listed in brief on the following pages.

In the Boolean case two sets of operations, duals of each other, are given which form the arithmetic of logic. We shall not attempt this since if we were to attempt it we would have to enlarge the number of theorems in Chapter One so as to cover the duals of many of the theorems that appear. However, it is well to remember that the dual case can be worked out as easily as the case which we have given.

NOTATION. We shall state our postulates for an abstract ring R*, as in B p. 599, in which the operations S,P,L,G,C,D,R may be read as, respectively, sum, product, L.C.M., G.C.D., divides, and residual. In L_n we use the same letters, avoiding the use of the small letters since many are in use as propositions. Various definitions will be introduced as they are needed.

POSTULATES FOR S,P,L,G,C,D,R. These postulates are from B and retain as much as possible of the notation used there.

Postulates for S and P.

R^*_1. If x,y are any two elements of R*, xSy, xPy are uniquely determined elements of R*, and

$$xSy = ySx, \qquad yPx = xPy.$$

R^*_2. If x,y,z are any three elements of R*,

$$(xSy)Sz = xS(ySz), \qquad (xPy)Pz = xP(yPz),$$

$$xP(ySz) = (xPy)S(xPz).$$

R^*_3. There exist in R* two distinct unique elements, denoted by $\underline{u}$ and $\underline{z}$, called the unity, zero of R*, such that if x is any element of R*, $xS\underline{z} = x$, $xP\underline{u} = x$.

Postulates for division in R*.

P1. xDx,

P2. $xDy.yDz:\supset:xDz$,

P3. $xDy.yDx:\supset:x = y$,

where xDy is uniquely significant for each $x \neq \underline{z}$ and y in R*, with the exception that $\underline{z}D\underline{z}$ is significant but indeterminate in R*.

Postulates for G and L in R*.

P4. $xGy = yGx$,

P5. $xG(yGz) = (xGy)Gz \equiv xGyGz$,

P6. $(xGy)Dx.(xGy)Dy$,

P7. $zDx.zDy:\supset:zD(xGy)$,

P8. $xLy = yLx$,

P9. $xL(yLz) = (xLy)Lz \equiv xLyLz$,

p10. $xD(xLy).yD(xLy)$,

P11. $xDz.yDz:\supset:(xLy)Dz$,

where x,y are any elements in R* and xLy and xGy are uniquely determined in R*.

Postulates for congruence in R*.

Let xCy be a relation in R* such that, if x,y,z,w are elements in R*, xCy is uniquely significant in R* and the postulates P12-P15 are satisfied:

P12. $xCy\supset yCx$,

P13. $xCy.yCz:\supset:xCz$,

P14. $xCy.zCw:\supset:(xSz)C(ySw)$,

P15. $xCy.zCw:\supset:(xPz)C(yPw)$.

Then C is called <u>abstract algebraic congruence</u>.

If R* is replaced by its instance A, an instance of xCy is $aCb \equiv (a \equiv b \bmod m)$, where a,b are integers ≥ 0 and <u>m</u> is an integer > 0.

In A C is said to be arithmetic congruence if to the postulates P12-P15 are added the postulates

P16. $(a \equiv 0 \bmod m) : :m$ divides a, $m \neq 0$;

P17. $(ka \equiv ka \bmod m) \quad (a \equiv b \bmod m'), \quad m \neq 0$,

where $qm' = m$, and q = G.C.D. of k,m;

P18. $a \equiv a \bmod m$.

<u>Abstraction of the residual.</u>

Let a,b,h,m for the moment denote elements of R*. Then, if m is uniquely determined by (<u>u</u> unity in R*),

P19. $\{aD(hPb)\}.\{mDh\}.\{m \neq \underline{u}$,

where h runs through all elements in R*, we shall call <u>m</u> the residual of b with respect to a, and we shall write m = bRa.

EQUIVALENCE. In addition to the sets of postulates listed by Dr. E.T. Bell, it is well to list a set of postulates for equality. In the Boolean case the properties of identity, symmetry, and transitivity are satisfied by Boolean equivalence. This is not the case where $n > 2$. Then, we can see by 4.22 that all of the types of equivalence which we defined are not transitive. Of the three types which are transitive (pE_2q, pE_3q, and pE_4q), we shall use pE_2q since this has the simplest truth-table of the three. Using our earlier interpretation for truth-values, pE_2q states that either p is equivalent to q, or p is not equivalent to q. This is not the case with pE_3q and pE_4q, as can readily be seen by theorems 6.7,6.8, and 6.9.

The set of postulates given below are those given by Dr. E.V. Huntington[20] with the exception that it has been necessary to change the operations listed in postulate D.

"An obvious set of postulates for = is as follows, where a,b,c,... are understood to be elements of the class K.

Postulate A. If a is in the class K, then a = a.

Postulate B. If a = b, then b = a.

Postulate C. If a = b, and b = c, then a = c.

Postulate D. If x = y, then f(x,a,b,c,..) = f(y,a,b,c...) where f(x,a,b,c,...) is any element of the class K built up from the elements x,a,b,c,..., by successive applications of the operator | and f(y,a,b,c,,..) is the element obtained from f(x,a,b,c,...) by writing y in place of x throughout."

pE_2q satisfies these postulates because of the following theorems:

Postulate A: 4.2.

Postulate B: 4.21. This is a stronger theorem than pEq.I.qEp.

Postulate C: 4.22.

Postulate D: Since we have defined operations as only operating on the truth-tables of a function, if two propostions have identical truth-tables one may replace the other. But our definition of equivalence implies that if two propositions are equivalent they will have identical truth-tables. Thus

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20. Huntington, <u>Trans</u>. <u>Amer</u>. <u>Math</u>. <u>Soc</u>., 35 (1933), p.280.

if xE_2y holds we can replace x by y wherever x occurs without changing the value of our function f(x,p,q,r,...). This satisfies Postulate D.

We shall drop the subscript on E_2 henceforth unless we are referring to some type of equivalence other than E_2.

THE SUM AND PRODUCT IN L_n. In working out the transforms of operators it is well to remember that = in R* transforms into E in L_n. Remembering this we are led to the following theory:

7.01 xSy.=.xVy Df

7.02 xPy.=.xy Df

7.1 xPy and xSy satisfy postulates R^*_1, R^*_2, R^*_3.

Proof: R^*_1 is satisfied since xSy and xPy are uniquely determined by 1.2 and 1.4. Also

xSy.E.ySx (4.31), xPy.E.yPx (4.3).

R^*_2 is satisfied since

(xSy)Sz.E.xS(ySz) (4.33), (xPy)Pz.E.xP(yPz) (4.32),

xP(ySz).E.(xPy)S(xPz) (4.4).

R^*_3 is satisfied where the zero and unity of L_n are respectively t_o and t_{n-1}, since

xSt_o.E.x (1.2), and xPt_{n-1}.E.x (1.4).

7.1 allows us to call xSy and xPy as defined the transforms of the sum and product in L_n

DIVISION IN L_n.

7.03 xDy.=.yI_2x Df

7.2 xDy in L_n satisfies postulates P1,P2,P3.

Proof: P1. xDx (4.2).

P2. xDy.yDz:I:xDz (3.33).

While this statement was proven where I is I_2, yet inspection of the proof will show that I may be any one of the five types of implication that we have defined.

P_3. xDy.yDx:I:xEy

xEy was defined in this fashion.

By 6.3 x divides y (xDy) when $[x]\geqq[y]$. xDy may be regarded as division in L_n.

THE L.C.M. AND G.C.D. IN L_n.

7.04 xGy.=.xVy Df

7.05 xLy.=.xy Df

7.3 xGy satisfies postulates P4, P5, P6 and P7.

Proof: P4. xGy.E.yGx (4.31)

P5. xG(yGz).E.(xGy)Gz (4.33)

P6. (xGy)Dx.(xGy)Dy This is true since

$[x.I_2.xVy]' =$ n-1 and $[y.I_2.xVy]' =$ n-1.

P7. zDx.zDy:I:zD(xGy). This may be written as

$xI_2z.yI_2z$:I:$(xVy)I_2z$ which is evidently true.

I may be anyone of the defined implications.

7.4 xLy satisfies postulates P8,P9,P10,P11.

Proof: P8. xLy.E.yLx (4.3)

P9. xL(yLz).E.(xLy)Lz (4.32)

P10. xD(xLy).yD(xLy). This may be written as

$xy.I_2.x:xyI_2.y$ which is obviously true.

P11. xDz.yDz:I:(xLy)Dz. This becomes

$zI_2x.zI_2y$:I:$z.I_2.xy$ which is true.

Then by 7.3 and 7.4 we can consider that xGy and xLy are transforms of the G.C.D. and L.C.M. in L_n.

RESIDUALS IN L_n. We can rewrite P19 in L_n in the form

$\{pD(xPq)\}.\{yDx\}.\{[y]\neq n-1\}$

where x is allowed to have any truth-value such that $\{pD(xPq)\}$ holds. These conditions are:

If $[q]\leqq[p]$ then x can assume any value and $[y]=n-1$.

If $[q]>[p]$ then $[x]\leqq[p]$. So if $[y]=[p]$, then yDx.

This leads us to the definition

7.06 $qRp.=.qI_3p$ Df

7.12 y.E.qRp

Proof: If y.E.qRp

when $[q]\leqq[p]$, $[y]=n-1$

$[q]>[p]$, $[y]=[p]$ (6.4)

and y satisfies the above condition for a residual.

CONGRUENCE IN L_n. No generalization of Hurwitz's congruence[21] $\alpha\mu'=\beta\mu'$ was found. The generalization $xP_o.E.yP_o$ of $xp'=yp'$ satisfies P12, P13, P14 and P15 but does not satisfy P16 or P17. Thus we could call $xP_o.E.yP_o$ algebraic congruence since they satisfy the four postulates given for algebraic congruence.

Keeping the definitions that we have made previously for the sum, product, divides, residual, G.C.D. and L.C.M., when $n>2$, if we accept Hurwitz's definition for congruence[22], it is impossible to satisfy P17. Assuming that $x\equiv y$ mod p when x.E.zVrp and y.E.zVqp, then if we take zx.E.zVrp and zy.E.zVqp where $[p]<[z]\leqq[xy]'$ and $[x]\neq[y]$, then

(1) $zx\equiv zy$ mod p, but

(2) $x\equiv y$ mod qRp only when $[qRp]'\geqq[xVy]'$.

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21. See H p. 421, (c).

22. H p. 421.

In case $[x] = n-1$, then $[qRp]' = n-1$ if (2) holds. But by P19 $[qRp]' \neq n-1$ in general. This leads us to the conclusion that defining the sum, product and residual as we have, we are unable to satisfy P17 by Hurwitz's definition of congruence.

However, we are able to prove the followingtheorem:

7.07 $(x \equiv y \bmod p) .=. xVp.E.yVp$ Df

7.13 xVp.E.yVp satisfies postulates P12,...,P16,P18.

Proof: P12,...,P15 are evident.

P16: $xVp.E.t_0Vp:E:pDx.([p] \neq 0)$.

If $[x] > [p]$ then $[xVp.E.t_0Vp]' = 0$, and $[pDx]' = 0$.

If $[x] \leqq [p]$ (or pDx) then $[xVp.E.t_0Vp]' = n-1$, or the statement is true.

Hurwitz interprets P16 to say:

$x \equiv 0 \bmod p$ if and only if pDx. This is too restrictive a statement in L_n since p could divide x when pDy and $y \neq 0$. This analogue occurs in A. We can amend Hurwitz's statement to say "if pDy in $x \equiv y \bmod p$, then pDx." Inspection of xVp.E.yVp verifies this statement.

P18: xVp.E.xVp is obviously true.

ANOTHER EXAMPLE OF THE ARITHMETIC OF L_n.

Deserting the operations defined in the earlier part of this chapter, we find that we can develope an arithmetic on the subscripts of the truth-values. This analogy with A is much more striking for $n > 2$ than for $n = 2$, when we only have two elements available. The operations which we shall use are defined below:

7.010 $p+q.=.p^{[q]}$ Df

7.011 $-p.=.(Np)^1$ Df

7.012 $poq.=.p^{[p(q-1)]}$ Df

They lead us to the following theorems:

7.20 $[p+q]' = [p+q]$

Proof: $[p+q]' = [p^{[q]}]' = [p+q]$.

7.21 $p+q.E.q+p$

Proof: $[p+q] = [q+p]$.

Note that we are retaining E from the previous section, meaning by E E_2.

7.22 $p+(q+r).E.(p+q)+r$

Proof: $[p+(q+r)]' = [p+q+r] = [(p+q)+r]'$.

7.013 $p+q+r.=.p+(q+r)$ Df

7.23 $[-p]' = [n-p]$

Proof: $[-p]' = [(Np)^1]' = [n-1-p+1] = [n-p]$.

7.014 $p-q.=.p+(-q)$ Df

7.24 $p-p.E.t_o$

Proof: $[p-p]' = [p+n-p] = [n] = 0$.

7.25 $[poq]' = [pq]$ (Notice absence of prime on $[pq]$)

Proof: $[poq]' = [p^{[p(q-1)]}]' = [p+pq-p] = [pq]$.

7.26 $poq.E.qop$

Proof: $[pq] = [qp]$.

7.27 po(qor).E.(poq)or

Proof: $[po(qor)]' = [pqr] = [(poq)or]'$

7.015 poqor.=.(poq)or Df

7.28 po(q+r).E.poq+por

Proof: $[po(q+r)]' = [p[q+r]'] = [p(q+r)] = [pq+pr] = [poq+por]'$

7.29 pot_1.E.p

Proof: $[pot_1]' = [1.p] = [p]$.

7.30 $p+t_o$.E.p

Proof: $[p+t_o]' = [p+0] = [p]$.

7.31 (-p)oq.E.-(poq)

Proof: $[(-p)oq]' = [(n-p)q] = [nq-pq] = [n-[pq]]$

7.32 (-p)o(-q).E.poq

Proof: $[(-p)o(-q)]' = [(n-p)(n-q)] = [n^2-np-nq+pq] = [pq]$.

7.33 p+x.E.q:E:x.E.q-p

Proof: Where $0 \leq [x] < n$ then there is evidently only one value of x such that $[p+x] = [q]$. But $[x] = [q+n-p]$ satisfies this relationship. Therefore the theorem holds.

7.34 L_n is a ring under operations p+q and poq.

Proof: (The following numerals refer to the postulates on p. 37 of B.L. van der Waerden: Moderne Algebra, Vol. I.)

I. Postulates on addition.

(a) (7.22).

(b) (7.21).

(c) (7.33).

II. Postulates on multiplication.

(a) (7.27)

III. Distributive laws.

(a) (7.28)

(b) (7.28,7.26).

$p \dotplus q$ and $p \circ q$ are generalizations of the two operations that Stone[23] used as a basis for his ring in L_2. It is well to note that $p \circ q$ is a generalization of pq in L_2.

Many other theorems could be developed about the above operations, but those given are sufficient to indicate a few of their properties and also the method to be used in working out new theorems.

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23. M.H. Stone, Proc. N.A.S., 21 (1935), pp. 103-105

CHAPTER FOUR

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GENERATION OF ANY N-VALUED LOGIC BY ONE BINARY OPERATION[1]

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A two-valued logic was shown in 1913 by Sheffer[2] to be obtainable by the iteration of a single binary operation. It was proved in 1925 by Zylinski[3] that Sheffer's function and its "dual," also introduced by Sheffer, are the only binary operations such that the iteration of either one will generate the two-valued logic of functions of two propositions. Zylinski's proof was by means of a truth table of 4 columns and 16 rows, corresponding to the two possible values (a or b) which an arbitrary function $\Phi(x, y)$ can assume in a two-valued system. While theoretically applicable to an n-valued system, the method of direct inspection of the truth table is impracticable. Following another method we prove that: *Any n-valued logic, where $n \geqq 2$, can be generated by the iteration of one binary operation.*

Designate the n truth values which an "elementary proposition" may take in an n-valued logic by the marks $a_0, a_1, \ldots, a_{n-1}$. For convenience, drop the a and retain only the subscript, so that our marks are now $0, 1, \ldots, n-1$. It is to be observed that these numbers denote merely n distinct marks without any arithemetical significance. Let p and q be any elementary propositions. Construct a truth table for two elementary propositions, p, q, of two columns and n^2 rows with the n marks, $0, 1, \ldots, n-1$ by assigning in the ith row of the table to p the value $[i-1-(i-1)']/n$, and to q the value $(i-1)'$, where $j' \equiv j \bmod n$, $j' \geqq 0$ and $i = 1, 2, \ldots, n^2$. Denote the statement, *p has the value i*, by $p = i$; let $p\beta q$ denote any function of p, q whose values are in the range $0, 1, \ldots, n-1$, when $p = i, q = j$ and i, j are in the same range; let $i\beta j = k$ denote that if $p = i$ and $q = j$ then $p\beta q = k$, where k is in the range.

Define the stroke, "|," function, $p|q$, by

$$i|j = 0 \text{ if } i \neq j; \quad i|i = (i+1)' \quad (i, j = 0, 1, \ldots, n-1).$$

From this binary operation we shall generate all functions of two variables in the n-valued logic. The proof will consist in exhibiting the particular general column of the p, q truth table in which the n^2 marks t_s ($s = 1, 2, \ldots, n^2$) are arbitrary elements of the set $0, 1, \ldots, n-1$ as a function constructed on p, q by means the stroke |.

As a notational definition we write

$$p^0 \equiv p, \quad p^i \equiv p^{i-1}|p^{i-1} \quad (i = 1, \ldots, n-1),$$

in which the exponents are superscripts. Define $R_{i,j}(p, q)$ to be a function of p and q such that in row j of the truth table, $R_{i,j}(p, q) = i$ and in every other row $R_{i,j}(p, q) = 0$ ($i = 0, 1, \ldots, n-1$; $j = 1, 2, \ldots, n^2$). For brevity we shall write $R_{i,j}$ for $R_{i,j}(p, q)$. Similarly, any function $F(p, q)$ of p, q will be written F.

From the definition of $p|q$ it is evident that p^1 can be obtained by operating with the cyclic substitution $S \equiv (0,1 \ldots, n-1)$ on the values of p^0. Similarly, p^i can be obtained by operating on the values of p^0 with the ith power, S^i, of S. Hence $R_{1,j}$ and $(p^a|q^b)^{n-1}|(p^0|p^1)$ are equivalent for the values of a, b indicated next, which we shall write

$R_{1,j} \cdot\equiv\cdot (p^a|q^b)^{n-1}|(p^0|p^1)$, $a =$ ~~$\{n - [i-1-(i-1)']/n\}', b = [n-(i-1)']'$.~~ *

By means of $R_{1,j}$ we obtain $R_{i,j}$ in terms of p, q and $p|q$. For,

$$R_{0,j} \cdot\equiv\cdot p^0|p^1, \; R_{i,j} \cdot\equiv\cdot (R_{1,j})^{i-2}|(R_{0,j})^{i-1} \quad (i = 2, 3, \ldots, n-1).$$

Define $p\alpha_1 q \equiv p^0|q^{n-1}$, $p\alpha_2 q \equiv R_{1,1}|R_{0,1}$, $p\alpha_i q \equiv N_{i-2}\alpha_1 M^1_{i-2}$ ($i = 3, 4, \ldots, n$). $M_0 \equiv R_{0,1}$, $M_h \equiv M_{h-1}\alpha_{h+1}(R_{h,h+2}\alpha_{h+1}R_{h,(h+1)n+1})$, $N_h \equiv M_h\alpha_{h+1}R_{1,1}$.

Then we see that $p\alpha_3 q$ has the property $0\alpha_3 i = i$; $i\alpha_3 0 = i$ ($i = 0, 1, 2$). Hence, by an easy mathematical induction, we prove that if

$p\alpha_n q \equiv N_{n-2}\alpha_1 M^1_{n-2}$, then $0\alpha_n i = i$; $i\alpha_n 0 = i$ ($i = 0, 1, \ldots, n-1$).

Let $T_i(p, q)$ be any function of p, q and define

$$\begin{aligned} T_1\alpha_n T_2\alpha_n T_3 &\equiv (T_1\alpha_n T_2)\alpha_n T_3 \\ T_1\alpha_n T_2\alpha_n T_3\alpha_n T_4 &\equiv ((T_1\alpha_n T_2)\alpha_n T_3)\alpha_n T_4 \\ \ldots & \quad \ldots \\ T_1\alpha_n T_2\alpha_n T_3\alpha_n \ldots \alpha_n T_j &\equiv (\ldots((T_1\alpha_n T_2)\alpha_n T_3)\alpha_n \ldots \alpha_n T_j. \end{aligned}$$

From the properties of $p\alpha_n q$ it follows that

$$0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n i\alpha_n 0\alpha_n \ldots \alpha_n 0 = i \quad (i = 0, 1, \ldots, n-1).$$

Hence we can construct any function $F(p, q)$ of p, q having the values t_i in row i of the truth table, where t_i is any one of the marks $0, 1, \ldots, n-1$, as follows:

$$F(p, q) \cdot\equiv\cdot R_{t_1,1}\alpha_n R_{t_2,2}\alpha_n \ldots \alpha_n R_{t_{n^2},n^2}$$

This follows immediately from $p\alpha_n q$ by inspection of the rows in the following table,

$$\begin{aligned} t_1\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n 0 &= t_1 \\ 0\alpha_n t_2\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n 0 &= t_2 \\ \ldots \quad & \quad \ldots \\ 0\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n t_i\alpha_n \ldots \alpha_n 0 &= t_i \\ \ldots \quad & \quad \ldots \\ 0\alpha_n 0\alpha_n 0\alpha_n \ldots \alpha_n 0\alpha_n \ldots \alpha_n t_{n^2} &= t_{n^2} \end{aligned}$$

Now $F(p, q)$ is any function defined by the truth table of the n-valued

* $a = \{n - [j - 1 - (j - 1)']/n\}'$, $b = [n - (j - 1)']'$.

logic. Hence, if $n \geqq 2$, *any* function of two propositions p, q can be constructed from p, q and $p|q$ in the manner just indicated. To extend this result to a function of m propositions, $p_1, p_2, \ldots, p_m$, we make the additional definitions and take the steps indicated below.

Define $i_{r,n} \equiv i \bmod n^r$ where i is any positive integer and $i_{r,n} \geqq 0$. Now we shall construct a truth table on $p_1, \ldots, p_k$ by assigning to p_s $(s = 1, 2, \ldots, k)$ the value

$$p_s = \{[(j-1) - (j-1)_{k-s,n}]/n^{k-s}\}_{1,n}$$

in the jth row $(j = 1, 2, \ldots, n^k)$. Let $R^k_{i,j} \equiv R_{i,j}(p_1, \ldots, p_k)$ be defined on a truth table of $p_1, \ldots, p_k$ as $R^k_{i,j} = i$ in row j, and $R^k_{i,j} = 0$ elsewhere. Let $S^k_j \equiv (R^k_{n-1,j})^1$ $(j = 1, 2, \ldots, n^k)$.

It is evident that a function $F(p_1, \ldots, p_k)$ will exist in a truth table on $p_1, \ldots, p_{k+t}$ and will be the function obtained from the $p_1, \ldots, p_k$ truth table by replacing each row by n^t consecutive identical rows. This, then, increases the number of rows from n^k to n^{k+t}.

Hence, we can prove

$$R^k_{i,j} \cdot\equiv\cdot R_{i,h}(S^{k-1}_l, p_k) \text{ where } h = (j-1)_{1,n} + 1,\ l = [j-1-(j-1)_{1,n}]/n + 1\ (j = 1, 2, \ldots, n^k;\ k = 3, 4, \ldots, m).$$

Having determined $R^m_{i,j}$, we proceed to construct any $F(p_1, \ldots, p_m)$ in terms of $p_1, \ldots, p_m$ and the stroke, "|", by using the same procedure as for $m = 2$. For if t_i is the value of $F(p_1, \ldots, p_m)$ in row i, then it follows that

$$F(p_1, \ldots, p_m) \cdot\equiv\cdot R^m_{t_1,1}\alpha_n R^m_{t_2,2}\alpha_n \ldots \alpha_n R_{t_{n^m}, n^m}.^*$$

Thus any function on any number of propositions $p_1, \ldots, p_m$ can be constructed by means of a single binary operation, the stroke, "|".

[1] The author is under obligations to Dr. E. T. Bell for his suggestions, aiding both in the solution of this problem and its presentation.

[2] Sheffer, H. M., *Trans. Amer. Math. Soc.*, **14**, 481–488 (1913).

[3] Zylinski, E., *Fund. Math.*, **7**, 203–209 (1925).

$$^* F(p_1, \ldots, p_m) \equiv\cdot R^m_{t_1,1}\alpha_n R^m_{t_2,2}\alpha_n \cdots \alpha_n R^m_{t_{n^m}, n^m}.$$

DEFINITION OF POST'S GENERALIZED NEGATIVE AND MAXIMUM IN TERMS OF ONE BINARY OPERATION.

By Donald L. Webb.

In 1921 Post[1] demonstrated that it was possible to construct a function for any order table in a system of m truth-values by the use of two primitive functions, $\sim_m p$ and $p \vee_m q$ which are generalizations of the functions $\sim p$ and $p \vee q$ in the two-valued case. Recently we[2] have been able to show that a function on m truth-values for any order table can be constructed in terms of one binary operation, using in this demonstration a negative that corresponds to Post's $\sim_m p$, a binary operator $p\, \alpha_m\, q$ which, for the value combinations used in the interpolation formula, corresponds to Post's $p \vee_m q$, and a binary operator $p \mid q$ which has no equivalent among the operators employed by Post. In the latter paper all operators were defined in terms of $p \mid q$. In this paper by redefining the truth-table of $p \mid q$ we are enabled to define Post's $\sim_m p$ and $p \vee_m q$ in terms of the " $\mid$ " function, thus greatly simplifying the proof that any m-valued logic can be generated by one binary operation. We find too that $p \mid q$ as so defined reduces in the two-valued case to one of Sheffer's functions,[3] as it evidently must.

The notation used in this paper is patterned after that of Post so as to avoid confusion.

Let $t_0, t_1, \cdots, t_{m-1}$, where m is any positive integer, signify the m truth values that an elementary proposition can assume in a m-valued logic. Denote by p, q elementary propositions. Let $p = t_i$ signify that the proposition p has the truth-value t_i. Make the two additional arithmetical definitions:

$$\begin{aligned} \min(i, j) &= i \quad \text{if} \quad i \leqq j \qquad\qquad (i, j = 0, 1, 2, \cdots) \\ &= j. \quad \text{if} \quad i \geqq j; \\ i \equiv i_n \bmod n, &\quad (i = 0, 1, 2, \cdots) \qquad 0 \leqq i_n < n. \end{aligned}$$

Hence, $p \mid q$ is defined: if $p = t_i$, $q = t_j$ $(i, j = 0, 1, \cdots, m-1)$, then $p \mid q = t_k$ where $k = [\min(i, j) + 1]_m$.

[1] E. L. Post, *American Journal of Mathematics*, vol. 43 (1921), pp. 163-185.

[2] D. L. Webb, *Proceedings of the National Academy of Sciences*, vol. 21 (1935), pp. 252-254.

[3] H. M. Sheffer, *Transactions of the American Mathematical Society*, vol. 14 (1913), pp. 481-488.

THEOREM 1. $\sim_m p \equiv p \mid p$.

If $p = t_i$, then $p \mid p = t_k$ where $k = (i+1)_m$. Thus $p \mid p$ cyclically permutes the truth-values t_i, giving $p \mid p$ and $\sim_m p$ the same truth-table. Therefore the two are equivalent.

Using Post's definition, $\sim_m^2 p = \sim_m \sim_m p$, etc., we may write

THEOREM 2. $p \vee_m q \equiv \sim_m^{m-1} (p \mid q)$.

By repeating the above process we find that if $p = t_i$, $\sim_m^h p = t_k$, where $k = (i+h)_m$ $(h = 2, 3, \cdots, m-1)$. Hence, if $p = t_i$, $q = t_j$, then $\sim_m^{m-1} (p \mid q) = t_k$ where $k = \{[\min(i,j)+1]_m + m - 1\}_m$, or $k = \min(i,j)$. But $p \vee_m q$[4] as given by Post has the same truth-table, making the two equivalent.

Since Post has shown that we can generate a function of any order in a m-valued truth system by means of $\sim_m p$ and $p \vee_m q$, then, by using the above theorems, we can generate a function of any order in a m-valued truth system in terms of "$\mid$".

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[4] This is called a maximum since the higher truth-value has the smaller subscript.