A COUNTEREXAMPLE IN THE THEORY OF FOURIER TRANSFORMS IN THE COMPLEX DOMAIN

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ABSTRACT

The Borel transform of an entire function of exponential type is defined outside a closed bounded convex set D. Paley and Wiener have given a necessary and sufficient condition on the entire function F(z) such that $\varphi(w)$, the Borel transform of F(z), is contained in $E^2(\mathbb{C} \setminus D)$ for the case when D is a line segment. Kacnel'son has shown that the natural extension of this result provides a necessary condition for a general closed bounded convex set D. Here, by counterexample, we show that the natural extension does not provide a sufficient condition.

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CHAPTER I

INTRODUCTION

1. For a given entire function of exponential type,

$F(z) = a_0 + \frac{a_1}{1!} z + \frac{a_2}{2!} z^2 + \cdots + \frac{a_n}{n!} z^n + \cdots$, the Borel transform $\varphi(w)$ of $F(z)$ is defined by $\varphi(w) = a_0 w^{-1} + a_1 w^{-2} + a_2 w^{-3} + \cdots + a_n w^{-n-1} + \cdots$.

Let D be the conjugate indicator diagram of $F(z)$ and let $h(\theta)$ be the indicator function of $F(z)$. A discussion of the indicator function and the conjugate indicator diagram is contained in Pólya [11, p. 571-597], and Boas [2, p. 66-77]. Then the following integral transforms relate $F(z)$ and $\varphi(w)$:

$$F(z) = \frac{1}{2\pi i} \int_\Gamma \varphi(w) e^{wz} dw$$

where Γ is a rectifiable Jordan curve homotopic to a positively oriented circle with D in its interior, and

$$\varphi(w) = \int_0^\infty F(re^{i\theta}) \exp(-wre^{i\theta}) e^{i\theta} dr$$

where $z = re^{i\theta}$ and $\operatorname{Re}(we^{i\theta}) > h(\theta)$. [11, p. 578-585], [2, p. 73-75].

Our work here begins with a consideration of the classical Paley-Wiener Theorem. The theorem as formulated by Paley and Wiener is the following:

(1.1) THEOREM. (Paley-Wiener) [9, p. 1-13], [10, p. 224-234], [7, p. 386-389]. The two following classes of entire functions are identical:

(1) the class of all entire functions $F(z)$ such that $F(z) = O(e^{A|z|})$ and $F(z) \in L_2(-\infty,\infty)$

(2) the class of all entire functions of the form

$$F(z) = \int_{-A}^{A} f(u)e^{iuz}du, \text{ such that } f(u) \in L_2(-A,A)$$

Note that from Plancherel's theorem we have the result

$$\int_{-\infty}^{\infty} |F(x)|^2 dx = 2\pi \int_{-A}^{A} |f(u)|^2 du.$$

In order to restate (2) in a more convenient form, we will need the following definition.

DEFINITION. [4, p. 168]. Let D be a simply connected domain with at least two boundary points. A function f analytic in D is said to be of class $E^p(D)$ if there exists a sequence of rectifiable Jordan curves, $C_1, C_2, \ldots, C_n, \ldots$ in D, tending to the boundary in the sense that C_n eventually surrounds each compact subdomain of D, such that

$$\int_{C_n} |f(z)|^p \, |dz| \leq M < \infty.$$

Restating (2), we have

(2′) the class of all entire functions $F(z)$ of exponential type such that the conjugate indicator diagram D is contained in the line segment $[-iA, iA]$, and $\varphi(w)$, the Borel transform of $F(z)$, is contained in $E^2(\mathbb{C} \setminus D)$. [10, p. 224-234].

We can extend the result of Paley and Wiener so that a conjugate indicator diagram different from a line segment along the imaginary axis can be considered. The first result is an extension to the case of any line segment, and the second result is an extension to the case of a convex polygon.

(1.2) THEOREM. The following two classes of entire functions are identical:

(1) The class of all entire functions $F(z)$ such that $e^{-\alpha z}F(z) = O(e^{A|z|})$, and $e^{-\alpha re^{i\theta}} F(re^{i\theta}) \in L_2(-\infty, \infty)$.

(2) The class of all entire functions $F(z)$ such that

$$F(z) = \int_{\alpha-iAe^{-i\theta}}^{\alpha+iAe^{-i\theta}} f(u)e^{uz}du,$$ where $f(u)$ is in L_2 of the line segment $[\alpha - iAe^{-i\theta}, \alpha + iAe^{-i\theta}]$.

For the next theorem, let D be a convex polygon with corners a_1, a_2, ..., a_ℓ numbered clockwise. Let $A_{ij} = |a_i - a_j|$ be the length of the side from a_i to a_j ($j = i + 1$ or $j = 1$, $i = \ell$). Let θ_{ij} be defined such that $\theta_{ij} \in [0,2\pi)$ and $a_i - a_j = A_{ij}e^{-i\theta_{ij}}$. We will also let $\alpha = \frac{1}{2}(a_i + a_j)$, $A = \frac{1}{2} A_{ij}$, and $\theta = -\frac{1}{2}\pi + \theta_{ij}$

(1.3) THEOREM. [7, p. 389-391] The following three classes of entire functions are identical:

(1) The class of all $F(z)$ such that $F(z)$ is entire, D is the conjugate indicator diagram of $F(z)$, and $F(z) = F_{12}(z) + F_{23}(z) + \cdots + F_{\ell 1}(z)$, where $F_{ij}(z)$ is in class (1) of Theorem 1.2, with α, A, θ as above.

(2) The class of all $F(z)$ such that $F(z)$ is entire, D is the conjugate indicator diagram of $F(z)$, and $F(z) = F_{12}(z) + F_{23}(z) + \cdots + F_{\ell 1}(z)$, where $F_{ij}(z)$ is in class (2) of Theorem 1.2 with α, A, θ as above.

(3) The class of all $F(z)$ such that $F(z)$ is entire, D is the conjugate indicator diagram of $F(z)$, and $\exp(-\alpha re^{i\theta})F(re^{i\theta}) \in L_2(0,\infty)$, with α, θ as above for i and j as above.

The Paley-Wiener theorem has been extended by Kacnel'son [6, p. 106] in one direction to the case of a general conjugate indicator diagram D. The result is:

(1.4) THEOREM. Let $F(z)$ be an entire function whose conjugate indicator diagram is the bounded closed convex set D. Let $\varphi(w)$ be the Borel transform of $F(z)$ where $\varphi(w)$ is contained in $E^2(\mathbb{C} \setminus D)$, and let Δ be the boundary of D traced counterclockwise. Then

$$\int_0^\infty |F(re^{i\theta})|^2 \exp(-2rh(\theta))dr \leq C_\Delta \int_\Delta |\varphi(w)|^2 |dw|$$

for all θ in $[0,2\pi)$, where C_Δ is a constant which depends only on Δ.

The question which we will answer here is whether the condition:

$\int_0^\infty |F(re^{i\theta})|^2 \exp(-2rh(\theta))dr \leq M$ for all $0 < \theta \leq 2\pi$, given in Theorem 1.4 as a necessary condition for $\varphi(w)$ to be in $E^2(\mathbb{C} \setminus D)$, is also a sufficient condition.

CHAPTER II

A COUNTEREXAMPLE

2. The question we ask here is whether the necessary condition given in Theorem 1.4 is also a sufficient condition. This would give us an extension of the classical Paley-Wiener Theorem for the case of a segment and for the case of a polygon. (Theorems 1.2 and 1.3). The conjecture would be:

(2.1) CONJECTURE. Let $F(z)$ be an entire function whose conjugate indicator diagram is the closed bounded convex set D and whose indicator function is $h(\psi)$. If

$$\int_0^\infty |F(re^{i\psi})|^2 \exp(-2rh(\psi))dr \leq M$$

for all ψ in $[0,2\pi)$, then

$$F(z) = \int_\Delta \varphi(w)e^{wz}dw$$

where $\varphi(w)$, the Borel transform of $F(z)$, is contained in $L_2(\Delta)$ and Δ is the boundary of D traced counterclockwise, or equivalently, $\varphi(w)$ is in $E^2(\mathbb{C} \setminus D)$.

The following counterexample will show that the conjecture is false for the case of a circle, even with the additional hypothesis that $\int_0^\infty |F(re^{i\psi})|^2 \exp(-2rh(\psi))dr$ is continuous in ψ.

DEFINITION. [4, p. 2]. A function $f(z)$ analytic in the unit disc is said to be of class $H^p (1 \leq p < \infty)$ if $\int_0^{2\pi} |f(re^{i\theta})|^p d\theta$ is bounded as r increases to one.

NOTE. [4, p. 2]. $H^p \supset H^q$ if $0 < p < q \leq \infty$; and if A is the unit disc, then for all $p > 0$ we have that $E^p(A) = H^p(A)$.

(2.2) THEOREM. [4, p. 17]. Given $f(z)$ in class H^p, then for almost all θ, there exist non-tangential limits for f. We will call these limits $f(e^{i\theta})$.

(2.3) THEOREM. [4, p. 8]. $\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$. Hence

$\int_0^{2\pi} |f(re^{i\theta})|^2 d\theta$ increases to $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta$ as r increases to one.

Hence $f(z)$ is in H^2 if and only if $\int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = 2\pi \sum_{n=0}^{\infty} |a_n|^2$ is finite.

(2.4) THEOREM. [4, p. 20]. Every function $f(z) \not\equiv 0$ of class H^p can be factored in the form $f(z) = B(z)g(z)$, where $B(z)$ is a Blaschke product and $g(z)$ is an H^p function which does not vanish in $|z| < 1$.

NOTE: [4, p. 19]. If $B(z)$ is a Blaschke product, then $|B(z)| < 1$ in $|z| < 1$ and $|B(z)| = 1$ almost everywhere for $|z| = 1$.

We shall now turn to the construction of the counterexample.

(2.5) COUNTEREXAMPLE. The main idea used in the construction of the counterexample is the fact that for each $0 \leq \psi < 2\pi$ the function $1/(1 - e^{-i\psi}z)$ is of class H^p, $0 < p < 1$, in the unit disc $D = \{z : |z| \leq 1\}$ and not of class H^1 in D. This will allow the construction of a function analytic in the unit disc D with the property that for all $0 \leq \psi < 2\pi$ it belongs to the class $H^1(D(\psi))$, where $D(\psi) = \{z : |z - \frac{1}{2} e^{i\psi}| < \frac{1}{2}\}$, but not to the class $H^1(D)$.

To accomplish this we shall first prove a series of lemmas. In the remainder of this chapter we shall denote the circle $\{z : |z - \frac{1}{2} e^{i\psi}| = \frac{1}{2}\}$, the boundary of $D(\psi)$, by $\Gamma(\psi)$. Furthermore, for each $0 < \lambda < 1$ we define

$$H_\lambda(\psi) = \frac{1}{2\pi} \int_{\Gamma(\psi)} |1 - z|^{-\lambda} \, |dz|, \quad 0 \leq \psi < 2\pi,$$

where the function $(1 - z)^{-\lambda}$ denotes that branch in $|z| < 1$ which for $z = 0$ is equal to 1.

The power series expansion of $(1 - z)^{-\lambda/2}$ in the neighborhood of the point $z = a$, $|a| < 1$, is of the form

$$(1 - z)^{-\lambda/2} = \sum_{n=0}^{\infty} \binom{-\lambda/2}{n} (-1)^n (z - a)^n/(1 - a)^{n+\lambda/2},$$

which holds for all $|z - a| < 1 - |a|$. In particular, for $z = (e^{i\psi} + e^{i\theta})/2$, $0 \leq \psi$, $\theta < 2\pi$, and $a = e^{i\psi}/2$ we obtain the expansion

$$(1 - z)^{-\lambda/2} = \sum_{n=0}^{\infty} \binom{-\lambda/2}{n} (-1)^n e^{in\theta}/2^n (1 - e^{i\psi}/2)^{n+\lambda/2}.$$

(2.6) LEMMA. If $0 < \lambda < 1$, then for all $0 \leq \psi < 2\pi$ we have $H_\lambda(0) \leq 1/(1-\lambda)$.

PROOF. If $0 < \lambda < 1$, by

$$|\sin \frac{\theta}{2}| \geq \frac{|\theta|}{\pi} \text{ for } |\theta| < \pi,$$

we obtain $H_\lambda(0)$

$$= \frac{1}{2\pi} \int_{|z-\frac{1}{2}|=\frac{1}{2}} \left| \, |-z|^{-\lambda} \, |dz| \right.$$

$$= \tfrac{1}{2} \cdot \frac{1}{2\pi} \int_0^{2\pi} |\sin(\tfrac{\theta}{2})|^{-\lambda} d\theta \leq \tfrac{1}{2}\, \pi^{\lambda-1} \int_0^{\pi} \theta^{-\lambda} d\theta$$

$$= \tfrac{1}{2}\, \pi^{\lambda-1} \, (1-\lambda)^{-1} \pi^{1-\lambda}$$

$$= \tfrac{1}{2} \, (1-\lambda)^{-1} < 1/(1-\lambda)$$

(2.7) LEMMA. For all $0 < \lambda < 1$, $H_\lambda(\psi)$ is a 2π-periodic even function of ψ which attains its maximum at $\psi = 0$ and its minimum at $\psi = \pi$. In particular, $H_\lambda(\psi) \leq H_\lambda(0) \leq 1/(1-\lambda)$ for all $0 \leq \psi < 2\pi$.

PROOF. From Parseval's relation it follows immediately that

$$H_\lambda(\psi) = \sum_{n=0}^{\infty} \binom{-\lambda/2}{n}^2 / 4^n \left(\tfrac{5}{4} - \cos \psi\right)^{n+\lambda/2} \tag{2.8}$$

Since $(\frac{5}{4} - \cos\psi)$ is a 2π-periodic continuous even function attaining its minimum at $\psi = 0$ and its maximum at $\psi = \pi$, the result follows easily from Lemma 2.6.

(2.9) LEMMA. If $\cos\psi > \frac{1}{4}$, then $H_\lambda(\psi)$ is increasing in λ, $0 < \lambda < 1$.

PROOF. If $0 < a < 1$, then a^λ is a decreasing function in λ for $0 < \lambda < 1$. For $\cos\psi > \frac{1}{4}$ we have that $(\frac{5}{4} - \cos\psi) < 1$, and so $(\frac{5}{4} - \cos\psi)^{-\lambda}$ is increasing in λ. If we then observe that $\binom{-\lambda/2}{n}^2$ is increasing in λ, the required result follows from (2.8).

(2.10) LEMMA. If $\psi \neq 0$ and $|\psi| < 1$, then for all $0 < \lambda < 1$ we have

$$H_\lambda(\psi) \leq H_1(\psi) \leq 3/\psi^2.$$

PROOF. For $z \in \Gamma(\psi)$ we have that $|1 - z|^{-1} \leq (\sqrt{\frac{5}{4} - \cos\psi} - \frac{1}{2})^{-1}$. Hence, if $\psi \neq 0$, $H_\lambda(\psi) \leq H_1(\psi) \leq \frac{1}{2}(\sqrt{\frac{5}{4} - \cos\psi} - \frac{1}{2})^{-1}$. From the elementary inequality $\cos\psi \leq 1 - \frac{\psi^2}{2} + \frac{\psi^4}{4!}$ and $|\psi| < 1$, it follows easily that $\sqrt{\frac{5}{4} - \cos\psi} - \frac{1}{2} \geq \frac{\psi^2}{2} - \frac{\psi^4}{3}$, and so, finally, since $|\psi| < 1$, we obtain $H_\lambda(\psi) \leq 1/(\psi^2 - 2\psi^2/3) \leq 3/\psi^2$; and the proof is finished.

We are now in a position to show that there exist functions in the unit disc of the type $F(z) = \sum_{n=1}^{\infty} \alpha_n f_n(z)$ where $f_n(z) = (1 - ze^{-i\psi_n})^{-\lambda_n}$

with $0 < \lambda_n < 1$, $0 < \psi_n < 2\pi$, and $f_n(0) = 1$ for all $n = 1, 2, \ldots$ such that $F \in H_1(D(\psi))$ for all $0 \leq \psi < 2\pi$ and $F \notin H_1(D)$. For this purpose, we will see that it is necessary that the sequence $\{\lambda_n\}$ increase sufficiently rapidly to 1 and the sequence $\{\psi_n\}$ decrease sufficiently slowly to 0. From $|f_n(z)| \leq (1 - |z|)^{-1}$ it follows immediately that, if

$\alpha_n > 0$ for all n and $\sum_{n=1}^{\infty} \alpha_n < \infty$, then F exists and is analytic in the

unit disc. In view of this fact we shall assume from now on that

$$\sum_{n=1}^{\infty} \alpha_n < \infty.$$

We begin with the following theorem.

(2.11) THEOREM. If $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} \alpha_n/\psi_n^2 < \infty$, where ψ_n is a sequence strictly decreasing to 0, then $H_1(\psi,F) = \frac{1}{2\pi}\int_{\Gamma(\psi)} |F(z)|\ |dz|$ is a 2π-periodic function of ψ, continuous for all $0 < \psi < 2\pi$ and continuous from below at $\psi = 2\pi$.

PROOF. For each $\delta > 0$ let $S_\delta = \{\psi : |\psi - \psi_n| > \delta,\ n = 1, 2, \ldots\}$. Then from $|f_n(z)| \leq (\sqrt{\frac{5}{4} - \cos(\psi - \psi_n)} - \frac{1}{2})^{-1}$ on $\Gamma(\psi)$, $n = 1, 2, \ldots$, it follows that $|F(z)|$ is bounded on the set $\cup(D(\psi) : \psi \in S_\delta)$. Hence, from the bounded convergence theorem it follows immediately that $H_1(\psi,F)$ is finite and continuous in ψ for all ψ except possibly at the points $\psi = \psi_n (n = 1, 2, \ldots)$ and $\psi = 0$. To show that $H_1(\psi,F)$ is continuous at $\psi = \psi_n (n = 1, 2, \ldots)$, let O_n be a small neighborhood of ψ_n such that $|\psi_k - \psi| \geq \delta$ for all $k \neq n$ and for all $\psi \in O_n$. Such a neighborhood exists since ψ_n is a sequence strictly decreasing to zero. Then for $\psi \in O_n$ and $z \in \Gamma(\psi)$ we have that $|F(z)| \leq \sum_{k \neq n} \alpha_k \, |f_k(z)| + \alpha_n |f_n(z)|$. For the same reason as above the function $\sum_{k \neq n} \alpha_k |f_k(z)|$ is bounded on $U(D(\psi) : \psi \in O_n)$, and $|f_n(z)| \in L^1(\Gamma(\psi))$ with the L^1 norm uniformly bounded for $\psi \in O_n$ by the L^1 norm for ψ_n by Lemma 2.7. Hence it follows again from the bounded convergence theorem that $H_1(\psi, F)$ is continuous at the points $\psi = \psi_n (n = 1, 2, \ldots)$. For $\psi = 2\pi$, we first note that it is clear from (2.8) that $H_\lambda(\psi)$ increases as $|\psi|$ decreases to zero. Hence,

$$\left| \int_{\Gamma(2\pi)} |F(z)| \, |dz| - \int_{\Gamma(\theta)} |F(z)| \, |dz| \right|$$

$$\leq \int_{-\pi}^{\pi} |F(\tfrac{1}{2} + \tfrac{1}{2} e^{i\Phi}) - F(\tfrac{1}{2} e^{i\theta} + \tfrac{1}{2} e^{i\Phi})| \, d\Phi$$

$$\leq \sum_{n=1}^{m} \alpha_n \int_{-\pi}^{\pi} |f_n(\tfrac{1}{2} + \tfrac{1}{2} e^{i\Phi}) - f_n(\tfrac{1}{2} e^{i\theta} + \tfrac{1}{2} e^{i\Phi})| \, d\Phi$$

$$+ 4\pi \sum_{n=m+1}^{\infty} \alpha_n H_{\lambda_n}(\psi_n)$$

$$\leq \sum_{n=1}^{m} \alpha_n \int_{-\pi}^{\pi} |f_n(\tfrac{1}{2} + \tfrac{1}{2} e^{i\Phi}) - f_n(\tfrac{1}{2} e^{i\theta} + \tfrac{1}{2} e^{i\Phi})| \, d\Phi$$

$$+ 12\pi \sum_{n=m+1}^{\infty} \alpha_n/\psi_n^2$$

The last step is from Lemma 2.10. Since the second term converges to zero as m tends to infinity by hypothesis, and for any fixed m the first sum clearly goes to zero as θ increases to 2π, $H_1(\psi,F)$ is continuous from below at 2π, completing the proof.

Note that if we prove $H_1(\psi,F)$ is continuous from above at 0, we then will have $H_1(\psi,F)$ continuous for all ψ and hence uniformly bounded in ψ.

So far, no particular hypotheses on the sequences $\{\alpha_n\}$, $\{\lambda_n\}$, and $\{\psi_n\}$ have been placed other than $\alpha_n > 0$, $\sum_{n=1}^{\infty} \alpha_n < \infty$, $0 < \lambda_n \uparrow 1$, $0 < \psi_n \downarrow 0$, and $\sum_{n=1}^{\infty} \alpha_n/\psi_n^2 < \infty$. We shall now show that there are sequences $\{\alpha_n\}$, $\{\lambda_n\}$ and $\{\lambda_n\}$ such that the resulting function $F \notin H^1(D)$, but $H_1(\psi,F)$ is continuous for all ψ. To this end, we shall first prove a lemma.

(2.12) LEMMA. Assume that $\psi_1 > \psi_2 > \cdots > \psi_n > \cdots \downarrow 0$, $0 < \lambda_n \uparrow 1$, and $\alpha_n > 0$, $n = 1, 2, \ldots$ with $\sum_{n=1}^{\infty} \alpha_n < \infty$. Let $p_1 > \psi_1$ and p_k be the midpoint between ψ_{k-1} and ψ_k $(k = 2, \ldots)$. Then $H_1(D,F) =$

$$\frac{1}{2\pi}\int_{|z|=1} |F(z)|\ |dz| \geq \frac{1}{2\pi}\sum_{k=1}^{\infty} \frac{\alpha_k}{1-\lambda_k}(\psi_k - p_{k+1})^{1-\lambda_k} -$$

$$-\frac{1}{4}\sum_{k=1}^{\infty}(p_k - p_{k+1})\sum_{n\neq k}\alpha_n(d(I_k,\psi_n))^{-\lambda_n},$$ where $d(I_k,\psi_n)$ is the distance of ψ_n from the interval $I_k = \{\psi : p_{k+1} < \psi \leq p_k\}$.

PROOF. With I_k as defined above we have $\int_{|z|=1} |F(z)|\ |dz| \geq$

$$\sum_{k=1}^{\infty}\int_{I_k} |F(z)|\ |dz| = \sum_{k=1}^{\infty}\int_{I_k}\Big(\sum_{n=1}^{\infty}|\alpha_n f_n(z)|\Big)\ |dz| \geq$$

$$\geq \sum_{k=1}^{\infty}\alpha_k\int_{I_k}|f_k(z)|\ |dz| - \sum_{k=1}^{\infty}\sum_{n\neq k}\alpha_n\int_{I_k}|f_n(z)|\ |dz|.$$

Observe that $\int_{I_k}|f_k(z)|\ |dz| = \int_{I_k}\left|2\sin\left(\frac{\theta-\psi_k}{2}\right)\right|^{-\lambda_k} d\theta \geq$

$$\geq \int_{p_{k+1}}^{p_k}|\theta-\psi_k|^{-\lambda_k}\,d\theta \geq \int_{p_{k+1}}^{\psi_k}|\theta-\psi_k|^{-\lambda_k}\,d\theta = \frac{|\psi_k - p_{k+1}|^{1-\lambda_k}}{1-\lambda_k}.$$ Now

consider the sum $\sum_{n\neq k} \alpha_n \int_{I_k} |f_n(z)|\,|dz|$. For this case observe that,

$$\int_{I_k} |f_n(z)|\,|dz| = \int_{I_k} \left|2\sin\left(\frac{\theta - \psi_n}{2}\right)\right|^{-\lambda_n} d\theta \leq$$

$$\leq \left(\frac{\pi}{2}\right)^{\lambda_n} \int_{I_k} |\theta - \psi_n|^{\lambda_n} d\theta \leq \frac{\pi}{2}\,(p_k - p_{k+1}) \cdot (d(I_k, \psi_n))^{-\lambda_n} \quad \text{where}$$

$d(I_k, \psi_n)$ is the distance of ψ_n to the interval $I_k (k \neq n)$; and the required result follows.

We shall now make the following selections: $\alpha_n = n^{-3}$, $\psi_n = 1/\sqrt{n}$, and $\lambda_n = 1 - (n+1)^{-2}$, $n = 1, 2, \ldots$. Then $\sum_{n=1}^{\infty} \alpha_n < \infty$, and so F exists; $\sum_{n=1}^{\infty} \alpha_n/\psi_n^2 = \sum_{n=1}^{\infty} n^{-2} < \infty$, and so Theorem 2.11 holds. What is left to show is that $F \notin H^1(D)$ and $H_1(\psi, F)$ is continuous from above at $\psi = 0$. To prove that $F \notin H^1(D)$ we shall use the result contained in Lemma 2.12.

(2.13) THEOREM. If $\alpha_n = n^{-3}$, $\psi_n = 1/\sqrt{n}$, and $\lambda_n = 1 - (n+1)^{-2}$ $(n = 1, 2, \ldots)$, then F is analytic in $|z| < 1$ satisfying the following properties: (i) $H_1(\psi, F)$ is a continuous function in ψ, $0 \leq \psi \leq 2\pi$, (ii) $F \notin H^1(D)$, and, (iii) $F \in H^p(D)$ for all $0 < p < 1$.

PROOF. From Theorem 2.11 it follows that to establish (i) we need only show that $H_1(\psi,F)$ is continuous from above at $\psi = 0$.

To this end, let $\theta \in (0,\delta)$, $\delta > 0$.

Then $\left| \int_{\Gamma(\theta)} |F(z)|\ |dz| - \int_{\Gamma(0)} |F(z)|\ |dz| \right|$

$$\leq \sum_{n=2}^{L} \alpha_n \int_{-\pi}^{\pi} |f_n(\tfrac{1}{2} e^{i\theta} + \tfrac{1}{2} e^{i\Phi}) - f_n(\tfrac{1}{2} + \tfrac{1}{2} e^{i\Phi})|\ d\Phi + \sum_{n=L+1}^{\infty} \alpha_n H_{\lambda_n}(\psi_n) +$$

$$\sum_{n=L+1}^{\infty} \alpha_n H_{\lambda_n}(\psi_n - \theta) = T_1 + T_2 + T_3.$$

For any fixed L, T_1 converges to zero as θ goes to zero. From Lemma 2.10 it follows immediately that T_2 converges to zero as L tends to infinity. Let N be defined such that $\theta \in (\psi_{N+1}, \psi_N]$, and let $A = [N - N^{3/4}]$, $B = [N + N^{3/4}]$. Then we write $T_3 = \sum_{n=L+1}^{A} \alpha_n H_{\lambda_n}(\psi_n - \theta)$

$$+ \sum_{n=A+1}^{B} \alpha_n H_{\lambda_n}(\psi_n - \theta) + \sum_{n=B+1}^{\infty} \alpha_n H_{\lambda_n}(\psi_n - \theta) = S_1 + S_2 + S_3 .$$

To estimate the parts T_1, T_2, and T_3 we observe first that, by Lemmas 2.7 and 2.10, $H_\lambda(\psi) \leq \min((1-\lambda)^{-1}, 3/\psi^2)$ and that

$(k - k^{3/4})^{-\frac{1}{2}} - k^{-\frac{1}{2}} \geq \frac{1}{2} k^{3/4} \cdot k^{-3/2} = \frac{1}{2} k^{-3/4}$ for $k \geq 1$. Hence, if $n \leq A$, then $H_{\lambda_n}(\psi_n - \theta) \leq 12\, n^{3/2}$, and if $n \geq B + 1$, then $H_\lambda(\psi_n - \theta) \leq 24(N+1)^{3/2}$, and for $A + 1 \leq n \leq B$ we have that $H_{\lambda_n}(\psi_n - \theta) \leq (N + N^{3/4})^2$. Observing that

$$\sum_{k=L+1}^{A} k^{-3} \cdot k^{3/2} \leq 2(L+1)^{-\frac{1}{2}}, \quad \sum_{k=A+1}^{B} k^{-3} \leq 2(N - N^{3/4})^{-3} \cdot N^{3/4}, \text{ and}$$

$\displaystyle\sum_{k=B+1}^{\infty} k^{-3} \leq (N + N^{3/4})^{-2}$ we obtain, finally, that $S_1 + S_2 + S_3 \leq$

$$24(L+1)^{-\frac{1}{2}} + 2(N - N^{3/4})^{-3} \cdot N^{3/4} \cdot (N + N^{3/4})^{2} + 24(N+1)^{3/2}(N+N^{3/4})^{-2}$$

which tends to zero as $N \to \infty$ and $L \to \infty$. For a given $\epsilon > 0$, first choose L so large that $T_2 < \epsilon/3$ and $S_1 < \epsilon/6$. Then with this L fixed choose $\delta > 0$ so small that $S_2 + S_3 < \epsilon/6$ and $T_1 < \epsilon/3$. Hence $H_1(\psi, F)$ is continuous from above at $\psi = 0$ and (i) holds. For (ii) we use Lemma 2.12. Observe that

$$\sum_{k=1}^{\infty} \frac{\alpha_k}{1-\lambda_k} (\psi_k - p_{k+1})^{1-\lambda_k} = \sum_{k=1}^{\infty} \frac{(k+1)^2}{k^3} \cdot \frac{1}{2}\left(\frac{1}{\sqrt{k}} - \frac{1}{\sqrt{k+1}}\right)^{(k+1)^{-2}} \geq$$

$$\geq \frac{1}{2} \sum_{k=1}^{\infty} \frac{(k+1)^2}{k^3} \left(\frac{1}{4(k+1)\sqrt{k+1}}\right)^{(k+1)^{-2}} \geq \frac{1}{4} \sum_{k=1}^{\infty} \frac{(k+1)^2}{k^3} = \infty.$$

Observing that for $k \neq n$, $d(I_k, \psi_n) \geq$

$$\geq (\psi_n - p_{n+1}) = \frac{1}{2}\left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right) = \frac{1}{2} \frac{1}{\sqrt{n}\sqrt{n+1}(\sqrt{n} + \sqrt{n+1})}$$

$\displaystyle\geq \frac{1}{4} \frac{1}{(n+1)\sqrt{n+1}}$, we obtain the result that

$$\sum_{k=1}^{\infty} (p_k - p_{k+1}) \sum_{n \neq k} \alpha_n (d(I_k, \psi_n))^{-\lambda_n} \leq \sum_{k=1}^{\infty} (p_k - p_{k+1}) \sum_{n=1}^{\infty} \frac{4(n+1)\sqrt{n+1}}{n^3} \leq$$

$\leq 4p_1 \sum_{n=1}^{\infty} \frac{(n+1)\sqrt{n+1}}{n^3} < \infty$. Hence, $F \notin H^1(D)$. In order to show that $F \in H_p(D)$ for all $0 < p < 1$ we observe that

$$\int_{|z|=1} |F(z)|^p \, |dz| \leq \sum_{n=1}^{\infty} \alpha_n{}^p \int_{|z|=1} |F_n(z)|^p \, |dz| .$$

From Lemma 2.7 it follows that $\int_{|z|=1} |f_n(z)|^p \, |dz| \leq 2\pi/(1 - p\lambda_n)$, and so, $\frac{1}{2\pi} \int_{|z|=1} |F(z)|^p \, |dz| \leq \sum_{n=1}^{\infty} \alpha_n{}^p \frac{1}{1 - p\lambda_n} \leq \frac{1}{1-p} \sum_{n=1}^{\infty} n^{-3p}$; and the latter sum converges for all $1 > p > 1/3$. Hence, from the Note (before Theorem 2.2) it follows that $F(z) \in H^p$ for all p satisfying $0 < p < 1$. This completes the proof of the theorem.

REMARK. It was shown by R. M. Gabriel and F. Carlson (see [5],[3]) that, if f is analytic in the unit disc D and γ is a closed rectifiable curve inside D, then for all $p > 0$

$$\int_{\gamma} |f(z)|^p \, |dz| \leq \frac{1}{\pi} \int_{|z|=1} |f(z)|^p \, V(z) \, |dz| ,$$

when $V(z) = \int_{\gamma} |d_x \arg(x - z)| dx$. If γ is the circle $\Gamma(\psi)$, then $V(z) \leq 2\pi$ for all $|z| = 1$, and so,

$$(*) \qquad \int_{\Gamma(\psi)} |f(z)|^p \, |dz| \leq 2 \int_{|z|=1} |f(z)|^p \, |dz| .$$

The function F of Theorem 2.13 shows that the inequality in (*) cannot be reversed, that is, even if the left-hand side of (*) is uniformly bounded in ψ, the right-hand side need not be finite.

We return now to the construction of the counterexample. From $F \in H^p$ $(0 < p < 1)$ it follows from Theorem 2.4 that $F(z) = B(z) \cdot G(z)$, $|z| < 1$, where B is a Blaschke product and $G \in H^p$ for all $0 < p < 1$ and $G(z) \neq 0$ for all $|z| < 1$. Let $g(z) = B(z) \cdot (G(z))^{\frac{1}{2}}$, where $(G(z))^{\frac{1}{2}}$ is a well-defined branch of the square root function. Then $|B| \leq 1$ shows that $g \in H^2(D(\psi))$ for all ψ and

$$\int_{\Gamma(\psi)} |g|^2 \, |dz| \leq \int_{\Gamma(\psi)} |F(z)| \, |dz| \text{ for all } 0 \leq \psi < 2\pi.$$

Furthermore, since $|B| \leq 1$, the fact that $\int_{\Gamma(\psi)} |g|^2 \, |dz|$ is continuous in ψ can be shown in exactly the same way as in the proof of Theorems 2.11 and 2.13. Since $F \notin H^1(D)$ it follows from $|B| = 1$ almost everywhere on $|z| = 1$, that $g \notin H^2(D)$.

Now let $\varphi(w) = \frac{1}{w} g\left(\frac{1}{w}\right)$. Then φ is analytic for $|w| > 1$, $|\varphi(w)| \to 0$ as $|w| \to \infty$, and $\varphi \in L^2$ along every line in the complex plane touching the unit circle (for each ψ, $1/w$ maps $\Gamma(\psi)$ into a line $\ell(\psi)$ tangent to $|w| = 1$ at $e^{-i\psi}$, and $\int_{\ell(\psi)} |\varphi(w)|^2 \, |dw| = \int_{\Gamma(\psi)} |g(z)|^2 \, |dz|$) with the L^2 norm continuous as the point of tangency is varied. However, $\varphi \notin H^2(\mathbb{C} \setminus D)$.

Let $f(z) = \frac{1}{2\pi i}\int_\gamma \varphi(w)e^{zw}dw$, where γ is a simple closed contour containing the unit circle in its interior, be the entire function of exponential type whose Borel transform is equal to φ. Then by Plancherel's theorem we have that for all $0 \leq \psi < 2\pi$

$$(2.14) \qquad 2\pi\int_0^\infty e^{-2r}|f(re^{i\psi})|^2 dr = \int_{-\infty}^\infty |\varphi(e^{-i\psi}(1+it))|^2\, dt$$

Hence, we have obtained the following result disproving the Conjecture 2.1.

(2.15) THEOREM. There exists an entire function f of exponential type such that the function $p(\theta) = \int_0^\infty e^{-2r}|f(re^{i\theta})|^2\, dr$ is a continuous function in θ, $0 \leq \theta \leq 2\pi$, whose indicator diagram is contained in the unit disc and whose Borel transform φ is not of class $H^2(\mathbb{C} \setminus D)$.

REMARKS 1. By means of a conformal mapping argument the above counterexample can be shown to exist for domains bounded by convex curves which have a continuously turning tangent. The result of Levin concerning convex polygons quoted in the introduction shows, however, that the conjecture holds for convex polygons.

2. To find an entire function f with indicator diagram equal to the unit disc and violating the conjecture one simply adds to f an entire function of exponential type of the form $\sum_{n=1}^\infty z^{n!}/n^2(n!)!$ whose

Borel transform has $|w| = 1$ as a natural boundary and is nevertheless in the class $H^2(\mathbb{C} \setminus D)$.

3. If we integrate (2.14) with respect to θ from 0 to 2π, then we obtain the formula

$$\int_0^\infty e^{-2r} \left(\int_0^{2\pi} |f(re^{i\theta})|^2 \, d\theta \right) dr =$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} |\varphi(e^{-i\theta}(1 + it))|^2 \, d\theta \, dt.$$

By means of the change of coordinates $x + iy = e^{-i\theta}(1 + it)$, the latter integral can be written in the form

$$\frac{1}{2\pi} \iint_{x^2+y^2 \geq 1} |\varphi(x + iy)|^2 \left(\sqrt{x^2 + y^2 - 1}\right)^{-1} dx \, dy =$$

$$= \frac{1}{2\pi} \int_1^\infty \frac{r}{\sqrt{r^2 - 1}} \int_0^{2\pi} \left| \sum_{n=0}^\infty a_n r^{-n-1} e^{-i(n+1)\theta} \right|^2 d\theta \, dr =$$

$$= \sum_{n=0}^\infty |a_n|^2 \int_1^\infty \frac{r}{\sqrt{r^2 + 1}} \frac{1}{r^{2n+2}} \, dr = \frac{|a_n|^2 \, \Gamma(n + \frac{1}{2})}{\Gamma(n+1)}, \text{ where}$$

$f(z) = \sum_{n=0}^\infty \frac{a_n z^n}{n!}$. Since $\Gamma(n + \frac{1}{2})/n! \sim 1/\sqrt{n}$, we obtain that, if

$\sum_{n=0} |a_n|^2/\sqrt{n+1} < \infty$, then $\int_0^\infty e^{-2r}|f(re^{i\theta})|^2\,dr \in L^1(0,2\pi)$. Since $\varphi \in H^2(C \setminus D)$ if and only if $\sum_{n=0} |a_n|^2 < \infty$, it follows now easily that $\int_0^\infty e^{-2r}\,|f(re^{i\theta})|^2\,dr \in L^1(0,2\pi)$ need not imply that $\varphi \in H^2(C \setminus D)$. Indeed, the function $\sum_{n=1}^{\infty} z^n/\sqrt{n} \cdot (n!)$ has the required properties.

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