CONTRIBUTIONS

TO THE

THEORY OF FUNCTIONALS

Thesis

by

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Part 1.

POLYNOMIALS AND ANALYTIC FUNCTIONS IN VECTOR SPACES

11. Introduction to Part 1.

Part 1 consists of three divisions. In division (12) vector spaces are defined and some of their immediate properties deduced. Division (13) discusses polynomials and deduces a number of their properites. In the final division the work of the preceding pages is applied to a generalization of the ordinary theory of analytic functions.

12. Vector Spaces

12.1 General Remarks.

Definitions have been given by Banach, Weiner, Frechet, and others, of abstract spaces having properties similar to those of the space of ordinary vectors. In order that this paper may be self contained and in order that there may be no confusion as to what is meant, we shall give our own formal statement of the postulates defining Vector Spaces. Our choice of the particular system has been guided by a desire to have, later on, as many analogies as possible to the ordinary theory of functions.

12.2 Vector Spaces and the Notation

The spaces with which we deal are systems consisting of a set E of elements or points, a number system A, and three operations $\oplus$, $\odot$, $\|\ldots\|$. These systems are assumed to satisfy some or all of the postulates given below.

The set E is called the support of the space. The elements of E are usually denoted by the letters x,y,z,... . In general no ambiguity aries when we refer to the space and its support by the same letter. To emphasize the fact that a space E involves a particular number system A_1, it will be referred to as an $E(A_1)$.

Functions, Mapping: The usual notion of mapping is assumed. If to each element x of a set E_0 there corresponds by some law a well determined x^1 of a second set E_0^1, the set E_0 is said to be mapped upon the set E_0^1. A function f is a symbol for such a mapping. The relation is denoted by $x^1 = f(x)$. Following the phrasing of E. H. Moore, we refer to f as a function on E_0 to E_0^1.

Composition of Classes: If $E_1, E_2, \ldots, E_n$ are sets of elements, then by $E_1 E_2 \cdots E_n$ or $\prod_{i=1}^{n} E_i$ we shall understand the multiplication class of ordered sets $(x_1, x_2, \cdots, x_n)$, where x_i runs through the set E_i. If $E_1 = E_2 = \cdots = E_n$, $\prod_{i=1}^{n} E_i$ is denoted simply by E^n. The set E_i is spoken of as the composite of the sets $E_1, \cdots, E_n$.

Functions Involving Parameters: If f is a function on E_0 to E_0^1 where $E_0 = \prod_{i=1}^{n} E_i$, it is sometimes convenient to consider the properties of the sub-mapping obtained by holding fixed, certain of the elements x_i, say $x_{k+1}, \cdots, x_n$, of the ordered sets $(x_1, x_2, \cdots, x_n)$, and regarding f quâ function of $x_1, \cdots, x_k$. In this f will be said to be a function on $E_1 \cdots E_k [E_{k+1} \cdots E_n]$ to E_o^1. The elements $x_1, \cdots, x_k$ of $E_1, \cdots, E_k$ will then be spoken of as variables; the elements $x_{k+1}, \cdots, x_n$ of $E_{k+1}, \cdots E_n$ as parameters.

12.3 The Postulates for Spaces E(R) and E(C).

A_1: E is a class consisting of at least two elements.

A_2: A is either R, the real number system, or C, the complex number system.

B_1: $z = x \oplus y$ is a function on E^2 to E. That is, to each ordered pair x,y of elements from E there corresponds a unique element z called their ~~norm~~ sum.

B_2: $x \oplus y = y \oplus x$

B_3: $(x \oplus y) \oplus z = x \oplus (y \oplus z)$

B_4: $z = \alpha \odot x$ is a function on AE to E. That is, to each ordered pair (α, x) consisting of an element α of A and an element x of E there corresponds an unique element z of E called the product of α and x.

B_5: The set E is included in the set $(\alpha \odot k)$. That is, to each element z of E there corresponds at least one pair (α, x) of AE such that $z = \alpha \odot x$.

B_6: $\alpha \odot (x \oplus y) = \alpha \odot x \oplus \alpha \odot y$.

B_7: $(\alpha + \beta) \odot x = \alpha \odot x \oplus \beta \odot x$. Where "+" denotes ordinary addition in the number system A.

B_8: $(\alpha \cdot \beta) \odot x = \alpha \odot (\beta \odot x)$ where "$\cdot$" denotes ordinary multiplication in A.

B_9: $\rho = \|x\|$ is a function on E to the non-negative real numbers; that is, to each x of E corresponds a non-negative number $\rho = \|x\|$. $\|x\|$ is called "norm of x".

B_{10}: There is a unique x* of E whose norm is zero.

B_{11}: $\|\alpha \odot x\| = |\alpha| \cdot \|x\|$ where $|\ldots|$ denotes the ordinary modulus

B_{12}: $\|x \oplus y\| \leqq \|x\| + \|y\|$.

Notation: When no ambiguity arises $\oplus$ will be written simply + and $\odot$ will be omitted entirely.

x - y will be defined as x + (-1)y.

x* which by B_{10} and B_{11} is seen to be identical with $0 \cdot x$ for every x will be denoted by 0.

Neighborhood: The points x for which $\|x - x_0\| < \rho$ are said to lie in the ρ neighborhood of x_0. The ρ neighborhood of x_0 is denoted by $(x_0)_\rho$.

Region: A set of points E_0 is a region in E if each x_0 of E_0 has a ρ neighborhood $(x_0)_\rho$ whose points are all in E_0.

Connected Region: A region E_0 is said to be connected if when x x^1 are any two of its points there exists a chain of neighborhoods finite in number, say $(x)_\rho$, $(x_1)_{\rho_1}, \cdots, (x_n)_{\rho_n}, (x^1)_{\rho 1}$, each having a point in common with the preceding.

Limit Point: A set of points E_0 of E is said to have a limit point x_0 if it contains points x distinct from x_0 in every $(x_0)_\rho$. An infinite sequ ence $(x_1, x_2, \cdots x_n, \ldots)$ of points of E is said to have a limit if there is a point x of E such that to each ρ there corresponds an integer N_ρ such that for $n > N_\rho$, x_n belongs to $(x)_\rho$. x is called the limit of the sequence.

C_1: The necessary and sufficient condition that a sequence (x_n) have a limit is that for each ρ there is an N_ρ such that for $n > N_\rho$ and $p > 0$, x_{n+p} belongs to $(x_n)_\rho$.

Spaces which satisfy the postulate C_1 are said to be complete. Many of the theorems which follow are proved without the postulate of completeness. When completeness is employed it will be noted.

12.4 Immediate consequences of the Postulates

T 1. $1 \cdot x = x$

Proof: There exists α, y such that $x = \alpha \cdot y$ (B_5)

Then $1 \cdot x = 1 \cdot (\alpha \cdot y) = (1 \cdot \alpha) \cdot y = \alpha \odot y$ (B_8)

$x = 1 \odot x$

T 2. $0 \cdot x = x^*$

Proof: $\|0 \cdot x\| = |0| \, \|x\| = 0$ (B_{11})

Hence $0 \cdot x = x^*$ (B_{10})

T 3. $x + x^* = x$

Proof: $x + x^* = 1 \cdot x + 0 \cdot x = (1 + 0)x = 1 \cdot x = x$

By T 1, T 2, B_7

T 4. $x - x = x^*$

Proof: $x - x = x + (-1)x = 1 \cdot x + (-1)x = (1-1)x = 0 \cdot x = x^*$

T 5. The set E forms an Abelian group under $\oplus$.

Proof: From B_1, B_2, B_3, follow closure under $\oplus$, associativity, and commutativity. From T 3 follows the existence of a unit; from T 4 the existence of an inverse.

From T 5 the next three standard theorems on Abelian groups may be taken over at once. Thus

T 6 The equation $x + z = y$ has the unique solution $z = y - x$.

T 7 The inverse to x is unique, i.e., for each x there is a unique x_0 (viz. $-x$) such that $x + x_0 = x^*$.

T 8 The unit is unique; i.e., there is but one x_0 (viz x^*) such that $x + x_0 = x$.

T 9 If $E(A)$ is complete and (x_n) is a sequence such that $\sum_0^n \|x_{n+1} - x_n\|$ converges then (x_n) has a limit.

Proof: Let $\rho > 0$ be arbitrary.

Then choose N so that for $n > N$

$$\|x_{n+1} - x_n\| + \cdots + \|x_{n+p} - x_{n+p-1}\| < \rho$$

(all $p > 0$). Thus

$$\|x_{n+p} - x_n\| \leq \|x_{n+1} - x_n\| + \cdots + \|x_{n+p} - x_{n+p-1}\| < \rho$$

Hence by C_1 there exists x such that $\lim_{n\to\infty} x_n = x$.

T 10 If a sequence (x_n) has a limit x then that limit is unique.

Proof: Let x^1 be another limit

Then $\|x^1 - x\| = \rho > 0$

Choose N such that for $n > N$ $\|x_n - x\| < \rho/2$

and $\|x_n - x^1\| < \rho/2$

then $\|x^1 - x\| < \rho$

T 11 If a sequence (x_n) has a limit x, then $\|x_n\|$ has a limit and

$$\left\| \lim_{n \to \infty} x_n \right\| = \lim_{n \to \infty} \|x_n\|$$

Proof: Let $\rho > 0$ be arbitrary.

Choose N such that for $n > N$ $\|x_n - x\| < \rho$

Then $\rho > \|x_n - x\| \geqq \left| \|x_n\| - \|x\| \right|$

Hence $\left\| \lim_{n \to \infty} x_n \right\| = \|x\| = \lim_{n \to \infty} \|x_n\|$

T 12 $\left| \|x\| - \|y\| \right| \leqq \|x \pm y\|$

Proof: $\|x\| = \|x \pm y \mp y\| \leqq \|x \pm y\| + \|\mp y\| = \|x \pm y\| + \|y\|$

So that $\|x\| - \|y\| \leqq \|x \pm y\|$

Similarly $\|y\| - \|x\| \leqq \|x \pm y\|$

Hence $\left| \|x\| - \|y\| \right| \leqq \|x \pm y\|$

T 13 If (α_n) is a sequence of numbers of A having α as its limit and x is an element of E(A) then the sequence $(\alpha_n x)$ has as its limit αx.

Proof: Choose $\rho > 0$ arbitrary.

Choose N such that for $n > N$ $|\alpha_n - \alpha| < \rho / \|x\|$

Then for $n > N$, $\|\alpha_n x - \alpha x\| = |\alpha_n - \alpha| \, \|x\|$

T 14 If $\|x - y\| = 0$, then $x = y$

Proof: $x - y = x^*$

$x = x^* + y = y + x^* = y$ $(B_2, T\ 3)$

T 15 If $\alpha x = x^*$ then either $\alpha = 0$ or $x = x^*$

Proof: $|\alpha| \, \|x\| = \|\alpha x\| = \|x^*\| = 0$

Hence either $|\alpha| = 0$ or $\|x\| = 0$

Hence $\alpha = 0$ or $x = x^*$

T 16 If $\alpha x = \beta x$ and $x \neq x^*$, then $\alpha = \beta$

Proof: $\alpha x - \beta x = x^*$ (T 4)

$(\alpha - \beta)x = x^*$ (B_7)

$\alpha - \beta = 0$ (T 15)

Consistency of the Postulates: The postulates A_1, A_2, B_1, ... ,B_{12}, C_1 are all satisfied if E is taken to be A itself, $\oplus$, and $\odot$ interpreted as ordinary addition and multiplication and $\|\ldots\|$ interpreted as ordinary modulus.

R and C as Vector Spaces: With the above interpretations of $\oplus$, $\odot$, $\|\ldots\|$, the number systems R and C may themselves be regarded as supports of vector spaces. There are, of course, other definitions of the three operations by which R and C may be taken as supports; however, for our purposes this definition is the simplest.

Function Space as a Vector Space: If the elements x of E are taken to be real continuous functions of one or finitely more real variables defined over some fixed domain D of the variables. If the sum of two functions is defined in the ordinary manner, and similarly the product of a function by a constant, and if $\|x\|$ be defined as the $\max_D |x|$, then it is easily verified that with suitable restrictions upon D the set of elements x constitutes a vector space E(R). In particular if D is taken to be the linear interval (a,b) and x is a function x(t) continuous for t in (a,b), then with the definition $\|x\| = \max_{(a,b)} |x(t)|$ the space E(R) of elements x is the well known space of continuous functions. It will in general be denoted by F(a,b).

Composition of Vector Spaces

T 17 Let $E_1(A), \cdots, E_n(A)$ be vector spaces with the number system A. Let $E = \prod_{i=1}^{n} E_i$ be the composite of $E_1, \cdots, E_n$ their respective supports. Let the sum of two elements $(x_1, x_2, \cdots, x_n)$, $(x_1', x_2', \cdots, x_n')$ of E be defined by

$$(x_1, x_2, \cdots, x_n) \oplus (x_1', x_2', \cdots, x_n') \equiv (x_1 \oplus x_1', x_2 \oplus x_2', \cdots, x_n \oplus x_n') \quad (1)$$

Let the product of α of A and $(x_1, x_2, \cdots, x_n)$ of E be defined by

$$\alpha \odot (x_1, x_2, \cdots, x_n) \equiv (\alpha \odot x_1, \alpha \odot x_2, \cdots, \alpha \odot x_n) \quad (2)$$

Let the norm of $(x_1, \cdots, x_n)$ be defined by

$$\|(x_1, x_2, \cdots, x_n)\| = \text{greatest of } \|x_1\|, \|x_2\|, \cdots, \|x_n\| \quad (3)$$

Then the system consisting of E, A, $\oplus$, $\odot$, $\|\ldots\|$ satisfies the postulates $A_1, \cdots, C_1$ and is therefore a vector space.

We shall verify the postulates for this system in order.

A_1: The multiplicative class of n classes each having at least 2 elements has not less than 2^n elements.

A_2: Trivial.

B_1: (1) uniquely determines the sum.

B_2: Clear.

B_3: Clear.

B_4: (2) uniquely determines the product.

B_5: $1 \odot (x_1, x_2, \cdots, x_n) = (1 \odot x_1, 1 \odot x_2, \cdots, 1 \odot x_n) = (x_1, x_2, \cdots, x_n)$

B_6: Clear.

B_7: Clear.

B_8: Clear.

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B_9: (3) defines $\rho = \|(x_1, \cdots, x_n)\|$ on E to non-negative real numbers.

B_{10}: $\|(x_1^*, \cdots, x_n^*)\| \equiv$ greatest of $\|x_1^*\|, \cdots, \|x_n^*\| = 0$

Moreover, if greatest of $x_1, \cdots, x_n = 0$ then

$x_1 = x_1^*,\ x_2 = x_2^*, \cdots, x_n = x_n^*$.

B_{11}: $\|\alpha \odot (x_1, \cdots, x_n)\| = \|(\alpha x_1, \cdots \alpha x_n)\| =$ greatest of

$\|\alpha x_1\|, \cdots, \|\alpha x_n\| =$ greatest of $|\alpha| \|x_1\|, \cdots, |\alpha| \|x_n\|$

$= |\alpha| \|(x_1 \cdots x_n)\|$

B_{12}: $\|(x_1, \cdots, x_n) \oplus (x_1', \cdots, x_n')\| = \|(x_1 + x_1', \cdots, x_n + x_n')\|$

$=$ greatest of $\|x_1 + x_1'\|, \cdots, \|x_n + x_n'\|$

$\leqq$ greatest of $\|x_1\|, \cdots, \|x_n\|$ + greatest of $\|x_1'\|, \cdots, \|x_n'\|$

C_1: Let $y_h \equiv (x_1^h, \cdots, x_n^h)$ $(h = 1, 2, \ldots)$ be a sequence. First assume (y_x) converges to y. Choose ρ and N_ρ so that for $m > N_\rho \|y - y_m\| < \rho/2$. Hence $\|y_m - y_{m+p}\| < \rho$

If for each ρ there corresponds N_ρ such that if $n > N_\rho$ $\|y_m - y_{m+p}\| < \rho$ then we have $\|x_i^m - x_i^{m+p}\| < \rho$. If therefore $E_1, \cdots, E_n$ are complete each sequence (x_i^m) converges to a limit x_i. The point $y \equiv (x_i, \cdots, x_n)$ is then the limit of (y_k)

12.5 <u>Continuity and Continuous Functions</u>

<u>Continuity</u>: Let E_0 be a region of E. Let f(x) be a function on E_0 to E'. f(x) is said to be continuous at a point x_0 of E_0 if to $\epsilon > 0$ there is determined $\rho > 0$ such thatif x is in $(x_0)_\rho$ then f(x) is in $[f(x_0)]_\epsilon$.

If f(x) is continuous at each point of E_0 it is spoken of as continuous throughout E_0.

Uniform Continuity: If $f(x)$ is on a region E_0 to E' and if $\epsilon > 0$ there is determined $\rho > 0$ such that for any x_0 of E_0, x in $(x_0)_\rho$ implies $f(x)$ in $[f(x_0)]_\epsilon$, then $f(x)$ is said to be uniformly continuous in E_0. It is clear from the very definition that if $f(x)$ is uniformly continuous in E_0 it is continuous throughout E_0.

Theorems on Continuity

Theorem (12.51) If $f_1(x)$, $f_2(x)$ are functions on E_0 to E' and if

(1) they are continuous at a point x_0 of E_0, their sum is continuous at x_0

(2) they are continuous throughout E_0, their sum likewise is continuous throughout E_0.

(3) they are uniformly continuous in E_0, their sum is uniformly continuous in E_0.

Proof: (1) Let $\epsilon > 0$ be given and let ρ_1 and ρ_2 be so determined that x in $(x_0)_{\rho_1}$ implies $f_1(x)$ in $[f_1(x_0)]_{\epsilon/2}$ and x in $(x_0)_{\rho_2}$ implies $f_2(x)$ in $[f_2(x_0)]_{\epsilon/2}$. Then if $\rho \leqq \rho_1, \rho_2$, x in $(x_0)_\rho$ implies $f_1(x) + f_2(x)$ in $[f_1(x_0) + f_2(x_0)]_\epsilon$ for if x is in $(x_0)_\rho$

$$\|f_1(x) - f_1(x_0)\| < \epsilon/2$$

$$\|f_2(x) - f_2(x_0)\| < \epsilon/2$$

$$\|f_1(x) + f_2(x) - [f_1(x_0) + f_2(x_0)]\| < \epsilon$$

(2) Apply the argument of (1) to each point x_0 of E_0.

(3) Apply the argument of (1) observing that the choice of ρ_1 and ρ_2 and hence of ρ does not involve the choice of the particular point x_0.

Corollary: By complete induction a similar theorem holds for the sum of a finite number of functions, $f_1(x), \cdots, f_n(x)$ on E_0 to E'.

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Theorem (12.52) (1) If E_0 and A_0 are regions of $E(A)$ and A respectively, $f(x)$ is a function on E_0 to $E'(A')$, $g(\alpha)$ is a function on A_0 to A' and if at x_0 of E_0 and α_0 of A_0 $f(x)$ and $g(\alpha)$ are continuous, then $g(\alpha)f(x)$ is a function on A_0E_0 to $E'(A')$ continuous at (α_0, x_0).
(2) If, moreover, $f(x)$, $g(\alpha)$ are uniformly continuous and $\|f(x)\|$, $|g(\alpha)|$ are bounded then $g(\alpha)f(x)$ is uniformly continuous.

Proof: (1) Let $\epsilon > 0$ be given. Choose ρ_1 so that if x is in $(x_0)_{\rho_1}$ $\|f(x) - f(x_0)\| < \dfrac{\epsilon}{2\|g(\alpha_0)\|}$. Choose ρ_2 so that if α is in $(\alpha_0)_{\rho_2}$ $|g(\alpha) - g(\alpha_0)| < \dfrac{\epsilon}{2\|f(x_0)\| + \epsilon/|g(\alpha_0)|}$. Let $\rho = \min(\rho_1 \rho_2)$. If then (α, x) is in $[(\alpha, x)]_\rho$ we shall have $|\alpha - \alpha_0| < \rho_2$ $\|x - x_0\| < \rho_1$. So

$$\|g(\alpha)f(x) - g(\alpha_0)f(x_0)\| \leqq |g(\alpha) - g(\alpha_0)|\,\|f(x)\| + |g(\alpha_0)|\,\|f(x) - f(x_0)\|$$

$$< \frac{\epsilon\,\|f(x)\|}{2\|f(x_0)\| + \epsilon/|g(\alpha_0)|} + \frac{\epsilon|g(\alpha_0)|}{2|g(\alpha_0)|} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

(2) Let M be the greater of the upper bounds of $\|f(x)\|$, $|g(\alpha)|$ in E_0 and A_0 respectively. Let $\epsilon > 0$ be given and let $\rho_1 \rho_2$ be so chosen that x in $(x_0)_{\rho_1}$ implies $f(x)$ in $[f(x_0)]_{\frac{\epsilon}{2M}}$ and α in $(\alpha_0)_{\rho_2}$ implies $g(\alpha)$ in $[g(\alpha_0)]_{\frac{\epsilon}{2M}}$ for all (α_0, x_0) of A_0E_0. Let $\rho = \min(\rho_1\rho_2)$. Then for (α, x) in $[(\alpha_0, x_0)]_\rho$,

$$\|g(\alpha)f(x) - g(\alpha_0)f(x_0)\| \leqq |g(\alpha) - g(\alpha_0)|\,\|f(x)\| + |g(\alpha_0)|\,\|f(x) - f(x_0)\|$$

$$< \frac{\epsilon}{2M}\|f(x)\| + \frac{\epsilon}{2M}\,g(\alpha_0) \leqq \epsilon$$

<u>Theorem (12.53)</u> If $f(x)$ is continuous on E_1E_2 to E' then it is continuous on $E_1 [E_2]$ to E'.

<u>Proof:</u> Let $x \equiv (x_1, x_2)$ $\quad x' \equiv (x_1', x_2)$

$$\|x - x'\| \equiv \|(x_1 - x_1', 0)\| = \|x_1 - x_1'\|$$

But $\quad \|f(x) - f(x')\| \to 0$ with $\|x - x'\|$ and hence with $\|x_1 - x_1'\|$.

<u>Functions of Functions:</u> If E, E', E'' are vector spaces and E_0, E_0', E_0'' are subspaces, proper or not, and if $y = f(x)$ is a function on E_0 to E_0' and $z = g(y)$ a function on E_0' to E_0'' then $z = g[f(x)]$ is a function on E_0 to E_0''.

<u>Theorem (12.54)</u> If E_0, E_0', E_0'' are regions and $y = f(x)$ is continuous on E_0 to E_0' and $z = g(y)$ is continuous on E_0' to E_0'', then $z = g(f(x))$ is continuous on E_0 to E_0''. Moreover, if $y = f(x)$ and $z = g(y)$ are uniformly continuous in their respective regions, then $z = g[f(x)]$ is uniformly continuous.

<u>Proof:</u> Let $\epsilon > 0$ be chosen, and consider a fixed point x_0 in E_0. Let $y_0 = f(x_0)$; $z_0 = g(y_0)$. Let $\rho_1 > 0$ be so selected that y in $(y_0)_{\rho_1}$ implies $z = g(y)$ in $(z_0)_\epsilon$. Let $\rho_2 > 0$ be so selected that x in $(x_0)_{\rho_2}$ implies $y = f(x)$ in $(y_0)_{\rho_1}$. Then if x is in $(x_0)_{\rho_2}$, z is in $(z_0)_\epsilon$. If $f(x)$, $g(y)$ are uniformly continuous in their respective regions, the choice of ρ_1 and in particular of ρ_2 does not involve the point x_0.

<u>Corollary:</u> By complete induction it follows that a similar theorem holds for any finite number of successive functional operations.

12.6 Other Special Notions

Successive Differences: Let $f(x)$ be a function on E to E'. Let Δx be an element of E. Then

$$\Delta f(x) \equiv f(x + \Delta x) - f(x)$$

is a function on E^2 to E' and is called the first difference of $f(x)$ with respect to x.

If $\Delta_1 x, \cdots, \Delta_n x$ denote elements of E then the n^{th} difference, $\overset{(n)}{\Delta} f(x) \equiv \Delta_n \Delta_{n-1} \cdots \Delta_1 f(x)$ of $f(x)$ with respect to $\Delta_1 x, \cdots, \Delta_n x$ is defined inductively by

$$\Delta_n \Delta_{n-1} \cdots \Delta_1 f(x) \equiv \Delta_n(\Delta_{n-1} \cdots \Delta_1 f(x)) = \Delta_{n-1} \cdots \Delta_1 f(x + \Delta_n x)$$
$$- \Delta_{n-1} \cdots \Delta_1 f(x)$$

The n^{th} difference is written in the form $\overset{(n)}{\Delta} f(x)$ in case there is no ambiguity as to what is meant. $\overset{(n)}{\Delta} f(x)$ is clearly a function on E^{n+1} to E'.

Theorem (12.61) $\overset{(n)}{\Delta} f(x)$ is symmetrical in the increments $\Delta_1 x, \cdots, \Delta_n x$.

Proof: For any $h(x)$ on E to E' and any pair $\Delta_i x$ and $\Delta_j x$ of the increments we have

$$\Delta_i \Delta_j h(x) = h(x + \Delta_i x + \Delta_j x) - h(x + \Delta_i x) - h(x + \Delta_j x)$$
$$+ h(x) = \Delta_j \Delta_i h(x)$$

Hence at any stage in the expansion of $\overset{(n)}{\Delta} f(x)$ we may exchange two adjacent Δ's. But any permutation of the Δ's may be produced in this manner.

Theorem (12.62) If $f_1(x),\cdots,f_s(x)$ are functions on E to E', then $\overset{(n)}{\Delta}$ regarded as an operator, operates distributively upon the sum $f_1(x) + f_2(x) + \cdots + f_s(x)$.

Proof: Let $f(x) \equiv f_1(x) + f_2(x) + \cdots + f_s(x)$, then for n = 1

$$\Delta f(x) = \sum f_r(x+\Delta x) - \sum f_r(x) = \sum f_r(x+\Delta x) - f_r(x)$$
$$= \sum \Delta f_r(x)$$

Using this result and assuming the theorem for (n-1).

$$\overset{(n)}{\Delta} f(x) = \Delta_n \overset{(n-1)}{\Delta} f(x) = \Delta_n \sum \overset{(n-1)}{\Delta} f_r(x) = \sum \overset{(n)}{\Delta} f_r(x)$$

Theorem (12.63) $\overset{(n)}{\Delta}$ regarded as an operator commutes with numerical multipliers.

Proof: Let f(x) be on E to E'(A) and α be in A. Then

$$\Delta \alpha f(x) = \alpha f(x+\Delta x) - \alpha f(x) = \alpha \Delta f(x)$$

Thus assuming the result for (n-1)

$$\overset{(n)}{\Delta} \alpha f(x) = \Delta_n \overset{(n-1)}{\Delta} \alpha f(x) = \Delta_n \alpha \overset{(n-1)}{\Delta} f(x) = \alpha \overset{(n)}{\Delta} f(x).$$

Theorem (12.64) If $\alpha(\lambda)$ is a function on A to A' and a is some fixed element of E(A') then for any increments $\Delta_1\lambda,\cdots,\Delta_n\lambda$ we have

$$\overset{(n)}{\Delta}(\alpha(\lambda)a) = \left(\overset{(n)}{\Delta}\alpha(\lambda)\right)a$$

Proof: Exactly similar to (12.63) except that the multiplier is now on the right.

Homogeneous Linear Continuous Functions: In order to make the next section intelligible we must define here a special case of homogeneous polynomials which will be discussed in 13. A function $f(x)$ on $E(A)$ to $E'(A)$ is homogeneous linear continuous if

(1) it is continuous.

(2) $f(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 f(x_1) + \alpha_2 f(x_2)$ for all x_1, x_2 in E and all α_1, α_2 in A.

Differentials: Let E_0 be a region of E. A function $f(x)$ on E_0 to E' is said to have a differential at a point x_0 of E_0 if there exists a function $f_x(x_0;\Delta x)$ on $E[E_0]$ to E' such that

(1) $f_x(x_0;\Delta x)$ is continuous homogeneous linear on $E[E_0]$ to E'.

(2) $\epsilon(\Delta x)$ defined by $\epsilon(\Delta x) \equiv \dfrac{f_x(x_0;\Delta x) - \Delta f(x)}{\|\Delta x\|}$

$(\Delta x \neq 0;\ (x_0 + \Delta x) \text{ in } E_0)$

$\epsilon(0) \equiv 0$

is continuous at $\Delta x = 0$. $f_x(x_0;\Delta x)$ is called the differential and will be denoted in this way or by the symbol $df(x_0)$, the argument Δx being understood.

Higher Differentials: If $df(x_0) \equiv f_x(x_0;\Delta_1 x)$ exists at each x_0 of E_0 then $df(x_0)$ is on EE_0 to E'. The second differential $f_{x^2}(x_0;\Delta_1 x,\Delta_2 x)$ is defined as the differential of $f_x(x_0;\Delta_1 x)$ quâ function on $[E]E_0$ to E'. The second differential will also be denoted by $d_1 d_2 f(x_0)$ or in case there is no ambiguity by $d^2 f(x_0)$. Higher differentials are defined inductively in the same manner.

Calculation of the Differential: This is simply the generalization to vector spaces of the theorem due to Gateaux.

Theorem (12.65) If $f(x)$ on E_0 to E' has a differential $df(x_0)$ at x_0 of E_0 then

$$df(x_0) = \lim_{\lambda \to 0} \frac{f(x_0 + \lambda \Delta x) - f(x_0)}{\lambda}$$

Proof: By definition we have that

$$df(x_0) \equiv f_x(x_0;\Delta x) = f(x_0 + \Delta x) - f(x_0) + \epsilon(\Delta x)\|\Delta x\|$$

Replace Δx by $\lambda \Delta x$ and divide by λ

$$f_x(x_0;\Delta x) = \frac{f(x_0 + \lambda\Delta x) - f(x_0)}{\lambda} + \epsilon(\lambda\Delta x)\|\Delta x\| \frac{|\lambda|}{\lambda}$$

As $\lambda \to 0$, $\epsilon(\lambda\Delta x) \to 0$. Hence the theorem.

The remaining necessary theorems on differentials we shall postpone until the discussion of the properties of linear functions has been reached.

13. Polynomials in a Vector Space

13.1 Introduction

Fréchet in his 1910 paper* discussed polynomials in the space of continuous functions. In a later paper**(1929) he generalized many of the theorems of the previous paper to a class of spaces which he calls "espaces algebrophiles", and of which our space is an instance. The essential result in both these papers is that a general polynomial is uniquely representable as a sum of homogeneous polynomials. The definition taken by Frechet in his later paper, while entirely elegant and satisfactory for spaces closed under multiplication by real numbers--which are the only ones he considers--runs afoul of the common definition of a polynomial in a single complex variable, if taken over bodily to spaces closed under multiplication by complex numbers. In order that later generalizations of functions of a complex variable may be made, it is essential that we formulate a definition which will be equally valid for spaces E(R) and E(C). The definition we have taken is equivalent for spaces E(R) to the definition of Frechet. Half the equivalence is really proved quite incidentally in Frechet's paper. We shall give an independent proof whose details are quite simple.

In the latter section of this work in polynomials, we shall discuss modular properties of polynomials and their related forms. We shall point out distinctions that exist between the modular properties of polynomials on E(R) and of those of polynomials on E(C).

This latter part of the work, we hardly need mention, does not presume to be in any way exhaustive of what looks to be a rather large subject.

*M. Frechet, Ann de l'Ecole Normal Sup. Ser 3. Vol 27, 1910.

**M. Frechet, Journal de Math. 1929.

13.2 Polynomials on A to E(A)

Definition: A function $p(\lambda) \not\equiv 0$ on A to E(A) is called a polynomial, if it is expressible in the form

$$p(\lambda) \equiv \sum \lambda^r a_r$$

where the summation extends over a finite number of terms, where λ^r stands for the ordinary r^{th} power of λ, and where each a_r is a fixed element of E(A). If n is the highest index for which $a_r \neq 0$, n is called the degree of $p(\lambda)$. If the function $p(\lambda) \equiv 0$ for all λ of A, then $p(\lambda)$ is called a "null" polynomial, and we add the gloss that the null polynomial may be regarded as having any degree whatever.

Theorem (13.201) If $p(\lambda)$ is a polynomial on A to E(A) it is continuous.

Proof: By Theorems (12.52), (12.51), $p(\lambda)$ is the sum of continuous functions each of which is continuous.

Theorem (13.202) If $p(\lambda)$ is a polynomial of degree n on A to E(A) and if $g(\lambda)$ is a polynomial of degree m on A to A, then $g(\lambda)p(\lambda)$ is a polynomial of degree n+m on A to E(A).

Proof:

Let $$p(\lambda) \equiv \sum_0^n \lambda^r a_r \qquad (a_n \neq 0)$$

$$g(\lambda) \equiv \sum_0^m \lambda^s \alpha_s \qquad (\alpha_m \neq 0)$$

Then $$g(\lambda)p(\lambda) = \sum_0^m \lambda^s \alpha_s \sum_0^n \lambda^r a_r = \sum_0^m \sum_0^n \lambda^s \alpha_s \lambda^r a_r$$

$$= \sum_{0}^{m} \sum_{0}^{n} \lambda^{s+r} \alpha_s a_r$$

$$= \sum_{p=0}^{m+n} \lambda^p \sum_{s \geq 0,\, p-n}^{s \leq p,\, n} \alpha_s a_{p-s} = \sum_{p=0}^{m+n} \lambda^p b_p$$

The first two steps follow from the associative and distributive laws. Now the coefficient b_{n+m} is simply $\alpha_m a_n$. But by definition $\alpha_m \neq 0$, $a_n \neq 0$ and hence $b_{n+m} \neq 0$.

If either or both $p(\lambda)$ and $g(\lambda)$ are null, their product is null and the theorem is still true.

Theorem (13.203) If $p_1(\lambda), \ldots, p_m(\lambda)$ are polynomials on A to E(A) each of degree $\leq$ n, then a linear combination $p(\lambda) \equiv \sum_{i=1}^{m} \alpha_i p_i(\lambda)$ with numerical coefficients α_i of A is a polynomial of degree $\leq$ n on A to E(A).

Proof: $p_i(\lambda)$, even though not of degree exactly n, may be written in the form $p_i(\lambda) = \sum_{r=0}^{n} \lambda^r a_{ir}$ by taking certain of a_{ir} to be null. Then

$$p(\lambda) = \sum_{i=1}^{m} \alpha_i \sum_{r=0}^{n} \lambda^r a_{ir} = \sum_{r=0}^{n} \lambda^r \sum_{i=1}^{m} \alpha_i a_{ir} \equiv \sum_{r=0}^{n} \lambda^r b_r$$

Theorem (13.204) Each coefficient a_r of a polynomial $p(\lambda) \equiv \sum_{r=0}^{n} \lambda^r a_r$ of degree n on A to E(A) is expressible as a linear combination $\sum_{s=0}^{n} A_{rs} p(s)$ of the n+1 vectors $p(0), p(1), \cdots, p(n)$, where A_{rs} are numbers independent of the choice of the polynomial $p(\lambda)$.

Proof: Consider the determinant $|\alpha_{sr}|$, where $\alpha_{sr} = s^r$ $(r, s = 0, 1, \cdots, n)$ $a_{00} = 1$. This is an instance of Vandermond's determinant, and its value is $\prod_{i>j}^{n} (i - j) \neq 0$.

Let A_{rs} be the typical term of the inverse determinant, so that

$$\sum_{s=0}^{n} A_{rs}\alpha_{sk} = \begin{cases} 0 & r \neq k \\ 1 & r = k \end{cases}$$

now

$$p(s) = \sum_{k=0}^{n} s^k a_k = \sum_{k=0}^{n} \alpha_{sk} a_k$$

$$\sum_{s=0}^{n} A_{rs} p(s) = \sum_{k=0}^{n} \left(\sum_{s=0}^{n} A_{rs}\alpha_{sk} \right) a_k = a_r$$

Theorem (13.205) Two polynomials $p(\lambda)$ and $q(\lambda)$ on A to E(A) are equal for all values of λ if and only if their coefficients are equal.

Proof: Let the degrees of $p(\lambda)$ and $q(\lambda)$ be $\leqq n$.

Let

$$r(\lambda) = p(\lambda) - q(\lambda)$$

Then by (13.203) $r(\lambda)$ is a polynomial of degree $\leqq n$. Moreover, if a_p, b_p, c_p are respectively the coefficients of λ^p in $p(\lambda)$, $q(\lambda)$, $r(\lambda)$, then again by (13.203)

$$c_p = a_p - b_p$$

By hypothesis $r(\lambda) = 0$ for all λ. Hence by (13.204) $c_p = 0$. Therefore

$$a_p \doteq b_p$$

13.3 Characterization of Polynomials on R to E(R)

We shall in this section give a new proof for Vector Space of a theorem proved by Frechet in his 1929 paper on abstract polynomials.

Theorem (13.31) A necessary and sufficient condition that a function $p(\lambda)$ on R to E(R) be a polynomial of degree n is that it satisfy the three conditions.

(1) $p(\lambda)$ is continuous.

(2) For all $\Delta_1\lambda, \cdots, \Delta_{n+1}\lambda$ and all λ in R we have $\overset{(n+1)}{\Delta} p(\lambda) \equiv 0$

(3) For some $\Delta_1\lambda, \cdots, \Delta_n\lambda$ and λ in R we have $\overset{n}{\Delta} p(\lambda) \neq 0$.

Proof:

A. Necessity: If $p(\lambda)$ is a polynomial then

(1) It is continuous by Theorem (13.201)

(2) Using theorems (12.62) and (12.64), and writing

$$p(\lambda) \equiv \sum_{r=0}^{n} \lambda^r a_r$$

we have
$$\overset{n+1}{\Delta} p(\lambda) \equiv \sum_{r=0}^{n} \overset{n+1}{\Delta}(\lambda^r a_r) = \sum_{r=0}^{n} (\overset{n+1}{\Delta} \lambda^r) a_r .$$

But, as is well known $(\overset{n+1}{\Delta} \lambda^r) = 0$ if $r < n+1$; hence

$$\overset{n+1}{\Delta} p(\lambda) = 0$$

(3) Again $\overset{n}{\Delta} p(\lambda) = \sum_{r=0}^{n} (\overset{n}{\Delta} \lambda^r) a_r = n!\, a_n \neq 0$, Since

$$\overset{n}{\Delta} \lambda^n = n!$$

B. Sufficiency: We prove by induction on n that the conditions (1), (2), (3) above imply that $p(\lambda)$ is a polynomial. For n = 0 the sufficiency is clear. Assuming the result for (n - 1) we have that if

$p(\lambda)$ satisfies the conditions (1), (2), (3), then $p(\lambda+\mu) - p(\lambda)$ is, for each value of μ, a polynomial of degree not greater than $(n - 1)$ in λ, and is for at least one value of μ of degree exactly $(n - 1)$ in λ. To see this we simply write $\mu = \Delta_1\lambda$, together with any other set of increments $\Delta_2\lambda, \cdots, \Delta_{n+1}\lambda$.

Then
$$p(\lambda+\mu) - p(\lambda) = \Delta_1 p(\lambda)$$

$$\Delta_{n+1}\Delta_n\cdots\Delta_2\big(p(\lambda+\mu) - p(\lambda)\big) = \overset{n+1}{\Delta} p(\lambda) = 0$$

Moreover, for some set of increments $\mu, \Delta_2\lambda, \cdots, \Delta_n\lambda$

we have
$$\overset{n}{\Delta} p(\lambda) \equiv \Delta_n\Delta_{n-1}\cdots\Delta_2\big(p(\lambda+\mu) - p(\lambda)\big) \neq 0$$

Hence
$$p(\lambda+\mu) - p(\lambda) = \sum_{r=0}^{n-1} \lambda^r a_r(\mu) \tag{1}$$

where $a_r(\mu)$ are functions on R to E whose exact nature has not shown up except that for some value of μ, $a_{n-1}(\mu) \neq 0$.

In expression (1), putting $\lambda = 0$, we obtain

$$p(\mu) - p(0) = a_0(\mu) \tag{2}$$

from which
$$p(\lambda+\mu) - p(\lambda) - p(\mu) + p(0) = \sum_{r=1}^{n-1} \lambda^r a_r(\mu) \tag{3}$$

Observing that the left hand side of (3) is symmetric in λ, μ we have

$$\sum_{r=1}^{n-1} \lambda^r a_r(\mu) = \sum_{r=1}^{n-1} \mu^r a_r(\lambda) \tag{4}$$

By (13.204) the coefficient $a_r(\mu)$ may be written as a linear combination with constant numerical coefficient of terms obtained by replacing λ in the right hand side of (4) successively by the values $0,1,2,\cdots,n$. Thus it follows that $a_r(\mu)$, being a linear combination of polynomials in μ, is also a polynomial in μ whose degree is at most $n-1$. Thus

$$p(\lambda+\mu) - p(\lambda) - p(\mu) + p(0) = \sum_{r,s=1}^{n-1} \lambda^r \mu^s a_{rs} \tag{5}$$

where a_{rs} are fixed elements of $E(R)$.

Suppose now first that λ_1 is a positive value of λ. Let μ be chosen so that $0<\mu<\lambda_1$. Let j be an integer. Put $\lambda = \mu j$ in (5) and sum from $j=1$ to $j = \left[\frac{\lambda_1}{\mu}\right]$, where $\left[\frac{\lambda_1}{\mu}\right]$ deonotes the greatest integer in $\frac{\lambda_1}{\mu}$.

We obtain

$$p\left(\mu\left\{\left[\frac{\lambda_1}{\mu}\right] + 1\right\}\right) - p(\mu) - \left[\frac{\lambda_1}{\mu}\right]\left\{p(\mu) - p(0)\right\}$$

$$= \sum_{r,s=1}^{n-1} \left\{ \sum_{j=1}^{\left[\frac{\lambda_1}{\mu}\right]} (j\mu)^r \mu \right\} \mu^{s-1} a_{rs} \tag{6}$$

Now since

$$\lim_{\mu\to+0} \left[\frac{\lambda_1}{\mu}\right] = \lambda_1$$

$$\lim_{\mu\to+0} \sum_{j=1}^{\left[\frac{\lambda_1}{\mu}\right]} (j\mu)^r \mu = \int_0^{\lambda_1} \mu^r d\mu = \frac{\lambda_1^{r+1}}{r+1}$$

it follows from the fact that $p(\lambda)$ is continuous and from T 13 (12.3), that as μ tends to zero from the positive side, the right hand side of (6) and the first two terms of the left hand side have well defined limits; and that, consequently, the last term on the left side also tends to a limit which we shall denote by K. The equation (6) then becomes

$$p(\lambda_1) - p(0) - K = \sum_{r=1}^{n-1} \frac{\lambda_1^{r+1}}{r+1} a_{r1} \tag{7}$$

where
$$K = \lim_{\mu \to +0} \left[\frac{\lambda_1}{\mu}\right]\left(p(\mu) - p(0)\right)$$
$$= \lim_{\mu \to +0} \mu\left[\frac{\lambda_1}{\mu}\right] \cdot \frac{p(\mu) - p(0)}{\mu}$$
$$= \lambda_1 \cdot \lim_{\mu \to +0} \frac{p(\mu) - p(0)}{\mu}$$

Hence, (7) gives for $\lambda_1 > 0$

$$p(\lambda_1) = p(0) + \lambda_1 \lim_{\mu \to +0} \frac{p(\mu) - p(0)}{\mu} + \sum_{r=1}^{n-1} \frac{\lambda_1^{r+1}}{r+1} a_{r1} \tag{8}$$

If λ_1 is taken to be negative, μ may also be restricted to be negative. The argument carries through in precisely the same fashion and we obtain for $\lambda_1 < 0$

$$p(\lambda_1) = p(0) + \lambda_1 \cdot \lim_{\mu \to -0} \frac{p(\mu) - p(0)}{\mu} + \sum_{r=1}^{n-1} \frac{\lambda_1^{r+1}}{r+1} a_{r1} \tag{9}$$

But by writing $\lambda = -\mu$ in (5) and making $\mu \to +0$ we have

$$\lim_{\mu \to +0} \frac{p(0) - p(-\mu)}{\mu} - \lim_{\mu \to +0} \frac{p(\mu) - p(0)}{\mu} = 0 ;$$

from which
$$\lim_{\mu \to -0} \frac{p(\mu) - p(0)}{\mu} = \lim_{\mu \to +0} \frac{p(\mu) - p(0)}{\mu} \tag{10}$$

Thus (8) and (9) coincide for $\lambda_1 \gtrless 0$ and either (8) or (9) is at once seen to be correct for $\lambda_1 = 0$. The work therefore shows that $p(\lambda)$ is a polynomial of degree at most n. If it were not exactly n, then the coefficient of λ^n would be zero and hence $\Delta^n p(\lambda) \equiv 0$ which by assumption is not true. The proof is therefore complete.*

13.4 Polynomials on E(A) to E'(A)

Definition (13.41): A function $p(x) \not\equiv 0$ on E(A) to E'(A) is called a polynomial if

(1) $p(x)$ is continuous at every x.

(2) There exists an integer n such that for each x,y, $P(\lambda;x,y) \equiv p(x+\lambda y)$ is a polynomial on $A[E^2]$ to E' of degree $\leqq$ n.

The least integer n saatisfying condition (2) is called the degree of $p(\lambda)$.

If $p(\lambda) \equiv 0$, we call it the null polynomial on E(A) to E'(A) and it is regarded as a member of the class of polynomials of degree m, where m is an integer.

Theorem (13.420) A polynomial $p(x)$ of degree zero is a constant.

Proof: $p(\lambda x)$ is a polynomial of degree zero in λ and therefore constant in λ. Hence putting λ successively equal to 0 and 1 we have

$$p(x) = p(0)$$

* It is worthwhile to mention here that the exact reason a similar theorem cannot be proven by a similar method for polynomials on C to E(C) is that the limit in (10) is in that case not necessarily unique. If to the conditions (1), (2), (3), we added the condition on $p(\lambda)$ of having a differential at $\lambda = 0$ the theorem just proved could be proved for polynomials C to E(C). To show that the theorem is not necessarily true for polynomials C to E(C), observe that

$$p(\lambda) \equiv \bar{\lambda}\text{ , the complex conjugate of } \lambda ,$$

satisfies the conditions (1), (2), (3) for n = 1 and is not a polynomial in the sense of (13.2)

Analogous to Theorem (13.203) we have:

Theorem (13.421) Let $p_1(x), p_2(x), \cdots, p_m(x)$ be m polynomials of degree $\leqq$ n on E(A) to E'(A). Then any linear combination $p(x) \equiv \sum_{r=0}^{m} \alpha_r p_r(x)$ with coefficients from A is a polynomial on E(A) to E'(A) whose degree is at most n.

Proof: p(x), being the sum of continuous functions, is continuous. Condition (1) of the definition (13.41) is therefore satisfied.

Let
$$P_i(\lambda; x, y) \equiv p_i(x + \lambda y)$$

Then $P_i(\lambda; x,y)$ is, by (13.41) and the hypotheses, for each x,y a polynomial in λ of degree at most n. Hence by Theorem (13.203)

$$p(x + \lambda y) = \sum_{r=0}^{m} \alpha_r P_i(\lambda; x, y)$$

is a polynomial of degree at most n in λ. The second condition is therefore satisfied.

Theorem (13.422) Let p(x) be a polynomial of degree n on E(A) to E'(A). Let c be a fixed number from A, different from zero, and x_0 be a fixed element of E(A). Then $\overline{p}(x) \equiv p(x_0 + cx)$ is a polynomial of degree n in x.

Proof: $x_0 + cx$ is clearly a continuous function on E(A) to E(A). $\overline{p}(x) \equiv p(x_0 + cx)$ ix therefore a continuous function on E to E' of a function continuous on E to E which, by (12.54), must be continuous.

Again, $\overline{p}(x + \lambda y) = p(x_0 + cx + \lambda cy) = P(\lambda; x_0 + cx, cy)$, which is a polynomial in λ.

Theorem (13.423) Let $p(X)$ be a polynomial of degree n on E to E'. Let $P(\lambda;x,y) = p(x+\lambda y) = \sum_{r=0}^{n} \lambda^r k_r(x,y)$. Then for each r the function $k_r(x,y)$ is unique and is a polynomial in x for each y and in y for each x*.

Proof: Let A_{rs} $(r,s = 0,1,\cdots,n)$ be the numbers defined in (13.204). Using that theorem we have

$$k_r(x,y) = \sum_{s=0}^{n} A_{rs}p(x+sy)$$

Now by (13.422) $p(x+sy)$ is a polynomial in x for fixed y and in y for fixed x. By Theorem (13.421) it follows that $k_r(x,y)$ is a polynomial in x for fixed y and a polynomial in y for fixed x.

Definition: A function $h(x)$ on E(A) to E'(A) is called homogeneous of degree n, if there exists an integer n such that for all x of E and all λ of A we have $h(\lambda x) = \lambda^n h(x)$

Theorem (13.424) If $h(x)$ is a polynomial on E(A) to E'(A) and is a homogeneous function of degree n, then it is a polynomial of degree n.

Proof: If $h(x) \equiv 0$, the theorem is true by definition. If $h(x) \not\equiv 0$, it must have some degree N. We must prove that N = n. By the definition of a polynomial of degree N we have

$$h(x+\lambda y) = \sum_{0}^{N} \lambda^r k_r(x,y) \qquad k_N(x,y) \not\equiv 0 \qquad (1)$$

where $k_r(x,y)$ is, according to Theorem (13.423), a uniquely determined polynomial in each of x and y. If in (1) we place x=0, then (1) must reduce to the known identity

$$h(\lambda y) = \lambda^n h(y) \qquad (2)$$

*A more exact statement as to the nature of $k_r(x,y)$ is given a little later; see Theorem (13.428)

Hence it follows that $k_n(0,y) = h(y)$ and therefore that $N \geqq n$. Assume $N > n$. We have $(\mu \neq 0)$

$$h(x+\lambda y) = \frac{1}{\mu^n} h(\mu x + \mu\lambda y) = \sum_{r=0}^{N} \lambda^r \mu^{r-n} k_r(\mu x, y) \tag{3}$$

Equating the coefficients of λ^N in (1) and (3), we obtain

$$k_N(\mu x, y) = \mu^{n-N} k_N(x,y) \tag{4}$$

Now if $N > n$, the exponent of μ on the right hand side of (4) is negative. Unless, therefore, $k_N(x,y) \equiv 0$, equation (4) is a contradiction of the fact that $k_N(x,y)$ is a polynomial.

We are, by this last theorem, justified in speaking either of a "polynomial homogeneous of degree n" or of a "homogeneous polynomial of degree n".

Theorem (13.425) If $h(x)$ is a homogeneous polynomial of degree n, and if $k_r(x,y)$ is defined as in the prededing two theorems, then $k_r(x,y)$ is homogeneous of degree r in y and of degree $n-r$ in x.

Proof: Let

$$h(x+\lambda y) = \sum_{r=0}^{n} \lambda^r k_r(x,y) \tag{1}$$

then $$h(x+\lambda y) = h\left(x + \frac{\lambda}{\mu}\mu y\right) = \sum_{r=0}^{n} \left(\frac{\lambda}{\mu}\right)^r k_r(x, \mu y) \tag{2}$$

and $$h(x+\lambda y) = \frac{1}{\mu^n} h(\mu x + \lambda\mu y) = \sum_{r=0}^{n} \lambda^r \mu^{r-n} k_r(\mu x, y) \tag{3}$$

Equating coefficients of λ^r in (1) and (2) we obtain

$$k_r(x,\mu y) = \mu^r k_r(x,y)$$

and doing a similar thing for the coefficients of λ^r in (1) and (3),

$$k_r(\mu x,y) = \mu^{n-r} k_r(x,y).$$

Theorem (13.426) Let $h(x) \equiv h_1(x) + \cdots + h_m(x)$ be the sum of m homogeneous polynomials of degree n. Then h(x) is a polynomial homogeneous of degree n.

Proof: h(x) is clearly homogeneous; for,

$$h(\mu x) = \mu^n h_1(x) + \cdots + \mu^n h_m(x) = \mu^n h(x).$$

It is also a polynomial by (13.421).

Theorem (13.427) A polynomial p(x) of degree n is uniquely representable as a sum of homogeneous polynomials of degrees $\leqq n$

Proof: By taking $x = 0$ in (13.423) we obtain

$$p(\lambda y) = \sum_{r=0}^{n} \lambda^r k_r(0,y) \equiv \sum_{r=0}^{n} \lambda^r h_r(y) \tag{1}$$

and
$$p(\frac{\lambda}{\mu} \cdot \mu y) = \sum_{r=0}^{n} \frac{\lambda^r}{\mu^r} h_r(\mu y) \tag{2}$$

therefore
$$h_r(\mu y) = \mu^r h_r(y) \tag{3}$$

Placing $\lambda = 1$ in (1) we have
$$p(y) = \sum_{r=0}^{n} h_r(y) \tag{4}$$

Suppose now that we had another representation of p(y) as a sum of homogeneous polynomials. Let it be a

$$p(y) = \sum_{r=0}^{N} \bar{h}_r(y) \qquad (5)$$

where $\bar{h}_r(y)$ is some polynomial homogeneous of degree r. Then for all

we have $$p(\lambda y) = \sum_{r=0}^{n} \lambda^r h_r(y) = \sum_{r=0}^{N} \lambda^r \bar{h}_r(y)$$

By theorem (13.205) we have $h_r(y) = \bar{h}_r(y)$. The apparent abundancy of terms in (5) is taken care of by observing that if $N > n$, $p(y)$ cannot satisfy the definition for a polynomial of degree n.

Theorem (13.428) Let $p(x)$ be a polynomial of degree n. As before let $p(x + \lambda y) = \sum_{r=0}^{n} \lambda^r k_r(x,y)$. Then $k_r(x,y)$ is a polynomial of degree n-r in x and is homogeneous of degree r in y.

Proof: By (13.427) we have

$$p(x) = \sum_{r=0}^{n} h_r(x) \qquad (1)$$

and by theorem (13.425) we have

$$h_r(x + \lambda y) = \sum_{s=0}^{r} \lambda^s k_{rs}(x,y) \qquad (2)$$

where $k_{rs}(x,y)$ is of degree s in y and r-s in x.

Hence $$p(x + \lambda y) = \sum_{r=0}^{n} \sum_{s=0}^{r} \lambda^s k_{rs}(x,y)$$

$$= \sum_{s=0}^{n} \lambda^s \sum_{r=s}^{n} k_{rs}(x,y) \qquad (3)$$

The coefficient of λ^s is by (13.426) homogeneous of degree s in y and by (13.421) of degree = (n-r) in x. This completes the proof.

13.5 Characterization of Polynomials on E(R) to E'(R)

13.51 The Frechet Definition

The definition given by Frechet of an abstract polynomial on an E(R) to an E'(R) is the following:

A function p(x) on E(R) to E'(R) is called a polynomial of degree n if (1) It is continuous at each x.

(2) For arbitrary increments $\Delta_1 x, \Delta_2 x, \cdots, \Delta_{n+1} x$ and arbitrary x we have

$$\overset{n+1}{\Delta} p(x) \equiv 0$$

(3) For some set of increments $\Delta_1 x, \cdots, \Delta_n x$ and some x we have $\overset{n}{\Delta} p(x) \neq 0$*

We propose now to prove two things: first, that any polynomial of degree n on E(A) to E'(A) in the sense of (13.41) satisfies the conditions (1), (2), and (3) above whether or not A = R; second, that the above definition of polynomials on E(R) to E'(R) implies the definition (13.41) for the special case A = R.

For convenience we shall refer to a function satisfying (1), (2), and (3) as a Frechet polynomial. The word "polynomial", alone, will refer to the definition of (13.41).

Theorem (13.521) Let p(x) be a polynomial of degree n on E(A) to E'(A). Then $\Delta p(x)$ is for all choices of Δx a polynomial in x of degree $\leq$ n-1, and for proper selection of Δx a non-null polynomial of degree exactly n-1.

* The notation is that explained in (12.6).

Proof: By application of Theorem (13.428) whose equations we shall here assume, we have

$$p(x + \Delta x) = p(\Delta x + \lambda x)\Big|_{\lambda = 1}$$

$$= \sum_{r=0}^{n} k_r(\Delta x, x) \tag{1}$$

where $k_r(\Delta x, x)$ is of degree r in x and $\doteq$ n-r in Δx. Since $k_n(\Delta x, x)$ is of degree 0 in Δx we have from (13.420) that

$$k_n(\Delta x, x) = k_n(0, x).$$

Therefore $$\Delta\, p(x) \equiv p(x + \Delta x) - p(x) = \sum_{r=0}^{n-1} \left(k_r(\Delta x, x) - k_r(0, x)\right) \tag{2}$$

is the sum of polynomials of degree $\leqq$ n-1 in x, which by (13.421) is a polynomial of degree $\leqq$ n-1.

To show the second part of the theorem we write p(x) as a sum of homogeneous polynomials. Theorem (13.427)

$$p(x) = \sum_{r=0}^{n} h_r(x) \qquad h_n(x) \not\equiv 0$$

so that $$\Delta\, p(\lambda x) = \sum_{r=0}^{n} h_r(\lambda x + \Delta x) - h_r(\lambda x) \tag{3}$$

If we write $\Delta x = x$, this becomes

$$\Delta\, p(\lambda x) = \sum_{r=0}^{n} h_r(\overline{1 + \lambda} \cdot x) - h_r(\lambda x)$$

$$= \sum_{r=0}^{n} \left\{(1 + \lambda)^r - \lambda^r\right\} h_r(x)$$

so that the coefficient of λ^{n-1} in $p(\lambda x)$ when $\Delta x = x$ is

$$n \cdot h_n(x) + h_{n-1}(x)$$

which, since $h_n(x) \not\equiv 0$, is a non-null polynomial of degree n.

Let x_1 be a value for which it does not vanish. Then with $x = x_1$, the coefficient of λ^{n-1} in the expansion of $\Delta p(\lambda x)$ is not null since it does not vanish in particular for $x = x_1$. Thus for suitably chosen Δx, $\Delta p(x)$ is a polynomial of degree exactly n-1.

Theorem (13.522) If $p(x)$ is a polynomial of degree n on $E(A)$ to $E'(A)$, then for all $\Delta_1 x, \Delta_2 x, \cdots, \Delta_{n+1} x$, we have

$$\overset{n+1}{\Delta} p(x) \equiv \Delta_{n+1} \cdots \Delta_1 p(x) = 0$$

Proof: If Theorem (13.521) is applied successively to $\Delta_1 p(x), \Delta_2 \Delta_1 p(x), \cdots$, then it follows that $\Delta_n \cdots \Delta_1 p(x)$ is of degree zero at most, and is therefore constant in x. Hence

$$\overset{n+1}{\Delta} p(x) \equiv \Delta_n \cdots \Delta_1 p(x + \Delta_{n+1} x) - \Delta_n \cdots \Delta_1 p(x) = 0.$$

Theorem (13.523) If $p(x)$ is a polynomial of degree n on $E(A)$ to $E'(A)$ then there exists a set of increments $\Delta_1 x, \Delta_2 x, \cdots, \Delta_n x$ such that

$$\overset{n}{\Delta} p(x) \equiv \Delta_n \cdots \Delta_1 p(x) \neq 0$$

Proof: We use the second part of Theorem (13.521), select $\Delta_1 x$ so that $\Delta_1 p(x)$ is a non-null polynomial of degree exactly n-1 in x. Select $\Delta_2 x$ so that $\Delta_2 \Delta_1 p(x)$ is a non-null polynomial of degree exactly n-2, and so on. With this selection we must have $\overset{n}{\Delta} p(x) \not\equiv 0$

Combining now the results of the last two theorems we have:

Theorem (13.53) If $p(x)$ is a polynomial of degree n on $E(A)$ to $E'(A)$, then it is a Frechet polynomial of degree n.

Proof: (1) A polynomial is continuous and therefore satisfies the Frechet condition (1).

(2) By Theorem (13.522) $\overset{n+1}{\Delta} p(x) = 0$ for all sets of increments.

(3) By Theorem (13.523) there exists a set of increments $\Delta_1 x, \cdots, \Delta_n x$ for which $\overset{n}{\Delta} p(x) \neq 0$.

We shall now prove a converse for spaces $E(R)$. Before proceeding, let us prove the following Lemma.

Lemma (13.540) If $f(\lambda)$ is a continuous function on R to $E(R)$, satisfying the condition $\overset{n+1}{\Delta} f(\lambda) \equiv 0$ for all choices of $\Delta_1\lambda, \Delta_2\lambda, \cdots, \Delta_n\lambda$ then $f(\lambda)$ is a polynomial on R to $E(R)$ of degree at most n.

Proof: Let m be the maximum integer for which there exists a choice of $\Delta_1\lambda, \Delta_2\lambda, \cdots, \Delta_m\lambda$ such that $\overset{m}{\Delta} f(\lambda) \not\equiv 0$. Clearly $m \leqq n$; for, since $\overset{n+1}{\Delta} f(\lambda) \equiv 0$, all differences of order greater than n must vanish identically. Then by the definition of m, for all choices of the Δ's we must have $\overset{m+1}{\Delta} f(\lambda) \equiv 0$. Therefore by Theorem (13.31), $f(\lambda)$ is a polynomial of degree $m \leqq n$.

Theorem (13.54) If $p(x)$ is a Frechet polynomial of degree n on $E(R)$ to $E'(R)$ then it is a polynomial of degree n on $E(R)$ to $E'(R)$ in the sense of (13.41).

Proof: Let $q(x)$ be any function on $E(R)$ to $E'(R)$. Let $g(\lambda) \equiv q(x + \lambda y)$. Let $\Delta\lambda$ be any increment of λ and define

$$\Delta x \equiv \Delta\lambda \cdot y$$

Then
$$\Delta g(\lambda) \equiv q(x+\lambda y+\Delta\lambda y) - q(x+\lambda y)$$
$$\equiv \Delta q(x+\lambda y) \qquad (1)$$

Now let
$$f(\lambda) \equiv p(x+\lambda y).$$

Let an arbitrary set of increments $\Delta_1\lambda, \Delta_2\lambda, \cdots, \Delta_{n+1}\lambda$ be chosen and let $\Delta_i x \equiv \Delta_i\lambda \cdot y$ $(i = 1,2,\cdots,n+1)$. Then applying the formula (1) successively to

$$\Delta_1 f(\lambda) = \Delta_1 p(x+\lambda y)$$
$$\Delta_2\Delta_1 f(\lambda) = \Delta_2\Delta_1 p(x+\lambda y)$$
$$\cdots\cdots\cdots\cdots$$

we obtain $\overset{n+1}{\Delta} f(\lambda) \equiv \overset{n+1}{\Delta} p(x+\lambda y)$. Now $f(\lambda)$ is evidently a function on R to E'(R) and is continuous since $p(x)$ is continuous. Hence, applying Lemma (13.540), we have the existence of $m \leqq n$ such that $f(\lambda)$ is a polynomial of degree m on R to E'(R). $p(x)$ is therefore a polynomial of degree m in the sense of (13.41). But m cannot be less than n; for, by theorem (13.522), we should have for all $\Delta_1 x, \Delta_2 x, \cdots, \Delta_{m+1} x$, $\overset{m+1}{\Delta} p(x) \equiv 0$, and hence have that $\overset{n}{\Delta} p(x) \equiv 0$. This would contradict the third Frechet condition.

This last work may be summed up in the theorem:

<u>Theorem (13.55)</u> For spaces E(R), E'(R) the definitions (13.41) and (13.51) of a polynomial of degree n are equivalent.

13.6 <u>Homogeneous Polynomials and their Polars. Multilinear forms.</u>

<u>Definition (13.61)</u> Let $h(x)$ be a homogeneous polynomial of degree n on E(A) to E'(A). Then the function $h(x_1,\cdots,x_n)$ on $E^n(A)$ to E'(A)

defined as $\frac{\overset{n}{\Delta}}{n!}h(0)$, where $\Delta_i x \equiv x_i$ will be called the complete polar or simply the polar of $h(x)$ with respect to $x_1, \cdots, x_n$.

Theorem (13.62) If $h(x_1, \cdots, x_n)$ is the polar of $h(x)$, then in each of its arguments it is a linear homogeneous polynomial. It is, furthermore, symmetric in the arguments.

Proof: The symmetry of $\overset{n}{\Delta} h(0)$ in the increments was noted in (12.63). It is therefore sufficient to consider one of them, say the n^{th}. By n-1 successive applications of Theorem (13.521), it is easily proved as in (13.521) that $\overset{n-1}{\Delta} h(x)$ is a polynomial in x of degree one, which, for suitable choices of $\Delta_1 x, \cdots, \Delta_{n-1} x$, is non-null. Therefore

$$f(x) \equiv \overset{n-1}{\Delta} h(x) - \overset{n-1}{\Delta} h(0). \tag{1}$$

is a polynomial of degree one on $E[E^{n-1}]$ to E'. Hence we may write

by (13.427) $$f(\lambda x) = k_0(x) + \lambda k_1(x) \tag{2}$$

Placing $x = 0$ in (1) we obtain $f(0) = 0$, from which $k_0(x) = 0$, $k_1(x) = f(x)$.

Thus (2) yields $$f(\lambda x) = \lambda f(x)$$

But $f(x)$ is by its definition $\overset{n}{\Delta} h(0)$ where $\Delta_n x = x$.

Definition: Any form which, like $h(x_1, \cdots, x_n)$, is linear homogeneous and continuous in each of its arguments is called a multilinear form.

The polar bears to the form a relation analogous to that which holds for ordinary algebraic forms. This is given by

Theorem (13.63) Let $h(x)$ be a homogeneous polynomial of degree n, and let $h(x_1, \cdots, x_n)$ be its polar. Then $h(x) = h(x, x, \cdots, x)$.

<u>Proof</u>: Take $\Delta_1 x = \Delta_2 x = \cdots = \Delta_n x = x$

Then $$\Delta_1 h(0) = h(x) - h(0)$$

$$\Delta^2 h(0) = h(2x) - 2h(x) + h(0)$$

$$\cdots\cdots\cdots\cdots$$

This is a well known interpolation formula whose n^{th} term is

$$\Delta^n h(0) = \sum (-)^{n-r} \binom{n}{r} h(\overline{n-r}\, x)$$

Using now the homogeneity of $h(x)$

$$\Delta^n h(0) = \sum (-)^{n-r} \binom{n}{r} (n - r)^n h(x) = n!\, h(x)$$

or $$h(x) = \frac{\Delta^n h(0)}{n!} = h(x, x, \cdots, x)$$

Conversely we have:

<u>Theorem (13.64)</u> For a given homogeneous polynomial $h(x)$ of degree n, there exists no symmetric multilinear form other than the polar of $h(x)$ which enjoys the property described in (13.63).

<u>Proof</u>: Let $h(x_1, \cdots, x_n)$ and $\bar{h}(x_1, x_2, \cdots x_n)$ be any two symmetric multilinear forms and let $h(x)$ be a homogeneous polynomial of degree n

such that $$h(x) = h(x, x, \cdots, x) = \bar{h}(x, x, \cdots, x) \qquad (1)$$

We shall employ induction on n to prove that

$$h(x_1, x_2, \cdots, x_n) = \bar{h}(x_1, x_2, \cdots, x_n)$$

In (1) put $x = x + \lambda y$, and make use of (13.425): and the distributivity of $h(x, \cdots, x)$ and $\bar{h}(x, \cdots, x)$ with respect to their arguments. This gives.

$$h(x+\lambda y) = \sum_{r=0}^{n} \lambda^r\, k_r(x,y) = \sum_{r=0}^{n} \lambda^r \binom{n}{r} h_r(x,y) = \sum_{r=0}^{n} \lambda^r \binom{n}{r} \bar{h}_r(x,y) \quad (2)$$

where $k_r(x,y)$ is a polynomial homogeneous of degree r in y and n-r in x, and where $h_r(x,y)$ and $\bar{h}_r(x,y)$ are the functions obtained by writing

$x_1 = x_2 = \cdots = x_{n-r} = x \qquad x_{n-r+1} = \cdots = x_n = y$ in $h(x_1, x_2, \cdots, x_n)$

respectively. Equating the coefficients of λ^1, we obtain

$$k_1(x,y) = n\ h(x,x,\cdots,x,y) = n\ \bar{h}(x,x,\cdots,x,h) \qquad (3)$$

Assuming the theorem true for (n-1) we have for any value of y that the two forms h and $\bar{h}$ are symmetric multilinear forms in (n-1) arguments x and are equal to a homogeneous polynomial of degree n-1 in x. Therefore we have for all $x_1, x_2, \cdots, x_{n-1}, y$

$$h(x_1, x_2, \cdots, x_{n-1}, y) = \bar{h}(x_1, x_2, \cdots, x_{n-1}, y)$$

To complete the induction we observe that the theorem is trivial for n=1.*

A general result on multilinear forms for complete spaces will be derived in a number of theorems.

<u>Theorem (13.65)</u> Hypotheses:

H_1. Let x be a typical element of a vector space E_1.

H_2. Let y be a typical element of a **complete** vector space E_2.

H_3. Let B(x,y) be continuous on $E_1[E_2]$ to E' at x = 0.

H_4. Let B(x,y) be continuous linear homogeneous on $[E_1] E_2$ to E'.

* A proof could be given for the above theorem without assuming that h(x,x,...,x) and $\bar{h}$(x,x,...,x) were each equal to a honogeneous polynomial in x, but it was thought of too late to be included.

H_5. Let $B(0,y) = 0$.

Conclusion: $B(x,y)$ is continuous on E_1E_2 to E' at the point $(x,y) = (0,0)$

Proof: We wish to prove that for $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that if (x,y) is in $(0,0)_{\delta_\epsilon}$, the δ_ϵ neighborhood of $(0,0)$, then

$$\| B(x,y) - B(0,0) \| \equiv \| B(x,y) \| < \epsilon$$

We shall denote the point $(0,0)$ by (0). If the theorem were not true then we should have $k > 0$ such that in every $(0)_\delta$ there exists at least one point (x,y) for which $\| B(x,y) \| > k$. We prove that this hypothesis leads to a contradiction.

Select an infinte sequence of points (x_n,y_n) as follows: Let (x_0,y_0) be in $(0)_{\delta_0 \equiv 1}$ and such that

$$\| B(x_0,y_0) \| > k$$

Now by H_3 and H_4 we may select $\delta_1 < 1/2$ such that for (x,y) in $(0)_{\delta_1}$ we have

$$\| B(x_0,y) \| < k/4; \| B(x,y_0) \| < k/4.$$

Now choose (x_1,y_1) in $(0)_{\delta_1}$ such that

$$\| B(x_1,y_1) \| > k$$

Continue this process. In general, having selected (x_p,y_p) in $(0)_{\delta_p}$ $(\delta_p < 1/2^p)$ such that

$$\| B(x_p,y_p) \| > k$$

choose $\delta_{p+1} < \frac{1}{2^{p+1}}$ such that for (x,y) in $(0)_{\delta_{p+1}}$

$$\| B(x_i,y) \| < \frac{k}{2^{p+2}} \qquad (i = 0,1,2,\cdots,p)$$

$$\| B(x,y_i) \| < \frac{k}{2^{i+2}} \qquad (i = 0,1,2,\cdots,p) \tag{1}$$

then select (x_{p+1},y_{p+1}) in $(0)\delta_{p+1}$ such that

$$\| B(x_{p+1},y_{p+1}) \| > k$$

Now let $Y_p \equiv \sum_{i=0}^{p} y_i$. Since $\| y_i \| < \delta_i < 1/2^i$ we have

$$\| Y_p - Y_{p+1} \| \leq \sum_{i=p}^{p+1} \| y_i \| < \sum_{i=p}^{p+1} \frac{1}{2^i} < \frac{1}{2^{p-1}}$$

Since E_2 is complete this is a sufficient condition for the existence of a limit Y for the sequence $\{ Y_p \}$.

Now, since $B(x,y)$ is by H_4 continuous and linear in y, we have

$$B(x_i,Y) = \lim_{p\to\infty} B(x_i,Y_p) = \lim_{p\to\infty} \sum_{j=0}^{p} B(x_i,y_j)$$

$$= B(x_i,y_i) + \lim_{p\to\infty} \sum_{j\neq i}^{p} B(x_i,y_j) \tag{2}$$

or

$$\| B(x_i,Y) \| \geq \| B(x_i,y_i) \| - \lim_{p\to\infty} \sum_{j\neq i}^{p} \| B(x_i,y_j) \| \tag{3}$$

But by the selection of (x_i,y_i) in (1) we have

$$\| B(x_i,y_j) \| < \frac{k}{2^{j+2}} \qquad (i \lessgtr j) \tag{4}$$

Hence the limit on the right hand side of (3) is less than $\sum_{j=0}^{\infty} \frac{k}{2^{j+2}}$ which is less than $k/2$. Remembering that $\| B(x_i,y_i) \| > k$ we have from (3)

$$\| B(x_i,Y) \| > k - k/2 > k/2$$

Since this holds for all i and since $\lim_{i\to\infty} x_i = 0$, we have a contradiction of H_3. This proves the theorem.

Theorem (13.66) Let $E_1, E_2, \cdots, E_n$ be a complete vector spaces, distinct or not. Let x_i be a representative point of E_i. Let $M(x_1, x_2, \cdots x_n)$ be a multilinear form on $E_1 E_2 \cdots E_n$ to E'. Then M is continuous at $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$

Proof: We prove the theorem by induction on n. For n=1 the theorem is true by definition. If it is true for n-1, then $M(x_1, \cdots, x_n)$ is continuous on $E_1 E_2 \cdots E_{n-1}[E_n]$ to E' at $(x_1, x_2, \cdots, x_{n-1}) = (0, 0, \cdots, 0)$ and is linear homogeneous and continuous on $[E_1 \cdot E_2 \cdots E_{n-1}] E_n$ to E'. Therefore by Theorem (13.65) M is continuous at $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$, on $E_1 \cdot E_2 \cdots E_n$ to E' which was to be proved.

The spaces F(a,b) are as is well known, instances of complete vector spaces. The result of (13.66) and a consequence of it will be assumed in part 2.

13.7 Modular Properties of Polynomials and Multilinear Forms.

Definition: By the modulus of a homogeneous polynomial of degree n we shall understand the upper bound of the expression $\frac{\|h(x)\|}{\|x\|^n}$ for all values of x. We denote this bound by mh.

To prove that mh exists we have:

Theorem (13.71) If h(x) is a homogeneous polynomial of degree n, then the expression $\frac{\|h(x)\|}{\|x\|^n}$ is bounded for all x.

Proof: By definition h(x) is continuous at x = 0. It follows that there exists δ_1 such that for $\|x\| < \delta_1$ $\|h(x)\| < 1$. Now if x is arbitrary, and if $\bar{x} \equiv \frac{\delta_1 x}{2\|x\|}$ we have

$$\|\bar{x}\| = \left\|\frac{\delta_1 x}{2\|x\|}\right\| = \frac{\delta_1}{2} < \delta_1$$

Therefore $$\|h(\bar{x})\| < 1$$

But $$\|h(\bar{x})\| = \left\|\left(\frac{\delta_1}{2\|x\|}\right)^n h(x)\right\| = \left(\frac{\delta_1}{2}\right)^n \frac{\|h(x)\|}{\|x\|^n}$$

Hence $$\frac{\|h(x)\|}{\|x\|^n} < \left(\frac{2}{\delta_1}\right)^n$$

The upper bound mh therefore exists.

Definition: By the modulus of a multilinear form $M(x_1, x_2, \cdots, x_n)$ we shall understand the upper bound, in case it exists, of the expression $\frac{\|M(x_1, x_2, \cdots, x_n)\|}{\|x_1\| \cdot \|x_2\| \cdots \|x_n\|}$ over all values of the x's. It is denoted by $m_n M$.

Theorem (13.72) If $E_1, E_2, \cdots, E_n$ are complete vector spaces, then the expression $\frac{\|M(x_1, x_2, \cdots, x_n)\|}{\|x_1\| \cdot \|x_2\| \cdots \|x_n\|}$ is bounded over all x's.

Proof: By theorem (13.66) M is continuous at the point $(x_1, x_2, \cdots, x_n) = (0, 0, \cdots, 0)$. There exists, therefore, δ_1, such that if $\|x_i\| < \delta_1$ $(i = 1, 2, \ldots, n)$ then $\|M\| < 1$. If $x_1, x_2, \cdots, x_n$ are arbitrary and if $\bar{x}_i = \frac{\delta_1 x_i}{2\|x_i\|}$ we have as before

$$M(\bar{x}_1, \bar{x}_2, \cdots, \bar{x}_n) = \frac{\delta_1^n}{2^n} \cdot \frac{\|M(x_1, \cdots, x_n)\|}{\|x_1\| \cdot \|x_2\| \cdots \|x_n\|} < 1$$

Whence $$\frac{\|M(x_1, x_2, \cdots, x_n)\|}{\|x_1\| \cdot \|x_2\| \cdots \|x_n\|} < \frac{2^n}{\delta_1^n}$$

It follows that $m_n M$ exists.

Theorem (13.721) If $h(x)$ is a homogeneous polynomial of degree n, and if $\| x\| < r$ then $\| h(x)\| < mhr^n$.

Proof: $\| h(x) \| < m\, h \|x\|^n < mh\, r^n$

Theorem (13.722) If M is a multilinear form in $x_1, \cdots, x_n$, if $\| x_i\| < r_i$ $(i = 1,2,\cdots,n)$, and if $m_n M$ exists, then $\| M \| < m_n M \cdot r_1 \cdot r_2 \cdots r_n$.

Proof: Similar to (13.721)

Theorem (13.723) If $h(x)$ is a homogeneous polynomial of degree n, then

(a) $mh = \frac{1}{r^n} \max\limits_{\|x\|=r} \| h(x) \|$ (b) $mh = \frac{1}{r^n} \max\limits_{\|x\|<r} \| h(x) \|$

(c) $mh = \frac{1}{r^n} \max\limits_{\|x\| \leq r} \| h(x) \|$

Proof:

$$mh = \max \frac{\| h(x)\|}{\|x\|^n} \geq \max_{\|x\|=r} \frac{\| h(x)\|}{r^n} \qquad (1)$$

since the second maximum is taken over a more restricted set of x's. Now let x_0 be a value for which

$$\frac{\| h(x_0)\|}{\|x_0\|^n} > \max_{\|x\|=r} \frac{\| h(x)\|}{r^n} \qquad (2)$$

Then put $\bar{x}_0 = \frac{r\, x_0}{\| x_0 \|}$ so that $\| \bar{x}_0 \| = r$

Thus $$\frac{\| h(\bar{x}_0)\|}{r^n} = \frac{\| h(x_0)\|}{\| x_0\|^n} > \max_{\|x\|=r} \frac{\| h(x)\|}{r^n}$$

This contradiction proves that the inequality in (1) is inadmissable. (b) is proved similarly; (c) follows immediately from (a) and (b)

Theorem (13.724) If $M(x_1, \cdots, x_n)$ is a multilinear form continuous at $(0,0,\cdots,0)$ then $m_n M$ exists.

Proof: An exact reproduction of (13.72); the only assumptions used there in the proof were the hypotheses of the present theorem.

Theorem (13.725) If $M(x_1,\cdots,x_n)$ is a multilinear form such that m_nM exists, then

$$m_nM = \frac{1}{r_1\cdot r_2\cdots r_n}\cdot \max_{\|x_i\|=r_i} M$$

Proof: Precisely similar to (13.723)

We shall now prove a special theorem on the modulus of the polar of a homogeneous polynomial.

Theorem (13.73) Let mh be the modulus of a homogeneous polynomial of degree n. Then the modulus m_nh of its polar exists and we have the relation

$$1 \leq \frac{m_nh}{mh} \leq \frac{n^n}{n!}$$

Proof: Since $\overset{n}{\Delta} h(x)$ is of degree 0 in x, we have $\overset{n}{\Delta} h(x) = \overset{n}{\Delta} h(0)$. Let $\Delta_1 x,\cdots,\Delta_n x$ be an arbitrary set of increments and take

$$x = -\frac{1}{2}\sum_{i=1}^{n}\Delta_i x.$$

Consider the manner in which the successive differences of h(x) are formed

$$\Delta_1\, h(x) = h(\Delta_1 x - \tfrac{1}{2}\textstyle\sum\Delta_i x) - h(-\tfrac{1}{2}\textstyle\sum\Delta_i x) \qquad (1)$$

At each stage, each of the terms on the right hand side gives rise to two (2) new ones. $\overset{n}{\Delta} h(x)$ is therefore the sum of 2^n terms of the type

$$h(\tfrac{1}{2}\textstyle\sum\epsilon_i\Delta_i x)$$

where $\epsilon_i = \pm 1$.

But $$\left\|\frac{1}{2}\sum \epsilon_i \Delta_i x\right\| \leqq \frac{1}{2}\sum \|\Delta_i x\| < \frac{n}{2}\cdot \max\|\Delta_i x\| \tag{2}$$

Therefore $$\left\|\frac{\Delta^n h(0)}{n!}\right\| = \left\|\frac{\Delta^n h(x)}{n!}\right\| \leqq \frac{2^n}{n!}\, mh \left(\frac{n}{2}\right)^n \cdot (\max\|\Delta_i x\|)^n \tag{3}$$

The existence of $m_n h$ follows at once from (3) and Theorem (13.724), for (3) shows that $h(x_1,x_2,\cdots,x_n)$ is continuous at $(0,0,\cdots,0)$.

If in (3) we require that $\|\Delta_i x\| = 1$ $(i = 1,2,\ldots,n)$, then (3) becomes $$\frac{\|\Delta^n h(0)\|}{n!} \leqq mh\,\frac{n^n}{n!} \tag{4}$$

By theorem (13.725) therefore

$$m_n h = \max_{\|\Delta_i x\|=1} \left\|\frac{\Delta^n h(0)}{n!}\right\| \leqq mh\,\frac{n^n}{n!} \tag{5}$$

Again we have $$m_n h = \max \frac{\|h(x_1,\cdots,x_n)\|}{\|x_1\|\cdot\|x_2\|\cdots\|x_n\|} \geqq \max\frac{\|h(x,x,\cdots,x)\|}{\|x\|^n} \tag{6}$$

Combining (5) and (6) we obtain

$$1 \leqq \frac{m_n h}{mh} \leqq \frac{n^n}{n!} \tag{7}$$

which was to be proved.

As a corollary to the last theorem we have.

Theorem (13.731) A necessary and sufficient condition that a homogeneous polynomial be null is that its polar be null.

Proof: By equation (7) of Theorem (13.73) mh and $m_n h$ must vanish together.

We have now a special result for spaces E(C) whose possibility seems to depend upon the fact that in the complex plane the functions $z, z^2, \cdots, z^n$, are orthogonal on the unit circle to the set $z^{-1}, z^{-2}, \cdots, z^{-n}$ with respect to the function $1/z$

<u>Theorem (13.74)</u> Let $p(\lambda) \equiv \sum_0^n \lambda^r a_r$ be a polynomial on C to E(C) Then for all s (s = 1,2,...,n)

$$\| a_s \| \leqq \max_{\|\lambda\|=1} \| p(\lambda) \|$$

<u>Proof:</u> Let h, k, s be integers $s \leqq n$. For convenience define

$$\lambda_h \equiv e^{2\pi i/h}$$

Then $$\left| \lambda_h^{k+1} - \lambda_h^k \right| = \left| 2 \sin \frac{\pi}{h} \right| < \frac{2\pi}{h}$$

Now define $$W_{hs}^r \equiv \frac{1}{2\pi i} \sum_{k=0}^{h-1} (\lambda_h^{k+1} - \lambda_h^k) \lambda_h^{(r-s-1)k}$$

So that $$\lim_{h\to\infty} W_{hs}^r = \frac{1}{2\pi i} \int_{\|\lambda\|=1} \lambda^{r-s-1} \, d\lambda = \begin{cases} 1 & (r = s) \\ 0 & (r \neq s) \end{cases}$$

Define $$U_{hs} \equiv \sum_{r=0}^{n} W_{hs}^r a_r$$

from which $$\lim_{h\to\infty} U_{hs} = a_s$$

But $$U_{hs} = \sum_{r=0}^{n} W_{hs}^r a_r = \sum_{k=0}^{h-1} \frac{(\lambda_h^{k+1} - \lambda_h^k) \, p(\lambda_h^k)}{2\pi i \, \lambda_h^{k(s+1)}}$$

Hence $$\| U_{hs} \| \leqq \frac{1}{2\pi} \sum_{k=0}^{h-1} \frac{| \lambda_h^{k+1} - \lambda_h^k | \, \| p(\lambda_h^k) \|}{| \lambda_h^{k(s+1)} |} < \max_{|\lambda|=1} \| p(\lambda) \|$$

Finally $$\| a_s \| = \| \lim_{h\to\infty} U_{hs} \| = \lim_{h\to\infty} \| U_{hs} \| \leqq \max_{|\lambda|=1} \| p(\lambda) \|$$

<u>Theorem (13.75)</u> Let p(x) be a polynomial of degree n on E(C) to E'(C). Let it be represented as a sum of homogeneous polynomials in the

form $p(x) \equiv \sum_{r=0}^{n} h_r(x)$, where $h_r(x)$ is of degree r. Then for all r

$$mh_r \leqq \max_{\|x\|=1} \| p(x) \|$$

Proof: Let $\|x\| = 1$

Write
$$p(\lambda x) = \sum_{r=0}^{n} \lambda^r h_r(x)$$

Then by Theorem (13.74)

$$\| h_r(x) \| \leqq \max_{\|\lambda\|=1} \| p(\lambda x)\| \leqq \max_{\|x\|=1} \|p(x)\| \tag{1}$$

Since this holds for all x for which $\| x \| = 1$

$$mh_r = \max_{\|x\|=1} \|h_r\| \leqq \max_{\|x\|=1} \| p(x) \| \tag{2}$$

which was to be proved.

These last two theorems are of some use in the theory of analytic functions to be introduced in the next division.

13.8 Differentials of Polynomials

Differentials were defined in (12.6). In this section we propose to show that a polynomial posesses differentials of all orders and that the differentials are symmetric in the increments. Furthermore, it will be shown that the differentials of a homogeneous polynomial may be conveniently expressed in terms of its polar.

At this point it is perhaps not amiss to prove the result we refrained from proving in section (12.6).

Theorem (13.80) Let $f_1(x)$, $f_2(x), \ldots, f_m(x)$ be functions on E_o to E' differentiable at a point x_o of E_o. Then $f(x) \equiv \sum_{r=0}^{m} f_r(x)$ is differentiable at x_o, and $df(x_o) = \sum_{r=0}^{m} df_r(x_o)$.

Proof: $df(x_0)$ is the sum of linear homogeneous polynomials in Δx and is therefore also linear homogeneous.

Let
$$\epsilon_r(\Delta x) \equiv \frac{f_r(x_0) - df_r(x_0)}{\|\Delta x\|}$$

so that by the definition of $df_r(x_0)$, $\epsilon_r(\Delta x) \to 0$ with Δx.

Now
$$\|\epsilon(\Delta x)\| \equiv \frac{\left\|\Delta \sum_{r=1}^{m} f_r(x_0) - \sum df_r(x_0)\right\|}{\|\Delta x\|}$$

$$\leqq \sum_{r=0}^{n} \frac{\|\Delta f_r(x_0) - df_r(x_0)\|}{\|\Delta x\|} \equiv \sum \|\epsilon_r(\Delta x)\|$$

Therefore as $\|\Delta x\| \to 0$ $\epsilon(\Delta x) \to 0$ and $df_r(x_0)$ is the differential.

Theorem (13.81) If $p(x)$ is a polynomial of degree n on E to E', then at every x of E the differential $dp(x)$ exists.

Employing Theorem (13.428) we write

$$p(x + \lambda y) = \sum_{r=0}^{n} \lambda^r k_r(x,y) \tag{1}$$

where $k_r(x,y)$ is homogeneous of degree r in y and of degree $\leqq n-r$ in x. Writing λ successively equal to 0 and 1 we obtain.

$$p(x) = k_0(x,y) \tag{2}$$

$$p(x + y) = \sum_{r=0}^{n} k_r(x,y) \tag{3}$$

We propose to show that if we take $\Delta x \equiv y$, then the differential of $p(x)$ is precisely $k_1(x,\Delta x)$ which is a linear function of Δx. Combining (2) and (3) and rearranging

$$p(x + \Delta x) - p(x) - k_1(x,\Delta x) = \sum_{r=2}^{n} k_r(x,\Delta x) \tag{4}$$

Regarding x as fixed we have

$$\| p(x+\Delta x) - p(x) - k_1(x,\Delta x)\| \leq \sum_{r=2}^{n} m\, k_r(x)\cdot\|\Delta x\|^r$$

Dividing by $\|\Delta x\|$ we have

$$\left\| \frac{p(x+\Delta x) - p(x) - k_1(x,\Delta x)}{\|\Delta x\|} \right\| \equiv \|\epsilon(\Delta x)\| = \sum_{r=2}^{n} m\, k_r(x)\cdot\|\Delta x\|^{r-1}$$

This is in exactly the form of (12.6) As $\|\Delta x\| \to 0$ the right hand member tends to zero and therefore so also does $\epsilon(\Delta x)$.

Theorem (13.82) If h(x) is a polynomial homogeneous of degree n then $dh(x) \equiv h_x(x;\Delta x)$ is given in terms of the polar of h by $nh(x,x,\cdots,x,\Delta x)$.

Proof: Employ the expansion used in Theorem (13.64) equation (2) with Δx written in place of y.

$$h(x+\lambda\Delta x) = \sum_{r=0}^{n} \lambda^r \binom{n}{r} h_r(x,\Delta x)$$

Using the Gateaux method of calculating dh(x) we have

$$dh(x) = \lim_{\lambda\to 0}\left\{\frac{h(x+\lambda\Delta x) - h(x)}{\lambda}\right\} = \sum_{r=1}^{n} \lambda^{r-1}\binom{n}{r} h_r(x,\Delta x)$$

$$= n\, h_1(x,\Delta x) = n\, h(x,x,\cdots,x,\Delta x).$$

Theorem (13.83) The r^{th} differential of a homogeneous polynomial h(x) of degree n ($r \leq n$) is given by

$$\frac{n!}{(n-r)!}\, h(\Delta_1 x,\cdots,\Delta_r x,x,x,\cdots,x)$$

Proof: For r = 1 this is Theorem (13.82). If the theorem is true for r - 1, then

$$d^{r-1}h(x) \equiv d_{r-1}\, d_{r-2}\cdots d_1 h(x) = \frac{n!}{(n - r + 1)!}\, h(\Delta_1 x, \cdots, \Delta_{r-1} x, x, \cdots, x) \tag{1}$$

is a homogeneous polynomial of degree $(n - r + 1)$ and the expression on the right of (1), being a multilinear form, must be precisely its polar. Therefore we may apply (13.82) again to (1) and obtain the result for r. This completes an induction on r.

An immediate consequence of the last theorem is.

Theorem (13.84) The r^{th} differential of a polynomial is symmetric in the increments.

Proof: From the symmetry of the polar of a homogeneous polynomial in its increments, and from Theorem (13.83) follows the symmetry in differentials. But since any polynomial is the sum of homogeneous polynomials and since the sum of the differentials is the differential of the sum, the symmetry holds for any polynomial.

13.9 Polynomials of Polynomials. Polynomials on Composite Spaces.

In this section we shall discuss rather briefly two situations in which we shall be interested in the next division.

First, let $q(x)$ be a polynomial on $E(A)$ to $E'(A)$ and $p(y)$ be a polynomial on $E'(A)$ to $E''(A)$. Let us consider the nature of the function $P(x) \equiv p\big(q(x)\big)$. We shall show that $P(x)$ is a polynomial on $E(A)$ to $E''(A)$ and that its degree is at most $m \cdot n$, where m is the degree of p and n is that of q.

Second, we shall examine the state of affairs when polynomials on a composite space E_1E_2 to E' are regarded as on $E_1[E_2]$ to E'. To proceed with the first part, let us prove:

Theorem (13.91) Let $p(x)$ be of degree n on $E(A)$ to $E'(A)$. Let $z(\lambda)$ be a polynomial of degree m on A to $E(A)$. Then $P(\lambda) \equiv p(z)$ is a polynomial of degree at most $m \cdot n$ on A to $E'(A)$.

Proof: We use induction over m. If $m = 0$, the theorem is trivial and true. Referring to (13.428) we have

$$p(x + y) = \sum_{r=0}^{n} k_r(x,y) \tag{1}$$

where $k_r(x,y)$ is of degree = $(n - r)$ in x and homogeneous of degree r in y.

Let
$$z(\lambda) \equiv \sum_{s=0}^{m} \lambda^s a_s$$

Assume the theorem for all polynomials z of degree < m. Replace y in (1) by $\lambda^m a_m$. This yields

$$p(x + \lambda^m a_m) = \sum_{r=0}^{n} \lambda^{mr} k_r(x,a_m)$$

If x is now replaced by $\sum_{s=0}^{m-1} \lambda^s a_s$ then, under the induction hypothesis, $k_r(x,a_m)$ becomes a polynomial of degree at most $(m - 1)(n - r)$. If this polynomial is multiplied by λ^{mr}, its degree will be at most

$$mr + (m - 1)(n - r) = mn - (n - r) \leqq mn$$

Therefore $P(\lambda) \equiv p(z)$ is the sum of polynomials of degrees $\leqq m \cdot n$.

Theorem (13.92) Let $q(x)$ be of degree m on $E(A)$ to $E'(A)$ and $p(u)$ of degree n on $E'(A)$ to $E''(A)$. Then $P(x) \equiv p(q(x))$ is a polynomial of degree at most $m \cdot n$ on $E(A)$ to $E''(A)$.

Proof: $P(x)$ is a continuous function of a continuous function, and therefore continuous. It remains to prove that $P(x + \lambda y)$ is a polynomial of degree $\leqq n \cdot m$ in λ. Now $q(x+\lambda y)$ is for fixed x,y a polynomial of degree $\leqq m$ on A to E'(A). Hence applying Theorem (13.91) $P(x+\lambda y) = p\big(q(x+\lambda y)\big)$ is of degree at most $m \cdot n$ in λ. The theorem is therefore proved.

Let us now consider a composite space $E_1E_2(A)$, of two spaces $E_1(A)$ and $E_2(A)$. We shall denote elements of E_1 by letters $x,y,z,\cdots$, elements of E_2 by letters $\bar{x},\bar{y},\bar{z},\cdots,$. The point $(x,\bar{x})$ of the composite support we shall denote by a capital letter X.

Theorem (13.93) If $P(X)$ is a polynomial of degree n on $E_1E_2(A)$ to $E'(A)$, then $p(x) \equiv P(X)$ is a polynomial in x on $E_1[E_2]$ to $E'(A)$.

Proof: By Theorem (12.53), $P(X)$ is continuous on $E_1[E_2]$ to E'. Now by the definition of a polynomial, $P(X+\lambda Y)$ is a polynomial in λ of degree $\leqq n$ on $A[(E_1E_2)^2]$ to E'. But if $Y \equiv (y,0)$

$$P(X+\lambda Y) = P\big[(x,\bar{x})+\lambda(y,\bar{0})\big] = P\big[(x+\lambda y,\ \bar{x}+\lambda\cdot 0)\big]$$
$$= p(x+\lambda y)$$

Therefore $p(x+\lambda y)$ is a polynomial of degree $\leqq n$ in λ. By a similar argument we could have proved that $\bar{p}(\bar{x}) \equiv P(X)$ is a polynomial of degree $\leqq n$ on $[E_1]E_2$ to E'.

14. Analytic Functions in Vector Space

14.1 Introduction

In this division of the work we propose to make investigations of the consequences of a certain formal generalization to vector spaces of the notion of an analytic function of a real or complex variable. A number of studies have been made in this direction*, but mostly from the stand point of remainder theorems. We attempt here to attack the problem from what is often called the "Weierstrass viewpoint". We shall define analytic functions in terms of a "power series" development in homogeneous polynomials and seek to derive the properties merely from the consideration of inequalities and identies among these polynomials. At the end, we shall apply some of our results to a differential equation theorem.

14.2 Convergence

We shall in general not require that all spaces with which we deal obey the postulate C_1 of completeness. When completeness is assumed in the present section it will be so specified in the theorem. The definition of the limit of a sequence of points of a space **E** was given in (12.3). Another way of stating the same thing is that a sequence $\{x_n\}$ has a limit if there exists in E a point x such that $\| x - x_n \|$ tends to zero with $1/n$. Using the same language, the postulate of completeness requires that the necessary and sufficient condition for the existence of the limit of a sequence $\{x_n\}$ is that $\max_{p>0} \| x_{n+p} - x_n \|$ tend to zero with $1/n$

* See, for example: L. M. Graves, Trans. Am. Math. Soc., V. 29, 1927, p 163.

(14.21) Convergent Sequences of Functions

The terms x_n of a sequence may depend in various ways upon some general parameter t whose range is T. In this case we say that the sequence $x_i(t)$ converges over T if it converges for every fixed value of t in T.

(14.23) Uniform Convergence

A sequence $\{x_n(t)\}$ depending upon a parameter t whose range is T is said to converge uniformly in t over T if there exists $x(t)$ such that $\max_{t \text{ in } T} \| x_n(t) - x(t) \|$ tends to zero with $1/n$.

(14.24) Convergence of Series

Let $\{x_n\}$ be a sequence of elements of E. Let $S_n = \sum_1^n x_r$. Then if S_n converges to a limit S we say that $\sum x_n$ converges to S. If the convergence of $\{S_n\}$ is uniform with respect to some parameter t over a range T, we say that $\sum x_n$ converges uniformly in t over T.

A series which does not converge is said to diverge.

(14.25) Absolute Convergence of Series

A series $\sum x_n$ is said to converge absolutely if the numerical series $\sum \| x_n \|$ is convergent in the ordinary sense.

Theorem (14.261) Let E_o be a region of E. Let $\{f_n(x)\}$ be a sequence of continuous functions on E_o to E' converging uniformly over E_o. Then $f(x)$ is continuous over E_o.

Proof: Select $e > 0$ and n_o so that $\| f_{n_o}(x) - f(x) \| < e/3$

Select δ so that if $\| x - x_o \| < \delta$ then $\| f_{n_o}(x_o) - f_{n_o}(x) \| < e/3$

Then for all x in $(x_o)_\delta$

$$\| f(x) - f(x_0)\| \leqq \|f(x) - f_{n_0}(x)\| + \|f_{n_0}(x) - f_{n_0}(x_0)\| + \|f_{n_0}(x_0) - f(x_0)\|$$
$$< \epsilon$$

which is the condition for continuity.

Theorem (14.262) If E is complete, and if a sequence $\{x_n\}$ from E is such that $\sum x_n$ is absolutely convergent, then $\sum x_n$ is convergent.

Proof: Let $S_n = \sum_1^n x_r$

Then $$\| S_{n+p} - S_n\| = \sum_{n+1}^{n+p} \| x_n \|$$

but since the right hand side is the partial remainder of a convergent series of positive terms its maximum value for all p tends to zero with 1/n. Then since E is complete this is a sufficient condition on $S_{n+p} - S_n$ for convergence.

Theorem (14.263) Let $f_n(x)$ be a sequence of functions on a region E_0 of E to E', where E' is complete. Let M_n denote the maximum of $f_n(x)$ over E_0. Then if $\sum M_n$ converges, $f_n(x)$ converges uniformly in x over E_0.

Proof:

Let $$S_n(x) = \sum_1^n f_r(x)$$

Then $$\| S_{n+p}(x) - S_n(x)\| = \sum_{n+1}^{n+p} \| f_r(x)\| = \sum_{n+1}^{n+p} M_n \qquad (1)$$

Thus $S_n(x)$ converges to $S(x)$.

Corollary: If the series $\sum f_n(x)$ is known to be convergent over E_0, then whether or not E' is complete the conclusions of Theorem (14.263) hold.

Proof: Completeness of E' is used only to show convergence of $\sum f_n(x)$.

If we calculate the limit of (1) as $p \to \infty$, n remaining fixed we have

$$\| S(x) - S_n(x) \| = \sum_{n+1}^{\infty} M_n \tag{2}$$

The right hand side tends to zero with $1/n$ and is independent of x.

Theorem (14.264) If $\{f_n(x)\}$ and $\{g_n(x)\}$ are sequences of functions on E_0 of E to E' such that $\sum f_n(x)$ and $\sum g_n(x)$ converge; then $\sum \{f_n(x) + g_n(x)\}$ converges and is equal to $\sum f_n(x) + \sum g_n(x)$.

Proof: Let $S_n f \equiv \sum_1^n f_n(x) \qquad S_n g \equiv \sum_1^n g_n(x)$

$$S_n(f + g) = \sum_1^n \left(f_n(x) + g_n(x) \right)$$

Then

$$\left\| S_n(f + g) - \left(\sum f_n(x) + \sum g_n(x) \right) \right\| = \left\| S_n f - \sum f_n(x) \right\| + \left\| S_n g - \sum g_n(x) \right\| \tag{1}$$

The fact that $\sum f_n$ and $\sum g_n$ converge shows that the right side of (1) tends to zero with $1/n$

It follows by induction that the sum of any finite number of convergent series is also convergent.

Theorem (14.265) Let $\sum f_n(x)$ be convergent on E_0 of E to E'(A). Then if α is a point of A, $\sum \alpha f_n(x)$ converges and is equal to $\alpha \sum f_n(x)$.

The proof is clear.

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14.3 Definition of Analytic Functions on E to E'. Some Definitions

Definition of Regularity: A function $f(x)$ on E_o, a region of E, to E' is said to be regular at a point x_o of E_o if there exists (1) a positive number r and (2) a sequence of homogeneous polynomials $h_n(x)$ -- $h_n(x)$ of degree n--on E to E', such that for $\|x - x_o\| < r$, the series $\sum_0^\infty h_n(x - x_o)$ converges to $f(x)$.

The maximum value of r satisfying these conditions we shall call the radius of regularity of $f(x)$ associated with the point x_o and shall denote it by $r(x_o)$.

Definition of Analyticity: A function $f(x)$ on E_o, a region of E, to E' is said to be analytic at the point x_o of E_o if: (1) it is regular at x_o. (2) the moduli mh_n of the polynomials $h_n(x)$ satisfy the condition $0 \leq \underline{\lim}_{n\to\infty} (mh_n)^{-\frac{1}{n}} \leqq r(x_o)$,

We shall define $r'(x_o)$ as the $\underline{\lim}_{n\to\infty} (mh_n)^{-\frac{1}{n}}$ and refer to it as the radius of analyticity of $f(x)$ associated with the point x_o. $f(x)$ will also be spoken of as "analytic $r'(x_o)$ at x_o".

We shall also have occasion to use the quantity $r''(x_o)$ defined as $\underline{\lim}_{n\to\infty} (m_nh_n)^{-\frac{1}{n}}$, where m_nh_n denotes the modulus of the polar of $h_n(x)$. We shall speak of $r''(x_o)$ as the radius of absolute analyticity of $f(x)$ associated with the point x_o.

Concerning this last we have incidentally:

Theorem (14.31) If $f(x)$ is on E_o to E', analytic $r'(x_o)$ at x_o, then $r''(x_o)$ satisfies the inequality

$$1 \leqq \frac{r''(x_o)}{r'(x_o)} \leqq e$$

where e is the base of natural logarithms.

Proof: By Theorem (13.73)

$$1 \leqq \frac{m_n h_n}{m h_n} \leqq \frac{n^n}{n!}$$

so that $\quad r''(x_0) = \lim_{n\to\infty} (m_n h_n)^{-\frac{1}{n}} = \lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} = r'(x_0)$

again. $\quad r''(x_0) = \lim_{n\to\infty} (m_n h_n)^{-\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{m_n h_n}{m h_n}\right)^{-\frac{1}{n}} (mh_n)^{-\frac{1}{n}}$

$$\geqq \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{-\frac{1}{n}} (mh_n)^{-\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n^n}{n!}\right)^{-\frac{1}{n}} \cdot \lim_{n\to\infty} (mh_n)^{-\frac{1}{n}}$$

$$= e^{-1} r'(x_0)$$

Dominants: Let $f(x) \equiv \sum_0^\infty h_n(x - x_0)$ be analytic $r'(x_0)$ at x_0. For a given number $\rho > 0$, a number M satisfying the condition

$$\| h_n(x) \| < M \frac{\|x\|^n}{\rho^n}$$

for all x and all n, will be called a ρ dominant of f at x_0. The lower bound of M's satisfying this condition we shall call the minimal ρ dominant of $f(x)$ at x_0. It will be denoted by $D_\rho f(x_0)$.

To show that dominants exist we have

Theorem (14.32) If $f(x)$ is analytic $r'(x_0)$ at x_0 and $\rho < r'(x_0)$ then $D_\rho f(x_0)$ exists.

Proof: Since $r'(x_0) = \lim_{n\to\infty} (mh_n)^{-\frac{1}{n}}$ we have from ordinary analysis that for $\rho < r'(x_0)$ $\sum_0^\infty mh_n \rho^n$ converges. There is therefore a greatest term in this series whose value let us call M. Thus

$$mh_n \rho^n \leqq M (\text{all } n)$$

But $$\frac{\|h_n(x)\|}{\|x\|^n} \leqq mh_n = \frac{mh_n\rho^n}{\rho^n} \leqq \frac{M}{\rho^n}$$

From which $$\|h_n(x)\| \leqq M\frac{\|x\|^n}{\rho^n}$$

One such positive number M existing, the lower bound certainly exists. As a matter of fact it is not difficult to see that the M we have just chosen is actually $B_\rho f(x_o)$.

14.4 Some properties of Analytic Functions on A to E(A).

Let us apply the definitions of (14.3) to the case where E(A) is A itself, and E'(A) is a vector space which in this section we shall write without the prime. The most general homogeneous polynomial of degree n on A to E(A) is simply $\lambda^n a$, where λ is of A and a is a fixed element of E(A). We are therefore led to consider functions which, in the neighborhood $(\lambda_o)_\rho$, are expressible as a power series in the form.

$$f(\lambda) \equiv \sum_0^\infty (\lambda - \lambda_o)^n a_n \qquad (a_n \text{ in E; } \lambda \text{ in } (\lambda_o)_\rho)$$

To simplify the work let us make the assumption that $\lambda_o = 0$, which amounts simply to writing λ for $\lambda - \lambda_o$.

Theorem (14.41) Let $\{a_n\}$ be an infinite sequence of elements of a complete vector space E(A). Let $r \equiv \lim_{n\to\infty} \|a_n\|^{-\frac{1}{n}}$ be positive. Then $\sum_0^\infty \lambda^n a_n$ converges or diverges accordingly as $|\lambda|$ is less than or greater than r.

Proof: Let $|\lambda| < r$. In Theorem (14.262) replace x_n by $\lambda^n a_n$ then $\sum_0^\infty \lambda^n a^n$ converges absolutely, since $\sum_0^\infty |\lambda|^n \|a_n\|$ is, by the definition of r and the well known Cauchy test, convergent. By the theorem already cited, therefore, $\sum_0^\infty \lambda^n a_n$ converges.

If $|\lambda| > r$, then from the definition of r and the ordinary theory of power series the sequence $\{|\lambda|^n \|a_n\|\}$ has no upper bound. Suppose for some value of λ, $(|\lambda| > r)$, $\sum_0^\infty \lambda^n a_n$ were convergent. Let S be its sum. Then for arbitrary e there is N_e such that $S_n \equiv \sum_0^n \lambda^n a_n$ satisfies $\|S_n - S\| < e$ for all $n > N_e$. Let n be chosen $> N_e$ so that

$$|\lambda|^{n+1}\|a_{n+1}\| > 3e$$

then

$$3e < |\lambda|^{n+1} \|a_{n+1}\| = |\lambda|^{n+1}\|a_{n+1}\| = \|S_{n+1} - S_n\|$$

$$\leq \|S_{n+1} - S\| + \|S_n - S\| < 2e$$

This contradiction proves the second part of the theorem.

It follows from this theorem that any sequence $\{a_n\}$ of elements of a complete space E such that $r \equiv \underline{\lim}_{n\to\infty} \|a_n\|^{-\frac{1}{n}} > 0$ defines a function,

$$f(\lambda) \equiv \sum_0^\infty \lambda^n a_n \text{ on } (0)_r \text{ of A to E which is}$$

by definition regular r(0) at $\lambda = 0$. Moreover, $\|a_n\|$ is precisely mh_n of the more general definiton so that the function $f(\lambda)$ defined by the sequende $\{a_n\}$ is also analytic $r'(0) = r(0)$ at $\lambda = 0$.

Conversely:

<u>Theorem (14.411)</u> If $f(\lambda)$ on $(0)_r$ of A to E(A) is regular r(0) at $\lambda = 0$ and if $f(\lambda) \equiv \sum_0^\infty \lambda^n a_n$ in $(0)_{r(0)}$, then $r' \equiv \underline{\lim}_{n\to\infty} \|a_n\|^{-\frac{1}{n}} > 0$.

<u>Proof</u>: If $r' = 0$, then by Theorem (14.41) $\sum_0^\infty \lambda^n a_n$ is divergent for $\lambda \neq 0$ which contradicts the definition of regularity.

<u>Theorem (14.412)</u> If in the above theorem $r(0) > r'$, then $f(\lambda)$ is analytic $r'(0) = r'$ at $\lambda = 0$.

Proof: The added assumption simply makes $f(\lambda)$ satisfy the condition $r'(0) < r(0)$ of the definition of analytic functions.

Theorem (14.92) If $f(\lambda) \equiv \sum_0^\infty \lambda^n a_n$ is analytic $r'(0)$ at $\lambda = 0$ on $(0)_{r'(o)}$ of A to E' and if $\rho < r'(0)$, then $\sum_0^\infty \lambda^n a_n$ converges uniformly in λ over $(0)_\rho$.

Proof: In the corollary of Theorem (14.263) take E_0 to be $(0)_\rho$ and $f_n(x)$ to be $\lambda^n a_n$. Then for λ in $(0)_\rho$ we have

$$\|\lambda^n a_n\| < \rho^n a_n$$

Since $\sum \rho^n \|a_n\|$ converges the conditions of the corollary are met.

Combining the results of this theorem and Theorem (14.261) we have the proof of

Theorem (14.420) If $f(\lambda)$ is analytic $r'(0)$ at $\lambda = 0$ on $(0)_{r'(o)}$ of A to E', and if $\rho < r'(0)$ then $f(\lambda)$ is continuous over $(0)_\rho$.

By Theorem (14.264) we have:

Theorem (14.422) If $f(\lambda) \equiv \sum_0^\infty \lambda^n a_n$ and $g(\lambda) \equiv \sum_0^\infty \lambda^n b_n$ are regular at $\lambda = 0$. Then $f(\lambda) + g(\lambda)$ is regular and is equal to $\sum_0^\infty \lambda^n(a_n + b_n)$.

Theorem (14.43) If $f(\lambda) \equiv \sum_0^\infty \lambda^n a_n$ is on $(0)_{r(o)}$ of A to E(A), regular $(0)_{r(o)}$ at $\lambda = 0$, and if for all λ in $(0)_{r(o)}$, $f(\lambda) \equiv 0$, then $a_n = 0$ $(n = 0,1,2,\cdots)$

Proof: Assume the contrary. Let a_m be the first coefficient which does not vanish, so that $0 = f(\lambda) = \sum_m^\infty \lambda^n a_n$. Using Theorem (14.265) we have for $0 < |\lambda| < r(0)$

$$0 = \frac{f(\lambda)}{\lambda^m} = \sum_m^\infty \lambda^{n-m} a_n = \sum_0^\infty \lambda^r a_{m+r}$$

Therefore $\sum_0^\infty \lambda^r a_{m+r}$ is a regular function of λ at $\lambda = 0$. And its value for $\lambda \neq 0$ is zero. By Theorem (14.421) itis continuous; so that its value at $\lambda = 0$ must also be zero. But $\left.\sum_0^\infty \lambda^r a_{m+r}\right|_{\lambda=0} = a_m \neq 0$. This contradiction proves the theorem.

Theorem (14.431) If $f(\lambda)$ is regular at $\lambda = 0$ its expansion about $\lambda = 0$ is unique.

Proof: Let $f(\lambda) = \sum_0^\infty \lambda^n a_n = \sum_0^\infty \lambda^n b_n$. Using Theorem (14.264)

$$0 = \sum_0^\infty \lambda^n a_n - \sum_0^\infty \lambda^n b_n = \sum_0^\infty \lambda^n (a_n - b_n)$$

and by the last theorem $a_n - b_n = 0$

or $a_n = b_n$

14.5 Theorems on Series of Polynomials.

We shall now deduce a result which it was not convenient to obtain in Division 13 because of the lack of adequate preliminaries on convergence.

Theorem (14.51) Let $h_n(x)$ be a sequence of homogeneous polynomials on E to E', each of degree k. Let the series $\sum_1^\infty h_n(x)$ converge uniformly in some neighborhood $(0)_r$. Then $h(x) \equiv \sum_0^\infty h_n(x)$ is a homogeneous polynomial of degree k.

Proof: It follows from Theorem (14.261) that h(x) is continuous over $(0)_r$.

Let $h_{np}(x) = \sum_{n+1}^{n+p} h_n(x)$. It is the sum of a finite number of homogeneous polynomials of degree k and is therefore one itself. The maximum of $\| h_{np}(x) \|$ over $(0)_r$ is $mh_{np} \cdot r^k$ and over any other $(0)_{\bar{r}}$ is $mh_{np} \cdot \bar{r}^k$. The condition for uniform convergence is equivalent to the condition that $mh_{np} \cdot r^k$ tend to zero with $1/n$ uniformly in p. Since convergence is assumed uniform over $(0)_r$ this condition is satisfied. But it is also satisfied for $(0)_{\bar{r}}$. Hence $\sum_1^{\infty} h_n(x)$ converges uniformly in $(0)_{\bar{r}}$. By taking $\bar{r}$ sufficiently large h(x) is proved continuous for every x.

Now by the definition of a polynomial $h_n(x + \lambda y)$ is a polynomial of degree k in λ and by Theorem (13.204) the coefficients of λ^s in this polynomial can be expressed in the form

$$\sum_{t=0}^{k} A_{st}\ h(x + ty)$$

We have therefore that

$$\sum_1^{\infty} h_n(x + \lambda y) = \sum_n^{\infty} \sum_{s=0}^{k} \lambda^s \sum_{t=0}^{k} A_{st}\ h_n(x + ty)$$

and using (14.264) we rearrange this sum so that

$$h(x + \lambda y) = \sum_{s=0}^{k} \lambda^s \sum_{t=0}^{k} A_{st}\ h(x + ty)$$

To show that h(x) is homogeneous we have

$$h(\lambda x) = \sum_1^{\infty} h_n(\lambda x) = \sum_1^{\infty} \lambda^k h_n(x) = \lambda^k h(x)$$

The seeming length of this latter portion of the argument may be explained on the grounds that completeness of E' was not assumed.

<u>Theorem (14.52)</u> Let $\{h_n(x)\}$ be a sequence of homogeneous polynomials each of degree k on E to E' where E' is complete. Let $\sum_1^{\infty} mh_n$

converge. Then $\sum_{1}^{\infty} h_n(x)$ converges to a homogeneous polynomial of degree k whose modulus is at most equal to $\sum_{1}^{\infty} mh_n$.

Proof: Theorem (14.263) establishes uniformity of convergence. Therefore by (14.51) above $h(x) \equiv \sum_{1}^{\infty} h_n(x)$ is a homogeneous polynomial of degree k. Again

$$mh = \max_{\|x\|=1} \|h(x)\| \leqq \sum_{1}^{\infty} \max_{\|x\|=1} \|h_n(x)\| = \sum_{1}^{\infty} mh_n$$

14.6 Fundamental Properties of Analytic Functions

In this section we shall deduce theorems analogous to some of those in section (14.4). As regards the point x_0 around which the functions are in the first instance assumed to be analytic, we observe that there is no real loss of generality in taking it to be the point 0. This amounts simply to writing x in place of $x - x_0$.

Theorem (14.61) Let $\{h_n(x)\}$ be a sequence of homogeneous polynomials on E to E' where E' is complete. Let the degree of $h_n(x)$ be n, and let $r'(0) \equiv \varliminf_{n\to\infty} \|mh_n\|^{-\frac{1}{n}} > 0$. Then $\sum_{0}^{\infty} h_n(x)$ converges and defines a function analytic $r'(0)$ at $x = 0$.

Proof: We have for $\|x\| < \rho < r'(0)$

$$\| h_n(x)\| < \rho^n \, mh_n$$

Since $\sum_{0}^{\infty} \rho^n \, mh_n$ converges, we have by Thoerem (14.263) that $\sum h_n(x)$ converges uniformly and absolutely over $(0)_\rho$. Since ρ is aribtrary within $0 < \rho < r'(0)$ the series converges at all points in $(0)_{r'(0)}$ and the function it defines is regular. By the hypothesis on $r'(0)$ the function is analytic.

Theorem (14.611) If $f(x) \equiv \sum h_n(x)$ is analytic $r'(0)$ on E to E' and if $\rho < r'(0)$ then $\sum h_n(x)$ converges uniformly and absolutely over $(0)_\rho$.

Proof: Completeness of the space was not used in (14.61) except to establish the esixtence of the limit. Therefore, since by hypothesis $\sum h_n(x)$ converges to f(x), the argument of (14.61) may be used.

Theorem (14.612) If $f_1(x) \equiv \sum_0^\infty h_n(x)$ and $f_2(x) \equiv \sum_0^\infty k_n(x)$ are regular at x = 0 then their sum is regular.

Proof: Theorem (14.264) gives

$$f_1(x) + f_2(x) = \sum_0^\infty h_n(x) + k_n(x)$$

But $h_n(x) + k_n(x)$ is homogeneous of degree n. By induction it follows that a similar thing is true for any finite number of functions.

Theorem (14.613) If $f(x) = \sum_0^\infty h_n(x)$ is regular $r(0)$ at x=0 and if f(x) = 0 throughout $(0)_{r(0)}$, then $h_n(x) = 0$. (all n)

Proof: Let x be any point in $(0)_{r(0)}$. Then if $|\lambda| < 1$

$$f(\lambda x) = \sum_0^\infty h_n(\lambda x) = \sum_0^\infty \lambda^n h_n(x) = 0$$

By Theorem (14.43) therefore $h_n(x) = 0$.

We shall now consider the question of differentiability of analytic functions. At this point it becomes neces ary to draw distinctions between theorems which hold for spaces E(C) but which we do not seem able to prove for spaces E(R).

Theorem (14.62) If $f(x)$ is analytic $r'(0)$ at 0 on $E(C)$ to $E'(C)$, and if $E'(C)$ is complete then the differential $f_x(x;\Delta x)$ exists at every point inside $(0)_{r'(o)}$.

Proof: Let $f(x) \equiv \sum_0^\infty h_n(x)$. Choose a fixed point x_o inside $(0)_{r'(o)}$ and select ρ, ρ' so that $x_o < \rho < \rho' < r'(0)$. Let $M = D_{\rho'}f(0)$. Equation (4) Theorem (13.81) gives

$$h_n(x_o + \Delta x) - h_n(x_o) - dh_n(x_o,\Delta x) = \sum_{r=2}^{n} k_{nr}(x_o, \Delta x) \quad (1)$$

If now Δx is restricted so that $\|\Delta x\| < \rho'' \equiv \rho_1 - \|x_o\|$ have by the definition of M that

$$h_n(x_o + \Delta x) \leqq M \frac{\|x_o + \Delta x\|}{\rho'^n} \leqq M \left(\frac{\rho}{\rho'}\right)^n \quad (2)$$

Applying to $h_n(x_o + \Delta x)$ result of the first part of inequality (1) of Theorem (13.741) we obtain

$$\| k_{nr}(x_o\Delta x)\| \leqq \max_{|\lambda|=1} \| h_n(x + \lambda \Delta x)\| \leqq M\left(\frac{\rho}{\rho'}\right)^n \quad (3)$$

and

$$\| dh_n(x_o;\Delta x)\| \leqq M\left(\frac{\rho}{\rho'}\right)^n \quad (4)$$

Since $k_{nr}(x_o;\Delta x)$ is homogeneous in Δx and since the inequality (3) holds for all Δx in $(0)_{\rho''}$ it follows that $\| k_{nr}(x_o,\Delta x)\| \leqq M\left(\frac{\rho}{\rho'}\right)^n \left(\frac{\|\Delta x\|}{\rho''}\right)^r$ (5)

thence that $$\left\| \sum_{r=2}^{n_0} k_{nr}(x_o,\Delta x)\right\| \leqq M\left(\frac{\rho}{\rho'}\right)^n \frac{\|\Delta x\|^2}{\rho''^2 - \rho''\|\Delta x\|} \quad (6)$$

Now simply sum the expression (1) from 0 to ∞. The two series whose general terms are $h_n(x_o + \Delta x)$ and $h_n(x)$ converge by the hypothesis of the theorem and the choice of Δx. The series $\sum dh_n(x,\Delta x)$ converges uniformly by (5) and Theorem (14.263) and therefore by (14.51) is linear homogeneous. Denote by $\|\Delta x\|\epsilon(\Delta x)$ the sum of the right hand side of (1)

Inequality (6) above gives that

$$\|\epsilon(\Delta x)\| \leqq \|\Delta x\| \frac{M}{(1 - \rho/\rho')(\rho''^2 - \rho''\|\Delta x\|)} \tag{7}$$

Writing down the sum of (1)

$$f(x_0 + \Delta x) - f(x_0) - \sum_0^\infty dh_n(x_0;\Delta x) = \|\Delta x\| \cdot \epsilon(\Delta x)$$

(7) and the linearity of $\sum_0^\infty dh_n(x_0;\Delta x)$ prove that $\sum_0^\infty dh_n(x_0;\Delta x)$ is the differential of $f(x)$ at (x_0).

<u>Theorem (14.621)</u> If $f(x)$ satisfies the conditions of Theorem (14.62), then the series $\sum h_n(x)$ may be differentiated repeatedly term by term.

<u>Proof</u>: Since equation (4) (14.62) holds for every x_0 in $(0)_\rho$ and since $dh_n(x;\Delta x)$ is of degree n-1, we have, by Theorem (13.723)

$$m(dh_n) = \frac{1}{\rho^{n-1}} \cdot \max_{\|x\| \leqq \rho} \|dh_n(x_0;\Delta x)\|$$

$$\leqq \frac{1}{\rho^{n-1}} \cdot M\left(\frac{\rho}{\rho'}\right)^n = \frac{\rho M}{\rho'^n}$$

Hence $$\bar{r} \equiv \lim_{n\to\infty} \left\{ m(dh_n)\right\}^{-\frac{1}{n-1}} \geqq \rho'$$

But ρ' is any number less than $r'(0)$. Thus $\bar{r} \geqq r'(0)$. We conclude that the differential $\sum dh_n(x)$ is analytic over $(0)_{r'(0)}$. It can therefore be differentiated again. We assert by induction that all differentials exist. Since the differential of a homogeneous polynomial is symmetric in the increments and since the analytic function is the sum of such functions it follows that the k^{th} differential of an analytic function is symmetric in the differentials.

The analogue of Theorem (14.62) for spaces E(R) cannot be proven in the same manner. As a matter of fact it is still an open question whether in general it can be proven at all. However, we shall prove.

<u>Theorem (14.63)</u> If f(x) is analytic r'(0) at 0 on E(R) to E'(R) and E'(R) is complete, then the differential of f(x) exists ~~and~~ at any point within $(0)_{r''(o)}$. It is given in this region by $\sum dh_n(x)$.

<u>Proof:</u> Employ an argument similar to that of (14.62). Starting with equation (1) of that section:

$$h_n(x+\Delta x) - h_n(x) - dh_n(x;\Delta x) = \sum_{r=2}^{n} k_{nr}(x;\Delta x) \tag{1}$$

we proceed by the use of Eq(2) in proof of 13.64) to write the inequality

$$\| k_{nr}(x;\Delta x) \| = \| \binom{n}{r} h_{nr}(x;\Delta x) \| \leq m_n h_n \binom{n}{r} \| x \|^{n-r} \cdot \| \Delta x \|^r$$
$$\| dh_n(x;\Delta x) \| = \| k_{n1}(x;\Delta x) \| \leq m_n h_n\, n \| x \|^{n-1} \cdot \| \Delta x \| \tag{2}$$

where $h_{nr}(x,\Delta x)$ is defined in (13.64). As an upper bound for the right hand side of (1) we have

$$\left\| \sum_{r=2}^{n} k_{nr}(x;\Delta x) \right\| \leq m_n h_n \sum_{r=2}^{n} \binom{n}{r} \| x \|^{n-r} \| \Delta x \|^r$$

$$= m_n h_n \| \Delta x \|^2 \sum_{s=0}^{n-2} \binom{n}{s+2} \| x \|^{n-2-s} \| \Delta x \|^s \tag{3}$$

$$= n(n-1) \cdot m_n h_n \| \Delta x \|^2 \cdot (\| x \| + \| \Delta x \|)^{n-2}$$

Let us now select $\rho < r''(0)$, and a fixed point x_o in $(0)_\rho$. Let Δx be so restricted that $\|x_0\| + \|\Delta x\| < \rho$. With these restrictions the right hand side of (3) is a convergent series whose sum is less than a certain constant M multiplied by $\|\Delta x\|^2$. From this point on the argument goes like that of Theorem (14.62). From equation (2) and Theorem (14.263) the convergence of $\sum dh_n(x;\Delta x)$ is established, and by (14.51) it is proved linear homogeneous in Δx. Thus

$$\left\| \Delta f(x) - \sum_1^\infty dh_n(x;\Delta x) \right\| \leq \|\Delta x\|^2 \cdot M \tag{4}$$

which proves the existence of the derivative, for it is given by $\sum_1^\infty dh_n(x;\Delta x)$. Furthermore, $\sum_1^\infty dh_n(x;\Delta x)$ is analytic; for we have, using the results of (13.82).

$$m(dh_n) \leqq n\, m_n h_n \cdot \|\Delta x\| \tag{5}$$

From this, (14.61), and the hypothesis, we have therefore that $\sum dh_n(x;\Delta x)$ is analytic $r''(0)$.

Theorem (14.631) If f(x) satisfies the hypotheses of Theorem (14.63), then it is differentiable of all orders, within $(0)_{r''(o)}$.

Proof: It is sufficient to show that the radius of absolute analyticity associated with a series is not greater than that of its derived series. If this is shown, then the theorem will follow by applying induction to (14.63). To prove the desired result we have from (13.82)

that
$$dh_n(x;\Delta x) = n\, h(\Delta x, x, x, \ldots, x)$$

and hence
$$m_{n-1}(dh_n) \equiv \max_{\|x_i\| \leq 1} n \cdot h(\Delta x, x_1, x_2, \cdots, x_{n-1})$$

$$\leqq n \cdot m_n h_n \cdot \|\Delta x\|$$

Now therefore we have

$$\lim_{n\to\infty} \left(m_{n-1}(dh_n)\right)^{-\frac{1}{n-1}} \geq \lim_{n\to\infty} (n\cdot m_n h_n \cdot \| x \|)^{-\frac{1}{n}} = r''(0)$$

The left hand side of this equation is the radius of absolute analyticity associated with $\sum_0^\infty dh_n(x;\Delta x)$. Applying what we have just proved to the (n-1)st derived series, which we assume to have a radius of analyticity not less than $r''(0)$, we prove that the same thing is true for the n^{th} derived series.

Let us now turn our attention to the question of series whose general term is an analytic function.

<u>Theorem (14.64)</u> Let $f_1(x), f_2(x), \cdots, f_i(x), \cdots$ be a sequence of functions on E(A) to E'(A), where E'(A) is complete. Let $f_i(x)$ be analytic $r_i'(0)$ at $x = 0$ and let R, the lower bound of all $r_i'(0)$ be positive. Let ρ be less than R, and define $M_i = D_\rho f_i(x)$. Then if $M \equiv \sum_1^\infty M_i$ converges, $f(x) \equiv \sum_1^\infty f_i(x)$ is an analytic function on E(A) to E'(A), whose radius of analyticity is not less than R.

<u>Proof:</u> Write $f_i(x) = \sum_0^\infty h_{in}(x)$. Let ρ_1 be selected so that $0 < \rho_1 < \rho$ and x restricted to be in $(0)_{\rho_1}$. Then by the definition of M_i

we have
$$\| h_{in}(x) \| \leq M_i \left(\frac{\rho_1}{\rho}\right)^n \tag{1}$$

So that the modulus of $h_{in}(x)$ is not greater than $M_i\left(\frac{1}{\rho}\right)^n$.

From (14.52) we have immediately that $h_n(x) \equiv \sum_{i=1}^\infty h_{in}(x)$ converges to a homogeneous polynomial of degree n whose modulus is not

greater than
$$\sum_{i=1}^\infty M_i\left(\frac{1}{\rho}\right)^n = \frac{M}{\rho^n}$$

From this it follows that

$$\lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} \geq \lim_{n\to\infty} \left(\frac{\rho^n}{M}\right)^{-\frac{1}{n}} = \rho \qquad (2)$$

and, by (14.61), that $\sum_0^\infty h_n(x)$ defines an analytic function whose radius of analyticity is not less than ρ.

It is now desired to prove that $f(x) = \sum_0^\infty h_n(x)$.

We have, remembering always that $\|x\| < \rho_1$,

$$\left\| f_i(x) - \sum_{n=0}^{p} h_{in}(x) \right\| = \left\| \sum_{p+1}^{\infty} h_{in}(x) \right\| \leq \sum_{p+1}^{\infty} M_i \left(\frac{\rho_1}{\rho}\right)^n$$

$$= M_i \frac{\left(\frac{\rho_1}{\rho}\right)^{p+1}}{1 - \frac{\rho_1}{\rho}} \qquad (3)$$

Now $$\| f_i(x) \| \leq \sum_{n=0}^{\infty} \| h_{in}(x) \| \leq \sum_{n=0}^{\infty} M_i \left(\frac{\rho_1}{\rho}\right)^n = \frac{M_i}{1 - \frac{\rho_1}{\rho}}$$

Therefore by (14.263 $\sum_1^\infty f_i(x)$ converges (uniformly) over $(0)\rho_1$. Now let ϵ be given and select p so that

$$\frac{\left(\frac{\rho_1}{\rho}\right)^{p+1}}{1 - \frac{\rho_1}{\rho}} < \frac{\epsilon}{3M}$$

Having chosen p, select k so that

$$\left\| f(x) - \sum_{i=1}^{k} f_i(x) \right\| < \frac{\epsilon}{3} ; \left\| h_n(x) - \sum_{i=1}^{k} h_{in}(x) \right\| < \frac{\epsilon}{3 \cdot 2^{n+1}} \qquad (4)$$

which may be done since all the series in (4) have been proved convergent.

then $$\left\| f(x) - \sum_{n=0}^{p} h_n(x) \right\| \leq \left\| f(x) - \sum_{i=1}^{k} f_i(x) \right\| + \left\| \sum_{i=1}^{k} f_i(x) - \sum_{i=1}^{k} \sum_{n=0}^{p} h_{in}(x) \right\|$$

$$+ \left\| \sum_{n=0}^{p} \sum_{i=1}^{k} h_{in}(x) - \sum_{n=0}^{p} h_n(x) \right\| \qquad (5)$$

But the second term on the right hand side of (5) is less than or equal to

$$\sum_{i=1}^{k} \left\| f_i(x) - \sum_{n=0}^{p} h_{in}(x) \right\| \leq \sum_{i=1}^{k} M_i \frac{\left(\frac{\rho_1}{\rho_i}\right)^{p+1}}{1 - \frac{\rho_1}{\rho_i}} < M \cdot \frac{\epsilon}{3M} = \frac{\epsilon}{3}$$

and as for the third term of (5) it is not greater than

$$\sum_{n=0}^{p} \left\| \sum_{i=1}^{k} h_{in}(x) - h_n(x) \right\| < \sum_{n=0}^{p} \frac{1}{2^{n+1}} \cdot \frac{\epsilon}{3} < \frac{\epsilon}{3}$$

These results, when placed in (5) show that

$$\left\| f(x) - \sum_{n=0}^{p} h_n(x) \right\| < \epsilon$$

Let us now apply the last theorem to **the proof of:**

<u>Theorem (14.65)</u> If $f(x)$ on $E_0(C)$ to $E'(C)$ $E'(C)$ complete is analytic $r'(0)$ at $x = 0$ it is analytic at every point x_0 in $(0)_{r'(0)}$.

<u>Proof</u>: Let $f(x) \equiv \sum_0^{\infty} h_n(x)$. Let x_0 be chosen in $(0)_{r'(0)}$. We wish to show that there exists a sequence of polynomials $\bar{h}_n(x)$, depending of course upon x_0, such that $\bar{f}(x) \equiv \sum_0^{\infty} \bar{h}_n(x - x_0)$ is analytic

at $x = x_0$ and equal to $f(x)$. To facilitate matters let us write $x - x_0 = y$, and let $F(y) \equiv f(y + x_0) = f(x)$, so that $F(y) \equiv \sum_0^{\infty} h_n(y + x_0)$ is analytic at $y = -x_0$ and has as its radius of analyticity $r'(0)$. We shall prove $F(y)$ is analytic at $y = 0$.

Let ρ, ρ_1 be selected such that $0 < x_0 < \rho_1 < \rho < r'(0)$. Let $M \equiv D\ f(0)$. It follows from the definition of M that

$$\| h_n(x) \| < M\left(\frac{\rho_1}{\rho}\right)^n \tag{1}$$

Let

$$f_i(y) = h_i(x_0 + y) = \sum_{n=0}^{i} h_{in}(y) \tag{2}$$

be the usual reduction of a polynomial in y to the sum of homogeneous polynomials. If y be restricted to satisfy $\| y \| < \rho_2 \equiv \rho_1 - \| x_0 \|$ we have from Theorem (13.74) that

$$\max_{\|y\| < \rho_2} \| h_{in}(y) \| \leqq \max_{|\lambda| = 1} \| h_i(x_0 + \lambda y) \| < M\left(\frac{\rho_1}{\rho}\right)^i \tag{3}$$

It follows that for all values of y

$$\| h_{in}(y) \| \leqq M\left(\frac{\rho_1}{\rho}\right)^i \left(\frac{\| y \|}{\rho_2}\right)^n \tag{4}$$

Now apply Theorem (14.64) observing that $f_i(x)$ is a polynomial in y and therefore a particular instance of an analytic function, and that from equation (4) $M\left(\frac{\rho_1}{\rho}\right)^i$ is a ρ_2 dominant of $f_i(y)$. Taking M_i in (14.64) to be $M\left(\frac{\rho_1}{\rho}\right)^i$ we see that $\sum M_i$ converges and that the conditions of the theorem are met.

Therefore, $F(y) = \sum_{n=0}^{\infty} \bar{h}_n(y)$, where $\bar{h}_n(y) = \sum_{i=0}^{\infty} h_{in}(y)$. Now the modulus of $\bar{h}_n(y)$ is seen by applying (14.52) to (4) to be not greater

$$\sum_{i=0}^{\infty} M\left(\frac{\rho_1}{\rho}\right)^i \frac{1}{\rho_2^n} = \frac{M}{1-\frac{\rho_1}{\rho}} \cdot \frac{1}{\rho_2^n}$$

Hence the radius $\bar{r}'(0)$ of nalyticity of $F(y)$ at $y = 0$ satisfies

$$\bar{r}'(0) \equiv \varliminf_{n\to\infty} (m\bar{h}_n)^{-\frac{1}{n}} \geqq \varliminf_{n\to\infty} \left(\frac{M}{1-\frac{\rho_1}{\rho}} \cdot \frac{1}{\rho_2^n}\right)^{-\frac{1}{n}} = \rho_2$$

Finally, go back to the original variables x. This gives

$$f(x) = F(y) = \sum_{n=0}^{\infty} \bar{h}_n(y) = \sum_0^{\infty} \bar{h}(x - x_0)$$

Everything is therefore proved. We note that by defining ρ suitably close to $r'(0)$ we prove that $r'(x_0) \geqq r'(0) - \|x_0\|$.

Theorem (14.66) Let $G(y)$ be a homogeneous polynomial of degree k in $E_0'(C)$ to $E''(C)$, and let $y = f(x)$ be a function analytic $r'(0)$ at $x = 0$ on $E_0(C)$ to $E_0'(C)$. Then $g(x) \equiv G(y)$ is analytic $\bar{r}'(0) \geqq r'(0)$ on $E(C)$ to $E''(C)$.

Proof: Let $f(x) \equiv \sum_{n=0}^{\infty} h_n(x)$; $H_s(x) \equiv \sum_{n=0}^{s} h_n(x)$.

Select ρ_1, ρ to satisfy $0 < \rho_1 < \rho < r'(0)$ and define $M \equiv D_\rho f(0)$, $\bar{M} \equiv \frac{M}{1-\frac{\rho_1}{\rho}}$. Thus for x in $(0)\rho_1$

$$\|h_n(x)\| < M\left(\frac{\rho_1}{\rho}\right)^n; \quad \|f(x)\| \sum_0^{\infty} M\left(\frac{\rho_1}{\rho}\right)^n = \bar{M}$$

$$\|H_s(x)\| < \bar{M}$$

Now since $G(y)$ is continuous and $\sum h_n(x)$ converges it follows, writing

$$G_s(x) \equiv G(H_s(x)),$$

$$\lim_{s\to\infty} G_s(x) = G\big(f(x)\big) = g(x) \tag{1}$$

$G_s(x)$ is, by Theorem (13.92), a polynomial in x of degree $\leq$ sk. Let $W_s(x)$ be the coefficient λ^s in $G_s(\lambda x)$ when it is expanded as the sum of homogeneous polynomials times powers of λ.

Let us first prove that if $p > 0$, $G_s(\lambda x) - G_{s+p}(\lambda x)$ contains no terms of degree $\leq$ s in λ. Write G(y + z) in the form

$$G(y + \mu z) = \sum_{r=0}^{k} \mu^r J_r(y,z) \tag{2}$$

where $J_r(y,z)$ is of degree r in z and k-r in y. In this formula write $y = H_s(\lambda x)$, $\mu = \lambda^{s+1}$, $\mu z = H_{s+p}(\lambda x) - H_s(\lambda x)$. Since the right side of this last expression contains no terms of degree $\leq$ s, z, as defined by it, will actually be a polynomial in λ. The coefficient $J_0(y,z)$ in (2) is, as has been verified, several times before, precisely G(y).

Therefore we have

$$G(y + \mu z) - G(y) \tag{3}$$

$$= G(H_{s+p}(\lambda x)) - G\big(H_s(\lambda x)\big)$$

$$= G_s(\lambda x) - G_{s+p}(\lambda x)$$

$$= \sum_{r=1}^{k} \lambda^{r(s+1)} G_r(h,z)$$

which is what we wanted.

It follows from this fact that if $r < s$, then the coefficient of λ^r in $G_s(\lambda x)$ must be precisely $W_r(x)$; for, if it were not, then $G_s(\lambda x) - G_r(\lambda x)$ would have a term of degree r. Therefore, the first s+1 terms in the expression of $G_s(\lambda x)$ as the sum of homogeneous polynomials are $W_0(x), W_1(x), \cdots, W_s(x)$.

To prove that $\sum_{s=0}^{\infty} W_s(x)$ converges to $g(x)$ let

$$G_s(\lambda x) = \sum_{r=0}^{k\cdot s} \lambda^r \dot{W}_{rs}(x)$$

so that by what we have just said $W_{rs}(x) = W_s(x)$ when $r < s$.

Now restrict x to be in $(0)\rho_1$. Using Theorem (13.74)

$$\| W_{rs}(x) \| \leqq \max_{|\lambda|=1} \| G(H_s(\lambda x)) \| \leqq mG\cdot \bar{M}^k \tag{4}$$

from which it follows that for all x

$$\| W_{rs}(x) \| < mG\cdot\bar{M}^k \frac{\|x\|^r}{\rho_1^r} \tag{5}$$

and therefore that if $\|x\| < \rho_1$

$$\| G_s(x) - \sum_{r=0}^{s} W_r(x) \| = \| G_s(x) - \sum_{r=0}^{s} W_{rs}(x) \|$$

$$\leqq mG\cdot\bar{M}^k \sum_{r=s+1}^{sk} \frac{\|x\|^r}{\rho_1^r} < \frac{mG\cdot\bar{M}^k}{1 - \frac{\|x\|}{\rho_1}} \frac{\|x\|^{s+1}}{\rho_1^{s+1}} \tag{6}$$

Now when $\|x\| < \rho_1$ $W_r(x)$ converges to $g(x)$; for if e is a given number we may take s so large that for $s_1 > s$, the right hand side of (6) is less than e/2 and then select s_1 so that $\| G_{s_1}(x) - g(x) \| < e/2$. Doing this we have

$$\| g(x) - \sum_{r=0}^{s} W_r(x) \| < e \tag{7}$$

To see that g(x) is analytic we have only to consider equation (5) which shows that the radius of analyticity must not be less than ρ_1. Since ρ_1 may be as close as we like to r'(0) it follows that the radius of analyticity of g(x) cannot be less than r'(0).

<u>Theorem (14.67)</u> Let $f(y)$ be analytic $r'(0)$ at 0 on $E_0'(C)$ to $E''(C)$ where $E''(C)$ is complete. Let $g(x)$ be analytic $\bar{r}'(0)$ at 0 on $E_0(c)$ to $E_0'(C)$. If there exist R and ϖ such that $R = \bar{r}'(0)$ and such that for all x in $(0)_R \| g(x)\| < \varpi < r'(0)$ then $F(x) \equiv f\big(g(x)\big)$ is analytic $R' \geqq R$ at $x = 0$.

<u>Proof</u>: Let $f(y) = \sum_{i=0}^{\infty} J_i(y)$

$$g(x) = \sum_{n=0}^{\infty} h_n(x)$$

Select ρ_1, ρ to satisfy $0 < \rho_1 < \rho < R$. Define $K_i(x) \equiv J_i\big(g(x)\big)$. By the preceding theorem $K_i(x)$ is an analytic function of x. Let us write

$$K_i(x) = \sum_{p=0}^{\infty} k_{ip}(x)$$

We wish to show now that $K_i(x)$ satisfies the requirements of Theorem (14.64). To do so we replace $G(y)$ in Theorem (14.66) by $J_i(y)$. Equation (5) of that theorem then becomes $\left\| k_{ip}(x)\right\| < mJ_i \cdot \varpi^i \dfrac{\|x\|^p}{\rho_1^p}$ (1)

This shows that $M_i \equiv mJ_i \cdot \varpi^i$ is a ρ_1 dominant of $K_i(x)$. But since $\varpi < r'(0)$, $\sum M_i$ is convergent. Thus the series $\sum K_i(x)$ of analytic functions $K_i(x)$ defines by Theorem (14.64) an analytic function. We have

$$F(x) = \sum_{i=0}^{\infty} J_i\big(g(x)\big) = \sum_{i=0}^{\infty} K_i(x)$$

$$= \sum_{p=0}^{\infty} H_p(x).$$

where $H_p(x)$ is the homogeneous polynomial of degree p defined by

$H_p(x) \equiv \sum_{i=0}^{\infty} k_{ip}(x)$

14.7 Analytic Functions and Differentials in Composite Spaces.

Although the results of the present section have considerable interest in themselves, especially as a starting point for further work, they are here directed almost entirely toward the differential equation problem which will be discussed in (14.8). The manipulations made possible by the notion of composite spaces seem to present a very glib way for proving theorems. We shall, of course, make free use of the notation introduced in Section (12.4).

Theorem (14.71) Let $p_i(x)$ $(i = 1,2,\cdots,k)$ be k polynomials, of degree n_i on E to E_i respectively. Then $p(x) \equiv \big(p_1(x),p_2(x),\cdots,p_k(x)\big)$ is a polynomial of degree n on E to $\prod_{i=1}^{k} E_i$, where $n = \max_i n_i$. In particular if $p_i(x)$ are all homogeneous of degree n, then so is $p(x)$.

Proof: If x,x' are two points then $\|p(x) - p(x')\|$ is the greatest of $\|p_i(x) - p_i(x')\|$. Hence $p(x)$ is continuous.

Let $p_i(x + \lambda y) = \sum_{r=0}^{n} \lambda^r k_{ir}(x,y)$, where, of course, if the degree of p_i is less than n some of the k's are zero. Then

$$p(x + \lambda y) = \left(\sum_{r=0}^{n} \lambda^r k_{1r}(x,y), \sum_{r=0}^{n} \lambda^r k_{2r}(x,y), \cdots, \sum_{r=0}^{n} \lambda^r k_{kr}(x,y)\right)$$

$$= \sum_{r=0}^{n} \lambda^r \big(k_{1r}(x,y), k_{2r}(x,y), \cdots, k_{kr}(x,y)\big) \qquad (1)$$

Hence $p(x + \lambda y)$ is a polynomial of degree $\leq n$ on $A[E]$ to $\prod_{i=1}^{k} E_i$. Let (x,y) take on values such that for some j $k_{jn} \neq 0$, then the coefficient of λ^n in the last member of (1) is different from the zero. Hence $p(x + \lambda y)$ is of degree exactly n.

If $p_i(x)$ are homogeneous of degree n we have

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$$p(\lambda x) = \left(p_1(\lambda x), \cdots, p_k(\lambda x) \right) = \lambda^n p(x)$$

Theorem (14.711) If $p_i(\lambda)$ in the last theorem are homogeneous of degree n, then $mp = \max_i mp_i$.

Proof:
$$mp = \max_x \left\| \frac{p(x)}{\|x\|^n} \right\|$$

$$= \max_x \left\| \frac{(p_1(x), \cdots, p_k(x))}{\|x\|^n} \right\|$$

$$= \max_x \max_i \left\| \frac{p_i(x)}{\|x\|^n} \right\| = \max_i mp_i$$

Theorem (14.72) If $f_i(x)$ $(i = 1,2,\cdots,k)$ are analytic $r_i'(x_o)$ at x_o on E to E_i respectively, then $f(x) \equiv \left(f_1(x), f_2(x), \cdots, f_k(x) \right)$ is analytic $r' \equiv \min_i r_i'(x_o)$ on E to $\prod E_i$.

Proof: For simplicity take $x_o = 0$. Let $f_i(x) = \sum_{n=0}^{\infty} h_{in}(x)$, and $h_n(x) \equiv \left(h_{1n}(x), \cdots, h_{kn}(x) \right)$. Then by Theorem (14.71) $h_n(x)$ is homogeneous of degree n and by (14.711) its modulus is the greatest of mh_{in}. Hence

$$\lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} = \lim_{n\to\infty} (mh_{in})^{-\frac{1}{n}} = r_i'(0) = r'$$

Now let ρ be selected so that $0 < \rho < r'$. Let $M \equiv \max_i D_\rho f_i(0)$. Then clearly $\|h_n(x)\| < M\left(\frac{\|x\|}{\rho}\right)^n$, so that $mh_n \leqq \frac{M}{\rho^n}$. From this we have

$$\lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{M}{\rho^n}\right)^{-\frac{1}{n}} = \rho \qquad (2)$$

Since ρ is at our disposal the inequality (2) proves in conjunction with (1) that
$$\lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} = r'$$

That $\sum h_n(x)$ converges to $f(x)$ is evident since the difference

$$\left\| f(x) - \sum_0^s h_n(x) \right\| = \max_i \left\| f_i(x) - \sum_0^s h_{in}(x) \right\|$$

Theorem (14.73) Let $h(x)$ be a homogeneous polynomial of degree n on E to E'; then the r^{th} differential of $h(x)$ considered as a function of the composite variable $(x, \Delta_1 x, \cdots, \Delta_r x)$ on E^{r+1} to E' is also a homogeneous polynomial of degree n. Its modulus is not greater than

$$\frac{n!}{(n-r)!} \, m_n h_n$$

Proof: Let us use the representation of (13.83) and write

$$d^r h(x) = \frac{n!}{(n-r)!} \, h(\Delta_1 x, \cdots, \Delta_r x, x, x, \cdots, x) \quad (1)$$

We wish first to prove that $d^r h(x)$ is continuous in the variable $(x, \Delta_1 x, \cdots, \Delta_r x)$. We observe that $d^r h(x)$ when expressed in terms of the polar is the sum of terms of the type

$$h(sx + \Delta_{i_1} x + \cdots + \Delta_{i_{n-s}} x) \quad (2)$$

where s is an integer. But the arguments $sx + \Delta_{i_1} x + \cdots + \Delta_{i_{n-s}} x$ are continuous in the variable $(x, \Delta_1 x, \cdots, \Delta_r x)$ and $d^r h(x)$ being the sum of continuous functions of continuous functions is continuous.

If in $d^r h$ the argument $(x, \Delta_1 x, \cdots, \Delta_r x)$ is replaced by $(x + \lambda x', \Delta_1 x + \lambda \Delta_1 x', \cdots, \Delta_r x + \lambda \Delta_1 x', \cdots, \Delta_r x + \lambda \Delta_r x')$ the result is a polynomial in λ, since the same thing is true for the expression (2).

Therefore $d^r h$ is a polynomial in $(x, \Delta_1 x, \cdots, \Delta_r x)$. That it is homogeneous of degree n may be proven by replacing $\Delta_i x$ by $\lambda \Delta_i x$ and x by λx in (1).

As for the modulus of $d^r h$ we observe that

$$m\, d^r h = \max \frac{\| d^r h \|}{\|(x, \Delta_1 x, \cdots, \Delta_r x)\|^r}$$

$$\leqq \max \frac{\| d^r h \|}{\|x\|^{n-r}\|\Delta_1 x\| \cdots \|\Delta_r x\|}$$

$$\leqq \frac{n!}{(n-r)!} \quad m_n h_n$$

Theorem (14.74) If $f(x) \equiv \sum_0^\infty h_n(x)$ is analytic at $x = 0$ on E_o of E to E', then its r^{th} differential is analytic in $(x, \Delta_1 x, \cdots, \Delta_r x)$ on $E_o E^r$ to E'.

Proof: Let us write $X \equiv (x, \Delta_1 x, \ldots, \Delta_r x)$ and $H_n(X) \equiv d^r h_n(x)$. Then by the last theorem $H_n(X)$ is a polynomial of degree n on E^{r+1} to E' and is therefore in particular on $E_o E^r$ to E'. Since mH_n is by the last theorem not greater than $\frac{n!}{(n-r)!}$ the radius of analyticity is not less than $\lim_{n\to\infty} (m_n h_n)^{-\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{n!}{(n-r)!} m_n h_n\right)^{-\frac{1}{n}} = r''(0)$.

Partial Differentials

If $f(X) \equiv f(x_1, x_2, \cdots, x_n)$ is a function on a region $(\prod_{i=1}^n E_i)_o$ to E' and if on $(E_k[\prod_{i \neq k}^n E_i])_o$ to E', f is differentiable quâ x_k at $X_o \equiv (x_1^o, x_2^o, \cdots, x_k^o)_o$. $f_{x_k}(X_o; \Delta x_k)$ is called the partial differential with respect to x_k at X_o

The existence of $f_{x_k}(X_o; \Delta x_k)$ for each k does not necessarily imply the existence of $f_X(X_o; \Delta X)$, the differential of f with respect to the composite variable X. However, the converse is true; the existence of the differential $f_X(X_o; \Delta X)$ at a point X_o implies the existence of all the partial differentials f_{x_k} at X_0. This is shown in:

Theorem (14.741) Let H(X) be a homogeneous polynomial of degree n in the variable $X \equiv (x_1,\cdots,x_n)$ on space $\prod_1^k E_i$ to E'. Then H(X) is a homogeneous polynomial of degree n on any subspace of $\prod_1^k E_i$ obtained by equating to zero certain of the variables x.

Proof: Let Y denote a typical element of the subspace of $\prod_1^k E_i$ obtained by replacing by zero certain of the x's, say x_{p+1}, $x_{p+2},\cdots,x_k$, so that $Y \equiv (x_1,\cdots,x_p) \equiv (x_1,\cdots,x_p,0,0,\cdots,0)$. Now since Y is defined as a point of a subspace of $\prod_1^k E_i$, H(Y) must be continuous, $H(Y+\lambda Y_0)$ must be a polynomial in λ and we must have $H(\lambda Y) = \lambda^n H(Y)$. These three properties characterize H(Y) as a homogeneous polynomial of degree n.

The same procedure may be carried out for any other selection of the subset of x's.

Theorem (14.75) Let f(X) be analytic r'(0) at X = 0, where X is an element $(x_1,x_2,\cdots,x_k)$ of $\left(\prod_1^k E_i(C)\right)_0$ to E'(C)$_\lambda$ complete then it is analytic on any subset of $\prod E_i$ obtained by giving certain of the x's fixed values inside the region of analyticity of X.

Proof: Let $x_{p+1}^0,\cdots,x_n^0$ be fixed values of $x_{p+1},\cdots,x_n$, where $x_i^0 < r'(0)$. Then by Theorem (14.65) f(X) may be expanded about the point $(0,0,\cdots,0,\ x_{p+1}^0,\cdots,x_n^0)$ in the form

$$f(X) = \sum H_n(x_1,\cdots,x_p,x_{p+1}-x_{p+1}^0,\cdots,x_k-x_k^0)$$

If now $x_{p+1},\cdots,x_n$ take on the values $x_{p+1}^0,\cdots,x_n^0$, H(X) will by (14.741) become a homogeneous polynomial of degree n in $(x_1,\cdots,x_p)$. Furthermore, denoting by $\overline{m}H_n$ the modulus of the polynomial obtained by fixing these variables we have

$$\overline{m}H_n \equiv \max_{\|(x_1,\cdots,x_{p+1},0,0,\cdots,0)\|=1} H(x_1,\cdots,x_p,0,0,\cdots)$$

$$\leq mH_n$$

<u>Theorem (14.76)</u> If $f(X) \equiv f(x_1,x_2,\cdots,x_n)$ on $\left(\prod_{i=1}^{n} E_i\right)_0$ to E' has a total differential at X_0, then the partial differentials f_{x_i} exist and are given by

$$f_{x_k}(X_0;\Delta x_k) = f_X(X_0;(\Delta X)_k)$$

where $(\Delta X)_k = (0,0,\cdots,\Delta x_k,0,\ldots,0)$.

<u>Proof:</u> The condition for the existence of $f_X(X_0;\Delta X)$ is that $\epsilon(\Delta x)$ defined by

$$\epsilon(\Delta X) \equiv \frac{1}{\|X\|}\; f(X_0+\Delta X) - f(X_0) - f_X(X_0;\Delta X) \qquad (1)$$

$$(\|\Delta X\| \neq 0)$$

$$\epsilon(0) \equiv 0$$

be continuous at $\Delta X = 0$, and that $f_X(X_0;\Delta X)$ be linear continuous in ΔX. This must be true for all ΔX and hence in particular for $(\Delta x_1,0,0,\cdots,0)$. The codntions above then give that $\epsilon_1(\Delta x_1)$ defined by $\epsilon_1(\Delta x_1) \equiv \epsilon(\Delta x_1,0,0,\cdots,0)$

is equal to $\dfrac{1}{\|\Delta x_1\|}\; f(x_1^0 + \Delta x_1, x_2^0,\cdots,x_n^0) - f(X) - f_{x_1}(X;\Delta x_1)$

and is equal to 0 for $\Delta x_1 = 0$. But since f_{x_1} is linear and continuous in Δx_1, it must be the partial differential.

Theorem (14.77) Let $g(x)$ be a function on E_0 of E to E'_0 of E', differentiable at x_0. Let $f(y)$ be a function in E'_0 to E'' differentiable at $y_0 \equiv g(x_0)$. Then $F(x) \equiv f(g(x))$ is differentiable at x_0 and the differential is given by $f_g[g; g_x(x_0;\Delta x)]$

Proof: Define

$$\epsilon_1(\Delta y) \equiv \frac{1}{\|\Delta y\|}\left[f(y_0+\Delta y) - f(y_0) - f_y(y_0;\Delta y)\right] \quad (\Delta y \neq 0)$$

$$\epsilon_1(0) \equiv 0$$

$$\epsilon_2(\Delta x) \equiv \frac{1}{\|\Delta x\|}\left[g(x_0+\Delta x) - g(x_0) - g_x(x_0;\Delta x)\right] \quad (\Delta x \neq 0)$$

$$\epsilon_2(0) \equiv 0$$

so that by the definition of differentials $\epsilon_1(\Delta y)$, $\epsilon_2(\Delta x)$ are continuous at $\Delta y = 0$ and $\Delta x = 0$, respectively.

Now
$$F(x_0+\Delta x) - F(x_0) = f(g(x_0+\Delta x)) - f(y_0)$$

$$= f\big(g(x_0+\Delta x) - g(x_0) + y_0\big) - f(y_0)$$

$$= f\big(y_0 + g_x(x_0;\Delta x) + \|\Delta x\|\cdot\epsilon_2(\Delta x)\big) - f(y_0) \qquad (1)$$

$$= f_y\big(y_0; g_x(x_0;\Delta x)\big)$$

$$+\|\Delta x\|\cdot\left\{\frac{\|\Delta y\|}{\|\Delta x\|}\ \epsilon_1(\Delta y) + f_y\big(y_0;\epsilon_2(\Delta x)\big)\right\}$$

where
$$\Delta y \equiv g_x(x_0;\Delta x) + \|\Delta x\|\cdot\epsilon_2(\Delta x)$$

Observing the linearity in Δx of $g_x(x_0;\Delta x)$ we see that $\frac{\|\Delta y\|}{\|\Delta x\|}$ remains bounded as δx tends to zero as does the bracketed expression in equation (1). Therefore the derivative of $F(x)$ exists at x_0 and we have

$$F_x(x_0;\Delta x) = f_y\big(g(x_0); g_x(x_0;\Delta x)\big)$$

<u>Theorem (14.78)</u> If $f(x) \equiv \left(f_1(x), f_2(x), \cdots, f_n(x)\right)$ is the composite of n differentiable functions then

$$df(x) = \left[df_1(x), df_2(x), \cdots, df_n(x)\right]$$

Proof: Let $\Delta f_i(x) = f_i(x + \Delta x) - f_i(x)$, and let

$$\Delta f_i(x) - df_i(x) \equiv \|\Delta x\| \epsilon_i(\Delta x)$$

Then clearly $\Delta f(x) - df(x) = \|\Delta x\| \left[\epsilon_1(\Delta x), \epsilon_2(\Delta x), \cdots, \epsilon_n(\Delta x)\right]$

$$= \|\Delta x\| \epsilon(\Delta x)$$

where $\|\epsilon(\Delta x)\| = \max_i \|\epsilon_i(\Delta x)\|$

14.8 <u>A Differential Equation Problem in Vector Space</u>

In this last section we shall devote our attention to the solution of the differential equation $dy = f(x, y; \Delta x)$. To state the problem exactly; let $f(x, y, \Delta x)$ be an function of the variable triplet $(x, y, \Delta x)$ analytic at $(x_0, y_0, 0)$, on $(E_1 E_2 E_1)_0$ to E_2 and linear in Δx. It is possible to define a function $y(x)$ analytic at x_0 on $(E_1)_0$ to E_2 such that $y(x_0) = y_0$ and such that

$$y_x(x; \Delta x) = f\left[x, y(x), \Delta x\right]$$

is satisfied identically in $x, \Delta x$ inside a suitably small neighborhood of the point $(x_0, y_0, \Delta x)$.

In the discussion we shall write $x_0 = y_0 = 0$ simply to save writing $x - x_0$ and $y - y_0$. <u>Until further mention is made of the matter, we shall consider that all the spaces are spaces $E(C)$.</u> In order to center the discussion better, let us derive a number of necessary conditions on the solutions of such equations.

Theorem (14.81) If $U(x,y)$ is analytic on $(E_1E_2)_0$ to E', and if $y(x)$ is analytic at $x = 0$, on $(E_1)_0$ to E_2, and is such that $y(0) = 0$, then $U(x,y(x))$ is analytic on $(E_1)_0$ to E'. Furthermore, its differential is given by $U_x[x,y;\Delta x] + U_y[x,y;y_x(x;\Delta x)]$

Proof: Consider the valuable composite variable $X \equiv (x,y)$ of E_1E_2. By Theorem (14.72) the function $X(x) \equiv [x,y(x)]$ is analytic at $x = 0$ on $(E_1)_0$ to E_1E_2. Since $y(0) = 0$, it is always possible to restrict x so that $\|X(x)\|$ is suitably bounded and less than the radius of analyticity of $U(X) \equiv U(x,y)$. This then satisfies the conditions of Theorem (14.67). The differential with respect to x of $U(X)$ is by Theorem (14.77) equal to

$$U_X[X;X_x(x;\Delta x)] \tag{1}$$

But since

$$X = [x,y(x)]$$

we have from Theorem (14.78)

$$X_x(x;\Delta x) = [\Delta x, y_x(x;\Delta x)]$$

Hence (1) becomes $U_x[x,y;\Delta x] + U_y[x,y;y_x(x;\Delta x)]$, which completes the proof.

Theorem (14.82) Let $f(x,y;\Delta x)$ be analytic in $(x,y;\Delta x)$ on $E_1E_2E_1$ to E_2 at $(0,0;0)$. Let it be linear in Δx. Then if there exists a function $y(x)$ analytic at $x = 0$ satisfying

$$y(0) = 0$$

$$y_x(x;\Delta x) = f[x,y(x);\Delta x]$$

identically in $(x,\Delta x)$ it must satisfy the condition that

$$f_x[x,y(x),\Delta_1 x;\Delta_2 x] + f_y[x,y(x),\Delta_1 x;f(x,y(x),\Delta_2 x)]$$

be symmetric in $\Delta_1 x, \Delta_2 x$.

Proof: By theorem (14.81) we have

$$y_{xy}(x,\Delta_1 x,\Delta_2 x) = f_x(x,y;\Delta_1 x,\Delta_2 x) + f_y[x,y;\Delta_1 x;f(x,y;\Delta_2 x)] \quad (1)$$

Since the left hand side is symmetric in $\Delta_1 x, \Delta_2 x$ the right hand side must be also.

As in the case of the classical theory it is, of course, not necessary that the relation (1) above be satisfied identically in $x,y,\Delta_1 x,\Delta_2 x$ but only in $(x,\Delta_1 x,\Delta_2 x)$ when y has been replaced as a function of x.

Let us now fix once and for all a function $f(x,y;\Delta x)$ which satisfies the restrictions placed on f in (14.82) and which in addition satisfies the equation (1) above identically in $x,y,\Delta_1 x,\Delta_2 x$.

The Symbol δ

Let $U(\mathfrak{X})$ denote any function differentiable with respect to $\mathfrak{X} \equiv (x,y)$. Let $U_x(x,y;\Delta x) + U_y(x,y;\Delta y)$ be its differential. We shall define the symbol δ by means of the equation

$$\delta U(x,y) \equiv U_x(x,y;\Delta x) + U_y\big(x,y;f(x,y;\Delta x)\big)$$

By $\delta_1 U$, $\delta_2 U$, etc. we shall naturally mean the same function with Δx replaced by $\Delta_1 x, \Delta_2 x, \ldots$ If U depends upon other increments $\Delta_i x$, they are to be treated as constant.

In terms of this symbol the symmetry ~~as constan~~t condition in (1) of the preceding theorem may be written

$$\delta_2 f(x,y,\Delta_1 x) = \delta_1 f(x,y,\Delta_2 x)$$

Theorem (14.83) If $U(x,y)$ is a function analytic in (x,y) at $(x,y) = (0,0)$ on $(E_1 E_2)_0$ to E', then $\delta U(x,y)$ is analytic in (x,y) at $(0,0)$

Proof: Let $X \equiv (x,y)$ denote a point of the composite space E_1E_2. Write $U(x,y) \equiv U(X)$. By Theorem (14.74) $U_X(X,\Delta X)$ is analytic in the composite pair $(X,\Delta X)$. Now according to the definition of δU, it is obtained simply by writing $f(X;\Delta_1 x)$ in lieu of $\Delta_1 y$ in $U_X(X,\Delta X)$. If we make this replacement in ΔX we have

$$\Delta X \Big|_{\Delta y = f(X;\Delta x)} = (\Delta x, \Delta y)\Big|_{\Delta y = f(X;\Delta x)} = [\Delta x, f(X;\Delta x)] \quad (1)$$

Now by (14.72) the right hand member of (1) is, in view of the fact that $f(X;\Delta x)$ is analytic and Δx is a constant, analytic in X. By (14.72) again, the variable pair $Z \equiv (X,\Delta X)\Big|_{\Delta y = f(X;\Delta x)}$ is analytic in X, so that

$$U_X(X,\Delta X)\Big|_{\Delta y = f(X;\Delta x)} = U_X(Z)$$

is an analytic function of an analytic function Z. Furthermore since

$$\|Z\| = \max(\|X\|, \|\Delta X\|)$$

we can suitably bound the quantity $\|X\|$ and, through the linearity of ΔX in Δx, suitably bound $\|Z\|$ so that it always lies inside the radius of analyticity, of the function $U_X(Z)$. Therefore we may apply Theorem (14.67) to $U_X[Z(X)]$ and prove $U_X(Z)$ is analytic in X.

Theorem (14.84) If $U(x,y)$ is analytic on $(E_1E_2)_0$ to E' at $(x,y) = 0$, then

$$\delta_2\delta_1 U = \delta_1\delta_2 U$$

Using the work of the preceding theorem, we have

$$\delta_1 U = U_X(X;\Delta_1 X)\Big|_{\Delta y = f(X,\Delta_1 x)}$$

<u>Proof:</u> Let us denote by Δ^*X the function obtained by replacing Δy in ΔX by $f(X,\Delta x)$. Let us first calculate the differential of Δ_1^*X with respect to X and an increment $\Delta_2 X$. We have

$$\Delta_1^*X = [\Delta_1 x, f(X,\Delta_1 x)]$$

and using (14.78) $(\Delta_1^*X)_X[\Delta_2 X] = [0, f_X(X,\Delta_1 x;\Delta_2 X)]$ (1)

Replacing $\Delta_2 y$ by $f(X,\Delta_2 x)$, we obtain

$$(\Delta_1^*X)_X[\Delta_2^*X] = [0, f_X(X,\Delta_1 x;\Delta_2^*X)] \qquad (2)$$

$$= [0, \delta_2 f(X), \Delta_1 x]$$

Now $\delta_1 U(X) = U_X(X;\Delta_1^*X)$

Taking the differential of this with respect to X and an increment $\Delta_2 X$

we have $\delta_1 U(X)_X = U_{XX}(X;\Delta_1^*X,\Delta_2 X) + U_X(X;[\Delta_1^*X]_X)$ (3)

The last step is justified on the grounds that $U_X(X;\Delta_1^*X)$ is linear in its second argument and quâ that argument is its own differential. Our result then becomes on starring $\Delta_2 X$ and using (2)

$$\delta_2\delta_1 U(X) = U_{XX}(X;\Delta_1^*X,\Delta_2^*X) + U_X[X;[0,\delta_2 f(X),\Delta_1 x]]$$

The first term is symmetric since it is the ordinary second differential of U and the second is symmetric because of the hypotheses on $f(X,\Delta_1 x)$.

<u>Theorem (14.85)</u> If $\delta_1\delta_2\cdots\delta_n$ represent operators associated with n increments $\Delta_1 x,\cdots,\Delta_n x$ then

$$\delta_n\delta_{n-1}\cdots\delta_3\delta_2\ f(X;\Delta_1 x)$$

is symmetric in all the increments $\Delta_1 x, \Delta_2 x, \cdots, \Delta_n x$.

Proof: We prove this by induction on n. The theorem is true by hypothesis on f for n = 1. Assume the theorem true for n-1. Take

$$U(X) \equiv \delta_{n-2} \cdots \delta_2 f(X;\Delta_1 x)$$

and apply Theorem (14.85). There results

$$\delta_n \delta_{n-1} U(X) = \delta_n \delta_{n-1} \cdots \delta_2 f(X;\Delta_1 x)$$

But since the left hand side is symmetric in $\delta_n \delta_{n-1}$, so is the right, for, by the induction hypothesis $\delta_{n-1} \cdots \delta_1$ were commutable.

Theorem (14.86) Let $f(x,y,\Delta x)$ be analytic in $(x,y\Delta x)$ at $(0,0,0)$ on $(E_1 E_2 E_1)(C)$ to a complete $E_2(C)$ and linear in Δx. Then if we have identically in $(x,y,\Delta x)$

$$\delta_2 f(x,y;\Delta_1 x) = \delta_1 f(x,y;\Delta_2 x)$$

there exists a unique analytic function $y(x)$ satisfying identically in $(x,\Delta x)$ the relation

$$y_x(x;\Delta x) = f\big(x,y(x);\Delta x\big) \tag{1}$$

and reducing to zero for $x = 0$.

Proof: A repeated application of Theorem (14.81) shows that if an analytic solution $y(x)$, the equations (1) exists, then its successive derivatives must satisfy the relations

$$y_x(x;\Delta_1 x) = f\big(x,y(x);\Delta_1 x\big)$$

$$y_{x^2}(x;\Delta_1 x,\Delta_2 x) = \delta_2 f(x,y;\Delta_1 x)$$

$$\cdots\cdots\cdots\cdots$$

$$y_{x^n}(x;\Delta_1 x,\Delta_2 x,\cdots,\Delta_n x) = \delta_n \delta_{n-1} \cdots \delta_2 f(x,y;\Delta_1 x) \tag{2}$$

Conversely if a function is analytic and satisfies the condition $y(0) = 0$ as well as the relation (2) then it is a solution of (1) since in particular it must satisfy the first equation (2).

Now
$$\delta^n f(X) \equiv \delta_n \delta_{n-1} \cdots \delta_2 f(x,y;\Delta_1 x)$$

is a symmetric multilinear form in $\Delta_1 x, \cdots, \Delta_n x$ since it is manifestly linear in $\Delta_n x$, and is symmetric by Theorem (14.85). We shall prove that a solution of (1) is given by

$$y(x) = \sum_1^\infty h_n(x) \qquad (3)$$

where
$$h_n(x) = \left.\frac{\delta^n f(0)}{n!}\right|_{\Delta_i x = x}$$

The hypothesis that $f(x,y;\Delta x)$ is analytic in the variable $(x,y,\Delta x)$ as a unit gives us easily that f is expansible in the form

$$f(x,y,\Delta x) \equiv f(X;\Delta x) = \sum_{n=1}^\infty H_n(X,\Delta x) \qquad (4)$$

where X is the composite (x,y) and where $H_n(X,\Delta x)$ is a homogeneous polynomial of degree n-1 in X and linear in Δx.

To see this, let $h_n(x,y,\Delta x)$ be the homogeneous polynomial of degree n in $(x,y,\Delta x)$ in the expansion of f. Then

$$h_n(\lambda x,\lambda y, \mu \Delta x) = h_n[\lambda(x,y,0) + \mu(0,0,\Delta x)]$$

is a homogeneous polynomial of degree n in (λ,μ). But since f and therefore h_n is linear in Δx, the only term possible in this polynomial is that in which the degree of μ is 1 and that of λ is n-1. Therefore we have

$$h_n(\lambda x,\lambda y, \mu \Delta x) = \lambda^{n-1}\mu\, h_n(x,y,\Delta x) \qquad (5)$$

By (13.93) $h_n(x,y,\Delta x)$ is a polynomial in $X \equiv (x,y)$ and by (5) it is of degree $(n-1)$. Hence the $H_n(X;\Delta x)$ of (5) is given by

$$H_n(X,\Delta x) = h_n(x,y,\Delta x)$$

Now let M be a dominant for the series (4) so that

$$\| H_n(X;\Delta x) \| < \frac{M}{\rho^{n-1}} \|X\|^{n-1} \|\Delta x\| \tag{6}$$

Let us define[§] $L_k^i(X)$ for $i > 1$ as the form of degree k in the expansion qua(X) of $\delta^i f$ about $X = 0$, and for the purposes of symmetry define $L_k^1(X)$ as the term of degree k in the expansion (4), so that $L_k^1(X) = H_{k+1}(X)$

Now since

$$\delta^i f = \sum_0^\infty L_k^i(X)$$

we have

$$d\delta^i f = \sum_1^\infty dL_k^i(X;\Delta X)$$

and

$$\delta^{i+1} f = \sum_1^\infty dL_k^i(X;\Delta^*_{i+1} X) \tag{7}$$

where $\Delta^*_{i+1} X$ is as defined previously the composite function

$$\left(\Delta_{i+1} x, f(X, \Delta_{i+1} x)\right)$$

We may write $\Delta^*_{i+1} X$ in the form of an analytic expansion as

$$\Delta^*_{i+1} X = \left(\Delta_{i+1} x, L_0(X)\right) + \sum_1^\infty \left(0, L_k(X)\right) \tag{8}$$

Using (8) we proceed to pick out the terms of degree k in (7). We obtain

$$L_k^{i+1}(X) = \sum_{r=1}^{k} dL_r^i \left[X; \left(0, L_{k-r+1}^1(X;\Delta_{i+1} x)\right)\right] + dL_{k+1}^i\left[X; \Delta_{i+1} x, L_0^1(X;\Delta_{i+1} x)\right] \tag{9}$$

[§] It must be remembered of course that the expression L(X) involves Δx as well as X.

Let us make the following observation on the differential of a homogeneous polynomial of degree r. We have by (13.82)

$$\|dh(x)\| = r\|h(\Delta x, x, \cdots, x)\| = r\, m_r h_r \,\|x\|^{r-1}\|\Delta x\|$$
$$= r\, mh_r \frac{r^r}{r!}\|x\|^{r-1}\|\Delta x\| < r\, e\, mh_r \|x\|^{r-1}\|\Delta x\| \tag{10}$$

In what follows we shall assume that the increments $\Delta_1 x, \Delta_2 x, \cdots, \Delta_h x, \cdots$ are fixed and that $\|\Delta_i x\|$ is bounded, say $\|\Delta_i x\| < \omega$. It will be understood that the modulus mL^i_k stands for the modulus quâ X. We shall write $A^i_k = mL^i_k$ for $(i \geq 1;\ k > 0)$ and for $(i > 1;\ k \geq 0)$ and shall define A^1_0 as the greater of ω and max $\frac{\|L^1_0\|}{\|\Delta x\|}\omega$

We now obtain from (9)

$$\| L^{i+1}_k(X) \| \leq \sum_{r=1}^{k+1} r\, e\, A^i_r\, A^1_{k-r+1} \|X\|^k \tag{11}$$

which gives
$$A^{i+1}_k \leq \sum_{r=1}^{k+1} e\, r\, A^i_r\, A^1_{k-r+1} \tag{12}$$

We now make the following observation: let B^1_k $(k = 0,1,2,\ldots)$ be positive numbers selected at random so that $B^1_k \geq A^1_k$. Let B^i_k $(i > 1)$ be defined inductively by (12) with its inequality sign replaced by equality. Then $B^i_k \geq A^i_k$ all (i,k). This is easily proved by induction; for, if we have up to some i, $B^i_k \geq A^i_k$, then

$$B^{i+1}_k = \sum_{r=1}^{k+1} e\, r\, B^i_r\, B^1_{k-r+1} \geq \sum_{r=1}^{k+1} e\, r\, A^i_r\, A^1_{k-r+1} \geq A^{i+1}_k \tag{13}$$

Consider now a set of power series* $\phi_i(z) = \sum_{r=0}^{\infty} B^i_r\, z^r$

*Their convergence does not concern us.

The modified expression (12) then gives that $\phi_{i+1}(z)$ is e times the Cauchy product of $\phi_1(z)$ and the derived series of $\phi_i(z)$; i.e.,

$$\phi_{i+1}(z) = e\, \phi_1(z)\cdot \phi_i'(z)$$

$$= e\left(\sum_{r=0}^{\infty} B_r^1 z^r\right)\cdot\left(\sum_{r=0}^{\infty} r\, B_r^i z^r\right)$$

$$= \sum_{k=0}^{\infty}\left(\sum_{r=1}^{k+1} e\, r\, B_r^i\, B_{k-r+1}^1\right) z^r$$

Now put $B_k^1 = \bar{M}\, b^k$ where $\bar{M}$ is the greater of A_0^1 and M of equation (4) and where $b = 1/\rho$, being that of equation (4). With this choice we have

$$\phi_1(z) = (1 - bz)^{-1} M$$

$$\phi_2(z) = (1 - bz)^{-3} (Mbe)\cdot M$$

$$\cdots\cdots$$

$$\phi_n(z) = 1\cdot 3\cdot 5\cdots 2n-1\cdot (1 - bz)^{-(2n+1)} (Mbe)^n M$$

From this we obtain in particular

$$B_0^n = 1\cdot 3\cdot 5\cdots(2n-1)\cdot(\bar{M}be)^n\bar{M}$$

$$< 2\cdot 4\cdot 6\cdots 2n\cdot(\bar{M}be)^n\bar{M}$$

$$= n!\,(2\bar{M}be)^n\bar{M}$$

Now going back to the original set up, the expressions $\delta^n f(0) = L_0^n(0)$ are multilinear in $\Delta_1 x, \Delta_2 x, \cdots, \Delta_n x$ and with $\|\Delta_i x\| < \omega$ we have

$$\| L_0^n(0) \| < A_0^n \leq B_0^n < n!\,(2\bar{M}be)^n\bar{M}$$

Therefore the modulus of $h_n(x)$ of equation (3) satisfies

$$mh_n < \bar{M}\left(\frac{2\bar{M}be}{\omega}\right)^n$$

and
$$\lim_{n\to\infty} (mh_n)^{-\frac{1}{n}} \geqq \frac{\omega}{2\bar{M}be} > 0$$

Therefore the series $\sum_0^\infty h_n(x)$ in (3) defines an analytic function $y(x)$ such that $y(0) = 0$ and such that its differentials satisfy the sequence of relations (2) at the point $x = 0$.

The function $g(x;\Delta x) \equiv y_x(x;\Delta x) - f\big(x,y(x);\Delta x\big)$ is an analytic function of x which together with all its differentials vanishes at $x=0$. Since in any analytic function $F(x) = \sum_0^\infty H_n(x)$, the differentials $d^nF(X)$ satisfy the relation

$$\left.\frac{d^nF(0)}{n!}\right]_{\Delta_1 x = x} = H_n(x)$$

the function $g(x;\Delta x)$ must be identically zero. Therefore $y(x) = \sum_0^\infty h_n(x)$ is an analytic solution. It is furthermore unique since the calculation of $h_n(x)$ is uniquely indicated by the sequence of relations (2), together with the condition $y(0) = 0$. The theorem is therefore proved.

It may be noted in concluding this section that the formal steps of Theorem (14.86) and the inequalities used are independent of the fact that the spaces are E(C), except in so far as Theorem (14.67) was used to justify the manner of obtaining (9). We believe that the analogue of (14.67) may be proved for spaces E(R) at least inside a suitably restricted region, but we have not yet succeeded in doing so.

Part 2.

VARIOUS THEOREMS ON THE REPRESENTATION OF FUNCTIONAL FORMS AND TRANSFORMATIONS BY MEANS OF STIELTJES INTEGRALS

21. Introduction to Part Two.

In this second part we shall give a proof of the representability by means of multiple Stieltjes integrals of the most general multilinear form on the space of continuous functions to the space of real numbers. We shall also give a proof of the generalization to Riemann-Stieltjes integrals of Arzela's theorem on necessary and sufficient conditions for term by term integrability of a convergent sequence of Riemann integrable functions.

The proofs given are by elementary methods; i.e., without the use of any but the more elementary theorems of point set theory. We shall assume familiarity with the definition of and the more or less standard properties of simple Riemann - Stieltjes integrals and of functions of limited variation in a single variable. Also we shall assume the fundamental theorem of Frederic Riesz on the representation of linear continuous functions by means of simple Riemann- Stieltjes integrals. The majority of the assumed theorems and definitions are stated in section 22.

22. Preliminaries on Notation and Results Assumed

22.1. Subdivisions of a linear interval:

Let (a,b) denote a fixed closed interval of the real x-axis. By a subdivision of (a,b) we shall understand division of (a,b) into a finite number, n, of closed intervals (x_i, x_{i+1}), where $(i = 0,1,\cdots,n-1)$, and where $a = x_0 < x_1 < \cdots < x_n = b$. Such a subdivision we shall designate by a single symbol Δ. In general, $\Delta, \Delta', \Delta'', \cdots$, etc. will denote distinct subdivisions of (a,b). Two subdivisions Δ and Δ' are said to be equal when they have identical points of division; this equality is expressed by $\Delta = \Delta'$. If every point of division of a subdivision Δ is also a point of division of a second subdivision Δ', then Δ is said to be included in Δ' and the fact is expressed by the formula $\Delta \leqq \Delta'$. If $\Delta \leqq \Delta'$ and if Δ' contains a point of division which is not a point of division of Δ, then Δ is said to be properly included in Δ' and the fact is expressed by $\Delta \subset \Delta'$. If Δ and Δ' are subdivisions and Δ'' is a subdivision which has as its points of division all the points of division of Δ and Δ', and those alone, Δ'' is called the superposition or sum of Δ and Δ'; we denote the superposition by $\Delta'' = \Delta + \Delta'$. By the norm or modulus of a subdivision Δ we mean the maximum of the lengths of its intervals. The symbol $\|\Delta\|$ is used to designate the norm of Δ. The necessary properties of these relationships among subdivisions of (a,b) are given in the following:

Theorem (22.11): Subdivisions of (a,b) being denoted by $\Delta, \Delta', \Delta''$, the following relations hold:

(I) If $\Delta \leqq \Delta'$ and $\Delta' \leqq \Delta''$ then $\Delta \leqq \Delta''$

(II) If $\Delta \subset \Delta'$ and $\Delta' \leqq \Delta''$ or if $\Delta \leqq \Delta'$ and $\Delta' \subset \Delta''$ then $\Delta \subset \Delta''$

(III) $\Delta + \Delta' = \Delta' + \Delta$

(IV) $(\Delta + \Delta') + \Delta'' = \Delta + (\Delta' + \Delta'')$

(V) If $\Delta'' = \Delta + \Delta'$ then $\Delta \subseteq \Delta''$ and $\Delta' \subseteq \Delta''$

(VI) If $\Delta \subseteq \Delta'$ then $\|\Delta\| \geqq \|\Delta'\|$

(VII) $\|\Delta + \Delta'\| \leqq \|\Delta\|$; $\|\Delta + \Delta'\| \leqq \|\Delta'\|$;

__Proof__: Properties (I),$\cdots$,(V) are immediate consequences of the definitions. To show property (VI) it is sufficient to observe that every interval of Δ' lies inside or coincides with some interval of Δ. Property (VII) is a consequence of properties (V) and (VI).

22.2. Subdivisions of a Rectangular Hyperparallelopiped.

Consider a rectangular cartesian hyperspace of n dimensions whose points are given by coordinates $x^1,\cdots,x^n$. Let H_n be a fixed rectangular hyperparallelopiped whose projection on the x^i-axis is the closed interval (a^i,b^i) $(a^i \quad b^i;\ i = 1,2,\cdots,n)$. In brief, H_n is the set of points whose coordinates satisfy the inequalities $a^i \leqq x^i \leqq b^i$. Now for each value of i $(i = 1,2,\cdots,n)$ let Δ^i designate a subdivision of the linear interval (a^i,b^i) by points $x^i_0,\cdots,x^i_{m_i}$, where $a^i = x^i_0 < x^i_1 < \cdots < x^i_{m_i} = b^i$. H_n is by this process divided into closed hyper-rectangular cells $m_1 \times m_2 \times \cdots \times m_n$ in number, a given cell being made up of those points whose coordinates satisfy the inequalities $x^i_{j_i} \leqq x^i \leqq x^i_{j_i+1}$, $(i = 1,2,\cdots,n;\ j_i = 1,2,\cdots,m_i)$. Such a subdivision of H_n we shall designate either by a single symbol Δ or by $\Delta^1\cdot\Delta^2\cdots\Delta^n$, where $\Delta^1, \Delta^2,\cdots,\Delta^n$ are the subdivisions of $(a^1,b^1),\cdots(a^n,b^n)$ which produce Δ. The equivalence of the symbolisms will be expressed by writing $\Delta \equiv \Delta^1\Delta^2\ldots\Delta^n$. Two subdivisions $\Delta \equiv \Delta^1\Delta^2\ldots\Delta^n$ and

$\Delta' \equiv \Delta^{1'} \cdot \Delta^{2'} \cdots \Delta^{n'}$ of H_n are said to be equal if $\Delta^i = \Delta^{i'}$ $(i = 1,2,\cdots n)$. A subdivision $\Delta \equiv \Delta^1 \cdot \Delta^2 \cdots \Delta^n$ is said to be included in $\Delta' \equiv \Delta^{1'}\Delta^{2'} \cdots \Delta^{n'}$ if $\Delta^i \leqq \Delta^{i'}$ $(i = 1,2,\cdots,n)$. This fact is denoted by $\Delta \leqq \Delta'$. Δ is said to be properly included in Δ' if at least one of the inclusions $\Delta^i \leqq \Delta^{i'}$ is proper; in this case we shall write $\Delta \subset \Delta'$. The sum or superposition Δ'' two subdivisions $\Delta \equiv \Delta^1 \cdot \Delta^2 \cdots \Delta^n$, $\Delta' \equiv \Delta^{1'}\Delta^{2'} \cdots \Delta^{n'}$ if H_n is defined by $\Delta'' \equiv (\Delta^1 + \Delta^{1'})(\Delta^2 + \Delta^{2'}) \cdots (\Delta^n + \Delta^{n'})$ and is written $\Delta'' = \Delta + \Delta'$. Finally the modulus or norm $\|\Delta\|$ of $\Delta \equiv \Delta^1 \Delta^n$ of H_n is defined as the greatest of $\|\Delta'\|, \cdots, \|\Delta^n\|$. We have, relative to these relationships

Theorem (22.21): If $\Delta \equiv \Delta^1 \cdots \Delta^n$; $\Delta' \equiv \Delta^{1'} \cdots \Delta^{n'}$; and $\Delta'' \equiv \Delta^{1''} \cdots \Delta^{n''}$ are subdivisions of H_n, then the following properties hold:

(1) If $\Delta \leqq \Delta'$ and $\Delta' \leqq \Delta''$ then $\Delta \leqq \Delta''$

(II) If $\Delta \subset \Delta'$ and $\Delta' \leqq \Delta''$ or $\Delta \leqq \Delta'$ and $\Delta' \subset \Delta''$ then $\Delta \subset \Delta''$.

(III) $\Delta + \Delta' = \Delta' + \Delta$

(IV) $(\Delta + \Delta') + \Delta'' = \Delta + (\Delta' + \Delta'')$

(V) If $\Delta'' = \Delta + \Delta'$ then $\Delta \leqq \Delta''$ and $\Delta' \leqq \Delta''$

(VI) If $\Delta \leqq \Delta'$ then $\|\Delta\| \geqq \|\Delta'\|$

(VII) $\|\Delta + \Delta'\| \leqq \|\Delta\|$; $\|\Delta + \Delta'\| \leqq \|\Delta'\|$

Proof: Each of the properties is an immediate consequence of the definitions and of the corresponding property shown for subdivisions of linear intervals in Theorem (22.11).

22.3. Differences of Functions with Respect to Subdivisions.

Let $\phi(x)$ be any real function of the real variable x defined for $a \leqq x \leqq b$; let Δ be an arbitrary subdivision of (a,b). Let $a = x_0 < x_1 < \cdots < x_m = b$ be the points of division of Δ. We shall define $\Delta_i\phi$, the difference of ϕ with respect to the interval (x_i, x_{i+1}) of Δ, by

$\Delta_i \phi \equiv \phi(x_{i+1}) - \phi(x_i)$.

Let $\phi(x^1,\cdots,x^n)$ be a real function of $x^1,\cdots,x^n$ defined over a hyperparallelopiped H_n. Let $\Delta \equiv \Delta^1\Delta^2\cdots\Delta^n$ be a subdivision of H_n. Using the notation of (22.2) we shall refer to the cell of whose projection on the x^i-axis is the interval $(x^i_{j_i}, x^i_{j_{i+1}})$ $(i = 1,2,\cdots,n;\ j_i = 0,1,\cdots,m_{i-1})$ as the cell $(j_1,j_2,\cdots,j_n)$. We shall define the difference, $\Delta_{j_1 j_2,\cdots j_n}\phi$, of $\phi(x^1,\cdots,x^n)$ with respect to the cell $(j_1,j_2,\cdots,j_n)$ by means of the formula

$$\Delta_{j_1,j_2,\cdots j_n}\phi \equiv \Delta^1_{j_1}\left[\Delta^2_{j_2}\left[\cdots\Delta^{n-1}_{j_{n-1}}\ \Delta^n_{j_n}\phi(x^1,\cdots,x^n)\right]\cdots\right]$$

where the symbol $\Delta^i_{j_i}$ is regarded as an instance of the operator Δ_j defined in the immediately preceding paragraph, and is interpreted as operating upon all that follows it quâ function of x^i. For example, if $n = 2$, we have

$$\Delta_{j_1 j_2}\phi \equiv \Delta^1_{j_1}\left[\Delta^2_{j_2}\phi(x^1,x^2)\right] = \Delta^1_{j_1}\left[\phi(x^1,x^2_{j_2+1}) - \phi(x^1,x^2_{j_2})\right]$$

$$= \phi(x^1_{j_1+1},x^2_{j_2+1}) - \phi(x^1_{j_1},x^2_{j_2+1}) - \phi(x^1_{j_1+1},x^2_{j_2}) + \phi(x^1_{j_1},x^2_{j_2}).$$

22.4. Sets of Linear Intervals

Consider again the closed linear interval (a,b). Let I_1, $I_2,\cdots,I_k$ denote k closed intervals contained in (a,b). We shall employ the common usage of point-set theory in defining J, the sum of $I_1,I_2,\cdots,I_k$, as the set of points contained in at least one of the intervals $I_1,\cdots,I_k$. It is shown as one of the elementary theorems of point-set theory that such a set J is made up of a finite number of closed, non-overlapping sub-intervals of (a,b). It is convenient to write $J = I_1 + I_2 + \cdots + I_k$ and the theorem cited states that there always exists a second set, $I'_1,\cdots,I'_m$ of non-overlapping closed intervals such that $J = I'_1 + \cdots + I'_m$.

Let J be the sum of the non-overlapping closed intervals $I_1, I_2, \cdots, I_m$, and let Δ be a subdivision of (a,b). If each end point of the intervals I is also a point of division of Δ then we shall say that J is included by Δ and write $J \subset \Delta$. It is clear that if $J \subset \Delta$ then J may be regarded as the sum of those intervals (x_i, x_{i+1}) of Δ which are contained in J.

By the sum J + J' of two such sets of intervals we shall mean, as is usual, the set of points occurring in either J or J'. The sum of two J's can be shown to be a J.

22.5. Functions of Limited Variation.

We shall now recall briefly the definition of a function of limited total variation and for convenience state some of the standard theorems on such functions to which it will be necessary to refer.

Let $\phi(x)$ be defined on the interval (a,b). Let J be a set of non-overlapping closed intervals $I_1, \cdots, I_m$. Let Δ be a subdivision of (a,b) such that $J \subset \Delta$. Then the number $V_{J_1\Delta}\phi$ defined by

$$V_{J_1\Delta}\phi \equiv \sum_J |\Delta_i \phi|$$

where the summation extends over all i and such that the interval (x_i, x_{i+1}) of Δ lies in J, is called the variation of ϕ over J with respect to Δ.

If there exists a positive number M such that for all Δ for which $J \subset \Delta$ we have $V_{J_1\Delta}\phi < M$, then ϕ is said to be of limited total variation over J. The lower bound of numbers M having this property is called the total variation of ϕ over J and is denoted by $V_J\phi$.

In these definitions the set J may in particular be taken to be the interval (a,b) itself. In this event we denote the variation of

ϕ over (a,b) with respect to Δ by $V_{(a,b),\Delta}\phi$ and the total variation of ϕ over (a,b), if it exists, by $V_{(a,b)}\phi$.

The following properties of functions of limited variation are standard.

Theorem (22.51) If $\phi(x)$ is of limited variation on (a,b), then there exist two bounded positive monotone increasing functions p(x) and n(x) such that $\phi(x) = p(x) - n(x)$.

Theorem (22.52) If $\phi(x)$ and $\psi(x)$ are of limited variation on (a,b) and c is a constant, then the functions $\phi(x) + \psi(x)$, $\phi(x)\cdot\psi(x)$ and $c\cdot\phi(x)$ are of limited variation and we have moreover

$$V_{(a,b)}(\phi + \psi) \leqq V_{(a,b)}\phi + V_{(a,b)}\psi$$

$$V_{(a,b)}c\cdot\psi \leqq (\max \phi)\cdot V_{(a,b)}\psi + (\max \psi)\cdot V_{(a,b)}\phi$$

$$V_{(a,b)}c\cdot\phi = c\, V_{(a,b)}\phi$$

Theorem (22.53) If $\phi(x)$ is of limited variation over (a,b) and if J is a finite set of closed non-overlapping sub-intervals of (a,b), then ϕ is of limited variation over J and $V_{(a,b)}\phi \geqq V_J\phi$

Theorem (22.54) If J and J' are two sets of closed non-overlapping sub-intervals of (a,b) and are mutually exclusive except possibly for certain of the end points of their respective intervals and if $\phi(x)$ is of limited variation on J and on J', then $\phi(x)$ is of limited variation on J + J' and we have $V_{J+J'}\phi = V_J\phi + V_{J'}\phi$.

<u>Theorem (22.55)</u> If $\phi(x)$ is of limited variation on (a,b), then at each point x $(a<x\leq b)$ $\phi(x-0)\equiv \lim_{h\to+0}\phi(x-h)$ and at each point x $(a\leq x<b)$ $\phi(x+0)\equiv \lim_{h\to+0}\phi(x+0)$ exist.

<u>Definition</u>. If $\phi(x)$ is of limited variation on (a,b) and if, (1) $\phi(a)=0$, and (2) $\phi(x)=\phi(x+0)$ $(a<x<b)$, then $\phi(x)$ is said to be regular.

22.6. <u>Simple Riemann-Stieltjes Integrals and Their Properties</u>

In terms of our notation the definition of the integral $\int_a^b f(x)\,d\phi(x)$ may be stated as follows:

Let $f(x)$ and $\phi(x)$ be defined over (a,b). If there exists a number I such that to $\epsilon>0$ corresponds $\delta_\epsilon>0$ such that for any subdivision Δ satisfying $\|\Delta\|<\delta_\epsilon$ we have

$$\Big|I-\sum_{(a,b)} f(\xi_i)\Delta_i\phi\Big|<\epsilon$$

where ξ_i is any point in the interval (x_i, x_{i+1}) of Δ, and where the summation extendes over all intervals of Δ, then the number I is called the Riemann-Stieltjes integral of $f(x)$ with respect to $\phi(x)$ and is denoted by $\int_a^b f(x)\,d\phi(x)$.

The following theorems concerning Riemann-Stieltjes integrals are well known and we shall not indicate the proofs.

<u>Theorem (22.61)</u> If $\int_a^b f(x)d\phi(x)$ exists and if c is an interior point of (a,b), then $\int_a^c f(x)d\phi(x)$ and $\int_c^b f(x)d\phi(x)$ exist and

$$\int_a^b f(x)d\phi(x) = \int_a^c f(x)d\phi(x) + \int_c^b f(x)d\phi(x)$$

As a corollary to Theorem (22.61) it follows that if $\int_a^b f(x)d\phi(x)$ exists then $\int_c^d f(x)d\phi(x)$ exists when c, d, are the end points of any closed interval inside (a,b).

Theorem (22.62) If f(x) and $\psi(x)$ are both integrable over (a,b) with respect to ϕ, and if c_1 and c_2 are real constants then $c_1 f(x) + c_2\psi(x)$ is integrable with respect to ϕ and we have

$$\int_a^b \left[c_1 f(x) + c_2\psi(x)\right] d\phi(x) = c_1\int_a^b f(x)d\phi(x) + c_2\int_a^b \psi(x)d\phi(x)$$

Theorem (22.63) If f(x) is integrable over (a,b) both with respect to $\phi(x)$ and with respect to X(x) and we have

$$\int_a^b f(x)d\left[c_1\phi(x) + c_2 X(x)\right] = c_1\int_a^b f(x)d\phi(x) + c_2\int_a^b f(x)dX(x).$$

Theorem (22.64) If $\int_a^b f(x)d\phi(x)$ exists then $\int_a^b \phi(x)df(x)$ exists and

$$\int_a^b f(x)d\phi(x) = \left[f(x)\phi(x)\right]_a^b - \int_a^b \phi(x)df(x).$$

Theorem (22.65) If $\int_a^b f(x)d\phi(x)$ exists, if $\phi'(x)$, the derivative of $\phi(x)$ exists, and if $\int_a^b f(x)\phi'(x)dx$ exists as a Riemann integral, then

$$\int_a^b f(x)d\phi(x) = \int_a^b f(x)\phi'(x)dx.$$

Theorem (22.66) If $f(x)$ is continuous and if $\phi(x)$ is of limited variation on (a,b) then $\int_a^b f(x)d\phi(x)$ exists.

By the maximum oscillation $\omega_\Delta f$ of a function $f(x)$ with respect to a subdivision Δ we mean the greatest of the numbers $(\overline{f}_i - \underline{f}_i)$ where $\overline{f}_i$ denotes the upper bound of $f(x)$ in the interval (x_i, x_{i+1}) and $\underline{f}_i$ the lower bound in the same interval.

Theorem (22.67) If $\phi(x)$ is of limited variation $V_{(a,b)}\phi$ and if $\int_a^b f(x)d\phi(x)$ exists then

$$\left| \int_a^b f(x)d\phi(x) - \sum_{(a,b)} f(\zeta_i)\Delta_i\phi \right| \leqq \omega_\Delta f \cdot V_{(a,b)}\phi$$

Theorem (22.68) If $\phi(x)$ is of limited variation $V_{(a,b)}\phi$ and if $\int_a^b f(x)d\phi(x)$ exists then

$$\left| \int_a^b f(x)d\phi(x) \right| \leqq \max|f(x)| \cdot V_{(a,b)}\phi$$

Theorem (22.69) If $\phi(x)$ is a regular function of limited variation then the upper bound over all continuous $f(x)$ of the expression

$$\frac{\int_a^b f(x)d\phi(x)}{\max|f(x)|} \quad \text{is equal to } V_{(a,b)}\phi$$

Theorem (22.610) If $\phi(x)$ is of limited variation on (a,b) and $f_n(x)$ is a uniformly convergent sequence of continuous functions then

$$\lim_{n\to\infty}\left\{\int_a^b f_n(x)d\phi\right\} = \int_a^b \left\{\lim_{n\to\infty} f_n(x)\right\} d\phi$$

22.7. The Riesz Theorem on Linear Continuous Functionals

In terms of definitions and notations of part I the definition of a linear-homogeneous continuous functional of a function continuous on (a,b) is simply that of a homogeneous polynomial $p(y)$ of degree one on $F(a,b)$ to R, where $F(a,b)$ denotes the space of real continuous functions y on (a,b), $\|y\|$ being defined as $\max |y|$. In the present discussion it seems convenient to employ the notation of the classical theory of functionals. The general linear-homogeneous continuous functional of a continuous function $y(t)$ $(a \leqq t \leqq b)$ we shall denote by $L[y(t)]$, where t is, so to speak, a dummy. The following important result is due to Frederic Riesz*.

Theorem (22.71) Let $L[y(t)]$ be a linear homogeneous continuous functional of a continuous function $y(t)$ $(a \leqq t \leqq b)$. Let $\eta_n(t;\tau)$ be a function of t depending upon two parameters n, $(n = 1,2,3,\ldots; a \leqq \tau \leqq b)$ and defined by

$$\eta_n(t;\tau) \equiv 1 \quad (a \leqq t \leqq \tau = b)^{**}$$

$$\eta_n(t;\tau) \equiv 1 - n(t - \tau) \quad (a \leqq \tau \leqq t \leqq \tau + 1/n \leqq b)^{**}$$

$$\eta_n(t;\tau) \equiv 0 \quad (a = \tau + 1/n = t = b)^{**}$$

then the following conclusions hold.

I. For $(a \leqq \tau \leqq b)$ $\lim\limits_{n \to \infty} L[\eta_n(t;\tau)]$ exists.

II. The function $\alpha(\tau)$ defined by

$$\alpha(\tau) \equiv \lim_{n \to \infty} L[\eta_n(t;\tau)] \quad a < \tau \leqq b$$
$$\alpha(a) \equiv 0$$

*F. Riesz--Ann. Scientifique de l'Ecole. Norm. Sup. Ser 3, Tome 28, p 33, 1911
Ibid, Tome 31, p. 9, 1914.

**It is unnecessary to define $\eta_n(t;\tau)$ for the case $\tau < b < \tau + 1/n$, since for fixed τ and sufficiently large n this inequality cannot hold.

is of limited variation on (a,b).

III. For any continuous y(t)

$$\int_a^b y(t)\,d\alpha(t) = L[y(t)]$$

IV. $V_{(a,b)}\alpha = \max_{y} \frac{|L[y(t)]|}{\|y\|}$, where $\|y\| \equiv \max_{a \leqq t \leqq b} y(t)$

As a corollary we may show that the function $\alpha(t)$ defined in I. is a regular function of limited variation. We have by II. and III. for $a < \tau < b$.

$$\alpha(\tau) = \lim_{n \to \infty} L[\eta_n(t;\tau)] = \lim_{n \to \infty} \int_a^b \eta_n(t;\tau)\,d\alpha(t) \qquad (1)$$

Using Theorems (22.65), (2..61), (22.65), we have

$$\int_a^b \eta_n(t;\tau)\,d\alpha(t) = \Big[\eta_n(t;\tau)\alpha(t)\Big]_{t=a}^{t=b} - \int_a^b \alpha(t)\,d\eta_n(t;\tau)$$

$$= n\int_{\tau}^{\tau+1/n} \alpha(t)\,dt = \frac{1}{n}\int_{\tau}^{\tau+h} \alpha(t)\,dt \qquad (2)$$

where we have put $h = 1/n$. Now since $\alpha(t)$ is of limited variation it follows that $\alpha(\tau + 0)$ exists and from the theory of ordinary Riemann integrals that the limit as $h \to 0$ of the last expression in (2) is precisely $\alpha(\tau + 0)$. Hence we have that for $(a < \tau < b)$ $\alpha(\tau) = \alpha(\tau + 0)$. and by the definition $\alpha(a) = 0$ so that $\alpha(\tau)$ is regular.

22.8. Functions of Limited Multilinear Variation.

Let $\phi(x^1,\dots,x^n)$ be defined over a hyperparallelopiped H_n, and let $\Delta \equiv \Delta^1 \cdot \Delta^2 \cdots \Delta^n$ be a subdivision of H_n. Let the number $V_{H_n,\Delta}\phi$ be defined by

$$V_{H_n,\Delta}\phi = \max_{\epsilon^i_{j_i} = \pm 1} \sum_{H_n} \epsilon^1_{j_1}, \cdots, \epsilon^n_{j_n} \Delta_{j_1 \cdots j_n} \phi$$

where the summation extends over all cells of Δ, and where the numbers $\epsilon^1_{j_1}, \cdots, \epsilon^n_{j_n}$ are taken to be 1 or -1 in such a way as to make the sum as large as possible. $V_{H_n,\Delta}\phi$ is called the multilinear variation of ϕ over H_n with respect to Δ.

If there exists a number $M > 0$ such that for all subdivisions Δ we have $M > V_{H_n,\Delta}\phi$, then ϕ is said to be of limited multilinear variation. The lower bound of numbers M having this property is called the multilinear variation of ϕ over H_n and is denoted by $V_{H_n}\phi$.

The properties of functions of limited multilinear variation which we shall have occasion to use will for the most part come as by products in the argument.

A function ϕ of limited multilinear variation is called regular* if it satisfied the conditions

$$(1)\ \phi(x^1, \cdots, x^i + 0, \cdots, x^n) = \phi(x^1, \cdots, x^n) \quad (i = 1, 2, \cdots, n)$$

except upon the boundaries of H_n

$$(2)\ \phi(a^1, x^2, \cdots x^n) = \phi(x^1, a^2, \cdots, x^n) = \cdots = \phi(x^1, \cdots, x^{n-1}, a^n) = 0$$

22.9. Definition of Multiple Riemann-Stieltjes Integrals

Let $f(x^1, \cdots, x^n)$ and $\phi(x^1, \cdots, x^n)$ be defined over H_n. If there exists a number I such that to every $\epsilon > 0$ corresponds $\delta_\epsilon > 0$ such that for any subdivision Δ of H_n satisfying $\|\Delta\| < \delta_\epsilon$ we have

$$\left| I - \sum_{H_n} f(\zeta^1_{j_1}, \zeta^2_{j_2}, \cdots, \zeta^n_{j_n}) \Delta_{j_1 \cdots j_n} \phi \right| < \epsilon$$

*This definition is due to C. A. Fischer.

where $(\zeta^1_{j_1}, \zeta^2_{j_2}, \cdots, \zeta^n_{j_n})$ is any point in the cell $(j_1, j_2, \cdots, j_n)$ of Δ, and where the summation extends over all cells of Δ, then the number I is called the multiple Riemann-Stieltjes integral of f with respect to ϕ over H_n and is denoted by $\int_{a^1}^{b^1}, \cdots, \int_{a^n}^{b^n} f(x^1, \cdots, x^n) d\phi(x^1, \cdots, x^n)$. The limits a^i, b^i, are of course simply the end points of the projection of H_n upon the x^i-axis.

Since we shall be interested only in a special case of the multiple Riemann-Stieltjes integral there seems to be no point in stating its more general properties.

23. Representation of Numerical Valued Functional Polynomials by Means of Multiple Stieltjes Integrals.

23.1. Outline of the Problem.

We wish to give a representation as a sum of certain multiple Stieltjes integrals on F(a,b) to R, where F(a,b) is the space of real functions $y(x)$ continuous on the linear range (a,b) and where $\|y\|$ is defined as $\max_{a \leq x \leq b} |y(x)|$. The results on abstract polynomials show the representability of every such polynomial as the sum of homogeneous polynomials. Theorem (13.427). The intimate connection shown in (13.62-3) between a homogeneous polynomial of degree n and its nth polar, which is a multilinear form, brings the problem of finding a representation for a polynomial down to the problem of finding a representation for a multilinear form. The representation for the case $n = 2$ was given by Frechet*, who showed that a bilinear form $B\left[y_1(s)\Big|_a^b, y_2(t)\Big|_{a^1}^{b^1}\right]$ could be represented by an integral in the formula

$$B\left[y_1(s)\Big|_a^b, y_2(t)\Big|_{a^1}^{b^1}\right] = \int_a^b \int_{a^1}^{b^1} y_1(s)y_2(t)du(s,t)$$

where the function $u(s,t)$ is of limited multilinear variation. A new proof of this result and a remark as to how the proof for general n could be carried out were given by C. A. Fischer**. We shall give a more or less independent proof starting from first principles. We do this for completeness and for the purpose of deriving some relationships we shall

*M. Frechet, Trans. Am. Ma. Soc., V. 16, 1915, p. 215.

**C. A. Fischer, Proc. National Acad. of Sciences, V. III, 1917, p. 640.

need in the characterization of the general polynomial on $F(a,b)$ to $F(c,d)$.

As in the statement of Riesz' theorem it seems convenient to employ the notation of the more classical theory of functionals.

23.2. Representation of Numerical Valued Multilinear Functional Forms by means of Multiple Stieltjes Integrals.

Theorem (23.21) Let $y_i \equiv y_i t$ $(a^i \leqq t \leqq b^i;\ i = 1,2,\cdots,n)$ be n continuous functions on the indicated ranges. Let $W \equiv W(y_1,\cdots,y_n)$ be a multilinear functional form in the arguments y. Then there exists a function $u(t^1,\cdots,t^n)$, defined for $(a^i \leqq t^i \leqq b^i)$, such that for all continuous y the integral

$$\int_{a^1}^{b^1}\cdots\int_{a^n}^{b^n} y_1(t^1)\ldots y_n(t^n)du(t^1,\cdots,t^n)$$

exists and is equal to W.

Proof: Let M be the modulus of W so that

$$|W| = M\|y_1\|\cdots\|y_n\|$$

where $\|y_i\| \equiv \max|y_i(t)|$. Define n continuous functions $\eta_{i,m}(t;\tau)$ $(i = 1,2,\cdots,n)$ of t depending on a continuous parameter and an integer m. By means of the formulas

$$\begin{aligned} \eta_{i,m}(t;\tau) &\equiv 1 \quad (a^i \leqq t \leqq \tau \leqq b^i) \\ &\equiv 1 - m(t-\tau) \quad (a^i \leqq \tau < t \leqq \tau + 1/m \leqq b^i) \\ &\equiv 0 \quad (a^i \leqq \tau < \tau + 1/m < t \leqq b^i) \end{aligned}$$

Put

$$\begin{aligned} W(m;\tau) &\equiv W(m_1,\cdots,m_n;\tau^1,\cdots\tau^n) \\ &\equiv W[\eta_{1,m_1}(t^1;\tau^1),\cdots,\eta_{n,m_n}(t^n;\tau^n)] \end{aligned}$$

We shall now prove by induction on n that

$$\lim_{m_1 \to \infty} \lim_{m_2 \to \infty} \cdots \lim_{m_n \to \infty} W(m, \tau)$$

exists and that $u \equiv u(\tau^1, \cdots \tau^n)$

defined by $u(\tau^1, \cdots, \tau^n) \equiv \lim_{m_1 \to \infty} \cdots \lim_{m_n \to \infty} W(m; \tau) \quad (a^i < \tau^i \leqq b^i)$

$$u(a^1, \tau^2, \cdots, \tau^n) = u(\tau^1, a^2, \cdots, \tau^n) = \cdots = u(\tau^1, \cdots, \tau^{n-1}, a^n) = 0 \tag{2}$$

is such that for any subdivision $\Delta \equiv \Delta^1 \cdot \Delta^2 \cdots \Delta^n$

$$\Big| W - \sum_{H_n} y_1(\tau^1_{j_1}) \cdots y_n(\tau^n_{j_n}) \Delta_{j_1 \cdots j_n} u \Big| \leqq M \sum_{i=1}^{n} \|y_1\| \cdots \|y_{i-1}\| \cdot \|y_{i+1}\| \cdots \|y_n\| \, \omega_{\Delta^i} y_i \tag{3}$$

where $\omega_{\Delta^i} y_i$ is the oscillation of y_i with respect to the simple subdivision Δ^i, and where $(\tau^1_{j_1}, \cdots, \tau^n_{j_n})$ is a point in the cell $(j_1 \cdots j_n)$ of .

We first observe that (2) and (3) are, for the case n = 1, true by virtue of Riesz' theorem. For n = 1, equation (2) becomes

$$u(\tau^1) = \lim_{m_1 \to \infty} W[\eta_{1,m_1}(t; \tau)]$$

$$u(a^1) = 0$$

and the existence of the limit is proven by Theorem (22.7). Inequality (3) becomes

$$\Big| W - \sum_{(a^1, b^1)} y_1(\tau^1_{j_1}) \Delta^1_{j_1} u \Big| \leqq M \omega_{\Delta^1} y_1$$

which is true by (22.7) and (22.67)

Now W quâ functional of y_n, is linear and continuous. From the Riesz theorem (22.7) we have that $v \equiv v(y_1,\cdots,y_{n-1};\tau)$ defined by

$$v \equiv \lim_{m\to\infty} W[y_1,\cdots,y_{n-1},\eta_{n,m}(t;\tau)] \qquad a^n < \tau \leqq b^n$$

$$v \equiv 0 \qquad a^n = \tau$$

exists, that $V_{(a^n b^n)} v = \max \frac{|W|}{\|y_n\|} \leqq M\|y_1\|\cdots\|y_{n-1}\|$ (4)

and that $$W = \int_{a^n}^{b^n} y_n(t)\,dv(y_1,\cdots,y_{n-1};t) \qquad (5)$$

Employing Theorem (22.67) and equations (4) and (5) above we obtain

$$\Big|W - \sum_{(a^n,b^n)} y_n(\tau^n_{j_n})\Delta^n_{j_n} v\Big| \leqq M\|y_1\|\cdots\|y_{n-1}\|\,\omega_{\Delta^n} y_n \qquad (6)$$

Now, for each value of τ^n, $v \equiv v(y_1,\cdots,y_{n-1};\tau^n)$ is a multilinear form in $y_1,\cdots,y_{n-1}$; for, if c and c' are constants, we have

$$\begin{aligned} &v[cy_1 + c'y_1', y_2\cdots,y_{n-1};\tau^n] \\ &= \lim_{m\to\infty} W[cy_1 + c'y_1', y_2,\cdots,y_{n-1};\eta_{n,m}] \\ &= \lim_{m\to\infty}\{c\,W[y_1,\cdots,y_{n-1},\eta_{n,m}] + c'W[y_1',\cdots,y_{n-1},\eta_{n,m}]\} \\ &= cv[y_1,\cdots,y_{n-1};\tau^n] + c'v[y_1',\cdots,y_{n-1};\tau^n] \qquad (7)\end{aligned}$$

and similarly for the other arguments y.

Also. $$|v| = \lim_{m\ \infty}|W[y_1,\cdots,y_{n-1},\eta_{n,m}]| \qquad (8)$$

$$= M\|y_1\|\cdots\|y_{n-1}\|\,\|\eta_{n,m}\| = M\|y_1\|\cdots\|y_{n-1}\|$$

Having proven by equations (7) and (8) that v is a multilinear functional in (n - 1) arguments we make use of the induction hypothesis. Writing out the equation for v analogous to (2) for W and assuming the induction correct for (n - 1), we have the existence of

$$\lim_{m_1\to\infty}\lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty} v[\eta_{1,m_1}(t;\tau^1),\cdots,\eta_{n-1,m_{n-1}}(t;\tau^{n-1});\tau^n]$$

But from the definition of v, this last expresses the iterated limit (2). Thus the existence of u is established.

We now proceed to prove the inequality (3).

Let $\quad \mathfrak{s} \equiv \sum_{(a^n,b^n)} y_n(\tau^n_{j_n})\Delta^n_{j_n} v$, which is simply the sum occurring in the left hand side of (6). Now $\mathfrak{s}$ is a linear combination of terms of the type $v(y_1,\cdots,y_n;t_j)$ and is hence a multilinear form in $y_1,\cdots,y_{n-1}$. Moreover, from the equation (4) we have that

$$|\mathfrak{s}| \leq \max_{j_n} |y_n(\tau^n_{j_n})| \cdot V_{(a^n b^n)} v \leq M\|y_1\|\cdots\|y_n\| \tag{9}$$

Hence $\mathfrak{s}$ quâ functional of $y_1,\cdots,y_{n-1}$ is of modulus lessthan or at most equal to $M\|y_n\|$. Assuming now the induction hypothesis we write an inequality analogous to (3) in which we replace W by $\mathfrak{s}$ and $\Delta \equiv \Delta^1\cdots\Delta^n$ by the division $\Delta^1\cdots\Delta^{n-1}$ defined over the hyperparallelopiped H_{n-1} whose points $(t^1,\cdots,t^{n-1})$ satisfy the inequalities $a^i \leqq t^i \leqq b^i$ $(i = 1,2,\cdots,(n-1))$. This gives

$$\left|\mathfrak{s} - \sum_{H_{n-1}} y_1(\tau^1_{j_1})\cdots y_{n-1}(\tau^{n-1}_{j_{n-1}})\Delta^1_{j_1}\cdots\Delta^{n-1}_{j_{n-1}} s\right|$$

$$= M\|y_n\| \sum_{i=1}^{n-1} \|y_1\|\cdots\|y_{i-1}\|\cdot\|y_{i+1}\|\cdots\|y_{n-1}\|\,\omega_{\Delta^i} y_i \tag{10}$$

Where $\quad s \equiv s(\tau^1,\cdots,\tau^n) \equiv \lim_{m_1\to\infty}\cdots\lim_{m_{n-1}\to\infty} \mathfrak{s}[\eta_{1,m_1}(t^1;\tau^1),\cdots\eta_{n-1,m_{n-1}}(t^{n-1};\tau^{n-1});\tau^n]$

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when $$\tau^i > a^i \ (i = 1,2,\cdots,n-1) \text{ and}$$

$$s = 0$$

when for any $i < n$ $\tau^i = a^i$.

Replacing s by its original defining expression and using the fact that the limit of a finite sum is the sum of the limits, we have that

$$s = \sum_{(a^n,b^n)} y_n(\tau^n_{j_n})\Delta^n_{j_n} u \tag{11}$$

Placing the result (11) in the inequality (1) and rearranging the multiple sum we have

$$\left| s - \sum_{H_n} y(\tau^1_{j_1})\cdots y(\tau^n_{j_n})\Delta^1_{j_1}\cdots\Delta^n_{j_n} u \right. \leqq M\|y_n\| \sum_{i=1}^{n-1} \|y_1\|\cdots\|y_{i-1}\|\cdot\|y_{i+1}\|\cdots\|y_{n-1}\|\omega_{\Delta^i} y_i \tag{12}$$

Combining the inequalities (12) and (6) we obtain the inequality (3). This completes the induction.

Now since $y_1,\cdots,y_n$ are continuous, we have that $\omega_{\Delta^i} y_i \to 0$ with $\|\Delta^i\|$, and hence, that the right hand side of (3) tends to zero with $\|\Delta\|$. Therefore, by the definition of a multiple integral we have

$$W = \int_{a^1}^{b^1}\cdots\int_{a^n}^{b^n} y_1(t)\cdots y_n(t)\, du(t^1,\cdots,t^n)$$

This completes the proof.

The function $u(\tau^1,\cdots,\tau^n)$ defined in equation (2) of the last theorem we shall call the kernel of the multilinear form W. Using the formulas derived in the last theorem it is easy to prove:

<u>Theorem (23.22)</u> The kernel $u(\tau^1,\cdots,\tau^n)$ of a multilinear form $W[y_1,\cdots,y_n]$ is of limited multilinear variation.

Proof: We shall prove this result by induction on n. For $n = 1$ the theorem is a trivial consequence of (22.7), for in that case the definition of limited multilinear variation coincides with the ordinary definition of limited variation for a linear interval, and by (22.7), we have $V_{(a^1,b^1)} u = M$. Assume now that for any multilinear form $\overline{W}$ in $(n - 1)$ arguments $y_1, \cdots, y_{n-1}$, the kernel $u(\tau^1, \cdots, \tau^{n-1})$ satisfies

$$\Big|\sum_{H_{n-1}} \epsilon_{j_1} \cdots \epsilon_{j_{n-1}} \Delta_{j_1 \ldots j_{n-1}} u\Big| \leqq M \quad (\epsilon_j = \pm 1),$$

where M is the modulus of the form $\overline{W}$. The functional $v \equiv v[y_1, \cdots, y_{n-1}; \tau^n]$ is for each value of τ^n a multilinear form in $y_1, \cdots, y_n$. Hence for any subdivision Δ^n of the interval (a^n, b^n) and for arbitrary choice of the numbers $\epsilon_{j_n} = \pm 1$ the expression $w \equiv \sum_{(a^n,b^n)} \epsilon_{j_n} \Delta^n_{j_n} v$ is a linear combination of v's and is therefore also multilinear in $y_1, \cdots, y_{n-1}$. Moreover, using equation (4) (23.21) we have

$$|w| = \Big|\sum_{(a^n,b^n)} \epsilon^n_{j_n} \Delta^n_{j_n} v\Big| \leqq V_{(a^n,b^n)} v \leqq M \|y_1\| \cdots \|y_{n-1}\| \qquad (1)$$

so that the modulus of w cannot be greater than M. The kernel of ω is given by replacing each argument y_i by $\eta_{i,m}(t; \tau^i)$ and passing to the iterated limit as in (23.21), equation (2). The kernel of is thus shown to be $\sum_{(a^n,b^n)} \epsilon^n_{j_n} \Delta^n_{j_n} u$. Under the induction hypothesis we have therefore that

$$\Big|\sum_{H_{n-1}} \epsilon^n_{j_1} \cdots \epsilon^n_{j_{n-1}} \Delta^1_{j_1} \cdots \Delta^{n-1}_{j_{n-1}} \Big(\sum_{(a^n,b^n)} \epsilon^n_{j_n} \Delta^n_{j_n} u\Big)\Big| \leqq M$$

Rearranging this sum we have, writing $\Delta \equiv \Delta^1 . \Delta^2 \ldots \Delta^n$,

$$\left|\sum_{H_n}\epsilon^n_{j_1}\cdots\epsilon^n_{j_n}\Delta_{j_1\cdots j_n}u\right| \leqq M$$

Hence the induction is complete, and we have that the kernel $u(\tau^1,\cdots,\tau^n)$ of a form $W[y_1,\cdots,y_n]$ is of total multilinear variation not greater than M, the modulus of W.

There is a certain property which holds for functions of limited multilinear variation defined over an H_n, which functions, like $u(\tau^1,\cdots,\tau^n)$ used in the last two theorems, satisfy the condition of vanishing whenever one of the arguments, say τ^i takes on the value a^i; namely, that if we keep certain of the arguments fixed and consider $u(\tau^1,\cdots,\tau^n)$ quâ function of the others, it is of limited multilinear in these latter. We shall prove this in:

<u>Theorem (23.23)</u> If $u(\tau^1,\cdots,\tau^n)$ is defined for H_n: $(a^i \leqq \tau^i \leqq b^i)$ and is of limited multilinear variation $V_{H_n}u$ over H_n, and if we have $u = 0$ when for any i, $\tau^i = a^i$, then u is of limited multilinear variation over the hyperparallelopiped described by any subset of the variables $\tau^1,\cdots,\tau^n$, the remaining variable being held fixed; moreover the variation over this subregion is always less than or equal to $V_{H_n}u$.

<u>Proof:</u> We shall prove this by reverse induction on n. It is clearly of no moment which of the variables we choose to hold fixed, since by renaming them we could put them in any preassigned order.

Let us first prove that if τ^1 is given a fixed value $\bar{\tau}^1$, then $u(\bar{\tau}^1,\tau^2,\cdots,\tau^n)$ is of limited multilinear variation over the H_{n-1} of the variables $\tau^2,\cdots\tau^n$.

From the definition of V_{H_n} we have for arbitrary $\Delta \equiv \Delta^1\cdot\Delta^2\cdots\Delta^n$.

$$\left|\sum_{H_n}\epsilon^1_{j_1}\cdots\epsilon^n_{j_n}\Delta_{j_1\cdots j_n}u\right| \leqq V_{H_n}u \qquad (\epsilon^i_{j_i} = \pm 1) \qquad (1)$$

If $\bar{\tau}^1 = a^1$ we have $u = 0$, and the theorem is trivial; if $\bar{\tau}^1 = b^1$, we take Δ^1 to consist simply of the two divisions a^1 and b^1. On summing out from (1) the range (a^1, b^1) we get two terms, one of which vanishes because of $u(a^1, \tau^2, \cdots, \tau^n) = 0$, and the other of which is

$$\sum_{H_{n-1}} \epsilon^1_1 \epsilon^2_{j_2} \cdots \epsilon^n_{j_n} \Delta_{j_2} \cdots \Delta_{j_n} u(b^1, \tau^2, \cdots, \tau^n)$$

Taking $\epsilon^1_1 = 1$ we have from (1)

$$\left| \sum_{H_{n-1}} \epsilon^2_{j_2} \cdots \epsilon^n_{j_n} \Delta^2_{j_2} \cdots \Delta^n_{j_n} u(b^1, \tau^2, \cdots, \tau^n) \right| \leqq V_{H_n} u \qquad (2)$$

For $a^1 < \bar{\tau}^1 < b^1$ we take Δ^1 to have the points of division a^1, $\bar{\tau}^1$, b^1. Summing out the range (a^1, b^1) from (1) we get (omitting the term which vanishes indentically)

$$\Big| \epsilon^1_0 \sum_{H_{n-1}} \epsilon^2_{j_2} \cdots \epsilon^n_{j_n} \Delta^2_{j_2} \cdots \Delta^n_{j_n} u(\bar{\tau}^1, \tau^2, \cdots, \tau^n)$$

$$+ \epsilon^1_1 \Delta^1_1 \left[\sum_{H_{n-1}} \epsilon^2_{j_2} \cdots \epsilon^n_{j_n} \Delta^2_{j_2} \cdots \Delta^n_{j_n} u \right] \Big| \leqq V_{H_n} u \qquad (3)$$

On choosing $\epsilon^1_0 = \pm 1$ and $\epsilon^1_1 = \pm 1$ so that both the terms on the left hand side of (3) are non negative, we have

$$\left| \sum_{H_{n-1}} \epsilon^2_{j_2} \cdots \epsilon^n_{j_n} \Delta^2_{j_2} \cdots \Delta^n_{j_n} u(\bar{\tau}^1, \tau^2, \cdots, \tau^n) \right| \leqq V_{H_n} u \qquad (4)$$

where $\epsilon^i_{j_i} = \pm 1$ are still at our disposal.

The general theorem follows at once from this special case, by fixing one more of the variables of $u(\bar{\tau}^1, \tau^2, \cdots, \tau^n)$ and applying the argument we have just given.

It is true in general that if $u(\tau^1,\cdots,\tau^n)$ is of limited multilinear variation over H_n ($a^i \leqq \tau^i \leqq b^i$) and if $y_i \equiv y_i(\tau^i)$ $(i = 1,2,\cdots,n)$ are continuous that the multiple integral

$$\int_{a^1}^{b^1}\cdots\int_{a^n}^{b^n} y_1(\tau^1)\cdots y_n(\tau^n)\,du(\tau^1,\cdots,\tau^n)$$

exists and is a multilinear form in $y_1,\cdots,y_n$. We shall not prove it simply for the case where u is restricted, as in theorem (23.23), to vanish when $\tau^i = a^i$ for any i. The more general case will not concern us.

As a preliminary, let us state the following lemma which is often used in the proof of the existence of an ordinary Riemann-Stieltjes integral of a continuous function with respect to a function of limited variation:

Lemma (23.231) If $f(x)$ is continuous on (a,b) and if $\phi(x)$ is of limited variation on the same interval, and if Δ and $\overline{\Delta}$ are two subdivisions of (a,b) such that $\Delta \leqq \overline{\Delta}$, then

$$\left|\sum_{(a,b)} f(\zeta_i)\Delta_i\phi - \sum_{(a,b)} f(\overline{\zeta}_i)\overline{\Delta}_i\phi\right| \leqq V_{(a,b)}\phi\cdot\omega_\Delta f \qquad (1)$$

ζ_i and $\overline{\zeta}_i$ denoting as usual any points in the i^{th} interval of Δ and $\overline{\Delta}$ respectively.

The proof comes out at once if we observe that $\Delta_i\phi$ is equal to the sum $\sum_{(x_i,x_{i+1})} \overline{\Delta}_j\phi$ extended over those subdivisions of $\overline{\Delta}$ lying in (x_i,x_{i+1}) of Δ. Thus for a typical term in the first sum we have

$$f(\zeta_i)\Delta_i\phi = \sum_{(x_i,x_{i+1})} f(\zeta_i)\overline{\Delta}_j\phi. \qquad (2)$$

which gives

$$\left| f(\zeta_i)\Delta_i\phi - \sum_{(x_i,x_{i+1})} f(\bar{\xi}_j)\bar{\Delta}_j\phi \right|$$

$$= \left| \sum_{(x_i,x_{i+1})} \{f(\zeta_i) - f(\bar{\xi}_j)\}\bar{\Delta}_j\phi \right| \leqq V_{(x_i,x_{i+1})}\phi \cdot \max_j \left| f(\zeta_i) - f(\bar{\xi}_j) \right|$$

$$\leqq V_{(x_i,x_{i+1})}\phi \cdot \omega_\Delta f \qquad (3)$$

Combining the inequalities (3) for all i we have

$$\left| \sum_{(a,b)} f(\zeta_i)\Delta_i\phi - \sum_{(a,b)} f(\bar{\xi}_j)\bar{\Delta}_j\phi \right| \leqq \sum_i V_{(x_i,x_{i+1})}\phi \cdot \omega_\Delta f$$

$$= V_{(a,b)}\phi \cdot \omega_\Delta f$$

which is the inequality (1).

A second lemma, which we shall not prove explicitly is the following.

Lemma (23.232) Let $F \equiv \sum_{j_1\cdots j_n} a_{j_1\cdots j_n}\theta^1_{j_1}\cdots\theta^n_{j_n}$ be a multilinear form with numerical coefficients $a_{j_1\cdots j_n}$ in the numerical components $\theta^i_{j_i}$ of n ordinary vectors θ^i $(i = 1,2,\cdots,n)$. Then if $\|\theta^i\|$ denotes $\max_{j_k}|\theta^i_{j_i}|$, we have

$$|F| \leqq \|\theta^1\|\cdot\|\theta^2\|\cdots\|\theta^n\|\cdot \max_{\epsilon^i_{j_i}=\pm 1} \sum_{j_1\cdots j_n} a_{j_1\cdots j_n}\epsilon^1_{j_1}\cdots\epsilon^n_{j_n}$$

We shall now prove the following:

Theorem (23.24) If $u \equiv u(\tau^1,\cdots,\tau^n)$ is defined over H_n: $(a^i \leqq \tau^i \leqq b^i)$ and is of limited multilinear variation over H_n, if $u = 0$ when for any i $\tau^i = a^i$, and if $y_i \equiv y_i(\tau^i)$ $(i = 1,2,\ldots,n)$ are continuous functions, then the integral $\int_{a^1}^{b^1} \cdots \int_{a^n}^{b^n} y_1(\tau^1)\cdots y_n(\tau^n)\, du(\tau^1,\cdots,\tau^n)$ exists.

Proof: Let $M \equiv V_{H_n} u$. Let $\Delta \equiv \Delta^1\cdot\Delta^2\cdots\Delta^n$ and $\bar{\Delta} \equiv \bar{\Delta}^1\cdot\bar{\Delta}^2\cdots\bar{\Delta}^n$ be two subdivisions of H_n arbitrary except that $\Delta \leqq \bar{\Delta}$. We wish first to establish the following inequality.

$$\left|\sum_{H_n} y_1(\tau^1_{j_1})\cdots y_n(\tau^n_{j_n})\Delta_{j_1\cdots j_n} u - \sum_{H_n} y_1(\bar{\tau}^1_{j_1})\cdots y_n(\bar{\tau}^n_{j_n})\bar{\Delta}_{j_1\cdots j_n} u\right|$$

$$\leqq M \sum_{i=1}^{n} \|y_1\|\cdots\|y_{i-1}\|\cdot\|y_{i+1}\|\cdots\|y_n\| \cdot \omega_{\Delta^i} y_i \tag{1}$$

where $\tau^i_{j_i}$ and $\bar{\tau}^i_{j_i}$ represent points in the cells of Δ and $\bar{\Delta}$ respectively, and where $\Delta \leqq \bar{\Delta}$.

We shall use induction on n. For n = 1 the inequality (1) is precisely the inequality proved in Lemma (23.231). Let assume (1) for n - 1. The subdivisions Δ and $\bar{\Delta}$ and the numbers $\tau^i_{j_i}$ and $\bar{\tau}^i_{j_i}$ will be kept fixed throughout the process; hence we shall denote $y_i(\tau^i_{j_i})$ by $y^i_{j_i}$ and $y_i(\bar{\tau}^i_{j_i})$ by $\bar{y}^i_{j_i}$. Now v denoting a function of $\tau^1,\cdots,\tau^n$ and possibly of other arguments we define

$$S^k_{\Delta}\, v \equiv \sum_{H_k} y^1_{j_1}\cdots y^k_{j_k}\Delta^1_{j_1}\cdots\Delta^k_{j_k} v$$

and $S^k_{\bar{\Delta}} v$ by a similar expression, in which bars are written. The sums in the left hand side of (1) are thus respectively $S^n_{\Delta}\, u$ and $S^n_{\bar{\Delta}}\, u$.

Let us first prove thay if we regard $S^{n-1}_{\Delta} u$ qua function of τ^n, then

$$V_{(a^n,b^n)} S^{n-1}_{\Delta} u \leqq M\|y_1\|\cdots\|y_{n-1}\| \tag{2}$$

If $\epsilon^n_{j_n} = \pm 1$ we have

$$\left|\sum_{(a^n,b^n)} \epsilon^n_{j_n} \Delta^n_{j_n} (S^{n-1}_\Delta u)\right| = \left|\sum_{H_n} y^1_{j_1} \cdots y^{n-1}_{j_{n-1}} \epsilon^n_{j_n} \Delta_{j_1 \ldots j_n} u\right|$$

Taking this last expression for $|F|$ in Lemma (23.232) and writing $a_{j_1 \ldots j_n} = \Delta_{j_1 \ldots j_n} u$, $\theta^i_{j_i} = y^i_{j_i}$ $(i = 1,2,\ldots,n-1)$, $\theta^n_{j_n} = \epsilon^n_{j_n}$ we have

$$\left|\sum_{(a^n,b^n)} \epsilon^n_{j_n} \Delta^n_{j_n} (S^{n-1}_\Delta u)\right| \leqq M\|y_1\| \cdots \|y_{n-1}\| \tag{3}$$

Since this is true for every choice of $\epsilon^n_{j_n}$ the equation (2) is established.

Now, observing that $S^n_\Delta u = \sum_{(a^n,b^n)} y^n_{j_n} \Delta^n_{j_n} (S^{n-1}_\Delta u)$, and using the Lemma (23.231) with $f(\tau^n) \equiv y_n(\tau^n)$, $\phi(\tau^n) \equiv S^{n-1}_\Delta u$ we obtain

$$\left|S^n_\Delta u - \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} (S^{n-1}_\Delta u)\right|$$

$$= \left|\sum_{(a^n,b^n)} y^n_{j_n} \Delta^n_{j_n} (S^{n-1}_\Delta u) - \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} (S^{n-1}_\Delta u)\right|$$

$$\leqq M\|y_1\| \cdots \|y_{n-1}\| \; \omega_\Delta n y^n \tag{4}$$

Now rearranging the sum in the first member of (4) we have

$$\sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} (S^{n-1}_\Delta u) = \sum_{H_n} y^1_{j_1} \cdots y^{n-1}_{j_{n-1}} \bar{y}^n_{j_n} \Delta^1_{j_1} \cdots \Delta^{n-1}_{j_{n-1}} \bar{\Delta}^n_{j_n} u$$

$$= S^{n-1}_\Delta \left\{ \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u \right\} \tag{5}$$

Similarly rearranging $S^n_{\bar{\Delta}} u$ we have

$$S^n_{\bar{\Delta}} u = S^{n-1}_{\Delta} \left\{ \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u \right\} \tag{6}$$

Subtracting (6) from (5) and taking the modulus

$$\left| \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} (s_\Delta^{n-1} u) - s_\Delta^n u \right|$$

$$= \left| s_\Delta^{n-1} \left\{ \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u \right\} - s_\Delta^{n-1} \left\{ \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u \right\} \right| \tag{7}$$

This last expression is in the same form as the left hand side of equation (1) except that n is replaced by n-1 and u is replaced by

$$\sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u$$

which we shall now show has a variation quâ $\tau^1, \cdots, \tau^{n-1}$ less than $M\|y_n\|$.

Using Lemma (23.232) with $\theta^i_{j_i} = \epsilon^i_{j_i}$ $(i = 1,2,\ldots,n-1)$, $\theta^n_{j_n} = \bar{y}^n_{j_n}$, $a_{j_1 \ldots j_n} = \bar{\Delta}_{j_1 \ldots j_n} u$ we have

$$\left| \sum_{H_{n-1}} \epsilon^1_{j_1} \cdots \epsilon^{n-1}_{j_{n-1}} \bar{\Delta}^1_{j_1} \cdots \bar{\Delta}^{n-1}_{j_{n-1}} \left\{ \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} u \right\} \right|$$

$$= \left| \sum_{H_n} \epsilon^1_{j_1} \cdots \epsilon^{n-1}_{j_{n-1}} \bar{y}^n_{j_n} \bar{\Delta}_{j_1 \ldots j_n} \right|$$

$$\leq \max_{\epsilon = \pm 1} \left| \sum_{H_n} \epsilon^1_{j_1} \cdots \epsilon^n_{j_n} \Delta_{j_1 \ldots j_n} \right| \cdot M\|y_n\| \tag{8}$$

The induction hypothesis now gives from (7) and (8) that

$$\left| \sum_{(a^n,b^n)} \bar{y}^n_{j_n} \bar{\Delta}^n_{j_n} (s_\Delta^{n-1} u) - s_\Delta^n u \right|$$

$$\leq M\|y_n\| \sum_{i=1}^{n-1} \|y_1\| \cdots \|y_{i-1}\| \cdot \|y_{i+1}\| \cdots \|y_{n-1}\| \, \omega_{\Delta^i} y_i \tag{9}$$

Combining the inequalities (4) and (9), there results (1) which was to be proved.

From inequality (1) the existence of the integral follows at once. Let $\Delta_1, \Delta_2, \cdots, \Delta_m, \cdots$ be an infinite sequence of subdivisions of H_n such that $\Delta_1 \leqq \Delta_2 \leqq \cdots \leqq \Delta_m \leqq \cdots$ and such that $\|\Delta_m\| \to 0$ with $1/m$. $\|y_i\|$ being fixed and y_i being continuous we have that $\omega_{\Delta^i} y_i$ and hence the right hand side of (1) tends to zero with $1/m$. Butthe inequality (1) is precisely the Cauchy condition for the existence of a unique limit for the sequence $S^n_{\Delta_1} u, S^n_{\Delta_2} u, \cdots, S^n_{\Delta_m} u, \cdots$. Call this limit S. We now prove that given $e > 0$ we may select $\delta > 0$ so that if $\|\Delta\| < \delta$ then $|S^n_\Delta u - S| < e$. Suppose e given. Choose δ so that if $\|\Delta\| < \delta$ the right hand side of (1) is less than $e/3$. Choose m so that $\|\Delta_m\| < \delta$ and $|S^n_{\Delta_m} u - S| < e/3$. Let $\bar{\Delta} \equiv \Delta + \Delta_m$. Then since $\Delta \leqq \bar{\Delta}$ and $\Delta_m \leqq \bar{\Delta}$ we have from (1)

$$|S^n_\Delta u - S^n_{\bar{\Delta}} u| < e/3$$

$$|S^n_{\bar{\Delta}} u - S^n_{\Delta_m} u| < e/3$$

and by the choice of m $|S^n_{\Delta_m} - S| < e/3$

or $$|S^n_\Delta u - S| < e$$

This completes the proof of the theorem and shows that S is the integral.

Theorem (23.25) The multiple integral defined in the previous theorem is a multilinear form in the continuous functions y_i.

Proof: In order to show that $W \equiv W(y_1, \cdots, y_n) \equiv \int_{a^1}^{b^1} \cdots \int_{a^n}^{b^n} y_1(\tau^1) \cdots y_n(\tau^n)\, du(\tau^1, \cdots, \tau^n)$ is a multilinear, it is sufficient to show that it is linear and continuous in each of its arguments.

The distributive property of the sum $S^n_\Delta u$ with respect to each of the functions y_i is evident from the explicit form of $S^n_\Delta u$. The limit as $\|\Delta\| \to 0$ of $S^n_\Delta u$ is therefore also distributive.

To show the continuity of a linear functional it is sufficient to show it at one point, say the origin. If we use Lemma (23.231), taking $F = S^n u$, $\theta^i_{j_i} = y^i_{j_k}$, $a_{j_1 \cdots j_n} = \Delta_{j_1 \cdots j_n} u$ we obtain

$$\left| S^n_\Delta u \right| \leqq M\|y_1\| \cdots \|y_n\|$$

allowing $\|\Delta\|$ to tend to zero we have

$$|W| \leqq M\|y_1\| \cdots \|y_n\|$$

which shows that W is continuous at $y_i = 0$.

It is of interest to show that the integral defined in (23.25) can be calculated as an iterated integral and we shall find the result useful in proving unicity of representation when the kernel $u(\tau^1, \cdots, \tau^n)$ is restricted to be regular.

Theorem (23.26) Let $u(\tau^1, \cdots, \tau^n)$ be of multilinear limited variation on H_n: $(a^i \leqq \tau^i \leqq b^i)$ and let $u = 0$ when $\tau^i = a^i$ for any i. Then for each $r < n$ the iterated integral

$$\int_{a^{r+1}}^{b^r} \cdots \int_{a^n}^{n} y_{r+1}(\tau^{r+1}) \cdots y_n(\tau^n)\, d\int_{a^1}^{b^1} \cdots \int_{a^r}^{b^r} y_1(\tau^1) \cdots y(\tau^r) du \quad (1)$$

has a meaning and is equal to the multiple integral

$$\int_{a^1}^{b^1} \cdots \int_{a^n}^{b^n} y_1(\tau^1) \cdots y_n(\tau^n) du \quad (2)$$

Proof: Making full use of the notation introduced in Theorem (23.24) we shall let S denote the multiple integral. The right hand side of equation (1) (23.24) is independent of the subdivision $\bar{\Delta}$. Letting $\|\Delta\| \to 0$ in that equation we have

$$\left| S - S^n_\Delta u \right| \leqq M \sum_{i=1}^{n} \|y_1\| \cdots \|y_{i-1}\| \cdot \|y_{i+1}\| \cdots \|y_n\| \quad \omega_{\Delta^i} y_i \qquad (3)$$

It is clear that the integral $U \equiv \int_{a^1}^{b^1} \cdots \int_{a^r}^{b^r} y_1(\tau^1) \cdots y_r(\tau^r) du$ exists, for, by (23.23) u is of limited variation in $\tau^1, \cdots, \tau^r$.

Let us rearrange the sum $S^n_\Delta u$

$$S^n_\Delta u = \sum_{H_{n-r}} y^{r+1}_{j_{r+1}} \cdots y^n_{j_n} \Delta^{r+1}_{j_{r+1}} \cdots \Delta^n_{j_n} (S^r_\Delta u) \qquad (4)$$

and place (4) in (3). Now by the definition of a multiple integral, U is equal to the limit as $\|\Delta_1 \Delta_2 \cdots \Delta_r\| \to 0$ of $S^r u$. If we pass to the limit with $\Delta_1, \cdots, \Delta_r$ and leave $\Delta_{r+1}, \cdots, \Delta_n$ untouched, the combination of (3) and (4) becomes

$$\left| S - \sum_{H_{n-r}} y^{r+1}_{j_{r+1}} \cdots y^n_{j_n} \Delta^{r+1}_{j_{r+1}} \cdots \Delta^n_{j_n} U \right|$$

$$\leqq M \sum_{i=r+1}^{n} \|y_1\| \cdots \|y_{i-1}\| \|y_{i+1}\| \cdots \|y_n\| \, \omega_{\Delta^i} y_i \qquad (5)$$

Now pass to the limit with the remainder of the Δ's. (5) then gives

$$S = \int_{a^{r+1}}^{b^{r+1}} y_{r+1}(\tau^{r+1}) \cdots y_n(\tau^n) dU$$

This completes the proof of the theorem. It may be observed incidentally that by a use of Lemma (23.232) we obtain

$$\left| \sum_{H_{n-r}} \epsilon^{r+1}_{j_{r+1}} \cdots \epsilon^{n}_{j_n} \Delta^{r+1}_{j_{r+1}} \cdots \Delta^{n}_{j_n} (S^r_\Delta u) \right| \leqq M \|y_1\| \cdots \|y_r\| \qquad (6)$$

The limit of (6) as $\|\Delta_1 \Delta_2 \cdots \Delta_r\| \to 0$ shows that

$$V_{H_{n-r}} U \leqq M \|y_1\| \cdots \|y_r\| \qquad (7)$$

Corollary (23.261) If $u(\tau^1, \cdots \tau^n)$ is of limited multilinear variation over H_n, then $U \equiv U(\tau^n) \equiv \int_{a^1}^{b^1} \cdots \int_{a^{n-1}}^{b^{n-1}} y_1 \cdots y_{n-1} du$ is regular in τ^n.

Proof: Take U to be the U of the preceding theorem with $r = n-1$. Then by equation (7) (23.26) U is of limited variation. The definition of U tells us that

$$U \equiv \lim_{\|\Delta_1 \cdots \Delta_{n-1}\| \to 0} S^{n-1}_\Delta u \qquad (1)$$

and, writing out the formula analogous to equation (3) Theorem (23.26) in which we replace $S^n_\Delta u$ by $S^{n-1}_\Delta u$, S by U, we have

$$\left| U - S^{n-1}_\Delta u \right| \leqq M \sum_{i=1}^{n-1} \|y_1\| \cdots \|y_{i-1}\| \cdot \|y_{i+1}\| \cdots \|y_n\| \, \omega_{\Delta i}\, y_i \qquad (2)$$

so that the approach is uniform with respect to τ^n, since the right hand side of (2) does not depend on τ^n.

Now since u is regular, we have for all τ^n that

$$\lim_{h \to +0} S^{n-1}_\Delta u(\tau^n + h) = S^{n-1}_\Delta u(\tau^n)$$

Since the limit in (1) is approached uniformly with respect to τ^n it follows that

$$\lim_{h \to +0} U(\tau^n + h) = U(\tau^n)$$

which was to be proved.

Theorem (23.28) Two distinct regular kernels cannot represent the same multilinear form.

Proof: We use induction on n. For n = 1, the result is well known, and is in fact a consequence of Theorem (22.67). For general n, let $u(\tau^1, \cdots \tau^n)$ and $\bar{u}(\tau^1, \cdots, \tau^n)$ be two distinct regular kernels on H_n. We shall prove that there is at least one set of continuous functions y_i $(i = 1,2,\ldots,n)$ such that

$$\int_{H_n} y_1 \cdots y_n du \neq \int_{H_n} y_1 \cdots y_n d\bar{u} \qquad (1)$$

By Theorem (23.27) and (23.271) the integrals in (1) may be written respectively as

$$\int_{a^n}^{b^n} y_n dU \quad \text{and} \quad \int_{a^n}^{b^n} y_n d\bar{U} \qquad (2)$$

where U and $d\bar{U}$ are defined as in (23.27) for the regular kernels u and $\bar{u}$ respectively. Now since U and $\bar{U}$ are regular it follows that if for all y_n

$$\int_{a^n}^{b^n} y_n dU = \int_{a^n}^{b^n} y_n d\bar{U}$$

then $U = \bar{U}$. Now using the induction hypothesis for n-1, it follows that since U and $\bar{U}$ are identical forms in $y_1, \cdots, y_{n-1}$, we have $u = \bar{u}$, which contradicts the original hypothesis.

24. A Theorem on the Limits of Stieltjes Integrals

In this division we shall give a proof by elementary methods of a generalization to Riemann-Stieltjes integrals of the well known Arzela theorem on sequences of Riemann integrable functions.

Let us first prove several lemmas:

Lemma (24.1) If ϕ is bounded and monotone increasing on (a,b), f is an function on (a,b), and if Δ and $\bar{\Delta}$ are subdivisions such that $\Delta \subset \bar{\Delta}$, then

$$\left| \sum_{(a,b)} f(\xi_i)\Delta_i\phi - \sum_{(a,b)} f(\bar{\xi}_i)\bar{\Delta}_i\phi \right| < \sum_{(a,b)} \omega_i f \Delta_i\phi \qquad (1)$$

and in general if $J \subset \Delta$,

$$\left| \sum_{J} f(\xi_i)\Delta_i\phi - \sum_{J} f(\bar{\xi}_i)\bar{\Delta}_i\phi \right| < \sum_{J} \omega_i f \Delta_i\phi \qquad (2)$$

where $\omega_i f$ is the oscillation of f in the interval (x_i, x_{i+1}).

Proof: Since

$$\Delta_i\phi = \sum_{(x_i,x_{i+1})} \bar{\Delta}_j\phi$$

we have

$$\left| f(\xi_i)\Delta_i\phi - \sum_{(x_i,x_{i+1})} f(\bar{\xi}_j)\bar{\Delta}_j\phi \right|$$

$$\leqq \sum_{(x_i,x_{i+1})} \left|\left(f(\xi_i) - f(\bar{\xi}_j)\right)\right| \Delta_j\phi \qquad (3)$$

$$\leqq \omega_i f \Delta_i\phi$$

Summing (3) over all intervals in J we obtain (2). If J = (a,b) we have (1).

<u>Lemma (24.2)</u>. If ϕ is bounded monotone increasing and if f is any function on (a,b), then in order $\int_a^b f\,d\phi$ exist it is necessary and sufficient that $\sum_{(a,b)} \omega_i f \Delta_i \phi \to 0$ with $\|\Delta\|$.

<u>Sufficiency</u>: Select a sequence of subdivisions $\{\Delta^n\}$ such that $\|\Delta^n\| \to 0$ and such that $\Delta^n \subset \Delta^{n+1}$. Denote by S_Δ and $S_{\bar\Delta}$ the expressions occuring in the left hand side of inequality (1) of the preceding lemma.

Thus
$$\left| S_{\Delta^n} - S_{\Delta^{n+p}} \right| < \sum \omega_i f \Delta_i \phi$$

and by hypothesis, the right hand side tends to zero with 1/n. The sequence $\{S_{\Delta^n}\}$ therefore has a limit. Callit S. If $e > 0$, choose δ so that if $\|\Delta\| < \delta$ $\sum \omega_i f \Delta_i \phi < e/3$. Choose n_o so that $|S_{\Delta^{n_o}} - S|$ e/3. Then if Δ is <u>any</u> subdivision satisfying $\|\Delta\| < \delta$ we have, since $\Delta \subset \bar\Delta + \Delta^n$ and $\Delta^n \subset \Delta + \Delta^n$

$$\left| S_\Delta - S_{\Delta + \Delta^n} \right| < e/3$$

$$\left| S_{\Delta^n} - S_{\Delta + \Delta^n} \right| < e/3$$

So that
$$\left| S - S_\Delta \right| < e$$

<u>Necessity</u>: For any division let $\bar f_i$ and $\underline{f}_i$ denote respectively the upper and lower bounds of $f(\xi_i)$ in (x_i, x_{i+1}). Given $e > 0$, select δ so that $\|\Delta\| < \delta$ implies

$$\left| \sum_{(a,b)} f(\xi_i) \Delta_i \phi - S \right| < e/2$$

then
$$\left| \sum_{(a,b)} \bar f_i \Delta_i \phi - S \right| \leq e/2$$

$$\left| \sum_{(a,b)} \underline{f}_i \Delta_i \phi - S \right| \leq e/2$$

or $$\sum \omega_i f \Delta_i \phi = \sum (\bar{f}_i - \underline{f}_i) \Delta_i \phi < e$$

Lemma (24.3) Let f,g be integrable with respect to bounded monotone increasing ϕ. Let m(f,g) denote the function whose value at each x is the greater of f(x) and g(x). Then m(f,g) is integrable with respect to ϕ.

Proof: Since $\int f\, d\phi$ and $\int g\, d\phi$ exist we have that

$$\sum_{(a,b)} \omega_i f \Delta_i \phi \text{ and } \sum \omega_i g \Delta_i \phi \to 0$$

with $\|\Delta\|$. But $\omega_i m(f,g) \leqq \omega_i f$ and $\leqq \omega_i g$. Therefore

$$\sum_{(a,b)} \omega_i m(f,g) \Delta_i \phi \to 0$$

with $\|\Delta\|$. Hence by the preceding lemma $\int m(f,g) d\phi$ exists.

Lemma (24.4) Let $f_1 \geqq f_2 \geqq \cdots = f_n \geqq \cdots \geqq 0$ be a sequence of functions having 0 as their limit. Then if $\int f_n d\phi$ exists for all n, $\lim_{n \to 0} \int f_n d\phi = 0$.

Proof: Select $e > 0$. Let $\{e_n\}$ be a sequence of positive terms such that $\sum e_n$ converges and is less than $e/4$. Select $\Delta^1 \subset \Delta^2 \subset \cdots \Delta^n \subset \cdots$ a sequence of divisions such that

$$\left| \sum_{(a,b)} f_n(\xi_i) \Delta_i^n \phi - \int_a^b f_n d\phi \right| < e_n \tag{1}$$

Now with each point x of (a,b) we may associate a closed interval containing x, as follows: Given x, choose n_x so that $f_n(x) < e/2V_{(a,b)}\phi$. Let I_x be the interval (or intervals in case x is an end point) of Δ^{n_x} which contains x.

We now apply the Heine-Borel result and establish the existence of a finite number of points $\bar{x}_1, \cdots, \bar{x}_k$, such that $I_{\bar{x}_1} \cdots I_{\bar{x}_k}$ entirely cover (a,b).

Arrange the distinct integers of the set $(n_{\bar{x}_1}, \cdots, n_{\bar{x}_k})$ in order of increasing mangitude, say $n_1, \cdots, n_s$.

Let J_1 denote the sum of all intervals $I_{\bar{x}_i}$ for which $n_1 = n_{\bar{x}_i}$. Now if (x_j, x_{j+1}) is one of the intervals of Δ^{n1} lying in J_1, it must, in order to have been included in some I, contain at least one of the points $\bar{x}_1, \cdots, \bar{x}_k$. Write (1) with the points ξ_i lying in J_1 replaced by members of the set $\bar{x}_1, \cdots, \bar{x}_k$, and the others unchanged, and add the resulting inequality to (1). There results

$$\left| \sum_{J_1} f_{n_1}(\bar{x}_i)\Delta_i^{n1}\phi - \sum_{J_1} f_{n_1}(\xi_i)\Delta_i^{n1}\phi \right| < 2e_{n_1}$$

which now must hold if in particular $f(\xi_i)$ are replaced by $\bar{f}_i$. Hence we

obtain $$\int_{J_1} f_{n_s} d\phi \leqq \int_{J_1} f_{n_1} d\phi < \sum_{J_1} \bar{f}_{n_1} \varphi_i^{n1}\phi < 2e_{n_1} + V_{J_1}\phi \cdot e/2V_{(a,b)}\phi.$$

Now take for J_2 the sum of the set of I's, those $I_{\bar{x}_i}$ for which $n_{x_i} = n_2$, but which are not included in J_1. Each interval of Δ^{n2} lying in J_2 has, as before, one point of the set $\bar{x}_1, \cdots, \bar{x}_k$. By a similar process to that used above we obtain the inequality

$$\int_{J_2} f_{n_s} d\phi \leqq \int_{J_2} f_{n_2} d\phi < \sum_{J_2} \bar{f}_{n_2} \Delta_i^{n2}\phi < 2e_{n_2} + V_{J_2}\phi \cdot e/2V_{(a,b)}\phi$$

continue this process s times. $J_1, J_2, \cdots, J_s$ completely cover (a,b) and are non-overlapping except for end points. Adding the results, we have

$$\sum_{r=1}^{s} \int_{J_r} f_{n_s} d\phi = \int_a^b f_{n_s} d\phi < e$$

This proves the lemma.

Corollary: If the sequence $f_1, f_2, \cdots, f_n, \cdots$ is monotone increasing and converges to zero, a similar theorem holds.

We are now in a position to prove the theorem:

Theorem (24.5) Let ϕ be bounded monotone increasing and let $\{f_n\}$ be a bounded sequence converging to zero. Let $\int f_n d\phi$ exist. Then

$$\lim_{n\to\infty} \int f_n d\phi = 0$$

Proof: Define, for $k > n$, functions $Y_{nk} \equiv Y_{nk}(x)$, $y_{nk} \equiv y_{nk}(x)$

$$Y_{nk}(x) \equiv \text{greatest of } f_n(x), f_{n+1}(x), \cdots, f_k(x)$$

$$y_{nk}(x) \equiv \text{least of } f_n(x), f_{n+1}(x), \cdots, f_k(x)$$

So that Y_{nk} is an increasing sequence in k and y_{nk} is decreasing in k. The sequence of numbers

$$\int_a^b Y_{nk} d\phi \qquad \int_a^b y_{nk} d\phi$$

are therefore respectively increasing and decreasing and are bounded. Let B_n be the upper bound of $\int_a^b Y_{nk} d\phi$ and b_n the lower bound of $\int_a^b y_{nk} d\phi$. We have, using the fact that ϕ is monotone increasing,

$$B_n \geq \int_a^b Y_{nk} d\phi \geq \int_a^b f_n d\phi \geq \int_a^b y_{nk} \geq b_n$$

Now since $Y_{n+p,k} = Y_{nk}$ we have $B_{n+p} \leqq B_n$ and similarly $b_{n+p} \geqq b_n$ $(p>0)$. If therefore we can show that: (I) $B_n \geqq 0 \geqq b_n$

(II) $\lim_{n\to\infty} (B_n - b_n) = 0$

the theorem will be proven. To prove (I), define

$$Z_{nk}(x) \equiv \text{lesser of } Y_{nk} \text{ and } 0$$

$$z_{nk}(x) \equiv \text{greater of } y_{nk} \text{ and } 0$$

so that $$B_n \geqq \int Y_{nk}d\phi \geqq \int Z_{nk}d\phi$$

and $$b_n \leqq \int y_{nk}d\phi \leqq \int z_{nk}d\phi$$

But Z_{nk} is monotone increasing in k and has 0 for its limit. By the corollary to Lemma (24.4) we have

$$\lim_{k\to\infty} \int Z_{nk}d\phi = 0$$

and similarly by (24.4) itself

$$\lim_{k\to\infty} \int z_{nk}d\phi = 0.$$

This proves (I).

To prove (II), let e be given. Select a sequence $\{e_n\}$ of positive numbers such that $\sum e_n$ converges and is less than $e/4$. Choose r_n so that

$$B_n - \int_a^b Y_{nk}d\phi < e_n$$

and $$\int_a^b y_{nk}d\phi - b_n < e_n \qquad \text{for all } k \geqq r_n.$$

Now define: $H_n(x) \equiv$ least of $(Y_{1r_1}, Y_{2r_2}, \cdots, Y_{nr_n})$

$h_n(x) \equiv$ greatest of $(y_{1r_1}, y_{2r_2}, \cdots, y_{nr_n})$.

Then certainly $H_n \geqq H_{n+e}$ and $h_n \leqq h_{n+e}$. Let $k > r_1, r_2, \cdots, r_n$. From the definitions we have that

$$Y_{ik} - Y_{ir_i} = 0 \qquad (i = 1,2,\cdots,n)$$

and that $Y_{nk} - H_n$ is less than or at most equal to at least one of them.

Therefore
$$Y_{nk} - H_n \leqq \sum_{i=1}^{n} Y_{ik} - Y_{ir_i}$$

Integrating with respect to ϕ and allowing k to increase without limit we have
$$B_n - \int_a^b H_n d\phi < \sum_1^n e_n < e/4$$

By a similar argument
$$\int_a^b h_n d\phi - b_n < e/4$$

$$B_n - b_n < e/2 + \quad (H_n - h_n) d\phi$$

But $H_n - h_n$ is monotone decreasing and its limit is nowhere positive.

Hence defining $K_n \equiv$ greater of $(H_n - h_n)$ and 0

we have that
$$B_n - b_n < e/2 + \int K_n d\phi$$

and applying Lemma (24.3) the right hand side tends to $e/2$ as $n \to \infty$.

Thus $\lim_{n\to\infty} (B_n - b_n) = 0$.

CONCLUSION AND SUMMARY

In part 1, vector spaces with closure under multiplication by real and complex numbers were defined. Polynomials were then defined and discussed for both types of spaces, and it was shown that for spaces E(R) the definition given was equivalent to that of Frechet. A special result of some interest was given on the continuity of a multilinear form in the ensemble of its variables. Modular properties of polynomials were discussed.

In division (14) application of these results was made in the definition of analytic functions in vector spaces. The field of this division is essentially virgin. It was pointed out that a distinction arises between spaces E(R) and E(C) in the methods necessary to prove certain theorems on analytic functions. Various theorems of a fundamental nature were proven for analytic functions on spaces E(C) and a fewer number for those on spaces E(R). It was shown that the generalization of the completely integrable Pfaffian System has, in spaces E(C), a unique analytic solution.

In part 2 there was given a generalization of Frechet's theorem on the representability by means of double Stieltjes integrals of the general bilinear continuous functional form on the space of continuous functions. It was proven that a multilinear form in n variables is representable by means of a multiple Stieltjes integral and that the kernel of the integral if regular is unique. A proof by elementary methods was also given of the generalization to Riemann Stieltjes integrals of Arzela's necessary and sufficient condition for the term by term integrability of a sequence of Riemann integrable functions.

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