Topics in equidistribution and exponential sums

Thesis by
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In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2022
Defended June 28, 2021

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ACKNOWLEDGEMENTS

First and foremost, I would like to thank my supervisor Maksym Radziwill for guiding me during my doctoral studies, for the large amount of helpful advice, smart ideas, and experience he shared with me, for introducing me to many interesting problems (especially the one about points on the sphere).

I thank all the Caltech professors and faculty I had a chance to interact with, to the staff, and the administration for being friendly and helpful.

I thank the faculty, staff, and administration of the Department of Mathematics and Statistics at McGill University where I started my doctoral research.

I thank my undergraduate and master's advisor Maxim Korolev for his support and some interesting problems shared with me.

I thank all my friends from Caltech, McGill, and UdeM for many fruitful discussions on math and just for the pleasant time spent together during my PhD.

I thank my family for their continued support.

ABSTRACT

In this thesis, we consider a few problems connected to the exponential sums which is one of the most important topics in analytic number theory.

In the first part, we study the distribution of prime numbers in special subsets of integers and, in particular, the distribution of these primes in arithmetic progressions, small gaps between them, the behavior of the corresponding exponential sums over primes, and related questions. Big progress was made on these questions in recent years. The famous works of Zhang and Maynard gave the proof of existence of bounded gaps between consecutive primes. Applying the sieve of Selberg-Maynard-Tao and an analogue of the Bombieri-Vinogradov theorem, we obtain similar results for a large class of subsets of primes and improve some of the previous results. The proof of the analogue of the Bombieri-Vinogradov theorem is also connected to a breakthrough work of Bourgain, Demeter, and Guth on the proof of Vinogradov Mean Value Conjecture via l^2-decoupling. Their result, in particular, has led to a significant improvement of the classical van der Corput estimates for a large class of exponential sums.

In the second part, we study the behavior of higher moments of Gauss sum twisted by a Mobius function. The moments of exponential sums are very important in number theory and harmonic analysis as they appear in many other problems. The sum with the Mobius function is of independent interest because of the famous Sarnak Conjecture which is on the edge of number theory, analysis, and dynamical systems. The bound we obtain for L^p-norm of the sum confirms that the Mobius function is uncorrelated with the quadratic phase αn^2 for most $\alpha \in [0; 1]$.

In the third part, we study the distribution of lattice points on the surface of 3-dimensional sphere, which is known as Linnik problem. It turns out that the variance for such points is closely related to the behavior of certain $GL(2)$ L-functions estimated at the central point $1/2$. To evaluate the moments of these L-functions, we apply similar techniques used to evaluate the moments of Riemann zeta function on the critical line in the breakthrough works of Soundararajan and Harper. Their results have led to the sharp upper bounds for all positive moments of zeta function conditionally on Riemann Hypothesis and similar bounds for a broad class of L-functions in families conditionally on the corresponding Grand Riemann Hypothesis. We apply similar methods to get sharp upper bound for the variance of

lattice points on the sphere. The connection of Weyl sums on the sphere to the sums of special values of $GL(2)$ L-functions is a big output of the Langlands program, which has also gotten a lot of attention in recent years.

PUBLISHED CONTENT AND CONTRIBUTIONS

[1] A. Shubin. Bounded gaps between primes of special form. *Dokl. Math.*, 101: 235–238, 2020. doi: 10.1134/S1064562420030199.

[2] A. Shubin. Fractional parts of noninteger powers of primes. *Math. Notes*, 108 (3-4):394–408, 2020. ISSN 0001-4346. doi: 10.1134/S0001434620090084.

[3] A. Shubin. Variance estimates in Linnik's problem. *arXiv:2108.00726*, 2021.

[4] A. Shubin. Fractional parts of noninteger powers of primes. II. *Q. J. Math.* *arXiv:2011.11790*, 2021.

Andrei Shubin conducted all of the research and authored all four manuscripts. Chapters 3, 4 and 6 are based on the papers [1] and [2], Chapter 5 is based on [4], Chapters 8 and 9 are related to the content of [3].

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Chapter 1

INTRODUCTION

The notion of exponential sums is the central subject in analytic number theory. The estimating of such sums is a key point in many famous problems related to equidistribution of various elements within natural domains, counting problems, distribution of prime numbers in different subsets of integers, calculating the number of solutions of Diophantine equations, distribution of lattice points, and many others.

A typical example of an exponential sum is

$$\sum_{n=1}^{N} e^{2\pi i a_n},$$

where a_n is a sequence of real numbers.

A well-known result providing a link between equidistribution of the elements of a sequence and the corresponding exponential is Weyl's criterion. It states that a sequence of real numbers $(a_1, \ldots, a_n, \ldots)$ from the unit interval $[0; 1]$ is equidistributed on this interval if for any subinterval $[a; b] \subset [0; 1]$, one has

$$\lim_{n \to +\infty} \frac{\left|\{a_1, \ldots, a_n\} \cap [a; b]\right|}{n} = b - a.$$

More generally, we say that the sequence $a_1, \ldots, a_n, \ldots$ of real numbers is *equidistributed modulo* 1 if the sequence of its fractional parts

$$\{a_n\} := a_n - \lfloor a_n \rfloor$$

is equidistributed on $[0; 1]$. Weyl's criterion states that the sequence a_n is equidistributed modulo 1 if and only if

$$\lim_{X \to +\infty} \frac{1}{X} \sum_{n \leqslant X} e^{2\pi i m a_n} = 0 \qquad \text{for any } m > 0. \tag{1.1}$$

In other words

$$\sum_{n \leqslant X} e^{2\pi i m a_n} = o(X),$$

so the sum is asymptotically small compared to its length. However many problems require a better upper bound than $o(X)$. In modern analytic number theory, there are various methods of getting such upper bounds. Studying them was initiated in the classical works of Weyl [105], van der Corput [93], and I. Vinogradov [98, 99] about a century ago. In this work, we apply some of these approaches together with the modern techniques to particular exponential sums and show the applications of the obtained bounds.

In the first part of the thesis (Chapters 3–5), we study the distribution of prime numbers from a special subset $\mathbb{E}(\alpha) \subset \mathbb{N}$ of integers in arithmetic progressions. We deal with the exponential sums over primes of the form

$$\sum_{\substack{p \leqslant X \\ p \equiv a \pmod q}} e^{2\pi i p^{\alpha}},$$

where p denotes a prime number in the arithmetic progression $qn + a$ such that a and q are coprime, $\alpha > 0$ is a fixed non-integer number.

As an application of the results about equidistribution of primes in arithmetic progressions in Chapter 6, we show the existence of bounded gaps between consecutive primes from $\mathbb{E}(\alpha)$.

Another well-known example of the exponential sum arising in many problems is

$$\sum_{n \leqslant X} a_n e^{2\pi i \alpha P(n)},$$

where $\alpha \in [0; 1]$ is fixed, $\{a_n\}$ is a sequence of real numbers, and $P(n)$ is a polynomial with real coefficients. In this case, it is clearly not possible to get the cancellation for all possible values of α. For example, if α is close to zero and all $a_n \approx 1$, the value of sum would be $\approx X$. However, in many cases it is enough to have a good upper bound for the value of this sum "on average" over $\alpha \in [0; 1]$. Many problems require an estimate for L^p-norm

$$\int_0^1 \left| \sum_{n \leqslant X} a_n e^{2\pi i \alpha P(n)} \right|^p d\alpha.$$

In Chapter 7, we obtain an upper bound for $p > 4$ in the particular case $a_n = \mu(n)$, $P(n) = n^2$, where $\mu(n)$ is the Mobius function (which is one if $n = 1$, zero if n is divisible by a square of a prime, and $(-1)^k$ if n is a product of k different primes). This is in turn related to the Fourier restriction theory for parabola and Sarnak's Conjecture, one of the most active areas of modern analysis, number theory, and dynamical systems.

Another example of exponential sum is related to the famous Riemann zeta function which is defined as

$$\zeta(s) = \sum_{n=1}^{+\infty} \frac{1}{n^s},$$

for Re $s > 1$, and can be analytically continued to the rest of the complex plane with a simple pole at $s = 1$. Riemann Hypothesis states that all non-trivial zeros of this function lie on the critical line $s = 1/2 + it$. Many problems in number theory require some knowledge about the behavior of zeta function on the critical line. This is in turn related to the behavior of the exponential sum

$$\sum_{n \leqslant x} \frac{1}{n^{1/2+it}}.$$

There is another bulk of methods to study the distribution of such sums as t varies within some interval, say $[T, 2T]$ for large fixed $T > 0$ or over a shorter one. The zeta function is also closely related to the distribution of prime numbers. Many of these techniques can also be applied to more general L-functions which are related to more general exponential sums

$$\sum_{n \leqslant x} \frac{a_n}{n^{1/2+it}}, \qquad L(a, s) = \sum_{n=1}^{+\infty} \frac{a_n}{n^s}.$$

In Chapters 8–9, we study the distribution of lattice points inside small balls on the surface of a 3-dimensional sphere when its radius grows to infinity. Counting the points inside such natural domains are related to another type of exponential sums called Weyl sums on the sphere. They in turn could be reduced to the oscillating sums over primes with the Fourier coefficients of holomorphic Hecke cusp forms

$$\sum_{p \leqslant x} \frac{\lambda_f(p)}{\sqrt{n}}.$$

Applying some of the recent moment techniques, we get sharp upper bounds on the first moment of corresponding $GL(2)$ L-functions conditional on Grand Riemann Hypothesis (in Chapter 8), and as an application, get a nearly sharp upper bound on the variance for lattice points on the sphere (in Chapter 9).

1.1 Distribution of primes in subsets

The questions about the distribution of prime numbers in subsets are among the most important and actively studied questions in analytic number theory. The first classical problem asks about the behavior of the prime counting function $\pi(X)$,

$$\pi(X) := \sum_{p \leqslant X} 1,$$

as $X \longrightarrow +\infty$. The answer is given by the Prime Number Theorem:

$$\pi(X) = \int_2^X \frac{du}{\log u} + R(X), \tag{1.2}$$

where the error term $R(X)$ is bounded from above by the function $Xe^{-c\sqrt{\log X}}$ with an absolute constant $c > 0$. This was proved independently by Hadamard and Poussin in 1896. A more precise bound of the error term

$$R(X) \ll Xe^{-c(\log X)^{3/5}(\log\log X)^{-1/5}}$$

was obtained by I. Vinogradov [100] and Korobov [53] in 1958. This result so far is essentially the best possible. The Riemann Hypothesis implies

$$R(X) \ll \sqrt{X}(\log X).$$

Here and later in the work, we use the Vinogradov notation $A \ll B$ to denote $A = O(B)$.

We study the distribution of primes inside the special subset of integers, which can be defined in terms of fractional parts in the following way:

$$\mathbb{E}(1/2) := \left\{n \in \mathbb{N} : \{\sqrt{n}\} < \sigma\right\}$$

for some fixed $0 < \sigma < 1$. For a more general case, we define the subset

$$\mathbb{E}(\alpha) := \left\{n \in \mathbb{N} : \{n^\alpha\} < \sigma\right\},$$

where $\alpha > 0$ is any fixed non-integer. The asymptotic formula for the proportion of primes from $\mathbb{E}(\alpha)$ in the case $0 < \alpha < 1$ was the first time obtained by Vinogradov [104] in 1940. Using his method of trigonometric sums, he proved the formula

$$\pi_{\mathbb{E}}(X) := \sum_{p \leqslant X, p \in \mathbb{E}} 1 = \sigma\pi(X) + O\left(X^{\vartheta(\alpha)+\varepsilon}\right), \tag{1.3}$$

where $\varepsilon > 0$ is arbitrarily small,

$$\vartheta(\alpha) = \max\left(\frac{4+\alpha}{5}, 1 - \frac{2}{15}\alpha\right) = \begin{cases} 1 - \frac{2}{15}\alpha, & \text{if } 0 < \alpha \leqslant \frac{3}{5}; \\ \frac{4+\alpha}{5}, & \text{if } \frac{3}{5} < \alpha < 1. \end{cases}$$

The asymptotic formula (1.3) meets the probabilistic expectations since the fraction of the coverage of the positive half of real line by $\mathbb{E}$ is σ. Later, Vinogradov proved a similar formula for arbitrarily fixed $\alpha > 6$ such that $\|\alpha\| \geqslant 3^{-\alpha}$, where $\|.\|$ denotes the distance to the nearest integer number. The exponent $\vartheta(\alpha)$ in the error term $R(X)$ is much weaker in this case:

$$\vartheta(\alpha) = 1 - \left(34 \cdot 10^6 \alpha^2\right)^{-1}$$

(see [101]). This result was later strengthened by Baker and Kolesnik [1] who obtained a similar formula for all $\alpha > 1$ with

$$\vartheta(\alpha) = 1 - \left(15 \cdot 10^3 \alpha^2\right)^{-1}.$$

The result was further improved for small values of $\alpha > 1$ by a number of authors (see, for example, [15]). The uniform bound on $R(X)$ for $\alpha > 1$ was obtained by Changa in 2003 [18].

I. Vinogradov gave an interesting interpretation of the subset of primes p satisfying the restriction $\{p^\alpha\} < \sigma$: all such primes lie in the intervals of the form

$$\left[k^{1/\alpha}; (k+\sigma)^{1/\alpha}\right), \qquad k = 1, 2, 3, \ldots.$$

So, for example, $\alpha = \sigma = 1/2$ corresponds to the intervals of the form $\left[k^2; (k + 1/2)^2\right)$. If $0 < \alpha < 1$, the length of such an interval clearly grows to infinity with k, whereas for $\alpha > 1$, it goes to zero. Thus, in the second case the intervals are short and most of them do not contain even a single integer. In some sense, one can think of them as of "random" primes chosen with probability $1/2$. This is the reason why the case $\alpha > 1$ seems to be more difficult.

In 1945, Linnik [56] suggested another approach to this problem based on the zero density theorems for the Riemann zeta function. Using this approach, Kaufman [50] in 1979 proved the existence of the infinite number of primes p from a very thin subset, precisely $\{\sqrt{p}\} < p^{-c+\varepsilon}$ for any fixed c,

$$0 < c < \frac{\sqrt{15}}{2(8+\sqrt{15})} = \frac{1}{6} - 0.00356\ldots$$

and arbitrarily small $\varepsilon > 0$ (in particular, this was an improvement of the earlier result of Vinogradov [100] corresponding to $c \leqslant 1/10$). Kaufman also showed that on RH, this result is valid for all $c \leqslant 1/4$. Later, Balog [2] and Harman [36] independently proved this result unconditionally. Finally, Harman and Lewis established the result for all $c \leqslant 0.262$ in [37].

In 1986, Gritsenko [34] had sharpened the bound for the error term in (1.3) for all $1/2 \leqslant \alpha < 1$ via Linnik's approach:

$$\vartheta(\alpha) = \begin{cases} 1 - \dfrac{\alpha}{2} + (\sqrt{3\alpha} - 1)^2, & \text{if } \frac{1}{2} \leqslant \alpha < \frac{3}{4}; \\ \dfrac{1+\alpha}{2}, & \text{if } \frac{3}{4} \leqslant \alpha < 1. \end{cases}$$

He also showed that for $\alpha = 1/2$, one can take $\vartheta(\alpha) = 4/5$. In the case of $0 < \alpha < 1/2$, the best known result is due to Ren [77]:

$$\vartheta(\alpha) = \max\left(\frac{2+\alpha}{3}, 1 - \frac{\alpha}{2}\right) = \begin{cases} 1 - \dfrac{\alpha}{2}, & \text{if } 0 < \alpha < \frac{2}{5}; \\ \dfrac{2+\alpha}{3}, & \text{if } \frac{2}{5} \leqslant \alpha < \frac{1}{2}. \end{cases}$$

1.2 Distribution of primes from subsets in arithmetic progressions

The next question, which arises naturally, is about the behavior of the function $\pi(X; q, a)$ counting the primes in arithmetic progression $qn + a$, $(a, q) = 1$:

$$\pi(X; q, a) := \sum_{\substack{p \leqslant X \\ p \equiv a \pmod q}} 1.$$

Since for fixed q the number of progressions is $\varphi(q)$ (where $\varphi(n)$ is the Euler totient function, which is the amount of numbers coprime with n not exceeding n), one can naturally expect the asymptotics of the form

$$\pi(X; q, a) \sim \frac{\pi(X)}{\varphi(q)}, \tag{1.4}$$

or, in other words, that the difference

$$R(X; q, a) = \pi(X; q, a) - \frac{\pi(X)}{\varphi(q)}$$

is small compared to the right hand side of (1.4). The last statement holds true not only for fixed q but also for slowly growing q so that $q \leqslant (\log X)^A$ with any fixed $A > 0$. This is known as Siegel-Walfisz theorem:

$$\pi(X; q, a) = \frac{\pi(X)}{\varphi(q)} + O(Xe^{-c_0(A)\sqrt{\log X}}).$$

The Grand Riemann Hypothesis implies a similar asymptotic formula for all $q \leqslant \sqrt{X}(\log X)^{-2}$. Unconditionally, this is only known for "almost all" $q \leqslant \sqrt{X}(\log X)^{-2}$. This statement is known as the theorem of Bombieri and A. Vinogradov [10, 96]. Precisely, it states that the following inequality holds true:

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \left| R(X; q, a) \right| = \sum_{q \leqslant Q} \max_{(a,q)=1} \left| \sum_{\substack{p \leqslant X \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{p \leqslant X} 1 \right| \leqslant \frac{c_1 X}{(\log X)^A} \tag{1.5}$$

for any $A > 0$, $0 < \varepsilon < 1/2$. Here $c_1 = c_1(A;\varepsilon)$, $Q = X^{\theta-\varepsilon}$, and the number $\theta = 1/2$ is usually called the "level of distribution" of a given sequence. The famous conjecture of Elliott and Halberstam states that for primes one can take any $\theta \leqslant 1$. The estimate (1.5) allows one to get the results comparable to the corollaries from GRH. One well-known application of the Bombieri-Vinogradov theorem is the Titchmarsh divisor problem. Another famous one concerns the existence of small gaps between consecutive primes. In the latter problem, a lot of progress has been achieved in recent years.

The next natural question is about the distribution of primes from aforementioned sets $\mathbb{E}(\alpha)$ in arithmetic progressions. In 1997 Tolev [92] obtained the analogue of the Bombieri-Vinogradov theorem for the primes from $\mathbb{E}(1/2)$ and all $q \leqslant X^{1/4-\varepsilon}$. In 2013, this result was improved by Gritsenko and Zinchenko [35]: they extended the theorem for the primes from $\mathbb{E}(\alpha)$ for all $1/2 \leqslant \alpha < 1$ and $q \leqslant X^{1/3-\varepsilon}$:

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\Bigg|\sum_{\substack{p\leqslant X, p\in\mathbb{E}\\ p\equiv a \pmod q}} 1 - \frac{1}{\varphi(q)}\sum_{p\leqslant X, p\in\mathbb{E}} 1\Bigg| \ll_{A,\varepsilon} \frac{X}{(\log X)^A}, \tag{1.6}$$

where $Q = X^{1/3-\varepsilon}$.

In this work, we show that a similar formula holds true for all non-integer $\alpha > 0$. This follows from the non-trivial estimate of the corresponding exponential sum over primes $p \equiv a \pmod q$:

Theorem 1.1. *Suppose that* $\alpha > 0$ *is a fixed non-integer,* θ, ε, D *are fixed constants satisfying the conditions* $0 < \varepsilon < \theta < 1/3$, $\varepsilon < \alpha/20$, $D > 1$, *and suppose that* $1 \leqslant h \leqslant (\log X)^D$, $2 < q \leqslant X^{\theta-\varepsilon}$, $1 \leqslant a \leqslant q-1$, $(a,q) = 1$. *Then the sum*

$$T = \sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} e^{2\pi i h p^{\alpha}}$$

satisfies the estimate

$$T \ll \frac{X^{1-\delta-\varepsilon^3/(3\alpha^2)}}{q},$$

where $0 < \delta \leqslant \varepsilon^3/(50\alpha^2)$, *and the implied constant depends on* α, ε, *and* D.

As a corollary, we obtain the asymptotic formula for the proportion of primes in arithmetic progressions from $\mathbb{E}(\alpha)$:

Corollary 1.1. *Let $\alpha > 0$ be a fixed non-integer $\varepsilon > 0$ an arbitrarily small number. Then for any $q \leqslant X^{1/3-\varepsilon}$, a, $(a, q) = 1$, and any given subinterval $I \subset [0; 1)$, the following asymptotic formula holds true:*

$$\pi_I(X; q, a) := \sum_{\substack{p \leqslant X \\ \{p^\alpha\} \in I \\ p \equiv a \pmod q}} 1 = |I| \cdot \pi(X; q, a) + O\left(\frac{\pi(X; q, a)}{(\log X)^A}\right)$$

for any fixed $A > 0$.

Another corollary of Theorem 1.1 is the analogue of the Bombieri-Vinogradov theorem with a level of distribution $1/3$:

Theorem 1.2. *Suppose that $\alpha > 0$ is fixed non-integer and let $\mathbb{E}$ be the set of integers n satisfying the condition $\{n^\alpha\} \in I = [c; d) \subset [0; 1)$ for given c and d. Further, let θ, ε and $A > 0$ be some fixed numbers such that $0 < \varepsilon < \theta < 1/3$, $\varepsilon < \alpha/20$, and let $2 < Q \leqslant X^{\theta-\varepsilon}$. Then the inequality*

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \Bigg| \sum_{\substack{X \leqslant p < 2X \\ p \equiv a \pmod q \\ p \in \mathbb{E}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{X \leqslant p < 2X \\ p \in \mathbb{E}}} 1 \Bigg| \leqslant \frac{\kappa X}{(\log X)^A}$$

holds for any $X \geqslant X_0(\alpha, \theta, \varepsilon)$ with some constant $\kappa > 0$ depending on $\alpha, \theta, \varepsilon$, and A.

The proof of Theorem 1.2 is contained in Section 3. The estimate given by Theorem 1.1 is proved in Section 4.

In the case of small values of α, we are able to improve these results. Precisely, we can get similar upper bounds for all $Q = X^{2/5-(3/5)\alpha}$:

Theorem 1.3. *Suppose that $0 < \alpha < 1/9$ is fixed non-integer, θ, ε, C are fixed constants satisfying the conditions $0 < \varepsilon < \alpha/100$, $\varepsilon < \theta < 2/5 - (3/5)\alpha$, $C \geqslant 1$, and suppose that $1 \leqslant h \leqslant (\log X)^C$, $2 < q \leqslant X^{\theta-\varepsilon}$, $1 \leqslant a \leqslant q - 1$, $(a, q) = 1$. Then, the sum*

$$T = \sum_{\substack{X \leqslant p < 2X \\ p \equiv a \pmod q}} e^{2\pi i h p^\alpha}$$

satisfies the estimate

$$T \ll \frac{X}{q} (\log X)^{-A} \tag{1.7}$$

with an arbitrarily large $A > 0$.

Corollary 1.2. *Let $0 < \alpha < 1/9$ be a fixed non-integer $\varepsilon > 0$ and arbitrarily small number. Then for any $q \leqslant X^{2/5-(3/5)\alpha-\varepsilon}$, a, $(a, q) = 1$, and any given subinterval $I \subset [0; 1)$, the following asymptotic formula holds true:*

$$\pi_I(X; q, a) := \sum_{\substack{p \leqslant X \\ \{p^\alpha\} \in I \\ p \equiv a \pmod q}} 1 = |I| \cdot \pi(X; q, a) + O\left(\frac{\pi(X; q, a)}{(\log X)^A}\right)$$

with any fixed $A > 0$.

Theorem 1.4. *Let $0 < \alpha < 1/9$ be fixed, $I = [c; d) \subset [0; 1)$, $\mathbb{E} = \{n \in \mathbb{N} : \{n^\alpha\} \in I\}$, and let θ, ε, A be fixed constants such that $0 < \varepsilon < \theta < 2/5 - (3/5)\alpha$, $\varepsilon < \alpha/100$, $A > 0$. Next, let $2 < Q \leqslant X^{\theta-\varepsilon}$. Then the following inequality holds true:*

$$\sum_{q \leqslant Q} \max_{(a,q)=1} \left| \sum_{\substack{p \leqslant X \\ p \in \mathbb{E} \\ p \equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{p \leqslant X \\ p \in \mathbb{E}}} 1 \right| \ll \frac{X}{(\log X)^A}.$$

The proof of Theorem 1.3 is contained in Chapter 4. Theorem 1.4 can be deduced from Theorem 1.3 in a similar way as Theorem 1.2 follows from Theorem 1.1 (see Chapter 3).

1.3 Van der Corput method

Proof of the estimates for the exponential sum over primes from both Theorem 1.1 and Theorem 1.3 is based on the same approach which goes back to the classical works of van der Corput [93]. The main idea is to transform the original sum, which is of the form

$$\sum_{\substack{X \leqslant p < 2X \\ p \equiv a \pmod q}} e^{2\pi i h p^\alpha} \tag{1.8}$$

to a smoothed sum over all integers, which is of the form

$$\sum_{Y < n \leqslant 2Y} e^{2\pi i t n^\alpha}, \tag{1.9}$$

where usually $Y \leqslant X$. The latter sum is less complicated and can be handled by van der Corput method. The main reason why the second sum is better is the absence of non-smooth weights $\mathbb{1}_{n=p}$, $\mathbb{1}_{n \equiv a \pmod q}$. The main difficulty is to attain a sufficiently large length Y in the new sum. In general, the larger length of the sum corresponds to the better saving in the final bound.

Van der Corput method is usually applied to smoothed sums of a more general form

$$\sum_{X\leqslant n<2X} e^{2\pi i f(n)},$$

where $f(x)$ is a real valued function having a certain number of derivatives on a given domain. If the length of the sum is X, one usually seeks for the upper bound of the form $X^{1-\beta}$ for any fixed $\beta > 0$ or at least a slowly decreasing function $\beta(X)$.

The main idea of the van der Corput approach reduces to the following two steps: the first is to replace the original phase $f(x)$ by its derivative $f'(x)$ by means of the following inequality:

$$\left|\sum_{X\leqslant n<2X} e^{2\pi i f(n)}\right| \ll \frac{X}{\sqrt{q}} + \left(\frac{X}{q} + \sum_{r=1}^{q-1}\left|\sum_{X\leqslant n<2X-r} e^{2\pi i (f(n+r)-f(n))}\right|\right)^{1/2}, \qquad (1.10)$$

where $q \leqslant X$. This can be easily proved by Cauchy inequality. The main idea is to increase the order of the derivative using mean value theorem $f(n+r) - f(n) = rf'(\xi)$ for some $\xi \in (n; n+r)$. This differencing process was first introduced by Weyl who applied it to polynomials.

The second step is based on the Poisson summation formula together with stationery phase estimate. Applying the Poisson summation, one replaces the original sum of length X by a new sum of length $\approx f'(X)$:

$$\sum_{n\sim X} e^{2\pi i f(n)} \approx \sum_{m\sim f'(X)} \int_{u\sim X} e^{2\pi i (f(u)-mu)} du.$$

It is often the case that the new sum is shorter so one can get a saving from just a trivial estimate of the new sum. Here and later, by "$n \sim X$" we mean "$X \leqslant n < 2X$." The oscillatory integral is estimated by the method of stationery phase: the idea is that the most contribution to the integral is coming from the neighborhood of the stationery point u_0 which is the zero of the first derivative of $f(u) - mu$. This follows from the fact that the integrand does not oscillate much in the neighborhood of the zero. This way, one gets the identity of the form

$$\sum_{n\sim X} e^{2\pi i f(n)} \approx e^{i\pi/4} \sum_{m\sim f'(X)} \frac{1}{\sqrt{f''(x_m)}} e^{2\pi i (f(x_m)-mx_m)}, \qquad f'(x_m) = m.$$

Here, one can gain from the shorter length of the new sum or from the large size of $f''(X)$.

The classical van der Corput k-derivative test consists of one application of Poisson summation and $(k-1)$ applications of van der Corput's differencing (1.10). For example, the second derivative test gives the estimate of the form

$$\sum_{n\sim X} e^{2\pi i f(n)} \ll X\lambda_2^{1/2} + \lambda_2^{-1/2},$$

where $\lambda_2 = f'(X)$ and the optimal choice of the parameter is $q = \lambda_2^{-1/2}$ (see also Lemma 2.3). The k-derivative test is given by the inequality

$$\sum_{n\sim X} e^{2\pi i f(n)} \ll X\lambda_k^{1/(2K-2)} + X^{1-2/K}\lambda_k^{-1/(2K-2)}, \qquad K = 2^{k-1}. \tag{1.11}$$

In general, one can apply both of these steps multiple times. This is the main idea of more advanced methods of exponent pairs.

The inequality (1.11) is closely connected to another important problem in analytic number theory called Vinogradov Mean Value Theorem. It was initiated by Vinogradov in 1935 (see [97]). The foundational conjecture in this area is stated as follows:

$$J_{s,k}(X) := \int_{[0;1]^k} \left| \sum_{n\leqslant X} e^{2\pi i(\alpha_1 n+\ldots+\alpha_k n^k)} \right|^{2s} d\alpha_1 \ldots d\alpha_k \ll_{s,k,\varepsilon} X^{\varepsilon}(X^s + X^{2s-\frac{1}{2}k(k+1)})$$

for all $X \geqslant 1$, $\varepsilon > 0$. Vinogradov's motivation was to obtain the bounds for individual sums from the bound for the mean value. Here, $J(s,k)$ can be interpreted as the number of integral solutions of the system of k equations

$$x_1^j + \ldots + x_s^j = x_{s+1}^j + \ldots + x_{2s}^j, \qquad 1 \leqslant j \leqslant k,$$

with $1 \leqslant x_i \leqslant X$ for $i = 1, \ldots, 2s$. The cases $k = 1, 2$ are trivial, for the proof see, for example, [69]. The case $k = 3$ was fully resolved by Wooley in a series of papers using his method of efficient congruencing (see [107]). In 2015 in a breakthrough work, Bourgain, Demeter, and Guth [12], using l^2-decoupling approach, resolved all the final cases of this conjecture for $k \geqslant 4$. This result impacts many famous problems in analytic number theory such as Waring's problem, Gauss circle problem, Dirichlet divisor problem, and many others. In particular, using these results Heath-Brown [52] got significantly more precise estimates for the Corput k-derivative test:

$$\sum_{n\sim X} e^{2\pi i f(n)} \ll X^{1+\varepsilon}\big(\lambda_k^{1/k(k-1)} + X^{-1/k(k-1)} + X^{-2/k(k-1)}\lambda_k^{-2/k^2(k-1)}\big).$$

These estimates allowed us to get better upper bound in Theorem 1.1 for the exponential sum T than the one which follows from the method of Gritsenko and Zinchenko [35].

1.4 Vinogradov-Vaughan decomposition

There are a few ways one can deal with non-smoothed sums of the form (1.8) to transform it to the smoothed form (1.9). The idea of the approach we apply in this work mostly goes back to I. Vinogradov, but later it was refined by Vaughan [94] and Heath-Brown [41]. In the Theorems 1.1 and 1.3, the original sum over primes by partial summation can be transformed to the form

$$\sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q}} \Lambda(n) e^{2\pi i h n^{\alpha}}, \tag{1.12}$$

where $\Lambda(n)$ is the Mangoldt function which equals $\log p$ if $n = p^k$ is a power of prime and zero otherwise. Roughly speaking, one can think of $\Lambda(n)$ as of the indicator function of primes. Due to some technical reasons, it is easier to work with the function defined in this way rather than with the actual indicator function of primes. Vaughan identity is the decomposition of the Mangoldt function of the form

$$\Lambda(n) = \sum_{\substack{b|n \\ b \leqslant y}} \mu(b) \log \frac{n}{b} - \sum_{b \leqslant y} \sum_{\substack{c \leqslant z \\ bc|n}} \mu(b)\Lambda(c) + \sum_{b > y} \sum_{\substack{c > z \\ bc|n}} \mu(b)\Lambda(c).$$

Then the sum (1.12) can be replaced by a double sum of the form

$$\sum_{M \leqslant m < 2M} \alpha_m \sum_{\substack{N \leqslant n < 2N \\ mn \equiv a \pmod q}} \beta_n e^{2\pi i h (mn)^{\alpha}},$$

where $MN \approx X$, α_m, β_n are real-valued coefficients (non-smooth in general). Depending on the relative sizes of M and N, there are two types of such sums. Type I sum corresponds to one "very long" variable (normally, of length $\geqslant X^{2/3}$) and one "short" variable (of length $\leqslant X^{1/3}$). Usually it is not hard to deal with such a sum because the corresponding weights are smooth, and one can directly apply Weyl's or van der Corput's methods. Type II sum is usually harder to estimate because both variables m and n are of similar sizes between $X^{1/3}$ and $X^{2/3}$ and, in particular, there is one critical range in the proof of Theorem 1.1 corresponding to $M = X^{1/3}$, $N = X^{2/3}$ (or vice versa), because in the inner sum over the progressions $qr + a$, there are only $\approx X^{1/3}/q$ terms. That means if q is large enough, the sum can only contain very few terms, and one cannot get any cancellation using the aforementioned approaches. This is the main reason why we cannot achieve a level of distribution better than $1/3$ using just Vaughan identity.

One can overcome these limitations by a more delicate combinatorial argument. This is the content of the so-called Heath-Brown identity which is essentially an iterated version of Vaughan identity. It allows us to get a better level of distribution in the case of small α (see Theorem 1.3). This time, the sum (1.12) can be written as the sum over k variables $d_1, \ldots, d_k$ where $d_1 \cdot \ldots \cdot d_k \approx X$. The parameters y and z can be adjusted so that the ranges of summation for the Type II sum get shorter: $X^{2/5+\varepsilon} \leqslant M, N \leqslant X^{3/5-\varepsilon}$, so that we avoid the critical range. The critical range now corresponds to the Type III sum which is basically a triple sum of the form

$$\sum_{M \leqslant m < 2M} f_1(m) \sum_{N \leqslant n < 2N} f_2(n) \sum_{\substack{K \leqslant k < 2K \\ mnk \equiv a \pmod q}} f_3(k) e^{2\pi i h (mnk)^\alpha},$$

where $M \approx N \approx K \approx X^{1/3}$, and the coefficients f_1, f_2, f_3 are smooth. Then the idea is to apply Poisson summation to replace each of the three sums of length $X^{1/3}$ by shorter sums and get a sufficient amount of saving estimating these new sums. More details on the combinatorics of Heath-Brown decomposition are given in Lemma 5.1 and Lemma 5.2.

1.5 Bounded gaps between primes

The problems about the behavior of the difference $p_{n+1} - p_n$ between consecutive primes p_n, p_{n+1} or, more generally, $p_{n+m} - p_n$ for fixed $m \geqslant 1$ are usually very hard. From prime number theorem, it follows that the difference $p_{n+1} - p_n$ is equal to $\log p_n$ "on average." However, there are many exceptional pairs with a much smaller or larger difference. The famous Twin Prime Conjecture states that there are infinitely many pairs p_n, p_{n+1} such that $p_{n+1} - p_n = 2$ (for example, 3 and 5, 5 and 7, 11 and 13, and so on).

In 1940, it was shown by Erdos that there exists $0 < c < 1$ such that the inequality

$$p_{n+1} - p_n \leqslant c \log p_n \tag{1.13}$$

holds true infinitely often. The result was sharpened by a number of authors. In 1988 Maier [60] showed that one can take $c = 0.2485\ldots$ In 2005 in a breakthrough work, Goldston, Pintz, and Yildirim [25] proved that the inequality (1.13) has infinitely many solutions for any arbitrarily small $c > 0$. In other words, they proved that

$$\liminf_{n \to +\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$

Later they were able to prove an even stronger result [28], namely,

$$\liminf_{n \to +\infty} \frac{p_{n+1} - p_n}{\sqrt{\log p_n}(\log \log p_n)^2} < +\infty.$$

In 2011 on the conference “Journees Arithmetiques — 27”, Pintz [70] announced the bound

$$\liminf_{n\to+\infty} \frac{p_{n+1}-p_n}{(\log p_n)^{3/7+\varepsilon}} < +\infty.$$

In 2013 in another breakthrough work. Zhang [108] showed the existence of infinitely many pairs p_{n+1}, p_n satisfying

$$p_{n+1} - p_n \leqslant C$$

for some absolute constant C. In particular, this result implies that there is some $k \leqslant C/2$ so that there are infinitely many solutions for the equation $p_{n+1} - p_n = 2k$. The first result stated $C = 7 \cdot 10^7$.

Further progress in this problem was achieved by Maynard [62] and Tao [72]. They significantly modified the Selberg sieve used in the previous works and were able to attain the value $C = 246$, which is the current record in this problem. They also showed that for any $m \geqslant 1$, one has the estimate

$$p_{n+m} - p_n \leqslant C_0 m^3 e^{4m}$$

for infinitely many pairs p_{n+m}, p_n. Here, C_0 is another absolute constant.

This result depends directly on the level of distribution of primes in arithmetic progressions. If one is able to get the level of distribution θ in the theorem of Bombieri-Vinogradov, then from the work of Maynard, it follows that one can prove an estimate of the form

$$p_{n+m} - p_n \leqslant C_1 m^3 e^{2m/\theta}. \tag{1.14}$$

For example, on Elliott–Halberstam conjecture, one can obtain the bound $p_{n+m} - p_n \leqslant C_1 m^3 e^{2m}$. In particular, Maynard showed that one can obtain $p_{n+1} - p_n \leqslant 12$.

The next question which arises naturally is what can be said about the small distances between consecutive primes from a given subset. A list of necessary conditions for this is given in [63]. The case of subset of primes related to the fractional parts of the polynomials $\{P(n)\}$ with the coefficients close to rational numbers with small denominators was considered in [7]. We explore the question about the bounded gaps for the set $\mathbb{E}(\alpha)$ with any $0 < \alpha < 1$ in Chapter 6 of this work. We use Theorem 1.2 as a corresponding analogue of the Bombieri-Vinogradov theorem.

Theorem 1.5. *Let* $\mathbb{E} = \{n \in \mathbb{N} : \{n^\alpha\} \in [c; d) \subset [0; 1)\}$ *for given* c *and* d, $0 < \alpha < 1$, $q_1, q_2, \dots, q_n, \dots$ *be all primes from* $\mathbb{E}$ *indexed in ascending order, and suppose that* $m \geqslant 1$ *is a fixed integer. Then*

$$\liminf_{n\to+\infty} (q_{n+m} - q_n) \leqslant 9\,700 m^3 e^{6m}.$$

This result is based on the fact that the sequence of primes from $\mathbb{E}(\alpha)$ has a level of distribution $1/3$. The result can be improved for $\alpha < 1/9$ if one replaces Theorem 1.2 by a stronger version of the analogue of Bombieri-Vinogradov theorem given by Theorem 1.4. A similar result can be proven for all non-integer $\alpha > 1$, but due to technicalities in this work, we restrict ourselves to the case $0 < \alpha < 1$.

1.6 Moments of exponential sums

Another class of important problems in analytic number theory and harmonic analysis relates to the behavior of the exponential sums on average. The most famous is probably the aforementioned Vinogradov Mean Value Theorem which deals with L^{2s}-norm over k-dimensional cube

$$J_{s,k}(X) = \int_{(0;1]^k} \left| \sum_{n \leqslant X} e^{2\pi i(\alpha_1 n + \ldots + \alpha_k n^k)} \right|^{2s} d\alpha_1 \ldots d\alpha_k \ll_{s,k,\varepsilon} X^{\varepsilon}\left(X^s + X^{2s-\frac{1}{2}k(k+1)}\right).$$

There are more general expressions of that sort related to the L^p-norms of various extension operators $E(\alpha, \beta)$ for various curves $(\mathbf{x}, \mathbf{y}(\mathbf{x}))$ applied to an arithmetic sequence a_n

$$\|Ea\|^p_{L^p(\mathbb{T}^k)} = \int_{\mathbb{T}^k} \left| \sum_{n \leqslant X} a_n e^{2\pi i(\alpha \mathbf{x} + \beta \mathbf{y}(\mathbf{x}))} \right|^p d\alpha d\beta.$$

Here, one usually seeks for the bound of the form

$$\|Ea\|_{L^p(\mathbb{T}^k)} \leqslant C_p\left(1 + X^{\frac{1}{2} - f(p)}\right) \|a\|_{l^2(\mathbb{Z})}.$$

In analytic number theory, one often deals with the one-dimensional L^p-norms

$$\int_0^1 \left| \sum_{n \leqslant X} a_n e^{2\pi i \alpha f(n)} \right|^p d\alpha,$$

where a_n is a given arithmetic sequence and $f(n)$ is a given phase. These moments appear in many other problems. The most studied examples in the literature are $a_n = 1, \Lambda(n), \mu(n), d(n)$ and $f(n) = n^k$ for $k \geqslant 1$. In terms of getting the asymptotic or the sharp upper and lower bounds, these problems are mostly very hard. Usually it is easier to deal with the even moments since in that case, the corresponding expression can be opened as a polynomial of $a_1, \ldots, a_n$. Thus, one natural approach is to apply Cauchy or Holder inequalities to bound the expression by the even moments.

Here we list a few known results of this type for the linear phase $f(n) = n$. For L^1-norm of $a_n = \Lambda(n)$, the works of Vaughan [95] and Goldston [24] give the bounds of the form,

$$\sqrt{X} \leqslant \int_0^1 \left|\sum_{n\leqslant X} \Lambda(n) e^{2\pi i\alpha n}\right| d\alpha \leqslant \left(\frac{\sqrt{2}}{2} + o(1)\right)\sqrt{X\log X}.$$

For the divisor function $a_n = \tau(n)$, the bounds

$$\sqrt{X} \ll \int_0^1 \left|\sum_{n\leqslant X} \tau(n) e^{2\pi i\alpha n}\right| d\alpha \ll \sqrt{X}\log X$$

are known from the work of Goldston and Pandey [26]. For higher moments and higher-order divisor function, see [67, 68]. For a_n, the indicator function of r-free numbers (the numbers which are not divisible by d^r for any d) and sharp upper and lower bounds were established in [5, 51]. For $a_n = \mu(n)$, the bounds

$$X^{\frac{1}{6}} \ll \int_0^1 \left|\sum_{n\leqslant X} \mu(n) e^{2\pi i\alpha n}\right| d\alpha \ll \sqrt{X}$$

are known from [3–5].

In this part of the work, we will focus on the case of higher moments of the Gauss sums ($f(n) = n^2$) with the Mobius function $a_n = \mu(n)$. In the case of the trivial sequence $a_n = 1$, the asymptotic formulas of the form

$$\int_0^1 \left|\sum_{n\leqslant X} e^{2\pi i\alpha n^2}\right|^p d\alpha \sim c_p X^{p-2}$$

for all real $p > 4$ and of the form

$$\int_0^1 \left|\sum_{n\leqslant X} e^{2\pi i\alpha n^2}\right|^p d\alpha \sim c_p X^{p/2}$$

for all real $0 < p \leqslant 4$ are known (see [46]). However, in the second case the constants c_p were not computed (except the case $p = 2$). For the explicit upper and lower bounds on c_1, see the recent work of Kalmynin [47].

In Chapter 7, we consider the L^p-norm of the Gauss sum with $a_n = \mu(n)$. The questions about Mobius correlations are of independent interest because of the famous Sarnak Conjecture which states that any sequence b_n observed by a deterministic dynamical system is uncorrelated to $\mu(n)$ or, in other words, one has

$$\sum_{n\leqslant X} b_n\mu(n) = o(X), \qquad X \to +\infty.$$

This is still far from being resolved in its full generality.

We show that the Mobius function is uncorrelated with the quadratic phase $e(n^2\alpha)$ on average:

Theorem 1.6. *For $p > 4$ and arbitrary $A > 0$, one has*

$$\int_0^1 \Big|\sum_{n\leqslant X} \mu(n) e^{2\pi i n^2 \alpha}\Big|^p d\alpha \ll_A \frac{X^{p-2}}{(\log X)^A}.$$

This result partially resolves one of the questions posted at the AIM Workshop "Arithmetic statistics, discrete restriction, and Fourier analysis" in February 2021.

The standard approach for estimating the moments of exponential sums is the circle method. The main idea is that the sum

$$\sum_{n\leqslant X} a_n e^{2\pi i \alpha n^2}$$

cannot be small for all real values of α, at least, if the sequence a_n does not oscillate. On the other hand, this sum cannot be always too large either. The circle method is based on the split of the integral to two roughly disjoint subsets called "major" and "minor" arcs. By the Dirichlet approximation theorem, each $\alpha \in [0; 1]$ can be approximated by a rational number a/q. If the denominator q is small, we say that α belongs to the major arcs. In that case, there is not much oscillation coming from the harmonic part $e(n^2\alpha)$, so we would essentially use the bound coming from the arithmetic side which follows from the prime number theorem

$$\sum_{n\leqslant X} \mu(n) \ll X \exp\big(-c\sqrt{\log X}\big)$$

with some absolute constant $c > 0$. If α is far from any rational with a small denominator, we say that it belongs to minor arcs, and in that case, we get the cancellation from the sums

$$\sum_{n\leqslant Y} e^{2\pi i \alpha n^2}$$

which can be obtained by Vinogradov-Vaughan decomposition similarly to the sums with $\Lambda(n)e(n^\alpha)$ from the previous sections. The key difference is that for the polynomial phase one cannot use van der Corput differencing any more since the derivatives of polynomials are either too large or vanish. We will apply Weyl differencing instead.

Apparently, the result of Theorem 1.6 cannot be significantly improved without assuming some strong conjectures about the zeros of Dirichlet L-functions.

1.7 Moments of L-functions

The Riemann zeta function is the central subject in analytic number theory. In the complex plane for Re$s > 1$, it is defined as

$$\zeta(s) := \sum_{n=1}^{+\infty} \frac{1}{n^s}.$$

In this region, this series converges absolutely. It has an analytic continuation to the rest of the complex plane with a simple pole at $s = 1$. Many problems in number theory require knowledge about the moments of zeta function on the critical line:

$$M_k(T) := \int_T^{2T} \left|\zeta\left(\frac{1}{2} + it\right)\right|^{2k} dt, \tag{1.15}$$

where k is a positive real number. Roughly speaking, the more moments are computed asymptotically, the more progress can be made towards proving the Riemann Hypothesis. But even on the assumption of the Riemann Hypothesis, getting the asymptotics for (1.15) is a very hard problem. Currently, the asymptotic formula is only known in the cases $k = 1$ due to Hardy and Littlewood and $k = 2$ due to Ingham (see [91]). These formulas meet the predictions from random matrix theory, which gives the formula

$$M_k(T) = c_k T (\log T)^{k^2} \big(1 + o(1)\big)$$

with c_k being the absolute constants.

Evaluating the moments of zeta function has a long history going back to classical works of Hardy, Littlewood, and Ingham. The sharp lower bounds of the form $M_k(T) \gg_k T(\log T)^{k^2}$ were established for all real $k \geqslant 1$ unconditionally by Radziwill and Soundararajan [73] and for all $k \geqslant 0$ by Ramachandra [74, 75] and Heath-Brown [40] on RH. Getting the upper bounds is a much harder problem. The Lindelof Hypothesis is equivalent to the estimate $M_k(T) \ll_{k,\varepsilon} T^{1+\varepsilon}$ for all natural numbers k, so it seems hard to get the upper bounds of the right order unconditionally. Currently, the bounds of the form $M_k(T) \ll T(\log T)^{k^2}$ are only known for $0 \leqslant k \leqslant 2$ due to the works of Soundararajan, Radziwill, Heap, Bettin and Chandee ([8, 39, 40]). On the assumption of RH, the sharp upper bounds of the form $T(\log T)^{k^2+\varepsilon}$ for arbitrarily small $\varepsilon > 0$ were established in a breakthrough work of Soundararajan [88]. This bound was sharpened to $C_k T(\log T)^{k^2}$ in the work of Harper [38]. Their methods extend to a wide class of L-functions.

In this work, we apply some of the techniques of Soundararjan and Harper to estimate the first moment of the product of standard $GL(2)$ L-functions corresponding to holomorphic Hecke cusp forms and show the application of these results to the distribution of lattice points on 2-sphere (also known as Linnik's Problem).

The idea of Soundararajan's on work conditional upper bounds goes back to Selberg who studied the distribution of values of $\log\zeta(1/2+it)$. His method works very well for the imaginary part of the logarithm, but leads to complications in the case of the real part because of zeros lying very close to the critical line. One of the ideas in Soundararajan's work is that, to get an upper bound for $\log|\zeta(1/2+it)|$, one actually does not need to explore the contribution from zeros since it is essentially negative. The following upper bound (see main Proposition in [88]) holds true on RH:

$$\log\left|\zeta(\frac{1}{2}+it)\right| \leqslant \mathrm{Re}\sum_{n\leqslant x}\frac{\Lambda(n)}{n^{1/2+it}\log(n)}+\frac{3}{4}\frac{\log T}{\log x}+R(x) \tag{1.16}$$

for any $t\in[T,2T]$, $2\leqslant x\leqslant T^2$, and $R(x)$ corresponds to the lower-order terms. Thus, the problem essentially reduces to the understanding of the behavior of the exponential sum over primes

$$\sum_{p\leqslant x}\frac{1}{p^{1/2+it}} \tag{1.17}$$

as t varies between T and $2T$. The key idea in both Soundararajan's and Harper's works is to split the latter Dirichlet polynomial to a sum of a few shorter polynomials and study its joint behavior. As soon as t gets relatively large, these polynomials behave roughly speaking as independent random variables due to the fact that one can think of p^{-it} as a random point on the unit circle, and for large t, these points behave almost independently for the different primes p. The idea of this approach is to show that the exponential sum (1.17) cannot be too large for too many $t\in[T,2T]$. So this is another example of an exponential sum which cannot have a very good pointwise upper bound, but rather can be estimated well on average.

In Soundararajan's approach, the expression for $M_k(T)$ was rewritten as

$$\int_T^{2T}\left|\zeta(\frac{1}{2}+it)\right|^{2k}dt=2k\int_{-\infty}^{+\infty}e^{2kV}\mathrm{meas}\left\{t\in[T,2T]:\log\left|\zeta(\frac{1}{2}+it)\right|\geqslant V\right\}dV.$$

In order to get an upper bound for $M_k(T)$, one needs to obtain an appropriate upper bound for the measure inside the integral. This measure depends on the size of V. This was achieved in Soundararajan's work by exploring the joint behavior of the real parts of two (short and long) Dirichlet polynomials

$$\sum_{p\leqslant z}\frac{1}{p^{1/2+it}},\qquad\sum_{z<p\leqslant x}\frac{1}{p^{1/2+it}}.$$

The additional saving comes from the right choice of the parameter z.

A slightly different approach leading to a sharper bound was elaborated in Harper's work. The long Dirichlet polynomial splits into the sum of many shorter ones of the form

$$\sum_{T^{\beta_{i-1}}<p\leqslant T^{\beta_i}} \frac{1}{p^{1/2+it}}, \qquad 0=\beta_0<\beta_1<\ldots<\beta_{I-1}<\beta_I,$$

and the better bound was obtained by exploring the joint distribution of the real parts of these polynomials.

Both approaches work well in the case of more general class of L-functions. Some variations of these methods were applied to the products of automorphic L-functions [64] and averages over fundamental discriminants of central values of quadratic twists [87].

In Chapter 8, we will follow the Harper's approach to prove the bound for the first moment of the product of $GL(2)$ L-functions:

Theorem 1.7. *Assuming the Grand Riemann Hypothesis, we have*

$$\sum_{f\in S_{2m+2}(\Gamma_0(2))} \frac{L\big(\frac{1}{2},f\big)L\big(\frac{1}{2},f\otimes\chi_{-n}\big)}{L\big(1,\mathrm{Sym}^2 f\big)} \leqslant (2m+2)L(1,\chi_{-n})\exp\big\{U(n,m)\big\}, \quad (1.18)$$

where $m>n^{\varepsilon}$ for some $\varepsilon>0$ and

$$U(n,m)=\max\left(1,\frac{1}{4}\frac{\log n}{\log m}\right)\exp\left\{600\max\left(1,\frac{1}{4}\frac{\log n}{\log m}\right)\right\}.$$

In this case, the expression analogous to the right hand side of (1.16) can also be obtained:

$$\log\Big|L\big(\frac{1}{2},f\big)L\big(\frac{1}{2},f\otimes\chi_{-n}\big)\Big| \leqslant \sum_{p\leqslant x}\frac{\lambda_f(p)}{\sqrt{p}}+c\frac{\log\big(d^2m^4\big)}{\log x}+R(x),$$

where $\lambda_f(.)$ are the normalized Fourier coefficients of the corresponding cusp forms. So we can further split that expression to a sum of many Dirichlet polynomials

$$\sum_{x^{1/3}<p\leqslant x}\frac{\lambda_f(p)}{\sqrt{p}}+\sum_{x^{1/9}<p\leqslant x^{1/3}}\frac{\lambda_f(p)}{\sqrt{p}}+\ldots+\sum_{x^{3^{-I}}<p\leqslant x^{3^{1-I}}}\frac{\lambda_f(p)}{\sqrt{p}}+\sum_{1<p\leqslant x^{3^{-I}}}\frac{\lambda_f(p)}{\sqrt{p}}$$

and study the joint behavior of all of them as the function f varies in a Hecke basis of holomorphic cusp forms.

The upper bound for the measure of the "exceptional sets" of $t \in [T; 2T]$ or $f \in S_k(\Gamma_0(2))$, where a given Dirichlet polynomial is large, is obtained by Markov-type inequalities. This gives the upper bounds containing high moments of the corresponding Dirichlet polynomials

$$\int_T^{2T} \left| \sum_{z<p\leqslant x} \frac{1}{p^{1/2+it}} \right|^{2N} dt, \qquad \sideset{}{^h}\sum_{f\in S_k(\Gamma_0(2))} \left| \sum_{z<p\leqslant x} \frac{\lambda_f(p)}{\sqrt{p}} \right|^{2N}.$$

To treat such moments, one needs an analogue of the mean-value theorem. In Harper's work, he gets the necessary upper bounds from the asymptotic formulas for the integrals of the form

$$\int_T^{2T} \prod_{i=1}^r \big(\cos(t\log p_i)\big)^{\alpha_i}\, dt.$$

Such integrals appear after the expanding of the moments from the application of Markov inequality. The analogous mean-value theorem in our case is based on the Petersson trace formula

$$\sideset{}{^h}\sum_{f\in S_k(\Gamma_0(2))} \lambda_f(n)\lambda_f(m) = c_1 k \mathbb{1}_{n=m} + E_k, \qquad E_k \leqslant c_2 k e^{-k},$$

where c_1, c_2 are absolute constants. To treat the high moments of corresponding sums over $f \in S_k(\Gamma_0(2))$, we prove the multidimensional analogue of Petersson's formula. This way, we would be able to compute the asymptotics of the sums of the form

$$\sideset{}{^h}\sum_{f\in S_k(\Gamma_0(2))} \prod_{j=1}^r \lambda_f(p_j)^{\beta_j}.$$

In particularly, we will show that the contribution from most tuples of primes $(p_1, \ldots, p_r)$ is small (for more details, see Lemma 8.3).

1.8 Variance estimates in Linnik's problem

We apply the moment result to the well-known problem about the distribution of lattice points on the surface of the 3-dimensional sphere. One can imagine each integer solution of the equation $x^2 + y^2 + z^2 = n$ as a point (x, y, z) on the surface of the sphere with a center at the origin and radius $\sqrt{n}$. What is known about the number of such solutions as n tends to infinity? Asymptotically, we have around $\sqrt{n}$ solutions if n is squarefree and is not 0, 4, or 7 modulo 8. The conjecture of Linnik states that these points become equidistributed on the surface of the sphere with growing n. First time it was proved by Linnik on Grand Riemann Hypothesis [56]. Unconditionally, the problem was solved by Duke [19] and independently by Golubeva and Fomenko [32] (both after a breakthrough work of Iwaniec [43]).

We are interested in evaluating the variance of this distribution in small caps that are randomly rotated along the sphere. The variance is given by the expression of the form

$$V\big(n;\Omega_n(\mathbf{x})\big) := \int_{SO(3)} \left|Z(n, g\Omega_n) - \sigma(\Omega_n)N_n\right|^2 dg,$$

where N_n is the total number of lattice points on the sphere of radius $\sqrt{n}$ for given n, Z is the number of points inside a given spherical cap $\Omega_n(\mathbf{x})$ with the center at $\mathbf{x}$, and the area $\sigma(\Omega_n)$ is normalized so that $\sigma(S^2) = 4\pi$. The integration goes over all random rotations of the sphere, and dg is the Haar probability measure.

Conjecture (Bourgain, Rudnick, Sarnak, [13])**.** *Let Ω_n be a sequence of spherical caps, or annuli. If $N_n^{-1+\varepsilon} \ll \sigma(\Omega_n) \ll N_n^{-\varepsilon}$ as $n \to +\infty$, $n \neq 0, 4, 7 \pmod 8$, then*

$$\int_{SO(3)} \left|Z(n; g\Omega_n) - N_n\sigma(\Omega_n)\right|^2 dg \sim N_n\sigma(\Omega_n).$$

In their paper, Bourgain, Rudnick, and Sarnak obtained the upper bound for the left hand side of (1.18) assuming the Generalized Lindelof Hypothesis for the corresponding class of $GL(2)$ L-functions.

Theorem (Bourgain, Rudnick, Sarnak, [13])**.** *Let Ω_n be a sequence of spherical caps, or annuli. Assume the Lindelof Hypothesis for standard $GL(2)/\mathbb{Q}$ L-functions. Then for squarefree $n \neq 7 \pmod 8$, we have*

$$\int_{SO(3)} \left|Z(n; g\Omega_n) - N_n\sigma(\Omega_n)\right|^2 dg \ll_\varepsilon n^\varepsilon N_n\sigma(\Omega_n), \qquad \forall\varepsilon > 0.$$

Assuming GRH for these L-functions, we obtain the upper bound of the right order of magnitude:

Theorem 1.8. *Assume the Grand Riemann Hypothesis for $GL(2)/\mathbb{Q}$ L-functions. Then for squarefree $n \neq 7 \pmod 8$, we have*

$$\int_{SO(3)} \left|Z(n; g\Omega_n) - N_n\sigma(\Omega_n)\right|^2 dg \leqslant c\sigma(N_n)N_n,$$

where $N_n^{-1+\varepsilon} \ll \sigma(\Omega_n) \ll N_n^{-\varepsilon}$ and c is an absolute constant.

The proof of this theorem is contained in Chapter 9. Counting the points inside the natural domains (such as ball or annuli) on the sphere is directly related to the Weyl sums on the sphere. One can choose an orthonormal basis of spherical harmonics with respect to the standard Haar measure so that these harmonics are the eigenfunctions of the Laplacian and a Hecke operator for the group of rotations of the sphere $SO(3)$ corresponding to given N_n lattice points (for more details, see the paper of Lubotzky, Phillips, and Sarnak [59]). This basis can be constructed from the Legendre polynomials $P_n(x)$,

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

The space of $L^2(S^2)$ functions decomposes under the Laplacian as

$$\bigoplus_{m=0}^{+\infty} H_m,$$

where H_m is the space of spherical harmonics of degree m. We denote them by $\phi_{j,m}(\mathbf{x})$. Then, the indicator function of a ball with a center at point $\mathbf{y}$ has a Fourier expansion of the form

$$\mathbb{1}_{\mathbf{x}\in\Omega_n(\mathbf{y})} = \sum_{m=0}^{+\infty} h(m)\sum_{j=1}^{2m+1}\phi_{j,m}(\mathbf{x})\phi_{j,m}(\mathbf{y}),$$

where $h(m)$ is Selberg-Harish-Chandra transform for the sphere (for more details, see Section 9.2 and [42]). Then, one can express the variance in terms of spherical Weyl sums as follows:

$$V\big(n;\Omega_n(\mathbf{x})\big) = \sum_{m=1}^{+\infty} h^2(m)\sum_{j=1}^{2m+1}\left|\sum_{i=1}^{N_n}\phi_{j,m}(\mathbf{x}_i)\right|^2,$$

where $\mathbf{x}_1,\ldots,\mathbf{x}_{N_n}$ are all the lattice points.

Instead of dealing with a rather complicated exponential sum on the sphere, we move directly to the sum of the central values of $GL(2)$ L-functions. By Jacquet-Langlands correspondence, there is a bijection between Hecke eigenfunctions $\phi_{j,m}$ and holomorphic newforms $f_{j,m}$ of weight $2m+2$ and level 2. This way, one can apply the bound

$$\sum_{m=1}^{+\infty} h^2(m)\sum_{j=1}^{2m+1}\left|\sum_{i=1}^{N_n}\phi_{j,m}(\mathbf{x}_i)\right|^2 \leqslant \sum_{m=1}^{+\infty} h^2(m)\sum_{j=1}^{2m+1}\frac{c\sqrt{n}L(\frac{1}{2},f_{j,m})L(\frac{1}{2},f_{j,m}\otimes\chi_{-n})}{L(1,\mathrm{Sym}^2 f_{j,m})}$$

and further deal with the first moment of the central values of L-functions. This transition is one of the outreaches of the Langlands program which is a very active research area in modern number theory, algebra, and geometry.

Chapter 2

NOTATION AND AUXILIARY LEMMAS

In this work, we will use the following standard notation:

p — prime numbers; $p_1 < p_2 < \ldots < p_n < \ldots$ — prime numbers enumerated in the ascending order;

$\pi(x)$ — number of primes less than x;

$\{x\} = x - \lfloor x \rfloor$ — fractional part of x;

$\|x\|$ — distance to the nearest integer;

$e(x) = e^{2\pi i x}$;

$\Lambda(n)$ — Mangoldt function, which is $\log p$ if $n = p^k$ and zero otherwise;

$\mu(n)$ — Mobius function, which is one if $n = 1$, zero if n is divisible by a square of some prime, and $(-1)^k$ if n is a product of k different primes;

$\tau(n)$ — divisor function n;

$\tau_k(n)$ — number of solutions of the equation $x_1 \ldots x_k = n$ in positive integers $x_1, \ldots, x_k$;

$\varphi(n)$ — Euler totient function, which is the amount of numbers coprime with n do not exceeding n;

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ — binomial coefficient;

$\binom{n}{n_1, \ldots, n_k} = \frac{n!}{n_1! \ldots n_k!}$ — multinomial coefficient;

$(\alpha)_k = \prod_{i=1}^{k} (\alpha - i + 1)$ — Pochhammer symbol;

$(n_1, \ldots, n_k)$ — greatest common divisor;

$\mathbb{Z}_{\geqslant 0}$ — non-negative integers;

$C^\infty(\mathbb{R})$ — smooth real values functions;

$A \ll B$ — Vinogradov sign, which means $A = O(B)$ or, in other words, there is $c > 0$ such that $|A| \leqslant cB$;

$A \asymp B$ — Hardy symbol, which means that both $A \ll B$ and $B \ll A$ hold true;

$\tau(\chi; n)$ — Gauss sum,

$$\tau(\chi; n) = \sum_{l=1}^{q-1} \chi(l) e\left(\frac{ln}{q}\right);$$

$S_q(n,m)$ — Kloostermann sum,

$$S_q(n,m) = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} e\left(\frac{ml + nl^*}{q}\right),$$

where $ll^* \equiv 1 \pmod q$;

$\mathbb{E}(\alpha)$ — the subset of natural numbers satisfying the condition $\{n^\alpha\} < 1/2$ or $\{n^\alpha\} < \sigma$;

$S_k(\Gamma_0(2))$ — orthonormal basis of holomorphic Hecke cusp forms corresponding to the congruence subgroup $\Gamma_0(2)$;

$\sum^h$ — the normalized sum over the cusp forms:

$$\sum_{f \in S_k(\Gamma_0(2))}^{h} := \sum_{f \in S_k(\Gamma_0(2))} \frac{1}{L(1, \mathrm{Sym}^2 f)}.$$

Lemma 2.1 (Partial summation)**.** *Let $c_n \in \mathbb{C}$ and $C(x) = \sum_{a<n\leqslant x} c_n$. Let $f(x)$ be a complex valued smooth on $[a; b]$ function. Then*

$$\sum_{a<n\leqslant b} c_n f(n) = C(b)f(b) - \int_a^b C(x) f'(x) dx.$$

The proof can be found in [17, Ch. 1].

Lemma 2.2 ("Vinogradov cups")**.** *Let $r \geqslant 1$ be integer, $0 < \Delta < 1/4$, α, β are real numbers such that $0 \leqslant \alpha < \beta < 1$, $\Delta \leqslant \beta - \alpha \leqslant 1 - \Delta$. Then there is the 1-periodic function $\psi(x)$, satisfying the conditions*

- $\psi(x) = 1$ *if* $\alpha + \frac{\Delta}{2} \leqslant x \leqslant \beta - \frac{\Delta}{2}$*;*
- $0 < \psi(x) < 1$ *if* $\alpha - \frac{\Delta}{2} < x < \alpha + \frac{\Delta}{2}$ *and* $\beta - \frac{\Delta}{2} < x < \beta + \frac{\Delta}{2}$;
- $\psi(x) = 0$ *in* $\beta + \frac{\Delta}{2} \leqslant x \leqslant 1 + \alpha - \frac{\Delta}{2}$*;*
- $\psi(x)$ *has a Fourier expansion of the form*

$$\psi(x) = \beta - \alpha + \sum_{\substack{m=-\infty \\ m\neq 0}}^{+\infty} g(m) e^{2\pi i m x},$$

where

$$|g(m)| \leqslant \min\left(\beta - \alpha, \frac{1}{\pi|m|}, \frac{1}{\pi|m|}\left(\frac{r}{\pi|m|\Delta}\right)^r\right).$$

For the proof, see [48, Ch. 1].

Lemma 2.3 (Van der Corput k-derivative test)**.** *Let $k \geqslant 2$, $K = 2^{k-1}$, $b - a \geqslant 1$, $f(x)$ is a real valued function satisfying on $[a; b]$ the inequalities*

$$0 < \lambda_k \leqslant f^{(k)}(x) \leqslant h\lambda_k.$$

Then the following estimate holds true:

$$\sum_{a<n\leqslant b} e\big(f(n)\big) \ll h^{2/K}(b-a)\lambda_k^{1/(2K-2)} + (b-a)^{1-2/K}\lambda_k^{-1/(2K-2)},$$

where the constant in $\ll$ is absolute.

The proof can be found in [48, Ch. 1].

Lemma 2.4 (Heath-Brown k-derivative test)**.** *With the assumptions of Lemma 2.3 when $k \geqslant 3$ and arbitrarily small $\varepsilon > 0$ we have a more precise estimate*

$$\sum_{a<x\leqslant b} e\big(f(n)\big) \ll_{h,k,\varepsilon} (b-a)^{1+\varepsilon}\big(\lambda_k^{1/(k(k-1))} + (b-a)^{-1/(k(k-1))} + (b-a)^{-2/(k(k-1))}\lambda_k^{-2/(k^2(k-1))}\big).$$

For the proof, see [52].

Lemma 2.5 (Mardzhanishvili inequalities)**.** *For the numbers*

$$T_k^{(l)} = \sum_{m=1}^{N} \tau_k^l(m)$$

for any $k, l \geqslant 1$, the following inequalities hold true:

$$T_k^{(l)} < A_{k,l}N(\log N + k^l - 1)^{k^l-1},$$

where

$$A_{k,l} = k^l(k!)^{-(k^l-1)/(k-1)}.$$

The proof is in [61].

Chapter 3

AN ANALOGUE OF THE BOMBIERI-VINOGRADOV THEOREM

In this chapter, we prove Theorem 1.2 and Theorem 1.4 assuming Theorem 1.1 and Theorem 1.3 correspondingly. We need the following pointwise estimate for the standard exponential sum over primes:

Lemma 3.1. *Let* $\alpha > 0$, $\alpha \notin \mathbb{N}$ *is a fixed number,* $0 < h < \min\bigl(X^{\alpha/3}, X^{10^{-7}}\bigr)$. *Then, for the sum*

$$S(X) = \sum_{X \leqslant p < 2X} e\bigl(hp^{\alpha}\bigr),$$

the following inequalities hold true:

- *(I. Vinogradov) for* $0 < \alpha < 1$

$$S(X) \ll_{\alpha} \min\left(X, \sqrt{h}X^{\alpha/2} + \frac{X^{1-\alpha/2}}{\sqrt{h}}\right);$$

- *(Changa) for* $\alpha > 1$,

$$S(X) \ll_{\alpha} X^{1-\gamma/\alpha^2}(\log X)^2,$$

where $\gamma = 6 \cdot 10^{-11}$.

The proof of the first estimate can be found in [104]. For the second one, see [18].

Denote by $\chi(x)$ the indicator function of the interval $I = \bigl[c; d\bigr)$. Fixing some constant $B > 0$, we set $\Delta = (\log X)^{-B}$, $r = \lfloor H\Delta \rfloor$, and $H = \Delta^{-1}\lceil \log_2 X \rceil$. Then by Lemma 2.2, there exists a 1-periodic function $\psi(x)$ such that $\psi(x) = 1$ if $c + \Delta \leqslant x \leqslant d - \Delta$, $\psi(x) = 0$ if $x \in [0; c] \cup [d; 1]$, $0 < \psi(x) < 1$ if $x \in \bigl(c; c+\Delta\bigr) \cup \bigl(d-\Delta; d\bigr)$; moreover, $\psi(x)$ has the Fourier expansion of the form

$$\psi(x) = d - c - \Delta + \sum_{\substack{h=-\infty \\ h \neq 0}}^{+\infty} g(h)e(hx),$$

$$|g(h)| \leqslant \min\left(d - c - \Delta, \frac{1}{\pi|h|}, \frac{1}{\pi|h|}\left(\frac{r}{\pi|h|\Delta}\right)^{r}\right). \tag{3.1}$$

Therefore, setting

$$\mathbb{E}_\Delta = \Big\{ n \in \mathbb{N} : \{n^\alpha\} \in \big(c; c+\Delta\big) \cup \big(d-\Delta; d\big) \Big\},$$

we obviously get

$$\sum_{q\leqslant Q} \max_{(a,q)=1} \Bigg| \sum_{\substack{X\leqslant p<2X \\ n\equiv a \pmod q}} \chi\big(\{p^\alpha\}\big) - \frac{1}{\varphi(q)} \sum_{X\leqslant p<2X} \chi\big(\{p^\alpha\}\big) \Bigg| \leqslant S^{(1)} + S^{(2)} + S^{(3)},$$

where

$$S^{(1)} = \sum_{q\leqslant Q} \max_{(a,q)=1} \Bigg| \sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q}} \psi(p^\alpha) - \frac{1}{\varphi(q)} \sum_{X\leqslant p<2X} \psi(p^\alpha) \Bigg|,$$

$$S^{(2)} = \sum_{q\leqslant Q} \max_{(a,q)=1} \sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q \\ p\in\mathbb{E}_\Delta}} 1, \qquad S^{(3)} = \sum_{q\leqslant Q} \frac{1}{\varphi(q)} \sum_{\substack{X\leqslant p<2X \\ p\in\mathbb{E}_\Delta}} 1.$$

Using (3), we find

$$S^{(1)} \leqslant \big(d-c-\Delta\big) \sum_{q\leqslant Q} \max_{(a,q)=1} \Bigg| \sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q}} 1 - \frac{1}{\varphi(q)} \sum_{X\leqslant p<2X} 1 \Bigg| + \left(\sum_{0<|h|\leqslant H} + \sum_{|h|>H} \right) |g(h)| \sum_{q\leqslant Q} \max_{(a,q)=1} \Bigg| \sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q}} e\big(hp^\alpha\big) - \frac{1}{\varphi(q)} \sum_{X\leqslant p<2X} e\big(hp^\alpha\big) \Bigg|. \quad (3.2)$$

By Bombieri-Vinogradov theorem, the first term in the right-hand side in (3.2) is estimated as $X(\log X)^{-C}$ for any fixed $C > 0$. Trivial estimate of the inner sums over p for $|h| > H$ together with (3) yields:

$$\sum_{q\leqslant Q} \max_{(a,q)=1} \sum_{|h|>H} |g(h)| \cdot \Bigg| \sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q}} e\big(hp^\alpha\big) - \frac{1}{\varphi(q)} \sum_{X\leqslant p<2X} e\big(hp^\alpha\big) \Bigg| \leqslant 2X \sum_{q\leqslant Q} \sum_{|h|>H} \frac{1}{\pi|h|} \left(\frac{r}{\pi\Delta|h|} \right)^r \leqslant 4X \sum_{q\leqslant Q} \frac{1}{\pi r} \left(\frac{r}{\pi\Delta(H-1)} \right)^r \leqslant X \sum_{q\leqslant Q} \frac{1}{\pi} \left(\frac{1}{2} \right)^r \leqslant Q. \quad (3.3)$$

Next, the contribution coming from $0 < |h| \leqslant H$ does not exceed

$$\sum_{q\leqslant Q}\max_{(a,q)=1}\sum_{0<|h|\leqslant H}|g(h)|\left(\left|\sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} e\big(hp^{\alpha}\big)\right|+\frac{1}{\varphi(q)}\left|\sum_{X\leqslant p<2X} e\big(hp^{\alpha}\big)\right|\right) = S^{(4)}+S^{(5)}.$$

Using the estimates of the sum over primes p, $X \leqslant p < 2X$, given in [104] (for $0 < \alpha < 1$) and [18] (for $\alpha > 1$), we get

$$S^{(5)} \leqslant \sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{0<|h|\leqslant H}\frac{1}{\pi|h|}\left|\sum_{X\leqslant p<2X} e\big(hp^{\alpha}\big)\right| \ll_{\alpha,\varepsilon_1}$$
$$X^{1-\upsilon(\alpha)+\varepsilon_1}(\log Q)(\log H) \ll_{\alpha,\varepsilon_1} X^{1-\upsilon(\alpha)+2\varepsilon_1}$$

for arbitrarily small $\varepsilon_1 > 0$ and

$$\upsilon(\alpha) = \begin{cases} \alpha/2, & \text{if } 0<\alpha<1;\\ \dfrac{6\cdot 10^{-11}}{\alpha^2}, & \text{if } \alpha>1,\ \alpha\notin\mathbb{N}. \end{cases}$$

Similarly, the estimates of Theorems 1.1 and 1.3 yield:

$$S^{(4)} \leqslant \sum_{0<|h|\leqslant H}\frac{1}{\pi|h|}\sum_{q\leqslant Q}\max_{(a,q)=1}\left|\sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} e\big(hp^{\alpha}\big)\right| \ll X^{1-\varepsilon^3/(3\alpha^2)}\log H.$$

Let $\Delta_1 = \Delta/10$, $r_1 = \lfloor H_1\Delta_1\rfloor$, $H_1 = \Delta_1^{-1}\lceil \log_2 X\rceil$ and denote by $\psi_1(x)$ and $\psi_2(x)$ Vinogradov's cups such that $\psi_1(x) = 1$ if $x \in (c; c+\Delta)$, $0 < \psi_1(x) < 1$ if $x \in (c-\Delta_1; c) \cup (c+\Delta; c+\Delta+\Delta_1)$, and $\psi_1(x) = 0$ otherwise; $\psi_2(x) = 1$ if $x \in (d-\Delta; d)$, $0 < \psi_2(x) < 1$ if $x \in (d-\Delta-\Delta_1; d-\Delta) \cup (d; d+\Delta_1)$, and $\psi_2(x) = 0$ otherwise. Let us denote by $g_1(h)$ and $g_2(h)$ its Fourier coefficients. Then,

$$S^{(2)} \leqslant \sum_{q\leqslant Q}\max_{(a,q)=1}\left|\sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} \big(\psi_1(p^{\alpha})+\psi_2(p^{\alpha})\big)\right| \leqslant$$
$$2(\Delta+\Delta_1)\sum_{q\leqslant Q}\max_{(a,q)=1}\sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} 1 + \sum_{h\neq 0}\Big(|g_1(h)|+|g_2(h)|\Big)\cdot$$
$$\sum_{q\leqslant Q}\max_{(a,q)=1}\left|\sum_{\substack{X\leqslant p<2X\\ p\equiv a \pmod q}} e\big(hp^{\alpha}\big)\right| \quad (3.4)$$

and, similarly,

$$S^{(3)} \leqslant 2(\Delta+\Delta_1)\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\sum_{X\leqslant p<2X}1+ \sum_{h\neq 0}\bigl(|g_1(h)|+|g_2(h)|\bigr)\sum_{q\leqslant Q}\frac{1}{\varphi(q)}\Bigl|\sum_{X\leqslant p<2X}e\bigl(hp^{\alpha}\bigr)\Bigr|. \quad (3.5)$$

Trivially, the first terms in the right hand side of (3.4) and (3.5) do not exceed $2\Delta X\log X$, and the second terms can be estimated similarly to $S^{(4)}$, $S^{(5)}$, and the sum in the left hand side of (3.3). To finish the proof, we choose $C=A$, $B=A+1$.

Remark. *In a similar way, one deduces corollaries 1.1 and 1.2 from Theorems 1.1 and 1.3.*

Chapter 4

THE EXPONENTIAL SUM ESTIMATE FOR NON-INTEGER α WITH THE LEVEL OF DISTRIBUTION $1/3$

In this chapter, we prove Theorem 1.1. The proof is based on the estimation of the sum over primes

$$\sum_{\substack{X\leqslant p<2X \\ p\equiv a \pmod q}} e\big(hp^{\alpha}\big),$$

which can be transformed into the sum over integers

$$\sum_{\substack{X\leqslant n<Y \\ n\equiv a \pmod q}} \Lambda(n)e\big(hn^{\alpha}\big)$$

by partial summation. This sum can be decomposed to the sum of several double sums using the standard tool called Vaughan identity. These double sums are of the form

$$W_I = \sum_{M\leqslant m<2M} \alpha_m \sum_{\substack{N\leqslant n<2N \\ mn\equiv a (\mathrm{mod}\ q)}} f(n)e\big(h(mn)^{\alpha}\big), \tag{4.1}$$

$$W_{II} = \sum_{M\leqslant m<2M} \alpha_m \sum_{\substack{N\leqslant n<2N \\ mn\equiv a (\mathrm{mod}\ q)}} \beta_n e\big(h(mn)^{\alpha}\big). \tag{4.2}$$

Here $MN \approx X$, α_m, β_n are real, and $f(x)$ is a smooth function. The estimation of $|W_I|$ is easier compared to $|W_{II}|$: in the first case, the inner sum is of size $\geqslant N$, which is $\gg X^{2/3}$, and the smooth weight $f(n)$ can be removed by partial summation. For W_{II}, the weights α_m and β_n are not smooth in general, so one should apply Cauchy inequality, but this time both inner and outer sums have lengths $\ll X^{2/3}$. The restriction $mn \equiv a \pmod q$ is removed by substitution $n = qr + l$ (or $m = qr + l$). The new "smooth" sums are of size N/q (or M/q) and they can be evaluated by Corput k-derivative test, where k depends on α. To handle type II sums (and type I if $\alpha > 1$), the classical estimates are not powerful enough. In this case, we would apply Lemma 2.4.

4.1 Vaughan identity

We use the following form of Vaughan identity:

Lemma 4.1. *Let* $1 \leqslant V \leqslant X$*. Then for any complex valued* $f(x)$*, one has the identity*

$$\sum_{V<n\leqslant X} \Lambda(n) f(n) = \sum_{d\leqslant V} \mu(d) \sum_{l\leqslant Xd^{-1}} (\log l) f(ld) - \sum_{d\leqslant V} \mu(d) \sum_{n\leqslant V} \Lambda(n) \sum_{r\leqslant X(dn)^{-1}} f(ndr) - \sum_{V<m\leqslant XV^{-1}} \Bigg(\sum_{d\mid m, d\leqslant V} \mu(d) \Bigg) \sum_{V<n\leqslant Xm^{-1}} \Lambda(n) f(nm).$$

The proof can be found in [45, Ch. 13].

Suppose that $1 \leqslant a < q \leqslant Q$, $(a, q) = 1$, and consider the sum

$$W = W(Y) = \sum_{\substack{X\leqslant n<Y \\ n\equiv a \pmod q}} \Lambda(n) e\big(hn^{\alpha}\big).$$

The application of Lemma 4.1 with $V = X^{1/3}$ yields:

$$W = -W_0 + W_1 - W_2 + W_3.$$

Here

$$W_0 = \sum_{m\leqslant V^2} a_m \sum_{\substack{X/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} e\big(h(mn)^{\alpha}\big), \qquad a_m = \sum_{\substack{uv=m \\ u,v\leqslant V}} \mu(u)\Lambda(v),$$

$$W_1 = \sum_{\substack{n\leqslant V \\ n\equiv a \pmod q}} \Lambda(n) e\big(hn^{\alpha}\big),$$

$$W_2 = \sum_{V<m\leqslant YV^{-1}} b_m \sum_{\substack{X/m\leqslant n<Y/m,\ n>V \\ mn\equiv a \pmod q}} \Lambda(n) e\big(h(mn)^{\alpha}\big), \qquad b_m = \sum_{\substack{u\mid m \\ u\leqslant V}} \mu(u),$$

$$W_3 = \sum_{m\leqslant V} \mu(m) \sum_{\substack{X/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} (\log n) e\big(h(mn)^{\alpha}\big).$$

Trivially, we have

$$|a_m| \leqslant \sum_{v\mid m} \Lambda(v) = \log m, \qquad |b_m| \leqslant \tau(m), \qquad |W_1| \leqslant \sum_{n\leqslant V} \Lambda(n) \ll V.$$

Next, we have

$$W_0 = \sum_{m\leqslant V} a_m \sum_{\substack{X/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} e\big(h(mn)^{\alpha}\big) + \sum_{V<m\leqslant V^2} a_m \sum_{\substack{X/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} e\big(h(mn)^{\alpha}\big) =: W_4 + W_5.$$

Thus, we get type I sums W_3, W_4 and type II sums W_2, W_5. Type I sums have the form

$$\sum_{m\leqslant V}\gamma_m \sum_{\substack{X/m\leqslant n<Y/m\\ mn\equiv a \pmod q}} \beta_n e\big(h(mn)^\alpha\big),$$

where $\gamma_m = \mu(m)$, $\beta_n = \log n$ for W_3 and $\gamma_m = a_m$, $\beta_n = 1$ for W_4; type II sums have the form

$$\sum_{V<m\leqslant U}\gamma_m \sum_{\substack{Z/m\leqslant n<Y/m\\ mn\equiv a \pmod q}} \beta_n e\big(h(mn)^\alpha\big),$$

where $\gamma_m = b_m$, $\beta_n = \Lambda(n)$, $U = YV^{-1}$, $Z = \max(Vm, X)$ for W_2 and $\gamma_m = a_m$, $\beta_n = 1$, $U = V^2$, $Z = X$ for W_5.

4.2 Type I estimate

We split the range of summation $1 \leqslant m \leqslant V$ to the dyadic intervals $M < m \leqslant M_1$, $M_1 = \min(2M, V)$. Then the initial sum splits into $\ll \log X$ sums of the form

$$W(M) = \sum_{M<m\leqslant M_1}\gamma_m \sum_{\substack{X/m\leqslant n<Y/m\\ mn\equiv a \pmod q}} \beta_n e\big(h(mn)^\alpha\big).$$

By Lemma 2.1, we get

$$\big|W(M)\big| \leqslant 2\|\beta\|_\infty \sum_{\substack{M<m\leqslant M_1\\ (m,q)=1}} \big|\gamma_m\big| \cdot \Bigg| \sum_{\substack{X/m\leqslant n<Y_1/m\\ mn\equiv a \pmod q}} e\big(h(mn)^\alpha\big)\Bigg| \leqslant$$

$$2\|\gamma\|_\infty\|\beta\|_\infty \sum_{\substack{M<m\leqslant M_1\\ (m,q)=1}} \Bigg| \sum_{\substack{X/m\leqslant n<Y_1/m\\ mn\equiv a \pmod q}} e\big(h(mn)^\alpha\big)\Bigg|, \quad (4.3)$$

where $Y_1 \in (X; Y]$ and $\|\omega\|_\infty = \max_{n\leqslant 2X} |\omega_n|$. Next, we fix $m \in (M; M_1]$ with $(m, q) = 1$ and define $l \equiv am^* \pmod q$, $1 \leqslant l \leqslant q - 1$. Setting $n = qr + l$, we obtain

$$\frac{X}{mq} \leqslant r + \xi < \frac{Y_1}{mq}, \qquad \xi = \frac{l}{q}.$$

The inner sum over n in (4.3) takes the form

$$\sum_{R_1-\xi\leqslant r<R_2-\xi} e\big(h(mq)^\alpha (r+\xi)^\alpha\big), \quad (4.4)$$

where $R_1 = X/mq$, $R_2 = Y_1/mq \leqslant 2R_1$.

To estimate aforementioned sum, we apply Theorem 2.4. Consider the function $f_I(x) = h(mq)^\alpha(x + \xi)^\alpha$. Then, for $R_1 - \xi \leqslant x < R_2 - \xi$,

$$f_I^{(k)}(x) = (\alpha)_k h(mq)^\alpha (x+\xi)^{\alpha-k} \asymp \frac{h(mq)^k}{X^{k-\alpha}} = \lambda_k,$$

where $(\alpha)_k = \prod_{i=1}^{k}(\alpha - i + 1)$ is the Pochhammer symbol. Applying Lemma 2.4 to (4.4), we get

$$\left|\sum_{R_1-\xi\leqslant r<R_2-\xi} e\big(f_I(r)\big)\right| \ll_{k,\delta} \left(\frac{X}{mq}\right)^{1+\delta} \cdot \left[\left(\frac{h(mq)^k}{X^{k-\alpha}}\right)^{1/(k(k-1))} + \left(\frac{mq}{X}\right)^{1/(k(k-1))} + \left(\frac{mq}{X}\right)^{2/(k(k-1))}\left(\frac{X^{k-\alpha}}{h(mq)^k}\right)^{2/(k^2(k-1))}\right] \ll h^{1/(k(k-1))}\left(\frac{X}{mq}\right)^{\delta}\Big(T_1+T_2+T_3\Big),$$

where

$$T_1 = \frac{X^{1-(k-\alpha)/(k(k-1))}}{(mq)^{1-1/(k-1)}}, \qquad T_2 = \left(\frac{X}{mq}\right)^{1-1/(k(k-1))}, \qquad T_3 = \frac{X^{1-2\alpha/(k^2(k-1))}}{mq}.$$

Since $\|\gamma\|_\infty\|\beta\|_\infty \leqslant \log 2X$, we get

$$\big|W(M)\big| \ll h^{1/(k(k-1))}(\log X)\left(\frac{X}{q}\right)^{\delta}\sum_{M<m\leqslant M_1} m^{-\delta}(T_1+T_2+T_3).$$

Now, we estimate the contribution from T_1, T_2, T_3 to the sum over all values of M. The contribution from T_1 does not exceed

$$h^{1/(k(k-1))}(\log X)\left(\frac{X}{q}\right)^{\delta}\frac{X^{1-(k-\alpha)/(k(k-1))}}{q^{1-1/(k-1)}}\sum_{M<V}{}' \sum_{M<m\leqslant M_1} m^{-1+1/(k-1)} \ll \left(\frac{X}{q}\right)^{2\delta}\frac{X^{1-(k-\alpha)/(k(k-1))}}{q^{1-1/(k-1)}}V^{1/(k-1)}. \quad (4.5)$$

The contribution from T_2 is less than

$$h^{1/k(k-1)}(\log X)\left(\frac{X}{q}\right)^{1-1/(k(k-1))+\delta}\sum_{M<V}{}' \sum_{M<m\leqslant M_1} m^{-1+1/(k(k-1))} \ll \left(\frac{X}{q}\right)^{1-1/(k(k-1))+2\delta} V^{1/(k(k-1))}. \quad (4.6)$$

Finally, the contribution from T_3 is bounded by

$$h^{1/(k(k-1))}(\log X)\left(\frac{X}{q}\right)^{\delta}\frac{X^{1-2\alpha/(k^2(k-1))}}{q}\sum_{M<V}{}' \sum_{M<m\leqslant M_1} \frac{1}{m} \ll \left(\frac{X}{q}\right)^{2\delta}\frac{X^{1-2\alpha/(k^2(k-1))}}{q}\log V. \quad (4.7)$$

4.3 Type II estimate

By definition of U, $V^2 \leqslant U \leqslant YV^{-1} \leqslant 2XV^{-1} \leqslant 2V^2$. We split W_2, W_5 into $\ll \log X$ sums of the type $W(M)$. Cauchy inequality yields:

$$\left|W(M)\right|^2 \leqslant \left(\sum_{M<m\leqslant M_1} |\gamma_m|^2\right)\left(\sum_{M<m\leqslant M_1}\left|\sum_{\substack{Z/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} \beta_n e\big(h(mn)^\alpha\big)\right|^2\right).$$

Next, by Mardzhanishvili's inequality (Lemma 2.5), we get

$$\left|W(M)\right|^2 \ll M(\log X)^{2+\kappa}\left(\sum_{M<m\leqslant M_1}\left|\sum_{\substack{Z/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} \beta_n e\big(h(mn)^\alpha\big)\right|^2\right), \tag{4.8}$$

where $\kappa = 1$ for W_2 and $\kappa = 0$ for W_5. Now, we rewrite the sum over m as follows:

$$\sum_{M<m\leqslant M_1} \sum_{\substack{Z/m\leqslant n_1,n_2<Y/m \\ mn_i\equiv a \pmod q), i=1,2}} \beta_{n_1}\beta_{n_2} e\big(hm^\alpha(n_1^\alpha - n_2^\alpha)\big) =$$

$$\sum_{M<m\leqslant M_1} \sum_{\substack{Z/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} \beta_n^2 + 2\mathrm{Re}(S(M)),$$

where

$$S(M) = \sum_{M<m\leqslant M_1} \sum_{\substack{Z/m\leqslant n_1<n_2<Y/m \\ mn_i\equiv a \pmod q), i=1,2}} \beta_{n_1}\beta_{n_2} e\big(hm^\alpha(n_1^\alpha - n_2^\alpha)\big).$$

The diagonal term does not exceed

$$\sum_{M<m\leqslant M_1} \sum_{\substack{Z/m\leqslant n<Y/m \\ mn\equiv a \pmod q}} \beta_n^2 \ll \sum_{M<m\leqslant M_1} (\log X)^{2\kappa}\left(\frac{X}{mq}+1\right) \ll$$

$$(\log X)^{2\kappa}\left(\frac{X}{q}+M\right). \tag{4.9}$$

Setting $m = qr + l$, we get

$$\frac{M}{q} - \eta < r \leqslant \frac{M_1}{q} - \eta, \qquad \eta = \frac{l}{q}$$

for given l, $(l, q) = 1$. Hence,

$$S(M) = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \sum_{\frac{M}{q}-\eta<r\leqslant\frac{M_1}{q}-\eta} \sum_{\substack{\frac{Z}{qr+l}\leqslant n_2<n_1<\frac{Y}{qr+l} \\ n_1,n_2\equiv e \pmod q}} \beta_{n_1}\beta_{n_2} e\big(h(n_1^\alpha - n_2^\alpha)q^\alpha(r+\eta)^\alpha\big),$$

where $e = al^* \pmod q$. Next, we change the order of summation. If $Z = X$, then

$$\frac{X}{qr+l} \leqslant n_2 < n_1 < \frac{Y}{qr+l},$$

so $X/M_1 \leqslant n_2 < n_1 < Y/M$, and for fixed n_1, n_2, we get

$$\frac{X}{qn_2} - \eta \leqslant r < \frac{Y}{qn_1} - \eta.$$

By definition, $Z = \max(Vm, X) = \max\bigl(V(qr+l), X\bigr)$, hence

$$\max\left(V, \frac{X}{qr+l}\right) \leqslant n_2 < n_1 < \frac{Y}{qr+l}.$$

Since

$$\max\left(V, \frac{X}{qr+l}\right) = \begin{cases} \dfrac{X}{qr+l}, & \text{if } r \leqslant \dfrac{XV^{-1}}{q} - \eta; \\ V, & \text{if } r > \dfrac{XV^{-1}}{q} - \eta, \end{cases}$$

we estimate $S(M)$ as follows:

$$S(M) = \Biggl\{ \sum_{\substack{\frac{M}{q} - \eta < r \leqslant \frac{M_1}{q} - \eta \\ r \leqslant XV^{-1}/q - \eta}} \sum_{\substack{\frac{X}{qr+l} \leqslant n_2 < n_1 < \frac{Y}{qr+l} \\ n_1, n_2 \equiv e \pmod q}} + \sum_{\substack{\frac{M}{q} - \eta < r \leqslant \frac{M_1}{q} - \eta \\ r > XV^{-1}/q - \eta}} \sum_{\substack{V \leqslant n_2 < n_1 < \frac{Y}{qr+l} \\ n_1, n_2 \equiv e \pmod q}} \Biggr\} \ldots =$$
$$\Biggl\{ \sum_{\substack{X/M_1 \leqslant n_2 < n_1 < Y/M \\ n_1, n_2 \equiv e \pmod q}} \sum_{R^{(1)} - \eta < r \leqslant R^{(2)} - \eta} + \sum_{\substack{V \leqslant n_2 < n_1 < Y/M \\ n_1, n_2 \equiv e \pmod q}} \sum_{R^{(3)} - \eta < r \leqslant R^{(4)} - \eta} \Biggr\} \ldots,$$

where

$$R^{(1)} = \max\left(\frac{M}{q}, \frac{X}{qn_2}\right), \qquad R^{(2)} = \min\left(\frac{M_1}{q}, \frac{Y}{qn_1}, \frac{XV^{-1}}{q}\right),$$
$$R^{(3)} = \max\left(\frac{M}{q}, \frac{XV^{-1}}{q}\right), \qquad R^{(4)} = \min\left(\frac{M_1}{q}, \frac{Y}{qn_1}\right).$$

Therefore,

$$\bigl|S(M)\bigr| \leqslant \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \sum_{\substack{X/M_1 \leqslant n_2 < n_1 < Y/M \\ n_1, n_2 \equiv e \pmod q}} |\beta_{n_1}||\beta_{n_2}| \Biggl| \sum_{R_1 - \eta < r \leqslant R_2 - \eta} e\bigl(f_{II}(r)\bigr) \Biggr|,$$

where (R_1, R_2) denotes the pair $(R^{(1)}, R^{(2)})$, $(R^{(3)}, R^{(4)})$ that corresponds to the maximum absolute value of the sum over r, $f_{II}(x) = h(n_1^\alpha - n_2^\alpha)q^\alpha(x+\eta)^\alpha$. Using the conditions $n_2 < n_1$, $n_1 \equiv n_2 \equiv e \pmod q$, we write $n_1 = n_2 + qs$ with $s \geqslant 1$.

On the other hand, $n_1 < Y/M$ implies $n_2 + qs < Y/M$. Hence, $s < Y/(Mq) = t$, and therefore

$$\left|S(M)\right| \ll \sum_{\substack{l=1\\(l,q)=1}}^{q} \sum_{1\leqslant s<t} \sum_{\substack{X/M_1\leqslant n<Y/M\\ n\equiv e \pmod q}} |\beta_n||\beta_{n+qs}|\left|\sum_{R_1\leqslant r<R_2} e\big(f_{II}(r)\big)\right|.$$

Obviously,

$$f_{II}^{(k)}(x) = \frac{(\alpha)_k h(n_1^\alpha - n_2^\alpha)q^\alpha}{(x+\eta)^{k-\alpha}} = \frac{(\alpha)_k D_1}{(x+\eta)^{k-\alpha}},$$

where

$$D_1 = h(n_1^\alpha - n_2^\alpha)q^\alpha, \qquad \text{so we have} \quad \left|f_{II}^{(k)}(x)\right| \asymp \frac{D_1}{R_1^{k-\alpha}} \asymp D_1\left(\frac{q}{M}\right)^{k-\alpha}.$$

Next, by Lagrange mean value theorem,

$$D_1 = hq^\alpha\big((n+qs)^\alpha - n^\alpha\big) = hq^\alpha\cdot\alpha(n+qs\theta')^{\alpha-1}\cdot qs \asymp hsq^{\alpha+1}\left(\frac{X}{M}\right)^{\alpha-1}, \qquad |\theta'| \leqslant 1,$$

and hence

$$\left|f_{II}^{(k)}(x)\right| \asymp \frac{hsq^2}{X^{1-\alpha}}\left(\frac{q}{M}\right)^{k-1} = \lambda_k.$$

Put $1-\alpha = \nu$. By Lemma 2.4, we get

$$\sum_{R_1<r\leqslant R_2} e\big(f_{II}(r)\big) \ll_{k,\delta} \left(\frac{M}{q}\right)^{1+\delta}\left\{\left(\frac{hsq^2}{X^\nu}\right)^{1/(k(k-1))}\left(\frac{q}{M}\right)^{1/k} + \left(\frac{M}{q}\right)^{-1/(k(k-1))} + \left(\frac{M}{q}\right)^{-2/(k(k-1))}\left(\frac{hsq^2}{X^\nu}\right)^{-2/(k^2(k-1))}\left(\frac{q}{M}\right)^{-2/k^2}\right\}.$$

The factor $|\beta_n|\cdot|\beta_{n+qs}|$ is bounded from above by $(X/q)^\delta$. The summation over $n \equiv e \pmod q$ for $X/M_1 < n \leqslant Y/M$ contributes the factor of at most X/Mq. Thus,

$$S(M) \ll \left(\frac{X}{q}\right)^\delta \sum_{\substack{l=1\\(l,q)=1}}^{q} \frac{X}{Mq}\left(\frac{M}{q}\right)^{1+\delta}\left\{\left(\frac{hq^2}{X^\nu}\right)^{1/(k(k-1))}\left(\frac{q}{M}\right)^{1/k}\sum_{1\leqslant s<t} s^{1/(k(k-1))} + \left(\frac{M}{q}\right)^{-1/(k(k-1))}\sum_{1\leqslant s<t} 1 + \left(\frac{hq^2}{X^\nu}\right)^{-2/(k^2(k-1))}\left(\frac{M}{q}\right)^{-2/(k^2(k-1))}\sum_{1\leqslant s<t} s^{-2/(k^2(k-1))}\right\}.$$

The inequalities $M < X$ and $t \leqslant 2X/Mq$ imply:

$$S(M) \ll \left(\frac{X}{q}\right)^{2\delta} \cdot \frac{X}{q}\Bigg\{\left(\frac{hq^2}{X^\nu}\right)^{1/(k(k-1))}\left(\frac{q}{M}\right)^{1/k}\left(\frac{2X}{Mq}\right)^{1+1/(k(k-1))} + \left(\frac{M}{q}\right)^{-1/(k(k-1))}\frac{2X}{Mq} + \left(\frac{hq^2}{X^\nu}\right)^{-2/(k^2(k-1))}\left(\frac{M}{q}\right)^{-2/(k^2(k-1))}\left(\frac{2X}{Mq}\right)^{1-2/(k^2(k-1))}\Bigg\} \ll \left(\frac{X}{q}\right)^{2\delta}\frac{X^2}{Mq^2}(T_4+T_5+T_6), \quad (4.10)$$

where

$$T_4 = (2h)^{1/(k(k-1))}X^{(1-\nu)/(k(k-1))}\left(\frac{q}{M}\right)^{1/(k-1)} = \left(\frac{2hq^kX^\alpha}{M^k}\right)^{1/(k(k-1))},$$

$$T_5 = \left(\frac{q}{M}\right)^{1/(k(k-1))},$$

$$T_6 = (2h)^{-2/(k^2(k-1))}X^{(2\nu-2)/(k^2(k-1))} = (2hX^\alpha)^{-2/(k^2(k-1))}.$$

Thus, the contribution from (4.10) to $|W(M)|^2$ does not exceed

$$(\log X)^{2+\kappa}\left(\frac{X}{q}\right)^{2+2\delta}\left\{\left(\frac{2hq^kX^\alpha}{M^k}\right)^{1/(k(k-1))} + \left(\frac{q}{M}\right)^{1/(k(k-1))} + \left(\frac{1}{2hX^\alpha}\right)^{2/(k^2(k-1))}\right\},$$

hence, combining with (4.8) and (4.9), we get

$$\left|W(M)\right| \ll_k (\log X)^{1+3\kappa/2}\left(\frac{X}{q}\right)^{\delta}\left\{M + \left(\frac{MX}{q}\right)^{1/2} + \frac{X}{q}\left(\left(\frac{hq^kX^\alpha}{M^k}\right)^{1/(2k(k-1))} + \left(\frac{q}{M}\right)^{1/(2k(k-1))} + \left(\frac{1}{hX^\alpha}\right)^{1/(k^2(k-1))}\right)\right\}.$$

The summation over all M in the range $V \leqslant M < 2V^2$ leads to the estimate

$$W \ll (\log X)^{1+3\kappa/2}\left(\frac{X}{q}\right)^{\delta}\left\{V^2 + \frac{V\sqrt{X}}{\sqrt{q}} + \frac{X}{q}\left(\left(\frac{hq^kX^\alpha}{V^k}\right)^{1/(2k(k-1))} + \left(\frac{q}{V}\right)^{1/(2k(k-1))} + \left(\frac{1}{hX^\alpha}\right)^{1/(k^2(k-1))}\right)\right\}. \quad (4.11)$$

4.4 Final bound

From (4.5), (4.6), (4.7), (4.11), we conclude that

$$W \ll \left(\frac{X}{q}\right)^{1+2\delta}\left\{\left(\frac{Vq}{X}\right)^{1/(k-1)}X^{\alpha/(k(k-1))} + \left(\frac{Vq}{X}\right)^{1/(k(k-1))} + X^{-2\alpha/(k^2(k-1))}\log V + \frac{V^2q}{X} + \frac{V\sqrt{q}}{\sqrt{X}} + \left(\frac{hq^kX^\alpha}{V^k}\right)^{1/(2k(k-1))} + \left(\frac{q}{V}\right)^{1/(2k(k-1))} + \left(\frac{1}{hX^\alpha}\right)^{1/(k^2(k-1))}\right\}.$$

We estimate the factors $(X/q)^{\delta}$ and $h^{1/(2k(k-1))}$ by X^{δ}. Thus,

$$W \ll X^{1+3\delta}\Biggl\{\left(\frac{Vq}{X^{1-\alpha/k}}\right)^{1/(k-1)} + \left(\frac{Vq}{X}\right)^{1/(k(k-1))} + X^{-2\alpha/(k^2(k-1))} + \frac{V^2q}{X} + \frac{V\sqrt{q}}{\sqrt{X}} + \left(\frac{q^kX^{\alpha}}{V^k}\right)^{1/(2k(k-1))} + \left(\frac{q}{V}\right)^{1/(2k(k-1))} + \left(\frac{1}{X^{\alpha}}\right)^{1/(k^2(k-1))}\Biggr\} \ll X^{1+3\delta}\sum_{i=1}^{8}\Delta_i,$$

where

$$\Delta_1 \leqslant X^{(3\alpha-2k)/(3k(k-1))}q^{1/(k-1)} \ll X^{(3\alpha-k-3\varepsilon k)/(3k(k-1))},$$

$$\Delta_2 = \left(\frac{q}{X^{2/3}}\right)^{1/(k(k-1))} \ll X^{-1/(3k(k-1))}, \qquad \Delta_3 = X^{-2\alpha/(k^2(k-1))},$$

$$\Delta_4 = \frac{V^2q}{X} \leqslant X^{-\varepsilon}, \qquad \Delta_5 = \frac{V\sqrt{q}}{\sqrt{X}} \leqslant X^{-\varepsilon/2},$$

$$\Delta_6 \leqslant \left(X^{\alpha/k-\varepsilon}\right)^{1/(2(k-1))}, \qquad \Delta_7 \leqslant X^{-\varepsilon/(2k(k-1))}, \qquad \Delta_8 \leqslant X^{-\alpha/(k^2(k-1))},$$

$$\max_{1\leqslant i\leqslant 8}\Delta_i \leqslant X^{-2\varepsilon^3/(5\alpha^2)}$$

if $k = \lfloor 1.1\cdot\alpha/\varepsilon\rfloor + 1$. Finally, choosing $\delta \leqslant \varepsilon^3/(50\alpha^2)$ and applying partial summation, we get the desired bound.

Chapter 5

THE EXPONENTIAL SUM ESTIMATE FOR SMALL α WITH THE LEVEL OF DISTRIBUTION $2/5 - (3/5)\alpha$

In this chapter, we prove Theorem 1.3. The first big difference from the proof of Theorem 1.1 is a different decomposition of the exponential sum

$$\sum_{\substack{X \leqslant n < Y \\ n \equiv a \pmod q}} \Lambda(n) e\big(hn^{\alpha}\big).$$

This time, we consider the sums of three different types. Type I and type II are similar to the ones in the previous chapter, the key difference for type II are the better ranges of M and N (in this case, they are separated from the critical ranges $M \approx X^{1/3}$, $N \approx X^{2/3}$, and vice versa). Due to the small size of α, the sums W_I and W_{II} are easier to treat compared to Chapter 4. Type III sum is of the form

$$W_{III} = \sum_{M\Theta^{-1} \leqslant m < M\Theta} f_1(m) \sum_{N\Theta^{-1} \leqslant n < N\Theta} f_2(n) \sum_{\substack{K\Theta^{-1} \leqslant k < K\Theta \\ mnk \equiv a \pmod q}} f_3(k) e\big(h(mnk)^{\alpha}\big),$$

where f_1, f_2, f_3 are smooth real functions, $1 < \Theta = \Theta(X) \leqslant 2$, and M, N, K are close to each other in size ($M \approx N \approx K \approx X^{1/3}$). The desired upper bound is obtained in three steps: we apply Poisson summation twice to replace two of the sums over m, n, k by shorter sums which can be estimated trivially. The additional saving of $\sqrt{q}$ can be obtained by Weil's bound for Kloosterman sum [21].

5.1 Auxiliary lemmas

Lemma 5.1 (Heath-Brown identity)**.** *For any fixed integer $k \geqslant 1$, $V = X^{1/k}$, and any complex valued function $f(x)$, the following identity holds true:*

$$\sum_{X \leqslant n < Y} \Lambda(n) f(n) = \sum_{j=1}^{k} (-1)^{j-1} \binom{k}{j} S_j,$$

where

$$S_j = \sum_{\substack{X \leqslant d_1 \ldots d_{2j} < Y \\ d_{j+1}, \ldots, d_{2j} \leqslant V}} (\log d_1) \mu(d_{j+1}) \ldots \mu(d_{2j}) f(d_1 \ldots d_{2j}).$$

The proof is in [41].

Lemma 5.2 (Combinatorial decomposition)**.** *Let $1/10 < \sigma < 1/2$, and let $t_1, \ldots, t_n$ be non-negative real numbers such that $t_1 + \ldots + t_n = 1$. Then at least one of the following three statements holds:*

Type I: *There is a t_i with $t_i \geqslant 1/2 + \sigma$;*

Type II: *There is a partition $\{1, \ldots, n\} = \mathbf{S} \cap \mathbf{T}$ such that*

$$\frac{1}{2} - \sigma < \sum_{i \in \mathbf{S}} t_i \leqslant \sum_{i \in \mathbf{T}} t_i < \frac{1}{2} + \sigma;$$

Type III: *There exist distinct i, j, k with $2\sigma \leqslant t_i \leqslant t_j \leqslant t_k \leqslant 1/2 - \sigma$ and*

$$t_i + t_j,\ t_i + t_k,\ t_j + t_k \geqslant \frac{1}{2} + \sigma.$$

If $\sigma > 1/6$, then the type III situation is impossible.

See [71, Lemma 3.1]

Lemma 5.3 (Smoothing function)**.** *Let a, b be fixed real numbers, $a < b$; $\Delta = (\log X)^{-A_0}$ with some fixed $A_0 > 0$. Then there exists smooth function $\Psi(x) : \mathbb{R} \to \mathbb{R}$ supported on $[a - \Delta; b + \Delta]$, which is one on $[a; b]$, and satisfying the inequalities $0 \leqslant \Psi(x) \leqslant 1$ if $x \in [a - \Delta; a) \cup (b; b + \Delta]$ and the following upper bounds*

$$|\Psi^{(m)}(x)| \ll_m \log^{mA_0} x$$

for any fixed $m \geqslant 0$.

The proof is in [23].

Lemma 5.4 (Poisson summation)**.** *Let $f(x)$ be a smooth finitely supported function. Then the following formula holds true*

$$\sum_{n=-\infty}^{+\infty} f(n) = \sum_{m=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f(u) e(-mu) du.$$

See, for example, [45, Theorem 4.4].

Lemma 5.5 (*Lemma 8.1, [9]*)**.** *Let $Y_I \geqslant 1$, $X_I, Q_I, V_I, R_I > 0$, $w(t)$ is a smooth function supported on some finite interval $\mathbb{J} \subset \mathbb{R}$ such that*

$$w^{(j)}(t) \ll_j X_I V_I^{-j}$$

for all $j \geqslant 0$. *Suppose that* $g(t)$ *is a smooth function such that* $|g'(t)| \geqslant R_I$, $g^{(j)}(t) \ll_j Y_I Q_I^{-j}$ *for* $j \geqslant 2$, $t \in \mathbb{J}$. *Then, the integral* I *defined by*

$$I = \int_{-\infty}^{+\infty} w(t) e\big(g(t)\big) dt$$

satisfies

$$I \ll_{A_I} |\mathbb{J}| X_I \big((Q_I R_I / \sqrt{Y_I})^{-A_I} + (R_I V_I)^{-A_I} \big)$$

with any fixed real $A_I > 0$.

This result gives a non-trivial upper bound for the integral I in the case if $R_I V_I$ and $Q_I R_I Y_I^{-1/2}$ are much bigger than 1.

Lemma 5.6 (*Proposition 8.2, [9]*). *Let* $0 < \delta_I < 1/10$, $X_I, Y_I, V_I, \tilde{V}_I$, $Q_I > 0$, $Z_I = Q_I + X_I + Y_I + \tilde{V}_I + 1$, *and assume that* $Y_I \geqslant Z_I^{3\delta_I}$,

$$\tilde{V}_I \geqslant V_I \geqslant \frac{Q_I Z_I^{\delta_I/2}}{Y_I^{1/2}}.$$

Suppose that $w(t)$ *is a smooth function supported on an interval* $\mathbb{J}$ *of length* $\tilde{V}_I$ *satisfying*

$$w^{(j)}(t) \ll_j X_I V_I^{-j}$$

for all $j \geqslant 0$. *Suppose that* $g(t)$ *is a smooth function such that there is a unique point* $t_0 \in \mathbb{J}$ *such that* $g'(t_0) = 0$. *Further,* $g(t)$ *satisfies the estimates* $g''(t) < 0$, $g''(t) \gg Y_I Q_I^{-2}$, $g^{(j)}(t) \ll_j Y_I Q_I^{-j}$, *for all* $j \geqslant 1$, $t \in \mathbb{J}$. *Then, the integral*

$$I = \int_{-\infty}^{+\infty} w(t) e\big(g(t)\big) dt$$

has an asymptotic expansion of the form

$$I = \frac{e\big(g(t_0)\big)}{|g''(t_0)|^{1/2}} \sum_{0 \leqslant n \leqslant 3\delta_I^{-1} A_I} p_n(t_0) + O_{A_I, \delta_I}(Z_I^{-A_I}),$$

$$p_n(t_0) = \frac{\sqrt{2\pi} e^{-\pi i/4}}{n!} \frac{(2i)^{-n}}{|g''(t_0)|^n} G^{(2n)}(t_0),$$

where $A_I > 0$ *is arbitrary, and*

$$G(t) = w(t) e\big(H(t)\big), \qquad H(t) = g(t) - g(t_0) - \frac{1}{2} g''(t_0)(t - t_0)^2.$$

Lemma 5.7 (Weil's bound)**.** *The Kloosterman sum* $S_q(m,n)$ *satisfies the upper bound*

$$|S_q(m,n)| \leqslant \tau(q) \sqrt{q} (m,n,q)^{1/2}.$$

For proof, see [45, Corollary 11.12].

Lemma 5.8 (Faa di Bruno formula)**.** *Let $\varphi(x), f(x)$ be r-times differentiable functions on $\mathbb{R}$. Then, one has the formula*

$$\frac{d^r\varphi(f(x))}{dx^r} = \sum_{\substack{m_1+2m_2+\ldots+rm_r=r\\ m_1,\ldots,m_r\geqslant 0}} \frac{r!}{m_1!\ldots m_r!}\varphi^{(m_1+\ldots+m_r)}(f(x))\prod_{j=1}^{r}\left(\frac{f^{(j)}(x)}{j!}\right)^{m_j}.$$

See, for example, [54].

5.2 Heath-Brown identity

In this section, we adjust the initial sum W to simplify the estimation of type III sum. This technique is also described in [71, Section 3]. Suppose that $1 \leqslant a < q \leqslant Q$, $(a,q) = 1$. We consider the sum

$$W = W(Y) = \sum_{\substack{X\leqslant n<Y\\ n\equiv a \pmod q}} \Lambda(n)e(hn^\alpha), \qquad X < Y \leqslant 2X.$$

Let us denote $y = Y/X > 1$. Fix $B_0 > 0$ and choose $\Delta = (\log X)^{-B_0}$. There exists function $\psi(x) \in C^\infty$, such that $\psi(x) = 1$ if $1 \leqslant x \leqslant y$, $0 \leqslant \psi(x) \leqslant 1$ if $1-\Delta \leqslant x \leqslant 1$ or $y \leqslant x \leqslant y+\Delta$ and $\psi(x) = 0$ otherwise, and its derivatives satisfy the estimates $\psi^{(j)}(x) \ll_j (\log X)^{jB_0}$. See, for example, [23]. Then, W can be rewritten as

$$W = \sum_{\substack{n=1\\ n\equiv a \pmod q}}^{+\infty} \psi\left(\frac{n}{X}\right)\Lambda(n)e\left(hn^\alpha\right) + O\left(\frac{X(\log X)^{-B_0+1}}{q}\right). \tag{5.1}$$

By partial summation to prove Theorem 1.3, it is enough to show that the sum in (5.1) is bounded by $X(\log X)^{-A}$. Thus, one can take $B_0 = A+1$.

Applying Heath-Brown identity with $k = 5$, $V = X^{1/5}$ (Lemma 5.1), we get

$$W = \sum_{j=1}^{5}(-1)^{j-1}\binom{5}{j}W_j,$$

where

$$W_j = \sum_{\substack{d_1,\ldots,d_{2j}=1\\ d_{j+1},\ldots,d_{2j}\leqslant V\\ d_1\ldots d_{2j}\equiv a \pmod q}} (\log d_1)\mu(d_{j+1})\ldots\mu(d_{2j})\psi\left(\frac{d_1\ldots d_{2j}}{X}\right)e\left(h(d_1\ldots d_{2j})^\alpha\right).$$

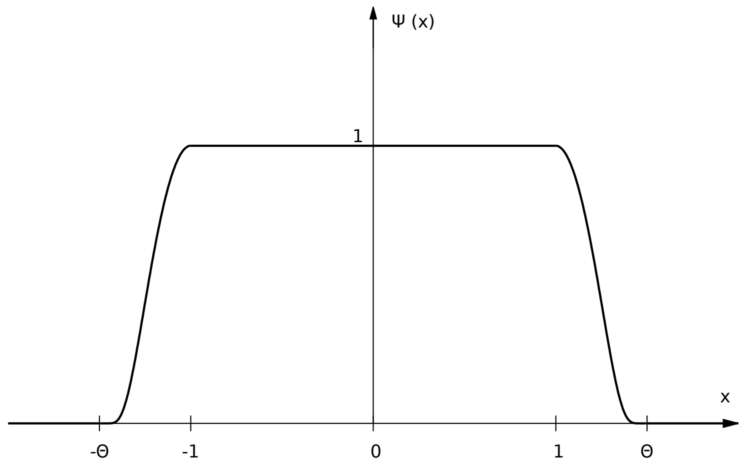

Figure 5.1: Smoothed indicator function of $[-1, 1]$.

The statement of the theorem clearly follows from the estimates $W_j \ll X(\log X)^{-A}$ for each $1 \leqslant j \leqslant 5$. We only provide the details for W_5. The sums $W_1, \ldots, W_4$ can be treated similarly.

We first split the summation over $d_1, \ldots, d_{10}$ to the "refined" dyadic intervals following the technique from [71]. Fix $A_0 > 0$ and $\Theta = 1 + (\log X)^{-A_0}$. Let $\Psi(x)$ be C^∞ function supported on $[-\Theta; \Theta]$ such that $\Psi(x) = 1$ on $[-1; 1]$ and $|\Psi^{(j)}(x)| \ll \log^{jA_0} x$ for all $j \geqslant 0$ (see Figure 5.1). For all $x \geqslant 1$, we have

$$1 = \sum_{D \in \mathbf{G}} \Psi_D(x),$$

where

$$\mathbf{G} = \left\{\Theta^l,\ l \in \mathbb{N} \cup \{0\}\right\}, \qquad \Psi_D(x) = \Psi\left(\frac{x}{D}\right) - \Psi\left(\frac{\Theta x}{D}\right).$$

Indeed, if $x \geqslant 1$, then

$$\sum_{D \in \mathbf{G}} \Psi_D(x) = \lim_{m \to +\infty} \Bigg(\Psi(x) - \Psi(\Theta x) + \Psi\big(\frac{x}{\Theta}\big) - \Psi(x) + \Psi\big(\frac{x}{\Theta^2}\big) - \Psi\big(\frac{x}{\Theta}\big) + \ldots$$

$$\ldots + \Psi\big(\frac{x}{\Theta^m}\big) - \Psi\big(\frac{x}{\Theta^{m-1}}\big)\Bigg) = \lim_{m \to +\infty} \left(-\Psi(\Theta x) + \Psi\big(\frac{x}{\Theta^m}\big)\right) = -0 + 1 = 1.$$

The function Ψ_D is supported on $[\Theta^{-1} D; \Theta D]$. Thus,

$$W_5 = \sum_{D_1, \ldots, D_{10} \in \mathbf{G}} \sum_{\substack{d_1, \ldots, d_{10} = 1 \\ d_6, \ldots, d_{10} \leqslant V \\ d_1 \ldots d_{10} \equiv a \pmod q}}^{+\infty} \log(d_1)\mu(d_6) \ldots$$

$$\ldots \mu(d_{10}) \Psi_{D_1}(d_1) \ldots \Psi_{D_{10}}(d_{10}) \psi\left(\frac{d_1 \ldots d_{10}}{X}\right) e\big(h(d_1 \ldots d_{10})^\alpha\big). \quad (5.2)$$

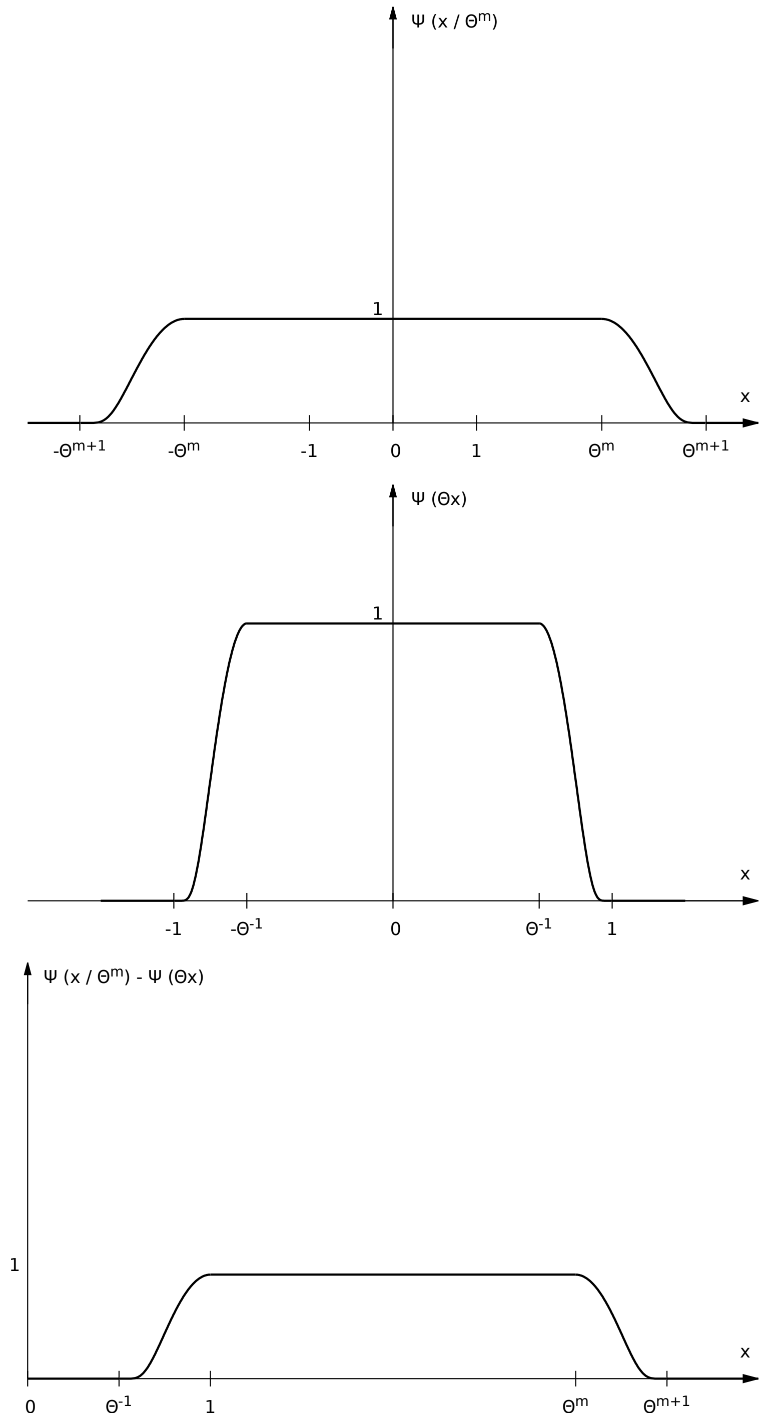

Figure 5.2: Smooth partition of unity. $\Psi(x/\Theta^m) - \Psi(\Theta x) \longrightarrow 1$ when $m \longrightarrow +\infty$ for all $x \geqslant 1$.

The non-zero contribution to W_5 is only coming from the terms satisfying

$$(1-\Delta)X \leqslant d_1 \dots d_{10} \leqslant (y+\Delta)X, \qquad \frac{D_i}{\Theta} \leqslant d_i \leqslant D_i\Theta \tag{5.3}$$

for $i = 1, \dots, 10$. From (5.3), we conclude that the non-zero contribution corresponds to the tuples $\mathbf{D} = \{D_1, \dots, D_{10}\}$ satisfying the inequality

$$X_1 \leqslant D_1 \dots D_{10} \leqslant Y_1, \qquad \text{where} \quad X_1 = (1-\Delta)\Theta^{-10}X, \qquad Y_1 = (y+\Delta)\Theta^{10}X,$$

and also satisfying

$$D_i \leqslant V\Theta \qquad \text{for} \quad i = 6, \dots, 10.$$

To split each W_j to the sums of three types, we apply combinatorial decomposition given by Lemma 5.2 with $\sigma = 1/10 + \varepsilon_1$, $\varepsilon_1 < 3\alpha/5$. Without loss of generality, consider the sum W_5. We have

$$W_5 \ll |W_I| + |W_{II}| + |W_{III}|,$$

where the sums correspond to the following cases:

Type I sum: there is one index $1 \leqslant i \leqslant 5$ such that $D_i \geqslant {X_1}^{3/5+\varepsilon_1}$.

Type II sum: there is a partition $\mathbf{S} \cup \mathbf{T} = \{1, \dots, 10\}$ such that

$${X_1}^{2/5-\varepsilon_1} < \prod_{i \in \mathbf{S}} D_i < {X_1}^{3/5+\varepsilon_1}.$$

Type III sum: there are three distinct indices $i, j, k \in \{1, \dots, 5\}$ such that

$$\begin{gathered} {X_1}^{1/5+2\varepsilon_1} \leqslant D_i \leqslant D_j \leqslant D_k \leqslant {X_1}^{2/5-\varepsilon_1}, \\ D_iD_j,\ D_iD_k,\ D_jD_k \geqslant {X_1}^{3/5+\varepsilon_1}. \end{gathered}$$

Remark. *Note that in the expression analogous to* (5.2) *for W_1 and W_2, the type III sum is empty.*

5.3 Type I estimate

For simplicity, we only consider the case $D_1 \geqslant {X_1}^{3/5+\varepsilon_1}$. The corresponding sum has the form

$$W_I = \sum_{\substack{U \leqslant Y_1 X_1^{-3/5-\varepsilon_1} \\ U \in \mathbf{G}}} \ \sum_{\substack{D_2 \dots D_{10} = U \\ D_2, \dots, D_{10} \in \mathbf{G}}} \cdots \sum \ \sum_{\substack{X_1^{3/5+\varepsilon_1} \leqslant D_1 \leqslant Y_1 U^{-1} \\ D_1 \in \mathbf{G}}} W(\mathbf{D}), \qquad \mathbf{D} = \big\{D_1, \dots, D_{10}\big\},$$

$$W(\mathbf{D}) = \sum_{U\Theta^{-9} \leqslant u \leqslant U\Theta^9} b(u) \sum_{\substack{d_1 = 1 \\ ud_1 \equiv a \pmod q}}^{+\infty} f(d_1) e\big(h(ud_1)^\alpha\big),$$

where

$$b(u) = \sum_{\substack{d_2\ldots d_{10}=u\\ d_6,\ldots,d_{10}\leqslant V}} \mu(d_6)\ldots\mu(d_{10})\Psi_{D_2}(d_2)\ldots\Psi_{D_{10}}(d_{10}), \qquad |b(u)| \leqslant \tau_9(u),$$

$$f(d_1) = (\log d_1)\Psi_{D_1}(d_1)\psi\left(\frac{ud_1}{X}\right).$$

Note that the sum over $U \in \mathbf{G}$ contains only $O\big((\log X)^{A_0+1}\big)$ terms. We have

$$\big|W(\mathbf{D})\big| \leqslant \|b\|_\infty \sum_{U\Theta^{-9}\leqslant u\leqslant U\Theta^9} \Bigg| \sum_{\substack{R_1<d_1\leqslant R_2\\ ud_1\equiv a \pmod q}} f(d_1)e\big(h(ud_1)^\alpha\big)\Bigg|,$$

where

$$\|b\|_\infty = \max_{n\leqslant Y_1} |b(n)|, \qquad R_1 = \max\left((1-\Delta)\frac{X}{u}, D_1\Theta^{-1}\right),$$
$$R_2 = \min\left((y+\Delta)\frac{X}{u}, D_1\Theta\right).$$

By partial summation,

$$\big|W(\mathbf{D})\big| \leqslant \|b\|_\infty \sum_{U\Theta^{-9}\leqslant u\leqslant U\Theta^9} \Bigg| f(R_2) \sum_{\substack{R_1<d_1\leqslant R_2\\ ud_1\equiv a \pmod q}} e\big(h(ud_1)^\alpha\big) - \int_{R_1}^{R_2}\Bigg(\sum_{\substack{R_1<d_1\leqslant v\\ ud_1\equiv a \pmod q}} e\big(h(ud_1)^\alpha\big)\Bigg)\frac{df(v)}{dv}dv\Bigg|.$$

Next,

$$\frac{d}{dv}\left(\log(v)\Psi_{D_1}(v)\psi\big(\frac{uv}{X}\big)\right) \ll \frac{1}{v} + \frac{\log v}{D_1}(\log X)^{A_0} + \log(v)\frac{u}{X}(\log X)^{B_0} \ll \frac{1}{D_1}(\log X)^{\max(A_0,B_0)+1},$$

and therefore

$$\int_{R_1}^{R_2}\Bigg(\sum_{\substack{R_1<d_1\leqslant v\\ ud_1\equiv a \pmod q}} e\big(h(ud_1)^\alpha\big)\Bigg)\frac{df(v)}{dv}dv \ll (\log X)^{\max(A_0,B_0)+1}\Bigg| \sum_{\substack{R_1<d_1\leqslant R_3\\ ud_1\equiv a \pmod q}} e\big(h(ud_1)^\alpha\big)\Bigg|,$$

where $R_1 < R_3 \leqslant R_2$. Thus, by the triangle inequality,

$$\left|W(\mathbf{D})\right| \ll \|b\|_\infty (\log X)^{\max(A_0,B_0)+1} \sum_{U\Theta^{-9}\leqslant u\leqslant U\Theta^{9}} \left| \sum_{\substack{R_1<d_1\leqslant R_3 \\ ud_1\equiv a \pmod q}} e\big(h(ud_1)^\alpha\big)\right|. \quad (5.4)$$

Due to the congruence restriction $ud_1 \equiv a \pmod q$, we can assume $(u,q)=1$ and define $l_1 \equiv au^* \pmod q$, $1 \leqslant l_1 \leqslant q-1$. Setting $d_1 = qr_1 + l_1$, we obtain

$$\frac{R_1}{q} \leqslant r_1 + \xi < \frac{R_3}{q}, \qquad \xi = \frac{l_1}{q}.$$

The inner sum over d_1 in (5.4) takes the form

$$\sum_{R_1/q-\xi\leqslant r_1<R_3/q-\xi} e\big(f_I(r_1)\big),$$

where $f_I(x) = h(uq)^\alpha (x+\xi)^\alpha$. Then, for $R_1/q - \xi \leqslant x < R_3/q - \xi$,

$$\left|f_I''(x)\right| \asymp \frac{hu^\alpha q^2}{{R_1}^{2-\alpha}} =: \lambda_2.$$

By van der Corput second derivative test (Lemma 2.3), we obtain

$$\left|\sum_{R_1/q-\xi\leqslant r_1<R_3/q-\xi} e\big(f_I(r_1)\big)\right| \ll \frac{R_3-R_1}{q}\lambda_2^{1/2} + \lambda_2^{-1/2} =$$

$$\frac{R_1}{q}\cdot\sqrt{h}\frac{u^{\alpha/2}q}{R_1^{1-\alpha/2}} + \frac{R_1^{1-\alpha/2}}{\sqrt{h}u^{\alpha/2}q}.$$

Since $(\log X)^{\max(A_0,B_0)+1}\|b\|_\infty \ll_{\delta_1} X^{\delta_1}$ for arbitrarily small $\delta_1 > 0$, we get

$$\left|W(\mathbf{D})\right| \ll_{\delta_1} X^{\delta_1} \sum_{U\Theta^{-9}\leqslant u\leqslant U\Theta^{9}} \left(\frac{R_1}{q}\sqrt{h}\frac{u^{\alpha/2}q}{R_1^{1-\alpha/2}} + \frac{R_1^{1-\alpha/2}}{\sqrt{h}u^{\alpha/2}q}\right) \ll$$

$$X^{\delta_1}\sqrt{h}R_1^{\alpha/2} \sum_{U\Theta^{9}\leqslant u\leqslant U\Theta^{-9}} u^{\alpha/2} + \frac{X^{\delta_1}R_1^{1-\alpha/2}}{\sqrt{h}q} \sum_{U\Theta^{-9}\leqslant u\leqslant \Theta U^{9}} \frac{1}{u^{\alpha/2}} \ll$$

$$X^{\delta_1}\sqrt{h}D_1^{\alpha/2}U^{\alpha/2+1} + \frac{X^{\delta_1}D_1^{1-\alpha/2}U^{1-\alpha/2}}{\sqrt{h}q}.$$

Thus,

$$W_I \ll \sum_{\substack{U\leqslant Y_1X_1^{-3/5-\varepsilon_1}\\ U\in\mathbf{G}}}\ \sum_{\substack{D_2\dots D_{10}=U\\ D_2,\dots,D_{10}\in\mathbf{G}}}\cdots\sum\ \sum_{\substack{X_1^{3/5+\varepsilon_1}\leqslant D_1\leqslant Y_1U^{-1}\\ D_1\in\mathbf{G}}}\Bigg(X^{\delta_1}\sqrt{h}D_1^{\alpha/2}U^{\alpha/2+1}+ \frac{X^{\delta_1}D_1^{1-\alpha/2}U^{1-\alpha/2}}{\sqrt{h}q}\Bigg) \ll \sum_{\substack{U\leqslant Y_1X_1^{-3/5-\varepsilon_1}\\ U\in\mathbf{G}}}\ \sum_{\substack{D_2\dots D_{10}=U\\ D_2,\dots,D_{10}\in\mathbf{G}}}\cdots\sum\Bigg(\sqrt{h}X^{\delta_1+\alpha/2}U\log(Y_1/U)^{A_0+1}+ \frac{X^{\delta_1+1-\alpha/2}}{\sqrt{h}q}\log(Y_1/U)^{A_0+1}\Bigg).$$

Finally, for fixed $U=\Theta^k$, $k\leqslant \log\big(Y_1X_1^{-3/5}\big)/\log\Theta$, using the trivial bound

$$\sum_{\substack{D_2\dots D_{10}=U\\ D_2,\dots,D_{10}\in\mathbf{G}}}\cdots\sum 1=\sum_{k_2+\ldots+k_{10}=k}\cdots\sum 1\leqslant k^9\ll(\log X)^{9(A_0+1)},$$

we get

$$W_I\ll \sum_{\substack{U\leqslant Y_1X_1^{-3/5-\varepsilon_1}\\ U\in\mathbf{G}}} X^{\delta_1}(\log X)^{10(A_0+1)}\Bigg(\sqrt{h}UX^{\alpha/2}+\frac{1}{\sqrt{h}}\frac{X^{1-\alpha/2}}{q}\Bigg)\ll X^{2\delta_1}\Bigg(X^{2/5+\alpha/2-\varepsilon_1}+\frac{X^{1-\alpha/2}}{q}\Bigg). \quad (5.5)$$

5.4 Type II estimate

For a fixed partition $\mathbf{S}\cup\mathbf{T}=\{1,\dots,10\}$, we use the notations

$$m=\prod_{i\in\mathbf{S}}d_i,\qquad n=\prod_{i\in\mathbf{T}}d_i,\qquad M=\prod_{i\in\mathbf{S}}D_i,\qquad N=\prod_{i\in\mathbf{T}}D_i.$$

Note that $MN\asymp X$. Then, the type II sum can be written as

$$W_{II}=\sum_{\substack{X_1^{2/5-\varepsilon_1}\leqslant M\leqslant X_1^{3/5+\varepsilon_1}\\ M\in\mathbf{G}}}\ \sum_{\substack{X_1/M\leqslant N\leqslant Y_1/N\\ N\in\mathbf{G}}}\ \sum_{\substack{\exists\mathbf{S},\mathbf{T}:\\ \prod_{i\in\mathbf{S}}D_i=M\\ \prod_{i\in\mathbf{T}}D_i=N}} W(\mathbf{D}),$$

where

$$W(\mathbf{D}) = \sum_{M\Theta^{-|\mathbf{S}|}\leqslant m\leqslant M\Theta^{|\mathbf{S}|}} \gamma(m) \sum_{\substack{n=1\\ mn\equiv a \pmod q}}^{+\infty} \beta(n)\psi\left(\frac{mn}{X}\right)e\left(h(mn)^{\alpha}\right),$$

$$\gamma(m) = \sum_{\substack{\prod_{i\in\mathbf{S}} d_i=m\\ d_i\leqslant V \text{ for } i\geqslant 6, i\in\mathbf{S}}} \left(\prod_{i\in\mathbf{S}} a_i(d_i)\Psi_{D_i}(d_i)\right),$$

$$\left|\gamma(m)\right| \leqslant (\log X) \sum_{\substack{\prod_{i\in\mathbf{S}} d_i=m\\ d_i\leqslant V \text{ for } i\geqslant 6, i\in\mathbf{S}}} 1 \leqslant (\log X)\tau_{|\mathbf{S}|}(m),$$

$$\beta(n) = \sum_{\substack{\prod_{i\in\mathbf{T}} d_i=n\\ d_i\leqslant V \text{ for } i\geqslant 6, i\in\mathbf{T}}} \left(\prod_{i\in\mathbf{T}} a_i(d_i)\Psi_{D_i}(d_i)\right), \qquad |\beta(n)| \leqslant (\log X)\tau_{|\mathbf{T}|}(n),$$

$$a_1(d) = \log d, \qquad a_2(d) = \ldots = a_5(d) = 1, \qquad a_6(d) = \ldots = a_{10}(d) = \mu(d).$$

By definition of $\beta(n)$, we have

$$W(\mathbf{D}) = \sum_{M_1\leqslant m\leqslant M_2} \gamma(m) \sum_{\substack{N_1\leqslant n\leqslant N_2\\ mn\equiv a \pmod q}} \beta(n)\psi\left(\frac{mn}{X}\right)e\left(h(mn)^{\alpha}\right),$$

$$M_1 = M\Theta^{-|\mathbf{S}|}, \qquad M_2 = M\Theta^{|\mathbf{S}|}, \qquad N_1 = N\Theta^{-|\mathbf{T}|}, \qquad N_2 = N\Theta^{|\mathbf{T}|}.$$

Cauchy inequality yields:

$$\left|W(\mathbf{D})\right|^2 \leqslant \left(\sum_{M_1\leqslant m\leqslant M_2} |\gamma(m)|^2\right)\left(\sum_{M_1\leqslant m\leqslant M_2}\left|\sum_{\substack{N_1\leqslant n\leqslant N_2\\ mn\equiv a \pmod q}} \beta(n)\psi\left(\frac{mn}{X}\right)e\left(h(mn)^{\alpha}\right)\right|^2\right).$$

Next, by Mardzhanishvili's inequality (Lemma 2.5), we get

$$\left|W(\mathbf{D})\right|^2 \ll M_1(\log X)^{2+\kappa}\left(\sum_{M_1\leqslant m\leqslant M_2}\left|\sum_{\substack{N_1\leqslant n\leqslant N_2\\ mn\equiv a \pmod q}} \beta(n)\psi\left(\frac{mn}{X}\right)e\left(h(mn)^{\alpha}\right)\right|^2\right), \quad (5.6)$$

where $\kappa = |\mathbf{S}|^2 - 1$. Rewrite the second factor as follows:

$$\sum_{M_1\leqslant m\leqslant M_2} \sum_{\substack{N_1\leqslant n_1,n_2\leqslant N_2\\ mn_i\equiv a \pmod q, i=1,2}} \beta(n_1)\beta(n_2)\psi\left(\frac{mn_1}{X}\right)\psi\left(\frac{mn_2}{X}\right)e\left(hm^{\alpha}(n_1^{\alpha}-n_2^{\alpha})\right) =$$

$$\sum_{M_1\leqslant m\leqslant M_2} \sum_{\substack{N_1\leqslant n\leqslant N_2\\ mn\equiv a \pmod q}} \beta^2(n)\psi^2\left(\frac{mn}{X}\right) + 2\mathrm{Re}\left(S(M,N)\right),$$

where

$$S(M,N) = \sum_{M_1 \leqslant m \leqslant M_2} \sum_{\substack{N_1 \leqslant n_1 < n_2 \leqslant N_2 \\ mn_i \equiv a \pmod q, i=1,2}} \beta(n_1)\beta(n_2)\psi\left(\frac{mn_1}{X}\right)\psi\left(\frac{mn_2}{X}\right)e\big(hm^\alpha(n_1^\alpha - n_2^\alpha)\big).$$

The diagonal term does not exceed

$$\sum_{M_1 \leqslant m \leqslant M_2} \sum_{\substack{N_1 \leqslant n \leqslant N_2 \\ mn \equiv a \pmod q}} \beta^2(n) \ll$$

$$(\log X)^2 \frac{MN}{q} (\log X)^{|\mathbf{T}|^2-1} \ll \frac{X}{q} (\log X)^{|\mathbf{T}|^2+1}. \quad (5.7)$$

Setting $m = qr + l$, we get

$$R_1 - \eta \leqslant r \leqslant R_2 - \eta, \qquad \eta = \frac{l}{q},$$

for given l, $(l,q) = 1$, $R_1 = M_1/q$, $R_2 = M_2/q$. Hence,

$$S(M,N) = \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \sum_{R_1 - \eta \leqslant r \leqslant R_2 - \eta} \sum_{\substack{N_1 \leqslant n_1 < n_2 \leqslant N_2 \\ n_1, n_2 \equiv e \pmod q}} \beta(n_1)\beta(n_2) \cdot$$

$$\psi\left(\frac{(qr+l)n_1}{X}\right)\psi\left(\frac{(qr+l)n_2}{X}\right)e\big(h(n_1^\alpha - n_2^\alpha)q^\alpha(r+\eta)^\alpha\big),$$

where $e = al^* \pmod q$. Changing the order of summation, we estimate $S(M,N)$ as follows:

$$\big|S(M,N)\big| \leqslant \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \sum_{\substack{N_1 \leqslant n_1 < n_2 \leqslant N_2 \\ n_1, n_2 \equiv e \pmod q}} |\beta(n_1)||\beta(n_2)| \cdot$$

$$\left|\sum_{R_1 - \eta \leqslant r \leqslant R_2 - \eta} \psi\left(\frac{(qr+l)n_1}{X}\right)\psi\left(\frac{(qr+l)n_2}{X}\right)e\big(f_{II}(r)\big)\right|,$$

where $f_{II}(x) = h(n_1^\alpha - n_2^\alpha)q^\alpha(x+\eta)^\alpha$. Using the conditions $n_1 < n_2$, $n_1 \equiv n_2 \equiv e \pmod q$, we set $n_2 = n_1 + qs$ with $s \geqslant 1$. On the other hand, $n_2 \leqslant N_2$ implies $n_1 + qs \leqslant N_2$. Hence, $s < (N_2 - N_1)/q = t$, and therefore

$$\big|S(M,N)\big| \ll \sum_{\substack{l=1 \\ (l,q)=1}}^{q} \sum_{1 \leqslant s < t} \sum_{\substack{N_1 \leqslant n \leqslant N_2 \\ n \equiv e \pmod q}} |\beta(n)||\beta(n+qs)| \cdot$$

$$\left|\sum_{R_1 - \eta \leqslant r \leqslant R_2 - \eta} \psi\left(\frac{(qr+l)n_1}{X}\right)\psi\left(\frac{(qr+l)n_2}{X}\right)e\big(f_{II}(r)\big)\right|.$$

By partial summation,

$$\left|S(M,N)\right| \ll (\log X)^{B_0} \sum_{\substack{l=1\\(l,q)=1}}^{q} \sum_{1\leqslant s<t} \sum_{\substack{N_1\leqslant n\leqslant N_2\\ n\equiv e \pmod q}} |\beta(n)||\beta(n+qs)| \Bigg| \sum_{R_1-\eta\leqslant r\leqslant R_3-\eta} e\big(f_{II}(r)\big)\Bigg|,$$

where $R_1 < R_3 \leqslant R_2$. Next,

$$f_{II}''(x) = \frac{\alpha(\alpha-1)h(n_2^\alpha - n_1^\alpha)q^\alpha}{(x+\eta)^{2-\alpha}}, \qquad \text{hence} \quad \left|f_{II}''(x)\right| \asymp h(n_2^\alpha - n_1^\alpha)q^\alpha \left(\frac{q}{M}\right)^{2-\alpha}.$$

By Lagrange mean value theorem,

$$hq^\alpha\big((n+qs)^\alpha - n^\alpha\big) = \alpha hsq^{\alpha+1}(n+qs\theta')^{\alpha-1} \asymp hsq^{\alpha+1}N^{\alpha-1} \asymp hsq^{\alpha+1}\left(\frac{X}{M}\right)^{\alpha-1},$$

where $|\theta'| \leqslant 1$. Hence,

$$\left|f_{II}''(x)\right| \asymp \frac{hsq^2}{X^{1-\alpha}}\frac{q}{M}.$$

Applying van der Corput second derivative test (Lemma 2.3), we get

$$\sum_{R_1-\eta\leqslant r\leqslant R_3-\eta} e\big(f_{II}(r)\big) \ll \big(R_3-R_1\big)\left(\frac{hsq^2}{X^{1-\alpha}}\frac{q}{M}\right)^{1/2} + \left(\frac{X^{1-\alpha}}{hsq^2}\frac{M}{q}\right)^{1/2}.$$

We have $R_3 - R_1 \ll M/q$. The factor $|\beta(n)|\cdot|\beta(n+qs)|$ is bounded from above by $(X/q)^{\delta_2}$ for arbitrarily small $\delta_2 > 0$. The summation over $n \equiv e \pmod q$ for $N_1 \leqslant n \leqslant N_2$ contributes the factor of at most $X/(Mq) > 1$ (since $M \ll X^{3/5+3\alpha/5}$, $q \leqslant X^{2/5-3\alpha/5}$). Thus,

$$\left|S(M,N)\right| \ll \left(\frac{X}{q}\right)^{\delta_2} \sum_{\substack{l=1\\(l,q)=1}}^{q-1} \sum_{1\leqslant s<t} \frac{X}{Mq}\left(\left(\frac{hsqM}{X^{1-\alpha}}\right)^{1/2} + \left(\frac{MX^{1-\alpha}}{hsq^3}\right)^{1/2}\right) \ll \left(\frac{X}{q}\right)^{\delta_2}\left(\frac{\sqrt{h}X^{2+\alpha/2}}{qM^2} + \frac{X^{2-\alpha/2}}{\sqrt{h}q^2M}\right). \quad (5.8)$$

Combining (5.6), (5.7), and (5.8), we get

$$\left|W(\mathbf{D})\right| \ll \sqrt{M}(\log X)^{1+\kappa/2}\left(\frac{X}{q}(\log X)^{|\mathbf{T}|^2+1} + \left(\frac{X}{q}\right)^{\delta_2}\left(\frac{\sqrt{h}X^{2+\alpha/2}}{qM_1} + \frac{X^{2-\alpha/2}}{\sqrt{h}q^2M}\right)\right)^{1/2} \ll X^{\delta_3}\left(\left(\frac{XM}{q}\right)^{1/2} + \frac{X^{1+\alpha/4}}{(qM)^{1/2}} + \frac{X^{1-\alpha/4}}{q}\right),$$

where we have used the inequality

$$\max\left((\log X)^{\kappa/2+|\mathbf{T}|^2+2}, (X/q)^{\delta_2}\sqrt{h}\right) \leqslant X^{\delta_3}$$

for some $\delta_3 \geqslant \delta_2$. For fixed $M = \Theta^k$ and $N = \Theta^l$ with $k + l = 10$, the number of corresponding tuples $\mathbf{S}$ and $\mathbf{T}$ does not exceed

$$\sum_{\substack{i+j=10\\ i+j\geqslant 1}} k^i l^j \ll \left(\frac{\log X}{\log \Theta}\right)^{10} \ll (\log X)^{10(A_0+1)}.$$

Thus,

$$W_{II} \ll \sum_{\substack{X_1^{2/5-\varepsilon_1}\leqslant M\leqslant X_1^{3/5+\varepsilon_1}\\ M\in\mathbf{G}}} \sum_{\substack{X_1/M\leqslant N\leqslant Y_1/N\\ N\in\mathbf{G}}} \sum_{\substack{\exists \mathbf{S},\mathbf{T}:\\ \prod_{i\in\mathbf{S}} D_i=M\\ \prod_{i\in\mathbf{T}} D_i=N}} \left|W(\mathbf{D})\right| \ll$$

$$(\log X)^{10(A_0+1)} \sum_{\substack{X_1^{2/5-\varepsilon_1}\leqslant M\leqslant X_1^{3/5+\varepsilon_1}\\ M\in\mathbf{G}}} \sum_{\substack{X_1/M\leqslant N\leqslant Y_1/N\\ N\in\mathbf{G}}} X^{\delta_3}\left(\left(\frac{XM}{q}\right)^{1/2} + \frac{X^{1+\alpha/4}}{(qM)^{1/2}} + \frac{X^{1-\alpha/4}}{q}\right) \ll$$

$$X^{2\delta_3}\left(\frac{X^{4/5+\varepsilon_1/2}}{\sqrt{q}} + \frac{X^{4/5+\alpha/4+\varepsilon_1/2}}{\sqrt{q}} + \frac{X^{1-\alpha/4}}{q}\right) \ll \frac{X^{4/5+\alpha/4+\varepsilon_1/2+2\delta_3}}{\sqrt{q}} + \frac{X^{1-\alpha/4+2\delta_3}}{q}. \tag{5.9}$$

5.5 Type III estimate

We apply the method of stationery phase to treat the type III sum. To deal with the oscillatory integrals arising after the Poisson summation, we use Lemmas 5.5 and 5.6.

Later in this section, we will use the following notation:

$$\beta = \frac{2-\alpha}{1-\alpha}, \qquad \gamma = \frac{\alpha}{1-\alpha}, \qquad \delta = \frac{1}{1-\alpha},$$
$$\xi = \frac{1}{1-\gamma} = \frac{1-\alpha}{1-2\alpha}, \qquad \eta = \frac{\alpha}{1-2\alpha}, \qquad \omega = \xi(2-\gamma) = \frac{2-3\alpha}{1-2\alpha}.$$

Let us denote as M, N, K the three indices from $\{D_1, \ldots, D_5\}$ satisfying type III conditions, as m, n, k the corresponding indices from $\{d_1, \ldots, d_5\}$, as i_1, i_2, i_3 the corresponding indices from $\{1, \ldots, 5\}$, and let $\mathbf{I}$ be the set of all remaining indices $\{1, \ldots, 10\}\backslash\{i_1, i_2, i_3\}$. Also let

$$U = \prod_{i\in\mathbf{I}} D_i, \qquad u = \prod_{i\in\mathbf{I}} d_i.$$

We get the sum of the form

$$W_{III} = \sum_{M,N,K\in\mathbf{G}}{}' \sum_{U\in\mathbf{G}}{}' \sum_{U\Theta^{-7}\leqslant u\leqslant U\Theta^{7}} F(U,u) \sum_{\substack{m,n,k=1\\ umnk\equiv a \pmod q}}^{+\infty} f_1(m)f_2(n)f_3(k)\cdot$$
$$\Psi_M(m)\Psi_N(n)\Psi_K(k)\psi\left(\frac{umnk}{X}\right)e\big(h(umnk)^{\alpha}\big), \quad (5.10)$$

where

$$F(U,u) = \left(\sum_{\prod_{i\in\mathbf{I}} D_i=U}\cdots\sum\right)\left(\sum_{\prod_{i\in\mathbf{I}} d_i=u}\cdots\sum\right)\left(\prod_{i\in\mathbf{I}} a_i(d_i)\Psi_{D_i}(d_i)\right),$$
$$a_1(d) = \log d, \qquad a_2(d) = \ldots = a_5(d) = 1, \qquad a_6(d) = \ldots = a_{10}(d) = \mu(d),$$

$f_i(x)$ are smooth functions such that $f_i(x) \equiv 0$ if $x \leqslant 0$ and $f_i(x) = 1$ or $f_i(x) = \log x$ for $x \geqslant 1$, $\sum'$ denotes the summation over $M, N, K, U \in \mathbf{G}$ satisfying the type III conditions. Without loss of generality, we can assume $M \leqslant N \leqslant K$. Then rewrite (5.10) in the following way:

$$W_{III} = \sum_{\substack{U\leqslant X^{1/10-3\varepsilon_1/2}\\ U\in\mathbf{G}}} \sum_{U\Theta^{-7}\leqslant u\leqslant U\Theta^{7}} F(U,u) \sum_{\substack{M_1\leqslant M\leqslant M_2\\ M\in\mathbf{G}}} \sum_{\substack{N_1\leqslant N\leqslant N_2\\ N\in\mathbf{G}}} \sum_{\substack{K_1\leqslant K\leqslant K_2\\ K\in\mathbf{G}}} W(M,N,K), \quad (5.11)$$

where

$$M_1 = X_1^{1/5+2\varepsilon_1}, \qquad M_2 = (Y_1U^{-1})^{1/3},$$
$$N_1 = \max\big(M, X^{3/5+\varepsilon_1}M^{-1}\big), \qquad N_2 = \min\left(X^{2/5-\varepsilon_1}, \left(\frac{Y}{MU}\right)^{1/2}\right),$$
$$K_1 = N, \qquad K_2 = \min\left(X^{2/5-\varepsilon_1}, \frac{Y}{UMN}\right),$$

and

$$W(M,N,K) = \sum_{\substack{m,n,k=1\\ mnku\equiv a \pmod q}}^{+\infty} f_1(m)f_2(n)f_3(k)\Psi_M(m)\Psi_N(n)\Psi_K(k)\psi\left(\frac{umnk}{X}\right)e\big(h(umnk)^{\alpha}\big).$$

Note that $f_i(.)\Psi_D(.)\psi(.)$ is smooth on $(0;+\infty)$, so one can apply Poisson summation (Lemma 5.4) to any of the sums over n, m, k. We also note that the number of terms in each sum over $U, M, N, K \in \mathbf{G}$ in (5.11) is $O\big((\log X)^{A_0+1}\big)$ and $|F(U,u)|$ can be bounded as follows:

$$\big|F(U,u)\big| \ll (\log X)\tau_7(u)\cdot \#\left\{(e_1,\dots,e_7)\in\mathbb{Z}^7_{\geqslant 0} : e_1+\ldots+e_7 = \frac{\log U}{\log\Theta}\right\} \ll (\log X)\tau_7(u)\left(\frac{\log U}{\log \Theta}\right)^6 \ll \tau_7(u)(\log X)^{6(A_0+1)+1}.$$

First iteration of Poisson summation

We first apply Poisson summation to the longest sum over k. By orthogonality of characters,

$$W(M,N,K) = \frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(ua^*)\sum_{m=1}^{+\infty}\chi(m)f_1(m)\Psi_M(m)\sum_{n=1}^{+\infty}\chi(n)f_2(n)\Psi_N(n)W_{m,n,\chi},$$

where

$$W_{m,n,\chi} = \sum_{k=1}^{+\infty}\chi(k)f_3(k)\Psi_K(k)\psi\left(\frac{umnk}{X}\right)e\big(h(umnk)^\alpha\big).$$

To remove the factor $\chi(k)$ in the last sum, we substitute $k = qr + l$:

$$W_{m,n,\chi} = \sum_{l=1}^{q-1}\chi(l)\sum_{r=-\infty}^{+\infty} f_3(qr+l)\Psi_K(qr+l)\psi\left(\frac{umn(qr+l)}{X}\right)e\big(h(umn(qr+l))^\alpha\big).$$

The function $(qr+l)^\alpha$ is extended by zero for $r < -l/q$. By Poisson summation,

$$W_{m,n,\chi} = \sum_{l=1}^{q-1}\chi(l)\sum_{s=-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_3(qv+l)\Psi_K(qv+l)\psi\left(\frac{umn(qv+l)}{X}\right)\cdot e\big(h(umn(qv+l))^\alpha\big)e\big(-vs\big)dv.$$

We can reduce the range of integration to $(-l/q;+\infty)$ due to the fact that $f_3(x)=0$ for $x \leqslant 0$. Then substituting

$$t = \frac{umn(qv+l)}{X},$$

we get

$$W_{n,m,\chi} = \frac{X}{qumn}\sum_{s=-\infty}^{+\infty}\tau(\chi;s)I_{m,n}(s),$$

where

$$I_{m,n}(s) = \int_0^{+\infty} f_3\left(\frac{Xt}{umn}\right)\Psi_K\left(\frac{Xt}{umn}\right)\psi(t)e\left(h(Xt)^\alpha - \frac{Xst}{qumn}\right)dt,$$

$$\tau(\chi; s) = \sum_{l=1}^{q-1} \chi(l)e\left(\frac{sl}{q}\right) \qquad \text{is a Gauss sum.}$$

Next, we verify the conditions of Lemma 5.5 and Lemma 5.6. Let

$$w(t) = f_3\left(\frac{Xt}{umn}\right)\Psi_K\left(\frac{Xt}{umn}\right)\psi(t),$$

$$g_n(t) = \begin{cases} h(Xt)^\alpha - \dfrac{Xst}{qumn}, & \text{if } 1-\Delta \leqslant t \leqslant y+\Delta; \\ 0 & \text{if } t \leqslant 1-2\Delta \quad \text{or } t \geqslant y+2\Delta, \end{cases}$$

and extend $g_n(t)$ to a smooth function on $[1-2\Delta, 1-\Delta]$ and $[y+\Delta, y+2\Delta]$. We now evaluate the derivatives. First, if $1-\Delta \leqslant t \leqslant y+\Delta$ and $j \geqslant 2$, then we have

$$g_n^{(j)}(t) = \frac{(\alpha)_j hX^\alpha}{t^{j-\alpha}}, \qquad (\alpha)_j = \prod_{i=1}^{j}(\alpha - i + 1), \qquad \left|g_n^{(j)}(t)\right| \asymp_{\alpha,j} hX^\alpha.$$

Thus, one can take $Y_I = hX^\alpha$, $Q_I = 1$. Now let us estimate $w^{(j)}(t)$ on $\mathbb{J}$. We have

$$\frac{d^j w(t)}{dt^j} = \sum_{j_1+j_2+j_3=j} \binom{j}{j_1, j_2, j_3} \frac{d^{j_1} f_3}{dt^{j_1}}\left(\frac{Xt}{umn}\right)\frac{d^{j_2}\Psi_K}{dt^{j_2}}\left(\frac{Xt}{umn}\right)\frac{d^{j_3}\psi(t)}{dt^{j_3}}.$$

Next,

$$\frac{d^{j_1} f_3}{dt^{j_1}}\left(\frac{Xt}{umn}\right) \ll \log X,$$

$$\frac{d^{j_2}}{dt^{j_2}}\Psi_K\left(\frac{Xt}{umn}\right) \ll \left(\frac{X}{Kumn}\right)^{j_2}(\log X)^{j_2A_0} \ll (\log X)^{j_2A_0},$$

$$\frac{d^{j_3}\psi(t)}{dt^{j_3}} \ll (\log X)^{j_3B_0}.$$

Thus, we find

$$w^{(j)}(t) \ll (\log X) \sum_{j_1+j_2+j_3=j} \binom{j}{j_1, j_2, j_3}(\log X)^{j_2A_0}(\log X)^{j_3B_0} \ll$$
$$(\log X)\left(1 + (\log X)^{A_0} + (\log X)^{B_0}\right)^j \ll (\log X)^{C_0 j+1}, \quad (5.12)$$

where $C_0 = \max(A_0, B_0)$. So one can take $X_I = \log X$, $V_I = (\log X)^{-C_0}$. From $Z_I = Q_I + X_I + Y_I + \tilde{V}_I + 1$, we get $Z_I \asymp Y_I \asymp hX^\alpha$, hence for any fixed δ_I, $0 < \delta_I < 1/10$, we have

$$V_I = (\log X)^{-C_0} \geqslant \frac{Q_I Z_I^{\delta_I/2}}{\sqrt{Y_I}} \asymp (hX^\alpha)^{-1/2+\delta_I/2}.$$

Now set

$$T_1 = \frac{1}{4}\frac{\alpha humNq}{X^{1-\alpha}}, \qquad T_2 = 4\frac{\alpha humNq}{X^{1-\alpha}}$$

and split the sum $W_{m,n,\chi}$ in the following way:

$$W_{m,n,\chi} = \frac{X}{qumn}\left\{\sum_{T_1 \leqslant s \leqslant T_2} + \sum_{|s|>T_2} + \sum_{-T_2 \leqslant s < T_1}\right\}\tau(\chi; s) I_{m,n}(s) =: \frac{X}{qumn}(S_1 + S_2 + S_3).$$

For S_2 and S_3, we apply Lemma 5.5 to estimate $I_{m,n}(s)$; for S_1, we compute $I_{m,n}(s)$ asymptotically using Lemma 5.6. We have

$$g_n'(t) = \alpha h X^\alpha t^{\alpha-1} - \frac{Xs}{qumn}.$$

If $|s| > T_2$, then

$$|g_n'(t)| \geqslant \frac{X|s|}{qumn}\left(1 - \frac{\alpha h t^{\alpha-1} qumn}{X^{1-\alpha} T_2}\right) \geqslant \frac{X|s|}{2qumn}.$$

If $-T_2 \leqslant s \leqslant 0$, then

$$g_n'(t) = \alpha h X^\alpha t^{\alpha-1} + \frac{X|s|}{qumn} \geqslant \alpha h X^\alpha t^{\alpha-1} \geqslant \frac{\alpha}{3} h X^\alpha.$$

Finally, if $1 \leqslant s < T_1$, then

$$g_n'(t) \geqslant \alpha h X^\alpha t^{\alpha-1}\left(1 - \frac{XT_1}{qumn}\frac{t^{1-\alpha}}{\alpha h X^\alpha}\right)\alpha h X^\alpha t^{\alpha-1}\left(1 - \frac{5}{8}\right) \geqslant \frac{\alpha}{6} h X^\alpha.$$

Thus, one can choose

$$R_I = \begin{cases} \dfrac{X|s|}{2qumn} & \text{if } |s| > T_2; \\ \dfrac{\alpha}{6} h X^\alpha & \text{if } -T_2 \leqslant s < T_1. \end{cases}$$

In the case $|s| > T_2$, we set

$$\Delta_1 = \frac{Q_I R_I}{\sqrt{Y_I}}, \qquad \Delta_2 = R_I V_I,$$

and get

$$\Delta_1 = \frac{X|s|}{2qumn}\frac{1}{\sqrt{hX^\alpha}} \geqslant \frac{X^{1-\alpha/2} T_2}{2\sqrt{h} qumn} \geqslant X^{\alpha/2} \cdot 2\alpha\sqrt{h}\frac{N}{n} \geqslant \alpha X^{\alpha/2},$$

$$\Delta_2 = \frac{X|s|}{2qumn}(\log X)^{-C_0} \geqslant \frac{XT_2(\log X)^{-C_0}}{2qumn} \geqslant X^\alpha \cdot \frac{2\alpha N}{n}(\log X)^{-C_0} \geqslant X^{\alpha/2}.$$

If $-T_2 \leqslant s < T_1$, then

$$\Delta_1 = \frac{\alpha h}{6}\frac{X^\alpha}{\sqrt{hX^\alpha}} \geqslant \frac{\alpha}{6}X^{\alpha/2}, \qquad \Delta_2 = \frac{\alpha h}{6}X^\alpha(\log X)^{-C_0} \geqslant X^{\alpha/2}.$$

Thus, by Lemma 5.5,

$$I_{m,n}(s) \ll_\alpha (\log X)\left\{\left(\frac{X|s|}{2qumn}\frac{1}{\sqrt{hX^\alpha}}\right)^{-A_I} + \left(\frac{X|s|}{2qumn}\frac{1}{(\log X)^{C_0}}\right)^{-A_I}\right\} \ll_\alpha (\log X)\left(\frac{X|s|}{2qumn}\frac{1}{\sqrt{hX^\alpha}}\right)^{-A_I} \ll_\alpha (\log X)\left(\frac{2qumn\sqrt{h}}{X^{1-\alpha/2}|s|}\right)^{A_I}$$

for $|s| \geqslant T_2$, and

$$I_{m,n}(s) \ll_\alpha (\log X)\left\{\left(\frac{\alpha}{6}\sqrt{h}X^{\alpha/2}\right)^{-A_I} + \left(\frac{\alpha h}{6}X^\alpha(\log X)^{-C_0}\right)^{-A_I}\right\} \ll_\alpha (\log X)X^{-\alpha A_I/2}$$

if $-T_2 \leqslant s \leqslant T_1$. Choose $A_I = 2D_0 + 1$, where $D_0 = D_0(\alpha) > 1$ is large enough. Going back to S_2 and S_3, we get

$$S_2 \ll \sum_{|s|>T_2} (\log X)\left(\frac{2qumn\sqrt{h}}{X^{1-\alpha/2}}\right)^{A_I}\frac{1}{|s|^{A_I}} \ll \frac{umNq}{X^{1-\alpha}}X^{-\alpha D_0},$$

$$S_3 \ll (T_1 + T_2 + 1)X^{-\alpha D_0 - \alpha/2}(\log X) \ll \left(\frac{umNq}{X^{1-\alpha}} + 1\right)X^{-\alpha D_0}.$$

From $qumn \gg XK^{-1}$, we find

$$W_{m,n,\chi} = \frac{X}{qumn}S_1 + O\left(\frac{X}{qumn}\left(\frac{umNq}{X^{1-\alpha}} + 1\right)X^{-\alpha D_0}\right) = \frac{X}{qumn}S_1 + O\big(X^{-\alpha(D_0+1)} + KX^{-\alpha D_0}\big) = \frac{X}{qumn}S_1 + O\big(KX^{-\alpha D_0}\big).$$

Now we compute S_1. Choosing $\delta_I = 1/20$, $A_I = D_0$, we apply Lemma 5.6 to $I(s)$ when $T_1 < s \leqslant T_2$. Let $g_n'(t_0) = 0$. Then

$$t_0 = \frac{1}{X}\left(\frac{\alpha hqumn}{s}\right)^{1/(1-\alpha)}.$$

Notice that for any $T_1 \leqslant s \leqslant T_2$, the point t_0 lies in $\mathbb{J} = \big[10^{-1}; 10\big]$. Thus,

$$I_{m,n}(s) = e\left(g_n(t_0) - \frac{1}{8}\right)\sum_{0\leqslant\nu\leqslant\nu_1}\frac{\sqrt{2\pi}}{\nu!}\frac{(2i)^{-\nu}}{|g_n''(t_0)|^{\nu+1/2}}\frac{d^{2\nu}G_n(t)}{dt^{2\nu}}\bigg|_{t=t_0} + O\big(X^{-\alpha D_0}\big),$$

where

$$G_n(t) = w(t)e\big(H_n(t)\big), \qquad \nu_1 = 60D_0, \qquad H_n(t) = g_n(t) - g_n(t_0) - \frac{1}{2}g_n''(t_0)(t-t_0)^2.$$

One can easily verify the identities

$$g_n(t_0) = (1-\alpha)(\alpha^\alpha h)^\delta \left(\frac{qumn}{s}\right)^\gamma, \qquad \left|g_n''(t_0)\right| = \alpha(1-\alpha)hX^2\left(\frac{s}{\alpha hqumn}\right)^\beta,$$

where $\gamma = \alpha/(1-\alpha), \delta = 1/(1-\alpha), \beta = (2-\alpha)/(1-\alpha)$. Then, if $1-\Delta \leqslant t_0 \leqslant y+\Delta$, we get

$$I_{m,n}(s) = e\left((1-\alpha)(\alpha^\alpha h)^\delta\left(\frac{qumn}{s}\right)^\delta\right)\sum_{0\leqslant\nu\leqslant\nu_1}\frac{c_\nu(\alpha)}{(hX^2)^{\nu+1/2}}\left(\frac{hqumn}{s}\right)^{\beta(\nu+1/2)}\cdot \left.\frac{d^{2\nu}G_n(t)}{dt^{2\nu}}\right|_{t=t_0} + O\big(X^{-\alpha D_0}\big), \quad (5.13)$$

with

$$c_\nu(\alpha) = \frac{\sqrt{2\pi}}{\nu!}\frac{(2i)^{-\nu}e^{-\pi i/4}}{\big(\alpha(1-\alpha)^{\nu+1/2}\big)}\alpha^{\beta(\nu+1/2)}.$$

Notice that (5.13) remains valid if $t_0 \notin \big[1-\Delta; y+\Delta\big]$ since $w(t) \equiv 0$, $G_n(t) \equiv 0$ for t close to t_0.

Going back to the sum $W_{m,n,\chi}$, we have

$$W_{m,n,\chi} = \frac{X}{qumn}S_1 + O\big(KX^{-\alpha D_0}\big) = \frac{X}{qumn}\sum_{T_1<s\leqslant T_2}\tau(\chi;s)\left\{e\left((1-\alpha)(\alpha^\alpha h)^\delta\left(\frac{qumn}{s}\right)^\gamma\right)\cdot \sum_{0\leqslant\nu\leqslant\nu_1}\frac{c_\nu(\alpha)}{(hX^2)^{\nu+1/2}}\left(\frac{hqumn}{s}\right)^{\beta(\nu+1/2)}\left.\frac{d^{2\nu}G_n(t)}{dt^{2\nu}}\right|_{t=t_0} + O\big(X^{-\alpha D_0}\big)\right\} + O\big(KX^{-\alpha D_0}\big).$$

The contribution from the error terms can be made arbitrarily small with the appropriate choice of D_0. The main term takes the form

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(ua^*)\sum_{m=1}^{+\infty}\chi(m)f_1(m)\Psi_M(m)\sum_{n=1}^{+\infty}\chi(n)f_2(n)\Psi_N(n)\cdot \frac{X}{qumn}\sum_{0\leqslant\nu\leqslant\nu_1}\frac{c_\nu(\alpha)}{(hX^2)^{\nu+1/2}}\sum_{T_1\leqslant s\leqslant T_2}\tau(\chi;s)\left(\frac{hqumn}{s}\right)^{\beta(\nu+1/2)}\left.\frac{d^{2\nu}G_n(t)}{dt^{2\nu}}\right|_{t=t_0}\cdot e\left\{(1-\alpha)(\alpha^\alpha h)^\delta\left(\frac{qumn}{s}\right)^\gamma\right\}.$$

We also note that for the small values of q, it is possible to get $T_2 < 1$. This case is not a problem since the sum S_1 is empty and the only contribution to the upper bound is coming from Lemma 5.5.

Second iteration of Poisson summation

We have:

$$W(M,N,K) = \frac{(qu)^{-1}}{\varphi(q)} \sum_{\chi \bmod q} \chi(ua^*) \sum_{m=1}^{+\infty} \chi(m) \frac{f_1(m)}{m} \Psi_M(m) \cdot \sum_{T_1<s<T_2} \tau(\chi; s) V_{\chi,m,s} + O\big(X^{-\alpha D_0/2}\big), \quad (5.14)$$

where

$$V_{\chi,m,s} = \sum_{n=1}^{+\infty} \chi(n) \frac{f_2(n)}{n} \Psi_N(n) \sum_{0\leqslant\nu\leqslant\nu_1} \frac{c_\nu(\alpha)}{X^{2\nu}} h^{\delta(\nu+1/2)} \left(\frac{qumn}{s}\right)^{\beta(\nu+1/2)} \cdot \left.\frac{d^{2\nu} G_n(t)}{dt^{2\nu}}\right|_{t=t_0} e\left\{(1-\alpha)(\alpha^\alpha h)^\delta \left(\frac{qumn}{s}\right)^\gamma\right\}.$$

Setting $n = q\rho + \lambda$, we get

$$V_{\chi,m,s} = \sum_{0\leqslant\nu\leqslant\nu_1} \frac{c_\nu(\alpha)}{X^{2\nu}} h^{\delta(\nu+1/2)} \left(\frac{qum}{s}\right)^{\beta(\nu+1/2)} \sum_{\lambda=1}^{q-1} \chi(\lambda) \cdot \sum_{\rho=-\infty}^{+\infty} f_2(q\rho+\lambda)\Psi_N(q\rho+\lambda)(q\rho+\lambda)^{\beta(\nu+1/2)-1} \left.\frac{d^{2\nu} G_{q\rho+\lambda}(t)}{dt^{2\nu}}\right|_{t=t_0} \cdot e\left\{(1-\alpha)(\alpha^\alpha h)^\delta \left(\frac{qum(q\rho+\lambda)}{s}\right)^\gamma\right\}.$$

Applying Poisson summation again, we obtain

$$V_{\chi,m,s} = \sum_{0\leqslant\nu\leqslant\nu_1} \frac{c_\nu(\alpha)}{X^{2\nu}} h^{\delta(\nu+1/2)} \left(\frac{qum}{s}\right)^{\beta(\nu+1/2)} \cdot \sum_{\lambda=1}^{q-1} \chi(\lambda) \sum_{\sigma=-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_2(qv+\lambda)\Psi_N(qv+\lambda)(qv+\lambda)^{\beta(\nu+1/2)-1} \cdot \left.\frac{d^{2\nu} G_{q\rho+\lambda}(t)}{dt^{2\nu}}\right|_{t=t_0} e\left\{(1-\alpha)(\alpha^\alpha h)^\delta \left(\frac{qum(qv+\lambda)}{s}\right)^\gamma - \sigma v\right\} dv. \quad (5.15)$$

Next, we substitute

$$\tau = \frac{\alpha h q u m(qv+\lambda)}{s X^{1-\alpha}}.$$

This implies

$$t_0 = \frac{1}{X}\left(\frac{\alpha h q u m(qv+\lambda)}{s}\right)^\delta = \tau^\delta, \qquad \left(\frac{qum(qv+\lambda)}{s}\right)^\gamma = \left(\frac{X^{1-\alpha}\tau}{\alpha h}\right)^\gamma,$$

$$(\alpha^\alpha h)^\delta \left(\frac{qum(qv+\lambda)}{s}\right)^\gamma = h X^\alpha \tau^\gamma.$$

For convenience we will further use a slightly different notation for functions G_n, H_n, and g_n: $G_n(t) = G(t,\tau), H_n(t) = H(t,\tau), g_n(t) = g(t,\tau)$. The integral in (5.15) takes the form

$$\frac{1}{q}\left(\frac{X^{1-\alpha}s}{\alpha hqum}\right)^{\beta(\nu+1/2)} e\left(\frac{\lambda\sigma}{q}\right)\int_0^{+\infty} f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right)\Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right)\tau^{\beta(\nu+1/2)-1}\cdot \left.\frac{d^{2\nu}G(t,\tau)}{dt^{2\nu}}\right|_{t=\tau^\delta} e\left\{(1-\alpha)hX^\alpha\tau^\gamma - \frac{X^{1-\alpha}s\sigma\tau}{\alpha hq^2um}\right\}d\tau.$$

Hence,

$$V_{\chi,m,s} = \sum_{0\leqslant\nu\leqslant\nu_1}\frac{c_\nu(\alpha)}{X^{2\nu}}h^{\delta(\nu+1/2)}\frac{1}{q}\left(\frac{X^{1-\alpha}}{\alpha h}\right)^{\beta(\nu+1/2)}\sum_{\sigma=-\infty}^{+\infty}\tau(\chi;\sigma)J(\sigma),$$

where the meaning of $J(\sigma)$ is clear. We further simplify the last expression by setting

$$b_\nu(\alpha) = \frac{c_\nu(\alpha)}{\alpha^{\beta(\nu+1/2)}},$$

which gives

$$V_{\chi,m,s} = \frac{X}{q}\sum_{0\leqslant\nu\leqslant\nu_1} b_\nu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\sum_{\sigma=-\infty}^{+\infty}\tau(\chi;\sigma)J(\sigma).$$

Let us denote

$$T_3 = \frac{(\alpha hq)^2um}{4sX^{1-2\alpha}}, \qquad T_4 = 16T_3 = \frac{4(\alpha hq)^2um}{sX^{1-2\alpha}},$$

and split the sum $V_{\chi,m,s}$ as follows:

$$V_{\chi,m,s} = \frac{X}{q}\sum_{0\leqslant\nu\leqslant\nu_1} b_\nu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\left(\sum_{T_3<\sigma<T_4} + \sum_{|\sigma|\geqslant T_4} + \sum_{-T_4<\sigma\leqslant T_3}\right)\tau(\chi;\sigma)J(\sigma) =: \frac{X}{q}\sum_{0\leqslant\nu\leqslant\nu_1} b_\nu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\big(C_1+C_2+C_3\big).$$

Similarly to above, we apply Lemma 5.5 to the integrals $J(\sigma)$ in C_2 and C_3 to estimate them from above and use Lemma 5.6 to compute $J(\sigma)$ in C_1. If q is small enough and $T_4 < 1$, the whole sum $V_{\chi,m,s}$ is estimated by Lemma 5.5.

Next, we verify the conditions of Lemma 5.5 and Lemma 5.6. Put

$$\tilde{w}(\tau) = f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right)\Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right)\tau^{\beta(\nu+1/2)-1}\left.\frac{d^{2\nu}G(t,\tau)}{dt^{2\nu}}\right|_{t=\tau^\delta},$$

$$\tilde{g}(\tau) = \begin{cases}(1-\alpha)hX^\alpha\tau^\gamma - \dfrac{X^{1-\alpha}s\sigma\tau}{\alpha hq^2um}, & \text{if } \tau\in\big[(1-\Delta)^{1/\delta};(y+\Delta)^{1/\delta}\big];\\ 0 & \text{if } \tau\leqslant(1-2\Delta)^{1/\delta} \text{ or } \tau\geqslant(y+2\Delta)^{1/\delta}\end{cases}$$

and define $\tilde{g}(\tau)$ on $(1-2\Delta)^{1-\alpha} \leqslant \tau \leqslant (1-\Delta)^{1-\alpha}$ and $(y+\Delta)^{1-\alpha} \leqslant \tau \leqslant (y+2\Delta)^{1-\alpha}$ appropriately.

Now we estimate $\tilde{w}^{(j)}(\tau)$ on $(1-\Delta)^{1-\alpha} \leqslant \tau \leqslant (y+\Delta)^{1-\alpha}$. We have

$$\tilde{w}^{(j)}(\tau) = \sum_{j_1+j_2+j_3+j_4=j} \binom{j}{j_1, j_2, j_3, j_4} \frac{d^{j_1}}{d\tau^{j_1}} f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right) \frac{d^{j_2}}{d\tau^{j_2}} \Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right) \cdot \frac{d^{j_3}}{d\tau^{j_3}} \tau^{\beta(\nu+1/2)-1} \frac{d^{j_4}}{d\tau^{j_4}} \left(\frac{d^{2\nu}G(t,\tau)}{dt^{2\nu}}\bigg|_{t=\tau^\delta}\right).$$

Next,

$$\frac{d^{j_1}}{d\tau^{j_1}} f_2\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right) \ll \log X,$$

$$\frac{d^{j_2}}{d\tau^{j_2}} \Psi_N\left(\frac{X^{1-\alpha}s\tau}{\alpha hqum}\right) \ll \left(\frac{X^{1-\alpha}s}{N\alpha hqum}\right)^{j_2} (\log X)^{A_0 j_2} \ll_{j_2} (\log X)^{A_0 j_2},$$

$$\frac{d^{j_3}}{d\tau^{j_3}} \left(\tau^{\beta(\nu+1/2)-1}\right) = \left(\beta\left(\nu+\frac{1}{2}\right)-1\right)_{j_3} \tau^{\beta(\nu+1/2)-1-j_3} \ll_{j_3} 1.$$

To estimate the last factor

$$\frac{d^{j_4}}{d\tau^{j_4}} \left(\frac{d^{2\nu}G(t,\tau)}{dt^{2\nu}}\bigg|_{t=\tau^\delta}\right)$$

we apply the formula of Faa di Bruno (Lemma 5.8). First, let us compute $(d^{2\nu}/dt^{2\nu})G(t,\tau)$. By the binomial theorem

$$\frac{d^{2\nu}}{dt^{2\nu}} G(t,\tau) = \sum_{l=0}^{2\nu} \binom{2\nu}{l} w^{(l)}(t) \frac{d^{2\nu-l}}{dt^{2\nu-l}} e(H(t,\tau)).$$

Put $f = H(t)$, $\phi = e(H)$. Then we get

$$\frac{d^{2\nu-l}}{dt^{2\nu-l}} e(H(t,\tau)) = \sum_{\substack{m_1+2m_2+\ldots+(2\nu-l)m_{2\nu-l}=2\nu-l \\ m_1,\ldots,m_{2\nu-l}\geqslant 0}} \frac{(2\nu-l)!}{m_1!\ldots m_{2\nu-l}!} \cdot (2\pi i)^{m_1+\ldots+m_{2\nu-l}} e(H(t,\tau)) \cdot \prod_{\kappa=1}^{2\nu-l} \left(\frac{H^{(\kappa)}(t,\tau)}{\kappa!}\right)^{m_\kappa}. \quad (5.16)$$

Note that the last expression is evaluated at $t = t_0 = \tau^\delta$. By definition,

$$H(t,\tau) = g(t,\tau) - g(t_0,\tau) - \frac{1}{2} g''(t_0,\tau)(t-t_0)^2, \quad (5.17)$$

and so $H(t_0,\tau) = H'(t_0,\tau) = H''(t_0,\tau) = 0$, $e(H(\tau^\delta,\tau)) = 1$. It means that the non-zero contribution to the right hand side of (5.16) when $t = t_0$ would only come from the tuples of the form $(0, 0, m_3, m_4, \ldots)$.

Thus, we need to apply the binomial theorem and Faa di Bruno formula again to compute

$$\frac{d^{j_4}}{d\tau^{j_4}}\left(w^{(l)}(t)\Big|_{t=\tau^\delta}\cdot\prod_{\kappa=3}^{2\nu-l}\left(\frac{1}{\kappa!}\frac{d^\kappa H(t,\tau)}{dt^\kappa}\Big|_{t=\tau^\delta}\right)^{m_\kappa}\right).$$

By the binomial theorem, the last derivative is a linear combination of the expressions

$$\frac{d^{j_5}}{d\tau^{j_5}}\left(w^{(l)}(t)\Big|_{t=\tau^\delta}\right)\cdot\frac{d^{j_6}}{d\tau^{j_6}}\left(\prod_{\kappa=3}^{2\nu-l}\left(\frac{1}{\kappa!}\frac{d^\kappa H(t,\tau)}{dt^\kappa}\Big|_{t=\tau^\delta}\right)^{m_\kappa}\right), \tag{5.18}$$

where $j_5+j_6=j_4$. First, we evaluate the factor

$$\frac{d^{j_5}}{d\tau^{j_5}}\left(w^{(l)}(t)\Big|_{t=\tau^\delta}\right).$$

Note that since $t_0=t$, we have

$$\frac{d^r}{dt_0^r}w^{(l)}(t)\Big|_{t=t_0}=\frac{d^{l+r}}{dt_0^{l+r}}w(t_0)=w^{(l+r)}(t_0).$$

Then by Faa di Bruno formula and (5.12)

$$\frac{d^{j_5}}{d\tau^{j_5}}\left(w^{(l)}(t)\Big|_{t=\tau^\delta}\right)=$$

$$\sum_{\substack{l_1+2l_2+\ldots+j_5l_{j_5}=j_5\\ l_1,\ldots,l_{j_5}\geqslant 0}}\frac{j_5!}{l_1!\ldots l_{j_5}!}w^{(l+l_1+\ldots+l_{j_5})}(\tau^\delta)\prod_{\kappa=1}^{j_5}\left(\frac{d^\kappa}{d\tau^\kappa}t_0(\tau)\right)^\kappa\ll_{j_5}$$

$$\sum_{\substack{l_1+2l_2+\ldots+j_5l_{j_5}=j_5\\ l_1,\ldots,l_{j_5}\geqslant 0}}w^{(l+l_1+\ldots+l_{j_5})}(\tau^\delta)\ll w^{(l+j_5)}(\tau^\delta)\ll(\log X)^{C_0(l+j_5)+1}.$$

Next, we evaluate the second factor in (5.18). For $\kappa\geqslant 3$, we have

$$\frac{d^r}{d\tau^r}\left(\frac{1}{\kappa!}\frac{d^\kappa H(t,\tau)}{dt^\kappa}\Big|_{t=\tau^\delta}\right)=\frac{1}{\kappa!}\frac{d^r}{d\tau^r}g^{(\kappa)}(\tau^\delta,\tau)=\frac{(\alpha)_\kappa hX^\alpha}{\kappa!}\frac{d^r}{d\tau^r}\tau^{\delta(\alpha-\kappa)}=$$

$$\frac{(\alpha)_\kappa hX^\alpha}{\kappa!}(\delta(\alpha-\kappa))_r\tau^{\delta(\alpha-\kappa)-r}\ll_\kappa hX^\alpha \tag{5.19}$$

and thus,

$$\frac{d^{j_6}}{d\tau^{j_6}}\left(\prod_{\kappa=3}^{2\nu-l}\left(\frac{1}{\kappa!}\frac{d^\kappa H(t,\tau)}{dt^\kappa}\Big|_{t=\tau^\delta}\right)^{m_\kappa}\right)\ll_\kappa(hX^\alpha)^{m_3+\ldots+m_\kappa}\ll(hX^\alpha)^{(2\nu-l)/3},$$

since $m_1+2m_2+3m_3+\ldots+\kappa m_\kappa\geqslant m_3+\ldots+m_\kappa$, so the expression (5.18) can be bounded from above by

$$(\log X)^{C_0(l+j_5)+1}(hX^\alpha)^{(2\nu-l)/3}.$$

The last expression reaches its maximum at $l = 0$, $j_5 = j_4$, $j_6 = 0$. We conclude

$$\frac{d^{j_4}}{d\tau^{j_4}}\left(\frac{d^{2\nu}G(t,\tau)}{dt^{2\nu}}\bigg|_{t=\tau^\delta}\right) \ll (\log X)^{C_0 j_4+1}(hX^\alpha)^{2\nu/3}.$$

Finally, we find

$$\tilde{w}^{(j)}(\tau) \ll (\log X)^2(hX^\alpha)^{2\nu/3}\sum_{j_1+\ldots+j_4=j}\binom{j}{j_1,\ldots,j_4}(\log X)^{A_0 j_2+C_0 j_4} \ll (\log X)^{2+E_0 j}(hX^\alpha)^{2\nu/3}, \quad (5.20)$$

where $E_0 = \max\{A_0, C_0\}$. So the inequality $\tilde{w}^{(j)}(\tau) \ll X_I Y_I^{-j}$ holds with $X_I = (\log X)^2(hX^\alpha)^{2\nu/3}$, $V_I = (\log X)^{-E_0}$. Next, we have

$$\tilde{g}'(\tau) = \gamma(1-\alpha)hX^\alpha\tau^{\gamma-1} - \frac{X^{1-\alpha}s\sigma}{\alpha hq^2um}, \qquad \tilde{g}''(\tau) = \gamma(\gamma-1)(1-\alpha)hX^\alpha\tau^{\gamma-2},$$
$$\tilde{g}^{(j)}(\tau) = (\gamma)_j(1-\alpha)hX^\alpha\tau^{\gamma-j} \asymp hX^\alpha,$$

so one can take $Y_I = hX^\alpha, Q_I = 1$. Put $\mathbb{J} = \left[10^{-1}; 10\right]$, $\tilde{V}_I = |\mathbb{J}|$, and $Z_I = Q_I + X_I + Y_I + \tilde{V}_I + 1 \asymp (\log X)^2(hX^\alpha)^{2\nu/3} + hX^\alpha$, which implies

$$Z_I \asymp \begin{cases} hX^\alpha, & \text{if } \nu = 0, 1; \\ (\log X)^2(hX^\alpha)^{2\nu/3}, & \text{if } \nu \geqslant 2. \end{cases}$$

We choose the constant $\delta_I > 0$ such that $Y_I > Z_I^{3\delta_I}$. If $\nu = 0, 1$, we get $hX^\alpha > (hX^\alpha)^{3\delta_I}$, which holds true for all $\delta_I < 1/3$. If $\nu \geqslant 2$, then

$$hX^\alpha \geqslant (\log X)^{6\delta_I}(hX^\alpha)^{2\nu\delta_I},$$

so one can take $\delta_I = 1/(121D_0)$. It is easy to check that

$$\frac{Q_I Z_I^{\delta_I/2}}{\sqrt{Y_I}} \leqslant V_I$$

holds true for all $\nu \leqslant \nu_1 = 60D_0$.

Next, if $|\sigma| \geqslant T_4$, then

$$|\tilde{g}'(\tau)| = \left|\frac{X^{1-\alpha}s\sigma}{\alpha hq^2um} - \gamma(1-\alpha)hX^\alpha\tau^{\gamma-1}\right| \geqslant \frac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um}\left(1 - \frac{\alpha^2h^2q^2um\tau^{\gamma-1}}{X^{1-2\alpha}sT_4}\right) \geqslant \frac{2}{3}\frac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um}.$$

If $-T_4 < \sigma \leqslant 0$, then

$$\tilde{g}'(\tau) = \gamma(1-\alpha)hX^{\alpha}\tau^{\gamma-1} - \frac{X^{1-\alpha}s\sigma}{\alpha hq^2um} = \alpha hX^{\alpha}\tau^{\gamma-1} + \frac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um} \geqslant \alpha hX^{\alpha}\tau^{\gamma-1} \geqslant \frac{1}{2}\alpha hX^{\alpha}.$$

Finally, if $1 \leqslant \sigma \leqslant T_3$, then

$$\tilde{g}'(\tau) = \alpha hX^{\alpha}\tau^{\gamma-1}\left(1 - \frac{X^{1-2\alpha}s\sigma\tau^{1-\gamma}}{(\alpha hq)^2um}\right) \geqslant \alpha hX^{\alpha}\tau^{\gamma-1}\left(1-\frac{3}{5}\right) \geqslant \frac{1}{6}\alpha hX^{\alpha}.$$

So one can choose

$$R_I = \begin{cases} \dfrac{2}{3}\dfrac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um} & \text{if } |\sigma| \geqslant T_4; \\ \dfrac{1}{6}\alpha hX^{\alpha} & \text{if } -T_4 < \sigma \leqslant T_3. \end{cases}$$

Again, setting

$$\Delta_1 = \frac{Q_I R_I}{\sqrt{Y_I}}, \qquad \Delta_2 = R_I V_I,$$

we show that $\Delta_1, \Delta_2 > 1$. Indeed, in the case $|\sigma| \geqslant T_4$, we have

$$\Delta_1 = \frac{2}{3}\frac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um}\frac{1}{\sqrt{hX^{\alpha}}} \geqslant \frac{2}{3\sqrt{h}}\frac{X^{1-3\alpha/2}s}{\alpha hq^2um}T_4 = \frac{8}{3}\alpha\sqrt{h}X^{\alpha/2} > 1,$$

and

$$\Delta_2 = \frac{2}{3}\frac{X^{1-\alpha}s|\sigma|}{\alpha hq^2um}(\log X)^{-E_0} \geqslant \frac{2}{3}\frac{X^{1-\alpha}s}{\alpha hq^2um}T_4(\log X)^{-E_0} = \frac{8}{3}\alpha\left(hX^{\alpha}\right)^{2/3}(\log X)^{-E_0} > 1.$$

If $-T_4 < \sigma \leqslant T_3$, then

$$\Delta_1 = \frac{1}{6}\alpha hX^{\alpha}\frac{1}{\sqrt{hX^{\alpha}}} = \frac{\alpha}{6}\sqrt{h}X^{\alpha/2} > 1,$$
$$\Delta_2 = \frac{1}{6}\alpha hX^{\alpha}(\log X)^{-E_0} > 1.$$

Thus, applying Lemma 5.5 with a large enough $F_0 = F_0(\alpha) > 1$, $\mathbb{J} = \left[(1-\Delta)^{1/\delta}; (y+\Delta)^{1/\delta}\right]$, for $|\sigma| \geqslant T_4$, we find

$$J(\sigma) \ll |\mathbb{J}|X_I(\Delta_1^{-F_0} + \Delta_2^{-F_0}) \ll (\log X)^2(hX^{\alpha})^{2\nu/3}\left\{\left(\frac{\alpha hq^2um\sqrt{hX^{\alpha}}}{2X^{1-\alpha}s|\sigma|}\right)^{F_0} + \left(\frac{3\alpha hq^2um(\log X)^{E_0}}{2X^{1-\alpha}s|\sigma|}\right)^{F_0}\right\} \ll (\log X)^2(hX^{\alpha})^{2\nu/3}\left(\frac{3\alpha}{2}h^{3/2}q^2um\frac{1}{X^{1-3\alpha/2}s|\sigma|}\right)^{F_0},$$

and for $-T_4 < \sigma \leqslant T_3$, we get

$$J(\sigma) \ll (\log X)^2 (hX^\alpha)^{2\nu/3} \left\{ \left(\frac{6}{\alpha\sqrt{h}X^{\alpha/2}} \right)^{F_0} + \left(\frac{6(\log X)^{E_0}}{\alpha h X^\alpha} \right)^{F_0} \right\}$$
$$\ll (\log X)^2 (hX^\alpha)^{2\nu/3} \left(\frac{6}{\alpha\sqrt{h}} X^{-\alpha/2} \right)^{F_0}.$$

Thus, the contribution from C_2, C_3 to $V_{\chi,m,s}$ can be made small enough with the appropriate choice of F_0. We get the formula

$$V_{\chi,m,s} = \frac{X}{q} \sum_{0 \leqslant \nu \leqslant \nu_1} b_\nu(\alpha) \left(\frac{h}{X^\alpha} \right)^{\nu+1/2} C_1 + O\big(X^{-\alpha F_0/10}\big), \tag{5.21}$$
$$C_1 = \sum_{T_3 < \sigma < T_4} \tau(\chi;\sigma) J(\sigma).$$

We are now ready to compute C_1 using Lemma 5.6. Let $T_3 < \sigma < T_4$, $\tilde{g}'(\tau_0) = 0$. Then $\tau_0 \in \mathbb{J} = \big[10^{-1}; 10\big]$. We find

$$\tau_0 = \frac{1}{X^{1-\alpha}} \left\{ \frac{(\alpha h q)^2 u m}{s\sigma} \right\}^\xi,$$

where $\xi = 1/(1-\gamma) = (1-\alpha)/(1-2\alpha)$;

$$\tilde{g}(\tau_0) = (1-\alpha) h X^\alpha \tau_0^\gamma - \frac{X^{1-\alpha} s\sigma\tau_0}{\alpha h q^2 u m} = (1-2\alpha) h \left\{ \frac{(\alpha h q)^2 u m}{s\sigma} \right\}^\eta,$$

where $\eta = \alpha/(1-2\alpha)$;

$$\tilde{g}''(\tau_0) = -\frac{\alpha(1-2\alpha)}{1-\alpha} h X^{2(1-\alpha)} \left(\frac{s\sigma}{(\alpha h q)^2 u m} \right)^\omega,$$

where $\omega = \xi(2-\gamma) = (2-3\alpha)/(1-2\alpha)$. Finally, take

$$\tilde{G}(\tau) = \tilde{w}(\tau) e^{2\pi i \tilde{H}(\tau)}, \qquad \tilde{H}(\tau) = \tilde{g}(\tau) - \tilde{g}(\tau_0) - \frac{\tilde{g}''(\tau_0)}{2}(\tau-\tau_0)^2.$$

Then Lemma 5.6 implies

$$J(\sigma) = e\left(\tilde{g}(\tau_0) - \frac{1}{8} \right) \sum_{0 \leqslant \mu \leqslant \mu_1} \frac{\sqrt{2\pi}}{\mu!} \frac{(2i)^{-\mu}}{|\tilde{g}''(\tau_0)|^{\mu+1/2}} \frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}} \bigg|_{\tau=\tau_0} + O\big(X^{-\alpha F_0}\big) =$$
$$e\left((1-2\alpha) h \left\{ \frac{(\alpha h q)^2 u m}{s\sigma} \right\}^\eta - \frac{1}{8} \right) \sum_{0 \leqslant \mu \leqslant \mu_1} \frac{\sqrt{2\pi}}{\mu!} \frac{(2i)^{-\mu}}{\big(\alpha(1-2\alpha)/(1-\alpha)\big)^{\mu+1/2}} \cdot$$
$$\frac{1}{(hX^{2(1-\alpha)})^{\mu+1/2}} \left(\frac{(\alpha h q)^2 u m}{s\sigma} \right)^{\omega(\mu+1/2)} \frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}} \bigg|_{\tau=\tau_0} + O\big(X^{-\alpha F_0}\big)$$

with $\mu_1 = 3F_0/\delta_I = 363D_0F_0$. Setting

$$c_\mu(\alpha) = e\left(\frac{1}{8}\right)\frac{\sqrt{2\pi}}{\mu!}(2i)^{-\mu}\left(\frac{1-\alpha}{\alpha(1-2\alpha)}\right)^{\mu+1/2}\alpha^{2\omega(\mu+1/2)},$$

we get

$$J(\sigma) = e\left((1-2\alpha)h\left\{\frac{(\alpha hq)^2um}{s\sigma}\right\}^\eta\right)\cdot \sum_{0\leqslant\mu\leqslant\mu_1}\frac{c_\mu(\alpha)}{\left(hX^{2(1-\alpha)}\right)^{\mu+1/2}}\left(\frac{(hq)^2um}{s\sigma}\right)^{\omega(\mu+1/2)}\frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}}\bigg|_{\tau=\tau_0} + O\left(X^{-\alpha F_0}\right).$$

Again, it is not hard to see that the O-term contributes at most $O\left(X^{-\alpha F_0/10}\right)$ to the sum $V_{\chi,m,s}$. This contribution can be made arbitrarily small. Hence, from (5.21) we have

$$V_{\chi,m,s} = \frac{X}{q}\sum_{T_3<\sigma\leqslant T_4}\tau(\chi;\sigma)\sum_{0\leqslant\nu\leqslant\nu_1}\sum_{0\leqslant\mu\leqslant\mu_1}b_\nu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\frac{c_\mu(\alpha)}{\left(hX^{2(1-\alpha)}\right)^{\mu+1/2}}\cdot \left(\frac{(hq)^2um}{s\sigma}\right)^{\omega(\mu+1/2)}\frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}}\bigg|_{\tau=\tau_0} e\left((1-2\alpha)h\left\{\frac{(\alpha hq)^2um}{s\sigma}\right\}^\eta\right) + O\left(X^{-\alpha F_0/10}\right).$$

Substituting this expression into (5.14) and changing the order of summation, we get

$$W(M,N,K) = \frac{X}{q^2}\sum_{m=1}^{+\infty}\frac{f_1(m)}{m}\Psi_M(m)\sum_{\substack{T_1<s<T_2\\ T_3<\sigma<T_4}}\sum_{\substack{0\leqslant\nu\leqslant\nu_1\\ 0\leqslant\mu\leqslant\mu_1}}b_\nu(\alpha)c_\mu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\frac{1}{\left(hX^{2(1-\alpha)}\right)^{\mu+1/2}}\cdot \left(\frac{(hq)^2um}{s\sigma}\right)^{\omega(\mu+1/2)}\frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}}\bigg|_{\tau=\tau_0} e\left((1-2\alpha)h\left\{\frac{(\alpha hq)^2um}{s\sigma}\right\}^\eta\right)\cdot \frac{1}{\varphi(q)}\sum_{\chi\ \mathrm{mod}\ q}\chi(mua^*)\tau(\chi;s)\tau(\chi;\sigma) + O\left(X^{-\alpha F_0/20}\right) + O\left(X^{-\alpha D_0/2}\right). \quad (5.22)$$

5.6 Final bound

Now we are ready to deduce the final bound. We only need to bound the Kloosterman sum which appears from the product of two Gauss sums.

Bound for the Kloosterman sum

We rewrite the inner sum in (5.22) as follows:

$$\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(mua^*)\tau(\chi;s)\tau(\chi;\sigma) = \sum_{l,r=1}^{q} e\left(\frac{ls+r\sigma}{q}\right)\frac{1}{\varphi(q)}\sum_{\chi \bmod q}\chi(lrmua^*). \quad (5.23)$$

By orthogonality of characters, the sum in the right hand side of (5.23) transforms into Kloosterman sum

$$\sum_{\substack{l=1\\(l,q)=1}}^{q} e\left(\frac{sl+\sigma a(mu)^* l^*}{q}\right) = S_q\big(s,\sigma a(mu)^*\big).$$

Thus,

$$W(M,N,K) = \frac{X}{uq^2}\sum_{\substack{m=1\\(m,q)=1}}^{\infty}\frac{f_1(m)}{m}\Psi_M(m)\sum_{\substack{T_1<s<T_2\\T_3<\sigma<T_4}}\sum_{\substack{0\leqslant\nu\leqslant\nu_1\\0\leqslant\mu\leqslant\mu_1}} b_\nu(\alpha)c_\mu(\alpha)\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\cdot$$
$$\frac{1}{\big(hX^{2(1-\alpha)}\big)^{\mu+1/2}}\left(\frac{(hq)^2um}{s\sigma}\right)^{\omega(\mu+1/2)}\frac{d^{2\mu}\tilde{G}(\tau)}{d\tau^{2\mu}}\bigg|_{\tau=\tau_0}\cdot$$
$$e\left((1-2\alpha)h\left\{\frac{(\alpha hq)^2um}{s\sigma}\right\}^{\eta}\right)S_q\big(s,\sigma a(mu)^*\big) + O\big(X^{-\alpha F_0/20}\big) + O\big(X^{-\alpha D_0/2}\big).$$

We can now estimate the multiple sum over m, s, σ, ν, and μ. Since $\tilde{H}(\tau_0) = \tilde{H}'(\tau_0) = \tilde{H}''(\tau_0) = 0$, similarly to (5.19), we get the upper bound

$$\frac{d^r}{d\tau^r}\big(e\big(\tilde{H}(\tau)\big)\big)\bigg|_{\tau=\tau_0} \ll (hX^\alpha)^{r/3}.$$

Together with (5.20), this implies

$$\tilde{G}^{(2\mu)}(\tau_0) \ll (hX^\alpha)^{(2/3)(\nu+\mu)}(\log X)^2.$$

Next, we apply Weil's bound (Lemma 5.7):

$$\big|S_q(s,\sigma a(mu)^*)\big| \leqslant \tau(q)\sqrt{q}(s,\sigma,q)^{1/2}.$$

Changing the order of summation, we get the inequality

$$W(M,N,K) \ll \frac{X\tau(q)}{uq\sqrt{q}}\frac{(\log X)^3}{M}\sum_{\substack{0\leqslant\nu\leqslant\nu_1\\0\leqslant\mu\leqslant\mu_1}}\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\frac{(hX^\alpha)^{(2/3)(\nu+\mu)}}{\big(hX^{2(1-\alpha)}\big)^{\mu+1/2}}(hq)^{\omega(2\mu+1)}\cdot$$
$$\sum_{M/\Theta\leqslant m\leqslant M\Theta}(mu)^{\omega(\mu+1/2)}\sum_{\substack{T_1<s<T_2\\T_3<\sigma<T_4}}\frac{(s,\sigma,q)^{1/2}}{(s\sigma)^{\omega(\mu+1/2)}} + O\big(X^{-\alpha F_0/20} + X^{-\alpha D_0/2}\big).$$

The sums over s and σ could be bounded as

$$\sum_{T_1<s<T_2}\sum_{T_3<\sigma<T_4}\frac{(s,\sigma,q)^{1/2}}{(s\sigma)^{\omega(\mu+1/2)}}\ll_\alpha\left(\frac{X^{1-2\alpha}}{(hq)^2um}\right)^{\omega(\mu+1/2)}\sum_{T_1<s<T_2}\sum_{T_3<\sigma<T_4}(s,\sigma)^{1/2}.$$

The last expression does not exceed

$$\left(\frac{X^{1-2\alpha}}{(hq)^2um}\right)^{\omega(\mu+1/2)}\sum_{1\leqslant d\leqslant\min(T_2,16T_4)}\sum_{\substack{T_1<s<T_2\\ s\equiv 0\pmod d}}\sum_{\substack{T_3/16<\sigma<16T_4\\ \sigma\equiv 0\pmod d}}\sqrt{d}\ll$$

$$\left(\frac{X^{1-2\alpha}}{(hq)^2um}\right)^{\omega(\mu+1/2)}\sum_{1\leqslant d\leqslant\min(T_2,16T_4)}\sqrt{d}\frac{T_2}{d}\frac{T_4}{d}\ll$$

$$T_2T_6\left(\frac{X^{1-2\alpha}}{(hq)^2um}\right)^{\omega(\mu+1/2)}\ll\left(\frac{X^{1-2\alpha}}{(hq)^2um}\right)^{\omega(\mu+1/2)-1}.$$

Next, the summation over m gives

$$\sum_{M/\Theta\leqslant m\leqslant M\Theta}(mu)^{\omega(\mu+1/2)}\cdot\left(\frac{X^{1-2\alpha}}{(hq)^2mu}\right)^{\omega(\mu+1/2)-1}\ll$$

$$M^2U\left(\frac{X^{1-2\alpha}}{(hq)^2}\right)^{\omega(\mu+1/2)-1}\frac{1}{(\log X)^{A_0}},$$

hence, if D_0 and F_0 are sufficiently large,

$$W(M,N,K)\ll\frac{X\tau(q)}{Uq\sqrt{q}}\frac{(\log X)^3}{M}\frac{M^2U}{(\log X)^{A_0}}\sum_{\substack{0\leqslant\nu\leqslant\nu_1\\ 0\leqslant\mu\leqslant\mu_1}}\left(\frac{h}{X^\alpha}\right)^{\nu+1/2}\frac{(hX^\alpha)^{(2/3)(\nu+\mu)}}{\left(hX^{2(1-\alpha)}\right)^{\mu+1/2}}\cdot$$

$$(hq)^{\omega(2\mu+1)}\left(\frac{X^{1-2\alpha}}{(hq)^2}\right)^{\omega(\mu+1/2)-1}\ll\frac{X\tau(q)}{q\sqrt{q}}(\log X)^{3-A_0}M\sum_{\substack{0\leqslant\nu\leqslant\nu_1\\ 0\leqslant\mu\leqslant\mu_1}}X^{\kappa_1}h^{\kappa_2}q^2,$$

where

$$\kappa_1=-\frac{\alpha\nu}{3}-\frac{\alpha\mu}{3}+\alpha-1,\qquad\kappa_2=2+\frac{5\nu}{3}-\frac{\mu}{3}.$$

Clearly the main contribution comes from the term $\nu=\mu=0$. We get

$$W(M,N,K)\ll X^\alpha\sqrt{q}\tau(q)(\log X)^{3-A_0}Mh^2.$$

Summing $W(M,N,K)$ over all admissible U,u,M,N,K, and using Lemma 2.5

$$\sum_{u\leqslant 2U}\tau_7(u)\ll U(\log U)^6,$$

finally, we find

$$W_{III} \ll X^{\alpha}\sqrt{q}\tau(q)(\log X)^{2C+3-A_0} \sideset{}{'}\sum_{U\in\mathbf{G}} \sum_{U\Theta^{-7}\leqslant u\leqslant U\Theta^{7}} \big|F(u,U)\big| \cdot \sideset{}{'}\sum_{\substack{M_1\leqslant M\leqslant M_2\\ M\in\mathbf{G}}} M \sideset{}{'}\sum_{\substack{N_1\leqslant N\leqslant N_2\\ N\in\mathbf{G}}} \sideset{}{'}\sum_{\substack{K_1\leqslant K\leqslant K_2\\ K\in\mathbf{G}}} 1 \ll$$

$$X^{\alpha}\sqrt{q}\tau(q)(\log X)^{2C+3-A_0} \sideset{}{'}\sum_{U\in\mathbf{G}} \sum_{u\leqslant 2U} \tau_7(u)(\log X)^{6(A_0+1)+1}\left(\frac{X}{U}\right)^{1/3}(\log X)^{3(A_0+1)} \ll$$

$$X^{1/3+\alpha}\sqrt{q}\tau(q)(\log X)^{2C+3-A_0+9(A_0+1)+1} \sideset{}{'}\sum_{U\in\mathbf{G}} U^{2/3}(\log U)^6 \ll$$

$$X^{1/3+\alpha}\sqrt{q}\tau(q)X^{(2/3)(1/10-3\varepsilon_1/2)}(\log X)^{8A_0+2C+19} \ll X^{2/5+\alpha-\varepsilon_1}\sqrt{q}\tau(q)(\log X)^{L_0}, \quad (5.24)$$

where $L_0 = 8A_0 + 2C + 19$.

Final bound

Combining together type I (5.5), type II (5.9), and type III (5.24) estimates, we get

$$W \ll X^{2/5+\alpha/2-\varepsilon_1+2\delta_1} + \frac{X^{1-\alpha/2+2\delta_1}}{q} + \frac{1}{\sqrt{q}}X^{4/5+\alpha/4+\varepsilon_1/2+3\delta_2} + \frac{1}{q}X^{1-\alpha/4+3\delta_2} + X^{2/5+\alpha-\varepsilon_1}\sqrt{q}\tau(q)(\log X)^{L_0}.$$

Further, the right hand side of the last inequality does not exceed

$$\frac{X}{q}\left(qX^{-3/5+\alpha/2-\varepsilon_1+2\delta_1} + X^{-\alpha/2+2\delta_1} + \sqrt{q}X^{-1/5+\alpha/4+\varepsilon_1/2+3\delta_2} + X^{-\alpha/4+3\delta_2} + q^{3/2}X^{-3/5+\alpha-\varepsilon_1+\delta_4}\right)$$

with an arbitrarily small $\delta_4 > 0$. Clearly

$$\max\left(X^{-\alpha/2+2\delta_1}, X^{-\alpha/4+3\delta_2}\right) \ll (\log X)^{-A}$$

if δ_1 and δ_2 are small enough. Next,

$$qX^{-3/5+\alpha/2-\varepsilon_1+2\delta_1} \ll X^{-1/5-\alpha/10+2\delta_1} \ll (\log X)^{-A}.$$

Then

$$W \ll \frac{X}{q}\left((\log X)^{-A} + \max\left(\sqrt{q}X^{-1/5+\alpha/4+\varepsilon_1/2+3\delta_2}, q^{3/2}X^{-3/5+\alpha-\varepsilon_1+\delta_4}\right)\right).$$

Thus, $W \ll (X/q)(\log X)^{-A}$ if

$$q \leqslant \min\left((\log X)^{-2A} X^{2/5-\alpha/2-\varepsilon_1-6\delta_2}, (\log X)^{-(2/3)A} X^{2/5-(2/3)\alpha+(2/3)\varepsilon_1-(2/3)\delta_4}\right).$$

The maximum of this bound is reached at $\varepsilon_1 = \alpha/10$. Thus, $q \leqslant X^{2/5-(3/5)\alpha-\varepsilon}$ with any $\varepsilon < \min(6\delta_2, (2/3)\delta_4)$. Finally, the desired bound (1.7) follows from partial summation.

Remark. *One can obtain a slightly better level of distribution, $q \leqslant X^{2/5-\alpha/2-\varepsilon}$, in Theorem 1.3 by iterating the Poisson summation for the third time (on the sum over m) and applying the bound for 2-dimensional Kloosterman sum [86].*

Chapter 6

BOUNDED GAPS BETWEEN PRIMES IN SUBSETS

In this section, we prove Theorem 1.5. We follow the well-known technique of Maynard [62] and Tao with a modified Selberg sieve and apply Theorem 1.2 in place of the Bombieri-Vinogradov theorem. We only consider the case $0 < \alpha < 1$, which is easier. With a little more effort, one can deduce a similar result for any non-integer $\alpha > 1$.

The set $\{h_1, \dots, h_k\}$ of integers is called an *admissible set* if for any prime p there is an a with $h_j \not\equiv a \pmod p$ for any $1 \leqslant j \leqslant k$. Consider the sum

$$S_\alpha = \sum_{\substack{X \leqslant n < 2X \\ n \in \mathbb{E} \\ n+h_j \in \mathbb{E}\ \forall j}} \left(\sum_{j=1}^{k} \chi_{\mathbb{P}}(n+h_j) - \rho \right) \omega_n = S_{2,\alpha} - \rho S_{1,\alpha},$$

where $\omega_n, \rho > 0$, $\chi_{\mathbb{P}}$ is the characteristic function of primes. Then Theorem 1.5 clearly follows from the inequality $S_{2,\alpha} - \rho S_{1,\alpha} > 0$ with $\rho = m$. Indeed, in this case the inner sum over $1 \leqslant j \leqslant k$ has at least $m+1$ positive terms for some n. Hence, there are at least $m+1$ primes from $\mathbb{E}$ between n and $n + h_k$ and

$$\liminf_{n \to +\infty} (q_{n+m} - q_n) \leqslant \max_{1 \leqslant i < j \leqslant k} |h_j - h_i|. \tag{6.1}$$

Thus, the problem reduces to choosing the appropriate weights ω_n maximizing the ratio $S_{2,\alpha}/S_{1,\alpha}$. We follow the choice made in [62]:

$$\omega_n = \left(\sum_{d_j | (n+h_j)} \lambda_{d_1,\dots,d_k} \right)^2,$$

where the sum is taken over all tuples of divisors $(d_1, \dots, d_k)$ and

$$\lambda_{d_1,\dots,d_k} = \left(\prod_{j=1}^{k} \mu(d_j) d_j \right) \sum_{\substack{r_1,\dots,r_k \\ d_j | r_j\ \forall j \\ (r_j, W)=1\ \forall j}} \frac{\mu^2(r_1 \dots r_k)}{\varphi(r_1) \dots \varphi(r_k)} F\left(\frac{\log r_1}{\log R}, \dots, \frac{\log r_k}{\log R} \right).$$

Here W is the product of all primes $\leqslant \log\log\log X$ (and so $W \leqslant (\log\log X)^2$ for large X), $R = X^{1/6-\delta_1}$ for some small fixed $\delta_1 > 0$, and $F(x_1, \dots, x_k)$ is a fixed piecewise continuous function supported on the set

$$\left\{ (x_1, \dots, x_k) \in [0;1]^k : \sum_{j=1}^{k} x_j \leqslant 1 \right\}.$$

We put $\omega_n = 0$ for all n except $n \equiv \nu_0 \pmod W$ for some fixed ν_0 such that $(\nu_0 + h_j, W) = 1$ for all j. We also put $\lambda_{d_1,\dots,d_k} = 0$ if $\big(\prod_{j=1}^k d_j, W\big) > 1$ for at least one j.

6.1 Proof of general case

We obtain the desired result using the following assertion (see [62]):

1) Under the above assumptions on ω_n, the following relations hold:

$$S_1 = \sum_{\substack{X \leqslant n < 2X \\ n \equiv \nu_0 \pmod W}} \omega_n = \frac{\big(1+o(1)\big)\varphi^k(W)X(\log R)^k}{W^{k+1}} I_k(F),$$

$$S_2 = \sum_{\substack{X \leqslant n < 2X \\ n \equiv \nu_0 \pmod W}} \left(\sum_{j=1}^k \chi_{\mathbb{P}}(n+h_j)\right)\omega_n = \frac{\big(1+o(1)\big)\varphi^k(W)X(\log R)^{k+1}}{W^{k+1}\log X} \sum_{i=1}^k J_k^{(j)}(F),$$

provided $I_k(F) \neq 0$ and $J_k^{(j)}(F) \neq 0$ for each j, where

$$I_k(F) = \int_0^1 \dots \int_0^1 F(t_1,\dots,t_k)^2 dt_1 \dots dt_k,$$

$$J_k^{(j)}(F) = \int_0^1 \dots \int_0^1 \left(\int_0^1 F(t_1,\dots,t_k)dt_j\right)^2 dt_1 \dots dt_{j-1}dt_{j+1}\dots dt_k.$$

2) Define

$$M_k = \sup_F \frac{\sum_{j=1}^k J_k^{(j)}(F)}{I_k(F)}.$$

Then for all $k \geqslant 600$, the following inequality holds true:

$$M_k > \log k - 2\log\log k - 1. \tag{6.2}$$

To apply Maynard's argument, we need an analogue of part 1) for $S_{1,\alpha}$ and $S_{2,\alpha}$. It would follow from the relations

$$S_{1,\alpha} = \Big(d-c+o(1)\Big)S_1, \qquad S_{2,\alpha} = \Big(d-c+o(1)\Big)S_2. \tag{6.3}$$

Here the coefficient $d-c<1$ corresponds to the density of $\mathbb{E}$ among the integers. Note that since $\alpha < 1$, the numbers $\{n^\alpha\}, \{(n+h_1)^\alpha\}, \dots, \{(n+h_k)^\alpha\}$ are close to each other. So, in order to verify that all $k+1$ conditions $n \in \mathbb{E}$, $n+h_j \in \mathbb{E}$, $1 \leqslant j \leqslant k$ hold true, it is sufficient to check only two of them: $n \in \mathbb{E}, n+h_k \in \mathbb{E}$. Thus, we define the new subset as

$$\mathbb{E}' = \big\{n \in \mathbb{N} : \{n \in \mathbb{E}\} \cap \{n+h_k \in \mathbb{E}\}\big\}.$$

Then

$$S_{2,\alpha} = \sum_{j=1}^{k} S_{2,\alpha}^{(j)} = \sum_{j=1}^{k} \sum_{\substack{X \leqslant n < 2X \\ n \in \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j)\omega_n.$$

Following [62], we change the order of summation in $S_{2,\alpha}^{(j)}$ and apply Chinese Remainder theorem. Thus we get

$$S_{2,\alpha}^{(j)} = \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ ([d_i,e_i],[d_j,e_j])=1 \ \forall i \neq j}} \lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k} \sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q \\ n \in \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j),$$

where $q = W \prod_{i=1}^{k} [d_i, e_i]$. Further,

$$S_{2,\alpha}^{(j)} = \frac{\pi_{\mathbb{E}}(2X) - \pi_{\mathbb{E}}(X) + O(1)}{\varphi(W)} \sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k \\ e_j=d_j=1 \\ ([d_i,e_i],[d_j,e_j])=1 \ \forall i \neq j}} \frac{\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}}{\prod_{i=1}^{k} \varphi\Big([d_i, e_i]\Big)} + O\Bigg(\sum_{\substack{d_1,\ldots,d_k \\ e_1,\ldots,e_k}} |\lambda_{d_1,\ldots,d_k}\lambda_{e_1,\ldots,e_k}| E^{(j)}(X, q)\Bigg), \quad (6.4)$$

where $\pi_{\mathbb{E}}$ is the counting function of primes from $\mathbb{E}$,

$$E^{(j)}(X, q) = 1 + \max_{(a,q)=1} \Bigg| \sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q \\ n \in \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j) - \frac{1}{\varphi(q)} \sum_{\substack{X \leqslant n < 2X \\ n \in \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j) \Bigg| \leqslant 1 + E_1^{(j)}(X, q) + E_2^{(j)}(X, q) + E_3^{(j)}(X, q),$$

where

$$E_1^{(j)}(X, q) = \max_{(a,q)=1} \sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q \\ n \in \mathbb{E} \setminus \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j),$$

$$E_2^{(j)}(X, q) = \frac{1}{\varphi(q)} \sum_{\substack{X \leqslant n < 2X \\ n \in \mathbb{E} \setminus \mathbb{E}'}} \chi_{\mathbb{P}}(n + h_j),$$

$$E_3^{(j)}(X, q) = \max_{(a,q)=1} \Bigg| \sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q \\ n \in \mathbb{E}}} \chi_{\mathbb{P}}(n + h_j) - \frac{1}{\varphi(q)} \sum_{\substack{X \leqslant n < 2X \\ n \in \mathbb{E}}} \chi_{\mathbb{P}}(n + h_j) \Bigg|.$$

The trivial upper bound for $E_1^{(j)}$, $E_2^{(j)}$, and $E_3^{(j)}$ is $X/\varphi(q)$. Similarly to (5.20) from [62], we apply Cauchy inequality to the error term in (6.4) to bound it from above by

$$\lambda_{\max}^2\Bigg(\sum_{q<R^2W}\mu^2(q)\tau_{3k}^2(q)\frac{X}{\varphi(q)}\Bigg)^{1/2}\cdot$$
$$\Bigg(\sum_{q<R^2W}\big(1+E_1^{(j)}(X,q)+E_2^{(j)}(X,q)+E_3^{(j)}(X,q)\big)\Bigg)^{1/2},$$

where $\lambda_{\max}=\max_{d_1,\dots,d_k}|\lambda_{d_1,\dots,d_k}|$. The sum in the second factor does not exceed $X(\log X)^{-B}$ for any fixed $B>0$. Indeed, Theorem 1.2 implies that the contribution coming from the term $E_3^{(j)}(X,q)$ is estimated by $X(\log X)^{-A}$ for any fixed $A>0$. We estimate $E_1^{(j)}(X,q)$ and $E_2^{(j)}(X,q)$ similarly to $S^{(2)}$ and $S^{(3)}$ in Chapter 3 (note that all the primes $n+h_j$ which belong to $\mathbb{E}$ but do not belong to $\mathbb{E}'$ lie in the subset $\mathbb{E}_\Delta$ defined earlier). The sum in the first factor does not exceed $X(\log X)^{c_k}$ for some $c_k>0$. So choosing A large enough, we get the bound of $\lambda_{\max}^2X(\log X)^{-B}$ for the error term in (6.4). Finally, we apply Vinogradov's result [104] to $\pi_{\mathbb{E}}(X)$ to get the second relation in (6.3).

We treat the sum $S_{1,\alpha}$ in a similar way. Thus we get

$$S_{1,\alpha}=\sum_{\substack{d_1,\dots,d_k\\e_1,\dots,e_k\\([d_i,e_i],[d_j,e_j])=1\ \forall i\neq j}}\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}\sum_{\substack{X\leqslant n<2X\\n\equiv a\pmod q\\n\in\mathbb{E}'}}1=$$
$$\frac{X}{2W}\sum_{\substack{d_1,\dots,d_k\\e_1,\dots,e_k\\([d_i,e_i],[d_j,e_j])=1\ \forall i\neq j}}\frac{\lambda_{d_1,\dots,d_k}\lambda_{e_1,\dots,e_k}}{\prod_{i=1}^k[d_i,e_i]}+O\Bigg(\sum_{\substack{d_1,\dots,d_k\\e_1,\dots,e_k}}\lambda_{\max}^2G(X,q)\Bigg),\quad(6.5)$$

where

$$G(X,q)=1+\max_{(a,q)=1}\Bigg|\sum_{\substack{X\leqslant n<2X\\n\equiv a\pmod q\\n\in\mathbb{E}'}}1-\frac{1}{q}\sum_{\substack{X\leqslant n<2X\\n\in\mathbb{E}'}}1\Bigg|\leqslant$$
$$1+G_1(X,q)+G_2(X,q)+G_3(X,q),$$

where

$$G_1(X,q)=\max_{(a,q)=1}\sum_{\substack{X\leqslant n<2X\\n\equiv a\pmod q\\n\in\mathbb{E}\setminus\mathbb{E}'}}1,\qquad G_2(X,q)=\frac{1}{q}\sum_{\substack{X\leqslant n<2X\\n\in\mathbb{E}\setminus\mathbb{E}'}}1,$$
$$G_3(X,q)=\max_{(a,q)=1}\Bigg|\sum_{\substack{X\leqslant n<2X\\n\equiv a\pmod q\\n\in\mathbb{E}}}-\frac{1}{q}\sum_{\substack{X\leqslant n<2X\\n\in\mathbb{E}}}1\Bigg|.$$

We estimate the error term in (6.5) similarly to the error term in (6.4). For G_3, we apply the arguments used in the proof of Theorem 1.2. This case is simpler since the summation goes over integers, so one deals with

$$\sum_{\substack{X \leqslant n < 2X \\ n \equiv a \pmod q}} e\big(hn^{\alpha}\big).$$

The congruence condition is removed by substitution $n = qr + l$. The sum over r is then estimated by Lemma 2.4 with $k = 3$. This concludes the proof of (6.3). We note that the estimates for $E_3^{(j)}$ and G_3 correspond to the conditions (2) and (1) in the Hypothesis 1 from [63].

We show that there are infinitely many n such that at least $\lceil M_k/6 \rceil$ numbers $n + h_i$ are primes from $\mathbb{E}$. By definition of M_k, there is a function F_0 such that

$$\sum_{j=1}^{k} J_k^{(j)}(F_0) > (M_k - \delta_1) I_k(F_0).$$

Then

$$S_\alpha > (d-c)\frac{\varphi^k(W)X(\log R)^k I_k(F_0)}{W^{k+1}}\left(\left(\frac{1}{6} - \delta_1\right)(M_k - \delta_1) - \rho + o(1)\right).$$

If $\rho = M_k/6 - \delta_2$ for some $\delta_2 > 0$ such that $\lceil M_k/6 \rceil = \lfloor \rho + 1 \rfloor$ and δ_1 is small enough (depending on δ_2), then there are infinitely many n such that at least $\lfloor \rho + 1 \rfloor = m + 1$ numbers among $n + h_j$ are primes.

To finish the computation, we estimate the right hand side of (6.1). In accordance with (6.2), we choose $k = \lfloor 390 m^2 e^{6m} \rfloor$ and take the tuple of primes $\{p_{\pi(k)+1}, \ldots, p_{\pi(k)+k}\}$ which is obviously admissible (indeed, there is no element of this set which is congruent to zero modulo any prime $p \leqslant k$; on the other hand, this set does not cover a complete residue system modulo any $p > k$ due to its size). Next,

$$\max_{1 \leqslant i < j \leqslant k} |h_j - h_i| \leqslant p_{\pi(k)+k} \leqslant p_{\lceil 1.1k \rceil}$$

for $k \geqslant 10^5$. Using the inequality $p_n < n(\log n + \log\log n + 8)$ (see, for example, [78]) with $n = \lceil 1.1 \cdot 390 m^2 e^{6m} \rceil$, we find that

$$\liminf_{n \to +\infty} (p_{n+m} - p_n) \leqslant 9\,700 m^3 e^{6m}. \tag{6.6}$$

6.2 Bound calculation for cases $m = 1, 2$

The bound (6.6) can certainly be sharpened for small m. For example, if $m = 1$, then we take $k = 157\,337$. Thus we get $M_k > 6$, and the precise computation of the right hand side of (6.1) gives

$$p_{\pi(k)+k} - p_{\pi(k)+1} = p_{171\,807} - p_{14\,471} = 2\,176\,652 < 2.18 \cdot 10^6.$$

For $m = 2$, we need $M_k > 12$, so we take $k = 157\,629\,323$ and

$$p_{\pi(k)+k} - p_{\pi(k)+1} = p_{166\,478\,324} - p_{8\,849\,002} = 3\,130\,607\,572 < 3.14 \cdot 10^9.$$

Chapter 7

MOBIUS RANDOMNESS MEETS DISCRETE RESTRICTION

In this chapter, we give a proof of Theorem 1.6 using Vaughan identity, Weyl's differencing for polynomials, and the circle method. We would need the following auxiliary lemmas:

Lemma 7.1 (Vaughan identity for Mobius function)**.** *For any V, one has the following decomposition for Mobius function:*

$$\mu(n) = 2\mu(n)\mathbb{1}_{n\leqslant V} - \sum_{\substack{mkd=n\\ k,d\leqslant V}} \mu(k)\mu(d) + \sum_{\substack{mk=n\\ m>V}} \mu(m)a_k, \qquad a_k = \sum_{\substack{d|k\\ d\leqslant V}} \mu(d).$$

The proof is similar to the standard Vaughan identity with Mangoldt function which can be found in [45].

Lemma 7.2 (Weyl's bound for linear phase)**.** *For fixed x, N such that $N \leqslant x$ and any real α, one has the inequality*

$$\left|\sum_{n=1}^{N} e(\alpha n)\right| \leqslant \min\left(N, \frac{1}{2\|\alpha\|}\right).$$

See [45, Chapter 8].

Lemma 7.3 (Weyl's bound for polynomial phase)**.** *Let $g(x) = cx^k + c_1x^{k-1} + \ldots$ with c and k positive integers. Suppose*

$$\left|\alpha - \frac{a}{q}\right| \leqslant \frac{1}{q^2}, \qquad \textit{with} \quad (a,q) = 1.$$

Then

$$\sum_{0<n\leqslant X} e\big(\alpha g(n)\big) \ll X^{1+\varepsilon}\left(\frac{c}{q} + \frac{c}{X} + \frac{q}{X^k}\right)^{2^{1-k}}$$

for any $\varepsilon > 0$ and $X \geqslant 1$, the implied constant depending only on ε. In particular for $k = 2$, one has the inequality

$$N\sqrt{\log q}\left(\sqrt{\frac{c}{q}} + \sqrt{\frac{c}{N}} + \frac{\sqrt{q}}{N}\right).$$

See [45, Chapter 8].

Lemma 7.4 (Discrete restriction estimate for parabola)**.** *For $p > 4$, one has*

$$\int_0^1 \Big|\sum_{n\leqslant X} e\big(n^2\alpha\big)\Big|^p d\alpha \leqslant C_p X^{p-2}$$

with an absolute constant $C_p > 0$.

For the proof, see [46] or [11].

Circle method

We fix the parameter $Q = X^2(\log X)^{-B}$ for some $B > 0$ which will be adjusted later. By Dirichlet approximation theorem, for any $\alpha \in [0;1]$, there is a, q such that $(a,q) = 1$, $q \leqslant Q$ and

$$\big|\alpha - \frac{a}{q}\big| \leqslant \frac{1}{qQ}.$$

For fixed a, q, we denote the corresponding set of α's by $\mathcal{E}_{a,q}$:

$$\mathcal{E}_{a,q} := \Big\{\alpha \in [0;1] : \big|\alpha - \frac{a}{q}\big| \leqslant \frac{1}{qQ}\Big\}.$$

The goal is to get the pointwise bound for the sum

$$\sum_{n\leqslant X} \mu(n) e\big(n^2\alpha\big) \ll \frac{X}{(\log X)^{A_1}} \tag{7.1}$$

for large enough $A_1 = A_1(A)$. The method would depend on the size of the denominator q. The following inequality holds true

$$\int_0^1 \Big|\sum_{n\leqslant X} \mu(n) e\big(n^2\alpha\big)\Big|^p d\alpha \leqslant$$
$$\sum_{q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q-1} \Big(\max_{\alpha\in\mathcal{E}_{a,q}} \Big|\sum_{n\leqslant X} \mu(n) e\big(n^2\alpha\big)\Big|\Big) \int_{\alpha\in\mathcal{E}_{a,q}} \Big|\sum_{n\leqslant X} \mu(n) e\big(n^2\alpha\big)\Big|^{p-1} d\alpha.$$

We split the interval $[0;1]$ the major and minor arcs as follows:

$$\mathfrak{M}_{a,q} := \Big\{\alpha \in [0;1] : \big|\alpha - \frac{a}{q}\big| \leqslant \frac{1}{qQ} \text{ for } q \leqslant (\log X)^C\Big\}$$
$$\mathfrak{m}_{a,q} := [0;1] \setminus \mathfrak{M}_{a,q},$$

where $C > 0$ is fixed. We treat the sum in the left hand side of (7.1) in different ways for $\alpha \in \mathfrak{M}_{a,q}$ and $\alpha \in \mathfrak{m}_{a,q}$.

Minor arcs

In this case, all such α can be approximated as $\alpha \approx a/q$ with $q \geqslant (\log X)^C$. By Vaughan identity (Lemma 7.1), the sum in (7.1) splits as follows:

$$\sum_{n\leqslant X} \mu(n)e\big(n^2\alpha\big) = 2W_0 - W_1 + W_2,$$

where

$$W_0 = \sum_{n\leqslant V} \mu(n)e\big(n^2\alpha\big), \qquad V := (\log X)^D,$$
$$W_1 = \sum_{n\leqslant V^2} a_n \sum_{m\leqslant X/n} e\big(\alpha n^2m^2\big), \qquad a_n = \sum_{\substack{d|n\\ d\leqslant V}} \mu(d),$$
$$W_2 = \sum_{V<n\leqslant X/V} \mu(n) \sum_{m\leqslant X/n} a_m e\big(\alpha n^2m^2\big),$$

and $D > 0$ is fixed.

The contribution from the sum W_0 is estimated trivially:

$$T_0 := \sum_{(\log X)^C<q\leqslant Q} \sum_{\substack{a=1\\(a,q)=1}}^{q-1} |W_0| \int_{a\in\mathfrak{m}_{a,q}} \left|\sum_{n\leqslant X} \mu(n)e\big(n^2\alpha\big)\right|^{p-1} d\alpha \ll$$
$$V\int_0^1 \left|\sum_{n\leqslant X} \mu(n)e\big(n^2\alpha\big)\right|^{p-1} d\alpha \leqslant C_0VX^{p-3} \ll X^{p-3}(\log X)^D, \quad (7.2)$$

where the bound for L_{p-1} norm of the sum comes from Lemma 7.4.

To estimate the contribution from W_1, as usual we split the outer sum to the dyadic intervals:

$$|W_1| \leqslant \sideset{}{'}\sum_{N\leqslant V^2} \sum_{n\sim N} |a_n| \left|\sum_{m\leqslant X/n} e\big(\alpha n^2m^2\big)\right|.$$

Then we apply Lemma 7.3 with $k = 2$:

$$|W_1| \ll \sideset{}{'}\sum_{N\leqslant V^2} \sum_{n\sim N} |a_n| \cdot \sqrt{\log q}\,\frac{X}{n}\left(\frac{n}{\sqrt{q}} + \frac{n\sqrt{N}}{\sqrt{X}} + \frac{\sqrt{q}N}{X}\right).$$

Then, by Mardzhanishvili inequality,

$$|W_1| \ll \sideset{}{'}\sum_{N\leqslant V^2} N(\log N)^{\kappa_1}\sqrt{\log q}\left(\frac{X}{\sqrt{q}} + \sqrt{XN} + \sqrt{q}\right) \ll$$
$$(\log X)^{\kappa_1+1+1/2}\left(\frac{V^2X}{\sqrt{q}} + \sqrt{X}V^3 + V^2\sqrt{q}\right) \ll$$
$$X(\log X)^{\kappa_1-3/2-C/2+2D} + \sqrt{X}(\log X)^{\kappa_1+3/2+3D} + X(\log X)^{\kappa_1+3/2-B/2+2D}.$$

Then the contribution from W_1 is

$$T_1 := \sum_{(\log X)^C < q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} |W_1| \int_{a \in \mathfrak{m}_{a,q}} \left| \sum_{n \leqslant X} \mu(n) e\big(n^2 \alpha\big) \right|^{p-1} d\alpha \ll$$
$$X^{p-2} (\log X)^{\kappa_1 + 2D} \left((\log X)^{-3/2 - C/2} + \frac{1}{\sqrt{X}} (\log X)^{3/2 + D} + (\log X)^{3/2 - B/2} \right). \quad (7.3)$$

The type II estimate is a little more delicate. Since q can be very large compared to the length N of the sum over n the Weyl's bound for polynomial phase is not directly applicable to that sum. We use Lemma 7.2 instead and split to two cases: when $am^2n^2 \equiv r \pmod{q}$ with $r \ll q$, we estimate the short sum trivially; otherwise we use the second bound from Lemma 7.2.

By Cauchy inequality, we have

$$|W_2| \leqslant \sum_{V < N \leqslant X/V}' \left(\sum_{n \sim N} \mu^2(n) \right)^{1/2} \left(\sum_{n \sim N} \sum_{m_1, m_2 \leqslant X/n} a_{m_1} \overline{a_{m_2}} e\big(\alpha (m_1^2 - m_2^2) n^2\big) \right)^{1/2} =:$$
$$D_1 + D_2. \quad (7.4)$$

We first evaluate the diagonal term:

$$D_1 \leqslant \sum_{V < N \leqslant X/V}' \sqrt{NX} \ll \frac{X \log X}{\sqrt{V}} = X (\log X)^{1 - D/2}$$

and the contribution from it

$$T_{2,1} := \sum_{(\log X)^C < q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} |D_1| \int_{a \in \mathfrak{m}_{a,q}} \left| \sum_{n \leqslant X} \mu(n) e\big(n^2 \alpha\big) \right|^{p-1} d\alpha \ll$$
$$X^{p-2} (\log X)^{1 - D/2}. \quad (7.5)$$

To evaluate the non-diagonal term, we swap the order of summation in the second factor of (7.4)

$$|D_2| \leqslant 2 \sum_{V < N \leqslant X/V}' \sqrt{N} \left(\sum_{\substack{m_1, m_2 \leqslant X/N \\ m_1 > m_2}} a_{m_1} \overline{a_{m_2}} \sum_{N \leqslant n < R} e\big(\alpha (m_1^2 - m_2^2) n^2\big) \right)^{1/2},$$
$$R := \min\big(2N; X/m_2\big),$$

and apply Cauchy and Mardzhanishvili inequalities one more time:

$$|D_2| \ll \sideset{}{'}\sum_{V<N\leqslant X/V} \sqrt{N}\Bigg(\Bigg(\sum_{\substack{m_1,m_2\leqslant X/N\\ m_1>m_2}} |a_{m_1}|^2|a_{m_2}|^2\Bigg)^{1/2}.$$

$$\Bigg(\sum_{\substack{m_1,m_2\leqslant X/N\\ m_1>m_2}} \sum_{N\leqslant n_1,n_2<R} e\big(\alpha(m_1^2-m_2^2)(n_1^2-n_2^2)\big)\Bigg)^{1/2}\Bigg)^{1/2} \ll$$

$$\sideset{}{'}\sum_{V<N\leqslant X/V} \sqrt{N}\bigg(\frac{X^2}{N^2}(\log X)^{2\kappa_1}\bigg)^{1/4}\Bigg(\sum_{\substack{m_1,m_2\leqslant X/N\\ m_1>m_2}} \sum_{N\leqslant n_1,n_2<R} e\big(\alpha(m_1^2-m_2^2)(n_1^2-n_2^2)\big)\Bigg)^{1/4}$$

which can be further bounded as

$$\sqrt{X}(\log X)^{\kappa_1/2} \sideset{}{'}\sum_{V<N\leqslant X/V} \Bigg(\sum_{\substack{m_1,m_2\leqslant X/N\\ m_1>m_2}} \sum_{N\leqslant n_1,n_2<R} e\big(\alpha(m_1^2-m_2^2)(n_1^2-n_2^2)\big)\Bigg)^{1/4} =:$$

$$D_3 + D_4.$$

The diagonal term corresponds to $n_1 = n_2$ and

$$|D_3| \ll \sqrt{X}(\log X)^{\kappa_1/2} \sideset{}{'}\sum_{V<N\leqslant X/V} \bigg(\frac{X^2}{N^2}N\bigg)^{1/4} \ll X(\log X)^{\kappa_1/2+1-D/4}. \tag{7.6}$$

Next, we evaluate the non-diagonal term. Denote the quadruple sum inside the second factor by $S(N)$:

$$S(N) := \sum_{\substack{m_1,m_2\leqslant X/N\\ m_1>m_2}} \sum_{\substack{N\leqslant n_1,n_2<R\\ n_1>n_2}} e\big(\alpha(m_1^2-m_2^2)(n_1^2-n_2^2)\big).$$

For convenience, we make the following substitutions: $n_1 =: n$, $n_2 =: n+s$, $m_1 =: m$, $m_2 =: m+t$. Then

$$S(N) = \sum_{s,t,m,n} e\big(\alpha st(2n+s)(2m+t)\big) =$$

$$\sum_{s,t,m} e\big(\alpha s^2 t(2m+t)\big)\sum_n e\big(2\alpha stn(2m+t)\big) \ll \sum_{s,t,m}\Big|\sum_n e\big(2\alpha stn(2m+t)\big)\Big|.$$

By Lemma 7.2,

$$|S(N)| \ll \sum_{s,t,m} \min\bigg(R, \frac{1}{\|2\alpha st(2m+t)\|}\bigg).$$

Due to the restrictions on α, the last sum can be bounded from above by

$$2\sum_{s,t,m} \min\bigg(R, \frac{1}{\|2(a/q)st(2m+t)\|}\bigg). \tag{7.7}$$

Next, we mimic the proof of the classical Weyl's bound for polynomial phase. We split the last sum into two parts according to the size of $r \leqslant q$ in the congruence restriction $2ast(2m+t) \equiv r \pmod q$ which is equivalent to $2st(2m+t) \equiv ra^* \pmod q$. First, assume $ra^* \leqslant (q/N)(\log X)^E =: L$ or $ra^* \geqslant q - L$, where $E > 0$ will be chosen later. Then, the triple sum over s, t, m is asymptotically at most

$$\sum_{l \leqslant L} \sum_{\substack{s,t,m \\ 2st(2m+t)=l}} R$$

if $2st(2m+t) < q$, and

$$\sum_{l \leqslant L} \sum_{s,t,m} \sum_{u \leqslant 2st(2m+t)/q} R \cdot \mathbb{1}_{2st(2m+t)=uq+l}$$

otherwise. By Mardzhanishvili inequality, both sums are bounded from above by

$$R\left(\frac{R \cdot (X/N)^2}{q} + 1\right) L (\log X)^{\kappa_2}. \tag{7.8}$$

In all the remaining cases, we use the second bound in (7.7). We get

$$\frac{1}{\|2(a/q)st(2m+t)\|} \leqslant \frac{q}{L}.$$

Combining (7.8) and the last bound, we obtain

$$S(N) \ll R\left(\frac{R \cdot (X/N)^2}{q} + 1\right) L (\log X)^{\kappa_2} + \sum_{s,t,m} \frac{q}{L} \ll$$
$$\left(\frac{X^2}{q} + R\right) L (\log X)^{\kappa_2} + \frac{q}{L} R \left(\frac{X}{N}\right)^2 \ll (\log X)^{\kappa_2 + E}\left(\frac{X^2}{N} + q\right) + \frac{X^2}{(\log X)^E}.$$

This leads to the estimate

$$|D_4| \ll \sqrt{X} (\log X)^{\kappa_1/2} \sum_{V < N \leqslant X/V}{}' \left[\frac{\sqrt{X}}{N^{1/4}} (\log X)^{\kappa_2/4 + E/4} + q^{1/4} (\log X)^{\kappa_2/4+E/4} + \frac{\sqrt{X}}{(\log X)^{E/4}}\right],$$

hence

$$|D_4| \ll X (\log X)^{\kappa_1/2+1}\left((\log X)^{-D/4+\kappa_2/4+E/4} + (\log X)^{-B/4+\kappa_2/4+E/4} + (\log X)^{-E/4}\right).$$

Together with (7.6), it gives

$$T_{2,2} := \sum_{(\log X)^C < q \leqslant Q} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} |D_2| \int_{a \in \mathfrak{m}_{a,q}} \left| \sum_{n \leqslant X} \mu(n) e(n^2 \alpha) \right|^{p-1} d\alpha \ll$$

$$X^{p-2} (\log X)^{\kappa_1/2+1} \Big((\log X)^{-D/4} + (\log X)^{-D/4+\kappa_2/4+E/4} +$$

$$(\log X)^{-B/4+\kappa_2/4+E/4} + (\log X)^{-E/4} \Big). \quad (7.9)$$

Major arcs

Let us introduce the notation

$$S(\alpha) := \sum_{n \leqslant X} \mu(n) e(n^2 \alpha).$$

We will first evaluate the sum

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left| S\left(\frac{a}{q}\right) \right|$$

and then obtain the upper bound for a more general case

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \max_{\alpha \in \mathfrak{M}_{a,q}} |S(\alpha)|.$$

Next, we have

$$\sum_{n \leqslant X} \mu(n) e\left(\frac{a}{q} n^2\right) = \sum_{r=0}^{q-1} e\left(\frac{r}{q}\right) \sum_{\substack{n \leqslant X \\ an^2 \equiv r \pmod q}} \mu(n).$$

Next, the relation $an^2 \equiv e \pmod q$ is equivalent to $an^2 \equiv s \pmod{q'}$, where $q' = q/(q,r)$, so that $(q', s) = 1$. Then, by the orthogonality of Dirichlet characters, the last sum is equivalent to

$$\sum_{r=0}^{q-1} e\left(\frac{r}{q}\right) \cdot \frac{1}{\varphi(q')} \sum_{\chi \pmod{q'}} \chi(s^* a) \sum_{n \leqslant X} \mu(n) \chi^2(n).$$

Note that $q' \leqslant q \leqslant (\log X)^C$, so using the standard techniques (see, for example, [65, Exercise 11.3.7]), we get the bound

$$\sum_{n \leqslant X} \mu(n) \chi^2(n) \ll_C X \exp\left(-c_1 \sqrt{\log X}\right)$$

with an absolute constant $c_1 > 0$. That implies

$$\left|S\left(\frac{a}{q}\right)\right| \ll_C \sum_{r=0}^{q-1} \frac{1}{\varphi(q')} \sum_{\chi \pmod{q'}} X \exp\bigl(-c_1\sqrt{\log X}\bigr) \ll qX \exp\bigl(-c_1\sqrt{\log X}\bigr).$$

Hence,

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left|S\left(\frac{a}{q}\right)\right| \ll_C \sum_{q \leqslant (\log X)^C} q\varphi(q) X \exp\bigl(-c_1\sqrt{\log X}\bigr) \ll X \exp\left(-\frac{c_1}{2}\sqrt{\log X}\right).$$

Next, by partial summation, we obtain

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \max_{\alpha \in \mathfrak{M}_{a,q}} |S(\alpha)| \leqslant \sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \left|S\left(\frac{a}{q}\right)\right| + \sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \int_1^X \left|\sum_{n \leqslant v} \mu(n) e\bigl(n^2\alpha\bigr)\right| |\theta_{a,q}| v\, dv,$$

where $|\theta_{a,q}| \leqslant 1/(qQ)$. Next, choose $Y = X(\log X)^{-F}$ for $F > 0$ and split the integral as

$$\int_1^X \left|\sum_{n \leqslant v} \mu(n) e\bigl(n^2\alpha\bigr)\right| |\theta_{a,q}| v\, dv = \left(\int_1^Y + \int_Y^X\right) \left|\sum_{n \leqslant v} \mu(n) e\bigl(n^2\alpha\bigr)\right| |\theta_{a,q}| v\, dv =: I_1 + I_2.$$

The contribution from I_1 can be estimated trivially by

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \int_1^Y \frac{v^2}{qQ} dv \ll \sum_{q \leqslant (\log X)^C} \frac{Y^3}{Q} \ll X(\log X)^{-3F+B}. \tag{7.10}$$

To estimate the contribution from I_2, we use a similar bound

$$\sum_{n \leqslant Z} \mu(n) e\bigl(n^2\alpha\bigr) \ll_q Z \exp\bigl(-c_1\sqrt{\log Z}\bigr)$$

for all $Y \leqslant Z \leqslant X$, which can be obtained in a similar way as for $S(a/q)$. Then

$$\sum_{q \leqslant (\log X)^C} \sum_{\substack{a=1 \\ (a,q)=1}}^{q-1} \int_Y^X \left|\sum_{n \leqslant v} \mu(n) e\bigl(n^2\alpha\bigr)\right| |\theta_{a,q}| v\, dv \ll \sum_{q \leqslant (\log X)^C} \frac{X^3}{Q} \exp\bigl(-c_1\sqrt{\log X}\bigr) \ll X \exp\left(-\frac{c_1}{2}\sqrt{\log X}\right).$$

Finally, together with (7.10), this bound gives

$$\sum_{q\leqslant(\log X)^C}\sum_{\substack{a=1\\(a,q)=1}}^{q-1}\left(\max_{\alpha\in\mathfrak{M}_{a,q}}\left|\sum_{n\leqslant X}\mu(n)e\big(n^2\alpha\big)\right|\right)\int_{\alpha\in\mathfrak{M}_{a,q}}\left|\sum_{n\leqslant X}\mu(n)e\big(n^2\alpha\big)\right|^{p-1}d\alpha \ll$$

$$X^{p-2}(\log X)^{-3F+B}+X^{p-2}\exp\left(-\frac{c_1}{2}\sqrt{\log X}\right). \quad (7.11)$$

Final bound

Combining together the bounds (7.2), (7.3), (7.5), (7.9), and (7.11), we get

$$\int_0^1\left|\sum_{n\leqslant X}\mu(n)e^{2\pi i n^2\alpha}\right|^p d\alpha \ll X^{p-3}(\log X)^D+X^{p-2}\exp\left(-\frac{c_1}{2}\sqrt{\log X}\right)\ll$$

$$X^{p-2}(\log X)^{1-D/2}+X^{p-2}(\log X)^{\kappa_1+2D}\left((\log X)^{-3/2-C/2}+(\log X)^{3/2-B/2}\right)+$$

$$X^{p-2}(\log X)^{\kappa_1/2+1}\Big((\log X)^{-D/4}+(\log X)^{-D/4+\kappa_2/4+E/4}+$$

$$(\log X)^{-B/4+\kappa_2/4+E/4}+(\log X)^{-E/4}\Big)+X^{p-2}(\log X)^{-3F+B}. \quad (7.12)$$

Choosing $E>4A+2\kappa_1+4$, $D>E+4A+2\kappa_1+\kappa_2+4$, $B,C\geqslant 10D$ and $F\geqslant 10B$, we get the desired result.

Chapter 8

MOMENTS OF $GL(2)$ L-FUNCTIONS

In this chapter, we prove Theorem 1.7 using the technique from the work of Adam Harper [38].

8.1 Auxiliary lemmas

For the sake of convenience, we change the notation for the weight of holomorphic Hecke cusp form $f_{j,m}$ to

$$k := 2m + 2.$$

To get the bound of Theorem 1.7, we need two key ingredients: the inequality analogous to (1.16) for $GL(2)$ L-functions and the analogue of mean-value estimate for the Fourier coefficients of the corresponding holomorphic cusp forms. The first ingredient is given by

Lemma 8.1. *Assume the Grand Riemann Hypothesis and suppose $f \in S_k(\Gamma_0(2))$ is a holomorphic Hecke cusp form. Then, for any $x \geqslant 2$, we have*

$$\log\left|L_1(f)L_2(f)\right| \leqslant \sum_{p\leqslant x} \frac{\lambda_f(p)\left[1+\chi_{-d}(p)\right]}{\sqrt{p}} W_p(x) + \sum_{p\leqslant\sqrt{x}} \frac{\lambda_f(p^2)-1}{p} W_{p^2}(x) + c_0 + \frac{3}{4}\frac{\log\left(d^2k^4\right)}{\log x},$$

where $c_0 \leqslant 2$ is an absolute constant,

$$W_n(x) = n^{-\lambda_0/\log x}\frac{\log(x/n)}{\log x},$$

$$e^{-\lambda_0} = \lambda_0 + \frac{\lambda_0^2}{2} \Longrightarrow \lambda_0 = 0.4912\ldots$$

For the proof, see [16, Theorem 2.1].

The second key ingredient is the multidimensional analogue of Petersson trace formula. We first state the classical version:

Lemma 8.2 (*Petersson trace formula*)**.** *Let k be a fixed positive integer, and m and n are natural numbers with $mn \leqslant k^2/10^4$. Then*

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \lambda_f(n)\lambda_f(m) = \frac{k-1}{2\pi^2}\mathbb{1}_{n=m} + E_k, \qquad E_k \leqslant c_1 k e^{-k},$$

where c_1 is an absolute constant and

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \lambda_f(n)\lambda_f(m) := \sum_{f\in S_k(\Gamma_0(2))} \frac{1}{L(1,\mathrm{Sym}^2 f)}\lambda_f(n)\lambda_f(m).$$

For the proof, see [80, Lemma 2.1].

Lemma 8.3 (*multidimensional Petersson's formula*)**.** *Suppose $N = p_1^{\beta_1}\dots p_r^{\beta_r} \leqslant \sqrt{k}$, where $p_1, \dots, p_r$ are distinct primes. Then*

$$1) \sum_{f\in S_k(\Gamma_0(2))}^{h} \prod_{j=1}^{r} \lambda_f(p_j)^{\beta_j}\left[1+\chi_{-d}(p_j)\right]^{\beta_j} = F(N) + E(N),$$

where

$$F(N) = \begin{cases} 0, \text{ if } \chi_{-d}(p_j) = -1 \text{ or } \beta_j \text{ is odd for at least one } j; \\ 2^{\beta_1+\dots+\beta_r}\dfrac{k-1}{2\pi^2}\displaystyle\prod_{j=1}^{r}\frac{2}{\beta_j+2}\binom{\beta_j}{\beta_j/2}, \text{ otherwise,} \end{cases}$$

and

$$|E(N)| \leqslant c_1 4^{\beta_1+\dots+\beta_r}\Big(\prod_{j=1}^{r}\beta_j\Big) k e^{-k}.$$

$$2) \sum_{f\in S_k(\Gamma_0(2))}^{h} \prod_{j=1}^{r} \lambda_f(p_j^2)^{\beta_j} = F_2(N) + E_2(N),$$

where

$$F_2(N) = \frac{k-1}{2\pi^2}\prod_{j=1}^{r}\sum_{k_j=0}^{\beta_j}(-1)^{k_j}\binom{\beta_j}{k_j}\binom{k_j}{\lfloor k_j/2\rfloor},$$

$$|E_2(N)| \leqslant c_1\Big(\prod_{j=1}^{r}\beta_j^2\Big)4^{\beta_1+\dots+\beta_r+r} k e^{-k}.$$

Proof. First, we obtain Hecke relations

$$\lambda_f(p)^n = \lambda_f(p^n) + \sum_{i=1}^{n/2}\left[\binom{n}{i} - \binom{n}{i-1}\right]\lambda_f(p^{n-2i})$$

if n is even, and

$$\lambda_f(p)^n = \lambda_f(p^n) + \sum_{i=1}^{(n-1)/2} \left[\binom{n}{i} - \binom{n}{i-1} \right] \lambda_f(p^{n-2i})$$

if n is odd. From the Euler product

$$\prod_p \left(1 + \frac{\lambda_f(p)}{p^s} + \frac{\lambda_f(p^2)}{p^{2s}} + \dots \right) = \prod_p \left(1 + \frac{\alpha_p}{p^s} + \frac{\alpha_p^2}{p^{2s}} + \dots \right) \left(1 + \frac{\alpha_p^{-1}}{p^s} + \frac{\alpha_p^{-2}}{p^{2s}} + \dots \right),$$

we get the identity

$$\lambda_f(p^{n-1}) = \frac{\alpha_p^n - \alpha_p^{-n}}{\alpha_p - \alpha_p^{-1}}, \tag{8.1}$$

and, in particular, $\lambda_f(p) = \alpha_p + \alpha_p^{-1}$. Next, we have

$$\lambda_f(p)^n = (\alpha_p + {\alpha_p}^{-1})^n = \frac{(\alpha_p + {\alpha_p}^{-1})^n (\alpha_p - {\alpha_p}^{-1})}{\alpha_p - {\alpha_p}^{-1}} =$$
$$\frac{\left(\alpha_p^n + n\alpha_p^{n-2} + \binom{n}{2}\alpha_p^{n-4} + \dots + \binom{n}{i}\alpha_p^{n-2i} + \dots + n\alpha_p^{2-n} + \alpha_p^{-n}\right)(\alpha_p - \alpha_p^{-1})}{\alpha_p - {\alpha_p}^{-1}}.$$

Opening the brackets in the numerator and applying (8.1) to each of the expressions

$$\frac{\binom{n}{i}\alpha_p^{n-2i+1} - \binom{n}{i}\alpha_p^{-(n-2i+1)}}{\alpha_p - \alpha_p^{-1}},$$

we get the identity

$$\lambda_f(p)^n = \lambda_f(p^n) + [n-1]\lambda_f(p^{n-2}) + \left[\binom{n}{2} - n \right] \lambda_f(p^{n-4}) + \\ \dots + \left[\binom{n}{i} - \binom{n}{i-1} \right] \lambda_f(p^{n-2i}) + \dots \tag{8.2}$$

First, assume that n is even and $\chi_{-d}(p) = 1$. Then using (8.2), we obtain

$$\sum_{f \in S_k(\Gamma_0(2))}^h \lambda_f(p)^n [1 + \chi_{-d}(p)]^n = 2^n \sum_{f \in S_k(\Gamma_0(2))}^h \lambda_f(p^n) + \\ 2^n \sum_{f \in S_k(\Gamma_0(2))}^h \sum_{i=1}^{n/2} \left[\binom{n}{i} - \binom{n}{i-1} \right] \lambda_f(p^{n-2i}) = F(p^n) + E(p^n), \tag{8.3}$$

where $F(p^n)$ is the term corresponding to $i = n/2$. By Lemma 8.2,

$$F(p^n) = 2^n \frac{k-1}{2\pi^2} \cdot \frac{2}{n+2} \binom{n}{n/2}. \tag{8.4}$$

The error term can be bounded from above in the following way:

$$E(p^n) \leqslant 2^n\left(1 + \sum_{i=1}^{n/2-1}\left[\binom{n}{i} - \binom{n}{i-1}\right]\right)c_1ke^{-k} \leqslant$$
$$2^n\frac{n}{2}\binom{n}{n/2}c_1ke^{-k} \leqslant c_1n4^nke^{-k}. \quad (8.5)$$

Now assume that n is odd and $\chi_{-d}(p) = 1$. The main term in the Petersson's formula vanishes, and for the error term by a similar argument, we get

$$E(p^n) \leqslant 2^n\frac{n-1}{2}\binom{n}{(n-1)/2}c_1ke^{-k} \leqslant c_1n4^nke^{-k}. \quad (8.6)$$

Combining (8.3), (8.4), (8.5), (8.6), Lemma 8.2 and using the multiplicativity of λ_f, we get the statement of part 1).

The proof of the second part is similar. We have

$$\lambda_f(p^2)^n = \frac{(\alpha_p^3 - \alpha_p^{-3})^n}{(\alpha_p - \alpha_p^{-1})^n} = \frac{(\alpha_p^2 + 1 + \alpha_p^{-2})^n(\alpha_p - \alpha_p^{-1})}{\alpha_p - \alpha_p^{-1}} =$$
$$\frac{\left(\sum_{l=0}^n \binom{n}{l}(\alpha_p^2 + \alpha_p^{-2})^l\right)(\alpha_p - \alpha_p^{-1})}{\alpha_p - \alpha_p^{-1}} = \frac{\sum_{l=0}^n \binom{n}{l}\sum_{i=0}^l \binom{l}{i}\alpha_p^{4i-2l}(\alpha_p - \alpha_p^{-1})}{\alpha_p - \alpha_p^{-1}} =$$
$$\frac{\sum_{l=0}^n \binom{n}{l}\sum_{i=0}^l \binom{l}{i}[\alpha_p^{4i-2l+1} - \alpha_p^{4i-2l-1}]}{\alpha_p - \alpha_p^{-1}}.$$

The last expression can be written as

$$\sum_{\substack{l=1\\ l \text{ odd}}}^n \binom{n}{l}\left[\sum_{i=0}^{(l-3)/2}\binom{l}{i}[\lambda_f(p^{2l-4i}) - \lambda_f(p^{2l-4i-2})] + \right.$$
$$\left.\binom{l}{(l-1)/2}[\lambda_f(p^2) - \lambda_f(1)]\right] +$$
$$\sum_{\substack{l=0\\ l \text{ even}}}^n \binom{n}{l}\left[\sum_{i=0}^{(l-2)/2}\binom{l}{i}[\lambda_f(p^{2l-4i}) - \lambda_f(p^{2l-4i-2})] + \binom{l}{l/2}\lambda_f(1)\right]. \quad (8.7)$$

The main contribution to the sum over $f \in S_k(\Gamma_0(2))$ after the application of Lemma 8.2 would come from the terms corresponding to $\lambda_f(1)$:

$$\sum_{f\in S_k(\Gamma_0(2))}^h \sum_{l=0}^n \binom{n}{l}(-1)^l\binom{l}{\lfloor l/2\rfloor}\lambda_f(1) = \frac{k-1}{2\pi^2}\sum_{l=0}^n(-1)^l\binom{n}{l}\binom{l}{\lfloor l/2\rfloor} + E_{k,n},$$

where

$$|E_{k,n}| \leqslant c_1 k e^{-k} \sum_{l=0}^{n} \binom{n}{l} \binom{l}{\lfloor l/2 \rfloor} \leqslant$$

$$c_1 k e^{-k}(n+1) \binom{n}{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} \leqslant c_1 (n+1) 4^n k e^{-k}.$$

The contribution to the sum over $f \in S_k(\Gamma_0(2))$ after the application of Lemma 8.2 from the remaining terms in (8.7) would not exceed

$$\sum_{l=0}^{n} \binom{n}{l} \sum_{i=0}^{l} 2 \binom{l}{i} c_1 k e^{-k} \leqslant c_1 k e^{-k} \cdot 2n^2 \binom{n}{\lfloor n/2 \rfloor} \binom{n}{\lfloor n/2 \rfloor} \leqslant 2n^2 4^n c_1 k e^{-k}.$$

Finally, combining together these bounds and the identity (8.7), we get

$$F_2(p^n) = \frac{k-1}{2\pi^2} \sum_{l=0}^{n} (-1)^l \binom{n}{l} \binom{l}{\lfloor l/2 \rfloor},$$

$$|E_2(p^n)| \leqslant c_1 n^2 4^{n+1} k e^{-k}.$$

The statement of part 2) follows from the multiplicativity of λ_f.

□

Remark. *In fact, we will only need a weaker form of the second part of Lemma 8.3, precisely, the bound*

$$|F_2(N)| \leqslant \frac{k-1}{2\pi^2} \prod_{j=1}^{r} \beta_j!, \tag{8.8}$$

which follows trivially from the inequality

$$\sum_{k=0}^{\beta} \frac{(-1)^k}{k!(\beta-k)!(\lfloor k/2 \rfloor !)^2} \leqslant 1.$$

Lemma 8.4. 1) *Assume A, B, x are large real numbers such that $A^2 < B \leqslant A^3$, $B \leqslant x$, and M is a positive integer such that* $\log A \geqslant M \log M$, $B^M \leqslant \sqrt{k}$. *Then, if M is even, the following inequality holds true:*

$$\sum_{f \in S_k(\Gamma_0(2))}^{h} \sum_{\substack{p_1,\dots,p_M \\ A < p_i \leqslant B}} \frac{\tilde{\lambda}_f(p_1)\dots\tilde{\lambda}_f(p_M)}{\sqrt{p_1 \dots p_M}} W_{p_1}(x) \dots W_{p_M}(x) \leqslant$$

$$2 \frac{k-1}{2\pi^2} \frac{M!}{(M/2)!} \Bigg(\sum_{\substack{A < p \leqslant B \\ \chi_{-d}(p) = 1}} \frac{2}{p} \Bigg)^{M/2}, \tag{8.9}$$

where $\tilde{\lambda}_f(p) := \lambda_f(p)[1+\chi_{-d}(p)]$*,* $p_1,\ldots,p_M$ *are primes,*

$$W_p(x) = p^{-\lambda_0/\log x}\frac{\log(x/p)}{\log x}, \qquad \lambda_0 = 0.4912\ldots$$

If M *is odd, then*

$$\left|\sum_{f\in S_k(\Gamma_0(2))}^{h}\sum_{\substack{p_1,\ldots,p_M\\ A<p_i\leqslant B}}\frac{\tilde{\lambda}_f(p_1)\ldots\tilde{\lambda}_f(p_M)}{\sqrt{p_1\ldots p_M}}W_{p_1}(x)\ldots W_{p_M}(x)\right| \leqslant$$

$$\frac{(B-A)^M}{A^{M/2}}\cdot 4^M \max_{\substack{p_1,\ldots,p_M\\ A<p_i\leqslant B}} g(p_1\ldots p_M)\cdot c_1 k e^{-k}, \quad (8.10)$$

where for distinct primes $q_1,\ldots,q_r$*, the function* $g(.)$ *is defined as*

$$g(q_1^{\beta_1}\ldots q_r^{\beta_r}) = \prod_{i=1}^{r}\beta_i.$$

2) *Let* $m, N > 1$ *be integers such that* $N \leqslant 2^m$*,* $(2^{m+1})^{2N} \leqslant \sqrt{k}$*. Then*

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\sum_{\substack{p_1,\ldots,p_{2N}\\ 2^m<p_i\leqslant 2^{m+1}}}\frac{\lambda_f(p_1^2)\ldots\lambda_f(p_{2N}^2)}{p_1\ldots p_{2N}}W_{p_1^2}(x)\ldots W_{p_{2N}^2}(x) \leqslant$$

$$\sqrt{2\pi}N4^N\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\left(\frac{1}{2^m}\right)^N.$$

Proof. The inequality (8.10) follows immediately from part 1) of Lemma 8.3 and the trivial upper bound

$$\sum_{A<p_1,\ldots,p_M\leqslant B}\frac{|W_{p_1}(x)\ldots W_{p_M}(x)|}{\sqrt{p_1\ldots p_M}} \leqslant \frac{(B-A)^M}{A^{M/2}}.$$

We now prove the inequality (8.9). Let $M := 2N$. We rewrite the product $p_1\ldots p_{2N}$ in the canonical form $q_1^{\beta_1}\ldots q_r^{\beta_r}$, where all q_i are distinct primes, all $\beta_i > 0$ and $\beta_1+\ldots+\beta_r = 2N$. We split the sums over $p_1,\ldots,p_{2N}$ corresponding to the different patterns $(\beta_1,\ldots,\beta_r)$. Then we apply Lemma 8.3 to each of them. We have

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\sum_{A<p_1,\ldots,p_{2N}\leqslant B}\frac{\tilde{\lambda}_f(p_1)\ldots\tilde{\lambda}_f(p_{2N})}{\sqrt{p_1\ldots p_{2N}}}W_{p_1}(x)\ldots W_{p_{2N}}(x) \leqslant$$

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\sum_{1\leqslant r\leqslant N}\sum_{\substack{\beta_1+\ldots+\beta_r=2N\\ \text{all even}}}\binom{2N}{\beta_1}\binom{2N-\beta_1}{\beta_2}\ldots\binom{2N-\beta_1-\ldots-\beta_{r-1}}{\beta_r}\cdot$$

$$\left(\frac{1}{r!}\sum_{\substack{A<q_1,\ldots,q_r\leqslant B\\ \text{all distinct}}}\frac{\tilde{\lambda}_f^{\beta_1}(q_1)\ldots\tilde{\lambda}_f^{\beta_r}(q_r)}{\sqrt{q_1^{\beta_1}\ldots q_r^{\beta_r}}}\right)+R(A,B,k), \quad (8.11)$$

where we have removed the weights $W_{p_i}(x)$ since for each i one has the inequalities $0 < W_{p_i}(x) \leqslant 1$, $\tilde{\lambda}_f^{\beta_1}(q_1)\dots\tilde{\lambda}_f^{\beta_r}(q_r) \geqslant 0$ when all β_i are even. The error term $R(A,B,k)$ can be bounded similarly to (8.10):

$$|R(A,B,k)| \leqslant 4^{2N} \max_{\substack{p_1,\dots,p_{2N}\\ A<p_i\leqslant B}} g(p_1\dots p_{2N})\frac{(B-A)^{2N}}{A^N}\cdot c_1 k e^{-k}$$

by part 1) of Lemma 8.3. The main contribution to the main term on the right hand side of (8.11) comes from the pattern $\beta_1 = \dots = \beta_r = 2$, $r = N$. By Lemma 8.3 this contribution is equal to

$$\frac{(2N)!}{2^N}\frac{1}{N!}\sum_{\substack{A<q_1,\dots,q_N\leqslant B\\ \chi_{-d}(q_i)=1\\ \text{all distinct}}}\frac{1}{q_1\dots q_N}\left(2^{2N}\frac{k-1}{2\pi^2}+E(q_1^2\dots q_N^2)\right)\leqslant$$

$$\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\left(\sum_{\substack{A<q\leqslant B\\ \chi_{-d}(q)=1}}\frac{2}{q}\right)^N + R_k,$$

where $R_k \leqslant |R(A,B,k)|$.

Next, we apply Lemma 8.3 to the sum over the rest of the squares. Without loss of generality, assume that $\beta_1,\dots,\beta_s > 2$, $\beta_{s+1} = \dots = \beta_r = 2$. The contribution from this pattern is

$$\frac{(2N)!}{\beta_1!\dots\beta_r!r!}\left(\sum_{\substack{A<q_1,\dots,q_r\leqslant B\\ \chi_{-d}(q_i)=1\\ \text{all distinct}}}\frac{1}{q_1^{\beta_1/2}\dots q_s^{\beta_s/2}q_{s+1}\dots q_r}\right).$$

$$2^{2N}\frac{k-1}{2\pi^2}\prod_{i=1}^r\frac{2}{\beta_i+2}\binom{\beta_i}{\beta_i/2}\big(1+o(1)\big). \quad (8.12)$$

Using the crude estimates

$$\sum_{\substack{A<q_i\leqslant B\\ \chi_{-d}(q_i)=1}}\frac{1}{q_i^{\beta_i/2}}\leqslant\frac{1}{(\beta_i/2)-1}\frac{1}{A^{\beta_i/2-1}}, \qquad i=1,\dots,s,$$

$$\sum_{\substack{A<q_i\leqslant B\\ \chi_{-d}(q_i)=1}}\frac{1}{q_i}\leqslant\frac{\log B}{\log A}+1\leqslant 4, \qquad i=s+1,\dots,r,$$

we bound the expression (8.12) by

$$2^{2N}\frac{k-1}{2\pi^2}\frac{(2N)!}{\beta_1!\dots\beta_r!r!}\left(\prod_{i=1}^s\frac{1}{(\beta_i/2-1)}\frac{1}{A^{\beta_i/2-1}}\right)4^{r-s}.$$

$$\left(\prod_{i=1}^r\frac{2}{\beta_i+2}\binom{\beta_i}{\beta_i/2}\right)\big(1+o(1)\big),$$

which does not exceed

$$2^{2N}\frac{k-1}{2\pi^2}\frac{(2N)!}{N!r!}\frac{N!}{(\beta_1/2)!^2\dots(\beta_r/2)!^2}\left(\prod_{i=1}^{s}\frac{1}{(\beta_i/2-1)}\frac{1}{A^{\beta_i/2-1}}\right)4^{r-s}\leqslant$$
$$2^{4N}\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\frac{N!}{A\prod_{i=1}^{r}(\beta_i/2)!^2}.$$

Next, applying the inequalities

$$\sum_{1\leqslant r\leqslant N-1}\sum_{\substack{\beta_1+\ldots+\beta_r=2N\\ \text{all even}}}1\leqslant 2^{N-1},\qquad \frac{1}{\prod_{i=1}^{r}(\beta_i/2)!^2}\leqslant 1,$$

we get an upper bound for the contribution to the main term on the right hand side of (8.11) from all the remaining patterns with even β_i's. That bound is

$$\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\frac{2^{5N}N!}{A}.$$

The desired bound follows from the inequality

$$\frac{2^{5N}N!}{A}\ll\Bigg(\sum_{\substack{A<q\leqslant B\\ \chi_{-d}(q)=1}}\frac{2}{q}\Bigg)^N$$

since $A\gg N^{N+o(N)}$.

Remark. *One can obtain a slightly worse bound in the right hand side of* (8.9) *in the case $A=O(1)$:*

$$\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\exp\left\{N\log N-\frac{1}{2}N\log\log N\right\}\tag{8.13}$$

using trivial inequality $(\beta_i/2)!^{-2}\leqslant(\beta_i/2)!^{-1}$, multinomial theorem, and Stirling formula:

$$\sum_{1\leqslant r\leqslant N-1}\frac{1}{r!}\sum_{\substack{\beta_1+\ldots+\beta_r=2N\\ \textit{all even}}}\frac{N!}{(\beta_1/2)!\dots(\beta_r/2)!}\leqslant\sum_{r\leqslant N}\frac{r^N}{r!}\leqslant$$
$$N\max_{1\leqslant r\leqslant N}\frac{r^N}{r!}\leqslant\exp\left\{N\log N-\frac{1}{2}N\log\log N\right\}$$

for large enough N.

The proof of part 2) is similar. The main contribution comes from the squarefull products $p_1 \dots p_{2N}$. We have

$$\sum_{2^m < p_1,\dots,p_{2N} \leqslant 2^{m+1}} \frac{W_{p_1^2}(x)\dots W_{p_{2N}^2}(x)}{p_1 \dots p_{2N}} \sum_{f \in S_k(\Gamma_0(2))}^h \lambda_f(p_1^2)\dots\lambda_f(p_{2N}^2) =$$
$$\sum_{\substack{2^m < p_1,\dots,p_{2N} \leqslant 2^{m+1} \\ \prod p_i \text{ squarefull}}} \frac{W_{p_1^2}(x)\dots W_{p_{2N}^2}(x)}{p_1 \dots p_{2N}} \sum_{f \in S_k(\Gamma_0(2))}^h \lambda_f(p_1^2)\dots\lambda_f(p_{2N}^2) + S(m,N),$$

where $S(m,N)$ can be bounded by Lemma 8.3 part 2) from above as follows:

$$|S(m,N)| \leqslant 4^{2N+2N} \max_{p_1,\dots,p_{2N}} h(p_1 \dots p_{2N}) \left(\frac{2^{m+1}-2^m}{2^m}\right)^{2N} \cdot c_1 k e^{-k} =$$
$$4^{4N} \max_{p_1,\dots,p_{2N}} h(p_1 \dots p_{2N}) \cdot c_1 k e^{-k} =: R(m,N),$$

where
$$h(p_1^{\alpha_1} \dots p_r^{\alpha_r}) = \prod_{i=1}^r \alpha_i^2.$$

Note that $F_2(p_1^{\beta_1} \dots p_r^{\beta_r}) = 0$ if $\beta_i = 1$ for at least one i. We rewrite the main term in a way similar to the proof of part 1):

$$\sum_{1 \leqslant r \leqslant N} \sum_{\substack{\beta_1+\ldots+\beta_r=2N \\ \beta_i \geqslant 2}} \binom{2N}{\beta_1}\binom{2N-\beta_1}{\beta_2}\dots\binom{2N-\beta_1-\ldots-\beta_{r-1}}{\beta_r}.$$
$$\left(\frac{1}{r!} \sum_{\substack{2^m < q_1,\dots,q_r \leqslant 2^{m+1} \\ \text{all distinct}}} \frac{W_{q_1^2}(x)^{\beta_1}\dots W_{q_r^2}(x)^{\beta_r}}{q_1^{\beta_1}\dots q_r^{\beta_r}} \sum_{f \in S_k(\Gamma_0(2))}^h \lambda_f^{\beta_1}(q_1^2)\dots\lambda_f^{\beta_r}(q_r^2)\right). \quad (8.14)$$

We bound this expression using the weaker form of Lemma 8.3 part 2) (see (8.8)). Thus, (8.14) does not exceed

$$\frac{k-1}{2\pi^2} \sum_{1 \leqslant r \leqslant N} \sum_{\substack{\beta_1+\ldots+\beta_r=2N \\ \beta_i \geqslant 2}} \frac{(2N)!}{\beta_1!\dots\beta_r! r!} \prod_{i=1}^r \beta_i! \sum_{2^m < q_i \leqslant 2^{m+1}} \frac{1}{q_i^{\beta_i}} + R(m,N) \leqslant$$
$$2\frac{k-1}{2\pi^2} \sum_{1 \leqslant r \leqslant N} \sum_{\substack{\beta_1+\ldots+\beta_r=2N \\ \beta_i \geqslant 2}} \frac{(2N)!}{N!} \frac{N!}{r!} \prod_{i=1}^r \frac{1}{\beta_i - 1}\left(\left(\frac{1}{2^m}\right)^{\beta_i-1} - \left(\frac{1}{2^{m+1}}\right)^{\beta_i-1}\right) \leqslant$$
$$2\frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\left(\frac{1}{2^m}\right)^N \sum_{1 \leqslant r \leqslant N} \sum_{\substack{\beta_1+\ldots+\beta_r=2N \\ \beta_i \geqslant 2}} \frac{N!}{r!}\left(\frac{1}{2^m}\right)^{N-r}.$$

By Stirling formula,

$$\frac{N!}{r!}\left(\frac{1}{2^m}\right)^{N-r} \leqslant \exp\left\{N\log N - N + \log N + \frac{1}{2}\log 2\pi - r\log r + r - (N-r)\log 2^m\right\}.$$

The maximum of the last expression is achieved at $r = N$. In this case, the only possible pattern is $(\beta_1, \ldots, \beta_N) = (2, \ldots, 2)$. Thus, it does not exceed $\sqrt{2\pi}N$. Using the crude bound

$$\sum_{1\leqslant r\leqslant N} \sum_{\substack{\beta_1+\ldots+\beta_r=2N \\ \beta_i\geqslant 2}} 1 \leqslant 2^{2N-1},$$

we get the desired result.

□

Using the multiplicativity of λ_f, we can now combine both parts of Lemma 8.4 together:

Lemma 8.5. *Suppose $(A_1, B_1], \ldots, (A_I, B_I], (2^m, 2^{m+1}]$ are disjoint intervals, $A_i^2 < B_i \leqslant A_i^3$. Let $M_0, M_1, \ldots, M_I$ be non-negative integers such that $A_i \geqslant M_i \log M_i$, $M_0 \leqslant 2^m$,*

$$(2^{m+1})^{M_0} \prod_{i=1}^{I} B_i^{M_i} \leqslant \sqrt{k},$$

and $c_0, c_1, \ldots$ be positive real numbers. Then

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \left(\prod_{i=1}^{I} \sum_{m_i\leqslant M_i} c_{m_i} \sum_{\substack{p_1,\ldots,p_{m_i} \\ A_i<p_j\leqslant B_i}} \frac{\tilde{\lambda}_f(p_1)\ldots\tilde{\lambda}_f(p_{m_i})}{\sqrt{p_1\cdots p_{m_i}}} W_{p_1}(x)\ldots W_{p_{m_i}}(x)\right)\cdot$$

$$\left(\sum_{2^m<p\leqslant 2^{m+1}} \frac{\lambda_f(p^2)}{p} W_{p^2}(x)\right)^{2M_0} \leqslant$$

$$3\frac{k-1}{2\pi^2}\left(\prod_{i=1}^{I} \sum_{n_i\leqslant M_i/2} c_{2n_i} \frac{(2n_i)!}{n_i!}\left(\sum_{\substack{A_i<p\leqslant B_i \\ \chi_{-d}(p)=1}} \frac{2}{p_i}\right)^{n_i}\right)\sqrt{2\pi}M_0 4^{M_0}\frac{(2M_0)!}{M_0!}\left(\frac{1}{2^m}\right)^{M_0}.$$

8.2 Notation and sketch of the proof

We basically follow the Harper's approach to prove Theorem 1.7. The expression analogous to the right hand side of (1.16) can be obtained for the general L-function (see [16] and Lemma 8.1). We get an expression

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \exp\left\{\log\left|L_1(f)L_2(f)\right|\right\} = \sum_{f\in S_k(\Gamma_0(2))}^{h} \prod_{i=1}^{I} \exp\left\{\sum_{x^{3^{-i}}<p\leqslant x^{3^{1-i}}} \frac{\lambda_f(p)}{\sqrt{p}}\right\},$$

where we use the notation

$$L_1(f) := L\big(\frac{1}{2}, f\big), \qquad L_2(f) := L\big(\frac{1}{2}, f \otimes \chi_{-d}\big).$$

We will show that, for most of $f \in S_k(\Gamma_0(2))$, we can retrieve a good upper bound expanding the exponent in a Taylor series because the corresponding error term is small. Thus, we will get the product of such Taylor series and then use the variations of mean-value theorems for zeta function to get the desired bound. For the analogue of mean-value theorem, we use the multidimensional analogue of Petersson trace formula.

The remaining functions f for which the error in the Taylor expansion is large form an exceptional set. We will essentially show that the number of that function is small combining Cauchy and Markov inequalities.

Let k be a weight and d be a modulus. We choose $x = k^{1/D}$, where D is a fixed large real number depending only on the size of $\log d / \log k$, which would be chosen later. Let us introduce the notation for Dirichlet polynomials:

$$G_1(f) := \sum_{2 < p \leqslant X_0} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}}, \qquad G_i(f) := \sum_{X_{i-1} < p \leqslant X_i} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p),$$

$$i \in \overline{1, I}, \qquad X_i := x^{3^{i-I}},$$

where $I = \lfloor \log\log\log x \rfloor$. We split the set $S_k\big(\Gamma_0(2)\big)$ into the union $\mathcal{F} \cup S(0) \cup S(1) \cup \ldots \cup S(I-1)$, where

$$\mathcal{F} = \big\{ f \in S_k(\Gamma_0(2)) : \big|G_i(f)\big| \leqslant B_i,\ 1 \leqslant i \leqslant I \big\},$$

$$S(j) = \Big\{ f \in S_k\big(\Gamma_0(2)\big) : \big|G_i(f)\big| \leqslant B_i,\ 1 \leqslant i \leqslant j, \big|G_{j+1}(f)\big| > B_{j+1} \Big\}.$$

The numbers B_i will be chosen later. We also introduce the notation for the second polynomial from Lemma 8.1 for all $m \geqslant 0$ and $2^{m+1} \leqslant \sqrt{x}$:

$$P_m(f) := \sum_{2^m < p \leqslant 2^{m+1}} \frac{\lambda_f(p^2)}{p} W(p^2),$$

$$\mathcal{P}(m) = \Big\{ f \in S_k(\Gamma_0(2)) : P_m(f) > \frac{1}{m^2}, P_n(f) \leqslant \frac{1}{n^2},\ \text{if } m < n \leqslant \frac{\log x}{2\log 2} \Big\}.$$

We first split the main sum as follows:

$$\sum_{f \in S_k(\Gamma_0(2))}^{h} \exp\big\{\log\big|L_1(f)L_2(f)\big|\big\} \leqslant$$

$$\sum_{f \in \mathcal{F}}^{h} \exp\big\{\log\big|L_1(f)L_2(f)\big|\big\} + \sum_{j=0}^{I-1} \sum_{f \in S(j)}^{h} \exp\big\{\log\big|L_1(f)L_2(f)\big|\big\},$$

and then additionally split them using subsets $\mathcal{P}(m)$:

$$\sum_{f\in\mathcal{F}}^{h}\exp\{\log|L_1(f)L_2(f)|\} \leqslant \sum_{0\leqslant m\leqslant\frac{\log x}{2\log 2}}\ \sum_{f\in\mathcal{F}\cap\mathcal{P}(m)}^{h}\exp\{\log|L_1(f)L_2(f)|\},$$

$$\sum_{f\in S(j)}^{h}\exp\{\log|L_1(f)L_2(f)|\} \leqslant \sum_{0\leqslant m\leqslant\frac{\log x}{2\log 2}}\ \sum_{f\in S(j)\cap\mathcal{P}(m)}^{h}\exp\{\log|L_1(f)L_2(f)|\}$$

for all $j \in \overline{0, I-1}$. Thus, we need an upper bound for

$$\Big(\sum_{0\leqslant m\leqslant\frac{\log x}{2\log 2}}\Big(\sum_{f\in\mathcal{F}\cap\mathcal{P}(m)}^{h} + \sum_{j=1}^{I-1}\ \sum_{f\in S(j)\cap\mathcal{P}(m)}^{h}\Big) + \sum_{f\in S(0)}^{h}\Big)\exp\{\log|L_1(f)L_2(f)|\}.$$

Applying Lemma 8.1 to this sum, we essentially get a product of two Dirichlet polynomials

$$\exp\Big\{\sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}\Big\}\exp\Big\{\sum_{p\leqslant\sqrt{x}}\frac{\lambda_f(p^2)}{p}\Big\}$$

which we further split into the product of polynomials over the intervals $(X_{i-1}, X_i]$ and $(2^m, 2^{m+1}]$ correspondingly. On the first step, we ignore the second polynomial and show that the main contribution comes from the set $f \in \mathcal{F}$. Using Taylor expansion, we get

$$\prod_{i=1}^{I}\exp\{G_i(f)\} = \prod_{i=1}^{I}\Big(\sum_{l\leqslant A_i}\frac{1}{l!}(G_i(f))^l + r(A_l, f)\Big),$$

where the error $r(A_l, f)$ is small due to the fact that $f \in \mathcal{F}$. To treat the main term, we expand the l-powers of $G_i(f)$ and the product over $1 \leqslant i \leqslant I$, and then apply Lemma 8.4 to the sums over primes we obtain. To treat the sum over the exceptional sets $S(j)$, we use a variation of Rankin's trick:

$$\sum_{f\in S(j)}^{h}\exp\{|L_1(f)L_2(f)|\} \leqslant \sum_{f\in S_k(\Gamma_0(2))}^{h}\exp\{L_1(f)L_2(f)\}\Big(\frac{G_{j+1}(f)}{B_{j+1}}\Big)^{2N},$$

where we gain from the fact that the last factor is much less than 1. The choice of N should follow the restrictions of Lemmas 8.2, 8.3 and 8.4. Smaller j's correspond to shorter Dirichlet polynomials, thus produce larger error terms in the Taylor expansion, so we need larger choices of A_{j+1} and B_{j+1}. As we will see, it is not enough in the case $S(0)$, so we combine Cauchy and Markov inequalities to gain from the fact that the measure of this set is small.

The same strategy works in the case of polynomials $P_m(f)$. If $m \leqslant 2\log\log x$, we use a crude bound

$$\sum_{p\leqslant 2^m}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)+\sum_{p\leqslant 2^m}\frac{\lambda_f(p^2)}{p}W(p^2)\leqslant 20\frac{2^{m/2}}{m}$$

and compensate this using Rankin's trick:

$$\exp\left\{\sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)+\sum_{p\leqslant\sqrt{x}}\frac{\lambda_f(p^2)}{p}W(p^2)\right\}\leqslant$$
$$\exp\left\{20\frac{2^{m/2}}{m}\right\}\cdot\sum_{f\in\mathcal{F}}^{h}\exp\left\{\sum_{2^m<p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)\right\}\cdot\left(\frac{P_m(f)}{1/m^2}\right)^{2M}.$$

In the case of large m, this is no longer enough, so we again apply Cauchy inequality to gain from the small size of the sets $\mathcal{P}(m)$.

8.3 First moment estimation

Now we are ready to give the details of the proof.

Computation for $f\in\mathcal{F}$

In this section, we will get the bound

$$\sum_{f\in\mathcal{F}}^{h}\exp\left\{\sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)\right\}\leqslant 0.11kL(1,\chi_{-d})\exp\{\log\log x\}.$$

In accordance with the notation, we may rewrite the first Dirichlet polynomial as follows:

$$\sum_{f\in\mathcal{F}}^{h}\exp\left\{\sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)\right\}=\sum_{f\in\mathcal{F}}^{h}\prod_{i=1}^{I}\exp\left\{\frac{1}{2}\sum_{X_{i-1}<p\leqslant X_i}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)\right\}^2\leqslant$$
$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\prod_{i=1}^{I}\left[\sum_{l\leqslant A_i}\frac{1}{l!}\left(\frac{1}{2}\,G_i(f)\right)^l+\frac{\exp\{1/2|G_i(f)|\}}{(A_i+1)!}\left(\frac{1}{2}\,|G_i(f)|\right)^{A_i+1}\right]^2,$$

where the error term in the Taylor expansion is written in the form of Lagrange. We choose $A_i=10B_i$ to make this error small. We rewrite the last expression as

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\prod_{i=1}^{I}(1+\varepsilon_i)^2\left(\sum_{l\leqslant A_i}\frac{1}{l!}\left(\frac{1}{2}\,G_i(f)\right)^l\right)^2=$$
$$(1+\varepsilon_0)\sum_{f\in S_k\left(\Gamma_0(2)\right)}^{h}\prod_{i=1}^{I}\left(\sum_{l\leqslant A_i}\frac{1}{l!}\left(\frac{1}{2}G_i(f)\right)^l\right)^2,\quad (8.15)$$

where

$$\varepsilon_i = \Big(\sum_{l\leqslant A_i}\frac{1}{l!}\Big(\frac{1}{2}|G_i(f)|\Big)^l\Big)^{-1}\frac{\exp\{1/2|G_i(f)|\}}{(A_i+1)!}\Big(\frac{1}{2}|G_i(f)|\Big)^{A_i+1} \leqslant$$
$$\frac{\exp\{B_i/2\}}{(A_i+1)!}\Big(\frac{B_i}{2}\Big)^{A_i+1},$$

and

$$1+\varepsilon_0 = \prod_{i=1}^{I}(1+\varepsilon_i)^2 \leqslant \exp\{2\sum_{i=1}^{I}\varepsilon_i\} =$$
$$\exp\Big\{2\sum_{i=1}^{I}\exp\Big\{\frac{B_i}{2} - \log(A_i+1)! + (A_i+1)\log\frac{B_i}{2}\Big\}\Big\},$$

which does not exceed

$$\exp\Big\{2\sum_{i=1}^{I}\exp\{\frac{B_i}{2} + (A_i+1)\log\frac{B_i}{2} - (A_i+1)\log(A_i+1) + A_i + 1\}\Big\} \leqslant$$
$$\exp\Big\{2\sum_{i=1}^{I}\exp\{-A_i\}\Big\}$$

by Stirling formula and our choice $A_i = 10B_i$.

Next, we expand the square and l-powers in the right hand side of (8.15) to get

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}(1+\varepsilon_0)\sum_{\tilde{l},\tilde{t}}\prod_{i=1}^{I}\Big(\frac{1}{l_i!t_i!2^{l_i+t_i}}\Big)\sum_{\tilde{p},\tilde{q}}\prod_{\substack{i\leqslant I\\ r\leqslant l_i\\ s\leqslant t_i}}\frac{\tilde{\lambda}_f(p_{i,r})\tilde{\lambda}_f(q_{i,s})}{\sqrt{p_{i,r}q_{i,s}}}W(p_{i,r})W(q_{i,s}),$$

where the first sum runs over all vectors of the form $\tilde{l} = (l_1,\dots,l_I)$, $\tilde{t} = (t_1,\dots,t_I)$ with $l_i, t_i \leqslant A_i$ and the second sum runs over all the vectors of primes

$$\tilde{p}(\tilde{l}) = (p_{1,1},\dots,p_{1,l_1},p_{2,1},\dots,p_{2,l_2},\dots,p_{I,1},\dots,p_{I,l_I}),$$

$$\tilde{q}(\tilde{t}) = (q_{1,1},\dots,q_{1,t_1},q_{2,1},\dots,q_{2,t_2},\dots,q_{I,1},\dots,q_{I,t_I}),$$

where $p_{i,r}, q_{i,s} \in (X_{i-1}, X_i]$. The contribution from these sums does not exceed

$$\sum_{f\in S_k(\Gamma_0(2))}^h (1+\varepsilon_0)\prod_{i=1}^I \sum_{l,t\leqslant A_i} \frac{1}{l!t!2^{l+t}}\cdot$$
$$\sum_{\substack{p_1,\dots,p_l\\ q_1,\dots,q_t\\ \in(X_{i-1},X_i]}} \frac{\tilde\lambda_f(p_1)\dots\tilde\lambda_f(p_l)\tilde\lambda_f(q_1)\dots\tilde\lambda_f(q_t)}{\sqrt{p_1\dots p_l q_1\dots q_t}} W(p_1)\dots W(q_t) \leqslant$$

$$\sum_{f\in S_k(\Gamma_0(2))}^h (1+\varepsilon_0)\prod_{i=1}^I \sum_{m\leqslant 2A_i} \frac{1}{2^m}\bigg(\sum_{l+t=m}\frac{1}{l!t!}\bigg)\cdot$$
$$\sum_{X_{i-1}<p_1,\dots,p_m\leqslant X_i} \frac{\tilde\lambda_f(p_1)\dots\tilde\lambda_f(p_m)}{\sqrt{p_1\dots p_m}} W(p_1)\dots W(p_m) \leqslant$$

$$\sum_{f\in S_k(\Gamma_0(2))}^h (1+\varepsilon_0)\prod_{i=1}^I \sum_{m\leqslant 2A_i} \frac{1}{m!}\cdot$$
$$\sum_{X_{i-1}<p_1,\dots,p_m\leqslant X_i} \frac{\tilde\lambda_f(p_1)\dots\tilde\lambda_f(p_m)}{\sqrt{p_1\dots p_m}} W(p_1)\dots W(p_m).$$

Since the sets of primes for each $i \leqslant I$ are distinct, we can apply Lemma 8.5 with $M_0 = 0, c_m = 1/m!$. This gives the bound

$$(1+\varepsilon_0)\cdot 2\frac{k-1}{2\pi^2}\prod_{i=1}^I \sum_{n\leqslant A_i} \frac{1}{(2n)!}\frac{(2n)!}{n!}\bigg(\sum_{\substack{X_{i-1}<p\leqslant X_i\\ \chi_{-d}(p)=1}} \frac{2}{p}\bigg)^n,$$

which does not exceed

$$(1+\varepsilon_0)\frac{k-1}{\pi^2}\exp\Big\{\sum_{p\leqslant x}\frac{1+\chi_{-d}(p)}{p}\Big\} \leqslant 0.11kL(1,\chi_{-d})\exp\big\{\log\log x\big\}.$$

Note that we have restrictions from Lemma 8.5 on the size of A_i's:

$$\prod_{i=1}^I X_i^{2A_i} \leqslant \sqrt{k}. \tag{8.16}$$

Computation for $f \in S(j)$

The goal of this section is to show that the contribution from each exceptional set $S(j)$ is small. Precisely, we will obtain the bound

$$\sum_{f\in S(j)}^h \exp\bigg\{\sum_{p\leqslant X_j}\frac{\tilde\lambda_f(p)}{\sqrt p}W(p) - \sum_{p\leqslant\sqrt{X_j}}\frac{1}{p}W(p^2) + 1.2 + \frac{3}{4}\frac{\log d^2k^4}{\log X_j}\bigg\} \leqslant$$
$$kL(1,\chi_{-d})\exp\bigg\{-\frac{3^{I-j-1}D}{32}\log\frac{D}{8}\bigg\}$$

for $1 \leqslant j \leqslant I-1$. Note that here we ignore the contribution from the second Dirichlet polynomial with coefficients $\lambda_f(p^2)$. Applying Lemma 8.1 with $x = X_j$, we get

$$\sum_{f\in S(j)}^{h} \exp\left\{\sum_{p\leqslant X_j} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p)\right\} = \sum_{f\in S(j)}^{h} \prod_{i=1}^{j} \exp\left\{\frac{1}{2} \sum_{X_{i-1}<p\leqslant X_i} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p)\right\}^2 \leqslant$$

$$\sum_{f\in S(j)}^{h} \prod_{i=1}^{j} \exp\left\{\frac{1}{2} G_i(f)\right\}^2 \left(\frac{G_{j+1}(f)}{B_{j+1}}\right)^{2N_{j+1}}.$$

Performing the same computation as in the previous case, we bound the last expression by

$$(1+\varepsilon_0^{(j)}) {B_{j+1}}^{-2N_{j+1}} \sum_{f\in S_k(\Gamma_0(2))}^{h} (G_{j+1}(f))^{2N_{j+1}} \prod_{i=1}^{j} \left(\sum_{l\leqslant A_i} \frac{1}{l!}\left(\frac{1}{2}G_i(f)\right)^l\right)^2 \leqslant$$

$$(1+\varepsilon_0^{(j)})\left({B_{j+1}}^{-2N_{j+1}}\right) \sum_{f\in S_k(\Gamma_0(2))}^{h} \left[\prod_{i=1}^{j} \sum_{m\leqslant 2A_i} \frac{1}{m!}\cdot\right.$$

$$\left.\sum_{X_{i-1}<p_1,\ldots,p_m\leqslant X_i} \frac{\tilde{\lambda}_f(p_1)\ldots\tilde{\lambda}_f(p_m)}{\sqrt{p_1\ldots p_m}} W(p_1)\ldots W(p_m)\right]\cdot$$

$$\left[\sum_{X_j<p_1,\ldots,p_{2N_{j+1}}\leqslant X_{j+1}} \frac{\tilde{\lambda}_f(p_1)\ldots\tilde{\lambda}_f(p_{2N_{j+1}})}{\sqrt{p_1\ldots p_{2N_{j+1}}}} W(p_1)\ldots W(p_{2N_{j+1}})\right].$$

Since both the sums are over disjoint subsets of primes, we can again apply Lemma 8.5. Then the last expression does not exceed

$$(1+\varepsilon_0^{(j)}) B_{j+1}^{-2N_{j+1}} \cdot 2\frac{k-1}{2\pi^2} \left(\prod_{i=1}^{j} \sum_{n\leqslant A_i} \frac{1}{(2n)!} \frac{(2n)!}{n!} \left(\sum_{\substack{X_{i-1}<p\leqslant X_i \\ \chi_{-d}(p)=1}} \frac{2}{p}\right)^n\right)\cdot$$

$$\frac{(2N_{j+1})!}{N_{j+1}!} \left(\sum_{\substack{X_j<p\leqslant X_{j+1} \\ \chi_{-d}(p)=1}} \frac{2}{p}\right)^{N_{j+1}}.$$

Combining this inequality and Stirling formula, we get

$$\sum_{f\in S(j)}^{h} \exp\left\{\sum_{p\leqslant X_j} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p)\right\} \leqslant (1+\varepsilon_0^{(j)}) \cdot 2\frac{k-1}{2\pi^2}\cdot$$

$$\exp\left\{-2N_{j+1}\log B_{j+1} + 2N_{j+1}\log 2N_{j+1} - 2N_{j+1} + \right.$$

$$\left.\log\left(2N_{j+1}\right) + \frac{1}{2}\log 2\pi - N_{j+1}\log N_{j+1} + N_{j+1} + \sum_{p\leqslant X_j} \frac{1+\chi_{-d}(p)}{p}\right\} \left(\log 3 + 1\right)^{N_{j+1}},$$

which can be bounded by

$$kL(1,\chi_{-d})\exp\big\{-2N_{j+1}\log B_{j+1}+N_{j+1}\log N_{j+1}+ N_{j+1}\big[2\log 2-1+\log(\log 3+1)\big]+\log N_{j+1}+\log\log X_j\big\}.$$

Here we get new restrictions similar to (8.16):

$$\left(\prod_{i=1}^{j} X_i^{2A_i}\right)X_{j+1}^{2N_{j+1}} \leqslant \sqrt{k}. \tag{8.17}$$

We choose

$$\log(X_1)^{2A_1} = \frac{1}{3\sqrt{3}}\log\sqrt{k},$$

$$\log(X_I)^{2A_I} = \frac{1}{9}\log\sqrt{k}, \qquad \log(X_{I-1})^{2A_{I-1}} = \frac{1}{9\sqrt{3}}\log\sqrt{k}, \qquad \dots,$$

$$\log\big(X_j\big)^{2A_j} = \frac{1}{(\sqrt{3})^{I-j+4}}\log\sqrt{k}, \qquad \dots, \qquad \log(X_2)^{2A_2} = \frac{1}{(\sqrt{3})^{I+2}}\log\sqrt{k}.$$

Then we get

$$A_1 = \frac{1}{12\sqrt{3}}\frac{\log k}{\log X_1} = \frac{D\cdot 3^{I-1}}{12\sqrt{3}},$$

$$A_2 = \frac{D}{36}\big(\sqrt{3}\big)^{I-2}, \qquad \dots, \qquad A_j = \frac{D}{36}\big(\sqrt{3}\big)^{I-j}, \qquad \dots, \qquad A_I = \frac{D}{36},$$

$$N_i = \lfloor B_i\rfloor = \left\lfloor\frac{A_i}{10}\right\rfloor = \left\lfloor\frac{D}{360}\big(\sqrt{3}\big)^{I-i}\right\rfloor \qquad \text{for all } i\in\overline{2,I},\ i\neq j+1,$$

$$N_{j+1} = \left\lfloor\frac{D}{8}3^{I-j-1}\right\rfloor.$$

This gives the bound

$$kL(1,\chi_{-d})\exp\Big\{-2\left\lfloor\frac{3^{I-j-1}D}{8}\right\rfloor\log\frac{\big(\sqrt{3}\big)^{I-j-1}D}{360}+\left\lfloor\frac{3^{I-j-1}D}{8}\right\rfloor\log\left\lfloor\frac{3^{I-j-1}D}{8}\right\rfloor+ 1.5\left\lfloor\frac{3^{I-j-1}D}{8}\right\rfloor+\log\log X_j\Big\} \leqslant kL(1,\chi_{-d})\exp\left\{-\frac{3^{I-j-1}D}{16}\log\frac{D}{8}+\log\log X_j\right\}$$

if $D/8>\exp\{2\log 45+3\}$. Finally, we get

$$\sum_{f\in S(j)}^{h}\exp\left\{\sum_{p\leqslant X_j}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)-\sum_{p\leqslant\sqrt{X_j}}\frac{1}{p}W(p^2)+1.2+\frac{3}{4}\frac{\log d^2k^4}{\log X_j}\right\} \leqslant$$

$$k\cdot L(1,\chi_{-d})\exp\left\{-\frac{3^{I-j-1}D}{16}\log\frac{D}{8}+6D\cdot 3^{I-j}\max\left\{1,\frac{1}{4}\frac{\log d}{\log k}\right\}\right\} \leqslant$$

$$kL(1,\chi_{-d})\exp\left\{-\frac{3^{I-j-1}D}{32}\log\frac{D}{8}\right\}$$

as soon as

$$D \geqslant 8\exp\left\{576\max\left\{1, \frac{1}{4}\frac{\log d}{\log k}\right\}\right\}.$$

Computation for $f \in \mathcal{F} \cap \mathcal{P}(m)$**,** $f \in S(j) \cap \mathcal{P}(m)$

In this section, we take into account the contribution from the second Dirichlet polynomial

$$\sum_{p \leqslant \sqrt{x}} \frac{\lambda_f(p^2)}{p} W(p^2)$$

and using the same trick, we will show that it is negligible for most $f \in S_k(\Gamma_0(2))$. First, note that the main contribution comes from subsets $f \in \mathcal{F} \cap \mathcal{P}(m)$ for small m. If $0 \leqslant m < 250$, we bound the second polynomial trivially:

$$\sum_{p \leqslant \sqrt{x}} \frac{\lambda_f(p^2)}{p} W(p^2) \leqslant \sum_{p \leqslant \sqrt{2^{m+1}}} \frac{3}{p} + \sum_{n=m+1}^{+\infty} \frac{1}{n^2} \leqslant 17. \tag{8.18}$$

Now suppose $m \geqslant 250$. Consider two cases:

1) $m \leqslant 2\log\log x$

Using partial summation and Ramanujan bounds $|\tilde{\lambda}_f(p)| \leqslant 4$, $|\lambda_f(p^2)| \leqslant 3$, we get

$$\sum_{p \leqslant 2^{m+1}} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p) + \sum_{p \leqslant 2^{m+1}} \frac{\lambda_f(p^2)}{p} W(p^2) \leqslant 4 \cdot \frac{15\sqrt{2^{m+1}}}{\log(2^{m+1})} \leqslant 124 \cdot \frac{2^{m/2}}{m}.$$

We denote by $X_i^{(m)}$ the largest X_i such that $X_i \leqslant 2^{m+1}$ and by $i = i(m)$ the corresponding index. We have

$$\begin{aligned} &\sum_{f \in \mathcal{F} \cap \mathcal{P}(m)}^{h} \exp\left\{\sum_{p \leqslant x} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p) + \sum_{p \leqslant \sqrt{x}} \frac{\lambda_f(p^2)}{p} W(p^2)\right\} \leqslant \\ &\quad \exp\left\{124 \cdot \frac{2^{m/2}}{m}\right\} \cdot \sum_{f \in \mathcal{F} \cap \mathcal{P}(m)}^{h} \exp\left\{\sum_{2^{m+1} < p \leqslant x} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p) + \sum_{n > m} \frac{1}{n^2}\right\} \leqslant \\ &\qquad \exp\left\{125 \cdot \frac{2^{m/2}}{m}\right\} \cdot \sum_{f \in \mathcal{F} \cap \mathcal{P}(m)}^{h} \left(\prod_{i=i(m)}^{I} \exp\left\{\sum_{\substack{X_i < p \leqslant X_{i+1} \\ p > 2^{m+1}}} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p)\right\}\right). \end{aligned}$$

The last expression does not exceed

$$\exp\left\{125 \cdot \frac{2^{m/2}}{m}\right\} \cdot \sum_{f \in \mathcal{F}}^{h} \left(\prod_{i=i(m)}^{I} \exp\left\{\sum_{\substack{X_i < p \leqslant X_{i+1} \\ p > 2^{m+1}}} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p)\right\}\right) \cdot \left(\frac{P_m(f)}{1/m^2}\right)^{2M}.$$

Repeating the same steps as in the previous section, we get

$$\exp\left\{125\cdot\frac{2^{m/2}}{m}\right\}(1+\varepsilon_0)m^{4M}\cdot \sum_{f\in S_k(\Gamma_0(2))}^{h}\left[\prod_{i=i_m}^{I}\sum_{n\leqslant 2A_i}\frac{1}{n!}\sum_{\substack{X_i<p_1,\dots,p_n\leqslant X_{i+1}\\ p>2^{m+1}}}\frac{\tilde\lambda_f(p_1)\dots\tilde\lambda_f(p_n)}{\sqrt{p_1\dots p_n}}W(p_1)\dots W(p_n)\right]\cdot \left[\sum_{2^m<p_1,\dots,p_{2M}\leqslant 2^{m+1}}\frac{\lambda_f(p_1^2)\dots\lambda_f(p_{2M}^2)}{p_1\dots p_{2M}}W(p_1^2)\dots W(p_{2M}^2)\right]. \tag{8.19}$$

The corresponding subsets of primes in all sums in the last expression are disjoint which means we can apply Lemma 8.5 with $M_0 = M$. The bound we get is

$$\exp\left\{125\cdot\frac{20^{m/2}}{m}\right\}(1+\varepsilon_0)m^{4M}\cdot 2\frac{k-1}{2\pi^2}\cdot \exp\left\{\sum_{2^{m+1}<p\leqslant x}\frac{1+\chi_{-d}(p)}{p}\right\}\cdot\sqrt{2\pi}M4^M\frac{(2M)!}{M!}\left(\frac{1}{2^m}\right)^M,$$

which does not exceed

$$(1+\varepsilon_0)2\frac{k-1}{2\pi^2}\cdot L(1,\chi_{-d})\exp\left\{125\cdot\frac{2^{m/2}}{m}+4M\log m+\log\log x- \log\log 2^{m+1}+0.5+M\log 4+M\log M+(\log 16-1)M+\log(4\pi)+2\log M-mM\log 2\right\}.$$

With the choice $M=\lfloor 2^{m/2}\rfloor$, this can be further bounded by

$$kL(1,\chi_{-d})\exp\{-m+\log\log x\} \tag{8.20}$$

if $m\geqslant 250$. Note that the restriction from Lemma 8.5 is

$$\log\left(2^{m+1}\right)^{2M}\leqslant\frac{1}{100}\log\sqrt{k}$$

equivalent to

$$m\frac{\log 2}{2}\leqslant\big(1-o(1)\big)\log\log k=\big(1-o(1)\big)\log\log(x)$$

which implies

$$m\leqslant\frac{2}{\log(2)}\log\log x$$

which is satisfied by the choice of m.

2) $m\geqslant 2\log\log x$

Combining Cauchy inequality and Ramanujan bound $|\lambda_f(p^2)| \leqslant 3$, we get

$$\sum_{f\in\mathcal{F}\cap\mathcal{P}(m)}^h \exp\left\{\sum_{p\leqslant x}\frac{\tilde\lambda_f(p)}{\sqrt{p}}W(p)+\sum_{p\leqslant\sqrt{x}}\frac{\lambda_f(p^2)}{p}W(p^2)\right\}\leqslant$$
$$\exp\{3\log m+0.5\}\sqrt{\text{meas}\mathcal{P}(m)}\cdot\left(\sum_{f\in\mathcal{F}}^h\exp\left\{2\sum_{p\leqslant x}\frac{\tilde\lambda_f(p)}{\sqrt{p}}W(p)\right\}\right)^{1/2}. \quad (8.21)$$

To estimate the measure of $\mathcal{P}(m)$, we use Markov inequality and Lemma 8.4, part 2):

$$\text{meas}(\mathcal{P}(m))\leqslant\left(\frac{1}{m^2}\right)^{-4}\sum_{f\in S_k(\Gamma_0(2))}^h\left(\sum_{2^m\leqslant p<2^{m+1}}\frac{\lambda_f(p^2)}{p}W(p^2)\right)^4\leqslant$$
$$m^8\sum_{f\in S_k(\Gamma_0(2))}^h\sum_{2^m<p_1,p_2,p_3,p_4\leqslant 2^{m+1}}\frac{\lambda_f(p_1^2)\lambda_f(p_2^2)\lambda_f(p_3^2)\lambda_f(p_4^2)}{p_1p_2p_3p_4}W(p_1^2)\dots W(p_4^2)\leqslant$$
$$m^8\cdot 2\sqrt{2\pi}\cdot 16\frac{k-1}{2\pi^2}\cdot 12\left(\frac{1}{2^m}\right)^2\leqslant 50k\exp\{8\log(m)-2m\log 2\}.$$

To treat the second factor on the right hand side of (8.21), we apply Lemma 8.5 with $c_m=2^m/m!$ and end up with the expression

$$\exp\{3\log m+0.5\}\sqrt{50k}\exp\{4\log m-m\log 2\}\cdot$$
$$\sqrt{0.11k}\cdot L(1,\chi_{-d})^2\exp\{2\log\log x\}\leqslant$$
$$5.5k\cdot L(1,\chi_{-d})^2\exp\{-0.01m+\log\log x\}. \quad (8.22)$$

Note that the error from the Taylor expansion is

$$1+\varepsilon_0=\exp\left\{2\sum_{i=1}^I\exp\{B_i-\log(A_i+1)!+(A_i+1)\log B_i\}\right\}$$

since we have $\exp\{G_i(f)\}$ instead of $\exp\{1/2G_i(f)\}$, but the same upper bound $\exp\left\{2\sum_{i=1}^I e^{-A_i}\right\}$ is valid. Combining (8.18), (8.20), and (8.22), we get

$$\sum_{0\leqslant m\leqslant\frac{\log x}{2\log 2}}\sum_{f\in\mathcal{F}\cap\mathcal{P}(m)}^h\exp\left\{\sum_{p\leqslant x}\frac{\tilde\lambda_f(p)}{\sqrt{p}}W(p)+\sum_{p\leqslant\sqrt{x}}\frac{\lambda_f(p^2)}{p}W(p^2)\right\}\leqslant$$
$$250\cdot 0.11kL(1,\chi_{-d})\exp\{17+\log\log x\}+\sum_{m\geqslant 250}kL(1,\chi_{-d})\exp\{-m+\log\log x\}+$$
$$\sum_{m\geqslant\log\log x}5.5kL(1,\chi_{-d})\exp\{-0.01m+\log\log x+\log\log\log d\}\leqslant$$
$$kL(1,\chi_{-d})\exp\{21+\log\log x\}.$$

The computation of the upper bound for the sum over $f \in S(j) \cap \mathcal{P}(m)$ is similar. Here we apply Lemma 8.5 to the product of the form

$$\sum_{f \in S_k(\Gamma_0(2))}^{h} \left(\prod_{i=i(m)}^{j} \exp\left\{\frac{1}{2} G_i(f)\right\}^2 \right) \left(\frac{G_{j+1}(f)}{B_{j+1}} \right)^{2N_{j+1}} \left(\frac{P_m(f)}{1/m^2} \right)^{2M}$$

and get an upper bound of the form

$$\sum_{0<m \leqslant \frac{\log X_j}{2\log 2}} \sum_{f \in S(j) \cap \mathcal{P}(m)}^{h} \exp\Bigg\{ \sum_{p \leqslant X_j} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p) + \\ \sum_{p \leqslant \sqrt{X_j}} \frac{\lambda_f(p^2) - 1}{p} W(p^2) + 1.2 + \frac{3}{4} \frac{\log d^2 k^4}{\log X_j} \Bigg\} \leqslant \\ k L(1, \chi_{-d}) \exp\left\{ 21 - \frac{3^{I-j-1} D}{32} \log \frac{D}{8} \right\}.$$

Computation for $f \in S(0)$

By Cauchy inequality, we get

$$\sum_{f \in S(0)}^{h} \exp\left\{\log\left|L_1(f) L_2(f)\right|\right\} \leqslant \\ \big(\mathrm{meas}(S(0))\big)^{1/2} \left(\sum_{f \in S(0)}^{h} \exp\left\{2\log\left|L_1(f) L_2(f)\right|\right\} \right)^{1/2}.$$

For the second factor, we use the crude upper bound $k(\log k)^7 L(1, \chi_{-d})$ which can be easily obtained using Soundararajan's method. The details are given in the end of this section. Thus, the whole sum does not exceed

$$\big(\mathrm{meas}(S(0))\big)^{1/2} \big(k(\log k)^7 L(1, \chi_{-d})^4\big)^{1/2}. \tag{8.23}$$

The upper bound for the first factor could be obtained by Markov inequality and weak version of Lemma 8.4 part 1) (see (8.13)):

$$\mathrm{meas}\big(S(0)\big) \leqslant B_1^{-2N_1} \sum_{f \in S_k(\Gamma_0(2))}^{h} \left(\sum_{p \leqslant X_1} \frac{\tilde{\lambda}_f(p)}{\sqrt{p}} W(p) \right)^{2N_1} \leqslant \\ k \frac{(2N_1)!}{N_1!} \exp\left\{ N_1\big(\log N_1 - \frac{1}{2}\log\log N_1\big) - 2N_1 \log B_1 \right\}. \tag{8.24}$$

Then (8.23) does not exceed

$$k(\log k)^{\frac{7}{2}} L(1, \chi_{-d})^2 \exp\left\{-N_1 \log B_1 + N_1 \log N_1\right\}. \tag{8.25}$$

Choose

$$N_1 = \left\lfloor \frac{B_1}{2} \right\rfloor = \left\lfloor \frac{D}{240\sqrt{3}} 3^{I-1} \right\rfloor,$$

then (8.25) does not exceed

$$kL(1,\chi_{-d}) \cdot \exp\left\{ \frac{7}{2} \log\log k + 2 \log\log\log d - \frac{D}{5500} (\log\log x)^{\log 3} \right\} \leqslant \\ kL(1,\chi_{-d}) \exp\{-\log\log k\}$$

as soon as $\log\log x = (1+o(1)) \log\log k$.

Finally, we get

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \exp\{\log|L_1(f)L_2(f)|\} \leqslant \\ \left[\sum_{0\leqslant m\leqslant \frac{\log x}{2\log 2}} \left(\sum_{f\in \mathcal{F}\cap\mathcal{P}(m)}^{h} + \sum_{j=1}^{I-1} \sum_{f\in S(j)\cap\mathcal{P}(m)}^{h} \right) + \sum_{f\in S(0)}^{h} \right] \exp\{\log|L_1(f)L_2(f)|\} \leqslant \\ kL(1,\chi_{-d}) \left[\exp\left\{ 21 + 1.2 + \frac{3}{4} \frac{\log d^2 k^4}{\log x} \right\} + \sum_{1\leqslant j\leqslant I-1} \exp\left\{ 21 - \frac{3^{I-j-1} D}{32} \log \frac{D}{8} \right\} \right] + \\ k \cdot L(1,\chi_{-d}) \exp\{-\log\log k\} \leqslant \\ 4.4 \cdot 10^9 kL(1,\chi_{-d}) \exp\left\{ 6D \max\left\{ 1, \frac{1}{4} \frac{\log d}{\log k} \right\} \right\} \leqslant \\ kL(1,\chi_{-d}) \exp\left\{ \max\left\{ 1, \frac{1}{4} \frac{\log d}{\log k} \right\} \exp\left\{ 600 \max\left\{ 1, \frac{1}{4} \frac{\log d}{\log k} \right\} \right\} \right\}.$$

To finish the section, we prove the bound

$$\sum_{f\in S_k(\Gamma_0(2))}^{h} \exp\{2\log|L_1(f)L_2(f)|\} \leqslant k(\log k)^7 L(1,\chi_{-d})^2$$

by Soundararajan's technique. We rewrite the second moment as follows:

$$\int_{-\infty}^{+\infty} e^{2V} \mathrm{meas}\{f \in S_k(\Gamma_0(2)) : \log|L_1(f)L_2(f)| > V\} dV$$

and split it into the sum of two integrals

$$I_1 + I_2 = \int_{-\infty}^{3\log\log k} e^{2V} \mathrm{meas}\{\ldots\} dV + \int_{3\log\log k}^{+\infty} e^{2V} \mathrm{meas}\{\ldots\} dV.$$

The first integral can be estimated directly:

$$I_1 \leqslant \frac{k-1}{2\pi^2} \int_{-\infty}^{3\log\log k} e^{2V} dV \leqslant k(\log k)^6 \leqslant \frac{k}{2} (\log k)^7 L(1,\chi_{-d})^2.$$

The measure in the second integral is evaluated in the usual way. By Lemma 8.1

$$\log\left|L_1(f)L_2(f)\right| \leqslant \sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p)+\sum_{p\leqslant\sqrt{x}}\frac{\lambda_f(p^2)-1}{p}W(p^2)+1.2+3D\max\left\{1,\frac{\log d}{\log k}\right\}$$

with $x = k^{2D/V}$. Then

$$\operatorname{meas}\left\{f\in S_k(\Gamma_0(2)) : \log\left|L_1(f)L_2(f)\right| > V\right\} \leqslant \operatorname{meas}\left\{f\in S_k(\Gamma_0(2)) : \sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}W(p) > \frac{V}{4}\right\},$$

as soon as $V > 3\log\log x$, but $\sum_{p\leqslant x}\lambda_f(p^2)/p < (1+o(1))\cdot 3\log\log x$. Combining Markov inequality and (8.13), we get

$$\sum_{f\in S_k(\Gamma_0(2))}^{h}\left(\frac{V}{4}\right)^{-2N}\left(\sum_{p\leqslant x}\frac{\tilde{\lambda}_f(p)}{\sqrt{p}}\right)^{2N} \leqslant \frac{k-1}{2\pi^2}\frac{(2N)!}{N!}\exp\left\{-2N\log(V/4)+N\log N-\frac{1}{2}N\log\log N\right\}.$$

With the choice $N = \lfloor V/8D\rfloor$ the right hand side of the last inequality does not exceed

$$kL(1,\chi_{-d})\exp\left\{-\frac{V}{20}\log\log V\right\} \leqslant kL(1,\chi_{-d})\exp\left\{-10V\right\},$$

hence

$$I_2 \leqslant kL(1,\chi_{-d})\int_{3\log\log k}^{+\infty} e^{2V-10V}\,dV \leqslant kL(1,\chi_{-d}) \leqslant \frac{k}{2}(\log k)^7 L(1,\chi_{-d})$$

which completes the proof.

Chapter 9

VARIANCE ESTIMATES IN LINNIK'S PROBLEM

This chapter is devoted to the proof of Theorem 1.8 on the assumption of Theorem 1.7. We start from considering the random model for points on the sphere. In the second section of this chapter, we do a necessary spherical analysis to reduce the problem to the estimation of the first of $GL(2)$ L-functions.

9.1 Heuristics from random model

In this section, we compute the expected value and the variance of random uniformly distributed points $\mathbf{x}_1, \ldots, \mathbf{x}_{N_n}$ on the sphere inside a cap $\Omega_n(\mathbf{x})$ centered at some fixed point $\mathbf{x}$. Denote by $Z(n; \Omega_n(\mathbf{x}))$ the number of points inside the cap. Then Z can be written as $\sum_{i=1}^{N} \xi_i(\mathbf{x})$, where

$$\xi_i(\mathbf{x}) = \begin{cases} 1, \text{if } \mathbf{x} \in \Omega_n(\mathbf{x_i}); \\ 0, \text{otherwise.} \end{cases}$$

Note that one can think of random point $\mathbf{x}_i$ as of random rotation of the sphere moving $\mathbf{x}$ to $\mathbf{x}_i$. Then the expected value of the number of points inside the cap can be computed as follows:

$$E[Z] = \int_{SO(3)} Z(n; g\Omega_n(\mathbf{x}))dg = \sum_{i=1}^{N_n} E[\xi_i] = \sum_{i=1}^{N_n} \int_{SO(3)} \mathbb{1}_{\mathbf{x} \in g\Omega(\mathbf{x}_i)} dg = \sigma(\Omega_n) N_n.$$

The variance is

$$V(n, \Omega_n(\mathbf{x})) = \int_{SO(3)} \left(Z(n, g\Omega_n) - \sigma(g\Omega_n)N_n\right)^2 dg =$$
$$E[Z^2] - 2\sigma(\Omega_n)N_n E[Z] + (\sigma(\Omega_n)N_n)^2 = E[Z^2] - (\sigma(\Omega_n)N_n)^2.$$

Next,

$$E[Z^2] = \sum_{i=1}^{N_n} E[\xi_i^2] + \sum_{\substack{1 \leqslant i,j \leqslant N_n \\ i \neq j}} E[\xi_i \xi_j] = \sigma(\Omega_n)N_n + \sigma^2(\Omega_n)N_n(N_n - 1)$$

since ξ_i and ξ_j are independent. Thus,

$$V(n; \Omega_n) = \sigma(\Omega_n)N_n - \sigma^2(\Omega_n)N_n \sim \sigma(\Omega_n)N_n$$

as soon as $\sigma(\Omega_n) \ll 1$.

9.2 Variance estimation

For fixed n, we choose a spherical cap Ω_n of spherical radius ρ on the sphere of radius $\sqrt{n}$, and denote the area of the cap by $\sigma(\Omega_n)$. Consider the point-pair invariant

$$K(\mathbf{x},\mathbf{y}) = \begin{cases} 1, \text{if } \mathbf{y} \in \Omega_n(\mathbf{x}); \\ 0, \text{otherwise.} \end{cases}$$

It has a Fourier series expansion of the form

$$K(\mathbf{x},\mathbf{y}) = \sum_{m=0}^{+\infty} h(m) \sum_{j=1}^{2m+1} \phi_{j,m}(\mathbf{x})\phi_{j,m}(\mathbf{y}).$$

The functions $\phi_{j,m}$ form an orthonormal basis with respect to an inner product

$$\langle f, g\rangle = \int_{S^2} f(\mathbf{x})\overline{g(\mathbf{x})}d\mu(\mathbf{x}),$$

and the coefficients $h(m)$ are given by Selberg-Harish-Chandra transform (see, [59] or [42]):

$$h(m) = 2\pi \int_0^{\pi} P_m(\cos\theta)k(\cos\theta)\sin\theta d\theta.$$

Here

$$P_m(x) = \frac{1}{2^m m!}\frac{d^m}{dx^m}(x^2-1)^m$$

are the Legendre polynomials and

$$k(\cos\theta) = \begin{cases} 1, \text{if } |\theta| \leqslant \rho; \\ 0, \text{otherwise} \end{cases}$$

is the point pair invariant written in the spherical coordinates.

Again, let us denote the lattice points as $\mathbf{x}_1, \dots, \mathbf{x}_{N_n}$. We next compute the variance as follows:

$$V\big(n, \Omega_n(\mathbf{x})\big) = \int_{S^2}\left(\sum_{i=1}^{N_n} K(\mathbf{x},\mathbf{x}_i) - N_n\sigma(\Omega_n)\right)^2 d\mu(\mathbf{x}) =$$

$$\int_{S^2}\left(\sum_{i=1}^{N_n}\sum_{m=0}^{+\infty} h(m)\sum_{j=1}^{2m+1}\phi_{j,m}(\mathbf{x})\phi_{j,m}(\mathbf{x}_i) - N_n\sigma(\Omega_n)\right)^2 d\mu(\mathbf{x}).$$

Note that

$$\sum_{i=1}^{N_n} h(0)\phi_{0,0}(\mathbf{x})\phi_{0,0}(\mathbf{x}_i) = N_n\sigma(\Omega_n).$$

Indeed, one can easily verify that

$$h(0) = 2\pi \int_0^{\pi} k(\cos\theta)\sin\theta d\theta = 2\pi \int_0^{\rho} \sin\theta d\theta = 2\pi(1-\cos\rho) = 4\pi\sigma(\Omega_n),$$

$$\phi_{0,0}(\mathbf{x}) = \frac{1}{2\sqrt{\pi}}.$$

Then we get

$$V\big(n,\Omega_n(\mathbf{x})\big) = \sum_{m_1=1}^{+\infty}\sum_{m_2=1}^{+\infty} h(m_1)h(m_2)\sum_{j_1=1}^{2m_1+1}\sum_{j_2=1}^{2m_2+1} W_{\phi_{j_1,m_1}}(n)W_{\phi_{j_2,m_2}}(n)\cdot \int_{S^2}\phi_{j_1,m_1}(\mathbf{z})\phi_{j_2,m_2}(\mathbf{z})d\mu(\mathbf{z}),$$

where

$$W_{\phi_{j,m}}(n) = \sum_{i=1}^{N_n}\phi_{j,m}(\mathbf{x}_i)$$

is the Weyl sum. By the orthogonality of spherical harmonics, we deduce

$$V\big(n,\Omega_n(\mathbf{x})\big) = \sum_{m=1}^{+\infty} h^2(m)\sum_{j=1}^{2m+1}\big|W_{\phi_{j,m}}(n)\big|^2. \tag{9.1}$$

Then by Jacquet-Langlands lift, we have

$$\big|W_{\phi_{j,m}}(n)\big|^2 \leqslant \frac{c\sqrt{n}L\big(\frac{1}{2},f_{j,m}\big)L\big(\frac{1}{2},f_{j,m}\otimes\chi_{-n}\big)}{L\big(1,\operatorname{Sym}^2 f_{j,m}\big)}$$

for squarefree n with an absolute constant $c>0$ independent of m, n, and $\phi_{j,m}$.

Next, we evaluate the expression

$$\sum_{m=1}^{+\infty} h^2(m)\sum_{j=1}^{2m+1}\frac{c\sqrt{n}L\big(\frac{1}{2},f_{j,m}\big)L\big(\frac{1}{2},f_{j,m}\otimes\chi_{-n}\big)}{L\big(1,\operatorname{Sym}^2 f_{j,m}\big)}$$

from above separately for small and large values of m. Let

$$M := \frac{1}{\sqrt{\sigma(\Omega_n)}}\exp\bigg\{-\frac{\log n}{\log\log n}+\frac{1}{2}\frac{\log\sigma(\Omega_n)}{\log\log n}\bigg\}. \tag{9.2}$$

First, consider the case $m \leqslant M$. By Hilb's formula (see [42] or [90, Theorem 8.21.6]),

$$P_m(\cos\theta) = \sqrt{\frac{\theta}{\sin\theta}}J_0\bigg(\Big(m+\frac{1}{2}\Big)\theta\bigg) + \begin{cases} O(\theta^2) & \text{for } m \leqslant \frac{1}{\theta}, \\ O_\varepsilon\big(\frac{\sqrt{\theta}}{m^{3/2}}\big) & \text{for } m \geqslant \frac{1}{\theta} \geqslant \frac{1}{\pi-\varepsilon} \end{cases} \tag{9.3}$$

for fixed $\varepsilon > 0$, $0 < \theta < \pi - \varepsilon$, and

$$J_0(x) = \begin{cases} 1 + O(x^2) & \text{for } |x| \leqslant 1, \\ \sqrt{\frac{2}{\pi|x|}} \cos\bigl(|x| - \frac{\pi}{4}\bigr) + O\bigl(\frac{1}{|x|^{3/2}}\bigr) & \text{for } |x| \geqslant 1. \end{cases} \tag{9.4}$$

For $m \leqslant M$ with the chosen value of M, we get $m \ll (\sigma(\Omega_n))^{1/2} \sim 1/\rho$, and hence by (9.3)

$$h(m) = 2\pi \int_0^\rho P_m(\cos\theta) \sin\theta d\theta =$$
$$\int_0^\rho \sqrt{\frac{\theta}{\sin\theta}} J_0\biggl(\Bigl(m + \frac{1}{2}\Bigr)\theta\biggr) \sin\theta d\theta + O\biggl(\int_0^\rho \theta^2 \sin\theta d\theta\biggr),$$

and since $(m + 1/2)\theta \leqslant (M + 1/2)\rho \ll 1$ by (9.4), we get

$$h(m) \ll \int_0^\rho \sqrt{\theta \sin\theta} d\theta + O\biggl(\int_0^\rho m^2\theta^2 \sqrt{\theta\sin\theta} d\theta\biggr) + O\biggl(\int_0^\rho \theta^2 \sin\theta d\theta\biggr) \ll$$
$$\rho^2 + O(m^2\rho^4) + O(\rho^4) \ll \sigma(\Omega_n).$$

Together with the conditional pointwise bound,

$$\frac{L\bigl(\frac{1}{2}, f_{j,m}\bigr) L\bigl(\frac{1}{2}, f_{j,m} \otimes \chi_{-n}\bigr)}{L\bigl(1, \mathrm{Sym}^2 f_{j,m}\bigr)} \ll \exp\biggl\{\frac{\log mn}{\log\log n}\biggr\}$$

that finally gives

$$\sum_{m \leqslant M} h^2(m) \sum_{j=1}^{2m+1} \frac{c\sqrt{n} L\bigl(\frac{1}{2}, f_{j,m}\bigr) L\bigl(\frac{1}{2}, f_{j,m} \otimes \chi_{-n}\bigr)}{L\bigl(1, \mathrm{Sym}^2 f_{j,m}\bigr)} \ll$$
$$\sqrt{n} \exp\biggl\{\frac{\log Mn}{\log\log n}\biggr\} \sum_{m \leqslant M} m h^2(m) \ll \sqrt{n} \exp\biggl\{\frac{\log Mn}{\log\log n}\biggr\} \sigma(\Omega_n)^2 M^2. \tag{9.5}$$

Now consider the case $m > M$. Applying Theorem 1.7, we get

$$\sum_{m > M} h^2(m) \sum_{j=1}^{2m+1} \frac{c\sqrt{n} L\bigl(\frac{1}{2}, f_{j,m}\bigr) L\bigl(\frac{1}{2}, f_{j,m} \otimes \chi_{-n}\bigr)}{L\bigl(1, \mathrm{Sym}^2 f_{j,m}\bigr)} \ll$$
$$\sqrt{n} \sum_{m > M} h^2(m)(2m + 2) L(1, \chi_{-n}) \exp\bigl\{U(n, M)\bigr\}$$

which can be further bounded from above by

$$2\sqrt{n} L(1, \chi_{-n}) \exp\bigl\{U(n, M)\bigr\} \sum_{m=1}^{+\infty} h^2(m)(2m + 1).$$

Next, apply the formula

$$\sum_{m=1}^{+\infty} h^2(m)(2m+1) = \sigma(\Omega_n)$$

which follows from the following computation:

$$\sigma(\Omega_n(\mathbf{x})) = \int_{S^2} \Big(\mathbb{1}_{\mathbf{z}\in\Omega(\mathbf{x})}(\mathbf{z}) \Big)^2 d\mu(\mathbf{z}) = \int_{S^2} \Big(\sum_{m=0}^{+\infty} h(m) \sum_{j=1}^{2m+1} \phi_{j,m}(\mathbf{z})\phi_{j,m}(\mathbf{x}) \Big)^2 d\mu(\mathbf{z}) =$$

$$\sum_{\substack{m_1,m_2\\ j_1,j_2}} h(m_1)h(m_2)\phi_{j_1,m_1}(\mathbf{x})\phi_{j_2,m_2}(\mathbf{x}) \int_{S^2} \phi_{j_1,m_1}(\mathbf{z})\phi_{j_2,m_2}(\mathbf{z}) d\mu(\mathbf{z}) =$$

$$\sum_{m=0}^{+\infty} h^2(m) \sum_{j=1}^{2m+1} 1 = \sum_{m=0}^{+\infty} (2m+1)h^2(m).$$

Thus, together with (9.5), it gives

$$\sum_{m=1}^{+\infty} h^2(m) \sum_{j=1}^{2m+1} \frac{c\sqrt{n}L\big(\frac{1}{2}, f_{j,m}\big)L\big(\frac{1}{2}, f_{j,m}\otimes\chi_{-n}\big)}{L\big(1, \mathrm{Sym}^2 f_{j,m}\big)} \ll$$

$$\sqrt{n}\exp\left\{\frac{\log Mn}{\log\log n}\right\}\sigma(\Omega_n)^2 M^2 + 2\sqrt{n}L(1,\chi_{-n})\exp\big\{U(n,M)\big\}\sigma(\Omega_n).$$

Hence

$$V(n,\Omega_n) \ll \sqrt{n}\sigma(\Omega_n)\left(\sigma(\Omega_n)M^2\exp\left\{\frac{\log Mn}{\log\log n}\right\} + L(1,\chi_{-n})\exp\big\{U(n,M)\big\}\right).$$

With the choice of M given by (9.2), we have

$$\sigma(\Omega_n)M^2\exp\left\{\frac{\log Mn}{\log\log n}\right\} \ll 1.$$

Next, if $\sigma(\Omega_n) = n^{-A}$ for some fixed real number $A > 0$, the bound of Theorem 1.7 gives

$$U(n,M) \ll_A 1.$$

Then, finally

$$V(n,\Omega_n) \ll \sqrt{n}\sigma(\Omega_n)L(1,\chi_{-n}) = \sigma(\Omega_n)N_n$$

as desired.

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