Electromagnetic Pulses at the Boundary of a Nonlinear Plasma

Thesis by

Edgar H. Satorius

In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology
Pasadena, California

1975

(Submitted June 26, 1974)

ACKNOWLEDGMENT

The author wishes to express his indebtedness to his advisor, Professor C. H. Papas, for his guidance and encouragement throughout the course of this research.

The author acknowledges with thanks a number of helpful discussions with Mr. F. C. Yang, Dr. J. N. Franklin, and Dr. T. C. Mo. Special thanks are extended to Ruth Stratton and Ginny Ginegaw who typed the manuscript.

The author is grateful for the generous financial support he received through a Northrop Corporation Fellowship. He wishes to thank the Naval Undersea Center for the use of their computer to make all the numerical calculations and graphs in this paper.

The patience, encouragement, and assistance of the author's wife, Helen, are gratefully acknowledged.

Electromagnetic Pulses at the Boundary of a Nonlinear Plasma

E. H. Satorius

ABSTRACT

This paper describes an investigation of the behavior of strong electromagnetic pulses at the boundary of a nonlinear, cold, collisionless, and uniform plasma. The nonlinearity considered here is due to the nonlinear terms in the fluid equation which is used to describe the plasma.

Two cases are studied. First, we consider the case where there is a voltage pulse applied across the plane boundary of a semi-infinite, nonlinear plasma. Two different voltage pulses are considered: a delta function pulse and a suddenly turned-on sinusoidal pulse. The resulting electromagnetic fields propagating in the nonlinear plasma are found in this case. In the second case, we consider the reflection of incident E-polarized and H-polarized, electromagnetic pulses at various angles of incidence from a nonlinear, semi-infinite plasma. Again, two forms of incident pulses are considered: a delta function pulse and a suddenly turned-on sinusoidal pulse. In case two, the reflected electromagnetic fields are found.

In both cases, the method used for finding the fields is to first solve the fluid equation (which describes the plasma) for the nonlinear conduction current in terms of the electric field using a perturbation method (since the nonlinear effects are assumed to be small). Next, this current is substituted into Maxwell's equations, and finally the electromagnetic fields which satisfy the boundary conditions are found.

TABLE OF CONTENTS

1. Introduction

Among the first observed nonlinear effects in a plasma was the cross modulation of broadcast signals in the ionosphere by the strong Luxembourg station signal. This was reported by Tellegan's paper [1] in 1933. In order to explain Tellegan's observations, Bailey and Martyn [2] considered the heating effect of a passing electromagnetic wave on the collision frequency which in turn affects the propagation of another wave in the disturbed medium. With known numerical values for the physical parameters of the ionosphere and the broadcast signals, their theory predicts a detectable cross modulation.

A more formal approach to the problems of nonlinear wave propagation in plasmas involves solving Boltzmann's kinetic equation for the electron distribution function in an ionized gas in the presence of disturbing electromagnetic waves. Then, from a knowledge of the distribution function, the current is obtained through the relation

$$\underline{j}(\underline{r};t;\underline{E}) = Ne \int \underline{v}\, f(\underline{r};t;\underline{v};\underline{E})\, d\underline{v} \tag{1.1}$$

where $f(\underline{r};t;\underline{v};\underline{E})$ is the single electron distribution function, N is the electron density of the plasma, and $\underline{E}$ is the electric field. It is noted here that throughout the rest of this thesis we assume that only electrons contribute to the conduction current.

Expression (1.1) is expanded in a power expansion in $\underline{E}$, viz.,

$$\hat{j}_i(\vec{k};\underline{E}) = \sum_{n=1}^{\infty} \int d\lambda^{(n)} \sigma_{ij_1 \cdots j_n}(\vec{k},\vec{k}_1,\ldots,\vec{k}_n)\, \hat{E}_{j_1}(\vec{k}_1)\cdots\hat{E}_{j_n}(\vec{k}_n) \qquad (1.2)$$

(sum on repeated indices)

where $\vec{k}$ is the four vector $(\underline{k},\omega)$*

$$\hat{j}_i(\vec{k};\underline{E}) = \frac{1}{(2\pi)^4} \int e^{-i\underline{k}\cdot\underline{r} + i\omega t}\, j_i(\underline{r},t)\, d^3\underline{r}\, dt$$

($\hat{j}_i$ is the i^{th} component of $\hat{\underline{j}}$),

and

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4\, \delta^4(k - k_1 - \cdots - k_n)$$

Note that the n^{th} order term in (1.2) defines the $(n+1)^{th}$ rank tensor $\sigma_{ij_1\cdots j_n}$, which is the n^{th} order conductivity tensor. Equation (1.2) is then substituted into Maxwell's equations and the resulting non-linear equations are solved for $\underline{E}$. This is the approach which will be carried out in Chapter 2.

Al'tshul and Karpman [3] derive the expansion (1.2) (for a collisionless plasma) in terms of the unperturbed Hamiltonian,

$$H_o = \frac{1}{2m}(\underline{P} - \frac{e}{c}\underline{A}_o)^2$$

where $\underline{P}$ is the canonical momentum and $\underline{B}_o = \nabla\times\underline{A}_o$ is the background

---

*In this thesis all four vectors will be written with an arrow. Also, a carat over a variable will denote the Fourier transform of that variable with respect to space and time. A tilde over a variable will denote the Fourier transform of that variable with respect to time only.

magnetic field. Melrose [4] also derives (1.2) (for a collisionless plasma)* using the relativistic, unperturbed Hamiltonian (which simplifies the formal calculation),

$$H_0 = [m^2c^4 + (\underline{P} - \frac{e}{c}\underline{A}_0)^2c^2]^{1/2}$$

Tsytovich [5,6] develops an expression for the lowest order, nonlinear conductivity tensor of a collisionless plasma $\sigma_{ij\ell}$, which simplifies considerably when it is assumed that the plasma is also cold.** These results of Tsytovich are further discussed and made use of in the next chapter. Chan [11] derives an expansion similar to (1.2) for the case of time harmonic, plane electromagnetic waves. Unlike the papers [3-6], Chan considers a nonlinear plasma for which the nonlinearity is due mainly to collisions. For a brief discussion of two different types of plasma nonlinearities (due to heating and due to the Lorentz force term) one is referred to Appendix D.

Analogous expansions to (1.2) expressing the polarization in terms of the electric field are utilized in the study of nonlinear optics, viz.,

---

* Actually, [3] and [4] derive expansions for the current in terms of the vector potential $\underline{A}$. But $\underline{A}$ can be expressed in terms of $\underline{E}$ through the gauge equation. See, for example, Ref. [14], page 14.

**As discussed in [5,6], a plasma is considered cold when the mean thermal velocity of the plasma electrons is much less than the phase velocity of the electromagnetic waves in the plasma.

$$P_i(\underline{r},\omega) = \sum_{n=1}^{\infty} \int \chi_{ij_1\ldots j_n}(\omega_1,\omega_2,\ldots\omega_n)\tilde{E}_{j_1}(\omega_1)\ldots\tilde{E}_{j_n}(\omega_n)dx^{(n)} \qquad (1.3)$$

where $\tilde{P}_i(\underline{r},\omega) = \frac{1}{2\pi}\int P_i(\underline{r},t)e^{i\omega t}dt$; ($P_i$ the i^{th} component of $\underline{P}$, the polarizability); and $dx^{(n)} = \delta(\omega-\omega_1-\ldots-\omega_n)\, d\omega_1\ldots d\omega_n$.

Equation (1.3) and its Fourier inverse are further discussed by Owyoung [7] and Butcher [8]. Bloembergen [9] derives the lowest order, nonlinear susceptibility tensors $\chi_{ij_1\ldots j_n}$. This derivation is quantum mechanical, that is, the electron distribution function in phase space is replaced by the density matrix ρ, and the variables $\underline{P}$ and $\underline{r}$ are replaced by their corresponding quantum mechanical operators. Kubo [10] gives a further discussion of the kinetic and quantum mechanical approaches to the study of nonlinear media.

After the tensors $\sigma_{ij_1\ldots j_n}$ have been derived (or $\chi_{ij_1\ldots j_n}$ in the case of a nonlinear crystal), one is faced with the problem of solving the nonlinear Maxwell's equations for the electromagentic field. Various approximate schemes have been devised for solving these equations. In the case of nonlinear plasmas reference is made to Chan [11], Bassaninni [12], Tsytovich [6], and the book by Ginzburg [13], especially Chapter 8. Ginzburg's book has a very thorough set of references dealing with electromagnetic wave propagation in a nonlinear plasma. Further references can be found in the paper by Bornatici and Engelmann [15]. In the case of nonlinear optics, reference is made to Armstrong, Bloembergen, Ducuing, and Pershan [16], Bloembergen and Pershan [17], Bloembergen [9], and Small [18].

An iterative scheme used most notably in [9], [11] and [17] and which will be used in Chapter 2 is to expand the electric field in a perturbation expansion

$$\hat{\underline{E}} = \lambda\hat{\underline{E}}^{(1)} + \lambda^2\hat{\underline{E}}^{(2)} + \ldots\ldots \qquad (1.4)$$

where $\hat{\underline{E}}^{(1)}$ is the solution to the linear Maxwell's equations with

$$\hat{j}_i(\vec{K};\underline{E}) = \sigma_{i\ell}(\vec{K},\vec{K})\hat{E}_\ell\,(\vec{K}) \quad (\text{or, } \tilde{P}_i(\underline{r},\omega) = \chi_{i\ell}\,(\omega)\tilde{E}_\ell(\omega))$$

Substitution of (1.4) into the nonlinear Maxwell's equations will yield a hierarchy of equations--the n^{th} equation being the wave equation for $\underline{E}^{(n)}$ with a source distribution depending on $\underline{E}^{(n-1)}, \underline{E}^{(n-2)}, \ldots, \underline{E}^{(1)}$. Therefore, by solving for $\underline{E}^{(1)}$ one can generate, in principle, all of the $\underline{E}^{(n)}$. In practice, however, one usually neglects all terms higher than n=3 in (1.4).

Another approximate scheme used in [6], [16] and [9] is to write the solution $\underline{E}$ in the form

$$\underline{E} = \underline{A}(\underline{r},t)\, e^{i\underline{k}\cdot\underline{r} - i\omega t} \qquad (1.5)$$

where $\underline{A}(\underline{r},\underline{t})$ is a slowly varying function of $\underline{r}$ and t, for which only the first derivatives with respect to $\underline{r}$ and t need be included. An equation for $\underline{A}$ is then derived which is solved. Whitham [19] has developed an interesting approximate method which is used in [18]. This method consists of deriving equations for averaged quantities (which are related to the general solution) from conservation equations, such as the energy equation.

With the exception of [18], harmonic time dependence is assumed in the above references, [11],[16],[17]. A study of the transient behavior of electromagnetic waves in nonlinear plasmas and crystals has not received quite so much attention. Among the first to study transients in nonlinear plasmas was Fejer [20]. His paper dealt with the interaction of short, pulsed radio waves in the ionosphere. Kroll [21] considered the transient build up of electromagnetic waves from initial noise levels in crystals which are under intense illumination from suddenly turned on lasers. Kroll's method is to linearize the nonlinear, coupled electromagnetic-elastic wave equations by writing the electric displacement field as

$$\underline{D} = \underline{D}_0 \cos(\omega t - \underline{k}\cdot\underline{r}) + \underline{D}_1(\underline{r},t)$$

where $\underline{D}_0$ is a constant and $|\underline{D}_0| \gg |\underline{D}_1|$. The resulting linear equations in $\underline{D}_1$ and $\underline{u}$ (the elastic displacement) are then solved. Kryukov and Letokhov [22] consider the nonlinear propagation of light pulses in a resonantly amplifying medium. They use the approach described above in connection with equation (1.5).

Aside from these papers, [20]-[22], the literature dealing with transient electromagnetic wave propagation in nonlinear media is relatively sparse. There are, however, several reasons why an investigation of nonlinear transient phenomena is desirable. At present, various parameters of a plasma are measured by observing the linear transient electromagnetic fields which propagate in a plasma and are reflected from a semi-infinite plasma.* However, it may be possible to obtain more information about plasma constants such as the plasma frequency by observing the lowest order nonlinear transient fields which propagate in the plasma. We remark here that there are a number of

---

* See, for example, reference [26]

papers concerning linear transient pulses in a plasma and pulse reflection from a plasma. Reference is made to Case [23], Knop [24], Wait [25], Chabries and Bolle [26], and Kenny [27]. The papers [23]-[26] are concerned with the linear response to electromagnetic waves incident upon half-spaces and slabs of isotropic plasmas. Kenny [27] considers the problem of radiation in isotropic and uniaxial plasmas by a suddenly turned-on, harmonic dipole.

Another reason for the study of transients, at least in the case of nonlinear optics, is the recent development of "mode-locked" lasers which are capable of generating intense light pulses with a duration on the order of a picosecond. Actually, M.J. Colles at Harvard University and Kaiser and his co-workers at the Technical University of Munich have discovered mode-locking procedures capable of generating pulses as short as .3 picoseconds. For a very good summary of the advances in this field, reference is made to the article by Alfano and Shapiro [28] . Reference is also made to the introduction of the paper by Kryukov and Letokhov [22].

It is the purpose of this thesis to determine the lowest order, nonlinear transient response as well as the steady state response, from an isotropic, cold, collisionless plasma. In Chapter 2 the hydrodynamic equations are used* to derive the lowest order nonlinear tensor $\sigma_{ik\ell}$. Also in Chapter 2 we derive the nonlinear conduction current. In Chapter 3 we solve the nonlinear Maxwell's equations for the case of transient propagation in an infinite nonlinear plasma. In this chapter we consider the case of an electromagnetic field where the

---

*The use of the hydrodynamic equations is justified in the limit of a collisionless plasma. For a further discussion of this point, see Ref. [29], page 52.

component of the electric field $\underline{E}$ perpendicular to the direction of propagation is originally a delta pulse. We also consider the case where $\underline{E}$ is originally in the form of a suddenly turned-on sinusiod, viz.,

$$\underline{E}\,(\underline{0},t) = \underline{E}_0\,\delta(t) \qquad \text{Case I}$$

$$\underline{E}\,(\underline{0},t) = \underline{E}_0 \cos(\omega_0 t)\, H(t) \qquad \text{Case II}$$

Then, writing $\underline{E} = \underline{E}^{(1)} + \underline{E}^{(2)}$ where $\underline{E}^{(1)}$ is the appropriate linear field with the given initial conditions, we solve for $\underline{E}^{(2)}$--the lowest order nonlinear response. In Chapter 4, a calculation of the lowest order nonlinear response is made for the case of reflection from a semi-infinite, nonlinear plasma. In this chapter we consider the case when the incident electric field is a delta function, and the case when the incident electric field is a suddenly turned-on sinusiod. The method used is essentially that used in [17], but adapted to the case of arbitrary time dependent fields.

2. Nonlinear Conduction Current and Maxwell's Equations in a Nonlinear Plasma

Throughout the rest of this thesis we consider an isotropic, cold, collisionless plasma and therefore will use the hydrodynamics equation in its description,* i.e.,

$$\frac{\partial \underline{v}}{\partial t} + (\underline{v}\cdot\underline{\nabla})\underline{v} = \frac{e}{m}\left(\underline{E} + \frac{1}{c}\,\underline{v} \times \underline{H}\right) \tag{2.1}$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\underline{v}) = 0 \tag{2.2}$$

$$\underline{j} = en\underline{v} \tag{2.3}$$

where, $\underline{v}$ is the electron velocity; n is the electron density; $\underline{j}$ is the conduction current; $\underline{E}$, $\underline{H}$ is the electromagnetic field; c is the velocity of light, and m and e are the electron mass and charge, respectively.

What is desired is an expansion much like (1.2) for $\hat{\underline{j}}\,(\vec{k})$. Therefore, the first step is to Fourier transform equations (2.1) - (2.3). It is noted that the following definitions will be used concerning Fourier transforms:

If $\phi(\underline{r},t)$ is some function, then its Fourier transform will be given by

---

* In most of what follows, we will be using the development found in reference [6], Chapter 2.

$$\hat{\phi}(\vec{K}) = \frac{1}{(2\pi)^4} \int \phi(\underline{r},t)\, e^{i(\omega t - \underline{k}\cdot\underline{r})} d\underline{r}dt$$

where $\vec{K}$ is the four vector $(\underline{k},\omega)$.

Of course, a function can always be recovered from its Fourier transform by the relation

$$\phi(\underline{r},t) = \int \hat{\phi}\,(\vec{K}) e^{-i(\omega t - k\cdot r)} d\vec{K}$$

Use will also be made of the fact that if $\phi(\underline{r},t) = g(\underline{r},t)h(\underline{r},t)$, then

$$\hat{\phi}\,(\vec{K}) = \int \hat{g}(\vec{K}_1)\hat{h}(\vec{K} - \vec{K}_1)\, d\vec{K}_1$$

$$\overset{\text{def}}{=} \hat{g}(\vec{K}) * \hat{h}(\vec{K})$$

Likewise, if $\hat{\phi}(\vec{K}) = \hat{g}(\vec{K})\hat{h}(\vec{K})$, then

$$\phi(\underline{r},t) = (2\pi)^4 \int g(\underline{r} - \underline{r}_0,\, t - t_0) h(\underline{r}_0,t_0) d\underline{r}_0 dt_0$$

$$\overset{\text{def}}{=} (2\pi)^4\, g(\underline{r},t) * h\,(\underline{r},t)$$

Therefore, the Fourier transforms of equations (2.1) - (2.3) are given by

$$-i\omega\hat{\underline{v}}(\vec{K}) + i \int (\hat{\underline{v}}(\vec{K}_1) \cdot \underline{k}_2)\hat{\underline{v}}(\vec{K}_2)\, d\lambda = \frac{e}{m}\hat{\underline{E}}(\vec{K}) + \frac{e}{mc}\int \hat{\underline{v}}(\vec{K}_1) \times \hat{\underline{H}}(\vec{K}_2)\, d\lambda \tag{2.4}$$

$$\omega\hat{n}(\vec{k}) = \underline{k}\cdot\left(\int \hat{n}(\vec{k}_1)\hat{\underline{v}}(\vec{k}_2) d\lambda\right) \tag{2.5}$$

$$\hat{\underline{j}}(\vec{K}) = e \int \hat{n}(\vec{K}_1)\hat{\underline{v}}(\vec{K}_2)\, d\lambda \tag{2.6}$$

where $d\lambda = d\vec{k}_1 d\vec{k}_2 \delta(\vec{k} - \vec{k}_1 - \vec{k}_2)$.

To obtain the desired expansion (1.2), we solve equations (2.4) - (2.6) for $\hat{\underline{j}}(\vec{k})$ in terms of $\hat{\underline{E}}(\vec{k})$ as follows

We first use the Maxwell equation

$$\frac{1}{c}\frac{\partial \underline{H}}{\partial t} = -\nabla \times \underline{E}$$

or, equivalently,

$$\frac{1}{c}\hat{\underline{H}}(\vec{k}) = \frac{\underline{k}}{\omega} \times \hat{\underline{E}}(\vec{k})$$

Then we can write the Lorentz force as

$$\frac{\hat{\underline{v}}(\vec{k}_1)}{c} \times \hat{\underline{H}}(\vec{k}_2) = \frac{1}{\omega_2}\hat{\underline{v}}(\vec{k}_1) \times (\underline{k}_2 \times \hat{\underline{E}}(\vec{k}_2))$$

$$= \frac{\underline{k}_2}{\omega_2}(\hat{\underline{v}}(\vec{k}_1)\cdot\hat{\underline{E}}(\vec{k}_2)) - \frac{\hat{\underline{E}}(\vec{k}_2)}{\omega_2}(\underline{k}_2\cdot\hat{\underline{v}}(\vec{k}_1)) \qquad (2.7)$$

Substituting (2.7) into (2.4), we have

$$\hat{\underline{v}}(\vec{k}) = \frac{ie}{\omega m}\hat{\underline{E}}(\vec{k}) + \frac{ie}{\omega m}\int \frac{\underline{k}_2}{\omega_2}(\hat{\underline{v}}(\vec{k}_1)\cdot\hat{\underline{E}}(\vec{k}_2))d\lambda + \frac{1}{\omega}\int(\underline{k}_2\cdot\hat{\underline{v}}(\vec{k}_1))\cdot$$

$$\cdot(\hat{\underline{v}}(\vec{k}_2) - \frac{ie}{m\omega_2}\hat{\underline{E}}(\vec{k}_2))d\lambda \qquad (2.8)$$

Note that all terms on the right hand side of (2.8) except the first describe nonlinear effects. The first, on the other hand, describes the familiar small oscillations of a free charge in a wave field $\hat{\underline{E}}(\vec{k})$. Assuming the nonlinear effects are small and neglecting them in the first approximation, we have

$$\hat{\underline{v}}(\vec{k}) \approx \hat{\underline{v}}^{(1)}(\vec{k}) = \frac{ie}{\omega m}\hat{\underline{E}}(\vec{k}) \tag{2.9}$$

To find the lowest order nonlinear effect, with respect to $\hat{\underline{E}}(\vec{k})$, we substitute (2.9) into the nonlinear term (2.8). We see at once that the last term on the right-hand side of (2.8) becomes zero, and obtain

$$\hat{\underline{v}}^{(2)}(\vec{k}) = -\frac{e^2}{m^2\omega}\int \frac{\underline{k}_2}{\omega_1\omega_2}\left(\hat{\underline{E}}(\vec{k}_1)\cdot\hat{\underline{E}}(\vec{k}_2)\right)d\lambda$$

$$= -\frac{e^2}{2m^2\omega}\int \frac{\underline{k}}{\omega_1\omega_2}\left(\hat{\underline{E}}(\vec{k}_1)\cdot\hat{\underline{E}}(\vec{k}_2)\right)d\lambda \tag{2.10}$$

Here we have symmetrized the result for $\underline{k}_2 \to (\underline{k}_1 + \underline{k}_2)/2 = \underline{k}/2$. This is always possible because the remaining expressions are symmetric with respect to the indices 1 and 2.

Substituting the expansion $\hat{\underline{v}}(\vec{k}) = \hat{\underline{v}}^{(1)}(\vec{k}) + \hat{\underline{v}}^{(2)}(\vec{k})$ into (2.5) and assuming that in the zeroth order approximation $n(\underline{r},t) \simeq n_o(\underline{r},t) \equiv n_o$ (where n_o is the electron density of a homogeneous plasma) we have:

$$\hat{n}(\vec{k}) = \hat{n}^{(o)}(\vec{k}) + \hat{n}^{(1)}(\vec{k}) \tag{2.11}$$

where,

$$\hat{n}^{(o)}(\vec{k}) = \frac{1}{(2\pi)^4}\int n_o(\underline{r},t)e^{i(\omega t-\underline{k}\cdot\underline{r})}d\underline{r}dt = n_o\delta(\vec{k}) \tag{2.12}$$

and,

$$\hat{n}^{(1)}(\vec{k}) = \frac{1}{\omega}\underline{k}\cdot\left(\int \hat{n}^{(0)}(\vec{k}_1)\hat{\underline{v}}^{(1)}(\vec{k}_2)d\lambda\right)$$

$$= \frac{n_o}{\omega} \underline{k} \cdot \left(\int \delta(\vec{k}_1) \hat{\underline{v}}^{(1)}(\vec{k}_2) \delta(\vec{k}-\vec{k}_2) d\vec{k}_1 d\vec{k}_2 \right)$$

$$= \frac{n_o}{\omega} \underline{k} \cdot \hat{\underline{v}}^{(1)}(\vec{k}) \qquad (2.13)$$

Substituting this expansion (2.11) into (2.6) gives the desired result.

$$\hat{\underline{j}}(\vec{k}) = \hat{\underline{j}}^{(1)} + \hat{\underline{j}}^{(2)} \qquad (2.14)$$

where,

$$\hat{\underline{j}}^{(1)}(\vec{k}) = e \int \hat{n}^{(0)}(\vec{k}_1) \hat{\underline{v}}(\vec{k}_2) d\lambda$$

$$= e\, n_o \int \delta(\vec{k}_1) \delta(\vec{k} - \vec{k}_1 - \vec{k}_2) \hat{v}^{(1)}(\vec{k}_2) d\vec{k}_1 d\vec{k}_2$$

$$= e\, n_o \hat{\underline{v}}^{(1)}(\vec{k})$$

Substitution of (2.9) into this relation gives,

$$\hat{\underline{j}}^{(1)}(\vec{k}) = \frac{ie^2 n_o}{\omega\, m} \hat{\underline{E}}(\vec{k}) = \frac{i\omega_p^{\,2}}{4\pi\omega} \hat{\underline{E}}(\vec{k}) \qquad (2.15)$$

where,

$$\omega_p^{\,2} = \frac{4\pi\, n_o e^2}{m} \quad \text{is the plasma frequency.}$$

We also have:

$$\hat{\underline{j}}^{(2)}(\vec{k}) = e \int \left(\hat{n}^{(1)}(\vec{k}_1) \hat{\underline{v}}^{(1)}(\vec{k}_2) + \hat{n}^{(o)}(\vec{k}_1) \hat{\underline{v}}^{(2)}(\vec{k}_2) \right) d\lambda$$

$$= e \int \left[\frac{n_o ie}{m\omega_1^{\,2}} (\underline{k}_1 \cdot \hat{\underline{E}}(\vec{k}_1)) \frac{ie}{\omega_2 m} \hat{\underline{E}}(\vec{k}_2) - \frac{n_o e^2 \underline{k}}{2m^2 \omega\omega_1\omega_2} \hat{\underline{E}}(\vec{k}_1) \cdot \hat{\underline{E}}(\vec{k}_2) \right] d\lambda$$

After symmetrization of the first term with respect to the indices 1 and 2 we finally obtain

$$\hat{\underline{j}}^{(2)}(\vec{k}) = -\frac{\omega_p^2 e}{8\pi m}\int \frac{d\lambda}{\omega_1\omega_2}\left[\frac{\underline{k}}{\omega}\hat{\underline{E}}(\vec{k}_1)\cdot\hat{\underline{E}}(\vec{k}_2) + \hat{\underline{E}}(\vec{k}_1)\frac{\underline{k}_2\cdot\hat{\underline{E}}(\vec{k}_2)}{\omega_2} + \hat{\underline{E}}(\vec{k}_2)\frac{\underline{k}_1\cdot\hat{\underline{E}}(\vec{k}_1)}{\omega_1}\right] \tag{2.16}$$

Equation (2.14) is the required expansion of $\hat{\underline{j}}(\vec{k})$ in terms of the electric field. It is not quite in the same form as (1.2). We can, however, obtain the lowest order conductivity tensors σ_{ij} and σ_{ijn} by writing

$$\hat{j}_i(\vec{k}) = \int \delta(\vec{k}-\vec{k}_1)\, d\vec{k}_1 \sigma_{ij}(\vec{k},\vec{k}_1)\hat{E}_j(\vec{k}_1) + \frac{1}{(2\pi)^4}\int d\vec{k}_1 d\vec{k}_2 \delta\,(\vec{k}-\vec{k}_1)\sigma_{ijn}(\vec{k},\vec{k}_1,\vec{k}_2)\cdot \cdot\hat{E}_j(\vec{k}_1)\hat{E}_n(\vec{k}_2) \tag{2.17}$$

Comparing the expansion (2.14) with (2.17) gives,

$$\sigma_{ij}(\vec{k},\vec{k}_1) = \frac{i\omega_p^2}{4\pi\omega_1}\,\delta_{ij} \tag{2.18}$$

and *

$$\sigma_{ijn}(\vec{k},\vec{k}_1,\vec{k}_2) = \frac{-e\omega_p^2(2\pi)^4}{8\pi m\omega_1\omega_2}\left(\delta_{ij}\frac{k_{2n}}{\omega_2} + \delta_{in}\frac{k_{1j}}{\omega_1} + \delta_{jn}\frac{k_i}{\omega}\right) \tag{2.19}$$

---

*The factor $(2\pi)^4$ in (2.19) is not present in reference [5] or [6]. This is due to the definition of $d\lambda^{(n)}$ in (1.2).

It is seen that the first term in (2.16) is associated with the nonlinear velocity, $\hat{\underline{v}}^{(2)}$. This electron velocity arises from two terms in (2.1). One term is the Lorentz force which interacts with the linear velocity $\underline{v}^{(1)}$ through the force $\frac{e}{mc}\ \underline{v}^{(1)} \times \underline{H}$. The other is the term $(\underline{v}\cdot\nabla)\ \underline{v}$ on the right hand side of (2.1). It is also seen from equation (2.10) that $\hat{\underline{v}}^{(2)}$ is in the direction of $\underline{k}$. Therefore, the first term in (2.16) represents longitudinal conduction currents in the plasma which are excited by the nonlinear interaction of either longitudinal or transverse waves. The last two terms in (2.16) correspond to the first order correction of the electron density $\hat{n}^{(1)}(\vec{k})$. These two terms can only be excited in the first approximation by longitudinal waves. Transverse waves, for which $\nabla\cdot\underline{E}=0$, cannot excite those terms. Therefore, we can attribute the second order, nonlinearity of a cold, collisionless plasma to the nonlinear interaction of the Lorentz force with the linear velocity $\hat{\underline{v}}^{(1)}$ as well as the first order correction term $\hat{n}^{(1)}$ to the equilibrium electron density.

Before turning to the nonlinear, Maxwell equations, we note that the effects of temperature and collisions have been neglected in deriving (2.16). Of course, throughout this thesis we will not consider these effects. However, to properly include these effects one must start with the kinetic equation. Using the Vlasov equation (2.20), Tsytovich [5] derives an expansion in the form of (1.2) for the conduction current of a collisionless plasma in which the equilibrium temperature is not zero

$$\frac{\partial f}{\partial t} + \underline{v}\cdot\nabla f + me\ [\underline{E} + \frac{1}{c}\ \underline{v} \times \underline{H}] \cdot \nabla_{\underline{p}} f = 0 \qquad (2.20)$$

Tsytovich substitutes the expansion $f = \sum_{i=o}^{\infty} f_i\gamma^i$ into the Vlasov equation. In this expansion, f is the electron distribution function and γ is the expansion parameter. A hierarchy of equations are obtained for the f_i. For instance, the Fourier transform of the equation for f_i is

$$i(\omega-\underline{k}\cdot\underline{v})\hat{f}_1(\underline{p},\vec{k}) = e\underline{\hat{F}}(\vec{k})\cdot\nabla_{\underline{p}} f_o(\underline{p}) \tag{2.21}$$

where $f_o(\underline{p})$ is the equilibrium distribution function.

$$f_o(\underline{p}) = n_o \frac{m}{(2\pi kT)^{3/2}} e^{-\frac{p^2}{2mkT}} \quad \begin{cases} \underline{p} = \text{electron mementum} \\ T = \text{temperature of electrons} \\ k = \text{Boltzmann's constant} \end{cases}$$

with $\nabla_{\underline{p}} = \frac{\partial}{\partial p_x}\underline{e}_x + \frac{\partial}{\partial p_y}\underline{e}_y + \frac{\partial}{\partial p_z}\underline{e}_z$, and $\underline{\hat{F}}(\vec{k}) = \underline{\hat{E}}(\vec{k})(1-\underline{k}\cdot\underline{v}/\omega) + \frac{1}{\omega} k\,(\underline{v}\cdot\underline{\hat{E}}(\vec{k}))$.

The Fourier transform of the equation for f_2 is

$$i(\omega-\underline{k}\cdot\underline{v})\hat{f}_2(p,\vec{k}) = e\int\underline{\hat{F}}(\vec{k}_1)\cdot\nabla_{\underline{p}}\hat{f}_1(\underline{p},\vec{k}_2)d\lambda \tag{2.22}$$

where

$$\hat{f}_1(\underline{p},\vec{k}) = \frac{e\underline{\hat{F}}(\vec{k})\cdot\nabla_p f_o(\underline{p})}{i(\omega-\underline{k}\cdot\underline{v})} \qquad \text{(from (2.21))}$$

and $d\lambda = d\vec{k}_1 d\vec{k}_2 \delta(\vec{k}-\vec{k}_1-\vec{k}_2)$.

Equations (2.21) and (2.22) yield for the linear and second order, nonlinear currents

$$\hat{\underline{j}}^{(1)}(\vec{k}) = \frac{e}{m} \int \underline{p}\, \hat{f}_1(\underline{p},\vec{k}) \frac{d\underline{p}}{(2\pi)^3}$$

$$= \frac{e^2}{mi} \int \underline{p}\, \frac{\hat{\underline{F}}(\vec{k})\cdot\nabla_{\underline{p}} f_o(\underline{p})}{\omega - \underline{k}\cdot\underline{v}} \frac{d\underline{p}}{(2\pi)^3} \qquad (2.23)$$

$$\hat{\underline{j}}^{(2)}(\vec{k}) = \frac{e}{m} \int \underline{p}\, \hat{f}_2(\underline{p},\vec{k}) \frac{d\underline{p}}{(2\pi)^3} \qquad (2.24)$$

As a matter of comparison with (2.19), Tsytovich derives the nonlinear conductivity tensor σ_{ijn} from (2.22) and (2.24).*

$$\sigma_{ijn} = -e^3(2\pi)^4 \int \frac{v_i}{\omega-\underline{k}\cdot\underline{v}} \left[\left(1 - \frac{\underline{k}_1\cdot\underline{v}}{\omega_1}\right)\frac{\partial}{\partial p_j} + \frac{v_j}{\omega_1}(\underline{k}_1\cdot\nabla_{\underline{p}})\right]\cdot$$

$$\frac{1}{\omega_2 - \underline{k}_2\cdot\underline{v}} \left[\left(1 - \frac{\underline{k}_2\cdot\underline{v}}{\omega_2}\right)\frac{\partial}{\partial p_n} + \frac{v_n}{\omega_2}(\underline{k}_2\cdot\nabla_{\underline{p}})\right]\cdot$$

$$f_o(\underline{p}) \frac{d_{\underline{p}}}{(2\pi)^3} \qquad (2.25)$$

Equation (2.25) represents the second order conductivity tensor for a warm, collisionless plasma. As shown in [5], this expression for σ_{ijn} reduces to that given by (2.19) when the mean thermal velocity of the

*See footnote (2) concerning the factor $(2\pi)^4$ in (2.25).

plasma electrons, $\sqrt{kT/m}$, is much smaller than the phase velocity of the electromagnetic waves in the plasma. It is noted that some authors have derived the nonlinear conductivity tensors from Boltzmann's kinetic equation by expanding the electrom distribution function in Legendre polynomials $P_k(\cos\alpha)$, where α is the angle between $\underline{E}$ and $\underline{v}$.*

We now proceed to derive Maxwell's equations for a cold, collisionless nonlinear plasma. The nonlinear current (2.14) is seen to be a function of the electric field. This current is in turn a source for the electromagnetic field in Maxwell's equations. These equations are written in cgs units (the rest of this thesis will use cgs units).

$$\nabla \times \underline{B} = \frac{1}{c}\frac{\partial \underline{E}}{\partial t} + \frac{4\pi}{c}\underline{j} \tag{2.26}$$

$$\nabla \times \underline{E} = -\frac{1}{c}\frac{\partial \underline{B}}{\partial t} \tag{2.27}$$

$$\nabla \cdot \underline{E} = 4\pi\rho \tag{2.28}$$

$$\nabla \cdot \underline{B} = 0 \tag{2.29}$$

The Fourier transforms of these equations are

$$\nabla \times \tilde{\underline{B}} = \frac{-i\omega}{c}\tilde{\underline{E}} + \frac{4\omega}{c}\tilde{\underline{j}} \tag{2.30}$$

---

*See for example [11] chapter 3. Ginzburg [13] (page 506) uses the expansion of f in terms of Legendre polynomials as a method to solve (approximately) Boltzmann's equation.

$$\nabla \times \tilde{\underline{E}} = \frac{i\omega}{c} \tilde{\underline{B}} \tag{2.31}$$

$$\nabla \cdot \tilde{\underline{E}} = 4\pi\tilde{\rho} \tag{2.32}$$

$$\nabla \cdot \tilde{\underline{B}} = 0 \tag{2.33}$$

Where the tilde above the variables in (2.30) - (2.33) denote the Fourier transform with respect to the time variable only, viz.

$$\tilde{\underline{E}}(\underline{r},\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underline{E}(\underline{r},t) e^{i\omega t} dt \quad .$$

Equations (2.26) and (2.27) can be combined to yield the wave equation for $\underline{E}$.

$$\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} = -\frac{4\pi}{c^2} \frac{\partial \underline{j}}{\partial t} \tag{2.34}$$

Upon inverse transforming (2.14) into the space time domain and substituting this result into (2.34) for $\underline{j}$ yields the equation

$$\nabla \times \nabla \times \underline{E} + \frac{1}{c^2} \frac{\partial^2 \underline{E}}{\partial t^2} + \frac{\omega_p^2}{c^2} \underline{E} = \frac{\omega_p^2 e}{2mc^2} \frac{\partial}{\partial t} \left\{ \underline{W} + 2\underline{S}\, \nabla \cdot \underline{T} \right\} \tag{2.35}$$

where $\underline{W}$, $\underline{S}$, and $\underline{T}$ are related to $\underline{E}$ by

$$\frac{\partial \underline{W}}{\partial t} = \nabla(\underline{S} \cdot \underline{S}) \tag{2.36}$$

$$\frac{\partial \underline{S}}{\partial t} = \underline{E} \tag{2.37}$$

$$\frac{\partial^2 \underline{T}}{\partial t^2} = \underline{E} \tag{2.38}$$

Fourier transforming (2.34), we have

$$\nabla \times \nabla \times \tilde{\underline{E}} - \frac{\omega^2 - \omega_p^2}{c^2} \tilde{\underline{E}} = \frac{-i\omega\,\omega_p^2 e}{2mc^2} \text{F.T.}\left\{ \underline{W} + 2\,\underline{S}\,\nabla\cdot\underline{T} \right\} \tag{2.39}$$

where F.T. $\{\phi\}$ is the Fourier transform of ϕ (with respect to the time variable).

It is equation (2.39) together with (2.36)-(2.38) which we will solve by a perturbation expansion in Chapter 3. This method is valid, however, only if the right hand side of (2.39) is very small, or, equivalently, if the expansion converges sufficiently fast. A further discussion of this point will be given at the end of Chapter 3. After obtaining $\tilde{\underline{E}}$ in Chapter 3, we will then inverse transform to obtain the time response $\underline{E}(\underline{r},t)$.

3. Wave Propagation in a Nonlinear Plasma

We wish to consider (2.39) which will be rewritten for convenience

$$\nabla \times \nabla \times \underline{\tilde{E}} - \frac{\omega^2 - \omega_p^2}{c^2} \underline{\tilde{E}} = \frac{-i\omega\omega_p^2 e}{2mc^2} \text{ F.T. } \{\underline{W} + 2\underline{S}\ \nabla \cdot \underline{T}\} \qquad (3.1)$$

where,

$$\frac{\partial \underline{W}}{\partial t} = \nabla(\underline{S} \cdot \underline{S}) \qquad (3.2)$$

$$\frac{\partial \underline{S}}{\partial t} = \underline{E} \qquad (3.3)$$

$$\frac{\partial^2 \underline{T}}{\partial t^2} = \underline{E} \qquad (3.4)$$

To solve (3.1), we assume that the right hand side is very small so that its solution $\underline{\tilde{E}}$ is very close to the solution of the linearized equation.

$$\nabla \times \nabla \times \underline{\tilde{E}} - \frac{\omega^2 - \omega_p^2}{c^2} \underline{\tilde{E}} = 0 \qquad (3.5)$$

Therefore, we expand $\underline{\tilde{E}}$ as follows:

$$\underline{\tilde{E}} = \lambda \underline{\tilde{E}}^{(1)} + \lambda^2 \underline{\tilde{E}}^{(2)} + \qquad (3.6)$$

where $\tilde{\underline{E}}$ is the solution to (3.5). The particular form of $\tilde{\underline{E}}^{(1)}$ which is considered in this chapter is

$$\tilde{\underline{E}}^{(1)} = A^{(1)}(\omega)\, e^{i \sqrt{\omega^2-\omega_p^2}\; x/c}\, \underline{e}_z \tag{3.7}$$

We will consider the two cases:

$$A^{(1)}(\omega) = A_o/2\pi \tag{3.8}$$

and,

$$A^{(1)}(\omega) = \frac{i\omega A_o}{2\pi\,(\omega^2 - \omega_o^2)} \tag{3.9}$$

Substituting (3.6) into (3.1) gives:

$$\lambda\, \nabla \times \nabla \times \underline{E}^{(1)} + \lambda^2\, \nabla \times \nabla \times \tilde{\underline{E}}^{(2)} - \lambda \frac{\omega^2-\omega_p^2}{c^2} \tilde{\underline{E}}^{(1)} -$$

$$\lambda^2 \frac{\omega^2 - \omega_p^2}{c^2} \tilde{\underline{E}}^{(2)} = \frac{-i\omega\omega_p^2 e}{2mc^2} \text{ F.T. } \{\lambda^2 \underline{W}^{(2)} + \lambda^3 \underline{W}^{(3)} +$$

$$\lambda^4 \underline{W}^{(4)} \} \tag{3.10}$$

where,

$$\frac{\partial \underline{W}^{(2)}}{\partial t} = \nabla \underline{S}^{(1)} \cdot \underline{S}^{(1)} + 2\underline{S}^{(1)} \nabla \cdot \underline{T}^{(1)} \tag{3.11}$$

$$\frac{\partial \underline{W}^{(3)}}{\partial t} = 2 \nabla \underline{S}^{(1)} \cdot \underline{S}^{(2)} + 2\underline{S}^{(1)} \nabla \cdot \underline{T}^{(2)} + 2\underline{S}^{(2)} \nabla \cdot \underline{T}^{(1)} \tag{3.12}$$

$$\frac{\partial \underline{W}^{(4)}}{\partial t} = \nabla \underline{S}^{(2)} \cdot \underline{S}^{(2)} + 2\underline{S}^{(2)} \nabla \cdot \underline{T}^{(2)} \tag{3.13}$$

and,

$$\frac{\partial \underline{S}^{(n)}}{\partial t} = \underline{E}^{(n)} \tag{3.14}$$

$$\frac{\partial^2 \underline{T}^{(n)}}{\partial t^2} = \underline{E}^{(n)} \tag{3.15}$$

Equating coefficients of λ in (3.10) gives,

$$\nabla \times \nabla \times \underline{\tilde{E}}^{(1)} - \frac{\omega^2 - \omega_p^2}{c^2} \underline{\tilde{E}}^{(1)} = 0 \tag{3.16}$$

$$\nabla \times \nabla \times \underline{\tilde{E}}^{(2)} - \frac{\omega^2 - \omega_p^2}{c^2} \underline{\tilde{E}}^{(2)} = \frac{i\omega\omega_p^2 e}{2mc^2} \underline{\tilde{W}}^{(2)} \tag{3.17}$$

Equation (3.16) is just the linear wave equation which is satisfied by (3.7). Equation (3.17) represents the wave equation with the source distribution $\frac{i\omega\omega_p^2 e}{2mc^2} \underline{\tilde{W}}^{(2)}$

With the help of the various relations (3.11), (3.14), (3.15), and (3.7), we have the following expression for $\underline{\tilde{W}}^{(2)}$.

$$\tilde{\underline{W}}^{(2)} = \frac{-i}{\omega} \frac{\partial}{\partial x} \left\{ \frac{\tilde{E}^{(1)}}{\omega} * \frac{\tilde{E}^{(1)}}{\omega} \right\} \underline{e}_x \tag{3.18}$$

where,

$$\tilde{E}^{(1)} = A^{(1)}(\omega)\, e^{i\sqrt{\omega^2-\omega_p^2}\; x/c}$$

and the symbol "*" denotes convolution. Therefore, (3.17) becomes

$$\nabla \times \nabla \times \tilde{\underline{E}}^{(2)} - \frac{\omega^2-\omega_p^2}{c^2} \tilde{\underline{E}}^{(2)} = \frac{\omega_p^2 e}{2mc^2} \frac{\partial}{\partial x} \left\{ \tilde{G}^{(1)} * \tilde{G}^{(1)} \right\} \underline{e}_x \tag{3.19}$$

where, for convenience, we define

$$\tilde{G}^{(1)} = \frac{\tilde{E}^{(1)}}{\omega} \tag{3.20}$$

The solution to (3.19) can be written as the sum of a particular solution and a homogeneous solution. In this chapter we are only interested in the particular solution to (3.19). However, since the particular solution is longitudinal (as will later be seen), the solution we obtain $\tilde{\underline{E}} = \tilde{\underline{E}}^{(1)} + \tilde{\underline{E}}^{(2)}$ is, to second order, that solution which, at x = 0, reduces to $A^{(1)}(\omega)\, \underline{e}_z$.

To find the particular solution to (3.19), we consider the two Maxwell's equations (2.30) and (2.31) which reduce to *

---

*The following method is the one used by Bassanini [12] to solve for the fields generated by a time harmonic dipole oscillating in a spherical cavity which is immersed in a non-linear plasma.

$$\nabla \times \underline{\tilde{H}} = \left(\frac{-i\omega}{c} + \frac{i\omega_p^{\,2}}{\omega c}\right) \underline{\tilde{E}} - \frac{\omega_p^{\,2} e}{2mc} \text{ F.T. } \{\underline{W} + 2\underline{S}\ \nabla\cdot\underline{T}\} \tag{3.21}$$

$$\nabla \times \underline{\tilde{E}} = \frac{i\omega}{c}\ \underline{\tilde{H}} \tag{3.22}$$

Equation (3.21) and (3.22) are obtained from (2.14), (2.30), and (2.31) by first inverse transforming $\underline{\hat{j}}$ given by (2.14) into the space frequency domain, and then substituting the result into (2.30).

Expanding for $\underline{\tilde{E}}$ and $\underline{\tilde{H}}$,

$$\underline{\tilde{E}} = \lambda\ \underline{\tilde{E}}^{(1)} + \lambda^2\ \underline{\tilde{E}}^{(2)} \tag{3.23}$$

$$\underline{\tilde{H}} = \lambda\ \underline{\tilde{H}}^{(1)} + \lambda^2\ \underline{\tilde{H}}^{(2)} \tag{3.24}$$

and substituting (3.23) and (3.24) into (3.21) and (3.22), we have the following equations for $\underline{\tilde{E}}^{(2)}$ and $\underline{\tilde{H}}^{(2)}$.

$$\nabla \times \underline{\tilde{H}}^{(2)} = \left(\frac{-i\omega}{c} + \frac{i\omega_p^{\,2}}{\omega c}\right) \underline{\tilde{E}}^{(2)} + \frac{\omega_p^{\,2} e}{2mci\omega} \frac{\partial}{\partial x} \{\tilde{G}^{(1)} * \tilde{G}^{(1)}\} \underline{e}_x \tag{3.25}$$

$$\nabla \times \underline{\tilde{E}}^{(2)} = \frac{i\omega}{c}\ \underline{\tilde{H}}^{(2)} \tag{3.26}$$

Noting that $\frac{\partial}{\partial x} \{\tilde{G}^{(1)} * \tilde{G}^{(1)}\} \underline{e}_x$ is irrotational, we find, upon taking the curl of both sides of (3.25),

$$\nabla \times \nabla \times \underline{\tilde{H}}^{(2)} = \left(\frac{-i\omega}{c} + \frac{i\omega_p^{\,2}}{\omega c} \right) \nabla \times \underline{\tilde{E}}^{(2)} \qquad (3.27)$$

Substituting (3.26) into (3.27) gives:

$$\nabla \times \nabla \times \underline{\tilde{H}}^{(2)} = \left(\frac{-i\omega}{c} + \frac{i\omega_p^{\,2}}{\omega c} \right) \frac{i\omega}{c} \underline{\tilde{H}}^{(2)}$$

$$= \frac{\omega^2 - \omega_p^{\,2}}{c^2} \underline{\tilde{H}}^{(2)} \qquad (3.28)$$

The particular solution for (3.28) is just,

$$\underline{\tilde{H}}^{(2)} = 0 \qquad (3.29)$$

Substituting (3.29) into (3.25) gives, for the particular solution $\underline{\tilde{E}}^{(2)}$:

$$\underline{\tilde{E}}^{(2)} = \frac{-1}{\omega^2 - \omega_p^{\,2}} \frac{\omega_p^{\,2} e}{2m} \frac{\partial}{\partial x} \{\tilde{G}^{(1)} * \tilde{G}^{(1)}\} \underset{\rightarrow}{e}_x \qquad (3.30)$$

It is noted that $\underline{\tilde{E}}^{(2)}$ is longitudinal. It is also seen that we can inverse transform (3.30) to obtain the following expression for $\underline{E}^{(2)}$,

$$\underline{E}^{(2)} = \frac{\omega_p^{\,2} e}{2m} \frac{\partial}{\partial x} \underset{\rightarrow}{e}_x \int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{\omega_p^{\,2} - \omega^2} \int_{-\infty}^{\infty} \tilde{G}\,(\omega-\omega')\, \tilde{G}\,(\omega')\, d\omega'\, d\omega$$

(3.31)

Using the properties of convolution integrals and Fourier transforms, we can change (3.31) to

$$\underline{E}^{(2)} = \frac{\omega_p e}{2m} \frac{\partial}{\partial x} \underline{e}_x \int_{-\infty}^{t} H(t-\tau)\, \omega_p\, (t-\tau)[G^{(1)}(\tau)]^2 d\tau \tag{3.32}$$

where*,

$$G^{(1)}(\tau) = \int_{-\infty}^{\infty} e^{-i\omega\tau} \frac{\tilde{E}^{(1)}(\omega)}{\omega} d\omega$$

The upper limit on the integral in (3.32) follows from the fact that the inverse Fourier transform of $-\frac{1}{\omega^2 - \omega_p^2}$ which represents a causal signal,** is given by

$$\frac{2\pi}{\omega_p} \sin \omega_p t\, H(t)$$

where H(t) is the Heaviside unit step function. The presence of $H(t-\tau)$ in the integrand of (3.32) cuts the integral off at $\tau=t$. There will also be a lower limit on this integral at $\tau=x/c$. This arises from the exponential factor $e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c}$ in (3.7). It is this factor

---

*Actually, $G^{(1)}(\tau)$ also depends on x, as can be seen from (3.7), but this variable is suppressed for sake of brevity.

**Throughout the rest of this thesis, we will only consider causal signals. Therefore, all singularities which lie on the real axis in the frequency domain will be displaced infinitesimally below the real axis.

which introduces a factor of $H(t-x/c)$ in the inverse transform $G^{(1)}(t)$ (as can be seen in Appendix A). Therefore, (3.32) can be written:

$$\underline{E}^{(2)} = \frac{\omega_p e}{2m} \frac{\partial}{\partial x} \underline{e}_x \left\{ \int_{x/c}^{t} \sin\omega_p (t-\tau) f^2(\tau,x)d\tau \right\} H(t-x/c)$$

$$= \frac{\omega_p e}{2mc} \underline{e}_x \left\{ 2c \int_{x/c}^{t} \sin \omega_p (t-\tau) f_x(\tau,x) f(\tau,x) d\tau - \right.$$

$$\left. \sin \omega_p (t-\frac{x}{c}) f^2(\frac{x}{c}, x) \right\} H(t-\frac{x}{c}) \qquad (3.33)$$

where we have written $G^{(1)}(\tau)$ as,

$$G^{(1)}(\tau) = f(\tau,x) H(\tau - \frac{x}{c}) \qquad (3.34)$$

It is noted from (3.33) that $\underline{E}^{(2)}$ turns on at $t = x/c$ as expected from causality, viz., none of the higher order, non-linear responses of the plasma should propagate faster than the speed of light.

We now consider case one (where $A^{(1)}(\omega)$ is given by (3.8)). Then, we have,

$$E^{(1)}(x,t) = \frac{A_o}{2\pi} \int_{-\infty}^{\infty} e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c} e^{-i\omega t} d\omega \underline{e}_z$$

$$= \underline{e}_z A_o \left\{ \delta(t-\frac{x}{c}) - \frac{x\omega_p J_1(\omega_p\sqrt{t^2-x^2/c^2})}{c\sqrt{t^2 - x^2/c^2}} H(t-\frac{x}{c}) \right\} \qquad (3.35)$$

where, $J_n(x)$ represents a Bessel function of the first kind. Equation (3.35) can be obtained using the integration formula (A.3) or (A.8). To compute $\underline{E}^{(2)}$, we must first compute $G^{(1)}(t)$ defined by,

$$G^{(1)}(t) = \frac{A_o}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c}}{\omega} e^{-i\omega t} d\omega \tag{3.36}$$

Using the integration formula (A.8), we have,

$$G^{(1)}(t) = \frac{A_o}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\omega^2-\omega_p^2}\, x/c}}{\omega} e^{-i\omega t} d\omega$$

$$= \frac{A_o}{2\pi i} \left(\int_0^{2\pi} \frac{1-\xi^2}{1+\xi^2} e^{iq\cos\psi} d\psi \right) H(t- \tfrac{x}{c})$$

$$= \frac{A_o}{i} \left(2U_o(\gamma iq,q) - J_o(q) \right) H(t- \tfrac{x}{c}) \tag{3.37}$$

where, $U_n(w,z)$ is the Lommel function of two variables*;

$q = \omega_p\sqrt{t^2-x^2/c^2}$; $\gamma = \sqrt{\frac{t-x/c}{t+x/c}}$. Therefore, from (3.34) we have, for $f(t,x)$,

$$f(t,x) = \frac{1}{i} \left(2U_o(\gamma iq,q) - J_o(q) \right) \tag{3.38}$$

---

*See Appendix A (equations (A.12)-(A.19)) for a more thorough discussion of Lommel functions of two variables.

Also, from formulas (A.9) and (A.10),

$$\frac{\partial f(t,x)}{\partial x} \equiv f_x(t,x) = \frac{\omega_p}{c} \left[2U_1(\gamma_i q,q) - \frac{iJ_1(q)t}{\sqrt{t^2-x^2/c^2}} \right] \tag{3.39}$$

Therefore, from equation (3.33), we have the following expression for $\underline{E}^{(2)}(x,t)$,

$$\underline{E}^{(2)}(x,t) = A_o^2 \frac{\omega_p e}{2mc} \underline{e}_x \left\{ \frac{2\omega_p}{i} \int_{x/c}^{t} \sin \omega_p(t-\tau)[2U_o(\gamma iq,q)-J_o(q)] \cdot \right.$$

$$\left. [2U_1(\gamma iq,q) - \frac{iJ_1(q)\,\tau}{\sqrt{\tau^2-x^2/c^2}}]d\tau + \sin \omega_p(t-x/c)\right\} \cdot H(t-\frac{x}{c}) \tag{3.40}$$

Before turning to case two, we wish to make some comments in connection with (3.40). First we are reminded that the sum $\underline{E} = \underline{E}^{(1)} + \underline{E}^{(2)}$, given by equations (3.35) and (3.40) represent, to second order, the propagation of a pulse in the x direction whose z component is originally a delta function.* A priori one might expect that since the intensity of a delta function, i.e., $[\delta(t)]^2$, is infinite, there would be difficulties in computing the second order, nonlinear field. However, it is seen from (3.32) that the quantity of interest in evaluating $\underline{E}^{(2)}$ is $G^{(1)}(\tau)$, not $E^{(1)}(x,\tau)$. That is, to evaluate $\underline{E}^{(2)}$, we must first find $G^{(1)}(\tau)$ from the equation:

---

* This corresponds to applying an impulse voltage $v(t)= \delta(t)$ across the plane, bounding face of a semi-infinite plasma.

$$-iG^{(1)}(\tau) = \int_0^{\tau} E^{(1)}(x,\tau')\, d\tau' \tag{3.40A}$$

Therefore, it is the area under the $E^{(1)}(x,t)$ curve from 0 to t which is important in evaluating $\underline{E}^{(1)}$ and not the instantaneous field strength, $E^{(1)}(x,t)$. Actually, this is expected since

$\frac{q}{m} \int_0^t E^{(1)}(x,\tau)\, d\tau$ is just the linear velocity of the electron,

and as this velocity becomes larger, the nonlinear Lorentz force also becomes larger.

To see that the expression (3.40) for the second order field due to a delta pulse is indeed valid, we could instead consider a rectangular pulse, viz.,

$$E^{(1)}(o,t) = h_T(t) = \begin{cases} 0 & t \leq 0 \\ \frac{1}{T} & 0 \leq t \leq T \\ 0 & t \geq T \end{cases} \tag{3.40B}$$

To compute $E^{(1)}(x,t)$, we note that $h_T(t)$ is the difference of two step functions displaced in time by T. Therefore, $E^{(1)}(x,t)$ is given by :

$$E^{(1)}(x,t) = \frac{i}{2\pi T} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\omega^2-\omega_p^2}\,\frac{x}{c}}}{\omega} \left\{ e^{-i\omega t} - e^{-i\omega(t-T)} \right\} d\omega \tag{3.40C}$$

Using (3.37), we have:

$$E^{(1)}(x,t) = \frac{1}{T}\left\{ \left(2U_0(\gamma\, iq,q) - J_0(q)\right) H(t - \tfrac{x}{c}) - \right.$$

$$\left. (2U_0(\gamma' iq',q') - J_0(q')) H(t-T- \tfrac{x}{c}) \right\} \qquad (3.40D)$$

where

$$\gamma' = \sqrt{\frac{t - T - x/c}{t - T + x/c}} \quad ; \qquad q' = \omega_p \sqrt{(t-T)^2 - x^2/c^2}$$

We have from (3.40D) the following result:

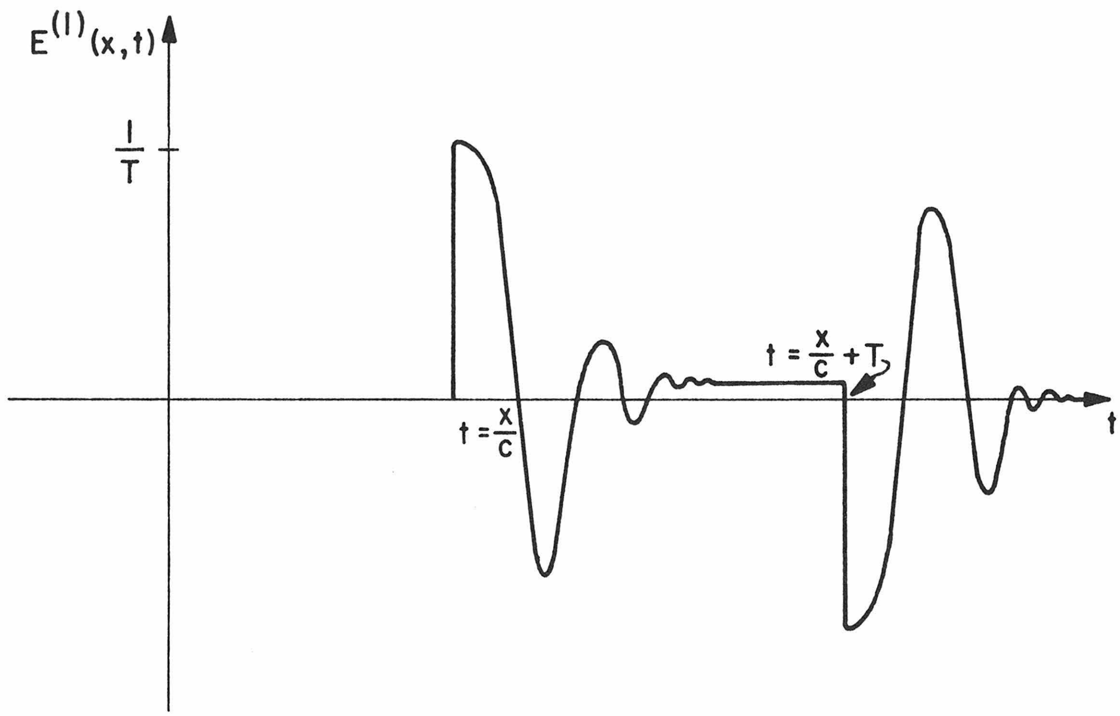
$$\lim_{T\to o} E^{(1)}(x,t) = \frac{\partial}{\partial t}\left[\left(2U_0 (\gamma\, iq,\, q) - J_0(q) \right) H(t - \tfrac{x}{c}) \right] \qquad (3.40E)$$

Of course, the expression in (3.40E) is just the linear field due to a delta function disturbance at $x = 0$, as expected. From (3.40E), we also have that:

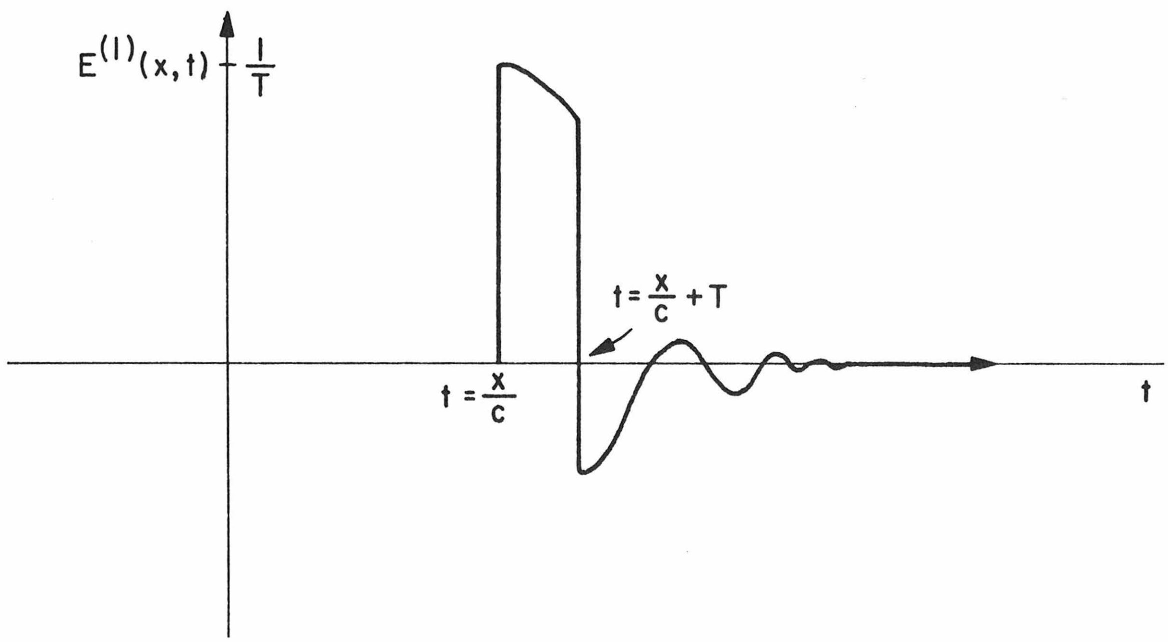
$$\lim_{T\to o} G^{(1)}(t) = \lim_{T\to o} \int_0^t E^{(1)}(x,\tau)\, d\tau = \{2U_0(\gamma\, iq,q) - J_0(q)\}\cdot H(t- \tfrac{x}{c})$$

$$(3.40F)$$

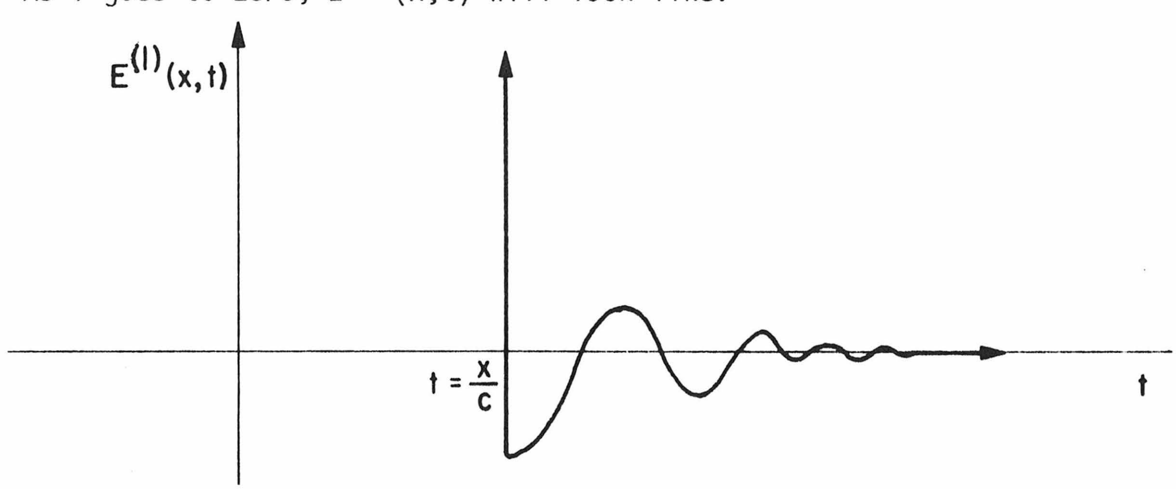
Therefore if T is small enough, $\underline{E}^{(2)}$ generated by a delta pulse at $x = 0$ is very similar to $\underline{E}^{(2)}$ which is generated by a rectangular pulse. To get an idea of how small T should be for our approximations to be valid, we can examine $E^{(1)}$ given by (3.40D). It consists of some oscillations at $t = x/c$ followed by more oscillations at $t = x/c + T$. $E^{(1)}(x,t)$ will look roughly as follows:



If T is so small that the first oscillations at $t = x/c$ and $t = x/c + T$ overlap, then $E^{(1)}(x,t)$ will be quite similar to the function given by (3.40E). That is, we wish to choose T so small that $E^{(1)}(x,t)$ looks as follows:



As T goes to zero, $E^{(1)}(x,t)$ will look like:



(In the previous figure, the arrow denotes a delta function)

Therefore, we wish to choose T smaller than the time of the first oscillation of the function, $2U_o(\gamma iq,q) - J_o(q)$. To determine this time, we note that for small t, we have:

$$2U_o(\gamma iq,q) - J_o(q) \approx J_o(q) = J_o(\omega_p \sqrt{t^2 - x^2/c^2}) \approx J_o(\omega_p \sqrt{2\Delta t\ x/c})$$

(3.40G)

Where $\Delta t = t - x/c$, and we note that for small Δt; $t + x/c \sim 2x/c$. Therefore, we must choose T smaller than Δt, where Δt is given by:

$$\omega_p \sqrt{2\Delta t\ x/c} = 2.42 \qquad (3.40H)$$

(2.42 is the first zero of $J_o(x)$). This means that T must be small enough so that:

$$\frac{2T\ \omega_p^2 x}{(2.42)^2 c} < 1 \qquad (3.40I)$$

Therefore, (3.40I) must hold in order that the results for $\underline{E}^{(2)}$ due to a rectangular pulse give a good approximation to the response of $\underline{E}^{(2)}$ due to a Dirac delta function.

Finally, we wish to comment that even though $\underline{E}^{(2)}$ is longitudinal and represents a discontinuity in E_x at x=0, $D_x=E_x+4\pi P_x$ is continuous at x=0. Actually, since $D_x=0$ for x<0, then we must have that $D_x=0$ in the nonlinear plasma. To see this, we write $\tilde{D}_x$ as:

$$\tilde{D}_x = \tilde{E}_x^{(1)} + \tilde{E}_x^{(2)} + 4\pi\ \chi\ (\tilde{E}_x^{(1)} + \tilde{E}_x^{(2)}) + 4\pi\ \tilde{P}_x^{NL} \tag{3.40J}$$

where in (3.40K), χ is the linear susceptibility and $\tilde{P}_x^{NL}$ is given by:

$$\tilde{P}_x^{NL} = \frac{i}{\omega}\ \tilde{J}_x^{NL} \tag{3.40 K}$$

Since $\tilde{E}_x^{(1)} = 0$, we must show that:

$$\tilde{E}_x^{(2)} + 4\pi\ \chi\ \tilde{E}_x^{(2)} = \varepsilon\ \tilde{E}_x^{(2)} = \frac{-4\pi i}{\omega}\ \tilde{J}_x^{NL} \tag{3.40 L}$$

where $\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}$

But (3.40L) is satisfied everywhere (x>o) by virtue of the fact that:

$$\nabla \times \nabla \times \underline{\tilde{E}}^{(2)} - \frac{\omega^2 - \omega_p^2}{c^2}\ \underline{\tilde{E}}^{(2)} = \frac{4\pi i\omega}{c^2}\ \underline{\tilde{J}}^{NL}$$

or, since $\underline{\tilde{E}}^{(2)}$ is longitudinal :

$$(1 - \frac{\omega_p^2}{\omega^2})\ \tilde{E}_x^{(2)} = \frac{-4\pi i\omega}{\omega^2}\ \tilde{J}_x^{NL} \tag{3.40 M}$$

which is the same as (3.40L).

We will now find the second order field, $\underline{E}^{(2)}$, for case two, i.e., where $A^{(1)}(\omega)$ is given by (3.9). We have, using (A.8)

$$\underline{E}^{(1)}(x,t) = \frac{A_o}{2\pi} \int_{-\infty}^{\infty} \frac{i\omega}{\omega^2 - \omega_o^2} e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c} e^{-i\omega t} d\omega \, \underline{e}_z$$

$$= \frac{A_o}{2\pi i} \left[\int_0^{2\pi} \frac{1 - \xi^4}{(\xi^2 - \xi_o^2)(\xi^2 - \xi_o^{-2})} e^{iq\cos\psi} d\psi \right] H(t - \frac{x}{c}) \, \underline{e}_z$$

$$= A_o \, \underline{e}_z \, \{-J_o(q) + U_o(\gamma \xi_o q, q) + U_o(\gamma \xi_o^{-1} q, q)\} \cdot$$

$$H\,(t - x/c) \qquad (3.41)$$

where, $\xi_o = (\omega_o + \sqrt{\omega_o^2 - \omega_p^2}\,) / \omega_p$; when $\omega_o > \omega_p$

or, $\xi_o = (\omega_o + i\sqrt{\omega_p^2 - \omega_o^2}\,) / \omega_p$; when $\omega_o < \omega_p$.

Proceeding as in the first case, we next compute (using (A.8)),

$$G^{(1)}(t) = \frac{A_o}{2\pi} \int_{-\infty}^{\infty} \frac{i e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c}}{\omega^2 - \omega_o^2} e^{-i\omega t} d\omega$$

$$= - \frac{A_o}{\omega_p \pi} \int_0^{2\pi} \frac{\xi(1-\xi^2)\, e^{iq\cos\psi}}{(\xi^2 - \varepsilon_o^2)(\xi^2 - \xi_o^{-2})} d\psi \;\; H(t - \frac{x}{c})$$

$$= - \frac{iA_o}{\omega_o} \{U_1 (\gamma \xi_o^{-1} q,q) + U_1 (\gamma \xi_o q,q)\} H(t- \frac{x}{c}) \tag{3.42}$$

This implies, from (3.34), that

$$f(t,x) = \frac{A_o}{i\omega_o} \{U_1 (\gamma \xi_o^{-1} q,q) + U_1 (\gamma \xi_o q,q)\} \tag{3.43}$$

Using formulas (A.9) and (A.10), we have,

$$f_x(t,x) = \frac{iA_o}{c} [J_o(q) - \frac{\xi_o^2 - 1}{\xi_o^2 + 1} \{U_o(\gamma \xi_o^{-1} q,q) - U_o(\gamma \xi_o q,q)\}] \tag{3.44}$$

Therefore, using (3.33), we have:

$$\underline{E}^{(2)}(x,t) = \underline{e}_x \frac{\omega_p e A_o^2}{\omega_o mc} [\int_{x/c}^{t} \sin\omega_p(t-\tau)\{U_1(\gamma \xi_o^{-1} q,q) + U_1(\gamma \xi_o q,q)\}$$

$$\cdot \{J_o(q) - \frac{\xi_o^2 - 1}{\xi_o^2 + 1} (U_o(\gamma \xi_o^{-1} q,q) -$$

$$- U_o(\gamma \xi_o q,q)\} d\tau] \cdot H(t- \frac{x}{c}) \tag{3.45}$$

The sum $\underline{E} = \underline{E}^{(1)} + \underline{E}^{(2)}$ given by equations (3.41) and (3.45) represent, to second order, the propagation of a wave whose z component is originally a suddenly turned-on sinusoid. We shall now obtain asymptotic formulas for the second order fields as $t-x/c \to 0$, corresponding to times just after the arrival of the wave. To do this, we will expand the integrands of (3.40) and (3.45) in powers of $(t-x/c)$. For case 1, we have from (3.40)

$$\underline{E}^{(2)}(x,t) \sim \frac{A_o^2 \omega_p e}{2mc} \underline{e}_x \left\{ \frac{2\omega_p}{i} \int_{x/c}^{t} \sin\omega_p(t-\tau) \left(\frac{-i\omega_p x}{2c} \right) d\tau + \sin\omega_p H- \frac{x}{c}) \right\}$$

$$\cdot H\ (t- \frac{x}{c})$$

$$= \frac{A_o^2 \omega_p e}{2mc} \underline{e}_x \left\{ \frac{-\omega_p x}{c} [\cos\omega_p(t- \frac{x}{c}) -1] + \sin\omega_p(t - \frac{x}{c}) \right\} H\ (t- \frac{x}{x})$$

$$= \frac{A_o^2 \omega_p e}{2mc} \underline{e}_x \{\omega_p(t - \frac{x}{c})\}\ H\ (t - \frac{x}{c}) \qquad (3.46)$$

For case 2, we have from (3.45):

$$E^{(2)}(x,t) \sim \underline{e}_x \frac{\omega_p e A_o^2}{mc} \left[\int_{x/c}^{t} \sin\omega_p(t-\tau)\ (\tau - \frac{x}{c})\ d\tau \right] = \underline{e}_x \frac{\omega_p^2 e A_o^2}{6\ mc} (t- \frac{x}{c})^3$$

$$(3.47)$$

The asymptotic behavior of the second order fields at a point x as $t \to \infty$ is now obtained. The method used is that which is outlined in Appendix C (equations (C.8) - (C.13)). Considering first case 1, we note that $\underline{E}^{(2)}$ can be written:

$$\underline{E}^{(2)} = A_o^2 \frac{\omega_p e}{2mc} \underline{e}_x \left\{ 2c \int_{x/c}^{t} Q(\tau) \sin\omega_p(t-\tau)\ d\tau + \sin\omega_p(t - \frac{x}{c}) \right\} \qquad (3.48)$$

where, $Q(t) = f(t,x)\, f_x(t,x)\, H\left(t-\frac{x}{c}\right)$

$$= \frac{1}{i}\left(2U_0(\gamma iq,q) - J_0(q)\right) \frac{\omega_p}{c} \left(2U_1(\gamma iq,q) - \frac{iJ_1(q)t}{\sqrt{t^2-x^2/c^2}}\right) H\left(t-\frac{x}{c}\right)$$

To asymptotically expand (3.48) (according to (C.13)), we first determine the asymptotic expansion of Q(t), which is the same as the product of the asymptotic expansion for f(t,x) and $f_x(t,x)$. To obtain the asymptotic expansion of f(t,x), we rewrite f(t,x) as (see (3.37)),

$$f(t,x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{i\sqrt{\omega^2-\omega_p^2}\,x/c}}{\omega}\, e^{-i\omega t}\, d\omega \qquad (3.49)$$

Since the integrand of (3.49) has a pole at $\omega=0$, the leading term in the asymptotic expansion for f(t,x) is:

$$f(t,x) \sim -i\, e^{-\omega_p x/c} \qquad (3.50)$$

The next term in the expansion for f(t,x) is a decreasing function of t and comes from the branch point singularities at $\omega = \pm\,\omega_p$ in (3.49). Differentiating (3.50), we have:

$$f_x(t,x) \sim \frac{\omega_p}{c}\, i e^{-\omega_p x/c} \qquad (3.51)$$

We therefore have, for the asymptotic expansion of Q(t),

$$Q(t) \sim \frac{\omega_p}{c} e^{-2\omega_p x/c} \tag{3.52}$$

Denoting the asymptotic expansion for Q(t) by $Q_{AS}(t)$, we have:

$$Q_{AS}(t) = \frac{\omega_p}{c} e^{-2\omega_p x/c} \tag{3.53}$$

Substituting (3.53) into (C.13), we have the following asymptotic expansion for $\underline{E}^{(2)}$,

$$\underline{E}^{(2)}(x,t) \sim \frac{\Lambda_o^2 \omega_p e}{2mc} \underline{e}_x \left\{ 2e^{-2\omega_p x/c} + \frac{\pi i c}{\omega_p} \left[\tilde{Q}(+\omega_p) e^{-i\omega_p t} - \tilde{Q}(-\omega_p) e^{i\omega_p t} \right] + \sin \omega_p (t - \frac{x}{c}) \right\} \tag{3.54}$$

for $t >> x/c$

where,

$$\tilde{Q}(\pm\omega_p) = \frac{1}{2\pi} \int_{\frac{x}{c}}^{\infty} f(t,x) f_x(t,x) e^{\pm i\omega_p t} dt$$

We proceed in a similar manner in obtaining the asymptotic expansion of $\underline{E}^{(2)}$ for case 2. In this case, we write $\underline{E}^{(2)}$ as,

$$\underline{E}^{(2)}(x,t) = \frac{A_o^2 \omega_p e}{2mc}\left[2c \int_{x/c}^{t} Q(t)\, \sin\omega_p(t-\tau)\, d\tau\right] \qquad (3.55)$$

where, for this case,

$$Q(t) = f(t,x)\, f_x(t,x)\, H\,(t - x/c)$$

$$= \frac{1}{\omega_o c}\left[U_1\,(\gamma\, \xi_o^{-1}\, q, q) + U_1\,(\gamma\, \xi_o q, q)\right] \cdot$$

$$\cdot \left[J_o(q) - \frac{\xi_o^2 - 1}{\xi_o^2 + 1}\{U_o\,(\gamma\, \xi_o^{-1} q, q) - U_o(\gamma\, \xi_o q, q)\}\right] H\,(t - \frac{x}{c})$$

To obtain the asymptotic expansion of Q(t), we first write f(t,x) as follows, (see 3.42))

$$f(t,x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{i e^{i\sqrt{\omega^2 - \omega_p^2}\, x/c}}{\omega^2 - \omega_o^2}\, e^{-i\omega t}\, d\omega \qquad (3.56)$$

Considering the poles in the integrand of (3.59) at $\omega = \pm\omega_o$, we have for the asymptotic expansion of f(t,x),

$$f(t,x) \sim -\frac{1}{2\pi} \cdot \frac{2\pi i}{2i\omega_o}\left[e^{-i\omega_o t - \frac{x}{c} i\sqrt{\omega_o^2 - \omega_p^2}} - e^{-i\omega_o t + \frac{x}{c} i\sqrt{\omega_o^2 - \omega_p^2}} \right]$$

$$= \frac{-i}{\omega_o} \sin\,(\omega_o t - \frac{x}{c}\sqrt{\omega_o^2 - \omega_p^2}) \qquad (3.57)$$

Differentiating (3.57), we have,

$$f_x(t,x) \sim \frac{i\sqrt{\omega_o^2 - \omega_p^2}}{\omega_o c} \cos\left(\omega_o t - \frac{x}{c}\sqrt{\omega_o^2 - \omega_p^2}\right) \tag{3.58}$$

Multiplying (3.60) and (3.61), we have for Q(t)

$$Q(t) \sim \frac{\sqrt{\omega_o^2 - \omega_p^2}}{2c\omega_o^2} \sin 2\left(\omega_o t - \frac{x}{c}\sqrt{\omega_o^2 - \omega_p^2}\right)$$

$$= \frac{\sqrt{\omega_o^2 - \omega_p^2}}{4ci\omega_o^2}\left[e^{2i\omega_o t - \frac{2ix}{c}\sqrt{\omega_o^2 - \omega_p^2}} - e^{-2i\omega_o t + \frac{2ix}{c}\sqrt{\omega_o^2 - \omega_p^2}}\right] \tag{3.59}$$

Equation (3.59) gives:

$$Q_{AS}(t) = \frac{\sqrt{\omega_o^2 - \omega_p^2}}{4ci\omega_o^2}\left[e^{2i\omega_o t - \frac{2ix}{c}\sqrt{\omega_o^2 - \omega_p^2}} - e^{-2i\omega_o t + \frac{2ix}{c}\sqrt{\omega_o^2 - \omega_p^2}}\right] \tag{3.60}$$

Proceeding in a similar manner as for case 1, we have the following asymptotic expansion for $\underline{E}^{(2)}$ in case 2:

$$\underline{E}^{(2)}(x,t) \sim \begin{cases} \dfrac{A_o^2 \omega_p^2}{2mc} \, \underline{e}_x \left[\dfrac{1}{4i} \dfrac{\sqrt{\omega_o^2 - \omega_p^2}}{\omega_o^2} \dfrac{2\cos\left(2\omega_o t - 2\frac{x}{c}\sqrt{\omega_o^2 - \omega_p^2}\right)}{\omega_p^2 - 4\omega_o^2} + \dfrac{2\pi i}{2\omega_p} \{\tilde{Q}(+\omega_p)\, e^{-i\omega_p t} - \tilde{Q}(-\omega_p) e^{i\omega_p t}\} \right] \\ \text{for } \omega_o > \omega_p \\ \\ \sim \dfrac{A_o^2 \omega_p^2 e}{2mc} \, \underline{e}_x \left[\dfrac{i\sqrt{\omega_p^2 - \omega_o^2}}{4i\,\omega_o^2} \dfrac{e^{-2\frac{x}{c}\sqrt{\omega_p^2 - \omega_o^2}}}{\omega_p^2 - 4\omega_o^2} \, 2\cos 2\omega_o t + \dfrac{2\pi i}{2\omega_p} \{\tilde{Q}\ (\omega_p)\, e^{-i\omega_p t} \ -\tilde{Q}(-\omega_p)\, e^{i\omega_p t}\} \right] \text{ for } \omega_o < \omega_p \end{cases} \tag{3.61}$$

We now wish to briefly summarize the results of the analysis presented on the preceding pages. For case 1, we see that at a point x, the nonlinear field $\underline{E}^{(2)}$ builds up from a zero value at $t = x/c$ to an oscillation at ω_p, $\sin \omega_p(t-x/c)$. This is borne out by computer plots of $E_x^{(2)}$ corresponding to case 1 (figures 3.1-3.2). For case 2, we find that $\underline{E}^{(2)}$ also builds up from a zero value at $t = x/c$. However, as $t \to \infty$, $\underline{E}^{(2)}$ approaches an oscillation at the second harmonic, $2\omega_o$ as well as an oscillation at ω_p. The computer plots of $E_x^{(2)}$ (figures 3.2-3.6) corresponding to case 2 with $\omega_o = 2\omega_p$ and $\omega_o = 10\omega_p$ show a strong beating of the two frequencies at $2\omega_o$ and ω_p. As a matter of comparison, computer plots of the linear fields $\underline{E}^{(1)}$ for case two (figures 3.7-3.10) are also given. It is noted that the method of evaluating $\underline{E}^{(2)}$ on the computer is to

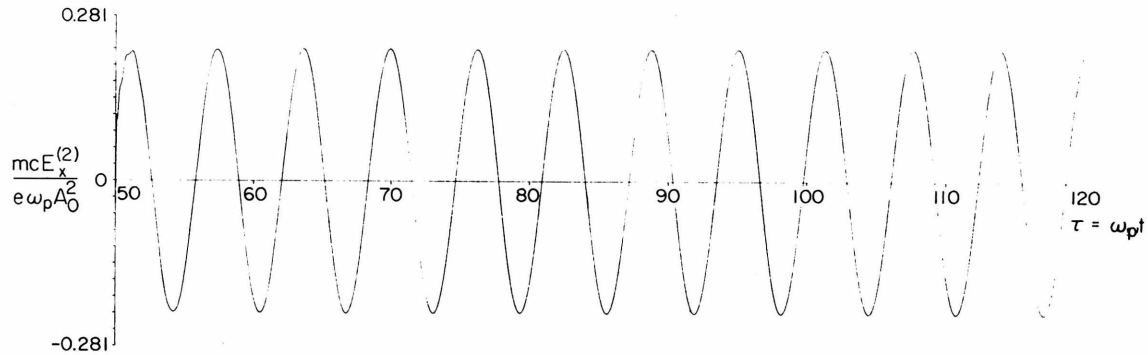


Figure 3.1: $E_x^{(2)}$ for $x = \frac{50c}{\omega_p}$ corresponding to case 1

($\underline{E}^{(1)}(0,t) = \delta(t)\ \underline{e}_z$).

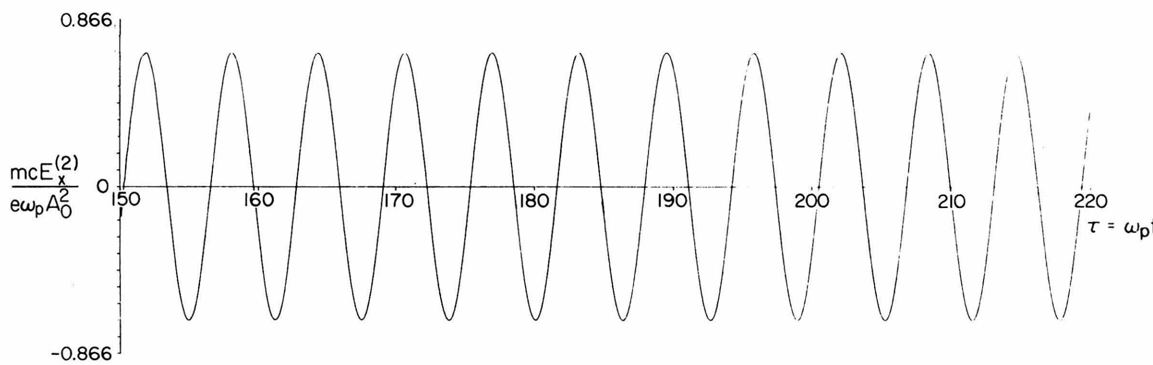


Figure 3.2: $E_x^{(2)}$ for $x = \frac{150c}{\omega_p}$ corresponding to case 1

($\underline{E}^{(1)}(0,t) = \delta(t)\ \underline{e}_z$).

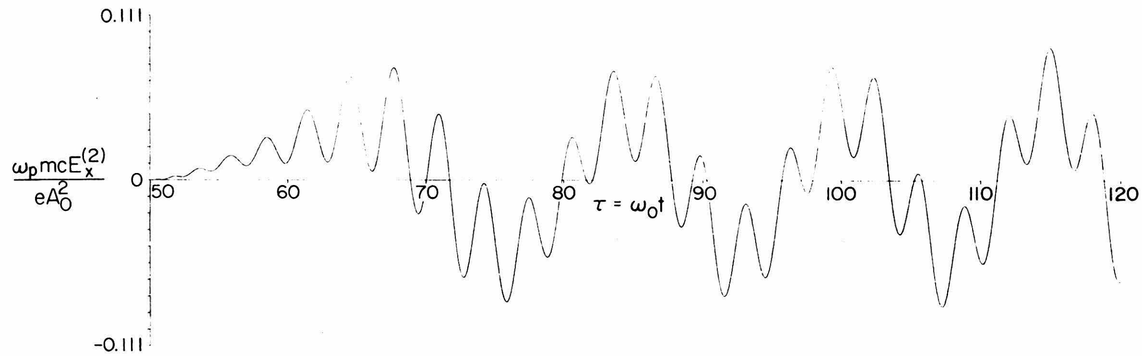


Figure 3.3: $E_x^{(2)}$ for $x = \frac{50c}{\omega_o}$, $\omega_o = 2\omega_p$ corresponding to case 2 ($\underline{E}^{(1)}(0,t) = \cos \omega_o t \cdot H(t)\ \underline{e}_z$).

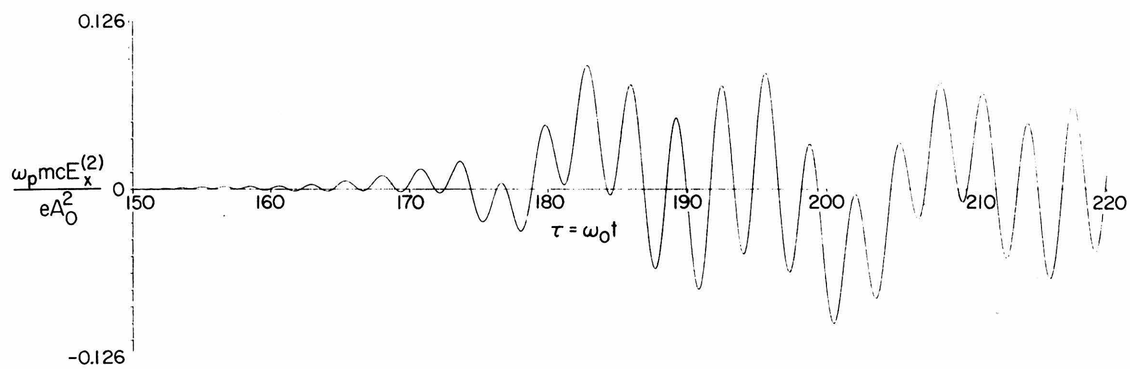


Figure 3.4: $E_x^{(2)}$ for $x = \frac{150c}{\omega_o}$, $\omega_o = 2\omega_p$ corresponding to case 2

($\underline{E}^{(1)}(0,t) = \cos \omega_o t \cdot H(t)\ \underline{e}_z$)

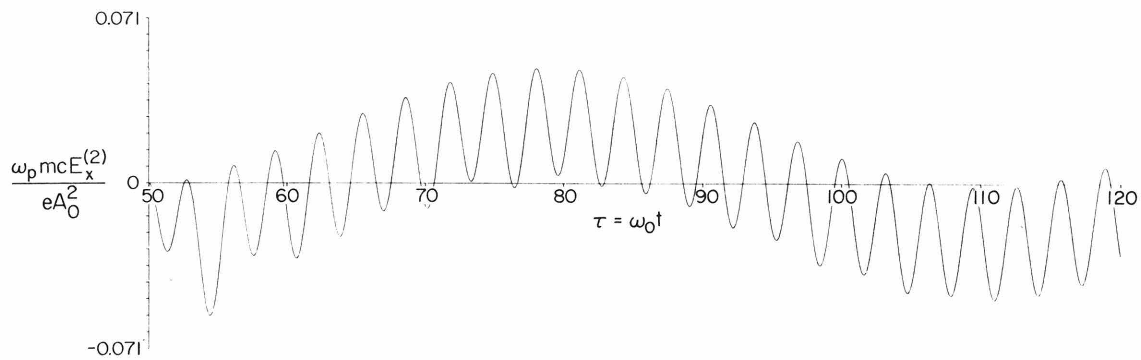


Figure 3.5. $E_x^{(2)}$ for $x = \frac{50c}{\omega_o}$, $\omega_o = 10\ \omega_p$ corresponding

to case 2 ($\underline{E}^{(1)}(0,t) = \cos\ \omega_o t \cdot H(t)\ \underline{e}_z$)

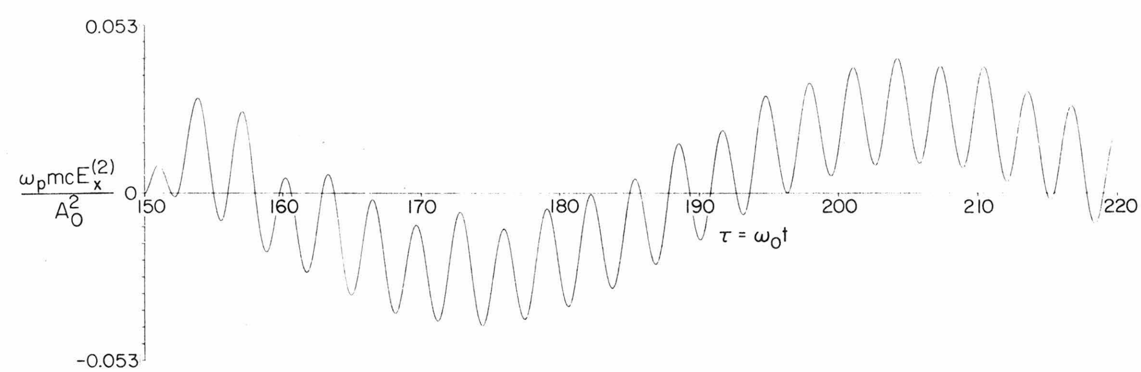


Figure 3.6: $E_x^{(2)}$ for $x = \frac{150c}{\omega_o}$, $\omega_o = 10\omega_p$ corresponding to case 2 ($\underline{E}^{(1)}(0,t) = \cos\ \omega_o t\ H(t)\ \underline{e}_z$)

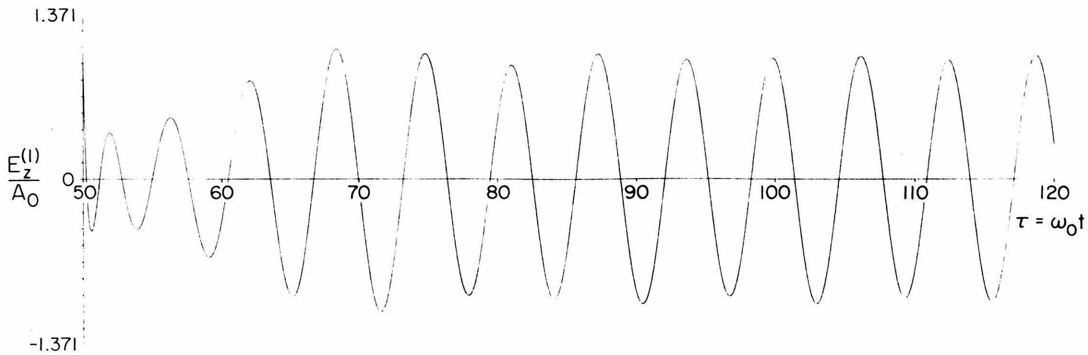


Figure 3.7: $E_z^{(1)}$ for $x = \frac{50c}{\omega_0}$, $\omega_0 = 2\omega_p$ corresponding to case 2 ($\underline{E}^{(1)}(0,t) = \cos \omega_0 t \, H(t) \, \underline{e}_z$)

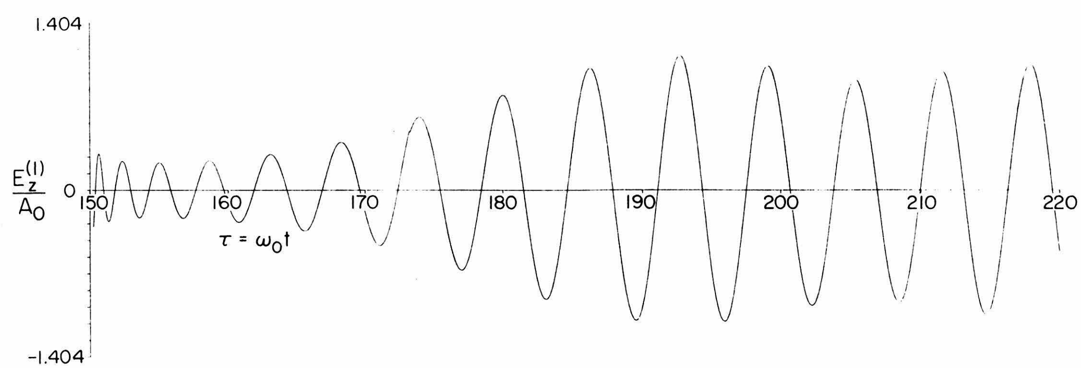


Figure 3.8: $E_z^{(1)}$ for $x = \frac{150c}{\omega_0}$, $\omega_0 = 2\omega_p$ corresponding to case 2 ($\underline{E}^{(1)}(0,t) = \cos \omega_0 t \, H(t) \, \underline{e}_z$)

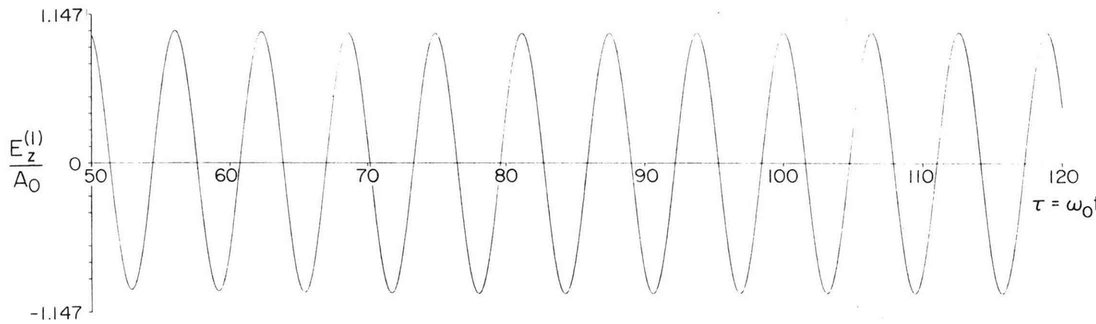


Figure 3.9: $E_z^{(1)}$ for $x = \frac{50c}{\omega_0}$, $\omega_0 = 10\omega_p$ corresponding to case 2
($\underline{E}^{(1)} = \cos \omega_0 t \; H(t) \; \underline{e}_z$)

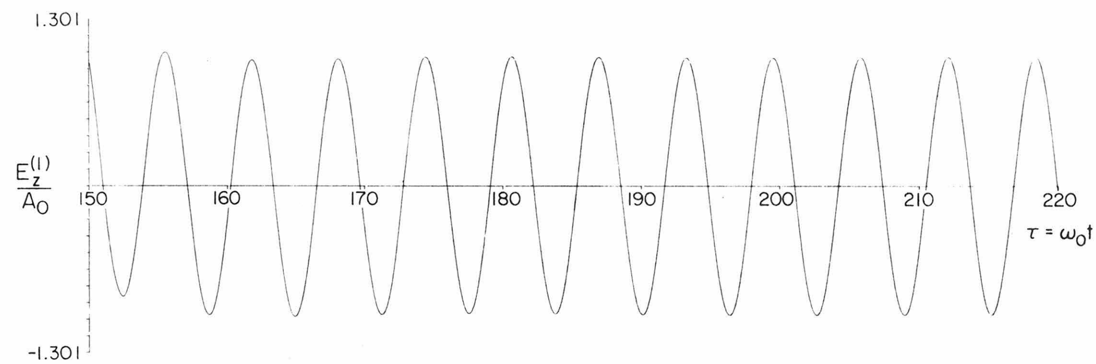


Figure 3.10: $E_z^{(1)}$ for $x = \frac{150c}{\omega_0}$, $\omega_0 = 10\omega_p$ corresponding to case 2
($\underline{E}^{(1)} = \cos \omega_0 t \; H(t) \; \underline{e}_z$)

numerically integrate the convolution integrals (3.40) and (3.45). This method is discussed in Appendix B.

Before turning to the case of reflections from a nonlinear, cold plasma, we will examine the requirement of the electric field strength so that the second order nonlinear effects in a cold plasma are appreciable. Certainly, by comparing $\hat{\underline{v}}^{(1)}$ and $\hat{\underline{v}}^{(2)}$ given by (2.9) and (2.10), one can determine the conditions of the electric field strength so that the nonlinear velocity $\hat{\underline{v}}^{(2)}$ is, say, 1% of the linear velocity. However, as a rough approximation, we will compare the Lorentz force $e \frac{\underline{v} \times \underline{H}}{c}$ with the force $e\underline{E}$. From Maxwell's equations, we have that

$$\underline{H} = \frac{c}{\omega} \underline{k} \times \underline{E}$$

This, of course, implies that the magnitude of $\underline{H}$ is the same order of magnitude as $\frac{c}{v_{phase}}$ ($\underline{E}$), where v_{phase} is the phase velocity of the electromagnetic wave. Assuming that $\frac{c}{v_{phase}} \sim 1$ (that is, we assume that the frequency of the electric field is greater than ω_p), we see that $|\underline{H}|$ is the same order of magnitude as $|\underline{E}|$. Therefore, the Lorentz force is on the same order of magnitude as $\frac{e|\underline{v}|}{c}|\underline{E}|$. If we assume that $\underline{v}$ is approximately the linear electron velocity, $\frac{e}{m\omega}\underline{E}$, we then have that the magnitude of the Lorentz force is approximately given by $\frac{e^2}{mc\omega}|\underline{E}|^2$. Requiring the Lorentz force to be 1% of the linear force $e\underline{E}$, we see that $|\underline{E}|$ must satisfy the following requirement,

$$\left(\frac{e^2}{mc\omega} |\underline{E}|^2 \right) / (e/\underline{E}) = \frac{e}{mc\omega} |\underline{E}| \sim .01$$

or, $$|\underline{E}| \sim .01 \frac{mc\omega}{e} \qquad (3.62)$$

Taking ω to be 10^8 $\sec^{-1}$, we have that

$$|\underline{E}| \sim (.01) \frac{3 \times 10^{10} \times 10^8}{5.3 \times 10^{17}} = 5.7 \times 10^{-2} \frac{\text{statvolts}}{\text{cm}} = 17 \text{ v/cm}$$

This gives the following energy density

$$u \sim \frac{1}{8\pi} |\underline{E}|^2 \sim 1.3 \times 10^{-4} \text{ ergs/cm}^3$$

In terms of field intensity, one has that,

$$I_c \sim u_c \sim 4 \times 10^6 \frac{\text{erg sec}^{-1}}{\text{cm}^2} = 4 \times 10^{-1} \frac{\text{watts}}{\text{cm}^2} = 400 \text{ mW/cm}^2$$

where I_c is the required intensity of the electric field for the nonlinear effects to be appreciable. Actually, in laboratory plasmas this value for I_c is easily obtained with the use of radio wave transmitters. Of course, it is remembered that the above value for I_c is obtained by comparing the Lorentz force with the force $e\underline{E}$. Actually, one should also consider the nonlinearity due to the $\underline{v} \cdot \nabla \underline{v}$ term. The above determination of I_c is, however, a fairly good order of magnitude estimate. It is also noted that the above analysis of obtaining $\underline{E}^{(2)}$ from $\underline{E}^{(1)}$ is invalid if the amplitude of the fields is such that $|\frac{v}{c}| \gtrsim \cdot 1$. When the electron velocities are this high, special relativistic effects come in to play, and the governing equations of the plasma (2.1) and (2.2) are invalid.

4. Reflection from a Nonlinear Plasma

Consider an E-polarized, electromagnetic signal incident on a linear, isotropic plasma (figure 4.1).

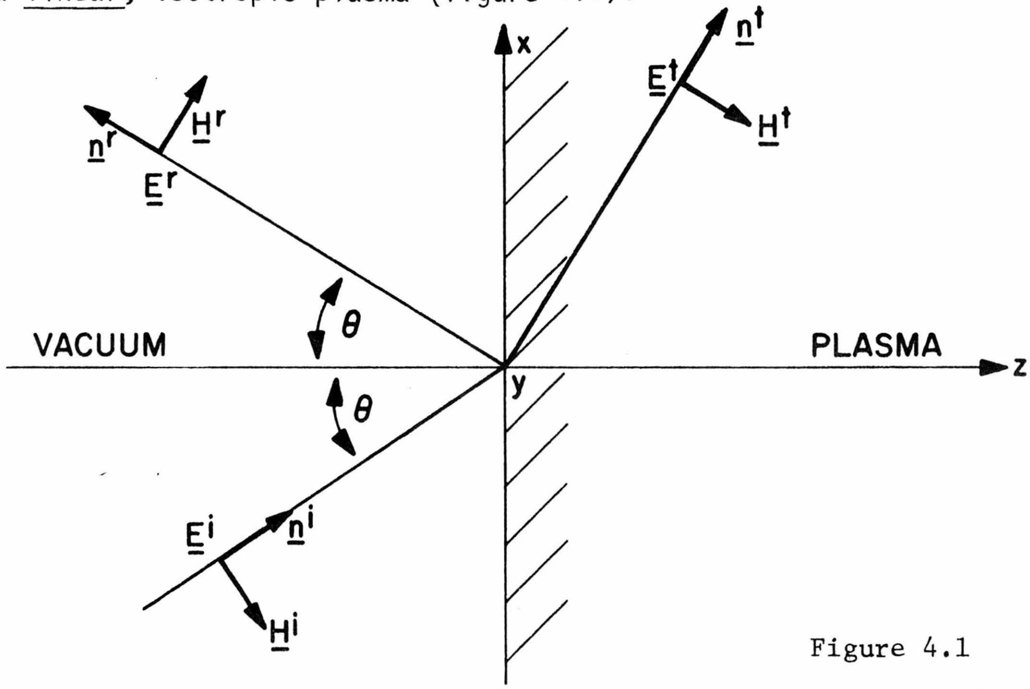


Figure 4.1

let,

$$\underline{E}^i = \underline{E}^i\,(t - \underline{r}\cdot\underline{n}^i)$$

$$\underline{E}^r = \underline{E}^r\,(t - \underline{r}\cdot\underline{n}^r) \qquad (4.1)$$

$$\underline{E}^t = \underline{E}^t\,(t, \underline{r})$$

Taking Fourier transforms of (4.1) on the time variable gives:

$$\tilde{\underline{E}}^i = \tilde{\underline{E}}^i\,(\omega)\, e^{i\frac{\omega}{c}\,\underline{r}\cdot\underline{n}^i} = \tilde{E}^i\,(\omega)\, e^{i\frac{\omega}{c}\,\underline{r}\cdot\underline{n}^i}\, \underline{e}_y$$

$$\tilde{\underline{E}}^r = \tilde{\underline{E}}^r\,(\omega)\, e^{i\frac{\omega}{c}\,\underline{r}\cdot\underline{n}^r} = \tilde{E}^r(\omega)\, e^{i\frac{\omega}{c}\,\underline{r}\cdot\underline{n}^r}\, \underline{e}_y \qquad (4.2)$$

$$\tilde{\underline{E}}^t = \tilde{\underline{E}}^t\,(\omega)\, e^{i\,\underline{k}^t\cdot\underline{r}} = \tilde{E}^t\,(\omega)\, e^{i\,\underline{k}^t\cdot\underline{r}}\, \underline{e}_y$$

where $\underline{n}^i$, $\underline{n}^r$ are unit vectors pointing in the direction of propagation and $\underline{k}^t$ is the propagation vector in the plasma. Continuity of E_y and H_x at z=0 gives,

$$\frac{\omega}{c} n_x^i = \frac{\omega}{c} n_x^r = \frac{\omega}{c} \sin\theta = k_x^t$$

$$n_z^i = \cos\theta = -n_z^r \tag{4.3}$$

Using (4.3), we have:

$$k_z^t = \sqrt{\underline{k}^t \cdot \underline{k}^t - (k_x^t)^2}$$

$$= \frac{\cos\theta}{c} \sqrt{\omega^2 - \alpha^2} \tag{4.4}$$

where,

$$\alpha^2 = \omega_p^2 \sec^2\theta$$

From the continuity of E_y and H_x at z=0, we also have:

$$\tilde{E}^r(\omega) = \frac{\omega - \sqrt{\omega^2 - \alpha^2}}{\omega + \sqrt{\omega^2 - \alpha^2}} \tilde{E}^i(\omega)$$

$$\tilde{E}^t(\omega) = \frac{2\omega}{\omega + \sqrt{\omega^2 - \alpha^2}} \tilde{E}^i(\omega) \tag{4.5}$$

Combining the results of (4.2) - (4.5) gives:

$$\tilde{\underline{E}}^{i} = \tilde{E}^{i}(\omega)\, e^{i\frac{\omega}{c}(x \sin\theta + z\cos\theta)}\, \underline{e}_y$$

$$\tilde{\underline{E}}^{r} = \tilde{E}^{i}(\omega)\, \frac{\omega - \sqrt{\omega^2 - \alpha^2}}{\omega + \sqrt{\omega^2 - \alpha^2}}\, e^{i\frac{\omega}{c}(x\sin\theta - z\cos\theta)}\, \underline{e}_y \qquad (4.6)$$

$$\tilde{\underline{E}}^{t} = \tilde{E}^{i}(\omega)\, \frac{2\omega}{\omega + \sqrt{\omega^2 - \alpha^2}}\, e^{i\frac{\omega}{c} x\sin\theta}\, e^{\frac{i\cos\theta}{c} z\sqrt{\omega^2-\alpha^2}}\, \underline{e}_y$$

Equation (4.6) represents the Fourier transformed fields due to the incidence of an E-polarized signal. As an example of what these fields might look like in the time domain, we consider the case where the incident pulse is a delta function. Then,

$$\tilde{E}^{i}(\omega) = \frac{1}{2\pi} \qquad (4.7)$$

The inverse Fourier transforms of the fields given by (4.6) (with $\tilde{E}^{i}(\omega)$ given by (4.7)) are *

---

* The following expressions are generated from the following integral

$$I(r,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\sqrt{\omega^2-\alpha^2}\, r}\, e^{-i\omega t}\, d\omega = \delta(t-r) - \frac{r\alpha J_1(\alpha\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} H(t-r)$$

by the appropriate differentiation of $I(r,t)$ with respect to t and r. Of course, these expressions could also be obtained directly from (4.6) and (4.7) by using the integration formulas (A.3) or (A.8).

$$\underline{E}^i(t,\underline{r}) = \delta(t - \frac{1}{c}\underline{n}^i \cdot \underline{r})\, \underline{e}_y$$

$$\underline{E}^r(t,\underline{r}) = \frac{-2J_2[\alpha(t - \frac{\underline{n}^r \cdot \underline{r}}{c})]}{t - \underline{n}^r \cdot \underline{r}/c}\, H(t - \frac{\underline{n}^r \cdot \underline{r}}{c})\, e_y \tag{4.8}$$

$$\underline{E}^t(t,\underline{r}) = \{\delta(Y-\beta) - \frac{\alpha}{2}[\frac{Y + \beta}{\sqrt{Y^2 - \beta^2}} J_1(\alpha\sqrt{Y^2 - \beta^2}) + \\ + \frac{(Y - \beta)^3}{(Y^2 - \beta^2)^{3/2}} J_3(\alpha\sqrt{Y^2 - \beta^2})]\; H(Y - \beta)\}\, \underline{e}_y$$

where

$$\underline{n}^i = \sin\theta\, \underline{e}_x + \cos\theta\, \underline{e}_z$$

$$\underline{n}^r = \sin\theta\, \underline{e}_x - \cos\theta\, \underline{e}_z$$

$$\alpha = \omega_p \sec\theta$$

$$Y = t - \frac{x}{c}\sin\theta$$

$$\beta = \frac{z}{c}\cos\theta$$

The above expression for $\underline{E}^r(t,r)$ given by (4.8) is also the one obtained by Chabries [26].

Suppose now that the plasma to the right of $z = 0$ is nonlinear, i.e., electromagnetic wave propagation in this plasma is described by (2.39), which is rewritten here:

$$\nabla \times \nabla \times \tilde{\underline{E}} - \frac{\omega^2 - \omega_p^2}{c^2} \tilde{\underline{E}} = \frac{-i\omega\, \omega_p^2\, e}{2mc^2} \text{ F.T. } \{\underline{W} + 2\underline{S}\, \nabla \cdot \underline{T}\} \tag{4.9}$$

where $\underline{W}$, $\underline{S}$, and $\underline{T}$ are related to $\underline{E}$ in equations (2.36)-(2.38). To solve (4.9) with the appropriate boundary conditions at $z = 0$, we proceed in the same manner as in reference [17],that is, in the non-linear plasma, we write

$$\tilde{\underline{E}} = \tilde{\underline{E}}^t + \tilde{\underline{E}}^s + \underline{E}_2^t \quad , \qquad z \geq 0 \tag{4.10}$$

where $\tilde{\underline{E}}^t$ is given by (4.6); $\tilde{\underline{E}}^s$ is the particular solution to (3.19) with $\tilde{\underline{E}}^{(1)} = \tilde{\underline{E}}^t$; and $\tilde{\underline{E}}_2^t$ is the homogeneous solution of (3.19) chosen to satisfy the boundary conditions at $z = 0$. In the region $z \leq 0$ we write $\tilde{\underline{E}}$ as follows:

$$\tilde{\underline{E}} = \tilde{\underline{E}}^i + \tilde{\underline{E}}^r + \tilde{\underline{E}}_2^r \quad , \qquad z \leq 0 \tag{4.11}$$

where $\tilde{\underline{E}}^i$ and $\tilde{\underline{E}}^r$ are given by (4.6) and $\tilde{\underline{E}}_2^r$ satisfies the wave equation:

$$\nabla \times \nabla \times \tilde{\underline{E}} - \frac{\omega^2}{c^2} \tilde{\underline{E}} = 0 \tag{4.12}$$

$\tilde{\underline{E}}_2^r$ is also chosen to satisfy the boundary conditions at $z = 0$. Analogous to the analysis in Chapter 3, ((3.19)-(3.31)), we obtain

$$\tilde{\underline{E}}^s = \frac{-1}{\omega^2 - \omega_p^2} \frac{\omega_p^2 e}{2m} \nabla[(\frac{\tilde{\underline{E}}^t}{\omega} e^{i\underline{k}^t \cdot \underline{r}}) * (\frac{\tilde{\underline{E}}^t}{\omega} e^{i\underline{k}^t \cdot \underline{r}})]$$

$$= - \frac{1}{\omega^2 - \omega_p^2} \frac{4}{\alpha^4} \frac{\omega_p^2 e}{2m} \nabla \int_{-\infty}^{\infty} d\omega' \; \tilde{E}^i(\omega')\tilde{E}^i(\omega-\omega')(\omega' - \sqrt{\omega'^2 - \alpha^2})$$

$$\cdot \quad (\omega-\omega' - \sqrt{(\omega-\omega')^2 - \alpha^2}) e^{i \frac{\omega}{c} \sin\theta x} e^{i \frac{\cos\theta}{c} z \sqrt{\omega'^2 - \alpha^2}}$$

$$\cdot \, e^{i \frac{\cos\theta}{c} z \sqrt{(\omega-\omega')^2 - \alpha^2}}$$

$$= \int_{-\infty}^{\infty} \underline{A}^s(\omega,\omega') e^{i\underline{k}^s(\omega,\omega')\cdot\underline{r}} \, d\omega' \tag{4.13}$$

where

$$\underline{A}^s(\omega,\omega') = -\frac{1}{\omega^2 - \omega_p^2} \frac{4}{\alpha^4} \frac{\omega_p^2 e}{2m} \tilde{E}^i(\omega') \tilde{E}^i(\omega-\omega') (\omega' - \sqrt{\omega'^2 - \alpha^2})$$

$$\cdot \quad (\omega-\omega' - \sqrt{(\omega-\omega')^2 - \alpha^2}) \, i\underline{k}^s(\omega,\omega')$$

and

$$\underline{k}^s(\omega,\omega') = \frac{\omega}{c} \sin\theta \, \underline{e}_x + \frac{\cos\theta}{c} \left(\sqrt{\omega'^2 - \alpha^2} + \sqrt{(\omega-\omega')^2 - \alpha^2} \right) \underline{e}_z$$

Equation (4.13) suggests that we write $\tilde{\underline{E}}_2^t$ and $\tilde{\underline{E}}_2^r$ as follows:

$$\tilde{\underline{E}}_2^t = \int_{-\infty}^{\infty} \underline{A}_2^t(\omega,\omega') \, e^{i\underline{k}_2^t(\omega,\omega')\cdot\underline{r}} \, d\omega' \tag{4.14}$$

$$\tilde{\underline{E}}_2^r = \int_{-\infty}^{\infty} \underline{A}_2^r(\omega,\omega') \, e^{i\underline{k}_2^r(\omega,\omega')\cdot\underline{r}} \, d\omega' \tag{4.15}$$

where $\underline{A}_2^t$, $\underline{A}_2^r(\omega,\omega')$, $\underline{k}_2^t(\omega,\omega')$, and $\underline{k}_2^r(\omega,\omega')$ are chosen to satisfy the boundary conditions at $z = 0$. Continuity of tangential components at $z = 0$ is satisfied if:

$$A_{2x}^t(\omega,\omega') e^{ik_{2x}^t(\omega,\omega')x} + A_x^s(\omega,\omega') e^{ik_x^s(\omega,\omega')x}$$

$$= A_{2x}^r(\omega,\omega') \, e^{ik_{2x}^r(\omega,\omega')x} \tag{4.16}$$

and

$$(k^r_{2x}A^r_{2z} - k^r_{2z}A^r_{2x})e^{ik^r_{2x}x} = (k^t_{2x}A^t_{2z} - k^t_{2z}A^t_{2x})e^{ik^t_{2x}x} \tag{4.17}$$

Equation (4.16) implies that

$$k^t_{2x} = k^s_x = k^r_{2x} = \frac{\omega}{c}\sin\theta \tag{4.18}$$

Since, $k^t_2 = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$

and $k^r_2 = \frac{\omega}{c}$

we have that

$$k^t_{2z} = \frac{\cos\theta}{c}\sqrt{\omega^2 - \alpha^2} \tag{4.19}$$

$$k^r_{2z} = -\frac{\omega}{c}\cos\theta \tag{4.20}$$

From (4.16) and (4.17), plus the fact that $\nabla\cdot\tilde{\underline{E}}^t_2 = \nabla\cdot\tilde{\underline{E}}^r_2 = 0$,

we have

$$A^r_{2x} = A^r_{2z}/\tan\theta = \frac{\omega^2 - \omega_p^2}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2}A^s_x \tag{4.21}$$

$$A^r_{2y} = 0 \tag{4.22}$$

$$A^t_{2x} = -\frac{\omega\sqrt{\omega^2 - \alpha^2}}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2}A^s_x \tag{4.23}$$

$$A^t_{2y} = 0 \tag{4.24}$$

$$A^t_{2z} = \frac{\omega^2 \tan\theta}{\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2} A^s_x \tag{4.25}$$

Equations (4.18)-(4.25) completely specify the second order reflected and transmitted fields. Since we are only interested in the second order reflected fields in this chapter, we will just work with the set of equations (4.21) and (4.22). Using (4.15), we see that (4.21) implies:

$$E^r_{2x} = E^r_{2z}/\tan\theta = \int_{-\infty}^{\infty} e^{-i\omega t}\, \tilde{E}^r_{2x}\, d\omega$$

$$= \int_{-\infty}^{\infty} d\omega\, e^{-i\omega t} \int_{-\infty}^{\infty} d\omega'\, A^r_{2x}(\omega,\omega')$$

$$= \frac{4}{\alpha^4} \frac{i \sin\theta}{c} \left(-\frac{\omega_p^2 e}{2m}\right) \int_{-\infty}^{\infty} d\omega'(\omega' - \sqrt{\omega'^2 - \alpha^2})\, \tilde{E}^i(\omega')$$

$$\int_{-\infty}^{\infty} \omega\, d\omega\, e^{-i\omega\tau} \tilde{E}^i(\omega-\omega') \frac{(\omega-\omega' - \sqrt{(\omega-\omega')^2 - \alpha^2})}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2} \tag{4.26}$$

where

$$\tau = t - \frac{x}{c}\sin\theta + \frac{z\cos\theta}{c}$$

From the form of the integrals appearing in (4.26), it is seen that we can express E^r_{2x} and E^r_{2y} in terms of the following convolution integrals:

$$E_{2x}^r = E_{2z}^r/\tan\theta = \frac{4}{\alpha^4}\frac{i\sin\theta}{c}\left(-\frac{\omega_p^2 e}{2m}\right)$$

$$\cdot \int_{s/c}^{t} d\tau\, f^2(t-\tau)\, g(\tau - \tfrac{s}{c})\, H(t - \tfrac{s}{c}) \qquad (4.27)$$

where* $s = x\sin\theta - z\cos\theta$

$$f(t) = \int_{-\infty}^{\infty} \tilde{E}^i(\omega)(\omega - \sqrt{\omega^2-\alpha^2})\, e^{-i\omega t}\, d\omega \qquad (4.28)$$

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega e^{-i\omega t}\, d\omega}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2} \qquad (4.29)$$

The expression for $g(t)$ (4.29) can be readily evaluated with the use of the integration formula (A.3). This formula gives:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\omega\, e^{-i\omega t}\, d\omega}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \frac{2(\cos\phi\sin\phi)e^{-i\alpha t\cos\phi}}{2i(\cos\phi\sin\phi) + 2(\cos^2\phi - \cos^2\theta)}\, H(t)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} d\phi\, \frac{i\sin 2\phi\, e^{-i\alpha t\cos\phi}}{i\sin 2\phi + \cos 2\phi - \cos 2\theta}\, H(t)$$

$$= \frac{1}{4\pi i}\int_{-\pi}^{\pi} d\phi\, \frac{1 - e^{-4i\phi}}{1 - \cos 2\theta\, e^{-2i\phi}}\, e^{-i\alpha t\cos\phi}\, H(t)$$

$$= \frac{1}{4\pi i}\sum_{n=0}^{\infty}\int_{-\pi}^{\pi} d\phi\, (1 - e^{-4i\phi})\cos^n 2\theta\, e^{-2in\phi}\, e^{-i\alpha t\cos\phi}\, H(t)$$

---

*For the rest of this chapter, s will always be defined as $s = x\sin\theta - z\cos\theta$.

$$^{*} \quad = \frac{1}{2i} \sum_{n=0}^{\infty} (-\cos 2\theta)^n \{J_{2n}(\alpha t) - J_{2n+4}(\alpha t)\} \quad H(t) \tag{4.30}$$

It is noted that we can also express $g(t)$ in terms of Lommel functions of two variables as follows:

$$g(t) = \frac{1}{2i} \{U_o(\sqrt{\cos 2\theta}\, \alpha t, \alpha t) - (\cos 2\theta)^{-2} U_4(\sqrt{\cos 2\theta}\, \alpha t, \alpha t)\} \tag{4.31}$$

As in Chapter 3, we now wish to consider two cases. These cases are as follows:

$$\text{(Case 1)} \qquad \tilde{E}^i(\omega) = \frac{1}{2\pi} \tag{4.32}$$

$$\text{(Case 2)} \qquad \tilde{E}^i(\omega) = -\frac{1}{2\pi} \frac{\omega_o}{\omega^2 - \omega_o^2} \tag{4.33}$$

The first case corresponds to a delta pulse incident on a nonlinear plasma. This is described mathematically as

$$\underline{E}^i(t,\underline{r}) = \delta(t - \frac{\underline{n}^i \cdot \underline{r}}{c})\, \underline{e}_y \tag{4.34}$$

(Case 1)

The second case corresponds to a suddenly turned on sinusoid incident on a nonlinear plasma, viz.,

$$\underline{E}^i(t,\underline{r}) = \sin \omega_o(t - \frac{\underline{n}^i \cdot \underline{r}}{c})\; H(t - \frac{\underline{n}^i \cdot \underline{r}}{c}) \tag{4.35}$$

(Case 2)

---

*This last equality follows from the Bessel definition (A.5).

Equation (4.27) (with $E^r_{2y} = 0$) represents the second order, nonlinear electric field response to any incident, E-polarized electromagnetic wave. Before considering the individual cases one and two above, we remark that $\underline{E}^r_2$ is perpendicular to both $\underline{E}^i$ and $\underline{E}^r$. We also note that for normal incidence (θ=0), there is no second order, nonlinear reflected field.

We now consider the first case. For this case, $f(t)$ is evaluated using (A.3) (with $\tilde{E}^i(\omega)$ given by (4.32)) as follows:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\omega - \sqrt{\omega^2 - \alpha^2})\, e^{-i\omega t}\, d\omega$$

$$= \frac{\alpha^2}{2\pi} \int_{-\pi}^{\pi} (\cos\phi - i\sin\phi) e^{-i\alpha t\cos\phi} \sin\phi\, d\phi\, H(t)$$

$$= \frac{\alpha^2}{4\pi i} \int_{-\pi}^{\pi} (1 - e^{-2i\phi})\, e^{-i\alpha\, t\cos\phi}\, d\phi\, H(t) \tag{4.36}$$

Again using (A.5) and the Bessel recursion relations, we have

$$f(t) = \frac{\alpha}{it} J_1(\alpha t)\, H(t) \tag{4.37}$$

(Case 1)

For case 2, we evaluate $f(t)$ (with $\tilde{E}^i(\omega)$ given by (4.33)) using (A.8):

$$f(t) = \frac{-\omega_0}{2\pi} \int_{-\infty}^{\infty} \frac{\omega - \sqrt{\omega^2 - \alpha^2}}{\omega^2 - \omega_0^2}\, e^{-i\omega t}\, d\omega$$

$$= \frac{\omega_0}{\pi} \int_0^{2\pi} \frac{\xi^2(1 - \xi^2)\, e^{iq\cos\psi}}{(\xi^2 - \xi_0^2)\,(\xi^2 - \xi_0^{-2})}\, d\psi$$

$$= \frac{\omega_o}{\pi} \int_0^{2\pi} d\psi \; e^{iq \cos\psi} \Big[\frac{1}{\xi_o^2 - \xi_o^{-2}} \Big(\frac{1}{\xi^2 - \xi_o^2} - \frac{1}{\xi^2 - \xi_o^{-2}} \Big) + \frac{1 - \xi_o^2 - \xi_o^{-2}}{\xi_o^2 - \xi_o^{-2}} \Big(\frac{\xi_o^2}{\xi^2 - \xi_o^2} - \frac{\xi_o^{-2}}{\xi^2 - \xi_o^{-2}} \Big) - 1 \Big]$$

$$= \frac{\alpha}{i} [\xi_o U_2(\xi_o^{-1} \alpha t, \alpha t) + \xi_o^{-1} U_2(\xi_o \alpha t, \alpha t)] \;\; H(t) \qquad (4.38)$$

(Case 2)

where

$$\xi_o = \frac{\omega_o + \sqrt{\omega_o^2 - \alpha^2}}{\alpha} \; , \qquad \text{for } \omega_o > \alpha$$

or

$$\xi_o = \frac{\omega_o + i\sqrt{\alpha^2 - \omega_o^2}}{\alpha} \; , \qquad \text{for } \omega_o < \alpha$$

Equation (4.37) together with (4.31) and (4.27) give the expression for $\underline{E}_2^r$ for case 1. Similarly (4.38), (4.31), and (4.27) give $\underline{E}_2^r$ for case 2. It is noted that E_{2x}^r and E_{2z}^r can be written as:

$$E_{2x}^r = E_{2z}^r / \tan\theta = \frac{2}{\alpha^4} \sin\theta \; \frac{\omega_p^2 e}{mc} \; I_1(s,t) \qquad (4.39)$$

(Case 1)

where

$$I_1(s,t) = \frac{\alpha^2}{2} \int_{s/c}^{t} \Big[\frac{J_1(\alpha(t-\tau))}{t-\tau} \Big]^2$$

$$\cdot \; \{U_o(\sqrt{\cos 2\theta}\; \alpha(\tau - \tfrac{s}{c}), \; \alpha(\tau - \tfrac{s}{c})) - (\cos 2\theta)^{-2}$$

$$\cdot\ U_4(\sqrt{\cos 2\theta}\ \alpha(\tau - \tfrac{s}{c}),\ \alpha(\tau - \tfrac{s}{c}))\}\ d\tau \tag{4.40}$$

$$E^r_{2x} = E^r_{2z}/\tan\theta = \frac{2}{\alpha^4}\sin\theta\ \frac{\omega_p^2 e}{mc}\ I_2(s,t) \tag{4.41}$$

(Case 2)

where

$$I_2(s,t) = \frac{\alpha^2}{2}\int_{s/c}^{t}\{\xi_o U_2(\xi_o^{-1}\alpha\tau,\alpha\tau) + \xi_o^{-1}U_2(\xi_o\alpha\tau,\alpha\tau)\}^2$$

$$\cdot\ \{U_o(\sqrt{\cos 2\theta}\ \alpha(\tau - \tfrac{s}{c}),\ \alpha(\tau - \tfrac{s}{c}))$$

$$-\ (\cos 2\theta)^{-2}\ U_4(\sqrt{\cos 2\theta}\ \alpha(\tau - \tfrac{s}{c}),\ \alpha(\tau - \tfrac{s}{c}))\}d\tau \tag{4.42}$$

We will consider the limiting forms of $\underline{E}^r_2$ for large and small t. However, before doing so, we will derive $\underline{E}^r_2$ for the case when the incident fields are H-polarized. In this case, the transforms of the linear transmitted, reflected, and incident fields are given by*

$$\underline{\tilde{E}}^t = \frac{2\omega\tilde{E}^i(\omega)}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2}\{-\cos\theta\sqrt{\omega^2-\alpha^2}\ \underline{e}_x$$

$$+\ \omega\sin\theta\ \underline{e}_z\}\ e^{i\frac{\omega}{c}\sin\theta x}\ e^{\frac{i\cos\theta}{c}z\sqrt{\omega^2-\alpha^2}}$$

---

*For a derivation of the following equations, see, for example, Reference [30], pp. 39-40.

$$\underline{\tilde{E}}^r = \tilde{E}^i(\omega)\left(1 - \frac{2\omega\sqrt{\omega^2-\alpha^2}}{\omega^2-\omega_p^2+\omega\sqrt{\omega^2-\alpha^2}}\right)(\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_z)$$

$$\times\; e^{i\frac{\omega}{c}\sin\theta x}\, e^{-i\frac{\omega}{c}\cos\theta z} \tag{4.43}$$

$$\underline{\tilde{E}}^i = \tilde{E}^i(\omega)(-\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_z)\, e^{i\frac{\omega}{c}x\sin\theta}\, e^{i\frac{\omega}{c}z\cos\theta}$$

The transmitted field given by (4.43) generates a second order pulse in the nonlinear plasma which is obtained as in the case of the E-polarized incident field (equation (4.13)). The result is

$$\underline{\tilde{E}}^s = -\frac{1}{\omega^2-\omega_p^2}\frac{\omega_p^2 e}{2m}\nabla\left(\left(\frac{\tilde{E}_j^t(\omega)}{\omega}\right) * \left(\frac{\tilde{E}_j^t}{\omega}\right)\right) \tag{4.44}$$

where $\tilde{E}_j^t(\omega)$ is the j^{th} component of $\underline{\tilde{E}}^t$, and the sum on the j in (4.44) is implied. From equation (4.44) and (4.43), we have:

$$\underline{\tilde{E}}^s = -\frac{4}{\omega^2-\omega_p^2}\frac{\omega_p^2 e}{2m}\nabla\left\{\int_{-\infty}^{\infty} d\omega'\,\tilde{E}^i(\omega')\,\tilde{E}^i(\omega-\omega')\right.$$

$$\cdot\;\frac{e^{i\frac{\omega}{c}\sin\theta x}\, e^{i\frac{\cos\theta}{c}z\sqrt{\omega'^2-\alpha^2}}\, e^{i\frac{\cos\theta}{c}z\sqrt{(\omega-\omega')^2-\alpha^2}}}{(\omega'\sqrt{\omega'^2-\alpha^2}+\omega'^2-\omega_p^2)((\omega-\omega')\sqrt{(\omega-\omega')^2-\alpha^2}+(\omega-\omega')^2-\omega_p^2)}$$

$$\left.\cdot\;[\cos^2\theta\sqrt{\omega'^2-\alpha^2}\sqrt{(\omega-\omega')^2-\alpha^2}+\sin^2\theta\,\omega'(\omega-\omega')]\right\}$$

$$= \int_{-\infty}^{\infty}\underline{A}^s(\omega,\omega')\,e^{i\underline{k}^s(\omega,\omega')\cdot\underline{r}}\,d\omega' \tag{4.45}$$

$$\underline{A}^s(\omega,\omega') = \frac{-4}{\omega^2 - \omega_p^2} \frac{\omega_p^2 e}{2m} \tilde{E}^i(\omega') \tilde{E}^i(\omega-\omega') \cdot i\underline{k}^s(\omega,\omega')$$

$$\frac{\{\cos^2\theta \sqrt{\omega'^2-\alpha^2} \sqrt{(\omega-\omega')^2-\alpha^2} + \sin^2\theta\ \omega'(\omega-\omega')\}}{(\omega' \sqrt{\omega'^2-\alpha^2} + \omega'^2 - \omega_p^2)((\omega-\omega') \sqrt{(\omega-\omega')^2-\alpha^2} + (\omega-\omega')^2 - \omega_p^2)}$$

and

$$\underline{k}^s(\omega,\omega') = \frac{\cos\theta}{c} \{ \sqrt{\omega'^2-\alpha^2} + \sqrt{(\omega-\omega')^2 - \alpha^2}\}\ \underline{e}_z + \frac{\sin\theta}{c}\ \omega\ \underline{e}_x$$

Analogous with the expressions obtained for the nonlinear reflected fields (4.26), we have:

$$E^r_{2x} = E^r_{2z}/\tan\theta = \int_{-\infty}^{\infty} d\omega' \int_{-\infty}^{\infty} d\omega\ e^{-i\omega\tau} \frac{\omega^2 - \omega_p^2}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2} A^s_x(\omega,\omega') \tag{4.46}$$

where

$$\tau = t - \frac{x}{c}\sin\theta + \frac{z}{c}\cos\theta$$

Equivalently, we can write (4.46) in terms of the convolution integral:

$$E^r_{2x} = E^r_{2z}\cot\theta = -4i\frac{\omega_p^2 e}{2m} \frac{\sin\theta}{c} \int_{s/c}^{t} [\frac{v^2(t-\tau)}{\cos^2\theta} + \frac{u^2(t-\tau)}{\sin^2\theta}] \cdot g(\tau - \frac{s}{c})\ d\tau \tag{4.47}$$

where $g(t)$ is given by either (4.29) or (4.31);

$$v(t) = \int_{-\infty}^{\infty} \frac{\tilde{E}^i(\omega) \cos^2\theta \sqrt{\omega^2-\alpha^2}}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2} e^{-i\omega t}\ d\omega \tag{4.48}$$

$$u(t) = \int_{-\infty}^{\infty} \frac{\tilde{E}^i(\omega)\ \sin^2\theta\ \omega\ e^{-i\omega t}}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2}\ d\omega \tag{4.49}$$

It is noted that

$$v(t) = \alpha^{-2}\ f(t) - u(t) \tag{4.50}$$

where $f(t)$ is defined by (4.28). In evaluating $u(t)$ we again consider the two cases specified by equations (4.32) and (4.33). When $\tilde{E}^i(\omega) = \frac{1}{2\pi}$, we have that

$$u(t) = \frac{\sin^2\theta}{2\pi} \int_{-\infty}^{\infty} \frac{\omega\ e^{-i\omega t}\ d\omega}{\omega\sqrt{\omega^2 - \alpha^2} + \omega^2 - \omega_p^2} \tag{4.51}$$

(Case 1)

We note the similarity between this integral and (4.29) for $g(t)$. We can therefore immediately write:

$$u(t) = \frac{\sin^2\theta}{2i} \{U_o(\sqrt{\cos 2\theta}\ \alpha t, \alpha t) - (\cos 2\theta)^{-2}\ U_4(\sqrt{\cos 2\theta}\ \alpha t, \alpha t)\} \tag{4.52}$$

(Case 1)

$v(t)$ for case 1 can be obtained from (4.50). When $\tilde{E}^i(\omega) = -\frac{\omega_o}{2\pi} \cdot \frac{1}{\omega^2 - \omega_o^2}$ we have that

$$u(t) = \frac{-\omega_o \sin^2\theta}{2\pi} \int_{-\infty}^{\infty} \frac{\omega\ e^{-i\omega t}\ d\omega}{(\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2)(\omega^2 - \omega_o^2)} \tag{4.53}$$

(Case 2)

To do this integral we can use (A.3) for the case when $\omega_o > \alpha$. Then we have:

$$u(t) = -\frac{\omega_o \sin^2\theta}{2\pi\alpha^2} \int_{-\pi}^{\pi} \frac{d\phi \cos\phi \sin\phi \, e^{-i\alpha t \cos\phi}}{(i \cos\theta \sin\phi + \cos^2\phi - \frac{\omega_p^2}{\alpha^2})(\cos^2\phi - \frac{\omega_o^2}{\alpha^2})} \tag{4.54}$$

(Case 2)

After expressing the integrand of (4.54) in terms of complex exponentials (as in (4.30)), and then partial fractioning and manipulating the integrand, we obtain the following for $u(t)$:

$$u(t) = \frac{2i\omega_o \sin^2\theta}{\alpha^2} \frac{1}{(\cos 2\theta - \xi_o^2)(\cos 2\theta - \xi_o^{-2})}$$

$$\cdot \quad [\cos 2\theta \, U_o(\sqrt{\cos 2\theta}\, \alpha t, \alpha t)$$

$$- \frac{1}{\cos 2\theta} U_4(\sqrt{\cos 2\theta}\, \alpha t, \alpha t) + \xi_o^2 \, U_4(\xi_o^{-1} \alpha t, \alpha t)$$

$$- \cos 2\theta \, U_o(\xi_o^{-1} \alpha t, \alpha t) + (\cos 2\theta - \xi_o^{-2})$$

$$\cdot \quad U_2(\xi_o \alpha t, \alpha t)] \, H(t) \tag{4.55}$$

(Case 2)

When $\omega_o < \alpha$, we can use (A.8), but the result is the same as (4.55). Therefore, $u(t)$ is given by (4.55) for all ω_o . Of course, $v(t)$ for case 2 can be obtained from (4.50) with $f(t)$ given by (4.38).

Equations (4.52), (4.50), and (4.47) give the expression for $\underline{E}_2^r$ for case 1. Equations (4.55), (4.50), and (4.47) give the expression for $\underline{E}_2^r$ for case 2. It is remembered that $\underline{E}_2^r$ as given in

(4.47) represents the second order, nonlinear electric field response to any incident, H-polarized electromagnetic wave. It is noted that E^r_{2x} and E^r_{2z} can be written as:

$$E^r_{2x} = E^r_{2z}\cot\theta = \frac{-4i\,\omega_p^2\,e}{2m}\;\frac{\sin\theta}{c}\;I_3(s,t) \tag{4.56}$$

(Case 1)

where

$$I_3(s,t) = \int_{s/c}^{t}\left[\frac{u_3^2(t-\tau)}{\sin^2\theta} + \frac{v_3^2(t-\tau)}{\cos^2\theta}\right] g(\tau - \tfrac{s}{c})\,d\tau \tag{4.57}$$

and $u_3(t) = u(t)$; $v_3(t) = f(t)$, with $u(t)$, $v(t)$ given by (4.52) and (4.50), respectively. For case 2, we have the following:

$$E^r_{2x} = E^r_{2z}\cot\theta = \frac{-4i\,\omega_p^2\,e}{2m}\;\omega_o^2\;\frac{\sin\theta}{c}\;I_4(s,t) \tag{4.58}$$

(Case 2)

where

$$I_4(s,t) = \int_{s/c}^{t}\left[\frac{u_4^2(t-\tau)}{\sin^2\theta} + \frac{v_4^2(t-\tau)}{\cos^2\theta}\right] g(\tau - \tfrac{s}{c})\,d\tau \tag{4.59}$$

and $u_4(t) = u(t)/\omega_o$; $v_4(t) = v(t)/\omega_o$, with $u(t)$, $v(t)$ given by (4.55) and (4.50), respectively.

We will now consider the limiting forms of $\underline{E}^r_2$ for large and small t for all the various cases (equations (4.39), (4.41), (4.56), and (4.58)). Actually, we need only consider the asymptotic expansions of the four integrals I_1-I_4, since these integrals are simply related to $\underline{E}^r_2$ by the equations (4.39), (4.41), (4.56), and (4.58). To simplify the following asymptotic analysis, we will write I_1 and

I_2 as:

$$I_1(s,t) = i \int_{s/c}^{t} f_1^2(t-\tau)\ g(\tau - \frac{s}{c})\ d\tau \tag{4.60}$$

$$I_2(s,t) = i \int_{s/c}^{t} f_2^2(t-\tau)\ g(\tau - \frac{s}{c})\ d\tau \tag{4.61}$$

where

$$f_1(t) = \alpha \frac{J_1(\alpha t)}{t} = \alpha^2(J_o(\alpha t) + J_2(\alpha t)) \tag{4.62}$$

and

$$f_2(t) = \alpha\{\xi_o U_2(\xi_o^{-1}\ \alpha t, \alpha t) + \xi_o^{-1} U_2(\xi_o\ \alpha t, \alpha t)\} \tag{4.63}$$

To carry out the expansions of I_1-I_4 for small t, we will make use of equation (C.6). This equation states the following:

If

$$\phi(t) = \int_{r}^{t} f^2(t-\tau)\ g(\tau-r)\ d\tau$$

and

$$f(t) = \sum_{n=0}^{\infty} a_n J_n(\alpha t)$$

$$g(t) = \sum_{n=0}^{\infty} b_n J_n(\alpha t)$$

then

$$\phi(t) \sim 2\alpha^{-1}\ b_o a_o^2\ J_1(\alpha(t-r)) \tag{4.64}$$

for small t. Making use of equations (4.30), (4.50), (4.52), (4.55), (4.62), and (4.63), and expressing all Lommel functions in terms of their Bessel sums (equations (A.12) and (A.13)), we have, using (4.64)*

$$I_1(s,t) \sim \frac{1}{4}\alpha^3 J_1(\alpha(t - \frac{s}{c})) \tag{4.65}$$

$$I_2(s,t) \sim \frac{4\omega_o^2}{\alpha} J_5(\alpha(t - \frac{s}{c})) \tag{4.66}$$

$$I_3(s,t) \sim \frac{i}{4\alpha} J_1(\alpha(t - \frac{s}{c})) \tag{4.67}$$

$$I_4(s,t) \sim \frac{4\omega_o^2 i}{\alpha^5} J_5(\alpha(t - \frac{s}{c})) \tag{4.68}$$

Equations (4.65)-(4.69) represent the expansion of the integrals for small t. We now consider these integrals for large t. They can all be expressed in the following form:

$$I(t) = \int_r^t f^2(t-\tau)\, g(\tau-r)\, d\tau \tag{4.69}$$

The form of (4.69) is quite similar to the integrals in Chapter 3 ((3.40) and (3.45)) in that the integrand of (4.69) contains both a complicated function squared, $f^2(t-\tau)$, and a transfer function $g(t-r)$. However, whereas the transfer function in the integrals of (3.40) and (3.45) is $\sin \omega_p \tau$, $g(\tau-r)$ in (4.69) is a complicated function. It is for this reason that an exact asymptotic analysis of the integrals I_1-I_4 is impossible. It is, however, reasonable to expect that the

---

*To obtain (4.66) and (4.68) one must use the total expansion (C.5). Equation (C.5) gives an expansion for $\phi(t)$ up to the 5th order Bessel function. This expansion is necessary as $a_o = a_1 = 0$, and $b_o \neq 0$ for I_2 and I_4.

integrals I_1 and I_3 will go to zero for large times, as they correspond to the second order reflected field due to a delta function incident wave. Indeed, (4.69) can be written as:

$$I_1(t) = 2\pi \int_{-\infty}^{\infty} F(\omega)\ G(\omega)\ e^{-i\omega\tau}\ d\omega \tag{4.70}$$

where

$$\tau = t - r\ ; \qquad F(\omega) = \frac{1}{2\pi} \int_0^{\infty} f^2(t)\ e^{i\omega t}\ dt$$

and

$$G(\omega) = \frac{1}{2\pi} \int_0^{\infty} g(t)\ e^{i\omega t}\ dt = \frac{1}{2\pi} \frac{\omega}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2} \tag{4.71}$$

Therefore, since $G(\omega)$ has no poles, we would expect that for the case of a delta impulse incidence, the reflected field will go to zero for large times. This is borne out by numerical calculations of E^r_{2x} for case 1 for both incident E and H polarization (Figures 4.2-4.7).

The integrals I_2 and I_4 correspond to a suddenly turned on sinusoidal incidence of frequency ω_o . We therefore expect I_2 and I_4 to asymptotically approach a wave at the second harmonic frequency. To get a better idea of the asymptotic behavior of I_2 and I_4 , we look at the steady state solution to the second order fields. First, examining I_2 (equation (4.61)), we see that in the steady state, $f_2(t)$ (4.63) is given by

$$f_2(t) = -\frac{\omega_o}{2\pi}\int_{-\infty}^{\infty}\frac{\omega-\sqrt{\omega^2-\alpha^2}}{\omega^2-\omega_o^2}e^{-i\omega t}\,d\omega$$

$$\sim i(\omega_o-\sqrt{\omega_o^2-\alpha^2})\cos\omega_o t \qquad \text{for } \omega_o > \alpha$$

$$\sim i(\omega_o\cos\omega_o t - \sqrt{\alpha^2-\omega_o^2}\sin\omega_o t) \quad \text{for } \omega_o < \alpha \qquad (4.72)$$

We will now assume that $f_2(t)$ is given by its asymptotic expansion (4.72) for all t. Then we have for $[f_2(t)]^2$

$$f_2^2(t) = -\frac{\omega_o^2}{4}\left(1-\sqrt{1-(\frac{\alpha}{\omega_o})^2}\,\right)^2\{2+e^{2i\omega_o t}+e^{-2i\omega_o t}\}$$

$$\text{for } \omega_o > \alpha$$

$$= -\frac{1}{2}[e^{2i\omega_o t}(\omega_o^2-\frac{\alpha^2}{2}-\frac{\omega_o}{i}\sqrt{\alpha^2-\omega_o^2})+e^{-2i\omega_o t}$$

$$\cdot(\omega_o^2-\frac{\alpha^2}{2}+\frac{\omega_o}{i}\sqrt{\alpha^2-\omega_o^2})+\frac{\alpha^2}{2}] \qquad \text{for } \omega_o < \alpha \qquad (4.73)$$

Substituting (4.73) into (4.61) and changing into the form (4.70) gives

$$I_2(s,t) = \frac{\omega_o^3\left(1-\sqrt{1-(\frac{\alpha}{\omega_o})^2}\,\right)^2\sin 2\omega_o(t-\frac{s}{c})}{2\omega_o\sqrt{4\omega_o^2-\alpha^2}+4\omega_o^2-\omega_p^2} \quad ; \ \omega_o > \alpha$$

$$= \frac{2\omega_o^3}{2\omega_o\sqrt{4\omega_o^2-\alpha^2}+4\omega_o^2-\omega_p^2}\{[1-\frac{1}{2}(\frac{\alpha}{\omega_o})^2]\sin 2\omega_o(t-\frac{s}{c})$$

$$-\sqrt{(\frac{\alpha}{\omega_o})^2-1}\,\cos 2\omega_o(t-\frac{s}{c})\} \ ; \ \alpha > \omega_o > \alpha/2$$

$$= i\omega_o^3\left(1 - \frac{1}{2}\left(\frac{\alpha}{\omega_o}\right)^2\right)\left\{\frac{e^{-2i\omega_o(t - \frac{s}{c})}}{2i\omega_o\sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2} - \frac{e^{2i\omega_o(t - \frac{s}{c})}}{-2i\omega_o\sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2}\right\}$$

$$+ \omega_o^3\sqrt{\left(\frac{\alpha}{\omega_o}\right)^2 - 1}\left\{\frac{e^{-2i\omega_o(t - \frac{s}{c})}}{2i\omega_o\sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2} + \frac{e^{2i\omega_o(t - \frac{s}{c})}}{-2i\omega_o\sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2}\right\}; \qquad \omega_o < \frac{\alpha}{2} \tag{4.74}$$

An analogous expression for $I_4(t)$ can also be derived:

$$I_4(t) = \frac{i\omega_o(\omega_p^2 - \omega_o^2)\sin 2\,\omega_o(t - \frac{s}{c})}{A(2\omega_o\sqrt{4\omega_o^2 - \alpha^2} + 4\omega_o^2 - \omega_p^2)} \qquad \text{for } \omega_o > \alpha$$

$$= \frac{\omega_o(\omega_p^2 - \omega_o^2)}{2(2\omega_o\sqrt{4\omega_o^2 - \alpha^2} + 4\omega_o^2 - \omega_p^2)}\left\{B e^{-2i\omega_o(t - \frac{s}{c})} - C\, e^{2i\omega_o(t - \frac{s}{c})}\right\} \quad \text{for } \frac{\alpha}{2} < \omega_o < \alpha$$

$$= \frac{\omega_o(\omega_p^2 - \omega_o^2)}{2}\left\{D\, e^{-2i\omega_o(t - \frac{s}{c})} - F\, e^{2i\omega_o(t - \frac{s}{c})}\right\} \quad \text{for } \omega_o < \frac{\alpha}{2}\,.$$

where in the above

$$A = (\omega_o \sqrt{\omega_o^2 - \alpha^2} + \omega_o^2 - \omega_p^2)^2 \; ; \qquad B = \frac{1}{(i\omega_o \sqrt{\alpha^2 - \omega_o^2} + \omega_o^2 - \omega_p^2)^2}$$

$$C = \frac{1}{(-i\omega_o \sqrt{\alpha^2 - \omega_o^2} + \omega_o^2 - \omega_p^2)^2} \; ; \qquad D = \frac{B}{(2i\omega_o \sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2)}$$

$$F = \frac{C}{-2i\omega_o \sqrt{\alpha^2 - 4\omega_o^2} + 4\omega_o^2 - \omega_p^2}$$

Expressions (4.74) and (4.75) are the values of I_2 and I_4 if the incident field were purely sinusoidal instead of a suddenly turned on sinusoid. It is reasonable to expect that (4.75) and (4.74) are also the asymptotic expansions of I_2 and I_4. This is borne out by computer calculations of E^r_{2x} for case 2, for both incident E and H polarization (Figures 4.8-4.13).

We now wish to summarize and comment on the results of the above asymptotic analysis. It is noted that from the results of formulas (4.65)-(4.68) that for short times, the second order, nonlinear reflected fields (for both E-polarized and H-polarized incident waves) represent oscillations whose frequency approaches $\alpha = \omega_p \sec \theta$. For the case when a delta pulse is incident on the nonlinear plasma, it was argued that $\underline{E}^r_2$ goes to zero for long times. From formulas (4.74) and (4.75) it was argued that when a suddenly turned on sinusoid of frequency ω_o is incident on the nonlinear medium, $\underline{E}^r_2$ is asymptotic to a sinusoid at the second harmonic frequency $2\omega_o$. It is also noted that in the two formulas (4.74) and (4.75), there are three ranges of ω_o which are of interest: $\omega_o > \alpha$, $\frac{\alpha}{2} < \omega_o < \alpha$, and $\omega_o < \frac{\alpha}{2}$. This is contrasted with the case of reflection from a linear, isotropic plasma where there are just two ranges of interest for ω_o: $\omega_o > \alpha$ and $\omega_o < \alpha$. The reason for this difference is that in the case of a nonlinear plasma, an incident wave whose frequency ω_o is below the cutoff frequency α , but yet greater than $\alpha/2$ can generate a second harmonic at $2\omega_o$ in the plasma which is above the cutoff frequency and therefore will propagate. This behavior is also noted in reference [17] where it is commented that, "The variety of nonlinear phenomena involving evanescent (exponentially decaying) waves is much wider than in the linear case."

In Figures 4.2 to 4.13 plots are made of $E^r_{2x}(0,t)$ corresponding to both E and H-polarized incidence for cases 1 and 2 for various values of ω_o and θ . Unfortunately, due to numerical difficulties no plots were made for the interesting case $\omega_o < \alpha$. For sake of comparison, plots are made of E^r_y (Figures 4.14-4.16) for the case of an E-polarized incident wave (equation (4.6)) where $E^i(\omega)$ is given by (4.33). According to (4.6) and (4.33), we can write E^r_y as $(E^r_x = E^r_z = 0)$:

$$E^r_y = -\frac{\omega_o}{2\pi}\int_{-\infty}^{\infty}\frac{\omega-\sqrt{\omega^2-\alpha^2}}{\omega+\sqrt{\omega^2-\alpha^2}}\;\frac{e^{i\frac{\omega}{c}s}\;e^{-i\omega t}}{\omega^2-\omega_o^2}\,d\omega \tag{4.76}$$

With the use of (A.8) and the indicated substitutions in Appendix A, we have

$$E_y^r = \frac{\omega_o}{\pi i \alpha} \int_0^{2\pi} \frac{\xi^3(1 - \xi^2)e^{iq \cos \psi}}{(\xi^2 - \xi_o^2)(\xi^2 - \xi_o^{-2})} d\xi \, H(t - \frac{s}{c})$$

$$= \frac{\omega_o}{\alpha} [\frac{\alpha}{\omega_o} (\xi_o^{-2} U_1(\alpha\xi_o\tau, \alpha\tau) + \xi_o^2 \, U_1(\alpha\xi_o^{-1}\tau, \alpha\tau))$$

$$-2J_1(\alpha\tau)] \, H(\tau) \qquad (4.77)$$

where

$$\tau = t - \frac{s}{c}$$

Corresponding plots of $\underline{E}^r$ given by (4.43) and (4.33) were not made due to numerical difficulties. Also, plots of $\underline{E}^r$ corresponding to E-polarized and H-polarized incident delta pulses were not made but can be found in Chabries' paper [26]. The numerical method used to compute all the graphs in this chapter can be found in Appendix B.

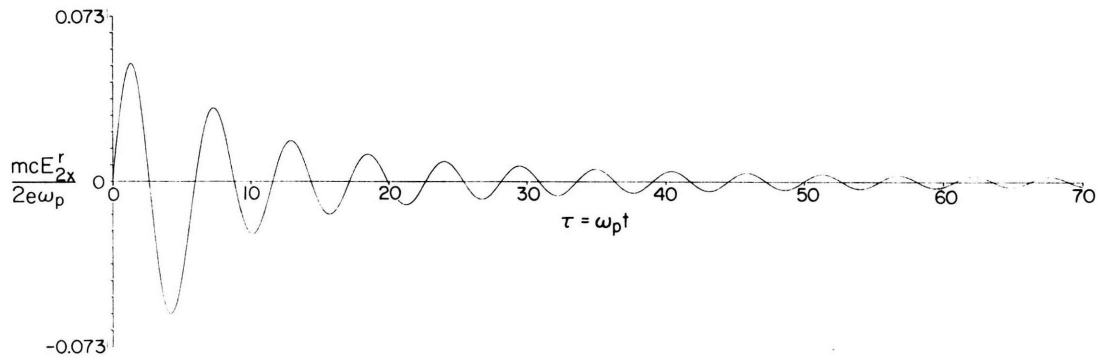


Figure 4.2: $E^r_{2x}(\underline{0},t)$ for $\theta = 30^o$, E-polarization incidence corresponding to case 1 ($\underline{E}^i(\underline{0},t) = \delta(t)\ \underline{e}_y$)

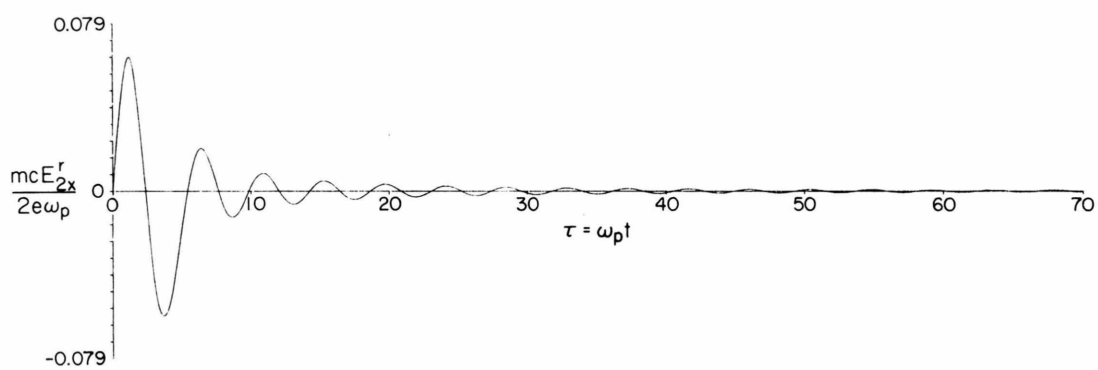


Figure 4.3: $E^r_{2x}(\underline{0},t)$ for $\theta = 45^o$, E-polarization incidence corresponding to case 1 ($\underline{E}^i(\underline{0},t) = \delta(t)\ \underline{e}_y$)

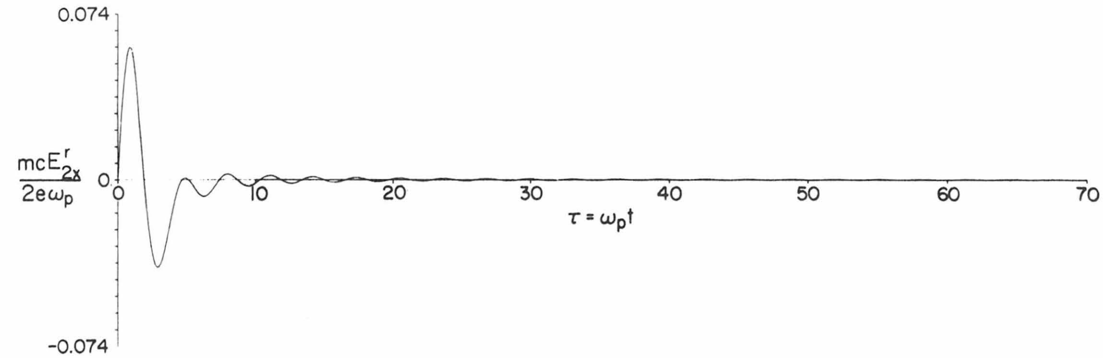


Figure 4.4: $E^r_{2x}(\underline{0},t)$ for $\theta = 60^o$, E-polarization incidence corresponding to case 1 $(E^i(\underline{0},t) = \delta(t)\ \underline{e}_y)$

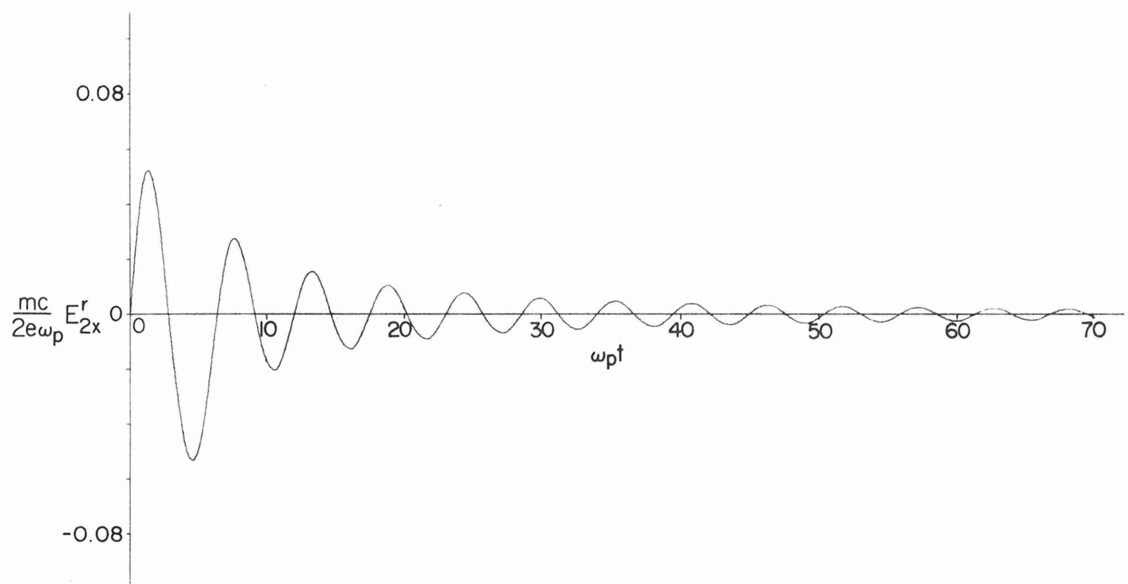
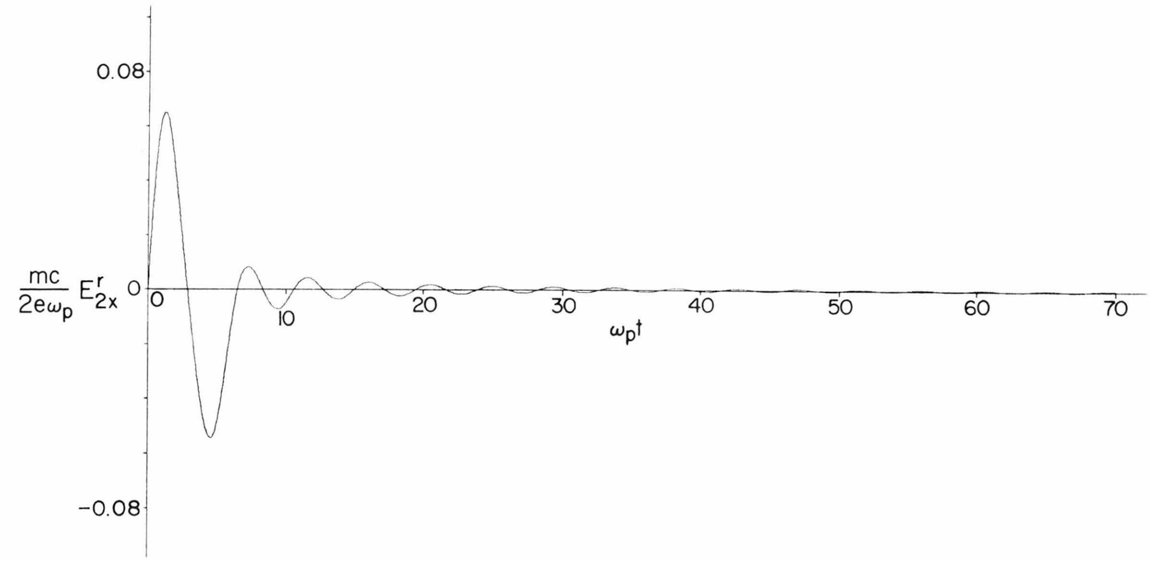


Figure 4.5: $E^r_{2x}(\underline{0},t)$ for $\theta = 30^o$, H-polarization incidence corresponding to case 1 $(\underline{E}^i(\underline{0},t) = \delta(t)\cdot(-\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_z))$



igure 4.6: $E^r_{2x}(\underline{0},t)$ for $\theta = 45^0$, H-polarization incidence corresponding to case 1 $(\underline{E}^i(\underline{0},t) = \delta(t)\cdot(-\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_z))$

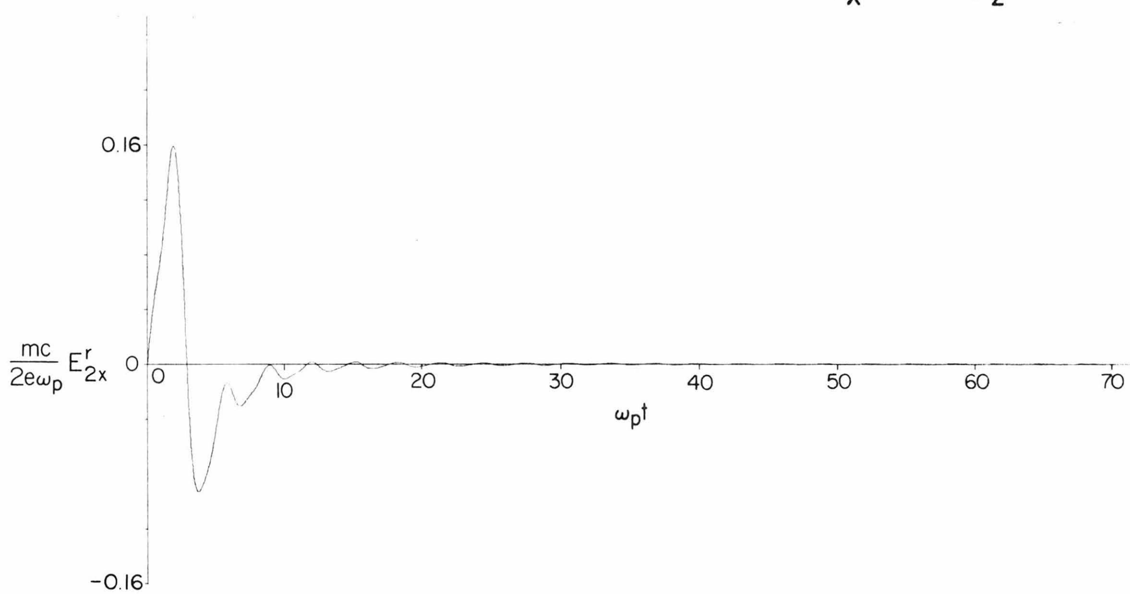


Figure 4.7: $E^r_{2x}(\underline{0},t)$ for $\theta = 60^0$, H-polarization incidence corresponding to case 1 $(\underline{E}^i(\underline{0},t) = \delta(t)\cdot(-\cos\theta\,\underline{e}_x + \sin\theta\,\underline{e}_z))$

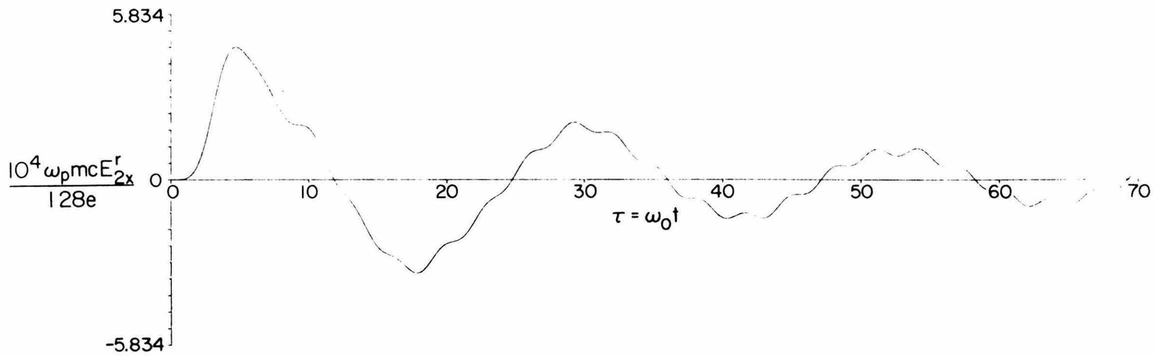


Figure 4.8: $E^r_{2x}(\underline{0},t)$ for $\theta = 30^0$, $\omega_o = 4\omega_p$, E-polarization incidence corresponding to case 2 $(\underline{E}^i(\underline{0},t) = \sin \omega_o t \, H(t) \, \underline{e}_y)$

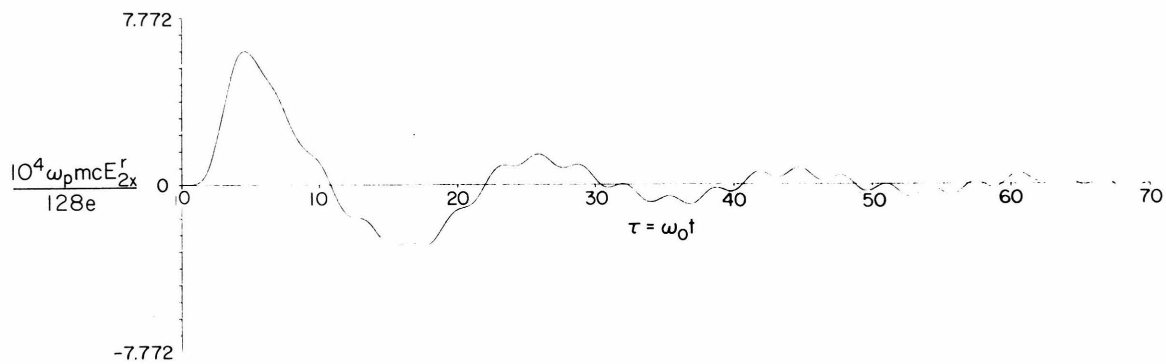


Figure 4.9: $E^r_{2x}(\underline{0},t)$ for $\theta = 45^0$, $\omega_o = 4\omega_p$, E-polarization incidence corresponding to case 2 $(\underline{E}^i(\underline{0},t) = \sin \omega_o t \, H(t) \, \underline{e}_y)$

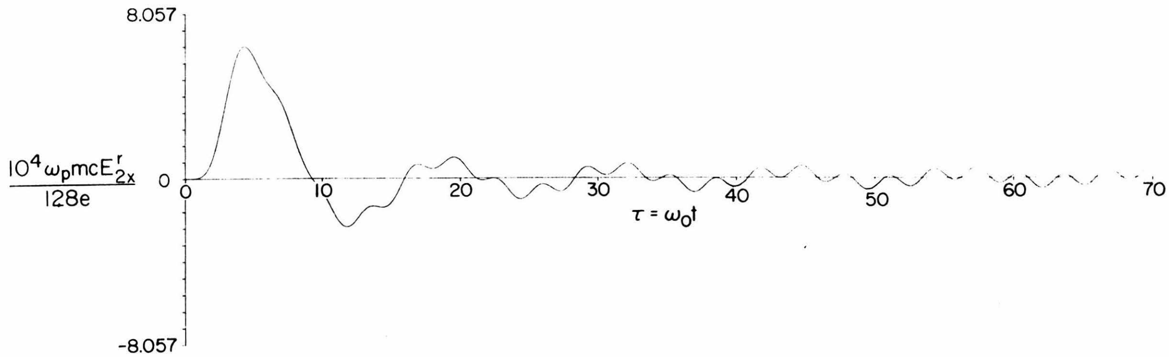


Figure 4.10: $E^r_{2x}(\underline{0},t)$ for $\theta = 60^0$, $\omega_0 = 4\omega_p$, E-polarization incidence corresponding to case 2 $(\underline{E}^i(\underline{0},t) = \sin \omega_0 t\ H(t)\ \underline{e}_y)$

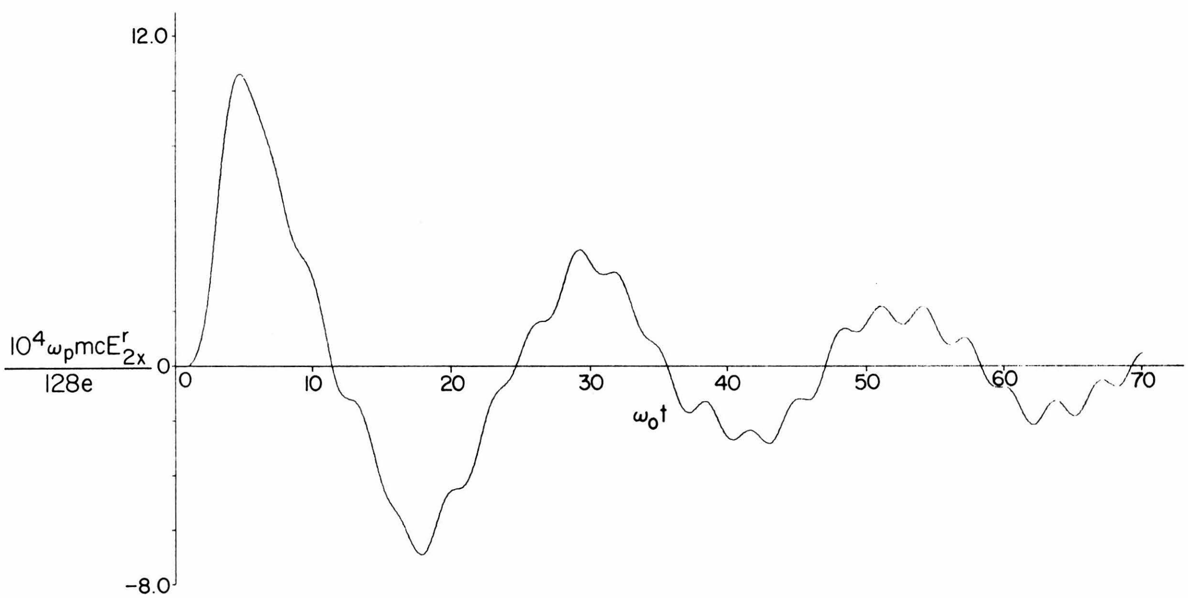


Figure 4.11: $E^r_{2x}(\underline{0},t)$ for $\theta = 30^0$, $\omega_0 = 4\omega_p$, H-polarization incidence corresponding to case 2 $(\underline{E}^i(\underline{0},t) = \sin \omega_0 t\ H(t)\cdot(-\cos\theta\ \underline{e}_x + \sin\theta\ \underline{e}_z))$

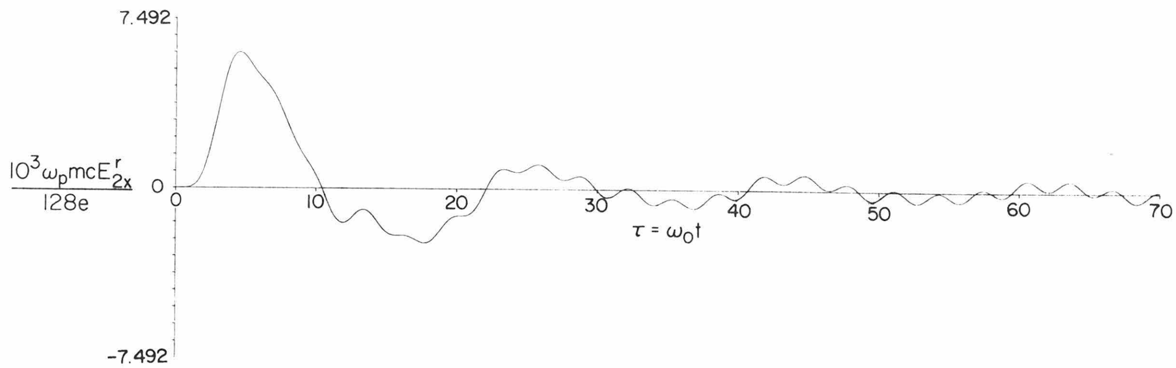


Figure 4.12: $E^r_{2x}(\underline{0},t)$ for $\theta = 45^o$, $\omega_o = 4\omega_p$, H-polarization incidence corresponding to case 2 ($\underline{E}^i(\underline{0},t) = \sin \omega_o t \cdot H(t) \cdot$ $(-\cos\theta\ \underline{e}_x + \sin\theta\ \underline{e}_z)$)

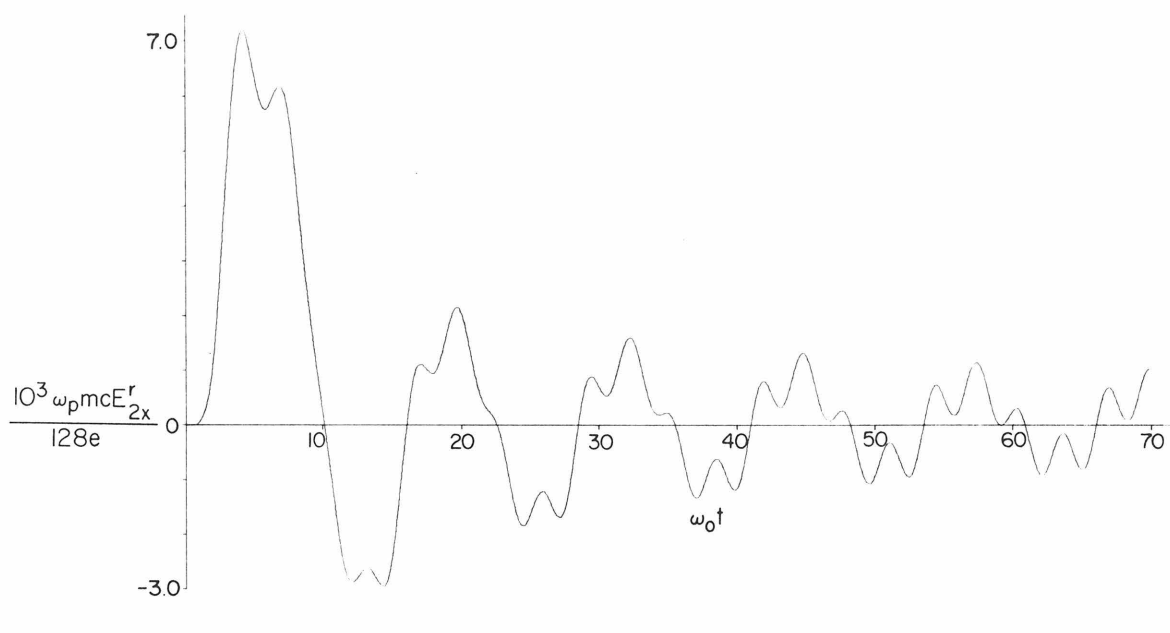


Figure 4.13: $E^r_{2x}(\underline{0},t)$ for $\theta = 60^o$, $\omega_o = 4\omega_p$, H-polarization incidence corresponding to case 2 ($\underline{E}^i(\underline{0},t) = \sin \omega_o t\ H(t) \cdot$ $(-\cos\theta\ \underline{e}_x + \sin\theta\ \underline{e}_z)$)

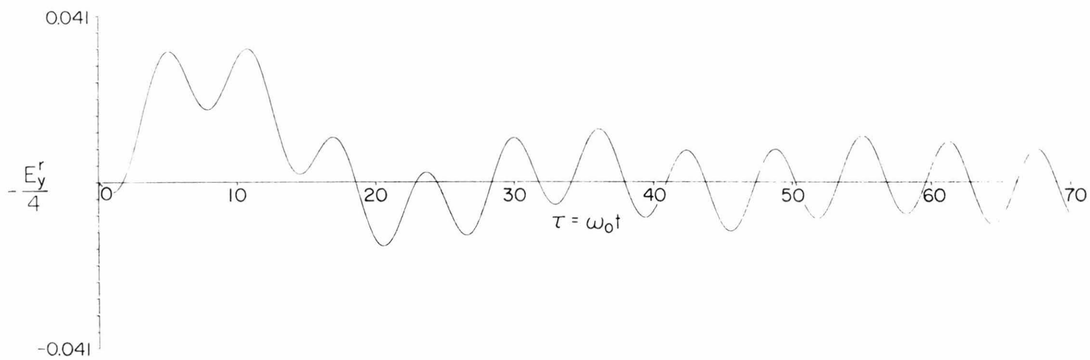


Figure 4.14: $E_y^r(\underline{0},t)$ for $\theta = 30^0$, $\omega_0 = 4\omega_p$, E-polarization incidence corresponding to case 2 ($\underline{E}^i(\underline{0},t) = \sin \omega_0 t\ H(t)\ \underline{e}_y$)

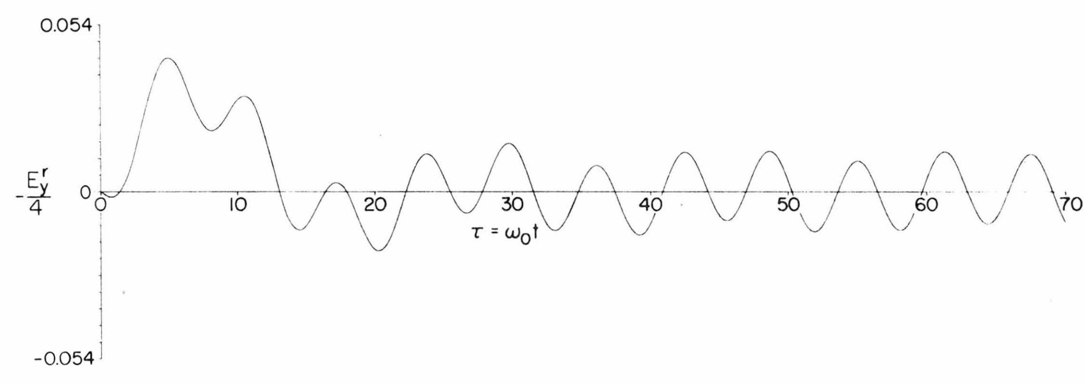


Figure 4.15: $E_y^r(\underline{0},t)$ for $\theta = 45^0$, $\omega_0 = 4\omega_p$, E-polarization incidence corresponding to case 2 ($\underline{E}^i(\underline{0},t) = \sin \omega_0 t\ H(t)\ \underline{e}_y$)

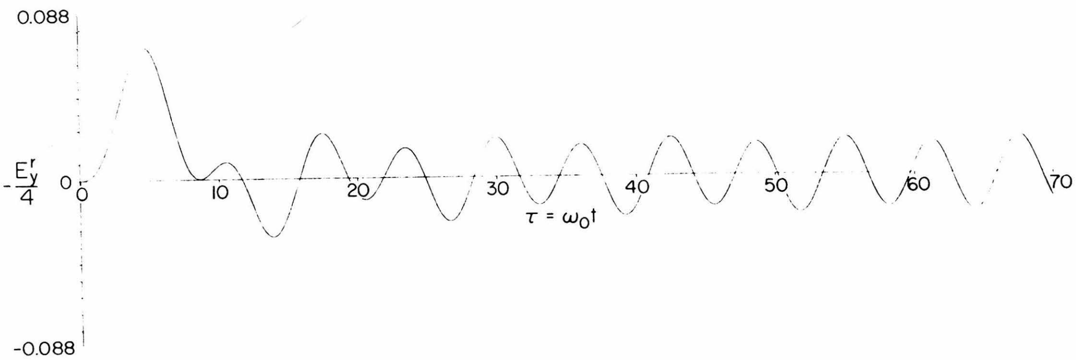


Figure 4.16: $E_y^r(\underline{0},t)$ for $\theta = 60^o$, $\omega_o = 4\omega_p$, E-polarization incidence corresponding to case 2 ($\underline{E}^i(\underline{0},t) = \sin \omega_o t \; H(t) \; \underline{e}_y$)

5. Conclusion

In this thesis expressions for the second order, nonlinear electric field which is generated in and reflected by a nonlinear, isotropic, cold, collisionless plasma have been found. In chapter 3 this second order field is generated in the plasma by a voltage applied across the boundary of the nonlinear plasma. We consider two types of voltages--a delta impulse voltage and a suddenly turned on sinusoidal voltage. The resulting second order field is found to be longitudinal and oscillates at the second harmonic frequency (for the case of the sinusoidal input) as well as the plasma frequency. It is also found in this chapter that field intensities on the order of 400 mW/cm^2 at an angular frequency of 10^8 rad/sec will produce a second order field which is on the order of 1% of the first order, linear electric field.

In chapter 4, we found the second order, nonlinear reflected field which is due to an E- or H-polarized wave incident on a semi-infinite, isotropic, cold, collisionless plasma. We consider the two cases where the incident wave is either a delta function or a suddenly turned on sinusoid. The resulting second order reflected field (for the case of the sinusoidal incident wave) oscillates at the second harmonic of the incident wave. The case of second order nonlinear reflection and transmission by a nonlinear plasma slab was not considered. This case is important, however, since in the laboratory the transmitted fields of a plasma slab are frequently of interest.*

---

*For the case of transmission and reflection by a slab of nonlinear crystal, see reference [9], page 83, or reference [17].

Appendix A

Integration Techniques and Lommel Functions

In this appendix we consider integrals of the type:

$$A(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega \, \frac{A_1(\omega, \sqrt{\omega^2-\alpha^2})}{A_2(\omega, \sqrt{\omega^2-\alpha^2})} \, e^{-i\omega t} \, e^{i\sqrt{\omega^2-\alpha^2}\, r} \qquad \text{(A.1)}$$

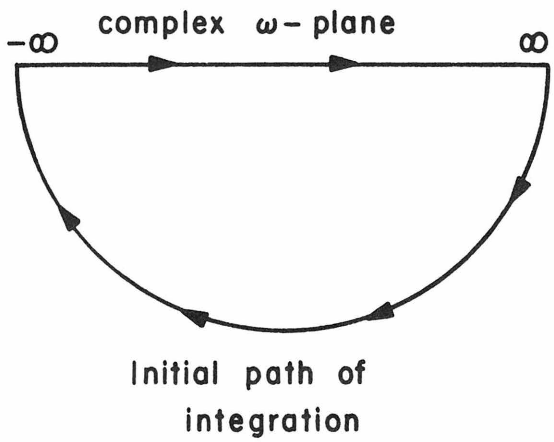
where A_1 and A_2 are polynomials in their arguments. As ω goes to infinity, $A_1/A_2 \sim a_o + \frac{a_1}{\omega} + \cdots$ in the instances considered here. If $a_o \neq 0$, then there is a Dirac delta function in the expression for $A(t)$. This can be subtracted away and inverted separately in view of the fact that the inverse transform of $e^{ir\sqrt{\omega^2-\alpha^2}}$ is $\delta(t-r) - r\alpha \frac{J_1(\alpha\sqrt{t^2-r^2})}{\sqrt{t^2-r^2}} H(t-r)$. Therefore, we only consider integrals where $A_1/A_2 \sim a_1/\omega + \cdots$.

We will consider two separate cases. The first case is when the integral in (A.1) has no poles in its integrand. In this case, we deform the initial path of integration into a path C which shrinks to the branch cut on the real line from $-\alpha$ to α (see Figure A.1)*. Therefore, equation (A.1 becomes)

$$A(t) = -\frac{1}{2\pi} \int_C \frac{A_1(\omega, \sqrt{\omega^2-\alpha^2})}{A_2(\omega, \sqrt{\omega^2-\alpha^2})} \, e^{-i\omega t} \, e^{i\sqrt{\omega^2-\alpha^2}\, r} \, d\omega \, H(t-r) \qquad \text{(A.2)}$$

---

*Similar integrals are treated by this change of contour method by Chabries [26].



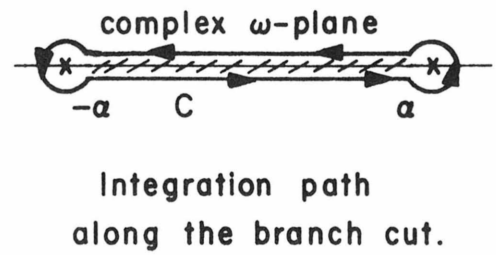


Figure A.1

Equation (A.2) reduces to

$$A(t) = -\frac{1}{2\pi}\left\{\int_{-\alpha}^{\alpha} \frac{A_1(\omega, i\sqrt{\alpha^2-\omega^2})}{A_2(\omega, i\sqrt{\alpha^2-\omega^2})} e^{-i\omega t} e^{-\sqrt{\alpha^2-\omega^2}\, r}\, d\omega\right.$$

$$\left. + \int_{-\alpha}^{\alpha} \frac{A_1(\omega, -i\sqrt{\alpha^2-\omega^2})}{A_2(\omega, -i\sqrt{\alpha^2-\omega^2})} e^{-i\omega t} e^{\sqrt{\alpha^2-\omega^2}\, r}\, d\omega\right\} H(t-r)$$

With the substitution $\omega = \alpha \cos\phi$, we have

$$A(t) = -\frac{\alpha}{2\pi}\left\{-\int_{0}^{\pi} \frac{A_1(\alpha\cos\phi, i\alpha\sin\phi)}{A_2(\alpha\cos\phi, i\alpha\sin\phi)} e^{-i\alpha(t\cos\phi - ir\sin\phi)} \sin\phi\, d\phi\right.$$

$$\left. - \int_{-\pi}^{0} \frac{A_1(\alpha\cos\phi, i\alpha\sin\phi)}{A_2(\alpha\cos\phi, i\alpha\sin\phi)} e^{-i\alpha(t\cos\phi - ir\sin\phi)} \sin\phi\, d\phi\right\} H(t-r)$$

$$= \frac{\alpha}{2\pi} \{ \int_{-\pi}^{\pi} \frac{A_1(\alpha \cos \phi, i\alpha \sin \phi)}{A_2(\alpha \cos \phi, i\alpha \sin \phi)} e^{-i\alpha \sqrt{t^2-r^2} \cos(\phi+\Phi)} \sin \phi \, d\phi \} H(t-r) \tag{A.3}$$

where

$$\Phi = \tan^{-1} \frac{ir}{t}$$

Expressing the sines and cosines in (A.3) in terms of complex exponentials and using the geometric series, one can express (A.3) in terms of a sum of Bessel functions, viz.,

$$A(t) = \sum_{n=0}^{\infty} C_n \left(\frac{t-r}{t+r}\right)^{\frac{f(n)}{2}} J_n(\alpha \sqrt{t^2-r^2}) H(t-r) \tag{A.4}$$

where $f(n)$ is a first degree polynomial in n. Equation (A.4) follows from the Bessel definition

$$J_n(x) = \frac{i^{-n}}{2\pi} \int_{-\pi}^{\pi} e^{i(x \cos \phi + n\phi)} \, d\phi \tag{A.5}$$

In deriving (A.4) from (A.3), use is also made of the relation $J_{-n}(x) = (-)^n J_n(x)$; and the identity $\tan^{-1} z = \frac{i}{2} \log \frac{i+z}{i-z}$.

When the integral in (A.1) has poles in its integrand, then we have to modify (A.4) to include the appropriate residue contributions. If these poles all lie outside the segment $|\omega| < \alpha$, then the formula for $A(t)$ becomes

$$A(t) = \sum_{n=0}^{\infty} C_n \left(\frac{t-r}{t+r}\right)^{\frac{f(n)}{2}} J_n(\alpha \sqrt{t^2-r^2}) H(t-r) - 2\pi i \sum_j \mathrm{Res}_j \left[\frac{A_1}{A_2} e^{ir\sqrt{\omega^2-\alpha^2}} e^{-i\omega t}\right] \tag{A.6}$$

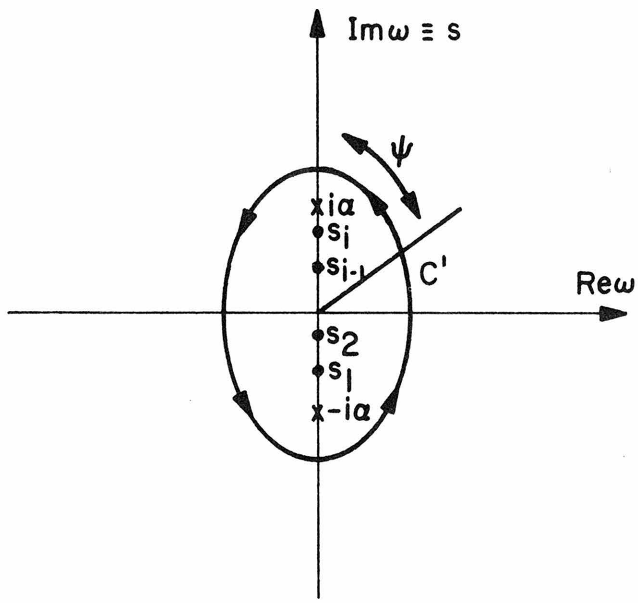
where $\mathrm{Res}_j[\Phi(\omega)]$ is the j^{th} residue of $\Phi(\omega)$. If the poles are all situated within the segment $|\omega| < \alpha$, then we must modify the initial path of integration into an ellipse*,** (as shown in Figure A.2) by the successive transformations:

$$\omega = is \quad ; \quad \sqrt{\omega^2-\alpha^2} = i\sqrt{s^2+\alpha^2}$$

$$s = \frac{\alpha i}{2\xi}(1+\xi^2) \qquad (s \text{ on } c')$$

$$\xi = \gamma e^{i\psi} \qquad (0 \leq \psi \leq 2\pi) \qquad (A.7)$$

where $\gamma = \sqrt{\frac{1-\beta}{1+\beta}}$, $\beta = r/t$



($s_1, s_2, \cdots, s_i$ are poles)

Figure A.2

*One is referred to Appendix B of Reference [27] for a fuller treatment of the following method.

**It is noted that the case where all the poles are situated outside the segment $|\omega| < \alpha$ could also be treated using the following method. The two different integration techniques are used for the sake of diversity.

Note that we can also write s on C' as

$$s = \frac{\alpha}{\sqrt{1-\beta^2}} (\beta \sin \psi + i \cos \psi)$$

From the above formula for s on C' we see that the ellipse has a semimajor axis of length $\frac{\alpha}{\sqrt{1-\beta^2}}$ and a semiminor axis of length $\frac{\alpha\beta}{\sqrt{1-\beta^2}}$. Also note from the transformation formulas (A.7) that we have the following relations:

$$\sqrt{s^2+\alpha^2} = \frac{\alpha i}{2\xi} (1 - \xi^2)$$

$$ds = \frac{\alpha}{2\xi}(1 - \xi^2) \, d\psi$$

$$s^2 + \omega_o^2 = -\frac{\alpha^2}{4\xi^2} (\xi^2 - \xi_o^2)(\xi^2 - \xi_o^{-2})$$

$$st - r\sqrt{s^2+\alpha^2} = iq \cos \psi \; ; \quad \text{where} \quad q = \alpha \sqrt{t^2-r^2}$$

Therefore, the integral in (A.1) becomes

$$A(t) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} \frac{A_1(is, i\sqrt{s^2+\alpha^2})}{A_2(is, i\sqrt{s^2+\alpha^2})} e^{st - r\sqrt{s^2+\alpha^2}} \, ds$$

$$= -\frac{\alpha}{2\pi i} \int_0^{2\pi} d\psi \, \frac{A_1(-\frac{\alpha}{2\xi}(1+\xi^2), -\frac{\alpha}{2\xi}(1-\xi^2))}{A_2(-\frac{\alpha}{2\xi}(1+\xi^2), -\frac{\alpha}{2\xi}(1-\xi^2))} e^{iq \cos \psi} \frac{(1-\xi^2)}{2\xi}$$

$$= -\frac{\alpha}{2\pi i} \int_0^{2\pi} \frac{P(e^{i\psi})}{Q(e^{i\psi})} e^{iq \cos \psi} \, d\psi \qquad \text{(A.8)}$$

where P and Q are polynomials in $e^{i\psi}$. By the method of partial fractioning, one can expand P/Q as:

$$\frac{P}{Q} = \sum_{n=0}^{N} a_n e^{in\psi} + \sum_{m=1}^{M} \sum_{\ell=1}^{L} \frac{b_{m,\ell}}{(C_{m,\ell} + e^{i\psi})^{\ell}} \qquad \text{(A.9)}$$

By expanding the double sum in a gemoetric series, one obtains for P/Q the infinite sum:

$$\frac{P}{Q} = \sum_{m=1}^{\infty} C_m e^{im\psi} \qquad \text{(A.10)}$$

Therefore, using the relationship:

$$2\pi i^n J_n(q) = \int_0^{2\pi} e^{\pm in\psi} e^{iq\cos\psi} d\psi \qquad \text{(A.11)}$$

one obtains for A(t) an infinite series of Bessel functions. This can be expressed in terms of Lommel functions of 2 variables, viz., $U_n(w,z)$ and $V_n(w,z)$ where*

$$U_n(w,z) = \sum_{m=0}^{\infty} (-)^m \left(\frac{w}{z}\right)^{n+2m} J_{n+2m}(z) \qquad \text{(A.12)}$$

$$V_n(w,z) = \sum_{m=0}^{\infty} (-)^m \left(\frac{w}{z}\right)^{-n-2m} J_{-n-2m}(z) \qquad \text{(A.13)}$$

The two types of Lommel functions defined by (A.12) and (A.13) are interrelated in the following ways:

---

*For a further discussion of Lommel functions, see Reference [32], pp. 537-550.

$$U_n(w,z) - V_{-n+2}(w,z) = \cos(\frac{w}{2} + \frac{z^2}{2w} - \frac{n\pi}{2}) \tag{A.14}$$

$$V_n(w,z) = (-)^n \, U_n(\frac{z^2}{w},z) \tag{A.15}$$

From the defining series (A.12) and (A.13) it is seen that these functions satisfy the recursion relations:

$$U_n(w,z) + U_{n+2}(w,z) = (\frac{w}{z})^n \, J_n(z) \tag{A.16}$$

$$V_n(w,z) + V_{n+2}(w,z) = (\frac{w}{z})^{-n} \, J_{-n}(z) \tag{A.17}$$

One also has:

$$\frac{\partial}{\partial z} U_v(w,z) = - \frac{z}{w} U_{v+1}(w,z) \tag{A.18}$$

$$\frac{\partial}{\partial w} U_v(w,z) = \frac{1}{2}(U_{v-1}(w,z) + (\frac{z}{w})^2 \, U_{v+1}(w,z)) \tag{A.19}$$

All of the integrals occurring in this thesis can be reduced from (A.8) to

$$I_{\ell,m}(q,K) = \int_0^{2\pi} \frac{\xi^{\ell} \, e^{iq \cos \psi}}{(\xi^2 - K^2)^m} \, d\psi \tag{A.20}$$

For a tabulation of these integrals (for small ℓ,m) one is referred to Appendix C of Reference [27]. The integrals encountered in this thesis are copied below from Appendix C in [27].

$$I_{o,1}(q,\xi_o^{\pm 1}) = -2\pi \, \xi_o^{\mp 2} \, U_o(\gamma \xi_o^{\mp 1} q,q) \tag{A.21}$$

$$I_{1,1}(q,\xi_o^{\pm 1}) = -2\pi i\ \xi_o^{\mp 1} U_1(\gamma\xi_o^{\mp 1}\ q,q) \qquad (A.22)$$

Appendix B

A Numerical Solution of Convolution Integrals Appearing in This Thesis*

In this appendix a method is described for evaluating the convolution integral:

$$x(t) = \int_0^t f(t-\tau)\ g(\tau)\ d\tau$$

$$= \int_0^t f(t-\tau)\ h'(\tau)\ d\tau \qquad \text{(B.1)}$$

where

$$g(t) = h'(t) \equiv \frac{dh}{dt}$$

To evaluate $x(t)$ we must first break up the interval $[0,t]$ into n subintervals of equal length Δt $(\Delta t = \frac{t}{n})$. Then $x(t)$ can be written as:

$$x(t = n\Delta t) = \int_0^t f(t-\tau)\ g(\tau)\ d\tau = \int_0^{n\Delta t} f(t-\tau)\ g(\tau)\ d\tau$$

$$= \int_0^{\Delta t} F_n\ g(\tau)\ d\tau + \int_{\Delta t}^{2\Delta t} F_{n-1}\ g(\tau)\ d\tau + \cdots$$

$$+ \int_{(n-1)\Delta t}^{n\Delta t} F_1\ g(\tau)\ d\tau \qquad \text{(B.2)}$$

where

$$F_i = f((i - \tfrac{1}{2})\ \Delta t)$$

*See Reference [33].

Since F_i remains constant in each integral, it may be taken out of the integral sign. Therefore,

$$x(n\Delta t) = \sum_{i=1}^{n} F_{n-i+1} \int_{(i-1)\Delta t}^{i\Delta t} g(\tau)\, d\tau$$

$$= \sum_{i=1}^{n} F_{n-i+1} \{h(i\Delta t) - h((i-1)\Delta t)\}$$

Let

$$H_i = h(i\Delta t) - h((i-1)\Delta t)$$

Then we have

$$x(t=n\Delta t) = \sum_{i=1}^{n} F_{n-i+1}\, H_i \qquad (B.3)$$

Thus the evaluation of $x(t)$ at $t = n\Delta t$ is reduced to the addition of n readings. By using (B.3), computer evaluations have been made of all convolution integrals in this paper. The form of the two integrals in Chapter 3 (equations (3.40) and (3.45)) are

$$\int_{x/c}^{t} \sin \omega_p(t-\tau)\; f(\tau)\; f_x(\tau)\; d\tau = \int_{0}^{t-x/c} f(t-\tau) f_x(t-\tau) \sin \omega_p \tau \; d\tau$$

$$= \omega_p^{-1} \int_{0}^{\omega_p(t-x/c)} f(t - \frac{u}{\omega_p})\; f_x(t - \frac{u}{\omega_p})$$

$$\cdot \frac{d}{du}(-\cos u)\; du \qquad (B.4)$$

where

$$f(t)\ f_x(t) = \omega_p(2U_o(\gamma iq,q) - J_o(q))\left(\frac{2U_1(\gamma iq,q)}{i} - \frac{J_1(q)t}{\sqrt{t^2 - x^2/c^2}}\right) \tag{B.5}$$

for case 1 in Chapter 3

and

$$f(t)\ f_x(t) = \frac{1}{\omega_o}\left[U_1(\gamma\xi_o^{-1}q,q) + U_1(\gamma\xi_o q,q)\right]$$

$$\cdot\ \left[J_o(q) - \frac{\xi_o^2 - 1}{\xi_o^2 + 1}\{U_o(\gamma\xi_o^{-1}q,q) - U_o(\gamma\xi_o q,q)\}\right] \tag{B.6}$$

for case 2 in Chapter 3

Comparing (B.4) with (B.1), we see that $h(t) = -\dfrac{\cos\omega_p t}{\omega_p}$. We also see that

$$H_i = \frac{1}{\omega_p}[\cos\omega_p(i-1)\Delta t - \cos\omega_p i\Delta t]$$

$$= \frac{2}{\omega_p}\sin\left(\frac{\omega_p\Delta t}{2}\right)\sin\left(\frac{\omega_p}{2}(2i-1)\Delta t\right) \tag{B.7}$$

Therefore, with (B.7) and (B.6) or (B.5), we can generate the two convolution integrals given by (B.4) by numerically summing (B.3). The form of the four integrals of interest in Chapter 4 (equations (4.40), (4.42), (4.57), and (4.59)) are

$$\int_{x/c}^{t} f_i^2(t-\tau)\ g(\tau - \frac{x}{c})d\tau = \int_0^{t-\frac{x}{c}} f_i^2(t-\tau-\frac{x}{c})\ g(\tau)d\tau$$

$$= \int_0^{\omega_p(t-\frac{x}{c})} f_i^2(t - \frac{u}{\omega_p} - \frac{x}{c})\ h'(\frac{u}{\omega_p})du \tag{B.8}$$

where

$$f_1(t) = \frac{\alpha^2}{2}\,(J_o(\alpha t) + J_2(\alpha t)) = \frac{\alpha J_1(\alpha t)}{t} \tag{B.9}$$

$$f_2(t) = \alpha[\xi_o U_2(\xi_o^{-1}\alpha t,\alpha t) + \xi_o^{-1} U_2(\xi_o \alpha t,\alpha t)] \tag{B.10}$$

$$f_3^2(t) = \sin^2\theta\; g^2(t) + \frac{1}{\cos^2\theta}\Big[\frac{1}{\alpha^2}\, f_1(t) - \sin^2\theta\; g(t)\Big]^2 \tag{B.11}$$

$$f_4^2(t) = \frac{u_4^2}{\sin^2\theta} + \frac{1}{\cos^2\theta}\Big(\frac{1}{\alpha^2}\, f_2(t) - u_4\Big)^2 \tag{B.12}$$

$$g(t) = \frac{1}{2}\,[U_o(\sqrt{\cos 2\theta}\;\alpha t,\alpha t) - (\cos 2\theta)^{-2}\; U_4(\sqrt{\cos 2\theta}\;\alpha t,\alpha t)] \tag{B.13}$$

and

$$u_4(t) = \frac{-2i\,\sin^2\theta}{\alpha^2}\,\omega_o\,\frac{1}{(\cos 2\theta - \xi_o^2)(\cos 2\theta - \xi_o^{-2})}$$

$$\cdot\;[\cos 2\theta\; U_o(\sqrt{\cos 2\theta}\;\alpha t,\alpha t) - (\cos 2\theta)^{-1} U_4(\sqrt{\cos 2\theta}\;\alpha t,\alpha t)$$

$$+\;\xi_o^2\; U_4(\xi_o^{-1}\alpha t,\alpha t) - \cos 2\theta\; U_o(\xi_o^{-1}\alpha t,\alpha t)$$

$$+\;(\cos 2\theta - \xi_o^{-2})\; U_2(\xi_o\alpha t,\alpha t)]\; H(t) \tag{B.14}$$

Comparing (B.8) with (B.1), we see that

$$\tilde{h} = \frac{i}{\omega}\,\tilde{g} = -\frac{1}{2}\,\frac{1}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2 - \omega_p^2} \tag{B.15}$$

or

$$h(t) = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega\; e^{-i\omega t}}{\omega\sqrt{\omega^2-\alpha^2} + \omega^2-\omega_p^2} =$$

$$= -\frac{\alpha}{2\pi} \int_{-\pi}^{\pi} \frac{e^{-i\alpha t \cos\phi} \sin\phi \, d\phi}{i\alpha^2 \cos\phi \sin\phi + \alpha^2 \cos^2\phi - \omega_p^2}$$

$$= \frac{1}{\alpha\sqrt{\cos 2\theta}} \{U_1(\sqrt{\cos 2\theta}\,\alpha t, \alpha t) + (\cos 2\theta)^{-1}$$

$$\cdot\; U_3(\sqrt{\cos 2\theta}\,\alpha t, \alpha t)\} \qquad (B.16)$$

Therefore, with (B.16) and either (B.9), (B.10), (B.11), or (B.12) we can generate the four convolution integrals (B.8) by summing (B.3).

Appendix C

Asymptotic Expansions of some Integrals in Chapters 3 and 4

In this appendix, we derive some asymptotic formulas for integrals which appear in chapters 3 and 4. Consider the integral:

$$\phi(t) = \int_r^t f^2(\tau - t)\, g(\tau - r)\, d\tau \qquad (C.1)$$

where,

$$f(t) = \sum_{n=0}^{\infty} a_n J_n(\alpha t) \qquad (C.2)$$

and,

$$g(t) = \sum_{n=0}^{\infty} b_n J_n(\alpha t) \qquad (C.3)$$

Then,

$$f^2(t) = \sum_{n,m=0}^{\infty} a_n a_m J_n(\alpha t) J_m(\alpha t)$$

$$= \sum_{n,m=0}^{\infty} a_n a_m \sum_{\ell=0}^{\infty} \frac{(-)^{\ell} (\frac{1}{2}\alpha t)^{n+2\ell}}{\ell!\, \Gamma(n+\ell+1)} J_m(\alpha t) \qquad (C.4)$$

Substituting (C.4) and (C.3) into (C.1) gives: *

$$\phi(t) = \frac{1}{\alpha} \sum_{s=0}^{\infty} b_s \sum_{n,m=0}^{\infty} \sum_{\ell=0}^{\infty} \frac{(-)^{\ell}(\frac{1}{2})^{n+2\ell}}{\ell!\ \Gamma(n+\ell+1)} \int_0^x (x-u')^{n+2\ell}$$

$$\cdot J_m(x-u')\ J_s(u')\ du' = \frac{1}{\alpha} \sum_{n,m,s=0}^{\infty} b_s a_n a_m \sum_{\ell=0}^{\infty}$$

$$\frac{(-)^{\ell}\ 2^{n+2\ell+1}}{\ell!\ 2^{n+2\ell}\ \Gamma(n+\ell+1)} \sum_{p=0}^{\infty} \frac{(-)^p\ \Gamma(n+2\ell+1+s+p)(\lambda)_p}{p!\ \Gamma(s+p+1)}$$

$$\cdot J_{2\ell+n+1+s+m+2p}(x) \qquad \text{(C.5)}$$

where

$$x = \alpha(t-r)\ ;\ \lambda = n + 2\ell + 1;\ (\lambda)_p = \frac{\Gamma(\lambda+p)}{\Gamma(\lambda)}$$

We note that (C.5) is an exact solution to the convolution integral, $\phi(t)$. We also note that for small (t-r), when we keep only the lowest order terms in (C.5), $\phi(t)$ is given approximately by:

---

* In evaluating (C.1) we have used the results found in reference [31], page 354 (equation (25)).

$$\phi(t) \sim \alpha^{-1} \, 2b_0 \, a_0^2 \, J_1 \, (\alpha(t-r)) \qquad (C.6)$$

The integral (C.1) occurs frequently in chapter 4.

We now wish to look at the asymptotic behavior for large t of integrals which occur frequently in chapter 3. Consider the integral:

$$I(t) = \int_r^t \phi\,(\tau,r) \sin \omega_p(t-\tau)\, d\tau \qquad (C.7)$$

where, $r = x/c$; and $\phi(t,r) = f(t,r)\, f_x\,(t,r)$.

What is desired is the asymptotic expansion of I(t) for large t. To expand (C.7) we rewrite it as follows:

$$I(t) = \frac{1}{2i}\left[e^{i\omega_p t} \int_r^t \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau - e^{-i\omega_p t} \int_r^t \phi(\tau,r)\, e^{i\omega_p \tau}\, d\tau \right] \qquad (C.8)$$

We will expand each term for I(t) in (C.8), however, since these terms are similar we will just consider the asymptotic expansion of one of them, i.e.,

$$I_1(t) = e^{i\omega_p t} \int_r^t \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau \qquad (C.9)$$

Equation (C.9) can be rewritten as:

$$I_1(t) = e^{i\omega_p t}\left\{ \int_r^\infty \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau - \int_t^\infty \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau \right\}$$

$$= e^{i\omega_p t} \int_r^\infty \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau - e^{i\omega_p t} \int_t^\infty \phi(\tau,r) e^{-i\omega_p \tau}\, d\tau$$

(C.10)

The first term on the right hand side of (C.10) is one term in the asymptotic expansion for $I(t)$ and represents a plasma oscillation. The magnitude of this oscillation is given by:

$$\int_r^\infty \phi(\tau,r)\, e^{+i\omega_p \tau}\, d\tau = 2\pi\, \tilde{\phi}\, (\omega_p,r) \qquad \text{(C.11)}$$

The second term on the right hand side of (C.10) can be further expanded by substituting the asymptotic expansion for $\phi(\tau,r)$ into the integral, viz.,

$$\int_t^\infty \phi(\tau,r)\, e^{-i\omega_p \tau}\, d\tau \sim \int_t^\infty \phi_{AS}\, (\tau,r)\, e^{-i\omega_p \tau}\, d\tau \qquad \text{(C.12)}$$

where $\phi_{AS}\, (\tau,r)$ is the asymptotic expansion of $\phi(\tau,r)$. Therefore, using the results of (C.11) - (C.12), we have that:

$$I(t) \sim \pi i \left\{ e^{-i\omega_p t} \tilde{\phi}(\omega_p, r) - e^{+i\omega_p t} \tilde{\phi}(-\omega_p, r) \right\} -$$

$$- \int_t^\infty \phi_{AS}(\tau, r) \sin \omega_p (t - \tau)\, d\tau \qquad (C.13)$$

(C.13) is the desired expansion for I(t).

Appendix D

Two Types of Nonlinearities in Plasmas

A plasma can become nonlinear under an intense electric field. This nonlinearity can be due to the raising of the plasma electron temperature by collisions, or the Lorentz force term in the equation of motion of the plasma particles*. This thesis has considered nonlinearities in a collisionless plasma and therefore, the nonlinearities considered here arise from the Lorentz force term (as well as the $(\underline{v} \cdot \nabla)\underline{v}$ term).

In this appendix we will discuss both types of nonlinearities. First, we will consider the nonlinearities in a collisionless plasma due to the Lorentz force acting on the plasma electrons. Since there are no collisions to prevent the plasma electrons from losing their energy (from the external electromagnetic field), the electrons all respond coherently to the external electromagnetic field, and therefore can reach very high velocities if the electric field is strong enough. In the linear approximation, this velocity $\underline{v}$ is given by $q\underline{E}/m\omega$, where q and m are the electron's charge and mass, $\underline{E}$ is the external field, and ω is the frequency of the external field. As this linear velocity becomes large (so that $v/c \sim .1$, say), there is an appreciable Lorentz force acting on the electron, $\frac{q}{c}\,\underline{v} \times \underline{B}$, which must be taken into account in the electron's equation of motion and therefore makes all the equations nonlinear. It is this nonlinearity which is considered in this paper.

---

*There is also nonlinearity due to the $\underline{v} \cdot \nabla\underline{v}$ term in the equation of motion. This term arises when one transforms from the plasma electron's rest frame to the laboratory rest frame.

Of course, if there are collisions present in the plasma, these collisions may prevent plasma electrons from obtaining high enough velocities so that the Lorentz force becomes appreciable.* However, even in this case, the plasma can still be nonlinear. This nonlinearity is due to the heating of the plasma electrons by the electric field. When there is no field present, the plasma electrons collide with other electrons, ions, or molecules present in the plasma. When these electrons are subject to an electric field, they try to speed up in response to the field but the net effect is an increase in the collision frequency (and thus electron temperature). It is this dependence of the collision frequency on the electric field which causes the plasma to be nonlinear.

Ginzburg** derives the dependence of the collision frequency on the electric field. His results are as follows***:

$$\text{collision frequency} \equiv \nu = \nu_{mo} \sqrt{T_e/T} \tag{D.1}$$

where ν_{mo} is the effective collision frequency for collisions with molecules; T = equilibrium temperature of the plasma; and T_e is the effective electron temperature when an external field is present****.

---

*Chan [11], pp. 25-26, under the assumption that the electron distribution remains Maxwellian for certain layers of the ionosphere subject to an external field, has shown that the plasma electrons in these layers can only reach a velocity which is three orders of magnitude less than c.

**See reference [13], pp. 498-505.

***The following results pertain to collisions of plasma electrons with molecules. Ginzburg also gives results for collisions of plasma electrons with ions.

****Since, when an external field is present the electron velocity distribution is by no means always Maxwellian, the temperature T_e is in general an effective electron temperature.

In a weak field $T_e = T$ and ν in (D.1) is an independent parameter. In a strong field, T_e and E_o are related as follows:

$$T_e = T \{1 + \frac{\omega^2 + \nu_{mo}^2}{2\nu_{mo}^2} (\sqrt{[1 + \frac{4\nu_{mo}^2}{\omega^2 + \nu_{mo}^2} (\frac{E_o}{E_p})^2] - 1})\} \tag{D.2}$$

where ω = frequency of the external field, E_o is the amplitude of the external field, and

$$E_p = 4.2 \times 10^{-10} \sqrt{\delta T(\omega^2 + \nu_{mo}^2)} \ \frac{\text{volts}}{\text{cm}} \tag{D.3}$$

In (D.3), δ is the effective (mean) relative fraction of energy transferred by the electron in a collision with a heavy particle (in elastic collisions $\delta = 2m/M$).

Equations (D.2) and (D.1) exhibit the nonlinear relationship between ν and E_o. It is this relationship which makes the equation of motion (and therefore Maxwell's equations) nonlinear. It should be mentioned that the concept of the effective electron temperature, as Ginzburg points out*, is based on an elementary picture of a plasma. To derive the nonlinear equations of the plasma in the general case, one must use the Boltzmann equation for the electron distribution function. However, this elementary picture is often of use even in strong fields.

We have briefly discussed the two types of nonlinearities--one due to heating (type I) and the other due to the Lorentz force acting

*See reference [13], page 498.

on the plasma electron (type II). In general type II will be observed in a plasma where the collision frequency is much smaller than the frequency of the external field, and the field strength of the external field is strong enough. In Chapter 3, it was found that a field strength of 1.7 x $10^{-7}\omega$ v/cm would cause type II nonlinearities to become appreciable. Type I nonlinearities will be observed in collision dominated plasmas where the external electric field is intense enough. As might be expected from (D.3) and (D.2), a field strength at which one might observe type I nonlinearities is E_p. Ginzburg* gives some typical values for E_p in the ionosphere and the solar corona. At low frequencies $(\omega^2 << \nu_{mo}^2) E_p \sim 10^{-5}$ to 10^{-7} v/cm in the E and F layers of the ionosphere. In the solar corona, $E_p \sim 10^{-7}$ v/cm. At high frequencies $(\omega^2 >> \nu_{mo}^2) E_p \sim 10^{-3}$ to 10^{-2} v/cm (in the ionosphere). In the solar corona, $E_p \sim 10$ to 10^4 v/cm.

Therefore, we see that there are cases when the critical field for type I nonlinearities is much less than that for type II nonlinearities and thus type I will be observed first, and vice versa. There are also cases when the critical field for type I and II nonlinearities are approximately the same, and one will see both nonlinear effects present simultaneously.

---

*Reference [13], page 497.

REFERENCES

1. B. D. H. Tellegen, "Interaction between radio waves?", Nature 131, 840 (1933)
2. V. A. Bailey and D. F. Martyn, "The influence of electric waves on the ionosphere", Phil. Mag. 18, 369 (1934)
3. L. M. Al'tshul'and V. I. Karpman, "The kinetics of waves in a weakly turbulent plasma", Soviet Phys. JETP 20, 1043 (1965)
4. D. B. Melrose, "Symmetry properties of nonlinear responses in a plasma", Plasma Physics 14, 1035 (1972)
5. V. N. Tsytovich, "Nonlinear effects in a plasma", Soviet Phys.--Usp. 9, 805 (1967)
6. V. N. Tsytovich, Nonlinear Effects in a Plasma (Plenum Press, New York, 1970)
7. A. Owyoung, "The origins of the nonlinear refractive indices of liquids and glasses", Ph.D. Thesis, California Institute of Technology, 1971
8. P. N. Butcher in Nonlinear Optical Phenomena, Bull. 200, Engineering Experiment Station, Ohio State University, Columbus (1965)
9. N. Bloembergen, Nonlinear Optics (W. A. Benjamin, Inc., New York, 1965)
10. R. Kubo, "Statistical-mechanical theory of irreversible processes, I", J. Phys. Soc. Japan 12, 570 (1957)
11. T. C. Chan, "A study of nonlinear phenomena in the propagation of electromagnetic waves in a weakly ionized gas", Ph.D. Thesis, California Institute of Technology, 1965
12. P. Bassanini, "On the electromagnetic field produced by an electric dipole in a spherical cavity immersed in a nonlinear plasma", Atti. Accadi. Sci. Torino 106, 399 (1972)

13. V. L. Ginzburg, The Propagation of Electromagnetic Waves in Plasmas (Pergamon Press, 1970)

14. C. H. Papas, Theory of Electromagnetic Wave Propagation (McGraw-Hill, New York, 1965)

15. M. Bornatici and F. Engelmann, "Nonlinear generation of transverse fluctuations from longitudinal fluctuations in an infinite stable plasma", Nuovo Cim. 56B, p. 220 (1968)

16. J. A. Armstrong, N. Bloembergen, J. Ducuing, and P. S. Pershan, "Interaction between light waves in a nonlinear dielectric", Phys. Rev. 127, 1918 (1962)

17. N. Bloembergen and P. S. Pershan, "Light waves at the boundary of nonlinear media", Phys. Rev. 128, 606 (1962)

18. R. D. Small, "Nonlinear dispersive waves in nonlinear optics", Ph.D. Thesis, California Institute of Technology, 1973

19. G. B. Whitham, "Nonlinear dispersive waves", Proc. Roy. Soc. A283, 238 (1965)

20. J. A. Fejer, "The interaction of pulsed radar waves in the ionosphere", J. Atmospheric and Terrestrial Phys. 7, 322 (1955)

21. N. M. Kroll, "Excitation of hypersonic vibrations by means of photo-elastic coupling of high-intensity light waves to elastic waves", JAP 36, 34 (1965)

22. P. G. Kryukov and V. S. Letokhov, "Propagation of a light pulse in a resonantly amplifying (absorbing) medium", Soviet Physics--Usp. 12, 641 (1970)

23. C. T. Case, "On transient wave propagation in a plasma", Proc. IEEE 53, 730 (1965)

24. C. M. Knop, "Pulsed electromagnetic wave propagation in dispersive media", IEEE Trans. Antenna Propagat. 12, 494 (1964)

25. J. R. Wait, "Reflection of a plane transient electromagnetic wave from a cold lossless plasma slab", Radio Science 4, 401 (1969)

26. D. M. Chabries and D. M. Bolle, "Impulse reflection with arbitrary angle of incidence and polarization from isotropic plasma slabs", Radio Science 6, 1143 (1971)

27. J. J. Kenny, "Electric dipole radiation in isotropic and uniaxial plasmas", Ph.D. Thesis, California Institute of Technology, 1968

28. R. R. Alfano and S. L. Shapiro, "Ultrafast phenomena in liquids and solids", Scientific American, June 1973.

29. T. J. M. Boyd and J. J. Sanderson, Plasma Dynamics (Barnes and Noble, Inc., 1969)

30. M. Born and E. Wolf, Principles of Optics (Pergamon Press, 1970)

31. A. Erdélyi, et al., Tables of Integral Transforms, Vol. II (McGraw Hill, New York, 1954)

32. G. N. Watson, A Treatise on the Theory of Bessel Functions (Cambridge University Press, 1966)

33. T. Mirsepassi, "Graphical evaluation of a convolution integral", Math Tables and Other Aids to Computation, 13, 202 (1959).