NUMERICAL RANGES OF POWERS OF OPERATORS

Thesis by

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To my Parents and Grandmother

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ABSTRACT

We study the relations between a Hilbert space operator and the numerical ranges of its powers in this thesis.

Let $\mathcal{B}(\mathcal{H})$ denote the set of bounded linear operators on a complex Hilbert space. For $T \in \mathcal{B}(\mathcal{H})$, let $\sigma(T)$ and $W(T)$ denote its spectrum and numerical range, respectively. The following are proved using von Neumann's theory of spectral sets:

(i) If $\sigma(T) \subset (\gamma,\infty)$ with $\gamma > 0$ and if T is not self-adjoint, then there is an index N such that $\{z \in \mathbb{C} : |z| \leq \gamma^n\} \subset W(T^n)$ whenever $n \geq N$.

(ii) T^n is accretive, $n = 1, 2, \ldots, k$, if and only if the closed sector $\{z \in \mathbb{C} : |\mathrm{Arg}\, z| \leq \pi/2k\} \cup \{0\}$ is spectral for T. In this case $\|\mathrm{Im}Tx\| \leq \tan(\pi/2k)\, \|\mathrm{Re}Tx\|$ for each $x \in \mathcal{H}$.

(i) remains valid if we replace T^n by T^nD, where D is a surjective operator commuting with T. Various situations in which the commutativity assumption is relaxed are examined.

A theorem for finite dimensional matrices by C. R. Johnson is generalized to the operator case: If $0 \notin \mathrm{Cl}(W(T^n))$, $n = 1, 2, 3, \ldots$, then an odd power of T is normal. Furthermore, if T is a convexoid, then T itself is normal; in fact, T is the direct sum of at most three rotated positive operators. Using these results, we prove: Let $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinite dimensional and separable. If $T^n \notin \{Y \in \mathcal{B}(\mathcal{H}) : Y = AX - XA,$ $A,X \in \mathcal{B}(\mathcal{H}),$ A positive$\}$, $n = 1, 2, 3, \ldots$, then there is an odd integer m and a compact operator K_o such that $T^m + K_o$ is normal. Moreover, T is a normal plus a compact if and only if $\bigcap \{\mathrm{Cl}(W(T + K)) : K \text{ compact}\}$ is a closed polygon (possibly degenerate).

TABLE OF CONTENTS

CHAPTER 1

In this chapter we shall state certain basic results, techniques and terminology that will be required later. Many of these results have appeared in the literature and are well-known.

1. NOTATION

We let $\mathbb{C}$ denote the set of complex numbers and $\mathbb{R}$ denote the set of real numbers. For $\Omega \subset \mathbb{C}$, $\mathrm{co}(\Omega)$ denotes the convex hull, $\mathrm{Cl}(\Omega)$ the closure, $\mathrm{Int}(\Omega)$ the interior and $\partial(\Omega)$ the boundary of Ω. We write $\Omega > (\geq) r$, $r \in \mathbb{R}$, if $\Omega \subset \mathbb{R}$, and each number in $\Omega > (\geq) r$.

Let $\Delta(r)$ denote the closed disc centered at the origin with radius r,

$$\Delta(r) = \{z \in \mathbb{C} : |z| \leq r\}.$$

Let $\Sigma(\varphi)$ denote the closed sector of the complex plane symmetric with respect to the real axis, with vertex at the origin and angular opening 2φ,

$$\Sigma(\varphi) = \{z \in \mathbb{C} : |\mathrm{Arg}\ z| \leq \varphi\} \cup \{0\}.$$

Note that $\Sigma(\pi/2)$ denotes the closed right half plane. For $\alpha, \beta \in \mathbb{C}$ and $\epsilon \in \mathbb{R}$, $0 < \epsilon \leq 1$, we let $\Theta(\alpha,\beta;\epsilon)$ denote the closed elliptical disc with foci at α and β and eccentricity ϵ,

$$\Theta(\alpha,\beta;\epsilon) = \{z \in \mathbb{C} : |z - \alpha| + |z - \beta| \leq |\alpha - \beta|/\epsilon\}.$$

Note that two degenerate cases are included in the definition:

(i) $\Theta(\alpha,\beta;1)$ is the line segment joining α and β,

(ii) $\Theta(\alpha,\alpha,\epsilon)$ is the singleton $\{\alpha\}$.

2. HILBERT SPACE OPERATORS

We let $\mathcal{H}$ denote a complex Hilbert space with inner product $(\cdot,\cdot)$ and $\mathcal{B}(\mathcal{H})$ the algebra of all bounded endomorphisms of $\mathcal{H}$. For $T \in \mathcal{B}(\mathcal{H})$, T^* denotes its adjoint and $\sigma(T)$ denotes its spectrum. $\sigma(T)$ is a nonempty compact set in $\mathbb{C}$. The spectral radius $r(T)$ of T is defined by

$$r(T) = \max\{|\lambda| : \lambda \in \sigma(T)\}.$$

It can be shown that

$$r(T) = \lim_{n\to\infty} \|T^n\|^{1/n} .$$

If T_1 and T_2 are two commuting operators in $\mathcal{B}(\mathcal{H})$, then

$$\sigma(T_1T_2) \subset \sigma(T_1)\cdot\sigma(T_2)$$

and

$$\sigma(T_1 + T_2) \subset \sigma(T_1) + \sigma(T_2).$$

These are simple consequences of the Gelfand representation for commutative Banach algebras. See Chapter 11 of [32].

$\sigma(\cdot)$ is an upper semicontinuous set function on $\mathcal{B}(\mathcal{H})$ with respect to the uniform operator topology, i.e., for $T \in \mathcal{B}(\mathcal{H})$ and $\epsilon > 0$, there exists $\delta > 0$ such that

$$\sup\{\text{dist}(\lambda,\sigma(T)) : \lambda \in \sigma(S)\} < \epsilon$$

whenever $\|S - T\| < \delta$. $\sigma(\cdot)$ is not continuous unless $\mathcal{X}$ is finite-dimensional. However, $\sigma(T)$ does change continuously with T if the perturbation commutes with T. See §IV.3 of [24].

For $T \in \mathcal{B}(\mathcal{X})$, an isolated point μ of $\sigma(T)$ is either a pole or an essential singularity of the resolvent $(\lambda - T)^{-1}$, depending on whether the Laurent development of the resolvent in powers of $\lambda - \mu$ has a finite or an infinite number of nonvanishing terms in negative powers of $\lambda - \mu$, respectively. See §5.8 of [38].

We let $\mathcal{F}(T)$ denote the family of functions analytic on some neighborhood of $\sigma(T)$. For $f \in \mathcal{F}(T)$, let Ω be an open set in $\mathbb{C}$, containing $\sigma(T)$, whose boundary $\partial(\Omega)$ consists of a finite number of rectifiable Jordan curves, oriented in the positive sense. If $Cl(\Omega)$ is contained in the domain of analyticity of f, then we define f(T) by the Dunford-Taylor integral

$$f(T) = \frac{1}{2\pi i}\int_{\partial(\Omega)} f(\lambda)(\lambda - T)^{-1}d\lambda.$$

f(T) does not depend on Ω as long as Ω satisfies the above conditions. We shall find the following facts useful:

(i) For $S \in \mathcal{B}(\mathcal{X})$, $ST = TS$, then $Sf(T) = f(T)S$.

(ii) SPECTRAL MAPPING THEOREM. $f(\sigma(T)) = \sigma(f(T))$.

(iii) If $g \in \mathcal{F}(f(T))$ and $h(z) = g(f(z))$, then $h \in \mathcal{F}(T)$ and $h(T) = g(f(T))$.

For further details of the operational calculus, refer to Chapter VII of [11]. A consequence of the above three properties is:

(1.1) PROPOSITION. Let $T \in \mathcal{B}(\mathcal{X})$ and $f \in \mathcal{J}(T)$. Suppose f is one-to-one on $\sigma(T)$ and that for each $\lambda \in \sigma(T)$ such that λ is not a simple pole of $(\lambda - T)^{-1}$, we have $f'(\lambda) \neq 0$. Then T and f(T) have identical commutants.

PROOF. It is sufficient to construct a function $g \in \mathcal{J}(f(T))$ such that $T = g(f(T))$. Let $\sigma_1 = \{\lambda \in \sigma(T) : f'(\lambda) \neq 0\}$. We assume both σ_1 and $\sigma(T) \setminus \sigma_1$ are nonempty. $\sigma(T) \setminus \sigma_1$ consists of finitely many points, say, $\lambda_1, \ldots, \lambda_k$. There exists an open neighborhood Ω_1 of σ_1 on which f is one-to-one and f' is nonzero, and $\partial(\Omega_1) \cap \sigma(T) = \emptyset$. Let N_j, $j = 1, \ldots, k$, be disjoint open sets in $\mathbb{C} \setminus \mathrm{Cl}(f(\Omega_1))$ and $f(\lambda_j) \in N_j$.

We put
$$g(z) = \begin{cases} f^{-1}(z) & z \in f(\Omega_1) \\ \lambda_j & z \in N_j \quad , \quad j = 1, \ldots, k . \end{cases}$$

Then $g \in \mathcal{J}(f(T))$ and $T = g(f(T))$. ∎

Proposition 1.1 generalizes the theorem in [14]. In Section 6 we shall show that its converse also holds if $\mathcal{X}$ is finite dimensional.

For $T \in \mathcal{B}(\mathcal{X})$, we let n(T) denote the nullity of T, i.e., the dimension of its nullspace. We let d(T) denote the defect of T, i.e., the dimension of the (algebraic) quotient space $\mathcal{X}/T\mathcal{X}$. T is called semi-Fredholm if T has closed range and either $n(T) < \infty$ or $d(T) < \infty$. T is called Fredholm if T has closed range and both n(T) and d(T) are finite.

(1.2) PROPOSITION. Let $T \in \mathcal{B}(\mathcal{X})$ with $0 \in \partial(\sigma(T))$. Then the following statements are equivalent.

(i) T is Fredholm.

(ii) $d(T)$ is finite.

(iii) T is semi-Fredholm.

Furthermore, if any one of these conditions hold, $n(T) = d(T)$ and 0 is a pole of $(\lambda - T)^{-1}$ with finite rank.

For a proof, see Theorem 2.7 in [26].

3. NUMERICAL RANGE

The numerical range of an operator $T \in \mathcal{B}(\mathcal{H})$ is the set

$$W(T) = \{(Tx,x) : x \in \mathcal{H}, \quad \|x\| = 1\}.$$

The numerical radius of T, $w(T)$, is the number

$$w(T) = \sup \{|\lambda| : \lambda \in W(T)\}.$$

A detailed discussion on numerical ranges may be found in Chapter 17 of [15]. The following list contains some of the well-known properties of numerical ranges:

(i) (Toeplitz-Hausdorff) $W(T)$ is convex.

(ii) $\sigma(T) \subset Cl(W(T))$.

(iii) If U is unitary, then $W(T) = W(U^*TU)$.

(iv) Let P be a nonzero (orthogonal) projection on $\mathcal{H}$. If $T_1 = PT|_{P\mathcal{H}}$, the compression of T to $P\mathcal{H}$, then $W(T_1) \subset W(T)$.

(v) If T is normal, then $Cl(W(T)) = co(\sigma(T))$.

(vi) T is Hermitian if and only if $W(T) \subset \mathbb{R}$.

If $W(T)$ is real and nonnegative, we say T is positive and write $T \geq 0$. By $T_1 \geq T_2$, we mean $(T_1 - T_2) \geq 0$. Note that a positive operator is necessarily Hermitian by (vi) and the set of all positive operators forms a cone under " $\geq$ ".

T is called accretive if $(T + T^*) \geq 0$, or equivalently, $W(T) \subset \Sigma(\pi/2)$.

The determination of the numerical range of an operator is often difficult. However, the following theorem describes the numerical ranges of all 2 x 2 matrices ([46],[43],[10],[42],[15, p.109]).

(1.3) THEOREM. Let A be the 2 x 2 upper triangular matrix $\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$. Then

$$W(A) = \begin{cases} \Theta(\alpha,\beta;\ (1 + |\gamma/(\alpha - \beta)|^2)^{-\frac{1}{2}}) & \alpha \neq \beta \\ \alpha + \Delta(|\gamma|/2) & \alpha = \beta . \end{cases}$$

(1.4) COROLLARY. For each positive integer n,

$$W(A^n) = \begin{cases} \Theta(\alpha^n,\beta^n;\ (1 + |\gamma/(\alpha - \beta)|^2)^{-\frac{1}{2}}) & \alpha \neq \beta \\ \alpha^n + \Delta(n|\gamma\alpha^{n-1}|/2) & \alpha = \beta . \end{cases}$$

PROOF.

$$\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}^n = \begin{cases} \begin{pmatrix} \alpha^n & \gamma(\alpha^n-\beta^n)/(\alpha-\beta) \\ 0 & \beta^n \end{pmatrix} & \alpha \neq \beta \\ \begin{pmatrix} \alpha^n & n\gamma\alpha^{n-1} \\ 0 & \alpha^n \end{pmatrix} & \alpha = \beta . \end{cases}$$

$$(1 + |(\gamma(\alpha^n - \beta^n)/(\alpha - \beta))/(\alpha^n - \beta^n)|^2)^{-\frac{1}{2}} = (1 + |\gamma/(\alpha - \beta)|^2)^{-\frac{1}{2}}. \quad \blacksquare$$

Remark: If A is a matrix with distinct eigenvalues α and β, then the numerical ranges of powers of A are elliptical discs with a constant eccentricity. For our purpose line segments and singletons are also elliptical discs. If $\alpha^n = \beta^n$ for some integer n, then $W(A^n) = \{\alpha^n\}$.

4. 2 x 2 OPERATOR MATRICES

Let $\mathcal{H} \oplus \mathcal{K}$ denote the direct sum of two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$. An operator on $\mathcal{H} \oplus \mathcal{H}$ is expressed as a 2 x 2 matrix whose entries are operators. See Chapter 7 of [15].

(1.5) LEMMA. Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$,

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Then

$$W(T) = \cup \left\{ W\left(\begin{pmatrix} (Ax,x) & (By,x) \\ (Cx,y) & (Dy,y) \end{pmatrix}\right) : x \in \mathcal{H}, y \in \mathcal{K}, \|x\| = \|y\| = 1 \right\}.$$

PROOF. Let $x \in \mathcal{H}$, $y \in \mathcal{K}$, $\alpha, \beta \in \mathbb{C}$.

Then

$$(T(\alpha x \oplus \beta y), \ \alpha x \oplus \beta y)$$

$$= \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix}, \ \begin{pmatrix} \alpha x \\ \beta y \end{pmatrix}\right)$$

$$= \left(\begin{pmatrix} (Ax,x) & (By,x) \\ (Cx,y) & (Dy,y) \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\right). \quad \blacksquare$$

(1.6) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ and

$$T = \begin{pmatrix} 0 & B \\ 0 & D \end{pmatrix}.$$

If $0 \notin \mathrm{Int}(W(T))$, then $B = 0$.

PROOF. Apply Theorem 1.3. ∎

Let $T \in \mathcal{B}(\mathcal{H})$ with disconnected spectrum, i.e., there are two disjoint, nonempty and closed sets σ_1 and σ_2 whose union is $\sigma(T)$. (Some authors, e.g. [11], [38], call σ_1 and σ_2 spectral sets, but we shall reserve this term for another concept.) Let Ω be an open set containing σ_1 such that $\mathrm{Cl}(\Omega) \cap \sigma_2 = \emptyset$ and $\partial(\Omega)$ consists of a finite number of positively oriented rectifiable Jordan curves. Put

$$E = \frac{1}{2\pi i} \int_{\partial(\Omega)} (\lambda - T)^{-1} d\lambda .$$

Then E is an idempotent, $ET = TE$ and

$$\sigma\,(T|_{E\mathcal{H}}) = \sigma_1,$$

$$\sigma\,(T|_{(I-E)\mathcal{H}}) = \sigma_2 .$$

Usually, the operator E is called a spectral projection. In order to emphasize that E is not necessarily Hermitian, we call it a spectral idempotent in this paper.

(1.7) PROPOSITION. Let T and E be as above and let P be the (orthogonal) projection on $E\mathcal{K}$. Then, with respect to the decomposition $E\mathcal{K} \oplus (E\mathcal{K})^{\perp}$, the operator matrix corresponding to T has the form $\begin{pmatrix} T_1 & T_1A-AT_2 \\ 0 & T_2 \end{pmatrix}$, where

$$\begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} = E - P$$

and

$$\sigma(T_i) = \sigma_i \quad , \quad i = 1,2.$$

Furthermore, $T_1A - AT_2 = 0$ if and only if $A = 0$.

PROOF. From the relations $EP = P$ and $PE = E$, we have

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix} .$$

Write
$$T = \begin{pmatrix} T_1 & B \\ C & T_2 \end{pmatrix} .$$

Since $TE = ET$, we get $C = 0$ and $B = T_1A - AT_2$. The facts that $\sigma(T_i) = \sigma_i$, $i = 1,2$, follow from the following two equations.

$$\begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} T_1 & T_1A \\ 0 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 & 0 \\ 0 & T_2 \end{pmatrix} \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & -AT_2 \\ 0 & T_2 \end{pmatrix} .$$

Suppose $T_1A = AT_2$, then $(\lambda - T_1)A = A(\lambda - T_2)$.

For $\lambda \in \partial(\Omega)$, $A(\lambda - T_2)^{-1} = (\lambda - T_1)^{-1}A$.

Hence $$A\int_{\partial(\Omega)} (\lambda - T_2)^{-1} d\lambda = \int_{\partial(\Omega)} (\lambda - T_1)^{-1} d\lambda\, A$$

or $$0 = I_{E\mathscr{H}}\, A = A.$$ ∎

5. ESSENTIAL NUMERICAL RANGE & ESSENTIAL SPECTRA

Let $\mathfrak{A}$ be a complex Banach algebra with unit 1. Let $\mathfrak{A}^*$ denote its dual space. For $a \in \mathfrak{A}$, $\sigma(a)$ denotes the spectrum of a and $V(\mathfrak{A}, a)$ denotes the algebra numerical range of a,

$$V(\mathfrak{A}, a) = \{f(a) : f \in \mathfrak{A}^*,\ f(1) = 1 = \|f\|\}.$$

$V(\mathfrak{A}, a)$ is a compact, convex set containing $\sigma(a)$. A detailed discussion of the numerical ranges of Banach algebras appears in [4] and [5].

Let $\mathfrak{A}$ be a C^*-algebra with unit, then by the Gelfand-Naimark theorem there exists an isometric $*$-isomorphism τ of $\mathfrak{A}$ onto a closed self-adjoint subalgebra of $\mathcal{B}(\mathscr{H})$, $\mathscr{H}$ a suitably chosen Hilbert space. See Theorem 12.41 of [32]. Furthermore, we have

$$V(\mathfrak{A}, a) = \mathrm{Cl}(W(\tau(a))). \quad ([3, \text{Theorem } 12], [2, \text{Theorem } 3])$$

For the rest of this section, we assume $\mathscr{H}$ is infinite dimensional and separable. We let $\mathcal{K}(\mathscr{H})$ denote the set of all compact operators and let $\mathfrak{A}(\mathscr{H}) = \mathcal{B}(\mathscr{H})/\mathcal{K}(\mathscr{H})$. $\mathfrak{A}(\mathscr{H})$ is a C^*-algebra called the Calkin algebra [8]. Let Π denote the canonical homomorphism from $\mathcal{B}(\mathscr{H})$ onto $\mathfrak{A}(\mathscr{H})$. The essential numerical range $W_e(T)$ of an operator $T \in \mathcal{B}(\mathscr{H})$ is by definition the algebra numerical range $V(\mathfrak{A}(\mathscr{H}), \Pi(T))$. It is shown in [35] that $W_e(T) = \cap\,\{\mathrm{Cl}(W(T + K)) : K \in \mathcal{K}(\mathscr{H})\}$.

$\sigma(\Pi(T))$ is called the Wolf (or Fredholm or Calkin) essential spectrum [39]. We define the Weyl essential spectrum $\sigma_W(T)$ to be the largest subset of $\sigma(T)$ which is invariant under compact perturbations of T,

$$\sigma_W(T) = \cap \{\sigma(T + K) : K \in \mathcal{K}(\mathcal{H})\}.$$

Stampfli [36] has shown that there exists $K_o \in \mathcal{K}(\mathcal{H})$, K_o depending on T, such that $\sigma_W(T) = \sigma(T + K_o)$.

It is proved in [13] that $\sigma_W(T)$ consists of $\sigma(\Pi(T))$ together with some of the bounded components of the complement of $\sigma(\Pi(T))$. Consequently, if $\sigma(\Pi(T))$ lies on a simple arc, $\sigma(\Pi(T)) = \sigma_W(T)$. Most of the operators to be discussed in the rest of this paper will have essential spectra lying on finitely many disjoint line segments.

(1.8) THEOREM. [44],[7, p.62] Let $T \in \mathcal{B}(\mathcal{H})$. Suppose $\Pi(T)$ is normal and $\sigma(\Pi(T))$ lies on a simple arc. Then, there exists a compact operator K_o such that $T + K_o$ is normal and $\sigma(T + K_o) = \sigma(\Pi(T))$.

NOTE: If the simple arc is a subset of the real axis, then Theorem 1.8 is obvious. Suppose $\Pi(T) = (\Pi(T))^*$. Then $T - T^* \in \mathcal{K}(\mathcal{H})$ and consequently, $T - \mathrm{Re}(T) \in \mathcal{K}(\mathcal{H})$.

6. N-TH ROOTS & COMMUTATIVITY

Given two n-th roots of an operator, we want to know when they are identical. The following theorem gives a sufficient condition.

(1.9) PROPOSITION. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$\sigma(A) \cap \omega^j\sigma(B) = \emptyset, \qquad 1 \leq j \leq n-1, \qquad \omega = \exp(2\pi i/n).$$

If $A^n = B^n$, then $A = B$.

Proposition (1.9) may be proved with the Dunford-Taylor integral, but it is a special case of

(1.10) THEOREM. Let $A, B \in \mathcal{B}(\mathcal{H})$ such that

$$\sigma(A) \cap \omega^j\sigma(B) = \emptyset, \qquad 1 \leq j \leq n-1, \qquad \omega = \exp(2\pi i/n).$$

If $A^nD = DB^n$ for some $D \in \mathcal{B}(\mathcal{H})$, then $AD = DB$.

With $A^n = B^n$, Theorem 1.10 is due to [12], and it is a special case of Proposition 1.1 with $f(z) = z^n$. We sketch two proofs for Theorem 1.10, the first one was suggested by De Prima and the second, Hille [18, I].

PROOF. For $C \in \mathcal{B}(\mathcal{H})$, define linear maps L_C and R_C on $\mathcal{B}(\mathcal{H})$ by $L_C(T) = CT$ and $R_C(T) = TC$, respectively.

I) Write $J = \sum_{j=0}^{n-1} L_{A^{n-1-j}} R_{B^j}$, then $0 = A^nD - DB^n = J(AD - DB)$.

Since $L_A R_B = R_B L_A$, we have

$$\sigma(J) \subset \left\{ \sum_{j=0}^{n-1} a^{n-1-j} b^j : a \in \sigma(A),\ b \in \sigma(B) \right\}.$$

(The above inclusion is actually an equality, see [27].)

By hypothesis, $0 \notin \sigma(A) \cap \sigma(B)$ and for $a \in \sigma(A)$, $b \in \sigma(B)$, $a \neq b$, then

$\sum_{j=0}^{n-1} a^{n-1-j}\, b^j = (a^n - b^n)/(a - b) \neq 0$. Consequently, $0 \notin \sigma(J)$ and $AD - DB = 0$.

II) Assume $0 \notin \sigma(A)$, then

$$0 = D - A^{-n}DB^n = \prod_{j=0}^{n-1} (\omega^j I_{\mathcal{B}(\mathcal{H})} - L_{A^{-1}} R_B)(D)$$

For $1 \leq j \leq n - 1$, $\omega^j \notin \sigma(B)/\sigma(A)$, hence $0 \notin \sigma(\omega^j I_{\mathcal{B}(\mathcal{H})} - L_{A^{-1}} R_B)$.

Therefore $D - A^{-1}DB = 0$. ■

The following theorem gives the promised finite-dimensional converse of Proposition 1.1.

(1.11) THEOREM. Let A and B be two operators on a finite dimensional Hilbert space. Then the following statements are equivalent.

(i) A and B have identical commutants.

(ii) A and B are polynomials of each other.

(iii) A is a polynomial of B and they have identical eigenvectors.

(iv) A is a polynomial of B, $A = p(B)$, p is one-to-one on the eigenvalues of B and p' is non-zero on those eigenvalues of B corresponding to nonlinear elementary divisors.

(v) A and B commute and have identical invariant subspaces.

PROOF. (i) ⇒ (ii) The double commutant of A is the polynomial ring generated by A ([19, p.113, Corollary 1]).

(ii) ⇒ (iii) and (ii) ⇒ (v) are obvious.

(iii) ⇒ (iv) is not hard to see if we first assume that B is in Jordan canonical form.

(iv) ⇒ (i) is Proposition 1.1.

(v) ⇒ (ii) follows from Theorem 10 of [6]. ■

The equivalence (iii) ⇔ (iv) is the theorem in [29].

The following example shows that Proposition 1.1 does not have a converse in an infinite dimensional case: Let $\nu = \exp(2\pi i\alpha)$, where α is an irrational real number. Consider $T \in \mathcal{B}(\ell_2)$,

$$T < \xi_0, \xi_1, \xi_2, \ldots > = < \xi_0, \nu\xi_1, \nu^2\xi_2, \ldots > .$$

Then the commutant of every power of T is the set of all diagonal operators.

CHAPTER 2

1. INTRODUCTION

The work in this chapter is motivated by the paper of DePrima and Richard [9]. Many of the results in [9] are extended and generalized here. Using von Neumann's theory of spectral sets, we show that if T is a non-Hermitian operator with positive spectrum, then for large integer n, 0 lies in the numerical range of T^n. Hence, any semigroup of accretive operators is necessarily a commutative semigroup of positive operators. Furthermore, the above theorem remains valid if we replace T^n by T^nD, where D is an invertible operator commutating with T. Various situations in which the commutativity assumption of T and D is relaxed are examined. In the last section some variants of these theorems, which are derived with Calkin algebra techniques, are given.

2. MAPPINGS OF SPECTRAL SETS

Let $T \in \mathcal{B}(\mathcal{H})$ and let Λ be a closed subset of $\mathbb{C}$ containing $\sigma(T)$. Λ is said to be spectral for T if, whenever q is a rational complex-valued function with poles outside Λ,

$$\|q(T)\| \leq \sup_{\lambda \in \Lambda} |q(\lambda)|.$$

Spectral sets were introduced by von Neumann. Chapter XI of [31] has a detailed discussion on spectral sets. We list some properties of spectral sets:

(i) If Λ is spectral for T, then any closed set containing Λ is spectral for T.

(ii) If Λ is spectral for T, then co $\Lambda \supset W(T)$.

(iii) $\Sigma(\pi/2)$ is spectral for T if and only if T is accretive.

(iv) $\mathbb{R}$ is spectral for T if and only if T is Hermitian.

If Λ, Λ_n, $n = 1, 2, 3, \ldots$ are closed convex subsets of $\mathbb{C}$ such that $\Lambda_n \supset \Lambda$, we say Λ_n tends to Λ, $\Lambda_n \to \Lambda$, whenever for each $\epsilon > 0$ and each compact set Γ, there exists a positive integer $n_o(\epsilon,\Gamma)$ such that $n \geq n_o(\epsilon,\Gamma)$ implies $\Lambda_n \cap \Gamma \subset (\Lambda \cap \Gamma) + \Delta(\epsilon)$.

The following two theorems about spectral sets are proved in [9]. They are the principal tools in the proof of the main theorem.

(2.1) THEOREM. Let Λ_n, Λ be closed convex subsets of $\mathbb{C}$ such that $\Lambda_n \supset \Lambda$ and $\Lambda_n \to \Lambda$. Let $T_n \in \mathcal{B}(\mathcal{H})$ with Λ_n spectral for T_n. If $T_n \to T$ (in the uniform operator topology), then Λ is spectral for T.

(2.2) THEOREM. Let f be an analytic function in $\mathrm{Int}(\Sigma(\pi/2))$. Suppose T is accretive and Re $\sigma(T) > 0$, then $\mathrm{Cl}(\mathrm{co}(f(\mathrm{Int}\ \Sigma(\pi/2))))$ is spectral for f(T).

3. THE MAIN THEOREM

In this section we investigate some of the relations between an operator and the numerical ranges of its powers. C. A. Berger proved the power inequality for numerical radii $w(T^n) \leq (w(T))^n$. [41] and [15] contain proofs, discussions and generalizations of the theorem. An important application appears in [45]. The power inequality indicates the maximum rate of the growth of the numerical ranges of the powers of an operator. The following theorem shows that the numerical ranges of large powers of a non-Hermitian operator with positive spectrum must have a certain minimum rate of growth.

(2.3) THEOREM. For $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) > \gamma > 0$, then either (i) $T \geq \gamma I$ or

(ii) there is a positive integer n_o such that $\Delta(\gamma^n) \subset W(T^n)$ whenever $n \geq n_o$.

Recall that we write $\sigma(T) > (\geq)\gamma$ if $\sigma(T)$ is real and for each $\lambda \in \sigma(T)$, $\lambda > (\geq)\gamma$.

(2.4) COROLLARY. For $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) > 1$, then either (i) $T \geq I$ or

(ii) for each bounded subset $\Omega \subset \mathbb{C}$, there is a positive integer $n_o(\Omega)$ such that $\Omega \subset W(T^n)$ whenever $n \geq n_o$.

The following lemma is needed to prove Theorem 2.3.

(2.5) LEMMA. Let T, T_n, $n = 1, 2, 3, \ldots \in \mathcal{B}(\mathcal{H})$ and T_n converge to T in the uniform operator topology. Let Λ be a neighborhood of $\sigma(T)$ and f_n, $n = 1, 2, 3, \ldots$, a sequence of functions analytic on Λ. Then there is an integer N such that $f_n \in \mathcal{J}(T_n)$ whenever $n \geq N$. Furthermore, if f_n converges uniformly to a function f on Λ, then $f_n(T_n)$ converges to $f(T)$ in the uniform operator topology.

PROOF. Let Ω be an open set containing $\sigma(T)$ such that $Cl(\Omega) \subset \Lambda$ and $\partial(\Omega)$ consists of a finite number of positively oriented rectifiable Jordan curves. Since $\sigma(\cdot)$ is an upper semi-continuous set function and T_n converges to T in the uniform topology, there is an integer N such that $\sigma(T_n) \subset \Omega$, $n \geq N$. Consequently, $f_n \in \mathcal{J}(T_n)$ whenever $n \geq N$.

Now, $\|f_n(T_n) - f(T)\| \leq \|f_n(T_n) - f_n(T)\| + \|f_n(T) - f(T)\| = I_1 + I_2$. Pick $\epsilon > 0$. Since $f_n \to f$ uniformly on $\partial\Omega$, there is an integer $N_1 \geq N$ such that $I_2 \leq \epsilon/2$ if $n \geq N_1$ ([11, Lemma VII. 3.13]).

Put $M = \sup \{|f_n(\lambda)| : \lambda \in \partial U, n \geq 1\}$ and $\ell(\partial\Omega)$ = length of $\partial\Omega$.

$$I_1 \leq \frac{1}{2\pi} \int_{\partial\Omega} |f_n(\lambda)| \, \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| \, |d\lambda|$$

$$\leq \frac{1}{2\pi} M \cdot \ell(\partial\Omega) \cdot \sup_{\lambda\in\partial\Omega} \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\|.$$

Applying Lemma VII. 6.3 of [11], we get another integer $N_2 \geq N$ such that

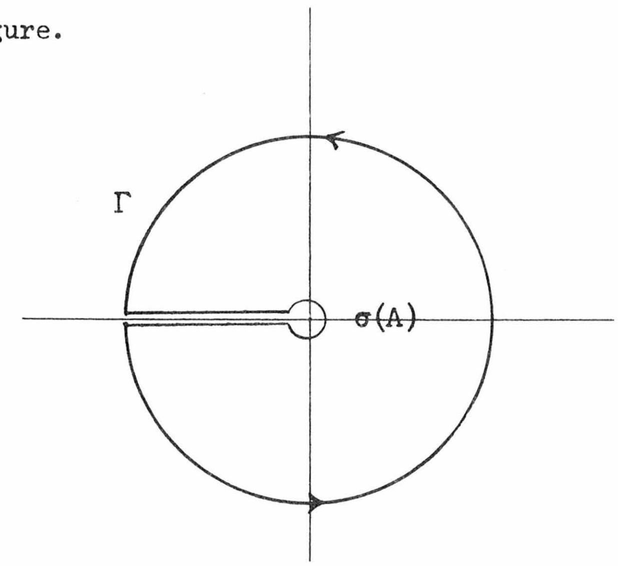
$$\sup_{\lambda\in\partial\Omega} \|(\lambda - T_n)^{-1} - (\lambda - T)^{-1}\| \leq \epsilon\pi/(M\,\ell(\partial\Omega))$$

if $n \geq N_2$. ■

For $A \in \mathcal{B}(\mathcal{H})$ with $(-\infty,0] \cap \sigma(A) = \emptyset$ and for $\beta \in \mathbb{R}$, define $A^\beta \in \mathcal{B}(\mathcal{H})$ by

$$A^\beta = \frac{1}{2\pi i} \int_\Gamma e^{\beta \mathrm{Log}\lambda} (\lambda - A)^{-1} d\lambda \qquad (1) ,$$

where Log λ is the principal logarithm of λ and Γ is the positively oriented curve containing $\sigma(A)$ in its interior domain as shown in the following figure.



If $\beta \in (-1,1)$, then $\sigma(A^{\beta}) \subset \mathrm{Int}(\Sigma(|\beta|\pi))$ by the spectral mapping theorem. It follows from Proposition 1.9 that for a positive integer m and $B \in \mathcal{B}(\mathcal{X})$, if $B^m = A$ and $\sigma(B) \subset \mathrm{Int}(\Sigma(\pi/m))$, then $B = A^{1/m}$.

Now we are ready to give the proof of Theorem 2.3.

PROOF. Assume there is an infinite subset M of the natural numbers and for each integer $m \in M$, there is a complex number k_m, $|k_m| \leq \gamma^m$ and $k_m \notin \mathrm{Cl}(W(T^m))$. We shall show that T is positive.

For each $m \in M$, $0 \notin \mathrm{Cl}(W(T^m - k_m))$; hence there is a real number $\alpha_m \in (-\pi,\pi)$ such that $A_m = e^{i\alpha_m}(T^m - k_m)$ is strictly accretive, i.e., $\mathrm{Cl}(W(A_m)) \subset \mathrm{Int}(\Sigma(\pi/2))$. By Theorem 2.2, the closed sector $\Sigma(\pi/2m)$ is spectral for $A_m^{1/m}$.

$$\limsup_{m \in M} \|k_m T^{-m}\|^{1/m} \leq \gamma/r(T) < 1.$$

Consequently, $\lim_{m \in M} \|k_m T^{-m}\| = 0$.

Put

$$B_m = (I - k_m T^{-m}).$$

By Lemma 2.5 $B_m^{1/m}$ is defined for large m

and $$\lim_{m \in M} \|B_m^{1/m} - I\| = 0.$$

For $m \in M$, set $C_m = e^{i\alpha_m} B_m = T^{-m} A_m$.

$$\sigma(A_m) \subset \mathrm{Int}(\Sigma(\pi/2)) \quad \text{and} \quad \sigma(T^{-n}) > 0,$$

therefore $\sigma(C_m) \subset \mathrm{Int}(\Sigma(\pi/2))$. Hence $C_m^{1/m}$ is well-defined. Since $\sigma(C_m^{1/m}) \subset \mathrm{Int}(\Sigma(\pi/2m))$, we have

$$A_m^{1/m} = T\, C_m^{1/m} .$$

We want to show $C_m^{1/m} \to I$ in the uniform topology and since $\Sigma(\pi/2m)$ is spectral for $A_m^{1/m} = T\, C_m^{1/m}$, the nonnegative real numbers form a spectral set for T according to Theorem 2.1.

$\lim_{m \in M} \|k_m T^{-m}\| = 0$ implies that there is a positive integer m_o such that for each $m \in M$, $m \geq m_o$, $\mathrm{Re}(\sigma(B_m)) > 0$.

Since $\alpha_n \in (-\pi, \pi)$ and $\mathrm{Re}(\sigma(C_m)) > 0$, we have

$$C_m^{1/m} = e^{i\alpha_m/m} B_m^{1/m} \qquad m \in M, \qquad m \geq m_o.$$

Therefore $\lim_{m \in M} \|C_m^{1/m} - I\| = 0.$ ∎

The following is an immediate consequence of Theorem 2.3.

(2.6) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) > 0$. If $0 \notin \mathrm{Int}(W(T^n))$ for infinitely many n's, then $T \geq 0$.

We note that if T is singular with $\sigma(T) \geq 0$, then Corollary 2.6 is not applicable. We give the following

CONJECTURE. Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \geq 0$. If $0 \notin \mathrm{Int}(W(T^n))$, $n = 1, 2, 3, \ldots$, then $T \geq 0$.

We shall prove this conjecture under the additional assumption that 0 is an isolated point of $\sigma(T)$. First let us state a simplified version of a theorem of Sinclair and Crabb [34].

(2.7) THEOREM. If $T \in \mathcal{B}(\mathcal{H})$ and $0 \notin \mathrm{Int}(W(T^{2^n}))$, $n = 0, 1, 2, \ldots$, then $\|T\| \leq 8\, r(T)$.

An elementary proof of Theorem 2.7 appears in [5, p.27]. One immediate corollary of Theorem 2.7 is: $T \in \mathcal{B}(\mathcal{H})$, T quasinilpotent, i.e., $\sigma(T) = \{0\}$, and $0 \notin \mathrm{Int}(W(T^{2^n}))$, $n = 0, 1, 2, \ldots$, then $T = 0$.

(2.8) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \geq 0$. If $0 \notin \mathrm{Int}(W(T^n))$, $n = 1, 2, 3, \ldots$, and if 0 is an isolated point of $\sigma(T)$, then $T \geq 0$.

PROOF. Let E be the spectral idempotent associated with 0. See Proposition 1.7. With respect to $E\mathcal{H} \oplus (E\mathcal{H})^{\perp}$,

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix},$$

and

$$T = \begin{pmatrix} Q & QA - AT_1 \\ 0 & T_1 \end{pmatrix}$$

where $\sigma(Q) = \{0\}$ and $\sigma(T_1) > 0$. Since

$$T^n = \begin{pmatrix} Q^n & Q^nA - AT_1^n \\ 0 & T_1^n \end{pmatrix},$$

$W(T^n)$ contains both $W(Q^n)$ and $W(T_1^n)$. By Theorem 2.7 and Corollary 2.6, we get $Q = 0$ and $T_1 \geq 0$, respectively. However, $0 \notin \mathrm{Int}(W(T))$ and $T = \begin{pmatrix} 0 & -AT_1 \\ 0 & T_1 \end{pmatrix}$, by Corollary 1.6, $-AT_1 = 0$. Consequently $T \geq 0$. ∎

4. A THEOREM OF JOHNSON, DEPRIMA AND RICHARD

The following theorem was first stated and proved by C. R. Johnson ([20, Chapter 2], [21]) for finite dimensional matrices and it was generalized by DePrima and Richard [9] for arbitrary bounded operators.

(2.9) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$. Then $T \geq 0$ if and only if T^n is accretive, $n = 1, 2, 3, \ldots$

PROOF. The necessity is clear. For the sufficiency, note that $\sigma(T) \geq 0$ by the spectral mapping theorem. For any $\gamma > 0$, $(T + \gamma)^n$ is also accretive, $n = 1, 2, 3, \ldots$ Applying Theorem 2.3, we get $(T + \gamma) \geq \gamma I$. ∎

At the end of this section we shall give an elementary proof of

Theorem 2.9. The proper way of viewing Theorem 2.9 appears to be:

(2.10) THEOREM. Let $T \in \mathcal{B}(\mathcal{X})$. Then T^n is accretive, $n = 1, 2, \ldots, k$, if and only if $\Sigma(\pi/2k)$ is spectral for T.

PROOF. The sufficiency follows from the definition of a spectral set. If T^n is accretive, $n = 1, \ldots, k$, then $\Sigma(\pi/2k) \supset \sigma(T)$; consequently, for each $\gamma > 0$, $((T + \gamma)^k)^{1/k} = T + \gamma$. By Theorem 2.2, $\Sigma(\pi/2k)$ is spectral for $T + \gamma$. We let γ tend to 0 and apply Theorem 2.1 ∎

We are going to give some results related to Theorem 2.9 and Theorem 2.10. Given an accretive operator $A \in \mathcal{B}(\mathcal{X})$ and $\alpha \in (0,1)$, we define the fractional power A^α by

$$A^\alpha x = \frac{\sin \pi\alpha}{\pi} \int_0^\infty \lambda^{\alpha-1} (A + \lambda)^{-1} Ax \, d\lambda \tag{2}$$

for each $x \in \mathcal{X}$. The integral in (2) is convergent in the Bochner or absolute sense; and if $0 \notin \sigma(A)$, then the fractional power defined by (2) is the same as the one defined by (1) on page 18. Furthermore, $\lim_{\gamma \to 0+} \|(A + \gamma)^\alpha - A^\alpha\| = 0$. See [24, §V. 3.11].

(2.11) THEOREM. For an accretive operator $A \in \mathcal{B}(\mathcal{X})$ and a positive integer k, there exists a unique operator B such that $A = B^k$ and $\Sigma(\pi/2k)$ is spectral for B.

Theorem 2.11 generalizes a theorem of Macaev and Palant [28]. Also see [25] and [37, Proposition 5.5]. The next theorem is a simplified version of [23, Theorem 1.1].

(2.12) THEOREM. Let $A \in \mathcal{B}(\mathcal{H})$ be accretive and let $\alpha \in (0,1/2]$. Put $H_\alpha = (A^\alpha + A^{*\alpha})/2$ and $K_\alpha = (A^\alpha - A^{*\alpha})/2i$. Then for each $x \in \mathcal{H}$

$\|K_\alpha x\| \leq \tan(\pi\alpha/2)\ \|H_\alpha x\|$.

(2.14) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$. If T^n is accretive for $n = 1, \ldots, k$, then $\|\operatorname{Im} T x\| \leq \tan(\pi/2k)\ \|\operatorname{Re} Tx\|$, $x \in \mathcal{H}$.

We conclude this section by giving the promised elementary proof of Theorem 2.9.

(2.13) LEMMA. Let $A, B \in \mathcal{B}(\mathcal{H})$. If $A = A^*$, $B \geq 0$ and $B^2 \geq A^2$, then $B \geq A$.

PROOF. Pick $\lambda < 0$ and $x \in \mathcal{H}$, $\|x\| = 1$.

$$\|(B - \lambda)x\|^2 - \|Ax\|^2 = ((B^2 - A^2)\ x,x) - 2\lambda(Bx,x) + \lambda^2 \geq \lambda^2.$$

Hence

$$\|(B - A - \lambda)x\| \geq \|(B - \lambda)x\| - \|Ax\|$$

$$\geq \frac{\lambda^2}{\|B - \lambda\| + \|A\|} > 0.$$

Since $B - A$ is Hermitian, we have $\sigma(B - A) \geq 0$. Consequently, $B \geq A$. ∎

(2.14) LEMMA. Let $T \in \mathcal{B}(\mathcal{H})$. If T and T^2 are accretive, then $W(T) \subset \Sigma(\pi/4)$.

PROOF. $(((\operatorname{Re} T)^2 - (\operatorname{Im} T)^2)\ x\ ,\ x)$

$$= \left(\left(\frac{T + T^*}{2}\right)^2 x, x\right) - \left(\left(\frac{T - T^*}{2i}\right)^2 x, x\right)$$

$$= \tfrac{1}{2}\,(T^2 x + T^{*2} x, x)$$

$$= \mathrm{Re}(T^2 x, x).$$

Hence T^2 is accretive if and only if $(\mathrm{Re}T)^2 \geq (\mathrm{Im}\,T)^2$. By Lemma 2.13 and the fact that T is accretive, we get $\mathrm{Re}(T) \geq \mathrm{Im}\,T$ and $\mathrm{Re}T \geq -\mathrm{Im}\,T$. Therefore, $W(T) \subset \Sigma(\pi/4)$. ■

(2.15) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$. If T is accretive and $W(T^2) \subset \Sigma(\alpha\pi/2)$, $0 \leq \alpha \leq 1$, then $W(T) \subset \Sigma(\alpha\pi/4)$.

PROOF. By Lemma 2.14, $\exp(\pm i\pi/4)T$ are accretive. Hence $\exp(\pm\, i(1-\alpha)\pi/4)T$ are accretive. The hypothesis $W(T^2) \subset \Sigma(\alpha\pi/2)$ implies that $\exp(\pm i(1 - \alpha)\pi/2)T^2$ are accretive. Applying Lemma 2.14 again, we get both $\exp(i\pi/4)(\exp(i(1 - \alpha)\pi/4)T)$ and $\exp(-i\pi/4)(\exp(-i(1 - \alpha)\pi/4)T)$ are accretive. Therefore, $\exp(\pm i(1 - \alpha/2)\pi/2)T$ are accretive and $W(T) \subset \Sigma(\alpha\pi/4)$. ■

(2.9′) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$. If T^{2^n} is accretive, $n = 0, 1, 2, \ldots,$ then $T \geq 0$.

PROOF. Corollary 2.15 shows that if

$$T^{2^n} \text{ is accretive, } \quad n = 0, 1, \ldots, k,$$

then $W(T) \subset \Sigma(\pi/2^{k+1})$. ■

5. PERTURBATIONS OF THE HYPOTHESIS OF THE MAIN THEOREM

The sequence T^nD, $n = 1, 2, 3, \ldots$ is studied in this section. We seek conditions which imply T is positive. This structure arises naturally in the study of multiplicative commutators. Let $A, D \in \mathcal{B}(\mathcal{H})$, put $T = ADA^{-1}D^{-1}$. If $AT = TA$, then $T^nD = A^nDA^{-n}$, $n \in \mathbb{Z}$. See [9, §4] and [30].

The following theorem generalizes Theorem 2.3.

(2.16) THEOREM. Let $T, D \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) > \gamma > 0$ and $TD = DT$. If there are infinitely many n's such that $\Delta(\gamma^n) \not\subset W(T^nD)$, then $\sigma(D) \subset e^{i\theta_o}\,\Sigma(\pi/2)$ for some $\theta_o \in \mathbb{R}$. Furthermore, if D is invertible, then $T \geq \gamma\, I$.

The proof of this theorem follows lines similar to that of Theorem 2.3.

PROOF. Assume there is an infinite subset M of the natural numbers and for each integer $m \in M$ there exists a complex number k_m, $|k_m| \leq \gamma^m$ and $k_m \notin \mathrm{Cl}(W(T^mD))$. For each $m \in M$, there is $\alpha_m \in (-\pi, \pi)$ such that $A_m = e^{i\alpha_m}(T^mD - k_m)$ is strictly accretive. Put $B_m = (D - k_mT^{-m})$, then

$$\lim_{m \in M} \|B_m - D\| = 0.$$

Since $TD = DT$, $\sigma(B_m) \subset \mathrm{Int}(e^{-i\alpha_m}\,\Sigma(\pi/2))$ and $\sigma(B_m) \to \sigma(D)$. Hence there is a real number θ_o such that $\sigma(D) \subset e^{i\theta_o}\,\Sigma(\pi/2)$.

If $0 \notin \sigma(D)$, we may assume $\sigma(D) \cap (-\infty, 0] = \emptyset$. By Lemma 2.5, there is a positive integer m_o such that $B_m^{1/m}$ is defined for $m \geq m_o$ and

$$\lim_{m \in M} \|B_m^{1/m} - I\| = 0.$$

For $m \in M$, $m \geq m_o$, we have

$$A_m^{1/m} = \exp(i(\alpha_m + 2\pi\epsilon(m))/m) \quad B_m^{1/m} T,$$

where $\epsilon(m) = 1$, 0 or -1.

Applying Theorem 2.1 and Theorem 2.2, we get $[0,\infty)$ as a spectral set for T. ■

As in §2.3 we do not know what conclusions can be drawn if we replace the hypothesis $\sigma(T) > 0$ by $\sigma(T) \geq 0$ in Theorem 2.6. However, if we assume 0 is a pole of the resolvent $(\lambda - T)^{-1}$, then we have the following

(2.17) THEOREM. Let $T, D \in \mathcal{B}(\mathcal{X})$ with $\sigma(T) \geq 0$ and D invertible. Suppose $0 \notin \mathrm{Int}(W(T^n D))$, $n = 1, 2, 3, \ldots$ If 0 is a pole of $(\lambda - T)^{-1}$ and if $TD = DT$, then $T \geq 0$.

REMARK. Since 0 is a boundary point of $\sigma(T)$, 0 is a pole of $(\lambda - T)^{-1}$ if either

(i) T is semi-Fredholm (Proposition 1.2), or

(ii) the ascent of T, $\alpha(T)$, is finite and $T^{\alpha(T)+k}(\mathcal{X})$ is closed for some positive integer k([26], Theorem 2.7).

Recall that $\alpha(T)$ is the smallest nonnegative integer p such that T^p and T^{p+1} have identical nullspaces.

PROOF. Let E be the spectral idempotent associated with $\{0\}$. See Proposition 1.7. With respect to $E\mathcal{X} \oplus (E\mathcal{X})^{\perp}$, we have

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix},$$

$$T = \begin{pmatrix} N & NA-AT_1 \\ 0 & T_1 \end{pmatrix}, \ \sigma(N) = \{0\}, \ \sigma(T_1) > 0,$$

and

$$D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}.$$

Let m be an integer such that $N^m = 0$. Since $T^m D = DT^m$, we get $D_3 = 0$. Thus $TD = DT$ implies $ND_1 = D_1 N$ and $T_1 D_4 = D_4 T_1$. Now the first half of Theorem 2.16 is applicable to the sequence $T_1^n D_4$, $n = 1, 2, 3, \ldots$, and we get $0 \notin \mathrm{Int}(\sigma(D_4))$. Furthermore, D_4 is onto since D is invertible. By Proposition 1.2, D_4 is invertible. Therefore $T_1 \geq 0$ by the second half of Theorem 2.16 and D_1 is also invertible [15, pp. 220-221]. The conditions that the operator N is nilpotent and commutes with D_1 imply ND_1 is also nilpotent. Since $0 \notin \mathrm{Int}\, W(ND_1)$ and D_1 invertible, we have $N = 0$. Hence

$$TD = \begin{pmatrix} 0 & -AT_1 D_4 \\ 0 & T_1 D_4 \end{pmatrix}.$$

By Corollary 1.6 and $0 \notin \mathrm{Int}\, W(TD)$, $-AT_1 D_4 = 0$. Consequently $A = 0$

and $T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}$ with $T_1 \geq 0$. ■

One may ask what happens to Theorem 2.16 if the commutativity assumption on T and D is dropped. Since $\sigma(T)$ lies on an open half ray originating from the origin, the condition $TD = DT$ is equivalent to that $T^mD = DT^m$ for some nonzero integer m by Theorem 1.10. In general, we cannot drop the commutativity condition as demonstrated in the following

(2.18) PROPOSITION. Let $T, D \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) > 0$, D invertible and $D \geq 0$. Suppose $0 \notin \operatorname{Int}(W(T^nD))$ for infinitely many n's, then $T \geq 0$ if and only if $TD = DT$.

PROOF. The sufficiency follows from Theorem 2.16. For the necessity note that

$$(T^nDx, x) = ((D^{-\frac{1}{2}}TD^{\frac{1}{2}})^n D^{\frac{1}{2}}x, D^{\frac{1}{2}}x).$$

By Corollary 2.6 we have $D^{-\frac{1}{2}}TD^{\frac{1}{2}} \geq 0$. $D \geq 0$ and $D(D^{-\frac{1}{2}}TD^{\frac{1}{2}}) = D^{\frac{1}{2}}TD^{\frac{1}{2}} \geq 0$ imply that D commutes with $D^{-\frac{1}{2}}TD^{\frac{1}{2}}$.

Hence $D^{\frac{1}{2}}TD^{\frac{1}{2}} = D^{-\frac{1}{2}}TD^{3/2}$, and we get $DT = TD$. ∎

Proposition 2.18 remains valid if we merely assume D is normal and restrict $\mathcal{H}$ to be finite dimensional.

(2.19) LEMMA: Let P_i, $i = 1, \ldots, k$, be k pairwise orthogonal projections on $\mathcal{H}$ and $\sum_{i=1}^{k} P_i = I$. Let $T = \sum_{i=1}^{k} \lambda_i P_i$ with $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$. Suppose $D \in \mathcal{B}(\mathcal{H})$ and $0 \notin \operatorname{Int}(W(T^nD))$ for infinitely many n's. Then

$$P_i D P_j = 0 \qquad 1 \leq i < j \leq k.$$

PROOF. It is sufficient to consider the case

$$T = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \qquad \alpha > \beta \geq 0$$

and $D = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We shall show that $b = 0$.

Since $T^n D = \begin{pmatrix} \alpha^n a & \alpha^n b \\ \beta^n c & \beta^n d \end{pmatrix}$ and $0 \notin \text{Int}\,(W(T^n D))$, there is a real number θ_n such that

$$0 \leq \det\,(\text{Re}(e^{i\theta_n}(T^n D)))$$

$$= \alpha^n \beta^n \,\text{Re}(e^{i\theta_n} a)\,\text{Re}(e^{i\theta_n} d)$$

$$- \frac{1}{4} \left| \alpha^n b e^{i\theta_n} + \beta^n \bar{c}\, e^{-i\theta_n} \right|^2 .$$

Thus

$$\frac{1}{4} \left| b\, e^{i\theta_n} + (\beta/\alpha)^n \,\bar{c}\, e^{-2i\theta_n} \right|^2 \leq (\beta/\alpha)^n \,\text{Re}(e^{i\theta_n} a)\,\text{Re}(e^{i\theta_n} d) .$$

Therefore $b = 0$ because $0 \leq \beta/\alpha < 1$ and there are infinitely many n's for which the above inequality holds. ∎

(2.20) PROPOSITION. Let $\mathcal{K}$ be finite-dimensional. Let $T, D \in \mathcal{B}(\mathcal{K})$ with $T \geq 0$ and D normal. If $0 \notin \text{Int}\,(W(T^n D))$ for infinitely many n's, then $TD = DT$.

PROOF. Let $T = \sum_{i=1}^{k} \lambda_i P_i$

where $\lambda_1 > \lambda_2 > \cdots > \lambda_k \geq 0$ and $\{P_i\}$ is a set of k pairwise orthogonal projections, $\Sigma P_i = I$.

By Lemma 2.19, $P_i D P_j = 0$, $1 \leq i < j \leq k$. Since $\mathcal{K}$ is finite dimensional and D is normal

$$P_i D P_j = 0 \qquad \forall i, j , \qquad i \neq j.$$

Therefore TD = DT. ∎

Proposition 2.20 is not true if $\mathcal{K}$ is infinite dimensional. The following counterexample is suggested by J. H. Anderson.

Let U denote the unilateral shift on ℓ_2,

$$U < \xi_0, \xi_1, \xi_2, \ldots > = < 0, \xi_0, \xi_1, \xi_2, \ldots > .$$

Let B denote the projection on ℓ_2 given by

$$B < \xi_0, \xi_1, \xi_2, \ldots > = < \xi_0, 0, 0, \ldots > .$$

Put

$$V = \begin{pmatrix} U^* & 0 \\ B & U \end{pmatrix} \quad \text{on } \mathcal{K} = \ell_2 \oplus \ell_2 ,$$

then $VV^* = V^*V = I$. Put $D = 2I + V$; D is normal. Let $T = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$,

then $T^n D = \begin{pmatrix} 2I+U^* & 0 \\ 0 & 0 \end{pmatrix}$ which is accretive, $n = 1, 2, 3, \ldots$ But $TD - DT = \begin{pmatrix} 0 & 0 \\ -B & 0 \end{pmatrix}$.

We conclude this section with the following

(2.21) THEOREM. Let $T, D \in \mathcal{B}(\mathcal{X})$ with the descent of T finite and D strictly accretive. Suppose $T^n D$ is accretive for n = 1, 2, 3, ... If $T^m D = DT^m$ for some positive integer m, then $T \geq 0$.

Recall that T has finite descent if there exists a nonnegative integer q such that $T^q\mathcal{X} = T^{q+1}\mathcal{X}$. This hypothesis is not necessary in Theorem 2.21 if m = 1. In fact we have

(2.22) PROPOSITION. (cf. [9], Theorem 3) Let $T, D \in \mathcal{B}(\mathcal{X})$ with $\sigma(D) \subset \text{Int}\,(\Sigma(\pi/2))$. Suppose $T^n D$ is accretive for n = 1, 2, 3, ... If TD = DT, then $T \geq 0$.

PROOF. First we show $\sigma(T) \geq 0$.

There is a real number $0 \leq \beta < 1$ such that $\sigma(D) \subset \text{Int}(\Sigma(\beta\pi/2))$. TD = DT implies that

$$(\sigma(T))^n = \sigma(T^n) \subset \sigma\ (T^n D)/\sigma(D) \subset \Sigma((1 + \beta)\pi/2).$$

We note that the hypotheses of the theorem are not changed if T is replaced by T + r, $r \geq 0$. Hence $(r + \sigma(T))^n = (\sigma(T + r))^n \subset \Sigma((1 + \beta)\pi/2)$ for all $r \geq 0$, n = 1, 2, 3, ... This is possible only if $\sigma(T) \geq 0$. For any $\gamma > 0$, $(T + \gamma)^n D$ is accretive, n = 1, 2, 3, ... Now Theorem 2.16 is applicable. ∎

Before we can give the proof of Theorem 2.21, we need one more lemma.

(2.23) LEMMA. [30, Theorem 4.18] Let $S, C \in \mathcal{B}(\mathcal{X})$. Suppose there is a vector $x_o \in \text{Ker}\ S^{*2} \setminus \text{Ker}\ S^*$ such that $(CS^* x_o, S^* x_o) \neq 0$. Then

$0 \in \text{Int}\,(W(SC))$.

PROOF. For $\alpha \in \mathbb{C}$,

$$(SC(x_o + \alpha S^* x_o)\ ,\ x_o + \alpha S^* x_o)$$

$$= (C\, x_o,\ S^* x_o) + \alpha(CS^* x_o,\ S^* x_o)\ .$$

Hence $\mathbb{C} = \cup \{(SC(x_o + \alpha S^* x_o),\ x_o + \alpha S^* x_o)\ :\ \alpha \in \mathbb{C}\}$.

Consequently, $0 \in \text{Int}\,(W(SC))$. ■

PROOF OF THEOREM 2.21. Since $T^m D = DT^m$ and $\sigma(D) \subset \text{Int}\,(\Sigma(\pi/2))$, we have $T^m \geq 0$ by Proposition 2.22. Hence the ascent of T, $\alpha(T) \leq m$. By hypothesis the descent of T is also finite, and applying Theorem 2.1 of [26], we have that 0 is a pole of $(\lambda - T)^{-1}$.

Let E be the spectral idempotent associated with $\{0\}$. With respect to $E\mathcal{H} \oplus (E\mathcal{H})^{\perp}$, put

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}, \quad \text{then}$$

$$T = \begin{pmatrix} N & NA-AT_1 \\ 0 & T_1 \end{pmatrix}, \text{ where N is nilpotent and } \sigma(T_1) > 0.$$

$T^m \geq 0$ implies $N^m = 0$, $T_1^m \geq 0$ and $-AT_1^m = 0$. Thus $A = 0$ and $T_1 \geq 0$.

Hence $T = N \oplus T_1$.

$0 \notin W(D)$, $0 \notin \text{Int}\,(W(TD))$ and by Lemma 2.23, we conclude that the ascent of N^*, $\alpha(N^*) < 2$. The operator N is nilpotent, therefore, $N = 0$. Hence $T \geq 0$. ■

6. OTHER RELATED RESULTS

In this section $\mathcal{H}$ is infinite dimensional and separable. For $T \in \mathcal{B}(\mathcal{H})$, let

$$W_e(T) = \cap \{Cl(W(T + K)) : K \text{ compact}\} \text{ and}$$

$$\sigma_W(T) = \cap \{\sigma(T + K) \quad : K \text{ compact}\}$$

as in §1.5. Corresponding to Theorem 2.3, Theorem 2.9 and Theorem 2.16, we have the following theorems:

(2.25) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma_W(T) > \gamma > 0$, then either (i) there is a compact operator K such that $T + K \geq \gamma I$, or

(ii) there is a positive integer n_o such that $\Delta(\gamma^n) \subset W_e(T^n)$ whenever $n \geq n_o$.

(2.26) THEOREM. (cf. [9, Theorem 4]) For $T \in \mathcal{B}(\mathcal{H})$, then $W_e(T^n) \subset \Sigma(\pi/2)$, n = 1, 2, 3, ..., if and only if there is a compact operator K such that $T + K \geq 0$.

(2.27) THEOREM. Let $T, D \in \mathcal{B}(\mathcal{H})$ with $\sigma_W(T) > \gamma > 0$ and $(T^mD - DT^m)$ compact for some positive integer m. If there are infinitely many n's such that $\Delta(\gamma^n) \not\subset W(T^nD)$, then $\sigma_W(D) \subset e^{i\theta_o}\Sigma(\pi/2)$ for some $\theta_o \in \mathbb{R}$. Furthermore, if D is semi-Fredholm, then there is a compact operator K such that $T + K \geq \gamma I$.

We sketch the proof of Theorem 2.27. As in §I.5, we let

$\pi : \mathcal{B}(\mathcal{H}) \to \mathfrak{A}(\mathcal{H})$, the quotient map, and

$$\tau : \mathfrak{A}(\mathcal{K}) \to \mathcal{B}(\mathfrak{h}), \quad \text{a faithful * representation.}$$

Since $\Pi(T^m D - DT^m) = 0$ and $\sigma(\Pi(T)) = \sigma_W(T) > 0$, $\tau \circ \Pi(T)$ and $\tau \circ \Pi(D)$ commute by Theorem 1.10. Furthermore, $W(\tau \circ \Pi(T^n D)) \subset \mathrm{Cl}(W(T^n D))$. Applying the first half of Theorem 2.16, we get

$$\sigma(\tau \circ \Pi(D)) \subset e^{i\theta_o} \Sigma(\pi/2) \text{ for some } \theta_o \in \mathbb{R}.$$

Consequently, $0 \notin \mathrm{Int}(\sigma(D))$. If D is semi-Fredholm, then D is Fredholm by Proposition 1.2. By a theorem of Atkinson, D is Fredholm if and only if $\Pi(D)$ is invertible ([40], [15, Problem 142]). Thus it follows from the second half of Theorem 2.16 that

$$\tau \circ \Pi(T) \geq \gamma I_{\mathfrak{h}} .$$

Applying Theorem 1.8, we conclude that there is a compact operator K such that $T + K \geq \gamma I$. ■

CHAPTER 3

1. INTRODUCTION

In this chapter we give some applications of the theory developed earlier. The main problem we study is the following: for $T \in \mathcal{B}(\mathcal{H})$, if each of the powers of T is not a commutator with a positive factor, then what conclusions can we draw about T? When $\mathcal{H}$ is an infinite dimensional separable Hilbert space, we show that there is a positive integer m and a compact operator K such that $T^m + K$ is normal. The main tools used to prove this fact are (i) a characterization of commutators with self-adjoint factor due to J. H. Anderson [1],

(ii) a number-theoretic result of C. R. Johnson and M. Newman [22], and

(iii) Theorem 2.3.

The author wishes to thank C. R. Johnson for informing him of the result in [22].

2. A THEOREM OF J. H. ANDERSON

A derivation on the algebra $\mathcal{B}(\mathcal{H})$ is a linear map δ from $\mathcal{B}(\mathcal{H})$ into itself with the property $\delta(XY) = \delta(X)Y + X\delta(Y)$ for every pair of operators X,Y in $\mathcal{B}(\mathcal{H})$. It is known that all derivations are inner, i.e., for each derivation δ, there is an operator A in $\mathcal{B}(\mathcal{H})$ such that $\delta(X) = \delta_A(X) = AX - XA$.

Let $\mathcal{R}$ denote the set

$$\cup\{\delta_A(\mathcal{B}(\mathcal{H})) : A \geq 0\} .$$

Thus the problem we are interested in is the following: what are the operators T with the property that $T^n \notin \mathcal{R}$, n = 1, 2, 3, ... ?

For the rest of this section $\mathcal{K}$ will denote an infinite-dimensional separable Hilbert space. J. H. Anderson in his thesis [1, Theorem 7.2] proved the following deep result:

(3.1) THEOREM. $\mathcal{R} = \{T \in \mathcal{B}(\mathcal{K}) : 0 \in W_e(T)\}$.

As in §1.5 we let Π denote the quotient map of $\mathcal{B}(\mathcal{K})$ onto the Calkin algebra and let τ denote a faithful * representation of the Calkin algebra onto a self-adjoint subalgebra of $\mathcal{B}(\mathfrak{H})$, $\mathfrak{H}$ some suitably chosen Hilbert space. For $T \in \mathcal{B}(\mathcal{K})$, $W_e(T) = \mathrm{Cl}(W(\tau \circ \Pi(T)))$.

Hence the hypothesis that $T^n \notin \mathcal{R}$, n = 1, 2, 3, ... is equivalent to $0 \notin \mathrm{Cl}(W(\tau \circ \Pi(T)^n))$, n = 1, 2, 3, ...

In §4 of this chapter we shall show:

if $0 \notin \mathrm{Cl}(W(T^n))$ n = 1, 2, 3, ..., then a power of T is normal.

3. A THEOREM OF JOHNSON AND NEWMAN

The following question is raised in [22]: how many distinct points $\alpha_1, \ldots, \alpha_\ell$ on the unit circle of $\mathbb{C}$ are in general required to insure that for some positive integer m, $0 \in \mathrm{co}\{\alpha_1^m, \ldots, \alpha_\ell^m\}$? A complete solution is given by

(3.2) THEOREM [22]. Let α, β, γ be distinct complex numbers with $|\alpha| = |\beta| = |\gamma| = 1$. Then there exists a positive integer m such that $0 \in \mathrm{co}\{\alpha^m, \beta^m, \gamma^m\}$ if and only if $\{\alpha, \beta, \gamma\}$ cannot be obtained from

$\{1, e^{2\pi i/7}, e^{6\pi i/7}\}$ or $\{1, e^{2\pi i/7}, e^{10\pi i/7}\}$ via any combination of permutation, reflection and simultaneous rotation.

NOTATION. Let $\mathbb{R}^+$ denote the set of strictly positive numbers, $\mathbb{R}^+ = (0,\infty)$. For $\mathcal{C} \subset \mathbb{C} \setminus \{0\}$, let $\#\operatorname{Arg} \mathcal{C}$ denote the cardinality of the set $\{\operatorname{Arg} \lambda : \lambda \in \mathcal{C}\}$, and let $\mathcal{C}^m$ denote the set $\{\gamma^m : \gamma \in \mathcal{C}\}$, m an integer.

(3.3) COROLLARY. Let $\mathcal{C}$ be a set of nonzero complex numbers such that $\mathcal{C} \cap \mathbb{R}^+ \neq \emptyset$. If $0 \notin co(\mathcal{C}^n)$, $n = 1, 2, 3, \ldots$, and if $\#\operatorname{Arg}\mathcal{C} \geq 3$, then $\#\operatorname{Arg}\mathcal{C} = 3$ and $\mathcal{C}^7 \subset \mathbb{R}^+$.

PROOF. For any three nonzero complex numbers α, β, γ, $0 \in co\{\alpha, \beta, \gamma\}$ if and only if $0 \in co\{\alpha/|\alpha|, \beta/|\beta|, \gamma/|\gamma|\}$. ∎

4. $0 \notin Cl(W(T^n))$

(3.4) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$ with $\sigma(T) \cap \mathbb{R}^+ \neq \emptyset$. Suppose $0 \notin Cl(W(T^n))$, $n = 1, 2, 3, \ldots$ We have the following cases:

(i) $\#\operatorname{Arg}\sigma(T) = 1$, then $T \geq 0$.

(ii) $\#\operatorname{Arg}\sigma(T) \geq 3$, then $\#\operatorname{Arg}\sigma(T) = 3$ and $T^7 \geq 0$.

(iii) $\#\operatorname{Arg}\sigma(T) = 2$, then either there is a positive odd number m such that $T^m \geq 0$ or there exists a closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ and positive operators T_1 and T_2 on $\mathcal{H}_1$ and $\mathcal{H}_1^\perp$ respectively such that $T = T_1 \oplus e^{i\theta}T_2$, θ being irrational modulo 2π.

PROOF. Case (i). Since $\sigma(T) > 0$, $T \geq 0$ by Corollary 2.6.

Case (ii). $0 \notin \mathrm{Cl}(W(T^n)) \supset \mathrm{co}(\sigma(T)^n)$, $n = 1, 2, 3, \ldots$ By Corollary 3.3, we have $\# \operatorname{Arg} \sigma(T) = 3$ and $\sigma(T^7) > 0$. Applying Corollary 2.6 again, we get $T^7 \geq 0$.

Case (iii). There exists a real number $\theta \in [0,2\pi)$ such that $\sigma(T) \subset \mathbb{R}^+ \cup e^{i\theta}\mathbb{R}^+$. If θ is rational modulo 2π, there is a positive odd integer m such that $\sigma(T^m) > 0$, thus $T^m \geq 0$. Before we can treat the case where θ is irrational modulo 2π, we need the following

(3.5) LEMMA. Let $\epsilon \in (0,1)$, $\alpha, \beta \in \mathbb{C}$, $\alpha \cdot \beta \neq 0$. If $|\operatorname{Arg}(\alpha/\beta)| \geq \arccos(-\epsilon^2)$, then $0 \in \Theta(\alpha, \beta; \epsilon)$.

PROOF. By definition of an ellipse, $0 \in \Theta(\alpha, \beta; \epsilon)$ if and only if

$$|\alpha| + |\beta| \leq |\alpha - \beta|/\epsilon.$$

Put $\psi = |\operatorname{Arg}(\alpha/\beta)|$. Then we have to show

$$(|\alpha| + |\beta|)^2 \leq (|\alpha|^2 + |\beta|^2 - 2|\alpha|\,|\beta| \cos\psi)/\epsilon^2,$$

or equivalently,

$$0 \leq (1/\epsilon^2 - 1)\,|\alpha|^2 - 2\,(1 + \cos\psi/\epsilon^2)|\alpha|\,|\beta| + (1/\epsilon^2 - 1)|\beta|^2.$$

The above inequalities hold if

$$0 \geq (1 + \cos\psi/\epsilon^2)^2 - (1/\epsilon^2 - 1)^2,$$

or $1/\epsilon^2 - 1 \geq |1 + \cos\psi/\epsilon^2|$,

or $1/\epsilon^2 - 1 \geq -(1 + \cos\psi/\epsilon^2)$ since $\psi \geq \arccos(-\epsilon^2)$.

But the last inequality is always true. ∎

We now come back to the proof of the last part of Theorem 3.4. Let E be the spectral idempotent associated with $\sigma(T) \cap \mathbb{R}^+$. See Proposition 1.7. With respect to $E\mathcal{H} \oplus (E\mathcal{H})^\perp$, put

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}, \text{ then}$$

$$T = \begin{pmatrix} T_1 & T_1A - e^{i\theta}AT_2 \\ 0 & e^{i\theta}T_2 \end{pmatrix},$$

where $T_1 \geq 0$, $T_2 \geq 0$ and θ is irrational modulo 2π. We note that

$$T^n = \begin{pmatrix} T_1^n & T_1^nA - e^{in\theta}AT_2^n \\ 0 & e^{in\theta}T_2^n \end{pmatrix}.$$

To show that $T = T_1 \oplus e^{i\theta}T_2$, we have to show $A = 0$. Assume $A \neq 0$. For a positive integer n and $y \in (E\mathcal{H})^\perp$, with $\|y\| = 1$ and $Ay \neq 0$, let $\Theta\,[n,y]$ denote the numerical range of the 2×2 matrix

$$\begin{pmatrix} (T_1^nAy,Ay)/\|Ay\|^2 & \Big((T_1^nAy,Ay) - e^{in\theta}(AT_2^ny,Ay)\Big)/\|Ay\| \\ 0 & e^{in\theta}(T_2^ny,y) \end{pmatrix}.$$

By Lemma 1.5, $\Theta\,[n,y] \subset W(T^n)$. By Theorem 1.3, $\Theta[n,y] = \Theta\,(\alpha,\beta;\ \epsilon[n,y])$ where $\alpha \in \mathbb{R}^+$, $\beta \in e^{in\theta}\mathbb{R}^+$ and

$$\epsilon[n,y] = \left(1 + \left(\frac{\Big((T_1^nAy,Ay) - e^{in\theta}(AT_2^ny,Ay)\Big)/\|Ay\|}{(T_1^nAy,Ay)/\|Ay\|^2 - e^{in\theta}(T_2^ny,y)} \right)^2 \right)^{-\frac{1}{2}}.$$

Let y_m, $m = 1, 2, 3, \ldots$ be a sequence in $(E\mathcal{H})^\perp$ such that $\|y_m\| = 1$ and

$\lim_{m\to\infty} \|Ay_m\| = \|A\|$. For each n,

$$\frac{\left((T_1^n Ay_m, Ay_m) - e^{in\theta}(T_2^n y_m, A^*Ay_m)\right)/\|Ay_m\|^2}{(T_1^n Ay_m, Ay_m)/\|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)}$$

$$= 1 + \frac{e^{in\theta}(T_2^n y_m, (\|Ay_m\|^2 - A^*A)y_m)}{(T_1^n Ay_m, Ay_m)/\|Ay_m\|^2 - e^{in\theta}(T_2^n y_m, y_m)} \to 1 \text{ as } m \to \infty.$$

Hence $\lim_{m\to\infty} \epsilon[n, y_m] = (1 + \|A\|^2)^{-\frac{1}{2}}$ (cf. Corollary 1.4). Thus for each integer n, there is an integer m(n) such that

$$\epsilon[n, y_{m(n)}] \leq (1 + \|A\|^2/2)^{-\frac{1}{2}} < 1.$$

Since θ is irrational modulo 2π, pick an integer N for which $|\text{Arg } e^{iN\theta}| \geq \arccos(-1/(1 + \|A\|^2/2))$. Then $0 \in \Theta[N, y_{m(N)}]$ by Lemma 3.5. However, $0 \notin W(T^N)$ by hypothesis; therefore, $A = 0$ and $T = T_1 \oplus e^{i\theta}T_2$. The proof of Theorem 3.4 is now completed. ∎

The following is an immediate consequence of Theorem 3.4.

(3.6) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$. If $0 \notin \text{Cl}(W(T^n))$, $n = 1, 2, 3, \ldots,$ then an odd power of T is normal.

With $\mathcal{H}$ finite dimensional Corollary 3.6 is first proved in [21].

5. $T^n \notin \mathcal{R}$

(3.7) THEOREM. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space and let $T \in \mathcal{B}(\mathcal{H})$. Suppose $T^n \notin \mathcal{R}$, $n = 1, 2, 3, \ldots$ Then we have the following cases:

(i) $\# \operatorname{Arg} \sigma_W(T) = 1$, then there exist $\theta \in [0,2\pi)$ and compact operator K such that $(e^{i\theta}T + K) \geq 0$.

(ii) $\# \operatorname{Arg} \sigma_W(T) \geq 3$, then $\# \operatorname{Arg} \sigma_W(T) = 3$ and there exist $\theta \in [0,2\pi)$ and compact operator K such that $(e^{i\theta}T^7 + K) \geq 0$.

(iii) $\# \operatorname{Arg} \sigma_W(T) = 2$, then either there exist a positive odd integer m, $\theta \in [0,2\pi)$ and compact operator K such that $(e^{i\theta}T^m + K) \geq 0$, or there exist a closed subspace $\mathcal{H}_1$ of $\mathcal{H}$ and positive operators T_1 and T_2 on $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact where $(\theta_1 - \theta_2)$ is a number irrational modulo 2π.

PROOF. By Theorem 3.1, $\mathcal{R} = \{S \in \mathcal{B}(\mathcal{H}) : 0 \notin \operatorname{Cl}(W(\tau \circ \Pi(S)))\}$. Now, most of the conclusions in the theorem follow directly from Theorem 3.4. However, a little more explanation is needed in the second half of case (iii). We know $\tau \circ \Pi(T) = e^{i\theta_1}V_1 \oplus e^{i\theta_2}V_2$ on $\mathfrak{H}_1 \oplus \mathfrak{H}_1^{\perp} = \mathfrak{H}$, where $V_1 \geq 0$ and $V_2 \geq 0$. Thus $\Pi(T)$ is normal and $\sigma(\Pi(T))$ lies on a simple arc. By Theorem 1.8, there is a compact operator K such that T + K is normal and $\sigma(T + K) = \sigma(\Pi(T))$. Consequently, there exist $\mathcal{H}_1$ closed subspace of $\mathcal{H}$ and positive operators T_1 and T_2 on $\mathcal{H}_1$ and $\mathcal{H}_1^{\perp}$ respectively such that $(T - e^{i\theta_1}T_1 \oplus e^{i\theta_2}T_2)$ is compact. ∎

(3.8) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinite-dimensional and separable. If $T^n \notin \mathcal{R}$, $n = 1, 2, 3, \ldots$, then an odd power of T is a normal plus a compact.

REMARK. One may ask if a stronger conclusion can be drawn when the hypothesis that each of the powers of T is not a commutator with a positive factor is replaced by the hypothesis that each of the powers of T is not a commutator with a normal factor. However, these two hypotheses are actually equivalent, i.e., $\mathcal{R} = \cup \{\delta_N(\mathcal{B}(\mathcal{H})) : N \text{ normal}\}$. It follows immediately from Theorem 3.1 that $\mathcal{R}$ is a norm closed subset of $\mathcal{B}(\mathcal{H})$. The theorem in [16] states that for each normal operator N, there is a Hermitian operator A and a function φ continuous on $\sigma(A)$ such that $\varphi(A) = N$. By the Weierstrass approximation theorem, $\delta_N(\mathcal{B}(\mathcal{H}))$ is a subset of the norm closure of $\delta_A(\mathcal{B}(\mathcal{H}))$ [1, Corollary 13.11]. Consequently, $\mathcal{R} = \cup \{\delta_N(\mathcal{B}(\mathcal{H})) : N \text{ normal}\}$.

6. SUFFICIENT CONDITIONS FOR NORMALITY

In this section we present some variants of the results in the last two sections. We give additional conditions which make the operator T itself normal or normal plus compact.

(3.9) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$ and $0 \notin \mathrm{Cl}(W(T^n))$, $n = 1, 2, 3, \ldots$ If T is a convexoid, i.e., $\mathrm{co}(\sigma(T)) = \mathrm{Cl}(W(T))$, then for $1 \leq j \leq k$, where k is some positive integer ≤ 3, there exist positive operators $T_j \in \mathcal{B}(\mathcal{H}_j)$ and real numbers θ_j such that $\mathcal{H} = \sum_{j=1}^{k} \oplus \mathcal{H}_j$ and $T = \sum_{j=1}^{k} \oplus e^{i\theta_j} T_j$.

Moreover, if k = 3, then $e^{i7\theta_1} = e^{i7\theta_2} = e^{i7\theta_3}$.

PROOF. Put $k = \# \operatorname{Arg} \sigma(T)$ and $k \leq 3$ by Theorem 3.4. First, we consider the case k = 2, i.e., there are two real numbers θ_1 and θ_2 such that $\sigma(T) \subset e^{i\theta_1} \mathbb{R}^+ \cup e^{i\theta_2} \mathbb{R}^+$. Let E be the spectral idempotent associated with $\sigma(T) \cap e^{i\theta_1} \mathbb{R}^+$. With respect to $E\mathcal{H} \oplus (E\mathcal{H})^\perp$, put

$$E = \begin{pmatrix} I & A \\ 0 & 0 \end{pmatrix}, \text{ then}$$

$$T = \begin{pmatrix} e^{i\theta_1} T_1 & e^{i\theta_1} T_1 A - A e^{i\theta_2} T_2 \\ 0 & e^{i\theta_2} T_2 \end{pmatrix},$$

where $T_1 \geq 0$ and $T_2 \geq 0$. Assume $A \neq 0$; thus there is a two-dimensional compression of T whose numerical range consists of an elliptical disc with foci on each of the two half-rays $e^{i\theta_j} \mathbb{R}^+$, j = 1, 2, and eccentricity strictly less than unity. However, T is a convexoid by hypothesis and $\operatorname{co}(\sigma(T))$ is a quadrilateral, a triangle or a line segment with all of its vertices lying on the two half-rays $e^{i\theta_j} \mathbb{R}^+$, j = 1, 2. Therefore, A = 0 and $T = e^{i\theta_1} T_1 \oplus e^{i\theta_2} T_2$.

The case that $\# \operatorname{Arg} \sigma(T) = 3$ is treated in a similar fashion. Nevertheless, we note that the above geometric argument fails if $\# \operatorname{Arg} \sigma(T) \geq 4$. Fortunately this case cannot arise. ∎

(3.10) COROLLARY. Let $T \in \mathcal{B}(\mathcal{H})$ and suppose $0 \notin \operatorname{Cl}(W(T^n))$, n = 1, 2,... Then T is normal if and only if $\operatorname{Cl}(W(T))$ is a closed polygon (possibly degenerate).

PROOF. $Cl(W(T))$ is a closed polygon implies that T is a convexoid ([17], [33, Corollary 2.1]). ∎

We note that the polygon mentioned in Corollary 3.10 may have at most six sides.

(3.11) THEOREM. Let $T \in \mathcal{B}(\mathcal{H})$, $\mathcal{H}$ infinite dimensional and separable. Suppose $T^n \notin \mathcal{R}$, $n = 1, 2, 3, \ldots$ Then T is a normal plus a compact if and only if $W_e(T)$ is a closed polygon (possibly degenerate).

PROOF. Apply Corollary 3.10 and Theorem 1.8. ∎

We conclude this chapter with the following theorem on finite dimensional matrices (cf. [21]).

(3.12) THEOREM. Let T be a finite dimensional square matrix. Suppose $0 \notin Int(W(T^n))$, $n = 1, 2, 3, \ldots$ If for each positive integer n, T and T^n have identical eigenvectors or identical commutants, then T is normal.

PROOF. By a theorem of Schur, there is a unitary matrix U such that U^*TU is upper triangular. We have to show that U^*TU is actually diagonal.

Assume U^*TU is not diagonal; then we can find, if necessary after applying a suitable simultaneous row and column permutation (which of course will preserve the upper triangular structure), a 2×2 submatrix $T_1 = \begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}$ along the main diagonal with $\gamma \neq 0$. For each n, $W(T^n)$

contains $W(T_1^n)$.

Suppose $\alpha = \beta = 0$, then $0 \in \mathrm{Int}(W(T_1))$. If $\alpha = \beta \neq 0$, $W(T_1^n)$ is the closed circular disc $\alpha^n + \Delta(n|\gamma\alpha^{n-1}|/2)$ by Corollary 1.4. Hence for $n > 2\cdot|\alpha/\gamma|$, $0 \in \mathrm{Int}(W(T_1^n))$.

Suppose $\alpha \neq \beta$, then $W(T_1^n)$ is the closed elliptical disc (possibly a singleton) Θ $(\alpha^n, \beta^n; \epsilon)$, where $\epsilon = (1+|\gamma/(\alpha - \beta)|^2)^{-\frac{1}{2}}$. Consequently, $0 \in \mathrm{Int}(T_1^n)$ if $|\alpha^n| + |\beta^n| < |\alpha^n - \beta^n|/\epsilon$. If $|\alpha| \neq |\beta|$, this inequality holds for large n since $\epsilon < 1$.

For the case $\alpha \neq \beta$ and $|\alpha| = |\beta|$, we apply the additional hypothesis and Proposition 1.11 to conclude that $\alpha^n \neq \beta^n$, $n = 1, 2, 3, \ldots$, or equivalently, $\alpha/\beta = \exp(2\pi i\theta)$ for some irrational real number θ. $W(T_1^n)$ is the ellipse with foci at α^n and β^n and constant eccentricity $\epsilon < 1$. Thus $0 \in \mathrm{Int}(W(T_1^n))$ for infinitely many n's.

We conclude that $\gamma = 0$ and T is normal. ∎

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