NETWORKS INVOLVING IDEAL TRANSFORMERS

Thesis by

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For the background material in Chapter I, I have relied on the work of Cherry and Kron; in Chapter II, Campbell, Foster and Mac-Neal; in Chapters III and IV, Belevitch and Cauer; Chapter V, Foster, Tellegen and Cauer; Chapter VI, Cauer; and Chapter VII, Dreher. The specific references are indicated in the text and listed at the end of the thesis. Some articles were made available through the courtesy of the library of the Bell Telephone Laboratory, and some others were furnished directly by the authors.

The line of attack leading to part of the results in Chapter VI and VII was visualized by my advisor, Professor Richard H. MacNeal. For orientation with regard to present-day developments in network theory, I am indebted to Bayard, Cherry, Tellegen, and to members of the network theory group at Bell Laboratory for the time each spent in conversation with me.

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ABSTRACT

The concept of the ideal transformer is presented in terms of flux relations contrasting a non-ideal transformer, which may be represented by an impedance matrix, to an ideal transformer which may not. The interchangeability of the considerations of a transformer, first, as a constraint on the currents (a "multiwinding" transformer) and, second, as a constraint on the voltages (a "multilimb" transformer) is formulated. Mesh and nodal analysis is extended to include networks involving ideal transformers by the use of Lagrange multipliers. These multipliers are eliminated from the equations by a procedure, in terms of compound matrices, that is facilitated by reduction of the transformers to a standard form. The procedure is also interpreted as a set of rules such that the mesh and nodal equations of a general network can be written by inspection. The possible degeneracies in network equations are considered, and a "scattering matrix" procedure presented to cover these cases. The orientation of the branches in a dual network is analyzed and the dual of an ideal transformer is given. The duality concept in electrical networks is considered in terms of matrices that describe the sets of branches belonging to the various meshes (connection matrix) and belonging to the various node-pairs (branch, node-pair matrix). Using the extension of the duality principle to non-planar networks, a procedure is presented for drawing a network diagram from its connection matrix. As an application, a general procedure is given for finding the electrical analog of a mechanical structure. Also, the role of gyrators and network duality is mentioned. The problem of minimizing the number of transformers in a network is approached by a circuit reduction technique. Networks uniformly dependent on frequency are first synthesized by Cauer's technique. The conditions are derived for then eliminating the transformers from this circuit, one by one, for the particular case of a network with three grounded terminal-pairs.

INTRODUCTION

An investigation of networks involving ideal transformers derives its timeliness from the rise of new, general network synthesis procedures, each of which involves the use of ideal transformers, and upon the increasing utilization of the electrical analog for the solution of problems in mechanics.

For example, if a light, stiff rod were connected between two of the variously moving parts of a mechanical system, then the possible types of motion within it would be constrained. If the inertia added to the system by the rod can be neglected, and if the change in the length of the rod due to forces applied to it can be neglected in comparison with other deformations taking place in the system, then the rod and a two-winding, ideal transformer are exact analogs, one of the other.

The type of constraints represented by ideal transformers is much more common in mechanical systems than is usually present in electrical networks because of the difficulty in building the ideal transformer. By careful utilization, however, of particularly adapted magnetic materials, two-winding transformers, with taps giving various turns ratios, have been built to represent ideal transformers for use in analog computers such as the Caltech Analog Computer.

There is direct application to analog computation, therefore, of research into: (1) the different manners in which ideal transformers can be connected within a system; (2) the elimination of transformers that represent superfluous constraints; and (3) the replacement, in certain situations, of ideal transformers by less expensive circuit elements, such as resistors.

In this thesis, the equations representing networks having a large -- or even unspecified -- number of meshes and nodes have been written in simplified notation by the use of "compound" or "partitioned" matrices. Also, as another mathematical tool, certain results from the field of topology have been used, some of which are quite familiar to those who work with electrical network problems.

One such property is the fact that each planar graph has associated with it a "dual" graph, the properties of the two being describable in the same manner. The concept of duality in electrical networks follows from the concept of duality in topology. Networks are electrical duals of each other if the equations of one are identical in form to those of the other -- provided the roles of current and voltage are interchanged.

To apply the concept of duality to non-planar networks, it is necessary to utilize ideal transformers. Hence, ideal transformers broaden the application of the principle of duality. This should be of interest not only to electrical engineers, but also to topologists.

Most efficient application of the theorems of topology to electrical networks can be made if these theorems are stated in the language of matrices, following the lead of Gabriel Kron. To describe the interconnection of the elements of an electrical network, that is, to describe its topological properties, various matrices can be written. The dual relation existing between these matrices is presented in the thesis, and through introduction of new notation, the fundamental matrices describing the graph of a network are reduced from three to two types.

Also, the principles of duality are broadened in another respect. Previously, networks described by mesh equations have not had an equal footing from the viewpoint of network topology with those described by node equations, since networks described by node equations can (in situations described in the text) be diagramed directly -- while, in comparison, the description in terms of mesh equations does not directly yield a network diagram. But now, in the section, "Drawing a Network Given its Connection Matrix," a procedure is given for obtaining the network diagram from the connection matrix (a matrix associated with mesh equations). An application of this procedure is given in the last chapter. Here, as far as the author knows, the step-by-step method of drawing the circuit for the electrical analog of a mechanical structure is given for the first time.

In presenting the material in this thesis, specific examples have preceded each general development. The circuit diagrams representing the examples have been included as well as equations written out term-by-term. This has been done in Chapter II, on the analysis of networks involving ideal transformers; in Chapter III, on the formation of a network impedance matrix; and in Chapter VII on the electrical analogs of mechanical structures.

The matrix formulas -- as in Chapter III -- have also been examined for a physical interpretation. Thus, from formulas, a procedure for writing the impedance matrix by inspection without the use of matrix methods is given which covers the case of networks involving ideal transformers. The rules owe their derivation to the matrix manipulation, and and are in simple form because of reductions that are first made on the circuit diagram.

Reduction of circuit diagrams is one of the most powerful tools of a network analyst. A simplification of the circuit diagram may represent the savings of a great deal of effort in the manipulation of equations, as well as the possible errors that may arise from these manipulations. Reduction of circuit diagrams, rather than an exclusive examination of the network equations, furnishes the key to the minimization of the number of ideal transformers needed in a particular network synthesis. For example, a Cauer network, consisting of ideal transformers and impedance elements all of one type (all resistors, all capacitors, or all self-inductors) may be drawn from a given set of equations. The question would then be the elimination of transformers from the Cauer network. A systematic procedure (given here, the author believes, for the first time) for stepwise elimination of the transformers and for the necessary requirements on the network parameters to permit such elimination, is presented in the case of a network with three terminal-pairs, each with a common ground. By this method, one finds the minimum number of transformers necessary in the given network.

In relating this last result to the work of others, there is the need to generalize, first, to include networks having a greater number of terminals, and secondly, to include networks containing various types of impedance elements. The synthesis procedures of Bayard, Belevitch, Leroy, MacMillan, Oono, and Tellegen would furnish the starting point for such a project. (See Bayard's summarization in the Proceedings of the Symposium on Modern Network Synthesis, Polytechnic Institute of Brooklyn (April, 1952), pages 66-83.)

TABLE OF CONTENTS

I. THE TWO-WINDING TRANSFORMER

THE GENERAL 4-POLE

The equations of a general 4-pole, written in reference to Figure 1, are:

$$v_1 = z_{11} i_1 + z_{12} i_2$$
$$v_2 = z_{21} i_1 + z_{22} i_2 \qquad (1)$$

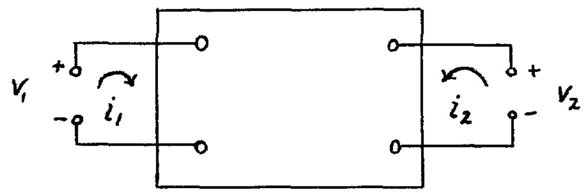


Figure 1

Note that the terminals are grouped in pairs; for any pair the current flowing into one terminal equals that flowing out the other. The parameters of the 4-pole, z_{ij}, can be determined experimentally by measuring voltages and currents at the terminals. The ideal transformer will be considered here as a particularized 4-pole (or later, 2n-pole) developed from magnetic considerations.

MAGNETIC CIRCUITS

Since the divergence of the induction, B, is zero, B is solenoidal. This permits the concept of circuitous tubes of flux, with the flux, ϕ, being a constant within each tube. By Ampere's Law (neglecting Maxwell's displacement currents), the line integral of the magnetic field intensity is equal to the product of the current interlinking the path of the line integral times the number of turns the current takes about, that is, interlinks this path. With the path selected along a tube of flux,

$$\int H_l \, dl = \int \frac{B_l}{\mu} \, dl = \int \frac{\phi}{\mu A} \, dl = \phi \int \frac{dl}{\mu A} = \phi R \qquad (2)$$

where R, the reluctance, represents the above line integral. This line integral is only a function of the geometry of the coils and the magnetic properties of the medium. Thus, Ampere's Law for a magnetic circuit is:

$$\phi \mathcal{R} = n\, i \tag{3}$$

(One may start from this point to derive magnetic circuit diagrams which turn out to be topologically the dual of the flux path configuration. See Cherry, R-1.)

INDUCED VOLTAGES

With the assumption that the magnetic permeability of the core is independent of the flux density, the equation for the magnetic circuit is linear. This being the case, if several electric circuits should encircle a tube of flux, a portion of the flux may be assigned to each. For example, the flux due to the jth current is

$$\phi_j \mathcal{R} = n_j\, i_j \tag{4}$$

From Faraday's Law, the voltage drop due to current in the jth circuit in the assigned direction is

$$v_{jj} = n_j \frac{d\phi_j}{dt} = \frac{n_j n_k}{\mathcal{R}} \frac{d i_k}{dt} \tag{5}$$

The voltage in that circuit due to the current in the kth circuit (assuming flux produced by i_k and i_j is in the same sense) is

$$v_{jk} = n_j \frac{d\phi_k}{dt} = \frac{n_j n_k}{\mathcal{R}} \frac{d i_k}{dt} \tag{6}$$

TRANSFORMER WITH NO LEAKAGE FLUX

In the case where all the flux links all the turns of both the windings, having turns n_1 and n_2 respectively, the voltages are

$$v_1 = \frac{n_1 n_1}{R} \frac{di_1}{dt} + \frac{n_1 n_2}{R} \frac{di_2}{dt}$$
$$v_2 = \frac{n_2 n_1}{R} \frac{di_1}{dt} + \frac{n_2 n_2}{R} \frac{di_2}{dt} \tag{7}$$

The ratio of the applied voltages is independent of the reluctance, R, of the core. It is

$$\frac{v_1}{n_1} = \frac{v_2}{n_2} \tag{8}$$

The expression for the total flux in the core yields

$$n_1 i_1 + n_2 i_2 = R(\phi_1 + \phi_2) \tag{9}$$

MAGNETIZING CURRENT

The equations may be simplified further if they are written in terms of new currents, linearly related to the old ones. In the parlance of mechanical systems, this is "transforming the coordinates." But what new coordinates shall be selected?

The linear combination, $n_1 i_1 + n_2 i_2$, occurs naturally in the system, as is seen in the section directly above. Let, therefore, this combination represent a new current. Or better still, in order to keep the new term in the same units as the old, let it be a current (i_m) times a number of turns, say, n_1. This would give

$$n_1 i_m = n_2 i_1 + n_2 i_2 \tag{10}$$

where i_m is called the "magnetizing current."

THE TRANSFORMED EQUATIONS

Writing the transformer equations in terms of i_m and i_2 (the magnetizing and load currents), instead of i_1 and i_2, the terms involving i_2 cancel, leaving

$$v_1 = \frac{n_1 n_1}{R} \; p \, i_m$$
$$v_2 = \frac{n_2 n_1}{R} \; p \, i_m \qquad (11)$$

where the symbol, p, is to stand for d/dt.

If this transformation had been carried out so as to keep the expression for power invariant (R-2), the resulting equations (easily obtained by matrix manipulations) would have been in symmetrical form, as follows:

$$v_1 = \frac{n_1 n_1}{R} \; p \, i_m + 0 \, i_2$$
$$v_2 - \frac{n_2}{n_1} v_1 = 0 \, i_m + 0 \, i_2 \qquad (12)$$

Note that there are no off-diagonal coupling terms, indicating that the coordinates chosen are the normal coordinates of the system.

THE IDEAL TRANSFORMER

Prompted by the fact that the reluctance of a path in iron is very small compared to one in air, the ideal concept is taken as a core with zero reluctance. This implies infinite values for v_1 and v_2 unless simultaneously i_m is set to zero. With both R and i_m zero, the values of v_1 and v_2 are indeterminate, and the equations reduce to

$$v_2 - \frac{n_2}{n_1} v_1 = 0$$
$$n_1 i_1 + n_2 i_2 = 0 \qquad (13)$$

THE "WINDINGS" OF THE IDEAL TRANSFORMER

A schematic representation of an ideal transformer is shown in Figure 2. The coils or windings should not be mis-

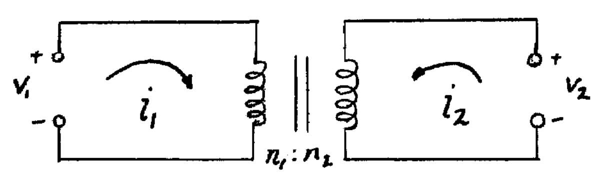


Figure 2

taken for inductors. There is no self or mutual impedance associated with an ideal transformer. The currents do not determine the value of the voltages. Such impedances were present, however, before i_m was set to zero.

A CIRCUIT FOR TRANSFORMER WITH NO LEAKAGE FLUX

A transformer in which all the flux links all the turns of both windings is represented by an ideal transformer and a shunt inductor as shown in Figure 3. The inductor has the value, $L_m = n_1 n_1/R$. The current through it equals $i_1 + n_2 i_2/n_1$, or i_m; the voltage drop across the inductor equals v_1, and this voltage "transferred" to the other side of the ideal transformer equals v_2, yielding equations (7).

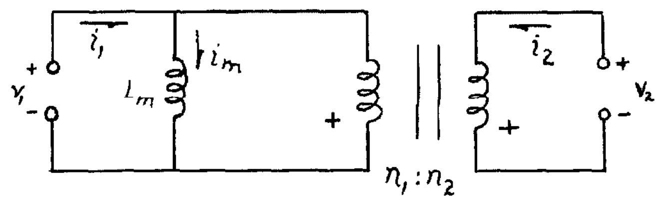


Figure 3

In terms of the circuit diagram, setting i_m and R equal to zero in the previous section, in effect, open-circuited the shunt inductor, L_m. This reduced the circuit from one with three meshes to one with two.

THE MULTIWINDING (IDEAL) TRANSFORMER

In the transformer of Figure 4, if there is no leakage flux, the fluxes, ϕ_1, ϕ_2, ..., of the various windings will be equal to each other. Then, assuming k windings in all, (R-1),

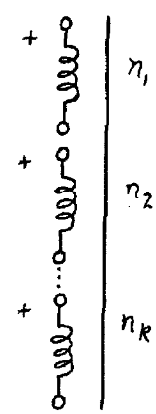


Figure 4

$$\dot{\phi}_1 = \dot{\phi}_2 = \cdots = \dot{\phi}_k \qquad (14)$$

where the dot above the letter indicates the time derivative. Equation (14) restated in terms of the voltage of each winding is

$$\frac{v_1}{n_1} = \frac{v_2}{n_2} = \ldots = \frac{v_k}{n_k} \tag{15}$$

where the turns on each winding, n_1, n_2, etc., may be positive or negative numbers.

If, in addition, the transformer core has zero reluctance, so that the magnetizing current is zero,

$$i_1 n_1 + i_2 n_2 + \ldots + i_k n_k = 0 \tag{16}$$

In this equation, v_1 may be substituted for n_1, v_2 for n_2, etc., since the v's are proportional to the n's, as per equation (15). The result is the expression for the total power flowing into the transformer.

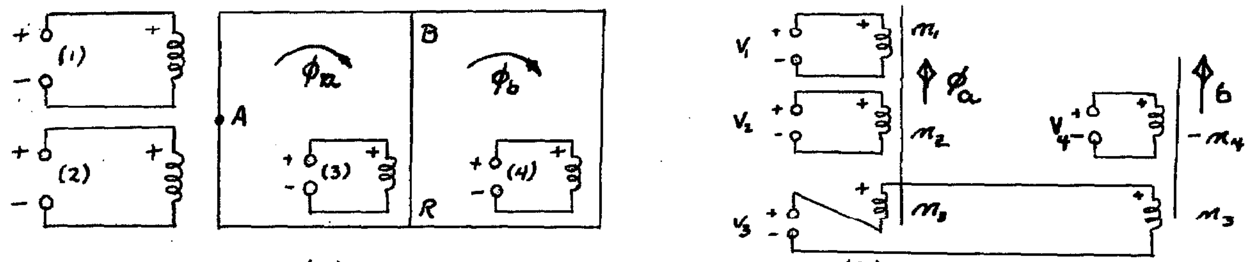
$$i_1 v_1 + i_2 v_2 + \ldots + i_k v_k = 0 \tag{17}$$

Thus, for a multiwinding, or "mesh-type" transformer,

1. The volts-per-turn for each winding are equal.
2. The sum of the ampere-turns of all the windings is zero.
3. The total power input is zero.

TRANSFORMER WITH SEVERAL MAGNETIC PATHS

The flux in every winding of a transformer (Figure 5 (a)) with more than one magnetic path will in general not be the same. For this type of transformer independent magnetic paths are selected in the same manner as independent meshes in an electrical circuit. For each of these magnetic paths the equations of the multiwinding transformer which had a single magnetic path may be applied, and an equivalent circuit using the type of multiwinding transformer drawn, as in Figure 5 (b).



(a) (b)

Figure 5

In reference to the example of Figure 5,

$$\frac{V_1}{n_1} = \frac{V_2}{n_2} = \dot{\phi}_a \; ; \; \frac{V_3}{n_3} = -\dot{\phi}_a + \dot{\phi}_b \; ; \; \frac{V_4}{n_4} = -\dot{\phi}_b \qquad (18)$$

II. ANALYSIS OF NETWORKS INVOLVING IDEAL TRANSFORMERS

THE EQUATIONS

The equations of the circuit in Figure 6 may be written in terms of mesh currents, i, and flux derivatives, $\dot{\phi}$. The network has three independent meshes and contains windings that are on two transformers.

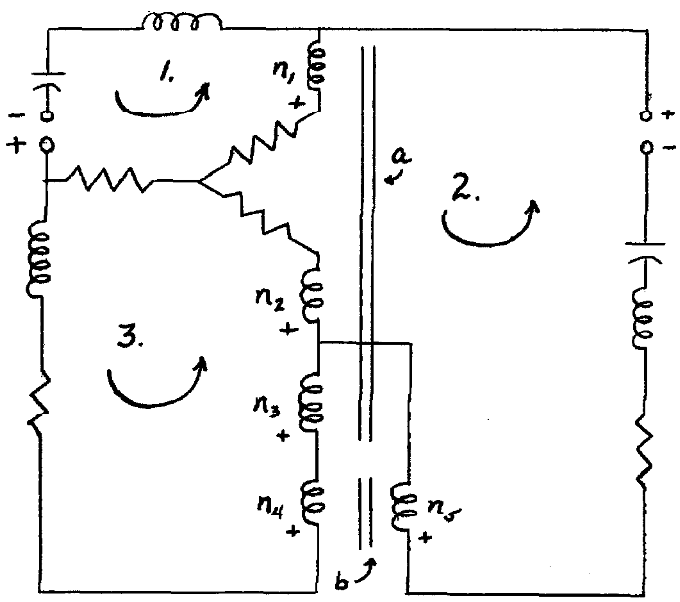


Figure 6

The equations are in the form, (R-3):

$$
\begin{aligned}
V_1 &= z_{11} i_1 + z_{12} i_2 + z_{13} i_3 + n_{1a}\dot{\phi}_a + n_{1b}\dot{\phi}_b \\
V_2 &= z_{21} i_1 + z_{22} i_2 + z_{23} i_3 + n_{2a}\dot{\phi}_a + n_{2b}\dot{\phi}_b \\
V_3 &= z_{31} i_1 + z_{32} i_2 + z_{33} i_3 + n_{3a}\dot{\phi}_a + n_{3b}\dot{\phi}_b \\
0 &= n_{1a} i_1 + n_{2a} i_2 + n_{3a} i_3 + 0\,\dot{\phi}_a + 0\,\dot{\phi}_b \\
0 &= n_{1b} i_2 + n_{2b} i_2 + n_{3b} i + 0\,\dot{\phi}_a + 0\,\dot{\phi}_b
\end{aligned}
\qquad (19)
$$

where for the circuit the winding-turns per mesh are

$n_{1a} = n_1$ $\qquad$ $n_{1b} = 0$

$n_{2a} = -n_1 - n_2$ $\qquad$ $n_{2b} = -n_5$

$n_{3a} = n_2 + n_3$ $\qquad$ $n_{3b} = n_4$

The first three equations are the respective voltage sums taken along the three meshes, while the last two equations are the respective ampere-turn relations for the two transformers. In summing the voltages, the voltage across a winding is taken as the number of turns in it times the derivative of the flux through it. The equations (19) would still have been written in terms of a $\dot{\phi}_a$ and a $\dot{\phi}_b$ if instead of two transformers the windings had been on one that had two independent magnetic paths.

THEIR SOLUTION

The solution for this system of five equations in five unknowns may be written in terms of the system determinant, D, (here, of order 5x5) and its signed cofactors. Thus, the solution for i_1 is

$$i_1 = V_1 \frac{d_{11}}{D} + V_2 \frac{d_{21}}{D} + V_3 \frac{d_{31}}{D} \tag{20}$$

If the determinant, D, is zero, the currents are not determinate but their ratios may be.

SOME PERMISSIBLE ALTERATIONS

The mesh currents will remain unchanged if equations are altered by:

1. Multiplying the turns for each winding of a transformer by the same constant, or

2. Replacing, for every mesh, the turns due to a transformer by the turns in that mesh due to two transformers. For instance, in

equations (19), n_{1b}, n_{2b}, and n_{3b} could be left as they are, and n_{1a}, n_{2a}, and n_{3a} replaced by, respectively, $n_{1a} + n_{1b}$, $n_{2a} + n_{2b}$, and $n_{3a} + n_{3b}$.

3. Performing a combination of the above two procedures.

The justification for these alterations is that they might simplify the analysis. They are permissible because they affect the signed cofactors appearing in the expressions for the mesh currents by the same factor that they affect the system determinant, thus leaving the mesh currents unchanged. A method for obtaining the circuit diagram for the altered equations will be given in the next chapter.

EXPANSION OF DETERMINANTS

Formula (20), the explicit solution for the mesh current, owes its derivation to the expansion of the system determinant, "D," by Cramer's Rule. This rule involves the selection of a column (or row) of the determinant, multiplication of the elements in that column by their cofactors, and then summation of these products. This is a special case of the development formulated by Laplace in which all the minors are formed from a selected set of rows or columns and the products of these minors times their algebraic complements summed. Instead of selecting a set of rows or columns with which to expand the determinant, there is a development, due to Cauchy, in which a set of rows along with the corresponding set of columns is selected for the expansion. An example, applying the Cauchy development to a determinant of the form of the network equations is given in the appendix. (R-4), (R-5)

LAGRANGE MULTIPLIERS

Recall the preceding analysis. In the network, which had three meshes and two transformers, there were used, in addition to the three mesh currents, two other variables, $\dot{\phi}_a$ and $\dot{\phi}_b$. Physically, $\dot{\phi}_a$ and $\dot{\phi}_b$ represent the derivative of fluxes in the transformer cores; mathematically, they take the role of Lagrange multipliers. (R-6).

Two changes were made in the network analysis to allow for the study of networks involving ideal transformers: terms other than the impedance drops or source voltages were added, and equations, namely the ampere-turn or "constraint" equations, were written supplementing the mesh equations. The voltage terms added to the mesh equations represent the voltages across transformer windings and involve the Lagrange multipliers, $\dot{\phi}$. These terms are subject to the following restriction. Namely, the power they represent must be zero, since the ideal transformers are constraints that consume and store no energy. In the example, this power is:

$$\begin{aligned} &i_1(n_{1a}\dot{\phi}_a + n_{1b}\dot{\phi}_b) + i_2(n_{2a}\dot{\phi}_a + n_{2b}\dot{\phi}_b) + i_3(n_{3a}\dot{\phi}_a + n_{3b}\dot{\phi}_b) \\ &= (n_{1a}i_1 + n_{2a}i_2 + n_{3a}i_3)\dot{\phi}_a + (n_{1b}i_1 + n_{2b}i_2 + n_{3b}i_3)\dot{\phi}_b \end{aligned} \qquad (21)$$

This power is zero if the currents do not violate the constraints represented by the ampere-turn equations.

The next sections are prerequisite to the application of the technique of Lagrange multipliers to the nodal analysis of networks involving ideal transformers.

THE MULTILIMB (IDEAL) TRANSFORMER

The development of the multilimb transformer is stepwise similar to that of the multiwinding transformer, but with the magnetic circuit considered from a nodal viewpoint.

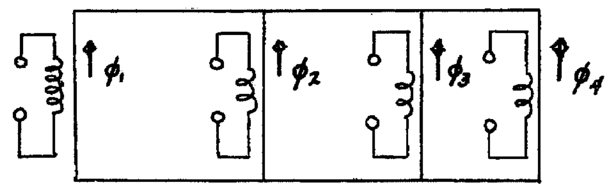


Figure 7

The limbs of a multilimb transformer, carrying fluxes, ϕ_1, ϕ_2, ..., ϕ_h, all meet at a magnetic "node" or "junction," and hence -- assuming no leakage -- the fluxes sum to zero. From this, differentiating, we have,(R-1),

$$\dot{\phi}_1 + \dot{\phi}_2 + \cdots + \dot{\phi}_n = 0 \tag{22}$$

Writing this in terms of the voltages across the respective windings,

$$\frac{v_1}{n_1} + \frac{v_2}{n_2} + \cdots + \frac{v_n}{n_n} = 0 \tag{23}$$

If u_1 is the reciprocal of n_1, u_2 the reciprocal of n_2, etc.,

$$v_1 u_1 + v_2 u_2 + \cdots + v_n u_n = 0 \tag{24}$$

If, in addition, the core has zero reluctance, the magnomotive force, M, measured in ampere-turns and summed about any magnetic mesh in the transformer core, has the value zero. This requires that the value of M for each of the parallel limbs be equal. Thus, in the notation of reciprocal-turns,

$$\frac{i_1}{u_1} = \frac{i_2}{u_2} = \cdots = \frac{i_n}{u_n} \tag{25}$$

Combining (24) and (25),

$$i_1 v_1 + i_2 v_2 + \cdots + i_n v_n = 0 \tag{26}$$

In summary, for a multilimb or "junction-type" transformer:

1. The magnomotive force, expressed in amperes per reciprocal-turn, is the same for each winding.

2. The (volts)x(reciprocal-turns) summed for all windings is zero.

3. The total power input is zero.

TRANSFORMER WITH SEVERAL MAGNETIC JUNCTIONS

As an example, consider the transformer of Figure 5. Independent node-pairs for the magnetic circuit, such as (A, R) and (B, R), are selected in the same manner that one selects node-pairs for an electric circuit. The number of "branches" of the magnetic circuit is taken equal to the number of windings on the transformer core. The magnetic circuit of the transformer core is without separate parts, and hence the number of magnetic node-pairs is equal to one plus the number of windings, minus the number of independent magnetic meshes in the core. For each magnetic node-pair, the equations of the multilimb transformer which had a single node-pair may be applied.

The equations for the circuit of Figure 5 (a) in terms of V_A, associated with magnetic node-pair (A, R) and V_B associated with magnetic node-pair (B, R), are:

$$\begin{aligned} 0 &= V_A (u_1 + u_2) - V_B u_1 \\ 0 &= -V_A u_1 + V_B (u_1 + u_3 + u_4) \end{aligned} \qquad (27)$$

But $V_1 = V_A - V_B$; $V_2 = V_A$; $V_3 = V_4 = V_B$. In terms of these voltages,

$$0 = -V_1 u_1 + V_2 u_2$$
$$0 = V_1 u_1 + V_3 u_3 + V_4 u_4 \qquad (28)$$

From these equations the equivalent circuit in terms of multilimb transformers is drawn, in Figure 8.

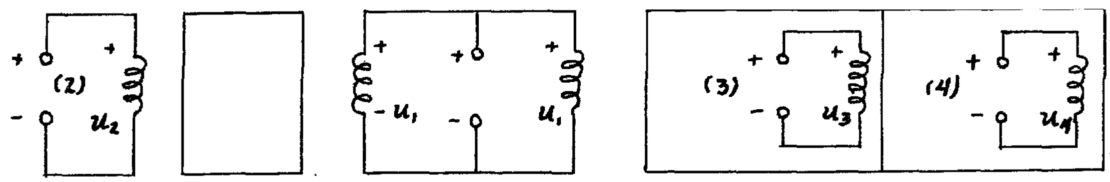


Figure 8

NODAL ANALYSIS

The equations for an electrical network may be written by summing the currents into each independent node-pair. The current through a branch is written in terms of the node voltages and the admittance of the branch. The current flowing from a node into the winding of an ideal transformer, is taken as the magnomotance for that multilimb transformer times the reciprocal-turns of the winding.

Consider, as an example, the circuit of Figure 9. This is a network with three independent node-pairs and two multilimb transformers having three node equations and one (volt)x(reciprocal-turn) equation for each of the two transformers.

$$\begin{aligned}
I_1 &= y_{11}V_1 + y_{12}V_2 + y_{13}V_3 + u_{1a}M_a + u_{1b}M_b \\
I_2 &= y_{21}V_1 + y_{22}V_2 + y_{23}V_3 + u_{2a}M_a + u_{2b}M_b \\
I_3 &= y_{31}V_1 + y_{32}V_2 + y_{24}V_3 + u_{3a}M_a + u_{3b}M_b \\
0 &= u_{1a}V_1 + u_{2a}V_2 + u_{3a}V_3 + 0\,M_a + 0\,M_b \\
0 &= u_{1b}V_1 + u_{2b}V_2 + u_{3b}V_3 + 0\,M_a + 0\,M_b
\end{aligned} \qquad (29)$$

In nodal analysis, the constraints due to ideal transformers are also represented by Lagrange multipliers. (R-7). Here, the multipliers are physically interpreted as the magnomotance of the various multilimb transformers.

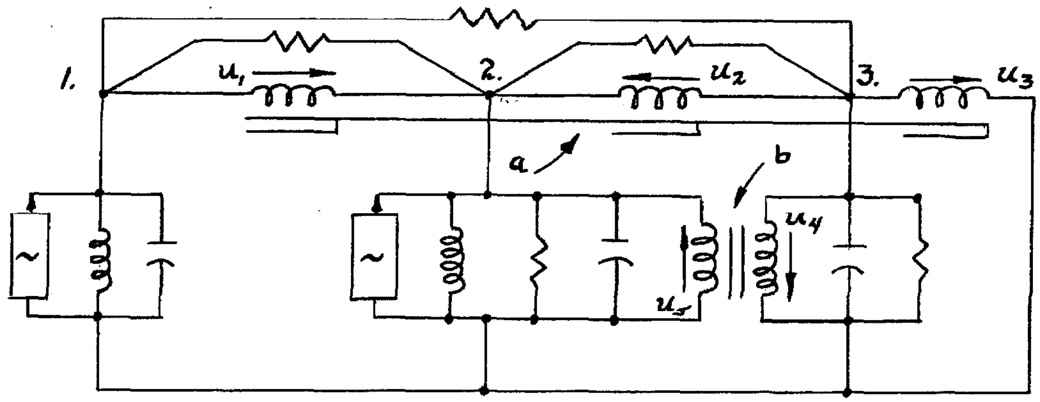


Figure 9

In Figure 9, the reciprocal-turns taken for the various nodes are:

$$u_{1a} = u_1 \qquad u_{1b} = 0$$

$$u_{2a} = -u_1 - u_2 \qquad u_{2b} = -u_5$$

$$u_{3a} = u_2 + u_3 \qquad u_{3b} = u_4$$

The same alterations are permissible on the reciprocal-turns of a multi-limb transformer as are on the winding-turns of a multiwinding transformer.

III. THE IMPEDANCE MATRIX IN MESH ANALYSIS

REDUCING THE EQUATIONS

The purpose of this section is to show how $\dot{\phi}_a$ and $\dot{\phi}_b$ can be eliminated as variables in the example given by equations (19). Whatever the number of windings on the transformers might be, this may be done.

Consider the last two equations in the set.

$$\begin{aligned} 0 &= n_{1a}\dot{i}_1 + n_{2a}\dot{i}_2 + n_{3a}\dot{i}_3 \\ 0 &= n_{1b}\dot{i}_1 + n_{2b}\dot{i}_2 + n_{3b}\dot{i}_3 \end{aligned} \tag{30}$$

From these equations, form the determinants by omitting one column of coefficients at a time.

$$\begin{vmatrix} n_{2a} & n_{3a} \\ n_{2b} & n_{3b} \end{vmatrix} \qquad \begin{vmatrix} n_{3a} & n_{1a} \\ n_{3b} & n_{1b} \end{vmatrix} \qquad \begin{vmatrix} n_{1a} & n_{2a} \\ n_{1b} & n_{2b} \end{vmatrix}$$

The columns of the second determinant have been interchanged to maintain a cyclic symmetry among the determinants.

One of these determinants at least must have a nonzero value if the equations in (30) are independent. If the equations were dependent, one equation would be a linear combination of the other; it could be scratched off and the transformer represented by it removed from the network without any resultant change in the system performance. In removal of the transformer, the windings are replaced by short-circuits. Figure 10 illustrates this point. Here, the ampere-turn equations are

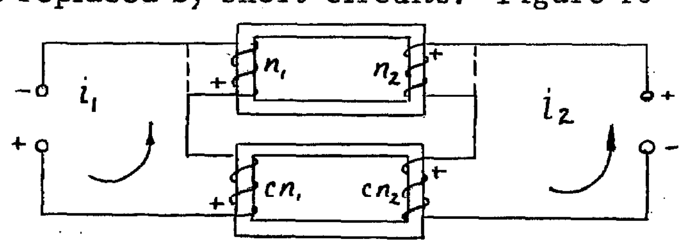


Figure 10

$$0 = m_1 i_1 + m_2 i_2$$

$$0 = c m_1 i_1 + c m_2 i_2$$

The dashed lines in Figure 10 may be replaced by short-circuits.

Assuming that the network has already been reduced so that the equations (30) are independent, they may be solved for i_2 and i_3 in terms of i_1 if the first of the above determinants is not zero. If the first determinant was zero, but the second was not, the equations could be solved for i_1 and i_3 in terms of i_2, etc. Because of the freedom to rearrange the order in writing the equations, no generality is lost, if the first of the above determinants is taken as not being zero and the expressions for i_2 and i_3 determined. If this is done, and if i_2 and i_3 are substituted in the first three equations of (19) the result is:

$$\begin{aligned} v_1 &= z_1 i_1 + n_{1a} \dot{\phi}_a + n_{1b} \dot{\phi}_b \\ v_2 &= z_2 i_1 + n_{2a} \dot{\phi}_a + n_{2b} \dot{\phi}_b \\ v_3 &= z_3 i_1 + n_{3a} \dot{\phi}_a + n_{3b} \dot{\phi}_b \end{aligned} \qquad (31)$$

where

$$\begin{aligned} z_1 &= z_{11} + \frac{N_{31}}{N_{23}} z_{12} + \frac{N_{12}}{N_{23}} z_{13} \\ z_2 &= z_{21} + \frac{N_{31}}{N_{23}} z_{22} + \frac{N_{12}}{N_{23}} z_{23} \\ z_3 &= z_{31} + \frac{N_{31}}{N_{23}} z_{32} + \frac{N_{12}}{N_{23}} z_{33} \end{aligned} \qquad (32)$$

for which the three determinants written below equations (30) have been abbreviated, respectively, by N_{23}, N_{31}, N_{12}.

The second two of the three equations in (31) can now be solved for $\dot{\phi}_a$ and for $\dot{\phi}_b$ in terms of i_1, v_2, and v_3. This solution exists because the determinant, N_{23}, has been taken as not being zero. The values found for $\dot{\phi}_a$ and $\dot{\phi}_b$ are substituted into the first equation of (31), giving

$$v_1 + \frac{N_{31}}{N_{23}} v_2 + \frac{N_{12}}{N_{23}} v_3 = Z\, i_1 \tag{33}$$

where

$$\begin{aligned} Z = \; & z_{11} + \frac{N_{31}}{N_{23}} z_{12} + \frac{N_{12}}{N_{23}} z_{13} \\ & + \frac{N_{31}}{N_{23}} z_{21} + \frac{N_{13}^2}{N_{23}^2} z_{22} + \frac{N_{12} N_{31}}{N_{23}^2} z_{23} \\ & + \frac{N_{12}}{N_{23}} z_{31} + \frac{N_{12} N_{31}}{N_{23}^2} z_{32} + \frac{N_{12}^2}{N_{23}^2} z_{33} \end{aligned} \tag{34}$$

In the process of eliminating $\dot{\phi}_a$ and $\dot{\phi}_b$ from equations (19), the set was reduced to one equation (33) written in terms of one impedence, Z, in (34). This is a system with three variables (i.e., three meshes) and two constraints (i.e., two transformers) and it has, therefore, only one degree of freedom. If there had been instead three meshes with only one transformer, the set would have reduced to two equations written in terms of two mesh currents. The coefficients of the currents would have the dimensions of impedance, and when placed in a 2x2 array, they would become the impedance matrix for that system.

A SECOND EXAMPLE OF REDUCTION

It will be instructive to consider a second example, the equations of which are the same as those of the preceding section except that particular values are chosen for the turns on some of the windings. Take, therefore,

$$\begin{aligned} v_1 &= z_{11} i_1 + z_{12} i_2 + z_{13} i_3 + n_a \dot{\phi}_a + n_b \dot{\phi}_b \\ v_2 &= z_{21} i_1 + z_{22} i_2 + z_{23} i_3 - \dot{\phi}_a \\ v_3 &= z_{31} i_1 + z_{32} i_2 + z_{33} i_3 - \dot{\phi}_b \\ 0 &= n_a i_1 - i_2 \\ 0 &= n_b i_1 - i_3 \end{aligned} \tag{35}$$

Solving for i_2 and i_3 from the last two equations and substituting into the first three,

$$\begin{aligned} v_1 &= (z_{11} + n_a z_{12} + n_b z_{13}) i_1 + n_a \dot{\phi}_a + n_b \dot{\phi}_b \\ v_2 &= (z_{21} + n_a z_{22} + n_b z_{23}) i_1 - \dot{\phi}_a \\ v_3 &= (z_{31} + n_a z_{32} + n_b z_{33}) i_1 \qquad\qquad - \dot{\phi}_b \end{aligned} \tag{36}$$

Then solving the last two equations of this set respectively for $\dot{\phi}_a$ and $\dot{\phi}_b$, and substituting into the first equation, there results

$$v_1 + n_a v_2 + n_b v_3 = Z\, i_1$$

where

$$\begin{aligned} Z = \; & z_{11} + n_a z_{12} + n_b z_{13} \\ & + n_a z_{21} + n_a^2 z_{22} + n_a n_b z_{23} \\ & + n_b z_{31} + n_a n_b z_{32} + n_b^2 z_{33} \end{aligned} \tag{37}$$

To show that the result is consistent with that of the previous section, form from the last two equations of (35) the determinants

$$\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} \quad \begin{vmatrix} 0 & n_a \\ -1 & n_b \end{vmatrix} \quad \begin{vmatrix} n_a & -1 \\ n_b & 0 \end{vmatrix}$$

The values of these determinants are $N_{23} = 1$, $N_{31} = n_a$, $N_{12} = n_b$, which substituted into (33) and (34) yield equation (37).

THE GENERAL PROCEDURE

Concerning the two examples given, the second was important because it had a simpler solution and yet was as general as the first. The equations of the first example can be reduced to those of the second by use of the "permissible alterations" described in the preceding chapter. This technique will be incorporated as part of the general procedure.

The results so far are:

1. An impedance matrix can be found for any network involving ideal transformers, so long as these transformers form independent constraints.

2. If a network contains transformers forming dependent constraints, some of these transformers can be removed, leaving the performance of the network unaltered and the remaining transformers independent.

3. The number of turns on the windings of the transformers may be altered into a standard form which facilitates the computations involved in finding the impedance matrix. The technique for carrying out this alteration is contained in the following sections.

NUMERICAL MANIPULATIONS

Consider again the network of Figure 6, and let the transformer windings have the following values:

$$n_1 = 2; \quad n_2 = -5; \quad n_3 = 1; \quad n_4 = 3; \quad n_5 = 2$$

The winding-turns, per transformer core, for the three meshes are then:

$$\begin{aligned} n_{1a} &= 2 & \qquad n_{1b} &= 0 \\ n_{2a} &= 3 & \qquad n_{2b} &= -2 \\ n_{3a} &= -4 & \qquad n_{3b} &= 3 \end{aligned}$$

These are the values that would be substituted in equation (30) and that would lead to the expression for the impedance in (34). To instead start with the equation (35) and to achieve the expression for the impedance more simply by means of equation (37), one must change the winding-turns for an equivalent set. To do this, Figure 6 is first redrawn as in Figure 11.

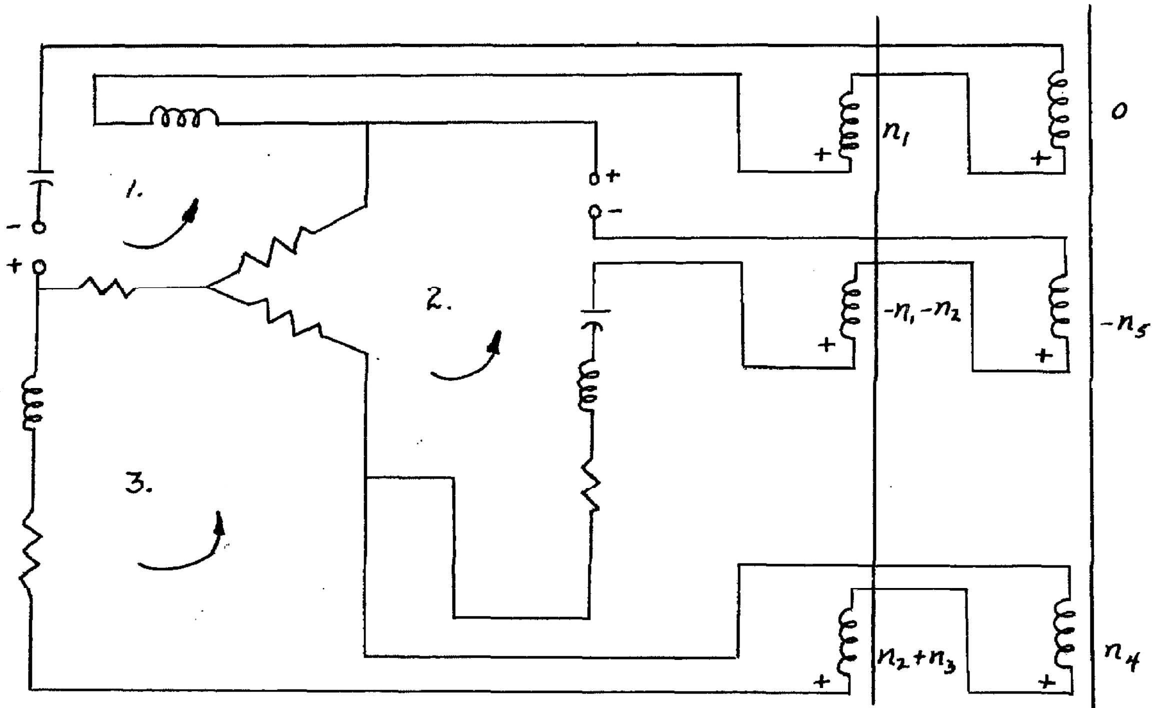


Figure 11

The procedure for separating the transformers from the rest of the network, as has been done in Figure 11, is discussed in the sections immediately following this.

The next series of steps is to change the winding-turns into the desired form. This procedure is shown in four steps in Figure 12. In this figure only the transformer cores of Figure 11 have been represented.

2	0	2	$\frac{4}{3}$	18	$\frac{4}{3}$	-6	-4
3	-2	3	0	3	0	-1	0
-4	3	-4	$\frac{1}{3}$	0	$\frac{1}{3}$	0	-1
The given set. 2/3rds turns "a" added to "b".		12 times turns "b" added to "a".		Multiply "a" by -1/3 "b" by -3.		The Standard Form	

Figure 12

This reduction may also be done in terms of matrix multiplication. A transformation matrix is formed by noting that multiplication of a matrix by the negative of its inverse results in a matrix with -1's along the main diagonal and zeros elsewhere. Postmultiplication of a matrix corresponds to operations upon the columns, rather than the rows, of that matrix. Noting that

$$\begin{bmatrix} 3 & -2 \\ -4 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 2 \\ 4 & 3 \end{bmatrix} \tag{38}$$

the above reduction is achieved by

$$\begin{bmatrix} 2 & 0 \\ 3 & -2 \\ -4 & 3 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -4 & -3 \end{bmatrix} = \begin{bmatrix} -6 & -4 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \tag{39}$$

TRANSFORMING THE WINDING-TURNS

The reduction in the preceding section depended on the existence of an inverse matrix, and this in turn depended on the independence of the rows in that matrix. A first step, then, in transforming the winding-turns to a standard form is to select a set of independent rows. This is the set of rows that, by means of the transformation, are replaced by rows having minus one as the diagonal element and zero elsewhere. The network meshes corresponding to these rows will be called the "magnetizing meshes" of the network. For the given matrix of numbers in the preceding example, any two meshes of the three meshes of this network could be chosen as the magnetizing meshes since it happens that any two rows of this matrix are independent. In every case it is possible to choose magnetizing meshes equal to the number of independent transformers as shown by the following reasoning.

If superfluous transformers are removed, the columns of winding-turn matrix will be independent. Furthermore, removal of the superfluous transformers requires that the number of transformers remaining be less than or equal to the number of meshes in the network, so that in the winding-turns matrix the number of columns is less than or equal to the number of rows. The columns being independent then requires that an equal number of the rows be independent, proving the point.

The matrix of the ampere-turn equations is the transpose of the matrix of the winding-turns appearing in the mesh equations. When replacing the winding-turns matrix in the mesh equations by a transformed matrix, the same substitution is made in the ampere-turn equations keeping the total set of network equations in symmetric form.

SELECTION OF THE MESHES IN A TRANSFORMERLESS NETWORK

A set of independent meshes is guaranteed to result if they are selected by assigning in a network a particular branch for each mesh which is a member of only that mesh. A branch of this type is called a "link." The remaining branches form, in each separate part of the network, what is called a "tree." The tree connects all the nodes of the separate part of the network to which it belongs, but, in itself, it contains no meshes. For any network a tree or trees can be drawn and a set of links determined. (R-8).

EQUIVALENT VOLTAGE SOURCES

If the equivalent of all the voltage sources encountered in a mesh is placed in the link belonging to that mesh, the equations describing the network will remain unchanged. (R-2).

BRANCHES AND TRANSFORMER WINDINGS

The number of mesh equations written for a given network depends on the degree to which detailed information is desired concerning the network performance. For instance, in the network of Figure 13, at most three independent mesh equations may be written. However, if the network is only to be considered as a transfer function between the two pairs of terminals, two equations will suffice. Another familiar instance involving the use of a reduced number of mesh equations occurs if two impedances in parallel are taken as a single branch.

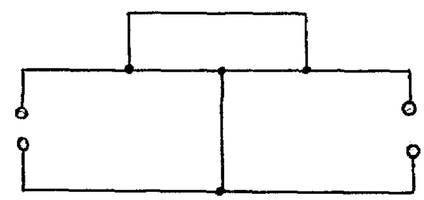


Figure 13

In selecting the meshes for a network involving ideal transformers, the following will be assumed:

1. A definite, definable mesh current (or several such currents) flows through each transformer winding. Thus for example two windings in parallel or a winding in parallel with an impedance element will not be considered as one equivalent current path.

2. The branches of the network will be separated from the

transformer windings. The network will therefore be considered as made of branches and windings, the branches including impedance elements and voltage sources.

SELECTION OF MESHES

In the selection of meshes for the analysis of a network involving ideal transformers, all the windings may be incorporated as part of the tree (or trees, if the network has separate parts), leaving only the branches as links. This can be done directly, unless there is a mesh consisting entirely of windings in the network. In such a situation, an impedanceless branch may be added in series with one of the windings in this mesh. The impedanceless branch may serve as the link for the mesh, all the windings being incorporated into the tree.

Now, in similar fashion to the method of equivalent voltage sources, an equivalent of all the windings encountered in a mesh may be placed in series with the link belonging to that mesh and the windings in the tree replaced by short-circuits, leaving the equations describing the network unchanged. In general, this equivalent will contain windings from various transformer cores. It will be convenient to assume that it contains windings from all of the cores, some perhaps with zero turns. Also, it is possible to schematically place the windings in a rectangular array corresponding to the arrangement of the terms in the winding-turn matrix.

Making use, therefore, of equivalent voltage sources and equivalent windings, the diagram for any network involving ideal transformers may be drawn as in Figure 14. As the diagram indicates, the network with "m" meshes and "k" multiwinding transformers has been separated into a passive 2m-pole and an array of winding-turns. As shown in the sample numerical manipulations, the array of winding-turns may be reduced to one in standard form. The equations of the equivalent, unconstrained 2(m - k)-pole may then be readily determined by elimination, as variables of the equations, the currents in the magnetizing meshes. One may choose to do this through substitution of equations as was illustrated in the second example given above, or by the techniques of matrix algebra, or, in simple cases, by inspection. The following sections are devoted to these techniques.

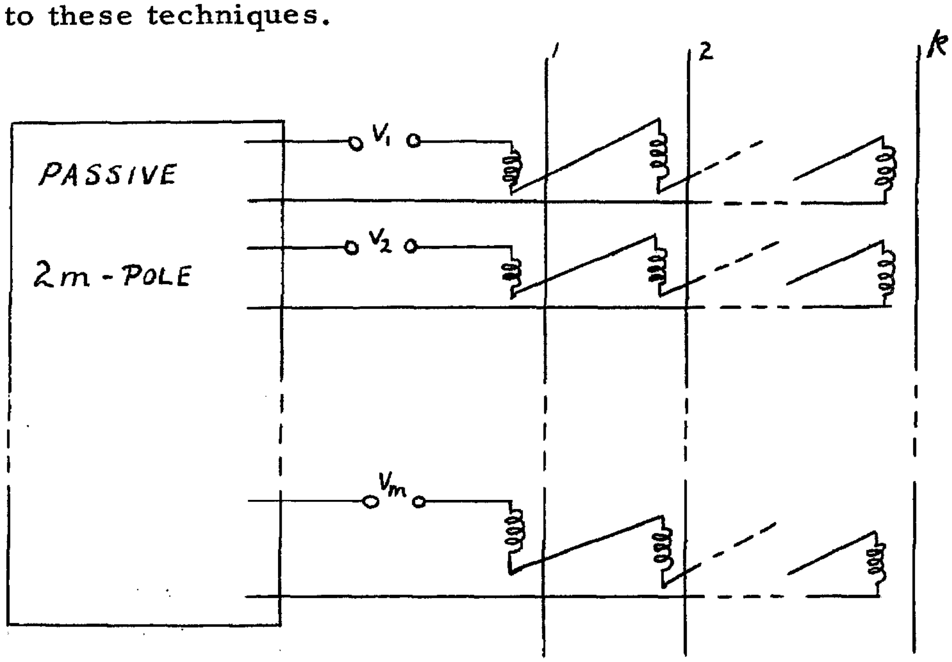


Figure 14

MATRIX PROCEDURE FOR TRANSFORMERLESS NETWORKS

Mesh equations were viewed by Gabriel Kron,(R-2), as the more simple branch equations that had been transformed, that is, undergone a "change of coordinates." Using the assumption that the power input is the same value in both systems of coordinates, he derived formulas given here in the following notation.

i_B, branch current matrix; i, mesh current matrix

$$i_B = C \; i \tag{40}$$

V_B, branch voltage source matrix; v, mesh voltage source matrix

$$v = C_t \; v_B \tag{41}$$

Z_B, branch impedance matrix; Z, mesh impedance matrix

$$Z = C_t \, Z_B \, C \tag{42}$$

$$v = Z \; i \tag{43}$$

The procedure in solving for the network performance is to number and assign a reference direction to each branch; select, number, and assign a reference direction to the meshes; and then write the matrices, C, v_B, i_B, Z_B, by inspection. The elements, c_{jk}, of C, the "connection" or "transformation" matrix (also called, in more general analysis, the "transformation tensor") are equal to

$$c_{jk} = \begin{cases} 1, & \text{if branch "j" is in mesh "k", directions aiding} \\ -1, & \text{if branch "j" is in mesh "k", directions opposing} \\ 0, & \text{if branch "j" is not in mesh "k"} \end{cases} \tag{44}$$

The transpose of C, written C_t, is obtained from C by interchanging the rows with the columns. Formulas (41) and (42) are then applied to yield the mesh equations represented by matrix equation (43). The mesh equations may be solved by the usual algebraic means, but the matrix procedure is to multiply (43) by the inverse of Z, Z^{-1}, giving

$$i = Z^{-1} v \tag{45}$$

MATRIX FORMULAS FOR NETWORKS INVOLVING TRANSFORMERS

The three procedures that have been given above for analyzing networks involving ideal transformers are: 1) solving instead a system with ordinary transformers, and then evaluating the expressions in the limit as the transformers become idealized; 2) using the ampere-turn and volt-per-turn, or the ampere-turn and the power equations descriptive of an ideal transformer; and 3) using Lagrange multipliers. The second and third of these procedures will be restated here in terms of matrix notation, and by the use of compound matrices formulas will result stated explicitly in terms of the winding-turns matrix. (R-9).*

First, the given network is separated as in Figure 14 and the 2m-pole (which may be found simply by replacing the windings by short-circuits, with the precaution that all-winding meshes are represented by impedanceless branches) analyzed as in the previous section. The mesh equations, represented by (43), may be separated in terms of the magnetizing meshes and the remaining or "load" meshes. In the notation of compound matrices, this separation is written:

$$\begin{bmatrix} v_L \\ \hline v_M \end{bmatrix} = \left[\begin{array}{c|c} z_{LL} & z_{LM} \\ \hline z_{ML} & z_{MM} \end{array}\right] \begin{bmatrix} i_L \\ \hline i_M \end{bmatrix} \qquad (46)$$

Reintroducing the ideal transformers imposes two conditions, the first being the ampere-turn equations. If the winding-turns have been reduced to the standard form, these equations matrixwise are

---

* The winding-turns matrix in (R-9) is the transpose of the one defined here, provided the sign of each of its elements is reversed.

(47)

In the analysis, the currents in the magnetizing meshes, i_M, are to be eliminated as variables from the equations. This elimination may be considered as a transformation from the old currents, i, which contain i_L and i_M, to the new currents, i', which are simply i_L. The transformation, found by means of (47), is:

$$i = \begin{bmatrix} i_L \\ \hline i_M \end{bmatrix} = \begin{bmatrix} 1 \\ \hline N_t \end{bmatrix} i_L = \begin{bmatrix} 1 \\ \hline N_t \end{bmatrix} i' = C_2 i' \qquad (48)$$

The second condition imposed on equation (46) by the ideal transformers is that the power input to the impedance elements of the network before and after the transformers are attached is the same. This condition must hold because no power is stored or consumed by the ideal transformers. But now, a transformation leaving the power invariant is described by formulas in the form (41), (42), and (43). Applying these,

$$v' = C_{2t}\, v = \left[1 \mid N \right] \begin{bmatrix} v_L \\ \hline v_M \end{bmatrix} = v_L + N v_M \qquad (49)$$

$$\begin{aligned} Z' &= C_{2t}\, Z\, C_2 \\ &= \left[1 \mid N \right] \begin{bmatrix} Z_{LL} & Z_{LM} \\ Z_{ML} & Z_{MM} \end{bmatrix} \begin{bmatrix} 1 \\ \hline N_t \end{bmatrix} \\ &= Z_{LL} + N Z_{ML} + Z_{LM} N_t + N Z_{MM} N_t \end{aligned} \qquad (50)$$

$$\begin{aligned} v' &= Z' i' \\ v_L + N v_M &= (Z_{LL} + N Z_{ML} + Z_{LM} N_t + N Z_{MM} N_t)\, i_L \end{aligned} \qquad (51)$$

Writing C_1 for the C of equation (40) and combining the transformations (42) and (50),

$$\begin{aligned} Z' &= C_{2t}\, C_{1t}\, Z_B\, C_1\, C_2 \\ &= C_t\, Z_B\, C \end{aligned} \tag{52}$$

where the transformation matrix, C, for a network involving ideal transformers is defined to be

$$C = C_1 C_2 \tag{53}$$

Partitioning C_1 according to the load and magnetization meshes,

$$C_1 = \left[C_L \;\vdots\; C_M \right] \tag{54}$$

and

$$\begin{aligned} C &= \left[C_L \;\vdots\; C_M \right] \left[\frac{1}{N_t} \right] \\ &= C_L + C_M N_t \end{aligned} \tag{55}$$

The formula is a basic result. It describes the connection between the impedances of the individual branches and the impedance matrix for a network involving ideal transformers. The network with no ideal transformers is a special case, corresponding to N = 0 in this formula.

If the impedance matrix is to be written in terms of the mesh impedances (of the network with the transformers removed), equation (51) is used. Thus, the equation may be derived without defining a connection matrix but instead by Lagrange multipliers.

To use the technique of Lagrange multipliers to analyze the network, the winding-turns are reduced to standard form and the network equations written:

$$\begin{aligned} V_L &= Z_{LL}\, i_L + Z_{LM}\, i_M + N \dot{\phi} \\ V_M &= Z_{ML}\, i_L + Z_{MM}\, i_M - \dot{\phi} \\ 0 &= N_t\, i_L - i_M \end{aligned} \tag{56}$$

The second two equations of this set are substituted into the first to yield equation (51).

THE DEVELOPMENT BY KRON

If the first procedure mentioned in the preceding section had been used instead of the second and the third, a shunt or "magnetizing" branch would, in effect, be added to each transformer, converting it (as shown in the first chapter of this thesis) into an impedance element, that is, a non-ideal transformer. The mesh equations for this system could then be written. From this viewpoint, the transformation of equation (50) is equivalent to the idealization of the transformers. Kron,(R-2), shows that the rectangular matrix, C_2, can be derived from a square, nonsingular matrix by omitting some of the columns in the square matrix, a process which corresponds to opening the "magnetizing branches" in the non-ideal system.

Kron does not separate the transformer windings from the network impedance branches. This enables him to write the ampere-turn equations in terms of branch currents. The equations may then be transformed, by C_1, to be in terms of mesh currents. The mesh currents are divided into the two categories of magnetizing and load currents, and the transformation matrix, C_2, is obtained by solving the ampere-turn equations for the magnetizing currents in terms of the load currents.

This last step involves the equivalent effort of reducing the winding-turns to the "standard form," but it does not bring C_2 to so simple a form. The difference in com plexity is the same as the difference of that involved in solving the examples of the first two sections of this chapter. In the Kron process, the branch impedance matrix will contain a row and column of zeros for each winding that does not have an impedance associated in series with it.

SUGGESTED MATRIX PROCEDURE

To determine the impedance matrix for a network involving ideal transformers, reduce the winding-turns to the standard form and proceed in the following steps:

1. Number, from 1 to b, and assign reference directions to the branches in the network, excluding the windings.

2. Number, from 1 to k, and assign reference directions to the magnetizing meshes. These meshes are numbered in the same order that the cores corresponding to them are numbered, and the current in their reference direction defines the reference direction for the flux in the associated transformer core; positive current creating negative flux since n = -1.

3. Number, from 1 to e, and assign reference directions to the load meshes. Since the load meshes are all those not chosen as magnetizing meshes,

$$e + k = m \tag{57}$$

Also, from network topology,

$$m = b + m - s \tag{58}$$

where "n" is the number of nodes in the network after the transformer windings have been replaced by short-circuits, and "s" is the number of separate parts of the network.

4. Write the connection matrix, C_L, relating the branches and the load meshes by using the procedure of (44). This matrix has "b" rows and "e" columns.

5. Likewise, write C_M, the connection matrix relating the branches to the magnetizing meshes and having "b" rows and "k" columns.

6. The winding-turns matrix, N, may be copied directly from the circuit diagram if the windings are arrayed as those in Figure 8. The matrix, N, has "e" rows and "k" columns. The element, n_{jk}, in the matrix is equal to the number of turns of the jth load mesh occurring on the kth transformer core. The sign of n_{jk} is positive if current in the reference direction of the mesh would create flux in the reference direction of the core, negative if the flux would be in the opposite direction.

7. Form the connection matrix, $C = C_L + C_M N_t$, having "b" rows and "e" columns.

8. Form the resulting mesh impedance matrix, $Z' = C_t Z_B C$. This matrix has "e" rows and columns, while the branch impedance matrix, Z_B, has "b" rows and columns. If there are no windings, Z' becomes equal to Z of (42).

THE IMPEDANCE MATRIX BY INSPECTION

For networks not containing ideal transformers, the mesh impedance matrix may be written, one row at a time, by superimposing the individual effects of each mesh current. The self impedance term of a mesh may be defined equal to the total voltage drop in the mesh due to a unit current in that mesh while all the other currents are zero. The mutual impedance term between one mesh and another may be defined as the voltage drop in the first mesh when the current in the second mesh is unity and all other currents are zero. These definitions apply to networks involving mutual inductance, but care must be taken to include the voltage drop, along with its proper polarity, in a coil if current is assumed to flow in another coil to which it is coupled. In networks containing several mutual inductances or an arrangement of impedance elements complicated by many wires crossing over each other, the analysis by means of matrices rather than direct use of these definitions becomes more convenient.

The definitions described cannot be immediately applied to networks involving ideal transformers. Due to the ampere-turn equations of the ideal transformers, it is not permissible to assume that each current may separately take the value of unity and all others be zero. If, however, only the load meshes are considered, the procedure may be applied, since it has been shown above that the mesh impedance equations for a network involving ideal transformers are a linear function of and only of the load currents. The self and mutual impedances due to transformer windings in the load meshes can be found by a set of rules to be given. The cases considered in the next few sections are those for which the matrices Z_{ML} and Z_{LM} of equation (51) would be zero.

SELF IMPEDANCE OF AN IDEAL WINDING

Consider the case of a two-winding transformer with a turns ratio, N, equal to the ratio of the turns on the primary winding to those on the secondary and with an impedance, z, shunting, that is, in parallel with the secondary. The magnetizing mesh will be taken as the mesh formed by the secondary and z. The remaining mesh, called the load mesh, is that of the primary winding. (Figure 15).

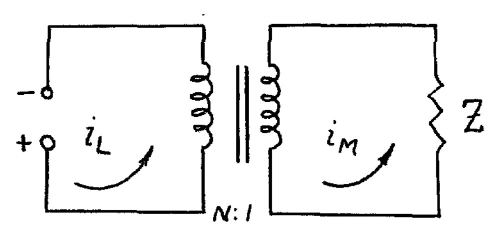


Figure 15

To determine the "self impedance" of the primary winding, consider a unit current flowing through it. The magnitude of the current in the secondary equals, by the ampere-turns equation, the magnitude of the turns ratio. The secondary or "magnetizing" current flows through z, producing a voltage drop equal, in magnitude, to Nz. This voltage, multiplied by N, gives the magnitude of the voltage of the primary winding as equal to N^2z, which thus is numerically the self impedance associated with the primary winding. This is the same value that one obtains by removing z from the secondary (leaving an open-circuit in its place) and putting an impedance equal to N^2z across the primary. If this were done, the secondary would become open-circuited; no current would flow in either winding and the transformer could be removed. Thus, the self impedance of a winding could be found by "transferring" the impedance of the magnetizing mesh to that winding.

If the windings had been reduced to standard form, N would have a magnitude equal to the number of turns of the primary winding. Assuming this, the above results may be stated in a rule which also applies to the load meshes of a multiwinding transformer.

Rule 1: The self impedance of a winding of "n" turns is the impedance, "z", of the transformer magnetizing mesh multiplied by n^2.

One should note that this rule applies only to cases when the windings have been reduced to standard form, when the load current does not flow through the impedance of the magnetizing mesh, and, furthermore, when the load and magnetizing meshes are not coupled by mutual inductance. The rule for the more general situation will be deferred to a later section.

MUTUAL IMPEDANCE BETWEEN IDEAL WINDINGS

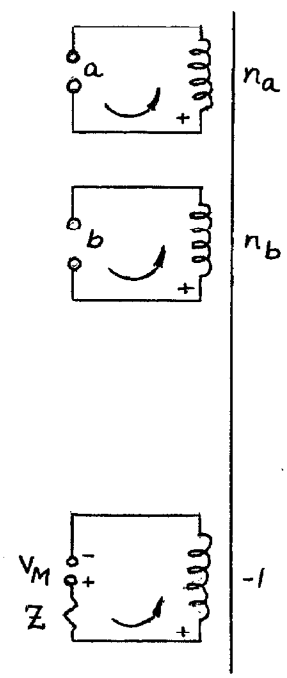


Figure 16

For the mutual impedance between windings "a" and "b" with turns n_a and n_b of a multiwinding transformer (Figure 16), a current of unit value will be assumed flowing in the reference direction of a load mesh containing winding "a", and the voltage drop along the reference direction in winding "b" will be calculated. All load currents other than this one will be assumed zero. As before, from the ampere-turn equations, the current in the magnetizing mesh is n_a, measured in units of current and producing a voltage drop in "z" equal to $n_a z$. Since the sum of the voltage drops in the magnetizing mesh must be zero, the voltage drop across the winding in this mesh is $-n_a z$, or if there is also in this mesh a voltage source, v_M, the voltage drop is $-n_a z + v_M$. Transferring this voltage to winding "b", that is, transferring the voltage from a winding of - 1 turns to one of n_b turns, the voltage becomes $n_a n_b z$, or if the source, v_M, is present, it is $n_a n_b z - n_a v_M$. This last expression contains, combined with the mutual impedance term, a transferred voltage term whose sign is changed to plus when it is considered as a voltage source term in the load mesh due to the winding and is placed on the left side of the equation. If the unit current had been applied to the load mesh containing "b" and the voltage drop measured in "a", its value would be, $n_b n_a z - n_b v_M$. From this we have:

Rule 2: The mutual impedance between a winding of n_a turns and one of n_b turns is the impedance of the transformer magnetizing mesh multiplied by $n_a n_b$.

Rule 3: The voltage source term due to a winding of "n" turns is the voltage source in the transformer magnetizing mesh multiplied by n.

A CAUER NETWORK

In order to give an example applying the above rules, the mesh impedance matrix for a network due to W. Cauer, (R-10), will be written.

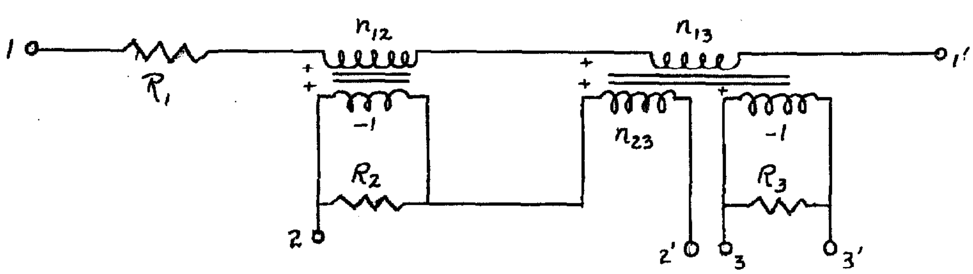


Figure 17

Using the notation of Cauer, the cores are numbered 2 and 3, 2 having two windings and 3 having three. As shown in Figure 17, the windings have been reduced to the standard form. Since the windings are incorporated in the "tree" of the network, or rather into three trees since there are three separate parts in this network, the meshes are as follows. The magnetizing mesh for transformer 2 consists of R_2 and the adjacent winding, and that for transformer 3 consists of R_3 and the winding adjacent to it. The three load meshes start, respectively, from the three sets of terminals, but R_2 and R_3 are not included since these resistors are the links of the magnetizing meshes.

The conditions of the preceding sections are fulfilled, and their rules may be applied. The self impedance of the first load mesh has the value R_1 plus the self impedance of the two windings contained in the mesh. By Rule 1, the total is then, $R_1 + n_{12}^2 R_2 + n_{13}^2 R_3$. The self impedance of the second mesh is that due to the two windings in it, giving the total to be $(-1)^2 R_2 + n_{23}^2 R_3$. That of the third load mesh is $(-1)^2 R_3$. Filling in the mutual impedance terms by Rule 2, the completed impedance matrix is:

$$\begin{bmatrix} R_1 + n_{12}^2 R_2 + n_{13}^2 R_3 & -n_{12} R_2 + n_{13} n_{23} R_3 & -n_{13} R_3 \\ -n_{12} R_2 + n_{13} n_{23} R_3 & R_2 + n_{23}^2 R_3 & -n_{23} R_3 \\ -n_{13} R_3 & -n_{23} R_3 & R_3 \end{bmatrix} \tag{59}$$

This matrix may be written in the general form:

$$\mathcal{R} = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} \tag{60}$$

Solving for the circuit parameters in terms of r_{jk}, there results:

$$R_3 = r_{33}; \quad R_2 = \frac{|r_{22}\ r_{23}|}{r_{33}}; \quad R_1 = \frac{|r_{11}\ r_{22}\ r_{33}|}{|r_{22}\ r_{33}|} \tag{61}$$

$$\frac{M_{13}}{-1} = \frac{r_{13}}{r_{33}}; \quad \frac{n_{23}}{-1} = \frac{k_{23}}{k_{33}}; \quad \frac{n_{12}}{-1} = \frac{|r_{12}\ r_{13}|}{|r_{22}\ r_{33}|}$$

where, in the above notation, determinants taken from R have been represented by their diagonal elements. Now, the condition that a matrix of numbers in the form of (60) represent a physically realizable network is that the determinant of the matrix and of each of its principal minors be greater or equal to zero. (R-5). (The matrix is assumed to be symmetrical, that is, $r_{jk} = r_{kj}$, and each of the elements a real number.) If, however, this condition holds, then by (61) the resistors in Cauer's network are either positive numbers or zero. Thus, any matrix in the form of (60) that satisfies the conditions that it be physically realizable can be represented by the Cauer network. The Cauer network for a matrix with more rows and columns is formed by adding additional multiwinding transformers to the network of Figure 17, as shown in Figure 18. Also, if each element in the matrix, R, is multiplied by the frequency variable, p, the resistors in the Cauer network are replaced by inductors; the resistors are replaced by capacitors if each element in the matrix is divided by p.

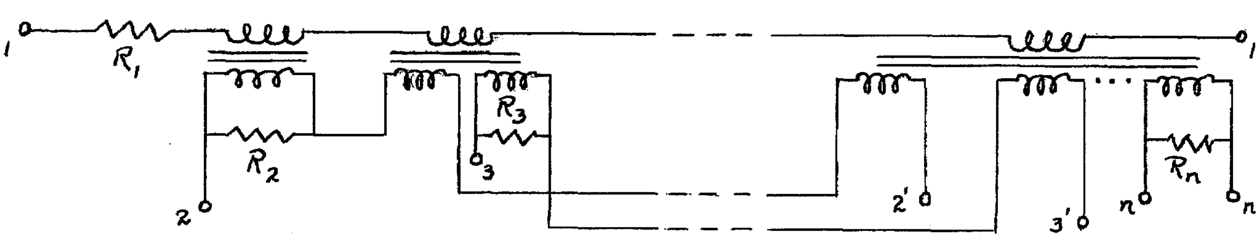


Figure 18

The impedance matrix, (59), which was written by inspection, can be obtained by matrix methods by the use of equation (51). The matrices, Z_{ML} and Z_{LM}, are zero in this case since there is no impedance coupling, either due to mutual inductance or to common impedance branches, between the load and magnetizing meshes. The matrix, v_M, is also zero, so that equation (51) may be written

$$v_L = (Z_{LL} + N Z_{MM} N_t) i_L \qquad (62)$$

or,

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \left\{ \begin{bmatrix} R_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} n_{12} & n_{13} \\ -1 & n_{23} \\ 0 & -1 \end{bmatrix} \begin{bmatrix} R_2 & 0 \\ 0 & R_3 \end{bmatrix} \begin{bmatrix} n_{12} & -1 & 0 \\ n_{13} & n_{23} & -1 \end{bmatrix} \right\} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix}$$

giving the same result as in equation (59).

MATRIX INTERPRETATION OF RULES 1 AND 2

The conditions of validity of the above rules were:

1. $Z_{ML} = 0$, $Z_{LM} = 0$; i.e., no impedance coupling between magnetizing and load meshes.

2. Z_{MM} be a diagonal matrix; i.e., no impedance coupling between the individual magnetizing meshes.

Under these conditions, as was shown in equation (62), $NZ_{MM}N_t$ represents the "impedances due to the windings." This expression for the impedance may be expanded as

$$N(Z_1 + Z_2 + \cdots + Z_k)N_t = NZ_1N_t + NZ_2N_t + \cdots + NZ_kN_t \qquad (63)$$

where

$$Z_1 = \begin{bmatrix} z_{11} & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & & & & \\ \cdot & \cdot & \cdot & \cdots & \cdot \end{bmatrix}, \quad Z_2 = \begin{bmatrix} \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & z_{22} & \cdot & \cdots & \cdot \\ \vdots & & & & \\ \cdot & \cdot & \cdot & \cdots & \cdot \end{bmatrix}, \ldots, Z_k = \begin{bmatrix} \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \vdots & & & & \\ \cdot & \cdot & \cdot & \cdots & z_{kk} \end{bmatrix} \tag{64}$$

The rows of zeros in, say, Z_i, nullify the effect of the corresponding rows of the following matrix, namely, N_t, and the columns of zeros in Z_i nullify the effect of the corresponding columns of the preceding matrix, namely, N. Therefore, for each diagonal element of Z_{MM}, say for z_{ii}, only one row of N and one row of N_t contribute nonzero terms. The role that z_{ii} plays in the impedance matrix is found by

$$z_{ii} \begin{bmatrix} n_{1i} \\ n_{2i} \\ \vdots \\ n_{ei} \end{bmatrix} \begin{bmatrix} n_{1i} & n_{2i} & \cdots & n_{ei} \end{bmatrix} = z_{ii} \begin{bmatrix} n_{1i}^2 & n_{1i} n_{2i} & \cdots & n_{1i} n_{ei} \\ n_{2i} n_{1i} & n_{2i}^2 & \cdots & n_{2i} n_{ei} \\ \vdots & & \vdots & \\ n_{ei} n_{1i} & n_{ei} n_{2i} & \cdots & n_{ei}^2 \end{bmatrix} \tag{65}$$

which is the statement, in matrix form, of Rules 1 and 2.

COMMON IMPEDANCE TO TWO MAGNETIZING MESHES

The above reasoning may be extended to the case where Z_{MM} is not a diagonal matrix. Here, the magnetizing meshes share impedances in common. Again, as in (63), the product, $NZ_{MM}N_t$, is broken into a sum of products. Now if there is an impedance, z_{ij}, common to the magnetizing meshes, i and j, it will appear in the ith and jth diagonal terms of Z_{MM} as well as in the off-diagonal positions, ij and ji (assuming the coupling is not due to mutual inductance), and the terms of the impedance matrix containing z_{ij} will be either in the form, below, of (66) or of (67). The first expression applies provided currents in the reference directions of the two magnetizing meshes aid each other as they pass through z_{ij}.

$$z_{ij}\begin{bmatrix} n_{1i} & n_{1j} \\ \vdots & \vdots \\ n_{ei} & n_{ej} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} n_{1i} & \cdots & n_{ei} \\ n_{1j} & \cdots & n_{ej} \end{bmatrix} = z_{ij}\begin{bmatrix} n_{1i}+n_{1j} \\ \vdots \\ n_{ei}+n_{ej} \end{bmatrix}\begin{bmatrix} n_{1i}+n_{1j}, & \cdots, & n_{ei}+n_{ej} \end{bmatrix} \qquad (66)$$

If the currentsoppose, the expression that must be used is:

$$z_{ij}\begin{bmatrix} n_{1i} & n_{1j} \\ \vdots & \vdots \\ n_{ei} & n_{ej} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} n_{1i} & \cdots & n_{ei} \\ n_{1j} & \cdots & n_{ej} \end{bmatrix} = z_{ij}\begin{bmatrix} n_{1i}-n_{1j} \\ \vdots \\ n_{ei}-n_{ej} \end{bmatrix}\begin{bmatrix} n_{1i}-n_{1j}, & \cdots, & n_{ei}-n_{ej} \end{bmatrix} \qquad (67)$$

The following rule summarizes these results:

Rule 4: If an impedance, z, is common to the magnetizing meshes of two transformers, form the sum or difference of the winding-turns in each mesh, respectively, and apply these by Rules 1 and 2.

If the two meshes pass through z in the same direction, the winding-turns are summed, if they pass in opposite directions, the winding-turns of one (it is immaterial which one) are subtracted from the other.

If two magnetizing meshes are coupled by mutual inductance or if more than two magnetizing meshes are coupled together by passing through a common impedance, the formation of the impedance matrix by rules, rather than by matrix multiplication, becomes increasingly complex.

ILLUSTRATION USING RULE 4

A network, illustrating the use of Rule 4, is given in Figure 19. Because z_1 appears in the first two magnetizing meshes, directions opposing, the difference of the winding-turns of the first and second transformer is taken as shown by the numbers encircled in the figure. The impedance, z_2, is in common to the second and third magnetizing meshes with their directions aiding. Winding-turn sums are formed, shown in the triangles.

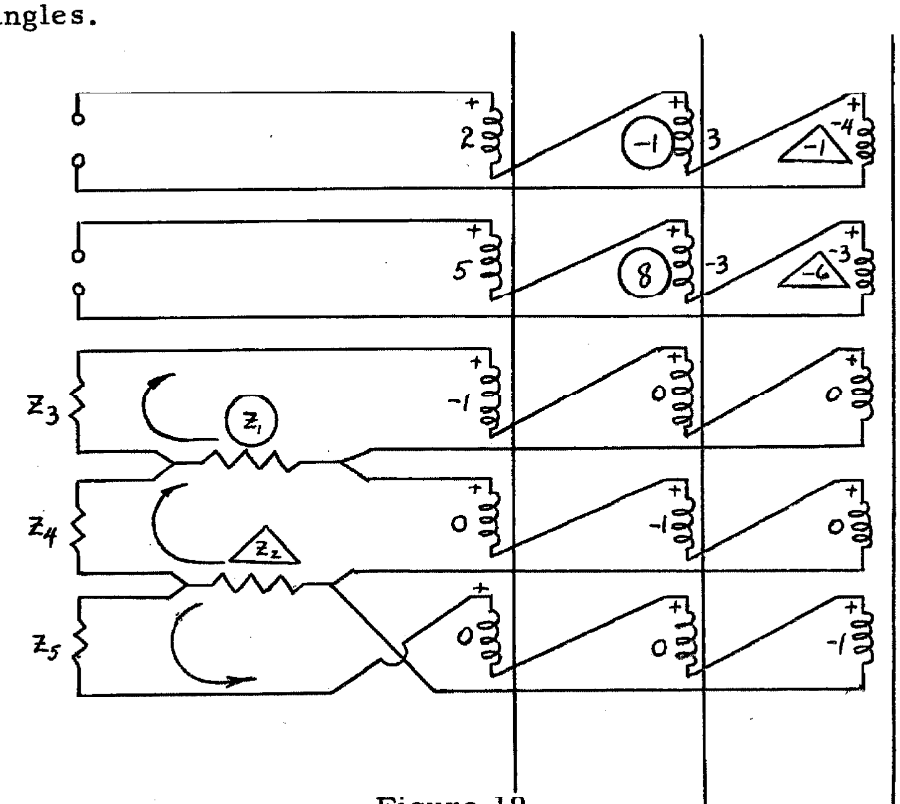


Figure 19

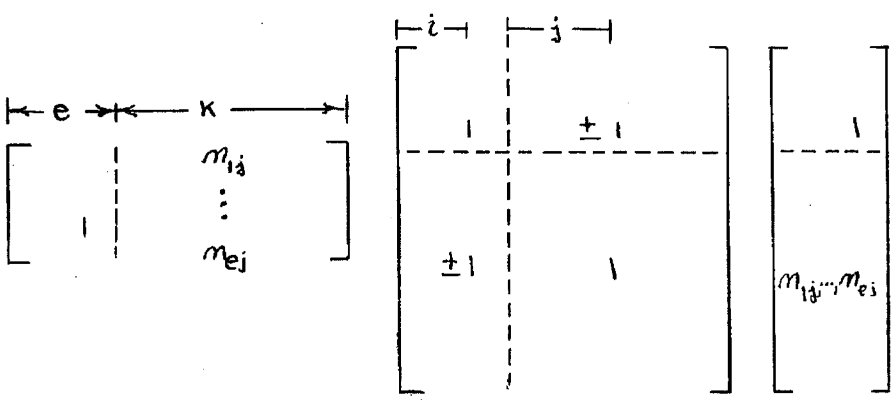
The contribution of z_3, z_4, and z_5 to the impedance matrix is found by Rules 1 and 2, while Rule 4 is applied for z_1 and z_2. The result, for example, for z_1, is:

$$\begin{bmatrix} (-1)^2 z_1 & (-1)(\gamma) z_1 \\ (-1)(\gamma) z_1 & (\gamma)^2 z_1 \end{bmatrix} \tag{68}$$

In the cases covered by the first four rules, the winding-turns enter as products taken two at a time. If the signs of all the winding-turns were changed, the result would be the same. If this were done in the example of the Cauer network, all the coefficients in the impedance matrix would become positive.

IMPEDANCE COUPLING BETWEEN LOAD AND MAGNETIZING MESHES

Assume impedance "z" is common to the ith load mesh and the jth magnetizing mesh, not due to mutual inductance but due to the two meshes passing through the same branch. To find the terms in the mesh impedance matrix containing z, all impedances except z may be replaced by zeros in the matrix Z of equation (50). This will make each element in all the rows except two of them equal to zero. Hence, in the matrix postmultiplying Z, each element in all rows except for two may be set to zero. Similarly, due to the columns of zeros in Z, each element in all columns, except two, in the matrix premultiplying Z are set to zero. Equations (50) thus become



(69)

where the zeros have not been written in. An equivalent matrix is

$$e\left\updownarrow\begin{bmatrix} n_{ij} \\ \vdots \\ n_{ij} \pm 1 \\ \vdots \\ n_{ej} \end{bmatrix}\right. \quad \underbrace{\begin{bmatrix} n_{ij}, \ldots, n_{ij} \pm 1, \ldots n_{ej} \end{bmatrix}}_{e} \tag{70}$$

where the upper sign is taken if the two meshes aid as they are traced through z, and the lower sign if they oppose. In summary, there is

Rule 5: To find the terms in the mesh impedance matrix containing an impedance "z" common to a given load mesh and to a transformer magnetizing mesh, consider the winding of that transformer in the given load mesh to be algebraically increased or decreased (according to whether the meshes aid or buck) by one turn, and, with the other windings unchanged, apply Rules 1 and 2.

Using the five rules given, the selection of the magnetizing meshes from those meshes having an independent set of winding-turns may be judged according to the relative complexity of the resulting impedance matrix.

IV. SINGULAR IMPEDANCE MATRICES AND SUPERFLUOUS TRANSFORMERS

POSSIBLE DEGENERACIES IN TRANSFORMERLESS NETWORKS

Given a network, the mesh equations can be found by one of the methods in the previous chapter. Quite likely, the equations will turn out to be nonsingular permitting one to solve for the mesh currents in terms of the applied voltages. Singular cases, however, are possible, and several will be discussed in this chapter starting with those networks not necessarily involving ideal transformers. It will be assumed that the equations are not singular because of improper selection of the meshes in the analysis, as this situation (for example, choosing the same path in the network as two independent meshes) may be avoided by use of the "tree technique."

CASE 1: The voltage sources are not independent, and therefore the sum of the corresponding rows in the impedance matrix would be zero. An example occurs if the voltage sources form a mesh so that the sum of their voltages must be zero, as would be the case if four voltage sources are applied to a four terminal network.

CASE 2: The voltage sources are dependent, but only at a particular frequency or set of frequencies. An interesting example of this type of circuit has been given by E. Achard (R-11). Although the circuit contains no transformers, it behaves as an ideal transformer for one frequency. The circuit is shown in Figure 20.

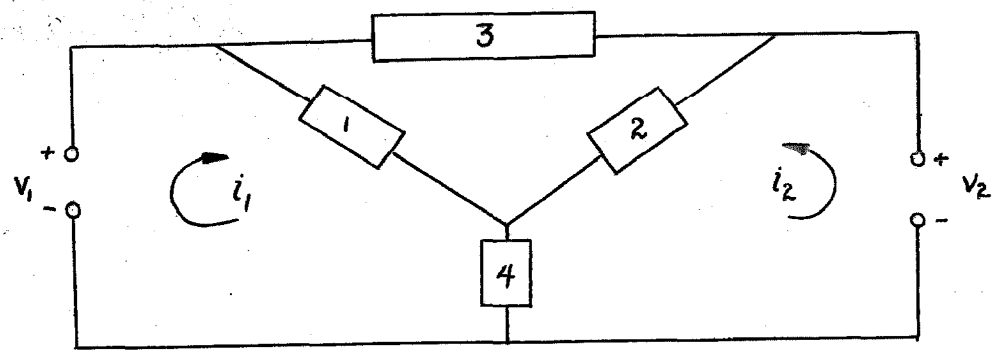


Figure 20

Achard's "Perfect Transformer"

The four elements comprising the network are pure reactances, two of them capacitors and the other two inductors. Their values are chosen so that at the frequency of operation the sum of the impedances of 1, 2, and 3 is zero and the sum of the admittances of 1, 2, and 4 is zero. The primary to secondary turns ratio equals the negative of the ratio of the impedances of elements 1 to 2. This turns ratio may be chosen either positive or negative.

CASE 3: The network is in resonance. If a voltage source is applied to the network, an infinite current will result. An example of a network containing a dissipative element but resonant at zero frequency is given in Figure 21.

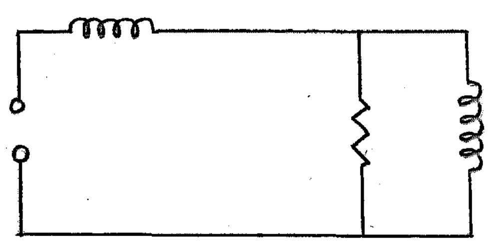
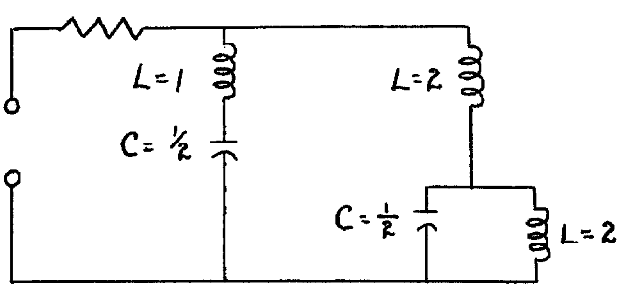


Figure 21

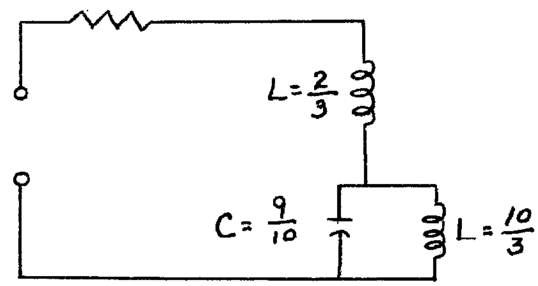
CASE 4: All the impedance terms for a mesh which contains no voltage source become zero at a particular frequency.

The networks under Case 1 are said to have "reducible" impedance matrices. That is, by means of a transformation necessitating ideal transformers, the impedance matrix may be replaced by one that is non-singular but of a lesser order, that is, with a smaller number of rows and columns. Cauer, (R-10), has treated the general case.

The usual analysis in Cases 2 and 3, in which the currents are indeterminant at discrete frequencies, is to solve for the relative amplitude of the currents. In Case 4, a non-singular impedance matrix may be found by eliminating as variables the currents of the meshes that contain no voltage sources, and thus also reducing the number of mesh equations. It may be possible to interpret these new equations as those of a network with fewer meshes than in the original. Figures 22 and 23 illustrate two examples of this.

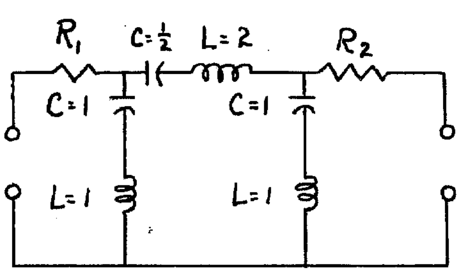


Non-accessible mesh degenerate at $w^2 = 1/2$.

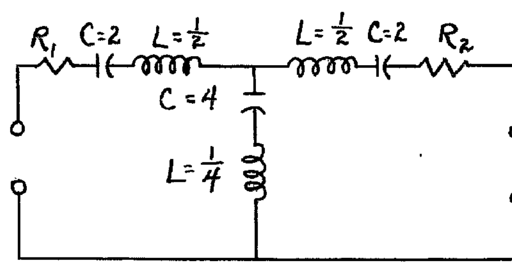


Equivalent network without this mesh.

Figure 22



Non-accessible mesh degenerate at w = 1.



The wye equivalent.

Figure 23

SINGULAR MESHES CONSISTING ONLY OF TRANSFORMER WINDINGS

Meshes which contain no impedance elements may be characterized in three ways. (R-3).

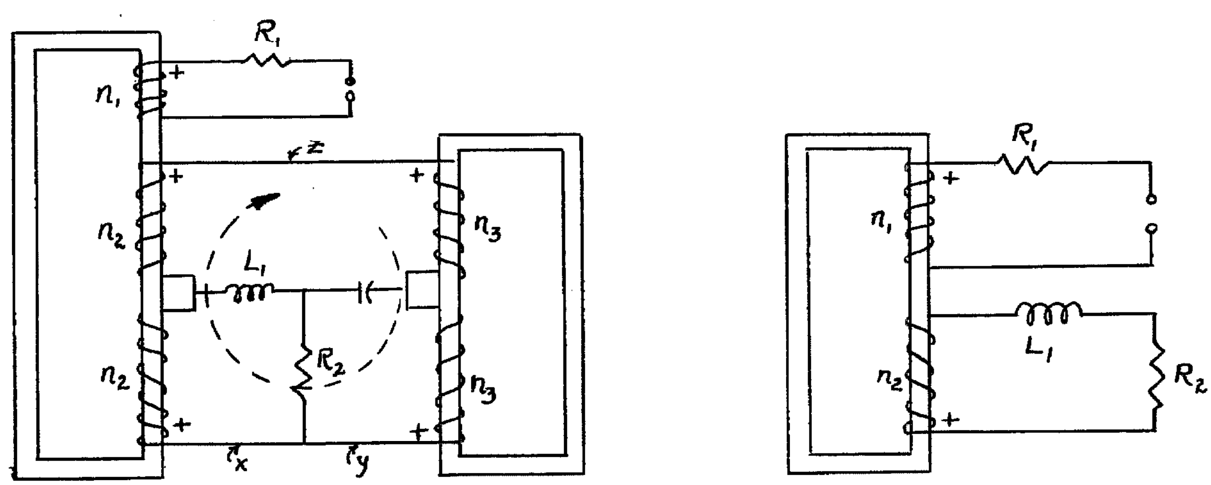
1. Of the windings in the mesh, the sum of the winding-turns due to each transformer is zero, and there are no voltage sources in the mesh.

2. Of the windings in the mesh, the sum of the winding-turns due to one transformer, at least, is not zero, and there are no voltage sources in the mesh.

3. The mesh consists of both windings and voltage sources.

Of these situations, the first will cause the impedance matrix to be singular, the second will not (but, instead, indicates the presence of superfluous transformers), and the third might possibly.

In the first situation, the network equations in terms of currents and flux derivatives will contain a row with nothing but zeros to describe the particular mesh. The current of this mesh is indeterminant. It may be set to zero by open-circuiting arbitrarily a branch in it. This is illustrated in Figure 24.



(a) (b)

Figure 24

The circuit in part (b) of the figure results when the mesh (indicated by a dashed line) is open-circuited either at point x, y, or z. In this type of circuit reduction, at least one winding can be removed, and the circuit equations written in terms of fewer variables. The removal of one winding may justify the removal of several, and, as in Figure 24, even the removal of a transformer.

SUPERFLUOUS TRANSFORMERS

For a mesh consisting only of windings, that is, containing no branches with impedance or voltage sources, there is a mesh equation in the form:

$$0 = 0\, i_1 + \cdots + 0\, i_m + n_a \dot{\phi}_a + \cdots + n_k \dot{\phi}_k \tag{71}$$

Now, as required in the second situation listed above, if the winding-turn sum for one of the transformers, say, n_a, is not zero, then the following "permissible alteration" can be made. The windings of transformer "a" may be first multiplied by $-n_b/n_a$ and then these turns added to those on "b", then by $-n_c/n_a$ and the turns added to those on "c", etc., finally reducing equation (71) to

$$0 = n_a \dot{\phi}_a \tag{72}$$

Thus, $\dot{\phi}_a$ will be zero, and all the windings of transformer "a" may be replaced by short-circuits. By this reduction, the number of meshes and the number of multiwinding transformers have both been reduced by one, leaving "e", the number of effective degrees of freedom, of the circuit unchanged.

REPLACING MULTIWINDING TRANSFORMERS BY TWO-WINDING TRANSFORMERS

A situation involving superfluous transformers occurs if it is desired to replaceeach multiwinding transformer by a set of two-winding transformers. In Figure 25, parts (b) and (c), two of the possible equivalent circuits for the multiwinding transformer of part (a) are given.

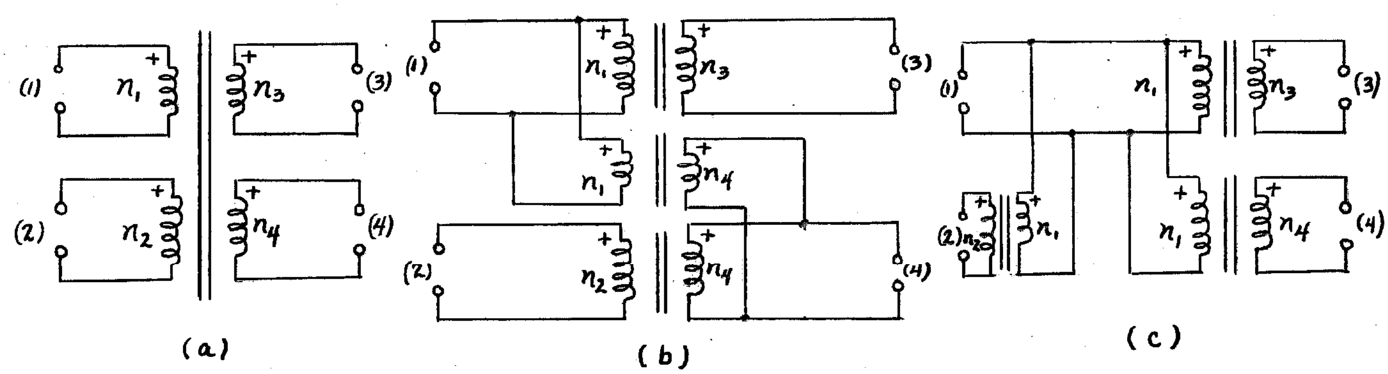


Figure 25

If the equivalent circuit is built up one transformer at a time, first two of the terminal pairs are magnetically linked; next, a third terminal pair is linked to these, and so on, each transformer (except the first) joining to the group one additional pair of terminals. The total number of transformers needed is one less than the number of terminal pairs, that is, windings on the multiwinding transformer. The two-winding transformers may be connected in any manner as long as they link all the terminal pairs together. (R-3).

The number of turns for the two-winding transformers are chosen by the rule that each winding (and there may be several) across a given terminal pair has the same number of turns that were on the winding originally across that terminal pair. The ampere-turn and volt-per-turn equations may be written to prove that this technique furnishes a valid equivalent circuit.

As an additional example, a five mesh Cauer network of the type previously given is drawn in Figure 26 so that it contains only two-winding transformers.

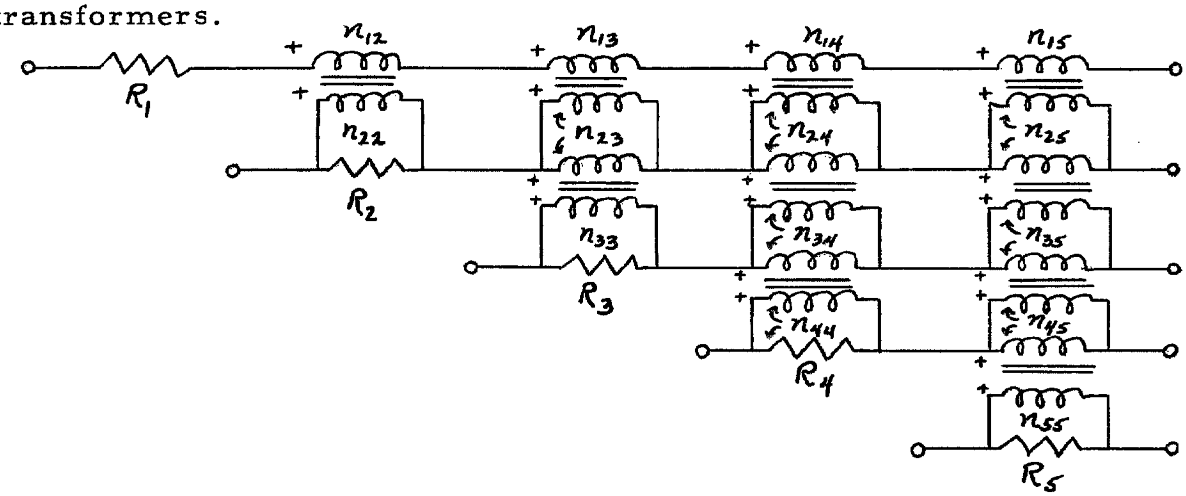


Figure 26

TRANSFORMERS INCIDENTALLY SUPERFLUOUS

The transformers that were described as superfluous in the previous two sections were so because of the particular way in which they were interconnected. It was shown that the flux deriviative for such a transformer was zero, allowing its windings to be replaced by short-circuits. An example, shown in Figure 27 (a), is now given in which the flux derivative of a transformer is zero due to the particular values assigned to the impedances and turns in the circuit.

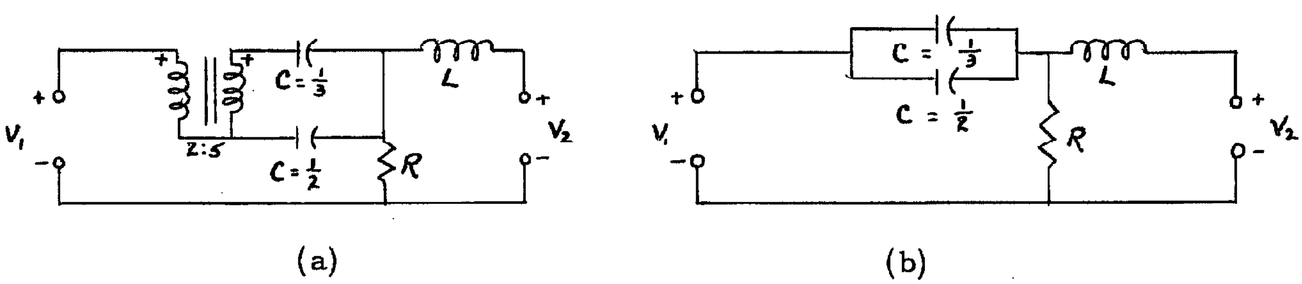


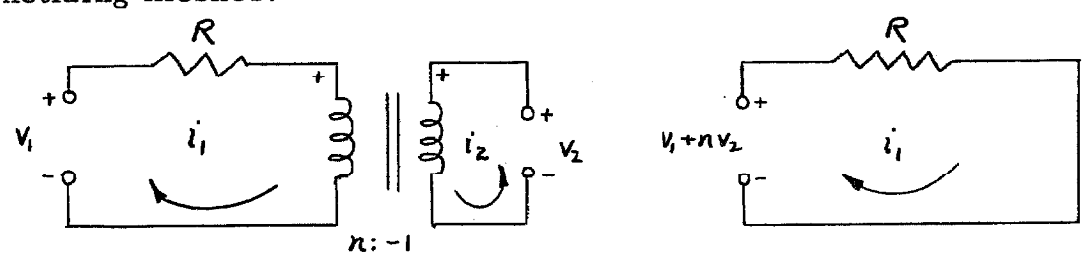
Figure 27

To check that $\dot{\phi}$ is zero, as claimed, one might write the network equations in terms of the three mesh currents and $\dot{\phi}$, and solve for $\dot{\phi}$. The value of $\dot{\phi}$ will be found to be zero for any frequency of operation, and the equivalent circuit of Figure 27 (b) will be justified. Other examples could be given in which the $\dot{\phi}$ of a transformer would be zero but for only one frequency of operation.

In the analysis of networks involving ideal transformers previously given, the flux derivatives were used to supplement the mesh currents as variables in order to facilitate the writing of the equations. In resolving these equations it was not necessary to solve for the flux derivatives. This, in fact, is one of the principal advantages in the representation of constraints by the use of Lagrange multipliers. However, to check if a transformer might be incidentally superfluous, it would be necessary to compute numerically whether or not the individual flux derivatives were zero. This effort would be warranted, it would seem, only in exceptional cases.

VOLTAGES APPLIED TO ALL-WINDING MESHES

A mesh consisting of a voltage source and transformer windings but no impedances (shown, for example, in mesh 2 in the circuit of Figure 28 (a)) may be taken as either a load mesh or a magnetizing mesh, unless, of the windings in this mesh, the sum of the winding-turns due to each transformer is zero. (If this latter situation were the case and the applied voltages not zero, then infinite currents would result.) Choosing the mesh with only windings and a voltage source as the magnetizing mesh for a transformer implies by Rules 1 and 2 in the chapter on analysis that the other windings of the transformer in the load meshes of the network would have no self or mutual impedance associated with them. Instead, these windings would appear only as voltage sources, the magnitude of which is found by Rule 3. One may physically represent on the circuit diagram the process of the elimination of the mesh currents as variables in the equation by replacing these windings by voltage sources of the proper value. This is done in Figure 28 (b) for the circuit in part (a) of that figure, and it may always be done in the case of impedanceless magnetizing meshes.



(a) (b)

Figure 28

If the mesh, with only a voltage source and transformer windings, is selected as a load mesh, the equation of that mesh is just written according to the rules, that is, by associating self and mutual impedances with the windings. For example, if mesh 2 in the circuit of Figure 28 (a) is chosen as the load mesh, rather than the magnetizing mesh, and the turns reduced to standard form, then the equation for this mesh would be

$$V_2 + NV_1 = N^2 R \tag{73}$$

where N = -1/n. Now if the resistance, R, is zero, no matter which of the two ways that the load and magnetizing meshes are selected, the result is a singular, in fact, null matrix.

Now, in general, through the process of choosing a set of magnetizing currents and then eliminating these as variables, it is possible to write an impedance matrix for any network involving ideal transformers. However, as in the case of networks without transformers, this matrix may be singular. Two limitations are thus implied on the formulation of network problems in terms of impedance matrices: first, the necessity of eliminating some of the mesh currents as variables, and, second, the possibility of the equations being singular. There are similar limitations as regards analysis on the nodal admittance basis. These limitations, however, may be overcome by formulation of the network problem in terms of its "scattering" matrix. This method will now be presented and applied to a particular system.

THE SCATTERING MATRIX

Consider the algebraic function

$$S = \frac{z-1}{z+1} = -\frac{y-1}{y+1} \tag{74}$$

where y is the reciprocal of z. In mathematical analysis, this function is called a "bilinear transformation." Note that if z is zero, s has the value minus one, and that if z becomes infinite, s remains finite and approaches the value plus one. In terms of complex variables, the function of equation (74) maps the right half z-plane into the area bounded by the unit circle in the s-plane.

The properties of this function suggest that if the impedance or admittance matrix of a network (or both) contain terms that are either infinite or undefinable, then a combination of these matrices might form a new function to which a value could be assigned. The "scattering" or "efficiency" matrix, "S", assumes this role. This matrix is usually defined in terms of the power transfer between the various terminal pairs of the network as in (R-12). An alternative definition, used when it is practical to form the impedance matrix of a network, is

$$S = (Z - 1)(Z + 1)^{-1} = (Z + 1)^{-1}(Z - 1) \tag{75}$$

By means of a limiting process, of the type shown in the next section, the definition (75) can be extended to describe all passive networks.

NETWORKS CONSISTING ONLY OF IDEAL TRANSFORMERS

To take an example that cannot be analyzed by the use of impedance or admittance matrices, the networks that consist only of ideal transformers will be considered and the expression for their scattering matrices determined. First, the transformers constituting such a network are considered non-ideal so that an impedance matrix can be written for the system. From this impedance matrix a scattering matrix is computed using the definition in equation (75), and the limit of the expression is taken as the transformers become idealized.

The model for the non-ideal transformer will be that of equation (9) in the first chapter. That is, the transformerswill be considered as multiwinding transformers that have no leakage flux, but for which the sum of the ampere-turns is not zero but equal to the reluctance, R, of the core times the core flux. The core flux may be written as $\dot{\phi}$ divided by the operator, "p".

The equations for a three mesh network consisting of two non-ideal multiwinding transformers are in the form, (R-3),

$$
\begin{aligned}
V_1 &= 0i_1' + 0i_2' + 0i_3' + n_{1a}\phi_a' + n_{1b}\phi_b' \\
V_2 &= 0i_1' + 0i_2' + 0i_3' + n_{2a}\phi_a' + n_{2b}\phi_b' \\
V_3 &= 0i_1' + 0i_2' + 0i_3' + n_{3a}\phi_a' + n_{3b}\phi_b' \\
0 &= n_{1a}i_1' + n_{2a}i_2' + n_{3a}i_3' - \frac{R_a}{p}\phi_a' + 0\phi_b' \\
0 &= n_{b}i_1' + n_{2b}i_2' + n_{3b}i_3' + 0\phi_a' - \frac{R_b}{p}\phi_b'
\end{aligned}
\qquad (76)
$$

Again, in equations of this form, the windings of the various cores may be multiplied and added to each other the same as for ideal transformers, since these alterations change only the value of the flux derivatives and not that of the voltages or the currents. Carrying out these alterations, the equations in (76) may be brought to the form:

$$
\begin{aligned}
V_1 &= 0i_1' + 0i_2' + 0i_3' + n_a\phi_a' + n_b\phi_b' \\
V_2 &= 0i_1' + 0i_2' + 0i_3' - \phi_a' + 0\phi_b' \\
V_3 &= 0i_1' + 0i_2' + 0i_3' + 0\phi_a' - \phi_b' \\
0 &= n_a i_1' - i_2' + 0i_3' + f_{aa}\phi_a' + f_{ab}\phi_b' \\
0 &= n_b i_1' + 0i_2' - i_3' + f_{ba}\phi_a' + f_{bb}\phi_b'
\end{aligned}
\qquad (77)
$$

where f_{aa} and f_{ab} are proportional to R_a/p, and f_{ba} and f_{bb} are proportional to R_b/p.

Equations (77) may be written in matrix form, in which, by use of compound matrices, the equations are divided as were the equations describing magnetizing and load meshes. The result is:

$$
\begin{aligned}
V_P &= 0 i_P + 0 i_Q + N \phi' \\
V_Q &= 0 i_P + 0 i_Q - \phi' \\
0 &= N_t i_P - i_Q + F \phi'
\end{aligned}
\qquad (78)
$$

This matrix equation has the same form for a network consisting of these non-ideal transformers, regardless of the number of meshes. The subscripts, "P" and "Q", are used to distinguish this from the case in which magnetizing and load meshes were selected, since, here, the transformers are non-ideal, and the flux derivatives may be eliminated from the equations without simultaneous elimination of the so-called "magnetizing" currents. The equations without $\dot{\phi}$ are:

$$
\begin{aligned}
V_P &= -NF^{-1} N_t i_P + NF^{-1} i_Q \\
V_Q &= +F^{-1} N_t i_P - F^{-1} i_Q
\end{aligned}
\qquad (79)
$$

giving the impedance matrix for a system of non-ideal transformers to be:

$$
Z = \left| \begin{array}{c|c} -NF^{-1} N_t & NF^{-1} \\ \hline F^{-1} N_t & -F^{-1} \end{array} \right| \qquad (80)
$$

The scattering matrix may also be partitioned, and written in terms of Z. By equation (75) it is:

$$
S = \left[\begin{array}{c|c} S_{PP} & S_{PQ} \\ \hline B_{QP} & S_{QQ} \end{array} \right] \qquad (81)
$$

$$
= \left[\begin{array}{c|c} -1-NF^{-1} N_t & NF^{-1} \\ \hline F^{-1} N_t & -F^{-1}-1 \end{array} \right] \left[\begin{array}{c|c} 1-NF^{-1} N_t & NF^{-1} \\ \hline F^{-1} N_t & 1-F^{-1} \end{array} \right]^{-1}
$$

Expanding (81), the four sub-matrices of S are explicitly determined, each in a form that permits the limit to be taken as F approaches zero, that is, permitting the transformers to become idealized. Thus, for example,

$$S_{PP} = (1 - N(F+1)^{-1}N_t)(1 + N(F+1)^{-1}N_t)^{-1} \tag{82}$$

and as F approaches zero,

$$\begin{aligned} S_{PP} &\rightarrow (1 - NN_t)(1 + NN_t)^{-1} \\ &= 1 - 2(1 + NN_t)^{-1} \end{aligned} \tag{83}$$

The other portions of the scattering matrix for a system composed only of ideal transformers are, (R-9),

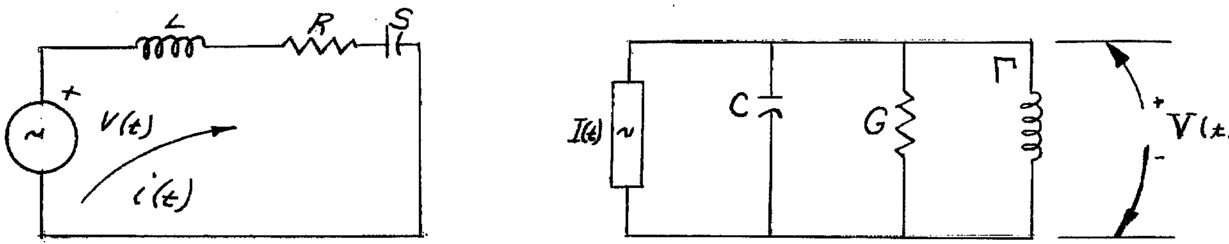
$$\begin{aligned} S_{PQ} &= -2(1 + NN_t)^{-1}N \\ S_{QP} &= -2(1 + N_tN)^{-1}N_t \\ S_{QQ} &= 2(1 + N_tN)^{-1} - 1 \end{aligned} \tag{84}$$

Belevitch has applied this scattering matrix formula to the design of ideal telephone conference networks, (R-9). He has also given the necessary and sufficient conditions on a matrix that it be the scattering matrix of an electrical network with physically realizable, passive elements (including ideal transformers), and has given a synthesis procedure based on the scattering matrix. (R-13). Belevitch's procedure, which applies even to networks for which both the impedance and admittance matrices are singular, has been summarized in English by Bayard, (R-14).

V. DUALITY AND TOPOLOGY

THE ELECTRICAL DUAL

Consider the circuits of Figure 29 (a) and (b).



(a) Figure 29 (b)

The equations for these circuits are, respectively,

$$V(t) = L\frac{di(t)}{dt} + Ri(t) + S\int_0^t i(t)\,dt \tag{85}$$

$$I(t) = C\frac{dV(t)}{dt} + GV(t) + \Gamma\int_0^t V(t)\,dt \tag{86}$$

These two equations are numerically equivalent, if, by multiplying the second one through by a constant, "r", (which is in the units of resistance), they could be equated term by term. This requires

$$V = rI, \qquad i = \frac{V}{r}$$
$$L = r^2 C, \qquad R = r^2 G, \qquad S = r^2 \Gamma \tag{87}$$

Two networks that are related in similar fashion as (a) is to (b), namely such that the equations of one are equivalent to the equations of the other, provided the roles of current and voltage are interchanged, are called the electrical dual of each other. In one (here network "a"), the equation is the sum of the voltage drops of a mesh, while for the other (here, "b"), the equation is the sum of currents at a node. This dual role of a mesh and a node (or more generally, a mesh and a node-pair) makes the subject of the geometry of the network prominent in the discussion of electrical duality.

PLANAR NETWORKS AND THE TOPOLOGICAL DUAL

A planar network is one that can be diagramed on a plane without wires (i.e., branches) crossing over each other. A property of a planar network is that it also can be drawn on the surface of a sphere without having wires crossing over each other. (R-15). If this were done, the surface of the sphere would be divided into a group of simply connected areas, each bounded by a mesh of the network made up of the branches of the network. Now, in each of these areas a dot may be placed, and for each branch of the network a line may be drawn crossing it and joining the dots of the adjacent areas. The dots represent the nodes and the lines represent the branches of a new network. This network is called the topological dual of the first. It is a property of all networks that can be drawn on a sphere without wires crossing over each other (in other words, for a planar network) that each has a dual. (R-16).

In transferring a network diagram from being drawn on a sphere to being drawn on a plane, one of the deliniated areas on the sphere (any one could be chosen) becomes identified with the area of the plane that surrounds the network. Thus, in finding the dual of a network drawn on a plane, a dot must also be placed in the area of the plane that surrounds the network. This process is illustrated in Figure 30. In part (a) of that figure, a network is given, and by means of dashed lines the method of forming the dual is shown. The dual network is drawn separately in part (b).

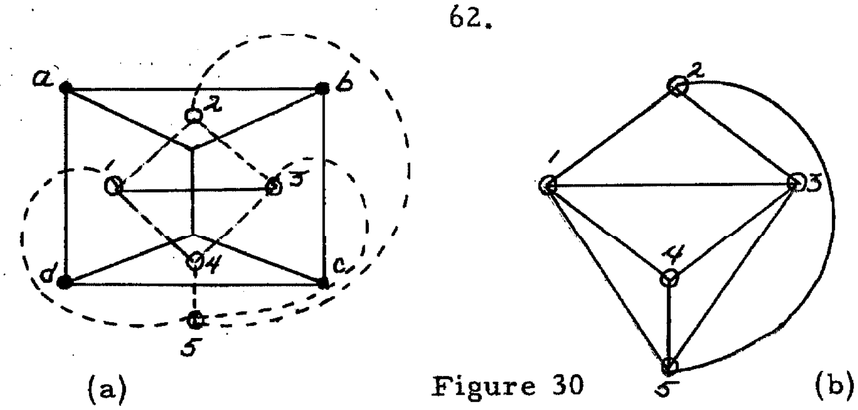


Figure 30

In the formation of duals, one separate part of a network is treated at a time. Two parts connected only at one point are separated and treated individually.

ORIENTED BRANCHES, MESHES, AND CUT-SETS

A branch is "oriented" by assigning a reference direction to it, while a mesh is oriented by assigning to it a reference direction in which it may be traced. For planar networks, a convention may be applied that the positive direction for each mesh bounding one of the simply connected areas is the direction a traveler takes to keep the bounded area on his left. For a network drawn on a plane (rather than on a sphere), this reference direction is counter-clockwise for all interior meshes and clockwise for the one exterior mesh. (It is assumed that there is one, non-separable part to the network.)

The group of branches comprising a mesh form a "set." It is useful to define as another type of set the group of branches that are the dual to those in a mesh. The property of this group of branches, as can be noted from the procedure in finding the dual, is that they, and only they, are all joined to a particular node. If the branches in this set were cut, they would separate the node from the rest of the network. More generally, a "cut-set" is defined as that set of branches which if cut would increase the number of separate parts of the network, provided that all the branches in the set must be cut in order to do this. (R-17). In analogy to the reference direction assigned to a mesh, a reference direction may be assigned to a cut-set. For the cut-set consisting of all the branches assigned to a particular node, the reference direction will be taken as the direction along the branches a traveler takes to go into the node.

ORIENTATION OF BRANCHES IN THE TOPOLOGICAL DUAL

The branches of a network may be arbitrarily oriented to establish the electrical reference directions for the sources, for the currents and voltages in transformer windings, etc. The question is to orient the corresponding set of branches in the dual in an equivalent manner.

In a planar network, each branch can be assigned as a member of two meshes, which -- according to the convention in the preceding section -- are traced through the branch in opposing directions. That is, the reference direction for each branch will be along the positive direction for one mesh to which it belongs and along the negative direction for the other. Each branch serves as a dividing line between two adjacent areas. The corresponding branch in the dual network (i.e., the "crossing" branch, if the dual is superimposed as in Figure 30 (a)) joins the node associated with one of these areas to that of the other. If the branch in the dual network is given a reference direction, the direction will point in toward one of these nodes and out from the other. That is, each branch in the dual network belongs to two cut-sets, and, according to the convention of the preceding section, the two cut-sets have opposing directions with respect to the branch.

In summary, one can say that each branch in a planar network has a "positive" and a "negative" mesh, and that with each of these meshes there is associated a node in the dual network which, for the purposes of a rule, may be taken, respectively, as the "positive" and "negative" node for the branch connecting these nodes. To illustrate, this rule may be applied to the network of Figure 31 (a). The branches of this network have been arbitrarily oriented. The dual of this network has been drawn in Figure 31 (b), and the orientation of its branches determined by the given rule. As a second illustration, the dual of the network in Figure 31 (b) has been drawn in Figure 31 (c), and the orientation of its branches determined by the same rule.

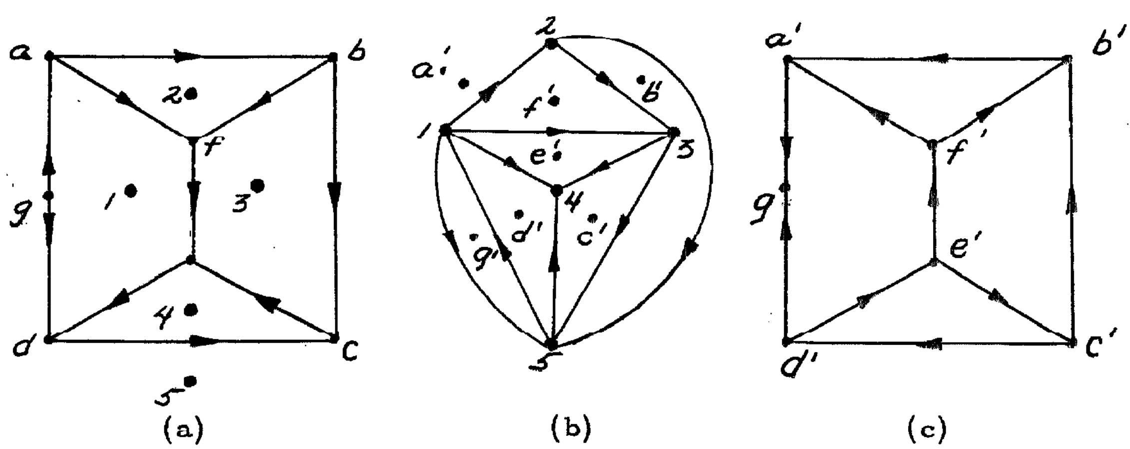


Figure 31

THE DUAL OF A DUAL

Comparing the networks in Figure 31 (a) and (b), one may note that the dual of the dual of a network with oriented branches is the same network with the orientation of the branches reversed. The proof that this is an inherent property of planar networks (and not an accident due to a peculiarity of the rule chosen for orientation of the branches in the dual) may be stated in terms of the diagram in Figure 32. This figure represents the oriented branch of a network upon which is superimposed the crossing branch of the dual network (shown as a dashed line).

Figure 32

A rule for orienting the dashed line with respect to the one it crosses, involves, at least implicitly, the rotation of a branch to be in line with its dual so that the orientation of the two branches may be compared. The rotation is analogous to the 90° rotation associated with the operator, "j" (the square root of minus one). In fact, each planar network may be said to belong to a group of four networks that is isomorphic to the numbers 1, j, -1, and -j which form a group under multiplication by the operator, j. Thus, for instance, if the dual of the dual of the network of Figure 31 (c) were taken, the network of Figure 31 (a) would result.

POLARITY NOTATION OF ELECTRICAL DUALS

Because of the property pointed out in the preceding section, it is necessary, in orienting with respect to each other the branches of dual networks, to distinguish one network as the dual and the other as the inverse dual. For electrical networks, the distinction can be achieved by calling those networks for which the mesh equations are to be written, the "voltage networks," and those for which the node equations are to be written, the "current networks." The reference direction for a branch in a voltage network will be indicated by placing a "-" and a "+" sign along the branch, with the reference direction taken from - to +. The reference direction for a branch in a current network will be marked by an arrowhead. For instance, a branch consisting of a voltage source has its reference polarity indicated by + and - marks if it is in a voltage network, and by an arrowhead if it is in a current network.

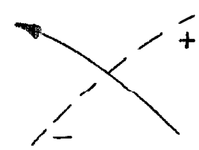


Figure 33

The rule for orienting the branches of the dual or the inverse dual of an electrical network may be given in terms of the diagram in Figure 33. The rule is, that taking the two networks, one superimposed on the other, the branch with an arrowhead is directed across the branch with the polarity marks in such a way as to keep the + polarity mark on its right. (R-1). With this convention, the dual of the network in Figure 34 (a) is given in Figure 34 (b), and the dual of the network in (b) is given in Figure 34 (c). Note that the networks in (a) and (c) are identical.

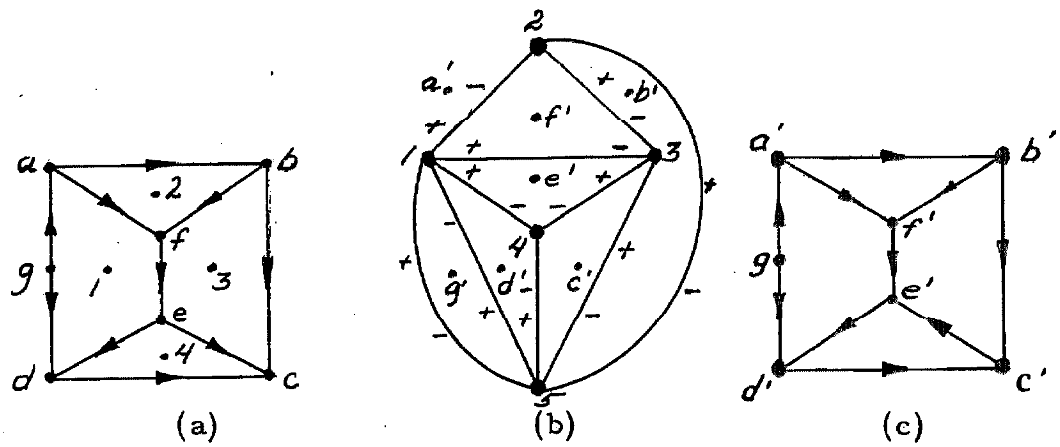


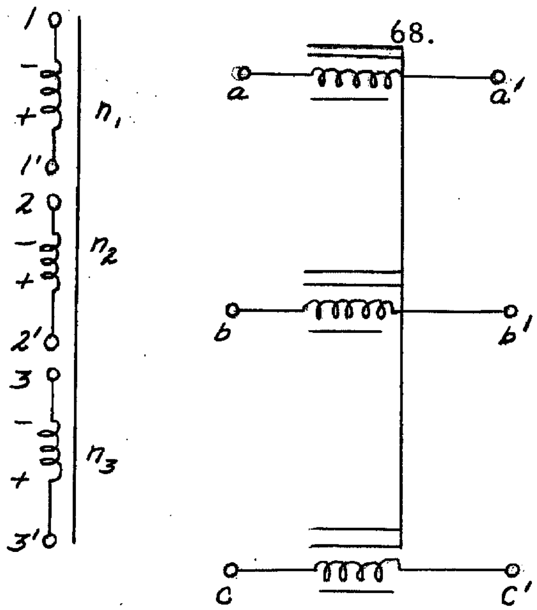
Figure 34

THE DUAL OF AN IDEAL TRANSFORMER

The equations (15) and (16) given in the first chapter for the multiwinding transformer are identical to the ones, (24) and (25), given for the multilimb transformer, provided:

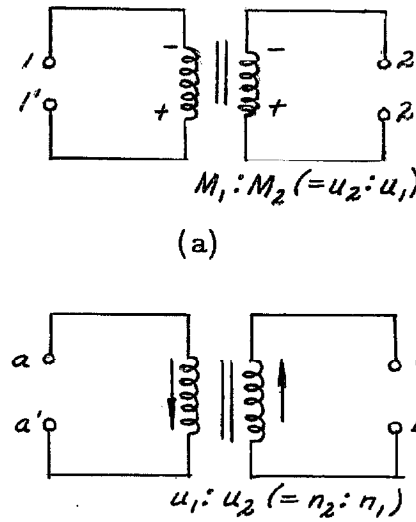
1. The roles of current and voltage are interchanged.
2. The roles of winding-turn and reciprocal-turn are interchanged.
3. There is a one-to-one correspondence between the windings on one transformer and the windings on the other.

Thus, one transformer is the dual of another. Assigning reference polarities by considering the multiwinding transformer as a voltage network and the multilimb transformer as a current network, the dual configurations for three-winding transformers are shown in Figure 35, and for two-winding transformers in Figure 36.



(a) (b)

Figure 35



(a)

(b)

Figure 36

Physically, the dual of a two-winding transformer is the same transformer with the primary and secondary terminals exchanged and the sign of the turns ratio reversed. Figures (6) and (9) are an example of two networks involving ideal transformers that are the dual of each other. Further examples are given in the following sections.

THE DUAL OF CAUER'S NETWORK

The dual of Cauer's network of Figure 18 is given in Figure 37, and the dual of Figure 26, which is the Cauer network made up only of two-winding transformers is given in Figure 38.

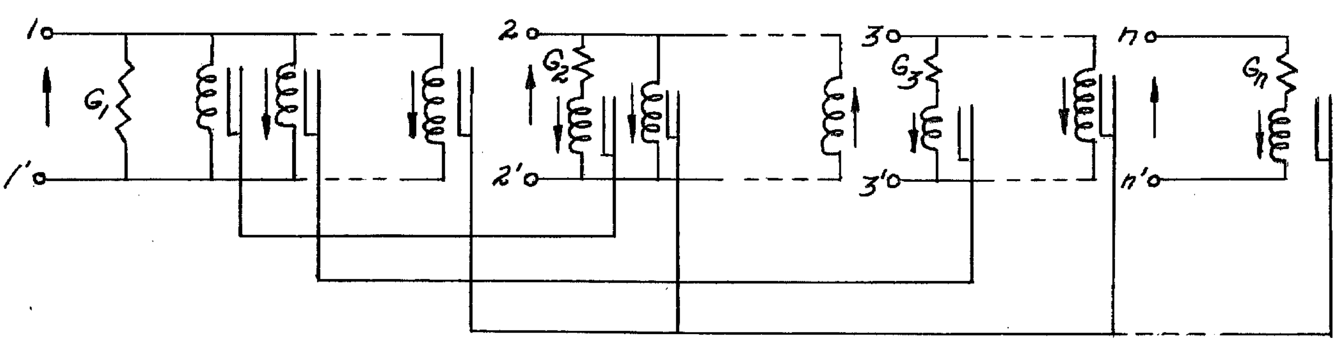


Figure 37

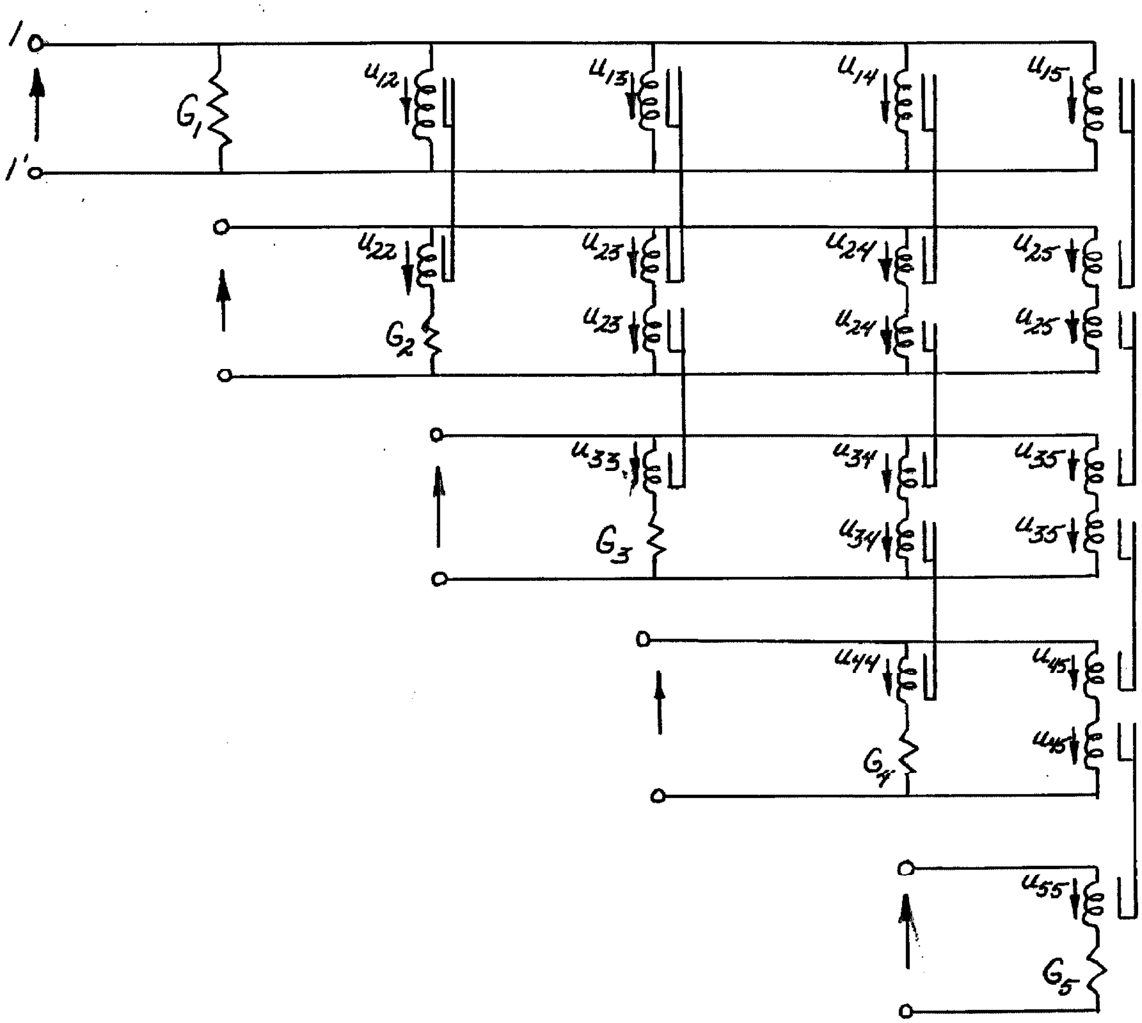


Figure 38

DUALITY AND MUTUAL INDUCTANCE

To find the dual of a pair of branches coupled by mutual inductance, Gardner and Barnes (R-18) essentially replace the coupled inductors by an equivalent "T" shaped circuit made up of three self inductances and then take the dual of the altered circuit. This is shown in Figure 39 (a), (b), and (c).

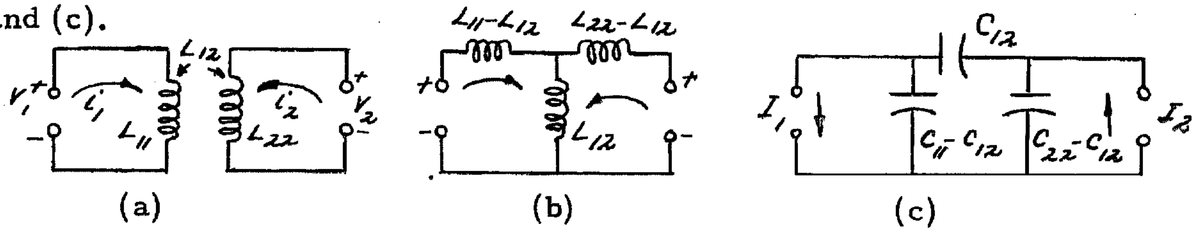


Figure 39

However, if the mutual inductance, L_{12}, is greater than either of the self inductances, an element in the circuit of (b), and its dual in (c), will have a negative parameter so that it will not be realizable.

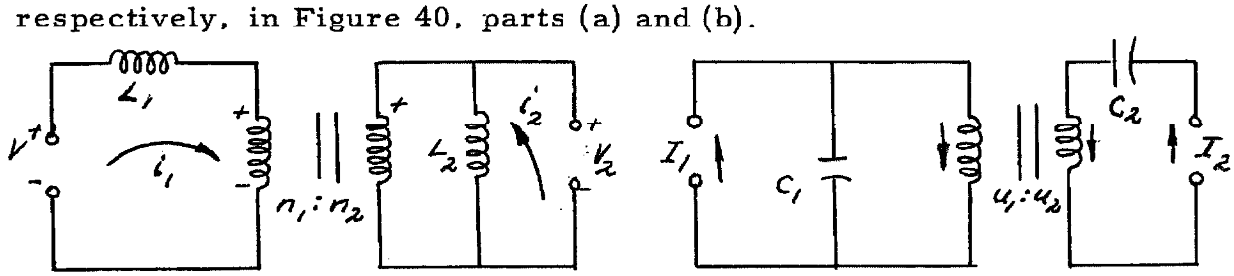
Another approach would be to replace the 4-pole in (a) by its equivalent Cauer network. First, the equations of the 4-pole are written:

$$\begin{aligned} v_1 &= L_{11}\, p i_1 + L_{12}\, p i_2 \\ v_2 &= L_{12}\, p i_1 + L_{22}\, p i_2 \end{aligned} \tag{88}$$

and then the formulas (61) applied

$$L_2 = L_{22}, \quad L_1 = \frac{L_{11}L_{22} - L_{12}^2}{L_{22}}, \quad \frac{n_1}{n_2} = \frac{L_{12}}{L_{22}} \tag{89}$$

giving the parameters of the Cauer network and its dual, shown, respectively, in Figure 40, parts (a) and (b).



(a) (b)

Figure 40

REDUCTIONS IN A PARTICULAR BEAM ANALOGY CIRCUIT

In the solution of aeroelastic problems by means of electrical analogies, circuits involving ideal transformers are used. (R-19). As an example, the circuit for a beam, in vertical bending coupled with torsion, has been used to represent an airplane wing in a vibration analysis as conducted on the Caltech Analog Computer. In such a circuit, particular groupings of elements occur repeatedly. A portion of the circuit, (R-19), is given, showing two of the groupings or "cells" along with the transformer circuit used as a "coordinate transformation" to connect the wing circuit to the fuselage circuit. (In the diagram here, the fuselage has been represented by a current generator.) As an application of the results given in the preceding section, one such group, containing the three capacitors and one transformer of terminals 2, 3, and 4, may be singled out. This grouping is the same as the circuit (Figure 39 (c)) given by Gardner and Barnes for the dual of mutually coupled coils, except that a transformer has been added to eliminate the possibility of negative elements.

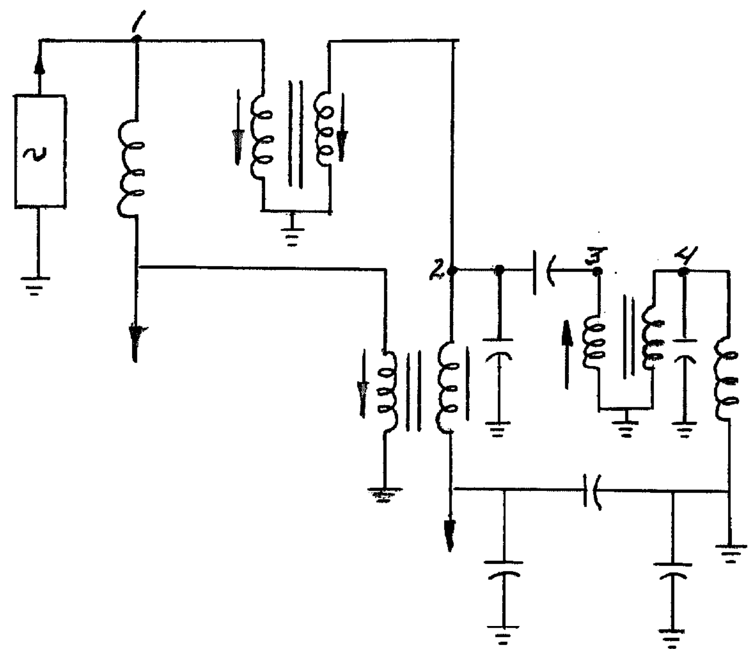


Figure 41

Now, provided that the voltage at terminal 3 is not one of the variables being measured, the grouping singled out may be replaced by the circuit of Figure 40 (b), which contains only two capacitors rather than three. In using this circuit one winding of the ideal transformer will be placed between terminal 2 and ground. However, there is another transformer (placed between terminals 1 and 2) also having a winding between terminal 2 and ground. The two transformers combined are equivalent to a single multiwinding transformer, which has three windings. However, if one of the windings is reversed and its ground terminal combined with that of the adjacent winding, the multiwinding transformer is reduced to a tapped, two-winding transformer. The resulting circuit is shown in Figure 42. Terminals 1, 2, and 4 are identical to those in the given circuit, only terminal 3 has been changed to 3'. Because this grouping of elements occurs repeatedly, the reduction described reduces in the over-all circuit the total number of transformers and the total number of capacitors each by about 10%.

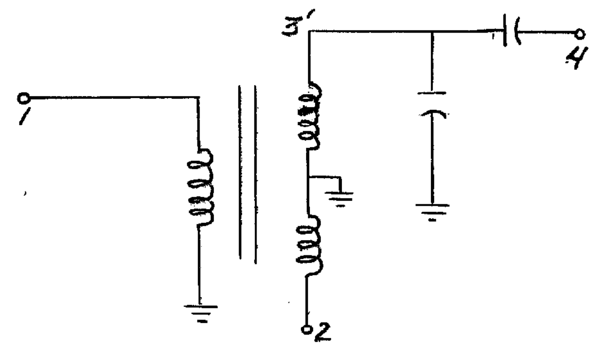


Figure 42

CONNECTION MATRIX AND THE STANDARD BRANCH, NODE MATRIX

So far, the dual circuits presented have relied upon finding the topological dual of the network configuration, a process applicable only to planar networks. The extent of this limitation will be discussed in this and the sections immediately following through the analysis of additional properties of planar networks. First, the connection matrix will be considered.

If the meshes in a non-separable, planar network are selected as suggested previously with each interior mesh traced counter-clockwise and the one exterior mesh traced clockwise, each branch will belong to two meshes, passing through the branch in opposing directions, and the number of meshes will be one greater than the number of independent meshes for the network. This last point can be verified by comparing Euler's Polyhedron Formula (R-16) with the standard formula for the number of independent meshes as given in equation (58). If the connection matrix is written as described in equation (44), with one row for each branch and one column for each mesh (the exterior mesh being included), each row will contain two non-zero entries, one of them +1 and the other -1. This will be considered the standard form for the connection matrix.

Another matrix that is used in describing the topology of a network is the "standard branch, node matrix." Each row in this matrix corresponds to a branch in the network, and each column to a node. The element, a_{ij}, in this matrix is +1 if branch i is directed into node j, -1 if it is directed away from node j, and 0 if it is not connected to node j. Two examples are presented: the matrix and network of Figure 44, and the matrix and network of Figure 45. In regard to notation, it could be mentioned that Foster (R-17) calls the transpose of the standard branch, node matrix the "vertex-element incidence matrix" and says that it was first presented by Kirchhoff, although not in matrix notation, in 1847. Also, Foster uses the label, "element-cut-set incidence matrix" for what is called here, the branch, node-pair matrix.

1	1			-1		
2	1				-1	
3	1					-1
4		1			-1	
5		1				-1
6		1		-1		
7			1			-1
8			1		-1	
9			1			

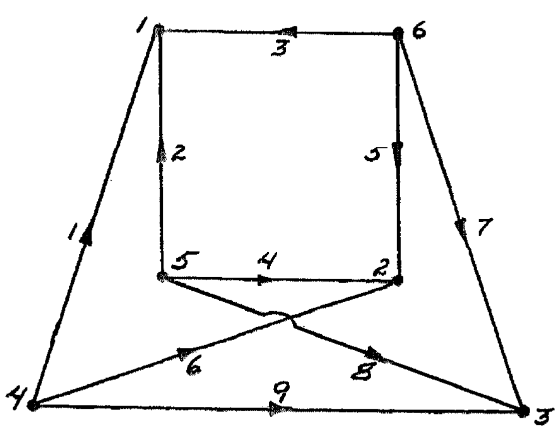


Figure 44

	1	2	3	4	5
1	1				-1
2	1			-1	
3	1		-1		
4	1	-1			
5		1			-1
6		1		-1	
7		1	-1		
8			1		-1
9			1	-1	
10				1	-1

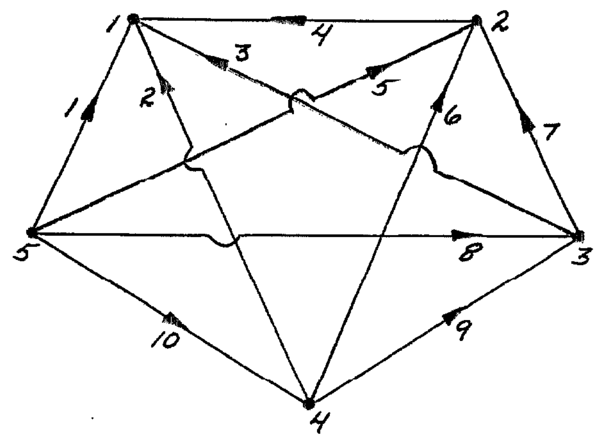


Figure 45

The standard branch, node matrix is a special case of the more general "branch, node-pair matrix," just as the standard connection matrix described for planar networks is a special form of the more general connection matrix. The characteristic of the standard branch, node matrix is that every row contains just two non-zero entries, one a +1 and the other a -1, and that, compared to the general branch node-pair matrix, it has one additional or "bordered" column, the elements of which are a linear combination of the elements of the other columns. The standard connection matrix is related to the more general connection matrix in exactly the same fashion, having just two non-zero entries for each row, a +1 and a -1, and having an additional, non-independent column. However, the standard branch, node matrix is not limited to planar networks, as is the case with the standard connection matrix. For every network, planar or not, a standard branch, node matrix can be formed. Also, for every such matrix, a network configuration can be found by simply drawing an array of dots, one for each node, and then connecting these nodes by branches, one for each row of the matrix, starting from the node having the -1 entry and ending on the node having the +1 entry. Now, since any standard connection matrix has the same form required for the standard branch, node matrix, it could take the place of a standard branch, node matrix, and, by the procedure just given, a network drawn. The matrix taken as a standard connection matrix describes one network, and the matrix taken as a standard branch, node matrix describes another; these two networks are related, one being the dual of the other. In summary, if a network is described by a connection matrix in the standard form, it must be a planar network, and its dual may be drawn by considering the connection matrix as a standard branch, node matrix.

This correspondence between a connection matrix in the standard form and a standard branch, node matrix permits the following test. If a matrix is given, but no corresponding network diagram is available, then, assuming the matrix is one with two non-zero entries in each row, a +1 and a -1 (i.e., the matrix is in the "standard" form), then it can be determined whether or not this matrix is the connection matrix for a planar network. Simply call it a standard branch, node matrix, and draw the corresponding network. Then, if this network can be drawn without wires crossing over each other, the original network was planar and it is equal to the dual of the network that has been drawn.

A test to see if a given network can be drawn without its wires crossing over each other (i.e., to see if the network is planar) without actually drawing all the possible circuit configurations may be devised from a theorem presented by Kuratowski and by Whitney, (R-20). The theorem states that if a network is non-planar, it can be reduced by removing, possibly, some of the branches from the circuit, to one of the two basic non-planar circuits. These two circuits are those chosen in the example of Figure 44 and of Figure 45. In the reduction process, branches in series are replaced by a single branch, which means that each node in the reduced network is connected to at least three nodes. By example, one may consider the network in Figure 46. By removing one branch (marked "x"), there result two branches in series. These are replaced by a single branch by removing a node (marked "y"); the resulting network is identical with that of Figure 44.

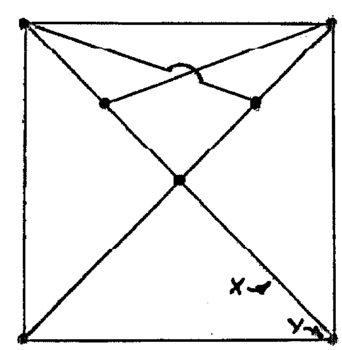


Figure 46

The Kuratowski-Whitney theorem may be restated to the effect that if a network is non-planar, its standard branch, node matrix must reduce to one equivalent to the standard branch, node matrix describing a basic, non-planar circuit. The standard branch, node matrices of the two basic, non-planar circuits are written out in Figures 44 and 45. The permissible steps in the reduction of a standard branch, node matrix are:

1. Striking out any row (removing a branch).
2. In case a column contains only two entries, the rows corresponding to these entries are combined into one (reducing two branches in series to one branch), either by adding or

subtracting them term by term, so that the two entries in question will cancel, leaving no entries in their column.

3. Multiplication of any column or row by -1.

The order in applying these steps is immaterial, and they may be applied repeatedly.

THE CONNECTION MATRIX NOT IN STANDARD FORM

If a network contains more than one separate part, the columns of its connection matrix may be divided into sets corresponding to each separate part. The property of these sets of columns is that no row, that is, no branch, contains entries in more than one set. Now, if a matrix not in the standard form but known to be the connection matrix for a network is given, it will first be examined for separate parts. If separate parts are found, it will be divided into these parts, each to be considered as an individual connection matrix.

Now, given the matrix containing only one separate part, it is tested to see that each column is linearly independent from the others. The condition for this independence is that the number of rows be at least as great as the number of columns and that there be a non-zero determinant that can be formed from all the columns and an equal number of rows. If some linearly dependent columns are discovered, they are eliminated from the matrix. To the matrix with independent columns, one column is added. This column corresponds to the "exterior" or "non-independent" mesh used as part of a connection matrix in the standard form. The condition on the elements of this column is that termwise they are linearly dependent on the elements of the other columns.

After these operations, the given connection matrix may perhaps still not be in the standard form. The possibilities are:

1. The network described by the connection matrix is planar, but the meshes were not selected according to the suggested procedure.
2. The network is non-planar.
3. The network contains ideal transformers.

For cases 1 and 2, the matrix consists only of elements that are +1, -1, or 0. For case 3, the restriction is lifted.

If the matrix was not in standard form due to the first cause listed, an alteration, as follows, can bring it into the standard form. Consider a particular row (any one) and the columns in which it has entries. The configuration described by the connection matrix will remain invariant if any of these columns is replaced by the termwise combination, that is, sum or difference, of it and one of the other columns, provided in doing so the resulting combination contains a zero entry in the row originally considered. In repeatedly forming such combinations, it is possible that some entries in the matrix will be given a value other than +1, -1, or 0. However, a necessary condition that a connection matrix represent a planar network is that it is possible to reduce it to the standard form by the operation just described, combined, possibly, with the operation of multiplying various rows and columns through by -1. Having reduced a matrix to the standard form, a sufficient condition that it be the connection matrix of a planar network is the condition given in the preceding section based on the Kuratowski-Whitney theorem, or the condition that the network found by considering the connection matrix as a standard branch, node matrix is drawn without wires (i.e., branches) crossing over each other. This being the case, the dual network may be drawn, yielding the network associated with the given connection matrix.

The remaining two of the three possibilities listed above will be treated in the section entitled, "Drawing a Network Given its Connection Matrix." In the next section, a relationship between non-planar networks and networks containing ideal transformers will be presented.

PLANAR NETWORKS ELECTRICALLY EQUIVALENT TO NON-PLANAR NETWORKS

In the diagram of a non-planar network, some wires must be drawn crossing over each other. If a pair of such wires is considered, as in Figure 47 (a), one of its wires may be separated from a terminal and re-joined passing through a terminal of the wire it crossed, provided a one-to-one ideal transformer is used, as shown in Figure 47 (b). This change

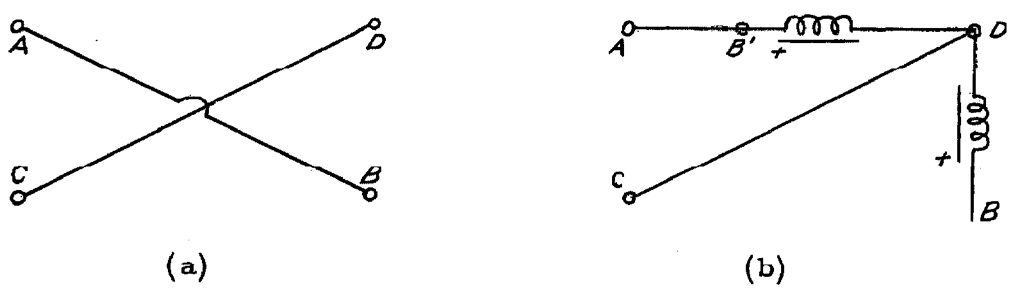


Figure 47

does not alter the electrical characteristics of the network because the voltage at B' (in the example given) equals that at B, and the current in branch AB'D equals the current in branch DB. By repeating addition of one-to-one transformers any non-planar network can be reduced to a planar network. The dual for this equivalent network can then be obtained by the method given for finding the dual of a planar network containing ideal transformers. (R-21). As an example, the planar equivalent of the network of Figure 45 is drawn in Figure 48 (a) and its dual in part (b) of that figure.

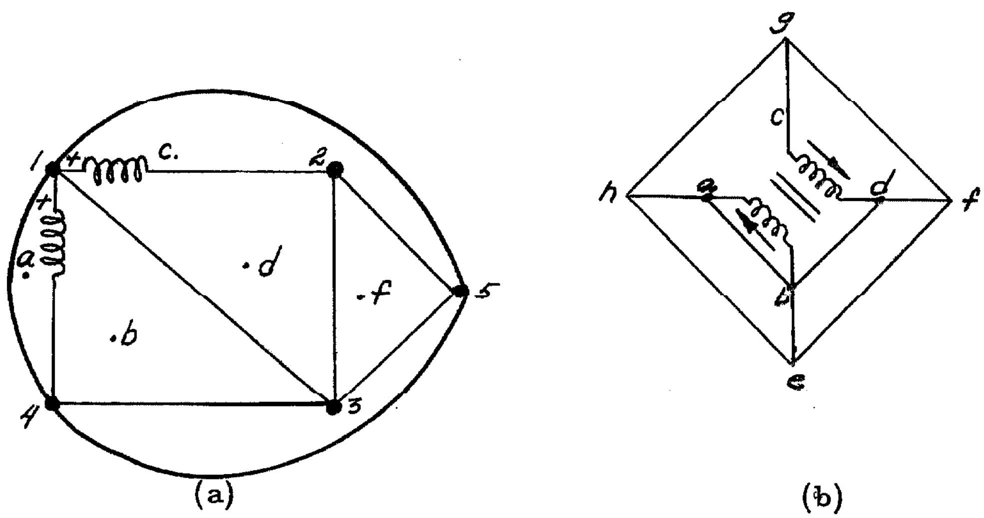


Figure 48

One may note that the windings of the one-to-one ideal transformer joining node-pairs (a, b) and (c, d) can be replaced by a one-to-one ideal transformer joining node-pairs (a, c) and (b, d). Also, because of the unity turns ratio, impedances across node-pair (a, b) may be transferred to node-pair (c, d), and impedances across (b, d) to (a, c) without any change in the impedance magnitude. Conversely to the process shown in Figure 47, one may replace a one-to-one ideal transformer by wires that cross each other if the transformer windings are joined to a common node with the polarity mark at the node being the same for each.

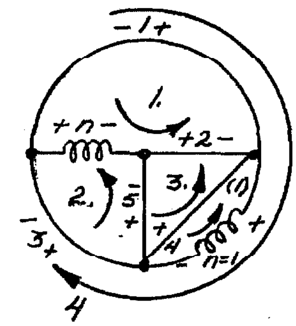
Another interesting feature in this reduction of a non-planar network to a planar network is that introduction of the one-to-one ideal transformers leaves the connection matrix unaltered. Referring to Figure 47, one may note that any mesh, that had contained branch (A, B) in the non-planar network, must contain branch (A, B') along with the windings from B' to D and D to B. Since the algebraic sum of these windings is zero, no winding-turns due to the one-to-one transformer appear in any of the equations.

ANALYSIS OF A PLANAR NETWORK CONTAINING TRANSFORMERS

In part (a) of Figure 49 the connection matrix for the network shown in part (b) is given. This network is a planar network containing an ideal transformer. A "bordering mesh" is included for the analysis.

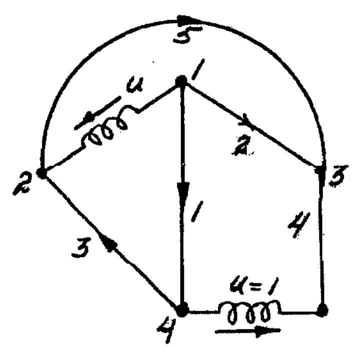
$$C = C_L + C_M N_t$$

$$= \begin{bmatrix} -1 & & & 1 \\ -1 & & 1 & \\ & 1 & & -1 \\ & & -1 & \\ & -1 & 1 & \end{bmatrix} + \begin{bmatrix} \\ \\ \\ 1 \\ \\ \end{bmatrix} \begin{bmatrix} -n & n & & \end{bmatrix}$$



(b)

$$= \begin{bmatrix} -1 & & & 1 \\ -1 & & 1 & \\ & 1 & & -1 \\ -n & n & -1 & 1 \\ & -1 & 1 & \end{bmatrix}$$



(a) (c)

Figure 49

As can be seen from this example, networks of this type have the properties:

1. Every branch belongs to two meshes; therefore, the sum of the entries, (1, -1, and 0), in a row of C_L plus the sum of the entries, (1, -1, and 0), for that row in C_M is zero.
2. Every winding belongs to two meshes; therefore, the sum of the winding-turns in any row of C is zero, and the sum of winding-turns per mesh for each transformer is zero.

The dual network, shown in part (c) of the figure, illustrates the properties:

1. In drawing the dual, the location of each branch may be determined by treating the combination of C_L and C_M as a standard branch, node matrix having a number of columns equal to the sum of the number of magnetizing and load meshes.
2. The location of each winding may be determined by treating the winding-turns matrix as a reciprocal-turns matrix. It should be recalled that each magnetizing node (the dual of a magnetizing mesh) is attached to a winding of value, -1, not appearing in the reciprocal-turns matrix. Therefore, for each transformer, the sum of the terms that do appear is +1.

DRAWING A NETWORK GIVEN ITS CONNECTION MATRIX

Given a matrix which may have entries other than 1, -1, or 0, but which has been reduced, by the procedure previously explained, to represent a network with one separate part and with independent meshes, the following procedure will yield the network configuration.

1. Border the matrix with an additional column so that the sum of the elements of the bordered matrix is now zero.

2. If possible, alter the matrix using the following steps so that each row contains two non-zero entries, a +1 and a -1.

   a) Multiply any column by -1.

   b) Replace a column by the sum or difference of two columns such that at least one original, non-zero entry now becomes zero.

3. If repeated application of (a) and (b) fail, separate the matrix into two parts, corresponding to C_L and $C_M N_t$. In the C_L matrix, leave unaltered any row that contains two entries, a +1 and a -1, but for rows not in that form, remove all the elements except one, of value +1 or -1, placing the elements that were removed into the matrix $C_M N_t$.

4. Form the matrix, C_M, with a single entry in each column such that the sum of the elements in every row for C_L and C_M combined is zero.

5. Form the matrix, N_t, so that when premultiplied by C_M, the product is $C_M N_t$. Now, add a bordering set of columns to N_t, one column for each magnetizing mesh (i. e., for each column of C_M), placing in each column a single entry such that the sum of the elements in any row of the matrix, N_t, together with this border, is zero.

6. Draw a network by considering the combined (not added) matrices, C_L and C_M, as a standard branch, node matrix in order to determine the position of the branches, and by considering the matrix, N_t, combined with its border, as a reciprocal-turns matrix to determine the position of the windings.

7. Obtain the desired result, namely the network described by the given connection matrix, by taking the dual of the network found in step 6. If the network of step 6 is not planar, to find the dual it will be necessary, as explained, to use additional transformers.

An application and example of this procedure is given in the chapter entitled, "Electrical Analogs of Mechanical Structures."

DUAL RESULTS TO THOSE IN CHAPTERS III AND IV

For each of the situations described on the mesh impedance basis in the previous chapters, there exists a dual of the circuit, concept, or procedure in terms of nodal equations written on the admittance basis.

First, for redrawing a circuit involving multilimb transformers (in order to physically interpret the permissible alterations of the reciprocal-turns) each winding is replaced by an open-circuit, and then across the various independent node-pairs selected for the analysis a winding from each of the transformers is placed, some, perhaps with zero reciprocal-turns. (Physically, a winding with zero reciprocal-turns is an open-circuit.) As an example, the circuit of Figure 9 is redrawn in Figure 50. This is the dual of the circuit given in Figure 11.

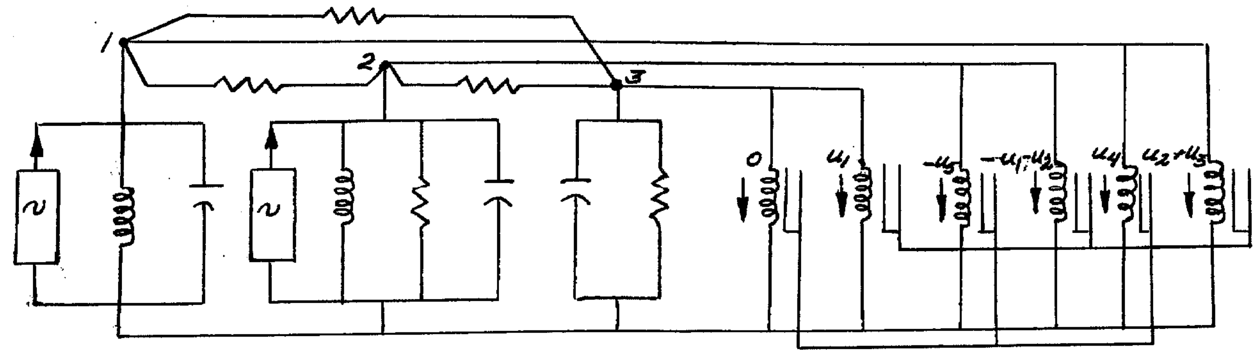


Figure 50

To insure independent reciprocal-turn equations, superfluous multilimb transformers are removed and their windings replaced by open-circuits. As an example, one might take the dual of the circuit in Figure 10. Reduction of the reciprocal-turns is identical to that of the winding-turns, with the nomenclature of "magnetizing node-pairs" and "load node-pairs" replacing that of the magnetizing and load meshes. Similarly, the selection of independent node-pairs is guaranteed, if, for each separate part of the network, they form a tree. The simplest form for a tree is found by selecting one of the nodes as reference (i.e., as the second member of each node-pair, so that a node-pair may then be designated by a single node).

The matrix procedure for the analysis on the nodal basis is given in the following notation:

V_B, branch voltage matrix; V, node-pair voltage matrix

$$V_B = A\ V \tag{90}$$

I_B, branch current source matrix; I, node-pair current source matrix

$$I = A_t\ I_B \tag{91}$$

Y_B, branch admittance matrix; Y, node-pair admittance matrix

$$Y = A_t Y_B A \tag{92}$$

$$I = \mathbf{Y}\ V \tag{93}$$

where, A, the branch, node-pair matrix, may be defined in terms of equation (90).

The relationship between analysis of a given network on the nodal basis and analysis of the same network on the mesh basis is that

$$Y_B = Z_B^{-1} \tag{94}$$

and, if the tree, consisting of the independent node-pairs in the analysis, is the same tree that is used in determination of the links in the mesh analysis,

$$A = C_t^{-1} \tag{95}$$

provided that the connection matrix is non-singular. Because of relation (95) the two matrices are called "orthogonal." Kron, (R-2), presents an analysis combining or "mixing" together the features of mesh and nodal analysis, based on this orthogonal property. He also shows that any singular (rectangular) connection matrix (or, branch, node-pair matrix) represents only a part of a non-singular (square) matrix that can be established by supplying missing meshes, which are inactive because they contain open-circuits, or for the branch, node-pair matrix, supplying missing node-pairs, which are inactive because they contain short-circuits. An interesting exercise is to write the orthogonal equations for two branches, each with both a current and a voltage source, and then by alternately setting sources to zero, to generalize the interchange of current and voltage sources to the case of mutual inductance in the generator branch.

For networks containing ideal, multilimb transformers, the branch, node-pair matrix becomes

$$A = A_L + A_M U_t \tag{96}$$

and the node-pair admittance matrix in terms of the reciprocal-turns is:

$$Y' = Y_{LL} + UY_{ML} + Y_{LM}U_t + UY_{MM}U_t \tag{97}$$

The same rules apply for writing the node-pair admittance matrix by inspection as those that govern the mesh impedance matrix, except that the dual terminology must be used in the appropriate places. In writing a node-pair admittance matrix by inspection, the special selection of the node-pairs, by which one node in each separate part of the network is selected as the reference node, considerably simplifies matters.

The case of superfluous multilimb transformers hinges on the presence of a cut-set occurring in the network made up entirely of transformer windings. As an example of such a cut-set, one may consider each node of the circuit in Figure 38 to which only transformer windings are attached, there being six such nodes. The superfluous transformers may be eliminated by replacing the ten two-winding transformers with four multiwinding transformers as shown in Figure 37 for the case where "n" equals 5. Thus, for the six "all-winding" nodes, there were six superfluous transformers.

Campbell and Foster, (R-3), show that if there is a cut-set made up of windings belonging to a single multiwinding transformer, then an arbitrary number of turns, n, may be added or subtracted from the windings in the cut-set, allowing, by proper choice of n, for one of the windings to be set to zero turns, that is, replaced by a short-circuit. The dual situation occurs if there is a mesh consisting of windings from the same multilimb transformer. Here, by proper choice of an additive number of reciprocal-turns, u, one of the windings may be replaced by an open-circuit.

The duality principle with regard to the scattering matrix is that, if two networks are duals of each other, the scattering matrix for one equals the negative of the scattering matrix for the other. Thus, dual networks give the same transmission losses. (R-12).

A dual situation also occurs in the analysis of the connection matrix compared to the analysis of the branch, node-pair matrix. Given a connection matrix, the branches in the various cut-sets of the network represented by the connection matrix may be picked out from the matrix as follows: a cut-set consists of those branches whose entries in the connection matrix sum, columnwise, to zero, and which, removed from the matrix, would increase the number of its separate parts. In analogy, a mesh consists of those branches whose entries in the branch, node-pair matrix sum columnwise to zero, and which removed would not increase the number of the separate parts in the branch, node-pair matrix. Note, that a connection matrix with one separate part corresponds to a "cyclically connected graph," that is, a network which cannot be separated into two parts having only a single node in common, while a branch, node-pair matrix with one separate part corresponds to a network whose nodes can be connected by means of one tree. In either case, the "separate parts" of a matrix are defined as those sets of columns in the matrix for which no row (that is, no branch) contains entries in more than one set.

THE GYRATOR

Before concluding the discussion on duality, another network element, named the "gyrator," should be mentioned. The concept of the gyrator may be developed by interchanging the role of current and voltage for <u>one</u> of the windings of a two-winding transformer. Making this substitution in equations (13), and replacing the turns ratio by the letter, g, the resulting equations are:

$$\begin{aligned} v_1 &= 0\, i_1 - g\, i_2 \\ v_2 &= g\, i_1 + 0\, i_2 \end{aligned} \qquad (98)$$

Since these equations relate the voltages to the currents, they are in the form of an impedance matrix. However, it is not symmetrical, and therefore networks containing gyrators do not obey the reciprocity relations. The total power into a gyrator is zero, as in the case of an ideal transformer, so that it is therefore not an active, but rather, passive element. It can be realized by means of the gyromagnetic effect in a ferromagnetic medium, but the construction of gyrators, except for the use in wave-guides, is still in the laboratory stage. Because of the partial interchange of the role of current and voltage which was made in arriving at the definition of the gyrator, it is possible to separate a circuit, take the dual of one of the parts, and recombine the circuit by connecting the parts through a gyrator. The gyrator was named, and its properties analyzed, by Tellegen. He has published several papers on the subject, for instance, (R-22), (R-23) and (R-24).

VI. MINIMIZING THE NUMBER OF IDEAL TRANSFORMERS IN NETWORK SYNTHESIS

THE NODAL ADMITTANCE MATRIX

If the node equations are written for a network by using one node (the "ground" node) for reference, the node-pair admittance matrix can be written by considering a node at a time. This being the case, the node-pair admittance matrix is given the special name, "nodal admittance matrix." The nodal admittance matrix for the network of Figure 51 is:

$$\begin{aligned} I_1 &= (Y_A + Y_B + Y_D)V_1 - Y_B V_2 - Y_A V_3 \\ I_2 &= -Y_B V_1 + (Y_B + Y_C + Y_E) - Y_C V_3 \\ I_3 &= -Y_A V_1 - Y_C V_2 + (Y_A + Y_C + Y_F)V_3 \end{aligned} \qquad (99)$$

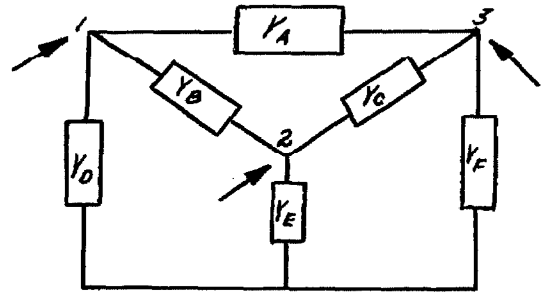


Figure 51

The necessary and sufficient condition that a node-pair admittance matrix be physically realizable by means of ordinary passive elements, including ideal transformers, is that the matrix be positive real. The definition of a "positive real matrix" is in terms of the quadratic form found by premultiplying the matrix by a row matrix of real, but otherwise arbitrary, variables and then postmultiplying it by the transpose of the row matrix. The resulting form must be a rational function which is real for real values of the variable and of which the real part is positive or zero when the real part of the complex frequency variable, p, is positive.

THE NODAL ADMITTANCE MATRIX FOR TRANSFORMERLESS NETWORKS

Sufficient conditions on a nodal admittance matrix, Y, with elements, $y_{ij}(p)$, that it can be synthesized without the use of ideal transformers are:

a) $y_{ij}(p) = y_{ji}(p)$

b) $-y_{ij}(p)$ is positive real, $i \neq j$ (100)

c) The sum of the $y_{ij}(p)$ in each row is positive real

The second and third conditions, together, imply that $y_{ii}(p)$ is positive real. The synthesis of the network consists of drawing an array of nodes, one for each row in the matrix, interconnecting them with 2-poles having the value, $-y_{ij}$, and connecting each node to ground by a 2-pole whose admittance is equal to the sum of the y_{ij} in a row. This is shown, schematically, in Figure 52. Each of the 2-poles may be synthesized without ideal transformers. (R-25).

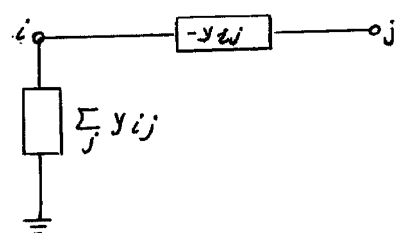


Figure 52

THE NODAL ADMITTANCE MATRIX FOR TRANSFORMERLESS CONDUCTANCE NETWORKS

The necessary and sufficient conditions on the elements, g_{ik}, of a nodal admittance matrix, G, of a conductance network that it be realized without ideal transformers are:

a) $g_{ij} = g_{jk}$

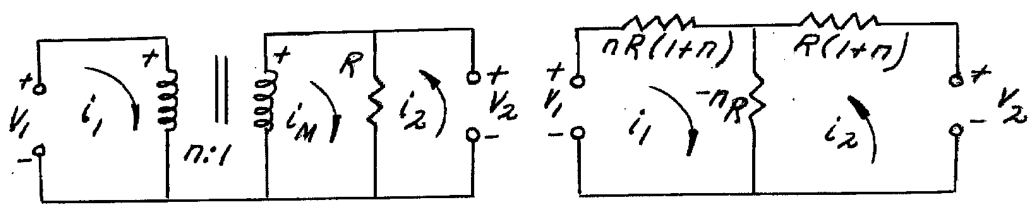
b) $-g_{ij}$ is greater than or equal to zero, $i \neq j$ (101)

c) The sum of the g_{ij} in each row is greater than or equal to zero.

The second and third conditions, together, imply that g_{ii} is greater than or equal to zero. The synthesis of the network is the same as shown in Figure 52, with the y's replaced by g's. These conditions are valid even if there are "hidden nodes" in the network, that is, even if a nodal admittance matrix of a given order is reduced to one of lower order by the elimination of some of the node voltages as variables.

ELIMINATION OF A TRANSFORMER

Consider the circuit in Figure 53 (a).



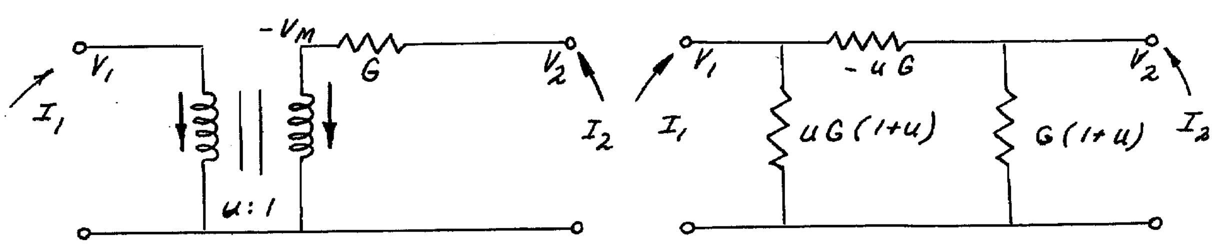
(a) (b)

Figure 53

By use of the rules given, the equations can be written by inspection.

$$
\begin{aligned}
V_1 &= n^2 R i_1 + nR i_2 \\
V_2 &= nR i_1 + R i_2
\end{aligned}
\qquad (102)
$$

These equations may be formally represented by the circuit of Figure 53 (b). The dual of these two circuits, shown with the reference directions altered for purposes of symmetry, are given in Figure 54 (a) and (b).



(a) (b)

Figure 54

In these circuits, G is a positive number. The elements in Figure 54 (b) are not realizable, except for the degenerate case where u = -1. However, this equivalent circuit for a transformer that has a common ground for its windings and which is taken together with a series conductance (furnishing the necessary magnetizing node) is quite useful, as shown in the following sections.

ELIMINATION OF A TRANSFORMER IN THE CAUER 4-POLE

The Cauer 4-pole on the nodal admittance basis is the same as the circuit of Figure 54 (a), provided an additional conductance, G_1, is placed from node 1 to ground, as shown in Figure 55 (a). Replacing the transformer yields the circuit of Figure 55 (b).

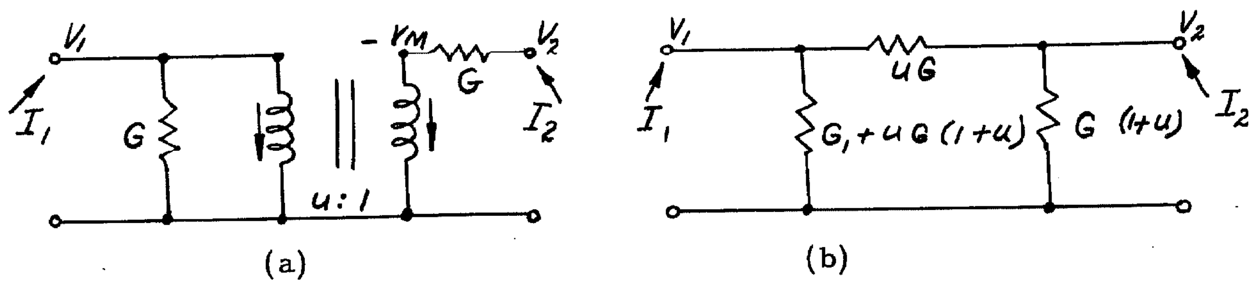


Figure 55

The requirements that the elements of this circuit be physically realizable, that is, that they be positive, are

$$\begin{aligned} &a) \quad -1 \leq u \leq 0 \\ &b) \quad \frac{G_1}{G} \geq -u(1+u) \end{aligned} \tag{103}$$

The values of the Cauer parameters in terms of the elements, g_{ij}, of the nodal admittance matrix, are, by formula (61),

$$G = g_{22}; \quad G_1 = \frac{g_{11}g_{22} - g_{12}^2}{g_{22}}; \quad u = \frac{g_{12}}{g_{22}} \tag{104}$$

If these expressions are substituted into (103), one obtains the same conditions as in (101), namely,

$$g_{12} \leq 0 ; \; g_{11} + g_{12} \geq 0 ; \; g_{12} + g_{22} \geq 0 \tag{105}$$

It could be noted, parenthetically, that by condition (103 (a)), condition (103 (b)) is automatically satisfied if

$$4G_1 \geq G \tag{106}$$

and that it is necessary for

$$G_1 \geq G \tag{107}$$

ELIMINATION OF TRANSFORMERS IN THE CAUER 6-POLE

Consider the Cauer 6-pole (Figure 56), drawn on the nodal admittance basis in terms of two-winding transformers, with the elements arranged to make the transformer elimination more obvious.

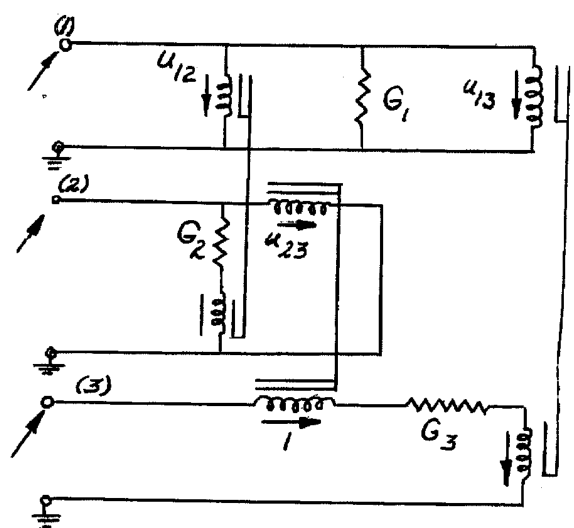


Figure 56

Now, if either u_{12} or u_{13} satisfies condition (103 (a)), the transformer to which they belong may be replaced, provided the magnetizing conductance, respectively, G_2 or G_3, is less than G_1 to the extent that condition (103 (b)) is satisfied. If one transformer is eliminated, and the negative conductance, $uG(1 + u)$, (Figure 54 (b)), paralleled with G_1, that is, added algebraically to the value of G_1, the result might still be large enough to allow the second transformer to be eliminated. If this is the case, only the transformer with the winding, u_{23}, remains, and the circuit is reduced to that of Figure 57.

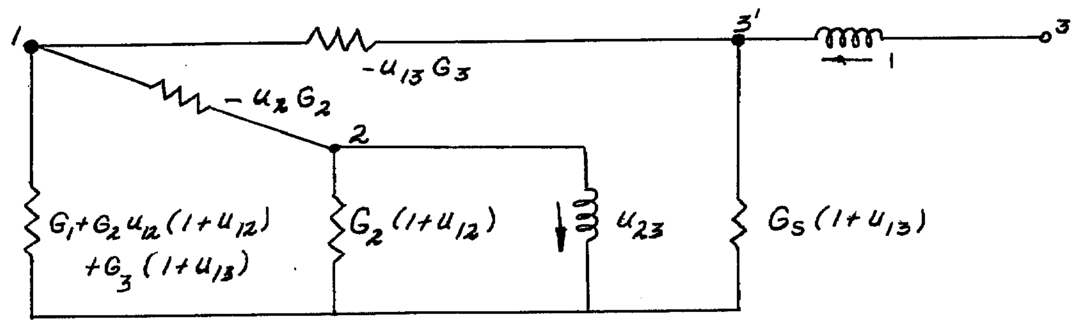


Figure 57

If the transformer is considered absent from the circuit (i.e., open-circuited), the equations in terms of the voltages at nodes 1, 2, and 3' are:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_{3'} \end{bmatrix} = \begin{bmatrix} G_1\ u_{12}^2\ G_2 + u_{13}^2\ G_3 & u_{12}\ G_2 & u_{13}\ G_3 \\ u_{12}\ G_2 & G_2 & 0 \\ u_{13}\ G_3 & 0 & G_3 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_{3'} \end{bmatrix} \qquad (108)$$

Now, if the transformer is replaced, and the equations are in terms of nodes 1, 2, and 3 instead of 1, 2, and 3', the following changes take place. Since

$$V_{3'} = V_3 + U_{23}V_2 \qquad (109)$$

the coefficients of $V_{3'}$ are the coefficients of V_3, but they also must be multiplied by u_{23} and added to the coefficients of V_2. Also, the current, I_2, that had previously flowed just through the conductances attached to node 2, is now partly diverted through the winding, u_{23}. Therefore, to keep the equations unchanged, the current through this winding must be added to I_2. Noting that the current entering node 3' is the same as the current entering node 3, the current in winding u_{23} to be added to I_2 is equal to $u_{23}I_3$. The total effect, therefore, of the transformer is to add to the members of the second column of the admittance matrix the elements in the third column, multiplied by u_{23}, and to add to the elements of the second row, the third row, multiplied by u_{23}. The resulting equations are:

$$\begin{bmatrix} I_1 \\ I_2 \\ I_3 \end{bmatrix} = \begin{bmatrix} G_1 + u_{12}^2 G_2 + u_{13}^2 G_3 & u_{12}G_2 + u_{23}u_{13}G_2 & u_{13}G_3 \\ u_{12}G_2 + u_{13}G_3 & G_2 + u_{23}G_3 & u_{23}G_3 \\ u_{13}G_3 & u_{23}G_3 & G_3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \tag{110}$$

which are identical to the equations written for the system before the transformers had been replaced. It may be pointed out that the method of using a transformer to alter the rows and columns of an impedance matrix in the manner just stated was the basis by which Cauer, (R-10), showed that any real, linear transformation (having constant coefficients) operating on an impedance or admittance matrix could be interpreted circuit-wise.

SUGGESTED SYNTHESIS PROCEDURE FOR 6-POLES

To synthesize a conductance 6-pole, having been given a nodal admittance matrix, the first check to make is to see if the matrix satisfies the conditions (101) to the effect that no transformers would be required. If it does not satisfy these conditions, but could be made to satisfy them by a simple alteration such as changing the sign in a column and the corresponding row, or by adding a multiple of one column to another, and doing likewise for the corresponding rows in order to maintain symmetry, then only one transformer would be required. It is assumed here that the current generators applied to the system are required to have a node (ground) in common. If this were not the case, sign changes could be made by reversing the polarity of some of the generators, and columns (and rows) could be combined to yield new numbers in the impedance matrix by changing the reference node-pairs so that all the generators would not return to the same node.

The test next in order would be to check the matrix to see if it is positive in the sense that the determinant and all its principal minors are non-negative. If the matrix is positive, then it is realizable, and the Cauer network gives the sufficient number of transformers. The calculation to see if the matrix is positive involves nearly the same operations as evaluation of the parameters (the three conductances and the three turns-ratios) of the Cauer network. Calculation of the Cauer parameters yields a network with the sufficient number of transformers, namely, in the case of a 3x3 matrix, one two-winding and one three-winding transformer. To test whether or not these transformers are necessary, the basic circuit alteration, shown in Figure 54, is to be applied, with reference to the conditions given in (103). There being a limited number of possibilities for the elimination of a transformer, these could be itemized and applied in order to assure a circuit with a minimum number of transformers.

VII. ELECTRICAL ANALOG OF MECHANICAL STRUCTURES

ELECTRICAL ANALOGS

If one is given a mechanical system consisting of lumped masses, springs, and dash-pots, the analogous electrical circuit may be found by drawing a diagram in the same form as that of the mechanical system. (R-18). However, strict topological correspondence is lost in the case of mechanical structures in which members are subject to bending, perhaps combined with tension and shear. The problem can be resolved by finding a connection matrix for the mechanical system, and then interpreting this matrix in terms of an electrical circuit diagram. The technique for obtaining the connection matrix for a mechanical system has been treated in the literature. (R-26), (R-27). In general, these matrices contain entries other than 1, -1, and 0. The technique for finding the network diagram, given such a matrix, has been stated in Chapter V. An example using this technique will now be given.

ANALOG FOR A CANTILEVER

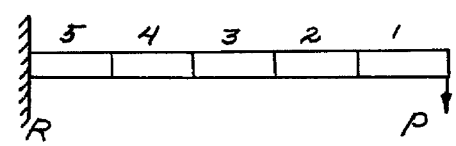


Figure 58

Consider the non-uniform beam, broken into five lumped sections, as shown in Figure 58. The connection matrix for this system will be written by computing the bending moment at each section (assumed constant over the section) due to a unit load applied at the end, P. Selecting an arbitrary reference direction for positive bending moment, it is seen that the moment for each section is in the same direction, but the magnitude of the moment increases with distance from P. This moment is balanced by an equal and opposite moment due to the reaction at the wall, point R. Writing in terms of these two points, a matrix analogous to the bordered connection matrix results. This is given in Figure 59 (a). The values for the moments given in the matrix are in arbitrary units. To find the electrical circuit diagram, this matrix is broken into the two parts, C_L and $C_M N_t$, according to step 3 of the procedure in Chapter V, and as shown in Figure 59 (b). The next step of decomposing $C_M N_t$ is given in Figure 59 (c), and the final step of bordering the matrix, N_t, is shown in 59 (d).

$C =$

1	2	-2
2	4	-4
3	6	-6
4	8	-8
5	10	-10
P	1	-1

(a)

$C =$

1	
1	
1	
1	
1	
1	

$+$

1	-2
3	-4
5	-6
7	-8
9	-10

(b)

-1				
	-1			
		-1		
			-1	
				-1

-1	2
-3	4
-5	6
-7	8
-9	10

(c)

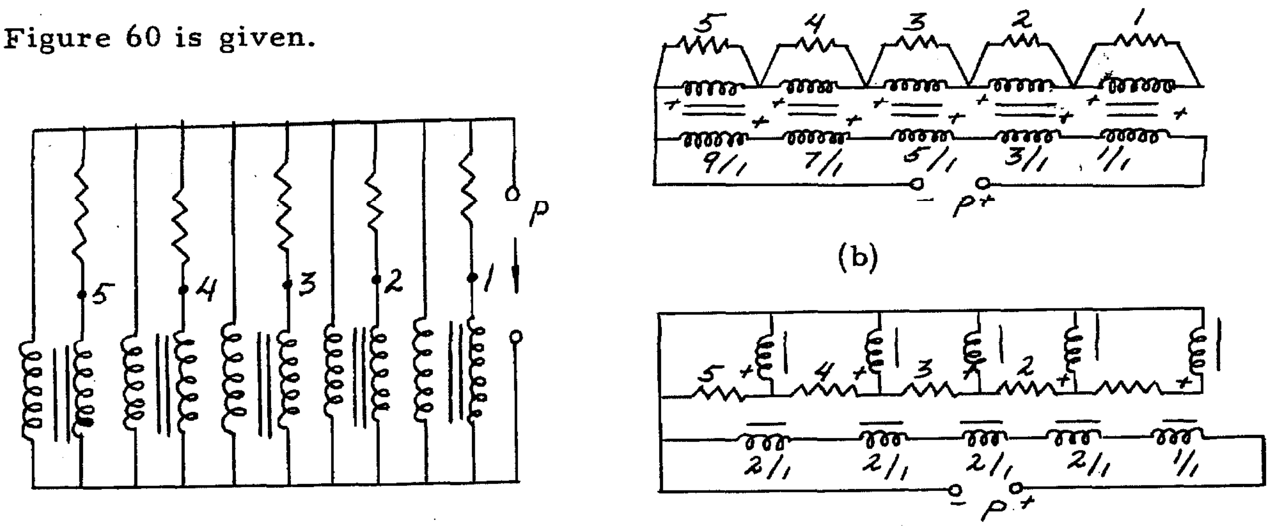
-1	2	-1				
-3	4		-1			
-5	6			-1		
-7	8				-1	
-9	10					-1

(d)

Figure 59

From the information of Figure 59, the dual network is drawn, Figure 59 (a). Then by the last step, step 7, the desired network is found, Figure 59 (b). The five windings that are placed in series indicated that all the transformers could be substituted for one multilimb transformer.

In Figure 59 (c), a second circuit that takes the role of the beam shown in Figure 60 is given.



(a) (b) (c)

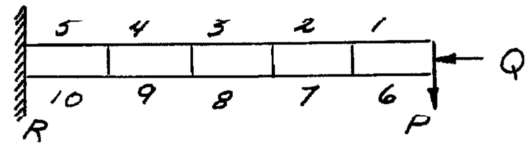
Figure 60

CANTILEVER BEAM IN BENDING AND COMPRESSION

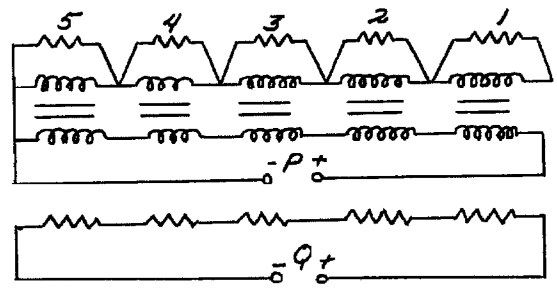
If the beam to be analyzed undergoes two types of deformation, bending and compression, then each section is considered as two branches, one under bending, the other under compression. All deflections that occur due to loading the beam will be considred very small, so that, for instance, compression forces do not set up moments due to the lever arm created in the bending. Only under this restriction is the principle of superposition, which is implicit in this analysis, valid. The branches numbered 1 through 5 will be taken as branches in bending, and 6 through 10 as the ones in compression. A force, Q, as well as P is applied to the beam. (Figure 61 (a)). The bordered connection matrix, and the network that is finally derived from it are given in (b) and (c) of Figure 61. In the connection matrix, two other branches have been added, one for P and one for Q in order to explicitly represent these sources as part of the network.

	P	Q	R
	2		-2
	4		-4
	6		-6
	8		-8
	10		-10
		1	-1
		1	-1
		1	-1
		1	-1
		1	-1
P	1		-1
Q		1	-1

(b)



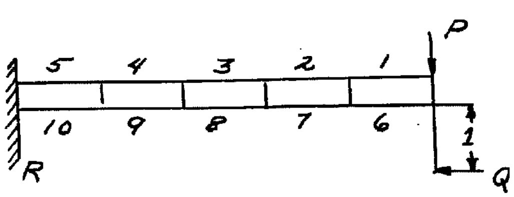
(a)



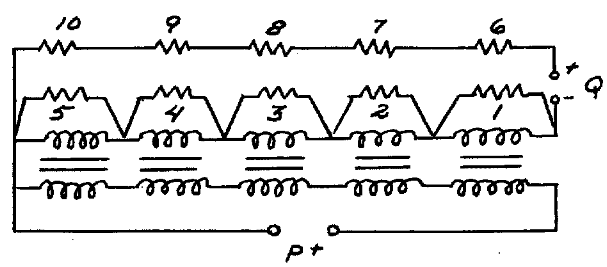
(c)

Figure 61

As may be seen, again with the limitations of small deflections, the compression force exerts no bending action, and the bending force, P, exerts no compressive action, so that the bending and compression circuits are separate from each other. However, if the force, Q, had been applied to the beam off-center, it would add a constant bending moment to each of the branches, 1 through 5. The entries in the branches, 6 through 10, would remain unchanged. The net effect on the circuit is now that both parts will be coupled together. The beam and the resulting circuit are shown in Figure 62 (a) and (b).



(a)



(b)

Figure 62

CANTILEVER BEAM WITH REDUNDANT SUPPORT

Consider, again, the beam shown in Figure 58 with each section to be considered under bending action only. Now, if at point,P, there is not only a force pushing downwards, but also there is a spring supporting the end, P, pushing it upwards, the problem becomes "statically indeterminate." That is, all the external forces acting on the beam are not known explicitly, as the force due to the spring is a function of position. As far as the electrical analog is concerned, addition of the redundant force due to the spring causes no change in the analysis. The result is to place in the circuit of Figure 60 (b), or 60 (c), another element with a stiffness corresponding to that of the spring, such that this element is joined from ground to the terminal of P.

Within the study of statics, problems of this type are often solved in terms of a "cut" structure and "redundance." In the example of the cantilever beam with a spring support, one might, for the purpose of analysis, cut the connection between the spring and the beam, and replace the effect of the spring by a redundant force. The beam deflection would then be solved without explicit information about the force due to the spring, but with the implicit information that the spring was only a passive element, it could store energy, but could not perform work on the beam like an active source, etc. The mathematical development of the analogous situation for electrical networks will be given, namely that the value of some of the sources in the network is not known, but what is known is that they contribute no power to the network.

CONSTRAINED SOURCES IN AN ELECTRICAL NETWORK

Consider the equations of an electrical network:

$$i_B = C\, i \tag{111}$$

$$v = C_t\, v_B \tag{112}$$

$$Z = C_t\, Z_B\, C \tag{113}$$

$$v = Z\, i \tag{114}$$

Now, with some of the sources known, and some unknown, compound matrices will be used for C, i, v, and Z. Quantities known will be signified by a subscript, "o", and the quantities unknown by "x".

$$i_B = \begin{bmatrix} C_o & C_x \end{bmatrix} \begin{bmatrix} i_o \\ \hline i_x \end{bmatrix} \tag{115}$$

$$\begin{bmatrix} V_o \\ \hline V_x \end{bmatrix} = \begin{bmatrix} C_{ot} \\ \hline C_{xt} \end{bmatrix} V_B \tag{116}$$

$$\begin{bmatrix} Z_{oo} & Z_{ox} \\ Z_{xo} & Z_{xx} \end{bmatrix} = \begin{bmatrix} C_{ot} \\ \hline C_{xt} \end{bmatrix} Z_B \begin{bmatrix} C_o & C_x \end{bmatrix} \tag{117}$$

$$\begin{bmatrix} V_o \\ \hline V_x \end{bmatrix} = \begin{bmatrix} Z_{oo} & Z_{ox} \\ Z_{xo} & Z_{xx} \end{bmatrix} \begin{bmatrix} i_o \\ \hline i_x \end{bmatrix} \tag{118}$$

If the sources labeled "x" contribute no power to the network, then a change in the currents, i_x, of these sources to $i_x + \delta i_x$, will produce no change in power. The expression for power, $i_{Bt}^{*} v_B$, is:

$$P = i_{Bt}^{*}\, Z_B\, i_B \tag{119}$$

(where the asterisk stands for the complex conjugate).

Substituting $i_B + C_x \delta i_x$ for i_B,

$$\delta P = (i_B + C_x\,\delta i_x)^*_t\, Z_B(i_B + C_x\,\delta i_x) - i^*_{Bt} Z_B i_B$$
$$= (C_x\,\delta i_x)^*_t\, Z_B i_B + i^*_{Bt} Z_B C_x\,\delta i_x \qquad (120)$$

Only the real part of Z_B will contribute to this power expression. Considering then only the real part of the branch impedance, the expression in (120) is the sum of a one-by-one matrix and the conjugate of its transpose. With this sum zero, the matrix must be zero, or

$$\delta i^*_{xt}\, C^*_{xt}\, Z_B i_B = 0 \qquad (121)$$

But this must be true for all values of i_x. Noting that C_x is real, the expression reduces to

$$C_{xt} Z_B i_B = 0 \qquad (122)$$

From equation (122) it is possible to express the branch currents in terms of the currents of the known sources. To do this, one substitutes for i_B in (122) and solves for i_x.

$$0 = C_{xt} Z_B C_o i_o + C_{xt} Z_B C_x i_x$$
$$i_x = -(C_{xt} Z_B C_x)^{-1} C_{xt} Z_B C_o i_o \qquad (123)$$

The explicit expression for i_B is:

$$i_B = C_o i_o + C_x i_x$$
$$= [1 - C_x (C_{xt} Z_B C_x)^{-1} C_{xt} Z_B] C_o i_o \qquad (124)$$
$$= [C_o + C_x Q_o]\, i_o$$

where,

$$Q_o = -(C_{xt} Z_B C_x)^{-1} C_{xt} Z_B C_o \qquad (125)$$

The voltages of the known sources are:

$$V_o = C_{ot} V_B$$
$$= C_{ot} Z_B i_B \qquad (126)$$

THE ANALOGOUS QUANTITIES

The equations of the preceding section are identical with those presented by Denke (R-26) for the solution of a mechanical structure by the principle of least work. Equating similar terms in the equations, the following interpretations are made:

Network branch: A flexible element undergoing one type of generalized displacement.

v_B: The generalized branch displacements.

i_B: The generalized force acting on each element.

Z_B: The flexibility coefficients for the branches. This is generally a diagonal matrix, although mutual mechanical coupling can occur.

i_x: The redundant (unknown) forces.

$C_o i_o$: The force on the elements due to external sources, but not including the force due to the redundants.

C_x: The internal force — redundant force connection matrix.

v_o: The deflection of the structure at the points of applied loads.

C_o: The internal force — applied force connection matrix.

The usual problem in structures is to solve for the forces acting on each element (i.e., the branch currents) in terms of the known external loads. Another application of the electrical analog is for the solution of the natural frequencies and the modes of vibration for statically indeterminate structures. This is accomplished by making up the analog of the structure entirely from inductors and ideal transformers (current analogous to force, voltage to velocity) and then to load this structure with capacitors. If such an analysis is not desired, the circuit may be made up entirely of resistors and ideal transformers (current analogous to force, voltage to displacement) and the analysis given earlier in this thesis for the application of Cauer networks, along with the minimization of the number of transformers, may be directly applied to the solution of mechanical systems.

APPENDIX

THE EXTENDED CAUCHY EXPANSION OF DETERMINANTS

A determinant of equations (19) is in the form

$$\left|\begin{array}{c:c} Z & N \\ \hdashline N_t & O \end{array}\right| \begin{array}{l} \updownarrow m \\ \updownarrow k \end{array}$$
$$\overset{\leftarrow m \rightarrow}{} \overset{\leftarrow k \rightarrow}{}$$

It may be expanded so that every term has three factors. The first is a minor of order "k" taken from N, the second is a minor of order "k" taken from N_t, and the third is a minor of order "m - k" = "e", taken from Z and involving those rows and columns of Z not included in the other two minors. The sign given to each term depends on the following argument.

The interchange of adjacent columns (or adjacent rows) changes the sign of a determinant. If a minor determinant (or simply, "minor) is formed by selecting the elements from a particular set of rows and of columns in the determinant, then, too, its sign would be changed if adjacent rows or adjacent columns in it were interchanged. A minor formed in a symmetrical fashion by selecting elements from a set of rows and corresponding columns is called a "principal minor." In a determinant with the elements, a_{ij}, the following would be examples of principal minors:

$$\begin{vmatrix} a_{44} \end{vmatrix} \qquad \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{21} & a_{22} & a_{24} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} \qquad \begin{vmatrix} a_{22} & a_{25} \\ a_{52} & a_{55} \end{vmatrix}$$

A "signed minor" is a minor multiplied by "+1" if the minor is a principal minor or if it is possible to change the position in the determinant of the selected elements by an even number of interchanges of adjacent rows and columns so that they would be in the position of a principal minor; the minor is multiplied by "-1" if it would take an odd number of such interchanges to bring it to the position of a principal minor.

If all the rows and columns of a determinant not selected in forming a minor are considered, another minor can be formed from them. This minor is called the "complement" of the first minor. If it is taken as a signed minor, it is called the "algebraic complement." The complement of a principal minor is also a principal minor. The "sign" of a signed minor is the same as that of its algebraic complement.

Laplace's rule for evaluating a determinant is, (R-5):

> If all the (unsigned) minors are formed from a selected set of rows or columns of a determinant and the products of these minors with their respective algebraic complements are added, the resulting sum is equal to the determinant.

The more familiar rule due to Cramer is a special case of Laplace's rule in which one column (or row) is selected and the sum taken of the elements of that column (or row) times their respective algebraic compliments, called in this case, "cofactors."

Consider the following determinants:

$$\begin{vmatrix} m_{11} & m_{12} & m_{13} & 0 & 0 & 0 \\ m_{21} & m_{22} & m_{23} & 0 & 0 & 0 \\ m_{31} & m_{32} & m_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & m_{44} & m_{45} & m_{46} \\ 0 & 0 & 0 & m_{54} & m_{55} & m_{56} \\ 0 & 0 & 0 & m_{64} & m_{65} & m_{66} \end{vmatrix} \qquad \begin{vmatrix} 0 & 0 & 0 & m_{14} & m_{15} & m_{16} \\ 0 & 0 & 0 & m_{24} & m_{25} & m_{26} \\ 0 & 0 & 0 & m_{34} & m_{35} & m_{36} \\ m_{41} & m_{42} & m_{43} & 0 & 0 & 0 \\ m_{51} & m_{52} & m_{53} & 0 & 0 & 0 \\ m_{61} & m_{62} & m_{63} & 0 & 0 & 0 \end{vmatrix}$$

In these determinants, there is only one non-zero minor which involves the first three columns. Applying Laplace's rule, the value of the determinant is this minor times its algebraic cofactor. In the first determinant, the sign of the algebraic minor, which is a principal minor, is "+1". In the second determinant, if each of the first three columns is shifted three times to the right, making a total of nine shifts, the elements in the selected minors would be in the positions of principal minors, that is, in the position of the non-zero elements of the first determinant. Since it took an odd number of shifts, namely nine, the sign of the algebraic complement in this case is "-1". If, instead of the second determinant, there had been one of similar form but with its non-zero minors containing four rows and columns instead of three, then four shifts of each of four columns, totaling sixteen shifts, would have been visualized instead of the nine, and the sign of the algebraic complement would be "+1". Thus, in general, a determinant in the form, (R-4):

$$\begin{vmatrix} A & B \\ C & 0 \end{vmatrix} \begin{matrix} K \\ K \end{matrix}$$
$$\quad K \quad K$$

has the value, $(-1)^k |B| |C|$. Because the complement of "A" is zero, "A" does not appear in the result.

Now, in the original determinant considered in this appendix, if a principal minor of Z with "e" rows and columns were formed, its complement would be a minor in the form directly above. By Laplace's rule, the product of the two would be the contribution of this minor to the value of the determinant. If the "e" by "e" minor had to be shifted to a principal position, its sign would be changed accordingly. Hence, in terms of the defined notation, there is the following rule:

> The sign given to the terms in the extended Cauchy expansion is that of the signed minor of Z of order "e by e", provided "k" is even. It is the opposite sign if "k" is odd.

The total number of possible terms in the Cauchy expansion is the number of ways in which "e" columns may be selected from "m" columns times the number of ways "e" rows may be selected from "m" rows, or $m!^2/(e!^2 k!^2)$.

As an example, in the determinant for equations (19), $m = 3$, $k = 2$, and therefore, $e = 1$, so that the term containing z_{11} is

$$z_{11} \left| n_{2a} \ \ n_{3b} \right|^2$$

and that of z_{12} and z_{21} is

$$-z_{12} \left| n_{1a} \ \ n_{3b} \right|^2 \quad - z_{21} \left| n_{1a} \ \ n_{3b} \right|^2$$

where the winding-turn minors have been abbreviated by their diagonal terms.

As a second example, (R-3)

$$\begin{vmatrix} A & 0 & 0 & 0 & a & a' \\ 0 & B & 0 & 0 & b & b' \\ 0 & 0 & C & 0 & c & c' \\ 0 & 0 & 0 & D & d & d' \\ a & b & c & d & 0 & 0 \\ a' & b' & c' & d' & 0 & 0 \end{vmatrix} = \begin{matrix} AB\,|cd'|^2 + CD\,|ab'|^2 \\ AC\,|bd'|^2 + BD\,|ac'|^2 \\ AD\,|bc'|^2 + BC\,|ad'|^2 \end{matrix}$$

The signs here are all positive because the number of bordering rows, "k", is even and also all the non-zero minors of Z are principal minors.

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