Discrete Deligne Cohomology and Discretized Abelian Chern-Simons Theory

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Mum, for “saving some math genes” for me. Dad, for helping me with math homework. Carole, for support. Matilde, for guidance.

If you have built castles in the air, your work need not be lost; that is where they should be. Now put foundations under them. Henry David Thoreau

In memory of Wojbor Woyczynski.

ABSTRACT

The differential cohomology groups of a smooth manifold M are discretized with respect to a triangulation X. The realization of differential cohomology used is Deligne cohomology. A discretized version of the smooth Deligne double complex is constructed from cochain groups defined on simplices of X. The total cohomology of this double complex is studied and shown to satisfy exact sequences analogous to the standard structural sequences satisfied by differential cohomology. In the degree corresponding to line bundles with connection, our cohomology classes are shown to correspond to isomorphism classes of an existing notion [17] of discrete line bundles with connection. Explicit examples of these discrete line bundles with connection are constructed. A ring structure is defined on the discrete Deligne cohomology groups; it is graded-commutative and non-associative (however, associativity is recovered in the continuum limit). The ring structure allows one to define a more general discrete Chern-Simons action than has previously appeared in the literature.

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BACKGROUND

We aim to demonstrate that the differential cohomology of a smooth manifold has a satisfactory discretization. We follow a simple rule that has had much success in discrete differential geometry: triangulate M by a simplicial complex X, replace $\Omega^k(M)$ by $C^k(X)$, and see which statements still make sense. In the case of differential cohomology, we will see that a great many statements still make sense. The "discrete Deligne cohomology" groups we define on a triangulation X of a manifold M allow one to carry over to X many essential aspects of the theory of complex line bundles with connection, such as Chern classes and their relation to curvature. This notion is then shown to be isomorphic to a pre-existing notion of discrete line bundle with connection. In addition to line bundles, we obtain the algebraic framework in which one would define the higher "discrete circle n-bundles with connection."

1.1 Homological Algebra

We will assume that the reader is familiar with a few common chain complexes: simplicial cochain complexes, the de Rham complex, and Čech complexes (otherwise see [7]). We will find it necessary to use multiple of these complexes simultaneously in the form of a *double complex*. Double complexes break up delicate mathematical structures into simpler pieces along two directions. Our motivating example is a complex line bundle with connection on a smooth manifold. This is an object with data occupying various geometric dimensions: its transition maps are encoded as smooth $\mathbb{C}$-valued functions, while its connection may be encoded as a 1-form. Before breaking the bundle and connection up into these pieces, it is necessary to restrict to the open sets of a cover; the global object is recovered by comparing local data on the intersections of these open sets. Thus a line bundle with connection is best understood by performing two simultaneous resolutions: the first being its restriction to open sets, the second its representation in terms of smooth functions and differential 1-forms. Double complexes keep track of this information and make it easy to say when a collection of smooth functions and 1-forms on open sets can be combined globally to form a bundle with connection, as well as to say when two such collections of local data define the same bundle with connection.

A generic double complex looks something like this:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & K^{0,2} & \xrightarrow{\delta} & K^{1,2} & \xrightarrow{\delta} & K^{2,2} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & K^{0,1} & \xrightarrow{\delta} & K^{1,1} & \xrightarrow{\delta} & K^{2,1} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
\cdots & \xrightarrow{\delta} & K^{0,0} & \xrightarrow{\delta} & K^{1,0} & \xrightarrow{\delta} & K^{2,0} & \xrightarrow{\delta} & \cdots \\
 & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \uparrow{\scriptstyle d} & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

It consists of bi-graded pieces $K^{p,q}$ that are abelian groups[1] and two differentials $d : K^{p,q} \longrightarrow K^{p,q+1}$ and $\delta : K^{p,q} \longrightarrow K^{p+1,q}$ that raise the vertical/horizontal degrees of elements. These satisfy $d^2 = 0$ and $\delta^2 = 0$, so that each row $K^{*,q}$ and each column $K^{p,*}$ of the double complex is itself a complex. The differentials commute[2]: $d\delta = \delta d$. In general, the degrees p, q may take any integer value, but here we will only use double complexes in which both degrees are bounded below.

Any double complex may be "rolled up" into a single complex, the associated *total complex*. Its degree n piece $C^n = \oplus_{p+q=n} K^{p,q}$ consists of formal sums of all elements whose bi-degrees (p, q) sum to n. In other words, one contracts the double complex by summing along its diagonals. The *total differential* $D : C^n \longrightarrow C^{n+1}$ acts on a piece $K^{p,q}$, $p + q = n$, as $D = \delta + (-1)^p d$. That means that if $a \in K^{p,q}$ then $Da = (\delta a, (-1)^p da) \in K^{p+1,q} \oplus K^{p,q+1}$. The sign $(-1)^p$ ensures that $D^2 = 0$, so (C^*, D) is a complex.

A classical example of a double complex is the Čech–de Rham complex. This complex is defined relative to an open cover of a smooth manifold using the de Rham complex vertically and the Čech complex horizontally. It appears in Weil's proof of the equivalence of de Rham and Čech cohomologies [22] and is treated extensively in [7]. We will not make use of it, but of a closely related double complex.

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[1] Or, more generally, objects in an abelian category; see [21]. We will only need abelian groups here.

[2] The reader should be aware that another common convention is to take $d\delta + \delta d = 0$, as is the case in [21]. In this case the total differential is taken to be $D = \delta + d$.

The Smooth Deligne Complex

The smooth Deligne complex plays a central role in this thesis; it will be our primary objective to show that there is a satisfactory discretization of it. Horizontally, it is a Čech complex relative to an open cover $\mathcal{U} = \{U_\alpha\}$ of a smooth manifold M. Vertically it consists of a modified version of the de Rham complex on an open set. The complex is

$$\Omega^0(U; U(1)) \xrightarrow{\text{dlog}} \Omega^1(U; \mathbb{R}) \xrightarrow{d} \Omega^2(U; \mathbb{R}) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^k(U; \mathbb{R}) \to 0 \to \ldots$$

Here Ω^0 is used to denote smooth functions and the map dlog is given by $\text{dlog}(f) = -idf/f$ (considering $U(1)$ as the unit circle in $\mathbb{C}$). The complex is also truncated after Ω^k for some positive k. The motivation for modifying the de Rham complex in this way is to describe connections on $U(1)$-bundles and gerbes; more on this in the next section.

We assume the cover $\mathcal{U}$ is *good* in the sense that each open set U_α and each intersection $U_{\alpha_0\ldots\alpha_k} = U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ is contractible. Such covers exist [22]. When U is contractible the above complex is exact in degrees 1 to $k-1$. In degrees greater than 1 this is the Poincaré lemma, while in degree 1 it is because if $\eta \in \Omega^1(U; \mathbb{R})$ has $d\eta = 0$ then by fixing some $x_0 \in U$ and letting $f(x) = \exp i \int_{x_0}^{x} \eta$ we have $\text{dlog} f = \eta$. (The integral is taken along any smooth path in U from x_0 to x; the particular path is irrelevant because U is contractible and η is closed.) The complex is not exact at degree k because we truncate it, making all k-forms on U closed, regardless of whether they are closed in the usual de Rham sense.

Explicitly, the smooth Deligne double complex of degree $(k+1)$ is

$$\begin{tikzcd}
0 & 0 & 0 & \\
\prod_\alpha \Omega^k(U_\alpha;\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1} \Omega^k(U_{\alpha_0\alpha_1};\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^k(U_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \ldots \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \\
\prod_\alpha \Omega^1(U_\alpha;\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1} \Omega^1(U_{\alpha_0\alpha_1};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^1(U_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \ldots \\
\prod_\alpha \Omega^0(U_\alpha;U(1)) \arrow[u, "\text{dlog}"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1} \Omega^0(U_{\alpha_0\alpha_1};U(1)) \arrow[u, "\text{dlog}"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^0(U_{\alpha_0\alpha_1\alpha_2};U(1)) \arrow[u, "\text{dlog}"] \arrow[r, "\delta"] & \ldots
\end{tikzcd}$$

It is common to rewrite this complex's columns using a trick of homological algebra. The exact sequence of coefficients $0 \to \mathbb{Z} \hookrightarrow \mathbb{R} \xrightarrow{\exp} U(1) \to 0$ induces a quasi-isomorphism of complexes

$$\begin{tikzcd}
0 \arrow[r] & \Omega^{-1}(U;\mathbb{Z}) \arrow[r, "\iota"] \arrow[d] & \Omega^0(U;\mathbb{R}) \arrow[r, "d"] \arrow[d, "\exp"] & \Omega^1(U;\mathbb{R}) \arrow[r, "d"] \arrow[d] & \dots \\
0 \arrow[r] & 0 \arrow[r] & \Omega^0(U;U(1)) \arrow[r, "\mathrm{dlog}"] & \Omega^1(U;\mathbb{R}) \arrow[r, "d"] & \dots
\end{tikzcd}$$

where $\Omega^{-1}(U;\mathbb{Z})$ denotes constant $\mathbb{Z}$-valued functions on U, the notation reminding us that these have cohomological degree -1. The differential $\iota : \Omega^{-1} \to \Omega^0$ denotes the inclusion of constant functions; we will sometimes denote it as d for convenience. Thus a double complex with the same cohomology is

$$\begin{tikzcd}
0 & 0 & 0 & \\
\prod_\alpha \Omega^k(U_\alpha;\mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \prod_{\alpha_0,\alpha_1} \Omega^k(U_{\alpha_0\alpha_1};\mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^k(U_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[r, "\delta"] \arrow[u] & \dots \\
\vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \vdots \arrow[u, "d"] & \\
\prod_\alpha \Omega^1(U_\alpha;\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0,\alpha_1} \Omega^1(U_{\alpha_0\alpha_1};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^1(U_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \dots \\
\prod_\alpha \Omega^0(U_\alpha;\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0,\alpha_1} \Omega^0(U_{\alpha_0\alpha_1};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^0(U_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[r, "\delta"] \arrow[u, "d"] & \dots \\
\prod_\alpha \Omega^{-1}(U_\alpha;\mathbb{Z}) \arrow[r, "\delta"] \arrow[u, "\iota"] & \prod_{\alpha_0,\alpha_1} \Omega^{-1}(U_{\alpha_0\alpha_1};\mathbb{Z}) \arrow[r, "\delta"] \arrow[u, "\iota"] & \prod_{\alpha_0,\alpha_1,\alpha_2} \Omega^{-1}(U_{\alpha_0\alpha_1\alpha_2};\mathbb{Z}) \arrow[r, "\delta"] \arrow[u, "\iota"] & \dots
\end{tikzcd}$$

We will find this form of the double complex more convenient for discretization because the differential dlog: $\Omega^0(U;U(1)) \to \Omega^1(U;\mathbb{R})$ is difficult to discretize. The justification that both forms of the double complex are equivalent can be found in [8] or [21].

1.2 Differential Cohomology and Gauge Theory

The differential cohomology group $\hat{H}^k(M)$ of a smooth manifold is a refinement of its singular cohomology. By "refinement" we mean roughly that it contains an additional layer of information beyond what singular cohomology captures; more precisely, this group fits into an exact sequence

$$0 \to \Omega^{k-1}(M;\mathbb{R})/\Omega^{k-1}_{\mathbb{Z}}(M;\mathbb{R}) \to \hat{H}^k(M) \to H^k(M;\mathbb{Z}) \to 0 \tag{1.1}$$

where on the right we have the singular cohomology group of M and on the left we use $\Omega^{k-1}_{\mathbb{Z}}$ to denote the closed $(k-1)$-forms with integral periods.

Smooth Deligne Cohomology

The group $\hat{H}^k(M)$ admits a few equivalent definitions. One of these is to take the smooth Deligne complex in degree k from the previous section, form its total complex, and take the degree k cohomology of that total complex. A representative of a class in $\hat{H}^k(M)$ is a collection $(\omega^{k-1}_{\alpha_0}, \omega^{k-2}_{\alpha_0\alpha_1}, \ldots, \omega^0_{\alpha_0\ldots\alpha_{k-1}}, \omega^{-1}_{\alpha_0\ldots\alpha_k})$, where $\omega^{k-j-1}_{\alpha_0\ldots\alpha_j}$ is a collection of $(k-j-1)$-forms on all non-empty j-fold intersections $U_{\alpha_0\ldots\alpha_j}$ satisfying the closure relations

$$\begin{aligned}
(\delta\omega^{k-1})_{\alpha_0\alpha_1} &= d\omega^{k-2}_{\alpha_0\alpha_1} \\
&\vdots \\
(\delta\omega^{k-j-1})_{\alpha_0\ldots\alpha_{j+1}} &= (-1)^j d\omega^{k-j-2}_{\alpha_0\ldots\alpha_{j+1}} \\
&\vdots \\
(\delta\omega^{-1})_{\alpha_0\ldots\alpha_{k+1}} &= 0
\end{aligned}$$

These relations are simply $D\omega = 0$. Such a collection represents the trivial class $0 \in \hat{H}^k(M)$ if there exists a collection $(\mu^{k-2}_{\alpha_0}, \mu^{k-3}_{\alpha_0\alpha_1}, \ldots, \mu^0_{\alpha_0\ldots\alpha_{k-2}}, \mu^{-1}_{\alpha_0\ldots\alpha_{k-1}})$ satisfying

$$\begin{aligned}
\omega^{k-1}_{\alpha_0} &= d\mu^{k-2}_{\alpha_0} \\
\omega^{k-2}_{\alpha_0\alpha_1} &= (\delta\mu^{k-2})_{\alpha_0\alpha_1} - d\mu^{k-3}_{\alpha_0\alpha_1} \\
&\vdots \\
\omega^{k-j-1}_{\alpha_0\ldots\alpha_j} &= (\delta\mu^{k-j-1})_{\alpha_0\ldots\alpha_j} + (-1)^j d\mu^{k-j-2}_{\alpha_0\ldots\alpha_j} \\
&\vdots \\
\omega^{-1}_{\alpha_0\ldots\alpha_k} &= (\delta\mu^{-1})_{\alpha_0\ldots\alpha_k}
\end{aligned}$$

These relations are simply $\omega = D\mu$. This description of the differential cohomology group $\hat{H}^k(M)$ thus has the familiar form $\ker(D)/\mathrm{im}(D)$. Because this realization of differential cohomology arises from the smooth Deligne complex, we also refer to $\hat{H}^k(M)$ as the *smooth Deligne cohomology* of M in degree k.

Line Bundles with Connection

The relations describing classes of $\hat{H}^k(M)$ are familiar from gauge theory. In degree $k = 2$ they are

$$(\delta\omega^1)_{\alpha_0\alpha_1} = d\omega^0_{\alpha_0\alpha_1}$$
$$(\delta\omega^0)_{\alpha_0\alpha_1\alpha_2} = -\omega^{-1}_{\alpha_0\alpha_1\alpha_2}$$

There is also the relation $(\delta\omega^{-1}) = 0$, but this one is a trivial consequence of $(\delta\omega^0) = -\omega^{-1}$; what is non-trivial is the fact that $\omega^{-1}_{\alpha_0\alpha_1\alpha_2} \in \Omega^{-1}(U_{\alpha_0\alpha_1\alpha_2};\mathbb{Z})$ is constant and $\mathbb{Z}$-valued.

The data $(\omega^1_{\alpha_0}, \omega^0_{\alpha_0\alpha_1}, \omega^{-1}_{\alpha_0\alpha_1\alpha_2})$ define a $\mathbb{C}$-line bundle with connection on M. The bundle L will have restriction $L|_{U_\alpha} \cong U_\alpha \times \mathbb{C}$ and on overlaps $U_{\alpha\beta}$ we may use the transition functions $\varphi_{\alpha\beta} = \exp 2\pi i\omega^0_{\alpha\beta}$; the cocycle condition for these $U(1)$-valued functions is a consequence of $(\delta\omega^0) = -\omega^{-1}$ being $\mathbb{Z}$-valued. The local 1-forms ω^1_α define a connection on L as follows: Over each U_α we have the local section $s_\alpha(x) = 1 \in \mathbb{C}$, and for a tangent vector X to M at $x \in U_\alpha$ we define a connection locally via $\nabla^\alpha_X s_\alpha = (d + 2\pi i\omega^1_\alpha)_X s_\alpha = 2\pi i\omega^1_\alpha(X)$. These local formulas agree on overlaps $U_{\alpha\beta}$ because the 1-forms satisfy $\omega^1_\beta = \omega^1_\alpha + d\omega^0_{\alpha\beta}$, and so

$$\begin{aligned}
\nabla^\beta_X(s_\alpha) &= \nabla^\beta_X(\varphi_{\alpha\beta}s_\beta)\\
&= d\varphi_{\alpha\beta}(X)s_\beta + \varphi_{\alpha\beta}2\pi i\omega^1_\beta(X)\\
&= \varphi_{\alpha\beta}(2\pi i)d\omega^0_{\alpha\beta}(X)s_\beta + \varphi_{\alpha\beta}(2\pi i)\omega^1_\beta(X)\\
&= \varphi_{\alpha\beta}(2\pi i)(d\omega^0_{\alpha\beta} + \omega^1_\beta)(X)\\
&= \varphi_{\alpha\beta}(2\pi i)\omega^1_\alpha(X)\\
&= \varphi_{\alpha\beta}\nabla^\alpha_X(s_\alpha)
\end{aligned}$$

That is, on $U_{\alpha\beta}$ either expression may be used for the connection and the results are related by the transition function $\varphi_{\alpha\beta}$.

Suppose that $\omega = D\mu$ as described in the previous section. Then $(\mu^0_{\alpha_0}, \mu^{-1}_{\alpha_0\alpha_1})$ may be used to trivialize the bundle L and its connection. We modify each local section as follows: $\tilde{s}_\alpha = \exp(-2\pi i\mu^0_\alpha)s_\alpha$. Then these local sections are restrictions of a global section, since on $U_{\alpha\beta}$ we have

$$\begin{aligned}
\tilde{s}_\beta &= e^{-2\pi i \mu^0_\beta} s_\beta \\
&= e^{-2\pi i \mu^0_\beta} \varphi_{\alpha\beta} s_\alpha \\
&= \exp\left(2\pi i(\omega^0_{\alpha\beta} - \mu^0_\beta)\right) s_\alpha \\
&= e^{-2\pi i \mu^0_\alpha} s_\alpha \\
&= \tilde{s}_\alpha
\end{aligned}$$

Moreover, this global section is flat:

$$\begin{aligned}
\nabla^\alpha \tilde{s}_\alpha &= \nabla^\alpha \left(e^{-2\pi i \mu^0_\alpha} s_\alpha\right) \\
&= (de^{-2\pi i \mu^0_\alpha}) s_\alpha + e^{-2\pi i \mu^0_\alpha} \nabla^\alpha s_\alpha \\
&= -(2\pi i) d\mu^0_\alpha e^{-2\pi i \mu^0_\alpha} + e^{-2\pi i \mu^0_\alpha} (2\pi i) \omega^1_\alpha \\
&= 0
\end{aligned}$$

Therefore the bundle and connection (L, ∇) are trivial. So we may associate to classes of $\hat{H}^2(M)$ isomorphism classes of $\mathbb{C}$-line bundle with connection. Moreover, the group structure on smooth Deligne 2-cohomology classes is easily shown to coincide with the tensor product of line bundles with connection. Thus we have an equivalent description of the the first differential cohomology group $\hat{H}^2(M)$ as the isomorphism classes of $\mathbb{C}$-line bundles with connection.

Cheeger-Simons Forms

Yet another equivalent way to define the differential cohomology of M is via its Cheeger-Simons forms, also known as "differential characters." These were defined in [9] as homomorphisms $f : Z_{k-1} \to U(1)$ on the group Z_{k-1} of *smooth* $(k-1)$-cycles in M for which there exists a k-form $\mathrm{curv}(f) \in \Omega^k(M;\mathbb{R})$ with $f(\partial C) = \exp(2\pi i \mathrm{curv}(f)(C))$ for all smooth k-submanifolds[3] of M. The abelian group structure on differential characters is evident and provides another definition of $\hat{H}^k(M)$.

The sequences

$$0 \to \Omega^{k-1}(M;\mathbb{R})/\Omega^{k-1}_{\mathbb{Z}}(M;\mathbb{R}) \to \hat{H}^k(M) \to H^k(M;\mathbb{Z}) \to 0$$

and

$$0 \to H^{k-1}(M;U(1)) \to \hat{H}^k(M) \to \Omega^k_{\mathbb{Z}}(M;\mathbb{R}) \to 0$$

[3] We are glossing over some subtleties regarding which submanifolds C ought to be considered here. For a careful discussion see [5].

were proven in [9]. The left-most term in each sequence is a divisible abelian group or quotient thereof, and so the sequences are split exact. Thus one may write (non-canonically)

$$\hat{H}^k(M) \cong H^k(M;\mathbb{Z}) \oplus \Omega^{k-1}(M;\mathbb{R})/\Omega^{k-1}_{\mathbb{Z}}(M;\mathbb{R})$$

$$\hat{H}^k(M) \cong \Omega^k_{\mathbb{Z}}(M;\mathbb{R}) \oplus H^{k-1}(M;U(1))$$

In the degree $k = 2$ case, which corresponds to line bundles with connection, the first isomorphism presents the overall isomorphism class in the form (topological sector of bundle, connection 1-form modulo gauge transformations). This presentation is non-canonical because identifying the connection with a global 1-form requires comparing it to some arbitrary reference connection. The second isomorphism presents the overall isomorphism class in the form (curvature 2-form, holonomies around 1-cycles). Again this is non-canonical because the identification of a class in $H^1(M;U(1))$ is only possible after comparing to some reference connection with the chosen curvature.

1.3 Discrete Differential Geometry

Discrete differential geometry encompasses diverse attempts to apply ideas from differential geometry to "discrete spaces" such as lattices[4] and simplicial complexes. A common theme is to take geometric statements involving the exterior derivative on k-forms and to study analogous statements involving k-cochains on a simplicial complex and the simplicial coboundary operator. For example, rather than computing the de Rham cohomology of M we may instead compute the simplicial cohomology using real-valued k-cochains on a triangulation of M; the two are isomorphic. Sometimes a limiting procedure is required to recover the smooth results; for example, [11] shows that the spectrum of the Laplacian on a Riemannian manifold can be recovered from the spectrum of a certain discrete Laplace operator in a limit over triangulations of M with mesh size approaching zero.

Discrete $\mathbb{C}$-Line Bundles

In [17], Knöppel and Pinkall develop a theory of vector bundles on simplicial complexes. The idea is to place an n-dimensional fiber over each vertex of X and replace connections with invertible maps along the edges. This idea was studied

[4] We will not treat rectangular lattices here as they are less suited to questions of global topology than simplicial complexes. However the physics literature on lattice gauge theories and the mathematical literature on finite difference approximations to PDEs are full of successful discretizations on lattices.

quite early on square lattices by K. Wilson to approximate the physical gauge theory of quarks [25]. However, Wilson's theory is local, making no attempt to capture the global topology of vector bundles on manifolds. The theory developed in [17] captures the topology of vector bundles with connection via the monodromy representation that the connection induces. All definitions and theorems in this subsection are from [17], sometimes paraphrased.

Definition 1.3.1 ([17])**.** A discrete Hermitian[5] line bundle with connection and curvature on a simplicial complex X is a collection (L, η, Ω) consisting of

- A 1-dimensional complex vector space L_v for each vertex $v \in X^0$, equipped with an inner product.
- For each oriented edge e of X an isometry $\eta_e : L_{s(e)} \longrightarrow L_{d(e)}$.
- A real-valued 2-cochain $\Omega \in C^2(X; \mathbb{R})$ with $\exp 2\pi i \Omega = d\eta$.

The notation $d\eta$ needs some explanation: for each oriented 2-simplex σ^2 of X its boundary $\partial\sigma^2$ defines a closed loop based at v, where v may be any of the vertices belonging to σ^2. Then the composition of η_e for each e belonging to $\partial\sigma^2$ defines a $\mathbb{C}$-linear isometry $\eta(\partial\sigma^2) : L_v \longrightarrow L_v$, which we may canonically identify with an element of $U(1)$. This element we define to be $d\eta(\sigma^2) \in U(1)$. The choice of basepoint is seen to be irrelevant because changing basepoint has the effect of conjugating $d\eta(\sigma^2)$ in the group $U(1)$, which is trivial.

The definition bears an obvious resemblance to its smooth counterpart and a discrete line bundle with connection and curvature could be obtained from a smooth line bundle with unitary connection as follows: Let X triangulate M. Then the simplices of X are identified with subsets of M via a homeomorphism. Suppose that (L, ∇) is a smooth Hermitian line bundle with connection. Then take L_v to be the fiber over $v \in M$ and take η_e to be the parallel transport map induced by ∇. The curvature $\mathrm{curv}(\nabla) \in \Omega^2(M; \mathbb{R})$ may be integrated over each 2-simplex of X to define a 2-cochain Ω. Then $\exp 2\pi i \Omega = d\eta$ is a consequence of the Ambrose-Singer theorem [3].

One striking difference between the discrete and smooth versions of line bundles is that discrete line bundles always have a non-vanishing global section, since we have

[5]The Hermitian structure can be dropped with some obvious modifications to this definition. However, we want the structure group of these bundles to be reduced from $\mathbb{C}^*$ to $U(1)$, so we will always consider the fibers to be equipped with a Hermitian inner product.

no way of imposing a continuity condition on sections. A smooth line bundle with a non-vanishing global section is called trivial, so one must modify the definition of "trivial" for discrete line bundles to avoid defining all such objects to be trivial. The key is to incorporate the connection into the definition. One defines morphisms of line bundles with connection, then defines a trivial object, then finally defines a trivial discrete line bundle to be one which is isomorphic to the trivial object.

Definition 1.3.2 ([17])**.** A morphism of discrete line bundles (L, η, Ω) and $(\tilde{L}, \tilde{\eta}, \tilde{\Omega})$ with connection and curvature over the same simplicial complex X is a collection of isometries $f_v : L_v \to \tilde{L}_v$ such that for each edge the diagram

$$\begin{array}{ccc} L_{s(e)} & \xrightarrow{f_{s(e)}} & \tilde{L}_{s(e)} \\ \downarrow{\scriptstyle \eta_e} & & \downarrow{\scriptstyle \tilde{\eta}_e} \\ L_{d(e)} & \xrightarrow{f_{d(e)}} & \tilde{L}_{d(e)} \end{array}$$

commutes. That is, one has $\tilde{\eta}_e \circ f_{s(e)} = f_{d(e)} \circ \eta_e$ for each edge e. Moreover, one requires that $\Omega = \tilde{\Omega}$.

Observe that all morphisms are invertible.

Definition 1.3.3 ([17])**.** A discrete line bundle with connection and curvature over X is trivial if it admits a morphism to the bundle $(\mathbb{C}, 1, 0)$ whose fibers are all $L_v = \mathbb{C}$, whose connection is $\eta_e = 1$ for all e, and whose curvature 2-cochain $\Omega = 0$.

Triviality of a bundle is proven in [17] to be equivalent to the existence of a *parallel section*, which is a section $\{\varphi_v \in L_v\}$ with $\eta_e(\varphi_{s(e)}) = \varphi_{d(e)}$. These definitions allow us to speak of isomorphism classes of discrete vector bundles with connection and curvature. The set $\mathcal{L}_X^{\mathbb{C}}$ of isomorphism classes has an abelian group structure: fiberwise one takes a tensor product $L_v \otimes \tilde{L}_v$, and then a connection is naturally induced. When using global sections to identify the connection with a $U(1)$-valued 1-form on X the connection on the tensor product corresponds to the sum of the two connections. Therefore adding the curvature 2-cochains preserves their defining relation. It is clear that this operation is abelian and that it turns the space of isomorphism classes of discrete line bundles with connection into an abelian group.

A structural result for this group that we will generalize is:

Proposition 1.3.4 ([17])**.** *The abelian group of isomorphism classes of discrete line bundles with connection (but not equipped with curvature 2-cochain) is isomorphic to* $C^1(X; U(1))/dC^0(X; U(1))$.

The set of 2-cochains Ω for which a discrete line bundle (L, η, Ω) exists is $C^2_{\mathbb{Z}}(X;\mathbb{R})$, the closed 2-cochains with integer periods.

The set of those discrete line bundles with connection admitting the same curvature 2-cochain is in one-to-one correspondence with $H^1(X;U(1))$.

Another way of saying this is that there is an exact sequence

$$0 \to H^1(X;U(1)) \to \mathcal{L}^{\mathbb{C}}_X \to C^2_{\mathbb{Z}}(X;\mathbb{R}) \to 0$$

(Note that in our notation $\mathcal{L}^{\mathbb{C}}_X$ denotes classes of discrete line bundles with connection *and curvature*; it is used in [17] to denote these classes without their curvature.)

With this sequence the relation between these discrete line bundles and the smooth differential characters is evident: both objects consist of a rule for assigning holonomy to the 1-cycles (simplicial or smooth) in a way that is compatible with some curvature. Moreover, they fit in analogous exact sequences. Therefore the definitions in [17] do a good job of discretizing smooth line bundles with connection.

1.4 Motivation

The parallel just noted between Knöppel and Pinkall's discrete line bundles with connection and Cheeger and Simons' differential characters motivated us to seek a discretization of smooth Deligne cohomology. Since smooth Deligne cohomology and differential characters provide independent realizations of the same differential cohomology groups, we ought to be able to show that our discrete Deligne cohomology groups are isomorphic to Knöppel and Pinkall's $\mathcal{L}^{\mathbb{C}}_X$.

These goals will be realized over the next few chapters. We will see that it is no more difficult to make these definitions in degree k, with degree 2 representing line bundles with connection and degree $k > 2$ offering a definition of "discrete principal $(k-1)$-bundles." In each degree the discretization will be seen to fit into exact sequences analogous to those proven in [9].

One of our initial motivations for studying these discretizations was to study discretized abelian Chern-Simons theory. Because our discretization of Deligne cohomology closely mimics familiar definitions from physical gauge theories, we will find it easy to write down a discrete Chern-Simons action that can be studied as a theory in its own right or can be used as a starting point for defining discrete theories of matter fields coupled to a Chern-Simons gauge field.

Chapter 2

DISCRETE DELIGNE COHOMOLOGY

Here we define for simplicial complexes a cohomology theory that plays the role of degree 2 Deligne cohomology for smooth manifolds. It discretizes the degree 2 differential cohomology of a smooth manifold in a way analogous to how real-valued k-cochains discretize the real-valued k-forms on a smooth manifold. Like its smooth counterpart, our degree 2 discrete differential cohomology will characterize discrete line bundles with curvature; this is the subject of Chapter 4.

In this chapter we will define cochains with notions of closedness and exactness. In the usual way we then obtain a cohomology group with an abelian group structure. We will demonstrate that this cohomology group fibers in two ways that are entirely analogous to the exact sequences satisfied by the degree 2 Deligne cohomology group $\hat{H}^2$:

$$0 \to \Omega^1/\Omega^1_{\mathbb{Z}} \to \hat{H}^2 \to H^2(\mathbb{Z}) \to 0$$

and

$$0 \to H^1(U(1)) \to \hat{H}^2 \to \Omega^2_{\mathbb{Z}} \to 0$$

(recall that $\Omega^1_{\mathbb{Z}}$ and $\Omega^2_{\mathbb{Z}}$ denote closed 1- and 2-forms with integer periods).

In the next section we review the usual smooth Deligne cohomology (with emphasis on the word *smooth*; our presentation makes no mention of sheaves and may look foreign to algebraic geometers who are familiar the sheaf-theoretic approach). We present some essential results from the theory and outline their proofs. The definitions and proofs of our discretized version of the theory will closely mimic their smooth counterparts.

2.1 Smooth Deligne Cohomology in Degree 2

Smooth Deligne cohomology is the offspring of the de Rham and Čech complexes over a smooth manifold M. As such it involves differential forms and an open cover of M. The Čech complex can work with any open cover of M, but it will work best with the de Rham complex if we take that cover $\mathcal{U} = \{U_\alpha\}_\alpha$ to consist only of *contractible* open sets with all k-fold intersections $U_{\alpha_0 \ldots \alpha_k} := U_{\alpha_0} \cap \ldots \cap U_{\alpha_k}$ also contractible. We call such a cover *good*. Any smooth manifold admits such a cover [22] and we fix such a cover for the remainder of this chapter.

A Čech–de Rham cocycle will be a collection of differential forms defined only locally on open sets of the cover and intersections thereof. They will satisfy gluing conditions familiar from gauge theory or principal bundles. These gluing conditions allow objects with non-trivial global topology to be described via local data, and such descriptions are often necessary to compute the quantities of interest in applications, notably holonomies of connections on principal bundles.

Before giving its formal definition, we motivate the Čech–de Rham complex with elementary considerations. The reader familiar with gauge theory may prefer to skip directly to the formal definitions of the next subsection. (A reader familiar with gauge theory but unfamiliar with Čech cohomology may wish to read the discussion beginning after Equation (2.1).)

Pedagogical Introduction

To ease ourselves into the definition of the Čech–de Rham complex we start by thinking about a familiar object: a closed 2-form $F \in \Omega^2(M;\mathbb{R})$. A *potential* for F is a 1-form $A \in \Omega^1(M;\mathbb{R})$ with $F = dA$, but as we know from differential topology there are many closed 2-forms F which do not admit any potential. Indeed, these are the most interesting ones as they reveal to us features of the global topology of M. Although F may not have a global potential, it does have *local* potentials. More precisely, the restriction $F|_{U_\alpha}$ to one of our contractible open sets U_α always has a potential by the Poincaré lemma: $F = dA_\alpha$ for some $A_\alpha \in \Omega^1(U_\alpha;\mathbb{R})$. (Denoting the restriction of F to U_α makes for cumbersome notation and it will be understood from context in the remainder.) Collecting these local potentials $\{A_\alpha\}$ over all the open sets of our cover is the first step towards replacing F by a Čech–de Rham cocycle.

If we cared only about finding a new representation of F then we would stop here, since F can already be recovered at each point of $x \in M$ by choosing an appropriate open set $U_\alpha \ni x$ and computing the exterior derivative dA_α at x. But what if F means more to us than just a closed 2-form? If we are describing electromagnetism on M then F means a great deal more to us; it is then the electromagnetic field strength tensor, and alone it is not sufficient to describe all the phenomena of electromagnetism on M. For example, the quantum mechanics of charged particles on M keeps track of a certain complex phase that is computed from a potential for F (we will not describe this in detail here but the reader can look up the Aharonov-Bohm effect). Interesting physics occurs when F does not admit global potentials, and a proper description of these particles requires dealing carefully with the local potentials along the particle's path. This situation and more have been understood mathematically to be manifestations of the geometry of (complex) line bundles. (In physics it goes by the name of "abelian gauge theory.")

To capture the geometric object that underlies F in these situations we compare the local potentials A_α and A_β belonging to overlapping sets of our cover. They can only be compared where they are both defined, so we restrict both to $U_{\alpha\beta}$ and consider their difference $A_\beta - A_\alpha$. Since both are potentials for the same 2-form F, their difference is closed: $d(A_\beta - A_\alpha) = F - F = 0$. The virtue of good covers is that $U_{\alpha\beta}$ is contractible and so their difference is *exact*: $A_\beta - A_\alpha = d\varphi_{\alpha\beta}$ for some smooth function $\varphi_{\alpha\beta} \in \Omega^0(U_{\alpha\beta}; \mathbb{R})$. Mathematicians call $\varphi_{\alpha\beta}$ a *transition function*, while in physics it is called a *gauge transformation*. The second piece of our Čech–de Rham cocycle is the collection $\{\varphi_{\alpha\beta}\}$ of smooth functions on all non-empty intersections $U_{\alpha\beta}$ of two sets of the open cover $\mathcal{U}$. They satisfy

$$d\varphi_{\alpha\beta} = A_\beta - A_\alpha =: (\delta A)_{\alpha\beta} \tag{2.1}$$

Some terminology: the Latin d is the exterior derivative on forms, while the Greek δ is the Čech differential. The Čech differential compares forms defined locally on open sets U_α and U_β on the region $U_{\alpha\beta}$ where they overlap. Its definition is given in Equation (2.1) for simple (1-fold) intersections, but the Čech complex considers arbitrary k-fold intersections. There are two notions of *degree*. A differential form has some degree k and the exterior derivative increases this degree by 1. Similarly, Čech cochains have a degree that is increased by the Čech differential δ. The *Čech degree* of a Čech cochain is 0 for objects like $\{A_\alpha\}$ that are defined on entire open sets of the cover; it is 1 for objects like $\{\varphi_{\alpha\beta}\}$ that are defined only on 1-fold intersections of open sets of the cover; it is k for objects $\{\eta_{\alpha_0\ldots\alpha_k}\}$ defined on k-fold intersections (and a common convention is to define the Čech degree of globally defined objects like F to be -1). Cochains of Čech degree k can be compared on any $(k+1)$-fold intersection $U_{\alpha_0\ldots\alpha_{k+1}}$ by the Čech differential:

$$(\delta\eta)_{\alpha_0\ldots\alpha_{k+1}} = \sum_{i=0}^{k+1}(-1)^i \eta_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_k} \tag{2.2}$$

with $\hat{\alpha}_i$ indicating that we omit the index α_i.

In a Čech–de Rham cochain each piece has two degrees, its Čech and its de Rham degrees. Its *total degree* is the sum of these two. Note that the pieces we have defined so far have the same total degree: $\{A_\alpha\}$ has Čech degree 0 and de Rham degree 1, while $\{\varphi_{\alpha\beta}\}$ has Čech degree 1 and de Rham degree 0. This total degree is what is referenced in the chapter name when we say “Degree 2 Deligne Cohomology." The prevailing convention is to let degree k Deligne cochains consist of pieces whose total degree is $(k-1)$. This convention is used in order for the degree to be consistent with a ring structure that we define in Chapter 6. Note that the closed 2-form F we started from has de Rham degree 2 and Čech degree -1, according to the convention we mentioned above for the Čech degree of globally defined objects. There is another degree 1 object worth mentioning.

If we compare the transition functions $\varphi_{\alpha\beta}$ on the intersection of 3 open sets (a 2-fold intersection) then we obtain an object of Čech degree 2:

$$(\delta\varphi)_{\alpha\beta\gamma} = \varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta}$$

A priori its de Rham degree is 0, but observe that $(\delta\varphi)$ is constant:

$$\begin{aligned} d(\delta\varphi)_{\alpha\beta\gamma} &= d\left(\varphi_{\beta\gamma} - \varphi_{\alpha\gamma} + \varphi_{\alpha\beta}\right) \\ &= A_\gamma - A_\beta - A_\gamma + A_\alpha + A_\beta - A_\alpha \\ &= 0 \end{aligned}$$

In the language of Čech–de Rham complexes it is a useful convention to define constant 0-forms to have de Rham degree equal to -1. We will let $\Omega^{-1}(U; A)$ denote the constant A-valued functions on U. If $(A_\alpha, \varphi_{\alpha\beta})$ derives from a general closed 2-form F then $(\delta\varphi)_{\alpha\beta\gamma}$ will be a real-valued Čech 2-cochain whose *total* degree in the Čech–de Rham complex is 1. Its value on a 2-fold intersection could be any real number. However, in the case where F is a curvature form on a line bundle (including the example of electromagnetism alluded to above), then $(\delta\varphi)_{\alpha\beta\gamma}$ can be taken to be *integer*-valued.

Our final remark before giving rigorous definitions regards the topological features present in a Čech–de Rham cocycle of degree 1. These features reside in the first and second cohomology of M. Assuming that F has integral periods, as is the case in the applications mentioned above, its de Rham cohomology class $[F] \in H^2_{\mathrm{dR}}(M) \cong H^2(M;\mathbb{R})$ is the image of a class $\tilde{F} \in H^2(M;\mathbb{Z})$ under the coefficient morphism induced by inclusion $\mathbb{Z} \hookrightarrow \mathbb{R}$. All classes in the free part of the abelian group $H^2(M;\mathbb{Z})$ have such a representation by a 2-form F and so can be described as Čech–de Rham 1-cochains. The *torsion* part of $H^2(M;\mathbb{Z})$ is killed under $H^2(M;\mathbb{Z}) \to H^2(M;\mathbb{R})$ and so cannot be detected by any differential

2-form. Nonetheless, it can be represented using integer-valued Čech 2-cocycles $n_{\alpha\beta\gamma}$. These are defined by the property that $\delta n = 0$ and represent non-trivial classes in $H^2(M;\mathbb{Z})$ precisely when the equation $n = \delta m$ has no solutions among $\mathbb{Z}$-valued Čech 1-cochains m. A non-obvious fact is that the equation $n = \delta\varphi$ *does* have solutions if we allow $\varphi_{\alpha\beta}$ to be a Čech 1-cochain of real-valued functions, assuming that $n_{\alpha\beta\gamma}$ represents a torsion class.[1] Thus we also have a way of encoding the torsion part of $H^2(M;\mathbb{Z})$ into the data of a degree 1 Čech–de Rham cocycle. In summary, any class of $H^2(M;\mathbb{Z})$ can be represented by a Čech–de Rham cocycle, with the local potentials A_α encoding its free part, and its free and torsion parts contained in the transition functions $\varphi_{\alpha\beta}$. This justifies the statement that degree 2 smooth Deligne cohomology is a *refinement* of $H^2(M;\mathbb{Z})$.

The Degree 2 Čech–de Rham Complex

Degree 2 smooth Deligne cohomology is the hypercohomology of the Deligne double complex. The Deligne complex is a bi-graded complex whose horizontal differential is Čech and whose vertical differential is de Rham. Each horizontal row is therefore a Čech complex valued in k-forms for some k. The vertical columns are not quite de Rham complexes, but they are closely related. The difference is that the de Rham complex is truncated after degree 1 and is extended to have a degree -1 piece consisting of constant *integer*-valued functions. Over an open set U the vertical columns form the complex

$$\Omega^{-1}(U;\mathbb{Z}) \xrightarrow{\iota} \Omega^0(U;\mathbb{R}) \xrightarrow{d} \Omega^1(U;\mathbb{R}) \to 0$$

(Here $\Omega^{-1}(U;\mathbb{Z})$ denotes constant $\mathbb{Z}$-valued functions on U and ι denotes inclusion of such functions into $\Omega^0(U;\mathbb{R})$.) We will not motivate these modifications to the de Rham complex except to say that they are what makes the theory effective at describing connections on complex line bundles.[2]

The relevant parts of the complex are

---

[1] We will not prove this fact here. It is a consequence of the statement that the sheaf cohomology of M valued in smooth functions is isomorphic to the de Rham cohomology and therefore has no torsion. See for example [8] Chapter 1, 1.4.7

[2] These modifications can be nicely motivated by first defining the vertical rows of the complex to be $\Omega^0(U;U(1)) \xrightarrow{dlog} \Omega^1(U;\mathbb{R}) \to 0$ and then considering the consequences of using the exponential sequence $0 \to Z \to \mathbb{R} \to U(1) \to 0$ to change coefficients; see [8], Chapter 1.

$$\begin{tikzcd}
0 & \cdots & \cdots \\
\prod_\alpha \Omega^1(U_\alpha;\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha,\beta} \Omega^1(U_{\alpha\beta};\mathbb{R}) & \cdots \\
\prod_\alpha \Omega^0(U_\alpha;\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha,\beta} \Omega^0(U_{\alpha\beta};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha,\beta,\gamma} \Omega^0(U_{\alpha\beta\gamma};\mathbb{R}) \\
\cdots & \prod_{\alpha,\beta} \Omega^{-1}(U_{\alpha\beta};\mathbb{Z}) \arrow[u, "d_{-1}"] \arrow[r, "\delta"] & \prod_{\alpha,\beta,\gamma} \Omega^{-1}(U_{\alpha\beta\gamma};\mathbb{Z}) \arrow[u, "d_{-1}"]
\end{tikzcd} \tag{2.3}$$

Degree 2 cochains come from the diagonal of degree 1 components of this double complex. They consist of the data

$$\big(\{A_\alpha\}_\alpha, \{\varphi_{\alpha\beta}\}_{\alpha,\beta}, \{n_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma}\big) \in \prod_\alpha \Omega^1(U_\alpha;\mathbb{R}) \bigoplus \prod_{\alpha,\beta} \Omega^0(U_{\alpha\beta};\mathbb{R}) \bigoplus \prod_{\alpha,\beta,\gamma} \Omega^{-1}(U_{\alpha\beta\gamma};\mathbb{Z}) \tag{2.4}$$

In order that the horizontal rows form Čech complexes we demand that each component of the cochain is alternating in its indices: $\varphi_{\alpha\beta} = -\varphi_{\beta\alpha}$ and similarly for $n_{\alpha\beta\gamma}$.

A cochain is called a *1-cocycle* if it is *closed*, meaning that

$$(\delta A)_{\alpha\beta} = d\varphi_{\alpha\beta} \tag{2.5}$$

$$(\delta\varphi)_{\alpha\beta\gamma} = -n_{\alpha\beta\gamma} \tag{2.6}$$

$$(\delta n)_{\alpha\beta\gamma\rho} = 0 \tag{2.7}$$

Equation (2.5) is the familiar condition from gauge theories, while Equation (2.6) places a condition on the gauge transformations that ensures they form the transition functions of a complex line bundle. Equation (2.6) ensures that the topological invariants that can be extracted from a 1-cocycle indeed lie in $H^2(M;\mathbb{Z})$. Observe that the closure condition says that if we follow the images of $\{A_\alpha\}$ and $\{\varphi_{\alpha\beta}\}$ along their respective horizontal/vertical differentials in the double complex that they are equal as elements of $\prod_{\alpha,\beta} \Omega^1(U_{\alpha\beta};\mathbb{R})$; Equation (2.6) can be interpreted analogously. Equation (2.7) states that $n_{\alpha\beta\gamma}$ defines an integer-valued Čech 2-cocycle.

A cochain is called *exact* if there exist $\{\psi_\alpha\} \in \prod_\alpha \Omega^0(U_\alpha;\mathbb{R})$ and $\{m_{\alpha\beta}\} \in$

$\prod_{\alpha,\beta} \Omega^{-1}(U_{\alpha\beta}; \mathbb{Z})$ with

$$A_\alpha = d\psi_\alpha \tag{2.8}$$

$$\varphi_{\alpha\beta} = (\delta\psi)_{\alpha\beta} - m_{\alpha\beta} \tag{2.9}$$

$$n_{\alpha\beta\gamma} = (\delta m)_{\alpha\beta\gamma} \tag{2.10}$$

Note that these conditions say that $(A_\alpha, \varphi_{\alpha\beta}, n_{\alpha\beta\gamma})$ are the images of objects from the diagonal of total degree 0 elements in the double complex under their respective horizontal/vertical differentials.

The abelian group structure on 1-cochains is inherited from pointwise addition of forms. Note that because $n_{\alpha\beta\gamma}$ is $\mathbb{Z}$-valued we have only an abelian group and not a vector space. Two cochains are called *cohomologous* if their difference is exact.

Lemma. *An exact 1-cochain is closed.*

Lemma. *The sum or difference of exact cochains is exact. The sum or difference of closed cochains is closed.*

These results will be proven carefully in the next section for the discretized version of these cochains. Since the proofs are nearly identical and the smooth version is well-known, we omit the proofs of these two lemmas here. They justify the following definition:

Definition 2.1.1. The *degree 2 Deligne cohomology* of M relative to the good cover $\mathcal{U}$ is the quotient of the subgroup of closed 2-cochains by the subgroup of exact 2-cochains. This abelian group is independent up to isomorphism of the cover $\mathcal{U}$ (see Proposition 2.1.2) and so we may denote it $H^2_D(M)$.

Proposition 2.1.2. *Given any good cover $\mathcal{U}$ of M, the degree 2 Deligne cohomology group defined relative to $\mathcal{U}$ in Definition 2.1.1 is isomorphic to the abelian group of isomorphism classes of smooth $\mathbb{C}$-line bundles on M with connection.*

This is a well-known result; see, for example, [8].

2.2 Discrete Deligne Cohomology in Degree 2

We now present our discretization of the Deligne cochain complex and the resulting cohomology theory that it associates to a simplicial complex X. The idea of the discretization is familiar: one replaces differential k-forms with k-cochains. Our objective in this section is to demonstrate that all the important results mentioned above for the smooth Deligne cohomology groups also have discrete counterparts.

Before the definitions we begin with a comment on the role of open covers in the two theories. In the smooth theory one takes a manifold M as the basic object and in order to define the cochain complex we use the type of "good" cover mentioned in the previous section. The resulting cohomology group turns out to be independent of the cover used, which we argued by way of relating this group to the isomorphism classes of complex line bundles with connection on M.

It is useful to think of a simplicial complex X as a space already equipped with a good cover. The geometric realization of X is the space, while the simplicial structure of X is akin to a good cover of that space. We assume throughout that X is a triangulation of some manifold. Then each top-dimensional simplex σ of X has an ϵ-neighborhood (defined using some auxiliary Riemannian metric that we don't actually care about) which is contractible. Because any non-empty intersection of two simplices is again a simplex, the intersections of two such neighborhoods of top-dimensional simplices is a neighborhood of some other simplex in the cover. This is again contractible. We see that the open neighborhoods of top-dimensional simplices of X form a good cover of the underlying manifold.

Our discrete theory differs from the smooth theory in a key aspect: even if two complexes X and X' triangulate the same manifold M, their discrete Deligne cohomology groups generally will not be isomorphic. This can be seen from the short exact sequence

$$0 \to \Omega^1(M)/\Omega^1_{\mathbb{Z}}(M) \to \hat{H}^2(M) \to H^2(M;\mathbb{Z}) \to 0$$

whose discrete analogue is

$$0 \to C^1(X)/C^1_{\mathbb{Z}}(X) \to H^1_{dD}(X) \to H^2(X;\mathbb{Z}) \to 0$$

Because X and X' triangulate the same space, the final terms of the sequence will be the same, $H^2(X;\mathbb{Z}) \cong H^2(X';\mathbb{Z})$. But it will rarely be possible to find an isomorphism between $C^1(X)/C^1_{\mathbb{Z}}(X)$ and $C^1(X')/C^1_{\mathbb{Z}}(X')$ because the two complexes will generally have different 1–skeleta. This is a familiar feature of discretizations and is not troubling. We think formally of both of these groups as "converging" in some sense to $\Omega^1(M)/\Omega^1_{\mathbb{Z}}(M)$ as we take successive refinements of our triangulations (see Section 2.6).

The Discrete Deligne Complex

Analogous to (2.3) we define the *discrete Deligne double complex in degree 2* as

$$
\begin{tikzcd}
0 & \cdots & \cdots \\
\prod_\alpha C^1(\sigma_\alpha;\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha,\beta} C^1(\sigma_{\alpha\beta};\mathbb{R}) & \cdots \\
\prod_\alpha C^0(\sigma_\alpha;\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha,\beta} C^0(\sigma_{\alpha\beta};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha,\beta,\gamma} C^0(\sigma_{\alpha\beta\gamma};\mathbb{R}) \\
\cdots & \prod_{\alpha,\beta} C^{-1}(\sigma_{\alpha\beta};\mathbb{Z}) \arrow[u, "d_{-1}"] \arrow[r, "\delta"] & \prod_{\alpha,\beta,\gamma} C^{-1}(\sigma_{\alpha\beta\gamma};\mathbb{Z}) \arrow[u, "d_{-1}"]
\end{tikzcd}
\tag{2.11}
$$

where α ranges over an index set for the top-dimensional simplices σ_α of X. Then (α, β) is taken to run over the pairs of indices whose intersection $\sigma_{\alpha\beta} = \sigma_\alpha \cap \sigma_\beta$ is non-empty. We define $\sigma_{\alpha\beta\gamma}$ similarly. Recall that all simplices of X have been given an arbitrary but fixed orientation. We will reserve the notation $\sigma_{\alpha_0\ldots\alpha_k}$ for an intersection of k *top*-dimensional simplices and use σ or τ when we wish to refer to a simplex of general dimension. We again take $C^{-1}(\sigma; A)$ to refer to constant A-valued 0-cochains on σ.

The vertical differential d is the simplicial coboundary operator while δ is a discrete analog of the Čech differential. Its definition is the obvious one: if $\psi_\alpha \in C^k(\tau_\alpha)$ and $\psi_\beta \in C^k(\tau_\beta)$ (here τ_α and τ_β are simplices of arbitrary dimension and non-empty intersection) then we define $(\delta\psi)_{\alpha\beta} \in C^k(\tau_\alpha \cap \tau_\beta)$ to be the difference of their restrictions:

$$(\delta\psi)_{\alpha\beta} := \psi_\beta|_{\tau_\alpha\cap\tau_\beta} - \psi_\alpha|_{\tau_\alpha\cap\tau_\beta}. \tag{2.12}$$

(Henceforth we will stop denoting the restriction of a cochain to a subcomplex; it will always be clear from context.) This definition is given in degree 1, but the generalization to arbitrary Čech degree is exactly as in (2.2).

The *degree* in each piece of this double complex is defined as it was in the smooth case to be the sum of that piece's "de Rham" degree (k for $C^k(\ldots)$) and that piece's "Čech" degree (l for $C^{\cdots}(\sigma_{\alpha_0\ldots\alpha_l})$). We adopt again the convention that constant cochains have de Rham degree -1 and globally defined cochains have Čech degree -1.

The *discrete Deligne 2-cochains* are the total degree 1 objects in this double complex:

$$\big(\{a_\alpha\}_\alpha, \{\varphi_{\alpha\beta}\}_{\alpha,\beta}, \{n_{\alpha\beta\gamma}\}_{\alpha,\beta,\gamma}\big) \in \prod_\alpha C^1(\sigma_\alpha;\mathbb{R}) \bigoplus \prod_{\alpha,\beta} C^0(\sigma_{\alpha\beta};\mathbb{R}) \bigoplus \prod_{\alpha,\beta,\gamma} C^{-1}(\sigma_{\alpha\beta\gamma};\mathbb{Z}) \tag{2.13}$$

The convention that 2-cochains have pieces of total degree 1 is admittedly confusing. Although in [9] the convention is that these degrees would match, the modern convention is to let the cochain degree match the de Rham degree of the corresponding curvature form. The reason to use this convention is that it is consistent with the ring structure we will describe in Chapter 6.

The condition to be *closed* is exactly analogous to Equations (2.5) - (2.6) above:

$$(\delta a)_{\alpha\beta} = d\varphi_{\alpha\beta} \tag{2.14}$$

$$(\delta\varphi)_{\alpha\beta\gamma} = -n_{\alpha\beta\gamma} \tag{2.15}$$

$$(\delta n)_{\alpha\beta\gamma\rho} = 0 \tag{2.16}$$

and *exact* cochains satisfy the analogous equations to (2.8) - (2.10) above:

$$a_\alpha = d\psi_\alpha \tag{2.17}$$

$$\varphi_{\alpha\beta} = (\delta\psi)_{\alpha\beta} - m_{\alpha\beta} \tag{2.18}$$

$$n_{\alpha\beta\gamma} = (\delta m)_{\alpha\beta\gamma} \tag{2.19}$$

for some $\{\psi_\alpha\} \in \prod_\alpha C^0(\sigma_\alpha;\mathbb{R})$ and $\{m_{\alpha\beta}\} \in \prod_{\alpha,\beta} C^{-1}(\sigma_{\alpha\beta};\mathbb{Z})$.

Each term in the double complex is an abelian group, so the abelian group structure on discrete Deligne cochains is apparent. Moreover d and δ are compatible with this structure and so we have

Lemma. *The sum or difference of exact cochains is exact. The sum or difference of closed cochains is closed.*

2.3 Discrete Deligne k-Cocycles

The theory just described in degree 2 is easily defined in degree k. The relevant double complex consists of the abelian groups $K^{p,q} = \prod_{\alpha_0,\dots,\alpha_p} C^q(\sigma_{\alpha_0\dots\alpha_p};\mathbb{R})$ (Čech degree p, geometric degree q) with horizontal differential δ and vertical differential d. Degree k means that we truncate the vertical columns at geometric degree $k-1$: $K^{p,q} = 0$ if $q > k-1$. As with the 2-cochains, we adopt the convention that a cochain with geometric degree -1 is a constant and that $d : C^{-1}(\sigma) \to C^0(\sigma)$ is the inclusion of constant cochains into the space of 0-cochains. The k-cochains are, as before, elements belonging to $\sum_{p+q=k-1} K^{p,q}$, and the total differential acts on elements of $K^{p,q}$ as $D = \delta + (-1)^p d$. The sign $(-1)^p$ is introduced to ensure $D^2 = 0$. A k-cochain has a piece in degree $(k, -1)$. A portion of this complex looks like:

$$\begin{tikzcd}
0 & \cdots & \cdots & \\
\prod_{\alpha_0} C^{k-1}(\sigma_{\alpha_0};\mathbb{R}) \arrow[u] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1} C^{k-1}(\sigma_{\alpha_0\alpha_1};\mathbb{R}) & \cdots & \\
\prod_{\alpha_0} C^{k-2}(\sigma_{\alpha_0};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1} C^{k-2}(\sigma_{\alpha_0\alpha_1};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1,\alpha_2} C^{k-2}(\sigma_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) & \\
\cdots & \prod_{\alpha_0,\alpha_1} C^{k-3}(\sigma_{\alpha_0\alpha_1};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \prod_{\alpha_0,\alpha_1,\alpha_2} C^{k-3}(\sigma_{\alpha_0\alpha_1\alpha_2};\mathbb{R}) \arrow[u, "d"] \arrow[r, "\delta"] & \cdots
\end{tikzcd} \tag{2.20}$$

The k-cochains reside on the middle diagonal (total degree $k-1$), the gauge transformations along the lower diagonal (total degree $k-2$). The top diagonal is where one checks the closure condition.

We used the notation $(a_\alpha, \varphi_{\alpha\beta}, n_{\alpha\beta\gamma})$ for the 2-cochains; in the case of k-cochains it would be cumbersome to assign each piece its own symbol and so we denote a generic k-cochain as $a = (a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \ldots a^{-1}_{\alpha_0\ldots\alpha_k})$. In this notation, the 2-cochains would be denoted as $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$.

Cocycles and coboundaries are determined by the total differential $D = \delta + (-1)^p d$ (where p is the Čech degree). Explicitly, the conditions are

Definition 2.3.1. A *discrete Deligne k-cocycle* is a k-cochain $a = (a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \dots a^{-1}_{\alpha_0\dots\alpha_k})$ satisfying

$$\begin{aligned}
(\delta a^{k-1})_{\alpha_0\alpha_1} &= da^{k-2}_{\alpha_0\alpha_1} \\
(\delta a^{k-2})_{\alpha_0\alpha_1\alpha_2} &= -da^{k-3}_{\alpha_0\alpha_1\alpha_2} \\
&\vdots \\
(\delta a^{k-j-1})_{\alpha_0\dots\alpha_{j+1}} &= (-1)^j da^{k-j-2}_{\alpha_0\dots\alpha_{j+1}} \\
&\vdots \\
(\delta a^{0})_{\alpha_0\dots\alpha_k} &= (-1)^{k-1} a^{-1}_{\alpha_0\dots\alpha_k} \\
(\delta a^{-1})_{\alpha_0\dots\alpha_{k+1}} &= 0
\end{aligned} \tag{2.21}$$

A k-cocycle a is a *discrete Deligne k-coboundary* if there exists a $(k-1)$-cochain $b = (b^{k-2}_{\alpha_0}, b^{k-3}_{\alpha_0\alpha_1}, \dots b^{-1}_{\alpha_0\dots\alpha_{k-1}})$ such that

$$\begin{aligned}
a^{k-1}_{\alpha_0} &= db^{k-2}_{\alpha_0} \\
a^{k-2}_{\alpha_0\alpha_1} &= (\delta b^{k-2})_{\alpha_0\alpha_1} - db^{k-3}_{\alpha_0\alpha_1} \\
&\vdots \\
a^{k-j-1}_{\alpha_0\dots\alpha_j} &= (\delta b^{k-j-1})_{\alpha_0\dots\alpha_j} + (-1)^j db^{k-j-2}_{\alpha_0\dots\alpha_j} \\
&\vdots \\
a^{-1}_{\alpha_0\dots\alpha_k} &= (\delta b^{-1})_{\alpha_0\dots\alpha_k}
\end{aligned} \tag{2.22}$$

Equations (2.21) and (2.22) are to be understood for all choices of indices α_i labeling a top-dimensional simplex of X and it is also implicit that we restrict cochains to a subset of their domain whenever necessary.

As one expects, a k-couboundary is always a k-cocycle. This is because $d\delta = \delta d$ and the sign in $D = \delta + (-1)^p d$ ensures that $D^2 = 0$. It is also evident that the sets of cocycles and coboundaries are abelian subgroups of the group of chains, and so we may define cohomology groups.

Definition 2.3.2. The *degree k discrete Deligne cohomology group* of a triangulation X is the quotient of the discrete Deligne k-cocycles by the discrete Deligne k-coboundaries. We denote it $H^k_{dD}(X)$.

A certain type of k-cocycle deserves special mention: given any global $(k-1)$-cochain $a^{k-1} \in C^{k-1}(X;\mathbb{R})$ we may form a discrete Deligne k-cocycle $(a^{k-1}_{\alpha}, 0, \dots, 0)$

by taking a^{k-1}_α to be the restriction of a^{k-1} to σ_α. Since the columns of the double complex are truncated beyond geometric degree $(k-1)$, this is a k-cocycle. It is interesting to ask when the class of this cocycle is trivial.

Proposition 2.3.3. *Let $C^k_{\mathbb{Z}}(X;\mathbb{R})$ denote the space of simplicial k-cochains on X that are closed and have integral periods over all simplicial k-cycles. Then for $a^k \in C^k_{\mathbb{Z}}(X;\mathbb{R})$ the induced discrete Deligne $(k+1)$-cocycle $(a^k_\alpha, 0, \ldots, 0)$ is exact.*

Proof. A "walk down the stairs" proof.

Because $da^k_\alpha = 0$ we may choose some b^{k-1}_α with $a^k_\alpha = db^{k-1}_\alpha$ for each top-dimensional simplex σ_α. Then we'll have

$$\delta db^{k-1} = \delta a^k = 0$$

so $d\delta b^{k-1} = 0$ and we may write $(\delta b^{k-1})_{\alpha_0\alpha_1} = db^{k-2}_{\alpha_0\alpha_1}$ for some b^{k-2}. It follows that $\delta b^{k-2} = 0$, and so one continues to decrease the geometric degree and increase the Čech degree until we arrive at $b^0_{\alpha_0\ldots\alpha_{k-1}}$ satisfying $d\delta b^0 = 0$. Thus $(\delta b^0)_{\alpha_0\ldots\alpha_k}$ is a constant 0-cochain. It is evidently δ-exact, but only within the full space of local 0-cochains; although δb^0 is locally constant, there is no reason it ought to equal δb^{-1} for some locally constant 0-cochain b^{-1}.

This is evocative of Čech cohomology with constant coefficients, a connection that shall be made rigorous in the next section. Proposition 2.4.1 of the next section[3] will imply that we may modify the above procedure to ensure that δb^0 be $\mathbb{Z}$-valued. This will then yield $(a^k_\alpha, 0, \ldots, 0) = D(b^{k-1}_{\alpha_0}, b^{k-2}_{\alpha_0\alpha_1}, \ldots, b^{-1}_{\alpha_0\ldots\alpha_k})$. □

The remainder of this thesis will be devoted to proving structural theorems concerning $H^k_{dD}(X)$, to relating $H^2_{dD}(X)$ to the discrete line bundles of Section 1.3, and to explaining the role of $H^4_{dD}(X)$ in formulating a discrete abelian Chern-Simons theory in the case where X triangulates a 3-manifold.

2.4 Relation to Simplicial Cohomology

The degree k discrete Deligne cohomology group is an analog of the differential cohomology group of a smooth manifold. It consists of bi-graded pieces, interwoven according to the closure relations (2.21). Some of these pieces have useful relations to the usual simplicial cohomology groups of the triangulation X.

[3] None of these results require the current proof.

The component $a^{k-1}_{\alpha_0}$, whose Čech degree is 0 and whose geometric degree is $(k-1)$, satisfies

$$\delta(da^{k-1})_{\alpha_0\alpha_1} = d(\delta a^{k-1})_{\alpha_0\alpha_1} = d^2 a^{k-2}_{\alpha_0\alpha_1} = 0$$

using the closure relation. Therefore the the local k-cochains da^{k-1}_{α} are in fact restrictions of a global k-cochain $\omega \in C^k(X;\mathbb{R})$ which satisfies $\omega|_{\sigma_\alpha} = da^{k-1}_{\alpha}$. Note that ω is closed but not necessarily exact. This cochain ω is unchanged by adding a k-coboundary to the original k-cocycle, and hence we may associate ω to the class $[a] \in H^k_{dD}(X)$. More on this in Section 3.2. In particular, there is a real simplicial k-cohomology class $[\omega]$ belonging to each discrete Deligne k-cocycle.

The component $a^{-1}_{\alpha_0\ldots\alpha_k}$ has Čech degree k and geometric degree -1 (which, by convention, means that it is locally constant). It is also $\mathbb{Z}$-valued, by definition. Its domain of definition is the k-fold intersection $\sigma_{\alpha_0} \cap \ldots \cap \sigma_{\alpha_k}$. Thinking of these top-dimensional simplices σ_α as analogues of open sets in a cover of X, we are reminded of the Čech cohomology of a manifold. The collection $a^{-1}_{\alpha_0\ldots\alpha_k}$ satisfies $\delta a^{-1} = 0$, suggesting that it ought to define something like a $\mathbb{Z}$-valued Čech k-cocycle. This can be made rigorous using a result of Borsuk:

Theorem ([6]). *If the simplicial complex K is a geometric realization of the nerve of a regular decomposition of a finite-dimensional space A then that space A and the polytope $|K|$ have the same homotopy type.*

We use the result as follows: the space A is taken to be $|X|$ (or, equivalently, the manifold M it triangulates); the decomposition is provided by its simplicial complex structure; the complex K is taken to be the *nerve* of X. The nerve $N(X)$ of X is a simplicial complex having a k-simplex for each non-empty k-fold intersection of simplices in X. By Borsuk's theorem, it has the same homotopy type as X (see [13] for further interesting results on $N(X)$).

Thus $a^{-1}_{\alpha_0\ldots\alpha_k}$ can be thought of as assigning an integer value to each k-simplex of $N(X)$; the alternating nature of $a^{-1}_{\alpha_0\ldots\alpha_k}$ is easily reinterpreted via orientations on the simplices of $N(X)$ and so we find that it corresponds precisely to an element of $C^k(N(X);\mathbb{Z})$. Moreover, under this correspondence δ corresponds to $d_{N(X)}$. Thus $a^{-1}_{\alpha_0\ldots\alpha_k}$ is re-interpreted as a closed, $\mathbb{Z}$-valued simplicial k-cochain on the nerve complex $N(X)$. Because $N(X)$ has the same homotopy type as X, we may therefore associate with $a^{-1}_{\alpha_0\ldots\alpha_k}$ a class $[a^{-1}] \in H^k(N(X);\mathbb{Z}) \cong H^k(X;\mathbb{Z})$. (We will not make explicit the isomorphism $H^k(N(X);\mathbb{Z}) \cong H^k(X;\mathbb{Z})$. However, we will develop in the next section a way of pairing $a^{-1}_{\alpha_0\ldots\alpha_k}$ with the simplicial k-cycles of X.) We summarize this discussion for later reference:

Proposition 2.4.1. *To a collection $a^{-1}_{\alpha_0\ldots\alpha_k} \in C^{-1}(\sigma_{\alpha_0\ldots\alpha_k}; A)$ of locally constant A-valued 0-cochains we may associate a simplicial k-cochain on the nerve complex $N(X)$. This correspondence is such that the Čech coboundary operator δ corresponds to the simplicial coboundary operator $d_{N(X)}$, and thus the two pieces of data are closed/exact together.*

The coefficient groups A we will use are $\mathbb{Z}$ and $U(1)$.

2.5 Discrete Deligne Cycles and Holonomy

Having described a theory of k-cochains, it is natural to ask what we shall integrate these over. Here we describe a notion of Deligne k-chains, as well as a boundary operator by which we decide which of these chains are cycles and which cycles are boundaries. Whereas our Definition 2.3.1 is original, the definition we are about to present for k-cycles has been in use for some time, having already found application in the smooth version of Deligne cohomology. It can be found in writings of Weil [22], then again later in [2] and [12]. Here we add only exposition to their definitions.

The Deligne k-chains, like their cochain counterparts, consist of bi-graded pieces having a geometric degree and a Čech degree. Like with the cochains, it is useful to use a degree convention in which the Deligne k-chains have total degree summing to $(k-1)$ as opposed to k. Each piece is a simplex σ^l of dimension $l \leq k-1$ together with Čech data that describe this piece as lying in an intersection of $k-l-1$ top-dimensional simplices. These data will be denoted as

$$\left[\sigma^l\,;\, \chi_0\chi_1\ldots\chi_{k-l-1}\right]$$

Here χ_i index top-dimensional simplices of X and must be chosen such that the geometric piece σ^l lies in the intersection of these top-simplices: $\sigma^l < \cap\sigma_{\chi_i}$. We declare this symbol to be alternating in its Čech indices. The general Deligne k-chain is a formal sum of such parts using integral coefficients:

$$C = \sum_{l=0}^{k-1} \sum_{\sigma^l\subset X} \sum_{\chi_0,\ldots\chi_{k-l-1}} c(\sigma^l\,;\, \chi_0\ldots\chi_{k-l-1}) \left[\sigma^l\,;\, \chi_0\ldots\chi_{k-l-1}\right] \tag{2.23}$$

In the next subsection we will explain how to interpret these data as an integration prescription. First we describe boundaries of these chains.

Unsurprisingly, the homological algebra of these chains comes from a double complex much like (2.11) that combines the simplicial boundary operator ∂ with the Čech differential δ. Here δ is the Čech differential that *decreases* Čech degree:

$$\delta\left[\sigma^l\,;\,\chi_0\chi_1\ldots\chi_{k-l-1}\right] = \sum_{j=0}^{k-l-1}(-1)^j\left[\sigma^l\,;\,\chi_0\ldots\check{\chi}_j\ldots\chi_{k-l-1}\right] \tag{2.24}$$

with $\check{\chi}_j$ denoting omission of the index χ_j. We will abuse notation and use the same symbols δ and D for the Čech differential and total differential on Deligne chains and cochains.

The total differential then becomes $D = \delta + (-1)^{\text{Čech degree}}\partial$, giving

$$\begin{aligned} D\left[\sigma^l\,;\,\chi_0\chi_1\ldots\chi_{k-l-1}\right] =& (-1)^{k-l-1}\sum_{\sigma^{l-1}<\sigma^l}\operatorname{sign}(\sigma^{l-1};\sigma^l)\left[\sigma^{l-1}\,;\,\chi_0\chi_1\ldots\chi_{k-l-1}\right] \\ &+\sum_{j=0}^{k-l-1}(-1)^j\left[\sigma^l\,;\,\chi_0\ldots\check{\chi}_j\ldots\chi_{k-l-1}\right] \end{aligned} \tag{2.25}$$

The sign on ∂ ensures that $D^2 = 0$. We therefore have a homology theory, with k-cycles and k-boundaries defined as the kernel and image of D.

Chain/Cochain Pairing

In each piece where the geometric and Čech degrees match, we may pair chains and cochains. For $\varphi_{\chi_0\cdots\chi_{k-l-1}} \in C^l(\sigma_{\chi_0\cdots\chi_{k-l-1}})$, part of a Deligne k-cochain, we define

$$\langle\varphi, \left[\sigma^l\,;\,\chi_{k-l-1}\ldots\chi_1\chi_0\right]\rangle = \varphi_{\chi_{k-l-1}\cdots\chi_0}(\sigma^l) \tag{2.26}$$

The data of the k-chain is an integration prescription: it gives the geometrical region σ^l of integration along with the Čech data $\chi_0\chi_1\ldots\chi_{k-l}$ that specifies which gauge potential to use for the integration.

This pairing is compatible with the total differentials D on discrete Deligne chains and cochains:

Lemma 2.5.1. *For a discrete Deligne* $(k-1)$*-cochain* φ *and a Deligne* k*–chain* C *we have*

$$\langle D\varphi, C\rangle = \langle\varphi, DC\rangle \tag{2.27}$$

Proof. Consider a piece of C concentrated in geometrical degree l and Čech degree $k-l-1$, denoted $\left[\sigma^l\,;\,\chi_{k-l-1}\ldots\chi_1\chi_0\right]$. The left-hand side of (2.27) is

$$\begin{aligned}
\langle D\varphi, \left[\sigma^l\,;\,\chi_{k-l-1}\ldots\chi_1\chi_0\right]\rangle &= \langle (-1)^{k-l-1} d\varphi^{l-1}_{\chi_{k-l-1}\cdots\chi_1\chi_0} + (\delta\varphi^l)_{\chi_{k-l-1}\cdots\chi_1\chi_0}, \left[\sigma^l\,;\,\chi_{k-l-1}\ldots\chi_1\chi_0\right]\rangle \\
&= (-1)^{k-l-1}\langle \varphi^{l-1}_{\chi_{k-l-1}\cdots\chi_1\chi_0}, \partial\sigma^l\rangle + \sum_{j=0}^{k-l-1}(-1)^j\varphi^l_{\chi_{k-l-1}\cdots\check{\chi}_j\cdots\chi_0}(\sigma^l) \\
&= \langle \varphi, D\left[\sigma^l\,;\,\chi_{k-l-1}\ldots\chi_1\chi_0\right]\rangle
\end{aligned}$$

□

Holonomy

Lemma 2.5.1 implies that the pairing on chains and cochains induces a pairing on homology and cohomology classes:

Corollary 2.5.2. *For a discrete Deligne k–cocycle φ and a Deligne k–cycle C, the pairing $\langle\varphi, C\rangle$ depends only on the (co)homology classes of φ and C.*

Proof. A direct consequence of (2.27). Adding to φ the exact piece $D\psi$ results in

$$\langle \varphi + D\psi, C\rangle = \langle\varphi, C\rangle + \langle\psi, DC\rangle$$

and adding to C the exact piece DB results in

$$\langle\varphi, C + DB\rangle = \langle\varphi, C\rangle + \langle D\varphi, B\rangle.$$

By hypothesis, $DC = 0$ and $D\varphi = 0$. □

Definition 2.5.3. The *holonomy* of a discrete Deligne k–cocycle φ over a Deligne k–cycle C is defined by exponentiating the pairing of φ and C. It is denoted

$$\mathrm{Hol}(\varphi, C) = \exp 2\pi i\langle\varphi, C\rangle \tag{2.28}$$

For $k = 2$, we will see (Theorem 4.2.4) that our definition coincides with the definition given in [17] for holonomy of paths in discrete $\mathbb{C}$-line bundles. This justifies the name "holonomy." It discretizes the higher holonomy formulas found by Alvarez and Gawedzki in the description of higher gauge theories such as WZW theories. We will not take up WZW theory in the present work, but our discussion of the discrete abelian Chern-Simons theory interprets the Chern-Simons integral as a higher holonomy term.

Constructing Cycles

We describe a prescription for constructing Deligne $(k+1)$-cycles from simplicial k-cycles.[4] This is a crucial step, as it allows us later to evaluate holonomies over simplicial cycles. The construction appears in [22, 2, 12].

Suppose $C = \sum_\alpha c(\sigma_\alpha^k)[\sigma_\alpha^k]$ is a simplicial k-cycle in X. The construction of a Deligne $(k+1)$-cycle associated to C starts by arbitrarily assigning to each simplex σ of X an index $\chi(\sigma) \in I$, where I is an index set for all top-dimensional simplices of X, such that σ is a facet of the top-dimensional simplex indexed by $\chi(\sigma)$. The function χ is to be thought of as choosing coordinate patches to which the various simplices of X as belong. We emphasize that these choices are arbitrary and other choices will affect our Deligne representation for C only by a boundary, thus leaving all holonomies unchanged.

Once the choices χ are fixed we represent C by letting the degree $(k,0)$ piece of the Deligne chain be $\sum_\alpha[\sigma_\alpha^k; \chi(\sigma_\alpha^k)]$. The degree $(k-l,l)$ piece of the chain will be

$$C^{(k-l,l)} = \sum_{\sigma^{k-l}<\sigma^{k-l+1}<\ldots<\sigma^{k-1}<\sigma^k\subset C} \mathrm{sign}(\sigma^{k-l};\sigma^{k-l}<\ldots<\sigma^k)\, c(\sigma^k)\, [\sigma^{k-l};\chi(\sigma^{k-l})\ldots\chi(\sigma^k)] \tag{2.29}$$

The sum is taken over all increasing chains of length l from a $(k-l)$-simplex contained in C up to a k-simplex of C. The dimensions involved constrain the containments to each be of codimension 1, and so at each step $\sigma^m < \sigma^{m+1}$ the orientation on σ^{m+1} induces a boundary orientation on σ^m; this orientation either agrees or disagrees with the orientations we arbitrarily assigned at the outset and each disagreement introduces a minus sign. This sign could therefore also be written as

$$\mathrm{sign}(\sigma^{k-l};\sigma^{k-l}<\ldots<\sigma^k) = \prod_{j=0}^{l-1}\mathrm{sign}(\sigma^{k-l+j};\sigma^{k-l+j+1})$$

The full Deligne $(k+1)$-cycle associated to C is then

$$C_D = \sum_{l=0}^{k} C^{(k-l,l)} \tag{2.30}$$

with $C^{(k-l,l)}$ defined by (2.29).

[4]Recall the naming convention that degree k Deligne cycles have pieces whose total degree is $(k-1)$.

Proposition 2.5.4. *The Deligne* $(k+1)$*-chain* C_D *defined via* (2.29) *and* (2.30) *is a cycle. It depends on the choices* χ*, but a different set of choices* $\tilde{\chi}$ *changes* C_D *by a boundary.*

Corollary 2.5.5. *The holonomy of a Deligne* $(k+1)$*–cocycle* φ *over a simplicial* k*-cycle* C *given as* $Hol(\varphi, C) = \langle \varphi, C_D \rangle$ *is well-defined.*

Proof of Proposition 2.5.4. The condition to be checked is that $\partial C^{(k-l,l)} = \delta C^{(k-l-1,l+1)}$. With our sign choices we see that the geometric boundary in question gives

$$\partial C^{(k-l,l)} = \sum_{\sigma^{k-l-1} < \ldots < \sigma^k \subset C} \mathrm{sign}(\sigma^{k-l-1}; \sigma^{k-l-1} < \ldots < \sigma^k)\, c(\sigma^k)\, [\sigma^{k-l}; \chi(\sigma^{k-l}) \ldots \chi(\sigma^k)]$$

This is to be compared to $\delta C^{k-l-1,l+1}$, which we write in short hand as

$$\delta C^{(k-l-1,l+1)} = \sum_{\sigma^{k-l-1} < \ldots < \sigma^k \subset C} \sum_{j=0}^{l+1} (-1)^j \mathrm{sign}(\ldots) c(\sigma^k) [\sigma^{k-l}; \chi_0 \ldots \check{\chi}_j \ldots \chi_{l+1}]$$

For each of the chains $\sigma^{k-l-1} < \ldots < \sigma^k \subset C$ we sum over, the corresponding term in $\partial C^{(k-l,l)}$ matches the $j = 0$ term in $\delta C^{(k-l-1,l+1)}$. So we are done if we show that the terms

$$\sum_{\sigma^{k-l-1} < \ldots < \sigma^k \subset C} \sum_{j=1}^{l+1} (-1)^j \mathrm{sign}(\ldots) c(\sigma^k) [\sigma^{k-l}; \chi_0 \ldots \check{\chi}_j \ldots \chi_{l+1}]$$

cancel amongst themselves. The key observation is that for $\sigma^j < \sigma^{j+2}$ a codimension 2 facet of a simplex σ^{j+2} there exist exactly two facets σ_1^{j+1} and σ_2^{j+1} with $\sigma^j < \sigma_i^{j+1} < \sigma^{j+2}$ and the orientation-induced signs $\mathrm{sign}(\sigma^j; \sigma^j < \sigma_i^{j+1} < \sigma^{j+2})$ are opposite. This "codimension 2 cancellation" results in the pairwise cancellation of all the above terms.[5] Therefore the chain C_D is indeed a cycle.

Consider now the effect of changing the chosen indices $\chi(\sigma)$. It suffices to change a single index $\chi(\sigma)$ to $\tilde{\chi}(\sigma)$. We'll suppose $\sigma = \sigma^j$, $0 \leq j \leq k$. Call the resulting Deligne k-cycle $\tilde{C}_D$ and consider $C_D - \tilde{C}_D$. It is the boundary of

$$\sum_{l=0}^{k} \sum_{\sigma^{k-l} < \ldots < \sigma^j < \ldots < \sigma^k} \mathrm{sign}(\ldots) c(\sigma^k) (-1)^j [\sigma^{k-l}; \chi(\sigma^{k-l}) \ldots \tilde{\chi}(\sigma^j) \chi(\sigma^j) \ldots \chi(\sigma^k)]$$

[5]This codimension 2 cancellation is one of the reasons why our theory must be defined on a simplicial complex, as opposed to more general cell complexes.

(the sign being as in (2.29)). This can be seen by careful inspection of

$$\begin{aligned}
\delta(-1)^j[\sigma^{k-l};\chi(\sigma^{k-l})\dots\tilde{\chi}(\sigma^j)\chi(\sigma^j)\dots\chi(\sigma^k)] =& \\
&(-1)^j[\sigma^{k-l};\chi(\sigma^{k-l+1})\dots\tilde{\chi}(\sigma^j)\chi(\sigma^j)\dots\chi(\sigma^k)]\\
&\vdots\\
&+[\sigma^{k-l};\chi(\sigma^{k-l+1})\dots\chi(\sigma^j)\dots\chi(\sigma^k)]\\
&-[\sigma^{k-l};\chi(\sigma^{k-l+1})\dots\tilde{\chi}(\sigma^j)\dots\chi(\sigma^k)]\\
&\vdots\\
&(-1)^j[\sigma^{k-l};\chi(\sigma^{k-l+1})\dots\tilde{\chi}(\sigma^j)\chi(\sigma^j)\dots\chi(\sigma^{k-1})]
\end{aligned}$$

The first of these terms cancels with ∂ applied to a piece in degree $(k-l+1, l)$; the middle two terms produce $C_D - \tilde{C}_D$; all other terms cancel pairwise via the same codimension 2 cancellation we encountered above. □

These holonomy calculations are to be thought of as analogous to the holonomy of a connection on a bundle over a cycle. In degree 2, this is precisely what we are discretizing. In higher degrees, these are discretizing the higher holonomies over $(k-1)$-cycles. The holonomies can be used to detect non-triviality of a Deligne cohomology class.

Theorem 2.5.6. *If a degree k Deligne cocycle $a = (a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \dots, a^{-1}_{\alpha_0\alpha_1\alpha_k})$ has $da^{k-1}_\alpha = 0$ for all α and has only trivial holonomies $\langle a, C\rangle = 1 \in U(1)$ over all simplicial $(k-1)$-cycles C, then $[a] = 0 \in H^k_{dD}(X)$.*

Proof. We will be explicit only in degree $k = 2$; the general case is not more difficult, but the essential idea of the proof can already be seen clearly in degree 2.

Since $da^1_\alpha = 0$ on each n-simplex σ_α it has a local 0-potential b^0_α with $a^1_\alpha = db^0_\alpha$. The closure relation implies that

$$(\delta db^0)_{\alpha\beta} = (\delta a^1)_{\alpha\beta} = da^0_{\alpha\beta}$$

and so $d(a^0 - \delta b^0) = 0$. That is, the cochain $b^{-1}_{\alpha\beta} := a^0_{\alpha\beta} - \delta b^0_{\alpha\beta}$ is constant. Moreover, it has $\delta b^{-1} = \delta a^0 - \delta^2 b^0 = a^{-1}$. Note that this does not quite imply that a^{-1} is exact because there is no reason why b^{-1} ought to be integer-valued. We will show that an integer-valued version of b^{-1} can be chosen.

The reason is that the degree 2 cocycle $(0, a^0_{\alpha_0\alpha_1} - (\delta b^0)_{\alpha_0\alpha_1}, (\delta b^{-1})_{\alpha_0\alpha_1\alpha_2})$ has the same holonomies as a, implying that its holonomies are trivial. To see that the two cocycles have the same holonomies, consider their difference:

$$\begin{aligned}(a^1_\alpha, a^0_{\alpha\beta}, a^{-1}_{\alpha\beta\gamma}) - (0, a^0_{\alpha\beta} - (\delta b^0)_{\alpha\beta}, (\delta b^{-1})_{\alpha\beta\gamma}) &= (a^1_\alpha, (\delta b^0)_{\alpha\beta}, 0)\\ &= (db^0_\alpha, (\delta b^0)_{\alpha\beta}, 0)\\ &= D(b^0_\alpha, 0)\end{aligned}$$

That is, their difference is exact. So by Lemma 2.5.1 we see that their holonomies are equal.

Refer back to Definition 2.5.3: the holonomy is defined as the mod $\mathbb{Z}$ value of the pairing; the fact that it is trivial tells us that the values of the pairing must all be integer-valued. So $a^0_{\alpha\beta} - \delta b^0_{\alpha\beta}$ has integer pairing with all cycles and can therefore be modified by an exact term to be integer-valued itself. This then makes a^{-1} exact, proving that the class $[a]$ is trivial in $H^2_{dD}(X)$. □

2.6 Approximation in the Continuum Limit

In the next chapter we will demonstrate algebraic similarities between the smooth and discrete versions of Deligne cohomology. Before doing so we briefly explore the analytic relation between the smooth and discrete theories. We outline here an argument that smooth classes can be well-approximated by discrete classes on triangulations with small mesh size. We will be explicit in the degree 2 case, though our arguments extend in an obvious way to higher degree.

We consider a closed Riemannian manifold (M, g). The metric introduces two useful notions of size: mesh of a triangulation and norm of a k-form. If X triangulates M then each simplex σ of X is identified with a subset of M; this subset is compact and therefore has a finite diameter. We define the *mesh* of X to be the maximum diameter of a simplex σ. The metric induces norms on the space $\Lambda^k T^*_p M$ over each point of M; by integration this induces a norm $\|\omega\|^2_{L^2} = \int_M \|\omega(p)\|^2 d\mathrm{Vol}$ on k-forms. We let $L^2(\Lambda^k T^* M)$ denote the completion of $\Omega^k(M; \mathbb{R})$ with respect to this metric. It contains the image of the Whitney map (see Section 6.1), since these forms are smooth except on the $(n-1)$-skeleton of X.

Dodziuk proved in [11] that smooth k-forms can be approximated arbitrarily well by k-cochains as follows: the de Rham map $R : \Omega^k(M) \to C^k(X)$ integrates a k-form over each k-simplex of X to form a k-cochain. Then $W : C^k(X) \to L^2(\Lambda^k T^* M)$ produces a piecewise-smooth k-form whose L^2-norm is close to the original k-form.

The approximation can be made arbitrarily good using the standard subdivisions S_nX. The mesh of S_nX approaches 0 as n grows (see [11, 23]). Let R_n and W_n denote the de Rham and Whitney maps with respect to the n^{th} standard subdivision S_nX. The precise approximation result is

Theorem ([11])**.** *Let f be a smooth k-form on M. There exists a constant C_f independent of n such that $\|f(p) - W_nR_nf(p)\|_p \leq C_f\mathrm{mesh}(S_nX)$ almost everywhere on X.*

To apply this to our situation we first fix a cover $\mathcal{U} = \{U_\alpha\}$ of M and take some smooth Deligne 2-cocycle representative $(A_\alpha, \Phi_{\alpha\beta}, N_{\alpha\beta\gamma})$ of a line bundle with connection. Note that since M is compact we may assume that the open cover $\mathcal{U}$ is finite. We then triangulate M in a way that is compatible with this cover. Specifically, starting from any triangulation X of M, we perform sufficiently many standard subdivisions S_nX to have mesh(S_nX) less than the Lebesgue number of $\mathcal{U}$. This implies that every simplex of S_nX is contained wholly within at least one of the open sets U_α of the open cover. Assign each top-dimensional simplex σ^m and index α such that $\sigma^m \subset U_\alpha$ and define a discrete Deligne 2-cocycle using the de Rham map $R_n : \Omega^k(M) \to C^k(S_nX)$:

- Over each top-dimensional simplex σ^m_α we let $a^1_\alpha = R_nA_\alpha$.
- Over each intersection $\sigma_{\alpha\beta}$ of top-dimensional simplices we let $\varphi_{\alpha\beta} = R_n\Phi_{\alpha\beta}$.
- Over each two-fold intersection $\sigma_{\alpha\beta\gamma}$ we let $n_{\alpha\beta\gamma} = R_nN_{\alpha\beta\gamma}$. Since $N_{\alpha\beta\gamma}$ is locally constant this is the same as saying $n_{\alpha\beta\gamma} = N_{\alpha\beta\gamma}$.

Then because the de Rham map is a chain map ($Rd_{\text{exterior}} = d_{\text{simplicial}}R$) the closure relations of (A, Φ, N) are also satisfied by (a, φ, n).

In what sense does (a, φ, n) approximate (A, Φ, N)? Focus for a moment on one of the local 1-forms A_α over U_α. By Dodziuk's approximation result we can apply the Whitney map $W_n : C^k(S_nX) \to L^2(\Lambda^kT^*M)$ and the result will be close to the original local data A_α. There is a constant C_{A_α} in the statement of the approximation theorem, which we can deal with using the compactness of M. Assuming the cover $\mathcal{U}$ to be finite, we let C be the maximum of all constants C_{A_α}, $C_{\Phi_{\alpha\beta}}$ for all open sets and intersections. Then when one wants to have $\|W_nR_nA_\alpha - A_\alpha\|_{L^2} < \epsilon$ it suffices to choose n large enough that mesh$(S_nX) < \epsilon/C$.

So we may approximate the local data (A, Φ, N) arbitrarily well by cochains on a sufficiently fine triangulation S_nX of M. Still there is a difficulty related to the choices $\sigma < U_\alpha$ we made to decide which open set of $\mathcal{U}$ to regard a given simplex σ as sitting within. The issue is that although we would like to say that each A_α may be recovered to within an error of ϵ from its discrete approximations, there are some simplices $\sigma \subset U_\alpha \cap U_\beta$ which we may have regarded as lying in U_β instead of U_α. So the values of A_α over this simplex are missing because the discrete approximation is keeping track of the values of A_β instead.

The way out is to refine the cover $\mathcal{U}$ and observe that the data of smooth Deligne 2-cocycles form a direct system with respect to refinement of covers. That is, we replace $\mathcal{U}$ with $\mathcal{U}_n$, an open cover whose sets $U_\delta(\sigma)$ are δ-expansions of a top-dimensional simplex σ of S_nX. For small enough δ, each $U_\delta(\sigma)$ lies within the U_α of which we chose to regard σ as a subset. Over $U_\delta(\sigma)$ we take the connection 1-form to be the restriction A_α; over an intersection of two $U_\delta(\sigma_1)$ and $U_\delta(\sigma_2)$ we take the gauge changing function $\tilde{\Phi}_{\alpha\beta}$ to be either the restriction of some $\Phi_{\alpha\beta}$ or 0, as appropriate. The refined integer-valued Čech 2-cocycle $\tilde{N}_{\alpha\beta\gamma}$ is then defined by the $\tilde{\Phi}_{\alpha\beta}$ and is a refined version of the original $N_{\alpha\beta\gamma}$.

This refinement $(\tilde{A}_\alpha, \tilde{\Phi}_{\alpha\beta}, \tilde{N}_{\alpha\beta\gamma})$ represents the same class in $\hat{H}^2(M)$ as (A, Φ, N) and is well-approximated by the discrete Deligne 2-cocycle (a, φ, n) in the sense that $||W_n a_\alpha - \tilde{A}_\alpha||_{L^2} < \epsilon$ and $||W_n \varphi_{\alpha\beta} - \tilde{\Phi}_{\alpha\beta}||_{L^2} < \epsilon$ on each top-dimensional simplex σ_α of S_nX. These arguments prove the following theorem.

Theorem 2.6.1. *Fix a finite open cover $\mathcal{U}$ of M, a triangulation X of M, and a representative $(A_\alpha, \Phi_{\alpha\beta}, N_{\alpha\beta\gamma})$ of a Deligne 2-cohomology class on M relative to $\mathcal{U}$. Then given $\epsilon > 0$, for sufficiently large n the n^{th} standard subdivision S_nX carries a discrete Deligne 2-cocycle (a, φ, n) that approximates the class of (A, Φ, N) in the sense that for a refinement $(\tilde{A}, \tilde{\Phi}, \tilde{N})$ to a finer cover $\mathcal{U}_n$ the relations $||W_n a_\alpha - \tilde{A}_\alpha||_{L^2} < \epsilon$ and $||W_n \varphi_{\alpha\beta} - \tilde{\Phi}_{\alpha\beta}||_{L^2} < \epsilon$ hold almost everywhere on each open subset of $\mathcal{U}_n$. The components of this discrete Deligne cocycle are obtained from the smooth cocycle via the de Rham map.*

It is evident that this prescription applies more generally to k-cocycles.

In [11], Dodziuk's proof of the approximation theorem shows that the constant C_f depends only on the magnitudes of first derivatives of the components of the k-form f in a coordinate system. Therefore we could loosen the hypotheses of Theorem 2.6.1 to say that the ϵ-approximation can be achieved on S_nX for all

Deligne 2-cohomology classes admitting a representative with the first derivatives of all components bounded by a fixed constant C.

Chapter 3

SHORT EXACT SEQUENCES

Here we prove two essential structural results concerning H^k_{dD}, which take the form of short exact sequences. They are the discrete analogues of two basic results, namely that the differential cohomology groups of a manifold M fit into the exact sequences

$$0 \to \Omega^{k-1}(M)/\Omega^{k-1}_{\mathbb{Z}}(M) \to \hat{H}^k(M) \to H^k(M;\mathbb{Z}) \to 0$$

$$0 \to H^{k-1}(M;U(1)) \to \hat{H}^k(M) \to \Omega^k_{\mathbb{Z}}(M) \to 0$$

These results were first proved in [9]. The first sequence says that each degree k Deligne class lies over some integral degree k class in singular cohomology; two classes lying over the same integral k class differ by a globally defined smooth $(k-1)$-form that is only unique up to a closed $(k-1)$-form with integral periods. In the case $k = 2$, the classes of $\hat{H}^2(M)$ correspond to isomorphism classes of $U(1)$-bundle on M with connection and the map $\hat{H}^2(M) \to H^2(M;\mathbb{Z})$ is the Chern class morphism. Two $U(1)$-bundles with the same Chern class are topologically equivalent, and their connections differ by a globally defined 1-form, defined up to a closed 1-form with integral periods.

The second sequence offers a different point of view. In the degree 2 case the morphism $\hat{H}^2(M) \to \Omega^2_{\mathbb{Z}}(M)$ is the map from a bundle with connection to its curvature 2-form. It is well known that this curvature form is closed with integer periods. The statement that its kernel is $H^1(M;U(1))$ means that a line bundle with connection is determined by its curvature and its $U(1)$-valued periods over the 1-cycles in M.

Each sequence has a discrete analogue obtained by replacing $\hat{H}^k$ by H^k_{dD} and $\Omega^k(M)$ by $C^k(X)$.

Theorem 3.0.1. *The degree k discrete Deligne cohomology fits in an exact sequence*

$$0 \to C^{k-1}(X;\mathbb{R})/C^{k-1}_{\mathbb{Z}}(X;\mathbb{R}) \to H^k_{dD}(X) \to H^k(X;\mathbb{Z}) \to 0 \tag{3.1}$$

Theorem 3.0.2. *The degree k discrete Deligne cohomology fits in an exact sequence*

$$0 \to H^{k-1}(X;U(1)) \to H^k_{dD}(X) \to C^k_{\mathbb{Z}}(X;\mathbb{R}) \to 0 \tag{3.2}$$

3.1 First Sequence

We'll first prove that the degree k discrete Deligne cohomology fits into the sequence

$$0 \to C^{k-1}(X;\mathbb{R})/C_{\mathbb{Z}}^{k-1}(X;\mathbb{R}) \to H_{dD}^{k}(X) \to H^{k}(X;\mathbb{Z}) \to 0$$

The first map is the inclusion map. The second map will be a "Chern class" map (in the degree 2 case it will indeed give the Chern class of a discrete complex line bundle with connection). This Chern class map is obtained using the relation to simplicial cohomology of Proposition 2.4.1.

Definition 3.1.1. The Chern class of a degree k discrete Deligne cocycle $a = (a_{\alpha_0}^{k-1}, \ldots a_{\alpha_0\ldots\alpha_k}^{-1})$ is the class in simplicial cohomology with $\mathbb{Z}$-coefficients obtained by associating to $a_{\alpha_0\ldots\alpha_k}^{-1}$ the corresponding simplicial k-cocycle on the nerve complex $N(X)$ and using the isomorphism between the simplicial cohomologies of X and $N(X)$.

The existence of our first exact sequence (3.1) will now follow by a "walking up the stairs" argument that is familiar from the usual Čech–de Rham complex [7]. We prove Theorem 3.0.1 in two parts.

Proposition 3.1.2. *The Chern class morphism $H_{dD}^{k}(X) \to H^{k}(X;\mathbb{Z})$ is surjective.*

Proof. By Proposition 2.4.1, to a class in $H^{k}(X;\mathbb{Z})$ we may associate a collection $a^{-1} \in \prod_{\alpha_0\ldots\alpha_k} C^{-1}(\sigma_{\alpha_0\ldots\alpha_k};\mathbb{Z})$. The question is whether this collection a^{-1} is the degree $(k,-1)$ piece of a discrete Deligne k-cocycle on X. This question is akin to asking whether an arbitrary smooth $\mathbb{C}$-line bundle admits a connection.

Our goal is to walk up the following staircase:

$$\begin{tikzcd}
C^{k-1}(\sigma_{\alpha_0};\mathbb{R}) & & & \\
{} \arrow[u] \arrow[r] & \ddots & & \\
& \ddots & & \\
& {} \arrow[u] \arrow[r] & C^{0}(\sigma_{\alpha_0\ldots\alpha_{k-1}};\mathbb{R}) & \\
& & {} \arrow[u] \arrow[r] & C^{-1}(\sigma_{\alpha_0\ldots\alpha_k};\mathbb{Z})
\end{tikzcd}$$

Taking a step up the staircase means taking data $a^l_{\alpha_0\ldots\alpha_{k-l-1}} \in C^l(\sigma_{\alpha_0\ldots\alpha_{k-l-1}};\mathbb{R})$ and producing $a^{l+1}_{\alpha_0\ldots\alpha_{k-l-2}} \in C^{l+1}(\sigma_{\alpha_0\ldots\alpha_{k-l-2}};\mathbb{R})$ satisfying

$$(\delta a^{l+1})_{\alpha_0\ldots\alpha_{k-l-1}} = (-1)^{k-l} da^l_{\alpha_0\ldots\alpha_{k-l-1}},$$

a relation that is to be understood as equality in $C^{l+1}(\sigma_{\alpha_0\ldots\alpha_{k-l-1}};\mathbb{R})$ for each set of indices $\alpha_0,\ldots,\alpha_{k-l-1}$.

Because the object we are producing has gauge transformations we expect the construction to be non-unique; therefore it is no surprise that we start by making choices $\chi(\sigma)$ that index a top-dimensional simplex $\sigma^n_{\chi(\sigma)} > \sigma$ for every simplex σ of X. We will need to demonstrate that different choices $\tilde{\chi}$ produce the same class in $H^k_{dD}(X)$.

Using the choices χ we define a^{l+1} on any $(l+1)$-simplex $\sigma < \sigma_{\alpha_0\ldots\alpha_{k-l-2}}$ by

$$a^{l+1}_{\alpha_0\ldots\alpha_{k-l-2}}(\sigma) = da^l_{\alpha_0\ldots\alpha_{k-l-2}\chi(\sigma)}(\sigma) \tag{3.3}$$

That is, we use the choice $\chi(\sigma)$ to supply the extra index needed to bring the Čech degree to $k-l-1$ and use the data that have already been defined. Observe that $a^l_{\alpha_0\ldots\alpha_{k-l-2}\chi(\sigma)}$ is always well-defined because we assumed that $\sigma < \sigma_{\alpha_0\ldots\alpha_{k-l-2}}$ and also that $\sigma < \sigma_{\chi(\sigma)}$, so we are certain that $\sigma < \sigma_{\alpha_0\ldots\alpha_{k-l-2}\chi(\sigma)}$. The closure relation is satisfied because

$$\begin{aligned}
(\delta a^{l+1})_{\alpha_0\ldots\alpha_{k-l-1}}(\sigma) &= \sum_{j=0}^{k-l-1}(-1)^j a^{l+1}_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_{k-l-1}}(\sigma)\\
&= \sum_{j=0}^{k-l-1}(-1)^j da^l_{\alpha_0\ldots\hat{\alpha}_j\ldots\alpha_{k-l-1}\chi(\sigma)}(\sigma)\\
&= (\delta da^l)_{\alpha_0\ldots\alpha_{k-l-1}\chi(\sigma)}(\sigma) + (-1)^{k-l-2} da^l_{\alpha_0\ldots\alpha_{k-l-1}}(\sigma)\\
&= (-1)^{k-l} da^l_{\alpha_0\ldots\alpha_{k-l-1}}(\sigma)
\end{aligned}$$

In the last line we use the fact that a^l itself satisfies $\delta a^l = da^{l-1}$, and so the term $\delta da^l = d\delta a^l = d^2 a^{l-1} = 0$. (Recall that at the bottom step $l = 0$ we define da^{-1} to be the inclusion of constant 0-cochains into the space of 0-cochains, so it remains true that $\delta da^0 = 0$.)

This produces a discrete Deligne cocycle $a = (a^{k-1}_{\alpha_0}\ldots a^{-1}_{\alpha_0\ldots\alpha_k})$ whose associated Chern class is the desired element of $H^k(X;\mathbb{Z})$. To make this well-defined at the level of discrete Deligne cohomology classes we must demonstrate that a different choice of indices $\tilde{\chi}$ will change the resulting cocycle by a coboundary. To be explicit about these choices we will denote the cocycle we just constructed a^{χ}, and we will show that $a^{\tilde{\chi}} - a^{\chi}$ is exact.

It suffices to suppose that $\tilde{\chi}$ differs from χ only on a single simplex σ^j of dimension j. The choice $\tilde{\chi}(\sigma^j)$ enters the above construction only when a collection $\alpha_0, \ldots \alpha_{k-j}$ of indices has $\sigma^j < \sigma_{\alpha_0 \ldots \alpha_{k-j}}$. When this happens we are interested in the difference

$$a^{j,\tilde{\chi}}_{\alpha_0 \ldots \alpha_{k-j-1}}(\sigma^j) - a^{j,\chi}_{\alpha_0 \ldots \alpha_{k-j-1}}(\sigma^j) = d\left(a^{j-1}_{\alpha_0 \ldots \alpha_{k-j-1}\tilde{\chi}(\sigma^j)} - a^{j-1}_{\alpha_0 \ldots \alpha_{k-j-1}\chi(\sigma^j)}\right)(\sigma^j)$$

Note that on the right-hand side there is no need for superscripts $\tilde{\chi}, \chi$ denoting choices because we assumed that $\tilde{\chi}$ differs from χ only on σ^j. The form of the right-hand side suggests that we define

$$b^{j-1}_{\alpha_0 \ldots \alpha_{k-j-1}}(\sigma^j) = (-1)^{k-j-1}\left(a^{j-1}_{\alpha_0 \ldots \alpha_{k-j-1}\tilde{\chi}(\sigma^j)} - a^{j-1}_{\alpha_0 \ldots \alpha_{k-j-1}\chi(\sigma^j)}\right)(\sigma^j)$$

When the total differential $D = \delta + (-1)^{\text{Čech degree}} d$ is applied to this b^{j-1} it will produce the difference $a^{j,\tilde{\chi}} - a^{j,\chi}$. However it will also produce a term δb^{j-1}:

$$\begin{aligned}
(-1)^{k-j-1}(\delta b^{j-1})_{\alpha_0 \ldots \alpha_{k-j}} &= \sum_{i=0}^{k-j}(-1)^i\left(a^{j-1}_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{k-j}\tilde{\chi}(\sigma^j)} - a^{j-1}_{\alpha_0 \ldots \hat{\alpha}_i \ldots \alpha_{k-j}\chi(\sigma^j)}\right)(\sigma^j) \\
&= (\delta a^{j-1})_{\alpha_0 \ldots \alpha_{k-j}\tilde{\chi}(\sigma^j)} - (-1)^{k-j} a^{j-1}_{\alpha_0 \ldots \alpha_{k-j}} \quad \ldots \\
&\quad - (\delta a^{j-1})_{\alpha_0 \ldots \alpha_{k-j}\chi(\sigma^j)} + (-1)^{k-j} a^{j-1}_{\alpha_0 \ldots \alpha_{k-j}} \\
&= (-1)^{k-j}\left(da^{j-2}_{\alpha_0 \ldots \alpha_{k-j}\tilde{\chi}(\sigma^j)} - da^{j-2}_{\alpha_0 \ldots \alpha_{k-j}\chi(\sigma^j)}\right)(\sigma^j) \\
&= (-1)^{k-j-1} db^{j-2}_{\alpha_0 \ldots \alpha_{k-j}}(\sigma^j)
\end{aligned}$$

provided that we define

$$b^{j-2}_{\alpha_0 \ldots \alpha_{k-j}}(\sigma^j) = (-1)^{k-j-1}\left(a^{j-2}_{\alpha_0 \ldots \alpha_{k-j}\tilde{\chi}(\sigma^j)} - a^{j-2}_{\alpha_0 \ldots \alpha_{k-j}\chi(\sigma^j)}\right)(\sigma^j).$$

It is clear now that we may continue down the staircase defining terms b^{j-l} as needed to realize the difference $a^{\tilde{\chi}} - a^{\chi}$ as a coboundary.

□

Lemma 3.1.3. *The δ-complex is exact in each* non-negative *degree.*

Proof. Suppose that $a^j_{\alpha_0 \ldots \alpha_{k-1}}$ has $(\delta a^j)_{\alpha_0 \ldots \alpha_k} = 0$. We make choices $\chi(\sigma)$ of indices for a top-dimensional simplex $\sigma^n_{\chi(\sigma)} > \sigma$ for every simplex σ of X. These choices allow us to define

$$b^j_{\alpha_0 \ldots \alpha_{k-1}}(\sigma) = (-1)^k a^j_{\alpha_0 \ldots \alpha_{k-1}\chi(\sigma)}(\sigma) \tag{3.4}$$

for a j-simplex $\sigma < \sigma_{\alpha_0\ldots\alpha_{k-1}}$. Then we have

$$\begin{aligned}(\delta b^j)_{\alpha_0\ldots\alpha_k}(\sigma) &= (-1)^k\sum_{i=0}^{k}(-1)^i a^j_{\alpha_0\ldots\hat{\alpha}_i\ldots\alpha_k\chi(\sigma)}(\sigma)\\ &= (-1)^k(\delta a^j)_{\alpha_0\ldots\alpha_k\chi(\sigma)}(\sigma) + a^j_{\alpha_0\ldots\alpha_k}(\sigma)\\ &= a^j_{\alpha_0\ldots\alpha_k}(\sigma)\end{aligned}$$

We note here that this result is only for non-negative degree $j \geq 0$. This proof would not work for $j = -1$, given our convention that $a^{-1}_{\alpha_0\ldots\alpha_k}$ denotes a locally constant cochain. □

Proposition 3.1.4. *The kernel of the Chern class morphism can be identified with $C^{k-1}(X;\mathbb{R})/C^{k-1}_{\mathbb{Z}}(X;\mathbb{R})$, with $C^{k-1}_{\mathbb{Z}}$ denoting closed $(k-1)$-cochains with integral periods.*

Proof. Suppose that $a = (a^{k-1}_{\alpha_0}, \ldots, a^{-1}_{\alpha_0,\ldots\alpha_k})$ is a discrete Deligne k-cocycle whose Chern class $[a^{-1}] = 0 \in H^k(X;\mathbb{Z})$. This means that there exists a degree $(k-1,-1)$ cochain $b^{-1}_{\alpha_0\ldots\alpha_{k-1}}$ with $\delta b^{-1} = a^{-1}$. By the closure relation for a we have $(\delta a^0)_{\alpha_0\ldots\alpha_k} = a^{-1}_{\alpha_0\ldots\alpha_k}$ and therefore

$$\delta(a^0 - b^{-1}) = 0.$$

By Lemma 3.1.3 this implies that $(a^0 - b^{-1})_{\alpha_0\ldots\alpha_{k-1}} = (\delta b^0)_{\alpha_0\ldots\alpha_{k-1}}$ for some degree $(k-2,0)$ cochain b^0.

Now we walk up the stairs: using again the closure relation and the fact that b^{-1} is a locally constant cochain we have

$$d\delta b^0 = d(a^0 - b^{-1}) = da^0 = (-1)^{k-1}\delta a^1,$$

implying that $\delta(db^0 - (-1)^{k-1}a^1) = 0$. By the exactness of the δ-complex we therefore have $db^0 - (-1)^{k-1}a^1 = \delta b^1$ for some degree $(k-3,1)$ cochain b^1. This same argument continues up the staircase until finally we write $\delta(a^{k-1} - db^{k-2})_{\alpha_0\alpha_1} = 0$. This implies that the collection of $(k-1)$-cochains $(a^{k-1} - db^{k-2})_{\alpha_0}$ can be glued to form a global $(k-1)$-cochain, which we will denote $a - db \in C^{k-1}(X;\mathbb{R})$.

Thus far we have seen that a cocycle $(a^{k-1}_{\alpha_0}, \ldots a^{-1}_{\alpha_0\ldots\alpha_k})$ whose Chern class is 0 is cohomologous to a cocycle of the form $((a-db)|_{\sigma_\alpha}, 0, \ldots, 0)$. That suggests that the kernel of the Chern class morphism would be something like $C^{k-1}(X;\mathbb{R})$, however we must recall that the some of these global $(k-1)$-cochains are gauge equivalent, *i.e.* induce the same class in $H^k_{dD}(X)$. As we saw in Proposition 2.3.3, gauge equivalence reduces the kernel from $C^{k-1}(X;\mathbb{R})$ to $C^{k-1}(X;\mathbb{R})/C^{k-1}_{\mathbb{Z}}(X;\mathbb{R})$. □

3.2 Second Sequence

The first sequence we demonstrated shows that $H^k_{dD}(X)$ fibers over the group $H^k(X;\mathbb{Z})$ via a Chern class map; the second sequence will show that it fibers also over the group $C^k_{\mathbb{Z}}(X;\mathbb{R})$ via a curvature map. The sequence is

$$0 \to H^{k-1}(X;U(1)) \to H^k_{dD}(X) \to C^k_{\mathbb{Z}}(X;\mathbb{R}) \to 0$$

Whereas for the first map we took the piece of $a = (a^{k-1}_{\alpha_0}, \dots, a^{-1}_{\alpha_0\dots\alpha_k})$ on the "bottom step," namely a^{-1}, the curvature map uses the piece on the "top step": $a^{k-1}_{\alpha_0}$. The key observation is that because of the closure condition we have

$$(\delta da^{k-1})_{\alpha_0\alpha_1} = d(\delta a^{k-1})_{\alpha_0\alpha_1} = d^2 a^{k-2}_{\alpha_0\alpha_1} = 0$$

and therefore the k-cochains da^{k-1}_{α} can be glued to form a global k-cochain that we will denote as $da^{k-1} \in C^k(X;\mathbb{R})$. We will refer to this cochain as the "curvature" of a.

Lemma 3.2.1. *The curvature da^{k-1} has integral periods.*

Proof. Consider the exact Deligne $(k+1)$-cocycle that we obtain by applying the total differential $D = \delta + (-1)^{\text{Čech degree}} d$ to $a = (a^{k-1}_{\alpha_0}, \dots, a^{-1}_{\alpha_0\dots\alpha_k})$. The closure relation on a implies that all components of Da are zero except its top component $da^{k-1}_{\alpha_0}$. When we compute holonomies of Da over simplicial k-cycles these are all equal to $1 \in U(1)$, since Da is exact (Lemma 2.5.1). On the other hand, since Da is the $(k+1)$-cocycle corresponding to the global k-cochain da^{k-1}, the holonomy over a cycle C is $\exp 2\pi i \langle da^{k-1}, C\rangle$. Therefore $\langle da^{k-1}, C\rangle \in \mathbb{Z}$. □

Therefore the map $a \mapsto da^{k-1}$ is an abelian group homomorphism $H^k_{dD}(X) \to C^k_{\mathbb{Z}}(X;\mathbb{R})$. We will call it the curvature map, since in the degree 2 case it corresponds to the curvature of a connection on a discrete line bundle.

Proposition 3.2.2. *The curvature map is surjective.*

Proof. This will be a "walking down the stairs" proof. It starts with a closed k-cochain $\omega \in C^k_{\mathbb{Z}}(X;\mathbb{R})$ with integer periods. Because ω is closed, its restriction to a top-dimensional simplex σ_α is exact; therefore we may choose a collection $a^{k-1}_\alpha \in C^{k-1}(\sigma_\alpha;\mathbb{R})$ with $da^{k-1}_\alpha = \omega$ on σ_α. These satisfy

$$d\delta a^{k-1} = \delta da^{k-1} = 0$$

and therefore $(\delta a^{k-1})_{\alpha_0\alpha_1} = da^{k-2}_{\alpha_0\alpha_1}$ on $\sigma_{\alpha_0\alpha_1}$ for some collection of $(k-2)$-cochains $a^{k-2}_{\alpha_0\alpha_1}$. We continue step-by-step until we reach $a^0_{\alpha_0\ldots\alpha_{k-1}}$ and see that $d(\delta a^0)_{\alpha_0\ldots\alpha_k} = 0$. This implies that $(\delta a^0)_{\alpha_0\ldots\alpha_k}$ is a constant 0-cochain. However for a Deligne cocycle we require that its locally constant piece a^{-1} be *integer*-valued. There is no reason why (δa^0) as we have just constructed it should be $\mathbb{Z}$-valued, but we will show that it is possible to choose it to be so.

For the moment we consider the k-cochain $a = (a^{k-1}_{\alpha_0}, \ldots a^0_{\alpha_0\ldots\alpha_{k-1}}, 0)$. Note that this is not a k-cocycle, as we have used 0 in its $(k,-1)$ degree piece. Apply the total differential to obtain $Da = (da^{k-1}_{\alpha_0}, 0, \ldots, 0, (\delta a^0)_{\alpha_0\ldots\alpha_k}, 0)$. This is an exact $(k+1)$-cocycle, and so its holonomies over the simplicial k-cycles of X are all equal to $1 \in U(1)$. However, because most components of Da are 0 and because the local cochains da^{k-1}_α glue to give our original $\omega \in C^k_{\mathbb{Z}}$ we find that the holonomies of Da consist of two terms: the pairing of ω with the cycle and the pairing of (δa^0) with the cycle. Because ω was assumed to have integral periods and the overall holonomies are trivial, we conclude that (δa^0) must also have integral periods.

Because δa^0 has integral periods, it belongs to the same cohomology class as some integer valued $\tilde{a}^{-1}_{\alpha_0\ldots\alpha_k}$; that is,

$$\tilde{a}^{-1}_{\alpha_0\ldots\alpha_k} - (\delta a^0)_{\alpha_0\ldots\alpha_k} = (\delta b^{-1})_{\alpha_0\ldots\alpha_k}$$

The locally constant cochains $b^{-1}_{\alpha_0\ldots\alpha_{k-1}}$ are $\mathbb{R}$-valued; when we modify the terms $a^0_{\alpha_0\ldots\alpha_{k-1}}$ that we found above by the values $b^{-1}_{\alpha_0\ldots\alpha_{k-1}}$ then we do not change the relation $(\delta a^1) = (-1)^k da^0$, but we do manage to make $a^{-1} = (\delta a^0)$ integer-valued, as desired.

□

Proposition 3.2.3. *The kernel of the curvature morphism can be identified with $H^{k-1}(X; U(1))$, the simplicial cohomology of X with $U(1)$-coefficients.*

Proof. A "walk down the stairs" proof.

Suppose that $a = (a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \ldots, a^{-1}_{\alpha_0\ldots\alpha_k})$ has $da^{k-1}_\alpha = 0$ for all α. This implies that a^{k-1}_α is an exact $(k-1)$-cochain on σ_α, and so we may choose b^{k-2}_α satisfying $a^{k-1}_\alpha = db^{k-2}_\alpha$ for each α. By the closure relation we have

$$(\delta db^{k-2})_{\alpha_0\alpha_1} = (\delta a^{k-1})_{\alpha_0\alpha_1} = da^{k-2}_{\alpha_0\alpha_1},$$

which implies that $d(\delta b^{k-2} - a^{k-2}) = 0$. Therefore we may choose $b^{k-3}_{\alpha_0\alpha_1}$ satisfying $\delta b^{k-2} - a^{k-2} = db^{k-3}$. We continue down the stairs, at each step observing that

$\delta b^j \pm a^j$ is exact and choosing b^{j-1} to realize that exactness. This continues until the bottom step, where one has chosen $b^0_{\alpha_0 \ldots \alpha_{k-2}}$ that satisfies

$$d(\delta b^0 \pm a^0) = 0.$$

So defining $b^{-1}_{\alpha_0 \ldots \alpha_{k-1}} = (\delta b^0 \pm a^0)_{\alpha_0 \ldots \alpha_{k-1}}$ we have a locally constant $\mathbb{R}$-valued 0-cochain on intersections $\sigma_{\alpha_0 \ldots \alpha_{k-1}}$. It is not δ-closed, but rather satisfies $\delta b^{-1} = \pm \delta a^0 = \pm a^{-1}$. Recalling that a^{-1} is $\mathbb{Z}$-valued we see that the $\mathbb{R}/\mathbb{Z}$-valued cochain $\tilde{b}^{-1}$ obtained from b^{-1} is δ-closed, and therefore it corresponds to an $\mathbb{R}/\mathbb{Z}$-valued Čech cocycle via Proposition 2.4.1.

The above steps are not unique; each b^j could have been modified by any closed cochain of matching degree without spoiling the relations used. Therefore the resulting Čech cocycle is defined only up to a Čech coboundary, and in this way the original discrete Deligne k-cocycle a is identified with the class $[\tilde{b}^{-1}]$ in $H^{k-1}(X; U(1))$. □

Chapter 4

EQUIVALENCE OF DISCRETIZATIONS IN DEGREE 2

The degree 2 discrete Deligne cohomology discretizes the notion of $\mathbb{C}$-line bundles with connection in a way that can describe topologically non-trivial bundles. We described the topology of these bundles precisely in Chapter 3. As we explained in Section 1.3 these were quite adequately discretized in [17]. Thus we are obliged to demonstrate that our definition is no worse than theirs; this chapter is devoted to constructing an isomorphism between the abelian group $\mathcal{L}_X^{\mathbb{C}}$ of [17] and our group $\hat{H}^2_{dD}(X)$ of degree 2 discrete Deligne cohomology classes.

The proof will be broken up in order to carefully define the two halves of the isomorphism. In the first section we produce a map from Knöppel and Pinkall's bundles to our discrete Deligne classes. One might call it the "geometric to algebraic" half of the isomorphism. This is section will not surprise a reader who feels comfortable describing a line bundle with connection locally by 1-forms and transition functions relative to an open cover. The other "algebraic to geometric" side is more interesting because one must decide how to reconcile the competing local descriptions of the connection to assign a single isometry to each edge. The difficulty is resolved by a spanning tree argument that we borrow from [17]; this argument is quite unlike the smooth version of "algebraic to geometric" and we recommend that it be read carefully.

4.1 Geometric to Algebraic: $\mathcal{L}_X^{\mathbb{C}} \to H^2_{dD}(X)$

We begin with the data that represent an isomorphism class of line bundle with connection and curvature on X (see Section 1.3). Thus we have 1-dimensional Hermitian vector spaces L_v for each vertex $v \in X$ and isometries $\eta_e : L_{s(e)} \to L_{d(e)}$ for each oriented edge e of X. These are compatible with some curvature 2-cochain $\Omega \in C^2(X;\mathbb{R})$ that satisfies $\exp 2\pi i\Omega = d\eta$. This compatibility relation implies that Ω is closed with integral periods.

Choose some local unit sections: in each n-simplex $\sigma_\alpha^{(n)}$ of X let $\varphi_\alpha(v) \in L_v$ with $\|\varphi_\alpha(v)\| = 1$ for each vertex v of σ_α. Now for $e < \sigma_\alpha$, since η_e is an isometry, the norms of $\eta_e\varphi_\alpha(s(e))$ and $\varphi_\alpha(d(e))$ are both 1; therefore they differ by a $U(1)$-element.

We may choose $a^1_\alpha \in C^1(\sigma_\alpha; \mathbb{R})$ to satisfy

$$\eta_e \varphi_\alpha(s(e)) = \exp\left(2\pi i a^1_\alpha(e)\right) \varphi_\alpha(d(e))$$

for each edge. Note that from this follows the relationship

$$\exp\left(2\pi i \Omega(f)\right) \varphi_\alpha(v) = d\eta(f)\varphi_\alpha(v) = \exp(2\pi i da^1_\alpha(f))\varphi_\alpha(v)$$

This implies that the difference $da^1_\alpha(f) - \Omega(f)$ is integer-valued. It is clearly closed, and since we are restricting our attention to a simplex σ_α it is therefore exact: $da^1_\alpha(f) - \Omega(f) = db^1_\alpha(f)$ for some integer-valued $b^1_\alpha \in C^1(\sigma_\alpha; \mathbb{Z})$. Subtracting b^1_α from a^1_α does not ruin $\eta_e \varphi_\alpha(s(e)) = \exp\left(2\pi i a^1_\alpha(e)\right) \varphi_\alpha(d(e))$ and results in $da^1_\alpha = \Omega$.

Assuming that a^1_α was chosen as above on each n-simplex σ_α, we turn our attention to the intersections $\sigma_{\alpha_0\alpha_1}$ of two n-simplices. Here we compare the two local sections $\varphi_{\alpha_0}(v)$ and $\varphi_{\alpha_1}(v)$ at each vertex $v \in \sigma_{\alpha_0\alpha_1}$. Since they are both unit length they differ by a $U(1)$ element that we represent as

$$\varphi_{\alpha_0}(v) = \exp\left(2\pi i a^0_{\alpha_0\alpha_1}(v)\right) \varphi_{\alpha_1}(v)$$

Then we have two ways of expressing the connection on edges $e < \sigma_{\alpha_0\alpha_1}$, and they yield

$$\begin{aligned}
\eta_e \varphi_{\alpha_0}(s(e)) &= \eta_e e^{2\pi i a^0_{\alpha_0\alpha_1}(s(e))} \varphi_{\alpha_1}(s(e)) \\
&= \exp 2\pi i \left[a^0_{\alpha_0\alpha_1}(s(e)) + a^1_{\alpha_1}(e)\right] \varphi_{\alpha_1}(d(e)) \\
&= \exp 2\pi i \left[a^0_{\alpha_0\alpha_1}(s(e)) + a^1_{\alpha_1}(e) - a^0_{\alpha_0\alpha_1}(d(e))\right] \varphi_{\alpha_0}(d(e)) \\
&= \exp 2\pi i \left[a^1_{\alpha_1}(e) - da^0_{\alpha_0\alpha_1}(e)\right] \varphi_{\alpha_0}(d(e))
\end{aligned}$$

On the other hand this must equal $\exp 2\pi i a^1_{\alpha_0}(e)\varphi_{\alpha_0)}(d(e))$, which implies that the difference $(\delta a^1)_{\alpha_0\alpha_1}(e) - da^0_{\alpha_0\alpha_1}(e)$ is an integer. Treated as a 1-cochain, $\delta a^1 - da^0$ is closed because $d\delta a^1 = \delta d a^1 = \delta\Omega = 0$. Since we are restricting our attention to $\sigma_{\alpha_0\alpha_1}$, which is a simplex, this makes $\delta a^1 - da^0$ exact: it equals $db^0_{\alpha_0\alpha_1}$, for some integer-valued b^0. We may subtract b^0 from a^0 without ruining the relationship $\varphi_{\alpha_0}(v) = \exp\left(2\pi i a^0_{\alpha_0\alpha_1}(v)\right) \varphi_{\alpha_1}(v)$, and so we will assume that we have done so. Making this choice we have $\delta a^1 - da^0 = 0$.

Where three simplices intersect we have three distinct local sections, which for

$v \in \sigma_{\alpha_0\alpha_1\alpha_2}$ implies that

$$\begin{aligned}\varphi_{\alpha_0}(v) &= \exp\left(2\pi i a^0_{\alpha_0\alpha_1}(v)\right)\varphi_{\alpha_1}(v)\\ &= \exp 2\pi i\left(a^0_{\alpha_0\alpha_1}(v) + a^0_{\alpha_1\alpha_2}(v)\right)\varphi_{\alpha_2}(v)\\ &= \exp 2\pi i\left(a^0_{\alpha_0\alpha_1}(v) + a^0_{\alpha_1\alpha_2}(v) + a^0_{\alpha_2\alpha_0}(v)\right)\varphi_{\alpha_0}(v)\end{aligned}$$

and therefore that $(\delta a^0)_{\alpha_0\alpha_1\alpha_2}$ is integer-valued. Moreover, it must be constant, since

$$\begin{aligned}d(\delta a^0) &= \delta d a^0\\ &= \delta^2 a^1\\ &= 0\end{aligned}$$

So we may define $a^{-1}_{\alpha_0\alpha_1\alpha_2}(v) = (\delta a^0)_{\alpha_0\alpha_1\alpha_2}(v)$ and this will be the final piece of data in our discrete Deligne 2-cocycle.

Proposition 4.1.1. *The above assignment $(L, \eta, \Omega) \mapsto (a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ maps discrete line bundles with connection and curvature to discrete Deligne 2-cocycles in such a way that isomorphic bundles are mapped to cohomologous cocycles. That is, it induces a well-defined map $\mathcal{L}^{\mathbb{C}}_X \to H^2_{dD}(X)$. Moreover, this map is a morphism of abelian groups.*

Proof. First note that in our construction of $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ we ensured that it satisfied the cocycle conditions. We made choices that need to be examined. In our definition of a^1_α there remains an ambiguity: the conditions $da^1_\alpha = \Omega$ and the compatibility with η_e only define a^1_α up to an exact, integer-valued 1-cochain db^0_α. Similarly, our conditions on $a^0_{\alpha_0\alpha_1}$ only define it up to a constant, integer-valued 0-cochain $b^{-1}_{\alpha_0\alpha_1}$. These two ambiguities are of the form $D(b^0_\alpha, b^{-1}_{\alpha_0\alpha_1})$, i.e. they define an exact 2-cocycle, and so any valid choices will lead to 2-cocycles belonging to the same class $[a] \in H^2_{dD}(X)$.

Suppose that we carry out the construction using a bundle $(\tilde{L}, \tilde{\eta}, \tilde{\Omega})$ that is isomorphic to (L, η, Ω) in the sense defined in Section 1.3. This will involve choosing local sections $\tilde{\varphi}_\alpha$, and the isometry ψ from fibers of $\tilde{L}$ to fibers of L will result in us having two local sections $\psi(\tilde{\varphi}_\alpha(v))$ and $\varphi_\alpha(v)$ over each n-simplex σ_α. Given an edge $e < \sigma_\alpha$ we have the commuting diagram

$$\begin{array}{ccc} \tilde{L}_{s(e)} & \xrightarrow{\psi_{s(e)}} & L_{s(e)} \\ \Big\downarrow \tilde{\eta}_e & & \Big\downarrow \eta_e \\ \tilde{L}_{d(e)} & \xrightarrow{\psi_{d(e)}} & L_{d(e)} \end{array}$$

Defining b^0_α by the relation $\exp(2\pi i b^0_\alpha(v))\varphi_\alpha(v) = \psi_v \tilde{\varphi}_\alpha(v)$ and following the section $\tilde{\varphi}_\alpha(s(e))$ along either side of the diagram to $L_{d(e)}$ yields the relation

$$\exp 2\pi i \left[a^1_\alpha(e) + b^0_\alpha(s(e))\right] = \exp 2\pi i \left[\tilde{a}^1_\alpha(e) + b^0_\alpha(d(e))\right]$$

which implies that $\tilde{a}^1_\alpha(e) - a^1_\alpha(e) = db^0_\alpha(e)$, at least up to an integer. This integer can be dealt with as before, by observing that $d(\tilde{a}^1_\alpha - a^1_\alpha) = 0$ (since $da^1_\alpha = \Omega = \tilde{\Omega} = d\tilde{a}^1_\alpha$) and redefining b^0 by integer values to have $\tilde{a}^1_\alpha - a^1_\alpha = db^0_\alpha$. It is then easily seen that the data $\tilde{a}^0_{\alpha_0\alpha_1}$ and $a^0_{\alpha_0\alpha_1}$ differ by δb^0 up to a constant, integer-valued 0-cochain. This yields a $b^{-1}_{\alpha_0\alpha_1}$ with $\tilde{a}^0 - a^0 = b^{-1}$, and so we have shown that $\tilde{a} - a = Db$. So the two isomorphic bundles give rise to cohomologous classes in $H^2_{dD}(X)$. Thus we have a well-defined map $\mathcal{L}^{\mathbb{C}}_X \to H^2_{dD}(X)$. It remains to show that it is a homomorphism.

The group operation in $\mathcal{L}^{\mathbb{C}}_X$ is the tensor product, and so we consider $(L, \eta, \Omega) \otimes (\tilde{L}, \tilde{\eta}, \tilde{\Omega})$. The local sections φ_α, $\tilde{\varphi}_\alpha$ used in constructing 2-cocycles $a, \tilde{a}$ yield local sections $(\varphi_\alpha \otimes \tilde{\varphi}_\alpha)_v \in L_v \otimes \tilde{L}_v$ and relative to these the connection acts as

$$(\eta \otimes \tilde{\eta})(e)(\varphi_\alpha \otimes \tilde{\varphi}_\alpha)(s(e)) = \exp 2\pi i \left[a^1_\alpha(e) + \tilde{a}^1_\alpha(e)\right] (\varphi_\alpha \otimes \tilde{\varphi}_\alpha)(d(e))$$

Moreover, transitions between the local sections are performed by

$$(\varphi_\alpha \otimes \tilde{\varphi}_\alpha)(v) = \exp 2\pi i \left[a^0_{\alpha\beta}(v) + \tilde{a}^0_{\alpha\beta}(v)\right] (\varphi_\beta \otimes \tilde{\varphi}_\beta)(v)$$

for $v \in \sigma_{\alpha\beta}$. This confirms that the 2-cocycle corresponding to the tensor product of discrete line bundles with connection is simply the sum of 2-cocycles corresponding to each individual bundle. This relation descends to isomorphism classes and cohomology classes and so we have a group homomorphism. □

4.2 Algebraic to Geometric: $H^2_{dD}(X) \to \mathcal{L}^{\mathbb{C}}_X$

The other half of the isomorphism is a recipe for translating the algebraic data of a discrete Deligne 2-cocycle into a discrete line bundle with connection and curvature. The idea is that every isomorphism class of discrete line bundle with connection has a standard model whose fibers are copies of $\mathbb{C}$ and whose curvature and connection must be specified by the holonomies of a Deligne 2-cocycle. These holonomies are extracted from the cocycle as described in Section 2.5. The difficulty one encounters is this: holonomies are $U(1)$ elements associated to *cycles*, but the data (L, η, Ω) requires that we associate a $U(1)$ element to *each edge*. Phrased another way, the difficulty is in resolving the fact that the Deligne cocycle represents $\eta(e)$ in multiple different ways: as $\exp 2\pi i a^1_\alpha(e)$ for each index α such that $e < \sigma_\alpha$. How are we to choose a single $U(1)$ element for e from among the many gauge-equivalent expressions a^1_α?

The solution is to take advantage of the fact that a finite, connected graph has a spanning tree. This was exploited as well by Knöppel and Pinkall [17], whose techniques we imitate in this section. Let $\mathcal{T}$ be such a spanning tree in X. Observe that each oriented edge $e \notin \mathcal{T}$ defines a cycle c_e by following the paths in $\mathcal{T}$ from $d(e)$ to the root, then from the root to $s(e)$.

The line bundle with connection is defined as follows: let each fiber L_v be a copy of $\mathbb{C}$ and for each edge $e \in \mathcal{T}$ let $\eta_e = \mathrm{Id}_{\mathbb{C}}$ be the identity map on $\mathbb{C}$. For $e \notin \mathcal{T}$ we let $\eta_e = \mathrm{Hol}(a; c_e) \in U(1)$, viewing $U(1)$ as the isometry group of $\mathbb{C}$. For the curvature 2-cochain it is clear that we must take $\Omega = da^1$. This is easily seen to be compatible with η because for any 2-simplex σ^2 the holonomy around $\partial\sigma^2$ may be calculated entirely in the gauge defined by one n-simplex $\sigma_\alpha > \sigma^2$. In this gauge that holonomy is $a^1_\alpha(\partial\sigma^2) = da^1(\sigma^2)$, showing that Ω and η are compatible on each 2-simplex, and thus are compatible overall.

Proposition 4.2.1. *The above assignment* $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2}) \mapsto (L, \eta, \Omega)$ *maps discrete Deligne 2-cocycles to discrete line bundles with connection and curvature in such a way that cohomologous 2-cocycles are mapped to isomorphic bundles. That is, it induces a well-defined map* $H^2_{dD}(X) \to \mathcal{L}^{\mathbb{C}}_X$*. Moreover, this map is a morphism of abelian groups.*

Proof. We will first argue that the assignment is additive, then we will show that trivial 2-cocycles are mapped to trivial line bundles. It will follow that cohomologous cycles are mapped to isomorphic bundles.

The additivity of the map follows from the fact that $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C}$ and that the holonomy as defined in Section 2.5 is a group homomorphism.

Suppose that the 2-cocycle a is trivial. Then by Lemma 2.5.1 its holonomy around all cycles is trivial, and so the connection η is the trivial connection and the curvature zero. Thus we obtain the trivial bundle of Definition 1.3.3. $\square$

Lemma 4.2.2. *Under the assignment* $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2}) \mapsto (L, \eta, \Omega)$ *the holonomies of the connection* η *around all 1-cycles in* X *agree with the holonomies computed from* $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$.

Proof. Recall that a spanning tree determines a *fundamental cycle basis:* a collection of cycles in X that span the cycle space of X. The fundamental cycles defined by $\mathcal{T}$ are precisely the cycles c_e for $e \notin \mathcal{T}$ that we used above. The connection η was engineered to have the same holonomies as the Deligne 2-cocycle over these fundamental cycles. Since the holonomy mapping of Section 2.5 is a group homomorphism, the connection produced using the tree $\mathcal{T}$ agrees with $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ on the holonomy it assigns to C. □

Corollary 4.2.3. *The discrete line bundles with connection and curvature produced from* $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ *using two different spanning trees* $\mathcal{T}$ *and* $\mathcal{T}'$ *are isomorphic.*

Proof. In [17] it is shown that two discrete line bundles with connection and curvature whose holonomies agree on all cycles and whose curvature cochains are equal are isomorphic. By Lemma 4.2.2 the two connections η, η' arising from trees $\mathcal{T}, \mathcal{T}'$ have the same holonomies as $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ on any cycle. Both have curvature da^1, so by the result of [17] they are isomorphic. □

Remark. The result of [17] referenced in the above proof is the geometric analogue of our Theorem 2.5.6. Both results can be summarized as saying that curvature and holonomies uniquely determine an isomorphism class, either in $\mathcal{L}^{\mathbb{C}}_X$ or in $H^2_{dD}(X)$.

Theorem 4.2.4. *The morphisms of Propositions 4.1.1 and 4.2.1 are inverse to each other. Therefore the group of isomorphism classes of discrete line bundles with connection and curvature is isomorphic to the degree 2 discrete Deligne cohomology group.*

Proof. We use the fact referenced above: the isomorphism class of a discrete line bundle with connection is uniquely determined by its holonomies around closed cycles. So if we begin with (L, η, Ω) representing a class in $\mathcal{L}^{\mathbb{C}}_X$, construct from it the discrete 2-cocycle $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$, and then construct from this cocycle a new bundle $(\tilde{L}, \tilde{\eta}, \tilde{\Omega})$, then it suffices to show that the holonomies of the two bundles agree on all cycles to show that the bundles are isomorphic. Lemma 4.2.2 reduces this to showing that the holonomies of $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ are those of (L, η, Ω).

To that end let $\gamma = e_0 e_1 \dots e_n$ be a cycle in X and choose a gauge for each edge e_i as we did in Chapter 2, meaning that we choose indices χ_i for n-simplices $\sigma_{\chi_i} > e_i$ for each $i = 0, 1, \dots n$. The construction of the cocycle a from the line bundle used local sections φ_α on each n-simplex σ_α, and the holonomy of the connection η around γ can be computed by comparing $\eta(e_n)\eta(e_{n-1})\dots\eta(e_0)\varphi_{\chi_0}(s(e_0))$ to $\varphi_{\chi_0}(s(e_0))$. That is, we parallel transport $\varphi_{\chi_0}(s(e_0))$ around γ using the connection η and compare. The result may also be expressed in terms of the local 1-cochains a^1_α and gauge transformations $a^0_{\alpha_0\alpha_1}$: it reproduces precisely the expression of Section 2.5 for the holonomy of a around γ. Explicitly,

$$\mathrm{Hol}(a, \gamma) = \exp 2\pi i \left(\sum_{i=0}^{n} a^1_{\chi_i}(e_i) + \sum_{i=0}^{n} a^0_{\chi_i \chi_{i+1}}(d(e_i)) \right) \tag{4.1}$$

So the holonomies of (L, η, Ω) and its associated 2-cocycle $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$ agree.

Now when we construct $(\tilde{L}, \tilde{\eta}, \tilde{\Omega})$ from $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$, it will have the same holonomies and curvature as the bundle (L, η, Ω) we started from. So the two are isomorphic. □

4.3 The Topology of Discrete Line Bundles

We now summarize the ways in which discrete line bundles with connection interact with the topology of the simplicial complex on which they are defined. All cohomology groups in this section should be understood as simplicial cohomology. Let us frame the discussion in terms of the decomposition of $H^2(X;\mathbb{Z})$ into its free and torsion parts:

$$H^2(X;\mathbb{Z}) \cong H^2_{\mathrm{free}}(X;\mathbb{Z}) \oplus H^2_{\mathrm{tor}}(X;\mathbb{Z})$$

Each bundle has a class $c(L)$ in $H^2(X;\mathbb{Z})$ that we referred to as its Chern class, by analogy with the Chern classes of smooth complex vector bundles. This is the class of Theorem 3.0.1.

The free part of $c(L)$ is easily read off: it is the class represented by the curvature 2-cochain Ω (or da^1 in terms of Deligne cocycles). Though Ω is a *real* 2-cochain, its periods are integral and so it represents a class that is in the image of $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$ under the coefficient morphism. That is, the real image of the Chern class $\overline{c(L)} \in H^2(X;\mathbb{R})$ is represented by the curvature 2-cochain; this is an analogue of the corresponding result for smooth $\mathbb{C}$-line bundles. The curvature of the connection determines the free part of the Chern class.

However, the curvature does not determine the torsion part of the Chern class. The kernel of the morphism $H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R})$ is precisely $H^2_{\mathrm{tor}}(X;\mathbb{Z})$. Two well-known results from algebraic topology help us to understand this torsion group. The first is the long exact sequence induced by $\mathbb{Z} \hookrightarrow \mathbb{R} \to U(1)$, which reads

$$\ldots \to H^1(X;\mathbb{Z}) \to H^1(X;\mathbb{R}) \to H^1(X;U(1)) \to H^2(X;\mathbb{Z}) \to H^2(X;\mathbb{R}) \to \ldots$$

It tells us that the torsion part of the Chern class can be understood in terms of $H^1(X;U(1))$ via the Bockstein homomorphism. The relevant part of the above long exact sequence is

$$0 \to \frac{H^1(X;\mathbb{R})}{H^1(X;\mathbb{Z})} \to H^1(X;U(1)) \to H^2_{\text{tor}}(X;\mathbb{Z}) \to 0$$

So the torsion part of the Chern class may be thought of in terms of a $U(1)$-valued cocycle that is determined only up to the image of $H^1(X;\mathbb{R})$ in $H^1(X;U(1))$. To think more geometrically about this, it is useful to look at the universal coefficient sequence:

$$0 \to \text{tor}(H_1(X;\mathbb{Z})) \to H^2(X;\mathbb{Z}) \to \text{Hom}(H_2(X),\mathbb{Z}) \to 0$$

The upshot is that $H^2_{\text{tor}}(X;\mathbb{Z}) \cong \text{tor}(H_1(X;\mathbb{Z}))$: the torsion in $H^2(X;\mathbb{Z})$ comes from torsion in $H_1(X;\mathbb{Z})$. So this interpretation of the torsion part of the Chern class in terms of $H^1(X;U(1))$–classes boils down to looking at the connection's holonomy over torsion 1–cycles in the complex.

In summary, the Chern class is specified by:

- The periods of the curvature 2-cochain over all of the 2-cycles.
- The holonomies of the connection over the torsion 1-cycles.

The Chern class does not tell us the holonomies of the connection over 1-cycles with a non-zero free component and it specifies the curvature 2-cochain only up to its cohomology class, meaning that it leaves an ambiguity that takes the form of an exact 2-cochain.

So discrete line bundles with curvature feel the topology of X in the following ways:

- A bundle may "twist" non-trivially around the 2-cycles of X. Should this occur, the bundle will not admit a flat connection, because the curvature 2-cochain will detect this twisting. (The bundle will not twist around torsion 2-cycles.)

- A bundle may twist non-trivially around the torsion 1-cycles of X. This twisting will not be detected by the curvature; a flat connection may exist despite such twisting. This will, however, be detected by the connection's holonomy.

- The connection may also have non-zero holonomy around the freely-generated 1-cycles of X. This does not require the bundle to be topologically non-trivial; it may occur even if the bundle's Chern class is trivial. A flat connection can still have non-zero holonomies around these free 1-cycles.

None of these statements are surprising, considering that the analogous statements about smooth $\mathbb{C}$-line bundles with connection are well-known. They ought to be viewed merely as confirmations that these discrete line bundles do indeed discretize their smooth counterparts. We should emphasize that all of these statements can be found in or deduced from the results proven in [17]; we have provided new proofs in the Deligne framework. In doing so, we obtained the generalizations of these results to arbitrary degree k.

Chapter 5

EXAMPLES

In this chapter we show some explicit examples of discrete Deligne classes, together with the corresponding discrete line bundles with connection. We give two-dimensional examples that illustrate all the relevant topological features of discrete line bundles: free 2-cohomology classes on S^2, torsion 2-cohomology on $\mathbb{R}P^2$, and free 1-cohomology classes on T^2.

We will use the notation $(a_\alpha, \varphi_{\alpha\beta}, n_{\alpha\beta\gamma})$ to denote a generic Deligne 2-cocycle. It means the same thing as $(a^1_{\alpha_0}, a^0_{\alpha_0\alpha_1}, a^{-1}_{\alpha_0\alpha_1\alpha_2})$.

5.1 The Sphere S^2

We will triangulate S^2 as the boundary of a single 3-simplex. We label the vertices a, b, c, d and number the faces $1, 2, 3, 4$ and orient the edges and faces as in Figure 5.1.

Over S^2 we expect to find examples of bundles with any desired Chern class $c(L) \in H^2(S^2; \mathbb{Z}) \cong \mathbb{Z}$. In Figure 5.2 we illustrate a Deligne 2-cocycle whose Chern class is a generator of $H^2(S^2; \mathbb{Z})$. This cocycle has all its 1-cochains $a^1_\alpha(e)$ taking the value $1/2$ except that $a_4([cd]) = -1/2$. The edge $[cd]$ is shared by faces σ_3 and σ_4, and the fact that $a_3([cd]) \neq a_4([cd])$ indicates that these faces must belong to different gauges. The gauge change is effected by letting $\varphi_{34}(c) = 0$ and $\varphi_{34}(d) = 1$. Since $[cd]$ is the only edge where a gauge change is needed we may let all other $\varphi_{\alpha\beta}(v) = 0$. Therefore the only non-zero $n_{\alpha\beta\gamma}$ term is $n_{234}(d) = -1$. It is easily checked that these data define a discrete Deligne 2-cocycle.

The Chern class of this 2-cocycle can be checked in two different ways. Since we know that $H^2(S; \mathbb{Z})$ has no torsion, it suffices to look at the curvature 2-cochain da. This 2-cochain has the values $da(\sigma_1) = da(\sigma_2) = da(\sigma_3) = -1/2$ and $da(\sigma_4) = 1/2$. Therefore $da(S^2) = -1$, and so the Chern class is a generator of $H^2(S^2; \mathbb{Z})$. Alternatively, we can look at the Čech 2-cochain corresponding to $n_{\alpha\beta\gamma}$. It can be seen to be non-trivial by pairing with the 0-skeleton of the triangulation. We will carefully demonstrate this calculation.

Recall from Section 2.5 that explicit calculations start with gauge choices: for each simplex σ we must choose an index $\chi(\sigma)$ such that $\sigma < \sigma^2_{\chi(\sigma)}$. We summarize these

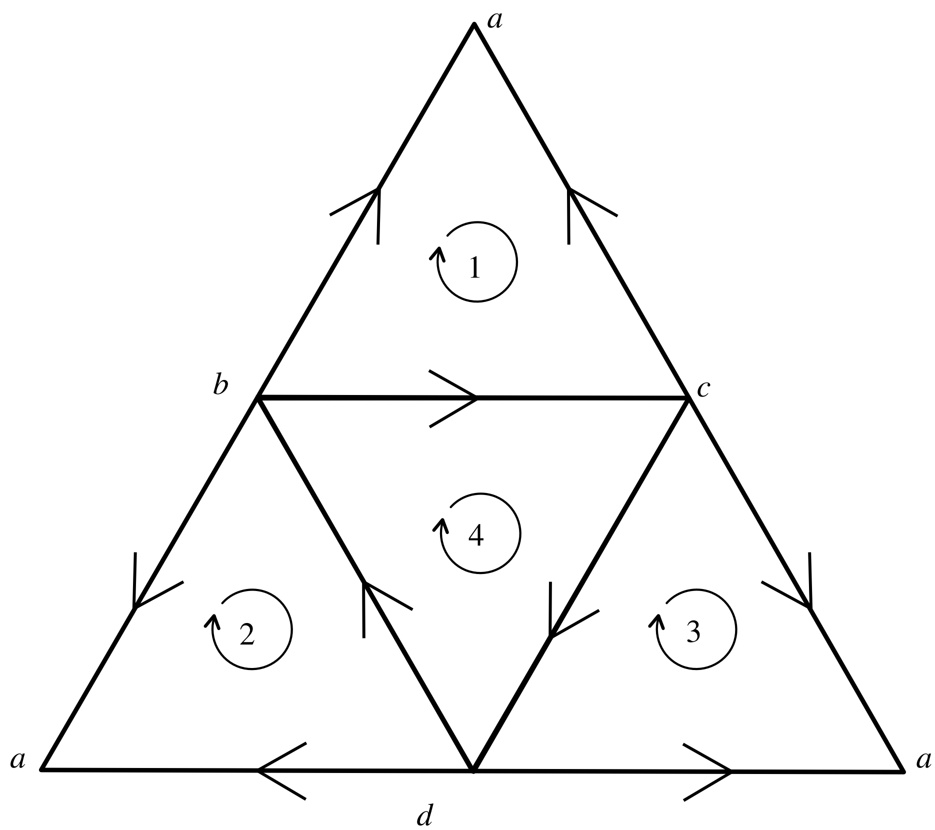

Figure 5.1: A triangulation of S^2 with arbitrarily chosen orientations. (All vertices are oriented as "+".)

choices in Table 5.1. Once the gauge choices are made, one looks at each increasing chain $\sigma^0 < \sigma^1 < \sigma^2$ and evaluates $n_{\chi_2\chi_1\chi_0}(\sigma^0)$. Since $n_{\alpha\beta\gamma}$ is alternating in its indices, most of these terms end up being 0. The increasing chains and corresponding values of $n_{\chi_2\chi_1\chi_0}(\sigma^0)$ are given in Table 5.2. In fact, it was unnecessary to list all the increasing chains of simplices, since we knew in advance that the only non-zero $n_{\alpha\beta\gamma}$ terms appeared at vertex d. It would have sufficed to look only at those chains that increase from vertex d. The result is the same as above; the pairing $\langle n_{\alpha\beta\gamma}, S^2\rangle = -1$ and so we see that the Chern class is a generator for $H^2(X;\mathbb{Z})$.

We emphasize that the computation of the Chern class from the curvature cochain alone is only possible when there is no torsion in $H^k(X;\mathbb{Z})$; in general it is necessary to compute the class of the Čech cocycle corresponding to $a^{-1}_{\alpha_0\ldots\alpha_k}$. We will see this in the case of $\mathbb{R}P^2$.

It is also interesting to see the discrete line bundle with connection that corresponds to

σ^0	$\chi(\sigma)$	σ^1	$\chi(\sigma)$
a	1	ab	1
b	1	ac	1
c	1	ad	2
d	2	bc	1
		bd	2
		cd	3

Table 5.1: Gauge choices χ for triangulation of S^2.

Vertex a	$n_{\chi_2\chi_1\chi_0}(a)$	Vertex b	$n_{\chi_2\chi_1\chi_0}(b)$
$a < ba < 1$	$n_{111}(a) = 0$	$b < -ba < -1$	$n_{111}(b) = 0$
$a < ba < -2$	$n_{211}(a) = 0$	$b < -ba < 2$	$n_{211}(b) = 0$
$a < ca < -1$	$n_{111}(a) = 0$	$b < -bc < 1$	$n_{111}(b) = 0$
$a < ca < 3$	$n_{311}(a) = 0$	$b < -bc < -4$	$n_{411}(b) = 0$
$a < da < 2$	$n_{221}(a) = 0$	$b < db < 4$	$n_{421}(b) = 0$
$a < da < -3$	$n_{321}(a) = 0$	$b < db < -2$	$n_{221}(b) = 0$
Vertex c	**$n_{\chi_2\chi_1\chi_0}(c)$**	**Vertex d**	**$n_{\chi_2\chi_1\chi_0}(d)$**
$c < ac < 1$	$n_{111}(c) = 0$	$d < ad < -2$	$n_{222} = 0$
$c < ac < -3$	$n_{311}(c) = 0$	$d < ad < 3$	$n_{322} = 0$
$c < bc < -1$	$n_{111}(c) = 0$	$d < bd < 2$	$n_{222} = 0$
$c < bc < 4$	$n_{411}(c) = 0$	$d < bd < -4$	$n_{422} = 0$
$c < dc < -4$	$n_{431}(c) = 0$	$d < cd < 4$	$n_{432} = -1$
$c < dc < 3$	$n_{331}(c) = 0$	$d < cd < -3$	$n_{332} = 0$

Table 5.2: Increasing chains of simplices from the vertices of S^2. Signs are used to express the consistency of orientations of neighboring pieces.

this 2-cocycle under the isomorphism of Theorem 4.2.4. Recall that the isomorphism class of this line bundle is represented by a bundle having all fibers $L_v \cong \mathbb{C}$; therefore the connection is represented by $U(1)$ elements attached to the edges. In Figure 5.3 we have denoted these elements as a, indicating $\exp 2\pi i a \in U(1)$. Recall that the curvature 2-cochain is an additional piece of data in this formulation; it is identical to the curvature da described earlier. Note that the values $\eta(e)$ of the connection are consistent with many possible curvature forms; any 2-cochain with $\omega(\sigma^2)$ differing from da by integer values on the faces would be a curvature cochain compatible with

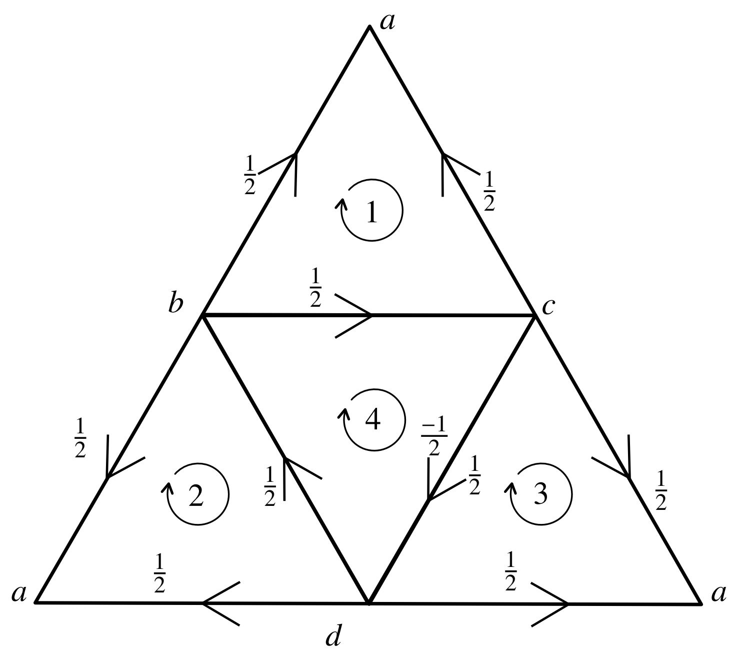

Figure 5.2: A Deligne 2-cocycle on S^2. The $(0, 1)$ pieces have $a_i(e) = \frac{1}{2}$ for all e except $a_4([cd]) = -\frac{1}{2}$. This implies a gauge change when crossing from σ_3 into σ_4. The gauge change is accomplished by the $(1, 0)$-component, which has all $\varphi_{ij}(v) = 0$ except $\varphi_{34}(d) = 1$.

this connection. Specifying the curvature 2-cochain is an essential part of specifying the topological class of a discrete line bundle with connection; it cannot be inferred from the connection's values. This is to be contrasted with the formulation in terms of discrete Deligne cocycles.

5.2 The Projective Space $\mathbb{R}P^2$

Here we see an example of a torsion Chern class. In degree 2, this can occur when the underlying space has a torsion 1-cycle, like the generator of $H_1(\mathbb{R}P^2; \mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$. Our triangulation of $\mathbb{R}P^2$, together with labellings and arbitrarily chosen orientations, is shown in Figure 5.4. There are triangulations with fewer 2-simplices, but we prefer this one for its simple structure.

We see that specifying a Deligne 2-cocycle on this triangulation will involve a lot of data. It is given in Figure 5.5. The values $a_\alpha(e)$ are zero on all horizontal edges and are $\pm 1/2$ on all non-horizontal edges. The signs are chosen in such a way that we have $da(\sigma_\alpha^2) = 0$ for each α. Still, there are 1-cycles with non-zero holonomy. This is typical of a bundle whose Chern class is a torsion element of $H^2(X; \mathbb{Z})$. In

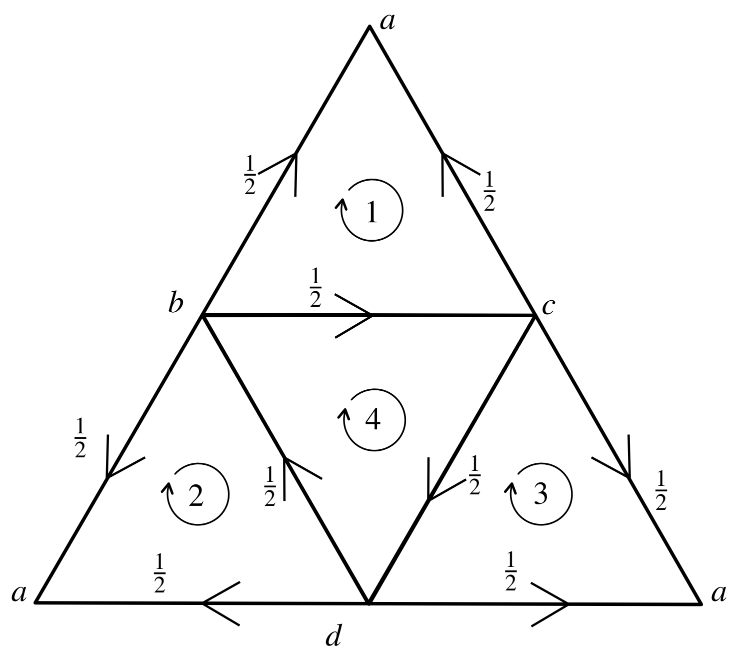

Figure 5.3: A discrete line bundle with connection on S^2. All the isometries $\eta(e)$ are equal to $-1 \in U(1)$. The real curvature 2-cochain has $\omega(\sigma_1) = \omega(\sigma_2) = \omega(\sigma_3) = -1/2$ and $\omega(\sigma_4) = 1/2$.

this case we know the second cohomology group to be $H^2(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ and so it will suffice for us to demonstrate that the Chern class is non-trivial.

The only places where a gauge transformation must occur are along $\sigma_{18,17}$, $\sigma_{12,11}$, and $\sigma_{6,5}$; in each case the value $(\delta a)_{\alpha\beta} = 1$, so these gauge transformations can all be taken care of by letting $\varphi_{6,5}(d) = 1$, $\varphi_{12,11}(c) = 1$, $\varphi_{18,17}(i) = 1$, and all other $\varphi_{\alpha\beta}(v) = 0$. Therefore the only non-zero components $n_{\alpha\beta\gamma}(v)$ are $n_{\alpha,6,5}(d)$, $n_{\alpha,12,11}(c)$, and $n_{\alpha,18,17}(i)$. We will only worry about the chains $\sigma^0 < \sigma^1 < \sigma^2$ that start from these three vertices. In Table 5.3 we indicate gauge choices, listing only those cells which are relevant to non-zero values of $n_{\alpha\beta\gamma}(v)$.

σ^0	$\chi(\sigma)$	σ^1	$\chi(\sigma)$	$\sigma^0 < \sigma^1 < \sigma^2$	$n_{\chi_2\chi_1\chi_0}(v)$
d	13	jd	5	$d < jd < \sigma^2_6$	$n_{6,5,13}(d) = -1$
c	8	ic	11	$c < ic < \sigma^2_{12}$	$n_{12,11,8}(c) = 1$
i	10	ai	18	$i < ai < \sigma^2_{17}$	$n_{17,18,10}(i) = -1$

Table 5.3: Gauge choices χ for triangulation of $\mathbb{R}P^2$. All those not listed lead to $n_{\chi_2\chi_1\chi_0}(v) = 0$.

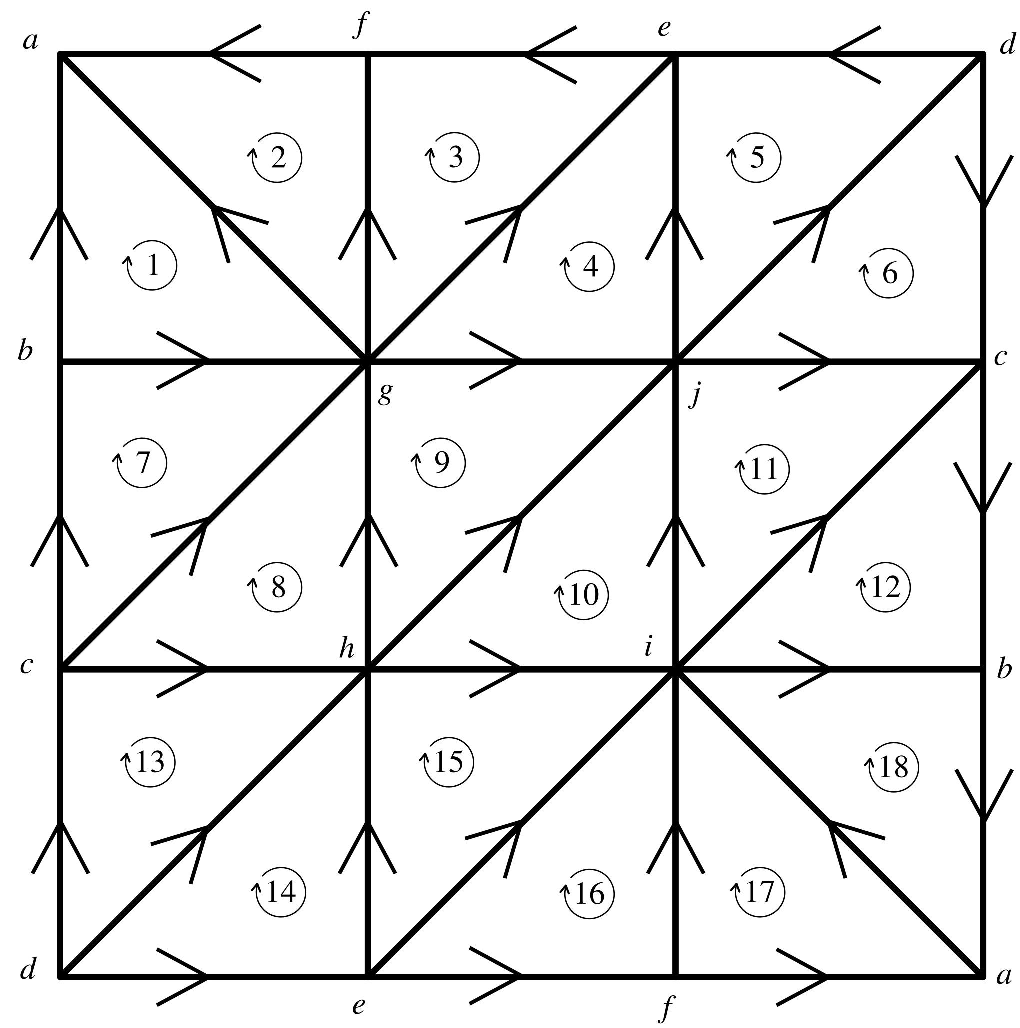

Figure 5.4: A triangulation of $\mathbb{R}P^2$. All simplices are given an arbitrary orientation, with all vertices oriented "+".

With the gauge choices indicated in Table 5.3 we may evaluate the pairing of the $n_{\alpha\beta\gamma}$ piece with the 0-skeleton of $\mathbb{R}P^2$ and we find $\langle n_{\alpha\beta\gamma}, \mathbb{R}P^2\rangle = -1$. The value itself is not what interests us, but rather the *parity*. Given that $H^2(\mathbb{R}P^2;\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$, the fact that we found an odd value for the pairing indicates that this 2-cocycle's Chern class is the non-trivial class in $H^2(\mathbb{R}P^2;\mathbb{Z})$. So we have an example of a flat connection on a topologically non-trivial bundle.

The corresponding discrete line bundle with connection is simple because our local connection 1-cochains $a_\alpha(e)$ take only the values 0 and $\pm 1/2$. Therefore the $U(1)$–valued connection takes the value 1 on horizontal edges and -1 on non-horizontal edges. Thus we see that there are many cycles with a non-trivial holonomy, such as

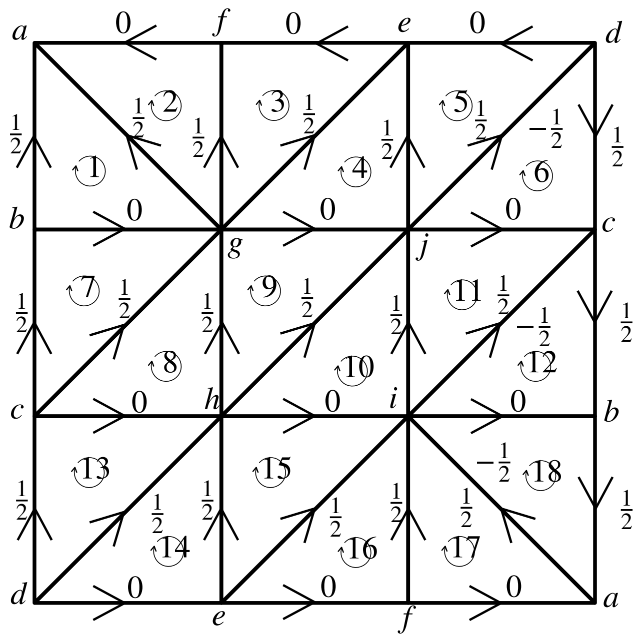

Figure 5.5: A discrete Deligne 2-cocycle on $\mathbb{R}P^2$. All horizontal edges have $a^1_\alpha(e) = 0$, all non-horizontal edges have $a^1_\alpha(e) = \pm 1/2$. When only one label is shown it is because $a^1_\alpha(e) = a^1_\beta(e)$ for both 2-simplices $\sigma^2_\alpha, \sigma^2_\beta$ containing the edge. Note that the only edges for which these expressions differ are ai, ic, and jd.

the path $e \to i \to j \to e$. The connection we have chosen here is somewhat special, as each cycle's holonomy is either 1 or -1. As a result this bundle is its own inverse under the group operation on line bundles with connection: we see that tensoring the bundle with itself produces a bundle where the connection's value is $1 \in U(1)$ on all edges.

5.3 The Torus T^2

We triangulate the torus as in Figure 5.6. In this case the labelings and orientations are less essential to understanding the cocycle, so we omit them. Opposite vertices and edges should be identified in the obvious way.

With this example we want to illustrate a Deligne 2-cocycle whose Chern class is trivial, whose curvature is flat, but which has nontrivial holonomies around 1-cycles. The local 1-cochains $a_\alpha(e)$ are all restrictions of a global 1-cochain $a \in C^1(T^2; \mathbb{R})$. The values $a(e)$ are indicated in Figure 5.6; the parameter $\theta \in \mathbb{R}$ is arbitrary. This is a closed 1-cochain: $da = 0$, and so the curvature 2-cochain is 0. Since the local 1-cochains are all restrictions of a global 1-cochain, the gauge transformations

$\varphi_{\alpha\beta}(v)$ may all be taken to be 0; therefore $n_{\alpha\beta\gamma} = 0$ as well and so the Chern class of the bundle is trivial.

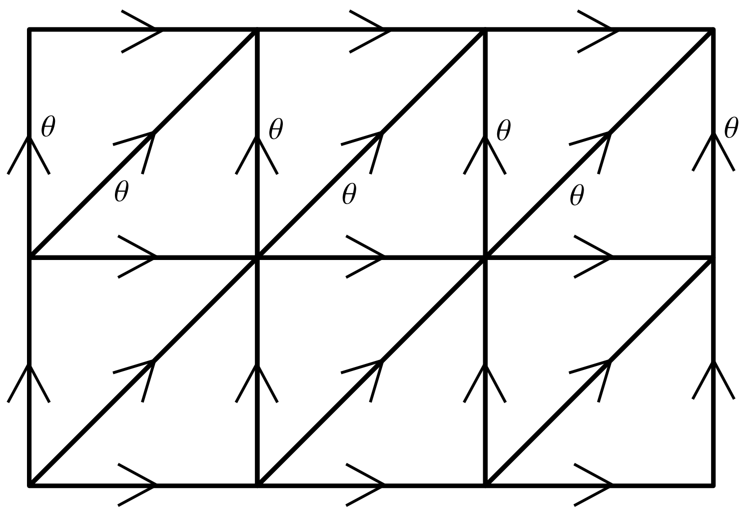

Figure 5.6: A discrete Deligne 2-cocycle on the torus. The local 1-cochains $a_\alpha(e)$ are all restrictions of a global 1-cochain, so no edge need be labeled more than once. All unlabeled edges have $a(e) = 0$.

Nonetheless, this Deligne 2-cocycle's holonomies are non-trivial; we easily find 1-cycles on the torus for which the holonomy is $\exp(2\pi i\theta)$. Of course, these 1-cycles are non-trivial. This is therefore an example of a 2-cocycle whose curvature and Chern class are 0 but whose corresponding class in $H^1(T^2; U(1))$ is non-trivial.

Chapter 6

RING STRUCTURE

In this chapter we describe a ring structure on the discrete Deligne cohomology groups. The product is a map $H^k_{dD}(X) \times H^l_{dD}(X) \to H^{k+l}_{dD}(X)$ that is graded-commutative. The corresponding product on the usual smooth Deligne cohomology groups is also associative, but due to our use of the Whitney product on cochains we have a non-associative product.

This product is compatible with the curvature and Chern class maps defined in Chapter 3 in the sense that for $a \in H^k_{dD}(X), b \in H^l_{dD}(X)$ the curvature cochains satisfy

$$\mathrm{curv}(a \star b) = \mathrm{curv}(a) \wedge \mathrm{curv}(b)$$

where $\wedge$ denotes the Whitney product, defined below. The Chern classes satisfy

$$\mathrm{ch}(a \star b) = \mathrm{ch}(a) \cup \mathrm{ch}(b)$$

where $\cup$ denotes the product in $H^*(X;\mathbb{Z})$.

Our definition is modeled on the well-known ring structure on the smooth version. Because that definition uses the exterior product on k-forms, we will need to use an appropriate replacement of this product for cochains. Such a replacement was defined by Whitney, so we first review his construction before giving the definition of the $\star$-product.

6.1 Whitney Product

In [23], Whitney proposed a product on cochains whose algebraic properties mimic those of the exterior product on k-forms. This Whitney product is a graded-commutative, non-associative operation $\wedge : C^k(X;\mathbb{R}) \times C^l(X;\mathbb{R}) \to C^{k+l}(X;\mathbb{R})$. The Whitney product is defined in terms of the *Whitney map*, which maps k-cochains on a triangulation of a smooth manifold to "piecewise linear k-forms" on the manifold. (Throughout this section we assume that X triangulates M and that M is equipped with some arbitrarily chosen volume form that allows us to speak of $L^2(M;\Lambda^k T^*M)$, square-integrable sections of T^*M that are not necessarily smooth.)

Definition 6.1.1 ([23])**.** The Whitney map $W_X : C^k(X;\mathbb{R}) \to L^2(M;\Lambda^k T^*M)$, where X triangulates the manifold M, acts on the cochain $a = [v_0, v_1, \ldots v_k]^*$ (the cochain whose value is 1 on the oriented k-simplex with vertices $v_0, \ldots, v_k$ and 0 on all other simplices) as

$$W_X a = \sum_{i=0}^{k} (-1)^i \mu_i d\mu_0 \wedge \ldots \wedge \widehat{d\mu_i} \wedge \ldots \wedge d\mu_k$$

where $\mu_i : M \to \mathbb{R}$ are the barycentric coordinates of the vertex v_i. Here $\widehat{d\mu_i}$ denotes that this term is to be omitted from the wedge product.

The barycentric coordinate functions are not smooth, but they are smooth almost everywhere. Thus $d\mu_i$ is not a k-form, but does belong to $L^2(M; \Lambda^k T^* M)$. It is smooth on the interior of all k-simplices. The key properties of this map are:

Proposition ([23])**.** *The Whitney map is a right-inverse to the de Rham map* $R_X : L^2(M; \Lambda^k T^* M) \to C^k(X; \mathbb{R})$ *that defines a cochain by integrating k-forms over k-simplices of the triangulation. That is,* $R_X \circ W_X = Id_{C^k(X)}$*. Moreover, the Whitney map is compatible with differentials in the sense that* $d_{\text{exterior}} \circ W_X = W_X \circ d_{\text{simplicial}}$.

This Whitney map allows one to translate the exterior product on forms to a product defined on cochains. One simply turns both cochains into forms, wedges these forms, and then uses the de Rham map to turn the resulting form back into a cochain:

Definition 6.1.2 ([23])**.** The Whitney product $\wedge : C^k(X; \mathbb{R}) \times C^l(X; \mathbb{R}) \to C^{k+l}(X; \mathbb{R})$ is defined as

$$a \wedge b = R_X \left(W_X a \wedge_{\text{exterior}} W_X b \right)$$

where R_X denotes the de Rham map.

Proposition ([23])**.** *The Whitney product $a \wedge b$ for $a \in C^k(X; \mathbb{R})$, $b \in C^l(X; \mathbb{R})$ satisfies:*

- *Bilinearity:* $C^k(X; \mathbb{R}) \times C^l(X; \mathbb{R}) \to C^{k+l}(X; \mathbb{R})$*;*
- *Graded-commutativity:* $a \wedge b = (-1)^{\deg(a)\deg(b)} b \wedge a$*;*
- *Leibniz rule:* $d(a \wedge b) = da \wedge b + (-1)^{\deg(a)} a \wedge db$*.*

These were originally proven by Whitney; we also recommend that the reader consult [11, 27] for proofs in more modern notation.

Remark. *The Whitney product is* not *associative.*

Though non-associativity is a drawback, the other algebraic properties are desirable enough that the Whitney product has found many applications. Moreover, the non-associativity of this product vanishes in an appropriate continuum limit; this is a consequence of:

Proposition ([26])**.** *For smooth forms ω_1, ω_2 on M there exists a constant C_{ω_1,ω_2} such that $\|\omega_1 \wedge \omega_2 - W_X(R\omega_1 \wedge R\omega_2)\|_{L^2} < C_{\omega_1,\omega_2}\mathrm{mesh}(X)$ for any X that triangulates M.*

In this sense, the Whitney product converges to the exterior product in the continuum limit and thus the non-associativity of the Whitney product becomes small together with the mesh of the triangulation.

6.2 Product on Deligne Cocycles

We now describe the construction of a product on discrete Deligne cocycles that will induce a graded-commutative ring structure on $H^*_{dD}(X)$. Here $\wedge$ denotes the Whitney product on cochains described in the previous section. We let $a = (a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \ldots a^{-1}_{\alpha_0\ldots\alpha_k})$ and $b = (b^{l-1}_{\alpha_0}, b^{l-2}_{\alpha_0\alpha_1}, \ldots b^{-1}_{\alpha_0\ldots\alpha_l})$ represent classes $[a] \in H^k_{dD}(X)$ and $[b] \in H^l_{dD}(X)$. The product can be obtained by starting in bi-degree $(0, k+l-1)$ with the local cochains $a^{k-1}_{\alpha_0} \wedge db^{l-1}_{\alpha_0}$. One proceeds to "walk down the stairs" of the degree $(k+l)$ discrete Deligne complex. That is, one engineers the components of $a \star b$ one-by-one to satisfy the closure relations. First we have

$$\begin{aligned}
\delta(a^{k-1} \wedge db^{l-1})_{\alpha_0\alpha_1} &= a^{k-1}_{\alpha_1} \wedge db^{l-1}_{\alpha_1} - a^{k-1}_{\alpha_0} \wedge db^{l-1}_{\alpha_0} \\
&= (\delta a^{k-1})_{\alpha_0\alpha_1} \wedge db^{l-1}_{\alpha_1} \\
&= da^{k-2}_{\alpha_0\alpha_1} \wedge db^{l-1}_{\alpha_1} \\
&= d(a^{k-2}_{\alpha_0\alpha_1} \wedge db^{l-1}_{\alpha_1})
\end{aligned}$$

In the second line we used the fact that $\delta db^{l-1} = 0$, since db^{l-1}_{α} is the restriction of a global l-cochain, $\mathrm{curv}(b) \in C^l(X;\mathbb{R})$. In the third line we used the closure relation for a. In the final line we used the compatibility of d and $\wedge$. These same steps are used repeatedly:

$$\begin{aligned}
\delta(a^{k-j} \wedge db^{l-1})_{\alpha_0\ldots\alpha_j} &= \sum_{i=0}^{j-1}(-1)^i a^{k-j}_{\alpha_0\ldots\widehat{\alpha_i}\ldots\alpha_j} \wedge db^{l-1}_{\alpha_j} + (-1)^j a^{k-j}_{\alpha_0\ldots\alpha_{j-1}} \wedge db^{l-1}_{\alpha_{j-1}} \\
&= (\delta a^{k-j})_{\alpha_0\ldots\alpha_j} \wedge db^{l-1}_{\alpha_j} - (-1)^j a^{k-j}_{\alpha_0\ldots\alpha_{j-1}} \wedge (\delta db^{l-1})_{\alpha_{j-1}\alpha_j} \\
&= (-1)^{j+1} da^{k-j-1}_{\alpha_0\ldots\alpha_j} \wedge db^{l-1}_{\alpha_j} \\
&= (-1)^{j+1} d\left(a^{k-j-1}_{\alpha_0\ldots\alpha_j} \wedge db^{l-1}_{\alpha_j}\right)
\end{aligned}$$

This shows that we satisfy the closure relation by choosing the components of $a \star b$ in bi-degrees $(0, k+l-1), (1, k+l-2), \ldots (k-1, l)$ to be the collections $a^{k-j-1}_{\alpha_0 \ldots \alpha_j} \wedge db^{l-1}_{\alpha_j}$. Then we hit a point where this argument breaks down and we need to continue differently:

$$\begin{aligned}
\delta(a^0 \wedge db^{l-1})_{\alpha_0 \ldots \alpha_k} &= \sum_{i=0}^{k-1} (-1)^i a^0_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_k} \wedge db^{l-1}_{\alpha_k} + (-1)^k a^0_{\alpha_0 \ldots \alpha_{k-1}} \wedge db^{l-1}_{\alpha_{k-1}} \\
&= (\delta a^0)_{\alpha_0 \ldots \alpha_k} \wedge db^{l-1}_{\alpha_k} - (-1)^k (\delta db^{l-1})_{\alpha_{k-1}\alpha_k} \\
&= (-1)^{k+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge db^{l-1}_{\alpha_k} \\
&= (-1)^{k+1} d \left(a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-1}_{\alpha_k} \right)
\end{aligned}$$

At this point it becomes impossible to write $a^{-1}_{\ldots}$ as $d(\ldots)$; however it is locally constant, and so we do have $d(a^{-1} \wedge b^{l-1}) = a^{-1} \wedge db^{l-1}$. Thus the next component of $a \star b$ ought to be $a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-1}_{\alpha_k}$. Next we have

$$\begin{aligned}
\delta(a^{-1} \wedge b^{l-1})_{\alpha_0 \ldots \alpha_{k+1}} &= \sum_{i=0}^{k} (-1)^i a^{-1}_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{k+1}} \wedge b^{l-1}_{\alpha_{k+1}} + (-1)^{k+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-1}_{\alpha_k} \\
&= (\delta a^{-1})_{\alpha_0 \ldots \alpha_{k+1}} \wedge b^{l-1}_{\alpha_{k+1}} - (-1)^{k+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-1}_{\alpha_{k+1}} + (-1)^{k+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-1}_{\alpha_k} \\
&= (-1)^{k+2} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge (\delta b^{l-1})_{\alpha_k \alpha_{k+1}} \\
&= (-1)^{k+2} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge db^{l-2}_{\alpha_k \alpha_{k+1}} \\
&= (-1)^{k+2} d \left(a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-2}_{\alpha_k \alpha_{k+1}} \right)
\end{aligned}$$

This suggests continuing with components $a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-j-1}_{\alpha_k \ldots \alpha_{k+j}}$. Doing so gives

$$\begin{aligned}
\delta(a^{-1} \wedge b^{l-j-1})_{\alpha_0 \ldots \alpha_{k+j+1}} &= \sum_{i=0}^{k} (-1)^i a^{-1}_{\alpha_0 \ldots \widehat{\alpha_i} \ldots \alpha_{k+1}} \wedge b^{l-j-1}_{\alpha_{k+1} \ldots \alpha_{k+j+1}} \\
&\quad + \sum_{i=k+1}^{k+j+1} (-1)^i a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{l-j-1}_{\alpha_k \ldots \widehat{\alpha_i} \ldots \alpha_{k+j+1}} \\
&= (\delta a^{-1})_{\ldots} \wedge b^{l-j-1}_{\alpha_{k+1} \ldots \alpha_{k+j+1}} + (-1)^{k+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge (\delta b^{l-j-1})_{\alpha_k \ldots \alpha_{k+j+1}} \\
&= (-1)^{k+1} (-1)^{j+1} a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge db^{l-j-2}_{\alpha_j \ldots \alpha_{k+j+1}}
\end{aligned}$$

as required by the closure relation. At the final step $a \star b$ has bi-degree $(k+l, -1)$ component $a^{-1}_{\alpha_0 \ldots \alpha_k} \wedge b^{-1}_{\alpha_k \ldots \alpha_{k+l}}$, which is easily seen to be a locally constant integer-valued $(k+l)$-cocycle. Thus we have proven:

Proposition 6.2.1. *Given discrete Deligne cocycles* $(a^{k-1}_{\alpha_0}, a^{k-2}_{\alpha_0\alpha_1}, \dots a^{-1}_{\alpha_0\dots\alpha_k})$ *and* $b = (b^{l-1}_{\alpha_0}, b^{l-2}_{\alpha_0\alpha_1}, \dots b^{-1}_{\alpha_0\dots\alpha_l})$*, the discrete Deligne* $(k+l)$*-cochain defined by*

$$\begin{aligned} a \star b = \Big(& a^{k-1}_{\alpha_0} \wedge db^{l-1}_{\alpha_0}, \dots, a^{k-j-1}_{\alpha_0\dots\alpha_j} \wedge db^{l-1}_{\alpha_j}, \dots a^{-1}_{\alpha_0\dots\alpha_k} \wedge b^{l-1}_{\alpha_k}, \\ & \dots, a^{-1}_{\alpha_0\dots\alpha_k} \wedge b^{l-j-1}_{\alpha_k\dots\alpha_{k+j}}, \dots, a^{-1}_{\alpha_0\dots\alpha_k} \wedge b^{-1}_{\alpha_k\dots\alpha_{k+l}} \Big) \end{aligned} \tag{6.1}$$

is a cocycle.

This expression for the cocycle $a \star b$ is somewhat cumbersome. It can be written instead as

$$a \star b = \begin{cases} a^j \wedge db^{l-1} & \text{for } j = 0, 1, \dots k-1 \\ a^{-1} \wedge b^j & \text{for } j = -1, 0, \dots l-1 \end{cases} \tag{6.2}$$

In this expression we have suppressed the indices that express the Čech degree of each piece; they are easily inferred in each case, as in the expression (6.1). This form of the product is analogous to that found in standard references like [8].

As a result of Proposition 6.2.1, we have a discrete Deligne $(k+l)$-class associated with $a \star b$, $[a \star b] \in H^{k+l}_{dD}(X)$. The compatibility of this $\star$ product with the familiar wedge and cup operations is easily seen:

Proposition 6.2.2. *The product* $\star : H^k_{dD}(X) \times H^l_{dD}(X) \to H^{k+l}_{dD}(X)$ *is compatible with the curvature map in the sense that* $curv(a \star b) = curv(a) \wedge curv(b)$ *and with the cohomology cup product in the sense that* $ch(a \star b) = ch(a) \cup ch(b)$*.*

Proof. The local expression for the curvature of $a \star b$ will be $d(a^{k-1}_\alpha \wedge db^{l-1}_\alpha) = da^{k-1}_\alpha \wedge db^{l-1}_\alpha$, which is indeed the Whitney product of the cochains curv(a) and curv(b).

Under the correspondence of Proposition 2.4.1 that associates integer-valued cochains on the nerve of X to our locally constant integer cochains on intersections, the expression $a^{-1}_{\alpha_0\dots\alpha_k} \wedge b^{-1}_{\alpha_k\dots\alpha k+l}$ is seen to map to the expression introduced by Whitney [24] to define the cup product on simplicial cohomology. This is because the Whitney product $\wedge$ of cochains reduces in the case of 0-cochains to pointwise multiplication at the vertices. □

6.3 Chern-Simons Theory

In [10], Chern and Simons initiated the study of certain Lie algebra–valued differential forms associated to connections on principal G-bundles over smooth manifolds.

These forms are related to the Chern-Weil representation of a G-bundle's characteristic classes in terms of the curvature of a connection. Since these Chern-Weil $2k$-forms are closed, they admit local $(2k-1)$-form potentials; these potentials are interesting to study in their own right.

Among the most studied of these Chern-Simons theories is three-dimensional abelian Chern-Simons theory. The setting for this theory is a principal $U(1)$-bundle P over a closed 3-manifold M. The theory concerns a single field: a connection on the bundle. As $U(1)$ is a 1-dimensional Lie group, its Lie algebra–valued local connection forms may be identified with real-valued 1-forms. When the bundle P is trivial, a global connection form $A \in \Omega^1(M;\mathbb{R})$ may be used. In this case, the $U(1)$-valued Chern-Simons action of the theory is

$$CS(A) = \exp 2\pi i k \int_M A \wedge dA \tag{6.3}$$

The integer k determines the "level" of the theory, which in the abelian theory can be thought of as determining the possible charges. The level does not play a role in the present work, and so we omit it; it can easily be added back in. Our primary concern is with the definition of this action when the bundle P has a non-zero Chern class. In this case the connection 1-form A cannot be defined globally; instead the connection on P is described as a smooth Deligne 2-cocycle $(A_\alpha, \varphi_{\alpha\beta}, n_{\alpha\beta\gamma})$. The density $A \wedge dA$ is no longer well-defined, so we must try to make sense of the local expressions $A_\alpha \wedge dA_\alpha$ and account for lower-degree terms arising from changing gauge. Guadagnini and Thuillier observed [16, 14, 18, 15] that the ring structure on smooth Deligne cohomology is perfectly suited for keeping track of these gauge change subtleties. In this formulation of Chern-Simons theory, we treat the bundle and connection as a single class $A \in \hat{H}^2(M)$ and define the Chern-Simons action to be the holonomy of $A \star A \in \hat{H}^4(M)$ over the 3-manifold M, considered as a 3-cycle. This definition coincides with (6.3) when the connection 1-form is global, and provides a natural generalization of the action to connections on non-trivial principal bundles.

Replacing the density $A \wedge dA$ with the Deligne 4-cocycle $A \star A$ offers a new geometric interpretation of Chern-Simons theory: it is a theory of connective structures on "2-bundle gerbes." At the time of writing there does not seem to be a consensus on the precise definition of a 2-bundle gerbe, but they ought to be the fourth step in the "categorical ladder" of objects that begins: $U(1)$–valued functions, principal $U(1)$–bundles with connection, abelian bundle gerbes, 2-bundle gerbes, etc. See [4, 20]. A discretization of these objects would be interesting.

One might object that on a 3-manifold there is no degree 4 cohomology and so the introduction of Deligne cohomology and its product structure to deal with Chern-Simons theory is overkill. After all, the exact sequence

$$0 \to \Omega^3(M)/\Omega^3_{\mathbb{Z}}(M) \to \hat{H}^4(M) \to H^4(M;\mathbb{Z}) \to 0$$

shows that for a 3-manifold, $\hat{H}^4(M) \cong \Omega^3(M)/\Omega^3_{\mathbb{Z}}(M)$. That is, the 4-cocycle whose holonomy we use to define the Chern-Simons action may be represented by a global 3-form. Therefore the full generality of smooth Deligne cohomology may not be needed to describe an object that is essentially just a 3-form defined up to closed 3-forms with integral periods.

This is not so, however, due to the effects of torsion, a point which is explained clearly in [15]. The argument is that if the bundle P which supports the connection has its Chern class in the torsion component $H^2_{\mathrm{tor}}(M;\mathbb{Z})$, then the bundle admits a flat connection. Moreover, this flat connection may be represented as a Deligne 2-cocycle of the form $(0, \varphi_{\alpha\beta}, n_{\alpha\beta\gamma})$ with all the local connection 1-forms $A_\alpha = 0$. Despite this, such connections contribute non-trivially to the Chern-Simons path integral. Their contribution ends up being related to linking numbers of homology 1-cycles corresponding to these torsion Chern classes. The utility of defining the Chern-Simons action in terms of the product $\hat{H}^2(M) \times \hat{H}^2(M) \to \hat{H}^4(M)$ is that it provides the framework needed for dealing effectively with the contributions of these torsion bundles to the path integral.

For a discrete abelian Chern-Simons action we will use the $\star$-product defined in the previous section together with the holonomy pairing of Section 2.5 to obtain an analogous definition.

Definition 6.3.1. The *Chern-Simons action* of a discrete line bundle with connection represented as a discrete Deligne 2-cocycle $a \in H^2_{dD}(X)$, where X triangulates a closed 3-manifold, is the higher holonomy $\mathrm{Hol}(a \star a, X) \in U(1)$.

Note that actions defined in terms of gauge fields ordinarily require a proof of gauge invariance. In our case, this is taken care of by Corollary 2.5.2, which states that the pairing of Deligne cocycles with simplicial cycles is well-defined.

6.4 Future Directions

Reshetikhin-Turaev Invariants

One of the successes of 3–dimensional Chern-Simons theory has been to compute invariants of 3–manifolds, and one would naturally hope to be able to do the same with our discrete Chern-Simons theory. A notable example of this is Adams' work [1], which successfully computed the Ray-Singer torsion invariant from a "doubled" Chern-Simons theory in which gauge fields lived on the edges of a primary and a dual lattice. In this section we outline how the abelian Witten-Reshetikhin-Turaev invariants [19, 28] of M might be extracted from our discrete theory, following an argument used in [15] for the continuum case.

The idea is to study the formal path integral partition function of the theory whose action is $S_{CS}[a] = \mathrm{Hol}(a \star a, X)$, where $a \in H^2_{dD}(X)$ and X triangulates an oriented 3-manifold M. The partition function of this theory would be some path integral over the space $H^2_{dD}(X)$ of field configurations of a $\mathbb{C}$-line bundle with connection of the holonomy $\mathrm{Hol}(a \star a, X)$. We expect this to be divergent and, as is standard for path integral computations, try instead to make sense of ratios of divergent quantities. In this case, observing that $H^2_{dD}(X)$ breaks up as

$$0 \to C^1(X;\mathbb{R})/C^1_{\mathbb{Z}}(X;\mathbb{R}) \to H^2_{dD}(X) \to H^2(X;\mathbb{Z}) \to 0 \tag{6.4}$$

it seems reasonable to normalize the partition function by the path integral over the subspace of those configurations which are connections on a trivial line bundle. That is, we formally define

$$Z(X) = \frac{\int_{H^2_{dD}(X)} \mathcal{D}a \ \mathrm{Hol}(a \star a, X)}{\int_{C^1(X;\mathbb{R})/C^1_{\mathbb{Z}}(X;\mathbb{R})} \mathcal{D}a \exp 2\pi i \langle a \wedge da, X\rangle} \tag{6.5}$$

The normalizing term in the denominator of this expression uses the fact that a configuration whose Chern class is zero may be represented by a globally defined 1-cochain a (determined only up to a closed 1-cochain with integer periods) and that in this case the $\mathrm{Hol}(a \star a, X)$ term is expressed in terms of the familiar $a \wedge da$ integrated over X. This is a useful normalizing factor because the full configuration space $H^2_{dD}(X)$ breaks up into fibers which, according to (6.4), are all non-canonically isomorphic to $C^1(X;\mathbb{R})/C^1_{\mathbb{Z}}(X;\mathbb{R})$. The identification of one of these fibers with $C^1(X;\mathbb{R})/C^1_{\mathbb{Z}}(X;\mathbb{R})$ requires a choice of origin in the fiber, i.e. a reference connection.

As suggested in [15], we pick convenient origins in each of the fibers over $H^2(X;\mathbb{Z})$, relative to which each configuration may be written in the form $a_0 + a$ for some global 1-cochain a. That is, the Chern class of the configuration is specified by the Deligne class a_0, which is like choosing a topological class of $\mathbb{C}$-line bundle; the particular connection to use over that bundle is specified by the global 1-cochain

a. Moreover, as we discussed in Section 4.3, it is useful to separate out the classes a_0 whose Chern class is torsion from those whose Chern class has a non-zero free component. According to the decomposition $H^2(X;\mathbb{Z}) \cong H^2_{\mathrm{tor}}(X;\mathbb{Z}) \oplus H^2_{\mathrm{free}}(X;\mathbb{Z})$ we choose representatives a_t having Chern class $t \in H^2_{\mathrm{tor}}(X;\mathbb{Z})$ and a_f having Chern class $f \in H^2_{\mathrm{free}}(X;\mathbb{Z})$. Writing a general element of $H^2_{dD}(X)$ as $a_t + a_f + a$ for $a \in C^1(X;\mathbb{R})$ we have

$$(a_t+a_f+a)\star(a_t+a_f+a) = a_t\star a_t + a_f\star a_f + a\wedge da + 2\left(a_t \star a_f + a_t \star a + a_f \star a\right) \tag{6.6}$$

One then argues that by carefully choosing the origins a_t and a_f, most of these terms can be arranged to have integer pairing with X, thus giving no contribution to the holonomy term in the partition function. For example, the pairings of $a_t \star a_t$, $a_t \star a_f$, $a_f \star a_f$ with X can be shown to be topological, computing various linking numbers, all of which are integer except those $a_t \star a_t$ terms. These are related to self-linking numbers of torsion 1-cycles in X, which may take non-integer rational values. The $\langle a \wedge da, X\rangle$ pairing is precisely what is needed to cancel with the normalization term. The $a_t \star a$ terms can be made integer by choosing representatives for a_t in which the connection is not only flat but in fact has all $a^1_\alpha = 0$.

This is done carefully in [15] for the smooth case, with the result that the normalized partition function of the theory is related to the abelian WRT invariant up to a constant factor that can be expressed in terms of the first homology of X. We hope in future work to adapt these arguments to the discrete setting, thereby showing that our discrete theory provides another means of calculating these same invariants.

Discrete Gerbes

Another promising direction for future work is to define the geometric counterpart of the groups $H^k_{dD}(X)$ for $k > 2$. We have shown that $k = 2$ corresponds to Knöppel and Pinkall's discrete line bundles with connection, and one naturally wonders if it would not be possible to raise the "categorical degree" of their construction to produce the discretizations of gerbes and k–gerbes. Actually, given that $k = 2$ corresponds already to the "0-gerbe" case of principal $U(1)$-bundles, $H^k_{dD}(X)$ ought to classify $(k-2)$–gerbes with connective structure. Let us suggest how this might work for $k = 3$.

Suppose that instead of attaching 1-dimensional $\mathbb{C}$-vector spaces to the vertices of X, we were to attach to each vertex a category C_v. Take each C_v to be a $U(1)$–groupoid, a category with all morphisms invertible and with all hom sets acted on by $U(1)$ in a way compatible with composition. Attach to each oriented edge e an invertible functor $\eta_e : C_{s(e)} \to C_{d(e)}$ compatible with the $U(1)$–groupoid structure. To each 2–simplex one attaches a natural transformation, which could be thought of in a few equivalent ways. For example, for two fixed vertices of the face, the face's boundary decomposes into two distinct paths between these vertices, and thus yields two invertible functors between the categories attached to these vertices. The face should be labeled with a natural transformation between these functors, using orientations to determine which is the source and which is the target.

Roughly speaking, the idea is then that if one were to choose as a "local section" over a top-dimensional simplex σ_α objects $s_\alpha(v) \in C_v$ as well as morphisms $\varphi_\alpha(e) \in \mathrm{Hom}_{d(e)}(\eta_e s_\alpha(s(e)), s_\alpha(d(e)))$, the effect of the natural transformation attached to a face f would be to specify an element of $\psi(f) \in \mathrm{Aut}(s_\alpha(v)) \cong U(1)$ for one of the vertices $v < f$. The choices leading to this identification would affect the result in a familiar "change-of-gauge-like" way. The value of $d\psi$ on a 3-simplex would be shown to be independent of the choices, providing a well-defined curvature 3-cochain for the structure. We hope in future work to identify an appropriate notion of isomorphism for these structures such that the isomorphism classes correspond to $H^3_{dD}(X)$. It is quite conceivable that by attaching k-categories to vertices and attaching appropriate functors-of-functors to the simplices of X one could realize each $H^k_{dD}(X)$ geometrically.

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