MANY-VALUED LOGICS

Thesis
by
Donald L. Webb

In Partial Fulfillment
of the
Requirements for the
Degree of Doctor of Philosophy

California Institute of Technology
Pasadena, California

1936
INTRODUCTION

This thesis is not intended to add any substance to the philosophical controversies being waged about "many-valued logics". Its purpose is to develop, for the first time as far as the writer knows, the interesting mathematical theory which lies behind these logics. It is an effort to provide a sane method of attack on discovering the properties of operations on a finite number of elements to a finite number of elements.

It is likely that the greatest practical application in the future of the study of these logics will lie in the discovery of relations between operations on a finite number of elements and the principles upon which they are based. Consider the case when \( n \) is 2, the Boolean Algebra, and all of the binary operations on two elements such that the result of the operation is again one of two elements. Study of all possible combinations shows that there are 16 operations definable that satisfy these conditions. The relationships between these operations and their properties can be fairly easily established by trial and error methods because of the smallness of their number. Now take \( n \) as 3. All of the possible binary operations on 3 elements to 3 elements are 16,183 in number. In case we wish to select a particular binary operation, the methods of trial and error are impractical. This problem was faced when the problem of finding a single binary operation which would generate each of the other binary operations was suggested. The analogous case for \( n \) is 2 was solved by inspection of the 16 operations. The number of binary operations on \( n \) elements to \( n \) elements is \( n^2 \).
Lewis and Langford have proven propositions in 3-valued logic. To do this they made a table in which they placed the results of applying each of 3 truth-values to the proposition. If the proposition had the truth-value "certainly true" in each case, then the proposition was said to be assertable. It is obvious that for the case of a general n we can not make out a table of this sort. Besides, this would involve the proving of a proposition for each value of n. Because of these difficulties we abandon this method and develop a more general one. Then we prove propositions which will hold for any finite integral value of n. These proofs hold as well for $n = 1,000,000$ as $n = 2$.

Summarizing briefly the results of the following chapters, we have:

1. Generalized many results of the Boolean Algebra, obtaining a single binary operations which will generate all of the remaining operations of the logic; generalized the Boolean expansion.

2. Obtained more completely the properties of five types of implication, one of which was defined by Lukasiewicz.

3. Discovered which of the important propositions of Whitehead and Russell carry over to $L_n$ for five different types of implication.

4. Developed two types of arithmetic of $L_n$.

In closing, I wish to express my appreciation to Professor E.T. Bell for his direction of this thesis and his many suggestions relating to this work.
CHAPTER ONE

THE DEVELOPMENT OF THE ALGEBRA OF N-VALED LOGIC

INTRODUCTION. In 1920 Lukasiewicz\(^1\) defined in terms of a matrix a "three-valued logic". A year later Post\(^2\) generalized two-valued truth systems, giving an m-valued system. This system was defined in terms of two operators which were generalizations of the negation and disjunction of two-valued logic. Lukasiewicz\(^1\) gave a short characterization of an n-valued system in 1922. This was followed by a paper\(^3\) in 1930 defining implication and negation for an n-valued system. Lewis\(^4\) and Langford extended the results concerning the three-valued logic given by Lukasiewicz and Tarski in their papers by using the truth-tables in terms of which Post had defined his n-valued system.


4. Lewis and Langford: Symbolic Logic, Century Co., 1932. See in particular Chapter VII. Henceforth called LL.
In this chapter we define the implication and negative of Lukasiewicz in terms of the negative and disjunctive of Post. These in turn are defined in terms of a single operator\textsuperscript{5}, \( p|q \). The operators of Post are introduced since they will allow us to generate the matrix of any order function on \( n \) truth-values. This statement may not be made about the implication and negation of Lukasiewicz since they are not symbolically complete\textsuperscript{6}.

To the above operators we add three others, equivalence and two products, \( pq \) and \( pxq \). With these relations we develop an extension of the algebra on two truth-values to \( n \) truth-values.

A large portion of the properties of Chapter Two of Symbolic Logic by Lewis and Langford have been generalized. By generalization we mean that in the case when \( n \) is two the generalized property becomes the Boolean property of which it is the generalization. In the next chapter this algebra is applied to the propositions listed by Whitehead and Russell as the most important in divisions 2, 3, 4, and 5 of the Principia Mathematica.

---

5. Webb, "Definition of Post's Generalized Negative and Maximum in Terms of one Binary Operation", \textit{Amer. Jour. of Math.}, LVIII (1936), pp. 193-194. Post was familiar with a result of this nature. See p. 133, \{15. His operator was defined by \( p|q = \neg_m(p \lor_m q) \). The definition in the paper cited above was \( p|q = \neg_m(p \lor_n q) \).

6. For this statement I am indebted to Mr. J.C.C. McKinsey of the University of California. For the definition of "symbolically complete" see LL page 251.
The notation used follows that of Whitehead and Russell. Np was introduced instead of \( \forall p \) to avoid confusion with Post's negation. It is more convenient to have the subscript of the truth-values, \( t_i \), range \( 0, 1, \ldots, n-1 \) than in the traditional manner since this allows the use of congruences.

**NOTATION AND DEFINITIONS.** Let \( L_n \) be a logic of \( n \), where \( n \) is a positive integer, truth-values \( t_0, t_1, \ldots, t_{n-1} \). The \( t_1 \) are marks such that to each of the \( t_i \) any one of the \( n \) truth-values of the system may be assigned. One interpretation that may be given to them is that \( t_1 \) is less likely to be true than \( t_j \) if \( i < j \), \( t_1 \) is as likely to be true as \( t_j \) if \( i = j \), and \( t_1 \) is more likely to be true than \( t_j \) if \( i > j \). Then \( t_{n-1} \) is taken to be certainly true and \( t_0 \) certainly false. During the remainder of this chapter \( t_{n-1} \) may be interpreted as being certainly true since we accept a proposition as being assertable when we can show that it has the truth-value \( t_{n-1} \), for all possible truth-values that the component elementary propositions may assume. Since these demonstrations depend upon the subscripts and not upon the truth-values correlated with the subscript, we can correlate any truth-value with \( t_{n-1} \) and obtain a series of propositions having the truth-value correlated to \( t_{n-1} \), for all possible truth-values of its component propositions.

Let \( L_n \) be the logic based on the implication and negation of Lukasiewicz and in the case of \( L_3 \), as modified by Lewis and

---

7. See LT and L.
Langford. $P_n$ represents the logic of Post. $p, q, r, x, y, z$ are elementary propositions in $L_n$. $p \in L_n$ signifies that $p$ is in $L_n$, etc.

In place of the matrices of Lewis and Langford, we adopt arithmetical methods of showing what values the matrix possesses. We denote this arithmetic, which includes the ordinary operations of $+, -, \cdot, =, >, <$, defined as usual, and the integers $a, b, ..., e, h, i, j, k$ by $\mathbb{A}$. To associate the truth-values $t_i \in L_n$ with $i \in \mathbb{A}$, we use the following two symbols:

If $p$ has the truth-value $t_i$, \([p]' = i (i = 0, 1, ..., n-1)\). Dr
\[ f(p, q, ..., r; a, b, ..., e) \]

1. $f(p, q, ..., r; a, b, ..., e)$ is considered as a convenient method of writing $f([a]', [b]', ..., [r]'; a, b, ..., e)$.

2. $f([a]', [b]', ..., [r]'; a, b, ..., e)$ may be any rational polynomial with arguments and operations in $\mathbb{A}$.

3. If for $[p]' = i, [q]' = j, ..., [r]' = k$
\[ f([a]', [b]', ..., [r]'; a, b, ..., e) = c, \text{ then } f(p, q, ..., r; a, b, ..., e) = d \]
where $c \equiv d \mod n, 0 \leq d < n$.

If we enclose a system of brackets in another set of brackets, we shall consider the expression to mean that we shall operate with the inner brackets before considering the outer set of brackets; e.g., by \([i+j]'\) we mean \([i+j]\).

---

8. See LL Chapter VII.

9. See P.
The chief difference between \([\ ]\) and \([\ ]'\) is that all operations indicated in \([\ ]'\) are in \(L_n\) while all operations indicated in \([\ ]\) are in \(A\). An example is \([p \triangleright q]' = [n-1+q-p]\) if \([p] \supset [q]\). This statement might be written as follows: If \(p\) has the truth-value \(t_i\) and \(q\) the truth-value \(t_j\), where \(i \geq j\), then the truth-value of \(t_i \lor t_j\) is \(t_k\) where \(k = n-1+j-i\).

It is convenient to define:

If \([p] = i, [q] = j\), then
\[
\begin{align*}
\max(p, q) &= j \text{ where } i \leq j \\
&= i \text{ where } i > j.
\end{align*}
\]

If \([p] = i, [q] = j\), then
\[
\begin{align*}
\min(p, q) &= j \text{ where } i \geq j \\
&= i \text{ where } i < j.
\end{align*}
\]

It is evident from the properties of congruences that \([\alpha + q]\) may be written as \([\alpha + b]\). Accordingly, we shall consider \([\alpha + \max(p, q)]\) to mean \([\alpha] + \max(p, q)]\), etc.

Dots are used here as in the two-valued logic for punctuation.

We shall define all operations of \(L_n\) in terms of \(p|q\). The truth-table for \(p|q\) is given by:

1.01 \([p|q]' = [1 + \max(p, q)]\) \(\text{Df}\)

Other operations in \(L_n\) to be used are defined as follows:

1.02 \(p^i = p\), \(p^{i+1} = p^i\) \((i = 0, 1, \ldots, n-2)\) \(\text{Df}\)

1.03 \(p \lor q = (p|q)^n\) \(\text{Df}\)

1.04 \(Np = p^{n-2} - p\) \(\text{Df}\)

1.05 \(pc = N(Np, V, Nq)\) \(\text{Df}\)
We shall at times find it more convenient to use $p.q$ than $pq$. In such instances $p.q$ will be considered as merely another way of writing $pq$.

1.06 $p \supset q = (pq)^{[n-1-p]}$  \hspace{1cm} \text{Df}

1.07 $p \equiv q : p \supset q, q \supset p$  \hspace{1cm} \text{Df}

1.08 $pq = (p^n \lor q^n)^{[1]}$  \hspace{1cm} \text{Df}

THEOREMS READILY DEDUCIBLE. From the preceding definitions we can readily show the following theorems concerning their properties.

1.1 $[p^n]' = [p^n + h]$  \hspace{1cm} (1.02, 1.01)

Proof: $[p^n]' = [p^{[h]}^{[n]}] = \left[1 + \max \left([p^n], [p^{[h]}]\right)\right] = [1 + [p^n]]$  \hspace{1cm} (1.02, 1.01)

Continuing this process

$[p^n]' = [1 + [p^n]] = [2 + [p^n]] = \cdots = [n + [p^n]]$  \hspace{1cm} (1.02, 1.01)

Or $[p^n]' = [n + p]$.

1.2 $[p \lor q]' = [\max(p, q)]$

Proof: $[p \lor q]' = [(p \lor q)^{(n-1)}] = [1 + \max(p, q) + n - 1] = [\max(p, q)]$  \hspace{1cm} (1.03, 1.1)

1.3 $[n p]' = [n-1-p]$  \hspace{1cm} (1.04, 1.1)

Proof: $[n p]' = [p + 2n - 2p - 1] = [n-1-p]$  \hspace{1cm} (1.05, 1.3, 1.2)

1.4 $[p q]' = [\min(p, q)]$

Proof: $[p q] = [n-1-\max(n-1-p, n-1-q)]$  \hspace{1cm} (1.05, 1.3, 1.2)

If $[p] < [q]$, $[n-1-p] > [n-1-q]$ and $[p q]' = [p]$

If $[p] > [q]$, $[n-1-p] < [n-1-q]$ and $[p q]' = [q]$

If $[p] = [q]$, $[n-1-p] = [n-1-q]$ and $[p q]' = [p] = [q]$

Hence $[p q]' = [\min(p, q)]$
1.5 \( [p \circ q]' = n-1 \) if \( [p] \leq [q] \)
\[ = [n-l+q-p] \text{ if } [p] > [q] \]

Proof: \( [p \circ q]' = [\min(p, q)+n-1-p] \) \hspace{1cm} (1.06, 1.1)

If \( [p] \leq [q] \), \( [p \circ q]' = [p+n-1-p] = n-1 \)
If \( [p] > [q] \), \( [p \circ q]' = [n-l+q-p] \)

1.6 \( [p \equiv q]' = n-1 \) if and only if \( [p] = [q] \)
\[ = [n-l+q-p] \text{ if } [p] > [q] \]
\[ = [n-l+p-q] \text{ if } [p] < [q] \]

Proof: \hspace{1cm} (1.07, 1.4, 1.5)

From this result we readily see that 1.07 implies that two propositions may be asserted as equivalent when and only when they have the same truth-table. Using 1.6 and the preceding theorems we immediately get the following relations of equivalence between the operations in \( L^n \) with those of \( P_n \).

The difference in notation of truth-values must be considered. \( t_m \) in \( L^n \) becomes \( \frac{m}{n-1} \) in \( L^n \) and \( t_{n-m} \) in \( P_n \). Thus
\[ [n-l+q-p] \text{ in } L^n \text{ becomes } [1+q-p] \text{ in } L^n \text{ and } [1+q-p] \text{ in } P_n. \]

1.7' \( p \circ q \equiv p \land q \); \( p \equiv q \in L^n \), \( p \land q \in L^n \).

1.8'' \( Np \equiv (Np) \); \( Np \in L^n \), \( (Np) \in L^n \).

1.9' \( p \equiv q \equiv p \Rightarrow q \); \( p \equiv q \in L^n \), \( p \Rightarrow q \in L^n \).

1.10 \( p \circ q \equiv q \equiv p \land q \)

Proof: \( [p \circ q]' = [n-l+q-(n-1)] = [q] \) if \( [p] \leq [q] \)

Since \( [n-l+q-p] > [q] \) if \( [p] \neq n-1 \)
\[ = [q] \text{ if } [p] = n-1, \text{ we have } \]

10. See LL p. 213 footnote or L p. 72.

11. See LL p. 214. \( p \Rightarrow q \) is undefined for \( L^n \).
\[ [p \supset \q \circ \theta] = [n-1+q-(n-1)-q+r] = [p] \text{ if } [\theta] = [n-1] \text{ and } [\theta] \neq n-1 \]

\[ [p \supset \q \circ \theta]' = n-1 \text{ if } [\theta] = n-1 \]

Hence, \[ [p \supset \q \circ \theta]' = [\max(p, q)] = [p \lor q]' \]

1.11 \[ p \lor q \supset p \lor q ; \ p \lor q \supset L_\theta, \ p \lor q \supset L_{\theta} \]

1.12 \[ p \lor q \supset p \lor q ; \ p \lor q \supset L_\theta, \ p \lor q \supset L_{\theta} \]

1.13 \[ \l_1 \supset \l_1 \lor \l_1, \ p_1 \lor \l_1 \supset \l_1 \lor \l_1 (i = 1, \ldots, n-1) ; \ p_1 \lor \l_1 \supset L_n ; \]

\[ \l_1 \lor \l_1 \lor \l_1 \supset P_n. \]

1.14 \[ p \lor q \supset p \lor q ; \ p \lor q \supset L_n, \ p \lor q \supset P_n. \]

1.15 \[ [p \lor q]' = 0 \text{ if } [\theta]' = 0 \]

\[ = [\max(p, q)] \text{ if } [\theta]' \neq 0 \]

Proof: \[ [p \lor q]' = [1 + \max([n-1 + p], [n-1 + q])] \] (1.08, 1.1, 1.2)

1.16 \[ p \lor q \supset p \lor q \]

1.17 \[ (p \lor q) \lor (q \lor r) \supset p \lor q \lor (q \lor r) \]

Proof: \[ [(p \lor q) \lor (q \lor r)]' = [\max(\max(p, q), 1)]' = [\max(\max(p, q, r))' \]

\[ = [p \lor q \lor (q \lor r)]' \]

1.18 \[ p \lor q \supset p \lor q \]

1.19 \[ (p \lor q) \lor (q \lor r) \supset p \lor q \lor (q \lor r) \]

1.20 \[ p \lor q \supset p \lor q \]

1.21 \[ p \lor (q \lor r) \supset (p \lor q) \lor r \]

Proof: If \[ [(p \lor q)' = 0 \text{ then } 1.21 \text{ reduces to } 1.18, \text{ if } [p \lor q]' \neq 0 \]

then \[ 1.21 \text{ reduces to } 1.17 \]


1.22 \( p(qVr) \equiv pq \cdot V \cdot pr \)

Proof: By 1.16 without loss of generality we can take \([q] \geq [r] \).

If \([r] \geq [q] \geq [r] \), then \([p(qVr)]' = [q] \) and \([pq \cdot V \cdot pr]' = [q] \)
If \([q] > [p] \geq [r] \), then \([p(qVr)]' = [q] \) and \([pq \cdot V \cdot pr]' = [p] \)
If \([q] \geq [r] > [p] \), then \([p(qVr)]' = [p] \) and \([pq \cdot V \cdot pr]' = [p] \).

1.23 \( pV(qr) \equiv (pVq)(pVr) \)

Proof: Similar to that of 1.22.

1.24 \( pV(qVr) \equiv pVq \cdot V \cdot pVr \)

Proof: \([\max\{p, \max(q, r)\}'] = \left[\max\{\max(p, q), \max(p, r)\}\right]' \).

1.25 \( px(qVr) \equiv pxq \cdot V \cdot pxr \)

Proof: If \([p] = 0\), \([px(qVr)]' = 0 \) and \([pxq \cdot V \cdot pxr]' = 0 \)
If \([r] = 0\), \([qVr]' = [r] \) and \([pxq \cdot V \cdot pxr]' = [pxr]' \)
Hence \([px(qVr)]' = [pxq \cdot V \cdot pxr]' \) if \([p] = 0 \).
Similarly for \([r] = 0 \).
If \([p] \neq 0\), \([q] \neq 0\), \([r] \neq 0 \), then we can replace \(x\) by \(V\) and 1.25 becomes 1.24.

1.26 \( p \equiv p \cdot V \cdot pq \)

Proof: \([p \cdot V \cdot pq]' = [\max\{p, \min(p, q)\}] \)
If \([p] \leq [q] \) then \([p \cdot V \cdot pq]' = [p] \)
If \([p] > [q] \) then \([p \cdot V \cdot pq]' = [p] \)

1.09 \( pVqVr \equiv pV(qVr), \quad pqr \equiv p(qr), \quad pxqxr \equiv px(qxr) \) \( \text{Df} \)

1.010 \( X_i \equiv x_{i,1}^1 \cdot x_{i,2}^2 \cdot \ldots \cdot x_{i,m}^m \) \( \text{Df} \)

1.011 \( \sum_{i=1}^{n} X_i = X_{1,0} ^1 \cdot V_{1,1} ^1 \cdot \ldots \cdot V_{1,m} ^1 \) similarly for \( \sum_{i=1}^{n} \sum_{j=1}^{k} \sum_{k=0}^{\ldots} \)
where \( X_i \) and \( X_{i,k,m} \) (\( i, j, \ldots, k = 0, 1, \ldots, n-1 \))
are any elementary propositions.

1.27 \( \left[ x_{i,0} \right]' = 1 \) if \([x] = n-i \)
= 0 if \([x] \neq n-i \)
Proof: If \([x] = n-j\) where \(j \neq 1\) then \([x^j] = [n-j+j] = 0\). But \(x^j\) occurs in \(X_1\), hence \([X_1] = 0\) if \([x] \neq n-i\) (1.4, 1.09, 1.010, 1.18). If \([x] = n-i\), \([x^i] = 0\), \([x^{i+j}] = 1\) and \([X_1] = 1\), since \(t_0\) does not occur in \(X_1\).

1.28 \([X_1^i x_1 x_2 \ldots x_n]^t = 1\) if \([x] = n-i, [y] = n-j, \ldots, [z] = n-k\) = 0 otherwise.

Proof: (1.27, 1.15).

1.29 \(X_1 x_1 = t_0\) \(i \neq j\)

Proof: (1.4, 1.27).

1.30 \(X_1 x_1 = t_0\) \(i \neq j\)

Proof: (1.15, 1.27).

1.31 \(\sum_{i=0}^{n-1} X_1 = t_1\)

Proof: If \([x] = n-i\), \([x^i] = 1\), \([x_0 v x_1 v x_2 \ldots x_{i+1} v x_{i+2} v \ldots x_{n-1} = 0\)

and \([\sum_{i=0}^{n-1} x_i] = 1\) \((i = 0, 1, \ldots, n-1\). (1.2, 1.27).

1.32 \(\sum_{i=0}^{n-1} X_1 \sum_{j=0}^{n-1} Y_j \sum_{k=0}^{n-1} Z_k\)

Proof: (1.31).

1.33 \(\sum_{i=0}^{n-1} X_1 \sum_{j=0}^{n-1} Y_j \sum_{k=0}^{n-1} Z_k\)

Proof: (1.28, 1.32, 1.2, 1.17).

1.012 By \(F(x, y, \ldots, z)\) we represent any function of \(x, y, \ldots, z\). Or to be more explicit, if \([x] = 1, [y] = j, \ldots, [z] = k\) then \(F(x, y, \ldots, z)\) has a definite value \(h\), \(F(t_i, t_j, \ldots, t_k)\) has the truth-value that \(F(x, y, \ldots, z)\) has when \(x, y, \ldots, z\) are replaced by \(t_i, t_j, \ldots, t_k\) respectively.

1.013 If \(F(x, y, \ldots, z) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} A_{i,j,k} X_{n-i} Y_{n-j} Z_{n-k}\)

then \(\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} A_{i,j,k} X_{n-i} Y_{n-j} Z_{n-k}\), where \(A_{i,j,k}\) represents some definite truth-value \(t_h\) which depends upon \(F(x, y, \ldots, z)\), is said to be the normal form for \(F(x, y, \ldots, z)\).
1.34 \( F(x) := \sum_{i=0}^{n-1} F(t_i) x^{n-i} \)

Proof: \( \left[ F(t_i) x^{n-i} \right]' = \left[ F(t_i) \right]' \) if \( \left[ x \right] = 1 \)

\( = 0 \) otherwise \((1.15, 1.27)\)

Hence \( \left[ \sum_{i=0}^{n-1} F(t_i) x^{n-i} \right]' = \left[ F(t_i) \right]' \) if \( \left[ x \right] = 1 \).

1.35 \( F(x, y, \ldots, z) := \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \sum_{k=0}^{n-1} F(t_i, t_j, \ldots, t_k) x^{n-i} y^{n-j} z^{n-k} \)

Proof: Generalization of that of 1.34.

---

The above is a proof that \( F(x, y, \ldots, z) \) may always be expressed in normal form. \( A_{i, j, \ldots, k} \) is \( F(t_i, t_j, \ldots, t_k) \).

1.36 \( x^i x_j \equiv t_0, \quad x^i x_j \equiv t_0 \)

Proof: If \( \left[ x \right] = n-1, \left[ x^i \right]' = 0 \). Hence \( \left[ x^i x_j \right]' = 0 \) and \( \left[ x^i x_j \right]' = 0 \)

for \( \left[ x \right] = 1 \) \((i = 0, 1, \ldots, n-1)\). \((1.4, 1.27)\).

\( x_1 \) becomes \( \sim x \) in \( L_2 \). This allows us to consider 1.36 as a generalization of \( x(\sim x) = 0 \) in \( L_2 \). 1.35 can be used as a proof that any function \( F(x, y, \ldots, z) \) can be generated from \( p \mid q \).

In such a proof we would substitute \( (t_0)^k \), where

\[ k = \left[ F(t_i, t_j, \ldots, t_h) \right]' \]

for \( F(t_i, t_j, \ldots, t_h) \) as the coefficient of \( x^{n-i} y^{n-j} \ldots z^{n-h} \).

Then, since we have obtained \( t_0 \) in terms of \( p \mid q \), we have obtained \( F(x, y, \ldots, z) \) in terms of \( p \mid q \). This may also be considered as a means of determining an expression for any single valued function on a finite number of elements to a finite number of elements.

We also see that in \( L_2 \) 1.35 becomes the Boolean expansion.
CHAPTER TWO

IMPLICATION

In $L_2$ we find that $p \rightarrow p$ and $p \rightarrow q \rightarrow r \rightarrow p \rightarrow r$ hold. However in $L_n$ the implication of Lukasiewicz fails to possess the latter property, that of transitivity. There are a great number of possible choices of matrices defining implication relations. If by $p \rightarrow q$ we mean "$p$ implies $q$" and if we take $[p \rightarrow q]' = n-1$ if and only if $[p] \leq [q]$ it is interesting to discover the necessary and sufficient conditions that must be imposed upon $p \rightarrow q$ before the proposition $p \rightarrow q, q \rightarrow r : l \rightarrow p \rightarrow r$ holds in $L_n$. Using our earlier interpretation for truth-values we see that the condition $[p \rightarrow q]' = n-1$ if and only if $[p] \leq [q]$ involves the principle "a proposition implies any which is equally or more probable; and is implied by any which is equally or less probable."

An investigation of the possibilities of $p \rightarrow q$ defined in the above manner leads us to the following theorem:

6.1 If $[p \rightarrow q]' = n-1$ when and only when $[q] \leq [q]$ then in order for $[p \rightarrow q, q \rightarrow r : l \rightarrow p \rightarrow r]' = n-1$ where $p, q, r$ may assume any truth-value $t_i, t_j, t_k$ it is necessary and sufficient that

1. If $[p] > [q] > [r]$ then $[p \rightarrow q] \geq [p \rightarrow r]'$ and $[p \rightarrow r] \leq [q \rightarrow r]'$

2. And if $p, q, r$ have particular truth-values, say $t_i, t_j, t_k$ respectively, where $i > j > k$ then either

15. See in particular LL pp. 229 and 230.
\[ [qIr] \geq [pIr] \]

and \[ [pIq] = [pIr] \]
or \[ [qIr] = [pIr] \]

and \[ [pIq] \geq [pIr] \]

but we cannot have both of the relations
\[ [qIr] > [pIr] \]

and \[ [pIq] > [pIr] \]

holding simultaneously.

Proof: 1. If \([q] \leq [r] \)

then \([pIr] = n-1\), making \([pIq.qIr:I.pIr] = n-1\).

2. \([q] > [r] \)

a. \([q] \geq [c] \).

Then \([pIq.qIr:I.pIr] \) becomes \([pIq.I.pIr] \) since \([qIr] = n-1\).

Therefore, if \([q] > [c] \geq [q] \) then
\[ [pIq] \leq [pIr] \]

in order for the proposition \(pIq.qIr:I.pIr \)

to hold since by hypothesis \([pIq] = n-1 \) if and only

if \([q] \leq [c] \).

b. \([q] \geq [c] > [r] \).

\([pIq.qIr:I.pIr] \) becomes \([qIr.I.pIr] \) since \([pIq] = n-1\).

Hence, as above, if \([c] \geq [q] > [r] \) then
\[ [qIr] \leq [pIr] \].

\(c. \ [p] > [c] > [r] \).

In \(pIq.qIr:I.pIr \) from \(a \) and \(b \) we see that the conditions

\[ [qIr] \geq [pIr] \]

\[ [pIq] \geq [pIr] \]

must hold.
If these conditions hold

\[ [p_{1q}.q_{1r}] \geq [p_{1r}]' \]

However, it is necessary that the equality sign hold in the above statement, otherwise by hypothesis \[ [p_{1q}.q_{1r}:p_{1r}]' \neq n-1 \]. Then, in order for the equality sign to hold, for a particular set of truth-values such that \([p] > [q] > [r] \), it is necessary and sufficient that either

\[ [q_{1r}]' = [p_{1r}]' \]
\[ [p_{1q}]' \geq [p_{1r}]' \]

or

\[ [q_{1r}]' \geq [p_{1r}]' \]
\[ [p_{1q}]' = [p_{1r}]' \]

The above conditions are evidently sufficient.

\[ p \supset q \]

does not satisfy the conditions for transitivity when \( n > 2 \) since if \([p] = 2 \), \([q] = 1 \), \([r] = 0 \)

\[ [p \supset q]' = [n-1+0-2] = n-3 \], \([p \supset c \supset r : \supset r : \supset r]' = n-2 \].

Hence \([p \supset c \supset r : \supset r : \supset r]' = [n-1+n-3-(n-2)] = n-2 \).

In other words, for these particular values of \( p, q, r \)

\[ [p \supset q]' > [p \supset r]' \]
\[ [q \supset r]' > [p \supset r]' \]

There are many operations which satisfy 6.1. In particular we shall study three of these operations.

Let us make the definitions:

6.01 \( p_{1q} = .p \supset q \) \hspace{1cm} \text{Df}
6.02 \( p_{1q} = .N[p_{1q} .I_{1} .(p_{1q})'] \) \hspace{1cm} \text{Df}
6.03 \( p_{1q} = p_{1q} .V_{q} \) \hspace{1cm} \text{Df}
6.04 \( p_{1q} = p_{1q} .V .N_{p} \) \hspace{1cm} \text{Df}
6.05 \( p_{I_5}q \).\( \equiv p_{I_2}q \cdot v \cdot q \cdot v \cdot Np \)

Df

From these definitions by means of the theorems of the preceding chapter, we determine their truth-tables.

6.2 \( [p_{I_1}q]' = n-1 \) if \( [p] \leq [q] \)

\[ = [n-1+t_{-1}] \text{ if } [p] > [q] \]

Proof: (1.5).

6.3 \( [p_{I_2}q]' = n-1 \) if \( [p] \leq [q] \)

\[ = 0 \text{ if } [p] > [q] \]

Proof: If \( [p] \leq [q] \), then \( [p_{I_1}q]' = n-1 \) and \( [p_{I_1}q \cdot \overline{I_1} \cdot (p_{I_1}q)'] = [t_{-1}I_1t_q]' = 0 \).

Hence \( [p_{I_2}q]' = [n[p_{I_1}q \cdot \overline{I_1} (p_{I_1}q)']].' = n-1 \) if \( [p] \leq [q] \).

If \( [p] > [q] \), then say \( [p_{I_1}q]' = i \) where \( i \neq n-1 \).

then \( [p_{I_2}q]' = [n[t_{-1}I_1t_{i+1}]'] = [n-1-(n-1)] = 0 \).

6.4 \( [p_{I_3}q]' = n-1 \) if \( [p] \leq [q] \)

\[ = [q] \text{ if } [p] > [q] \]

Proof: \( [p_{I_3}q]' = n-1 \) if \( [p] \leq [q] \), hence \( [p_{I_3}q \cdot v \cdot q]' = n-1 \).

\( [p_{I_2}q]' = 0 \) if \( [p] > [q] \), hence \( [p_{I_2}q \cdot v \cdot q]' = [q] \) in this case.

6.5 \( [p_{I_4}q]' = n-1 \) if \( [p] \leq [q] \)

\[ = [n-1-p] \text{ if } [p] > [q] \]

Proof: Same type as in 6.4.

6.6 \( [p_{I_5}q]' = n-1 \) if \( [p] \leq [q] \)

\[ = [Np \cdot v \cdot q]' \text{ if } [p] > [q] \]

Proof: (6.3,1.3)

By theorems 6.3,6.4,6.5 it is evident that \( p_{I_3}q \), \( p_{I_2}q \), and \( p_{I_4}q \) satisfy the necessary and sufficient conditions of 6.1 making each of these types of implication transitive. If we check the conditions for \( p_{I_5}q \) in 6.1 for the particular values
of \([p] = n-1, [q] = n-2, [r] = 0\), we find when \(n > 2\) that
\[
[p_{5q}] > [p_{5r}]
\]
\[
[q_{5q}] > [q_{5r}]
\]
contrary to conditions 6.1 part 2.

Thus, we can say that \(p_{5q}\) is not transitive.

It is interesting to determine which of the propositions listed as the most important by Whitehead and Russell in divisions 2, 3, 4, and 6 of *Principia Mathematica* hold in \(L_n\). We list these important propositions in the *Principia Mathematica* below:

2.302 q.I.pIQ
2.303 pINq.I.qIMp
2.155 NpIQ.I.qIQp
2.166 pIQ.I.qINp
2.177 NqINp.I.qIQ
2.24 p.I.qIR:1:q.I.pIR
2.25 qIR.I:pIQ.I.pIR
2.26 pIQ.I.qIR.I.pIR
2.23 pIp
2.215 Np.I.pIQ
3.2 p.I:q.I.pq
3.26 pq.I.p
3.27 pq.I.q
3.3 pq.I.r:1:p.I.qIR
3.31 p.I.qIR:1:pq.I.r
3.33 pq.qIR:1:pIR
3.43 pIQ.pIR:1:p.I.qR
3.35 p.pIQ.I.q
3.45 $pIq.I:pr.I.qr$
3.47 $pIr.qis.I:pq.I.rs$
4.01 $pEq.=pq.qIp$
4.11 $pEq.E.NpENq$
4.13 $p.E.N(Np)$
4.2 $pEp$
4.21 $pEq.E.qEp$
4.22 $pEq.qEr:I:pEr$
4.24 $p.E.pp$
4.25 $p.E.pVp$
4.3 $pq.E.qp$
4.31 $pVq.E.qVp$
4.32 $(pq)r.E.p(qr)$
4.33 $(pVq)Vr.E.pV(qVr)$
4.4 $p.qVr:E:pq.V.pr$
4.41 $p.V.qr:E:pVq.pVr$
4.71 $pIq.E:p.E.pq$
4.73 $c.I:p.E.pq$
5.1 $pq.I.pEq$
5.32 $p.I.qEr:E:pq.E.pr$
5.6 $p.Nq.I.r:E:p.I.qVr$

Utilizing the results of theorems 6.2, 6.3, 6.4, 6.5, and 6.6 we obtain the following table of results. A stands for assertable and N for not assertable. Thus, in the table we find A opposite 2.02 and under $pIq$. That means if we replace I by $l_1$ in 2.02 above, the proposition is assertable in $L_n$. The rest of the table reads in the same manner.
<table>
<thead>
<tr>
<th>p_{i1q}</th>
<th>p_{i2q}</th>
<th>p_{i3q}</th>
<th>p_{i4q}</th>
<th>p_{i5q}</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.02</td>
<td>A</td>
<td>N</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>2.03</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>2.15</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>2.16</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>2.17</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>3.04</td>
<td>A</td>
<td>N</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.05</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.06</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.08</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.21</td>
<td>A</td>
<td>N</td>
<td>N</td>
<td>A</td>
</tr>
<tr>
<td>3.2</td>
<td>A</td>
<td>N</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.26</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.27</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.3</td>
<td>A</td>
<td>N</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.31</td>
<td>N</td>
<td>A</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.33</td>
<td>N</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.35</td>
<td>N</td>
<td>A</td>
<td>A</td>
<td>N</td>
</tr>
<tr>
<td>3.43</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.45</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>3.47</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>4.1</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>4.11</td>
<td>A</td>
<td>A</td>
<td>N</td>
<td>N</td>
</tr>
<tr>
<td>4.13</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>4.2</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>4.21</td>
<td>A</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
<tr>
<td>4.22</td>
<td>N</td>
<td>A</td>
<td>A</td>
<td>A</td>
</tr>
</tbody>
</table>
\begin{tabular}{|c|c|c|c|c|c|}
\hline
& \(p_{1q}\) & \(p_{2q}\) & \(p_{3q}\) & \(p_{4q}\) & \(p_{5q}\) \\
\hline
4.73 & A & N & A & N & A \\
5.1 & A & N & A & N & A \\
5.32 & N & N & A & N & N \\
5.6 & N & N & N & N & N \\
\hline
\end{tabular}

Total of A's: 33 30 31 21 21 33

The above table is verified on succeeding pages.

The five propositions which Lukasiewicz \(14\) states as being a sufficient condition that the system \(L_n\) may be put on a postulational basis hold in \(L_n\) for \(p_{1q}\). They follow:

\[
p_{1q}\cdot q_{1p}, \quad p_{1q}\cdot q_{1r}\cdot I_1\cdot p_{1r}, \quad p_{1q}\cdot I_1\cdot q_{1p}\cdot I_{1r}, \quad p_{1q}\cdot q_{1p}\cdot I_1\cdot q_{1r}, \quad N_{p_{1q}}\cdot q_{1p}\cdot I_{1q}.
\]

By utilizing definition 4.01, \(p_{E_1q} = \cdot p_{1q}\cdot q_{1p}\) (\(i = 0, 1, \ldots, 5\)) we obtain the following theorems concerning the truth-tables of \(p_{E_1q}\):

\[
6.7 \quad \left[ p_{E_1q} \right]' = n-1 \text{ if } [p] = [q] = 0 \text{ otherwise.}
\]

\(14\). See LT p. 41 (following Theorem 26).
6.8 \([pE_3q]' = n-1\) if \([p] = [q]
= [pq]' otherwise

6.9 \([pE_4q]' = n-1\) if \([p] = [q]
= [Nh.Nq]' otherwise

6.10 \([pE_5q]' = n-1\) if \([p] = [q]
= [(Nh.V.q)(Nq.V.p)]' otherwise

**VERIFICATION OF THE RESULTS OF THE TABLE.**

Below we have proven the results of the table. We first list the proposition number and then have five sub-numbers under this heading. The first sub-number refers to the verification of the result for \(pI_1q\), the second for \(pI_2q\), etc.

2.02 \(q.I.pIq\)

True for \(pIq\) when \([p] \leq [q]\).

1. \([p] > [q]\).

\([pI_1q]' = [n-1+c-p]\)

But \([q] \leq [n-1+c-p]\) since \([n-1-p] \geq 0\).

Hence the proposition holds for all \([p], [q]\).

2. Let \([p] = 2, [q] = 1\), then

\([q.I_2.pI_2q]' = 0\), so does not hold when \(n > 2\).

3. If \([p] > [q]\) then \([pI_3q]' = [q]\), therefore theorem holds.

4. Let \([q] = n-2, [p] = n-1\), then theorem does not hold for \(n > 2\).

5. \([p] > [q]\).

\([q] \geq [Nh]\)' evidently holds.

Similarly for \([q] < [Nh]\), when \([q.I_5.pI_5q]' becomes [q.I_5.Np]'\)

2.03 \(pINq.I.qINp\)

True if \([n-1-p] \geq [q]\).
1. \([n-1-p] < [q]\).

Then \([p_{[p]}n_{[q]}'] = [n-l+n-1-c-p]\) since \([p] > [n-l-q]\)
and also \([q_{[p]}n_{[q]}'] = [n-l+n-1-c-p]\) making the proposition hold.

2. Holds since \([p_{[p]}n_{[q]}'] = 0 = [n_{[p]}n_{[q]}]\) when \([n-l-p] < [q]\).

3. Proposition does not hold when \(n > 2\) for \([p] = n-1, [q] = n-2\).

4. Proposition does not hold when \(n > 2\) for \([p] = n-2, [q] = n-1\).

5. When \([n_{[p]} < [q]]\), then \([n_{[q]} < [p]]\)
   and \([q_{[p]}n_{[q]}'] = [n_{[p]}v_{[q]}n_{[q]}']\), \([p_{[p]}n_{[q]}'] = [n_{[p]}v_{[q]}n_{[q]}']\). Theorem holds.

2.15 \(n_{[p]}q_{[p]}i_{[q]}n_{[q]}p_{[p]}\).

True if \([n_{[q]}] < [p]\).

Proof follows same lines as in 2.03.

2.16 \(p_{[q]}i_{[q]}n_{[q]}i_{[p]}\).

Proof same type as in 2.03.

2.17 \(n_{[q]}i_{[q]}n_{[q]}p_{[q]}i_{[p]}\).

Proof same type as in 2.03.

2.04 \(p_{[q]}i_{[q]}r_{[q]}i_{[q]}c_{[q]}i_{[p]}r_{[p]}\).

True if \([p] < [r]\).

1. \([p] > [r]\).

True if \([q] < [n-1+r-p]\)

Take \([q] > [n-1+r-p]\), then \([q] > [r]\) and \([n-1-q+r] < [p]\).

Accordingly, \([q_{[q]}i_{[q]}p_{[p]}i_{[q]}r_{[q]}'] = [n-l+r-1+p-q]\)
and \([q_{[q]}i_{[q]}p_{[p]}i_{[q]}r_{[q]}'] = [n-l+r-1+p-q]\)
and the theorem holds for this case.

2. Let \([p] = 2, [q] = 1,\) and \([r] = 1\). For this case the theorem does not hold for \(n > 2\).

3. \([p] > [r]\).

Holds if \([q] < [r]\).
Let $[q] \geq [p] > [r]$, then $[q \cdot I_{q} \cdot p_{I_{q}}]^{1} = [r] = [p \cdot I_{q} \cdot q_{I_{q}}]^{1}$. We get the same result when $[p] > [q] > [r]$, making the theorem hold.


5. $[p] > [r]$.

Then $[p_{I_{q}}]^{1} = [n \cdot p \cdot r]^{1}$.

Holds if $[q] \leq [n \cdot p \cdot r]^{1}$

Take $[q] > [n \cdot p \cdot r]^{1}$, then $[q] > [r]$, $[q] > [n \cdot p]^{1}$, and $[p] > [n \cdot q]^{1}$.

Hence $[q \cdot I_{q} \cdot p_{I_{q}}]^{1} = [n \cdot q \cdot n \cdot p \cdot r]^{1}$

and $[p \cdot I_{q} \cdot q_{I_{q}}]^{1} = [n \cdot q \cdot n \cdot p \cdot r]^{1}$ making theorem hold.

2.35 \textit{qir}.1:pIq.1.pIr.

True if $[p] \leq [r]$.

1. $[p] > [r]$.

$[q] \geq [p] > [r]$, $[p_{I_{q}}]^{1} = n-1$ and $[p_{I_{q}}]^{1} = [n-1 \cdot r-p]$ making $[p_{I_{q}} \cdot I_{q} \cdot p_{I_{q}}]^{1} = [n-1 \cdot r-p]$ $[q_{I_{q}}]^{1} = [n-1 \cdot r-q]$, but $[n-1 \cdot r-q] < [n-1 \cdot r-p]$, making theorem hold for this case.

$[p] > [q] > [r]$,

$[p_{I_{q}} \cdot I_{q} \cdot p_{I_{q}}]^{1} = [n-1 \cdot r-q] = [q_{I_{q}}]^{1}$, so theorem holds.

$[p] > [r] > [q]$, $[p_{I_{q}}]^{1} > [p_{I_{q}}]^{1}$, so theorem holds.

2. Holds.

3. $[p] > [r]$, $[p_{I_{q}}]^{1} = [r]$

$[q] \leq [r]$, holds since then $[q_{I_{q}}]^{1} = [r]$.

$[q] > [p] > [r]$, $[p_{I_{q}} \cdot I_{q} \cdot p_{I_{q}}]^{1} = [r] = [q_{I_{q}}]^{1}$.

$[p] > [q] > [r]$, $[p_{I_{q}} \cdot I_{q} \cdot p_{I_{q}}]^{1} = [r] = [q_{I_{q}}]^{1}$.


5. $[p] > [r]$, then $[p_{I_{q}}]^{1} = [n \cdot p \cdot r]^{1}$.
\([q] \leq [r] \) holds since \([pq_5q] = [np_vq] \) and \([np_vq] \leq [np_vr] \).

\([p] > [q] > [r] \), \([np_vq] \geq [np_vr] \).

Theorem holds if equality sign is true.

Let then \([np_vq] > [np_vr] \), or \([q] > [np_vr] \).

Then \([pi_{5q}, pi_{5r}] = [np_vnp_vr] \).

But \([qi_{5r}] = [np_vr] \), and \([np_vr] \leq [np_vnp_vr] \) so the theorem holds.

\([q] > [p] > [r] \), \([pi_{5q}, pi_{5r}] = [np_vr] \) and \([qi_{5r}] = [np_vr] \)

\([np_vq] \leq [np_v] \), therefore \([np_vq] \leq [np_vr] \) and the theorem holds.

2.06 \(pi_{5s}, pi_{5r}, pi_{5r} \)

True for \([0] \leq [r] \).

1. \([q] > [p] > [r], [pi_{1s}] = [n-ltr-p], [qi_{1s}] = [n-ltr-q] \)

since \([n-ltr-p] \geq [n-ltr-q] \), \([qi_{1s}, pi_{1s}] = n-1. \)

\([p] > [q] > [r] \)

\([qi_{1s}, pi_{1s}] = [n-ltr-p-qr(n-1)-r+q] \)

\([n-ltr-p-q] \)

But \([pi_{1q}] = [n-ltr-p] \) so theorem holds.

\([p] > [q] > [r] \), \([pi_{1s}] = [n-ltr-p] \)

and \([qi_{1s}, pi_{1s}] = [n-ltr-q] \)

But \([n-ltr-p] \leq [n-ltr-q] \) so theorem holds.

2. Evident.

3. \([p] > [r], [pi_{5r}] = [r] \).

\([q] \leq [r], then [qi_{5r}, pi_{5r}]' = [r] \) and \([pi_{5q}] = [q] \leq [r] \).

\([q] > [r], then [qi_{5r}, pi_{5r}]' = n-1. \)

4. Not assertable for \(n > 2 \) if \([p] = 1, [q] = 0, [r] = 0 \).

5. \([p] > [r], [pi_{5r}] = [npvr] \).

\([q] \leq [r], [qi_{5r}] = n-1, [qi_{5r}, pi_{5r}]' = [np_vr] \).
\[ [p_{i5}q]' = [np.v.q] \leq [np.v.r], \text{ so theorem holds for this case.} \]
\[ [q] \geq [p] \geq [r], \quad [q_{i5}r]' = [nq.v.r]', \quad [p_{i5}q]' = [np.v.r]', \quad [q_{i5}r]' \leq [p_{i5}r]' \text{ making this case hold.} \]
\[ [p] > [q] > [r], \quad [q_{i5}r]' = [nq.v.r]', \quad [p_{i5}q]' = [np.v.r]' \]

Hence theorem holds if equality is true in

\[ [q_{i5}r]' \geq [p_{i5}r]' \]

If \([q_{i5}r]' > [p_{i5}r]'\), \([p_{i5}r.i_{5}.p_{i5}r]' = [np.v.np.v.r]' \]
but \([p_{i5}q]' = [nq.v.q]' \), so theorem holds for all cases.

3.03 \( p_ip_p \)

Evident.

2.31 \( np.i.pq \)

True for \([p] \leq [q] \).

1. \([p] > [q], \quad [p_{i1}q]' = [n-l+q-p] \geq [np] \).
2. Not assertable for \([p] = 1, \quad [q] = 0 \) when \( n > 2 \).
3. Not assertable for \([p] = 1, \quad [q] = 0 \) when \( n > 2 \).
4. True since when \([p] > [q], \quad [p_{i4}q]' = [np]' \).
5. \([p] > [q], \quad [p_{i5}q]' = [np.v.q] \geq [np]' \).

3.2 \( p.i.q.i.pq \)

True when \([q] \geq [q] \).

1. \([p] < [q], \text{ then } [q.i_{1}.pq]' = n-1 \).
2. Not assertable for \([p] = 1, \quad [q] = 2 \) when \( n > 2 \).
3. \([p] < [q], [p_{i5}q.i_{5}.pq]' = [p_{i5}p]' = n-1 \).
4. Not assertable for \([p] = 1, \quad [q] = 2 \) when \( n > 2 \).
5. \([p] < [q], \quad [p.i_{5}.q_{i5}r]' = [p.i_{5}.nq.v.p]' = n-1 \).

3.26 \( pq.i.p \)

True if \([p] \leq [q] \).
If \( [p] > [q] \) then we have \( qlp \) which for these values is assertable for all five types of implication.

3.37 \( pq.I.q \)

Same as 3.26.

3.3 \( pq.I.r; I.p.I.q.r \)

True for \( [q] > [r] \).

1. \( [q] > [r] \), \( [qI_1r]' = [n-1+r-q] \)
   \( [p] \leq [n-1+r-q] \) true
   \( [p] > [n-1+r-q] \), then \( [p.I_1.qI_1r]' = [n-1+r-n-1+r-q-p] \)
   Since \([p] > [r] \), \([pq.I_1.r]' = [n-1+r-min(p,q)]'\)
   Therefore \( [pq.I_1.r]' = [p.I_1.qI_1r]' \) and theorem holds.

2. Not assertable for \( n > 2 \) when \( [p] = 1, [q] = 2, [r] = 1 \).

3. \( [q] > [r] \), \( [qI_3r]' = [r] \)
   Holds if \( [p] \leq [r] \).
   \( [p] > [r] \), \( [p.I_3.qI_3r]' = [r] \), \( [pq.I_3.r]' = [r] \), so theorem holds.

4. \( [q] > [r] \), \( [qI_4r]' = [Nq]' \)
   \( [p] \leq [Nq]' \) holds.
   \( [p] > [Nq]' \), \( [p.I_4.qI_4p]' = [Np]' \)
   For \( n > 2 \) the theorem is not assertable for \( [p] = 2, [q] = 1, [r] = 0 \).

5. \( q > [r] \), \( [qI_5r]' = [Nq.V.r]' \)
   \( [p] \leq [Nq.V.r]' \) holds.
   \( [p] > [Nq.V.r]' \), so \( [pq.I_5.r]' = [Np.V.Nq.V.r]' \), making proposition hold.
5.31 p.I.qIr:I:pq.I.r

True if \([pq] \leq [r]\).
1. Not assertable for \(n > 2\) when \([p] = 1, [q] = 1, [r] = 0\).
2. If \([p] > [r], [q] > [r]\) then \([p.I_{q}.r'] = 0\), and \([pq.I_{q}.r'] = 0\), making the theorem hold.
3. Holds since for \([p] > [r], [q] > [r]\), \([pq.I_{q}.r'] = [r]\) and \([p.I_{q}.qI_{q}.r'] = [r]\).
4. Not assertable for \(n > 2\) when \([p] = 1, [q] = 1, [r] = 0\).
5. Not assertable for \(n > 2\) when \([p] = 1, [q] = 1, [r] = 0\).

5.33 pIq.qIr:I:pIr

For proofs see earlier part of the chapter.
1. Not assertable.
2. Assertable.
3. Assertable.
4. Assertable.
5. Not assertable.

5.35 p.pIq:I.q

True when \([p] \leq [q]\)
1. Not assertable for \(n > 2\) when \([p] = 1, [q] = 0\).
2. Holds since when \([p] > [q], [pI_{q}q'] = 0\).
3. \([p] > [q], [p.pI_{q}.q.I_{q}q'] = [pq.I_{q}.q'] = n-1\).
4. Not assertable for \(n > 2\) when \([p] = 1, [q] = 0\).
5. Not assertable for \(n > 2\) when \([p] = 1, [q] = 0\).

5.43 pIq.pIr:I:p.I.qr

True when \([p] \leq [q]'\)
1. \([q] \geq [r] > [p]\), then \([p_{11}q]' = [n-l+p-q], [p_{11}r]' = [n-l+p-r]\)
   or \([p_{11}q] \leq [p_{11}r]'
   \[p_{11}.qr]\), so the theorem holds for this case.

2. \([q] > [r] > [p]\), same as above since \(q\) and \(r\) are symmetrical.

3. \([p] > [qr] \), \([p_{12}.qr]' = [qr]'\), and either \([p_{12}q]' = q\) or
   \([p_{12}r]' = r\), so \([p_{12}q].p_{12}r]' = [qr]'\) making the theorem hold.

4. \([p] > [qr] \), \([p_{14}.qr]' = [np]' = [p_{14}q].p_{14}r]'\).

5. \([p] > [qr] \) and \([q] \leq [r]\). Then \([p_{15}.qr]' = [np.v.qr]' = [np.v.q]'\)
   \([p_{15}q]' = [np.v.q]'\), hence \([p_{15}q].p_{15}r]' = [np.v.q]'\.
   The case is the same for \([q] > [r]\).

3.45 \(p_{1q}.I: p_{11}.q.r\)

True when \([p] \leq [q], [r] \geq [pq]'\)

1. \([p] > [r] > [q], [pr_{11}.cr]' = [r_{11}q]' = [n-l+q-r]\)
   \([p_{11}q]' = [n-l+q-p] < [n-l+q-r]\).

2. \([p] > [r] > [q], [pr_{12}.qr]' = [qr]' = [p_{12}q]'\).

3. \([p] > [r] > [q], [pr_{13}.qr]' = [q] = [p_{13}q]'\).

4. \([p] > [r] > [q], [pr_{14}.qr]' = [nr]'\), \([p_{14}q]' = [np]'\)
   But \([np] \leq [nr]'\).

5. \([p] > [r] > [q], [pr_{15}.qr]' = [np.v.q]'\)
   \([p_{15}q]' = [np.v.q] \leq [np.v.q]'\)

3.47 \(p_{1q}.q.s.I: p_{11}.q.r.s\)

True if \([pq] \leq [rs]'\).

1. \([pq'] > [rs]', and take \([p] \leq [q]\).
   \([q] \leq [r], [s] < [p]\), then \([p_{11}.q.rs]' = [n-l+s-p]\)
\[
[qI_1s]' = [n-1+s-q] \leq [n-1+s-p]
\]

Hence \([pI_1r.qI_1s]' \leq [pq.I_1.rs]'.
\[
r > [s], \ [r] < [p], \text{ then } [pq.I_1.rs]' = [n-1+r-p]
\]

and \([pI_1r]' = [n-1+r-p]'\), so \([pI_1.qI_1s]' \leq [pq.I_1.rs]'

Since the proposition is symmetric in \(p\) and \(q\), the treatment for \([p] \geq [q]\) is the same as in the above case, making the proposition hold.

2. If \([pq] \geq [rs]'\) then either \([pI_2r]' = 0\) or \([qI_2s]' = 0\)

making the proposition hold.

3. \([pq] > [rs]'\), \([s] \leq [r]'\), then \([pq.I_2.rs]' = [s]'\),

\([pI_2r.qI_2s]' = [s]'\).

The treatment is the same for \([s] > [r]'\).

4. \([pq]' > [rs]'\), \([s] \leq [r]'\), then \([pq.I_3.rs]' = [Np.V.Nq]'\)

and \([qI_3s]' = [Nq]'\), making \([pI_3r.qI_3s]' \leq [Np.V.Nq]'\).

The treatment is the same for \([s] > [r]'\).

5. \([pq]' > [rs]'\), \([s] \leq [r]'\), then \([pq.I_5.rs]' = [Np.V.Nq.V.s]'\)

but \([qI_5s]' = [Nq.V.s]' \leq [Np.V.Nq.V.s]'\), making the theorem hold for this case.

The treatment for \([s] > [r]'\) is of the same type.

4.1 \(pq.E:Nq.I_1.Np\)

True for \([p] \leq [q]\), when \([Nq] \leq [Np]'\).

1. \([p] > [q]'\), making \([pI_1q]' \geq [Np]'\), \([pI_1q]' = [n-1+q-p]' = [Nq.I_1.Np]'\).

2. Evident.

3. Not assertable for \(n > 2\) when \([p] = 1\), \([q] = 0\).

4. Not assertable for \(n > 3\) when \([p] = 1\), \([q] = 0\).

5. \([p] > [q]'\), making \([pI_5q]' \geq [Np.V.q]'\), \([pI_5q]' = [Np.V.q]' = [Nq.I_5.Np]'\).
4.11 pEq:E:Np.E.Nq

True when \([p] = [q]\).

1. \([p] > [q]\), then \([pE_1q]' = [n-1+p- q] \), and \([Np.E_1.Nq]' = [n-1+ q-p]\).
2. \([p] < [q]\), then \([pE_1q]' = [n-1+p-q] \), and \([Np.E_1.Nq]' = [n-1+p-q]\).
3. Not assertable for \(n>2\) when \([p] = 0\), and \([q] = 1\).
4. Not assertable for \(n>2\) when \([p] = 0\), and \([q] = 1\).

5. \([p] > [q]\), then \([pE_5q]' = [Np.V.q] \), and \([Np.E_5.Nq]' = [Np.V.q] \).
6. \([p] < [q]\), then \([pE_5q]' = [Nq.V.p] \), and \([Np.E_5.Nq]' = [Nq.V.p] \).

4.13 p.E.N(Np)

True since \([N(Np)]' = [n-1-(n-1)+p] = [p]\).

4.2 pEp

Obviously true.

4.21 pEq.E.qEp

By inspection of theorems 1.6, 6.7, 6.8, 6.9, and 6.10 it is evident that \([pE_1q]' = [qE_1p]' (i = 1, 2, 3, 4, 5)\).

4.22 pEq.qEr:I:pEr

True when \([p] = [q]\), or \([p] = [r]\), or \([q] = [r]\).

1. Not assertable for \(n>2\) when \([p] = 1, [q] = 2, [r] = 0\).
2. If \([p] \neq [r]\) then either \([p] \neq [q]\), or \([q] \neq [r]\), in which case \([pE_2r]' = 0 = [pE_2q.qE_2r]'\) making the theorem hold.
3. \([p] \neq [q] \neq [r], [p] \neq [r] \neq [q]\), then \([pE_2q.qE_2r]' = [pqr]'\)
   and \([pE_3r]' = [pr]'\), thus making \([pE_2q.qE_2r]' \leq [pE_3r]'\).
4. \([p] \neq [q], [q] \neq [r], [p] \neq [r]\), then \([pE_4q.qE_4r]' = [Np.qNq.qNq]'\)
   and \([pE_4r]' = [Np.qNq]'\), thus making \([pE_4q.qE_4r]' \leq [pE_4r]'\).

5. Not assertable for \(n>2\) when \([p] = 0, [q] = 1, [r] = n-1\).

If we prove that a proposition \(pE_iq\) \((i = 1, 2, 3, 4, 5)\) is assertable for some particular \(i\) then \(pE_jq\), where the truth-values of \(p\) and \(q\) do not change as \(j\) changes, is assertable for all \(j\).
(j = 1, 2, ..., 5), since \([p_{E_k} q]' = n-1\) \((k = 1, 2, ..., 5)\) if and only if \([p] = [q]\). We have utilized this fact in propositions 4.34, 4.25, 4.3, 4.31, 4.32, 4.33, 4.4, 4.41 giving in each case the proof for \(p_{E_1} q\), or where the proof for \(p_{E_1} q\) may be found.

4.24 p.E.pp
\[
[p_{pp}'] = [p].
\]

4.25 p.E.pVp
\[
[p_{pVp}'] = [p].
\]

4.3 pq.E.qp
\[
(1.18).
\]

4.31 pVq. E.qVp
\[
(1.16).
\]

4.32 (pq)r. E.p(qr)
\[
(1.19).
\]

4.33 (pVq)Vr.E.pV(qVr)
\[
(1.17).
\]

4.4 p.qVr:E.pq.V.pr
\[
(1.22).
\]

4.41 p.V.qr:E.pVq.pVr
\[
(1.23).
\]

4.71 pIq.E:p.E.pq

Obviously true when \([p] \leq [q]\).

1. \([p] > [q]\), then \([p_{I_1} q]' = [n-1+q-p]\), and \([p_{E_1} pq]' = [n-1+q-p]\)
2. \([p] > [q]\), then \([p_{I_2} q]' = 0\), and \([p_{E_2} pq]' = 0\).
3. \([p] > [q]\), then \([p_{I_3} q]' = [q]\), and \([p_{E_3} pq]' = [q]\).
4. \([p] > [q]\), then \([p_{I_4} q]' = [Np]\), and \([p_{E_4} pq]' = [Np]\).
5. \([p] > [q]\), then \([p_{I_5} q]' = [Np.V.q]'\), and \([p_{E_5} pq]' = [Np.V.q]'\).
4.73 q.I:p.E.pq

True when \([p \leq q]\).
1. \([p] > [q]\), then \([p.E_1.pq]' = [n-l+q-p]\). But \([q] \leq [n-l+q-p]\).
2. Not assertable for \(n > 2\) when \([p] = 2, [q] = 1\).
3. \([p] > [q]\), then \([p.E_2.pq]' = [q]\).
4. Not assertable for \(n > 2\) when \([p] = n-1, [q] = 1\).
5. \([p] > [q]\), then \([p.E_3.pq]' = [Np.V.q]'\). But \(q Np.V.q\).

5.1 pq.I:pEq

True when \([p] = [q]\).
1. \([p] > [q]\), \([pE_1q]' = [n-l+q-p]\), but \([q] \leq [n-l+q-p]\).
   \([p] < [q]\), \([pE_1q]' = [n-l+p-q]\), but \([p] \leq [n-l+p-q]\).
2. Not assertable for \(n > 2\) when \([p] = 1, [q] = 2\).
3. \([p] \neq [q]\), then \([pE_3q]' = [pq]'\).
4. Not assertable for \(n > 2\) when \([p] = n-1, [q] = n-2\).
5. \([p] < [q]\), \([pE_5q]' = [p.V.Nq]'\), and \([pq] \leq [p.V.Nq]'\).
   \([p] > [q]\), similar to \([p] < [q]\).

5.32 p.I.qE'r:E.pq.E.pr

True if \([q] = [r]\).
1. Not assertable for \(n > 2\) when \([p] = 1, [q] = 1, [r] = 0\).
2. Not assertable for \(n > 2\) when \([p] = 1, [q] = 2, [r] = 1\).
3. \([p] \geq [q] > [r]\), then \([p.I_3.qE_3r]' = [r]\)
   and \([pq.E_3.pr]' = [qE_3r]' = [r]\).
   \([q] > [p] > [r]\), then \([p.I_3.qE_3r]' = [r]\)
   and \([pq.E_3.pr]' = [pqE_3r]' = [r]\).
   Similarly we can show that \([p] \geq [r] > [q]\), and \([r] > [p] > [q]\)
   hold.
\([p] \leq [qr]'\), then proposition becomes \([p.I_3.qE_3r:E.pE_3p]'\)
This obviously holds, making the proposition hold for
\(p, q, r\) ranging over all possible truth-values.

4. Not assertable for \(n > 2\) when \(\lceil p \rceil = n-1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

5. Not assertable for \(n > 2\) when \(\lceil p \rceil = 1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

5.6 \(p \cdot \lnot q \cdot I \cdot r \cdot \overline{I} \cdot p \cdot I \cdot q \cdot \text{Vr}\)

1. Not assertable for \(n > 2\) when \(\lceil p \rceil = 1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

2. Not assertable for \(n > 2\) when \(\lceil p \rceil = 1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

3. Not assertable for \(n > 2\) when \(\lceil p \rceil = n-1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

4. Not assertable for \(n > 2\) when \(\lceil p \rceil = n-1, \lceil q \rceil = 1, \lceil r \rceil = 0\).

5. Not assertable for \(n > 2\) when \(\lceil p \rceil = n-1, \lceil q \rceil = n-2, \lceil r \rceil = 1\).

Considering now the five propositions of Lukasiewicz:

\(p \cdot I \cdot q \cdot I \cdot p, \quad p \cdot I \cdot q \cdot I \cdot r \cdot I \cdot q \cdot p, \quad p \cdot I \cdot q \cdot I \cdot p, \quad p \cdot I \cdot q \cdot I \cdot p, \quad p \cdot I \cdot q \cdot I \cdot p,
\)

The first, second and fifth of these propositions we have proven already. They are respectively, 2.02, 2.06 and 2.17. Considering the remaining two we find that they both hold.

The proofs follow:

\(p \cdot I \cdot q \cdot I \cdot p \cdot I \cdot q \cdot I \cdot p\)

\(\lceil p \rceil \leq \lceil q \rceil, \quad [q] \leq [p], \quad [I \cdot q \cdot I \cdot p] = [n-1+p-(n-1)] = [p]\)
and \(\lceil p \rceil \leq \lceil q \rceil \leq [n-1+p-q] \geq [q] \) so that \([p \cdot I \cdot q \cdot I \cdot q] = [p] \).

\(\lceil p \rceil \leq [q], \quad [p \cdot I \cdot q \cdot I \cdot q] = [q] = [I \cdot q \cdot I \cdot p].\)

Thus the proposition holds.

\(p \cdot I \cdot q \cdot I \cdot q \cdot I \cdot p \cdot I \cdot q \cdot I \cdot p\)

\([q] \leq [p], \quad \text{obviously true.}\)

\([q] \geq [p], \quad [q \cdot I \cdot p] = [n-1+p-q] = [p \cdot I \cdot q \cdot I \cdot q \cdot I \cdot p].\)
CHAPTER THREE

THE ARITHMETIC OF N-VALUED LOGIC

INTRODUCTION. This chapter is based largely upon efforts to extend the results given in the papers "Arithmetic of Logic" by Dr. E.T. Bell and "On Bell's Arithmetic of Boolean Algebra" by W.A. Hurwitz. Most of the operations that we shall use are generalizations of those given by Bell and Hurwitz. However, in a few cases the obvious generalization is found to be defective, and we have to seek more obscure generalizations, or even at times, operations that are not generalizations of the corresponding operations in the above mentioned papers.

This chapter is meant to be merely a short exploration of the possibilities of this topic. Many continuations of some of the items of this section will probably be evident. However, lack of time and space prevent a more thorough treatment of this subject. Later the writer intends to more thoroughly explore the generalizations of the ideas in the papers of Bell and Hurwitz, as well as attempting to adapt the results in the papers of Bernstein, Stone, and Von Neumann to a logic of n-values.


We shall say that we have developed the arithmetic of n-valued logic if we can find operations in \( L_n \) which satisfy the abstraction of the postulates covering the corresponding operations in the rational arithmetic. For these postulate systems we shall consult the papers of Bell and Hurwitz. They are listed in brief on the following pages.

In the Boolean case two sets of operations, duals of each other, are given which form the arithmetic of logic. We shall not attempt this since if we were to attempt it we would have to enlarge the number of theorems in Chapter One so as to cover the duals of many of the theorems that appear. However, it is well to remember that the dual case can be worked out as easily as the case which we have given.

**NOTATION.** We shall state our postulates for an abstract ring \( R^* \), as in B p. 53\#, in which the operations \( S, P, L, G, C, D, R \) may be read as, respectively, sum, product, LCM, GCD, divides, and residual. In \( L_n \) we use the same letters, avoiding the use of the small letters since many are in use as propositions. Various definitions will be introduced as they are needed.

**POSTULATES FOR S, P, L, G, C, D, R.** These postulates are from B and retain as much as possible of the notation used there.

**Postulates for \( S \) and \( P. \)**

**R\(^*_1\).** If \( x, y \) are any two elements of \( R^* \), \( xS y \), \( xP y \) are uniquely determined elements of \( R^* \), and

\[
xS y = yS x, \quad yP x = xP y.
\]

**R\(^*_2\).** If \( x, y, z \) are any three elements of \( R^* \),

\[
(xS y)S z = xS(yS z), \quad (xP y)P z = xP(yP z),
\]

\[
xP(yS z) = (xP y)S(xP z).
\]
There exist in $R^*$ two distinct unique elements, denoted by $u$ and $z$, called the unity, zero of $R^*$, such that if $x$ is any element of $R^*$, $x \cdot z = x$, $x \cdot u = x$.

**Postulates for division in $R^*$:**

P1. $x \cdot D \cdot x,$

P2. $x \cdot Dy \cdot zD \cdot xDz,$

P3. $x \cdot Dy \cdot zEx : x = y,$

where $x \cdot Dy$ is uniquely significant for each $x \neq z$ and $y$ in $R^*$, with the exception that $zDz$ is significant but indeterminate in $R^*$.

**Postulates for $G$ and $L$ in $R^*$:**

P4. $x \cdot Gy = yGx,$

P5. $xG(yGz) = (xGy)Gz \equiv xGyz,$

P6. $(xGy)Dx \cdot (xGy)Dy,$

P7. $zEx \cdot zDy : zD(xGy),$

P8. $xLy = yLx,$

P9. $xL(yLz) = (xLy)Lz \equiv xLyz,$

P10. $xD(xLy) \cdot yD(xLy),$

P11. $xDz \cdot yDz : (xLy)Dz,$

where $x, y$ are any elements in $R^*$ and $xLy$ and $xGy$ are uniquely determined in $R^*$.

**Postulates for congruence in $R^*$:**

Let $xCy$ be a relation in $R^*$ such that, if $x, y, z, w$ are elements in $R^*$, $xCy$ is uniquely significant in $R^*$ and the postulates P12-P15 are satisfied:

P12. $xCy : yCx,$

P13. $xCy \cdot yCz : xCz,$

P14. $xCy \cdot zCw : (xSz)C(ySw),$
Then $C$ is called **abstract algebraic congruence**.

If $R^*$ is replaced by its instance $A$, an instance of $xCy$ is $aCb \equiv (c \equiv b \mod m)$, where $a, b$ are integers $\geq 0$ and $m$ is an integer $> 0$.

In $A$, $C$ is said to be arithmetic congruence if to the postulates $P12$-$P15$ are added the postulates

**P16.**  
$$ (a \equiv 0 \mod m) : m \text{ divides } a, \ m \neq 0; $$

**P17.**  
$$ (k \equiv ka \mod m) (a \equiv b \mod m^t), \ m \neq 0, $$

where $qm^t = m$, and $q = G.C.D.$ of $k, m$;

**P18.**  
$$ a \equiv a \mod m. $$

**Abstraction of the residual.**

Let $a, b, h, m$ for the moment denote elements of $R^*$. Then, if $m$ is uniquely determined by $(u \text{ unity in } R^*)$,

**P19.**  
$$ \{aD(hPb) \}, \{aDh \}, \{m \neq u, $$

where $h$ runs through all elements in $R^*$, we shall call $\overline{m}$ the residual of $b$ with respect to $a$, and we shall write $m = bRa$.

**EQUIVALENCE.** In addition to the sets of postulates listed by Dr. E.T. Bell, it is well to list a set of postulates for equality. In the Boolean case the properties of identity, symmetry, and transitivity are satisfied by Boolean equivalence. This is not the case where $n > 2$. Then, we can see by 4.22 that all of the types of equivalence which we defined are not transitive. Of the three types which are transitive ($pE_2q$, $pE_3q$, and $pE_4q$), we shall use $pE_2q$ since this has the simplest truth-table of the three. Using our earlier interpretation for truth-values, $pE_2q$ states that either $p$ is equivalent to $q$, or $p$ is not equivalent to $q$. This is not the case with $pE_3q$ and $pE_4q$, as can readily be
seen by theorems 6.7, 6.8, and 6.9.

The set of postulates given below are those given by Dr. E.V. Huntington\textsuperscript{20} with the exception that it has been necessary to change the operations listed in postulate D.

"An obvious set of postulates for $=$ is as follows, where $a, b, c, \ldots$ are understood to be elements of the class $K$.

Postulate A. If $a$ is in the class $K$, then $a = a$.
Postulate B. If $a = b$, then $b = a$.
Postulate C. If $a = b$, and $b = c$, then $a = c$.
Postulate D. If $x = y$, then $f(x, a, b, c, \ldots) = f(y, a, b, c, \ldots)$ where $f(x, a, b, c, \ldots)$ is any element of the class $K$ built up from the elements $x, a, b, c, \ldots$, by successive applications of the operator $|$ and $f(y, a, b, c, \ldots)$ is the element obtained from $f(x, a, b, c, \ldots)$ by writing $y$ in place of $x$ throughout."

$pE_{q}q$ satisfies these postulates because of the following theorems:

Postulate A: 4.8.
Postulate B: 4.21. This is a stronger theorem than $pE_{q}l.qE_{p}$.
Postulate C: 4.22.
Postulate D: Since we have defined operations as only operating on the truth-tables of a function, if two propositions have identical truth-tables one may replace the other. But our definition of equivalence implies that if two propositions are equivalent they will have identical truth-tables. Thus

if $x E_2 y$ holds we can replace $x$ by $y$ wherever $x$ occurs without changing the value of our function $f(x,y,q,r,\ldots)$. This satisfies Postulate D.

We shall drop the subscript on $E_2$ henceforth unless we are referring to some type of equivalence other than $E_2$.

The sum and product is $L_n$. In working out the transforms of operators it is well to remember that $=$ in $R^*$ transforms into $E$ in $L_n$. Remembering this we are led to the following theory:

7.01 $xS\bar{y} = .xV\bar{y}$  
7.02 $xP\bar{y} = .xy$  

7.1 $xP\bar{y}$ and $xS\bar{y}$ satisfy postulates $R^*_1$, $R^*_2$, $R^*_3$.

Proof: $R^*_1$ is satisfied since $xS\bar{y}$ and $xP\bar{y}$ are uniquely determined by 1.2 and 1.4. Also

\[ xS\bar{y}.E.ySx \quad (4.31), \quad xP\bar{y}.E.yP\bar{x} \quad (4.3). \]

$R^*_2$ is satisfied since

\[ (xS\bar{y})Sz.E.xS(ySz) \quad (4.23), \quad (xP\bar{y})Pz.E.xP(yPz) \quad (4.32), \]
\[ xP(ySz).E.(xP\bar{y})S(xPz) \quad (4.4). \]

$R^*_3$ is satisfied where the zero and unity of $L_n$ are respectively to and $t_{n-1}$, since

\[ xSt_{o}.E.x \quad (1.2), \quad xP_{t_{n-1}}.E.x \quad (1.4). \]

7.1 allows us to call $xS\bar{y}$ and $xP\bar{y}$ as defined the transforms of the sum and product in $L_n$.

Division in $L_n$.

7.03 $xD\bar{y} = .yI_{x\bar{y}}$  

7.3 $xD\bar{y}$ in $L_n$ satisfies postulates $P_2, P_3, P_4$.

Proof: Pl. $x\bar{D}x \quad (4.2)$.

P3. $xD\bar{y}.ySz.L.xDz \quad (3.32)$.

While this statement was proven where $I$ is $I_2$, yet
inspection of the proof will show that I may be any
one of the five types of implication that we have de-
defined.

P₂. \(xDy . yDx : I : xEy\)

xEy was defined in this fashion.

By 6.3 x divides y (\(xDy\)) when \([\frac{x}{y}] \geq \frac{y}{y}\). xDy may be regarded
as division in \(L_n\).

7.04 \(xGy = .xVy\) \(\text{Df}\)

7.05 \(xLy = .xy\) \(\text{Df}\)

7.3 xGy satisfies postulates P4, P5, P6 and P7.

Proof: P4. \(xGy . E . yGx\) (4.31)

P5. \(xG(yGz) . E . (xGy)Gz\) (4.32)

P6. \((xGy)Dx . (xGy)Dy\). This is true since

\([\frac{x}{y}] I_2 . xVy' = n-1 \text{ and } [\frac{y}{y}] I_2 . xVy' = n-1.\)

P7. \(zDx . zDy : I : zD(xGy)\). This may be written as

\(xI_2 . yI_2 . z : I : (xVy)I_2 . z\) which is evidently true.

I may be anyone of the defined implications.

7.4 xLy satisfies postulates P8, P9, P10, P11.

Proof: P8. \(xLy . E . yLx\) (4.3)

P9. \(xL(yLz) . E . (xLy)Lz\) (4.32)

P10. \(xD(xLy) . yD(xLy)\). This may be written as

\(xy . I_2 . x . xyI_2 . y\) which is obviously true.

P11. \(xDz . yDz : I : (xLy)Dz\). This becomes

\(zI_2 x . zI_2 y : I : z . I_2 . xy\) which is true.

Then by 7.3 and 7.4 we can consider that xGy and xLy are
transforms of the G.C.D. and L.C.M. in \(L_n\).
RESIDUALS I: $L_n$. We can rewrite Pl9 in $L_n$ in the form

\[ \{pD(xpq)\}, \{ydx\}, [y] \neq n-1 \]

where $x$ is allowed to have any truth-value such that

\[ \{pD(xpq)\} \]

holds. These conditions are:

- If $[a] \leq [p]$ then $x$ can assume any value and $[y] = n-1$.
- If $[a] > [p]$ then $[x] \leq [p]$. So if $[y] = [x]$, then $ydx$.

This leads us to the definition

7.36 $qRp \equiv q1qRp$

7.12 $y.E.qRp$

Proof: If $y.E.qRp$

when $[a] \leq [p]$, $[y] = n-1$

$[a] > [p]$, $[y] = [x]$ (6.4)

and $y$ satisfies the above condition for a residual.

CONGRUENCE IN $L_n$. No generalization of Hurwitz's congruence $xP' \equiv \beta' \mu'$ was found. The generalization $xpE.yP_0$

of $xp' = yp'$ satisfies Pl2, Pl3, Pl4 and Pl5 but does not satisfy Pl6 or Pl7. Thus we could call $xpE.yP_0$ algebraic congruence since they satisfy the four postulates given for algebraic congruence.

Keeping the definitions that we have made previously for the sum, product, divides, residual, G.C.D. and L.C.M., when $n > 2$, if we accept Hurwitz's definition for congruence $22$, it is impossible to satisfy Pl7. Assuming that $x \equiv y \mod p$ when $x.E.zVrp$ and $y.E.zVqp$, then if we take $x.E.zVrp$ and $zy.E.zVqp$ where $[a] \leq [x] \leq [y]$ and $[x] \neq [y]$, then

1. $x \equiv y \mod p$, but

2. $x \equiv y \mod qRp$ only when $[qRp] \leq [xVy]$.


In case \([x] = n-1\), then \([q\mathbb{R}]' = n-1\) if (2) holds. But by Pl9 \([q\mathbb{R}]' \neq n-1\) in general. This leads us to the conclusion that defining the sum, product and residual as we have, we are unable to satisfy Pl7 by Hurwitz's definition of congruence.

However, we are able to prove the following theorem:

7.07 \((x \equiv y \mod p) : = x\mathbb{V}_p \cdot E \cdot y\mathbb{V}_p\) \(Df\)

7.13 \(x\mathbb{V}_p \cdot E \cdot y\mathbb{V}_p\) satisfies postulates Pl2, ..., Pl6, Pl8.

Proof: Pl2, ..., Pl5 are evident.

Pl6: \(x\mathbb{V}_p \cdot E \cdot t_1 \mathbb{V}_p \cdot E \cdot p\mathbb{D}_x \cdot \lfloor x \rfloor \neq 0\).

If \([x] > [p]\) then \([x\mathbb{V}_p \cdot E \cdot t_1 \mathbb{V}_p]' = 0\), and \([p\mathbb{D}_x]' = 0\).

If \([x] \leq [p]\) (or \(p\mathbb{D}_x\)) then \([x\mathbb{V}_p \cdot E \cdot t_0 \mathbb{V}_p]' = n-1\), or the statement is true.

Hurwitz interprets Pl6 to say:

\(x \equiv 0 \mod p\) if and only if \(p\mathbb{D}_x\). This is too restrictive a statement in \(\mathbb{L}_n\) since \(p\) could divide \(x\) when \(p\mathbb{D}_y\) and \(y \neq 0\). This analogue occurs in \(\mathbb{A}\). We can amend Hurwitz's statement to say "if \(p\mathbb{D}_y\) in \(x \equiv y \mod p\), then \(p\mathbb{D}_x\)." Inspection of \(x\mathbb{V}_p \cdot E \cdot y\mathbb{V}_p\) verifies this statement.

Pl8: \(x\mathbb{V}_p \cdot E \cdot x\mathbb{V}_p\) is obviously true.
ANOTHER EXAMPLE OF THE ARITHMETIC OF $L_n$.

Deserting the operations defined in the earlier part of this chapter, we find that we can develop an arithmetic on the subscripts of the truth-values. This analogy with $A$ is much more striking for $n > 2$ than for $n = 2$, when we only have two elements available. The operations which we shall use are defined below:

7.010 $p + q = p^{[q]}$  
7.011 $-p = (Np)^1$  
7.012 $pq = p^{[p(q-1)]}$

They lead us to the following theorems:

7.20 $[p + q]' = [p + q]$  
Proof: $[p + q]' = [p^{[q]}]' = [p + q'].$

7.21 $p + q. E. q + p$  
Proof: $[p + q] = [q + p]$.  

Note that we are retaining $E$ from the previous section, meaning by $E$ $E_{\mathcal{D}}$.

7.22 $p + (q + r). E. (p + q) + r$  
Proof: $[p + (q + r)]' = [p + q + r] = [(p + q) + r]'$.

7.013 $p + q + r = p + (q + r)$  
7.23 $[-p]' = [n-p]$  
Proof: $[-p]' = [(Np)^1]' = [n-1-p+1] = [n-p]$.  
7.014 $p - q = p + (-q)$  
7.24 $p - p = 0$  
Proof: $[p - p]' = [p + n - p] = [n] = 0$.

7.25 $[pq]' = [pq]$ (Notice absence of prime on $[pq]$)  
Proof: $[pq]' = [p^{[pq-1]}]' = [p + pq - p] = [pq]$.  
7.26 $pq. E. qop$  
Proof: $[pq] = [qp]$.
7.27 \( \text{po(qor).E.(poq)or} \)

Proof: \( [\text{po(qor)}]' = [\text{pqr}] = [\text{(poq)or}]' \)

7.015 \( \text{poqor} = \text{(poq)or} \) \( \text{Df} \)

7.28 \( \text{po(q+r).E.poq+por} \)

Proof: \( [\text{po(q+r)}]' = [\text{p}[q+r]]' = [\text{p(q+r)}] = [\text{pq + pr}] = [\text{poq + por}]' \)

7.29 \( \text{pot} = \text{E.p} \)

Proof: \( [\text{pot}]' = [\text{l.p}] = [\text{p}] \).

7.30 \( \text{p+t0 = E.p} \)

Proof: \( [\text{p+t0}]' = [\text{p+0}] = [\text{p}] \).

7.31 \( \text{(-p)oq = E.-(poq)} \)

Proof: \( [\text{(-p)oq}]' = [\text{n-p}] = [\text{nq-pq}] = [\text{n-[pq]}] \)

7.32 \( \text{(-p)o(-q) = E.poq} \)

Proof: \( [\text{(-p)o(-q)}]' = [\text{n-p(n-q)}] = [\text{n^2-np-nq+pq}] = [\text{pq}] \).

7.33 \( \text{p+x.E.q:E.x.E.q-p} \)

Proof: Where \( 0 \leq [x] < n \) then there is evidently only one value of \( x \) such that \( [\text{p+x}] = [\text{q}] \). But \( [x] = [\text{q+n-p}] \) satisfies this relationship. Therefore the theorem holds.

7.34 \( \text{Ln is a ring under operations p+q and poq.} \)

Proof: (The following numerals refer to the postulates on p. 37 of B.L. van der Waerden: Moderne Algebra, Vol. I.)

I. Postulates on addition.

(a) \( (7.22) \).

(b) \( (7.21) \).

(c) \( (7.33) \).

II. Postulates on multiplication.

(a) \( (7.27) \)

III. Distributive laws.

(a) \( (7.28) \)

(b) \( (7.28, 7.26) \).
p+q and poq are generalizations of the two operations that Stone\textsuperscript{23} used as a basis for his ring in \( L_2 \). It is well to note that poq is a generalization of pq in \( L_2 \).

Many other theorems could be developed about the above operations, but those given are sufficient to indicate a few of their properties and also the method to be used in working out new theorems.

CHAPTER FOUR

GENERATION OF ANY N-VALUED LOGIC BY ONE BINARY
OPERATION

BY DONALD L. WEBB

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY

Communicated April 1, 1935

A two-valued logic was shown in 1913 by Sheffer\(^2\) to be obtainable by
the iteration of a single binary operation. It was proved in 1925 by Zylinski\(^3\) that Sheffer’s function and its “dual” also introduced by Sheffer,
are the only binary operations such that the iteration of either one will
generate the two-valued logic of functions of two propositions. Zylinski’s
proof was by means of a truth table of 4 columns and 16 rows, correspond-
ing to the two possible values \((a \text{ or } b)\) which an arbitrary function \(\Phi(x, y)\)
can assume in a two-valued system. While theoretically applicable to an
\(n\)-valued system, the method of direct inspection of the truth table is
impracticable. Following another method we prove that: Any \(n\)-valued logic,
where \(n \geq 2\), can be generated by the iteration of one binary operation.

Designate the \(n\) truth values which an “elementary proposition” may
take in an \(n\)-valued logic by the marks \(a_0, a_1, \ldots, a_{n-1}\). For convenience,
drop the \(a\) and retain only the subscript, so that our marks are now 0, 1, \ldots,
\(n-1\). It is to be observed that these numbers denote merely \(n\) distinct
marks without any arithmetical significance. Let \(p\) and \(q\) be any ele-
mentary propositions. Construct a truth table for two elementary proposi-
tions, \(p, q\), of two columns and \(n^2\) rows with the \(n\) marks, 0, 1, \ldots, \(n-1\)
by assigning in the \(i\)th row of the table to \(p\) the value \([i-1-(i-1)']/n,\)
and to \(q\) the value \((i-1)'\), where \(j' = j \text{ mod } n, j' \geq 0\) and \(i = 1, 2, \ldots, n^2\).
Denote the statement, \(p\) has the value \(i\), by \(p = i\); let \(p \Delta q\) denote
any function of \(p, q\) whose values are in the range 0, 1, \ldots, \(n-1\), when
\(p = i, q = j\) and \(i, j\) are in the same range; let \(i \Delta j = k\) denote that if \(p = i\)
and \(q = j\) then \(p \Delta q = k\), where \(k\) is in the range.

Define the stroke, “\(\Delta\)” function, \(p|q\), by

\[i|j = 0 \text{ if } i \neq j; \quad i|i = (i + 1)' \quad (i, j = 0, 1, \ldots, n-1).\]

From this binary operation we shall generate all functions of two variables
in the \(n\)-valued logic. The proof will consist in exhibiting the particular
general column of the \(p, q\) truth table in which the \(n^2\) marks \(i, (s = 1, 2, \ldots, n^2)\)
are arbitrary elements of the set 0, 1, \ldots, \(n-1\) as a function constructed
on \(p, q\) by means the stroke |.

As a notational definition we write

\[p^0 = p, \quad p^i = p^{i-1}|p^{i-1} \quad (i = 1, \ldots, n-1),\]
in which the exponents are superscripts. Define \(R_{i,j}(p, q)\) to be a function
of \(p\) and \(q\) such that in row \(j\) of the truth table, \(R_{i,j}(p, q) = i\) and in every
other row \( R_{i,j}(p, q) = 0 \) \((i = 0, 1, \ldots, n-1; j = 1, 2, \ldots, n^2)\). For brevity we shall write \( R_{i,j} \) for \( R_{i,j}(p, q) \). Similarly, any function \( F(p, q) \) of \( p, q \) will be written \( F \).

From the definition of \( p|q \) it is evident that \( p^1 \) can be obtained by operating with the cyclic substitution \( S = (0,1,\ldots, n-1) \) on the values of \( p^0 \). Similarly, \( p^i \) can be obtained by operating on the values of \( p^0 \) with the \( i \)th power, \( S^i \), of \( S \). Hence \( R_{i,j} \) and \( (p^0|q)^{n-1}|(p^0|p^1) \) are equivalent for the values of \( a, b \) indicated next, which we shall write

\[
R_{i,j} = (p^0|q)^{n-1}|(p^0|p^1), \quad a = \left\{ n - \left\lfloor \frac{i-1}{(i-1)!} \right\rfloor \right\}^n, \quad b = \left\lfloor \frac{n}{(i-1)!} \right\rfloor.
\]

By means of \( R_{i,j} \) we obtain \( R_{i,j} \) in terms of \( p, q \) and \( p|q \). For,

\[
R_{i,j} = \frac{p^0}{q} | p^1, \quad R_{i,j} = (R_{i,j})^{i-1} \cdot (R_{i,j})^{i-1} (i = 2, 3, \ldots, n-1).
\]

Define \( p \alpha q = p^0|q^{n-1}, p \alpha_2 q = R_{0,1}|R_{0,2}, p \alpha_3 q = N_{i-2}a_1M_{i-2} \) \((i = 3, 4, \ldots, n)\). \( M_0 = R_{0,1}, M_h = M_{h-1}a_{h+1}R_{h, h+1}a_{h+1}R_{h, (h+1)n+1}, N_h = M_{h+1}a_{h+1}R_{1,1}\).

Then we see that \( p \alpha q \) has the property \( 0 \alpha q = i; \ i \alpha q = i (i = 0, 1, 2) \). Hence, by an easy mathematical induction, we prove that if

\[
p \alpha q = N_{n-1}a_1M_{n-2}, \text{ then } 0 \alpha q = i; \ i \alpha q = i (i = 0, 1, \ldots, n-1).
\]

Let \( T_i(p, q) \) be any function of \( p, q \) and define

\[
T_1a_2a_3 = (T_1a_2a_3)a_3T_3
\]

\[
T_1a_2a_3T_3a_3T_4 = ((T_1a_2T_3)a_3T_4)
\]

\[
T_1a_2T_3a_3T_4 = \ldots (T_1a_2T_3)a_3T_n.
\]

From the properties of \( p \alpha q \) it follows that

\[
0 \alpha_0a_n = \ldots a_n0 \alpha_0 = i (i = 0, 1, \ldots, n-1).
\]

Hence we can construct any function \( F(p, q) \) of \( p, q \) having the values \( t_i \) in row \( i \) of the truth table, where \( t_i \) is any one of the marks \( 0, 1, \ldots, n-1 \), as follows:

\[
F(p, q) = R_{0,1}a_nR_{0,2}a_n \ldots a_nR_{n, n}.
\]

This follows immediately from \( p \alpha q \) by inspection of the rows in the following table,

\[
t_0a_0a_n = \ldots a_n0 \alpha_0 = t_0
\]

\[
t_0a_0a_n = \ldots a_n0 \alpha_0 = t_0
\]

\[
0 \alpha_0a_n = \ldots a_n0 \alpha_0 = t_i
\]

\[
0 \alpha_0a_n = \ldots a_n0 \alpha_0 = t_i
\]

\[
0 \alpha_0a_n = \ldots a_n0 \alpha_0 = t_{n-1}
\]

Now \( F(p, q) \) is any function defined by the truth table of the \( n \)-valued

\[
a = \left\{ n - \left\lfloor \frac{n-1}{(j-1)!} \right\rfloor \right\}, \quad b = \left\{ n - \left\lfloor \frac{n-1}{(j-1)!} \right\rfloor \right\}.
\]
logic. Hence, if \( n \geq 2 \), any function of two propositions \( p, q \) can be constructed from \( p, q \) and \( p \lor q \) in the manner just indicated. To extend this result to a function of \( m \) propositions, \( p_1, p_2, \ldots, p_m \), we make the additional definitions and take the steps indicated below.

Define \( i_{r,s} = i \mod n^s \) where \( i \) is any positive integer and \( i_{r,s} \geq 0 \). Now we shall construct a truth table on \( p_1, \ldots, p_k \) by assigning to \( p_s \) (\( s = 1, 2, \ldots, k \)) the value

\[
p_s = \left\lfloor (j - 1) - (j - 1)_{k-s+1} \right\rfloor / n^{k-1},
\]

in the \( j \)th row (\( j = 1, 2, \ldots, n^k \)). Let \( R_{i,j}^k \equiv R_{i,j}(p_1, \ldots, p_k) \) be defined on a truth table of \( p_1, \ldots, p_k \) as \( R_{i,j}^k = i \) in row \( j \), and \( R_{i,j}^k = 0 \) elsewhere. Let \( S_j^k \equiv (R_{i,j}^k)\) (\( j = 1, 2, \ldots, n^k \)).

It is evident that a function \( F(p_1, \ldots, p_k) \) will exist in a truth table on \( p_1, \ldots, p_k \), and will be the function obtained from the \( p_1, \ldots, p_k \) truth table by replacing each row by \( n^1 \) consecutive identical rows. This, then, increases the number of rows from \( n^k \) to \( n^{k+1} \).

Hence, we can prove

\[
R_{i,j}^k \equiv R_{i+h}^l(p_k), \quad \text{where} \quad h = (j-1)_{k+1} + 1, \quad l = (j-1)_{k+1} / n + 1, \quad j = 1, 2, \ldots, n^k; \quad k = 3, 4, \ldots, m.
\]

Having determined \( R_{i,j}^k \), we proceed to construct any \( F(p_1, \ldots, p_m) \) in terms of \( p_1, \ldots, p_m \) and the stroke, \( \lor \), by using the same procedure as for \( m = 2 \). For if \( i_l \) is the value of \( F(p_1, \ldots, p_m) \) in row \( l \), then it follows that

\[
F(p_1, \ldots, p_m) \equiv R_{i_{l,1}}^m \alpha_1 R_{i_{l,2}}^m \alpha_2 \ldots \alpha_n R_{i_{l,n}}^m. \quad (*)
\]

Thus any function on any number of propositions \( p_1, \ldots, p_m \) can be constructed by means of a single binary operation, the stroke, \( \lor \).

---

\(^1\) The author is under obligations to Dr. E. T. Bell for his suggestions, aiding both in the solution of this problem and its presentation.


\(*\) \( F(p_1, \ldots, p_m) \equiv R_{i_{1,1}}^m \alpha_1 R_{i_{2,2}}^m \alpha_2 \ldots \alpha_n R_{i_{n,n}}^m \).
DEFINITION OF POST'S GENERALIZED NEGATIVE AND 
MAXIMUM IN TERMS OF ONE BINARY OPERATION.

By Donald L. Webb.

In 1921 Post\textsuperscript{1} demonstrated that it was possible to construct a function for any order table in a system of $m$ truth-values by the use of two primitive functions, $\neg_m p$ and $p \lor_m q$ which are generalizations of the functions $\neg p$ and $p \lor$ in the two-valued case. Recently we\textsuperscript{2} have been able to show that a function on $m$ truth-values for any order table can be constructed in terms of one binary operation, using in this demonstration a negative that corresponds to Post's $\neg_m p$, a binary operator $p \vee_m q$ which, for the value combinations used in the interpolation formula, corresponds to Post's $p \lor_m q$, and a binary operator $p \mid q$ which has no equivalent among the operators employed by Post. In the latter paper all operators were defined in terms of $p \mid q$. In this paper by redefining the truth-table of $p \mid q$ we are enabled to define Post's $\neg_m p$ and $p \lor_m q$ in terms of the "\mid" function, thus greatly simplifying the proof that any $m$-valued logic can be generated by one binary operation. We find too that $p \mid q$ as so defined reduces in the two-valued case to one of Sheffer's \(^\dagger\) functions,\textsuperscript{3} as it evidently must.

The notation used in this paper is patterned after that of Post so as to avoid confusion.

Let $t_0, t_1, \cdots, t_m$, where $m$ is any positive integer, signify the $m$ truth values that an elementary proposition can assume in a $m$-valued logic. Denote by $p, q$ elementary propositions. Let $p = t_i$ signify that the proposition $p$ has the truth-value $t_i$. Make the two additional arithmetical definitions:

$$\min (i, j) = \begin{cases} i & \text{if } i \leq j \\ j & \text{if } i \geq j \\ i \equiv i \mod n, (i = 0, 1, 2, \cdots) & 0 \leq i < n. \end{cases}$$

Hence, $p \mid q$ is defined: if $p = t_i$, $q = t_j$ ($i, j = 0, 1, \cdots, m - 1$), then $p \mid q = t_k$ where $k = \min (i, j) + 1 \mod n$.

\textsuperscript{1} E. L. Post, American Journal of Mathematics, vol. 43 (1921), pp. 163-185.
THEOREM 1. \( \sim_m p \equiv p \mid p \).

If \( p = t_i \), then \( p \mid p = t_k \) where \( k = (i+1)_m \). Thus \( p \mid p \) cyclically permutes the truth-values \( t_i \), giving \( p \mid p \) and \( \sim_m p \) the same truth-table. Therefore the two are equivalent.

Using Post's definition, \( \sim^2_m p \equiv \sim_m \sim_m p \), etc., we may write

THEOREM 2. \( p \lor_m q \equiv \sim_m^{-1}(p \mid q) \).

By repeating the above process we find that if \( p = t_i \), \( \sim_m^k p = t_k \), where \( k = (i + h)_m \) (\( h = 2, 3, \ldots, m - 1 \)). Hence, if \( p = t_i, q = t_j \), then \( \sim_m^{-1}(p \mid q) = t_k \) where \( k = \{\lfloor \min(i, j) + 1 \rfloor + m - 1\}_m \), or \( k = \min(i, j) \). But \( p \lor_m q \) as given by Post has the same truth-table, making the two equivalent.

Since Post has shown that we can generate a function of any order in a \( m \)-valued truth system by means of \( \sim_m p \) and \( p \lor_m q \), then, by using the above theorems, we can generate a function of any order in a \( m \)-valued truth system in terms of \( "\mid " \).

CALIFORNIA INSTITUTE OF TECHNOLOGY.

---

\(^1\) This is called a maximum since the higher truth-value has the smaller subscript.