

# New Examples and Monotonicity Formula for Mean Curvature Flow

Thesis by  
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## ABSTRACT

The first main result of this thesis is the proof of the superconvexity of the heat kernel on hyperbolic space. We prove a conjecture of Bernstein that the heat kernel on hyperbolic space of any dimension is superconvex in a suitable coordinate and, hence, there is an analog of Huisken's monotonicity formula for mean curvature flow in hyperbolic space of all dimensions.

In the second part of the thesis, we construct an ancient solution to planar curve shortening. The solution is at all times compact and embedded. For  $t \ll 0$  it is approximated by the rotating Yin-Yang soliton, truncated at a finite angle  $\alpha(t) = -t$ , and closed off by a small copy of the Grim Reaper translating soliton.

## PUBLISHED CONTENT AND CONTRIBUTIONS

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Yongzhe Zhang organized and wrote the full proof of the article under the supervision of professor Lu Wang.

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## Chapter 1

### INTRODUCTION

#### 1.1 Motivation and history

Mean curvature flow is a generalization of curve shortening flow and they can both be defined for a family of smoothly embedded hypersurfaces: [Mean curvature flow] For a family of smoothly embedded hypersurface  $(M_t)_{t \in I}$  in  $\mathbb{R}^n$  is moving by mean curvature if

$$(x_t)^\perp = \vec{H}(x)$$

for all  $x \in M_t$  and  $t \in I$ ,  $I \subset \mathbb{R}$  is an open interval. Here,  $(x_t)^\perp$  is the normal part of the velocity and  $\vec{H}(x)$  is the mean curvature vector of the hypersurface  $M_t$  (curvature vector of a curve if the  $M_t$  is a family of smooth curve). As summarized in (T. Colding, W. Minicozzi, Pedersen, et al., 2015), mean curvature flow is a nonlinear partial differential equation for the evolving hypersurface that is formally similar to the ordinary heat equation, with some important differences. Mean curvature flow behaves like the heat equation for a short time with the solution becoming smoother and small-scale variations averaging out. However, after more time, the nonlinearities dominate and the solution becomes singular. To understand the flow, one must understand the singularities it goes through.

Therefore, there is a type of solutions that plays the key role of understanding the flow: Solitons. Those solutions are characterized by being self-similar under the flow and they are very useful for understanding the singularity of the flow.

From the point of view of geometric variational problem, mean curvature flow is the negative gradient flow for area. There is another important tool called Huiskin's monotonicity formula.

If  $\{\Sigma_t\}_{t \in [0, T]}$  is a mean curvature flow of  $n$ -dimensional complete submanifolds  $\Sigma_t \subset \mathbb{R}^{n+1}$  and  $M_0$  has polynomial volume growth at infinity, then, for  $t_0 \in (0, T)$ ,  $p_0 \in \mathbb{R}^{n+1}$  and  $t \in (0, t_0)$ ,

$$\frac{d}{dt} \int \Phi_n(x_0, t_0) dVol_{\Sigma_t}(p) \leq 0$$

and the inequality is strict unless  $\Sigma_t$  is a shrinker. Here,  $\Phi_n(x_0, t_0)$  is the heat kernel



defined as

$$\Phi_n(x_0, t_0)(x, t) = \frac{1}{2\pi(t_0 - t)^{n/2}} e^{-\frac{|x-x_0|^2}{t_0-t}}.$$

## 1.2 Structure of the thesis

In (Bernstein, 2007), Bernstein proved an analogue of Huisken's monotonicity formula on Hyperbolic space (2.2) for small  $n$  and conjectured it for all dimensions.

In the first part of this paper, we use observations from (Davies and Mandouvalos, 1988) and (Yu and Zhao, 2020) to prove a convexity estimate for heat kernels on hyperbolic space and to show its application to mean curvature flow as explained in (Zhang, 2021).

In (Daskalopoulos, Hamilton, and Sesum, 2010) Daskalopoulos, Hamilton, and Sesum showed that any compact, convex, and embedded ancient solution to Curve Shortening in the plane is either a shrinking circle or the ancient paperclip solution. Qian You et.al.(You, 2014; Angenent and You, 2021) showed that there exist many other ancient solutions that are either embedded and not compact, or otherwise compact, convex, but not embedded. Therefore, it would be interesting if we can construct an ancient solution to Plane Curve Shortening that is embedded and compact at all times, but not convex.

In the second part of this paper, we are going to use gluing method to glue two solitons of Plane Curve Shortening and use perturbation method to construct an ancient solution with the desired properties.

*Chapter 2*

SUPERCONVEXITY OF THE HEAT KERNEL ON  
HYPERBOLIC SPACE WITH APPLICATIONS TO MEAN  
CURVATURE FLOW

### 2.1 Introduction

Throughout this chapter, let  $\mathbb{H}^n$  be the hyperbolic space of dimension  $n$  and let  $H_n(t, p; t_0, p_0)$  be the heat kernel on  $\mathbb{H}^n$  with singularity at  $p = p_0$  at time  $t = t_0$ . Thus,  $H_n$  is the unique positive solution to

$$\begin{cases} \left( \frac{\partial}{\partial t} - \Delta_{\mathbb{H}^n} \right) H_n = 0 & \text{for } t > t_0, \\ \lim_{t \downarrow t_0} H_n = \delta_{p_0}. \end{cases}$$

By the symmetries of  $\mathbb{H}^n$ , there is a positive function  $K_n(t, \rho)$  on  $(0, \infty) \times (0, \infty)$  such that

$$H_n(t, p; t_0, p_0) = K_n(t - t_0, \text{dist}_{\mathbb{H}^n}(p, p_0)) > 0,$$

where  $\rho = \text{dist}_{\mathbb{H}^n}(p, p_0)$  is the hyperbolic distance between  $p$  and  $p_0$ . As remarked in (Bernstein, 2007), although  $K_n$  can be explicitly computed, the formulas become unmanageable for large  $n$ ; see (Davies and Mandouvalos, 1988) for more details.

In this short note we use observations from (Davies and Mandouvalos, 1988) and (Yu and Zhao, 2020) to prove the following convexity estimate for  $K_n$ .

If  $\sigma = \cosh \rho$ , then  $K_n$  is superconvex in  $\sigma$ , i.e., for any  $t > 0$  and  $\rho > 0$ ,

$$\frac{\partial^2}{\partial \sigma^2} \log K_n = \left( \frac{1}{\sinh \rho} \frac{\partial}{\partial \rho} \right)^2 \log K_n > 0. \quad (2.1)$$

Observe that by the chain rule (2.1) is equivalent to

$$\partial_\rho^2 \log K_n(t, \rho) - \coth(\rho) \partial_\rho \log K_n(t, \rho) > 0. \quad (2.2)$$

In (Bernstein, 2007), Bernstein proved (2.2) for small  $n$  and conjectured it for all  $n$ . Hence we confirm this conjecture in Theorem 2.1.

In (Ishige, Salani, and Takatsu, 2020), Ishige, Salani and Takatsu prove that the heat kernel  $H_n(t, \cdot; 0, p_0) = K_n(t, \text{dist}_{\mathbb{H}^n}(\cdot, p_0))$  is a log-concave function on  $\mathbb{H}^n$  for  $t > 0$  and  $p_0 \in \mathbb{H}^n$ . Equivalently, for  $\rho > 0$  and  $t > 0$ ,

$$\partial_\rho^2 \log K_n < 0 \text{ and } \coth(\rho) \partial_\rho \log K_n < 0.$$

However, Theorem 2.1 says that, performing a change of variable by  $\sigma = \cosh \rho$ ,  $K_n$  is a log-convex function of  $\sigma \in (1, \infty)$  for  $t > 0$ . As a consequence of 2.2 and (Ishige, Salani, and Takatsu, 2020), one gets

$$0 > \partial_\rho^2 \log K_n > \coth(\rho) \partial_\rho \log K_n.$$

The proofs in (Ishige, Salani, and Takatsu, 2020) employ standard ODE theory and maximum principle while our proof of Theorem 2.1 uses an effective expression of the heat kernel in (Davies and Mandouvalos, 1988) (see Lemma 2.2) and some basic fact for function  $\frac{\rho}{\sinh \rho}$  in (Yu and Zhao, 2020) (see Lemma 2.2).

We also give an application of Theorem 2.1 to the mean curvature flow in hyperbolic space. We say an  $n$ -dimensional submanifold  $\Sigma \subset \mathbb{H}^{n+k}$  has *exponential volume growth*, provided that there is a constant  $M > 0$  and a point  $p_0 \in \mathbb{H}^{n+k}$  so that for any  $R > 0$

$$\text{Vol}_{\mathbb{H}^{n+k}}(\Sigma \cap B_R^{\mathbb{H}^{n+k}}(p_0)) \leq M e^{MR},$$

where  $B_R^{\mathbb{H}^{n+k}}(p_0)$  is the (open) geodesic ball in  $\mathbb{H}^{n+k}$  centered at  $p_0$  with radius  $R$ . As noted in (Bernstein, 2007, Remark 1.2), we can use Theorem 2.1 to extend (Bernstein, 2007, Theorem 1.1), an analog of Huisken's monotonicity formula (Huisken, 1990) for mean curvature flow in hyperbolic space in low dimensions, to higher dimensions.

If  $\{\Sigma_t\}_{t \in [0, T]}$  is a mean curvature flow of  $n$ -dimensional complete submanifolds  $\Sigma_t \subset \mathbb{H}^{n+k}$  that have exponential volume growth, then, for  $t_0 \in (0, T]$ ,  $p_0 \in \mathbb{H}^{n+k}$  and  $t \in (0, t_0)$ ,

$$\frac{d}{dt} \int_{\Sigma_t} K_n(t_0 - t, \text{dist}_{\mathbb{H}^{n+k}}(p, p_0)) d\text{Vol}_{\Sigma_t}(p) \leq 0$$

and the inequality is strict unless  $\Sigma_t$  is a minimal cone over  $p_0$ .

In (Bernstein, 2007), Bernstein introduced a notion of hyperbolic entropy for submanifolds in hyperbolic space, which is analogous to the one introduced by Colding-Minicozzi for hypersurfaces in Euclidean space (T. H. Colding and W. P. Minicozzi, 2012). Using Corollary 2.1 and some observations of (Bernstein, 2007), one may adapt the arguments of (Jacob Bernstein and Wang, 2016; Jacob Bernstein and Wang, 2017; Jacob Bernstein and Wang, 2018; Jacob Bernstein and Wang, 2020; Ketover and Zhou, 2018; Zhu, 2020) to prove that closed hypersurfaces in hyperbolic space with small hyperbolic entropy are simple in various senses.

Another consequence of Corollary 2.1 is that the second part of (Bernstein, 2007, Theorem 1.5) (i.e., "If, in addition,  $\Sigma$  is minimal and  $n < N \dots$ ") holds true for all

dimensions  $n$ . Thus there is a natural relationship between the hyperbolic entropy of an asymptotic regular submanifold of hyperbolic space and the conformal volume of its asymptotic boundary, which is analogous to the relationship between the entropy of an asymptotically conical self-expander and the entropy of its asymptotic cone (Jacob Bernstein and Wang, 2021, Lemma 3.5).

## 2.2 Proof of Theorem 2.1

Set  $\sigma = \cosh \rho$ . Let

$$f_1(\rho) = \frac{\rho}{\sinh \rho} \quad \text{for } \rho > 0$$

and

$$f_{l+1}(\rho) = -\frac{df_l}{d\sigma} = (-1)^l \frac{d^l f_1}{d\sigma^l} \quad \text{for } \rho > 0.$$

It is shown in (Davies and Mandouvalos, 1988) that

$$K_n(t, \rho) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(n-1)^2}{4}t} e^{-\frac{\rho^2}{4t}} \alpha_n(t, \rho)$$

and  $\alpha_n(t, \rho)$  satisfies the following recurrence relation:

$$\alpha_n = f_1 \alpha_{n-2} - 2t \frac{\partial \alpha_{n-2}}{\partial \sigma}.$$

As  $\mathbb{H}^1$  is the one-dimensional Euclidean space, we have

$$K_1 = (4\pi)^{-\frac{1}{2}} e^{-\frac{\rho^2}{4t}},$$

so  $\alpha_1 = 1$ .

Using the definition of  $f_l$  and the recurrence relation for  $\alpha_n$ , Davies and Mandouvalos prove the following properties for  $f_l$  and  $\alpha_n$  for odd  $n$ .

[(Davies and Mandouvalos, 1988)] The following is true:

1. For each  $l \geq 1$ ,  $f_l$  is positive and decreasing.
2. For all  $m \geq 1$ ,

$$\alpha_{2m+1} = \sum_{i=0}^{m-1} t^i P_{m,i}(f_1, f_2, \dots, f_m) > 0,$$

where  $P_{m,i}$  are all polynomials with nonnegative coefficients.

We will also need the following fact proved by C. Yu and F. Zhao.

[(Yu and Zhao, 2020, Proposition 3.1)] For all  $l \geq 1$ ,

$$\frac{d}{d\sigma} \left( -\frac{f_{l+1}}{f_l} \right) \geq 0.$$

We are now ready to prove Theorem 2.1.

[Proof of Theorem 2.1] By (Bernstein, 2007, Proposition 2.1), it is sufficient to prove the claim for odd  $n \geq 3$ . To that end, suppose  $n = 2m + 1$  for some  $m \geq 1$ .

We first compute  $\partial_\rho \log K_n(t, \rho)$ :

$$\partial_\rho \log K_n = \frac{\partial_\rho K_n}{K_n}$$

and

$$\partial_\rho K_n = (4\pi t)^{-\frac{n}{2}} e^{-\frac{(n-1)^2}{4}t} \left( -\frac{\rho}{2t} \alpha_n + \partial_\rho \alpha_n \right) e^{-\frac{\rho^2}{4t}}.$$

Thus

$$\partial_\rho \log K_n = \frac{-\frac{\rho}{2t} \alpha_n + \partial_\rho \alpha_n}{\alpha_n} = -\frac{\rho}{2t} + \partial_\rho \log \alpha_n.$$

Then we differentiate the above identity with respect to  $\rho$ :

$$\partial_\rho^2 \log K_n = -\frac{1}{2t} + \partial_\rho^2 \log \alpha_n.$$

Thus, using the chain rule,

$$\begin{aligned} \sinh^2(\rho) \partial_\sigma^2 \log K_n &= \partial_\rho^2 \log K_n - \coth(\rho) \partial_\rho \log K_n \\ &= \left( -\frac{1}{2t} + \partial_\rho^2 \log \alpha_n \right) - \coth(\rho) \left( -\frac{\rho}{2t} + \partial_\rho \log \alpha_n \right) \\ &= \frac{\rho \coth(\rho) - 1}{2t} + \left( \partial_\rho^2 \log \alpha_n - \coth(\rho) \partial_\rho \log \alpha_n \right) \\ &= \frac{\rho \coth(\rho) - 1}{2t} + \sinh^2(\rho) \partial_\sigma^2 \log \alpha_n. \end{aligned}$$

Using  $x > \tanh(x)$  for  $x > 0$ , it is easy to see that the first term is always positive and independent of  $n$ . Therefore, it suffices to show that

$$\partial_\sigma^2 \log \alpha_n \geq 0$$

for  $n = 2m + 1$ . Since

$$\partial_\sigma^2 \log \alpha_n = \frac{(\partial_\sigma^2 \alpha_n) \alpha_n - (\partial_\sigma \alpha_n)^2}{\alpha_n^2}$$

it would be sufficient if we proved the following claim:

$$A_n \stackrel{def}{=} (\partial_\sigma^2 \alpha_n) \alpha_n - (\partial_\sigma \alpha_n)^2 \geq 0 \quad \text{for } n = 2m + 1.$$

To see this, we need to use Lemma 2.2 and Lemma 2.2 to compute the  $\sigma$ -derivatives of  $\alpha_n$  where  $n = 2m + 1$  for some  $m \geq 1$ . Since

$$\alpha_{2m+1} = \sum_{i=0}^{m-1} t^i P_{m,i}(f_1, \dots, f_m)$$

it follows that

$$\begin{aligned} A_{2m+1} &= \left[ \sum_{i=0}^{m-1} t^i P_{m,i} \right] \left[ \sum_{i=0}^{m-1} t^i \frac{d^2 P_{m,i}}{d\sigma^2} \right] - \left[ \sum_{i=0}^{m-1} t^i \frac{dP_{m,i}}{d\sigma} \right]^2 \\ &= \sum_{i=0}^{2m-2} t^i \sum_{\substack{\alpha+\beta=i \\ 0 \leq \alpha, \beta \leq m-1}} \left[ P_{m,\alpha} \frac{d^2 P_{m,\beta}}{d\sigma^2} - \frac{dP_{m,\alpha}}{d\sigma} \frac{dP_{m,\beta}}{d\sigma} \right]. \end{aligned}$$

To show  $A_{2m+1} \geq 0$ , it is sufficient to show that for each  $0 \leq i \leq 2m - 2$

$$B_{m,i} \stackrel{def}{=} \sum_{\substack{\alpha+\beta=i \\ 0 \leq \alpha, \beta \leq m-1}} \frac{d^2 P_{m,\beta}}{d\sigma^2} - \frac{dP_{m,\alpha}}{d\sigma} \frac{dP_{m,\beta}}{d\sigma} \geq 0.$$

By Lemma 2.2, we know that  $P_{m,r}(y_1, \dots, y_m)$  is a polynomial in  $y_1, \dots, y_m$  with nonnegative coefficients, so we can assume that

$$P_{m,r}(y_1, \dots, y_m) = \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} y_1^{j_1} \dots y_m^{j_m},$$

where all  $a_{m,r,j_1 \dots j_m} \geq 0$  with only finitely many nonzero. Then, applying chain rule, one gets

$$\begin{aligned} \frac{dP_{m,r}}{d\sigma} &= \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} \sum_{s=1}^m f_1^{j_1} \dots \left( j_s f_s^{j_s-1} \frac{df_s}{d\sigma} \right) \dots f_m^{j_m} \\ &= \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right) \end{aligned}$$

and from  $P_{m,r} \geq 0$  and Lemma 2.2, one gets

$$\begin{aligned} \frac{d^2 P_{m,r}}{d\sigma^2} &= \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right)^2 \\ &\quad + \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \frac{d}{d\sigma} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right) \\ &\geq \sum_{j_1, \dots, j_m \geq 0} a_{m,r,j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right)^2. \end{aligned}$$

Now, we realize that we can symmetrize the expression of  $B_{m,i}$ :

$$2B_{m,i} = \sum_{\substack{\alpha+\beta=i \\ 0 \leq \alpha, \beta \leq m-1}} P_{m,\alpha} \frac{d^2 P_{m,\beta}}{d\sigma^2} + P_{m,\beta} \frac{d^2 P_{m,\alpha}}{d\sigma^2} - 2 \frac{dP_{m,\alpha}}{d\sigma} \frac{dP_{m,\beta}}{d\sigma}$$

and from the previous computation we know that

$$\begin{aligned} C_{m,i,\alpha,\beta} &\stackrel{\text{def}}{=} P_{m,\alpha} \frac{d^2 P_{m,\beta}}{d\sigma^2} + P_{m,\beta} \frac{d^2 P_{m,\alpha}}{d\sigma^2} - 2 \frac{dP_{m,\alpha}}{d\sigma} \frac{dP_{m,\beta}}{d\sigma} \\ &\geq \left( \sum_{j_1, \dots, j_m \geq 0} a_{m,\alpha, j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \right) \left( \sum_{k_1, \dots, k_m \geq 0} a_{m,\beta, k_1 \dots k_m} f_1^{k_1} \dots f_m^{k_m} \left( \sum_{s=1}^m -k_s \frac{f_{s+1}}{f_s} \right)^2 \right) \\ &\quad + \left( \sum_{j_1, \dots, j_m \geq 0} a_{m,\alpha, j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right)^2 \right) \left( \sum_{k_1, \dots, k_m \geq 0} a_{m,\beta, k_1 \dots k_m} f_1^{k_1} \dots f_m^{k_m} \right) \\ &\quad - 2 \left( \sum_{j_1, \dots, j_m \geq 0} a_{m,\alpha, j_1 \dots j_m} f_1^{j_1} \dots f_m^{j_m} \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right) \right) \\ &\quad \quad \times \left( \sum_{k_1, \dots, k_m \geq 0} a_{m,\beta, k_1 \dots k_m} f_1^{k_1} \dots f_m^{k_m} \left( \sum_{s=1}^m -k_s \frac{f_{s+1}}{f_s} \right) \right) \\ &\geq \sum_{j_1, \dots, j_m \geq 0} \sum_{k_1, \dots, k_m \geq 0} a_{m,\alpha, j_1 \dots j_m} a_{m,\beta, k_1 \dots k_m} f_1^{j_1+k_1} \dots f_m^{j_m+k_m} \\ &\quad \times \left[ \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right)^2 + \left( \sum_{s=1}^m -k_s \frac{f_{s+1}}{f_s} \right)^2 - 2 \left( \sum_{s=1}^m -j_s \frac{f_{s+1}}{f_s} \right) \left( \sum_{s=1}^m -k_s \frac{f_{s+1}}{f_s} \right) \right]. \end{aligned}$$

Therefore, by completing squares and the facts that  $f_s > 0$  and  $a_{m,\alpha, j_1 \dots j_m} \geq 0$  and  $a_{m,\beta, k_1 \dots k_m} \geq 0$ , we conclude  $C_{m,i,\alpha,\beta} \geq 0$  and prove the claim.

## NONCONVEX ANCIENT SOLUTIONS TO CURVE SHORTENING FLOW

### 3.1 Introduction

Our construction begins with the Yin–Yang soliton, i.e. the rotating soliton that is invariant with respect to reflection in the origin—see Figure 3.1 (left). This spiral shaped curve divides the plane into two congruent parts, and under Curve Shortening evolves by rotating with unit speed in the counterclockwise direction. Each of the two branches of the Yin–Yang spiral is a graph in polar coordinates, given by

$$r = \left(\theta + \frac{\pi}{2}\right), \quad r = \left(\theta - \frac{\pi}{2}\right)$$

respectively. The Yin–Yang spiral is asymptotic to a Fermat spiral which in polar coordinates is given by  $r = a\sqrt{\theta}$ . The Yin–Yang curve satisfies

$$r(\theta) = \sqrt{2\theta} + (\theta^{-3/2}), \quad r'(\theta) = \frac{1}{\sqrt{2\theta}} + (\theta^{-5/2}) \quad (\theta \rightarrow \infty). \quad (3.1)$$

We review the properties and derive expansions for  $r$  in section A.1.

The Yin–Yang soliton is itself an example of an embedded ancient (and in fact eternal) solution. It is however not compact. Our goal in this paper is to construct a compact embedded ancient solution which, for  $t \rightarrow -\infty$  converges to the rotating Yin–Yang solution. In section 1 of this paper we construct an approximate solution to CSF, by truncating the Yin–Yang solution at polar angle  $\alpha(t)$ , and connecting the two ends with a small cap whose shape is approximately that of a rotated and rescaled copy of the Grim Reaper soliton. See Figure 3.1 (right).

To determine how to choose the angle  $\alpha(t)$  we estimate the speed  $V$  of the cap at time  $t$ . Since the cap is approximately a Grim Reaper, its speed is related to its width  $w$  by  $V = \pi/w$ . The width  $w$  is approximated by

$$w \approx \left(\alpha(t) - t + \frac{\pi}{2}\right) - \left(\alpha(t) - t - \frac{\pi}{2}\right) \approx' (\alpha(t) - t) \pi,$$

so that the cap moves with speed

$$V \approx \frac{\pi}{w} \approx \frac{1}{r'(\alpha(t) - t)}.$$



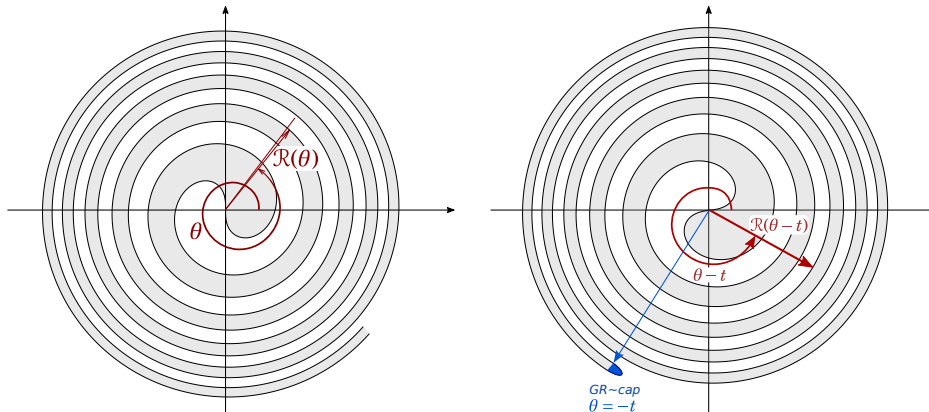


Figure 3.1: **Left:** The Yin Yang soliton in polar coordinates. **Right:** The Truncated Yin Yang soliton at time  $t \ll 0$ . The bulk of the curve consists of an initial segment of the Yin–Yang curve rotated counterclockwise by an angle  $t$ , which has two ends. These ends are connected by a small cap in the shape of a translating soliton (the so-called “Grim Reaper”) located at polar angle  $\theta = -t$ .

On the other hand, far away from the center, the arms of the Yin-Yang spiral are close to circular. The cap, which moves with angular velocity  $-\alpha'(t)$  along a near circle with radius  $(\alpha(t) - t)$ , therefore has velocity  $V \approx -\alpha'(t)(\alpha(t) - t)$ . It follows that  $\alpha'(t) \approx -1/(\cdot)$ . Since the asymptotics (3.1) imply  $\alpha' \approx 1$ , we end up with

$$\alpha'(t) = -1 + o(1), \quad \alpha(t) = -t + o(t). \quad (3.2)$$

For any  $t < 0$  we let  $\Omega(t) \subset \mathbb{C}^2$  be the region given in polar coordinates by

$$\Omega(t) = \left\{ (r \cos \theta, r \sin \theta) \mid t \leq \theta \leq -t, (\theta - t - \frac{\pi}{2}) \leq r \leq (\theta - t + \frac{\pi}{2}) \right\}. \quad (3.3)$$

We will call the boundary curve  $\partial\Omega(t)$  the *truncated Yin-Yang curve*. It consists of a segment of the rotating Yin-Yang curve and a straight line segment that connects the two ends at  $\theta = -t$ .

**Main Theorem.** There exists a compact ancient solution  $\{C(t) \mid t < 0\}$  to Curve Shortening which for  $t \rightarrow -\infty$  is uniformly close to the truncated Yin-Yang curve in the sense that if  $\Lambda(t)$  is the bounded region enclosed by  $C(t)$  then<sup>1</sup>

$$\text{Area}(\Lambda(t) \Delta \Omega(t)) \lesssim |t|^{-1+\delta} \quad (t \rightarrow -\infty)$$

for every  $\delta > 0$ .

<sup>1</sup>For two sets  $A$  and  $B$  we denote their symmetric difference by  $A \Delta B = (A \setminus B) \cup (B \setminus A)$ .

The construction follows a pattern similar to the construction of ancient solutions in (Angenent and You, 2021; You, 2014), namely, construct a sequence of “really old solutions”  $\{n(t) \mid -n \leq t \leq t_0\}$  of Curve Shortening and extract a convergent subsequence whose limit is the desired ancient solution. In section 3.2 we first construct an ancient approximate solution  $\{*(t) \mid -\infty < t < 0\}$  of Curve Shortening, i.e. a family of curves for which

$$\int_{-\infty}^{t_0} \int_{*(t)} |V - \kappa| ds dt < \infty. \quad (3.4)$$

Then we consider a sequence of solutions  $\{n(t) \mid -n \leq t < t_0\}$  of Curve Shortening whose initial curves  $n(-n)$  are chosen increasingly close to the approximate solution  $*(-n)$  at time  $-n$ , in the sense that the area between  $*(-n)$  and  $n(-n)$  tends to zero as  $n \rightarrow \infty$ . Arguing as in (Angenent and You, 2021; You, 2014) we observe that the area between the solutions  $n(t)$  and the approximate solution  $*(t)$  is bounded in terms of the “error” and the area between the initial curves  $*(-n)$  and  $n(-n)$ . Since the curves we deal with in this paper are not graphical the area estimate is a bit more complicated than in (Angenent and You, 2021; You, 2014). In section 3.3 we present a more general estimate that generalizes the Altschuler-Grayson (S. J. Altschuler and Grayson, 1992) area bounds for Space Curve Shortening. Finally, in section 3.4 we show how this bound allows us to extract a convergent subsequence of the very old solutions  $n(t)$ , and provides enough control to conclude that the resulting limit satisfies the description in the Main Theorem.

In appendix A.1 we recall the derivation of the Yin-Yang soliton, and obtain its asymptotic expansion at infinity.

### 3.2 Construction of the cap

#### Parametrized curves and the Curve Shortening deficit

An evolving family of curves is a map  $X : \times(t_0, t_1) \rightarrow^2$  for which  $X_p(t, p) \neq 0$  for all  $(p, t)$ . For such a family we define

$$ds = \|X_p(t, p)\| dp, \quad \frac{d}{ds} = \frac{1}{\|X_p(t, p)\|} \frac{d}{dp}.$$

The normal velocity and curvature of the family  $X$  are

$$V = \langle X_t, JX_s \rangle, \quad \kappa = \langle X_{ss}, JX_s \rangle = \left\langle \frac{X_{pp}}{\|X_p\|^2}, \frac{JX_p}{\|X_p\|} \right\rangle,$$

where  $J = 0 - 1$

10 represents counterclockwise rotation by  $\frac{\pi}{2}$ . By definition, the parameterized

family of curves  $X$  satisfies Curve Shortening if  $V = \kappa$ . If it does not, then we measure the “discrepancy with curve shortening” in terms of the form

$$|V - \kappa| ds = \left| \left\langle X_t - \frac{X_{pp}}{\|X_p\|^2}, JX_p \right\rangle \right| dp. \quad (3.5)$$

The error is obtained by integrating this form over the curve and in time.

We write  $1 = (1, 0)$ ,  $2 = (0, 1)$  for the standard basis for  $\mathbb{R}^2$ . In this basis counter-clockwise rotation by  $\theta$  is given by

$$e^{\theta J} = \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix}.$$

We will use the rotated frame  $\{1(\theta), 2(\theta)\} = \{e^{\theta J}1, e^{\theta J}2\}$ , which satisfy

$$1'(\theta) = 2(\theta), \quad 2'(\theta) = -1(\theta).$$

### Z or $(u, v)$ -coordinates

We expect the tip of the ancient solution to be located near the point  $R(t)1(-t)$ , and to have width  $\sim R^{-1}$ , where

$$R = R(t)(-2t).$$

Thus we introduce new, time dependent, coordinates  $Z = u1 + v2$  related to the cartesian coordinates  $X = (x_1, x_2)$  via

$$X = e^{-tJ}\{R1 + R^{-1}Z\}, \text{ i.e. } Z = R\{e^{tJ}X - R1\}. \quad (3.6)$$

### The inner and outer Yin-Yang arms

The region on one side of the Yin-Yang curve is foliated by rotated copies of the curve. At time  $t \in \mathbb{R}$  the leaves of this foliation are parametrized by

$$Y(\theta, t, y) = (\theta - t + y)1(\theta) \quad (3.7)$$

where  $y \in [-\pi/2, \pi/2]$  determines the leaf, and  $\theta \in (t - y, \infty)$  is the polar angle on the leaf. The inner and outer arms of the region that contains our ancient solution correspond to  $y = \pm\pi/2$ . See Figure 3.2.

### Lemma

For any  $M > 0$  there is a  $t_M < 0$  such that if  $t \leq t_M$ , then the segments of the Yin-Yang leaves  $Y(\theta, t, y)$  with  $|\theta + t| \leq M\tau^{-1} \ln \tau$  are graphs in  $(u, v)$  coordinates of the form  $u = U_y(t, v)$ , at least if  $t \leq t_M$ . Moreover, the functions  $U_y(t, v)$  satisfy

$$U_y(t, v) = y - \frac{y^2 + v^2 - 2v}{2\tau} + (\tau^{-2+\delta}), \quad (3.8)$$

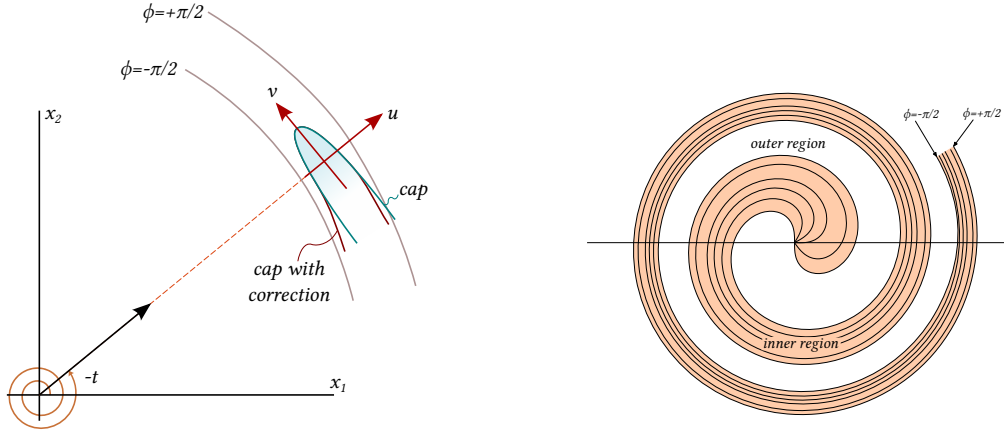


Figure 3.2: **Left:** The  $(u, v)$  coordinates. **Right:** The Yin Yang foliation at time  $t = 0$ .

where  $\tau = -4t$ . We begin with the defining equations

$$(\theta - t + y)1(\theta) = e^{-tJ} \{R1 + R^{-1}Z\} = e^{-tJ} \{(R + u/R)1 + v/R2\}. \quad (3.9)$$

Multiply with  $e^{tJ}$  on both sides:

$$(\theta - t + y)1(\theta + t) = R1 + R^{-1}Z = (R + u/R)1 + v/R2.$$

Here the left hand side is the polar form of the right hand side. Under our assumptions  $R^2 \sim \tau$  and  $|u| \ll \tau$ , so  $1 + u/R^2 > 0$ , and hence we have

$$(\theta - t + y)^2 = R^2 + 2u + \frac{u^2 + v^2}{R^2}, \quad \theta + t = \arctan \frac{v/R^2}{1 + u/R^2}. \quad (3.10)$$

Since  $\alpha \mapsto (\alpha)^2$  is a monotone function, the first equation in (3.10) can be solved for  $y$ . Using the asymptotic expansion in section A.1 we get

$$2(\theta - t + y) = \sqrt{R^2 + 2u + \frac{u^2 + v^2}{R^2}} + \mathcal{O}(R^{-4}), \quad (3.11)$$

where  $\sqrt{\cdot} = (\theta - t + y)$ . Replace  $\sqrt{\cdot}$  by the expression in (3.10), to get

$$2(\theta - t + y) = R^2 + 2u + \frac{u^2 + v^2 + 2}{R^2} + \mathcal{O}(R^{-4}).$$

To eliminate  $\theta$  we expand the second equation in (3.10),

$$\theta + t = \frac{v}{R^2} + \mathcal{O}(R^{-4}).$$

Hence

$$\begin{aligned} 2y &= 2(\theta - t + y) - 2(\theta + t) + 4t & (-4t = \tau) & \quad (3.12) \\ &= R^2 - \tau + 2u + \frac{u^2 + v^2 - 2v + 2}{R^2} + (z^2 R^{-4}). \end{aligned}$$

Expand  $R = R(t) = (-2t)$  in powers of  $\tau = -4t$  using (A.7):

$$R^2 = (-2t)^2 = \tau \left( 1 - 2\tau^{-2} + (\tau^{-3}) \right)$$

and substitute this in (3.12) to get

$$2y = 2u + \frac{u^2 + v^2 - 2v}{\tau} + (z^2 R^{-4}).$$

By assumption we have  $|z| \lesssim \ln \tau$  and  $R \sim \sqrt{\tau}$ , so  $z^2 R^{-4} = (\tau^{-2+\delta})$  for any  $\delta > 0$ .

If  $\tau$  is sufficiently large then the above equation has a unique solution  $u = U_y(t, v)$  with

$$u = y - \frac{u^2 + v^2 - 2v}{2\tau} + (\tau^{-2+\delta}) = y - \frac{y^2 + v^2 - 2v}{2\tau} + (\tau^{-2+\delta}).$$

### General *ansatz* for the cap

We now construct the cap by assuming that it is given by

$$X(t, p) = e^{-tJ} \{ R1 + \varepsilon Z(t, p) \}, \text{ with } R(t) = (-2t), \quad \varepsilon(t) = \frac{1}{R(t)}, \quad (3.13)$$

and by computing its deviation from Curve Shortening (3.5)

$$(V - \kappa)ds = \langle X_t - X_{ss}, JX_s \rangle ds = \left\langle X_t - \frac{X_{pp}}{\|X_p\|^2}, JX_p \right\rangle dp W_p dp.$$

In the following computations it will be convenient to abbreviate

$$R_\theta(t) = '(-2t),$$

so that  $R'(t) = -2R_\theta(t)$ .

The space derivatives of  $X$  are

$$X_p = e^{-tJ} \varepsilon Z_p \quad X_{pp} = e^{-tJ} \varepsilon Z_{pp}.$$

The time derivative has a few more terms:

$$X_t = e^{-tJ} \{ -2R_\theta 1 - R2 + \varepsilon Z - \varepsilon JZ + \varepsilon Z_t \}.$$

Express  $W = \langle X_t - X_{pp}/\|X_p\|^2, JX_p \rangle$  in terms of  $Z$ , keeping in mind that  $\varepsilon = R^{-1}$ ,

$$W = \left\langle \varepsilon^2 Z_t - \frac{Z_{pp}}{\|Z_p\|^2}, JZ_p \right\rangle - \langle 1, Z_p \rangle + 2\varepsilon R_\theta \langle 2, Z_p \rangle + \varepsilon \varepsilon' \langle Z, JZ_p \rangle - \varepsilon^2 \langle Z, Z_p \rangle. \quad (3.14)$$

We look for a cap in the form of a normal perturbation of the Grim Reaper curve, i.e. we assume

$$Z(t, p) = G(p) + f(t, p)JG_p(p) + \alpha 2,$$

where  $\alpha \in \mathbb{R}$  is a constant and

$$G(p) = -\arcsin(\tanh p)1 - \ln(\cosh p)2$$

is the arclength parametrization of the Grim Reaper.

For now  $\alpha$  can be any real constant. Later, when we compare the cap that we are now constructing with the inner and outer Yin-Yang arms, we will be forced to choose  $\alpha = 2$  to make the cap match with the inner and outer Yin-Yang arms. At this point the choice  $\alpha = 2$  does not seem obvious, so we leave  $\alpha$  undetermined for now.

Since  $G$  is an arclength parametrization  $G_p, JG_p$  are unit tangent and normal to the Grim Reaper. Specifically,

$$G_p = -\frac{1}{\cosh p}1 - \tanh p 2.$$

The parametrization  $G(p)$  traces the Grim Reaper out from right to left. Furthermore, the curvature vector of the Grim Reaper is

$$G_{pp}(p) = \kappa(p)JG_p, \text{ where } \kappa(p) = \frac{1}{\cosh p}.$$

### Detailed computation of $W$ on the cap

We have

$$\begin{aligned} Z_t &= f_t JG_p \\ Z_p &= G_p + f_p JG_p + f JG_{pp} = (1 - \kappa f)G_p + f_p JG_p \\ JZ_p &= -f_p G_p + (1 - \kappa f)JG_p \\ \|Z_p\|^2 &= (1 - \kappa f)^2 + f_p^2 = 1 - 2\kappa f + \kappa^2 f^2 + f_p^2 \\ Z_{pp} &= -(\kappa_p f + 2\kappa f_p)G_p + (f_{pp} + \kappa - \kappa^2 f)JG_p. \end{aligned}$$

Substituting in (3.14), and using  $\kappa = 1/\cosh p$ , we get

$$\begin{aligned}\langle \varepsilon^2 Z_t, JZ_p \rangle &= \varepsilon^2(1 - \kappa f) f_t \\ \left\langle \frac{Z_{pp}}{\|Z_p\|^2}, JZ_p \right\rangle &= \frac{(1 - \kappa f)(f_{pp} + \kappa(1 - \kappa f)) + \kappa_p f f_p + 2\kappa f_p^2}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} \\ \langle 1, Z_p \rangle &= -\kappa(1 - \kappa f) + f_p \tanh p.\end{aligned}$$

Thus after substituting and expanding we find

$$\begin{aligned}W &= \varepsilon^2(1 - \kappa f) f_t - \frac{\kappa + f_{pp} - 2\kappa^2 f - \kappa f f_{pp} + \kappa_p f f_p + \kappa^3 f^2 + 2\kappa f_p^2}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} + \kappa(1 - \kappa f) \\ &\quad - f_p \tanh p - \varepsilon^2 \langle Z, Z_p \rangle + 2\varepsilon R_\theta \langle 2, Z_p \rangle + \varepsilon \varepsilon' \langle Z, JZ_p \rangle.\end{aligned}\tag{3.15}$$

We will choose  $f(t, p)$  so as to make  $W$  integrable in space and time. To find  $f$  we linearize the expression for  $W$  and solve the resulting first order equation for  $f$ . It turns out that one solution is of the form  $f(t, p) = \tau^{-1} F(p)$  for a function  $F$  that is of polynomial growth for  $|p| \rightarrow \infty$ . We restrict our attention to the region

$$|p| \leq 2K \ln \tau, \quad \tau \gg 1,\tag{3.16}$$

where  $K = 100$  is a fixed, largish, constant. We will assume that  $f$  and its derivatives are bounded by

$$|f| + |f_p| + |f_{pp}| + \tau |f_t| \lesssim \tau^{-1+\delta} \text{ for } |p| \leq 2K \ln \tau.\tag{3.17}$$

Here, and in what follows, when we write estimates for remainder terms of the form  $(\tau^{-m+\delta})$ , the estimate is implicitly meant to hold “for all  $\delta > 0$ .”

The bound (3.17) will certainly hold if  $f(t, p) = \tau^{-1} F(p)$  for some function  $F(p)$  for which  $F(p)$ ,  $F'(p)$ , and  $F''(p)$  grow polynomially as  $p \rightarrow \pm\infty$ .

We now consider the many terms in (3.15) that add up to  $W$ . To begin, we have for  $|p| \leq 2K \ln \tau$ ,

$$\varepsilon^2(1 - \kappa f) f_t = (\tau^{-2+\delta}),$$

and also

$$\frac{-\kappa f f_{pp} + \kappa_p f f_p + \kappa^3 f^2 + 2\kappa f_p^2}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} = (\tau^{-2+\delta}) \text{ for any } \delta > 0.$$

It follows from  $Z = G + fJG_p$  that

$$Z = G + \alpha 2 + (\tau^{-1+\delta}), \quad Z_p = G_p + (\tau^{-1+\delta}).$$

Hence, for  $|p| \leq 2K \ln \tau$ ,

$$\begin{aligned} \|G(p)\| &\lesssim \ln \tau & \varepsilon^2 \langle Z, Z_p \rangle &= \tau^{-1} \langle G, G_p \rangle + (\tau^{-2+\delta}) \\ \varepsilon R_\theta \langle 2, Z_p \rangle &= \tau^{-1} \langle 2, G_p \rangle + (\tau^{-2+\delta}) & \varepsilon \varepsilon' \langle Z, JZ_p \rangle &= (\tau^{-2+\delta}). \end{aligned}$$

In this computation we have used the expansions  $\varepsilon = R(t)^{-1} = (-2t)^{-1} = \tau^{-1/2} + (\tau^{-3/2})$  and  $R_\theta = '(-2t) = \tau^{-1/2} + (\tau^{-3/2})$  that follow from the expansions of  $(\theta)$  and  $'(\theta)$  in section A.1.

So far we have

$$\begin{aligned} W &= (\tau^{-2+\delta}) - \frac{\kappa + f_{pp} - 2\kappa^2 f}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} + \kappa(1 - \kappa f) - f_p \tanh p \\ &\quad + \tau^{-1} \langle (2 - \alpha)2 - G, G_p \rangle. \end{aligned}$$

We can simplify the fraction (for  $|p| \leq 2K \ln \tau$ ) by using

$$\frac{1}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} = 1 + 2\kappa f - \kappa^2 f^2 - f_p^2 + \frac{(-2\kappa f + \kappa^2 f^2 + f_p^2)^2}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} = 1 + 2\kappa f + (\tau^{-2+\delta})$$

which implies

$$\frac{\kappa + f_{pp} - 2\kappa^2 f}{1 - 2\kappa f + \kappa^2 f^2 + f_p^2} = \kappa + 2\kappa^2 f + f_{pp} - 2\kappa^2 f + (\tau^{-2+\delta}) = \kappa + f_{pp} + (\tau^{-2+\delta}),$$

and hence

$$W = -f_{pp} - \tanh(p)f_p - \kappa^2 f + \frac{1}{\tau} \langle (2 - \alpha)2 - G, G_p \rangle + (\tau^{-2+\delta}).$$

### Computation of the correction term

We look to perturb the Grim Reaper with a term of the form

$$f(t, p) = \frac{F(p)}{\tau},$$

where  $F$  is a solution of

$$FF_{pp} + \tanh(p)F_p + \kappa^2 F = \langle (2 - \alpha)2 - G, G_p \rangle.$$

The linear operator can be factored

$$= \frac{d^2}{dp^2} + \tanh(p) \frac{d}{dp} + \frac{1}{\cosh^2(p)} = \frac{d}{dp} \circ \frac{1}{\cosh p} \circ \frac{d}{dp} \circ \cosh(p)$$

while we also have

$$\langle (2 - \alpha)2 - G, G_p \rangle = \frac{d}{dp} \langle (2 - \alpha)2 - \frac{1}{2}G, G \rangle,$$



all of which allows us to solve the equation for  $F$ :

$$F(p) = \frac{A}{\cosh p} + B \tanh p + \int_0^p \frac{\cosh r}{\cosh p} \langle (2 - \alpha)2 - \frac{1}{2}G(r), G(r) \rangle dr. \quad (3.18)$$

It appears that the first term is of no use, so we set  $A = 0$ . The resulting function  $F(p)$  is an odd function of  $p$ . We use the asymptotic behavior of  $\langle (2 - \alpha)2 - \frac{1}{2}G, G \rangle$  for large  $p$  to find an expansion for the integral as  $p \rightarrow +\infty$ .

Consider

$$I \int_0^p \frac{\cosh r}{\cosh p} \langle (2 - \alpha)2 - \frac{1}{2}G(r), G(r) \rangle dr.$$

The explicit expression for  $G$  implies

$$\langle 2, G(r) \rangle = -\ln \cosh r$$

$$\|G(r)\|^2 = (\ln \cosh r)^2 + (\arcsin \tanh r)^2 = (\ln \cosh r)^2 + \frac{\pi^2}{4} + (e^{-r}) \quad (r \rightarrow \infty).$$

To compute  $I$  we substitute  $\lambda = \ln \cosh p$ ,  $\mu = \ln \cosh r$ , which leads to

$$I = - \int_0^\lambda e^{\mu-\lambda} \left\{ (2 - \alpha)\mu + \frac{1}{2}\mu^2 + \frac{\pi^2}{8} + (e^{-\mu}) \right\} \frac{d\mu}{\sqrt{1 - e^{-2\mu}}}.$$

The integrand is singular but integrable at  $\mu = 0$ . To deal with this singularity split

$$(1 - e^{-2\mu})^{-1/2} = 1 + [(1 - e^{-2\mu})^{-1/2} - 1], \quad \text{with } 0 \leq (1 - e^{-2\mu})^{-1/2} - 1 \lesssim \frac{1}{\sqrt{\mu}} e^{-2\mu}.$$

Replacing  $(1 - e^{-2\mu})^{-1/2}$  by 1 therefore introduces an extra term that is bounded by  $(e^{-\lambda})$ . Hence we have

$$\begin{aligned} I &= (e^{-\lambda}) - \int_0^\lambda e^{\mu-\lambda} \left\{ (2 - \alpha)\mu + \frac{1}{2}\mu^2 + \frac{\pi^2}{8} + (e^{-\mu}) \right\} d\mu \\ &= (\lambda^3 e^{-\lambda}) - \int_0^\lambda e^{\mu-\lambda} \left\{ (2 - \alpha)\mu + \frac{1}{2}\mu^2 + \frac{\pi^2}{8} \right\} d\mu \\ &= (\lambda^3 e^{-\lambda}) - \left\{ (2 - \alpha)\lambda + \frac{1}{2}\lambda^2 + \frac{\pi^2}{8} \right\} + \left\{ (2 - \alpha) + \lambda \right\} - \{1\} \\ &= -\frac{1}{2}\lambda^2 - (1 - \alpha)\lambda - \frac{\pi^2}{8} + 1 - \alpha + (\lambda^3 e^{-\lambda}). \end{aligned}$$

Thus, for large  $p$  we get

$$F(p) = B - \frac{1}{2}(\ln \cosh p)^2 - (1 - \alpha) \ln \cosh p - \frac{\pi^2}{8} + 1 - \alpha + (p^3 e^{-p}) \quad (p \rightarrow +\infty).$$

Since  $F(p)$  is an odd function we also have

$$F(p) = - \left\{ B - \frac{1}{2}(\ln \cosh p)^2 - (1 - \alpha) \ln \cosh p - \frac{\pi^2}{8} + 1 - \alpha \right\} + (p^3 e^{-p}) \quad (p \rightarrow -\infty).$$

Applying this to  $Z = G + \tau^{-1}F(p)JG_p + \alpha 2$  we get for the two components  $u$  and  $v$  of  $Z$  as  $p \rightarrow \pm\infty$ :

$$\begin{aligned} u(t, p) &= -\arcsin \tanh p + \frac{1}{\tau}F(p) \tanh p = \mp \frac{\pi}{2} + \frac{1}{\tau}F(p) \tanh p + (e^{-p}) \\ v(t, p) &= -\ln \cosh p - \frac{1}{\tau} \frac{F(p)}{\cosh p} + \alpha = -\ln \cosh p + \alpha + (p^3 e^{-p}) \end{aligned}$$

We can again eliminate  $p$  when  $p$  is large by using

$$\ln \cosh p = -v + \alpha + (p^2 e^{-|p|}), \quad (p \rightarrow \pm\infty)$$

which leads to

$$\begin{aligned} u(t, p) &= \mp \frac{\pi}{2} - \frac{v^2 + 2(1 - \alpha)v + \pi^2/4 + 2(1 - \alpha + B)}{2\tau} \\ &\quad + (|p|^3 e^{-|p|}) \quad (p \rightarrow \pm\infty) \quad (3.19) \end{aligned}$$

We now determine  $\alpha$  and  $B$  by matching (3.19) with the representation of the Yin-Yang arms in  $(u, v)$  coordinates that we found in (3.8). Setting  $y = \mp\pi/2$  in (3.8) we find for the outer and inner Yin-Yang arms

$$u = \mp \frac{\pi}{2} - \frac{\pi^2/4 + v^2 - 2v}{2\tau} + (\tau^{-2+\delta}) \quad (3.20)$$

If  $|p| \geq \frac{K}{2} \ln \tau$  then  $p^3 e^{-|p|} = (\tau^{-K/2+\delta}) = o(\tau^{-2+\delta})$ , and therefore the two expansions (3.19),(3.20) match if

$$\alpha = 2 \text{ and } B = 1. \quad (3.21)$$

This choice of  $\alpha$  actually simplifies the expression for  $F(p)$  somewhat.

To summarize, we choose the cap to be given by

$$X(t, p) = e^{-tJ} \left\{ R(t)1 + \frac{1}{R(t)} Z(t, p) \right\}$$

with

$$Z(t, p) = G(p) + \frac{F(p)}{\tau} JG_p(p) + 22$$

and, from (3.18),

$$F(p) = \tanh p - \frac{1}{2} \int_0^p \frac{\cosh r}{\cosh p} \|G(r)\|^2 dr.$$

**Definition of the smooth interpolation of cap and arms**

In the  $(u, v)$  coordinates, according to (3.8), the Yin-Yang arms are given by

$$u = U_{\pm}(t, v) \text{ for } |v| \leq 3K \ln \tau,$$

where we abbreviate  $U_{\pm}(t, v) = U_{\pm\pi/2}(t, v)$ . In the same  $(u, v)$  coordinates the ends of the cap are given by (3.19) with  $\alpha = 2$ ,  $B = 1$ , i.e.

$$u = h_{\pm}(t, v) \text{ for } \frac{1}{2}K \ln \tau \leq v \leq 2K \ln \tau.$$

We constructed the cap so that both  $U_{\pm}$  and  $h_{\pm}$  have the same asymptotic behavior, namely,

$$U_{\pm}(t, v), h_{\pm}(t, v) = \pm \frac{\pi}{2} - \frac{1}{\tau} \left\{ \frac{1}{2}v^2 - v + \pi^2/8 \right\} + (\tau^{-2+\delta}). \quad (3.22)$$

Choose a smooth nondecreasing function  $\eta : \mathbb{R} \rightarrow [0, 1]$  with  $\eta(u) = 0$  for  $u \leq \frac{1}{2}$  and  $\eta(u) = 1$  for  $u \geq 2$ , and define

$$\begin{aligned} k_{\pm}(t, v) &= \eta\left(\frac{v}{\ln \tau}\right) U_{\pm}(t, v) + \left\{ 1 - \eta\left(\frac{v}{\ln \tau}\right) \right\} h_{\pm}(t, v) \\ &= U_{\pm}(t, v) + \eta\left(\frac{v}{\ln \tau}\right) (U_{\pm}(t, v) - h_{\pm}(t, v)). \end{aligned}$$

The graphs of these two functions are  $Z$ -coordinate representations of curve segments that smoothly interpolate between the two ends of the cap and the two Yin-Yang arms. The two segments are parametrized by

$$X_{\pm}(t, v) = e^{-tJ} \{ R1 + R^{-1} Z_{\pm}(t, v) \} \text{ with } Z_{\pm}(t, v) = k_{\pm}(t, v)1 + v2.$$

It follows from (3.14) that the Curve Shortening Deficit for such curves is given by  $(\kappa - V)ds = W_v dv$  with

$$W_v[k] = -R^{-2}k_t + \frac{k_{vv}}{1+k_v^2} - k_v + 2\frac{R_{\theta}}{R} + 2\frac{R_{\theta}}{R^3}(vk_v - k) - R^{-2}(v + kk_v).$$

**Derivative bounds for  $U_{\pm}, h_{\pm}, k_{\pm}$**

Careful scrutiny of the construction of  $U_{\pm}(t, v)$  and  $h_{\pm}(t, v)$  shows that the remainder terms  $(\tau^{-2+\delta})$  in (3.22) may be differentiated. This implies that the functions  $U_{\pm}$  and  $h_{\pm}$  satisfy

$$|k_{vv}| + |k_v| + |k_t| \lesssim \frac{\ln \tau}{\tau} \lesssim \tau^{-1+\delta} \quad (3.23)$$

for  $\frac{1}{2} \ln \tau \leq v \leq 2K \ln \tau$ , and large enough  $\tau$ .

The derivatives of the gluing function  $\eta(v/\ln \tau)$  are

$$\eta_v = \frac{\eta'(v/\ln \tau)}{\ln \tau}, \quad \eta_{vv} = \frac{\eta''(v/\ln \tau)}{(\ln \tau)^2}, \quad \eta_t = \frac{4v\eta'(v/\ln \tau)}{(\ln \tau)^2},$$

so they are bounded by

$$|\eta_v| + |\eta_{vv}| + |\eta_t| \lesssim (\ln \tau)^{-1}$$

for  $\frac{1}{2} \ln \tau \leq v \leq 2K \ln \tau$ , and large enough  $\tau$ .

It follows that the interpolating functions  $k_{\pm} = \eta U_{\pm} + (1 - \eta)h_{\pm}$  also satisfy (3.23).

### Estimating $W_v[k_{\pm}]$

We show that

$$|W_v[k_{\pm}]| \lesssim \tau^{-2+\delta} \quad (3.24)$$

holds in the region  $\frac{1}{2}K \ln \tau \leq v \leq 2K \ln \tau$ , for sufficiently large  $\tau$ .

If  $k$  is any of the functions  $U_{\pm}, h_{\pm}, k_{\pm}$  then we have

$$\frac{|k_v^2 k_{vv}|}{1 + k_v^2} \leq |k_v^2 k_{vv}| \lesssim \tau^{-3+\delta}, \quad R^{-2}|k k_v| \lesssim \tau^{-3+\delta}, \quad \text{and} \quad -R^{-2}|k_t| \lesssim \tau^{-2+\delta}.$$

Furthermore  $|k| \lesssim 1$ ,  $|v k_v| \lesssim \tau^{-1+\delta}$ , and  $R_{\theta} \sim R^{-1} \lesssim \tau^{-1/2}$  lead to

$$\frac{R_{\theta}}{R^3}|v k_v - k| \lesssim \tau^{-2}.$$

Hence

$$W_v[k] = k_{vv} - k_v + 2\frac{R_{\theta}}{R} - vR^{-2} + (\tau^{-2+\delta})$$

holds for all six functions  $k \in \{U_{\pm}, h_{\pm}, k_{\pm}\}$ .

To simplify our notation we drop the  $\pm$  subscript for now and expand the derivatives of  $k = \eta U + (1 - \eta)h$  (with  $U = U_{\pm}$ ),

$$\begin{aligned} k_v &= \eta U_v + (1 - \eta)h_v + \eta_v(U - h), \\ k_{vv} &= \eta U_{vv} + (1 - \eta)h_{vv} + 2\eta_v(U - h)_v + \eta_{vv}(U - h). \end{aligned}$$

Since we have matched the two cap ends with the Yin-Yang arms, it follows from (3.22) that the difference  $U - h$  and its derivatives are bounded by

$$|U - h| + |(U - h)_v| + |(U - h)_{vv}| \lesssim \tau^{-2+\delta}.$$

Together these inequalities give us the desired estimate for  $W_v[k_{\pm}]$ , namely

$$\begin{aligned} W[k] &= W[\eta U + (1 - \eta)h] \\ &= (\eta U + (1 - \eta)h)_{vv} - (\eta U + (1 - \eta)h)_v + 2\frac{R_{\theta}}{R} - vR^{-2} + (\tau^{-2+\delta}) \\ &= \eta W[U] + (1 - \eta)W[h] + (\tau^{-2+\delta}) \\ &= (\tau^{-2+\delta}). \end{aligned}$$

### 3.3 Area decreasing property of Space Curve Shortening

In 1991 Altschuler and Grayson (S. J. Altschuler and Grayson, 1992) observed that for two solutions of space curve shortening the area of the minimal surface spanning them is non increasing. Here we elaborate on this and prove a similar result without using the existence of the minimal surface.

#### Moving space curves

For an immersed curve  $X : \rightarrow^n$  one defines the arc length one-form  $ds$  and the arc length derivative  $\partial_s$  of any quantity  $f : \rightarrow$  by

$$ds = \|X_p\| dp, \text{ and } \frac{\partial f}{\partial s} = \frac{1}{\|X_p\|} \frac{\partial f}{\partial p}.$$

The unit tangent and curvature of the curve are  $X_s = \frac{X_p}{\|X_p\|}$  and  $X_{ss}$ .

A moving family of space curves is a map  $X : (t_0, t_1) \times \rightarrow^n$ . The family evolves by Curve Shortening if it satisfies  $X_t^\perp = X_{ss}$ , i.e., if for some smooth function  $\lambda(t, p)$  one has

$$X_t = X_{ss} + \lambda X_s = \frac{1}{\|X_p\|} \frac{\partial}{\partial p} \left( \frac{X_p}{\|X_p\|} \right) + \lambda \frac{X_p}{\|X_p\|}. \quad (3.25)$$

Since  $X_s \perp X_{ss}$ , one can always find  $\lambda$  from  $\lambda = \langle X_{ss}, X_s \rangle$ .

#### Evolution of arc length and the commutator $[\partial_t, \partial_s][dt, ds]$

The following are commonly used relations. We record them here for completeness, and also because we allow the velocity  $X_t$  of the parametrizations to have a nonvanishing tangential component. Assuming that  $X_t = X_{ss} + \lambda X_s$  one has

$$\partial_t \|X_p\| = (\lambda_s - \kappa^2) \|X_p\|, \quad \frac{\partial}{\partial t} ds = (\lambda_s - \kappa^2) ds = d\lambda - \kappa^2 ds, \quad (3.26)$$

and

$$[\partial_t, \partial_s] = (-\lambda_s + \kappa^2) \partial_s. \quad (3.27)$$

We have

$$\frac{\partial}{\partial t} \|X_p\| = \left\langle \frac{X_p}{\|X_p\|}, X_{tp} \right\rangle = \langle X_s, X_{tp} \rangle = \langle X_s, X_{ts} \rangle \|X_p\|,$$

and hence

$$\frac{\partial}{\partial t} ds = \frac{\partial}{\partial t} (\|X_p\| dp) = \langle X_s, X_{tp} \rangle dp = \langle X_s, X_{ts} \rangle ds.$$

The evolution equation  $X_t = X_{ss} + \lambda X_s$  then implies

$$\langle X_s, X_{ts} \rangle = \langle X_s, X_t \rangle_s - \langle X_{ss}, X_t \rangle = \lambda_s - \langle X_{ss}, X_{ss} + \lambda X_s \rangle = \lambda_s - \|X_{ss}\|^2 = \lambda_s - \kappa^2,$$

which directly implies (3.26). Using (3.26) we get

$$[\partial_t, \partial_s] = [\partial_t, \|X_p\|^{-1} \partial_p] = -\frac{\partial_t \|X_p\|}{\|X_p\|^2} \partial_p = -(\lambda_s - \kappa^2) \partial_s.$$

### Dependence on a parameter

Let  $[\epsilon_0, \epsilon_1] \subset \mathbb{R}$  be some parameter interval, and let  $X : [\epsilon_0, \epsilon_1] \times (t_0, t_1) \times \mathbb{R} \rightarrow \mathbb{R}^n$  be a family of moving curves that depends on a parameter  $\epsilon \in [\epsilon_0, \epsilon_1]$ . We compute the evolution of the first variation

$$X_\epsilon = \frac{\partial X}{\partial \epsilon}(\epsilon, t, p).$$

Throughout the computation we will assume that the parametrization  $X$  is such that

$$X_\epsilon \perp X_p(\epsilon, t, p) \text{ for all } (\epsilon, t, p). \quad (3.28)$$

For any given parametrization  $\tilde{X}$  one can find a reparametrization  $\varphi(\epsilon, t, p)$  so that  $X(\epsilon, t, p) = \tilde{X}(\epsilon, t, \varphi(\epsilon, t, p))$  satisfies (3.28).

If  $X : [\epsilon_0, \epsilon_1] \times [0, 1] \rightarrow \mathbb{R}^n$  is injective with  $X_\epsilon \perp X_p$ , then the double integral

$$\int_{\epsilon_0}^{\epsilon_1} \int_{p=0}^1 \|X_\epsilon\| ds d\epsilon \quad (\text{where } ds = \|X_p\| dp)$$

is the area (2-dim Hausdorff measure) of the image  $X([\epsilon_0, \epsilon_1] \times [0, 1])$ . If  $X$  merely satisfies  $X_\epsilon \perp X_p$ , without necessarily being injective, then the area formula implies that the double integral is bounded from below, by

$$\ell(X) \int_{\epsilon_0}^{\epsilon_1} \int_{p=0}^1 \|X_\epsilon\| ds d\epsilon \geq \mathcal{H}^2(X([\epsilon_0, \epsilon_1] \times [0, 1])). \quad (3.29)$$

We will call the integral  $\ell(X)$  the *length of the homotopy*  $X$ , and we will show that Curve Shortening decreases the length of homotopies.

The following improvement of the inequality  $\|X_\epsilon\|_s \leq \|X_{\epsilon s}\|$  (which follows from the triangle inequality) will be useful.

### Lemma

Assuming (3.28) we have

$$(\|X_\epsilon\|_s)^2 \leq \|X_{\epsilon s}\|^2 - \langle X_{\epsilon s}, X_\epsilon \rangle^2.$$

Split  $X_{\epsilon s}$  into tangential and orthogonal components:

$$X_{\epsilon s} = P + \langle X_{\epsilon s}, X_s \rangle X_s.$$

Since  $X_\epsilon \perp X_s$  we can rewrite this as

$$X_{\epsilon s} = P - \langle X_\epsilon, X_{ss} \rangle X_s,$$

and thus

$$\|X_{\epsilon s}\|^2 = \|P\|^2 + \langle X_\epsilon, X_{ss} \rangle^2.$$

On the other hand

$$\|X_\epsilon\|_s = \left\langle \frac{X_\epsilon}{\|X_\epsilon\|}, X_{\epsilon s} \right\rangle = \left\langle \frac{X_\epsilon}{\|X_\epsilon\|}, P \right\rangle \leq \|P\|,$$

where we have again used that  $X_s \perp X_\epsilon$ . Combining these observations we arrive at

$$(\|X_\epsilon\|_s)^2 - \|X_{\epsilon s}\|^2 \leq \|P\|^2 - \|X_{\epsilon s}\|^2 = -\langle X_\epsilon, X_{ss} \rangle^2,$$

as claimed.

### The commutator $[\partial_\epsilon, \partial_s]$ [deps, ds]

Assuming (3.28) one has

$$[\partial_\epsilon, \partial_s] = \langle X_{ss}, X_\epsilon \rangle \partial_s \text{ and } [\partial_\epsilon, \partial_s^2] = 2\langle X_{ss}, X_\epsilon \rangle \partial_s^2 + \langle X_{ss}, X_\epsilon \rangle_s \partial_s$$

The computation follows the same pattern as the derivation of (3.27). Here we have no equation for  $X_\epsilon$ , but we do know that  $X_\epsilon \perp X_s$ . Thus

$$\partial_\epsilon \|X_p\| = \left\langle \frac{X_p}{\|X_p\|}, X_{p\epsilon} \right\rangle = \langle X_s, X_{\epsilon p} \rangle = \langle X_s, X_{\epsilon s} \rangle \|X_p\| = -\langle X_{ss}, X_\epsilon \rangle \|X_p\|.$$

Apply this to  $\partial_s = \|X_p\|^{-1} \partial_p$  to get the commutator  $[\partial_\epsilon, \partial_s]$ . The other commutator follows from expanding  $[\partial_\epsilon, \partial_s^2] = [\partial_\epsilon, \partial_s] \partial_s + \partial_s [\partial_\epsilon, \partial_s]$ .

### Lemma

The length of the first variation  $X_\epsilon$  satisfies the differential inequality

$$\partial_t \|X_\epsilon\| - \lambda \partial_s \|X_\epsilon\| \leq \partial_s^2 \|X_\epsilon\| + \kappa^2 \|X_\epsilon\|.$$

Differentiating the evolution equation (3.25) for  $X$  we get

$$\begin{aligned} \partial_t X_\epsilon &= \partial_\epsilon X_t = \partial_\epsilon (X_{ss} + \lambda X_s) \\ &= X_{\epsilon ss} + 2\langle X_{ss}, X_\epsilon \rangle X_{ss} + \langle X_{ss}, X_\epsilon \rangle_s X_s + \lambda_\epsilon X_s + \lambda \partial_\epsilon X_s. \end{aligned}$$

Hence

$$\begin{aligned} \partial_t X_\epsilon - \lambda \partial_s X_\epsilon &= X_{\epsilon ss} + 2\langle X_{ss}, X_\epsilon \rangle X_{ss} + \langle X_{ss}, X_\epsilon \rangle_s X_s + \lambda_\epsilon X_s + \lambda [\partial_\epsilon, \partial_s] X \\ &= X_{\epsilon ss} + 2\langle X_{ss}, X_\epsilon \rangle X_{ss} + \{\langle X_{ss}, X_\epsilon \rangle_s + \lambda_\epsilon + \lambda \langle X_{ss}, X_\epsilon \rangle\} X_s. \end{aligned}$$

We next compute the evolution of  $\|X_\epsilon\|^2$ , keeping in mind that  $X_\epsilon \perp X_s$ :

$$\begin{aligned}\partial_t \|X_\epsilon\|^2 &= 2\langle X_\epsilon, X_{\epsilon t} \rangle \\ &= 2\langle X_\epsilon, X_{\epsilon ss} \rangle + 4\langle X_{ss}, X_\epsilon \rangle^2 \\ &= (\|X_\epsilon\|^2)_{ss} - 2\|X_{\epsilon s}\|^2 + 4\langle X_{ss}, X_\epsilon \rangle^2 \\ &= 2\|X_\epsilon\| \|X_\epsilon\|_{ss} + 2(\|X_\epsilon\|_s)^2 - 2\|X_{\epsilon s}\|^2 + 4\langle X_{ss}, X_\epsilon \rangle^2.\end{aligned}$$

At this point we use Lemma 3.3, to get

$$\partial_t \|X_\epsilon\|^2 \leq 2\|X_\epsilon\| \|X_\epsilon\|_{ss} + 2\langle X_{ss}, X_\epsilon \rangle^2.$$

Since  $\partial_t \|X_\epsilon\|^2 = 2\|X_\epsilon\| \partial_t \|X_\epsilon\|$ , we have

$$\partial_t \|X_\epsilon\| \leq \|X_\epsilon\|_{ss} + \left\langle X_{ss}, \frac{X_\epsilon}{\|X_\epsilon\|} \right\rangle^2 \|X_\epsilon\| \leq \|X_\epsilon\|_{ss} + \kappa^2 \|X_\epsilon\|.$$

### Contractive property of Curve Shortening

If  $X^0, X^1 : [t_0, t_1] \times [0, 1] \rightarrow^n$  are two solutions of Curve Shortening (3.25), then a homotopy  $\{X^\epsilon : 0 \leq \epsilon \leq 1\}$  of solutions to Curve Shortening connecting them is, by definition, a map  $X : [0, 1] \times [t_0, t_1] \times [0, 1] \rightarrow^n$  such that  $(t, p) \mapsto X(\epsilon, t, p)$  is a solution of Curve Shortening with  $X^\epsilon(t, p) = X(\epsilon, t, p)$  for  $\epsilon \in \{0, 1\}$ .

Given any homotopy  $X^\epsilon$  between solutions  $X^0, X^1$  of Curve Shortening, one can always find a reparametrization  $\tilde{X}^\epsilon(t, p) = X^\epsilon(t, \varphi(\epsilon, t, p))$  for which  $\partial_\epsilon \tilde{X}^\epsilon \perp \partial_s \tilde{X}^\epsilon$  holds pointwise. We will call such a homotopy a *normal homotopy*.

Our main observation in this section is the following: *if  $X^\epsilon$  is a normal homotopy between solutions  $X^0, X^1$  of Curve Shortening, then one has for each  $\epsilon \in [0, 1]$  and  $t \in (t_0, t_1)$*

$$\frac{d}{dt} \int_{p=0}^1 \|X_\epsilon(\epsilon, t, p)\| ds \leq 0 \quad (3.30)$$

and

$$\int_{\epsilon=0}^1 \int_{p=0}^1 \|\partial_\epsilon X(\epsilon, t_1, p)\| ds d\epsilon \leq \int_{\epsilon=0}^1 \int_{p=0}^1 \|\partial_\epsilon X(\epsilon, t_0, p)\| ds d\epsilon \quad (3.31)$$

We use (3.26) and Lemma 3.3 to differentiate under the integral:

$$\begin{aligned}\frac{d}{dt} \int_{p=0}^1 \|X_\epsilon\| ds &\leq \int \{ \|X_\epsilon\|_{ss} + \kappa^2 \|X_\epsilon\| + \lambda \|X_\epsilon\|_s + (\lambda_s - \kappa^2) \|X_\epsilon\| \} ds \\ &= \int \{ \|X_\epsilon\|_s + \lambda \|X_\epsilon\| \}_s ds = 0.\end{aligned}$$

This implies (3.30). Integration in  $\epsilon$  and in time then leads to (3.31).



### Deviation from an approximate solution

We now consider the case of two moving curves  $X^0, X^1 : [t_0, t_1] \times [0, 1] \rightarrow^n$  with the same initial value, i.e. with  $X^0(t_0, p) = X^1(t_0, p)$  for all  $p \in [0, 1]$ . We assume that  $X^1$  is a solution of Curve Shortening but allow  $X^0$  to be a general moving curve. We measure its deviation from Curve Shortening in terms of

$$\Delta \int_{t_0}^{t_1} \int_{p=0}^1 \|(\partial_t X^0)^\perp - \partial_s^2 X^0\| ds dt. \quad (3.32)$$

In the case of plane curves  $X : [t_0, t_1] \times [0, 1] \rightarrow^2$ , we have  $\partial_t(X^0)^\perp = VJX_s^0$ , where  $V$  is the normal velocity of the curve  $X^0$ . Therefore the integrand in (3.32) is

$$\|(\partial_t X^0)^\perp - \partial_s^2 X^0\| ds = |V - \kappa| ds.$$

The quantity  $\Delta$  therefore coincides with the “error” defined in (3.4).

Assume that for each  $\epsilon \in [t_0, t_1]$  there is a smooth solution  $(t, p) \mapsto X(\epsilon, t, p)$  of Curve Shortening that is defined for  $t \in [\epsilon, t_1]$ , and that has initial value  $X(\epsilon, \epsilon, p) = X^1(\epsilon, p)$ . After reparametrizing we may assume that  $X_\epsilon \perp X_p$  holds point-wise. Then the final values of these solutions, i.e. the curves  $p \mapsto X(\epsilon, t_1, p)$  form a normal homotopy from  $X^0(t_1, \cdot)$  to  $X^1(t_1, \cdot)$ . We will now show that

$$\int_{\epsilon=t_0}^{t_1} \int_{p=0}^1 \|\partial_\epsilon X\| ds dt \leq \Delta. \quad (3.33)$$

Our argument is a nonlinear version of the Variation of Constants Formula, or of Duhamel’s principle.

For  $\epsilon \in [t_0, t_1]$  we consider

$$E(\epsilon) \int_{\bar{\epsilon}=t_0}^{\epsilon} \int_0^1 \|\partial_\epsilon X(\bar{\epsilon}, t_1, p)\| ds d\bar{\epsilon}.$$

Then

$$E'(\epsilon) = \int_0^1 \|\partial_\epsilon X(\epsilon, t_1, p)\| ds.$$

The contraction property (3.30) implies

$$\int_0^1 \|\partial_\epsilon X(\epsilon, t_1, p)\| ds \leq \int_0^1 \|\partial_\epsilon X(\epsilon, \epsilon, p)\| ds.$$

We compute  $X_\epsilon(\epsilon, \epsilon, p)$  by differentiating the relation  $X(\epsilon, \epsilon, p) = X^0(\epsilon, p)$  with respect to  $\epsilon$ :

$$(\partial_t X^0)(\epsilon, p) = \frac{\partial X^0(\epsilon, p)}{\partial \epsilon} = \frac{\partial X(\epsilon, \epsilon, p)}{\partial \epsilon} = X_\epsilon(\epsilon, \epsilon, p) + X_t(\epsilon, \epsilon, p).$$

Since  $(t, p) \mapsto X(\epsilon, t, p)$  evolves by Curve Shortening, we have  $X_t(\epsilon, \epsilon, p) = X_{ss}(\epsilon, \epsilon, p) + \lambda X_s$ . By definition of  $X$  we have  $X(\epsilon, \epsilon, p) = X^0(\epsilon, p)$ , so  $X_{ss}(\epsilon, \epsilon, p) = X_{ss}^0(\epsilon, p)$ .

We have parameterized the homotopy  $X(\epsilon, t, p)$  so that  $X_\epsilon \perp X_p$ , and therefore

$$\partial_\epsilon X(\epsilon, \epsilon, p) = (\partial_t X^0(\epsilon, p))^\perp - X_{ss}^0(\epsilon, p).$$

Hence

$$E'(\epsilon) \leq \int_0^1 \|\partial_\epsilon X(\epsilon, \epsilon, p)\| ds = \int_0^1 \left\| (\partial_t X^0(\epsilon, p))^\perp - X_{ss}^0(\epsilon, p) \right\| ds.$$

Integrate over  $\epsilon \in [t_0, t_1]$  to recover (3.33).

### Application to plane Curve Shortening

Let  $X^0, X^1 : [t_0, t_1] \times [0, 1] \rightarrow \mathbb{R}^2$  be two moving curves that are embedded at all time. Assume  $X^1$  evolves by Curve Shortening, and assume that initially  $X^1$  lies in the closed region enclosed by  $X^0$ , i.e. for all  $p \in [0, 1]$  the point  $X^1(t_0, p)$  lies in the region enclosed by the simple curve  $p \mapsto X^0(t_0, p)$ .

Assume furthermore that at each time  $t_* \in [t_0, t_1]$  the area enclosed by  $p \mapsto X^0(t_*, p)$  is at least  $2\pi(t_1 - t_*)$ . By the Gage-Hamilton-Grayson theorem this guarantees that the solution to Curve Shortening starting at  $X^0(t_*, \cdot)$  exists until time  $t_1$ .

We now consider two homotopies  $A$  and  $B$  of evolving curves. The first is the homotopy defined in the proof in the previous section 3.3, i.e. for each  $\epsilon \in [t_0, t_1]$  we consider the solution  $(t, p) \mapsto X(\epsilon, t, p)$  to Curve Shortening defined for  $t \in [\epsilon, t_1]$  and starting from  $X(\epsilon, \epsilon, p) = X^0(\epsilon, p)$ . Our first homotopy is then the family of final curves  $A(\epsilon, p) = X(\epsilon, t_1, p)$  of these solutions. In section 3.3 we showed that the length of the homotopy  $A$  is bounded from above by

$$\ell(A) = \int_{t_0}^{t_1} \int_0^1 \|A_\epsilon(\epsilon, p)\| ds d\epsilon \leq \int_{t_0}^{t_1} \int_0^1 \|(\partial_t X^0)^\perp - X_{ss}^0\| ds dt.$$

The second homotopy is constructed by evolving a homotopy between the two initial curves  $p \mapsto X^j(t_0, p)$  ( $j = 0, 1$ ). Since  $X^1$  initially lies inside  $X^0$  we can choose the homotopy  $(\epsilon, p) \mapsto \bar{X}(\epsilon, t_0, p)$  so that its length is exactly the area of the region between the two initial curves, and so that the curve  $p \mapsto \bar{X}(\epsilon, t_0, p)$  lies inside the curve  $p \mapsto \bar{X}(\epsilon', t_0, p)$  if  $0 \leq \epsilon \leq \epsilon' \leq 1$ . Given this initial homotopy let  $(t, p) \mapsto \bar{X}(\epsilon, t, p)$  be the solution to Curve Shortening starting at  $\bar{X}(\epsilon, t_0, p)$ . Since

all initial curves enclose  $X^1(t_0, p)$  the corresponding solutions exist for  $t \in [t_0, t_1]$ , and possibly longer. Our second homotopy is now  $B(\epsilon, p) = \bar{X}(\epsilon, t_1, p)$ . As explained in section 3.3, the length of the homotopy  $B$  is bounded by the length of the initial homotopy  $(\epsilon, p) \mapsto \bar{X}(\epsilon, t_0, p)$ . We had chosen this initial homotopy so that its length is exactly the area between the two curves  $p \mapsto X^0(t_0, p)$  and  $p \mapsto X^1(t_0, p)$ .

By concatenating the two homotopies  $A$  and  $B$  we obtain a combined homotopy  $A\#B$  between  $p \mapsto X^0(t_1, p)$  and  $p \mapsto X^1(t_1, p)$ . The length of this homotopy is bounded by

$$\begin{aligned} \ell(A\#B) = \ell(A) + \ell(B) &\leq \text{Area between } X^0(t_0, \cdot) \text{ and } X^1(t_1, \cdot) \\ &+ \int_{t_0}^{t_1} \int_0^1 \|(\partial_t X^0)^\perp - X_{ss}^0\| \, ds \, dt. \end{aligned}$$

If  $\Omega^0(t)$  and  $\Omega^1(t)$  are the regions enclosed by  $X^0(t, \cdot)$  and  $X^1(t, \cdot)$ , then the homotopy  $A\#B$  between the two curves at time  $t_1$  must pass through each point in the interior of the symmetric difference  $\Omega^0(t_1) \Delta \Omega^1(t_1)$ , as one sees by considering the winding numbers of the curves in the homotopy around any point in  $\Omega^0(t_1) \Delta \Omega^1(t_1)$ . It follows that the area of  $\Omega^0(t_1) \Delta \Omega^1(t_1)$  is a lower bound for the length of the homotopy  $A\#B$ , and thus we conclude that

$$\begin{aligned} \text{Area of } \Omega^0(t_1) \Delta \Omega^1(t_1) &\leq \\ \text{Area between } X^0(t_0, \cdot) \text{ and } X^1(t_1, \cdot) &+ \int_{t_0}^{t_1} \int_0^1 \|(\partial_t X^0)^\perp - X_{ss}^0\| \, ds \, dt. \quad (3.34) \end{aligned}$$

### 3.4 Convergence

In this section, we obtain uniform curvature bounds on a sequence of “really old solutions”  $\{j(t)\}$  and extract a subsequence of solutions that converges locally smoothly to an ancient solution of curve shortening flow (CSF).

There exists a  $T$  such that for any  $T^* > T$ , the curvatures  $|\kappa_j|$  of the “really old solutions”  $\{j(t)\}_j$  are bounded independently of  $j$  on the interval  $[-T^* - 1/4, -T^*]$ .

The strategy to obtain these bounds is as follows: a) decompose an element  $j(t)$  of this sequence into the union of several graph representations, b) use the  $L^1$  bound on the error to obtain  $L^\infty$  estimates for these graphs, and c) apply the standard estimates for divergence-form quasilinear parabolic equations to establish a uniform curvature bound.

Let  $T^* > 0$  be a large positive number, which may be increased as necessary throughout this section. The obvious candidates for the sequence of “really old solutions”  $\{j(t)\}$  are the CSF solutions defined on  $[-j, -T^*]$  with the initial condition  $j(-j) =_* (-j)$ . At any time  $t \in [-j, -T^*]$ , the unsigned area enclosed by the curves  $_*(t)$  and  $j(t)$  is bounded by the quantity

$$\int_{-j}^t \int_{_*(\tau)} |V - \kappa| ds d\tau.$$

By the results of the previous sections, this quantity is in  $L^1$  and given  $\epsilon > 0$ , we can find  $T^* > 0$  such that

$$(T^*) = \int_{-\infty}^{-T^*} \int_{_*(\tau)} |V - \kappa| ds d\tau < \epsilon.$$

This estimate gives a uniform bound on the unsigned area between  $j(t)$  and  $_*(t)$  for any sufficiently large  $j$  and  $t < -T^*$ , whenever defined.

For simplicity of calculation, we consider an alternative sequence of “really old solutions”  $\{j(t)\}$ , called “square-profile approximations”, which do not satisfy the initial condition  $j(-j) =_* (-j)$ . They are constructed as follows: Let  $[\vartheta_*^-(t), \vartheta_*^+(t)]$  be the interval of polar angles that contain the approximate solution at a given time  $t$ . Let  $r = (\theta)$  be the central branch of the Yin-Yang foliation, so the Yin-Yang solution is given by the two branches  $r = (\theta + \pi/2 - t)$  and  $r = (\theta - \pi/2 - t)$ . We define  $j(t)$  to be the solution of curve shortening which at time  $t = -j$  is given by

- Two arms of the Yin-Yang soliton  $r = (\theta - \pi/2 - t) = (\theta - \pi/2 + j)$ , and  $r = (\theta + \pi/2 - t) = (\theta + \pi/2 + j)$  truncated at  $\theta = \vartheta_*^+(-j)$ ;
- a straight line segment connecting the two arms of the Yin-Yang soliton. This segment is part of the ray  $\theta =_*^+(-j)$ .

Notice that the square-profile approximations  $j(t)$  enclose the CSF solutions starting from  $_*(-j)$ , and that the area bounded by these solutions stays constant along the flow, for all  $t \in [-j, -T^*]$ . The area  $A_j$  between  $j(-j)$  and  $_*(-j)$  is small and goes to zero as  $j \rightarrow \infty$ . Thus, the area between the old solution  $j(t)$  with the “square initial data” and the approximate solution  $_*(t)$  is bounded by

$$\text{Area}(j(t),_*(t)) \leq A_j + \int_{-j}^t \int_{_*(\tau)} |V - \kappa| ds d\tau.$$

In order to improve these area bounds to  $L^\infty$  bounds, we will use the geometry of the  $j(t)$  and several properties of CSF. In particular, we often appeal to the maximum principle and the following Sturmian property for intersections of curve shortening flows.

Consider two CSF solutions  $\gamma_0, \gamma_1 : [T_1, T_2) \times [0, 1] \rightarrow \mathbb{R}^2$ , for which

$$\partial\gamma_0(t) \cap \gamma_1(t) = \partial\gamma_1(t) \cap \gamma_0(t) = \emptyset$$

holds for any  $t \in [T_1, T_2)$ . Then the number of intersections of  $\gamma_0(t)$  and  $\gamma_1(t)$  is a finite and non-increasing function of  $t \in (T_1, T_2)$ . It decreases whenever  $\gamma_0$  and  $\gamma_1$  have a tangency.

There is a useful related theorem for inflections points.

Let  $\gamma : [T_1, T_2) \times S^1 \rightarrow \mathbb{R}^2$  be a solution of CSF. Then, for any  $t \in (T_1, T_2)$ ,  $\gamma(t)$  has at most a finite number of inflection points, and this number does not increase with time. In fact, it drops whenever the curvature  $\kappa$  has a multiple zero.

While the curves  $j(t)$  are not convex, we do have a one sided curvature bound. If  $\kappa$  is the curvature of a counterclockwise oriented parametrization  $X$  of the curves  $j(t)$ , then

$$\kappa - \langle X, X_s \rangle > 0.$$

Assuming that the parametrization  $X$  is normal ( $X_t \perp X_s$ ), the curvature evolves by

$$\kappa_t = \kappa_{ss} + \kappa^3.$$

A short computation using  $X_t = X_{ss}$  and  $\|X_s\| = 1$  shows that

$$(\partial_t - \partial_s^2)\|X\|^2 = 2\langle X_t, X \rangle - \langle X_{ss}, X \rangle - 2\|X_s\|^2 = -2.$$

Differentiating with respect to arclength, using the commutator  $[\partial_t, \partial_s] = \kappa^2 \partial_s$ , and also  $\partial_s \|X\|^2 = 2\langle X, X_s \rangle$  we get

$$\partial_t \langle X, X_s \rangle = \partial_s^2 \langle X, X_s \rangle + \kappa^2 \langle X, X_s \rangle.$$

Hence  $\kappa$  and  $\langle X, X_s \rangle$  satisfy the same linear equation. Therefore  $\sigma = \kappa - \langle X, X_s \rangle$  also satisfies

$$\sigma_t = \sigma_{ss} + \kappa^2 \sigma.$$

The quantity  $\sigma$  vanishes on the rotating soliton (see the appendix).

The square-profile initial curves  $j(-j)$  consist of two arcs. One is the Yin-Yang soliton, so on this arc we have  $\sigma = 0$ . The other arc is the radial line segment on the ray  $\theta = \vartheta_*^+(-j)$ . On this segment we clearly have  $\kappa = 0$ . Since we orient  $j$  counterclockwise,  $X$  and  $X_s$  are parallel with opposite directions; i.e.  $\langle X, X_s \rangle > 0$ . Hence  $\sigma > 0$  on the line segment. Finally, the initial curve  $j(-j)$  is not smooth, having two corners where the line segment and Yin-Yang arms meet. If one rounds these corners off by replacing them with small circle arcs with radius  $\rho \ll 1$ , then the curvature of these arcs will be  $\kappa = \rho^{-1} \gg 1$ , so that  $\sigma > 0$  on the circular arcs, provided  $\rho$  is sufficiently small. The resulting curve has  $\sigma = 0$  on the Yin-Yang arms, and  $\sigma > 0$  on the line segment, as well as the small circular arcs. The solution to CS starting from the modified initial curve therefore has  $\sigma > 0$ . Letting  $\rho \searrow 0$  we conclude that  $\sigma > 0$  also holds on  $j(t)$ .

With Theorem 3.4, we can decompose the solutions  $j(t)$  into exactly two graphs over the polar angle parameter.

For any  $t \in (-j, -T^*]$ , there is an interval  $[\bar{j}(t), \bar{j}^+(t)]$  such that the curve  $j(t)$  can be written as the union of two graphs of polar functions,  $R_j^-(\theta, t)$  and  $R_j^+(\theta, t)$  defined for  $\theta \in [\bar{j}(t), \bar{j}^+(t)]$ . The functions  $t \mapsto \bar{j}^-(t)$  and  $t \mapsto \bar{j}^+(t)$  are strictly increasing and decreasing, respectively.

By the maximum principle, the “really old solutions”  $j(t)$  will be contained inside of the Yin-Yang curve. The Sturmian property, Theorem 3.4, tells us that the number of intersections of  $j(t)$  and the rays  $\theta = \theta_0 \in$  is non-increasing, and only decreases when there is a tangency. This implies that the desired graph decomposition exists. These two graphs are bounded above and below by the branches of the Yin-Yang soliton on their polar interval of definition,  $[\bar{j}(t), \bar{j}^+(t)]$ .

Similarly, we can always write each  $j(t)$  as a union of two graphs taking values in  $\theta$ , the polar angle. Recall that the images of  $(\theta - t)$  for  $t \in [-\pi, \pi)$  foliate the punctured plane  $\mathbb{R}^2 \setminus \{0\}$ . See Figure 3.2.

For all  $t$ ,  $j(t)$  can be decomposed into two graphs of two functions which take leaves of the foliation as inputs and have their range in the set of polar angles. More specifically, for  $T \ll 0$  there exist  $y_{j,1}, y_{j,2} : (-\infty, T] \rightarrow (-\frac{1}{2}\pi, \frac{1}{2}\pi)$  and functions

$$\Theta_j^\pm : \{(t, y) : t < T, y_{j,1}(t) < y < y_{j,2}(t)\} \rightarrow$$

such that the very old solution  $j(t)$  is the union of the two curves

$$Y(\Theta_j^\pm(t, y), t, y) = (\Theta_j^\pm(t, y) - t + y)1(\theta)$$

((3.7)).

The initial square-profile curve  $j(-j)$  is tangent to the graphs of  $(\theta \pm \pi/2 + j)$  and intersects the graphs of  $(\theta + y + j)$ ,  $y \in (-\pi/2, \pi/2)$  twice: once at the origin and once on the line segment connecting the two branches of  $j(-j)$ . Then, by the Sturmian theorem, for all subsequent  $t > -j$ ,  $j(t)$  can be split into two graphs corresponding to the “upper” and “lower” intersection points with the leaves of the Yin-Yang foliation. At each time  $t$ , these graphs split at two unique leaves of the foliation, marked by values  $y_{j,1}(t), y_{j,2}(t) \in (-\pi/2, \pi/2)$ , so that  $j(t)$  is tangent to the curves  $\{r = (\theta + y_{j,1}(t) - t)\}$  and  $\{r = (\theta + y_{j,2}(t) - t)\}$ . We know that these two points are unique since a greater number of tangencies would introduce more than two intersection points for other curves  $\{r = (\theta + y - t)\}$ . We call the coordinate system  $(y, \theta) \in (-\pi, \pi) \times (0, \infty)$  the “Yin-Yang polar coordinate system” and denote the two functions giving the upper and lower graphs comprising  $j(t)$  by  $\Theta_j^-(y, t)$  and  $\Theta_j^+(y, t)$  respectively, defined on the interval  $(y_{j,1}(t), y_{j,2}(t)) \subset (-\pi/2, \pi/2)$ .

There exist  $T < 0$  and  $C > 0$  such that  $\vartheta_j^+(t) \leq \vartheta_*^+(t) + C$  for all  $j \in \mathbb{Z}$  and all  $t \in [-j, -T]$ .

Assume that  $\epsilon < \pi/16$ .

For any  $t \in [-j, T]$  at which  $\vartheta_j^+(t) >_*^+(t)$  we consider the area  $j(t)$  of the “really old solution”  $j(t)$  inside the polar interval  $[_*^+(t), \vartheta_j^+(t)]$ , where  $\vartheta_j^+(t)$  and  $_*^+(t)$  are the endpoints of the intervals of definition of the approximate solution  $_*(t)$  and  $j(t)$  respectively. This area measures the “tail” of the  $j(t)$  that may form between the tip of  $j(t)$  and the tip of  $_*(t)$ . Note that the area  $j(t)$  is bounded above by the error

$$j(t) \leq A_j + (T^*) = A_j + \int_{-\infty}^{-T^*} \int_{_*(\tau)} |V - \kappa| ds d\tau < \epsilon.$$

To calculate this area, first consider the function  $\Theta_{t,j}(y) := \max\{\Theta_j^+(y, t) - \vartheta_*^+(t), 0\}$  over the interval  $(y_{j,1}(t), y_{j,2}(t))$ . Then in the  $(\theta, y)$  “Yin-Yang coordinates,” we can integrate to find the area:

$$j(t) = \int_{y_{j,1}(t)}^{y_{j,2}(t)} \int_{\vartheta_*^+}^{\Theta_{t,j}(y) + \vartheta_*^+} (\theta + y - t) \det(DT) d\theta dy,$$

where  $T : (0, \infty) \times (-\pi/2, \pi/2) \rightarrow (0, \infty) \times (0, \infty)$  is the coordinate transformation given by  $T(\theta, y) = (\theta, (\theta + y - t))$ . Clearly,  $\det T =' (\theta + y - t)$ , so

$$j(t) = \int_{y_{j,1}(t)}^{y_{j,2}(t)} \int_{\vartheta_*^+}^{\Theta_{t,j}(y) + \vartheta_*^+} (\theta + y - t)' (\theta + y - t) d\theta dy \approx \int_{y_{j,1}(t)}^{y_{j,2}(t)} \Theta_{t,j}(y) dy,$$

by the asymptotic expansions in (3.1).

We argue that given a small  $\delta > 0$ , it is possible to pick an angle  $\theta_0$  independent of  $j$  such that the measure  $|\{y : \Theta_{t,j}(y) > \theta_0\}| < \delta$ . Indeed, it follows from

$$\theta_0 |\{y : \Theta_{t,j}(y) > \theta_0\}| \leq \int_{y_{j,1}(t)}^{y_{j,2}(t)} \Theta_{t,j}(y) dy < \epsilon$$

that if  $\theta_0 < \frac{\epsilon}{\delta}$ , then  $|\{y : \Theta_{t,j}(y) > \theta_0\}| < \delta$  holds for all  $t, j$ .

The two points intersection of  $j(t)$  with the ray  $\theta = \vartheta_*^+(t) + \theta_0$  are

$$P^\pm(t) = R_j^\pm(\vartheta_*^+(t) + \theta_0, t).$$

Let  $\gamma(t)$  be the arc on  $j(t)$  on which  $\theta \geq \vartheta_*^+(t) + \theta_0$ , and whose endpoints therefore are  $P^\pm(t)$ . Consider the area  $A(t)$  of the region enclosed by  $\gamma(t)$  and the line segment connecting  $P^\pm(t)$ . This area changes because the arc  $\gamma(t)$  moves, and also because the line segment  $P^-P^+$  moves. The rate of change is therefore the sum of  $-\int_{\gamma(t)} \kappa ds$  and the rate at which the segment  $P^-P^+$  sweeps out area.

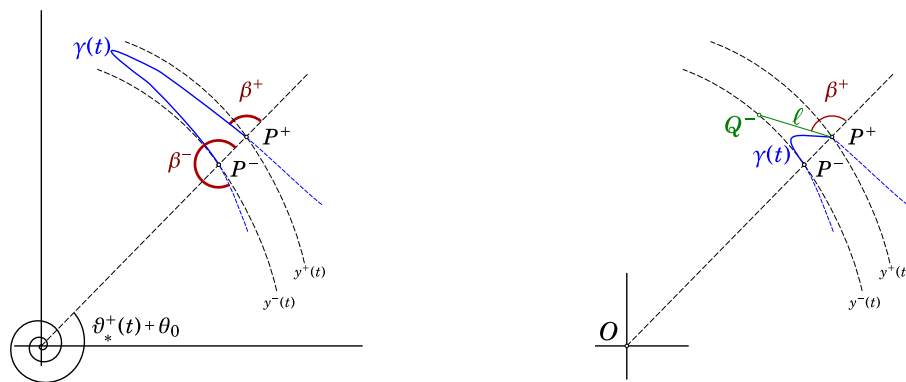


Figure 3.3: **Left:** The arc  $\gamma(t)$ . **Right:** the angles  $\beta^\pm$

If  $\phi : \gamma(t) \rightarrow$  is the tangent angle along the arc (i.e.  $X_s = 1(\phi)$ ), then the curvature integral is

$$\int_{\gamma(t)} \kappa ds = \phi_{P^-(t)} - \phi_{P^+(t)}.$$

The line segment  $P^+P^-$  moves with angular velocity  $\frac{d}{dt}\vartheta_*^+(t)$  and therefore adds area to the region enclosed by  $\gamma(t)$  at the rate

$$\frac{1}{2} \left\{ (R_j^+)^2 - (R_j^-)^2 \right\} \frac{d\vartheta_*^+(t)}{dt}$$



in which  $R_j^\pm$  are evaluated at  $\theta = \vartheta_*^+(t) + \theta_0$ . Our construction of the cap implies that  $\vartheta_*^+(t) = -t + o(1)$ , and that this relation may be differentiated:  $\frac{d}{dt}\vartheta_*^+(t) = -1 + o(t)$ .

The radii  $R_j^\pm(\vartheta_*^+(t) + \theta_0, t)$  are given in terms of their Yin-Yang coordinates  $y^\pm(t)$  via

$$R_j^\pm(\vartheta_*^+(t) + \theta_0, t) = (\vartheta_*^+(t) + \theta_0 - t + y^\pm).$$

It follows that at  $\vartheta_*^+(t) + \theta_0$

$$\begin{aligned} \frac{1}{2} \left\{ (R_j^+)^2 - (R_j^-)^2 \right\} &= (\vartheta_*^+(t) + \theta_0 - t + y^+)^2 - (\vartheta_*^+(t) + \theta_0 - t + y^-)^2 \\ &= '(y^+ - y^-) \end{aligned}$$

in which  $'$  are evaluated at  $\vartheta_*(t) + \theta_0 + \tilde{y}$  for some  $\tilde{y} \in [y^-, y^+]$  that is provided by the mean value theorem. The asymptotics of imply that  $' = 1 + o(1) < 2$ . Our choice of  $\theta_0$  was such that  $0 < y^+ - y^- \leq \delta$ . Hence

$$\left| \frac{1}{2} \left\{ (R_j^+)^2 - (R_j^-)^2 \right\} \frac{d\vartheta_*^+(t)}{dt} \right| \leq 2\delta.$$

In total, the rate at which the area  $A(t)$  enclosed by the arc  $\gamma(t)$  grows is bounded by

$$\frac{dA}{dt} \leq -(\phi_{P^-(t)} - \phi_{P^+(t)}) + 2\delta.$$

We estimate the change in tangent angle across the arc  $\gamma(t)$ . Let  $\beta^+$  be the counterclockwise angle from the ray  $\theta = \vartheta_*^+(t) + \theta_0$  to the tangent  $X_s$  to  $\gamma$  at  $P^+$ , and similarly, let  $\beta^-$  be the counterclockwise angle from the same ray to the tangent to  $\gamma$  at  $P^-$  (see Figure 3.3). We have  $0 < \beta^+ < \pi < \beta^-$  and  $\phi_{P^-} - \phi_{P^+} = \beta^- - \beta^+$ .

Recall that  $\kappa - \langle X, X_s \rangle > 0$  along  $j(t)$ . Since  $\kappa = \phi_s$  and  $\langle X, X_s \rangle = \frac{1}{2} \frac{d}{ds} r^2$ , where  $r = \|X\|$ , it follows that  $\phi - \frac{1}{2} r^2$  increases as one traverses  $\gamma$  from  $P^+$  to  $P^-$ . Thus, at any point with polar coordinates  $(\theta, r)$  on  $\gamma$  one has

$$\phi > \phi_{P^+} + \frac{1}{2}(r^2 - r_{P^+}^2).$$

The lowest value  $r$  has on  $\gamma$  occurs at the point  $P_-$ , and we have just shown that  $r_{P^+}^2 - r_{P_-}^2 \leq 2\delta$ . Hence  $\phi > \phi_{P^+} - 2\delta$  on the entire arc  $\gamma$ .

It follows that if  $\beta^+ > \frac{3}{4}\pi$ , then the angle between the tangent to  $\gamma$  and the ray  $OP^+$  ( $O$  is the origin) will always be at least  $\frac{3}{4}\pi - 2\delta$ , i.e. more than  $\frac{5}{8}\pi$ , provided we choose  $\delta < \frac{\pi}{16}$ .

Consider the line  $\ell$  through  $P^+$  whose angle with  $OP^+$  is  $\frac{5}{8}\pi$ . The euclidean distance between  $P^-$  and  $P^+$  is  $r_{P^+} - r_{P^-} \leq 2\delta/(r_{P^-} + r_{P^+}) \leq C\delta|t|^{-1/2}$ , since  $r_{P^+} > r_{P^-} \gtrsim |t|^{1/2}$ .

At this scale the Yin-Yang leaves will be almost straight lines near  $P^\pm$ , so that the line  $\ell$  then intersects the Yin-Yang leaf with  $y = y_-(t)$  at a point  $Q^-(t)$ , also at a distance  $d(P^-, Q^-) \lesssim \delta|t|^{-1/2}$ .

$$\vartheta_j^+(t) - \vartheta_*^+(t) - \theta_0 \sim \frac{d(P^-, Q^-)}{r_{P^-}} \lesssim \delta|t|^{-1} \ll \theta_0.$$

Hence the largest polar angle on  $\gamma$  will be at most

$$\vartheta_j^+(t) \leq \vartheta_*^+(t) + \theta_0 + \delta|t|^{-1} \leq \vartheta_*^+(t) + 2\theta_0.$$

Thus we find that if  $|t|$  is sufficiently large, then either  $\vartheta_j(t) < \vartheta_*(t)$ , or else  $\beta^+ < \frac{5}{8}\pi$ .

In the latter case the area enclosed by  $\gamma(t)$  decreases faster than

$$\frac{dA}{dt} \leq -\beta^- + \beta^+ + 2\delta \leq -\pi + \frac{5}{8}\pi + 2\delta = -\frac{3}{8}\pi + 2\delta < -\frac{\pi}{4},$$

again, assuming  $\delta < \pi/16$ .

We now finally prove that  $\vartheta_j^+(t) - \vartheta_*^+(t)$  is uniformly bounded for all  $t \in [-j, T]$  and  $j$ .

At  $t = -j$  we have  $\vartheta_j^+(t) < \vartheta_*^+(t) + \theta_0$ , by definition of the initial curve  $j(-j)$ . Hence, if at any time  $t_1 < T$  one has  $\vartheta_j^+(t) > \vartheta_*^+(t) + 2\theta_0$ , then there is a largest interval  $(t_2, t_3) \ni t_1$  on which  $\vartheta_j^+(t) > \vartheta_*^+(t) + 2\theta_0$ . In particular, at  $t = t_2$  one has  $\vartheta_j^+(t) = \vartheta_*^+(t) + 2\theta_0$ .

Define the arc  $\gamma(t)$  as above. Its enclosed area is at most  $\varepsilon$ , where we may assume that  $\varepsilon < \pi/4$ . During the time interval  $(t_2, t_3)$  the area decreases at a rate of at least  $\pi/4$ , and therefore the length  $t_3 - t_2$  of the time interval cannot exceed  $\varepsilon/(\pi/4) = 1$ . At time  $t = t_2$  we had  $\vartheta_j^+(t) = \vartheta_*^+(t) + 2\theta_0$ . Since  $\vartheta_j^+(t)$  is a nonincreasing function, we have throughout  $(t_2, t_3)$

$$\begin{aligned} \vartheta_j^+(t) - \vartheta_*^+(t) + 2\theta_0 &\leq \vartheta_j^+(t_2) - \vartheta_*^+(t) + 2\theta_0 \\ &\leq \vartheta_j^+(t_2) - \vartheta_*^+(t_2) + 2\theta_0 + \vartheta_*^+(t) - \vartheta_*^+(t_2) \\ &\leq \vartheta_*^+(t_3) - \vartheta_*^+(t_2). \end{aligned}$$

Since  $\frac{d}{dt}\vartheta_*^+(t) = 1 + o(1)$  we find that

$$\vartheta_j^+(t) - \vartheta_*^+(t) + 2\theta_0 \leq 1 + o(1) < 2$$

for all  $t \in (t_2, t_3)$ .

To summarize, we can now decompose every very-old solution  $j(t)$ , for  $t \in [-T^* - 1/2, -T^*]$  into four graphs in two different coordinate systems,  $R_j^\pm(\theta, t)$  in polar coordinates, and  $\Theta_j^\pm(y, t)$  in Yin-Yang polar coordinates.

[Curvature bounds] There exist  $T < 0$  such that for any  $T' < T$  the lengths  $L_j(t)$  and curvatures of  $j(t)$  are uniformly bounded for all  $j$  and all  $t \in [T', T - 2]$ . The length bounds follow from the fact that in  $(y, \theta)$  coordinates each  $j(t)$  is contained in a uniformly bounded rectangle  $|y| \leq \pi/2$ ,  $-t \leq \theta \leq \vartheta_*^+(t) + 2\theta_0 + 2$ , and the fact that  $j(t)$  decomposes into four segments on each of which both  $y$  and  $\theta$  are monotone.

Consider a given value  $T' < T$ . Assume that our Lemma fails, and that along some subsequence  $j$  the maximal curvature of  $j(t)$  with  $t \in [T', T - 2]$  becomes unbounded.

For  $t \in [T' - 2, T]$  the lengths  $L_j(t)$  of  $j(t)$  are uniformly bounded by some  $L > 0$ .

It follows that

$$\int_{T'-2}^{T'-1} \int_{j(t)} \kappa^2 ds dt = [-L_j(t)]_{T'-2}^{T'-1} < L.$$

Therefore, there is a sequence  $t_j \in [T' - 2, T' - 1]$  such that

$$\int_{j(t_j)} \kappa^2 ds < L.$$

By a Sobolev embedding theorem this implies that the curves  $j(t_j)$  are uniformly  $C^{1,1/2}$ , i.e. they are continuously differentiable, and their tangent angles  $\phi_j$  are uniformly Hölder continuous—in fact, for any two points at arclength coordinates  $s_1, s_2$  in  $j(t_j)$  one has

$$|\phi(s_2) - \phi(s_1)| = \left| \int_{s_1}^{s_2} \kappa ds \right| \leq \sqrt{s_2 - s_1} \sqrt{\int_{s_1}^{s_2} \kappa^2 ds} \leq \sqrt{L} \sqrt{s_2 - s_1}.$$

It follows that all  $j(t_j)$  are uniformly locally Lipschitz curves. Now consider the solutions to curve shortening with  $j(t_j)$  as initial data, i.e. consider  $\tilde{j}(t) = j(t_j + t)$ . These solutions all exist for  $0 \leq t \leq T - t_j \geq T - T'$ . Supposing that along some subsequence of  $t_j$  the curvatures of the  $\tilde{j}$  are not bounded for  $1 \leq t \leq T - T'$ , we pass to a further subsequence for which the initial curves  $\tilde{j}(0)$  converge in  $C^1$  to some limit curve  $\tilde{*}$ . The enclosed areas of the  $\tilde{j}(0)$  then also converge, and hence, by Grayson's theorem (Grayson, 1986) the evolution by Curve Shortening  $*(t)$  starting

from  $*$  exists for  $0 \leq t \leq T - T'$ . By continuous dependence on initial data it follows that the solutions  $\tilde{y}_j(t)$  converge in  $C^\infty$  to  $*(t)$  on any time interval  $[\delta, T - T']$  with  $\delta > 0$ . This implies that the curvatures of the  $\tilde{y}_j(t)$  are uniformly bounded for  $t \in [1, T - T']$ , which then implies that the curvatures of  $y_j(t)$  are uniformly bounded after all for  $t \in [t_j + 1, t_j + T - T'] \subset [T', T - 2]$ .

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*Appendix A*

SUPPORTIVE MATERIALS FOR CHAPTER III

**A.1 The Yin-Yang Soliton**

Hungerbühler and Smoczyk (Hungerbühler and Smoczyk, 2000) proved uniqueness and existence of a rotating soliton for curve shortening that contains the origin. See also Halldorson (Halldorsson, 2012) and Altschuler et.al. (D. J. Altschuler et al., 2013). Here we derive its more detailed asymptotic behavior, which we use in our construction of the approximate solution.

**The Yin–Yang soliton in polar coordinates**

For an evolving family of curves written in polar coordinates  $X(t, \theta) = r(t, \theta)1(\theta)$  the Curve Shortening Deficit is

$$(V - \kappa) ds = \left\langle X_t - \frac{X_{\theta\theta}}{\|X_\theta\|^2}, JX_\theta \right\rangle d\theta = \left\{ -rr_t + \frac{r(r_{\theta\theta} - r) - 2r_\theta^2}{r^2 + r_\theta^2} \right\} d\theta.$$

It follows that  $X$  is a solution of CSF if and only if  $r(\theta, t)$  satisfies

$$r \frac{\partial r}{\partial t} = \frac{rr_{\theta\theta} - r_\theta^2}{r^2 + r_\theta^2} - 1 = \frac{\partial}{\partial \theta} \left( \arctan \frac{r_\theta}{r} \right) - 1. \quad (\text{A.1})$$

If we look for solutions of the form  $r(\theta, t) = (\theta - t)$  we get an ODE for  $(\theta)$

$$-\theta' = \frac{\theta'^2}{2 + \theta'^2} - 1. \quad (\text{A.2})$$

Hungerbühler and Smoczyk (Hungerbühler and Smoczyk, 2000) observed that this equation can be integrated once. By suitably rotating the curve around the origin we can ensure that the resulting integration constant vanishes, and we therefore have

$$\frac{1}{2}(\theta')^2 - \theta + \arctan \frac{\theta'}{\theta} = 0. \quad (\text{A.3})$$

We consider the soliton that passes through the origin. When this happens  $\theta \rightarrow 0$  and  $\theta' \rightarrow \infty$ , so that (A.3) implies  $\theta \rightarrow \pi/2$ . The function  $(\theta)$  is therefore defined for all  $\theta > \pi/2$ , and, as proved by Hungerbühler and Smoczyk,  $\theta'(\theta) > 0$  for all  $\theta > \pi/2$ .

### Asymptotic expansion of

We now show that  $(\theta)$  has an asymptotic expansion of the form

$$(\theta) = (2\theta)^{1/2} \left\{ 1 + \frac{c_1}{2\theta} + \frac{c_2}{(2\theta)^2} + \cdots + \frac{c_N}{(2\theta)^N} + (\theta^{-N-1}) \right\} \quad (\theta \rightarrow \infty) \quad (\text{A.4})$$

for any  $N \in \mathbb{N}$ . These expansions can be differentiated any number of times. The coefficients  $c_j$  can be computed by substituting the expansions in (A.2) and recursively solving for  $c_j$ . In particular, one finds  $c_1 = 0$ ,  $c_2 = -1$ , and  $c_3 = \frac{11}{3}$  so that

$$(\theta) = \sqrt{2\theta} - \frac{1}{(2\theta)^{3/2}} + \frac{11}{3} \frac{1}{(2\theta)^{5/2}} + (\theta^{-7/2}) \quad (\theta \rightarrow \infty) \quad (\text{A.5})$$

$$'(\theta) = \frac{1}{\sqrt{2\theta}} + \frac{3}{(2\theta)^{5/2}} - \frac{55}{3} \frac{1}{(2\theta)^{7/2}} + (\theta^{-9/2}) \quad (\theta \rightarrow \infty). \quad (\text{A.6})$$

This implies that the quantities  $R = R(t) = (-2t)$ ,  $\varepsilon = 1/R$ , and  $R_\theta = R_\theta(t) = '(-2t)$ , which we use in the construction of the cap, have expansions in powers of  $\tau = -4t$ , given by

$$\left\{ \begin{array}{l} R = (-2t) = \sqrt{\tau} \left\{ 1 - \tau^{-2} + \frac{11}{3} \tau^{-3} + (\tau^{-4}) \right\} \\ \varepsilon = \frac{1}{R} = \frac{1}{\sqrt{\tau}} \left\{ 1 + \tau^{-2} - \frac{11}{3} \tau^{-3} + (\tau^{-4}) \right\}. \\ R_\theta = '(-2t) = \frac{1}{\sqrt{\tau}} \left\{ 1 + 3\tau^{-2} - \frac{55}{3} \tau^{-3} + (\tau^{-4}) \right\}. \end{array} \right. \quad (\text{A.7})$$

*Proof of (A.4)* Consider the quantity  $u(\theta) = \theta - \frac{1}{2}(\theta)^2$ . Since  $'(\theta) > 0$  for all  $\theta$  it follows from (A.3) that

$$0 < u(\theta) < \frac{\pi}{2} \text{ for all } \theta > \frac{\pi}{2}. \quad (\text{A.8})$$

Directly differentiating  $u = \theta - \frac{1}{2}\theta^2$  and using (A.3) we find an equation for  $u$ ,

$$u'(\theta) = 1 - 2(\theta - u(\theta)) \tan u(\theta), \quad (\text{A.9})$$

i.e.

$$u' + 2\theta u = F(\theta, u)1 + 2\theta(u - \tan u) + 2u \tan u.$$

We use induction to show that  $u(\theta)$  has an expansion of the form

$$u(\theta) = \frac{u_1}{\theta} + \frac{u_2}{\theta^2} + \cdots + \frac{u_N}{\theta^N} + (\theta^{-N-1}) \quad (\text{A.10})$$

for any  $N \in \mathbb{N}$ .



Begin with the case  $N = 0$ . We know that  $0 < u < \pi/2 < \theta$ , so that  $u < \tan u$ , and hence

$$u'(\theta) = 1 - 2(\theta - u) \tan u \leq 1 - 2(\theta - u)u = 1 - 2\theta u + 2u^2$$

which implies

$$u' + 2\theta u \leq 1 + 2u^2 \leq 1 + \frac{\pi^2}{2} = C_0.$$

Multiply with  $e^{\theta^2}$ , and integrate

$$0 < u(\theta) \leq u(\theta_0)e^{\theta_0^2 - \theta^2} + C_0 \int_{\theta_0}^{\theta} e^{\xi^2 - \theta^2} d\xi = O(\theta^{-1}) \quad (\theta \rightarrow \infty),$$

Thus the case  $N = 0$  holds.

For the induction step we expand  $\tan u$  in a Taylor series,

$$\tan u = u + \frac{u^3}{3} + \dots = u + \sum_{k \geq 1} c_k u^{2k+1}$$

and rewrite the equation for  $u$  as

$$F(\theta, u) = 1 + 2u^2 + \sum_{k \geq 1} 2(\theta - u)c_k u^{2k+1}.$$

Multiplying with  $e^{\theta^2}$  and integrating from some fixed  $\theta_0 > \pi/2$  we get

$$u(\theta) = e^{\theta_0^2 - \theta^2} u(\theta_0) + \int_{\theta_0}^{\theta} e^{\xi^2 - \theta^2} F(\xi, u(\xi)) d\xi. \quad (\text{A.11})$$

Repeated integration by parts leads to

$$\begin{aligned} \int_{\theta_0}^{\theta} e^{\xi^2 - \theta^2} \xi^{-k} d\xi = \\ \frac{1}{2} \theta^{-k-1} + \frac{k+1}{2} \theta^{-k-3} + \dots + \frac{(k+1) \cdots (k+2m-1)}{2^m} \theta^{-k-2m-1} + (\theta^{-k-2m-3}) \end{aligned} \quad (\text{A.12})$$

for all  $k, m \in \mathbb{N}$ . If we assume that  $u$  has an expansion up to  $(\theta^{-(N-1)})$ , then we also have expansions for  $u^2$  and  $F(\theta, u)$  up to  $(\theta^{-(N-1)})$ . Substitute these expansions in the integral equation (A.11) and use (A.12) to conclude that  $u$  has an expansion up to  $(\theta^{-N})$ , as claimed.

The expansion (A.10) for  $u$ , which we now have proved, implies  $(\theta) = \sqrt{2(\theta - u(\theta))}$  also has an asymptotic expansion in powers of  $\theta^{-1}$ .

Finally, while one cannot in general differentiate asymptotic expansions, one can integrate them. Thus if a function  $f(\theta)$  and its derivative  $f'(\theta)$  both have asymptotic expansions in powers of  $\theta^{-1}$ , then by integrating the expansion of  $f'$  one should get the expansion for  $f$ , up to a constant: this implies that the expansion of  $f'$  can be found by differentiating the expansion for  $f$ . We therefore only have to show that all derivatives of  $u(\theta)$  have expansions in powers of  $\theta^{-1}$ , which will then imply that the expansions (A.10) can be differentiated.

To find expansions for  $u^{(m)}$ , note that if  $u$  has an expansion with remainder  $(\theta^{-N-1})$ , then simple substitution in the differential equation (A.9) leads to an expansion for  $u'(\theta)$  with remainder  $(\theta^{-N})$ . Going further, one can differentiate (A.9)  $m - 1$  times and express  $u^{(m)}(\theta)$  in terms of  $u, u', u'', \dots, u^{(m-1)}$ . This implies that if one has an expansion in powers of  $\theta^{-1}$  of the first  $m - 1$  derivatives of  $u$ , then one also has an expansion for  $u^{(m)}$ . By induction it follows that all derivatives of  $u$  have such expansions.

Similar arguments also apply to the expansions of  $(\theta)$ .

### **Inversion of the expansion of**

The expansion (A.4), which expresses  $\theta$  as a function of  $(\theta)$ , implies that one can invert the function  $\theta \mapsto (\theta)$ , and that the inverse has an asymptotic expansion. It follows from (A.4) that

$$(\theta)^2 = (2\theta) \left\{ 1 + \frac{\bar{c}_1}{2\theta} + \frac{\bar{c}_2}{(2\theta)^2} + \cdots + \frac{\bar{c}_N}{(2\theta)^N} + (\theta^{-N-1}) \right\}$$

and hence

$$2\theta =^2 \left\{ 1 + \frac{\bar{c}_1}{2\theta} + \frac{\bar{c}_2}{(2\theta)^2} + \cdots + \frac{\bar{c}_N}{(2\theta)^N} + (\theta^{-N-1}) \right\}^{-1}.$$

Repeated substitution of this expansion in itself allows one to convert all powers of  $(2\theta)$  on the left into powers of  $^2$ , so that we have an expansion

$$2\theta =^2 \left\{ 1 + \frac{\tilde{c}_1}{2} + \frac{\tilde{c}_2}{4} + \cdots + \frac{\tilde{c}_N}{2^N} + ({}^{-2N-2}) \right\}$$

for certain coefficients  $\tilde{c}_i$ .