AN ANALYSIS OF SERVOMECHANISMS
CONTAINING A
DEPENDENT VARIABLE NONLINEARITY

Thesis by

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ABSTRACT

A method is developed for obtaining the transient response of an automatic control system containing a nonlinearity in which one dependent variable may be expressed as a unique function of another variable. This method involves obtaining a mathematical expression for the nonlinear characteristic by an expansion in Legendre polynomials, introducing this expression into the equations describing the control system behavior thus obtaining a nonlinear equation in a power series of a dependent variable and solving this nonlinear equation by means of an assumed infinite series solution technique. The rules governing the application of the method are discussed.

A saturation type nonlinearity is used to illustrate the application of the method. A second order system is employed to illustrate the accuracy of the method and to present a numerical technique for solving the series of equations arising from the infinite series method of solving the nonlinear system equation. The stability of a fourth order missile control equation with a saturation limit on the control surface is investigated by the method.

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I. INTRODUCTION

Although an extensive development of the theory of linear servomechanism analysis and synthesis has been made, no general method applicable to nonlinear systems has been devised. Much work has been done in the field of topology as applied to second order differential equations, and this phase plane approach has been successfully employed in the analysis of nonlinear second order servomechanism systems. Unfortunately, this technique involves three-dimensional space for systems described by third order differential equations and so on for higher order systems. Since the second order servomechanism is almost a trivial case, a method applicable to higher order systems is extremely important.

One method for accomplishing analyses for higher order systems has been the frequency response or equivalent linearization technique developed by Kochenburger (Ref. 1). This involves an open-loop steady-state transfer function of the system including the nonlinearity. It appears to the author that this is at best a qualitative approach since nonlinear functions of a variable are dependent on the magnitude of this variable, which necessitates a transient analysis involving the actual closed-loop environment to obtain quantitative results.

The only other method which is known to the author is that of step-by-step analysis in which several sets of equations, that describe the nonlinear servomechanism for several ranges of the magnitude of the dependent variable of which the nonlinearity is a function, are successively solved. This is effected by using the final conditions (position, velocity, acceleration, etc., of the dependent variable) from the equation describing the system to the magnitude limit just reached as the corresponding initial conditions for the equation of the next range. Oldenbourg and Sartorius (Ref. 2) include a good exposition of this method for two types of nonlinearities in a second order system. This method gives an exact transient solution for those types of discontinuous nonlinearities in which the nonlinear function may be accurately described by several linear ranges, but is very lengthy in application especially if the number of range changes during the transient solution exceeds three or four. The method is not applicable for continuous nonlinear functions of a dependent variable.

Since neither of these two methods is satisfactory for the general solution of a nonlinear system, a more general method was sought. The remainder of this thesis describes a technique which it is felt provides this more general method for a certain class of nonlinearities which will be termed <u>dependent variable nonlinearities</u>. This class of functions, which is defined in the next section, includes the majority of the nonlinearities encountered in servomechanisms or automatic control systems.

II. EXPLICIT MATHEMATICAL EXPRESSION FOR DEPENDENT VARIABLE NONLINEARITY

A dependent variable nonlinearity will be defined as any non-linear function in which one dependent variable may be expressed as a unique function of another dependent variable for the entire range of the independent variable considered. By this definition the dependent variable nonlinearity class includes every type of nonlinear function encountered in control systems analysis except non-unique multivalued functions such as hysteresis together with nonlinearities composed of products of two or more differential functions of dependent variables, e.g. $\left(\frac{dx}{dt}\right)^2$ or $\left(\frac{dx}{dt} \cdot \frac{dy}{dt}\right)$.

Thus this thesis will be concerned with the development of a technique for analyzing control systems including a dependent variable nonlinearity. This involves first finding an explicit mathematical expression for the dependent variable nonlinearity. While some of the nonlinearities encountered will have an explicit functional relationship, this will not be true in the general case. Hence consider the general functional relationship

$$\theta_b = f(\theta_a)$$

which may represent a numerical or graphical expression obtained by theoretical or experimental consideration. The usual treatment to obtain an explicit mathematical expression of such a function is to express the relationship as a Fourier series. The orthogonal functions involved in the Fourier series are commonly trigonometric functions. Unfortunately, these result in transcendental equations which must be solved to obtain the solution of the system with which we are here concerned. However, other orthogonal functions which are not transcendental can be used such as Legendre polynomials.

The use of orthogonal polynomials requires only that the first derivative of the functional relationship exists and that the function be continuous in the interval considered. Thus, an expansion in Legendre polynomials is not as restrictive as a Taylor or power series expansion which is valid only where all derivatives of the function exist. Moreover, Legendre polynomials afford the advantage of superposition of additional terms in the expansion, to increase the accuracy of the expression, without affecting the coefficients of previous terms.

Since the expansion interval for Legendre polynomials is +1 to -1, a suitable change of variable must be made such that the entire range of interest of the dependent variable is included.

Let

$$\sigma_a = \frac{\theta_a}{NF} \tag{1}$$

$$\sigma_b = \frac{\theta_b}{NF} \tag{2}$$

where NF = normalizing factor which must be greater than the maximum magnitude of θ_a or θ_b. Then the functional relationship may be expressed to q terms as

$$\sigma_b = \frac{\theta_b}{NF} = f\left(\frac{\theta_a}{NF}\right) = \sum_{n=0}^{q} A_n P_n(\sigma_a) \tag{3}$$

where the coefficients A_n are given by the expression

$$A_n = \frac{2n+1}{2} \int_{-1}^{+1} f(\sigma_a) P_n(\sigma_a)\, d\sigma_a \tag{4}$$

and where the Legendre polynomials are

$$\begin{aligned}
P_0(\sigma_a) &= 1 \\
P_1(\sigma_a) &= \sigma_a \\
P_2(\sigma_a) &= \frac{1}{2}(3\sigma_a^2 - 1) \\
P_3(\sigma_a) &= \frac{1}{2}(5\sigma_a^3 - 3\sigma_a) \\
&\vdots \\
P_n(\sigma_a) &= \frac{1}{2^n n!} \frac{d^n(\sigma_a^2-1)^n}{d\sigma_a^n} \\
&= \frac{(2n-1)(2n-3)\cdots 1}{n!}\left[\sigma_a^n - \frac{n(n-1)}{2(2n-1)}\sigma_a^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}\sigma_a^{n-4} - \cdots\right]
\end{aligned} \tag{5}$$

where power of σ_a is > 0.

This expansion in Legendre polynomials may next be expressed as a finite series of powers of the dependent variable σ_a by combining coefficients of like powers of σ_a when (5) is substituted in (3). Thus

$$\sigma_b = \sum_{n=0}^{q} C_n \sigma_a^n \tag{6}$$

Whereas the A_n coefficients are independent of the number of terms of the expansion, q, the C_n constants are not. However, as q changes the new C_n's are merely the original C_n's with superimposed terms arising from the inclusion of additional Legendre polynomials.

As an example of this expansion consider the saturation nonlinearity that is commonly encountered in control systems analysis which has a limited output designated here as Θ_{NL}.

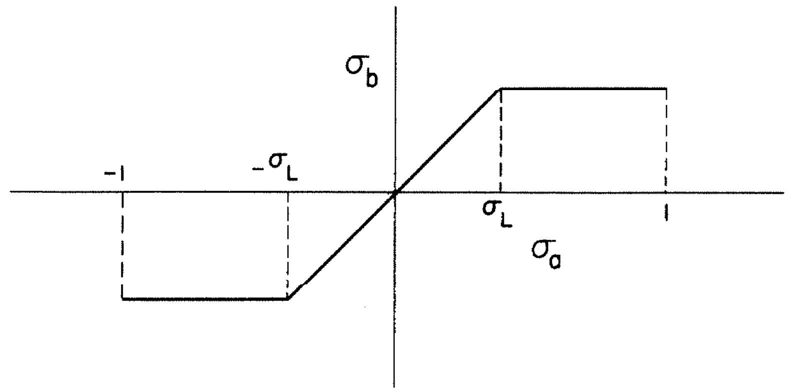


Figure 1

Figure 1 shows this type of nonlinear function expressed in normalized form where as in eqs. (1) and (2) $\sigma_L = \frac{\Theta_{NL}}{NF}$.

From eq. (4)

$$A_n = \frac{2n+1}{2}\left[\int_{-1}^{-\sigma_L} -\sigma_L P_n(\sigma_a)\,d\sigma_a + \int_{-\sigma_L}^{+\sigma_L} \sigma_a P_n(\sigma_a)\,d\sigma_a + \int_{\sigma_L}^{1} \sigma_L P_n(\sigma_a)\,d\sigma_a\right] \qquad (7)$$

for this saturation nonlinearity. The A_n constants may be readily calculated by substitution of (5) and evaluation of the integrals indicated. A summary of the results is given in Appendix A. The power series coefficients C_n, which result from the expansion of eqs. (3) and (5) with the A_n values included, are plotted in Figures 2, 3 and 4, 5 and 6 for q = 5, 13 and 17 respectively.

The accuracy of this representation for various σ_L limits is shown for Legendre polynomial expansions including 5 and 17 terms in Figures 7 and 8. As would be expected the accuracy increases with the number of terms included, especially as σ_L decreases. At the low σ_L limits the main error occurs in the representation of

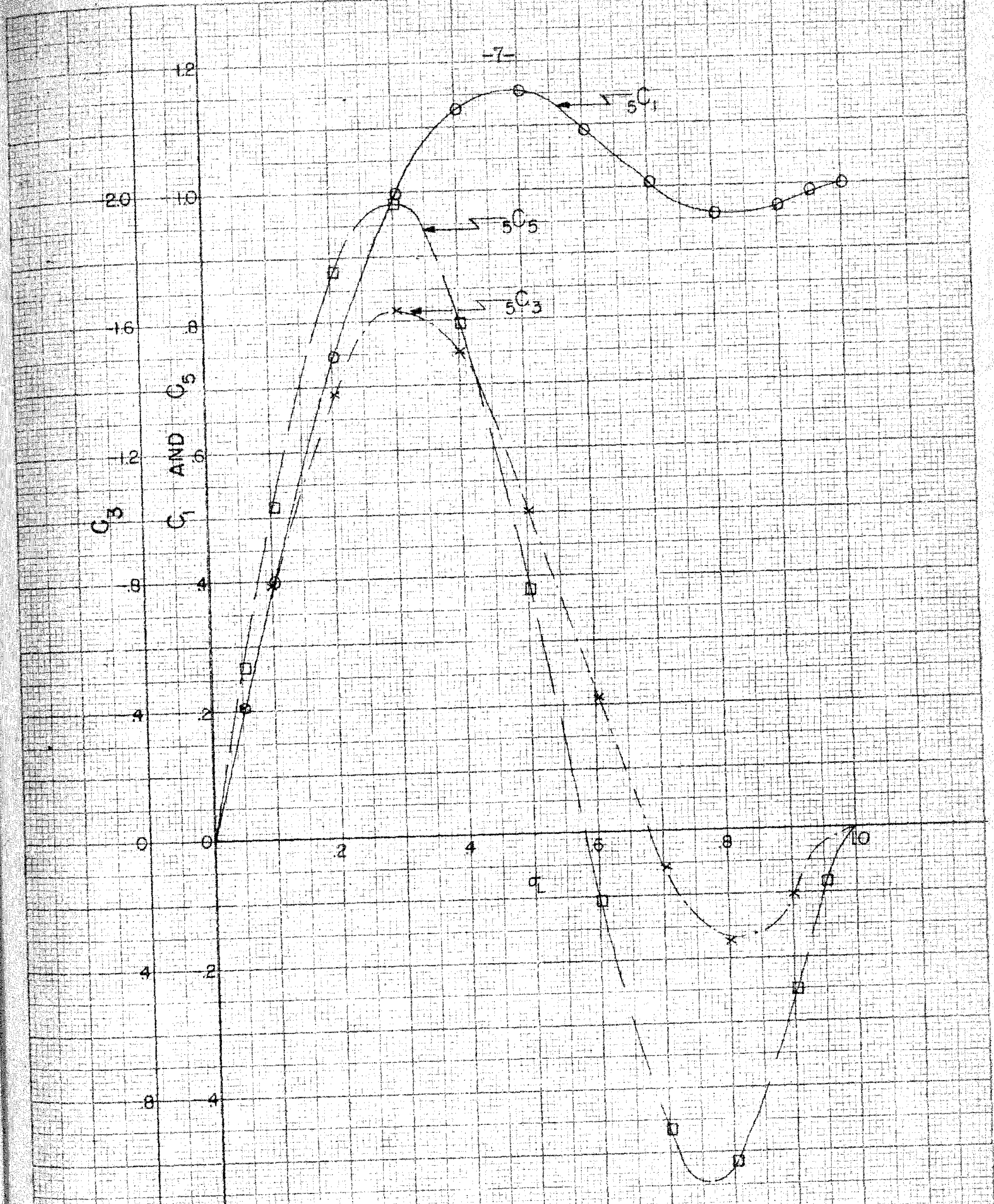


FIGURE 2. $_qC_n$ CONSTANTS FOR $q=5$

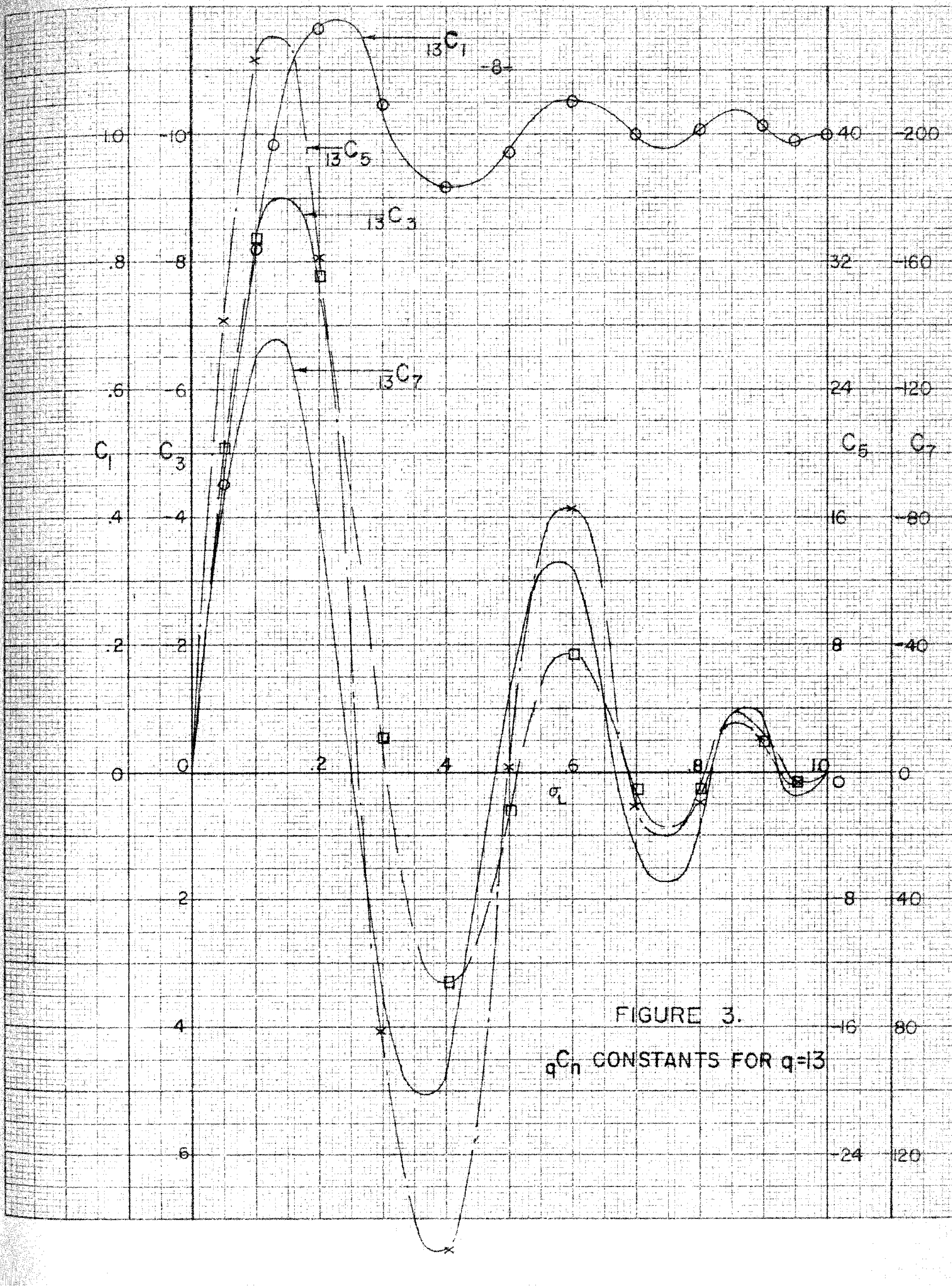


FIGURE 3.

$_qC_n$ CONSTANTS FOR q=13

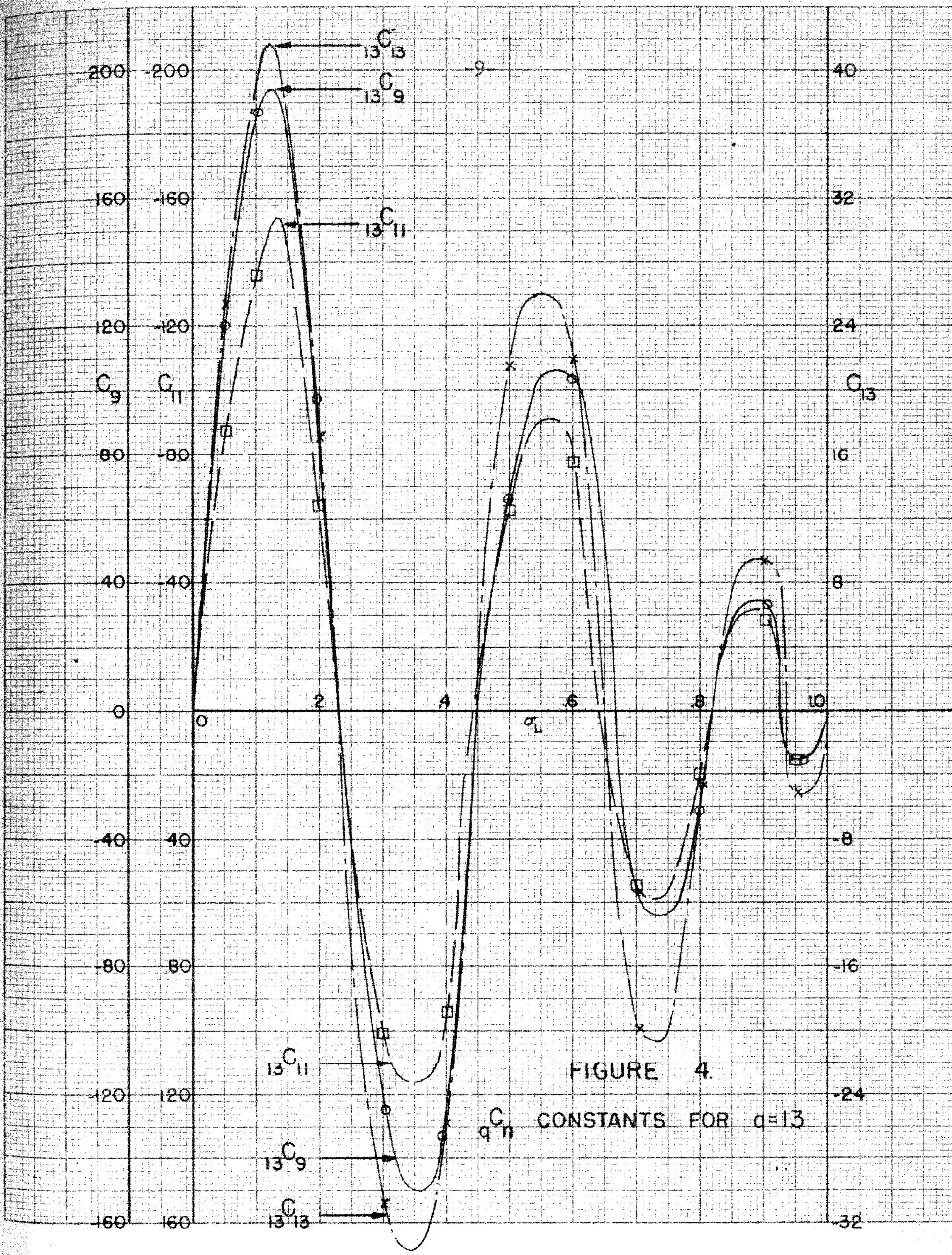


FIGURE 4.

$_qC_n$ CONSTANTS FOR $q=13$

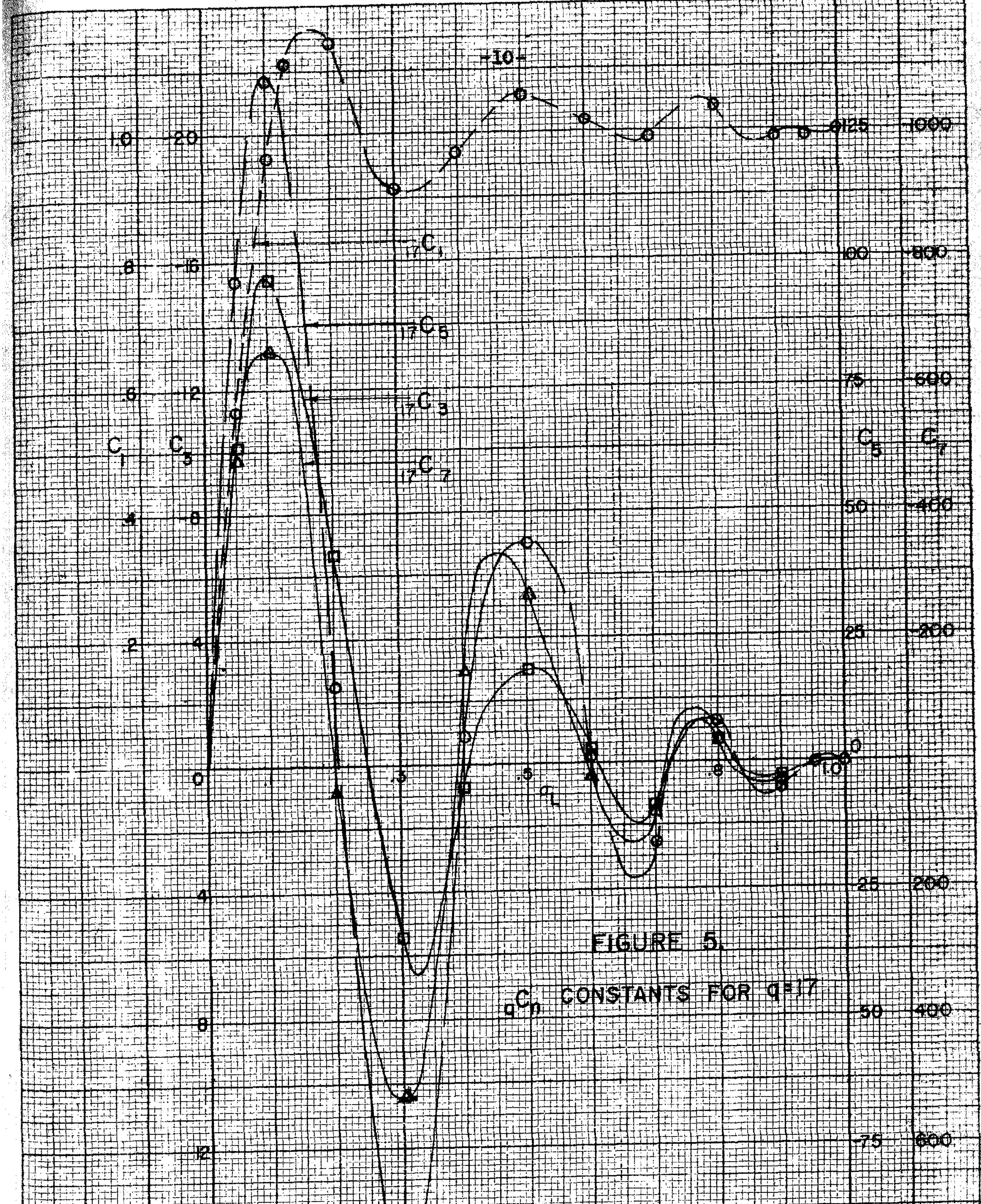


FIGURE 5.

$_qC_n$ CONSTANTS FOR q=17

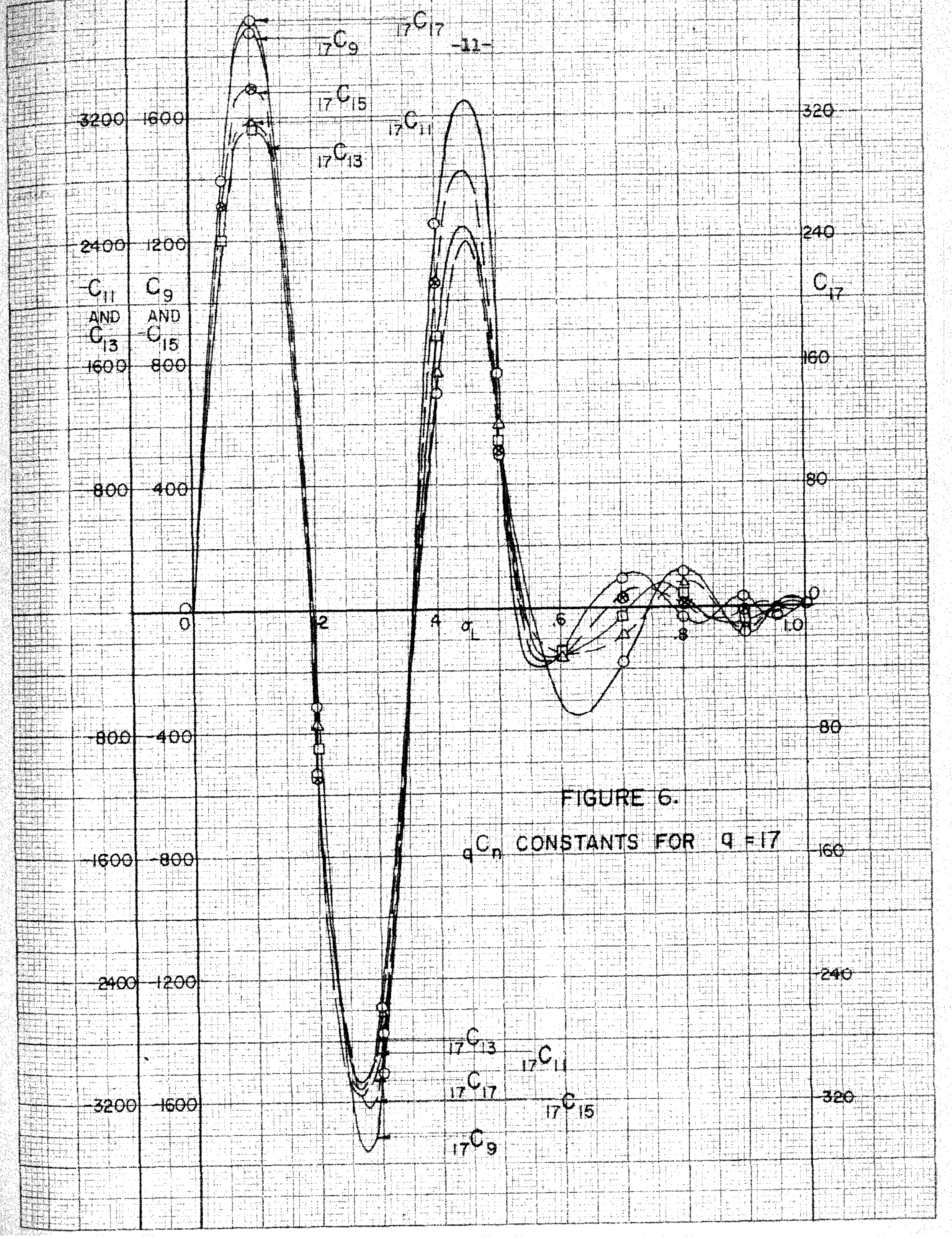


FIGURE 6.

$_qC_n$ CONSTANTS FOR $q = 17$

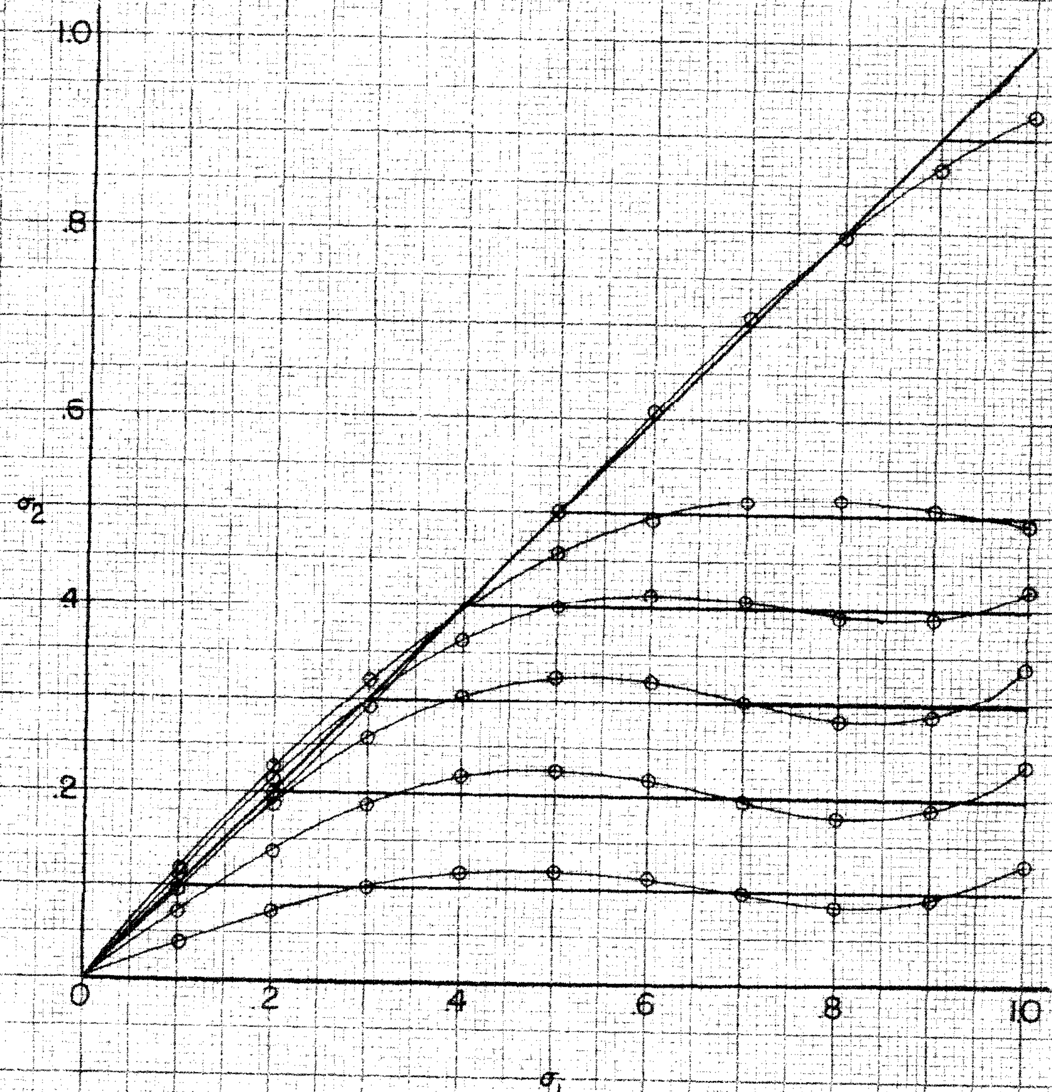


FIGURE 7 σ_2 vs σ_1 for $q = 5$

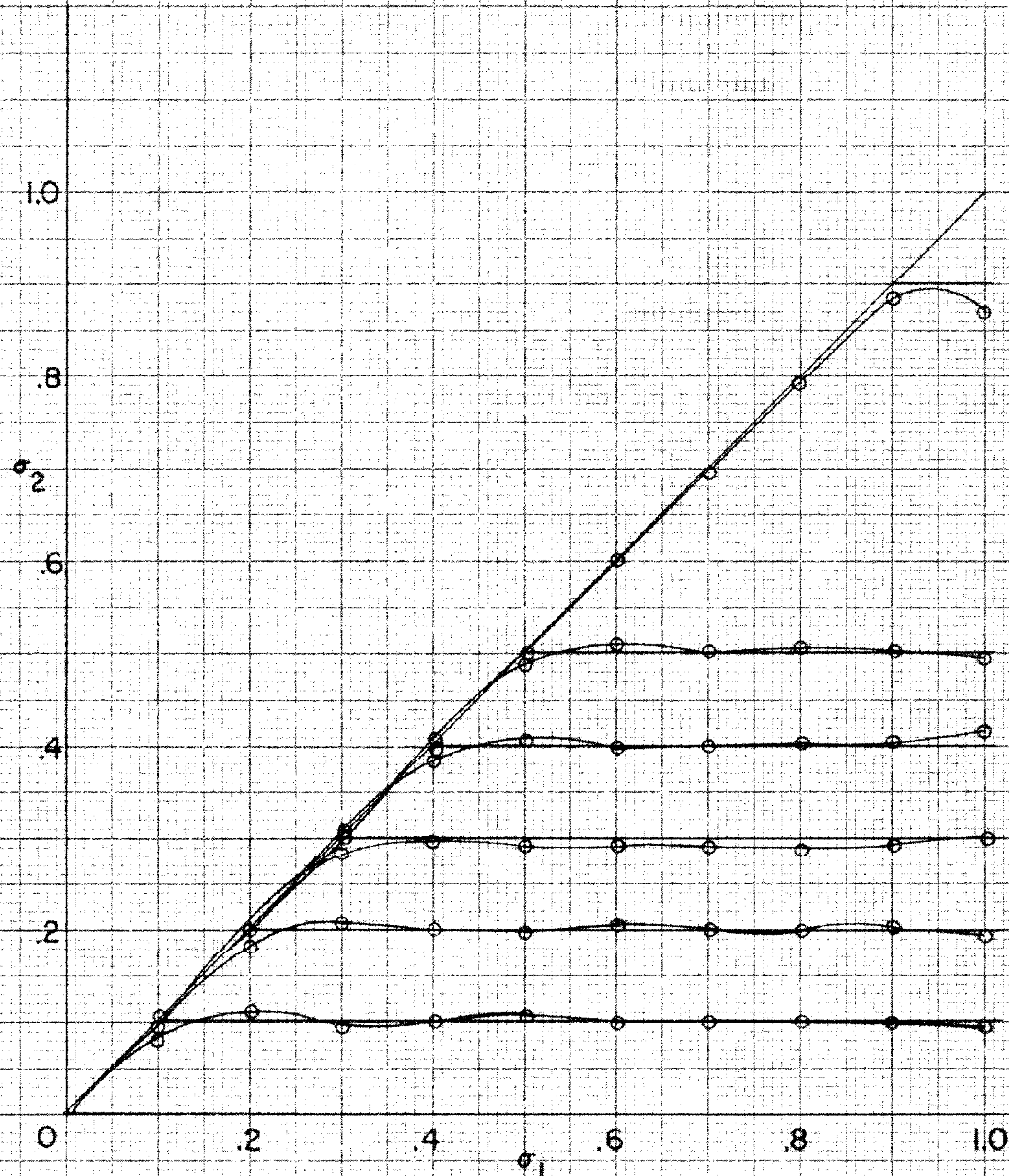


FIGURE 8. σ_2 vs σ_1 for q = 17

the linear region from 0 to σ_L. From the C_n curves and the resulting descriptions of the nonlinearity, it can be noted that the representation of the linear region has a fairly constant accuracy from $\sigma_L = 1$ to the value of σ_L for which C_1 is last equal to one as σ_L decreases. This minimum value of σ_L for which C_1 is equal to one decreases as the number of included polynomial terms increases, which accounts for the improved accuracy at lower σ_L for higher q. For q = 17, C_1 equals one at a minimum $\sigma_L = .102$ and the maximum error of the linear portion of the presentation for $\sigma_L = .1$ is 19% with a mean error of 10.3%. The saturation part of the nonlinear function is described within a maximum error of .01 which gives an increasing percentage as σ_L decreases reaching 10% at $\sigma_L = .1$.

For most engineering work q = 17 should give adequate accuracy. When the σ_L is of necessity less than 0.1, the linear region is of relatively little importance except near the steady-state of the transient response since σ_a will be almost entirely in the saturated region. Hence the effect of the linear region inaccuracy will be minimized.

III. GENERAL EQUATIONS FOR A NONLINEAR SERVOMECHANISM

With an explicit mathematical expression for the dependent variable nonlinearity one can now obtain a general equation describing the transient behavior of a closed loop control system containing such a nonlinearity. Since the nonlinearity may exist in either the main control loop or a subsidiary feedback loop within the closed loop, these two cases are considered separately. In both cases the equations are expressed in terms of the dependent variable describing the input to the nonlinearity. This is chosen since from it one may easily obtain the output of the nonlinearity from the nonlinear characteristic, and any other variable by solving a simple linear differential equation. It is obvious that the equation of the system may just as well be expressed in terms of any variable in the loop which is of particular interest.

A general schematic for a servomechanism containing a nonlinearity in the main control loop is shown in Figure 9.

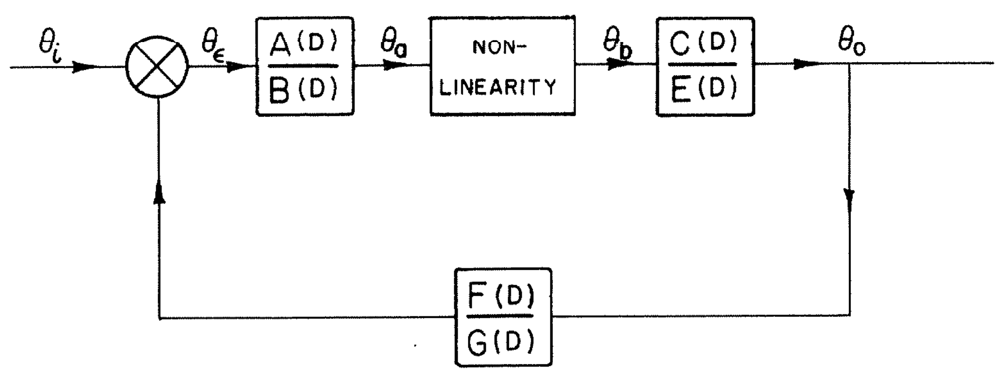


Figure 9

The equations describing the system are as follows where $D = \frac{d}{dt}$ and the functions A(D), B(D), etc., are polynomials in D.

$$B(D)\theta_a = A(D)\,\theta_\epsilon \tag{8}$$

$$\theta_\epsilon = \theta_i - \frac{F(D)}{G(D)}\theta_o \tag{9}$$

$$E(D)\theta_o = C(D)\theta_b \tag{10}$$

The nonlinearity may be expressed in the Legendre polynomial expansion which has been shown to yield

$$\sigma_b = \sum_{n=0}^{q} C_n \sigma_a^n \tag{6}$$

in terms of the normalized quantities. Substituting equations (1) and (2) into (6) permits writing the relationship describing the nonlinearity as

$$\theta_b = \sum_{n=0}^{q} \alpha_n \theta_a^n \tag{11}$$

where $$\alpha_n = \frac{C_n}{NF^{n-1}} \tag{12}$$

Now multiplying the linear equation (8) by G(D)E(D) and substituting (9) and (10) gives

$$B(D)G(D)E(D)\,\theta_a = A(D)\,G(D)E(D)\,\theta_i - A(D)\,F(D)\,C(D)\,\theta_b \tag{13}$$

Substitution of eq. (11) and rearranging yields

$$A(D)F(D)C(D)\sum_{n=0}^{q}\alpha_n\theta_a^n + B(D)G(D)E(D)\,\theta_a = A(D)G(D)E(D)\,\theta_i \tag{14}$$

which is a nonlinear equation governing the transient response of the control loop. It should be noted here that this result is perfectly general, an equation of identical form being obtained if the nonlinearity occurs in the feedback portion of the loop or if the designated input and output functions, θ_i and θ_o, are shifted with respect to the nonlinearity. Of course, the input function θ_i may be an external disturbance as well as the command to the control system.

For the case of a nonlinearity in a subsidiary feedback loop of the system, the general schematic will be as in Figure 10. While the configuration considered has the nonlinearity in the direct portion of the subsidiary loop, similar equations may be readily developed when the feedback portion contains the nonlinear element.

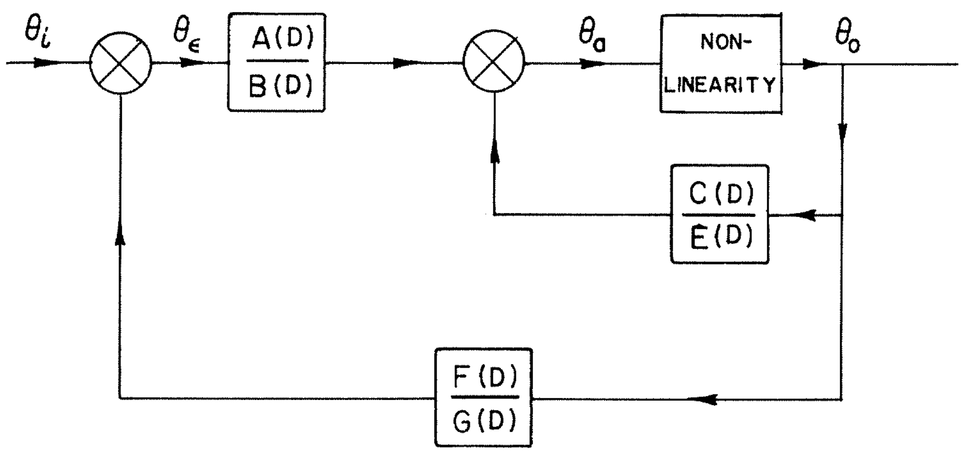


Figure 10

The equations of the system are

$$\theta_a = \frac{A(D)}{B(D)}\theta_\varepsilon - \frac{C(D)}{E(D)}\theta_o \tag{15}$$

$$\theta_\varepsilon = \theta_i - \frac{F(D)}{G(D)}\theta_o \tag{16}$$

$$\theta_o = \sum_{n=0}^{q} \alpha_n \theta_a^n \tag{17}$$

Multiplication of eq. (15) by B(D)E(D)G(D) and substitution of (16) gives

$$B(D)E(D)G(D)\Theta_a = A(D)E(D)G(D)\Theta_i - \left[A(D)E(D)F(D) + C(D)B(D)G(D)\right]\Theta_o \qquad (18)$$

Introducing (17) results in the final equation

$$\left[A(D)E(D)F(D) + C(D)B(D)G(D)\right]\sum_{n=0}^{q} \alpha_n \Theta_a^n + B(D)E(D)G(D)\Theta_a = A(D)E(D)G(D)\Theta_i \qquad (19)$$

The introduction of a second nonlinear function into the control system leads to an equation involving a power series in a function of differential operators acting upon a dependent variable. This type of equation is not solvable by either the technique to be used in the remainder of the discussion or any other practical technique known. Hence our considerations will be limited to systems involving a single nonlinearity.

IV. SOLUTION OF NONLINEAR EQUATIONS BY INFINITE SERIES

In the solution of linear differential equations many equations do not fall into one of the standard types and may not, in fact, yield solutions in terms of a finite number of the elementary functions. One then resorts to a solution in the form of an infinite series or an approximate solution involving a finite number of series terms when numerical computations are the main concern. Thus a method of solution by infinite series similar to that of Frobenius for linear differential equations has been employed here for the solution of the type of nonlinear equation given in equations (14) and (19). The method is parallel to that described by A. C. Sim (Ref. 3), the slightly different approach being made to facilitate the establishment of the convergence criterion which Sim leaves undiscussed.

Given an equation

$$a_n(D)\Theta^n + a_{n-1}(D)\Theta^{n-1} + \cdots a_2(D)\Theta^2 + a_1(D)\Theta = F(t) + I \qquad (20)$$

where the coefficients $a_n(D)$ contain no negative powers of D and

D = the differential operator $\frac{d}{dt}$

I = the initial conditions

a solution may be found by assuming

$$\Theta = \Theta_1 + \Theta_2 + \Theta_3 + \cdots \Theta_n = \sum_{n=1}^{\infty} \Theta_n \qquad (21)$$

where

$$\Theta_1 = \Theta_1$$

$$\Theta_2 = F_2(D)\Theta_1^2$$

$$\theta_3 = F_3(D)\,\theta_1^3$$

$$\vdots$$

$$\theta_n = F_n(D)\,\theta_1^n$$

Substitution of (21) into (20) and equating functional terms having like powers of θ_1 to zero, yields the following set of differential equations.

$$a_1(D)\theta_1 = F(t) + I \tag{22}$$

$$a_1(D)\theta_2 = -a_2(D)\,\theta_1^2$$

$$a_1(D)\theta_3 = -\left[a_3(D)\theta_1^3 + 2\,a_2(D)\,\theta_1\theta_2\right]$$

$$a_1(D)\theta_4 = -\left[a_4(D)\theta_1^4 + a_3(D)3\theta_1^2\theta_2 + a_2(D)\left(\theta_2^2 + 2\theta_1\theta_3\right)\right]$$

$$a_1(D)\theta_5 = -\left[a_5(D)\theta_1^5 + a_4(D)4\theta_2\theta_1^3 + a_3(D)\left(3\theta_2^2\theta_1 + 3\theta_1^2\theta_3\right) + a_2(D)\left(2\theta_2\theta_3 + 2\theta_1\theta_4\right)\right]$$

$$a_1(D)\theta_6 = -\left[a_6(D)\theta_1^6 + a_5(D)5\theta_1^4\theta_2 + a_4(D)\left(6\theta_1^2\theta_2^2 + 4\theta_1^3\theta_3\right) + a_3(D)\left(3\theta_1^2\theta_4 + 6\theta_1\theta_2\theta_3 + \theta_2^3\right) + a_2(D)\left(\theta_3^2 + 2\theta_1\theta_5 + 2\theta_2\theta_4\right)\right]$$

$$a_1(D)\theta_7 = -\left[a_7(D)\theta_1^7 + a_6(D)6\theta_1^5\theta_2 + a_5(D)\left(10\theta_1^3\theta_2^2 + 5\theta_1^4\theta_3\right) + a_4(D)\left(4\theta_1^3\theta_4 + 12\theta_1^2\theta_2\theta_3 + 4\theta_1\theta_2^3\right) + a_3(D)\left(3\theta_1\theta_3^2 + 3\theta_1^2\theta_5 + 6\theta_1\theta_2\theta_4 + 3\theta_2^2\theta_3\right) + a_2(D)\left(2\theta_1\theta_6 + 2\theta_2\theta_5 + 2\theta_3\theta_4\right)\right]$$

$$a_1(D)\theta_8 = -\left[a_8(D)\theta_1^8 + a_7(D)7\theta_1^6\theta_2 + a_6(D)\left(15\theta_1^4\theta_2^2 + 6\theta_1^5\theta_3\right) + a_5(D)\left(5\theta_1^4\theta_4 + 20\theta_1^3\theta_2\theta_3 + 10\theta_1^2\theta_2^3\right) + a_4(D)\left(4\theta_1^3\theta_5 + 6\theta_3^2\theta_1^2 + 12\theta_1^2\theta_2\theta_4 + 12\theta_1\theta_2^2\theta_3 + \theta_2^4\right) + a_3(D)\left(6\theta_1\theta_2\theta_5 + 3\theta_1^2\theta_6 + 6\theta_1\theta_3\theta_4 + 3\theta_2\theta_3^2 + 3\theta_2^2\theta_4\right) + a_2(D)\left(\theta_4^2 + 2\theta_1\theta_7 + 2\theta_2\theta_6 + 2\theta_3\theta_5\right)\right]$$

$$a_1(D)\theta_9 = -\Big[a_9(D)\theta_1^9 + a_8(D)8\theta_1^7\theta_2 + a_7(D)(21\theta_1^5\theta_2^2 + 7\theta_1^6\theta_3)$$
$$+ a_6(D)(6\theta_1^5\theta_4 + 30\,\theta_1^4\theta_2\theta_3 + 20\,\theta_1^3\theta_2^3)$$
$$+ a_5(D)(10\,\theta_3^2\theta_1^3 + 5\theta_1^4\theta_5 + 20\,\theta_1^3\theta_2\theta_4 + 30\,\theta_1^2\theta_2^2\theta_3 + 5\theta_1\theta_2^4)$$
$$+ a_4(D)(4\theta_1^3\theta_6 + 12\theta_1^2\theta_2\theta_5 + 12\theta_1^2\theta_3\theta_4 + 12\theta_1\theta_2\theta_3^2$$
$$+ 12\theta_1\theta_2^2\theta_4 + 4\,\theta_2^3\theta_3)$$
$$+ a_3(D)(3\theta_1\theta_4^2 + 3\theta_1^2\theta_7 + 6\,\theta_1\theta_2\theta_6 + 6\,\theta_1\theta_3\theta_5$$
$$+ 6\,\theta_2\theta_3\theta_4 + 3\theta_5\theta_2^2 + \theta_3^3)$$
$$+ a_2(D)(2\theta_2\theta_7 + 2\theta_1\theta_8 + 2\theta_3\theta_6 + 2\,\theta_4\theta_5)\Big]$$

Additional terms are found by substitution in the same manner.

The restriction that the coefficients $a_n(D)$ contain no negative powers of D is necessary in order that a valid solution of equation (20) results from the commutation of operators implied in the substitution of eq. (21) in (20). It is to be remembered that a fundamental law of operators states that while

$$DD^{-1}y = y$$

is always correct,

$$D^{-1}Dy = y$$

is correct only when y and its first m-1 derivatives vanish at some fixed value of the independent variable, m being the order of the equation defining y. Hence, while the terms of the assumed series solution involve, in general, both operators and inverse operators, the substitution of the series into equation (20) does not violate the above law providing the a_n coefficients contain no inverse operators.

Thus the original nonlinear equation is transformed into an infinite set of equations, which though nonlinear in nature, may be reduced to linear differential equations when solved in successive order. That is, the solution of the equations preceding the one whose solution is sought, when multiplied and grouped according to eq. (22) form the forcing function for that equation. This forcing function may, in general, be linearized so that the solution of the equation may be effected by linear differential equation techniques.

1. Introduction of Initial Conditions into the Solution

From the set of differential equations (eq. 22) determining the dependent variables of the series solution of the nonlinear differential equation, it is obvious that only one complementary solution is required. This is true since the complementary or homogeneous solution of each equation in the set is identical and will give rise to the same number of arbitrary constants. Thus, to avoid re-evaluation of the constants to satisfy the initial conditions upon the inclusion of one more term in the series solution, as would be necessary if each equation of the set generated a complementary solution, the only complementary solution will be determined by the solution of the prime (first) equation of the set. Hence, all other solutions are particular solutions.

Thus, to obtain a general solution of the nonlinear equation, the prime equation must have the same order as the nonlinear equation so that sufficient arbitrary constants are introduced. Moreover, the particular solutions of the successive equations and their derivatives to the order of the linear prime equation minus one must vanish when the independent variable equals zero. The arbitrary initial conditions may then be introduced by whatever technique is employed in solving the prime equation. In this manner one may study the transient behavior of a control system from any initial state.

2. Convergence of the Assumed Series Solution

This solution of the nonlinear equation is valid providing the assumed series of equation (21) converges. Thus, in applying this method a general convergence criterion is necessary. Since the series is one of terms which are continuous functions of the independent variable, time, it is necessary to define the convergence in a given interval of time. Moreover, the sum of this convergent series must be a continuous function of time, for which a sufficient condition is uniform convergence of the series. The Weierstrauss M-test for uniform convergence satisfies these requirements as follows:

Let $$\sum_{n=1}^{\infty} \theta_n(t) \equiv \theta_1(t) + \theta_2(t) + \cdots \theta_n(t) + \cdots \tag{21}$$

be a series each of whose terms is a continuous function of t in an interval 0 - T. If the series of positive constants

$$M_1 + M_2 + M_3 + M_4 + \cdots M_n + \cdots \tag{23}$$

is convergent, and if

$$|\theta_n(t)| \leqq M_n \tag{24}$$

for every n and for all values of t in the interval 0 - T, then the original series is uniformly convergent in the interval. Moreover, the series of θ's is absolutely convergent in the interval 0 - T.

Hence, the criterion of validity for this infinite series solution of the nonlinear equation becomes the criterion of minimal convergence of the assumed series. This may be defined by the well known p-series

$$1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{n^p} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^p} \tag{25}$$

which is convergent for $p > 1$ and divergent for $p \leqq 1$. Thus, setting $p = 1+\varepsilon$ gives the dominant M-series for minimal convergence as

$$1 + \frac{1}{2^{1+\varepsilon}} + \frac{1}{3^{1+\varepsilon}} + \cdots \frac{1}{n^{1+\varepsilon}} + \cdots \tag{26}$$

which must be term by term equal to or greater than the corresponding term of the series

$$1 + \frac{|\theta_2(t)|}{|\theta_1(t)|} + \frac{|\theta_3(t)|}{|\theta_1(t)|} + \cdots \frac{|\theta_n(t)|}{|\theta_1(t)|} + \cdots \tag{27}$$

where the magnitude of each θ_n is its peak magnitude, either positive or negative, over the time interval considered.

Unfortunately, the question of convergence of the series of time dependent functions defined by the set of nonlinear equations (eq. 22) is not amenable to general mathematical treatment. Only when the system is numerically characterized can the peak magnitude of the solution of each equation of the set, and consequently the convergence, be determined. Even this appears prohibitive since the transient solution found by a series subject to this convergence criterion must be used to determine the criterion. However, in applying this technique to nonlinear control systems one develops from the convergence criterion, the variation in the behavior of the system as a function of the magnitude of system disturbance with respect to the nonlinearity critical magnitude. Thus, the difficulty arising from the interdependence of transient solution and convergence is avoided for the general analysis of a nonlinear system. Moreover, in determining a specific transient response from a given set of conditions a little insight enables one to obtain a sufficiently rapidly converging solution for practical calculations.

V. GENERAL RULES GOVERNING USE OF THE METHOD TOGETHER WITH THE SPECIFIC RULES FOR SATURATION TYPE NON-LINEARITY

We may now summarize the procedure developed for analyzing control systems containing a dependent variable nonlinearity as follows:

1. The nonlinear characteristic is normalized by means of an arbitrary constant termed NF (normalizing factor). A mathematical expression of the characteristic may then be found by an expansion in a series of Legendre polynomials, a sufficient number of polynomials being included to obtain the desired accuracy of the expression. The number of polynomials required to give a certain accuracy will vary with NF, either directly or indirectly depending upon the type of nonlinear characteristic.

2. The complete equation for the system including the nonlinearity is developed in terms of the dependent variable of interest.

3. A solution for the resulting nonlinear equation is obtained by an infinite series technique. The validity of this solution is dependent upon the convergence of the assumed series.

In the above procedure certain conditions are explicit or implied. These conditions appear as restrictions on the arbitrary normalizing factor. Thus, NF is subject to the following general rules:

a. NF must be greater than the peak transient magnitude of the input to the nonlinearity. This arises because of the Legendre polynomial expansion interval -1 to +1.

b. NF must be of a magnitude such as to permit a sufficiently accurate description of the nonlinear function by the number of Legendre polynomials chosen.

c. NF must be chosen to assure at least minimal convergence of the assumed series solution comprising the transient response when the system is subjected to a defined disturbance or command.

It is necessary that all three rules be satisfied simultaneously, if possible. However, under certain conditions b may have to be violated in order to assure the irrevocable restrictions a and c. This leads to the development of specific rules for individual nonlinear characteristics. Since for the remainder of this thesis we will concern ourselves with application of this technique to the saturation nonlinearity of Figure 1, the specializations of the general rules will be developed for this nonlinear function.

a'. This will be the same as the general rule for all dependent variable nonlinearities.

b'. As developed in Section II, NF must be less than that for which C_1 is last equal to one as σ_L decreases (NF increases). This restriction arises from the inaccuracy of the linear region description when $C_1 < 1$ which occurs as $NF = \frac{\theta_{NL}}{\sigma_L}$ increases beyond the above value. This accuracy limited maximum value of the normalizing factor, ${}_{Acc.}NF_{Max.}$, will increase as the number of Legendre polynomials, q, increases.

c'. In order to quantitatively determine the convergence of the assumed series solution it is necessary to consider a specific system. However, certain general qualitative conditions may be formulated. Referring to equations (14) and (19) it is seen that for a control loop containing a dependent variable nonlinearity at any point in the system, equation (20) may be written as

$$\sum_{n=2}^{\infty} A(D)\alpha_n\theta^n + [A(D)\alpha_1 + B(D)]\theta = F(t) + I \tag{28}$$

i.e., the differential operator dependent coefficient is constant for each power of the dependent variable greater than one.

Thus, the subsequent set of differential equations defining the solution to (28) becomes

$$\begin{aligned}
&[A(D)\alpha_1 + B(D)]\theta_1 = F(t) + I \\
&[A(D)\alpha_1 + B(D)]\theta_2 = -A(D)\alpha_2\theta_1^2 \\
&[A(D)\alpha_1 + B(D)]\theta_3 = -A(D)[\alpha_3\theta_1^3 + 2\alpha_2\theta_1\theta_2] \\
&[A(D)\alpha_1 + B(D)]\theta_4 = -A(D)[\alpha_4\theta_1^4 + 3\alpha_3\theta_1^2\theta_2 + \alpha_2(\theta_2^2 + 2\theta_1\theta_3)] \\
&\quad\vdots
\end{aligned} \tag{29}$$

For a symmetrical dependent variable nonlinearity the Legendre expansion produces $\alpha_n=0$ for n odd only. Hence, a further reduction can be made for the symmetrical saturation nonlinearity.

$$\begin{aligned}
&[A(D)\alpha_1 + B(D)]\theta_1 = F(t) + I \\
&[A(D)\alpha_1 + B(D)]\theta_3 = -A(D)\alpha_3\theta_1^3 \\
&[A(D)\alpha_1 + B(D)]\theta_5 = -A(D)[\alpha_5\theta_1^5 + 3\alpha_3\theta_3\theta_1^2] \\
&[A(D)\alpha_1 + B(D)]\theta_7 = -A(D)[\alpha_7\theta_1^7 + 5\alpha_5\theta_1^4\theta_3 + \alpha_3(3\theta_1\theta_3^2 + 3\theta_1^2\theta_5)] \\
&\quad\vdots
\end{aligned} \tag{30}$$

From equation 30 it is evident that for any given system all of the factors operating on the dependent variables are fixed with the

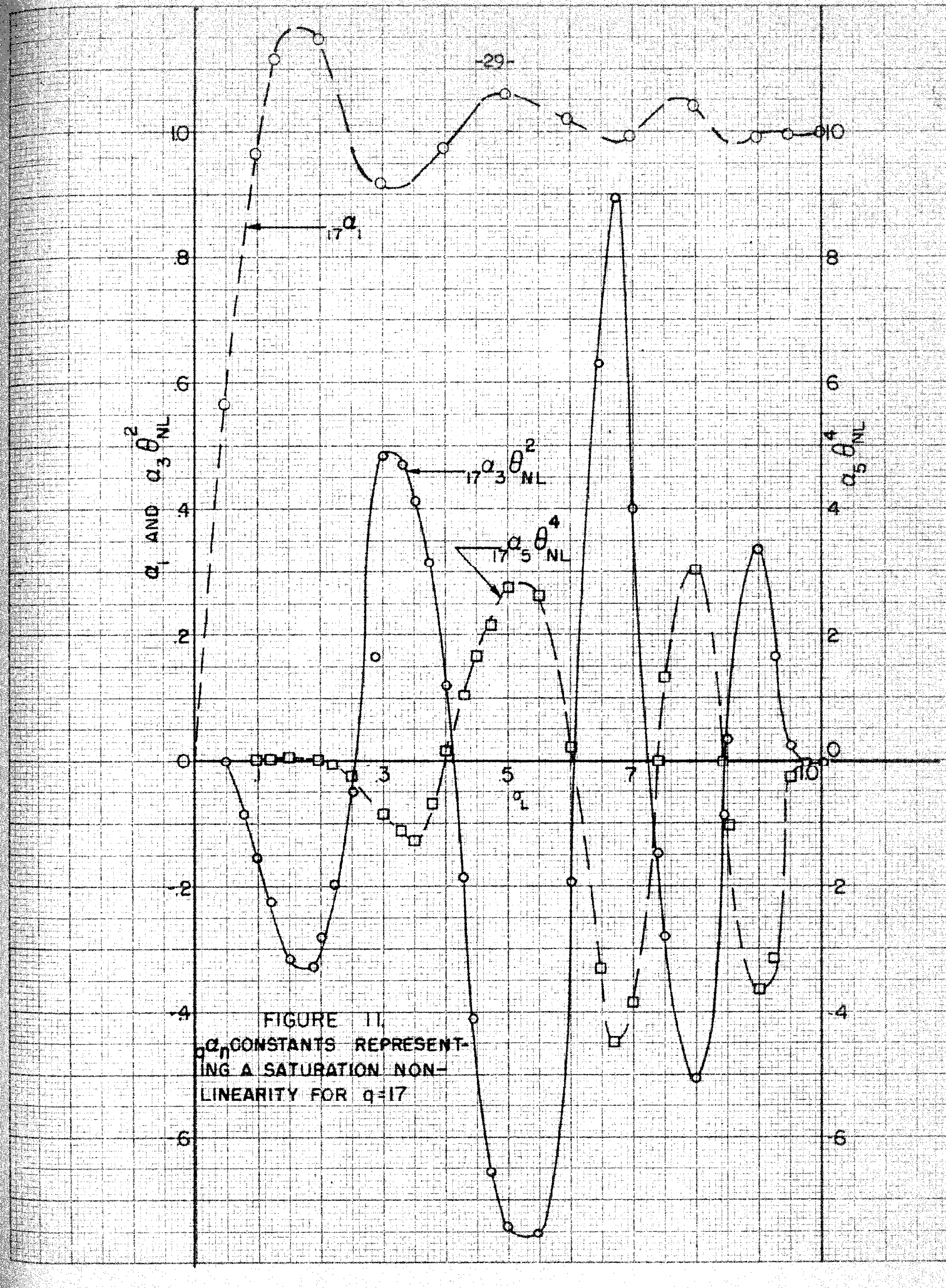


FIGURE 11.
$_q a_n$ CONSTANTS REPRESENTING A SATURATION NON-LINEARITY FOR q=17

exception of the α_n's which are functions of the normalizing factor. The α_n constants directly determine the magnitude of the solution of each successive equation and hence the convergence of the assumed series solution to the nonlinear equation. We may compute these convergence determining constants α_n from the expression

$$\alpha_n = \frac{C_n}{NF^{n-1}} = \frac{C_n \sigma_L^{n-1}}{\Theta_{NL}^{n-1}} \tag{12}$$

and the curves of C_n given in Section II. Figure 11 presents α_1, α_3 and α_5 as a function of σ_L for q = 17. The α_n factors for $n > 5$ have the same characteristics as n = 3 and 5, namely that the α_n increase in magnitude as σ_L increases above the minimum value for which $\alpha_1 = C_1 = 1$. Hence, for a given time function Θ_1 in equation 30, the degree of convergence of the series $\sum_n \Theta_n$ decreases as σ_L increases (NF decreases). It should be noted that the expected unlimited convergence at $\sigma_L = 1$ or linear condition is achieved since here all of the $\alpha_n = 0$ for $n > 1$ and $\alpha_1 = 1$. Although the α_n's oscillate with σ_L in the region where α_1 varies about unity, valid solutions for the nonlinear equation can be found for a normalizing factor which causes σ_L to lie anywhere in this region. However, maximum solution accuracy with the fewest number of successive equations results when the α_n's are at or near the peaks of their oscillations.

Since we have shown that the convergence for a given system decreases as NF decreases, it becomes possible to define a maximum magnitude of F(t)+I which will just permit convergence at the maximum NF for which $\alpha_1 = 1$. This, then constitutes an upper limit of F(t)+I for which the prime equation of (30) can remain the same as that for the linear system and hence restriction b' be satisfied. If the magnitude of F(t)+I increases beyond this limit, the convergence criterion requires that α_1 decrease so that the accompanying reduction in the α_n's permits a convergent series. Although the restriction b' must be violated in this region in order that the necessary convergence be assured, it is of relatively less importance since the transient solution is within the saturated region almost entirely except near the steady state.

Hence, it is seen that the convergence criterion provides a quantitative measure of the system conditions under which the closed loop will experience the gain reduction that is obviously the effect of a saturation type nonlinearity. In fact, the convergence criterion can be used to determine the effective gain reduction as a function of the magnitude of the input function or other parameter and the nonlinear saturation limit. In this manner, the study of the instability or stability brought about by the nonlinearity to an otherwise stable or unstable linear system should be possible. This type of study is performed with satisfactory success in Section VIII.

In order that this convergence criterion be unique, it must result in a consistent determination of the value of α_1 for convergence in this region of $0 < \alpha_1 \leqq 1$ which is independent of q, the number of Legendre polynomials included in the expansion. The conditions for this to be true may be stated as follows:

As n increases

$$\left(\frac{{}_q\Theta_n}{{}_q\Theta_{n-2}}\right)_{{}_q\alpha_1=a} = \left(\frac{{}_{q'}\Theta_n}{{}_{q'}\Theta_{n-2}}\right)_{{}_{q'}\alpha_1=a} \tag{31}$$

where $q' < q$

$0 < a \leqq 1$

Now $({}_q\alpha_n)_{{}_q\alpha_1=a} \neq ({}_{q'}\alpha_n)_{{}_{q'}\alpha_1=a}$ since the additive contribution to the high n α_n's as q increases is considerably greater than that to the low n constants. Hence, an identical set of equations is impossible. However, a consistent convergence determination can exist within the factor K if

$$({}_q\alpha_n)_{{}_q\alpha_1=a} = K^{\frac{n-1}{2}} ({}_{q'}\alpha_n)_{{}_{q'}\alpha_1=a} \tag{32}$$

This can be seen by substitution of (32) in (30), giving

$$\frac{{}_q\Theta_3}{{}_{q'}\Theta_3} = \frac{{}_q\alpha_n A(D)\,\Theta_1^3}{{}_{q'}\alpha_n A(D)\,\Theta_1^3} = K$$

$$\frac{{}_q\Theta_5}{{}_{q'}\Theta_5} = \frac{A(D)[{}_q\alpha_5\,\Theta_1^5 + 3\,{}_q\alpha_3\,{}_q\Theta_3\,\Theta_1^2]}{A(D)[{}_{q'}\alpha_5\,\Theta_1^5 + 3\,{}_{q'}\alpha_3\,{}_q\Theta_3\,\Theta_1^2]} = \frac{A(D)[K^2{}_{q'}\alpha_5\Theta_1^5 + 3\,{}_{q'}\alpha_3 K^2{}_{q'}\Theta_3\,\Theta_1^2]}{A(D)[{}_{q'}\alpha_5\,\Theta_1^5 + 3\,{}_{q'}\alpha_3\,{}_q\Theta_3\,\Theta_1^2]} = K^2 \tag{33}$$

$$\vdots$$

which results in

$$\left(\frac{{}_q\Theta_n}{{}_q\Theta_{n-2}}\right)_{{}_q\alpha_1=a} = K\left(\frac{{}_{q'}\Theta_n}{{}_{q'}\Theta_{n-2}}\right)_{{}_{q'}\alpha_1=a} \tag{34}$$

where $K \rightarrow 1$ as q' increases. Though considerable effort was made, no formal manner of proving or disproving this was found. Hence, we must assume for the present the validity of equation 34 from the following demonstration.

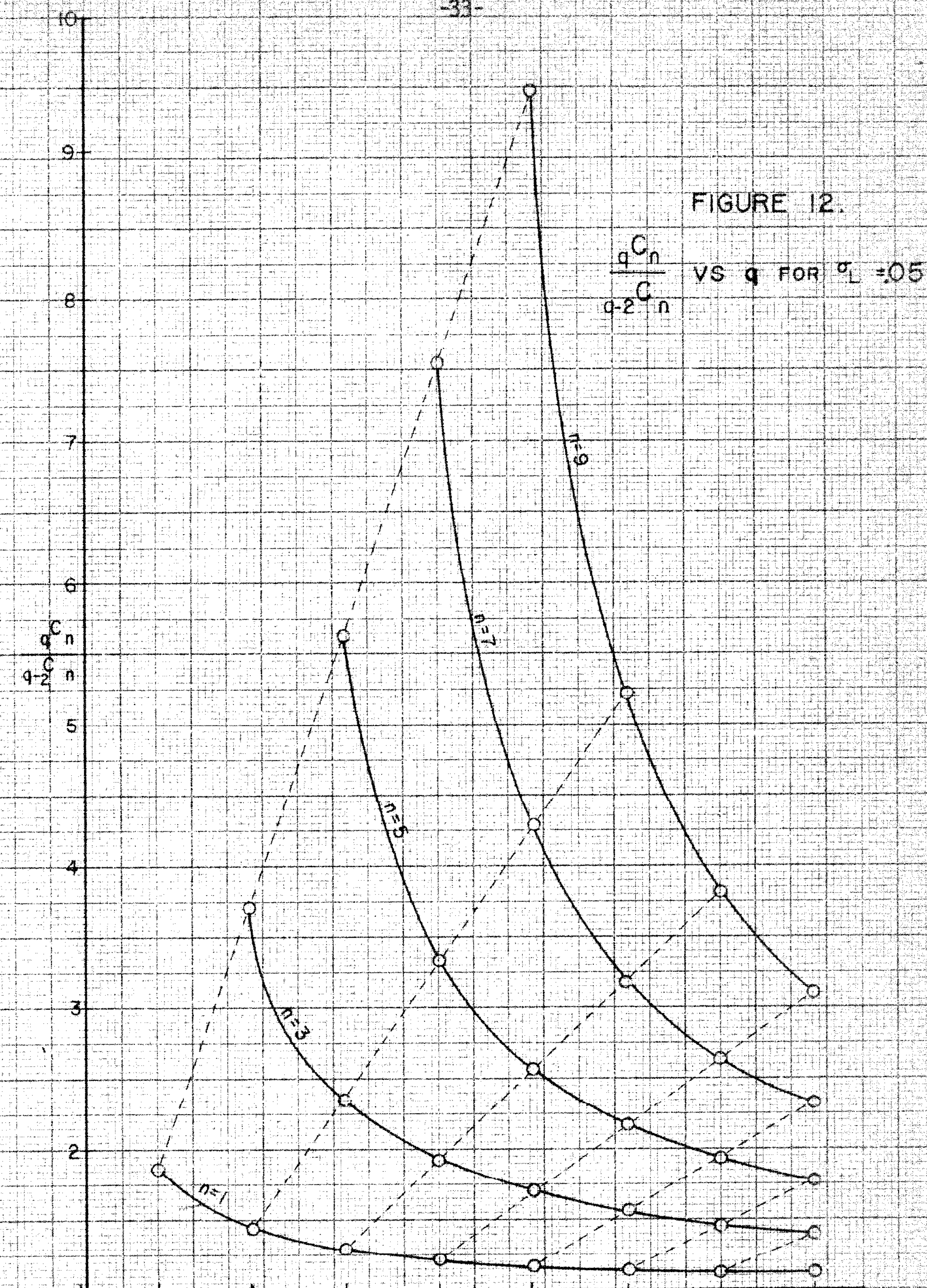


FIGURE 12.
$\frac{{}_qC_n}{{}_{q-2}C_n}$ VS q FOR σ_L =.05
n=1
n=3
n=5
n=7
n=9
1
2
3
4
5
6
7
8
9
10
3
5
7
9
11
13
15
17
NUMBER OF TERMS IN THE LEGENDRE EXPANSION, q

From the curves of C_n the ratios of $\frac{{}_qC_n}{{}_{q-2}C_n}$ as a function of the number of Legendre polynomials included in the expansion may be plotted as in Figure 12 for a value of σ_L in the linear portion of the C_1 curve between 0 and 1. Since all of the C_n curves are linear in this region, this point is typical. It may be noted that a straight line can be drawn through appropriate points on the family of curves for n. From these lines one can easily extrapolate to define succeeding curves of the family as q increases and from which one may formulate

$$\frac{{}_qC_n}{{}_{q-2}C_n} = \delta_n + \Delta_n(n-1) \tag{35}$$

where as $q \to \infty$, $\delta_n \gtrsim 1$ and $\Delta_n \gtrsim 0$

Now

$$\Theta_{NL}^{n-1}\;{}_q\alpha_n = {}_qC_n\;{}_q\sigma_L^{n-1} \tag{12}$$

and hence

$$\frac{{}_q\alpha_n}{{}_{q-2}\alpha_n} = \frac{{}_qC_n}{{}_{q-2}C_n}\left(\frac{{}_q\sigma_L}{{}_{q-2}\sigma_L}\right)^{n-1} \tag{36}$$

Since from the C_n curves

$${}_qC_n = {}_qK_n \cdot \sigma_L \tag{37}$$

then

$$\frac{{}_qC_n}{{}_{q-2}C_n} = \frac{{}_qK_n}{{}_{q-2}K_n} \tag{38}$$

and

$$\frac{({}_q\sigma_L)_{{}_q\alpha_1=a}}{({}_{q-2}\sigma_L)_{{}_{q-2}\alpha_1=a}} = \frac{\dfrac{a}{{}_qK_1}}{\dfrac{a}{{}_{q-2}K_1}} = \frac{{}_{q-2}K_1}{{}_qK_1} = \frac{1}{\delta_1} \tag{39}$$

Therefore

$$\frac{\left({}_{q}\alpha_n\right)_{{}_{q}\alpha_1 = a}}{\left({}_{q-2}\alpha_n\right)_{{}_{q-2}\alpha_1 = a}} = \frac{{}_{q}C_n}{{}_{q-2}C_n}\left(\frac{1}{\delta_1}\right)^{n-1} = \left[\left(\frac{{}_{q}C_n}{{}_{q-2}C_n}\right)^{\frac{2}{n-1}} \cdot \frac{1}{\delta_1^2}\right]^{\frac{n-1}{2}} \qquad (40)$$

$$= K^{\frac{n-1}{2}}$$

where $K \rightarrow 1$ AS $q \rightarrow \infty$

VI. TECHNIQUES FOR SOLVING THE SET OF NONLINEAR EQUATIONS

In effecting the solution of the set of nonlinear equations given in equation (29), several methods were investigated. Of these, two were found best suited, the Laplace Transformation and the P-Transformation. The first method gives, of course, an exact solution to each equation of the set and is accordingly very difficult to use, the complexity of the method increasing with each successive equation of the set. The P-Transformation, on the other hand, is an approximate numerical method which is much quicker than the Laplace Transformation. This is especially true for the successive equations since the complexity of the P-Transformation is essentially constant for each equation of the set. A general discussion of each method will be made here prior to the application of both to a specific example in the next section.

The Laplace Transformation method of solving linear differential equations is in sufficient general use that the principles of the method will be omitted. Thus transforming the prime equation of (29) we have

$$\mathcal{L}[a_1(D)\,\theta_1(t)] = \mathcal{L}[F(t)] + I \qquad (41)$$

hence

$$a_1(s)\,\theta_1(s) = F(s) + I \qquad (42)$$

from which

$$\theta_1(t) = \mathcal{L}^{-1}\left[\frac{F(s)}{a_1(s)} + \frac{I}{a_1(s)}\right] \qquad (43)$$

The second approximation equation may then be transformed giving

$$a_1(s)\,\Theta_2(s) = -A(s)\left\{\alpha_2 \mathcal{L}\left[\Theta_1(t)\right]^2\right\} \tag{44}$$

and

$$\Theta_2(t) = -\alpha_2 \mathcal{L}^{-1}\left\{\frac{A(s)}{a_1(s)} \mathcal{L}\left[\Theta_1(t)\right]^2\right\} \tag{45}$$

where $\left[\Theta_1(t)\right]^2$ must be linearized in order that the Laplace Transformation may be made.

The process continues in like manner, the forcing function for each successive approximation equation being generated by a linearized combination of powers of the preceding solutions in the time domain. In general, the characteristic of the prime equation may be factored as

$$a_1(s) = \prod_{r=1}^{r} (s - s_r) \tag{46}$$

which gives a prime equation solution which may be written as

$$\Theta_1(t) = K\,F(t) + \sum_{r=1}^{r} K_r e^{s_r t} + K_I I \tag{47}$$

where S_r will be complex in the general case.

In this form raising the solution to a given power may be facilitated by means of the multinomial expansion theorem. Given:

$$\left(1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots\right)^n = \left(1 + B_1 x + B_2 x^2 + B_3 x^3 + \cdots\right) \tag{48}$$

where the b_i's are constants

then

$$\begin{aligned}
B_1 &= n_1 b_1 \\
B_2 &= n_1 (b_2 + b_1^2) \\
B_3 &= n_1 b_3 + 2n_2 b_1 b_2 + n_3 b_1^3 \\
B_4 &= n_1 b_4 + 2n_2 \left(\tfrac{1}{2} b_2^2 + b_1 b_3\right) + 3n_3 b_1^2 b_2 + n_4 b_1^4
\end{aligned} \tag{49}$$

$$B_5 = n_1 b_5 + 2n_2(b_1 b_4 + b_2 b_3) + 3n_3(b_1 b_2^2 + b_1^2 b_3)$$
$$+ 4n_4 b_1^3 b_2 + n_5 b_1^5$$

etc.

where $n_r = \dfrac{n!}{(n-r)!r!}$ = binomial coefficient

Since the accuracy required in most engineering problems does not exceed that attainable by practical techniques, an approximate numerical method may be used in solving the set of equations describing the nonlinear system. In a paper titled "A New Linear Operational Calculus", Frank W. Bubb (Ref. 4) presents an operational method applicable to the approximate analysis of linear physical systems. In contrast to the methods based upon the transform theories of Fourier and Laplace, this calculus does not depart from the time domain. Since it is not our purpose to develop this method but only to present a usable account of it, the following brief resume' will be made without proof. For proofs of the theorems and additional details the reader is referred to Reference 4.

Due to the fact that the superposition, convolution or faltung integral embraces all of linear system analysis and synthesis, it provides a basis for any form of operational calculus. Hence, the following definitions are made such that a simple relation exists between the P-transforms of the three functions involved in the convolution integral

$$H(t) = \int_{-\infty}^{t} F(\tau)\, M(t-\tau)\, d\tau \qquad (50)$$

Given a time function F(t) and a set of its equally spaced ordinates $F(nv) = F_n$, we define a polynomial

$$\tilde{F}(x) = \sum_{n=-K}^{L} F_n x^n \tag{51}$$

called the Polynomial or P-Transform of F(t). One can think here of the n as an index for the time ordinate nv. The inverse transform is defined as the interpolation

$$f(t) = P^{-1}\tilde{F}(x) = \sum_{n=-K}^{L} F_n L(t-nv) \tag{52}$$

where L(t - nv) is an interpolating function of the cardinal type such as (but not limited to):

$$L(t-nv) = \frac{\sin \frac{\pi}{v}(t-nv)}{\frac{\pi}{v}(t-nv)} \tag{53}$$

The time function f(t) is an approximation to F(t) which for continuous F(t) becomes exact in the limit as the time interval $v \to 0$. The meaning of this interpolation is fully explained in the reference mentioned and will be omitted here.

The principal theorem concerns the transformation of the superposition integral of equation (50). The P-transform of this integral being given by

$$\tilde{H}(x) = v\ \tilde{F}(x) \cdot \tilde{M}(x) \tag{54}$$

It is noted that when v = 1, the relation between (50) and (54) is of precisely the same form as in Laplace transform theory. Equation (54) may be written in more explicit form as

$$\sum_{n=-N}^{M} H_n x^n = v \sum_{j=-J}^{I} F_j x^j \sum_{k=-K}^{L} M_k x^k \tag{55}$$

where M = I + L and N = J + K.

By ordinary polynomial multiplication of the two polynomials on the right-hand side of (55), one finds when the coefficients of equal indices x^n on each side of equation (55) are equated the recurrence relationship

$$H_n = v \sum_{j=-J}^{I} F_j M_{n-j} \qquad (56)$$

This expresses then that the ordinates of H(t) at the respective time points nv are H_n which can be found by the combination of ordinates of F(t) and M(t) at the time points jv and (n - j)v respectively as stated in equation (56). From this one fairs in a time curve h(t) through these points and accepts this curve as an adequate interpolation of the required response H(t).

In connection with the application of this method certain theorems are very useful and will be stated here without proof.

Th. 1 If $H(t) = F(t) + K(t)$ (57)

Then $\widetilde{H}(x) = \widetilde{F}(x) + \widetilde{K}(x)$

Th. 2 If $H(t) = cF(t)$ (58)

Then $\widetilde{H}(x) = c\widetilde{F}(x)$

Th. 3 If $H(t) = F(t-mv)$ (59)

Then $\widetilde{H}(x) = x^m \widetilde{F}(x)$

Th. 4 If $H(t) = F(-t)$ (60)

Then $\widetilde{H}(x) = \sum_n F(nv)\ x^{-n}$

Th. 5 If $H(t) = \Delta F(t) = F(t+v) - F(t)$

Then $\widetilde{H}(x) = \left(\frac{1}{x} - 1\right)\widetilde{F}(x)$ (61)

Th. 6 If $H(t) = t\,F(t)$

$$\text{Then } \tilde{H}(x) = xv\,\frac{d\tilde{F}(x)}{dx} \tag{62}$$

Th. 7 If $H(t) = \frac{F(t)}{t+v}$

$$\text{Then } \tilde{H}(x) = \frac{1}{vx}\int_0^x \tilde{F}(x)\,dx \tag{63}$$

$$\text{Where } \tilde{F}(x) = \sum_{n=0}^{\infty} F(nv)\,x^n$$

Th. 8 If $H(nv) = \int_0^{nv} F(t)\,dt$

$$\text{Then } \tilde{H}(x) = \frac{v(1+x)}{2(1-x)}\tilde{F}(x) - \frac{v}{2}F_0\sum_{k=1}^{\infty} x^k \tag{64}$$

$$\text{Where } \tilde{F}(x) = \sum_{k=0}^{\infty} F_k\,x^k$$

$$\text{And } \tilde{H}(x) = \sum_{n=0}^{\infty} H_n x^n$$

Th. 9 If $\dot{H}(t) = \frac{dH(t)}{dt}$

$$\text{Then } \tilde{\dot{H}}(x) = \frac{2(1-x)}{v(1+x)}\tilde{H}(x) \tag{65}$$

$$\text{Where } \tilde{\dot{H}}(x) = \sum_{k=0}^{\infty} \dot{H}(kv)\,x^k$$

The detailed application of this method is best illustrated by an example which will be postponed until the next section. However, a general description of the procedure may be made in the following steps:

1. Apply to the differential equation the definite integration $\int_0^t (\ )dt$ until it is an integral equation. This introduces the initial conditions and puts the equation into the proper form for the P-transformation.

2. Apply to the result of Step 1 the P-transformation making use of Theorems 1 to 8.

3. By ordinary polynomial multiplication manipulate the equation until only zero or positive powers of x appear on each side of the equation. Equate the coefficients of like powers of x thus forming a recurrence formula for the time ordinates of the dependent variable, e.g., $H(x) = H_0 + H_1 x + H_2 x + \ldots$ etc.

4. Pick off the time ordinates H_0, H_1, H_2, H_3, etc.; plot at the successive points 0, v, 2v, 3v, etc.; fair a curve through these points to obtain an approximation to the required solution H(t).

VII. SOLUTION OF A SECOND ORDER SYSTEM WITH A SATURATION TYPE NONLINEARITY

Although, as has been previously mentioned, the second order system is almost a trivial case, its simplicity does enhance its use in illustrating the technique for analyzing nonlinear control loops. The technique is applicable to systems of any order and its extension to the general case will be obvious from this simple study. In addition, this second order system will be used to demonstrate and compare the Laplace Transform and P-Transform solution methods described in the previous section.

Consider the following system containing a saturation

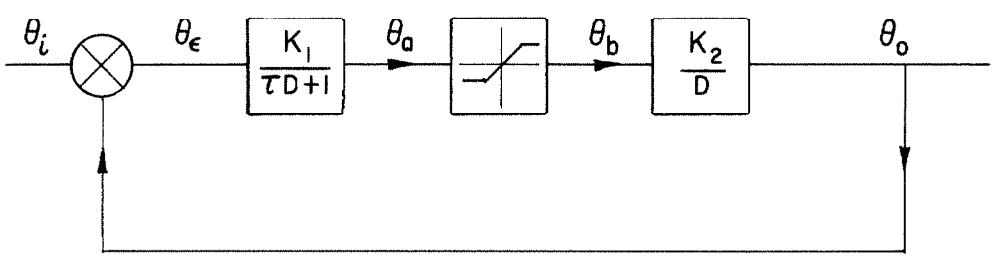


Figure 13

nonlinearity. The loop equations are

$$(\tau D+1)\Theta_a = K_1(\Theta_i - \Theta_o) \qquad (66)$$

$$\Theta_b = \sum_{n=1}^{8} \alpha_n \Theta_a^n \qquad (67)$$

$$\Theta_o = \frac{K_2}{D}\Theta_b \qquad (68)$$

Solving for the input to the nonlinearity θ_a from (66) and (68) gives

$$D(\tau D+1)\Theta_a = K_1 D\Theta_i - K_1 D\Theta_o = K_1 D\Theta_i - K_1 K_2 \Theta_b \qquad (69)$$

Substituting (67) in (69) and rearranging

$$(\tau D^2 + D)\theta_a + K_1 K_2 \sum_{n=1}^{8} \alpha_n \theta_a^n = K_1 D \theta_i \tag{70}$$

Since from Section II the expansion of the saturation nonlinearity yields only odd n α_n's, the following set of equations are obtained from equation (70) when the infinite series solution technique of Section IV is applied. Thus

$$\theta_a = \sum_{n=1}^{\infty} \theta_n \tag{71}$$

where the θ_n are defined by the equations

$$\begin{aligned}
(\tau D^2 + D + K_1 K_2 \alpha_1)\theta_1 &= K_1 D \theta_i(t) \\
(\tau D^2 + D + K_1 K_2 \alpha_1)\theta_3 &= -K_1 K_2 \alpha_3 \theta_1^3 \\
(\tau D^2 + D + K_1 K_2 \alpha_1)\theta_5 &= -K_1 K_2 \left[\alpha_5 \theta_1^5 + 3\alpha_3 \theta_1^2 \theta_3\right] \\
(\tau D^2 + D + K_1 K_2 \alpha_1)\theta_7 &= -K_1 K_2 \left[\alpha_7 \theta_1^7 + 5\alpha_5 \theta_1^4 \theta_3 + 3\alpha_3 (\theta_1 \theta_3^2 + \theta_1^2 \theta_5)\right] \\
&\vdots
\end{aligned} \tag{72}$$

The equation set (72) may be generalized in terms of a ratio of the magnitude of the input function θ_i and the saturation limit θ_{NL} by substituting for α_n and $\theta_i(t)$ as

$$\alpha_n = \frac{C_n \sigma_L^{n-1}}{\theta_{NL}^{n-1}} \tag{12}$$

and $\theta_i(t) = |\theta_i| \cdot F_i(t)$ (73)

Hence

$$\begin{aligned}
(\tau D^2 + D + K_1 K_2 C_1)\left(\frac{\theta_1}{|\theta_i|}\right) &= K_1 D F_i(t) \\
(\tau D^2 + D + K_1 K_2 C_1)\left(\frac{\theta_3 \theta_{NL}^2}{|\theta_i|^3}\right) &= -K_1 K_2 C_3 \sigma_L^2 \left(\frac{\theta_1}{|\theta_i|}\right)^3 \\
(\tau D^2 + D + K_1 K_2 C_1)\left(\frac{\theta_5 \theta_{NL}^4}{|\theta_i|^5}\right) &= -K_1 K_2 \left[C_5 \sigma_L^4 \left(\frac{\theta_1}{|\theta_i|}\right)^3 + 3C_3 \sigma_L^2 \left(\frac{\theta_1}{|\theta_i|}\right)^2 \left(\frac{\theta_3 \theta_{NL}^2}{|\theta_i|^3}\right)\right] \\
&\vdots
\end{aligned} \tag{74}$$

In this way a single calculation of each equation in terms of the generalized parameters enables one to obtain a transient response of the system for any value of $\frac{|\theta_i|}{\theta_{NL}}$ up to the convergence limit. This applies, of course, for a given value of C_1. Now we have already demonstrated that the convergence limit for $\frac{|\theta_i|}{\theta_{NL}}$ increases as C_1 decreases. Thus, the starting point of an analysis logically begins by defining the ratio of $\frac{|\theta_i|}{\theta_{NL}}$ for which the system departs from a prime equation equal to the linear system equation, i.e., $C_1 = 1$. This occurs at the minimum σ_L for which $C_1 = 1$ and affords the maximum amount of information from a single solution of the set of equations. The study of the system behavior under larger input to saturation limit ratios may then be accomplished by reducing C_1 until a convergent series results for that ratio.

In order to demonstrate the detailed application of this method, let us study the quadratic system of Figure 13 when subject to a step command, the system parameters being as follows:

$K_1 = 1.99$

$\tau = 0.199$

$K_2 = 3.944$

Then choosing $q = 17$ in the Legendre polynomial expansion of the nonlinearity, $C_1 = 1$ at a minimum σ_L of .102. The higher C_n's at this σ_L are

$C_3 = -15.36$

$C_5 = 140$

$c_7 = -630$

$c_9 = 1890$

Let us calculate the solutions of $\frac{\theta_1}{|\theta_i|}$ and $\frac{\theta_3\theta^2_{NL}}{|\theta_i|^3}$ for the step input command, θ_i, by the Laplace Transform and P-Transform methods in order to compare the accuracy and time involved for each.

a. Laplace transformation method.

The Laplace transformed equation for $\frac{\theta_1}{|\theta_i|}$ and $\frac{\theta_3\theta^2_{NL}}{|\theta_i|^3}$ are, assuming zero initial conditions

$$(\tau s^2 + s + K_1K_2C_1)\frac{\theta_1(s)}{|\theta_i|} = K_1 \tag{75}$$

$$(\tau s^2 + s + K_1K_2C_1)\frac{\theta_3(s)\theta^2_{NL}}{|\theta_i|^3} = -K_1K_2C_3\sigma_L^2\mathcal{L}\left(\frac{\theta_1(t)}{|\theta_i|}\right)^3 \tag{76}$$

The solution of the prime equation (75) is evidently

$$\frac{\theta_1(t)}{|\theta_i|} = \frac{K_1}{\tau\beta}e^{-\gamma t}\sin\beta t \tag{77}$$

where $\gamma = \frac{1}{2\tau}$

$$\beta = \frac{\sqrt{4K_1K_2C_1\tau - 1}}{2\tau}$$

Then linearizing $\left(\frac{\theta_1(t)}{|\theta_i|}\right)^3$ so that it may be transformed

$$\left(\frac{\theta_1(t)}{|\theta_i|}\right)^3 = \left(\frac{K_1}{\tau\beta}\right)^3 e^{-3\gamma t}\sin^3\beta t = \left(\frac{K_1}{\tau\beta}\right)^3\frac{e^{-3\gamma t}}{4}(3\sin\beta t - \sin 3\beta t) \tag{78}$$

Hence equation (76) becomes

$$(\tau s^2 + s + K_1K_2C_1)\frac{\theta_3(s)\theta^2_{NL}}{|\theta_i|^3} = -\frac{K_1^4K_2C_3\sigma_L^2}{4\tau^3\beta^3}\left[\frac{3\beta}{(s+3\gamma)^2+\beta^2} - \frac{3\beta}{(s+3\gamma)^2+9\beta^2}\right] \tag{79}$$

the solution of which may be written

$$\frac{\theta_3(t)\,\theta_{NL}^2}{|\theta_i|^3} = \frac{K_1^4 K_2 C_3 \sigma_L^2 \Lambda_1}{4\tau^4\beta^3}\left[3e^{-\gamma t}\left\{\sin(\beta t+\varphi_1) - \frac{\Lambda_3}{\Lambda_1}\sin(\beta t+\varphi_3)\right\} + e^{-3\gamma t}\left\{\sin(3\beta t+\varphi_2) - \frac{3\Lambda_3}{\Lambda_1}\sin(\beta t+\varphi_4)\right\}\right] \tag{80}$$

where

$$\Lambda_1 = \frac{1}{4[\gamma^4+4\beta^4+5\gamma^2\beta^2]^{\frac{1}{2}}}$$

$$\Lambda_3 = \frac{1}{4[\gamma^4+\gamma^2\beta^2]^{\frac{1}{2}}}$$

$$\varphi_1 = -\tan^{-1}\frac{\gamma\beta}{\gamma^2+2\beta^2}$$

$$\varphi_2 = -\tan^{-1}\frac{-3\gamma\beta}{\gamma^2-2\beta^2}$$

$$\varphi_3 = -\varphi_4 = -\tan^{-1}\frac{\beta}{\gamma} \quad .$$

Further terms may be found in a similar manner. For the case at hand substitution of the numerical values into equations (77) and (80) gives

$$\frac{\theta_1(t)}{|\theta_i|} = 17.37\, e^{-2.512t}\sin 5.756t \tag{81}$$

$$\frac{\theta_3(t)\,\theta_{NL}^2}{|\theta_i|^3} = -2.0304\Big(3e^{-2.512t}\big[\sin(5.756t-11.27^\circ) - 4.69\sin(5.756t-66.43^\circ)\big] + e^{-7.536t}\big[\sin(17.267t+144.1^\circ) - 14.07\sin(5.756t+66.43^\circ)\big]\Big) \tag{82}$$

b. P-Transform Method.

Applying the procedure set down in Section VI to the prime equation of (74)

$$(\tau D^2 + D + K_1K_2C_1)\left(\frac{\theta_1(t)}{|\theta_i|}\right) = K_1 D F_i(t) \tag{74a}$$

we twice integrate both sides of the equation from 0 to t.

$$\left[\tau D+1+K_1K_2C_1\int_0^t dt\right]\frac{\theta_1(t)}{|\theta_i|} = K_1F_i(t)+\tau\frac{\dot{\theta}_1(0)}{|\theta_i|}+\frac{\theta_1(0)}{|\theta_i|} \tag{83}$$

$$\left[\tau+\int_0^t dt+K_1K_2C_1\int_0^t\int_0^t dt^2\right]\frac{\theta_1(t)}{|\theta_i|} = K_1\int_0^t F_i(t)dt+\left(\frac{\tau\dot{\theta}_1(0)}{|\theta_i|}+\frac{\theta_1(0)}{|\theta_i|}\right)\int_0^t dt+\tau\frac{\theta_1(0)}{|\theta_i|} \tag{84}$$

For the present case both the position and velocity initial conditions will be assumed zero and thus applying Theorem 8 we have

$$\left[\tau+\frac{v}{2}\frac{(1+x)}{(1-x)}+K_1K_2C_1\frac{v^2}{4}\frac{(1+x)^2}{(1-x)^2}\right]\frac{\tilde{\theta}_1(x)}{|\theta_i|} = K_1\frac{v}{2}\frac{(1+x)}{(1-x)}\tilde{F}_i(x)-\frac{v}{2}\frac{x}{1-x}\tilde{F}_i(0) + \frac{v^2}{4}\frac{(1+x)x}{(1-x)^2}\frac{\tilde{\theta}_1(0)}{|\theta_i|} \tag{85}$$

For $F_i(t)$ equal to a unit step $S_1(t) = 1$ for $t \geq 0$ and $S_1(t) = 0$ for $t<0$ then by equation (51)

$$\tilde{S}_i(x) = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \tag{86}$$

Now introducing (86) into (85) and noting that $\dfrac{\tilde{\theta}_1(0)}{|\theta_1|} = 0$ by initial condition the equation may be manipulated to give

$$\left[4\tau(1-x)^2+2v(1-x^2)+K_1K_2C_1v^2(1+x)^2\right]\frac{\tilde{\theta}_1(x)}{|\theta_i|} = 2vK_1+2v(K_1-1)x+2vx^2 \tag{87}$$

Substituting for $\dfrac{\tilde{\theta}_1(x)}{|\theta_1|}$ according to the P-transform definition and rearranging yields

$$\left[1+\frac{2(K_1K_2C_1v^2-4\tau)x}{(4\tau+2v+K_1K_2C_1v^2)}+\frac{(4\tau-2v+K_1K_2C_1v^2)}{(4\tau+2v+K_1K_2C_1v^2)}x^2\right]\sum_{n=0}^{\infty}\frac{\theta_{1_n}x^n}{|\theta_i|} = \frac{2vK_1+2v(K_1-1)x+2vx^2}{(4\tau+2v+K_1K_2C_1v^2)} \tag{88}$$

or

$$\left[1+bx+cx^2\right]\sum_{n=0}^{\infty}\frac{\theta_{1_n}x^n}{|\theta_i|} = a_0+a_1x+a_2x^2 \tag{88'}$$

Now equating like powers of x and introducing the numerical values of the system parameters we have for $v = .05$

$$\frac{\theta_{1_0}}{|\theta_i|} = a_0 = .2173$$

$$\frac{\theta_{1_1}}{|\theta_i|} = a_1 - b\frac{\theta_{1_0}}{|\theta_i|} = .1082 + 1.696\frac{\theta_{1_0}}{|\theta_i|}$$

$$\frac{\theta_{1_2}}{|\theta_i|} = a_2 - b\frac{\theta_{1_1}}{|\theta_i|} - c\frac{\theta_{1_0}}{|\theta_i|} = .1092 + 1.696\frac{\theta_{1_1}}{|\theta_i|} - .7816\frac{\theta_{1_0}}{|\theta_i|}$$

$$\frac{\theta_{1_3}}{|\theta_i|} = -b\frac{\theta_{1_2}}{|\theta_i|} - c\frac{\theta_{1_1}}{|\theta_i|} = 1.696\frac{\theta_{1_2}}{|\theta_i|} - .7816\frac{\theta_{1_1}}{|\theta_i|} \tag{89}$$

$$\vdots$$

$$\frac{\theta_{1_n}}{|\theta_i|} = -b\frac{\theta_{1_{n-1}}}{|\theta_i|} - c\frac{\theta_{1_{n-2}}}{|\theta_i|} = 1.696\frac{\theta_{1_{n-1}}}{|\theta_i|} - .7816\frac{\theta_{1_{n-2}}}{|\theta_i|}$$

from which we may compute the $\frac{\theta_{1_n}}{|\theta_i|} = \frac{\theta_1(nv)}{|\theta_i|}$ and hence obtain the time response of the system. It will be noted that $\frac{\theta_{1_0}}{|\theta_i|} = a_0$ which is not zero as $\frac{\theta_1(0)}{|\theta_i|}$. The value of $\frac{\theta_{1_0}}{|\theta_i|}$ is due to the interpolation function and is for this singular point not equal to the value of $\frac{\theta_1(nv)}{|\theta_i|}$ at the corresponding point in time.

Applying the above procedure to the second equation of the set we find since the initial conditions for all equations succeeding the first are always zero

$$\left[T + \frac{v}{2}\frac{(1+x)}{(1-x)} + K_1K_2C_1\frac{v^2}{4}\frac{(1+x)^2}{(1-x)^2}\right]\frac{\tilde{\theta}_3(x)\,\theta_{NL}^2}{|\theta_i|^3} = -K_1K_2C_3\sigma_L^2\frac{v^2}{4}\frac{(1+x)^2}{(1-x)^2}\frac{\tilde{\theta}_1^3(x)}{|\theta_i|^3} \tag{90}$$

where since in the time domain the product of two time functions is equal to the product of the values of the functions at identical points in time and because the P-transform remains in the time domain then

$$\frac{\tilde{\Theta}_1^3(x)}{|\Theta_i|^3} = \sum_{n=0}^{\infty} \frac{\Theta_1^3(nv)\,x^n}{|\Theta_i|^3} \tag{91}$$

Hence equation (90) may be put in the form of (88') as

$$\left(1+bx+cx^2\right)\sum_{n=0}^{\infty} \frac{\Theta_{3_n}\Theta_{NL}^2}{|\Theta_i|^3}\,x^n = -\frac{K_1K_2C_3\sigma_L^2 v^2(1+2x+x^2)}{(4\tau+2v+K_1K_2C_1v^2)} \sum_{n=0}^{\infty} \frac{\Theta_{1_n}^3 x^n}{|\Theta_i|^3} \tag{92}$$

where b and c are the same in (92) and (88'). Letting

$a_3 = \dfrac{-K_1K_2C_3\,\sigma_L^2 V^2}{(4\tau+2V+K_1K_2C_1V^2)}$ and equating like powers of x^n we obtain

the recurrence relationship

$$\frac{\Theta_{3_n}\Theta_{NL}^2}{|\Theta_i|^3} = a_3\left[\frac{\Theta_{1_n}^3}{|\Theta_i|^3} + 2\frac{\Theta_{1_{n-1}}^3}{|\Theta_i|^3} + \frac{\Theta_{1_{n-2}}^3}{|\Theta_i|^3}\right] - b\,\frac{\Theta_{3_{n-1}}\Theta_{NL}^2}{|\Theta_i|^3} - c\,\frac{\Theta_{3_{n-2}}\Theta_{NL}^2}{|\Theta_i|^3} \tag{93}$$

With the system values and v = .1 second this becomes

$$\frac{\Theta_{3_n}\Theta_{NL}^2}{|\Theta_i|^3} = .01168\left[\frac{\Theta_{1_n}^3}{|\Theta_i|^3} + 2\frac{\Theta_{1_{n-1}}^3}{|\Theta_i|^3} + \frac{\Theta_{1_{n-2}}^3}{|\Theta_i|^3}\right] + 1.337\frac{\Theta_{3_{n-1}}\Theta_{NL}^2}{|\Theta_i|^3} - .6278\frac{\Theta_{3_{n-2}}\Theta_{NL}^2}{|\Theta_i|^3} \tag{94}$$

where all quantities equal zero for zero or negative subscripts.

The succeeding equations of the set are solved in the same manner, each resulting in an equation identical to eq. (93) except in the bracketed forcing function expression.

From the application of the Laplace and P-Transformation methods to this quadratic example we may draw certain conclusions. First, in the solution of the linear prime equation the Laplace method requires finding the roots of the characteristic equation which for higher order systems can be extremely lengthy. The P-Transformation method avoids this and if the time increment v is chosen small enough, the straight line approximations to the actual response of time length v give sufficient accuracy. Consequently, the P-transform takes less time to give a solution even when small time increments are used. Secondly, for the solutions of the successive equations the Laplace method requires linearization of the product forcing functions before they may be transformed. This takes a prohibitive amount of time above the third successive equation. On the other hand, each equation after the first can be solved by the P-transform numerical method in the same order of time as the first. In general, one may increase the time increment v for the equations of the set after the first and obtain sufficient accuracy because of the more slowly rising forcing function for these equations as compared with that of the first equation due to the time filtering action of most physical control systems. Thus as would be expected, the numerical P-transform method is very much faster than the Laplace Transform method which gives an exact solution, and there remains only to compare the accuracy of the numerical method.

In Figures 14 and 15 are shown the solutions of the generalized equations of the set given by (74). For the $\frac{\theta_1}{|\theta_1|}$ and $\frac{\theta_3 \theta_{NL}^2}{|\theta_1|^3}$ solutions the solid curves are calculated by the Laplace transform method and the circled points by the P-transform method. The accuracy of the approximate numerical method is seen to be very good. The time interval used was v = .05 sec. for calculation of $\frac{\theta_1}{|\theta_1|}$ and

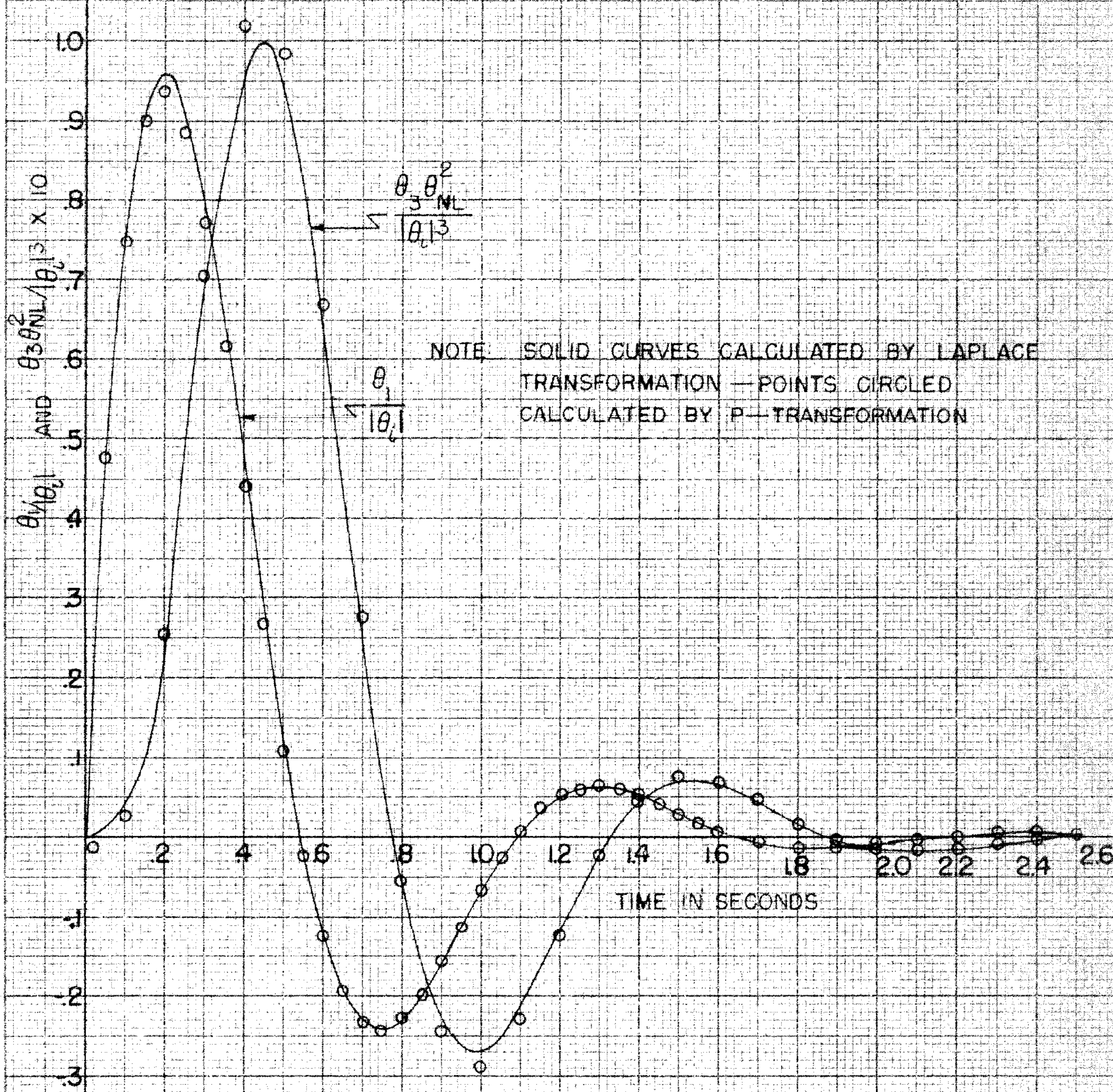


FIGURE 14.

GENERALIZED SOLUTIONS FOR QUADRATIC SYSTEM CONTAINING A SATURATION NONLINEARITY

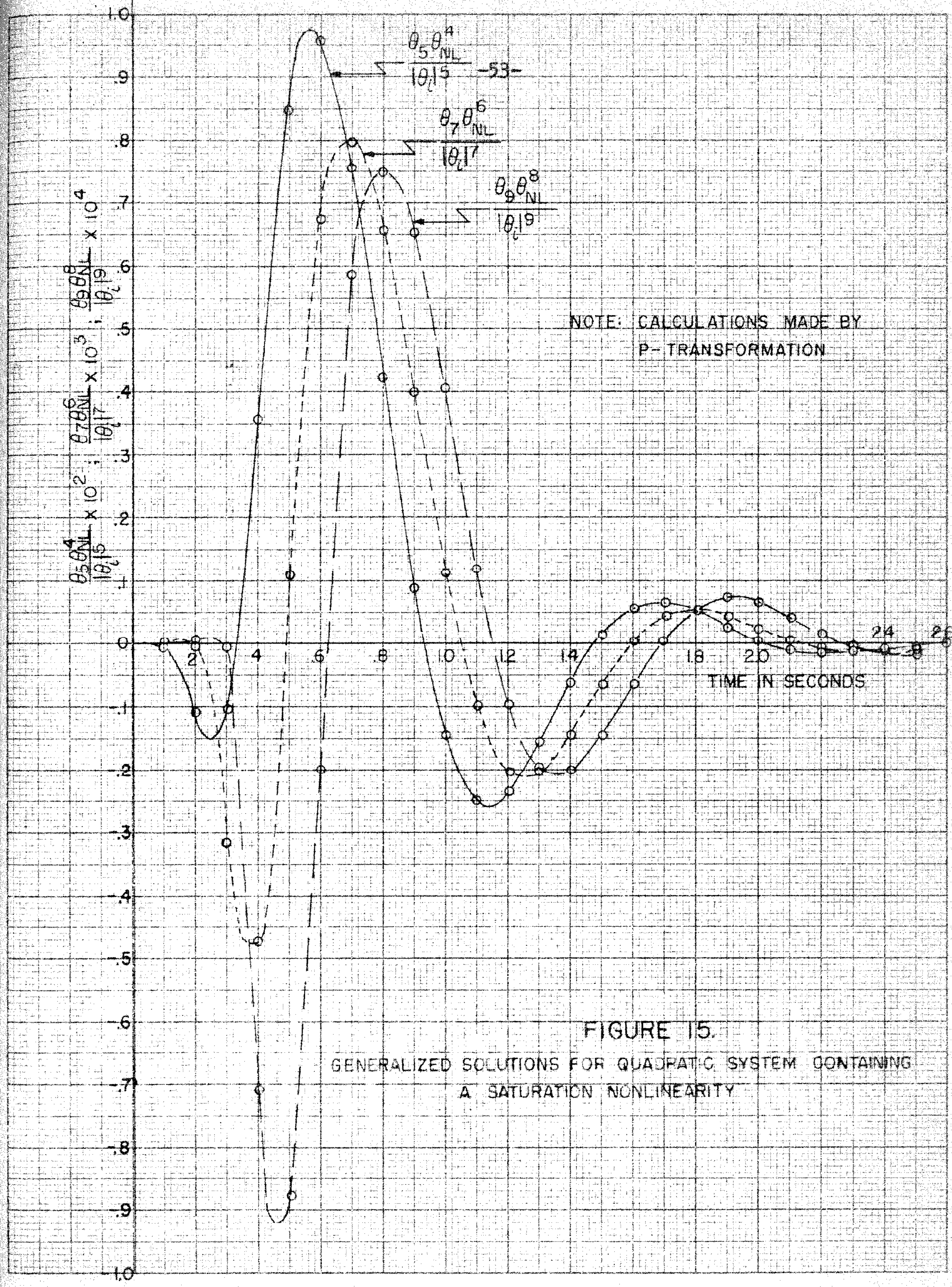


FIGURE 15.

GENERALIZED SOLUTIONS FOR QUADRATIC SYSTEM CONTAINING A SATURATION NONLINEARITY v = .1 sec. for $\frac{\theta_3 \theta_{NL}^2}{|\theta_i|^3}$. It is felt that the relative accuracy demonstrated in $\frac{\theta_3 \theta_{NL}^2}{|\theta_i|^3}$ is typical of the solution accuracy of each successive equation since each is found by an identical recurrence equation involving product forcing functions. Hence the further approximations were calculated by the P-Transform only.

We may now determine the convergence for this case where $C_1 = 1$ at a minimum σ_L, which will define the maximum ratio of $\frac{\theta_1}{\theta_{NL}}$ that will permit a linear prime equation. Since our assumed series solution consists of only odd terms due to the symmetrical saturation nonlinearity, the convergence criterion of Section IV-2 establishing the limit of minimum convergence becomes a comparison of the solution series

$$1 + \frac{|\theta_3(t)|}{|\theta_1(t)|} + \frac{|\theta_5(t)|}{|\theta_1(t)|} + \frac{|\theta_7(t)|}{|\theta_1(t)|} + \dots \frac{|\theta_n(t)|}{|\theta_1(t)|} + \dots \qquad (95)$$

which must be term by term less than or equal to the corresponding term of the minimal convergent p-series

$$1 + \frac{1}{2^{1+\varepsilon}} + \frac{1}{3^{1+\varepsilon}} + \frac{1}{4^{1+\varepsilon}} + \dots \frac{1}{m^{1+\varepsilon}} + \dots \qquad (26)$$

where the magnitude of each $\theta_n(t)$ is its peak magnitude, either positive or negative, over the time interval t from 0 to ∞. Hence from the generalized solutions of Figures 14 and 15 we have, letting ε be negligibly small so that for the first m terms the comparison term becomes effectively $\frac{1}{m}$.

$$\frac{1}{2} > \frac{|\theta_3(t)|}{|\theta_1(t)|} = \left[\left(\frac{\theta_3\,\theta_{NL}^2}{|\theta_i|^3}\right)_{PEAK} \div \left(\frac{\theta_1}{|\theta_i|}\right)_{PEAK}\right]\left(\frac{\theta_i}{\theta_{NL}}\right)^2 = \frac{.1055}{.96}\left(\frac{\theta_i}{\theta_{NL}}\right)^2 \tag{96}$$

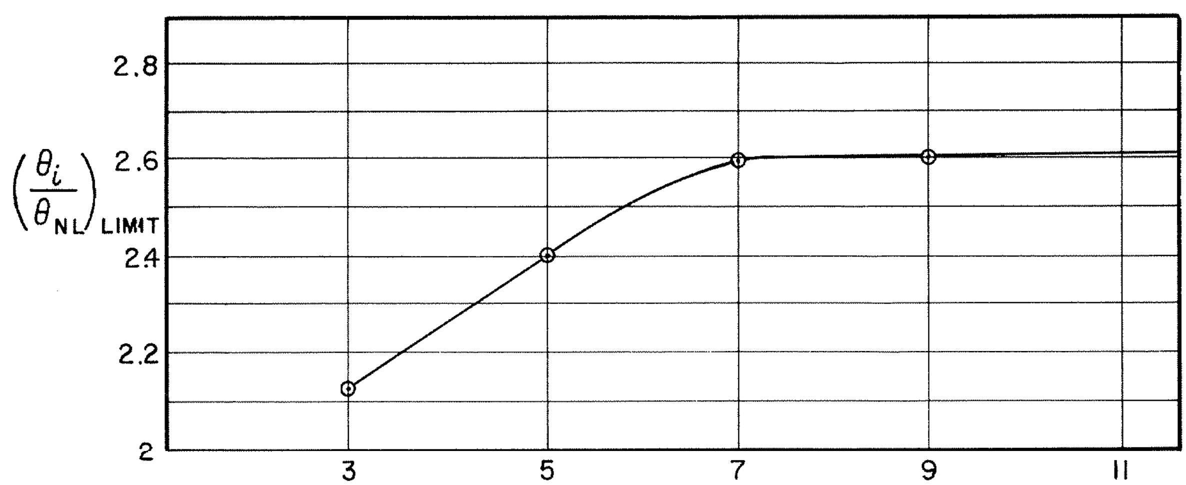
$$\frac{1}{3} > \frac{|\theta_5(t)|}{|\theta_1(t)|} = \left[\left(\frac{\theta_5\,\theta_{NL}^4}{|\theta_i|^5}\right)_{PEAK} \div \left(\frac{\theta_1}{|\theta_i|}\right)_{PEAK}\right]\left(\frac{\theta_i}{\theta_{NL}}\right)^4 = \frac{9.6\times10^{-3}}{.96}\left(\frac{\theta_i}{\theta_{NL}}\right)^4 \tag{97}$$

and in a similar manner

$$\frac{1}{4} > \frac{|\theta_7(t)|}{|\theta_1(t)|} = \frac{8.0\times10^{-4}}{.96}\left(\frac{\theta_i}{\theta_{NL}}\right)^6 \tag{98}$$

$$\frac{1}{5} > \frac{|\theta_9(t)|}{|\theta_1(t)|} = \frac{9.2\times10^{-5}}{.96}\left(\frac{\theta_i}{\theta_{NL}}\right)^8 \tag{99}$$

From these equations the upper limit of $\frac{\theta_1}{\theta_{NL}}$ which just permits each inequality to exist may be computed. These limits are plotted in Figure 16 vs. the index of the term of the series from which the limit was calculated. The higher terms



TERM INDEX

Figure 16

of the series are seen to define a trend toward an absolute convergence limit of $\frac{\theta_i}{\theta_{NL}} \approx 2.62$ for the entire series solution. Since for the present there is no known way to determine the n^{th} term convergence limit except by calculation of the n^{th} transient solution, it must be assumed that the limit defined by the first few terms of the series is equal to that defined by the n^{th} term as n increases. From the iterative form of the equations of the set whose solutions comprise the assumed series solution of the original nonlinear equation, the validity of this assumption seems qualitatively indicated.

Thus we may now obtain the transient solution for the nonlinear system of Figure 13 for any ratio of $\frac{\theta_i}{\theta_{NL}}$ up to the convergence limit of 2.62 from the generalized curves of Figure 14 and Figure 15. This is done by merely multiplying the scale of each θ_n curve by the constant $\frac{\theta_i^{\,n}}{\theta_{NL}^{\,n-1}}$ and adding the respective curves point by point in time. A typical transient solution for $\theta_i = 10$ and $\theta_{NL} = 4$ is shown in Figure 17 together with the transient solution for this case found by studying the nonlinear system on an analogue computer. The details of the analogue of the system are given in Appendix B. In this example the system is in the saturated region for the major portion of the first oscillation, the nonlinearity output being limited at a value of 4. The main deviation of the calculated from the experimental curve occurs in the transition from saturated to unsaturated region of the nonlinearity. This is to be expected since the Legendre expansion of the nonlinear characteristic also deviates in this region (Figure 8). If additional accuracy is required, it could be obtained by increasing the number of Legendre polynomials, q, included in the expansion plus including the solutions of additional successive equations. However, the main interest in control system analysis is the magnitude of the peak oscillation and settling-down time both of which are defined by this example within the required engineering accuracy. Moreover, on the limited number of systems studied, engineering accuracy has always been obtained for q = 17 and $\Theta_a(t) = \sum_{n=1}^{q} \Theta_n(t)$.

The transient solution for any other ratio of $\frac{\theta_i}{\theta_{NL}}$ within the convergence limit may be quickly found in the same way. For $\frac{\theta_i}{\theta_{NL}}$ greater than this convergence limit it is necessary to choose C_1 less than one in order that a convergent series solution results. It has been found that the necessary reduction can be approximately determined by assuming that the θ_1 peak varies inversely with C_1, from which the second term of the convergence determining series for $C_1 < 1$ becomes in terms of the same term for $C_1 = 1$

$$\frac{1}{2} > \underbrace{\left(\frac{|\Theta_3(t)|}{|\Theta_1(t)|}\right)}_{\text{PEAKS FOR } C_1<1} \approx \left[\left(\frac{C_3\sigma_L^2}{C_1^2}\right)_{C_1<1} \div \left(C_3\sigma_L^2\right)_{C_1=1}\right] \underbrace{\left(\frac{|\Theta_3(t)|}{|\Theta_1(t)|}\right)}_{\text{PEAKS FOR } C_1=1} \cdot \left(\frac{\Theta_i}{\Theta_{NL}}\right)^2 \qquad (100)$$

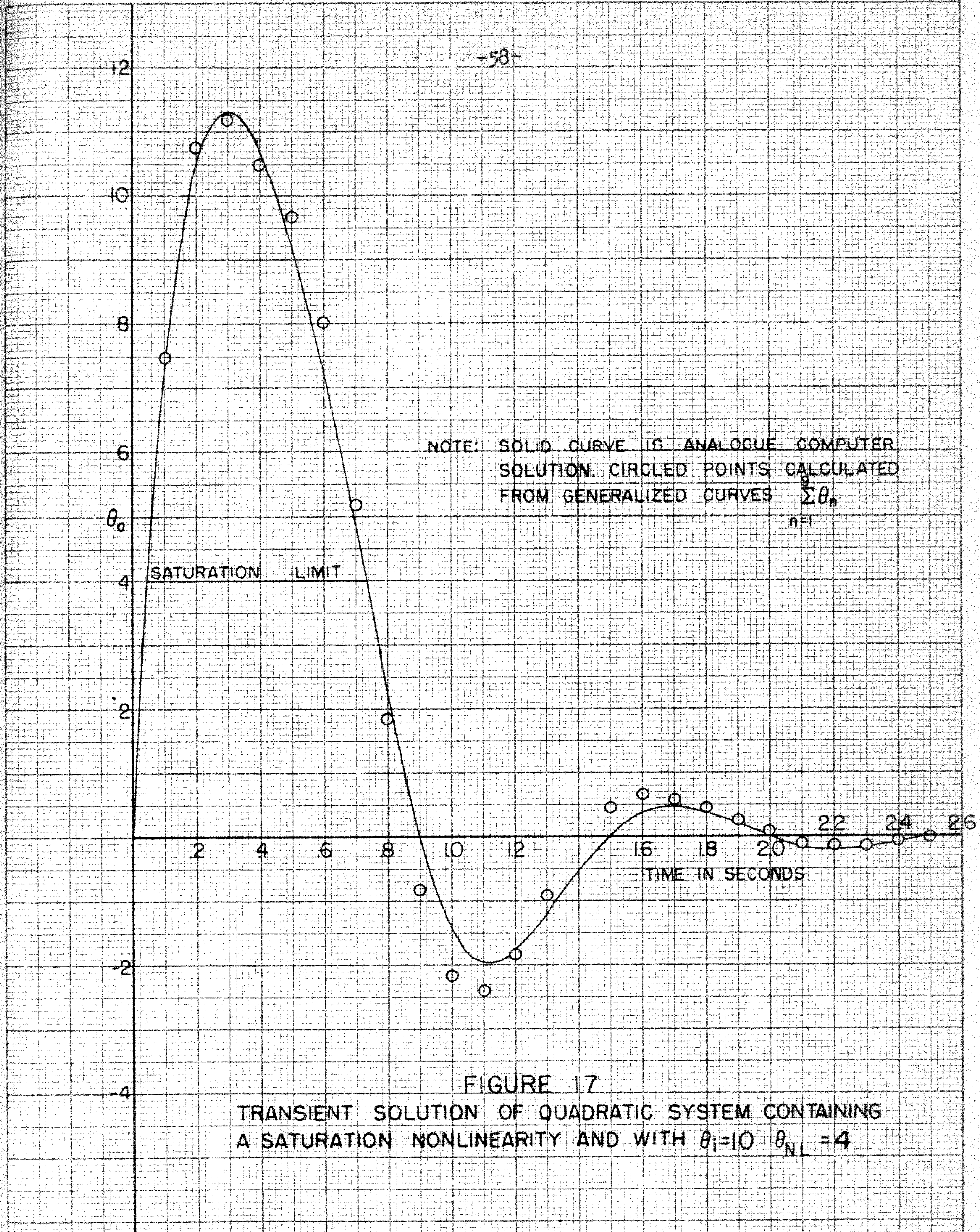


FIGURE 17

TRANSIENT SOLUTION OF QUADRATIC SYSTEM CONTAINING A SATURATION NONLINEARITY AND WITH $\theta_i = 10$ $\theta_{NL} = 4$

From this expression the C_1 for convergence for any $\frac{\theta_i}{\theta_{NL}}$ can be approximately calculated by the curves for C_n. This avoids the difficulty of having to find the complete transient solution before one can determine if that solution converges for the input of interest.

Whereas the generalized curves here presented are the solutions for a step function input, generalized curves for any time varying forcing function may be found by introducing this forcing function instead of the step into the prime equation of the set.

VIII. CALCULATION OF INSTABILITY OF A SERVOMECHANISM CONTAINING A SATURATION NONLINEARITY

While this analysis technique provides a means of calculation of the transient response of a nonlinear control system it has the complexity and consequent time expenditure which is inherent in any method of analyzing nonlinear systems. For this reason, it is much more efficient to study a nonlinear system on some electronic computer if a wide range of transient behavior as a function of system parameters is required. However, many times the problem is simply to determine the possibility of instability of a system containing a nonlinearity. Often it is not practical to perform a computer study to obtain this answer alone and thus a means of manual determination of instability is worth-while.

In this connection, the infinite series analysis method will be applied to the case of an automatically controlled missile containing a saturation nonlinearity on the control surface. Consider the system given by the block diagram of Figure 18.

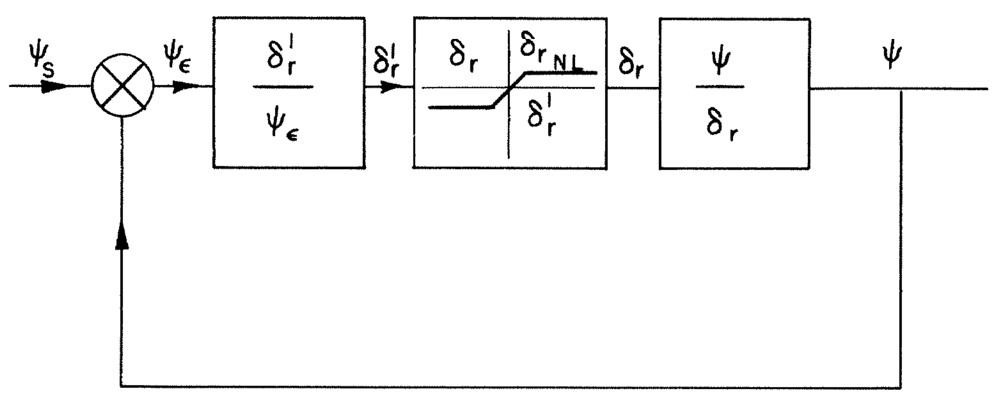


Figure 18

The dynamic motion is described by the equation

$$\frac{\psi}{\delta_r} = -\frac{K_H(\tau_H D+1)}{D(D^2+aD+b)} \tag{101}$$

and the control equation as

$$(\tau_\ell D+1)\delta_r' = -K_\psi \psi_s + (K_\psi + K_{\dot\psi} D)\psi \tag{102}$$

where

$$\delta_r = \sum_{n=1}^{8} \alpha_n \delta_r'^{\,n} \tag{103}$$

Multiplying (102) by $D(D^2 + aD + b)$ and substituting (101) gives

$$D(D^2+aD+b)(\tau_\ell D+1)\delta_r' = -K_\psi D(D^2+aD+b)\psi_s - (K_\psi + K_{\dot\psi} D)K_H(\tau_H D+1)\delta_r \tag{104}$$

Transposing the last expression and substituting (103) yields

$$\left[D^4 + \left(\frac{a\tau_\ell+1}{\tau_\ell}\right)D^3 + \left(\frac{\tau_\ell b+a}{\tau_\ell}\right)D^2 + \frac{bD}{\tau_\ell}\right]\delta_r' + \frac{K_H}{\tau_\ell}\left[K_{\dot\psi}\tau_H D^2 + (K_\psi \tau_H + K_{\dot\psi})D + K_\psi\right]\sum_{n=1}^{8}\alpha_n \delta_r'^{\,n} = -\frac{K_\psi}{\tau_\ell}(D^2+aD+b)\psi_s(t) \tag{105}$$

The system parameters will be defined as

$K_H = 33$

$\tau_H = .22$

$a = 10.22$

$b = -7.5$

$\tau_\ell = .13$

In Figures 19 and 20 are shown the Nyquist diagrams for the above system with control gains of $K_\psi = 6$ Rad/Rad, $K_{\dot\psi} = 2$ rad/rad/sec. and $K_\psi = 6$, $K_{\dot\psi} = 1$. In both cases it is seen that either decreasing or

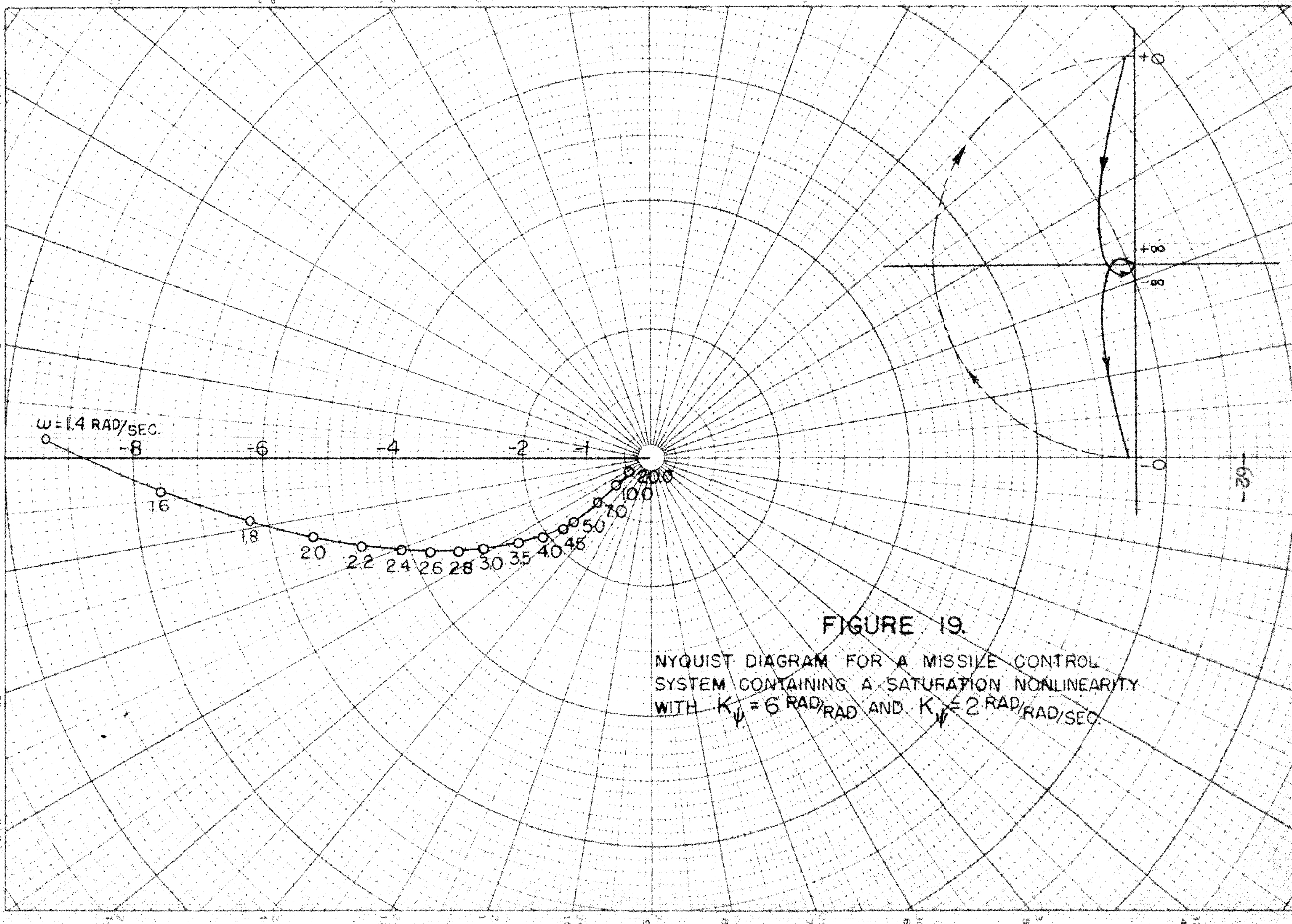


FIGURE 19.

NYQUIST DIAGRAM FOR A MISSILE CONTROL SYSTEM CONTAINING A SATURATION NONLINEARITY WITH $K_{\psi} = 6$ RAD/RAD AND $K_{\dot{\psi}} = 2$ RAD/RAD/SEC.

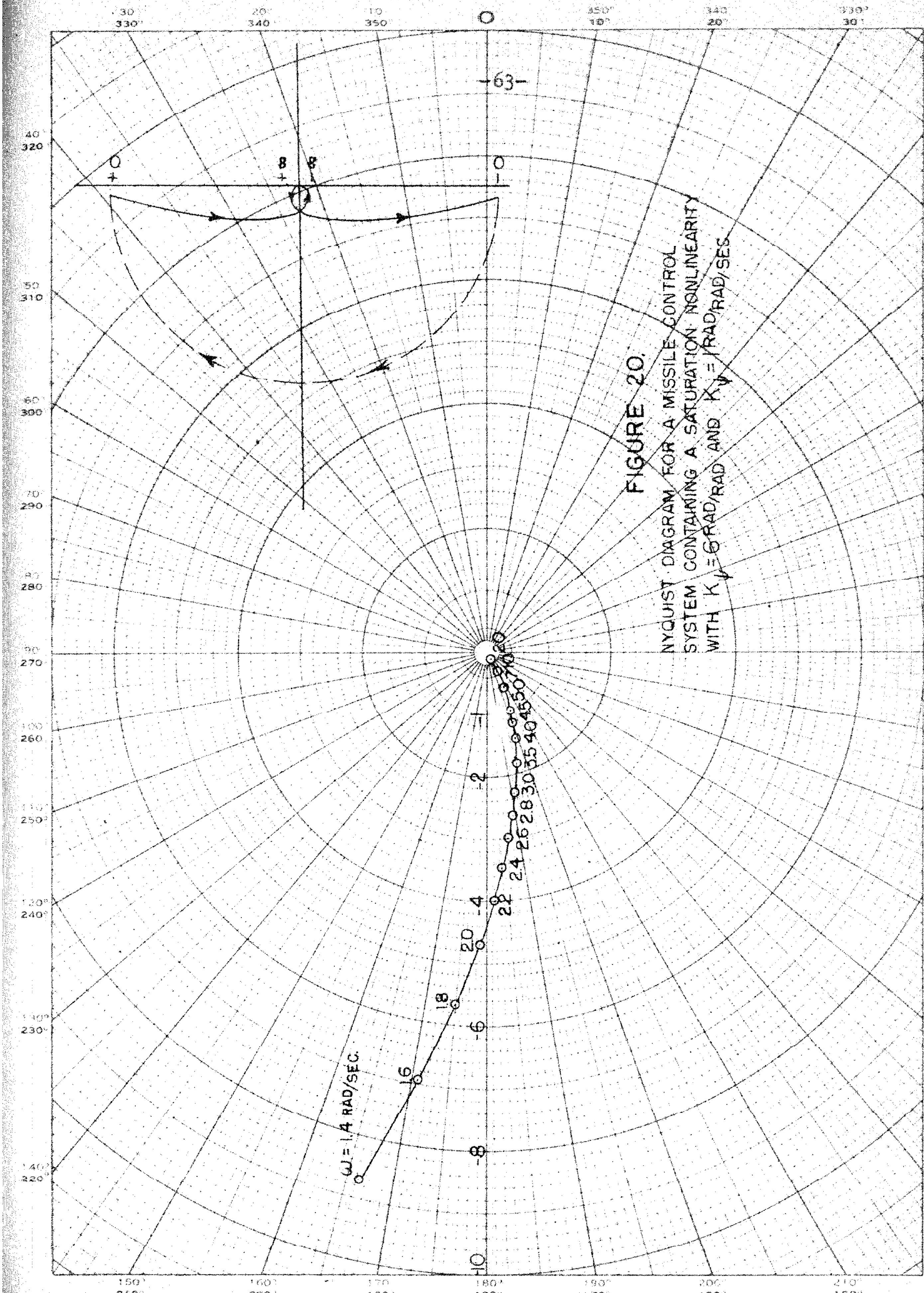


FIGURE 20.

NYQUIST DIAGRAM FOR A MISSILE CONTROL SYSTEM CONTAINING A SATURATION NONLINEARITY WITH K_{ψ} = 6 RAD/RAD AND $K_{\dot{\psi}}$ = 1 RAD/RAD/SES increasing the direct loop gain can result in instability. It remains now to investigate what magnitude of forcing function Ψ_s can produce sufficient saturation of the control surface that the effective loop gain decreases to the point of system instability. While the study may be made for any time varying forcing function we will concern ourselves with a step function of magnitude Ψ_s.

The set of generalized equations comprising the solution of the nonlinear equation (105) are, referring to (22) and including the values of the system parameters

$$\left[D^4+17.95D^3+(71.1+55.85K_{\dot\psi}C_1)D^2+(55.85K_\psi C_1+253.8K_{\dot\psi}C_1-57.65)D+253.8K_\psi C_1\right]\frac{\delta'_{r_1}}{\psi_s}$$

$$=-\frac{K_\psi}{.13}(D^2+10.22D-7.5)F_i(t)$$

$$\left[D^4+17.95D^3+(71.1+55.85K_{\dot\psi}C_1)D^2+(55.85K_\psi C_1+253.8K_{\dot\psi}C_1-57.65)D+253.8K_\psi C_1\right]\frac{\delta'_{r_3}\delta^2_{r_{NL}}}{\psi_s^3}$$

$$=-C_3\sigma_L^2\left[55.85K_{\dot\psi}D^2+(55.85K_\psi+253.8K_{\dot\psi})D+253.8K_\psi\right]\left(\frac{\delta'_{r_1}}{\psi_s}\right)^3 \qquad (106)$$

$$\left[D^4+17.95D^3+(71.1+55.85K_{\dot\psi}C_1)D^2+(55.85K_\psi C_1+253.8K_{\dot\psi}C_1-57.65)D+253.8K_\psi C_1\right]\frac{\delta'_{r_5}\delta^4_{r_{NL}}}{\psi_s^5}$$

$$=-\left[55.85K_{\dot\psi}D^2+(55.85K_\psi+253.8K_{\dot\psi})D+253.8K_\psi\right]\left[C_5\sigma_L^4\left(\frac{\delta'_{r_1}}{\psi_s}\right)^5+3C_3\sigma_L^2\left(\frac{\delta'_{r_3}\delta^2_{r_{NL}}}{\psi_s^3}\right)\left(\frac{\delta'_{r_1}}{\psi_s}\right)^2\right]$$

$$\vdots$$

Employing the P-Transform in solving these equations, we obtain in the same manner as in the previous section the recurrence relationships for the respective equations as

$$\frac{\delta'_{r_{1_n}}}{\psi_s}=a_0+a_1+a_2+a_3+a_4-b_1\frac{\delta'_{r_{1_{n-1}}}}{\psi_s}-b_2\frac{\delta'_{r_{1_{n-2}}}}{\psi_s}-b_3\frac{\delta'_{r_{1_{n-3}}}}{\psi_s}-b_4\frac{\delta'_{r_{1_{n-4}}}}{\psi_s} \qquad (107)$$

$$\frac{\delta'_{r_{3_n}}\,\delta^2_{r_{NL}}}{\psi_s^3} = f_0 F_{3_n} + f_1 F_{3_{n-1}} + f_2 F_{3_{n-2}} + f_3 F_{3_{n-3}} + f_4 F_{3_{n-4}} - \frac{\delta^2_{r_{NL}}}{\psi_s^3}\left(b_1 \delta'_{r_{3_{n-1}}} + b_2 \delta'_{r_{3_{n-2}}} + b_3 \delta'_{r_{3_{n-3}}} + b_4 \delta'_{r_{3_{n-4}}}\right) \tag{108}$$

where

$$F_{3_n} = -C_3\,\sigma_L^2\left(\frac{\delta'_{r_{1_n}}}{\psi_s}\right)^3$$

$$\frac{\delta'_{r_{5_n}}\,\delta^4_{r_{NL}}}{\psi_s^5} = f_0 F_{5_n} + f_1 F_{5_{n-1}} + f_2 F_{5_{n-2}} + f_3 F_{5_{n-3}} + f_4 F_{5_{n-4}} - \frac{\delta^4_{r_{NL}}}{\psi_s^5}\left(b_1 \delta'_{r_{5_{n-1}}} + b_2 \delta'_{r_{5_{n-2}}} + b_3 \delta'_{r_{5_{n-3}}} + b_4 \delta'_{r_{5_{n-4}}}\right) \tag{109}$$

where

$$F_{5_n} = -\left[C_5\,\sigma_L^4\left(\frac{\delta'_{r_{1_n}}}{\psi_s}\right)^5 + 3C_3\,\sigma_L^2\left(\frac{\delta'_{r_{3_n}}\,\delta^2_{r_{NL}}}{\psi_s^3}\right)\left(\frac{\delta'_{r_{1_n}}}{\psi_s}\right)^2\right]$$

and where the constants are defined as

$$a_0 = \frac{1}{B}\left[-\frac{K_\psi}{.13}\,\frac{v^2}{4}\left(1 + 10.22\frac{v}{2} - 7.5\frac{v^2}{4}\right)\right]$$

$$a_1 = \frac{1}{B}\left[-\frac{K_\psi}{.13}\,\frac{v^3}{4}\left(10.22 - 7.5v\right)\right]$$

$$a_2 = \frac{1}{B}\left[-\frac{K_\psi}{.13}\,\frac{v^2}{2}\left(1 + \frac{3v^2}{4}\times 7.5\right)\right]$$

$$a_3 = \frac{1}{B}\left[-\frac{K_\psi}{.13}\,\frac{v^3}{4}\left(10.22 + 7.5v\right)\right]$$

$$a_4 = \frac{1}{B}\left[-\frac{K_\psi}{.13}\,\frac{v^2}{4}\left(1 - 10.22\frac{v}{2} - 7.5\frac{v^2}{4}\right)\right]$$

$$f_0 = \frac{1}{B}\left[55.85K_{\dot\psi}\frac{v^2}{4} + (55.85K_\psi + 253.8K_{\dot\psi})\frac{v^3}{8} + 253.8\,K_\psi\frac{v^4}{16}\right]$$

$$f_1 = \frac{1}{B}\left[(55.85K_\psi + 253.8\,K_{\dot\psi})\frac{v^3}{4} + 253.8\,K_\psi\frac{v^4}{4}\right]$$

$$f_2 = \frac{1}{B}\left[253.8\,K_\psi\frac{3v^4}{8} - 55.85\,K_{\dot\psi}\frac{v^2}{2}\right]$$

$$f_3 = \frac{1}{B}\left[253.8\,K_\psi\frac{v^4}{4} - (55.85\,K_\psi + 253.8\,K_{\dot\psi})\frac{v^3}{4}\right]$$

$$f_4 = \frac{1}{B}\left[253.8\,K_\psi\frac{v^4}{16} - (55.85\,K_\psi + 253.8\,K_{\dot\psi})\frac{v^3}{8} + 55.85\,K_{\dot\psi}\frac{v^2}{4}\right]$$

$$b_1 = \frac{1}{B}\left[253.8\,K_\psi C_1 \frac{v^4}{4} + (55.85\,K_\psi C_1 + 253.8\,K_{\dot\psi} C_1 - 55.65)\frac{v^3}{4} - 17.95\,v - 4\right]$$

$$b_2 = \frac{1}{B}\left[6 - (71.1 + 55.85\,K_{\dot\psi} C_1)\frac{v^2}{2} + 253.8\,K_\psi C_1 \frac{3v^4}{8}\right]$$

$$b_3 = \frac{1}{B}\left[17.95\,v - 4 - (55.85\,K_\psi C_1 + 253.8\,K_{\dot\psi} C_1 - 55.65)\frac{v^3}{4} + 253.8\,K_\psi C_1 \frac{v^4}{4}\right]$$

$$b_4 = \frac{1}{B}\left[1 - 17.95\frac{v}{2} + (71.1 + 55.85\,K_{\dot\psi} C_1)\frac{v^2}{4} - (55.85\,K_\psi C_1 + 253.8\,K_{\dot\psi} C_1 - 55.65)\frac{v^3}{8} + 253.8\,K_\psi C_1 \frac{v^4}{16}\right]$$

with

$$B = \left[1 + 17.95\frac{v}{2} + (71.1 + 55.85\,K_{\dot\psi} C_1)\frac{v^2}{4} + (55.85\,K_\psi C_1 + 253.8\,K_{\dot\psi} C_1 - 55.65)\frac{v^3}{8} + 253.8\,K_\psi C_1 \frac{v^4}{16}\right]$$

In the two cases $K_\psi = 6$, $K_{\dot\psi} = 2$ and $K_\psi = 6$, $K_{\dot\psi} = 1$ these recurrence relationships were used to find the solutions of $\frac{\delta'_{r_1}}{\psi_s}$, $\frac{\delta'_{r_3}\,\delta^2_{r_{NL}}}{\psi_s^3}$ and $\frac{\delta'_{r_5}\,\delta^4_{r_{NL}}}{\psi_s^5}$ for C_1 ranging from 1 down to the value at which the prime equation became unstable, the C_n's being those for q = 17 terms in the Legendre expansion of the nonlinearity. From the peaks of these transient solutions the upper limit of $\frac{\psi_s}{\delta_{r_{NL}}}$ which just permits convergence of the assumed series solution may be calculated by

$$\frac{1}{2} > \frac{\left(\dfrac{\delta'_{r_3}\,\delta^2_{r_{NL}}}{\psi_s^3}\right)_{PEAK}}{\left(\dfrac{\delta'_{r_1}}{\psi_s}\right)_{PEAK}} \cdot \left(\frac{\psi_s}{\delta_{r_{NL}}}\right)^2 \tag{110}$$

and

$$1/3 > \frac{\left(\dfrac{\delta'_{r_5}\,\delta^4_{r_{NL}}}{\psi_s^5}\right)_{PEAK}}{\left(\dfrac{\delta'_{r_1}}{\psi_s}\right)_{PEAK}} \cdot \left(\frac{\psi_s}{\delta_{r_{NL}}}\right)^4 \tag{111}$$

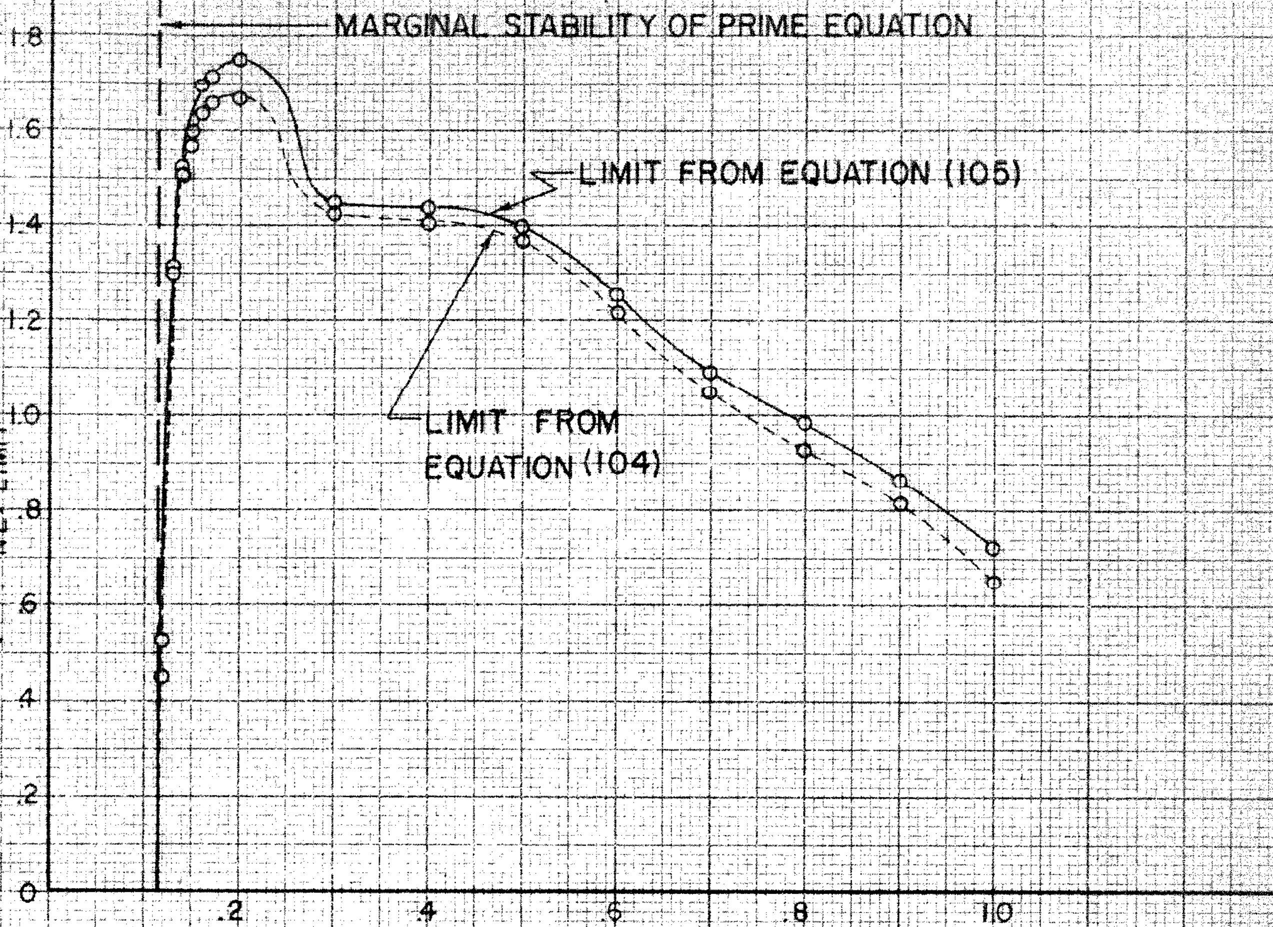


FIGURE 21. CONVERGENCE LIMITS FOR $K_\psi=6$ $K_{\dot{\psi}}=2$

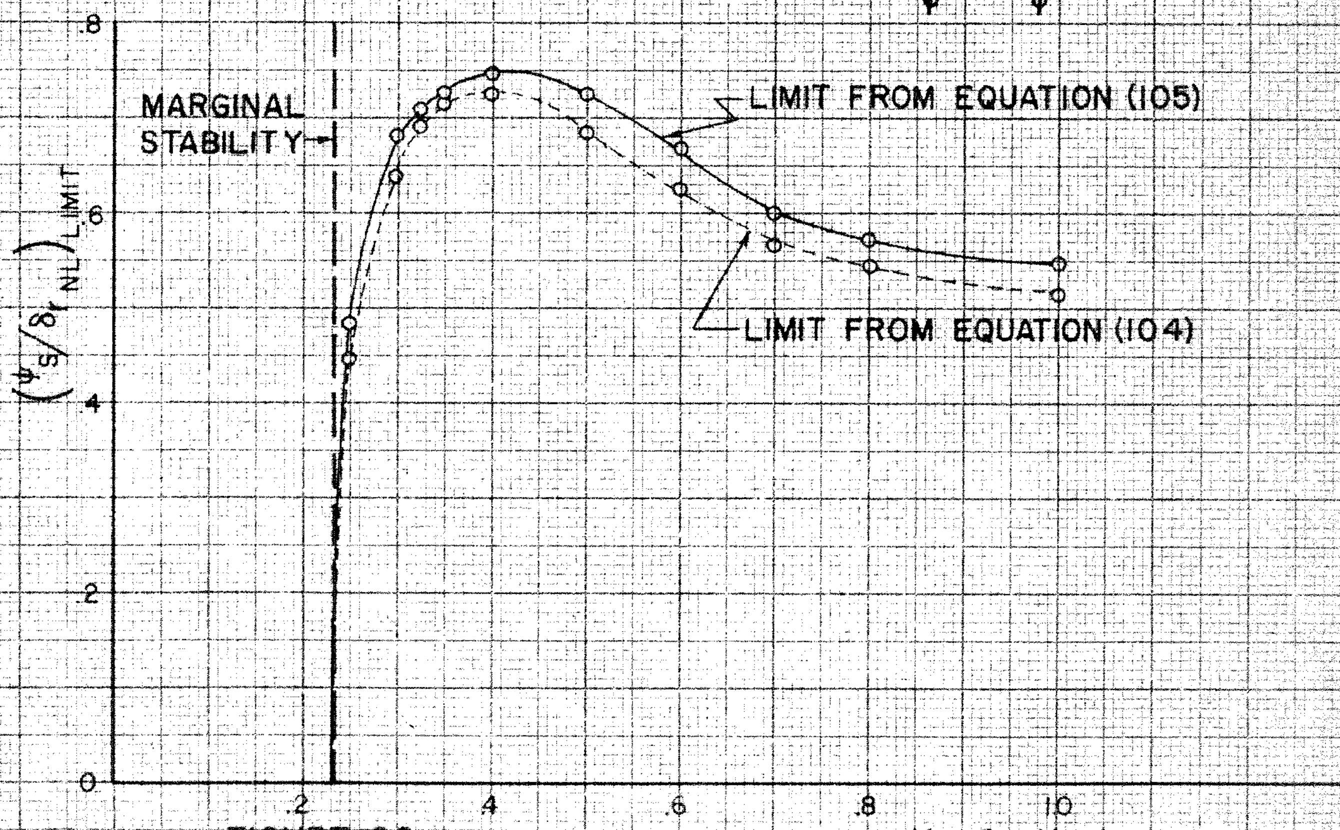


FIGURE 22. CONVERGENCE LIMITS FOR $K_\psi=6$ $K_{\dot{\psi}}=1$

The variation in the $\frac{\psi_s}{\delta_{r_{NL}}}$ upper bound with C_1 is shown in Figure 21 for $K_\psi = 6$, $K_{\dot\psi} = 2$ and Figure 22 for $K_\psi = 6$, $K_{\dot\psi} = 1$. The dashed curve in each case has the limits defined by equation (110), and the solid curve by equation (111).

Both cases of control gains demonstrate an increase in the upper limit of $\frac{\psi_s}{\delta_{r_{NL}}}$ for convergence up to a maximum as C_1 decreases from unity, and then a sharp decrease to zero at the value of C_1 for which a marginally stable prime equation occurs. The reason the convergence limit of $\frac{\psi_s}{\delta_{r_{NL}}}$ is zero at the point of a marginally stable prime equation may be seen by the following. The characteristic equation for each successive equation of the set defining the solution of the nonlinear system is identical to that of the prime equation. In general, the roots of this characteristic equation are in Laplace Transform notation $\sum_k (s+\zeta_k)^2 + \beta_k^2$. The forcing function or right-hand side of the successive equations is formed by time domain multiplications of the inverse transforms of the above roots. Since these roots give time characteristics as $A e^{-\zeta_k t} \sin(\beta_k t + \varphi)$, the forcing functions generated by raising these terms to the n^{th} power include terms as $B e^{-n\zeta_k t} \sin(\beta_k t + \varphi)$ plus others with higher harmonics of $(\beta_k t + \varphi)$. In the case of marginal stability of the prime equation one set of roots has zero real part, i.e., $\zeta_k = 0$ and thus the characteristic equation and forcing function of successive equations of the set result in repeated roots. These repeated roots give terms in the time domain of $Ct^n \sin(\beta_k t + \varphi)$ which, increasing with time, force $\frac{\psi_s}{\delta_{r_{NL}}}$ for convergence to equal zero. Thus it is seen that for this saturation nonlinearity and a system which becomes unstable as the effective control gain is decreased it is impossible to have a nonlinear limit cycle, i.e., the nonlinear system either converges or diverges in time.

The indication of the convergence limit curves is that a maximum $\frac{\psi_s}{\delta_{r_{NL}}}$ exists, given by the curve peak, above which this nonlinear analysis technique cannot result in a convergent assumed series solution, and hence, cannot give a valid solution. If we assume that this technique must be capable of giving a valid solution for any condition under which the system is time convergent then this maximum $\frac{\psi_s}{\delta_{r_{NL}}}$ limit defines the stability limit for this system. Since the limit curves defined by (110) and (111) exhibit a uniform relative trend it is reasonable to expect the convergence limits defined by further successive generalized equation solutions to maintain a uniform trend. Thus we need only concern ourselves with the peak point. Computing $\frac{\delta'_{r_7}\,\delta^6_{r_{NL}}}{\psi_s^7}$ and $\frac{\delta'_{r_9}\,\delta^8_{r_{NL}}}{\psi_s^9}$ and applying the convergence criterion to their respective peaks we find the convergence limit trend of Fig. 23 for $K_\psi = 6$, $K_{\dot\psi} = 2$ and Figure 24 for $K_\psi = 6$, $K_{\dot\psi} = 1$. Now if our above assumption is true, the maximum $\frac{\psi_s}{\delta_{r_{NL}}}$ step command to which this missile system may be exposed and have the resulting transient converge in time is 1.803 for $K_\psi = 6$, $K_{\dot\psi} = 2$ and .776 for $K_\psi = 6$, $K_{\dot\psi} = 1$.

In order to prove the validity of the results obtained by this assumption, the nonlinear system was studied on an analogue computer. The analogue details are given in Appendix C. The analogue study demonstrated that for control surface limits $\delta_{r_{NL}}$ of 20 degrees

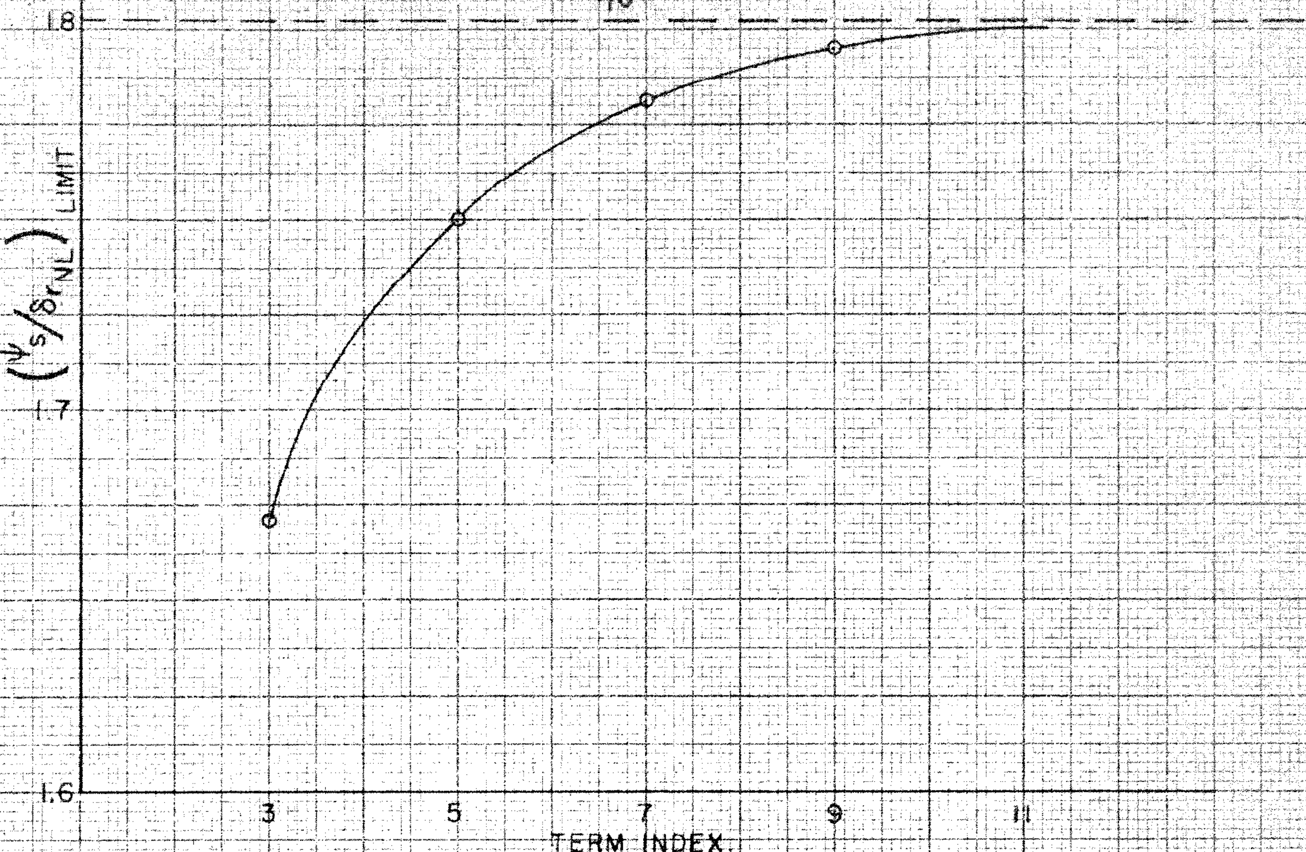


FIGURE 23. CONVERGENCE TREND FOR $K_{\psi}=6$, $K_{\dot{\psi}}=2$ WITH $C_1=.2$

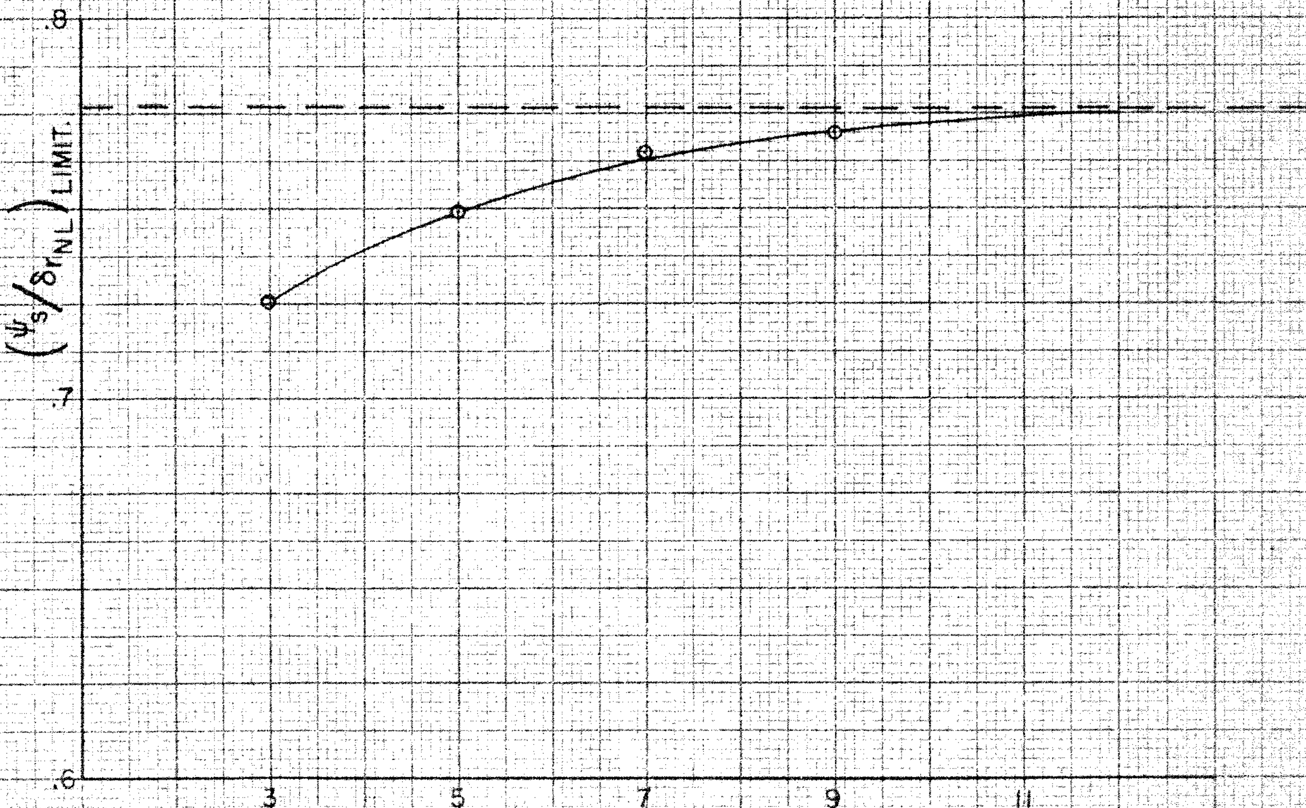


FIGURE 24. CONVERGENCE TREND FOR $K_{\psi}=6$, $K_{\dot{\psi}}=1$ WITH C_1 =4

the maximum step commands ψ_s for which the system response did not build up with time were 32.5 degrees for $K_\psi = 6$, $K_{\dot\psi} = 2$, and 15.25 degrees with control gains of $K_\psi = 6$, $K_{\dot\psi} = 1$. These results are equivalent to $\frac{\psi_s}{\delta_{R_{NL}}}$ limits of 1.625 and .7625 respectively, which check our calculated results within 10% and 1.8%. Since the accuracy of the analogue computer is of the order of 1 to 2 per cent, the agreement of the results is notably good.

Although two isolated examples do not absolutely prove our assumption, the above two cases are sufficiently different in their characteristics to suggest that further studies of other systems will bear out the results found here. This will require, of course, many examples which can be supplied by practicing control analysis engineers.

In the application of this method to stability determination it is not necessary to perform the many calculations made here to demonstrate the technique. Since the solutions of the successive generalized equations result in the same trend of the convergence limit, one need only calculate the solution of the second equation of the set to determine the peak of the convergence limit curve. Then at this point only, the further equations may be solved to establish the convergence limit as in Figures 23 and 24. Moreover, a pessimistic and hence safer design limit can be obtained by using the convergence limit defined by the second equation of the set which can be determined in approximately four hours for a fourth order system.

IX. CONCLUSIONS

A method of obtaining the transient response of an automatic control system containing a dependent variable nonlinearity has been developed together with the rules governing its use. The method consists of the following steps:

a. obtaining a mathematical expression for the nonlinear characteristic where necessary by an expansion in Legendre polynomials.

b. introducing this expression into the equations describing the control system behavior thus obtaining a nonlinear equation involving a power series of a dependent variable.

c. solving this nonlinear equation by means of an assumed infinite series solution requiring that the assumed series, each term of which is defined by a differential equation, be uniformly convergent by the Weierstrauss M criterion.

A saturation type nonlinearity was chosen to illustrate the technique and the method was applied to a second order system containing this nonlinearity in order to illustrate the accuracy of the method and present a numerical technique for solving the series of equations arising from the infinite series method of solving the nonlinear equation. A fourth order missile control equation with a saturation limit on the control surface was investigated by the method and by means of the convergence criterion the maximum step function command which could be imposed on the control system with resultant time convergent response was defined. The results calculated for both the second and fourth order systems were checked by solutions obtained on an analogue computer.

The discussion has been presented in such a manner that the application of this analysis method can be applied to any physical control system containing any type of dependent variable nonlinearity. It is hoped that the method will prove useful in the furtherance of nonlinear systems analysis and consequent understanding. Unfortunately, nonlinear systems calculations are quite laborious and for this reason it is much more efficient to find transient solutions by electronic computer methods. However, too often one obtains little insight into the problem when it is studied by a computer alone. For this reason and since a computer is not always accessible, some manual analysis of nonlinear systems is needed.

The use of the method developed in this thesis should prove most valuable in stability determination. Whereas only instability brought about by the nonlinearity in an otherwise stable linear system was investigated here, similar determination of nonlinearity produced stability; i.e., limit cycle, to an unstable linear system should be possible. It is the study of this together with the many other dependent variable nonlinearities found in physical systems which future work with this method should include. However, the principal need of the method now is a means of determining the convergence of the assumed series solution by mathematical technique rather than by calculating the transient solutions of each term in the series. In fact any method of solving nonlinear equations will involve a similar convergence determination. Some mathematical work is being carried on in this field presently by various people. If an analytic method for predicting convergence is found the utility of the technique presented in this thesis for analyzing nonlinear systems will be greatly enhanced.

REFERENCES

1. "A Frequency - Response Method for Analyzing and Synthesizing Contactor Servomechanisms" by R. J. Kochenburger, AIEE Transactions, Vol. 69, 1950.

2. "The Dynamics of Automatic Controls" by R. C. Oldenbourg and H. Sartorious, published by the American Society of Mechanical Engineers.

3. "A Generalization of Reversion Formulae With Their Application to Nonlinear Differential Equations" by A. C. Sim, Philosophical Magazine, Vol. XLII, 1951.

4. "A New Linear Operational Calculus" by Frank W. Bubb, A. F. Tech. Rept. No. 6581.

APPENDIX A

The A_n constants may be found from the following table by summing horizontally the tabulated constants times the columnar heading.

	σ_L	σ_L^3	σ_L^5	σ_L^7	σ_L^9	σ_L^{11}	σ_L^{13}	σ_L^{15}	σ_L^{17}	σ_L^{19}
1	1.5	.5								
3	−.875	1.75	−.875							
5	.6875	−3.4375	4.8125	−2.0625						
7	−.58594	5.46875	−14.76563	15.46875	−5.58594					
9	.51953	−7.79297	34.28906	−63.67969	53.06641	−16.40234				
11	−.471679	10.37694	−67.45008	192.71452	−273.01224	188.62664	−50.78410			
13	.435059	−13.19677	118.77096	−480.73961	1014.8947	−1162.5158	685.5862	−163.23482		
15	−.405851	16.23405	−193.1852	1048.7194	−3058.7649	5116.4795	−4919.6918	2530.1272	−539.5124	
17	.361846	−19.47413	296.0068	−2072.0478	7942.850	−18051.931	24994.982	−20710.128	9441.382	−1822.021

APPENDIX B

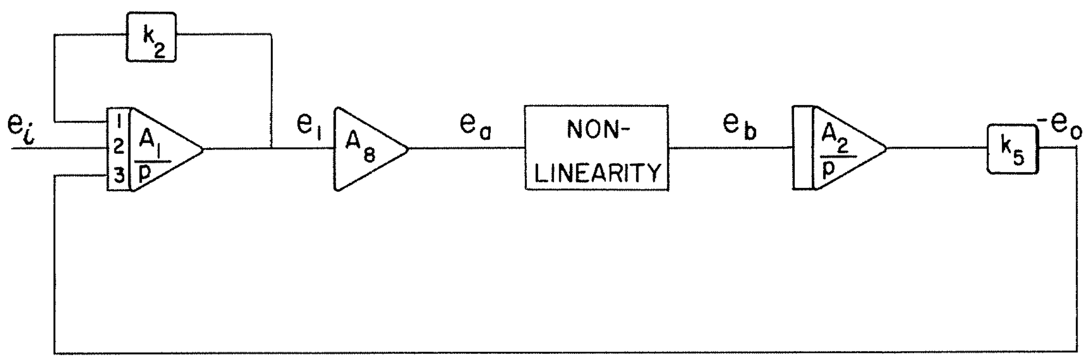
The quadratic system equations are

$$(\tau D+1)\theta_a = K_1(\theta_i-\theta_o) \tag{B1}$$

$$\theta_b = \theta_a \times \left[\text{Saturation Nonlinearity Characteristic}\right] \tag{B2}$$

$$\theta_o = \frac{K_2}{D}\theta_b \tag{B3}$$

An analogue of the system is



which is described by the equations

$$e_1 = \frac{-A_{1.2}e_i}{p} - \frac{A_{1.1}k_2e_1}{p} + \frac{A_{1.3}e_o}{p} \tag{B4}$$

$$e_a = -A_8e_1 \tag{B5}$$

$$e_o = \frac{e_bA_2k_5}{p} \tag{B6}$$

Combining (B4) and (B5) gives

$$e_a = \frac{A_8A_{1.2}e_i}{p} - \frac{A_{1.1}k_2e_a}{p} - \frac{A_8A_{1.3}e_o}{p} \tag{B7}$$

Let $a_1e_a = \theta_a$ and hence $a_1e_b = \theta_b$

$$a_2e_o = \theta_o \tag{B8}$$

$$a_3 e_i = \theta_i \tag{B8}$$

$np = D$ or $t_{computer} = nt_{actual}$

Substituting these definitions into the system equations gives when (B1) is divided by τD

$$a_1 e_a = \frac{K_1 a_3 e_i}{\tau np} - \frac{a_1 e_a}{\tau np} - \frac{K_1 a_2 e_o}{\tau np} \tag{B9}$$

$$a_2 e_o = \frac{K_2 a_1 e_b}{np} \tag{B10}$$

Equating the constants of (B6) and (B7) with (B9) and (B10) results in

$$\frac{K_1 a_3}{\tau n a_1} = A_8 A_{1.2}$$

$$\frac{1}{\tau n} = A_{1.1} k_2$$

$$\frac{K_1 a_2}{a_1 \tau n} = A_8 A_{1.3} \tag{B11}$$

$$\frac{K_2 a_1}{n a_2} = A_2 k_5$$

For the study made the following values were used:

$n = 1 \qquad a_1 = a_2 = a_3 = 1$

$A_8 = A_{1.1} = 10$

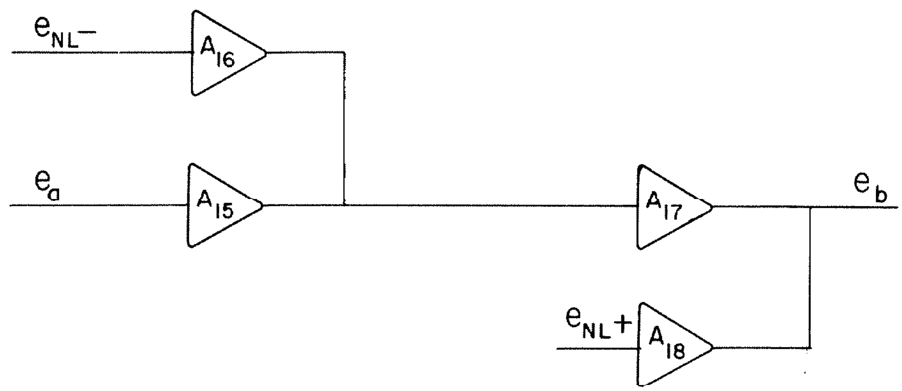
$A_2 = 4$

$A_{1.2} = A_{1.3} = 1$

$k_2 = .5024$

$k_5 = .987$

The saturation nonlinearity was generated by



where $A_{15} = A_{16} = A_{17} = A_{18} = 1$ and all are without automatic DC amplifier drift balancing. The e_{NL} for both positive and negative limits were set at $a_1 e_{NL} = \theta_{NL} = 4$.

APPENDIX C

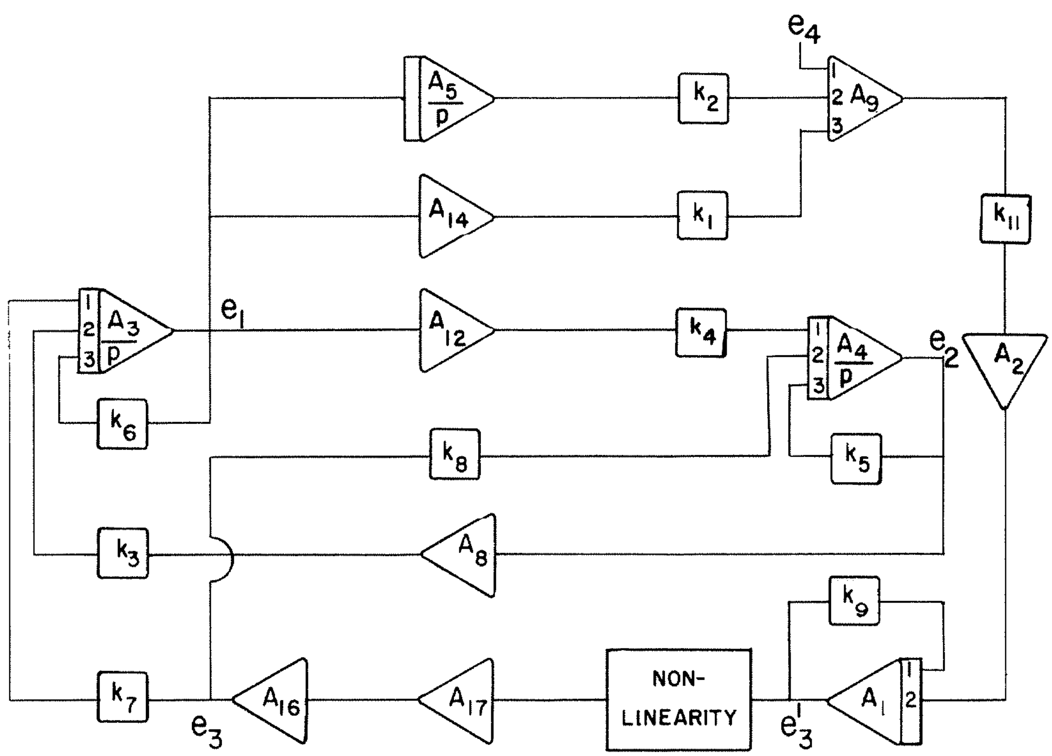
The system equations for the missile dynamical motion are

$$(D^2+aD+b)D\psi = -K_H(\tau_H D+1)\delta_r \qquad (C1)$$

$$\delta_r = \delta_r' \times \left[\text{Saturation Nonlinearity Characteristic}\right] \qquad (C2)$$

$$(\tau_\ell D+1)\delta_r' = -K_\psi \psi_s + (K_\psi + K_{\dot\psi} D)\psi \qquad (C3)$$

An analogue of the system is



The analogue equations are

$$e_1 = -A_{3.1}k_7\frac{e_3}{p} + A_{3.2}A_8k_3\frac{e_2}{p} - A_{3.3}k_6\frac{e_1}{p} \qquad (C4)$$

$$e_2 = A_{4.1}A_{12}k_4\frac{e_1}{p} - A_{4.2}k_8\frac{e_3}{p} - A_{4.3}k_5\frac{e_2}{p} \qquad (C5)$$

$$\frac{1}{1+\frac{}{A_{1.1}k_9}} \; pe_3' = \frac{A_{16}A_{17}A_{1.2}A_{2.2}k_{11}}{A_{1.1}k_9}\left[-A_{9.1}e_4+A_{9.2}A_5k_2\frac{e_1}{p}+A_{9.3}A_{14}k_1e_1\right] \quad (C6)$$

Solving (C4) and (C5) for e_1 yields

$$\left[p^2+(A_{3.3}k_6+A_{4.3}k_5)p+(A_{3.3}k_6A_{4.3}k_5-A_{3.2}A_8k_3A_{4.1}A_{12}k_4)\right]e_1 =$$

$$-\left[A_{3.1}k_7p+(A_{3.1}k_7A_{4.3}k_5+A_{3.2}A_8k_3A_{4.2}k_8)\right]e_3 \qquad (C7)$$

Letting $a_1e_1 = \dot{\psi}$

$a_3e_3 = \delta_r$ and hence $a_3e_3' = \delta_r'$

$a_4e_4 = \psi_s$

$np = D$

The analogue quantities are given by

$$n\left[A_{3.3}k_6+A_{4.3}k_5\right] = a = 10.22$$

$$n^2\left[A_{3.3}k_6A_{4.3}k_5-A_{3.2}A_8k_3A_{4.1}A_{12}k_4\right] = b = -7.5$$

$$\frac{na_1}{a_3}A_{3.1}k_7 = K_H\tau_H = 7.26$$

$$n^2\frac{a_1}{a_3}\left[A_{3.1}k_7A_{4.3}k_5+A_{3.2}A_8k_3A_{4.2}k_8\right] = K_H = 33$$

$$\frac{1}{A_{1.1}k_9n} = \tau_\ell = .13$$

$$\frac{a_3n}{a_1}\;\frac{A_{16}A_{17}A_{1.2}A_{2.2}k_{11}A_{9.2}A_5k_2}{A_{1.1}k_9} = K_\psi = 6$$

$$\frac{a_3}{a_1}\;\frac{A_{16}A_{17}A_{1.2}A_{2.2}k_{11}A_{9.3}A_{14}k_1}{A_{1.1}k_9} = K_{\dot{\psi}} = 1 \text{ or } 2$$

and

$$\frac{A_{16}A_{17}A_{1.2}A_{2.2}k_{11}A_{9.1}}{a_4A_{1.1}k_9} = K_{\psi}$$

$$\psi_s = a_4e_4 = \frac{A_{16}A_{17}A_{1.2}A_{2.2}k_{11}A_{9.1}}{K_{\psi}A_{1.1}k_9} \quad e_4$$

The values used are

$$n = 5 \quad \frac{a_1}{a_3} = 2$$

$A_{1.1} = A_{1.2} = 10$ $\quad A_{3.3} = A_8 = 4$ $\quad$ All other amplifier gains = 1

$k_1 \sim K_{\dot{\psi}}$ $\quad k_3 = .69$ $\quad k_5 = .453$ $\quad k_7 = .726$ $\quad k_9 = .1539$

$k_2 \sim K_{\psi}$ $\quad k_4 = .369$ $\quad k_6 = .398$ $\quad k_8 = .120$

The nonlinear characteristic was generated as described in Appendix B.