THE STRUCTURE AND STABILITY OF RELATIVISTIC, DIFFERENTIALLY ROTATING STARS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1975

(Submitted September 20, 1974)

ACKNOWLEDGEMENTS

A very large amount of counseling with James Gunn has been useful to me during my tenure as a graduate student and during my execution of the researches contained in this thesis. Others who have been helpful are Kip Thorne, Douglas Eardley, Rafael Sorkin, Marek Demianski, Peter Goldreich, William Fowler, Martin Schwarzschild, Don Page, and Saul Teukolsky.

Financial support for this research has come from National Science Foundation research grants GP-27304, GP-28027, and GP-36687X.

ABSTRACT

The stability of axisymmetric, differential rotation in non-magnetic stars of uniform chemical composition is studied in the context of general relativity theory. Criteria are found for stability against local, linear, axisymmetric perturbations in conducting, viscous stars and in perfect fluid models. When stated in the proper language, the relativistic stability conditions have the same forms as the non-relativistic conditions. When thermal conduction is much more efficient than viscosity, a star must be barytropic (the level surfaces of the pressure, P, and the total density of mass energy, ϵ , must coincide) and the gradient of the geometrical angular momentum ($L = - U_\phi/U_o$) must never point toward the interior of the quasi-cylindrical level surfaces of L. When viscosity dominates thermal conductivity by a large margin a star must be barytropic and must have an entropy (per baryon, S) gradient which is parallel to the vector $(\partial\epsilon/\partial S)_P \nabla P$. When conduction and viscosity have comparable efficiencies or are absent the criteria are only slightly more complex. Applications of the stability conditions to models of specific astrophysical objects are discussed.

The equations of hydrodynamics in the post-Newtonian approximation to general relativity are applied to differentially rotating, barytropic stars. In this approximation the equation of hydrodynamic equilibrium can be integrated to yield a simple algebraic equation, and the gravitational field equations can be written in easily handled integral forms; these facts make possible an iterative scheme of the "self-consistent field method" type which can be used to construct numerical models.

TABLE OF CONTENTS

PART ONE

INTRODUCTION

This thesis consists of three papers whose common goal is to determine, in the framework of Einstein's general theory of relativity, some of the properties of rotating, self gravitating fluid bodies. An investigation of these properties can be properly motivated and justified purely by a desire to explore the formal, theoretical consequences of Einstein's theory of gravitation. But since a number of classes of observed astrophysical objects (white dwarf stars, pulsars, and quasi-stellar objects) seem likely to be associated with fluid systems that cannot be adequately described with non-relativistic gravitation theory, it is clear that knowledge of the behaviors of relativistic systems has a direct application to specific astrophysical problems.

In part two (paper I) we find a set of constraints on the structures of fully relativistic, rotating systems by deriving criteria which must be satisfied if such a system is to be stable against the onset of convective motions. Of course, a stellar system can exist in a convecting state, but a rotating system which is unstable to convection will likely have its structure altered by a redistribution of angular momentum as a result of the motion of convective cells. If it is desired to model a rotating system whose structure persists on a time scale longer than the rotation time, it is necessary to build models which do not violate the conditions for convective stability.

While it is fine to have conditions for the stability of equilibrium configurations, such conditions are useless without configurations to apply them to. A certain amount of progress has been made by various authors in an effort to find methods of constructing fully relativistic, rapidly rotating equilibrium configurations, but a completely general method for doing so has thus far been elusive. One of the methods which several authors have used has been the application of the equations of the post-Newtonian approximation to general relativity to schemes of constructing certain types of equilibrium configurations which could model objects in which relativistic effects are significant but not too large. In parts three and four (papers II and III) a method of this sort is presented: one which is applicable to configurations with barytropic fluid distributions (in which the level surfaces of pressure and total density of mass energy coincide) with any (equilibrium) distribution of angular momentum.

Paper II, which describes the equations governing polytropic configurations, was written before paper I, and uses notations which are not always consistent with the fully relativistic notation of paper I. In both papers the structure of space-time is described with reference to the same coordinate system, but the coordinate distance from the axis of symmetry is called r in paper I and ϖ in paper II. The total density of mass energy, which is called ϵ in paper I, is not referred to directly in paper II but is equal to $\rho(c^2 + \Pi)$ in the notation of that paper. The quantity U_φ which appears in paper II is not to be confused with U_ϕ, a covariant component of the four velocity in the notation of paper I. Similarly, the symbol v means different things in the two papers. The coordinate angular velocity of a fluid element, called Ω in paper I, is called Ω^* in paper II.

In paper III the formalism of paper II is generalized to apply to any barytropic pressure dependence. In this paper the notation of Paper I is once again adopted, in order to make the post-Newtonian formalism more easily identified with that of the fully relativistic theory.

PART TWO

THE STABILITY OF NON-UNIFORM ROTATION IN RELATIVISTIC STARS

(To be submitted to the *Astrophysical Journal*)

I. INTRODUCTION

Very powerful necessary and sufficient conditions for the pulsational stability of perfect fluid stellar models have been derived in the context of the general theory of relativity by Schutz (1972), by Chandrasekhar and Friedman (1972), and by Will (1974). Any viable perfect fluid stellar model must satisfy these stability conditions. Unfortunately, most of these criteria can be applied only by making rather complicated mathematical tests on a model after it has been constructed. It would be nice if some unstable models could be eliminated on the basis of a stability criterion which was easier to apply, and it would be even nicer if some of these models could be eliminated <u>before</u> the trouble has been taken to construct them.

It is to this end that we turn in this paper to the derivation of criteria for local (or convective) stability in differentially rotating stars. We will consider models of non-magnetic stellar regions of homogeneous chemical composition which are more physically realistic than perfect fluid models by virtue of taking into account transport phenomena: energy transport via thermal conduction (radiative or molecular diffusion) and momentum transfer via radiative or molecular viscosity. We will study perturbations which are axisymmetric and small in size and extent (linear, local perturbations), finding necessary (though not necessarily sufficient) criteria for stability in realistic stellar interiors and (by neglecting the transport phenomena) in perfect fluid models. These criteria will be specific physical constraints which must be satisfied if a stellar model is to be stable against convective motions which would change the distribution of angular momentum.

The results of this investigation will not, of course, eliminate the need for performing the more complicated pulsational stability tests on those stellar models which satisfy the convective stability criteria, but they will eliminate large classes of models which, therefore, may just as well not be given further consideration. In some cases it should be possible to build the convective stability constraints into methods of constructing models to guarantee that all models which are constructed are convectively stable.

In order to set the stage for our calculations, and to prepare us with a feeling for the physical principles involved, it is helpful to consider for a moment what is known about convection in Newtonian theory. The Newtonian condition for stability against adiabatic convective motions in non-rotating stars has been derived by Schwarzschild (1906): the temperature gradient must not be superadiabatic (in other words, the radial derivative of the entropy per unit rest mass must not be negative). If this condition is violated, a fluid element which is displaced in the radial direction will be driven farther in the same direction by bouyant forces, because it will be less dense than the surrounding fluid at its new location by virtue of having the same pressure and greater entropy (we will call this a buoyant instability). This is a purely dynamical instability: it occurs without the help of any dissipative processes. Dissipative processes can only inhibit this type of instability in non-rotating configurations. Viscosity inhibits the motion. Thermal conduction tends to equalize the entropy of a displaced fluid element with that of its surroundings, and can therefore make the system tend toward neutral buoyant stability but cannot make it cross over the point of neutral stability.

In rotating stars, on the other hand, dissipative processes can cause purely secular instabilities: instabilities which would not exist in the absence of transport phenomena. Intuitively, we expect the following picture. The convective stability of a rotating star should depend on a combination of buoyant and rotational effects. If a star is in a state of neutral buoyant stability, then its overall stability depends entirely on rotational effects. If a fluid element in this star is displaced to a new location, stability will depend on whether the angular momentum per unit rest mass, $j = r^2\Omega$ (where r is the distance from the rotation axis and Ω is the angular velocity), of the displaced element is greater or smaller than that of the ambient fluid. If it came from a location closer to the rotation axis, and has a larger j than the surrounding fluid, it will be driven further outward, resulting in an instability. If the state of neutral buoyant stability exists by virtue of a spatially constant entropy per unit rest mass, then the resultant overall instability is a dynamical one whether thermal conduction is present or not. If thermal conduction were absent, and if the entropy gradient were adjusted to give a margin of buoyant stability, this effect could counterbalance the rotational instability and make the system stable overall. On the other hand, if thermal conductivity were then introduced, the margin of buoyant stability would be reduced and the rotational effects could dominate again and make the star unstable overall. This would be a purely secular instability.

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The analytical derivations of the Newtonian criteria for local stability in differentially rotating stars have been executed by Goldreich and Schubert (1967) and Fricke (1969). They have found that stellar models in which thermal conductivity is much more efficient than viscosity are stable against axisymmetric perturbations if the angular momentum per unit rest mass satisfies the two conditions

$$j,_r \geq 0 \tag{1}$$

and

$$j,_z = 0, \tag{2}$$

where z is the coordinate pointing along the axis of rotation. The conditions for stability in the absence of dissipative effects are

$$\frac{2\Omega}{r} j,_r - \frac{1}{C_P} \underset{\sim}{g} \cdot \underset{\sim}{\nabla} s \geq 0 \tag{3}$$

and

$$- g_z(j,_r s,_z - j,_z s,_r) \geq 0 . \tag{4}$$

(Fricke 1969, 1971), where $\underset{\sim}{g}$ is the acceleration due to gravity and motion, s is the entropy per unit rest mass, and C_P is the specific heat per unit rest mass at constant pressure.

A direct consequence of stability condition (2) is the requirement that inviscid, conducting stars be barytropic: the fluid quantities must be distributed in such a way that the pressure, P, and the rest mass density, ρ, have coincident level surfaces. That this is true can easily be demonstrated by taking the curl of the hydrodynamic equilibrium equation (the Euler equation),

$$\rho^{-1}\nabla P = \nabla U + r^{-3} j^2 \hat{r} \ , \tag{5}$$

where U is the gravitational potential, and finding that $j_{,z} = 0$ if and only if $\nabla\rho \times \nabla P = 0$. This is, of course, just the theorem of von Zeipel (1924).

So much for Newtonian theory. In the context of general relativity theory, Thorne (1966) has found the generalization of Schwarzschild's adiabatic convection criterion and has shown that it is essentially identical to the Newtonian criterion (neutral stability obtains when the gradient of the entropy per baryon vanishes). But the relativistic convection criteria for <u>rotating</u> stars have not been found, except for the special case of isentropic perfect fluid stars. Bardeen (1970) and Abramowicz (1974) have argued that, in this special case, neutral stability occurs when the angular momentum per baryon is not a function of position.

The more general set of relativistic criteria for local secular and dynamical stability will be derived in the remainder of this paper by using a method modeled after the one used by Goldreich and Schubert (1967) in their treatment of Newtonian stars. We will begin by writing out in §II the equations describing the general time dependent behavior of a non-magnetic fluid of uniform chemical composition with viscosity and thermal conduction. In §III these equations will be specialized to describe a time independent, axisymmetric, differentially rotating star. In §IV the unperturbed stellar model will be given a gentle pinch, changing the fluid quantities slightly in a small, localized region. Then we will stand back and let the time dependent equations of §II determine the time evolution of the perturbation. By restricting the perturbation in size and extent, we will be able to assume that the gravitational field (the metric) doesn't change, and we therefore will be spared the necessity of perturbing the field equations and dealing with the difficulties associated with gravitational radiation. The final result of §IV will be an equation ("dispersion relation") relating the temporal frequency of the perturbation to certain characteristics of the perturbation and the unperturbed model. Sections V and VI will use this equation to find the relativistic conditions for stability. While these two sections focus on the mathematical details of the derivations, §VII discusses the physical bases of the criteria and provides heuristic derivations which illustrate the important physical processes. The approximations that go into the various calculations will be tabulated in §VIII along with a discussion of the effects they are likely to have on the applicability of the results. Finally, §IX will contain a brief discussion of the application of the stability criteria to models of specific astrophysical objects.

In all sections of this paper except for §IX we will use units in which $G = c = 1$. In addition, we will adopt a mathematical convention in which the gradient operator $\underset{\sim}{\nabla}$ refers to the three dimensional spatial gradient operating as if the spatial coordinates described a flat space. In other words, if A is a spatial coordinate and X is any quantity (even a vector), then the A component of $\underset{\sim}{\nabla} X$ is equal to $X_{,A}$ rather than the covariant derivative $X_{;A}$. When we write the dot product $\underset{\sim}{C} \cdot \underset{\sim}{D}$

we will mean the sum over all spatial coordinates A of the terms $C_A D_A$. Similarly, we will from time to time use the curl operator and the cross product, and we will use them as though the covariant spatial components of any vector used in such a context resided in a flat three-space. These are only mathematical definitions, and are used in order to simplify the notation in certain calculations. All other mathematical notations will be conventional or self-explanatory, including the Einstein summation convention.

II. RELATIVISTIC FLUID DYNAMICS WITH VISCOSITY AND THERMAL CONDUCTION

In the context of general relativity theory, the state of any fluid system with a specified, uniform chemical composition and no macroscopic electromagnetic fields can be described by the following set of eighteen parameters:

a) the number density of baryons as measured in the rest frame of the fluid, N;

b) the isotropic pressure, P, measured in the fluid rest frame;

c) the total density of mass energy as measured in the rest frame of the fluid, ϵ;

d) the temperature, T, measured in the fluid rest frame;

e) the four components of the four velocity $\underset{\sim}{U}$, which describes the macroscopic motion of the fluid;

f) and the ten independent components of the metric tensor g, which describes the gravitational field.

Each of these parameters is, in general, a function of all four space-time coordinates x^α (where $x^0 = t$ and the remaining components are spatial). The complete specification of the behavior of the fluid consists in specifying the values assumed by all eighteen parameters at all values of the coordinates x^α.

The components of the metric tensor are determined by the Einstein field equations, which we will not discuss in full generality; in the next section we will discuss the form that they assume for the special system with which we will be working. Once the metric is known, the remaining quantities are determined by an equation specifying the normalization of the four velocity,

$$U^\alpha U_\alpha = -1; \tag{6}$$

the equation describing the conservation of baryons,

$$(NU^\alpha)_{;\alpha} = NU^\alpha{}_{;\alpha} + U^\alpha N_{,\alpha} = 0; \tag{7}$$

two equations of state describing the thermodynamic properties of the fluid material,

$$P = P(N, T) \tag{8}$$

and

$$\epsilon = \epsilon(N, T); \tag{9}$$

and the four equations of motion of the fluid,

$$T^{\alpha\beta}{}_{;\beta} = 0, \tag{10}$$

where T is the stress-energy tensor of the fluid (to be described shortly). The four equations (10) are most conveniently divided into two groups: one equation ensuring the conservation of energy,

$$U_\alpha T^{\alpha\beta}{}_{;\beta} = 0, \tag{11}$$

and three equations guaranteeing the conservation of momentum,

$$H_{\mu\alpha} T^{\alpha\beta}{}_{;\beta} = 0. \tag{12}$$

The tensor $\mathbf{H}$ is the so-called projection tensor,

$$H_{\alpha\beta} = U_\alpha U_\beta + g_{\alpha\beta}, \tag{13}$$

which, when contracted with any four vector, projects that vector into the three-surface orthogonal to the four velocity $\underset{\sim}{U}$.

One additional parameter which characterizes the thermodynamic state of the fluid is the entropy per baryon as measured in the fluid rest frame, S. This quantity is related to the other thermodynamic parameters through the first law of thermodynamics,

$$NT\, dS = d\epsilon - \frac{\epsilon + P}{N}\, dN. \tag{14}$$

The fluid which will concern us presently is a fluid in which the transport of thermal energy and the dissipative effects of viscosity and thermal conduction are important. The stress-energy tensor for a fluid of this type can be written in the following form (Eckart 1940):

$$T^{\alpha\beta} = (\epsilon + P)U^\alpha U^\beta + Pg^{\alpha\beta} - 2\eta\Sigma^{\alpha\beta} - \zeta U^\mu{}_{;\mu}\, H^{\alpha\beta} + U^\alpha q^\beta + U^\beta q^\alpha. \tag{15}$$

The first two terms in this expression comprise the standard perfect fluid stress-energy tensor. The next two terms arise from viscous effects. η is the coefficient of dynamic viscosity, ζ is the coefficient of bulk viscosity, and Σ is the shear tensor with covariant components

$$\Sigma_{\alpha\beta} = \frac{1}{2}\left(H^{\mu}{}_{\alpha}U_{\beta;\mu} + H^{\mu}{}_{\beta}U_{\alpha;\mu}\right) - \frac{1}{3}U^{\mu}{}_{;\mu}H_{\alpha\beta}. \tag{16}$$

The last two terms in the stress-energy tensor are due to the diffusion of heat. The vector $\underset{\sim}{q}$ is the heat flux vector, with contravariant components

$$q^{\alpha} = -KH^{\alpha\beta}\left(T_{,\beta} + Ta_{\beta}\right), \tag{17}$$

where K is the coefficient of thermal conductivity and $\underset{\sim}{a}$ is the four acceleration of the fluid with components $a_{\beta} = U^{\alpha}U_{\beta;\alpha}$.

By substituting this form for the stress-energy tensor into equations (11) and (12), we obtain the explicit forms of the equations of energy conservation,

$$\begin{aligned} U_{\alpha}T^{\alpha\beta}{}_{;\beta} = &- U^{\alpha}\epsilon_{,\alpha} - (\epsilon + P)\,U^{\alpha}{}_{;\alpha} - \left(\zeta - \frac{2}{3}\eta\right)\left(U^{\alpha}{}_{;\alpha}\right)^{2} \\ &+ \eta\left[U_{\alpha;\beta}\left(U^{\alpha;\beta} + U^{\beta;\alpha}\right) + a_{\alpha}a^{\alpha}\right] \\ &- q^{\alpha}{}_{;\alpha} - q^{\alpha}a_{\alpha} + U^{\alpha}{}_{;\alpha}\,U^{\beta}q_{\beta} = 0, \end{aligned} \tag{18}$$

and momentum conservation,

$$H^{\mu\alpha}T^{\alpha\beta}{}_{;\beta} = (\epsilon + P)a_\mu + P_{,\mu} + U_\mu U^\beta P_{,\beta} - \frac{2}{3} U_\mu \eta \, (U^\alpha{}_{;\alpha})^2$$

$$+ \eta \left[U_{\alpha;\beta} \, (U^{\alpha;\beta} + U^{\beta;\alpha}) + a_\alpha a^\alpha \right] U_\mu - \zeta U^\alpha{}_{;\alpha} a_\mu$$

$$- (\zeta U^\alpha{}_{;\alpha})_{,\mu} - U_\mu U^\beta \, (\zeta U^\alpha{}_{;\alpha})_{,\beta} - 2g_{\mu\alpha} \, (\eta \, \Sigma^{\alpha\beta}{}_{;\beta} + \Sigma^{\alpha\beta} \eta_{,\beta})$$

$$+ U^\alpha{}_{;\alpha} q_\mu + q^\alpha U_{\mu;\alpha} + U^\alpha q_{\mu;\alpha} - U_\mu a_\alpha q^\alpha = 0. \qquad (19)$$

By combining equation (18) with the baryon conservation equation and the first law of thermodynamics we can rewrite it in the form

$$NTU^\alpha S_{,\alpha} = NT \frac{dS}{d\tau} = - \left(\zeta - \frac{2}{3} \eta\right)(U^\alpha{}_{;\alpha})^2 + \eta \left[U_{\alpha;\beta} \, (U^{\alpha;\beta} + U^{\beta;\alpha}) + a_\alpha a^\alpha \right]$$

$$- q^\alpha{}_{;\alpha} - q^\alpha a_\alpha + U^\alpha{}_{;\alpha} \, U^\beta q_\beta \, , \qquad (20)$$

which relates the time derivative of S along the world line of a fluid element to various terms depending on viscous and thermal conduction effects.

III. THE UNPERTURBED STAR

Now, with equations in hand for describing the general, time dependent fluid system, we can discuss the nature of the stationary equilibrium model whose stability we wish to investigate. Stationary, in this context, will of course mean quasi-stationary, since no star with dissipative processes can be truly time independent. In order for our analysis to make sense, it is necessary only for the unperturbed stellar model to appear stationary on the time scale of the perturbations that we will study in the next section.

With this in mind, we will write the line element appropriate to the metric of a stationary, axisymmetric, differentially rotating configuration in the form

$$ds^2 = - e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + dz^2), \tag{21}$$

where r is the coordinate distance from the axis of rotation (which points in the z direction), ϕ is the aximuthal angle, and t is the coordinate time. The potentials ν, ψ, ω, and μ are functions of r and z only. These functions and their first derivatives are everywhere continuous, and are determined by a set of field equations that are written in appendix A. Bardeen (1970) gives a good description of the physical significance of the potentials. For our purposes it is necessary to know only that $\omega(r, z)$ is equal to the coordinate angular velocity $d\phi/dt$, or the angular velocity as seen by a distant observer, of any particle with zero angular momentum at position (r, z), and is called the angular velocity of the zero angular momentum observer at location (r, z).

So much for the gravitational field of our star. Next we must consider the physical state of the stellar fluid. The general motion of the fluid will be in the ϕ direction, i.e., around the z axis. Since energy transport by thermal conduction is to be taken into account, we should, strictly speaking, allow for the possibility of meridional currents, since we are familiar with the necessity of their existence in rotating stars from Newtonian theory (Eddington 1929). This necessity is connected with the fact that motion which is strictly azimuthal is inconsistent with the assumption of stationary equilibrium if energy transport occurs only through thermal conduction. It will be demonstrated in appendix C that the time scale over which this inconsistency manifests itself is much longer than the time scale we will be interested in; accordingly, we will ignore meridional motions and restrict the fluid to motions in the ϕ direction.

If we define Ω to be the angular velocity $d\phi/dt$ of a fluid element as seen by a distant observer, then we can write the contravariant components of the fluid four velocity $\underset{\sim}{U}$ as follows:

$$
\begin{aligned}
U^{0} &= e^{-\nu}(1 - v^{2})^{-\frac{1}{2}}, \\
U^{\phi} &= \Omega U^{0}, \\
U^{r} &= U^{z} = 0,
\end{aligned}
\tag{22}
$$

where $v = e^{\psi - \nu}(\Omega - \omega)$ is the velocity of a fluid element as measured by a zero angular momentum observer (Bardeen 1970). For future reference, we can also note the covariant components of $\underset{\sim}{U}$:

$$
\begin{aligned}
U_{0} &= -\left[e^{2\nu} + \omega(\Omega - \omega)e^{2\psi}\right] U^{0}, \\
U_{\phi} &= e^{2\psi}\,(\Omega - \omega)U^{0}, \\
U_{r} &= U_{z} = 0.
\end{aligned}
\tag{23}
$$

With the metric and four velocity that we have adopted, together with the assumption that our stellar model is stationary in time, we find that equation (19) gives us the following equation of hydrodynamic equilibrium (the relativistic Euler equation):

$$P_{,A} + (\epsilon + P)a_A = 0 \tag{24}$$

(where we have now adopted a notation in which indices written as capital Latin letters will indicate that only the coordinates r and z are to be considered). The four acceleration can be evaluated in terms of the four velocity:

$$a_A = U^o U_\phi \, \Omega,_A - (\ell n \, U^o),_A \, . \tag{25}$$

By substituting this expression for a_A into equation (24), and dividing by $(\epsilon + P)$, we obtain the Euler equation in a form which will be handy for us later:

$$(\epsilon + P)^{-1} \underset{\sim}{\nabla} P + U^o U_\phi \underset{\sim}{\nabla} \Omega - \underset{\sim}{\nabla} (\ell n \, U^o) = 0. \tag{26}$$

In a configuration with the characteristics we have specified, the heat flux vector $\underset{\sim}{q}$ has the following components (from eq. [17]):

$$\begin{aligned} q_r &= - K(T,_r + Ta_r) \, , \\ q_z &= - K(T,_z + Ta_z) \, , \\ q_o &= q_\phi = 0. \end{aligned} \tag{27}$$

As far as the specific thermodynamic properties of the fluid are concerned, we will leave the equations of state (8) and (9) in their general forms and consider them to be totally arbitrary for our present purposes.

The specific values of the coefficients of thermal conductivity and viscosity will be left unspecified for the time being also, but we will assume that they depend only on the thermodynamic characteristics of the fluid, so that their functional dependences can be indicated by writing them as $K(N, T)$, $\eta(N, T)$, and $\zeta(N, T)$. The thermal conductivity can occur, in general, by either radiative or molecular diffusion. The viscosity can be due to radiative or molecular velocity, but not to turbulent viscosity, which depends on other parameters in addition to the thermodynamic ones.

In closing this section, we will mention some quantities of general physical interest and some general characteristics of rotating equilibrium configurations. The quantity $\epsilon + P$ is the inertial mass per unit volume as measured in the rest frame of the fluid. It is this mass which determines how a fluid element responds to a force acting on it. The inertial mass per baryon is $(\epsilon + P)/N$. The quantity

$$E = - \frac{\epsilon + P}{N} U_o \tag{28}$$

is, in a sense, the inertial mass per baryon in the fluid referred to infinity (if a machine in the star which was comoving with the fluid were to take a parcel of energy $[\epsilon + P]/N$ and throw it out of the star "to infinity" [the energy needed to bring the parcel out of the gravitational field being taken from the energy in the parcel] with its angular momentum unchanged, a stationary observer at infinity would measure the energy content of the parcel to be E). The angular momentum per baryon,

$$J = \frac{\epsilon + P}{N} U_\phi , \tag{29}$$

is conserved in any axisymmetric, inviscid, adiabatic fluid motion (Bardeen 1970), but is not conserved in non-adiabatic motions (see appendix B). The quantity

$$L = J/E = - U_\phi / U_o \tag{30}$$

is, in a sense, the angular momentum per unit inertial mass of a fluid element. It is a purely geometrical quantity in the sense that $j = r^2 \Omega$ is in the Newtonian theory; it is independent of the thermodynamic properties of the fluid. In general, L is not conserved even in axisymmetric, inviscid, adiabatic motions (see appendix C), but can, under some circumstances, be conserved more nearly than J when thermal conduction occurs (see appendix C). Another quantity which could be called an angular momentum per unit inertial mass, and which is a purely geometrical quantity, is U_ϕ. But we will see that the quantity L will enter naturally into our calculations and will have some nice properties. Accordingly, we will henceforth refer to L as the geometrical angular momentum.

Now, in the Newtonian theory of rotating stellar equilibrium configurations, the level surfaces of the angular velocity Ω always coincide with the level surfaces of the angular momentum per unit rest mass $j = r^2 \Omega$ if the fluid is barytropic; i.e., if the level surfaces of pressure and rest mass density coincide (von Zeipel's theorem). If we want to know what the equivalent situation is in the relativistic theory, we must be careful about which relativistic quantities we identify with which Newtonian quantities. In the non-relativistic limit, ϵ, $\epsilon + P$, and $-(\epsilon + P)U_o$ all reduce to the rest mass density ρ, while J and L both reduce to the angular momentum per unit rest mass j. If we compare the relativistic and non-relativistic Euler equations, we might be tempted to identify the Newtonian rest mass density with the quantity $\epsilon + P$, and to define the appropriate relativistic analogue of the barytropic fluid as one in which the level surfaces of pressure coincide with the level surfaces of the inertial mass density $\epsilon + P$ (or, equivalently, one in which $P = P[\epsilon]$). If we then take the relativistic Euler equation,

$$(\epsilon + P)^{-1} \underset{\sim}{\nabla} P + U^o U_\phi \underset{\sim}{\nabla}\Omega - \underset{\sim}{\nabla} (\ell n \, U^o) = 0, \tag{31}$$

and take its curl, we obtain

$$\underset{\sim}{\nabla}(\epsilon + P)^{-1} \times \underset{\sim}{\nabla} P + \underset{\sim}{\nabla}(U^o U_\phi) \times \underset{\sim}{\nabla}\Omega = 0. \tag{32}$$

The quantity $\underset{\sim}{\nabla}(U^o U_\phi)$ can be rewritten in the form

$$\underset{\sim}{\nabla}(U^o U_\phi) = (U^o U_o)^2 (\underset{\sim}{\nabla} L + L^2 \underset{\sim}{\nabla}\Omega), \tag{33}$$

and substituted into equation (32) to yield

$$\underset{\sim}{\nabla}(\epsilon + P)^{-1} \times \underset{\sim}{\nabla} P + (U^o U_o)^2 \underset{\sim}{\nabla} L \times \underset{\sim}{\nabla}\Omega = 0, \tag{34}$$

from which we see that the level surfaces of L and Ω coincide if and only if the fluid in the equilibrium configuration is barytropic. If we consider the special subclass of barytropic configurations in which $\underset{\sim}{\nabla} S = 0$, we obtain the well known result that the level surfaces of J and Ω coincide (Bardeen 1970), since

$$\underset{\sim}{\nabla} L = \frac{1}{U^o U_o^2} \frac{N}{\epsilon + P} (\underset{\sim}{\nabla} J + U_\phi T \underset{\sim}{\nabla} S) \tag{35}$$

in an equilibrium fluid. In non-isentropic stars $\underset{\sim}{\nabla} J \times \underset{\sim}{\nabla} \Omega \neq 0$.

Abramowicz (1974) has proven a number of interesting theorems about the topological properties of rotating, barytropic stars, including one which will be of use to us a bit later: the level surfaces of Ω, and therefore the level surfaces of L, have the topology of a cylinder.

IV. THE PERTURBED STAR

a) Perturbations in the Fluid Quantities

Now that we've chosen an equilibrium stellar model, we can perturb it a bit and see what happens. We will take the fluid variables N, T, U^r, U^z, and U^ϕ, and change them by a very small amount (maintaining axisymmetry) in a very small region of the star at some initial time, say $t = 0$. If X is one of these quantities, we will write

$$X_*(r, z, t = 0) = X(r, z) + \delta X(r, z, t = 0), \tag{36}$$

where X is the value of the variable in the unperturbed configuration, X_* is the perturbed value of X at the same coordinate position, and δX is the Eulerian perturbation in X. The perturbation in U^o can be found from equation (6), which tells us that

$$\delta U^o = - (U_\phi / U_o) \, \delta U^\phi = L \delta U^\phi . \tag{37}$$

The perturbations $\delta\epsilon$ and δP will be determined by δN and δT through the equations of state.

Our procedure will be to substitute the perturbed variables into the various equations governing their behaviors, and to disregard all terms which are non-linear in the perturbed quantities. This process will give us a set of linear equations describing the evolution of the perturbation. If we restrict the perturbation to a region whose dimension, say λ, is much smaller than the scale height of the star, R, then the coefficients of the perturbed variables in our linear equations will be essentially constant over the perturbed region. It is therefore convenient to do a Fourier decomposition of each perturbed quantity,

$$\delta X(r,z,t=0) = \int_{-\infty}^{\infty} dk_r \int_{-\infty}^{\infty} dk_z \; \delta X(k_r,k_z) \; e^{-i(k_r r + k_z z)} , \tag{38}$$

and to follow the time evolution of only one Fourier component, because each component will evolve independently of the others. Our results will be quite general if we use the following form for the initial perturbation:

$$X_*(r,\; z,\; t=0) = X(r,\; z) + \delta X \; e^{i(k_r r + k_z z)} , \tag{39}$$

where δX is now a constant.

The square of the total "wave number" of the perturbation is

$$k_T^2 = g^{AB} k_A k_B = e^{-2\mu} \; (k_r^2 + k_z^2). \tag{40}$$

Since the perturbation occurs only in a region whose dimension is much smaller than the scale height R, we will be interested primarily in perturbations for which $k_T^{-1} \sim \lambda \ll R$, so that from now on we will make the assumption that

$$(k_T R)^{-1} \ll 1. \tag{41}$$

Now we will allow our perturbations to have a time dependence:

$$X_*(r, z, t) = X(r, z) + \delta X e^{\sigma t} e^{-i(k_r r + k_z z)}. \tag{42}$$

This done, we can easily find the perturbations in the components of the four acceleration; they are

$$\delta a^o = (2U^o{}_{;r} - U^o{}_{,r})\delta U^r + (2U^o{}_{;z} - U^o{}_{,z})\delta U^z + \sigma L U^o \delta U^\phi,$$

$$\delta a^\phi = (2U^\phi{}_{;r} - U^\phi{}_{,r})\delta U^r + (2U^\phi{}_{;z} - U^\phi{}_{,z})\delta U^z + \sigma U^o \delta U^\phi,$$

$$\delta a^r = \sigma U^o \delta U^r + e^{2(\psi + \nu - \mu)} U_o^{-3} \gamma_r \delta U^\phi,$$

$$\delta a^z = \sigma U^o \delta U^z + e^{2(\psi + \nu - \mu)} U_o^{-3} \gamma_z \delta U^\phi, \tag{43}$$

where the symbols γ_r and γ_z represent the functions

$$\gamma_r = 2e^{2(\mu - \psi - \nu)} U^o U_o^3 \left[L\Gamma^r_{oo} + (1 + \Omega L)\Gamma^r_{oo} + \Omega\Gamma^r_{oo} \right]$$

and

$$\gamma_z = 2e^{2(\mu - \psi - \nu)} U^o U_o^3 \left[L\Gamma^z_{oo} + (1 + \Omega L)\Gamma^z_{oo} + \Omega\Gamma^z_{oo} \right], \tag{44}$$

the symbols $\Gamma^\alpha_{\mu\nu}$ being the connection coefficients calculated in the usual way from the metric. The functions γ_r and γ_z will turn out to be very important. They can be written (as we discover by calculating the connection coefficients) as the components of the spatial vector

$$\underset{\sim}{\gamma} = (U^o U_o)^2 \left[2(\Omega - \omega)(\underset{\sim}{\nabla}\psi - \underset{\sim}{\nabla}\nu) - (1 + v^2)\, \underset{\sim}{\nabla}\omega \right]. \tag{45}$$

It is possible to verify that $\underset{\sim}{\gamma}$ can also be written in the form

$$\underset{\sim}{\gamma} = (U^o U_o)^2 \left[(1 - v^2) e^{-2\psi} U_o^2 \underset{\sim}{\nabla} L - \underset{\sim}{\nabla}\Omega \right] , \qquad (46)$$

which will turn out to be useful.

b) The Perturbed Equations

Now we have the general form for the perturbations in all of the parameters describing our fluid system except for the metric (which will be discussed shortly), including their specific values at the initial time $t = 0$. In order to judge whether or not a given equilibrium configuration is stable, we must ask the fully time-independent equations of §II how our perturbation will evolve in time. To do this, we will take an initial perturbation with some specific but arbitrary "wave vector" (specific values of k_r and k_z), and ask whether the temporal frequency σ can have a positive real part. Clearly, if σ has a positive real part the perturbation will grow in time, which means that the stellar model is unstable against perturbations with the given "wave vector." On the other hand, if σ has no positive real part, we conclude that our equilibrium configuration is stable against the perturbation at hand.

Schematically, the mathematics will work like this. Let's represent one of the equations governing the system as $\mathcal{E}(X) = 0$, where $\mathcal{E}$ is some function of all the parameters, which we represent now by the single parameter X. If we substitute into this equation the perturbed, time dependent value of X as written in equation (42), we will find

$$0 = \mathcal{E}(X_*) = \mathcal{E}(X) + F(X, \delta X, \sigma, k_r, k_z)\, e^{\sigma t - i(k_r r + k_z z)} + \mathcal{O}\left[(\delta X)^2 \right] , \qquad (47)$$

where F is some function of X, δX, σ, k_r, and k_z which is linear in δX. The condition that the unperturbed value of X describes an equilibrium configuration is the condition that $\mathcal{E}(X) = 0$. If we take this into account and neglect all terms in equation (47) of order $(\delta X)^2$ or higher, we are left with the equation that governs our linear perturbation:

$$F(X, \delta X, \sigma, k_r, k_z) = 0. \tag{48}$$

Our immediate task is to calculate in this manner the perturbed versions of the baryon conservation equation, the equations of motion, and the equations of state.

That we will be spared the necessity of perturbing the field equations can be seen in the following way. The role played by these equations is that of supplying us with the metric. Since the metric is generated by the stress-energy in the fluid, the metric will necessarily change as a result of the perturbations in the fluid quantities. Perturbations in the metric will contribute to perturbations in the equations of motion, since the metric makes its presence known through the covariant derivatives which appear in these equations. But these perturbations will not concern us, for the following reasons. Let π be one of the fluid quantities which contributes to the stress-energy of the fluid (for example ϵ). Let $\delta\pi/\pi = \beta$. The assumption that the perturbation is small in amplitude means that $\beta \ll 1$. Therefore, the smallest terms which are retained in the linear perturbation of the equations of motion are terms of order β with respect to the unperturbed terms. If $\delta\pi/\pi = \beta$ everywhere in the star, then we would expect the metric functions, for example ψ, to change by an amount $\delta\psi \sim \beta\psi$. But since the perturbation in π vanishes everywhere outside of a region of size $\lambda \ll R$, we can expect that the potential ψ will change by an amount $\delta\psi \sim \alpha\beta\psi$, where $\alpha \ll 1$, since, by analogy with the Newtonian theory, the gravitational potentials should depend on integrals over all of their sources in the star. This means that $\delta\psi/\psi \ll \delta\pi/\pi$; the perturbations in the metric functions will not contribute to the first order perturbations in the equations of motion, so that the metric of the unperturbed star can be used in all of our calculations.

Following the procedure outlined above, we can find the relevant perturbed equations. Since the equations will be linear, it will be possible to write each of them in the form

$$\sum_i a_i \delta X_i = 0, \tag{49}$$

where, in general, each coefficient a_i will be a sum of a very large number of terms built from σ, k_r, k_z, and the parameters describing the unperturbed configuration. If we compare all of the terms in each a_i, we will find that many of them will be negligible with respect to the others by virtue of the fact that $(k_T R)^{-1} \ll 1$. Taking this into account, and doing the required calculations, we find the perturbed equations. The r component of the equation $H_{\mu\beta}T^{\mu\nu}{}_{;\nu} = 0$ becomes

$$\Big\{ e^{2\mu}\sigma U^{o}(\epsilon + P) + 2ik_r q_r + ik_z q_z - e^{2\mu}\sigma^2 U^{o2} KT$$

$$+ (\zeta + \tfrac{1}{3}\eta)k_r^2 + e^{2\mu}\eta \left[k_T^2 - \sigma^2(g^{oo} + U^{o2}) \right] \Big\} \delta U^r$$

$$+ \left[ik_z q_r + (\zeta + \tfrac{1}{3}\eta)k_r k_z \right] \delta U^z$$

$$+ \left[(\epsilon + P) - \tfrac{1}{2}\sigma U^{o}KT)\, e^{2(\psi + \nu)}\, U_o^{-3}\gamma_r + \sigma L q_r - i\sigma L\, (\zeta + \tfrac{1}{3}\eta)k_r \right] \delta U^{\phi}$$

$$+ ik_r \delta P + q_r \delta\epsilon - ik_r \sigma U^{o} K \delta T$$

$$\sigma U^{o} q_r K^{-1} \delta K + g^{o\alpha}\, (U_\alpha a_r + U_{\alpha;r} + U_{r;\alpha})\delta\eta = 0. \tag{50}$$

The z component gives us a similar equation. The ϕ component gives

$$\Big\{ (\epsilon + P) \left[g_{\phi o}\, (2U^{o}{}_{;r} - U^{o}{}_{,r}) + g_{\phi\phi}\, (2U^{\phi}{}_{;r} - U^{\phi}{}_{,r}) \right] + U_\phi P_{,r}$$

$$+ 2\sigma U^{o}KT\, (U_\phi a_r - U_{\phi,r}) - i(\zeta + \tfrac{1}{3}\eta)k_r \sigma U^{o} U_\phi \Big\} \delta U^r$$

$$+ \Big\{ (\epsilon + P) \left[g_{\phi o}\, (2U^{o}{}_{;z} - U^{o}{}_{,z}) + g_{\phi\phi}\, (2U^{\phi}{}_{;z} - U^{\phi}{}_{,z}) \right] + U_\phi P_{,z}$$

$$+ 2\sigma U^{o}KT\, (U_\phi a_z - U_{\phi,z}) - i(\zeta + \tfrac{1}{3}\eta)k_z \sigma U^{o} U_\phi \Big\} \delta U^z$$

$$\Big[-e^{2(\psi + \nu)} U^{o} U_o^{-1} \Big\{ \sigma U^{o}(\epsilon + P) + ik_A q^A - \sigma^2 U^{o2} KT + \eta \left[k_T^2 - \sigma^2\, (g^{oo} + U^{o2}) \right] \Big\}$$

$$- (\zeta + \tfrac{1}{3}\eta)\, U^{o} U_\phi \sigma^2 L \Big] \delta U^{\phi} \tag{51}$$

(cont)

$$+ \sigma U^o U_\phi \delta P - K(\sigma^2 U^{o2} U_\phi + ik^A U_{\phi;A})\delta T$$

$$+ q^A(U_{\phi,A} - U_\phi a_A)K^{-1}\delta K - ik^A\,(U_\phi a_A + U_{\phi;A} + U_{A;\phi})\delta\eta = 0. \tag{51}$$

The equation $U_\mu T^{\mu\nu}{}_{;\nu} = 0$ becomes

$$\left[- NTS_{,A} + ik_A \sigma U^o KT + 2\eta\sigma g^{o\alpha}\,(U_\alpha a_A + U_{\alpha;A} + U_{A;\alpha})\right]\delta U$$

$$\Big[(\sigma^2 U^o L + ie^{2(\psi+\nu)}\, U_o^{-3} k_A \gamma^A)$$

$$+ 2i\eta k^A \left\{(U_{\phi;A} + U_{A;\phi}) + L(U_{o;A} + U_{A;o})\right\}\Big]\,\delta U^\phi$$

$$- \sigma U^o \delta\epsilon + \sigma U^o\,(\epsilon + P)N^{-1}\delta N - K\left[k_T^2 - \sigma^2\,(g^{oo} + U^{o2})\right]\delta T$$

$$- ik_A q^A K^{-1}\delta K + \left[U_{\alpha;\beta}\,(U^{\alpha;\beta} + U^{\beta;\alpha}) + a_A a^A\right]\delta\eta = 0. \tag{52}$$

The baryon conservation equation yields

$$ik_r\,\delta U^r + ik_z\,\delta U^z + \sigma L\,\delta U + \sigma U^o N^{-1}\delta N = 0 \tag{53}$$

and the equations of state become

$$\delta P = (\partial P/\partial N)_T \delta N + (\partial P/\partial T)_N \delta T \tag{54}$$

and

$$\delta\epsilon = (\partial\epsilon/\partial N)_T \delta N + (\partial\epsilon/\partial T)_N \delta T. \tag{55}$$

The notation $(\quad)_A$ indicates that the derivative in parentheses is to be executed with the quantity A held constant.

These equations can be simplified a bit. To begin with, we are looking at instabilities that are caused by the star's rotation, so we might as well restrict ourselves to perturbations which have time scales (σ^{-1}) related to the rotational time scale (Ω^{-1}):

$$\sigma \sim \Omega. \tag{56}$$

Secondly, the perturbations in the fluid four velocity should be related to the ratio of the characteristic size of the perturbation and the characteristic time of the perturbation:

$$\delta U^r \sim \delta U^z \sim R\,\delta U^\phi \sim \Omega/k_T. \tag{57}$$

This condition will allow direct comparison of the coefficients of the different components of $\delta \underset{\sim}{U}$ in each equation, and the elimination of some terms via the approximation $(k_T R)^{-1} \ll 1$.

Since we will be interested in the ways in which thermal conduction and viscosity will affect stability, we will want to look at perturbations whose sizes are such that the time scale of interest is comparable to the time scale for thermal diffusion ($t_T \sim NT(\partial S/\partial T)_P/Kk_T^2$; see appendix B) or the time scale for the diffusion of momentum ($t_M \sim (\epsilon + P)/\eta k_T^2$; see eq. [19]), whichever is larger. Accordingly, we will assume, for the present, that

$$\Omega \sim Kk_T^2(\partial T/\partial S)_P/NT \sim \eta k_T^2/(\epsilon + P). \tag{58}$$

Because perturbations in the pressure will be carried away by sound waves, they will tend to dissipate with a time scale t_P of order

$1/k_T c_s$, where c_s is the sound speed. This means that the ratio of the time scales for the dissipation of perturbations in T and P is

$$t_T/t_P \sim c_s k_T/\Omega \gtrsim k_T R \gg 1, \tag{59}$$

since $\Omega R \lesssim c_s$. This means that the perturbations in P always die away much faster than those in T, so that we can write

$$\delta P/P \sim (k_T R)^{-1} \, \delta T/T \tag{60}$$

and use this relationship to compare the coefficients of δP with the coefficients of δT. Given equation (60), we can appeal to equation (54) to see that

$$\delta N/N \sim \delta T/T. \tag{61}$$

The perturbations in η and K cannot be determined explicitly, since we do not know the details of their dependences on N and T, but we can say that

$$\delta\eta \sim \text{the maximum of } [(\partial\eta/\partial N)_T \delta N, \ (\partial\eta/\partial T)_N \delta T] \tag{62}$$

and

$$\delta K \approx \text{the maximum of } [(\partial K/\partial N)_T \delta N, \ (\partial K/\partial T)_N \delta T]. \tag{63}$$

By using these approximations to order the various terms in each equation, and by using the easily verified equality

$$(\epsilon + P)\left\{ g_{0\phi}(2U^0{}_{;A} - U^0{}_{,A}) + g_{\phi\phi}(2U^\phi{}_{;A} - U^\phi{}_{,A}) \right\} + U_\phi P_{,A}$$

$$= N(J_{,A} - U_\phi TS_{,A}) = U^0 U_0^2 \, (\epsilon + P) L_{,A} \quad , \tag{64}$$

we can rewrite the perturbed equations in much simpler forms:

$$\left[\sigma U^o e^{2\mu}(\epsilon + P) + \eta e^{2\mu} k_T^2 + (\zeta + \tfrac{1}{3}\eta)k_r^2\right] \delta U^r + (\zeta + \tfrac{1}{3}\eta)k_r k_z \delta U^z$$

$$+ U_o^{-3} e^{2(\psi+\nu)} \gamma_r \, \delta U^\phi + ik_r \delta P + a_r \delta\epsilon = 0, \tag{65}$$

$$(\zeta + \tfrac{1}{3}\eta)k_r k_z \delta U^r + \left[\sigma U^o e^{2\mu}(\epsilon + P) + \eta e^{2\mu} k_T^2 + (\zeta + \tfrac{1}{3}\eta)k_z^2\right] \delta U^z$$

$$+ U_o^{-3} e^{2(\psi+\nu)} \gamma_z \, \delta U^\phi + ik_z \delta P + a_z \delta\epsilon = 0, \tag{66}$$

$$U^o U_o^2(\epsilon + P)L,_r \delta U^r + U^o U_o^2(\epsilon + P)L,_z \delta U^z$$

$$- U^o U_o^{-1} e^{2(\psi+\nu)} \left[\sigma U^o(\epsilon + P) + \eta k_T^2\right] \delta U^\phi = 0, \tag{67}$$

$$-NTS,_r \delta U^r - NTS,_z \delta U^z - \sigma U^o \delta\epsilon + \sigma U^o(\epsilon + P)N^{-1}\delta N - Kk_T^2 \delta T = 0, \tag{68}$$

$$ik_r \delta U^r + ik_z \delta U^z + \sigma U^o N^{-1} \delta N = 0, \tag{69}$$

$$(\partial P/\partial N)_T \delta N + (\partial P/\partial T)_N \delta T = 0, \tag{70}$$

and

$$\delta\epsilon - (\partial\epsilon/\partial N)_T \delta N - (\partial\epsilon/\partial T)_N \delta T = 0. \tag{71}$$

Now, equations (65) through (71) are a set of seven linear, homogeneous equations for the seven quantities δN, δT, $\delta\epsilon$, δP, δU^r, δU^z, and δU^ϕ, and thus the determinant of the coefficients in these equations should vanish. We can therefore confidently calculate this determinant, eliminate some of the terms with the use of condition (41), equate the result to zero, and divide by the quantity $- k_T^2 e^{2(2\mu + \psi + \nu)} U^{o^4} U_o^{-1} (\epsilon + P)^2 NT(\partial S/\partial T)_P$ to obtain the following equation relating the perturbation frequency σ to k_r, k_z, and certain features of the unperturbed equilibrium configuration:

$$\sigma^3 + \sigma^2 U^{o^{-1}} k_T^2 \left[\frac{2\eta}{\epsilon + P} + \frac{K}{NT}\left(\frac{\partial T}{\partial S}\right)_P\right]$$

$$+ \sigma U^{o^{-2}} \left\{ e^{-4\mu} \left(\frac{k_z}{k_T}\right)^2 \left(L,_r - \frac{k_r}{k_z} L,_z\right)\left(\gamma_r - \frac{k_r}{k_z}\gamma_z\right) \right.$$

$$- e^{-4\mu}\left(\frac{k_z}{k_T}\right)^2 \frac{1}{\epsilon + P}\left(\frac{\partial \epsilon}{\partial S}\right)_P \left(a_r - \frac{k_r}{k_z} a_z\right)\left(S,_r - \frac{k_r}{k_z} S,_z\right)$$

$$\left. + k_T^4 \left(\frac{\eta}{\epsilon + P}\right)\left[\frac{\eta}{\epsilon + P} + \frac{2K}{NT}\left(\frac{\partial T}{\partial S}\right)_P\right] \right\}$$

$$+ e^{-4\mu} U^{o^{-3}} k_z^2 \frac{K}{NT}\left(\frac{\partial T}{\partial S}\right)_P \left(L,_r - \frac{k_r}{k_z} L,_z\right)\left(\gamma_r - \frac{k_r}{k_z}\gamma_z\right)$$

$$- e^{-4\mu} U^{o^{-3}} k_z^2 \left(\frac{\eta}{\epsilon + P}\right) \frac{1}{\epsilon + P}\left(\frac{\partial \epsilon}{\partial S}\right)_P \left(a_r - \frac{k_r}{k_z} a_z\right)\left(S,_r - \frac{k_r}{k_z} S,_z\right)$$

$$+ U^{o^{-3}} k_T^6 \left(\frac{\eta}{\epsilon + P}\right)^2 \frac{K}{NT}\left(\frac{\partial T}{\partial S}\right)_P = 0. \qquad (72)$$

V. STABILITY IN THE PRESENCE OF TRANSPORT PHENOMENA

Equation (72) is just a cubic equation for σ, and in principle the roots of this equation determine the time evolution of a perturbation with given values of k_r and k_z. We cannot solve explicitly for the three roots of this equation, but we can specify a necessary condition for the absence of unstable roots: those with positive real parts. If the three roots have the values a, b, and c, then equation (72) can be written

$$(\sigma - a)(\sigma - b)(\sigma - c) = 0, \tag{73}$$

from which we see that the constant term is equal to - abc. If this quantity is negative, then at least one of the roots has a positive real part. Therefore, a necessary condition for the stability of a stellar model is that the inequality

$$\begin{aligned} & e^{-4\mu} U^{o^{-3}} k_z^2 \frac{K}{NT} \left(\frac{\partial T}{\partial S}\right)_P \left(L,_r - \frac{k_r}{k_z} L,_z\right)\left(\gamma_r - \frac{k_r}{k_z} \gamma_z\right) \\ & - e^{-4\mu} U^{o^{-3}} k_z^2 \left(\frac{\eta}{\epsilon + P}\right) \frac{1}{\epsilon + P} \left(\frac{\partial \epsilon}{\partial S}\right)_P \left(a_r - \frac{k_r}{k_z} a_z\right)\left(S,_r - \frac{k_r}{k_z} S,_z\right) \\ & + U^{o^{-3}} k_T^6 \left(\frac{\eta}{\epsilon + P}\right)^2 \frac{K}{NT} \left(\frac{\partial T}{\partial S}\right)_P \geq 0 \end{aligned} \tag{74}$$

be satisfied everywhere in the configuration for all values of k_r and k_z.

In a given stellar model, the dimensionless ratio

$$Z \equiv \frac{K}{\eta} \frac{(\epsilon + P)}{NT} \left(\frac{\partial T}{\partial S}\right)_P , \tag{75}$$

which measures the relative efficiencies of conduction and viscosity, will have some specific value (or range of values). Stability condition (74) is most easily interpreted if we consider separately the three different kinds of circumstances in which $Z \gg 1$ (thermal conduction is much more efficient than viscosity), $Z \sim 1$ (thermal conduction and viscosity have comparable efficiencies), and $Z \ll 1$ (conduction is much less efficient than viscosity).

When $Z \gg 1$, a fluid element will be able to exchange heat with its surrounding on the time scale of interest but will not be significantly affected by viscous interactions. In this case we find, with the help of equation (58), that inequality (74) can be written

$$(L,_r + \xi L,_z)(\gamma_r + \xi\gamma_z) \geq 0, \tag{76}$$

where $\xi \equiv - k_r/k_z$. Since we would like this condition to hold for perturbations with all reasonable values of k_r and k_z, we must insist that it hold for all values of ξ. To aid us in imposing this condition, we will first rewrite inequality (76) in the form

$$L,_z\gamma_z\xi^2 + (L,_r\gamma_z + L,_z\gamma_r)\xi + L,_r\gamma_r \geq 0. \tag{77}$$

This condition just says that a quadratic function of ξ, $f(\xi) = a\xi^2 + b\xi + c$, must never be negative. In order to insure that this be true, we must insist that $a + c \geq 0$ and $b^2 - 4ac \leq 0$:

$$L,_r\gamma_r + L,_z\gamma_z \geq 0, \tag{78}$$

$$(L,_r\gamma_z - L,_z\gamma_r)^2 \leq 0. \tag{79}$$

Inequality (79) can be satisfied only if the quantity in parentheses vanishes; the necessary conditions for stability can thus be written

$$\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} L \geq 0 \tag{80}$$

and

$$\underset{\sim}{\gamma} \times \underset{\sim}{\nabla} L = 0. \tag{81}$$

In other words, the vector field $\underset{\sim}{\nabla} L$ must always be parallel to the vector field $\underset{\sim}{\gamma}$.

We can easily find a direct physical consequence of condition (81) by referring to the Euler equation and asking under what circumstances an equilibrium configuration can have a distribution of L with the required property. To this end, we will first rewrite equation (81) using the expression (46) for $\underset{\sim}{\gamma}$, and we will find

$$\underset{\sim}{\nabla} L \times \underset{\sim}{\nabla} \Omega = 0, \tag{82}$$

which, as we know from §III, is consistent with the Euler equation if and only if the stellar model is barytropic.

Now that we know that a stable star must be barytropic, we can use this fact to help us interpret condition (80). We know that the two surfaces orthogonal to the vector field $\underset{\sim}{\gamma}$ coincide with the level surfaces of L, by condition (81), and we know that in a barytropic star these level surfaces have the topology of cylinders (as discussed in §III). Condition (80) tells us that the gradient of L must point in the same direction as $\underset{\sim}{\gamma}$, so a knowledge of whether the topological orientation of $\underset{\sim}{\gamma}$ is inward with respect to its quasi-cylindrical orthogonal surfaces (pointing toward the rotation axis from the surfaces) or outward (pointing toward infinity) will help us determine how L must vary between its level surfaces.

In fact, the orientation of $\underset{\sim}{\gamma}$ is always outward in the interior of a stable, barytropic stellar model, as we will demonstrate in appendix C.

By combining this fact with condition (80), we see that $\underset{\sim}{\nabla}L$ must always be oriented outward.

Putting together the foregoing results, we see that <u>two necessary conditions for stability in regions of stellar models in which viscosity is negligible with respect to thermal conduction are that i) the level surfaces of the pressure and the total mass-energy density must coincide and ii) the gradient of the geometrical angular momentum must never point inward from a surface of constant L.</u>

Now we will turn our attention to the situation in which $Z \ll 1$, where the perturbation will be affected by viscosity but not by heat conduction. Inequality (74) can then be written

$$- (\partial\epsilon/\partial S)_P (a_r + \xi a_z)(S_{,r} + \xi S_{,z}) \geq 0 \tag{83}$$

and treated as we treated condition (76) to find that the necessary stability conditions are

$$- (\partial\epsilon/\partial S)_P \, \underset{\sim}{a} \cdot \underset{\sim}{\nabla} S \geq 0 \tag{84}$$

and

$$- (\partial\epsilon/\partial S)_P \, \underset{\sim}{a} \times \underset{\sim}{\nabla} S = 0. \tag{85}$$

If we refer to equations (24), (34), and (46), we find that

$$\underset{\sim}{\gamma} \times \underset{\sim}{\nabla} L = (\epsilon + P)^{-1} \, (\partial\epsilon/\partial S)_P \, \underset{\sim}{a} \times \underset{\sim}{\nabla} S. \tag{86}$$

By combining conditions (84) and (85) with equations (24), and (86), we see that <u>a stable star in which viscosity is more efficient than thermal conduction is barytropic and has a pressure gradient which is parallel to the vector field</u> $\underline{(\partial\epsilon/\partial S)_P \underset{\sim}{\nabla} S}$.

When thermal conduction and viscosity are both important ($Z \sim 1$), condition (74) can be written in the form

$$Z\,(L_{,r} + \xi L_{,z})(\gamma_r + \xi\gamma_z)$$

$$- \frac{1}{\epsilon + P}\left(\frac{\partial\epsilon}{\partial S}\right)_P (a_r + \xi a_z)(S_{,r} + \xi S_{,z})$$

$$+ e^{2\mu} k_T^4 (1 + \xi^2)\, Z \left(\frac{\eta}{\epsilon + P}\right)^2 \geq 0 \tag{87}$$

and treated in the usual way to yield the stability criteria

$$Z\nabla L \cdot \gamma - (\epsilon + P)^{-1} (\partial\epsilon/\partial S)_P\, a \cdot \nabla S \geq - e^{2\mu} k_T^4\, Z \left(\frac{\eta}{\epsilon + P}\right)^2 \tag{88}$$

and

$$4Z\,(\epsilon + P)^{-1} (\partial\epsilon/\partial S)_P\, (a \times \gamma) \cdot (\nabla S \times \nabla L) + (Z - 1)^2\, (\gamma \times \nabla L)^2$$

$$\leq e^{2\mu} k_T^4\, Z \left(\frac{\eta}{\epsilon + P}\right)^2 \left[e^{2\mu} k_T^4\, Z \left(\frac{\eta}{\epsilon + P}\right)^2 + Z\,\gamma \cdot \nabla L \right.$$

$$\left. - (\epsilon + P)^{-1} (\partial\epsilon/\partial S)_P\, a \cdot \nabla S \right] . \tag{89}$$

These conditions can be interpreted in two ways. They indicate, given a particular equilibrium configuration, what sizes of disturbances will be damped out. Or they tell us that if we want stability against disturbances with any value of k_T <u>the necessary conditions for stability for any Z are</u>

$$Z\gamma \cdot \nabla L - (\epsilon + P)^{-1} (\partial\epsilon/\partial S)_P\, a \cdot \nabla S \geq 0 \tag{90}$$

<u>and</u>

$$4Z\,(\epsilon + P)^{-1} (\partial\epsilon/\partial S)_P\, (a \times \gamma) \cdot (\nabla S \times \nabla L) + (Z - 1)^2\, (\gamma \times \nabla L)^2 \leq 0. \tag{91}$$

VI. THE STABILITY OF PERFECT FLUID MODELS

We can investigate the possibilities for instabilities in perfect fluid models by looking at the characteristics of equation (72) with all of the dissipative terms neglected. In this case we set $K = \eta = 0$, and the equation becomes

$$\sigma^2 - e^{-4\mu}\left(\frac{k_z}{k_T}\right)^2 U^{o^{-2}} \frac{1}{\epsilon + p}\left(\frac{\partial\epsilon}{\partial s}\right)_p \left(a_r - \frac{k_r}{k_z} a_z\right)\left(S,_r - \frac{k_r}{k_z} S,_z\right)$$
$$+ e^{-4\mu}\left(\frac{k_z}{k_T}\right)^2 U^{o^{-2}}\left(L,_r - \frac{k_r}{k_z} L,_z\right)\left(\gamma_r - \frac{k_r}{k_z}\gamma_z\right) = 0 \quad . \tag{92}$$

Following the same reasoning that we applied to the cubic equation discussed in the last section, we conclude that the constant term in equation (84) must be positive if there are to be no unstable solutions for σ. By insisting that this be true for any value of the quantity $-k_r/k_z$, we find that <u>two conditions which must be fulfilled for local stability in perfect fluid stellar models are</u>

$$\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} L + (\partial\epsilon/\partial S)_P \underset{\sim}{\nabla} P \cdot \underset{\sim}{\nabla} S \geq 0 \tag{93}$$

<u>and</u>

$$(\partial\epsilon/\partial S)_P (\underset{\sim}{\gamma} \times \underset{\sim}{\nabla} P) \cdot (\underset{\sim}{\nabla} S \times \underset{\sim}{\nabla} L) \geq 0 \quad . \tag{94}$$

In isentropic models the only condition is that

$$\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} L \geq 0 \quad . \tag{95}$$

VII. A PHYSICAL ANALYSIS OF THE STABILITY CRITERIA

a) Stability in the Presence of Transport Phenomena

The constraints which stability imposes on an equilibrium configuration can be understood with the help of a heuristic derivation that will illustrate the physical processes involved and simultaneously provide a mathematical check on the results of the more rigorous analytical derivations that we have already executed. For simplicity, let us first limit our attention to the equatorial plane of a stellar model and consider the situation in which we have thermal conduction but not viscosity. What would happen to a small ring of fluid if it were displaced slowly, in an axisymmetric manner, and with L held fixed (in appendix C it is demonstrated that L is conserved in the type of motion we are interested in), from its original position ("location 1," with coordinates $r = r_1$, $z = 0$) to a new, nearby position ("location 2," with coordinates $r = r_2 = r_1 + \Delta r$, $z = 0$), held momentarily stationary at this position, and then released? Its subsequent motion would depend on its acceleration relative to the surrounding fluid at its new location (let's call this acceleration A_r). An acceleration in the direction of the displacement ($A_r/\Delta r > 0$) would indicate a convective instability, while an acceleration back toward the original location of the fluid ring ($A_r/\Delta r \leq 0$) would indicate stability.

Let us denote the value of each quantity describing the fluid that was originally at location 1 by the subscript 1, the value of the quantities describing that same fluid after it has been displaced to location 2 by the subscript 2, and the quantities describing the fluid originally at location 2 by no subscript. If π is one of these quantities, then

$$\delta\pi = \pi_2 - \pi \tag{96}$$

is the Eulerian change in π at position 2, and

$$\Delta\pi = \pi_2 - \pi_1 \tag{97}$$

is the Lagrangian change in π for the displaced fluid ring. The Eulerian and Lagrangian changes are related to each other by the equation

$$\delta\pi = \Delta\pi - \pi,_r\Delta r \quad , \tag{98}$$

where $\pi,_r$ is the radial derivative of π in the original configuration.

Now, the radial acceleration of any fluid element, as measured by an observer in a local Lorentz frame momentarily comoving with the fluid, consists of an acceleration due to gravitational and centrifugal effects, namely $- e^{-\mu}a_r$, and a buoyant acceleration, $- e^{-\mu}(\epsilon + P)^{-1}P,_r$. The total radial acceleration of an equatorial fluid element in an equilibrium configuration is

$$A_r = -e^{-\mu}a_r - e^{-\mu}(\epsilon + P)^{-1}P,_r = 0 \quad , \tag{99}$$

and therefore the acceleration of the displaced fluid element can be written

$$\begin{aligned} A_{r2} = A_{r2} - A_r &= -e^{-\mu}(a_{r2} - a_r) - e^{-\mu}\left[(\epsilon_2 + P)^{-1} - (\epsilon + P)^{-1}\right]P,_r \\ &= -e^{-\mu}\left[\delta a_r - (\epsilon + P)^{-2}P,_r\delta\epsilon\right] \end{aligned} \tag{100}$$

(where δP has been equated to zero, since the displacement occurred subsonically, allowing the pressure of the displaced fluid to adjust to that of its new surroundings).

The Eulerian change in a_r was calculated in §IV. If we set $\delta U^r = 0$ (since we are here holding the displaced fluid element momentarily stationary), we find

$$\delta a_r = e^{2(\psi+\nu)} U_o^{-3} \gamma_r \delta U^\phi = -(U^o U_o^2)^{-1} \gamma_r \delta U_\phi \quad . \tag{101}$$

According to equation (91), the Eulerian change in U^ϕ is

$$\delta U_\phi = \Delta U_\phi - U_{\phi,r} \Delta r \quad . \tag{102}$$

The value of $U_{\phi,r}$ can be found through a straightforward differentiation of L:

$$U_{\phi,r} = U^o U_o^2 L,_r + U_\phi a_r \quad . \tag{103}$$

The value of ΔU_ϕ can be found by using the fact that ΔL vanishes. By combining equations (6) and (30), and setting $U_r U^r + U_z U^z$ equal to zero (it vanishes in our linear approximation), we can write

$$U_\phi = L(g^{oo} - 2Lg^{o\phi} + L^2 g^{\phi\phi})^{-2} \tag{104}$$

and calculate ΔU_ϕ, setting $\Delta L = 0$, to find

$$\Delta U_\phi = U_\phi a_r \Delta r \quad . \tag{105}$$

By combining equations (101), (102), (103), and (105), we find that the Eulerian change in a_r is

$$\delta a_r = \gamma_r L,_r \Delta r \quad .$$

In stellar models with thermal conduction, we considered perturbations whose sizes were small enough to allow efficient thermal coupling between the perturbed and unperturbed fluids. In the present case, that is essentially equivalent to setting $\delta T = 0$. Since $\delta P = 0$ also, and since any two thermodynamic parameters completely determine the thermodynamic state of a fluid, it follows that $\delta\epsilon = 0$. In this case, then, the total radial acceleration of the displaced fluid ring is just

$$A_{r2} = -e^{-\mu}\delta a_r = -e^{-\mu}\gamma_r L,_r \Delta r \quad . \tag{107}$$

In order to avoid instability it is necessary that $A_{r2}/\Delta r \leq 0$, so the criterion for stability is

$$\gamma_r L,_r \geq 0 \quad , \tag{108}$$

which is identical to condition (80). We know from appendix D that γ_r is always positive in a stable configuration, so our stability criterion means that $L,_r$ cannot be negative.

Why should we expect that stability should depend on L in this manner? Why might it not depend, for example, on the distribution of J, instead? We have displaced a fluid element, forcing it to always be thermodynamically indistinguishable from the ambient fluid at its new location. This means that the fluid is essentially in a state of neutral buoyant stability. Except for rotational effects, a fluid element can move about freely without feeling any net forces. Overall stability, or the lack of it, depends, therefore, only on the rotational behavior of the fluid. If the angular momentum per baryon in the displaced fluid is different from that in the surrounding fluid, all other aspects of the two fluids being identical, we would expect the displaced fluid to feel an excess centrifugal force and to move in the direction of that force. Therefore a comparison of J between the ambient and displaced fluids determines whether the model is stable in the present case. But J was not conserved in the motion of the displaced element from its original position (see appendix C), so we can't expect to judge stability by looking at the distribution of J in the equilibrium configuration. The value of J in the ambient fluid must be compared with the value of J which the displaced fluid has by virtue of having conserved L during its displacement. Since the displaced and ambient fluids are thermodynamically identical, comparing J at the final location is equivalent to comparing L; thus the dependence on the gradient of L.

When viscosity is important and thermal conduction can be neglected, it is appropriate to assume that the displaced fluid will assume the same value of Ω as that of the ambient fluid. This means that $\delta U^{\phi} = 0$, which means in turn that $\delta a_r = 0$. According to equation (52), there is no first order change in the entropy of the fluid element, which means that the Lagrangian change in ϵ is

$$\Delta\epsilon = (\partial\epsilon/\partial P)_S P,_r \Delta r \quad . \tag{109}$$

The radial derivative of ϵ can be written

$$\epsilon,_r = (\partial\epsilon/\partial P)_S P,_r + (\partial\epsilon/\partial S)_P S,_r \tag{110}$$

and combined with equations (91) and (102) to yield a value for the Eulerian change in ϵ:

$$\delta\epsilon = -(\partial\epsilon/\partial S)_P S,_r \Delta r \quad . \tag{111}$$

The total radial acceleration of the displaced fluid ring is then

$$A_{r2} = e^{-\mu}(\epsilon + P)^{-1}(\partial\epsilon/\partial S)_P a_r S,_r \Delta r \quad , \tag{112}$$

according to equation (100). The stability condition $A_{r2}/\Delta r \leq 0$ duplicates the analytically derived criterion (84).

When viscosity and thermal conduction are both very efficient, we expect that the viscosity will eliminate differences in Ω in between the displaced and ambient fluids while the thermal conduction will eliminate differences in thermodynamic properties. These effects should damp out any disturbance regardless of the nature of the angular momentum and entropy distributions. If we look at the analytically derived condition (88) we will see that when $Z \sim 1$ the term on the right-hand side, which is always negative, provides a larger margin of stability the larger k_T becomes; the smaller the disturbance, the harder it is to make the angular momentum and entropy gradients bad enough to cause an instability.

b) The Stability of Perfect Fluid Models

In the case of perfect fluid flow, the quantities S, L, and J are all conserved during the motion of our displaced fluid element. The value of $\delta\epsilon$ is given by equation (111) and δa_r can be found from equation (106). The total radial acceleration is

$$A_{r2} = -e^{-\mu}\left[\gamma_r L,_r - (\epsilon + P)^{-1}(\partial\epsilon/\partial S)_P a_r S,_r\right]\Delta r \quad , \tag{113}$$

and the stability condition $A_{r2}/\Delta r \leq 0$ becomes identical to criterion (85) for dynamical stability in perfect fluid stars.

In this case the overall stability of the system depends on a balancing of buoyant and rotational effects. It is particularly clear here that it is the geometrical angular momentum rather than the angular momentum per baryon which determines the effect of rotation on stability. The stability criterion indicates that when the ambient and displaced fluids have different thermodynamic properties it is the value of L which an element has that determines its rotational acceleration.

c) How to Apply the Stability Criteria

In order to apply the stability criteria to a particular model, it is first necessary to determine the magnitudes of the transport coefficients. Then, for convenience, we can make the definitions

$$\lambda_T \equiv \left[K(\partial T/\partial S)_P/NT\Omega\right]^{1/2} \tag{114}$$

and

$$\lambda_v \equiv \left[\eta/(\epsilon + P)\Omega\right]^{1/2} \quad . \tag{115}$$

The quantity λ_T is the maximum size of a disturbance for which thermal dissipation occurs on the time scale of interest, while λ_v is the analogous quantity for viscous dissipation. Disturbances on a large enough scale,

$$\lambda \gg \text{ the maximum of } (\lambda_T, \lambda_V) \quad , \tag{116}$$

will be essentially unaffected by the transport phenomena. As long as λ is still much smaller than the scale height R, stability against these disturbances is determined by the perfect fluid stability criteria. Disturbances on a small enough scale,

$$\lambda \ll \text{ the minimum of } (\lambda_T, \lambda_V) \quad , \tag{117}$$

will always be damped out by the dissipative effects.

If $\lambda_V \ll \lambda_T$, then disturbances in the range $\lambda_V \ll \lambda \lesssim \lambda_T$ will be governed by the conditions that were derived under the assumption that $Z \gg 1$. The conditions for disturbances of smaller λ become gradually weaker as λ decreases through the range $\lambda \sim \lambda_V$. If $\lambda_T \ll \lambda_V$, then disturbances in the range $\lambda_T \ll \lambda \lesssim \lambda_V$ are governed by the $Z \ll 1$ criteria. If $\lambda_T \sim \lambda_V$ $(Z \sim 1)$, it is necessary to apply the stability conditions (90) and (91).

In general, the conditions which govern stability against a particular size of disturbance can be regarded as necessary criteria for the stability of the star, but the star is really subject to the strongest stability constraints which follow from a consideration of all relevant $(\lambda \ll R)$ sizes of disturbances.

d) The Effects of Relativity

When cast in the proper language, the relativistic criteria for local stability are identical in form to the non-relativistic criteria. Thorne (1966) discovered that this was true in non-rotating configurations in the absence of dissipation. He pointed out that the non-linear effects of the relativistic theory of gravity should manifest themselves only over finite distances and should not, therefore, affect local processes such as those governing the onset of convection.

VIII. LIMITATIONS ON THE APPLICABILITY OF THE STABILITY CRITERIA

At this point we might pause and consider the limitations which might be placed on our results by the assumptions that have gone into our calculations. First of all we should recall that, because of the nature of their derivations, our stability criteria are necessary but not necessarily sufficient. In particular, we have considered only axisymmetric processes; it is possible that unstable non-axisymmetric modes might be available to stars which are stable against all axisymmetric distrubances. In addition to this limitation, the calculations are vulnerable to the following approximations and assumptions:
i) The perturbations which we have studied were short in wavelength (local) and were not self-gravitating (did not affect the metric). Some of the consequences of these assumptions have been investigated in Newtonian theory, and may give a partial indication of what could be expected in the relativistic theory. Fricke (1971) has shown that the global (long wavelength) criteria for stability in rotating perfect fluid stars are the same as the local criteria as long as the gravitational field remains unperturbed. He discussed the possibility that gravitational perturbations might have a destabilizing influence, but his results were not conclusive. For local perturbations in non-rotating

stars, Lebovitz (1965) has shown that gravitational perturbations do not alter the conditions for stability.

ii) Our results were derived using linear perturbation theory, which governs the behavior of disturbances only as long as they are small in amplitude. Thus, when our treatment indicates that a mode is unstable and grows exponentially in time, we can't predict how rapidly it will evolve once it enters the non-linear regime. James and Kahn (1970) have indicated that, in the Newtonian theory, the growth rate for local instabilities in rotating stars is diminished by non-linear effects when the amplitudes become large. But since we have really been interested in the conditions for the onset of instability, rather than the details of the consequences of instability, non-linear effects will not change our results. The same can probably be said of gravitational perturbations.

iii) Only regions of stars in which the chemical composition is homogeneous were properly represented in our derivations. Goldreich and Schubert (1967) have found in Newtonian theory that gradients of chemical composition can strongly affect the conditions for stability, and can, in particular, stabilize some configurations which would otherwise be unstable.

iv) We have completely ignored magnetic fields. Fricke (1969) has studied the effects of magnetic fields on the stability of Newtonian stars, and has found that toroidal fields supply a stabilizing influence in rotating stars.

IX. APPLICATIONS TO MODELS OF ASTROPHYSICAL OBJECTS

a) Non-relativistic Stars

For application to stars in which relativistic structure effects are unimportant, we can take the Newtonian limits of our results, and we will find that they reduce to the standard Newtonian criteria, as they should. In the non-relativistic limit the quantities which appear in the relativistic equations behave in the following manner:

$$L \rightarrow r^2\Omega = j \quad , \tag{118}$$

$$\underset{\sim}{\gamma} \rightarrow 2\Omega r^{-1}\hat{r} \quad , \tag{119}$$

$$\underset{\sim}{a} \rightarrow -\underset{\sim}{g} \quad , \tag{120}$$

$$NT(\partial S/\partial T)_P \rightarrow \rho^{-1}C_P \quad , \tag{121}$$

$$(\epsilon + P)(\partial S/\partial \epsilon)_P \rightarrow \rho(\partial S/\partial \rho)_P \quad , \tag{122}$$

$$Z = \frac{K}{\eta}\,\frac{\epsilon + P}{NT}\left(\frac{\partial T}{\partial S}\right)_P \rightarrow \frac{1}{C_P}\,\frac{K}{\eta} \quad . \tag{123}$$

When, in addition to being non-relativistic, the fluid being described obeys the ideal gas equation of state

$$P = \frac{\rho\kappa T}{m} \tag{124}$$

(where m is the mean molecular weight and κ is the Boltzmann constant), equation (122) reduces to

$$(\epsilon + P)(\partial S/\partial \epsilon)_P \rightarrow -\rho^{-1}C_P \quad . \tag{125}$$

Substitution of these quantities into the relativistic stability criteria for $Z \gg 1$ fluids and perfect fluids reproduces the Newtonian criteria written in §I (those equations all assume the ideal gas equation of state).

In the radiative zones of non-massive non-relativistic stars the value of Z is always very large (Goldreich and Schubert 1967, Fricke 1969), being equal, for example, to about 10^6 in the sun. This means that in realistic Newtonian stellar models the convective stability criteria assume the simple forms of the $Z \gg 1$ stability conditions.

b) Supermassive Stars

Supermassive stars have nearly Newtonian structures (Fowler 1964) which are essentially isentropic (Wagoner 1969). The pressure in a supermassive star includes the gas pressure of the ideal gas equation of state and a contribution from radiation:

$$P = \frac{\rho \kappa T}{m} + \frac{a}{3} T^4 \quad . \tag{126}$$

This equation is valid for stellar regions in which $T \lesssim 10^9$, in which case the electrons are non-relativistic and there is no pair production (Wagoner 1969). The ratio of gas pressure to total pressure, β, is quite small in supermassive stars, and is approximately

$$\beta \simeq 4.28 \; (m_p/m) \; \mathcal{M}^{-1/2} \tag{127}$$

(Wagoner 1969), where $\mathcal{M}$ is the mass of the star in units of the solar mass and m_p is the mass of a proton.

Under these circumstances both thermal conduction and viscosity are dominated by radiative diffusion, and the appropriate transport coefficients are

$$K = \frac{16aT^3}{3\kappa\rho} \tag{128}$$

(Schwarzschild 1958) and

$$\eta = \frac{16aT^4}{15c^2\kappa} \tag{129}$$

(Thomas 1930), where a is the Stefan-Boltzmann constant and c is the speed of light. The ratio (123) is therefore

$$Z = \frac{5c^2}{TC_P} \quad . \tag{130}$$

Chandrasekhar (1939) has calculated the values of the specific heats for fluids with radiation, and has found that

$$C_P = \frac{(\gamma - 1)(4 - 3\beta)\Gamma_2}{\beta^2(\Gamma_2 - 1)} c_V \quad , \tag{131}$$

where γ is the ratio of specific heats for the ideal gas part of the fluid, c_V is the specific heat per mass at constant volume for an ideal gas,

$$c_V = \frac{\kappa}{m(\gamma - 1)} \quad , \tag{132}$$

and Γ_2 is defined by the equation

$$\frac{\Gamma_2}{\Gamma_2 - 1} = \frac{P}{T}\left(\frac{\partial T}{\partial P}\right)_S \quad . \tag{133}$$

For small β, Γ_2 is approximately 4/3. If we assume, for simplicity, that we are dealing with a hydrogen plasma, then $m = m_p/2$, $\gamma = 5/3$, and C_p is approximately

$$C_p = \frac{24}{m_p} \frac{\kappa}{\beta^2} \quad . \tag{134}$$

Combining equations (127), (130), and (133) and substituting the values of the physical constants yields

$$Z = \frac{1.7 \times 10^{-3}}{T_9 \mathfrak{M}_8} \quad , \tag{135}$$

where T_9 is temperature in units of 10^9 Kelvin and $\mathfrak{M}_8$ is the mass of the star in units of 10^8 solar masses. Evidently, $Z \lesssim 1$ for most simple supermassive star models of the type we are discussing. When $Z \sim 1$, the stability criteria (90) and (91) indicate that stability will depend equally on the distributions of L and S. But $\nabla S \simeq 0$, and so stability will in this case depend essentially on ∇L; a well chosen angular momentum gradient could make the star stable against convection. For large enough masses and temperatures, $Z \ll 1$. In this case we probably can't simply use the $Z \ll 1$ stability criteria, because for small enough ∇S the terms depending on $Z\nabla L$ and ∇S could be comparable in equations (90) and (91); a careful examination of the specific qualities of the model must then be done.

c) White Dwarf Stars

White dwarfs evolve through stages in which they are relatively hot (directly following their formation) and settle down into cold, nearly isentropic states. The time scale for cooling is at least of order $NT(\partial S/\partial T)_P/KR^2$, which is longer than the time scale of interest to us by a factor of $(Rk_T)^2 \gg 1$. This means that any non-magnetic, chemically homogeneous, differentially rotating white dwarf model must satisfy our stability conditions.

In white dwarfs the viscosity is dominated by electron diffusion because of the long mean free paths of the degenerate electrons (Durisen 1973). The thermal conductivity due to electron diffusion becomes larger than the conductivity due to radiative diffusion for $T \lesssim 10^7$ Kelvin and $\rho \gtrsim 10^4$ grams/cm^3 (Cox and Giuli 1968). Under these circumstances, the value of $\mathcal{Z}$ can be calculated with the conductivity and viscosity of the electron gas only, using the fact that

$$\frac{K_e M_e}{\eta_e \mathcal{C}_V^e} \simeq 1 \tag{136}$$

(Durisen 1973), where K_e and η_e are the electron gas conductivity and viscosity, M_e is the mean mass-energy per electron in the fluid rest frame, and $\mathcal{C}_V^e$ is the specific heat per electron at constant volume. If we define M to be the mean mass-energy per particle (including ions and electrons), then we can write

$$\mathcal{Z} = \frac{K}{\eta}\,\frac{\epsilon + P}{NT}\left(\frac{\partial T}{\partial S}\right)_P \simeq \frac{K}{\eta}\,\frac{M}{\mathcal{C}_P} \simeq \frac{K_e}{\eta_e}\,\frac{M_e}{\mathcal{C}_V^e}\quad\frac{M}{M_e}\,\frac{\mathcal{C}_V^e}{\mathcal{C}_V}\,\frac{\mathcal{C}_V}{\mathcal{C}_P} \simeq \frac{M}{M_e}\,\frac{\mathcal{C}_V^e}{\mathcal{C}_V}\,\frac{\mathcal{C}_V}{\mathcal{C}_P}\ , \tag{137}$$

where $\mathcal{C}_V$ and $\mathcal{C}_P$ are the specific heats per particle at constant volume and pressure, respectively. When the electrons are degenerate, the pressure of the fluid is essentially dependent only on the electron number density and is independent of the temperature. Since the specific heat at constant volume is the same as the specific heat at constant electron number density, we can deduce that $\mathcal{C}_V \simeq \mathcal{C}_P$ and

$$Z \simeq \frac{M}{M_e} \frac{\mathcal{C}_V^e}{\mathcal{C}_V} . \tag{138}$$

For simplicity, let us assume now that the fluid is composed of helium ions with number density n_i and electrons with number density $n_e = 2n_i$. Then, if M_i is the mean mass-energy per ion, we have

$$M = \frac{n_i M_i + n_e M_e}{n_i + n_e} \simeq \frac{M_i}{3} \simeq \frac{4m_p}{3} , \tag{139}$$

where m_p is the rest mass energy of a proton. Although the ions are not relativistic, the electrons may be. The relativity parameter

$$x \equiv \frac{p_f c}{m_e} \tag{140}$$

(where m_e is the rest mass energy of an electron and p_f is the Fermi momentum of the electron gas) can be expressed in terms of the total mass density of our fluid:

$$\rho = 1.96 \times 10^6 \, x^3 \text{ gm cm}^{-3} \tag{141}$$

(Chandrasekhar 1939). In a highly degenerate electron gas the number density of electrons as a function of momentum is proportional to p^2 for $p < p_f$ and vanishes for $p > p_f$ (Chandrasekhar 1939). The mean total mass-energy per electron is therefore

$$M_e = \overline{(p^2c^2 + m_e^2)^{1/2}} \simeq m_e(1 + \frac{3}{5} x^2)^{-1/2} . \tag{142}$$

This implies that

$$\frac{M}{M_e} \simeq \frac{4}{3} \left(1 + \frac{3}{5} x^2\right)^{-1/2} \frac{m_p}{m_e} = 2.44 \times 10^3 \left(1 + \frac{3}{5} x^2\right)^{-1/2} . \tag{143}$$

All that remains to be done in order to calculate Z is to find the ratio $\mathcal{C}_V/\mathcal{C}_V^e$. According to Chandrasekhar (1939), the appropriate value for $\mathcal{C}_V^e$ is

$$\mathcal{C}_V^e = \pi^2 \frac{\kappa T}{m} \frac{(1 + x^2)^{1/2}}{x^2} = 1.67 \times 10^{-3} \frac{(1 + x^2)^{1/2}}{x^2} T_6 \, \kappa , \tag{144}$$

where T_6 is the temperature in units of 10^6 Kelvin. By writing

$$\mathcal{C}_V = \frac{n_i \mathcal{C}_V^i + n_e \mathcal{C}_V^e}{n_i + n_e} , \tag{145}$$

where $\mathcal{C}_V^i = (3/2)\,\kappa$ is the specific heat per ion at constant volume, we can see that

$$\frac{\mathcal{C}_V}{\mathcal{C}_V^e} = \frac{3\mathcal{C}_V^e}{\mathcal{C}_V^i + 2\mathcal{C}_V^e} = \frac{T_6}{3 \times 10^2 \, x^2 \, (1 + x^2)^{-1/2} + .67\, T_6} . \tag{146}$$

The result of combining equations (137), (141), (143), and (146) is

$$Z \simeq 2.4\left(1 + 2.2 \times 10^{-3} \frac{(1 + x^2)^{1/2}}{x^2} T_6\right) \left(\frac{1 + x^2}{1 + (3/5)\, x^2}\right)^{1/2} \rho_7^{-2/3} \, T_6 , \tag{147}$$

where ρ_7 is the mass density in units of 10^7 gm cm^{-3}. The quantity in the first set of parentheses is always of order unity as long as $T_6 \leq 10$ and $\rho \geq 10^4$ gm cm^{-3}. If $T_6 \leq 1$ and $\rho \geq 10^5$ gm cm^{-3}, it is equal to 1 to within 1.6 percent, in which case

$$Z \simeq 2.4\left(\frac{1 + x^2}{1 + (3/5)\, x^2}\right)^{1/2} \rho_7^{-2/3} \, T_6 \quad (T_6 \leq 1,\ \rho_7 \geq 10^{-2}) . \tag{148}$$

The function $[(1 + x^2)/(1 + (3/5)x^2)]^{1/2}$ is a slowly varying function of the mass density (through eq. [141]), equal to 1.03 at $\rho = 10^5$ gm cm^{-3} and 1.26 at $\rho = 10^8$ gm cm^{-3}. Roughly, then,

$$Z \sim 2.4\, \rho_7^{-2/3}\, T_6 \qquad (T_6 \leq 10,\ \rho_7 \geq 10^{-3}) \quad . \tag{149}$$

Since the pressure is relatively insensitive to the temperature in the domain of interest, the hydrodynamic structure of the white dwarf will not change very much as the star cools. This means that the value of Z will essentially decrease monotonically with the temperature. In low density configurations ($\rho \lesssim 10^5$ gm cm^{-3}) Z will be considerably larger than unity as long as the temperature is high. In this situation, our calculations indicate that stability will depend primarily on ∇L. As the star cools, Z will become smaller and give ∇S a larger role in determining stability. But S, and therefore ∇S, will decrease monotonically with decreasing T; this means that ∇L will probably continue to be the dominant factor in determining stability as the star cools.

In higher density stars Z never exceeds unity by a large margin. A detailed application of the stability conditions (90) and (91) must then be made in order to determine whether a given model is stable.

d) Neutron Stars

The transport properties of the material which constitutes neutron stars are not well understood. If a region of a neutron star is a degenerate neutron fluid with no superfluid properties, then Z will most likely be of order unity since both energy and momentum will be transported by the same particles. If superfluidity occurs, the situation is more speculative.

APPENDIX A

THE GRAVITATIONAL FIELD EQUATIONS

The Einstein field equations which determine the potentials that appear in the time independent metric whose line element is

$$ds^2 = -e^{2\nu}dt^2 + e^{2\psi}(d\phi - \omega dt)^2 + e^{2\mu}(dr^2 + dz^2) \qquad (A1)$$

can be written in the forms

$$e^{-\psi}\nabla^2 e^{\psi} + \nabla^2\mu + \tfrac{1}{4}\, e^{2(\psi - \nu)}\underset{\sim}{\nabla}\omega\cdot\underset{\sim}{\nabla}\omega = -\,8\pi e^{2\mu}\left[\frac{\epsilon + P}{1 - v^2} - P\right] , \qquad (A2)$$

$$e^{-\nu}\nabla^2 e^{\nu} + \underset{\sim}{\nabla}\nu\cdot\underset{\sim}{\nabla}\psi - \tfrac{1}{2}\, e^{2(\psi - \nu)}\underset{\sim}{\nabla}\omega\cdot\underset{\sim}{\nabla}\omega = 4\pi e^{2\mu}\left[(\epsilon + P)\,\frac{1 + v^2}{1 - v^2} + 2P\right] , \qquad (A3)$$

$$e^{-\nu}\nabla^2 e^{\nu} + \nabla^2\mu - \tfrac{3}{4}\, e^{2(\psi - \nu)}\underset{\sim}{\nabla}\omega\cdot\underset{\sim}{\nabla}\omega = 8\pi e^{2\mu}\left[(\epsilon + P)\,\frac{v^2}{1 - v^2} + P\right] , \qquad (A4)$$

$$\underset{\sim}{\nabla}\cdot\left[e^{3\psi - \nu}\underset{\sim}{\nabla}\omega\right] = -\,16\pi e^{2(\psi + \mu)}\,\frac{(\epsilon + P)v}{1 - v^2} , \qquad (A5)$$

and

$$e^{-(\psi + \nu)}\nabla^2 e^{\psi + \nu} = 16\pi e^{2\mu}P , \qquad (A6)$$

which have been adapted from Chandrasekhar and Friedman (1972). The gradient operator and the dot product are calculated in the flat r, z two-space. Bardeen and Wagoner (1971) write the line element and the field equations in slightly different forms which, for some calculations, might be more convenient.

APPENDIX B

THE TIME SCALE FOR THERMAL DISSIPATION

It is possible to find the time scale which characterizes the dissipation of perturbations in the temperature in the following way. Let Q be the heat content per unit volume as measured in the rest frame of the fluid. Then, if we adopt a Lorentz frame of reference which is momentarily comoving with the fluid, and if we assume that there is no energy generation in the fluid element under consideration, we can write a continuity equation for Q:

$$\frac{dQ}{dt} = - \underset{\sim}{\nabla} \cdot \underset{\sim}{q} \ , \tag{B1}$$

where $\underset{\sim}{q}$ is our heat flux vector. The spatial components of $\underset{\sim}{q}$ in our frame of reference are just

$$q_{,A} = - KT_{,A} \ . \tag{B2}$$

Now let us add a small amount of heat, δQ, to a fluid ring of size λ in the unperturbed configuration. How long will it take for the heat δQ to dissipate? In order to answer that question, we note first that, to first order, the Lagrangian change in position of our fluid element will vanish, because we have started in a state in which $U^r = U^z = 0$. This means that the pressure of the fluid element will remain constant, since $\delta P = 0$ to first order. Therefore, the change with time of δQ can be written

$$\frac{d}{dt}\,\delta Q = \left(\frac{dQ}{dT}\right)_P \frac{d}{dt}\,\delta T = NT\left(\frac{\partial S}{\partial T}\right)_P \frac{d}{dt}\,\delta T. \tag{B3}$$

The diffusion equation which governs the temperature perturbation can be found by combining equations (B1), (B2), and (B3):

$$\frac{d}{dt}\,\delta T = -\,\frac{K}{NT}\left(\frac{\partial T}{\partial S}\right)_P \nabla^2 \delta T \sim -\,\frac{1}{\lambda^2}\,\frac{K}{NT}\left(\frac{\partial T}{\partial S}\right)_P \delta T. \tag{B4}$$

The time scale for the dissipation of δT is clearly $\lambda^2 NT(\partial S/\partial T)_P/K$.

APPENDIX C

NOTES ON THE CONSERVATION OF ANGULAR MOMENTUM

If viscosity is neglected, the azimuthal and time components of the equation $T_{\alpha\ ;\beta}^{\ \beta} = 0$ can each be calculated and combined with the baryon conservation equation (7) to yield, respectively,

$$NU^{\beta}\left(\frac{\epsilon + P}{N} U_{\phi}\right)_{,\beta} = NdJ/d\tau$$

$$= \frac{1}{2}\left(\frac{\epsilon + P}{N}\right) U^{\alpha}U^{\beta}g_{\alpha\beta,\phi} - P_{,\phi} - q^{\beta}U_{\phi;\beta} - U_{\phi}q^{\beta}_{\ ;\beta} - U^{\beta}q_{\phi;\beta} - q_{\phi}U^{\beta}_{\ ;\beta} \tag{C1}$$

and

$$NU^{\beta}\left(\frac{\epsilon + P}{N} U_{o}\right)_{,\beta} = -NdE/d\tau$$

$$= \frac{1}{2}\left(\frac{\epsilon + P}{N}\right) U^{\alpha}U^{\beta}g_{\alpha\beta,o} - P_{,o} - q^{\beta}U_{o;\beta} - U_{o}q^{\beta}_{\ ;\beta} - U^{\beta}q_{o;\beta} - q_{o}U^{\beta}_{\ ;\beta} \ . \tag{C2}$$

Equation (C1) tells us that the time derivative of J along the world line of a fluid element in an axisymmetric system can be written

$$dJ/d\tau = \frac{1}{N}\left(q^{\beta}U_{\phi;\beta} + U_{\phi}q^{\beta}_{\ ;\beta} + U^{\beta}q_{\phi;\beta} + q_{\phi}U^{\beta}_{\ ;\beta}\right) . \tag{C3}$$

If there is no thermal conduction, then $dJ/d\tau = 0$; the angular momentum per baryon is conserved. But if thermal conduction can occur, the unperturbed motion ascribed to the models discussed in §III are not consistent with a time independent angular momentum per baryon. Using the unperturbed fluid parameters, we find from equation (C3) that

$$\frac{dJ}{d\tau} \sim \frac{1}{N}\frac{KT}{R^2} U_{\phi} = J\ \frac{KT}{(\epsilon + P)R^2} \ , \tag{C4}$$

which indicates an apparent change of J on a time scale of order $R^2(\epsilon + P)/KT$. But the time scale we are interested in is the time scale associated with the dissipation of heat in the perturbation, which was found in appendix B, and which is smaller than the present time scale by a factor of $(\lambda^2/R^2)\ NT^2(\partial S/\partial T)_P/(\epsilon + P)$. The quantity $NT^2(\partial S/\partial T)_P$ is essentially the thermal energy density; this means that $NT^2(\partial S/\partial T)_P/(\epsilon + P) \lesssim 1$. The ratio of the time scale over which the inconsistency of the constancy of J manifests itself to the time scale $\lambda^2(\epsilon + P)/KT$ is therefore at most of order $(\lambda^2/R^2) << 1$. For our purposes here, there is no contradiction because J appears constant.

For the perturbed motion, on the other hand, equation (C3) indicates that the time scale for changes in J is of order $\lambda^2(\epsilon + P)/KT$. The ratio of this time scale to the time scale of interest is $NT^2(\partial S/\partial T)_P(\epsilon + P) \lesssim 1$. Thus, we cannot assume that the angular momentum per baryon is conserved in the perturbed motions we are studying if thermal conduction can occur.

The geometrical angular momentum, on the other hand, <u>is</u> conserved in the perturbed motion on the time scale of interest, as long as viscosity is absent. In order to demonstrate this, we can note that

$$dL/d\tau = d(J/E)/d\tau = E^{-2}(EdJ/d\tau - JdE/d\tau)$$

$$= (\epsilon + P)^{-1}\left[U^{\beta}(Lq_{0;\beta} + q_{\phi;\beta})U_0^{-1} - q^{\beta}L_{,\beta} + U^{\beta}{}_{;\beta}(Lq_0 + q_{\phi})U_0^{-1}\right], \qquad \text{(C5)}$$

where, in evaluating $dE/d\tau$ with equation (C2), use has been made of the fact that the metric and pressure fields are essentially undisturbed by the perturbed motion. According to equation (C5),

$$\frac{dL}{d\tau} \sim \frac{KT}{\epsilon + P} \frac{L}{R\lambda} \tag{C6}$$

in the perturbed motion; the time scale for changes in L in the perturbed flow is of the order of $R\lambda(\epsilon + P)/KT$, which is longer than the time scale of interest by a factor of $(R/\lambda)(\epsilon + P)/NT^2(\partial S/\partial T)_P \gg 1$.

APPENDIX D

THE ORIENTATION OF $\underset{\sim}{\gamma}$

In §VI it was found that the stability criteria for stellar regions in which $\mathcal{Z} \gg 1$ are

$$\underset{\sim}{\gamma} \cdot \underset{\sim}{\nabla} L \geq 0 \tag{D1}$$

and

$$\underset{\sim}{\gamma} \times \underset{\sim}{\nabla} L = 0 \ , \tag{D2}$$

where

$$\underset{\sim}{\gamma} = (U^{0}U_{0})^{2}\left[2(\Omega - \omega)(\underset{\sim}{\nabla}\psi - \underset{\sim}{\nabla}\nu) - (1 + v^{2})\underset{\sim}{\nabla}\omega\right] . \tag{D3}$$

Equation (D2) implies that the fluid is barytropic and that the level surface of L and Ω coincide and have the topology of a cylinder. Equations (D1) and (D2) together imply that $\underset{\sim}{\gamma}$ is orthogonal to these surfaces.

We can determine the global behavior of $\underset{\sim}{\gamma}$ in the following way. Since the topology of each surface orthogonal to $\underset{\sim}{\gamma}$ is that of a cylinder, every such surface must intersect the stellar surface. The definition of $\underset{\sim}{\gamma}$ involves the quantities Ω and v, which are defined only in the fluid, and so these surfaces cannot, strictly speaking, be thought of as extending into the region beyond the surface of the star. But if we surround the stellar model with a fictitious, stable, equilibrium barytropic fluid envelope (joined to the surface of the model in such a way that Ω and all thermodynamic quantities are continuous across the interface), and if we make the radius of the envelope very much larger than the size of the model itself, insisting that the envelope fluid density be so small that it does not change the metric of the original configuration (i.e., consider the limit in which the density approaches zero), we will be able to extend our surfaces into this region. Very far from the original stellar surface (but still within the fictitious envelope), at a distance $\mathfrak{R}$, say, the potential ω varies as $\mathfrak{R}^{-3}$, while $\underset{\sim}{\nabla}\psi = r^{-1}\hat{r} + \mathcal{O}(\mathfrak{R}^{-2})$ and $\underset{\sim}{\nabla}\nu = \mathcal{O}(\mathfrak{R}^{-2})$ (Chandrasekhar and Friedman 1972). At this distance, then, the vector field is asymptotically approaching the vector field $2r^{-1}\Omega\hat{r}$. Far from the stellar surface, therefore, the surfaces orthogonal to $\underset{\sim}{\gamma}$ approach simple cylinders, and $\underset{\sim}{\gamma}$ points outward from these surfaces. Following the surfaces back into the stellar interior, and noting that condition (D1) guarantees that $\underset{\sim}{\gamma}$ has the same topological orientation everywhere on a given surface, we see that $\underset{\sim}{\gamma}$ is outwardly oriented everywhere in the interior of our original model.

This analysis makes use of equation (D1), which applies only to stable barytropic configurations in which $Z \gg 1$. A complete understanding of some of the stability criteria for the cases when Z is not very large will depend on finding a stronger indication of how $\underset{\sim}{\gamma}$ behaves in arbitrary configurations.

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PART THREE

THE POST-NEWTONIAN STRUCTURE OF POLYTROPIC STARS

(This paper has been published in

THE ASTROPHYSICAL JOURNAL, 179:289–308, 1973 January 1

I. INTRODUCTION

A detailed study of the equilibrium configurations of rapidly rotating stars in the full theory of general relativity has not been possible thus far, although certain special cases have been investigated. Hartle and Thorne (1968) have studied uniformly rotating neutron stars, white dwarfs, and supermassive stars in the limit of slow rotation. Wilson (1972) has developed a method of finding the velocity fields which will bring certain specified mass density distributions into hydrostatic equilibrium. Bardeen and Wagoner (1971) have studied rotating disks, and Chandrasekhar (1965*b*) and Bardeen (1971) have studied homogeneous, uniformly rotating fluids in the post-Newtonian approximation to general relativity.

Another special case is studied in the subsequent sections of this paper, where the formalism developed by Chandrasekhar (1965*a*) for describing the hydrodynamics of perfect fluids in the post-Newtonian approximation is applied to axisymmetric, differentially rotating, polytropic stars with no viscosity or magnetic fields. The results include a method for constructing models of such stars, without having to impose special constraints on the mass-density profile or angular-momentum distribution, and some qualitative information about the properties of such models.

II. THE NEWTONIAN APPROXIMATION

Before considering the post-Newtonian theory of differentially rotating polytropes, it will be helpful to look at the problem in the context of Newtonian theory. In this case the equations of hydrodynamics are

$$\frac{\partial}{\partial t}(\rho v_\alpha) + \frac{\partial}{\partial x_\mu}(\rho v_\alpha v_\mu) = -\frac{\partial P}{\partial x_\alpha} + \rho\frac{\partial U}{\partial x_\alpha}, \tag{1}$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_\mu}(\rho v_\mu) = 0\,, \tag{2}$$

$$\nabla^2 U = -4\pi G\rho\,. \tag{3}$$

Assuming that the system under consideration is in a state of static equilibrium and rotates in an axisymmetric manner about the x_3 axis, the velocity field can be written in the form

$$\begin{aligned} v_1 &= -\Omega^* x_2\,, \\ v_2 &= \Omega^* x_1\,, \\ v_3 &= 0\,, \end{aligned} \tag{4}$$

where Ω^* is the angular velocity of a fluid element about the x_3 axis. The continuity equation (2) is satisfied identically by this velocity field:

$$\nabla\cdot(\rho \boldsymbol{v}) = (\boldsymbol{v}\cdot\nabla)\rho + \nabla\cdot\boldsymbol{v} = 0\,.$$

Equation (1) can be written in a cylindrical coordinate system ($z = x_3$, $\varpi^2 = x_1{}^2 + x_2{}^2$, $\varphi =$ the polar angle) as

$$\frac{1}{\rho}\nabla P - \nabla U - \varpi\Omega^{*2}\hat{\varpi} = 0\,. \tag{5}$$

If the fluid is assumed to obey a polytropic equation of state,

$$P = a\rho^{1+1/n}\,, \tag{6}$$

equation (5) can be written in the form

$$\nabla[\partial(n+1)\rho^{1/n} - U] - \varpi\Omega^{*2}\hat{\varpi} = 0\,. \tag{7}$$

This equation has a solution only if

$$\frac{\partial}{\partial z}(\varpi\Omega^{*2}) = 0\,, \tag{8}$$

which means that the angular velocity Ω^* must be a function of ϖ only. If new potentials B and H are defined by

$$\nabla B \equiv \varpi\Omega^{*2}(\varpi)\hat{\varpi}\,, \tag{9}$$

$$H \equiv U + B\,, \tag{10}$$

equation (7) can be integrated to give

$$\rho = \left[\frac{H - H_c + a(n+1)\rho_c{}^{1/n}}{a(n+1)}\right]^n\,, \tag{11}$$

where H_c and ρ_c are the values of H and ρ at the center of the star.

The system of equations (3), (9), (10), and (11) completely determines the structure of an equilibrium configuration if the function $\Omega^*(\varpi)$ and the constants a, n, and ρ_c are specified. Ostriker and Mark (1968) have used an iterative technique of the "self-consistent field method" type to find equilibrium configurations. They express equations (3) and (9) in their integral forms,

$$U(x') = G\int \frac{\rho(x)}{|x - x'|}\, d^3x\,, \tag{12}$$

$$B(\varpi') = \int_0^{\varpi'} \varpi\Omega^{*2}(\varpi)d\varpi\,, \tag{13}$$

and specify the angular velocity $\Omega^*(\varpi)$ by specifying the angular momentum per unit rest mass,

$$j(\varpi) = \varpi^2\Omega^*(\varpi)\,, \tag{14}$$

as a function of a Lagrangian mass coordinate

$$m(\varpi') \equiv \left[\int_0^{\varpi'} d\varpi\varpi \int_{-\infty}^{\infty} dz\rho(\varpi, z)\right] \Big/ \left[\int_0^{\infty} d\varpi\varpi \int_{-\infty}^{\infty} dz\rho(\varpi, z)\right], \tag{15}$$

which is the fractional mass interior to the cylinder of radius ϖ. Their method of constructing a model is to specify $j[m(\varpi)]$, make an initial guess for the mass density $\rho(x)$, use this ρ to calculate the potentials U and B through equations (12) and (13), use these potentials to calculate a new ρ with equation (11), and so on. They found that this scheme would converge to a good $\rho(x)$ within 10 to 100 iterations, depending on the magnitude of the angular momentum.

III. THE POST-NEWTONIAN EQUATIONS

Chandrasekhar (1965*a*) has derived a system of equations describing inviscid hydrodynamic systems, analogous to the standard system of Newtonian equations (1), (2), and (3), which take into account the effects of general relativity to second order in the parameter $1/c$. His derivation is based on the assumption of a stress-energy tensor of the form

$$T_{ij} = [\rho(c^2 + \Pi) + P]u_iu_j - Pg_{ij}\,, \tag{16}$$

where ρ is the invariant rest mass density, Π is the internal energy per unit rest mass, u_i is the four velocity, g_{ij} is the metric tensor, and P is the pressure. The resultant equations are as follows, where the Greek indices assume the values 1, 2, and 3:

$$\begin{aligned}\frac{\partial}{\partial t}(\sigma v_\alpha) + \frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu) + \frac{\partial}{\partial x_\alpha}\left[\left(1 + \frac{2U}{c^2}\right)P\right] - \rho\frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2}\rho\frac{d}{dt}(v_\alpha U - U_\alpha) \\ + \frac{4}{c^2}\rho v_\mu\frac{\partial}{\partial x_\alpha}U_\mu + \frac{\rho}{2c^2}\frac{\partial}{\partial t}(U_\alpha - U_{\mu;\alpha\mu}) - \frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha} + \frac{\partial\Phi}{\partial x_\alpha}\right) = 0\,,\end{aligned} \tag{17}$$

$$\nabla^2 U = -4\pi G\rho\,, \tag{18}$$

$$\nabla^2 U_\alpha = -4\pi G\rho v_\alpha\,, \tag{19}$$

$$\nabla^2\Phi = -4\pi G\rho\phi\,, \tag{20}$$

and

$$\frac{\partial}{\partial t}\rho^* + \frac{\partial}{\partial x_\alpha}(\rho^* v_\alpha) = 0\,, \tag{21}$$

where

$$\sigma \equiv \rho\left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi + \frac{P}{\rho}\right)\right], \tag{22}$$

$$\phi \equiv v^2 + U + \frac{\Pi}{2} + \frac{3P}{2\rho}, \tag{23}$$

$$U_{\mu;\alpha\mu} \equiv G \int \frac{\rho(x')v_\mu(x')(x_\alpha - x_\alpha')(x_\mu - x_\mu')}{|x - x'|^3} d^3x', \tag{24}$$

$$\rho^* \equiv \rho\left[1 + \frac{1}{c^2}\left(\frac{v^2}{2} + 3U\right)\right], \tag{25}$$

and the v_μ are the components of the ordinary velocity.

He was also able to find expressions for conserved quantities corresponding to linear momentum per unit coordinate volume,

$$\pi_\alpha = \sigma v_\alpha + \frac{1}{2c^2}\rho(U_\alpha - U_{\mu;\alpha\mu}) + \frac{4}{c^2}\rho(v_\alpha U - U_\alpha), \tag{26}$$

angular momentum per unit coordinate volume,

$$J_{\alpha\beta} = x_\beta \pi_\alpha - x_\alpha \pi_\beta, \tag{27}$$

and energy per unit coordinate volume,

$$\begin{aligned}\mathfrak{E} = (\sigma - \tfrac{1}{2}\rho^*)v^2 + \rho^*\Pi - \tfrac{1}{2}\rho^* U^* \\ + \frac{1}{c^2}\rho\Big(-\tfrac{1}{8}v^4 + \tfrac{1}{2}U^2 - \Pi U - \tfrac{1}{2}v^2\Pi \\ + \frac{5}{2}v^2 U - \frac{7}{4}v_\alpha U_\alpha - \tfrac{1}{4}v_\alpha U_{\mu;\alpha\mu}\Big),\end{aligned} \tag{28}$$

where

$$\nabla^2 U^* = -4\pi G\rho^*. \tag{29}$$

The metric in this approximation is

$$\begin{aligned} g_{00} &= 1 - \frac{2U}{c^2} + \frac{1}{c^4}(2U^2 - 4\Phi) + O\left(\frac{1}{c^6}\right), \\ g_{0\alpha} &= \frac{1}{c^3}\left(4U_\alpha - \frac{1}{2}\frac{\partial^2 x}{\partial t \partial x_\alpha}\right) + O\left(\frac{1}{c^5}\right), \\ g_{\alpha\beta} &= -\left(1 + \frac{2}{c^2}U\right)\delta_{\alpha\beta} + O\left(\frac{1}{c^4}\right), \end{aligned} \tag{30}$$

where the function χ is defined through the equation

$$\nabla^2\chi = -2U.$$

IV. THE EQUATIONS WHICH GOVERN EQUILIBRIUM CONFIGURATIONS

a) Introduction

Equations (17) through (30) can be applied to a polytropic, differentially rotating, axisymmetric fluid whose axis of rotation will be assumed to correspond to the z-axis of a cylindrical coordinate system defined as in § II. Under these circumstances the equation of state and the velocity distribution take the forms

$$P = a\rho^{1+1/n}\,, \tag{31}$$

$$\rho\Pi = nP\,; \tag{32}$$

$$\begin{aligned} v_1 &= -\Omega^* x_2\,, \\ v_2 &= \Omega^* x_1\,, \\ v_3 &= 0\,, \end{aligned} \tag{33}$$

where $\Omega^* = d\varphi/dt$ is the angular velocity of a fluid element as measured by an observer located at infinity.

In the Newtonian theory it turned out that the hydrodynamic equilibrium equation (1) had no static solutions for equation of state (6) and velocity field (4) unless the angular velocity was a function only of $\tilde{\omega}$. It will turn out in part (*b*) of this section that the post-Newtonian hydrodynamic-equilibrium equation (17) will have static solutions for equation of state (31) and (32) and velocity field (33) only when the angular velocity satisfies a certain condition; in particular it will turn out to have a z dependence. For this reason, the square of the angular velocity, Ω^{*2}, will be chosen to be of the form

$$\Omega^{*2}(\tilde{\omega}, z) = \Omega^2(\tilde{\omega}) + \frac{1}{c^2} h^2(\tilde{\omega}, z)\,. \tag{34}$$

Exactly how this function is to be determined will be discussed in part (*d*) of this section. Since accuracy only to order $1/c^2$ will be required, Ω^* will be replaced by Ω whenever it appears with the coefficient $1/c^2$.

In view of expressions (31) through (34), the quantities (22) and (23) become

$$\begin{aligned} \sigma &= \rho\left[1 + \frac{1}{c^2}\left(v^2 + 2U + \Pi + \frac{P}{\rho}\right)\right] \\ &= \rho\left[1 + \frac{1}{c^2}\left(\tilde{\omega}^2\Omega^2 + 2U + a(n+1)\rho^{1/n}\right)\right], \end{aligned} \tag{35}$$

$$\phi = v^2 + U + \tfrac{1}{2}\Pi + \frac{3P}{2\rho} = \tilde{\omega}^2\Omega^{*2} + U + \frac{a(n+3)}{2}\rho^{1/n}\,. \tag{36}$$

Equation (21), the continuity equation, is identically satisfied by the velocity (33), so it can be ignored from now on.

b) The Hydrodynamic Equilibrium Equation

Equation (17) becomes, under stationary conditions,

$$\begin{aligned} &\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu) + \frac{\partial}{\partial x_\alpha}\left[\left(1 + \frac{2U}{c^2}\right)P\right] - \rho\frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2}\rho v_\mu \frac{\partial U_\mu}{\partial x_\alpha} \\ &\qquad - \frac{2}{c^2}\rho\left(\phi\frac{\partial U}{\partial x_\alpha} + \frac{\partial \Phi}{\partial x_\alpha}\right) + \frac{4}{c^2}\rho v_\mu \frac{\partial}{\partial x_\mu}(v_\alpha U - U_\alpha) = 0\,. \end{aligned} \tag{37}$$

Dividing equation (37) by ρ and substituting into it expressions (31) and (36) for P and ϕ transforms it to

$$\frac{1}{\rho}\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu) + \frac{a}{\rho}\frac{\partial}{\partial x_\alpha}\left[\left(1 + \frac{2U}{c^2}\right)\rho^{1+1/n}\right] - \frac{\partial U}{\partial x_\alpha} + \frac{4}{c^2} v_\mu \frac{\partial U_\mu}{\partial x_\alpha}$$

$$- \frac{2}{c^2}\left[\bar{\omega}^2\Omega^2 + U + \frac{a(n+3)}{2}\rho^{1/n}\right]\frac{\partial U}{\partial x_\alpha} - \frac{2}{c^2}\frac{\partial \Phi}{\partial x_\alpha} + \frac{4}{c^2} v_\mu \frac{\partial}{\partial x_\mu}(v_\alpha U - U_\alpha) = 0\,. \quad (38)$$

The various terms in this equation can be simplified as follows:

$$\frac{\hat{x}_\alpha}{\rho}\frac{\partial}{\partial x_\mu}(\sigma v_\alpha v_\mu) = \frac{1}{\rho}[\sigma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \sigma\boldsymbol{v}(\nabla\cdot\boldsymbol{v}) + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla)\sigma]$$

$$= \frac{1}{\rho}\sigma(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} = -\frac{\sigma}{\rho}\bar{\omega}\Omega^{*2}\hat{\bar{\omega}}\,; \quad (39)$$

$$\hat{x}_\alpha v_\mu \frac{\partial}{\partial x_\mu}(v_\alpha U) = U(\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + \boldsymbol{v}(\boldsymbol{v}\cdot\nabla)U = -\bar{\omega}\Omega^{*2}U\hat{\bar{\omega}}\,; \quad (40)$$

$$\frac{1}{c^2}\hat{x}_\alpha\left[v_\mu\frac{\partial U_\mu}{\partial x_\alpha} - v_\mu\frac{\partial U_\alpha}{\partial x_\mu}\right]$$

$$= \frac{\Omega}{c^2}\{(\nabla\times U)_3(x_1\hat{x}_1 + x_2\hat{x}_2) - [x_2(\nabla\times U)_2 + x_1(\nabla\times U)_1]\hat{x}_3\}$$

$$= \bar{\omega}\frac{\Omega}{c^2}[\hat{\bar{\omega}}(\nabla\times U)\cdot\hat{z} - \hat{z}(\nabla\times U)\cdot\hat{\bar{\omega}}]$$

$$= \bar{\omega}\frac{\Omega}{c^2}\left[\frac{\hat{\bar{\omega}}}{\bar{\omega}}\frac{\partial}{\partial\bar{\omega}}(\bar{\omega}U_\varphi) + \hat{z}\frac{\partial}{\partial z}U_\varphi\right]$$

$$= \frac{1}{c^2}\nabla(\bar{\omega}\Omega U_\varphi) - \frac{\bar{\omega}}{c^2}U_\varphi\frac{d\Omega}{d\bar{\omega}}\hat{\bar{\omega}}\,. \quad (41)$$

Substituting expressions (34), (35), (39), (40), and (41) into equation (38) yields

$$\frac{a}{\rho}\nabla\left[\left(1 + \frac{2U}{c^2}\right)\rho^{1+1/n}\right] - \left[1 + \frac{1}{c^2}(2\bar{\omega}^2\Omega^2 + 2U + a(n+3)\rho^{1/n})\right]\nabla U$$

$$- \bar{\omega}\left[\Omega^2 + \frac{\Omega^2}{c^2}(\bar{\omega}^2\Omega^2 + 6U + a(n+1)\rho^{1/n}) + \frac{h^2}{c^2}\right]\hat{\bar{\omega}}$$

$$- \frac{2}{c^2}\nabla\Phi + \frac{4}{c^2}\nabla(\bar{\omega}\Omega U_\varphi) - \frac{4}{c^2}\bar{\omega}U_\varphi\frac{d\Omega}{d\bar{\omega}}\hat{\bar{\omega}} = 0\,. \quad (42)$$

In order to simplify equation (42) further, it will be necessary to use the following definitions and relationships:

$$\nabla B \equiv \bar{\omega}\Omega^2\hat{\bar{\omega}}\,, \quad (43)$$

$$H \equiv U + B\,, \quad (44)$$

$$\nabla W \equiv \bar{\omega}^2\Omega^2\nabla B = \bar{\omega}^3\Omega^4\hat{\bar{\omega}}\,, \quad (45)$$

$$2U\nabla U = \nabla U^2 \,, \tag{46}$$

$$\begin{aligned} \tilde{\omega}^2\Omega^2\nabla U &= \nabla(\tilde{\omega}^2\Omega^2 U) - U\nabla(\tilde{\omega}^2\Omega^2) \\ &= \nabla(\tilde{\omega}^2\Omega^2 U) - 2\tilde{\omega}\Omega\left(\Omega + \tilde{\omega}\frac{d\Omega}{d\tilde{\omega}}\right)U\hat{\tilde{\omega}} \,, \end{aligned} \tag{47}$$

$$U\nabla B = \tilde{\omega}\Omega^2 U\hat{\tilde{\omega}} \,, \tag{48}$$

$$\frac{1}{\rho}\nabla(U\rho^{1+1/n}) = \rho^{1/n}\nabla U + \frac{U}{\rho}\nabla\rho^{1+1/n} \,. \tag{49}$$

Equation (42) reads

$$\frac{a}{\rho}\nabla\rho^{1+1/n} - \nabla(U + B) + O\left(\frac{1}{c^2}\right) = 0 \,, \tag{50}$$

and implies, to the required accuracy, the equality

$$\frac{a}{\rho}\frac{U}{c^2}\nabla\rho^{1+1/n} = \frac{U}{c^2}\nabla(U + B) = \frac{1}{2c^2}\nabla U^2 + \frac{U}{c^2}\nabla B \,,$$

which leads, in conjunction with relation (50), to the expression

$$\frac{a}{c^2}\rho^{1/n}\nabla U = \frac{a}{\rho}\frac{1}{c^2}\nabla(U\rho^{1+1/n}) - \frac{1}{2c^2}\nabla U^2 - \frac{U}{c^2}\nabla B \,. \tag{51}$$

Similarly,

$$\frac{a}{\rho}\frac{B}{c^2}\nabla\rho^{1+1/n} = \frac{B}{c^2}\nabla(U + B) = \frac{B}{c^2}\nabla U + \frac{1}{2c^2}\nabla B^2 \,,$$

so that

$$\frac{a}{c^2}\rho^{1/n}\nabla B = \frac{1}{c^2}\frac{a}{\rho}\nabla(\rho^{1+1/n}B) - \frac{1}{2c^2}\nabla B^2 - \frac{1}{c^2}\nabla(UB) + \frac{U}{c^2}\nabla B \,. \tag{52}$$

By using relations (43) through (52), it is possible to write equation (42) as

$$\begin{aligned} &\frac{a}{\rho}\nabla\left[\left(1 - \frac{(n+1)}{c^2}H\right)\rho^{1+1/n}\right] \\ &\qquad - \nabla\left[H + \frac{1}{c^2}(2\Phi - \tfrac{1}{2}(n+1)H^2 + W + 2\tilde{\omega}^2\Omega^2 U - 4\tilde{\omega}\Omega U\varphi)\right] \\ &\qquad\qquad + \frac{1}{c^2}\tilde{\omega}\left[4\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U\varphi) - h^2\right]\hat{\tilde{\omega}} = 0 \,. \end{aligned} \tag{53}$$

It can easily be seen, by expanding both sides, that

$$\frac{1}{\rho}\nabla[\xi(x)\rho^{1+1/n}] = (n+1)\xi^{n/(n+1)}\nabla(\rho^{1/n}\xi^{1/(n+1)}) \,. \tag{54}$$

If two new functions ξ and $\boldsymbol{\beta}$ are defined by

$$\xi(\boldsymbol{x}) \equiv 1 - \frac{(n+1)}{c^2} H \tag{55}$$

and

$$\boldsymbol{\beta}(\boldsymbol{x}) \equiv -\nabla\left[H + \frac{1}{c^2}\left(2\Phi - \frac{(n+1)}{2}H^2 + W + 2\tilde{\omega}^2\Omega^2 U - 4\tilde{\omega}\Omega U_\varphi\right)\right]$$
$$+ \frac{1}{c^2}\tilde{\omega}\left[4\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U_\varphi) - h^2\right]\hat{\tilde{\omega}} \quad , \tag{56}$$

then equation (53) can be written as

$$\frac{a}{\rho}\nabla[\xi\rho^{1+1/n}] + \boldsymbol{\beta} = 0\,, \tag{57}$$

and transformed with the help of equation (54) to

$$a(n+1)\nabla(\rho^{1/n}\xi^{1/(n+1)}) + \boldsymbol{\beta}\xi^{-n/(n+1)} = 0\,. \tag{58}$$

From definition (55) it is clear that

$$\xi^{-n/(n+1)} = \left[1 - \frac{(n+1)}{c^2}H\right]^{-n/(n+1)} = 1 + \frac{n}{c^2}H + O\left(\frac{1}{c^4}\right) \tag{59}$$

and

$$\xi^{1/(n+1)} = \left[1 - \frac{(n+1)}{c^2}H\right]^{1/(n+1)} = 1 - \frac{H}{c^2} + O\left(\frac{1}{c^4}\right)\,, \tag{60}$$

and with these expressions equation (53) can be brought into the form

$$\nabla\left[a(n+1)\left(1 - \frac{H}{c^2}\right)\rho^{1/n} - H - \frac{1}{c^2}(2\Phi - \tfrac{1}{2}H^2 + W + 2\tilde{\omega}^2\Omega^2 U - 4\tilde{\omega}\Omega U_\varphi)\right]$$
$$+ \frac{\tilde{\omega}}{c^2}\left[4\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U_\varphi) - h^2\right]\hat{\tilde{\omega}} = 0\,. \tag{61}$$

This equation has a solution only if

$$\frac{\partial}{dz}\left[4\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U_\varphi) - h^2(\tilde{\omega}, z)\right] = 0\,, \tag{62}$$

which means that $h^2(\tilde{\omega}, z)$ must take the form

$$h^2(\tilde{\omega}, z) = 4\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U_\varphi) + \alpha(\tilde{\omega})\,, \tag{63}$$

where α is any function of $\tilde{\omega}$. Since $\alpha(\tilde{\omega})$ is an arbitrary function, it can be chosen to be $\alpha(\tilde{\omega}) = 0$, and then the last term in brackets in equation (61) vanishes. The equation can then be integrated directly to become

$$a(n+1)\left(1 - \frac{H}{c^2}\right)\rho^{1/n} = H + \frac{1}{c^2}\left[2\Phi - \frac{H^2}{2} + W + 2\tilde{\omega}^2\Omega^2 U - 4\tilde{\omega}\Omega U_\varphi\right] + K\,, \tag{64}$$

where K is the constant of integration. Multiplying equation (64) by the function

$$(1 - H/c^2)^{-1} = 1 + H/c^2 + O(1/c^4) \tag{65}$$

brings it into the form

$$a(n+1)\rho^{1/n} = H + K + \frac{1}{c^2}\left[2\Phi + \frac{H^2}{2} + W + 2\bar{\omega}^2\Omega^2 U - 4\bar{\omega}\Omega U_\varphi + KH\right] \cdot \tag{66}$$

The constant of integration can be chosen so that the central density will always have a prescribed value, ρ_c. Equation (66) evaluated at the center of the star becomes

$$a(n+1)\rho_c^{1/n} = U_c + K + \frac{1}{c^2}\left[2\Phi_c + \frac{U_c^2}{2} + KU_c\right],$$

and K is therefore given by the expression

$$K = a(n+1)\rho_c^{1/n} - U_c + \frac{1}{c^2}\left[\frac{U_c^2}{2} - 2\Phi_c - a(n+1)\rho_c^{1/n}U_c\right] \equiv K_0 + \frac{1}{c^2}K_1 . \tag{67}$$

Now it is possible to solve for $\rho(x)$ by taking the nth power of equation (66) and dividing it by $[a(n+1)]^n$, to get

$$\rho(x) = \frac{(H+K)^n + (n/c^2)(H+K)^{n-1}\Lambda^*(x)}{[a(n+1)]^n}, \tag{68}$$

where

$$\Lambda^*(x) \equiv 2\Phi + \tfrac{1}{2}H^2 + W + 2\bar{\omega}^2\Omega^2 U - 4\bar{\omega}\Omega U_\varphi + KH . \tag{69}$$

Finally, equation (68) can be brought into a more illuminating form:

$$\rho(x) = \frac{(H+K_0)^n + (n/c^2)(H+K_0)^{n-1}(\Lambda^* + K_1)}{[a(n+1)]^n}$$

$$= \left[\frac{H+K_0}{a(n+1)}\right]^n\left[1 + \frac{n}{c^2}\left(\frac{\Lambda^* + K_1}{H+K_0}\right)\right],$$

$$\rho(x) = \left[\frac{H - U_c + a(n+1)\rho_c^{1/n}}{a(n+1)}\right]^n\left[1 + \frac{n}{c^2}\Lambda(x)\right], \tag{70}$$

where

$$\Lambda(x) \equiv \frac{2(\Phi - \Phi_c) + \frac{1}{2}(H - U_c)^2 + W + a(n+1)\rho_c^{1/n}(H - U_c) + 2\bar{\omega}\Omega(\bar{\omega}\Omega U - 2U_\varphi)}{H + a(n+1)\rho_c^{1/n} - U_c} . \tag{71}$$

The equation of hydrostatic equilibrium, which looked very complicated in the form of equation (37), has thus been reduced to a simple algebraic equation which is, in principle, no more complicated than the analogous Newtonian equation (11).

c) *The Potential Equations*

The system of equations described thus far contains a bewildering array of twelve potential functions: the six scalar functions U, U^*, Φ, B, W, and χ, the three components of the vector potential $\boldsymbol{U}$, and the three components of the function $U_{\mu;\alpha\mu}$. Fortunately, this array can be decreased somewhat in size.

The potential χ will not be used, since it appears only in a time-dependent term of the metric. The potentials B and W are easily expressed in integral form as

$$B(\tilde{\omega}') = \int_0^{\tilde{\omega}'} \tilde{\omega}\Omega^2(\tilde{\omega})d\tilde{\omega}\,, \tag{72}$$

$$W(\tilde{\omega}') = \int_0^{\tilde{\omega}'} \tilde{\omega}^3\Omega^4(\tilde{\omega})d\tilde{\omega}\,. \tag{73}$$

The functions U, U^*, and Φ can all be written formally in integral form:

$$U(\boldsymbol{x}') = G\int \frac{d^3x}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\rho(\boldsymbol{x})\,, \tag{74}$$

$$U^*(\boldsymbol{x}') = G\int \frac{d^3x}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\rho^*(\boldsymbol{x})\,, \tag{75}$$

$$\Phi(\boldsymbol{x}') = G\int \frac{d^3x}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\rho(\boldsymbol{x})\,\phi(\boldsymbol{x})\,. \tag{76}$$

Since it will eventually be necessary to calculate these functions, the integrals should be written in forms more explicitly conducive to computation:

$$\begin{aligned} U(\tilde{\omega}', z') &= G\int \frac{d\tilde{\omega}dz}{\alpha}\,\tilde{\omega}\rho(\tilde{\omega}, z)\int d\varphi(1 - \beta\cos\varphi)^{-1/2} \\ &= 4G\int_0^{\infty} d\tilde{\omega}\int_{-\infty}^{\infty} dz\,\frac{\tilde{\omega}K(k)}{\alpha(1 + \beta)^{1/2}}\,\rho(\tilde{\omega}, z)\,, \end{aligned} \tag{77}$$

$$U^*(\tilde{\omega}', z') = 4G\int_0^{\infty} d\tilde{\omega}\int_{-\infty}^{\infty} dz\,\frac{\tilde{\omega}K(k)}{\alpha(1 + \beta)^{1/2}}\,\rho^*(\tilde{\omega}, z)\,, \tag{78}$$

$$\Phi(\tilde{\omega}', z') = 4G\int_0^{\infty} d\tilde{\omega}\int_{-\infty}^{\infty} dz\,\frac{\tilde{\omega}K(k)}{\alpha(1 + \beta)^{1/2}}\,\rho(\tilde{\omega}, z)\,\phi(\tilde{\omega}, z)\,, \tag{79}$$

where $K(k)$ is the complete elliptic integral of the first kind, and the functions α, β, and k are defined by

$$\alpha(\tilde{\omega}, \tilde{\omega}', z - z') \equiv [\tilde{\omega}^2 + \tilde{\omega}'^2 + (z - z')^2]^{1/2}\,, \tag{80}$$

$$\beta(\tilde{\omega}, \tilde{\omega}', z - z') \equiv \frac{2\tilde{\omega}\tilde{\omega}'}{\tilde{\omega}^2 + \tilde{\omega}'^2 + (z - z')^2}\,, \tag{81}$$

$$k \equiv 2\beta/(1 + \beta). \tag{82}$$

It can be seen by rewriting equation (19) in the form

$$\nabla^2\boldsymbol{U} = -4\pi G\rho\boldsymbol{v} \tag{83}$$

that, because of the form of the velocity field, the only component of $\boldsymbol{U}$ with a non-vanishing particular solution is U_φ. Equation (83) can most easily be solved in Cartesian coordinates. Clearly

$$U_\varphi(\tilde{\omega}, z) = U_1(x_1 = \tilde{\omega}, x_2 = 0, x_3 = z)\,,$$

so that

$$\begin{aligned} U_\varphi(\tilde{\omega}', z') &= G \int \frac{d^3x}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \Omega(\tilde{\omega}) x_1 \rho(\boldsymbol{x}) \\ &= G \int \frac{d^3x}{|\boldsymbol{x} - \boldsymbol{x}'|}\, \tilde{\omega}\Omega(\tilde{\omega}) \cos\varphi \rho(\boldsymbol{x}) \\ &= G \int d\tilde{\omega} dz \tilde{\omega}^2 \Omega(\tilde{\omega}) \rho(\boldsymbol{x}) \frac{1}{\alpha} \int_0^{2\pi} \frac{d\varphi \cos\varphi}{(1 - \beta\cos\varphi)^{1/2}} \\ &= \frac{2G}{\tilde{\omega}'} \int_0^\infty d\tilde{\omega} \int_{-\infty}^{\infty} dz \Omega(\tilde{\omega}) \rho(\tilde{\omega}, z) \tilde{\omega} \alpha \lambda(\beta)\,, \end{aligned} \tag{84}$$

where

$$\lambda(\beta) \equiv (1 + \beta)^{-1/2} K(k) - (1 + \beta)^{1/2} E(k) \tag{85}$$

and $E(k)$ is the complete elliptic integral of the second kind.

The function $U_{\mu;\alpha\mu}$ will turn out to appear only in the form

$$x_1 U_{\mu;2\mu} - x_2 U_{\mu;1\mu} \equiv \tilde{\omega} A_\varphi \tag{86}$$

(where this expression defines the new function A_φ). In view of expression (24), A_φ can be calculated as follows:

$$\begin{aligned} A_\varphi(\tilde{\omega}', z') &= G \int \frac{\rho(\boldsymbol{x}) v_\mu(\boldsymbol{x}) x_2 (x_\mu - x_\mu')}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d^3x \\ &= \tilde{\omega}' G \int \frac{\rho(\boldsymbol{x}) \Omega(\tilde{\omega}) x_2^2}{|\boldsymbol{x} - \boldsymbol{x}'|^3}\, d^3x \\ &= \tilde{\omega}' G \int d\tilde{\omega} dz \rho(\boldsymbol{x}) \tilde{\omega}^3 \Omega(\tilde{\omega}) \frac{1}{\alpha^3} \int_0^{2\pi} \frac{d\varphi \sin^2\varphi}{(1 - \beta\cos\varphi)^{3/2}} \\ &= \frac{2G}{\tilde{\omega}'} \int_0^\infty d\tilde{\omega} \int_{-\infty}^{\infty} dz \Omega(\tilde{\omega}) \rho(\tilde{\omega}, z) \tilde{\omega} \alpha \lambda(\beta)\,. \end{aligned} \tag{87}$$

Thus we have the pleasant result that

$$A_\varphi \equiv U_\varphi\,, \tag{88}$$

and the total number of potentials that must be dealt with has been reduced to six.

d) The Angular Momentum

When an equilibrium configuration is constructed, it will be necessary to specify the angular velocity distribution. As Ostriker and Mark (1968) point out, it is better to do this by specifying an angular momentum distribution than by giving the angular velocity directly. In order to accomplish this, it will be necessary to have at hand an expression for the angular momentum.

In light of expression (27), the angular momentum per unit coordinate volume about the z axis is

$$J(\tilde{\omega}, z) = J_{21} = \sigma\Omega^*(x_1{}^2 + x_2{}^2) + \frac{\rho}{c^2}\left[\frac{x_2}{2}(U_1 - U_{\mu;1\mu}) - \frac{x_1}{2}(U_2 - U_{\mu;2\mu}) + 4\Omega U(x_1{}^2 + x_2{}^2) - 4(x_2 U_1 - x_1 U_2)\right]$$

$$= \sigma\tilde{\omega}^2\Omega^* + \frac{\rho}{c^2}[4\tilde{\omega}^2\Omega U - 4\tilde{\omega}U_\varphi], \tag{89}$$

$$J(\tilde{\omega}, z) = \tilde{\omega}^2\rho\Omega + \frac{\rho}{c^2}\left[\tilde{\omega}^4\Omega^3 + 2\tilde{\omega}^2\Omega U + a(n+1)\tilde{\omega}^2\Omega\rho^{1/n} + \frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)(\tilde{\omega}\Omega U - U_\varphi)\right]$$

$$= \tilde{\omega}^2\rho\Omega + \frac{\rho}{c^2}\left[\tilde{\omega}^4\Omega^3 + \tilde{\omega}^2\Omega(B - U_c + a(n+1)\rho_c{}^{1/n}) + 3\tilde{\omega}^2\Omega U + \frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)(\tilde{\omega}\Omega U - U_\varphi)\right], \tag{90}$$

where use has been made of equation (70) in evaluating the term $a(n+1)\rho_c{}^{1/n}/c^2$ to order $1/c^2$.

Expression (90) can be converted to an angular momentum per unit rest mass if it is divided by the rest mass per unit coordinate volume. To this end, it can be noted from expression (30) that the space part of the metric is

$$g_{\alpha\beta} = -\left(1 + \frac{2}{c^2}U\right)\delta_{\alpha\beta}, \tag{91}$$

so that the relationship between an element of proper volume, dV, and an element of coordinate volume, d^3x, is

$$dV = \left(1 + \frac{\tilde{\omega}^2\Omega^2}{2c^2} + \frac{3}{c^2}U\right)d^3x. \tag{92}$$

Since ρ is a rest mass per unit proper volume, dividing expression (90) by $\rho\, dV / d^3x$ yields an expression for the angular momentum per unit rest mass:

$$j(\tilde{\omega}, z) = \tilde{\omega}^2\Omega + \frac{1}{c^2}\left[\frac{\tilde{\omega}^4\Omega^3}{2} + \tilde{\omega}^2\Omega(B - U_c + a(n+1)\rho_c{}^{1/n}) + \frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)(\tilde{\omega}\Omega U - U_\varphi)\right] \equiv \tilde{\omega}^2\Omega(\tilde{\omega}) + \frac{1}{c^2}\Gamma(\tilde{\omega}, z). \tag{93}$$

The most convenient way to specify the angular-velocity distribution is to specify the angular momentum per unit rest mass on the equatorial plane as a function of the fractional radius $\tilde{\omega}/R$, where R is the value of $\tilde{\omega}$ on the surface of the model at the equator. If this function is called $j(\tilde{\omega}/R)$, the function $\Omega(\tilde{\omega})$ can be found in the following manner. First calculate

$$\Omega'(\tilde{\omega}) = \frac{j(\tilde{\omega}/R)}{\tilde{\omega}^2}, \tag{94}$$

which is in error only by a quantity of order $1/c^2$. Then calculate $\Gamma(\tilde{\omega}, z = 0)$ in terms of $\Omega'(\tilde{\omega})$. This Γ will also be in error by a quantity of order $1/c^2$, but $(1/c^2)\Gamma$ will be accurate to the proper order. Then the expression

$$\Omega(\tilde{\omega}) = \frac{j(\tilde{\omega}/R)}{\tilde{\omega}^2} - \frac{1}{c^2\tilde{\omega}^2}\,\Gamma(\tilde{\omega}, z = 0) \tag{95}$$

will also be correct to the proper order. This $\Omega(\tilde{\omega})$ determines $j(\tilde{\omega}, z)$ and $\Omega^*(\tilde{\omega}, z)$ over the entire star through equations (93), (34), and (63).

e) The Method of Constructing Models

If the constants a and n in the equation of state (31), the central density ρ_c, and the angular momentum density $j(\tilde{\omega}/R)$ are specified, there is a unique equilibrium configuration which is determined by the six equations (70), (77), (72), (73), (79), and (84) relating the rest-mass density distribution ρ and the five potentials U, B, W, Φ, and U_φ. The rest-mass density distribution $\rho(\tilde{\omega}, z)$ which satisfies this set of coupled equations can be found with the same kind of iterative method that was described in § II. First a guess is made for the function $\rho(\tilde{\omega}, z)$, which is used in conjunction with $\Omega(\tilde{\omega})$ [as determined from $j(\tilde{\omega}/R)$ through eq. (93)] to calculate the five potentials. These potentials are then used to calculate a new ρ through equation (70). At this point a convergence test is made by comparing the new ρ with the old ρ. If their difference is small enough, then $\Omega^*(\tilde{\omega}, z)$ is calculated through equations (34) and (63), and $\rho(\tilde{\omega}, z)$ together with $\Omega^*(\tilde{\omega}, z)$ completely describes the equilibrium configuration. If the difference is too large, the new ρ is used to calculate a new set of potential functions, and the cycle is started again. A flow chart illustrating the procedure is shown in figure 1.

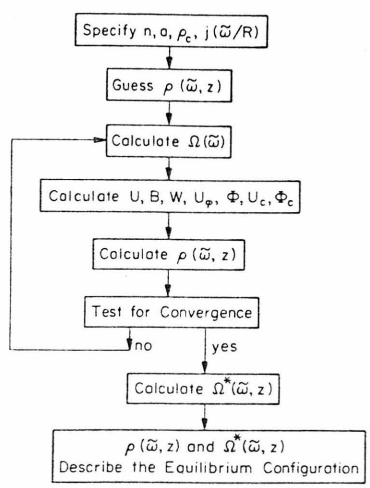

FIG. 1.—Flow diagram for the iterative method of finding equilibrium configurations

f) *The Mass and Binding Energies of a Model*

Once an equilibrium configuration has been constructed, its mass and binding energies can be found. The total rest mass energy of a star (the energy of its constituents when dispersed to infinity) is given by

$$M_0c^2 = c^2 \int \rho dV = c^2 \int \rho\left(1 + \frac{\tilde{\omega}^2\Omega^2}{2c^2} + \frac{3}{c^2}U\right)d^3x\,. \tag{96}$$

Expression (28) is an energy per unit coordinate volume, and includes all of the energy of the system except for that which comes from the rest mass. Substituting into this expression the values of v^2, Π, and ρ^* and contracting on repeated indices brings it into the form

$$\begin{aligned}\mathfrak{E} &= \tfrac{1}{2}\,\tilde{\omega}^2\Omega^2\rho + an\rho^{1+1/n} - \tfrac{1}{2}U^*\rho \\ &+ \frac{1}{c^2}\,\rho\left[\frac{5}{8}\tilde{\omega}^4\Omega^4 + \frac{11}{4}\,\tilde{\omega}^2\Omega^2U + a(n+1)\tilde{\omega}^2\Omega^2\rho^{1/n} + 2anU\rho^{1/n} - U^2 - 2\tilde{\omega}\Omega U_\varphi\right].\end{aligned} \tag{97}$$

The total mass energy of the star, the mass which governs the Keplerian orbits of distant particles, is then

$$Mc^2 = \int\left[\rho c^2\left(1 + \frac{\tilde{\omega}^2\Omega^2}{2c^2} + \frac{3}{c^2}\,U\right) + \mathfrak{E}\right]d^3x\,. \tag{98}$$

The binding energy of a star is defined as the difference between the rest mass energy and the total energy, and is given in this case by

$$E_b = -\int \mathfrak{E}d^3x\,. \tag{99}$$

V. THE GEOMETRICAL FEATURES OF THE ROTATION

It was demonstrated in § IV*b* that the angular velocity Ω^* must in general have a z dependence. In view of equation (93), the same is true of j, the angular momentum per unit rest mass. Since this indicates a possible qualitative departure from the situation in the Newtonian theory, it is of interest to take a closer look at the properties of the rotation.

First, however, it may be noted that in the relativistic theory there is a third function of physical interest associated with the rotation. This is the function $\Omega^* - \omega$, where $\omega(\tilde{\omega}, z)$ is the angular velocity, as measured by an observer located at infinity, of the local nonrotating frame at the location $(\tilde{\omega}, z)$ (Bardeen 1970). In the Newtonian theory, of course, the relativistic phenomenon of the dragging of inertial frames does not exist ($g_{\varphi t} \equiv 0$), so that $\Omega^* - \omega$ is always the same as Ω^*.

The function ω can be found from the metric, and is equal to

$$\omega = -\,g_{\varphi\varphi}/g_{\varphi t}\,. \tag{100}$$

In the present case the metric is given by equations (30), and can be written in cylindrical coordinates as

$$\begin{aligned}ds^2 &= \left(c^2 - 2U + \frac{2}{c^2}\,U^2 - \frac{4}{c^2}\,\Phi\right)dt^2 + \frac{8\tilde{\omega}}{c^2}U_\varphi d\varphi dt \\ &- \left(1 + \frac{2}{c^2}\,U\right)(d\tilde{\omega}^2 + dz^2 + \tilde{\omega}^2d\varphi^2)\,,\end{aligned} \tag{101}$$

so that

$$\omega = \frac{4}{c^2}\frac{U_\varphi}{\tilde{\omega}} + O\left(\frac{1}{c^4}\right) \cdot \tag{102}$$

The significance of this function is that any particle in orbit in the field described by the metric (101) has zero angular momentum about the z-axis if its trajectory satisfies the condition $d\varphi/dt = \omega$, a fact which can easily be verified to order $1/c^2$ by substituting $\Omega^* = \omega$ into equation (89) for the angular momentum. This means that if an observer at infinity were to throw a particle toward the $\tilde{\omega} = 0$ axis he would observe it to have an angular velocity of $d\varphi/dt = \omega(\tilde{\omega}, z)$ when its location was described by the coordinates $(\tilde{\omega}, z)$.

Now, in order to compare the geometrical features of different functions, it will be necessary to have a method for describing the level surfaces of various functions of the form

$$A(\tilde{\omega}, z) = X(\tilde{\omega}) + \frac{1}{c^2}Y(\tilde{\omega}, z)\,, \tag{103}$$

where it will be assumed that

$$A(\tilde{\omega}, z = 0) = X(\tilde{\omega})\,. \tag{104}$$

It is possible to solve for that surface $\tilde{\omega}(z)$ on which $A(\tilde{\omega}, z)$ is constant and has the value $A(\tilde{\omega}_0, z = 0)$ by writing

$$\tilde{\omega}(z) = \tilde{\omega}_0 - \frac{1}{c^2}f(\tilde{\omega}, z)\,, \tag{105}$$

where $\tilde{\omega}(z = 0) = \tilde{\omega}_0$, and evaluating expression (103) at the position $[\tilde{\omega}(z), z]$:

$$A(\tilde{\omega}_0, z = 0) = X(\tilde{\omega}_0) - \frac{1}{c^2}f(\tilde{\omega}_0, z)\left.\frac{\partial X}{\partial \tilde{\omega}}\right|_{\tilde{\omega}=\tilde{\omega}_0} + \frac{1}{c^2}Y(\tilde{\omega}_0, z) + O\left(\frac{1}{c^4}\right) \cdot \tag{106}$$

From this equation and equalities (104) and (105) it follows that

$$\tilde{\omega}(z) = \tilde{\omega}_0 - \frac{1}{c^2}Y(\tilde{\omega}_0, z)\left(\left.\frac{\partial X}{\partial \tilde{\omega}}\right|_{\tilde{\omega}=\tilde{\omega}_0}\right)^{-1} \cdot \tag{107}$$

This derivation is valid only when the level surfaces of A differ from the level surfaces of $\tilde{\omega}$ by small amounts, or when

$$\frac{\partial A/\partial z}{\partial A/\partial \tilde{\omega}} \leq O\left(\frac{1}{c^2}\right) \cdot \tag{108}$$

This method can now be used to find the level surfaces of Ω^*, j, and $\Omega^* - \omega$. In the case of Ω^*, it follows from expressions (34) and (65) that

$$\begin{aligned}\Omega^*(\tilde{\omega}, z) &= \Omega(\tilde{\omega}) + \frac{1}{c^2}\frac{2}{\Omega}\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U - U_\varphi)\\ &= \Omega(\tilde{\omega}) + \frac{1}{c^2}\frac{2}{\Omega}\frac{d\Omega}{d\tilde{\omega}}(\tilde{\omega}\Omega U_0 - U_{\varphi 0})\\ &\quad + \frac{1}{c^2}\frac{2}{\Omega}\frac{d\Omega}{d\tilde{\omega}}\left[\tilde{\omega}\Omega(U - U_0) - (U_\varphi - U_{\varphi 0})\right],\end{aligned} \tag{109}$$

where U_0 and $U_{\varphi 0}$ are defined as $U_0 = U(\tilde{\omega}, z = 0)$ and $U_{\varphi 0} = U_\varphi(\tilde{\omega}, z = 0)$. These functions are introduced in order that the expression for Ω^* comply with condition (103). According to equation (107), then, the level surfaces of Ω^* are described by the function

$$\tilde{\omega}(z) = \tilde{\omega}_0\left[1 + \frac{1}{c^2}\frac{2}{\tilde{\omega}_0\Omega}(U_\varphi - U_{\varphi 0}) - \frac{2}{c^2}(U - U_0)\right] + O\left(\frac{1}{c^4}\right) \cdot \qquad (110)$$

In the case of j,

$$\begin{aligned} j(\tilde{\omega}, z) &= \tilde{\omega}^2\Omega + \frac{1}{c^2}\left[\frac{\tilde{\omega}^4\Omega^3}{2} + \tilde{\omega}\Omega(B - U_c + a(n+1)\rho_c^{1/n})\right] \\ &\quad + \frac{1}{c^2}\frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)(\tilde{\omega}\Omega U - U_\varphi) \\ &= \tilde{\omega}^2\Omega + \frac{1}{c^2}\left[\frac{\tilde{\omega}^4\Omega^3}{2} + \tilde{\omega}\Omega(B - U_c + a(n+1)\rho_c^{1/n}) \right. \\ &\quad \left. + \frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)(\tilde{\omega}\Omega U_0 - U_{\varphi 0})\right] \\ &\quad + \frac{1}{c^2}\frac{2}{\Omega}\frac{d}{d\tilde{\omega}}(\tilde{\omega}^2\Omega)[\tilde{\omega}\Omega(U - U_0) - (U_\varphi - U_{\varphi 0})], \end{aligned} \qquad (111)$$

so that

$$\tilde{\omega}(z) = \tilde{\omega}_0\left[1 + \frac{1}{c^2}\frac{2}{\tilde{\omega}_0\Omega}(U_\varphi - U_{\varphi 0}) - \frac{2}{c^2}(U - U_0)\right] + O\left(\frac{1}{c^4}\right) \cdot \qquad (112)$$

In view of expression (102), $\Omega^* - \omega$ is equal to

$$\Omega^*(\tilde{\omega}, z) - \omega(\tilde{\omega}, z) = \Omega(\tilde{\omega}) + \frac{2}{c^2}\left[\tilde{\omega}\frac{d\Omega}{d\tilde{\omega}}U - \left(\frac{1}{\Omega}\frac{d\Omega}{d\tilde{\omega}} - \frac{2}{\tilde{\omega}}\right)U_\varphi\right], \qquad (113)$$

and

$$\tilde{\omega}(z) = \tilde{\omega}_0\left\{1 + \frac{2}{c^2}\left[\frac{1}{\tilde{\omega}_0\Omega} + \frac{2}{\tilde{\omega}_0^{\,2}}\left(\frac{d\Omega}{d\tilde{\omega}}\right)^{-1}\right](U_\varphi - U_{\varphi 0}) - \frac{2}{c^2}(U - U_0)\right\} \cdot \qquad (114)$$

This particular expression is not valid for small values of $d\Omega/d\tilde{\omega}$, because when $d\Omega/\tilde{\omega}$ approaches zero the level surfaces of $\Omega^* - \omega$ approach the level surfaces of ω, and this function does not satisfy condition (108).

The behaviors of the functions (110), (112), and (114) are, of course, dependent on the properties of the coordinate system in which they are written, so that the functions themselves do not necessarily exhibit explicitly the physical behaviors of the surfaces. The surfaces should be compared with surfaces which are chosen for physical properties. To this end it is desirable to find the functions $\tilde{\omega}(z)$ of surfaces whose intrinsic geometries are, in some sense, that of a cylinder. The only surfaces which qualify reasonably as "proper" cylinders, and are physically related to the system under consideration, are those axisymmetric surfaces whose circumferences are constant as a function of z as measured by observers who are (1) stationary with respect to infinity, (2) rotating with $d\varphi/dt = \omega$, (3) rotating with $d\varphi/dt = \Omega^*$, or (4) rotating with $d\varphi/dt =$

$\Omega^* - \omega$. In the case of the stationary observer, it is clear from the metric (101) that the proper circumference of a circle about the z-axis is equal to

$$C(\tilde{\omega}, z) = 2\pi\tilde{\omega}(1 + U/c^2)\,, \tag{115}$$

so that surfaces of constant C are described by the function

$$\tilde{\omega}(z) = \tilde{\omega}_0\left[1 - \frac{1}{c^2}(U - U_0)\right]\cdot \tag{116}$$

For an observer rotating with some $d\varphi/dt \neq 0$, the circumference is related to expression (115) through the usual special relativistic transformation:

$$\begin{aligned} C(\tilde{\omega}, z) &= 2\pi\tilde{\omega}(1 + U/c^2)(1 - v^2/c^2)^{-1/2} \\ &= 2\pi\tilde{\omega}\left(1 + \frac{U}{c^2}\right)\left(1 + \frac{\tilde{\omega}^2(d\varphi/dt)^2}{2c^2}\right) \\ &= 2\pi\tilde{\omega}\left(1 + \frac{U}{c^2} + \frac{\tilde{\omega}^2(d\varphi/dt)^2}{2c^2}\right)\cdot \end{aligned} \tag{117}$$

When $d\varphi/dt = \omega = 4U_\varphi/\tilde{\omega}c^2$, expression (115) is unchanged to order $1/c^2$, and therefore the surfaces of constant circumference are given by the function (116). When $d\varphi/dt = \Omega^*$ or $\Omega^* - \omega$,

$$C(\tilde{\omega}, z) = 2\pi\tilde{\omega}(1 + U/c^2 + \tilde{\omega}^2\Omega^{*2}/2c^2) + O(1/c^4)\,, \tag{118}$$

and the level surfaces are

$$\begin{aligned} \tilde{\omega}(z) &= \tilde{\omega}_0\left[1 - \frac{1}{c^2}(U - U_0) - \frac{\tilde{\omega}_0{}^2}{2c^2}(\Omega^{*2} - \Omega_0{}^{*2})\right] \\ &= \tilde{\omega}_0\left[1 - \frac{1}{c^2}(U - U_0)\right] + O\left(\frac{1}{c^4}\right), \end{aligned} \tag{119}$$

equal once again to expression (116).

The surface (116) is not the same as either of the surfaces (110), (112), or (114), so that, in contrast to the situation which occurs for rotating polytropes in the Newtonian theory, the level surfaces of the functions describing the rotation in the relativistic theory are not surfaces which can be called cylinders in any nice physical sense.

Another interesting consequence of the above derivations is the fact that in equilibrium configurations the surfaces of constant angular momentum per unit rest mass coincide with the surfaces of constant angular velocity $d\varphi/dt = \Omega^*$ (since the functions [110] and [112] describing these surfaces are the same). This is not true, of course, for arbitrary, nonequilibrium distributions of angular velocity. This result may be slightly surprising, since an intuitive guess might have been more likely to associate the level surfaces of j with the level surfaces of $\Omega^* - \omega$, if with anything at all, because it is the quantity $\Omega^* - \omega$, rather than Ω^*, which is the physically significant local quantity. On the other hand, the angular momentum is a global quantity, so it is not unreasonable that it should be associated in some way with the global quantity Ω^*.

Although it has been demonstrated here to be true only in the post-Newtonian approximation, it is tempting to conjecture that the coincidence of the level surfaces of j and Ω^* may also be true for polytropes in the full theory of general relativity. This seems likely because the post-Newtonian approximation already includes, to the order to which it is accurate, all of the physical effects, such as the dragging of inertial frames, which are important in relativistic, stationary, axisymmetric systems and do not appear in the Newtonian treatment.

VI. THE DOMAIN OF VALIDITY OF THE APPROXIMATION

Since the post-Newtonian approximation is an approximation, it is important to know something about the conditions under which it is a good approximation to general relativity. For nonrotating stellar configurations, the important parameter in the expansion that is used to find the post-Newtonian equations is the parameter U/c^2. The errors in this approximation would then be expected to go as the next higher order terms (the post-post-Newtonian terms) in the expansion, or as $(U/c^2)^2$. This parameter is largest at the center of the star, where it is of the order of

$$(U/c^2)^2 \simeq (4GM/c^2R)^2 \,, \tag{120}$$

(for an $n = 3$ polytrope; slightly less when $n < 3$). If the radius of a star is m Schwarzschild radii,

$$R = m\frac{2GM}{c^2} \,, \tag{121}$$

then the error should go as

$$(U/c^2)^2 \simeq (2/m)^2 \,, \tag{122}$$

which is quite small even for reasonably small radii (1 percent for $m = 20$, 4 percent for $m = 10$).

When rotation is added, the relevant parameter is v^2/c^2. The error in the rotational terms should then go as $(v^2/c^2)^2$. The maximum angular velocity which can occur on the surface of the star at the equator is no larger than the angular velocity for which rotational shedding will occur,

$$\Omega^2 \simeq GM/R^3 \,, \tag{123}$$

so that on the surface at the equator the error goes at most as

$$\left(\frac{v^2}{c^2}\right)^2 \sim \left(\frac{GM}{c^2R}\right)^2 . \tag{124}$$

This is smaller than expression (120), so that whenever the post-Newtonian approximation is good for a nonrotating star it will be good for a rotating star with the same approximate parameter m whose angular velocity is arbitrarily high on its surface at the equator, and whose angular velocity doesn't increase toward the axis of rotation faster than $\Omega \sim 1/\tilde{\omega}$ (since $v^2 = \tilde{\omega}^2\Omega^2$). Care must be exercised when the angular velocity increases faster than this.

VII. A METHOD OF CALCULATION

The calculations which are necessary in order to execute the program described in § IV can be done numerically on the computer. The following is a brief outline of one method of doing the calculations. This is a very straightforward treatment. It works quite well, although it requires careful programing in order to minimize the necessary computer storage space.

The stellar model will have an axis of rotation coinciding with the z-axis, and its equatorial plane will correspond to the plane $z = 0$. Since the system is symmetrical with respect to reflections through the equatorial plane, it will be necessary to record the values of functions only in the region $\tilde{\omega} \geq 0$, $z \geq 0$. This region can be divided into a grid of mesh points extending far enough away from the $\tilde{\omega}$- and z-axes to contain the model completely and to allow for the changes of size and shape which will occur during the iterative process. Functions can then be represented in the form

$$f(I, J) = f(\tilde{\omega} = I, z = J)\,, \tag{125}$$

where the indices I and J are integers which vary from 0 to maximum values of M and N, respectively, and where lengths are expressed in dimensionless form for convenience.

The one-dimensional integrals (72) and (73) for $B(\tilde{\omega})$ and $W(\tilde{\omega})$ can then be done with the usual Simpson's rule method,

$$B(I) = \sum_{L=0}^{I} A(L) L \Omega^2(L)\,, \tag{126}$$

$$W(I) = \sum_{L=0}^{I} A(L) L^3 \Omega^4(L)\,, \tag{127}$$

with the coefficients $A(L)$ appropriately chosen. Integrals over both $\tilde{\omega}$ and z, such as the integral (99) for the binding energy, can be done with a two-dimensional equivalent of the Simpson's rule,

$$\int d\tilde{\omega} dz f(\tilde{\omega}, z) = \sum_{I=0}^{M} \sum_{J=0}^{N} AA(I, J) f(I, J)\,, \tag{128}$$

with appropriate coefficients $AA(I, J)$.

In order to do the integrals for U, U^*, Φ, and U_φ, define the arrays

$$C(I, J, L) \equiv 4GJK(k)/\alpha(1 + \beta)^{1/2} \tag{129}$$

and

$$D(I, J, L) \equiv 2GJ\alpha\lambda(\beta)/I\,, \qquad I \neq 0\,, \tag{130}$$

where k, α, β, and $\lambda(\beta)$ are the functions defined in § IVc, and are evaluated at $\tilde{\omega} = I$, $\tilde{\omega}' = J$, and $z - z' = L$. If there were no singularities in the integrands of integrals (77) and (84), the potentials U and U_φ could be written in the forms

$$U(I, J) = \sum_{K=0}^{M} \sum_{L=0}^{N} AA(K, L)[C(I, K, |J - L|) + C(I, K, J + L)]\rho(K, L)\,, \tag{131}$$

$$\begin{aligned} U_\varphi(I, J) &= 0\,, \qquad I = 0\,, \\ &= \sum_{K=0}^{M} \sum_{L=0}^{N} AA(K, L)[D(I, K, |J - L|) + D(I, K, J + L)] \\ &\quad \times \Omega(K)\rho(K, L)\,, \qquad I > 0\,, \end{aligned} \tag{132}$$

where the integral for U_φ has been written in terms of Ω rather than Ω^* because U_φ always appears with a coefficient of $1/c^2$. Since there is one pole in each integral, when

$I = K$ and $J = L$, the contributions to the above sums corresponding to the integrals over the squares in the $(\tilde{\omega}, z)$-plane whose corners are $K = I \pm 1$, $L = J \pm 1$ can be dropped, and the integrals U' and U_φ' over these regions can be done separately:

$$U' = \rho(I, J)F(I)\,, \tag{133}$$

$$U_\varphi' = \rho(I, J)\Omega(I)G(I)\,, \tag{134}$$

where $F(I)$ and $G(I)$ are the values of the integrals

$$F(I) = 4G \int_{I-1}^{I+1} \tilde{\omega} d\tilde{\omega} \int_{-1}^{1} dz \int_{0}^{2\pi} d\varphi/(I^2 + \tilde{\omega}^2 + z^2 - 2I\tilde{\omega}\cos\varphi)^{1/2}\,, \tag{135}$$

$$G(I) = \frac{2G}{I} \int_{I-1}^{I+1} \tilde{\omega} d\tilde{\omega} \int_{-1}^{1} dz \int_{0}^{2\pi} \frac{d\varphi \cos\varphi}{(I^2 + \tilde{\omega}^2 + z^2 - 2I\tilde{\omega}\cos\varphi)^{1/2}}\,, \qquad I > 0\,, \tag{136}$$

which can be calculated numerically once and for all, and the approximation is made in writing in equations (133) and (134) that, over the regions involved, the values of ρ and Ω are constant and equal to their values at the locations of the singularities.

The integrals for Φ and U^* are calculated by replacing $\rho(I, J)$ by $\rho(I, J)\phi(I, J)$ and $\rho^*(I, J)$, respectively, in equations (131) and (133).

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PART FOUR

THE POST-NEWTONIAN STRUCTURE OF BARYTROPIC STARS

I. INTRODUCTION

It has been shown (Seguin 1974, "paper I") that convective stability in non-magnetic, differentially rotating relativistic stars requires that the stellar fluid be barytropic ($P = P[\epsilon]$, where P is the pressure and ϵ is the total density of mass-energy as measured in the rest frame of the fluid) whenever either thermal conductivity or viscosity is more efficient than the other by a large margin. It is therefore of interest to have at hand a scheme for constructing relativistic, barytropic stellar models. Seguin (1973, "paper II") has exhibited a scheme for constructing post-Newtonian models of differentially rotating, isentropic stars in which the pressure depends on the density of rest mass as measured in the fluid rest frame, ρ, through the polytropic relation

$$P = a\rho^{1 + 1/n} , \tag{1}$$

where a and n are constants. Fluids with this property are a subclass of the class of barytropic fluids. The same formalism can be extended to cover the completely general barytropic fluid. In this paper we will do just that, changing the notation of paper II slightly in order to make the connection between the post-Newtonian equations and the fully relativistic equations of paper I more clear.

An appendix closes this paper by deriving the post-Newtonian hydrodynamic equilibrium equation directly from the fully relativistic equation and the post-Newtonian metric. The result is the same as the equation derived from Chandrasekhar's post-Newtonian hydrodynamic equilibrium equation, indicating consistency in the post-Newtonian formalism.

II. THE GEOMETRY

We will assume that the geometry of space-time in the vicinity of our axisymmetric, stationary stellar model can be described by the asymptotically flat ("at infinity") metric whose line element is

$$ds^2 = - e^{2\nu}c^2dt^2 + e^{2\psi}(d\phi - \omega cdt)^2 + e^{2\mu}(dr^2 + dz^2) , \tag{2}$$

where t is the coordinate time and r, z, and ϕ are polar cylindrical spatial coordinates. For computational purposes, it is convenient to re-express several of the metric coefficients in the forms

$$e^{2\nu} = 1 - \frac{2}{c^2}U + \frac{2}{c^4}(U^2 - 2\Phi) , \tag{3}$$

$$e^{2\psi} = r^2(1 + \frac{2}{c^2}U) , \tag{4}$$

and

$$e^{2\mu} = 1 + \frac{2}{c^2}U \tag{5}$$

(paper II), and to calculate U, Φ , and ω by using the integral forms of the post-Newtonian limits of the Einstein field equations as derived by Chandrasekhar (1965). Under the present circumstances these equations can be written in the forms

$$U(r', z') = 4G \int_0^\infty rdr \int_{-\infty}^\infty dz \frac{K(k)}{\alpha(1 + \beta)^{1/2}} \rho , \tag{6}$$

$$\Phi(r', z') = 4G \int_0^\infty rdr \int_{-\infty}^\infty dz \frac{K(k)}{\alpha(1 + \beta)^{1/2}} (r^2\Omega^2 + U + \frac{\epsilon + 3P}{2\rho} - \frac{c^2}{2}), \tag{7}$$

and

$$\omega(r', z') = \frac{8G}{r^2c^2}\int_0^{\infty} r dr \int_{-\infty}^{\infty} dz \, \alpha \, \lambda(\beta) \rho \Omega \quad , \tag{8}$$

where G is the Newtonian gravitational constant, Ω is the angular velocity $d\phi/dt$ of a fluid element in the star as seen by an observer "at infinity," K(k) and E(k) are the complete elliptic integrals of the first and second kinds, respectively, and α, β, k, and λ are defined as

$$\alpha(r, r', z - z') \equiv \left[r^2 + r'^2 + (z - z')^2\right]^{1/2}, \tag{9}$$

$$\beta(r, r', z - z') \equiv \frac{2rr'}{r^2 + r'^2 + (z - z')^2} \quad , \tag{10}$$

$$k(\beta) \equiv 2\beta/(1 + \beta), \tag{11}$$

and

$$\lambda(\beta) \equiv (1 + \beta)^{-1/2} K(k) - (1 + \beta)^{1/2} E(k) \tag{12}$$

(see paper II).

III. THE FLUID MOTION

The rotational velocity of a fluid element in the stellar model that would be measured by a local observer with zero angular momentum ($d\phi/dt = \omega$) is

$$v = e^{\psi - \nu}(\Omega - \omega) \tag{13}$$

(Bardeen 1970), which, to post-Newtonian accuracy, is

$$v = r(\Omega - \omega)(1 + \frac{2}{c^2}U). \tag{14}$$

The four velocity of a fluid element has contravariant components

$$U^o = e^{-\nu}(1 - v^2)^{-1/2} = 1 + \frac{1}{c^2}(U + \frac{1}{2} r^2\Omega^2) \quad , \tag{15}$$

$$U^\phi = \Omega U^o \quad ,$$

$$U^r = U^z = 0 \quad , \tag{15}$$

and covariant components

$$U_o = -\left[e^{2\nu} + \frac{\omega}{c^2}(\Omega - \omega)e^{2\psi}\right]U^o = -1 + \frac{1}{c^2}(U - \frac{1}{2}r^2\Omega^2) \ ,$$

$$U_\phi = e^{2\psi}(\Omega - \omega)U^o = r^2(\Omega - \omega)\left[1 + \frac{1}{c^2}(3U + \frac{1}{2}r^2\Omega^2)\right] \quad ,$$

$$U_r = U_z = 0, \tag{16}$$

where all terms beyond the post-Newtonian terms have been dropped (as they will be from this point onward). The angular momentum per unit rest mass in the stellar fluid is

$$j = \frac{\epsilon + P}{\rho c^2} U_\phi = \frac{\epsilon + P}{\rho c^2} r^2 (\Omega - \omega) \left[1 + \frac{1}{c^2}(3U + \frac{1}{2}r^2 \Omega^2)\right] , \qquad (17)$$

while the geometrical angular momentum (see paper I) is

$$L = - \frac{U_\phi}{U_o} = r^2(\Omega - \omega)(1 + \frac{4}{c^2}U) . \qquad (18)$$

IV. THE EQUATION OF HYDRODYNAMIC EQUILIBRIUM

Chandrasekhar (1965) has derived the relativistic equations of motion $T^{\alpha\beta}{}_{;\beta} = 0$ for a perfect fluid in the post-Newtonian approximation. The spatial components of this equation can be specialized to describe the time-independent system under consideration here and written

$$\frac{1}{\rho}\nabla\left[\left(1+\frac{2}{c^2}U\right)P\right] - \hat{r}\, r\,\Omega^2\left[\frac{\epsilon + P}{\rho c^2} + \frac{1}{c^2}(r^2\Omega^2 + 2U)\right]$$

$$- \nabla U + \left[1 - \frac{\epsilon + P}{\rho c^2} - \frac{2}{c^2}\left(r^2\Omega^2 + U + \frac{P}{\rho}\right)\right]\nabla U$$

$$- \frac{2}{c^2}\nabla\Phi - \frac{4}{c^2}\, r\,\Omega^2 U\,\hat{r} + \Omega\nabla r^2\omega = 0 \qquad (19)$$

(by combining equations 35, 36, 37, 39, 40, 41, and 102 of paper II and using the fact that, in the notation of that paper, $\epsilon = \rho(c^2 + \Pi)$, where the gradient operator ∇ is the simple (not covariant) gradient operator in the r, z, ϕ three-space. If we now combine terms in equation (19), multiply through be the quantity $1 - 2U/c^2$, and divide by the quantity $(\epsilon + P)/\rho c^2 = 1 + \mathcal{O}(1/c^2)$, we obtain

$$\frac{c^2}{\epsilon + P}\nabla P - \nabla\left[U - r^2\omega\Omega + \frac{1}{c^2}(2\Phi + 2r^2\Omega^2 U)\right]$$

$$- \hat{r}\, r\,\Omega^2\left(1 + \frac{1}{c^2}r^2\Omega^2\right) + r^2\left(-\omega + \frac{4}{c^2}\Omega U\right)\nabla\Omega = 0 \; . \qquad (20)$$

V. SOLUTIONS FOR BARYTROPIC FLUIDS

In the Newtonian approximation, the level surfaces of Ω always coincide with the surfaces of constant r in barytropic equilibrium configurations (von Zeipel 1924). It is safe to assume, then, that post-Newtonian barytropic equilibrium configurations will have level surfaces of Ω which deviate only slightly from surfaces of constant r in such a way that it will be possible to write

$$\Omega^2(r, z) = \tilde{\Omega}^2(r) + \frac{1}{c^2}h^2(r, z) \quad , \tag{21}$$

where the function h(r, z) must be found. If we substitute equation (21) into equation (20), we find that the equation of hydrodynamic equilibrium can be written

$$\frac{c^2}{\epsilon + P}\nabla P - \nabla\left[B + U - r^2\omega\,\Omega + \frac{1}{c^2}(W + 2\Phi + 2r^2\Omega^2 U)\right]$$
$$- \hat{r}\left[r^2\,\omega\,\Omega,_r + \frac{1}{c^2}(rh^2 - 4r^2 U\,\Omega\,\Omega,_r)\right] = 0 \quad , \tag{22}$$

where we have introduced the definitions

$$B(r) = \int_0^r r'\,\tilde{\Omega}^2(r')dr' \tag{23}$$

and

$$W(r) = \int_0^r r'^3\,\tilde{\Omega}^4(r')dr' \quad . \tag{24}$$

Now let us restrict ourselves to fluid distributions in which $P = P(\epsilon)$. Once this pressure dependence has been specified, it will be possible to write

$$\frac{c^2}{\epsilon + P}\nabla P = \nabla F(P) \tag{25}$$

where F is some function of P only. Substituting equation (25) into equation (22), we find

$$\nabla\left[F(P) - U - B + r^2\omega\Omega - \frac{1}{c^2}(W + 2\Phi + 2r^2\Omega^2 U)\right]$$

$$- \hat{r}\, r\left[r\,\omega\Omega,_r + \frac{1}{c^2}(h^2 - 4rU\,\Omega\Omega,_r)\right] = 0 \quad . \tag{26}$$

This equation has a solution only if

$$\frac{\partial}{\partial z}\left[r\,\omega\Omega,_r + \frac{1}{c^2}(h^2 - 4rU\,\Omega\Omega,_r)\right] = 0 \quad , \tag{27}$$

which can be true only if

$$\frac{1}{c^2}h^2 = (\frac{4}{c^2}U\Omega - \omega)r\,\Omega,_r + f(r) \quad , \tag{28}$$

where f is some function of r only. But if f is a function of r only, it is possible to formally absorb it into the function $\tilde{\Omega}(r)$ in equation (21) and to equate it to zero in equation (28). Equation (26) can then be integrated to yield a simple algebraic equation:

$$F(P) - U - B + r^2\omega\Omega - \frac{1}{c^2}(W + 2\Phi + 2r^2\Omega^2 U) + C = 0 \quad , \tag{29}$$

where C is a constant of integration.

Once a barytropic pressure dependence has been specified, it should be possible to use the integrated equation of hydrodynamic equilibrium (29) together with the machinery described in paper II to construct, numerically, equilibrium configurations to model relativistic stars.

APPENDIX

ALTERNATIVE DERIVATION OF THE HYDRODYNAMIC EQUILIBRIUM EQUATION

The fully relativistic hydrodynamic equilibrium equation for stationary, axisymmetric stellar models can be written

$$(\epsilon + P)^{-1}\nabla P + U^{o}U_{\phi}\nabla\Omega - \nabla(\ell n\, U^{o}) = 0 \tag{A1}$$

(paper I, equation 26). If we evaluate the second and third terms in this equation in terms of the fully relativistic metric potentials (with the help of equations 15 and 16) and then substitute the post-Newtonian limits of the metric potentials from equations (3), (4), and (5), we find

$$\begin{aligned}
&\frac{c^2}{\epsilon + P}\nabla P + \frac{1}{2}(1 - v^2/c^2)^{-1}(c^2 e^{-2\nu}\nabla e^{2\nu} + v^2 e^{2\psi}\nabla e^{-2\psi} + 2e^{\psi - \nu} v\nabla\omega) \\
&\quad = \frac{c^2}{\epsilon + P}\nabla P - \nabla U - \frac{2}{c^2}\nabla\Phi - \frac{2}{c^2} r^2\Omega^2\nabla U - r^2\Omega\nabla\omega \\
&\quad\quad - \hat{r}\, r\left[(\Omega - \omega)^2 + \frac{4}{c^2} r\Omega^2 U + \frac{1}{c^2} r^3\Omega^4\right] \\
&\quad = \frac{c^2}{\epsilon + P}\nabla P - \nabla\left[U - r^2\omega\Omega + \frac{1}{c^2}(2\Phi + 2r^2\Omega^2 U)\right] \\
&\quad\quad - \hat{r}\, r\Omega^2\left(1 + \frac{1}{c^2} r^2\Omega^2\right) + r^2\left(-\omega + \frac{4}{c^2}\Omega U\right)\nabla\Omega = 0 \ ,
\end{aligned} \tag{A2}$$

which is identical to the post-Newtonian equation (20).

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