Black holes and entanglement entropy

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ABSTRACT

We study the deformation of the thermofield-double (TFD) under evolution by a double-traced operator by computing its entanglement entropy. After saturation, the entanglement change leads to the temperature change. In Jackiw-Teitelboim gravity, the new temperature can be computed independently from two-point function by considering the Schwarzian modes. We will also derive the entanglement entropy from the Casimir associated with the SL(2,R) symmetry. From AdS/CFT correspondence, where TFD is dual to a two-sided black hole, such deformations correspond to the *coherent* shrinking or expansion of the black hole.

Next, we compute the entanglement entropy after coupling a system to the bath perturbatively as a function of κ, the system-bath coupling. At very early times where the entanglement entropy is a logarithmic function of time, the leading contribution is due to the terms of order 2s in the coupling where s is the number of replicas. In the middle time, the entanglement goes linear as a function of time. Assuming saturation at a later time, we will study the effect of an external perturbation to the entropy at an early time where it is related to the OTOCs. A major simplification appears when the system saturates the chaos bound.

PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Pouria Dadras and Alexei Kitaev. "Perturbative calculations of entanglement entropy". In: (Nov. 2020). DOI: 10.1007/JHEP03(2021)198. eprint: 2011.09622. URL: https://arxiv.org/pdf/2011.09622.pdf.
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[2] Pouria Dadras. "Disentangling the Thermofield-double state". In: (May 2019). eprint: 1905.02305. URL: https://arxiv.org/pdf/1905.02305.pdf.
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Chapter 1

INTRODUCTION

Soon after Albert Einstein developed the general theory of relativity, an exact solution to Einstein's equation was found by Karl Schwarzschild, known as the Schwarzschild metric [41, 15]. The metric describes a vacuum solution in asymptotically flat spacetime with spherical symmetry. However, it exhibits a peculiar behavior at the Schwarzschild radius, where some metric components become singular. This hypersurface is called the event horizon and the inside region is called the black hole. The event horizon has the unusual property that, according to an outside observer, it will take an infinite time for an arbitrary object to reach the horizon, while the particle itself only experiences a finite amount of time. In addition, an event horizon is like a one-way membrane; namely, an object which has already passed the horizon is never able to reach out to the outside, and ultimately ends up hitting the physical singularity located at the black hole's center. Nevertheless, the singularity on the black hole's event horizon is not physical and can be removed by choosing an appropriate coordinate system, e.g. the Eddington-Finkelstein (EF) coordinate system.

A somewhat surprising fact about the Schwarzschild and EF coordinate systems is that they only describe a portion of the spacetime. In other words, it is possible to opt for a coordinate system, for example, the Kruskal-Szekeres coordinate, that covers the whole spacetime where the geodesics parametrized by the affine parameter either extend to infinity or terminate by hitting a physical singularity. It turns out that the maximally extended spacetime, in addition to the black hole region, also has a white hole region (its time reversal), and they indeed have two sides which are connected by a non-traversable wormhole called the Einstein-Rosen (ER) bridge. Such extended horizons are called the bifurcate horizons.

There are also other solutions to the Einstein's equation corresponding to the rotating and charged black holes. Such solutions, although more complicated, capture many features of the Schwarzschild black hole. For example, the maximally extended spacetime contains a bifurcate horizon. Indeed, as is pointed out by Racz and Wald [36, 37], any stationary spacetime which has a "one sided black hole" but no white hole with its Killing vector's orbits to be diffeomorphic to $\mathbb{R}$ can always be locally extended to a spacetime with a bifurcate horizon provided that the horizon's surface gravity is a nonzero constant, see Appendix C. Such horizons are usually identified by their mass, electric charge, and angular momentum [19, 18, 7]. Further developments by Hawking, Carter, and Bardeen [5] showed that assuming the spacetime is a solution to the Einstein's equation and the matter satisfies the dominant energy condition, it is possible to associate a well-defined temperature to the black hole's horizon. Moreover, the black hole satisfies the laws of thermodynamics with the energy of the system being equal to the black hole's mass and entropy proportional to the horizon's area.

A powerful tool to understand the nature of bifurcate horizons is the AdS/CFT correspondence [31, 42] which conjectures that, roughly speaking, a gravity theory in the bulk of Anti de Sitter (AdS) spacetime, which may include black holes as well, is equivalent to a specific conformal field theory living on the boundary. As a part of the dictionary between the bulk and the boundary [32], the existence of a two-sided black hole in the bulk is dual to the case where two copies of the boundary theory act on each side of the thermofield-double state (TFD) prepared as the initial state [1] :

$$|\text{TFD}\rangle = \frac{1}{\mathcal{Z}^{1/2}} \sum_n e^{-\frac{\beta E_n}{2}} |E_n^*\rangle_L |E_n\rangle_R, \tag{1.1}$$

where $|E_n\rangle_R$s are the energy eigenstates [2] of the right Hamiltonian, $|E_n^*\rangle_L$ are the eigenstates of H_R^*, and β is the black hole's inverse temperature. Note that TFD is invariant under the symmetry generated by $1 \otimes H_R - H_R^* \otimes 1 \equiv H_R - H_L$. A direct computation shows that for "large" AdS-Schwarzschild black holes in d+1 dimensions ($r_H \gg \ell_{AdS}$), the dependence of black hole's entropy on temperature is:

$$S_B(T) \propto T^{d-1}. \tag{1.2}$$

The AdS/CFT duality implies that the black hole's partition function is equal to that of the boundary theory and, consequently, the entanglement entropy associated with one side equals the black hole's entropy in the bulk. Entanglement, in general, has a non-local quantum nature; the action of a unitary operator on one side of an entangled state does not change the amount of entanglement between the two sides of the state.

However, it is possible to change the states' entanglement entropy by coupling their two sides. The simplest example capturing this idea is the Bell pair:

$$\frac{|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle}{\sqrt{2}} \tag{1.3}$$

with entanglement entropy equal to $\ln 2$. While the entanglement entropy remains unchanged under a local unitary operator, it is easy to construct a unitary operator which transforms the state to a pure state $|\uparrow\uparrow\rangle$.

Similarly, one expects the entanglement entropy of the TFD to change under the unitary evolution $U(t) = e^{-iHt}$ where the Hamiltonian is[3]:

$$H = H_0 + H_{int}, \qquad H_{int} = \frac{g}{N} \sum_{i=1}^{N} \int d^{d-1}x \; \phi_L^i(\vec{x}) \phi_R^i(\vec{x}), \tag{1.4}$$

[1] The correspondence between thermofield-double state and a two-sided black hole was first observed by Israel in asymptotically flat spacetime [17].

[2] For rotating black holes the states are labeled by energy and angular momentum eigenvalues.

[3] This Hamiltonian was used in [12] as a model for traversable wormholes; see also [29, 25]

where $\phi_{L(R)}$ acts on the left (right) side of the black hole in the bulk. To compute the amount of entanglement change, we will prepare the TFD at $t = 0$ and evolve it with U(t):

$$|\widetilde{\text{TFD}}(t)\rangle = U(t)\ |\text{TFD}\rangle. \tag{1.5}$$

The entanglement entropy can be computed using the replica trick:

$$S = -\partial_s \ln \text{Tr}\,\rho^s\Big|_{s=1}. \tag{1.6}$$

Direct computation showing ΔS_{EE} to leading order in g is:

$$\Delta S_{EE}(t) = ig\mathcal{S}_{d-1}\int_0^t du \quad \frac{d}{ds}\left(G_{s\beta}\Big(2iu+\frac{\beta}{2}\Big) - G_{s\beta}\Big(-2iu+\frac{\beta}{2}\Big)\right)\Bigg|_{s=1}, \tag{1.7}$$

where $G_{s\beta}\Big(2iu+\frac{\beta}{2}\Big) = \Big\langle \phi^i(u,\vec{x})\phi^i(-u+i\frac{\beta}{2},\vec{x})\Big\rangle_{s\beta}$ is the two-point function at inverse temperature $s\beta$. Notice that the two point function in 1.7 is space independent, and so the integral over the spacial part gives the volume of the transverse direction, denoted by $\mathcal{S}_{d-1}$. As a result, since the entropy of TFD is a function of its temperature, i.e. relation 1.2, one expects the temperature to change, and so does the black hole's size [4]. On the other hand, when the system reaches the equilibrium, the new temperature associated with $\widetilde{\text{TFD}}$ can be read from the two- point function whose computation needs information about the theory's higher point functions.

This observation motivated us to study the dynamics of $\widetilde{\text{TFD}}$ explicitly for a simple model, the Jackiw-Teitelboim (JT) gravity [40, 20] (see chapter 4). The JT gravity appears as the near horizon limit of the four dimensional charged black holes. The bulk is fixed to be AdS_2, and the dynamical degrees of freedom correspond to the reparametrization of the boundary whose dynamics is given by the Schwarzian action (see Chapter 2). The action initially appeared as the low energy limit of the SYK model [23, 27, 38]. Computing the quantity 1.7 in this model yields 4.14:

$$\Delta S_{EE}(t) = \frac{\pi b g}{2J}\left(\frac{\pi}{\beta J}\right)^{2\Delta-1}\left(1-\frac{1}{\left(\cosh(\frac{2\pi t}{\beta})\right)^{2\Delta}}\right), \tag{1.8}$$

which, after $t \sim \frac{2\pi}{\beta}$, implies the new temperature (4.16):

$$\tilde{\beta} = \beta\left(1-\frac{\pi b g}{2JS}\left(\frac{\pi}{\beta J}\right)^{2\Delta-1}\right). \tag{1.9}$$

On the other hand, the temperature of $\widetilde{\text{TFD}}$ can be read from the two-point function after reaching the equilibrium. In computing the two-point function to leading order in the coupling, the four-point function of the theory is needed. However, since 1.8 and 1.9 are general and independent of the fields,

[4] For simplicity, we assume the black hole is parametrized by one parameter, its mass.

the contributing modes must be generic and couple to all the fields, which in our case are the Schwarzian modes. Such modes are, indeed, the Goldeston modes assocaited with spontanuous breaking of the reparametrization symmetry. In fact, the computation of the two-point function in section 4 confirms this.
Our microscopic computation of the entanglement change 1.8 can be rederived from a coarse-grained quantity, the Casimir associated with the $SL(2,\mathbb{R})$ generators 2.37. More precisely, the entanglement entropy, 4.44, to leading order in the coupling can also be computed [5] from [22]

$$S(u) = 2\pi\sqrt{Q_R(u)}, \tag{1.10}$$

which is computed over the solution to the equation of motion 4.37. The exact match between the two quantities to the second order in the coupling 4.23 and 4.44 may confirm that 1.10 renders the entanglement entropy dynamics after the quench by the interaction Hamiltonian 1.4. Assuming so, we can get the following physical picture: In general, one expects the coarse grained entropy to be bigger than or equal to the entanglement entropy for typical states. In our case, for example, perturbing the TFD with a local positive Hamiltonian will increase the black hole's entropy, while its entanglement entropy remains unchanged. Our results show that 1.4 acts *coherently*. While our computation is for a two-dimensional model, 1.7 is true at any dimension leading us to conjecture that there are soft modes in higher dimensions, similar to the Schwarzian modes which are responsible for temperature change and also entanglement change at higher orders. Understanding such modes may be significant in developing a constructive holographic theory.

Another problem that we will study in this thesis is the evolution of the entanglement entropy when we couple two "many-body" pure states, which represents generic features of information transfer between a system coupled to the bath. While not an observable, entropy is useful as an abstract measure of active degrees of freedom and correlations between the system and bath. The general form of such correlations was predicted by Don Page [34]. Relevant quantities that play an important role in the process of information transfer are OTOCs [39]. However, their physics is relevant on short time scales and explains correlations present not in the radiation itself but relative to a purifying system [16]. The recent breakthrough in understanding the correlations developing over the Page time [3, 35] required a careful formulation of the problem, which we will now summarize.

We consider the entanglement entropy between the system and the emitted radiation at a particular time t. So let $\rho = \rho(t)$ be the system's density matrix; we will compute the von Neumann entropy using 1.6. For integer s, the expression $\operatorname{Tr}\rho^s$ may be interpreted as the partition functions of s replicas of the system.

Now, it turns out that the transition from the early phase (when the radiation is uncorrelated as the naive theory predicts) to the later phase (when the entanglement entropy equals the system's coarse-grained

[5] I am grateful to Alexei Kitaev for pointing this out to me.

entropy) is first order. The later phase corresponds to a new type of spacetime geometry, the replica wormhole [3, 35]. Although choosing the correct solution of the two is a global problem, each of them can be examined locally. We will study some properties of both solutions for general many-body systems.

The von Neumann and Renyi entropies are nonlinear functions of the quantum state, which is why they are not observables. However, the logarithmic nonlinearity is mild, such that in the thermodynamic limit, $S(\rho)$ is determined by typical microstates that contribute to the mixed state ρ. In contrast, Renyi entropies are often dominated by a fraction of microstates of tiny overall weight. This distinction is also evident from the replica wormhole picture. The s-Renyi entropy is related to an s-fold cover of spacetime, whose metric is different from the physical one. But when we analytically continue the solution in s and take s to 1, we get the standard metric with an additional piece of data, the branching surface. Thus, the $s \to 1$ limit is essential for compatibility with the usual (non-entropic) physics. Our main technical advance is how to take this limit in some specific cases.

To study the early phase of the entanglement growth, we adopt a simple variant of the problem, where instead of radiating energy, the system comes into contact with a heat bath at the same temperature. Turning the system-bath interaction on represents a slight change in the Hamiltonian and results in a brief period of non-equilibrium dynamics. Then a steady state is achieved such that all simple correlation functions are thermal. However, if the system's initial state was pure (though mimicking the thermal state), its von Neumann entropy will grow at a constant rate. We focus on this regime as well as the very beginning of quantum evolution. The entropy growth eventually saturates at the thermal (i.e. coarse-grained) entropy, but that is not captured by our method.

Our calculation is perturbative in the system-bath coupling strength κ. Note that the von Neumann entropy has a logarithmic singularity at the unperturbed state, which is pure. This is reflected by the fact that in addition to terms of order κ^2 (or any constant power of κ), terms of order κ^{2s} (where s is the number of replicas) play an important role.

More precisely, we consider two copies of thermofield-double $|\mathrm{TFD}\rangle_B \otimes |\mathrm{TFD}\rangle_b$ with inverse temperature β associated with the system, denoted by B, and the bath, denoted by b and model the interaction by:

$$H_{Bb} = \kappa \sum_{i=1}^{N} O_B^i O_b^i. \tag{1.11}$$

Then the initial growth of the entanglement entropy will be described by 5.15:

$$\ln\Big(\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s\Big) = N\Big(s\kappa^2\ \mathrm{Tr}\big(\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big) + \kappa^{2s}\,\mathrm{Tr}\Big(\big(-\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big)^s\Big)\Big). \tag{1.12}$$

At very early time, the above function behaves as:

$$S\big(\rho_{\mathrm{B^*B}}(t)\big) \approx c\kappa^2 t^2 \left(-\ln(c\kappa^2 t^2) + 1\right). \tag{1.13}$$

In the intermediate time, for systems with continuous excitation spectrum such as conformal field theories, the entanglement entropy will take the following form:

$$\begin{aligned} &S\Big(\rho_{\mathrm{B^*B}}(t)\Big) \approx -NA'(1)\, t, \\ &A'(1) = \kappa^2 \int \tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\Big(-\ln\Big(-\kappa^2 \tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\Big) + 1\Big)\frac{d\omega}{2\pi}. \end{aligned} \tag{1.14}$$

Next, we will consider the system at the saturated phase. We would like to study how the system's radiation to the bath will be affected by a small perturbation. For that, we will model the perturbation by a unitary operator acting on the system's density matrix ρ_0

$$\rho_1 = V\rho_0 V^{\dagger}. \tag{1.15}$$

We also model the system's radiation by a superoperator R, so that the final density matrix is:

$$\rho = R(\rho_1). \tag{1.16}$$

Our interest is computing the entanglement entropy $S(\rho)$ from the Renyi entropy. It turns out that at early time, the computation of entanglement entropy is dominated by OTOCs through the following quantity:

$$\langle \mathbf{B}, A_j(\beta + it), X(\beta), X(0), A_k^{\dagger}(it)\rangle, \tag{1.17}$$

where the operator X is the generator of the unitary V, and the operator A_is are the constituents of the superoperator R; see section 5.1. We will compute this quantity and discuss its possible relation to holography.

The thesis is organized as follows: In Chapter 2, we will study the near horizon limit of a charged black hole in four dimensions and derive the JT theory. Then we study the Schwarzian action, its properties, and its contribution to the four-point function. In Chapter 3, we will give an overview of the SYK model. especially, We study how the Schwarzian action can be derived as its low energy effective action. In Chapter 4, we will compute the entanglement entropy after deforming the TFD with the double traced operator and study various aspects, and finally, in Chapter 5, we study the entanglement dynamics when we couple a system to the bath. In Appendix A, we will briefly mention AdS_2 in different coordinates. Appendix B is devoted to the construction of TFD and its properties. Finally, in Appendix C, we will review the Racz-Wald construction of bifurcate horizon.

Chapter 2

GEOMETRY OF THE CHARGED BLACK HOLES IN FOUR DIMENSIONS

2.1 The charged black holes

The Reissner-Nordstrom (RN) black holes are the static solutions to the Einstein-Maxwell theory

$$\mathcal{S} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-g}\,\Big[\mathcal{R} - \mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Big], \qquad \mathcal{F}_{tr} = \frac{Q}{r^2} \tag{2.1}$$

with the following metric:

$$\begin{aligned} ds^2 &= -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ f(r) &= \Big(1 - \frac{r_+}{r}\Big)\Big(1 - \frac{r_-}{r}\Big), \qquad r_\pm = M \pm \sqrt{M^2 - Q^2}. \end{aligned} \tag{2.2}$$

The metric 2.2 has two types of singularities, one located at $r = r_\pm$, known as inner and outer horizon, and the other at $r = 0$. While the second is physical, i.e. observer-independent, the singularities at $r_\pm$ are coordinate dependent and can be removed by changing the coordinate system. For example, if we take

$$\begin{aligned} V &= \exp\Big(\frac{f'(r_+)(t+r)}{2}\Big)\Big(\frac{r}{r_+} - 1\Big)^{\frac{1}{2}}\Big(\frac{r}{r_-} - 1\Big)^{-\frac{\alpha}{2}} \\ U &= -\exp\Big(\frac{f'(r_+)(-t+r)}{2}\Big)\Big(\frac{r}{r_+} - 1\Big)^{\frac{1}{2}}\Big(\frac{r}{r_-} - 1\Big)^{-\frac{\alpha}{2}} \\ \alpha &= \Big(\frac{r_-}{r_+}\Big)^2, \end{aligned} \tag{2.3}$$

we can remove the singularity at $r = r_+$. The Hawking temperature associated to the outer horizon is equal to

$$T = \frac{f'(r_+)}{4\pi} = \frac{\Delta}{4\pi r_+^2}, \quad \Delta = r_+ - r_-. \tag{2.4}$$

To proceed, it is suitable to define r_E so that $r_\pm = r_E \pm \frac{\Delta}{2}$. In particular, we are interested in the near extremal limit where $\frac{\Delta}{r_E} \ll 1$ and can be regarded as the perturbation parameter. Perturbation around r_E leads to:

$$ds^2 = -\Big(1 - \frac{r_E}{r}\Big)^2 dt^2 + \frac{dr^2}{\Big(1 - \frac{r_E}{r}\Big)^2} + r^2\,d\Omega + O\Big(\big(\frac{\Delta}{r_E}\big)^2\Big). \tag{2.5}$$

For $\frac{r - r_E}{r_E} \ll 1$, the near horizon limit, the metric will take the form :

$$ds^2 \approx -\frac{1}{r_E^2}\frac{dt^2}{z^2} + \frac{r_E^2 dz^2}{z^2} + r^2 d\Omega. \tag{2.6}$$

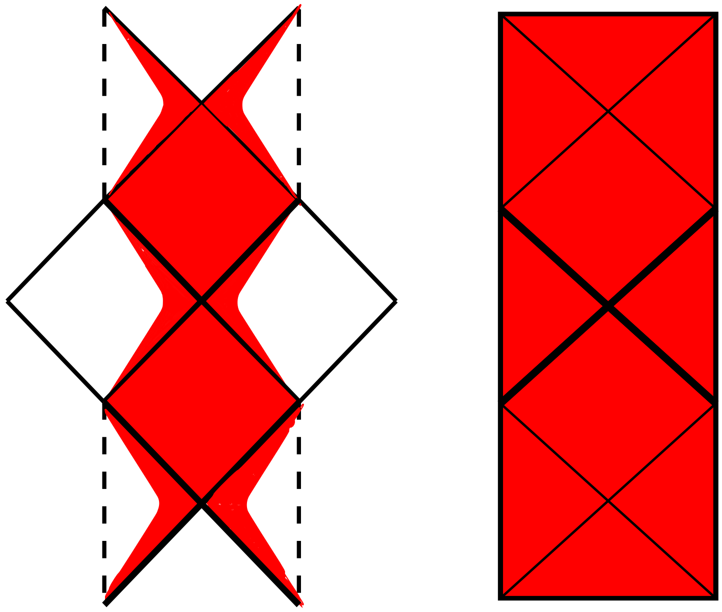


Figure 2.1: Penrose diagram of a near extremal charged black hole. (a) The near horizon limit $\frac{r_+ - r_-}{r_+} \ll 1$ is colored red.(b) This region has an approximate constant negative curvature and can be mapped global AdS_2 spacetime

Therefore, the above has the geometry of $AdS^2 \times S^2$. The Bekenstein-Hawking entropy associated to the black hole is proportional to the area of the black hole's horizon. In the near extremal limit it will take the form

$$S = \frac{\pi r_+^2}{G_N} = \frac{\pi r_E^2}{G_N} + \frac{4\pi^2 r_E^3}{G_N} T + \cdots , \tag{2.7}$$

where the first term is the zero temperature entropy associated with the extremal black hole and the second term is due to the $O\left(\frac{\Delta}{r_E}\right)$ correction to the horizon's area. On the other hand, for $r \gg r_E$ the spacetime becomes flat.Therefore, as we move from the outer horizon to infinity, one can think that the geometry will change from $AdS_2 \times S^2$ to the flat spacetime $R^{1,3}$.

2.2 The spherical reduction and the JT gravity

To study the near horizon limit of the black hole[1] we will take the metric to be

$$ds^2 = g_{mn} dx^m dx^n + e^{2\phi} d\Omega, \qquad m, n = 0, 1. \tag{2.8}$$

The four dimensional scalar curvature will take the form:

$$\mathcal{R} = R - 6\left(\nabla\phi\right)^2 - 4\nabla^2\phi + 2e^{-2\phi}. \tag{2.9}$$

Plug it into 2.1 with

$$\mathcal{F}_{mn} = Q\ e^{-2\phi}\ \epsilon_{mn} \tag{2.10}$$

[1]Here we follow [11].

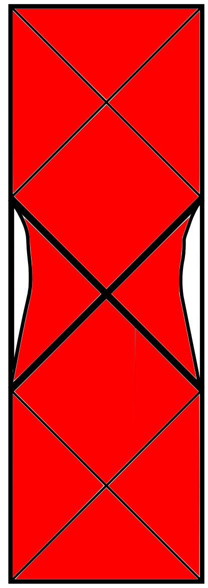


Figure 2.2: Here we assume that there is a sharp cutoff that separates the AdS region from the rest of the spacetime. This boundary can have arbitrary fluctuations

gives :

$$S = \frac{1}{4G} \int d^2x\sqrt{-g}\, e^{2\phi} \left[R + 2\left(\nabla\phi\right)^2 + 2e^{-2\phi} - 2Q^2 e^{-4\phi} \right]. \tag{2.11}$$

The near extremal limit corresponds to expanding the terms around $e^{\phi_s} = r_E \approx Q$. Dropping the higher order terms in ϕ, and rescaling $g_{mn} \to Q^2 g_{mn}$:

$$S = \frac{\phi_s}{16\pi G}\left[\int d^2x\sqrt{-g}\, R \ + 2\int \mathcal{K} d\ell\right] + \frac{1}{16\pi G}\left[\int d^2x\sqrt{-g}\left(R+2\right)\phi + 2\int \phi_b \mathcal{K} d\ell\right]. \tag{2.12}$$
$$\phi_s = 4\pi Q^2.$$

where ϕ_s is the zero temperature entropy. We also added the Hawking-Gibbons term to have a consistent variation. This action is called the Jackiw-Teitelboim (JT) theory [40, 20, 2]. The first term is completely topological and contributes to the zero temperature entropy. The second term renders the dynamics.The dynamical action in the Euclidean time is given by:

$$S = -\frac{1}{16\pi G}\left[\int d^2x\sqrt{g}\left(R+2\right)\phi + 2\int \phi_b \mathcal{K} d\ell\right] + \mathcal{I}_M. \tag{2.13}$$

We also added the matter field. Variation with respect to the dilaton field yields:

$$R + 2 = 0, \tag{2.14}$$

while the variation with respect to the metric gives the equation of motion for the dilaton field:

$$\nabla_\mu \nabla_\nu \phi - g_{\mu\nu}\nabla^2\phi + \phi g_{\mu\nu} = 8\pi G\, T_{\mu\nu}. \tag{2.15}$$

Suppose there is no matter field. Intergating out the dilaton, only the Hawking-Gibbons term remains.

$$-\frac{1}{8\pi G}\int \phi_b\,\mathcal{K}d\ell \tag{2.16}$$

For the case of the zero temperature which corresponds to the Poincare half plane,

$$ds^2=\frac{dt^2+dz^2}{z^2}. \tag{2.17}$$

The boundary of this space is located at $z=0$, and as we approach it, the affine length will grow as $\frac{1}{z}$. Therefore, it is convenient to define the regularization parameter ε and define the "physical boundary" to be parametrized by the affine parameter u so that

$$\frac{t'^2+z'^2}{z^2}=\frac{1}{\varepsilon^2}, \tag{2.18}$$

where derivative is with respect to u. To leading order in ε, 2.18 implies that the equation for the physical boundary will take the form:

$$\Big(t(u),z(u)\Big)=\Big(t(u),\varepsilon t'(u)\Big). \tag{2.19}$$

Now, to compute the extrinsic curvature, we take $\vec{n}=\frac{z(z',-t')}{(t'^2+z'^2)^{\frac{1}{2}}}$. From the definition of the extrinsic curvature $K=\frac{\langle\gamma'(u),\mathcal{D}_u n\rangle}{\langle\gamma'(u),\gamma'(u)\rangle}$, and then we have:

$$\begin{aligned}\nabla_u n^t &= \frac{t'^2z'^2+t'^4+zz''t'^2-zz't't''}{(t'^2+z'^2)^{\frac{3}{2}}}\\ \nabla_u n^z &= \frac{z'^3t'+z't'^3-zt''z'^2+zt'z'z''}{(z'^2+t'^2)^{\frac{3}{2}}}\\ \Rightarrow \mathcal{K} &= \frac{t'\Big(t'^2+z'^2+zz''-zz't''/t'\Big)}{\Big(t'^2+z'^2\Big)^{\frac{3}{2}}}=1+\varepsilon^2\,\mathrm{Sch}(t,u).\end{aligned} \tag{2.20}$$

On the other hand, ϕ is field with dimension two, and close to the boundary, it behaves as $\phi_b=\frac{\phi_r(u)}{\varepsilon}$. We also have $d\ell=\frac{du}{\epsilon}$. We define the boundary as a curve on which the value of the dilaton is the constant ϕ_r. Plugging into 2.16, dropping the constant term in 2.21, we will get the regularized action for the Poincare patch [28]:

$$S=\frac{-\phi_r}{8\pi G}\int du\ \mathrm{Sch}(t(u),u), \tag{2.21}$$

where the integrand is the Schwarzian derivative $\mathrm{Sch}(t(u),u)=\left(\frac{t''}{t'}\right)'-\frac{1}{2}\left(\frac{t''}{t'}\right)^2$. The equation of motion together with 2.19 determines the location of the boundary. One can think of the variable u as the physical time and t as its arbitrary reparametrization. In the absence of the matter field in the bulk one can solve 2.22:

$$\phi(t,z)=\frac{\alpha(z^2+t^2)+\beta t+\gamma}{z}. \tag{2.22}$$

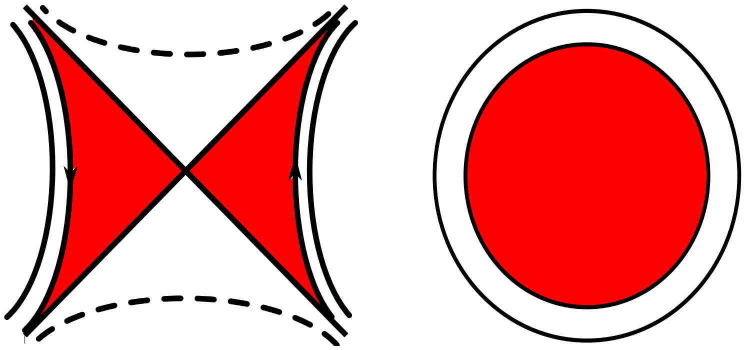


Figure 2.3

On the boundary, it will take the form:

$$\phi_r = \frac{\alpha t^2(u) + \beta t(u) + \gamma}{t'(u)}. \tag{2.23}$$

The solution associated to zero temperature is $t(u) = u$, $\phi(t, z) = \frac{\phi_r}{z}$. The wick rotated action 2.21 is invariant under the $SL(2, R)$ symmetry:

$$\mathrm{Sch}(\frac{at(u) + b}{ct(u) + d}, u) = \mathrm{Sch}(t(u), u). \tag{2.24}$$

The Noether procedure can be used to find out the conserved charges:

$$Q = \frac{2t''}{t'^2}\,\delta t' - \left(\frac{t''}{t'^2}\right)'\delta t - \frac{t''^2}{t'^3}\,\delta t - \frac{\delta t''}{t'}. \tag{2.25}$$

The charges associated with the $SL(2, R)$ generators $\delta t = 1, t, t^2$ are [28]

$$\begin{aligned} Q_{-1} &= \frac{t''^2}{t'^3} - \frac{t'''}{t'^2} \\ Q_0 &= t\left(\frac{t''^2}{t'^3} - \frac{t'''}{t'^2}\right) + \frac{t''}{t'} \\ Q_1 &= t^2\left(\frac{t''^2}{t'^3} - \frac{t'''}{t'^2}\right) + t\frac{2t''}{t'} - 2t'. \end{aligned} \tag{2.26}$$

Clearly, the solutions 2.22 and 2.23 break this symmetry. Therefore, we should take quotient with respect to such solutions, or equivalently, we can solve for the solution that satisfies $Q = 0$.

The finite temperature solution is associated with compactifying the time coordinate,

$$t = e^{i\varphi}. \tag{2.27}$$

Under this transformation, the action 2.21 becomes:

$$S = \frac{-\phi_r}{8\pi G}\int du\,\left(\mathrm{Sch}(\varphi(u), u) + \frac{1}{2}\varphi'^2\right). \tag{2.28}$$

We are interested in the solution

$$\varphi(u) = \frac{2\pi u}{\beta} \tag{2.29}$$

with the free energy given by:

$$F = -\frac{\pi\phi_r}{4G}\,T^2 \tag{2.30}$$

and other thermodynamic quantities equal to

$$S = S_0 + \frac{\pi\phi_r}{2G}\,T, \qquad E = \frac{\pi\phi_r}{4G}\,T^2. \tag{2.31}$$

Here, S_0 comes from the topological term in 2.12. The SL(2,R) charges in this coordinate will take the form:

$$\begin{aligned} Q_{-1} &= e^{-\varphi}\Big(-\frac{\varphi''}{\varphi'} - \frac{\varphi'''}{\varphi'^2} + \frac{\varphi''^2}{\varphi'^3}\Big) \\ Q_0 &= \varphi' + \frac{\varphi''^2}{\varphi'^3} - \frac{\varphi'''}{\varphi'^2} \\ Q_1 &= e^{\varphi}\Big(\frac{\varphi''}{\varphi'} - \frac{\varphi'''}{\varphi'^2} + \frac{\varphi''^2}{\varphi'^3}\Big). \end{aligned} \tag{2.32}$$

Note that the black hole has two sides. Therefore, the conserved charge is the sum of the above charges for the left and right sides. This is the thermofield-double solution TFD. To understand the geometry of such a configuration, we will go to the global coordinate. The boundary coordinate times are related by:

$$t = \tanh\frac{\pi u}{\beta} = \tan\frac{\eta_R}{2}, \qquad -\frac{\pi}{2} \le \eta_R \le \frac{\pi}{2}. \tag{2.33}$$

In global coordinates, the thermofield-double will be represented by

$$\Big(\eta\ ,\ \sigma_r\Big) = \Big(2\,\mathrm{arctan}\,\tanh\frac{\pi u}{\beta}\ ,\ \frac{\pi}{2} - \frac{2\pi\epsilon}{\beta}\frac{1}{\cosh\frac{2\pi u}{\beta}}\Big). \tag{2.34}$$

Defining $\eta' = e^{\phi}$ with the lagrange multiplier P_η the action will transform to

$$\begin{aligned} &-\int Sch\Big(\tan\frac{\eta}{2},\tilde{u}\Big)\,d\tilde{u} = S = \int d\tilde{u}\,\Big(\frac{1}{2}(\phi'^2 - e^{2\phi}) + P_\eta(\eta' - e^{\phi})\Big) = \int du\,\Big[P_\eta\eta' + P_\phi\phi' - H\Big] \\ &H = \frac{P_\phi^2}{2} + \frac{1}{2}e^{2\phi} + P_\eta e^{\phi}, \qquad \tilde{u} \equiv \frac{8\pi G\,u}{\phi_r}. \end{aligned} \tag{2.35}$$

Here derivative is with respect to $\tilde{u}$. Also (ϕ, P_ϕ) and (η, P_η) are conjugate variables, and H is the Hamiltonian. The equation of motion is given by:

$$\begin{aligned} &P'_\phi = -(e^{2\phi} + P_\eta e^{\phi}), \qquad \phi' = P_\phi, \\ &\eta' = e^{\phi}, \qquad P'_\eta = 0. \end{aligned} \tag{2.36}$$

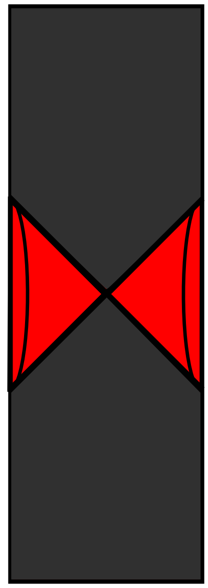


Figure 2.4: The embedding of the two-dimensional black hole in global AdS_2

The Schwarzian action also has the $SL(2,R)$ symmetry. The conserved charges are:

$$\begin{aligned} Q_1 &= \cos\eta(P_\eta + e^\phi) - \sin\eta P_\phi \\ Q_2 &= \sin\eta(P_\eta + e^\phi) + \cos\eta P_\phi \\ Q_3 &= P_\eta. \end{aligned} \tag{2.37}$$

They satisfy $\{Q, H\} = 0$, where the Poisson bracket is defined with respect to the conjugate variables (ϕ, P_ϕ) and (η, P_η).

Contribution of the Schwarzian modes to the four- point function

In this section, we assume that we have a large N field theory, bosonic or fermionic, whose low energy limit is described by the Schwarzian action. More precisely, we assume that the action has the form:

$$S = S_0 - N\alpha \int_0^{2\pi} d\theta\, Sch\Big(e^{i\varphi(\theta)}, \theta\Big) \qquad \alpha = \frac{\phi_r}{4N\beta G}, \tag{2.38}$$

where S_0 is conformal and has the reparametrization symmetry[2] which fixes the two-point function of operators of dimension Δ:

$$\begin{aligned} \tilde{G}(\theta_1,\theta_2) &= \left\langle \phi^i(\theta_1)\phi^i(\theta_2)\right\rangle = b\, \frac{\varphi'^\Delta(\theta_1)\varphi'^\Delta(\theta_2)}{\left(\sin\frac{\varphi(\theta_1)-\varphi(\theta_2)}{2}\right)^{2\Delta}} = G(\theta_1,\theta_2)\Big(1+\frac{\delta G}{G}\Big) \\ G(\theta_1,\theta_2) &= b\Big(\sin\frac{\theta_1-\theta_2}{2}\Big)^{-2\Delta}, \quad \frac{\delta G}{G} = \Delta\Big(\delta\varphi'(\theta_1)+\delta\varphi'(\theta_2) - \frac{\delta\varphi(\theta_1)-\delta\varphi(\theta_2)}{\tan\frac{\theta_1-\theta_2}{2}}\Big). \end{aligned} \tag{2.39}$$

[2]In the next chapter, we will introduce the SYK model as an example with such a theory.

$\delta\varphi$ can be expanded in terms of the Fourier modes $\delta\varphi = \sum_m \ \delta\varphi_m \ e^{im\theta}$, and so one gets:

$$\frac{\delta_m G}{G} = 2\, i\, \Delta\, e^{im\frac{\theta_1+\theta_2}{2}} \Big(\frac{\sin\frac{m(\theta_1-\theta_2)}{2}}{\tan\frac{\theta_1-\theta_2}{2}} - m\cos\frac{m(\theta_1-\theta_2)}{2}\Big) = 2i\Delta m(m^2-1)\int_{\theta_2}^{\theta_1} d\theta\ \frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}}\ e^{im\theta}. \tag{2.40}$$

In the rest of the section we compute the leading contribution of the Schwarzian action to the four point function. Consider the fields ϕ_1 with dimension Δ_1 and ϕ_2 with dimension Δ_2. Their four-point function has the following form:

$$\begin{aligned}&\frac{1}{N^2}\sum_{i,j}\ \big\langle\phi_1^i(\theta_1)\phi_1^i(\theta_2)\phi_2^j(\theta_3)\phi_2^j(\theta_4)\big\rangle = \\ &G_1(\theta_1-\theta_2)\ G_2(\theta_3-\theta_4)\begin{cases}\Big(1+\mathcal{F}^{TO}(\theta_1,\theta_2,\theta_3,\theta_4)\Big) & 0\le\theta_1\le\theta_2\le\theta_3\le\theta_4<2\pi\\ \pm\Big(1+\mathcal{F}^{OTO}(\theta_1,\theta_2,\theta_3,\theta_4)\Big) & 0\le\theta_1\le\theta_3\le\theta_2\le\theta_4<2\pi\end{cases},\end{aligned} \tag{2.41}$$

where $\pm$ is for bosons and fermions, respectively, and $\mathcal{F}$ denotes the connected part of the four-point function and is equal to:

$$\mathcal{F}(\theta_1,\theta_2,\theta_3,\theta_4) = \Big\langle\frac{\delta G_1(\theta_1,\theta_2)}{G_1}\ \frac{\delta G_2(\theta_3,\theta_4)}{G_2}\Big\rangle_c = \sum_{m,n}\frac{\delta_m G_1(\theta_1,\theta_2)}{G_1}\frac{\delta_n G_2(\theta_3,\theta_4)}{G_2}\big\langle\delta\varphi_m\delta\varphi_n\big\rangle. \tag{2.42}$$

We can read the two-point function $\langle\delta\varphi_m\delta\varphi_n\rangle$ from the quadratic terms in the Schwarzian action:

$$S = -N\alpha\int_0^{2\pi} d\theta\left(\frac{1}{2}-\frac{1}{2}\Big(\delta\varphi''^2-\delta\varphi'^2\Big)\right)\Rightarrow\big\langle\delta\varphi_m\delta\varphi_n\big\rangle = \frac{\delta_{m+n}}{2N\pi\alpha}\frac{1}{m^2(m^2-1)},\qquad m\neq 0,\pm1. \tag{2.43}$$

Plugging 2.40 into 2.42 and using 2.43, the expression for four point function will take the following form:

$$\frac{2\Delta_1\Delta_2}{N\pi\alpha}\int_{\theta_2}^{\theta_1} d\theta\int_{\theta_4}^{\theta_3} d\theta'\sum_{m\neq0}(m^2-1)\ \frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}}\ e^{im(\theta-\theta')}\ \frac{\sin\frac{\theta_3-\theta'}{2}\sin\frac{\theta'-\theta_4}{2}}{\sin\frac{\theta_3-\theta_4}{2}}. \tag{2.44}$$

We can also take the sum to get

$$\sum_{m\neq0}(m^2-1)\ e^{im(\theta-\theta')} = -2\pi\Big(\delta''(\theta-\theta')+\delta(\theta-\theta')\Big)+1. \tag{2.45}$$

In the time ordered case, i.e. $\theta_1>\theta_2>\theta_3>\theta_4$, the delta functions will not contribute and we simply have:

$$\begin{aligned}\mathcal{F}^{TO}\Big(\theta_1,\theta_2,\theta_3,\theta_4\Big) &= \frac{2\Delta_1\Delta_2}{N\pi\alpha}\left(\int_{\theta_2}^{\theta_1} d\theta\ \frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}}\right)\left(\int_{\theta_4}^{\theta_3} d\theta'\ \frac{\sin\frac{\theta_3-\theta'}{2}\sin\frac{\theta'-\theta_4}{2}}{\sin\frac{\theta_3-\theta_4}{2}}\right)\\ &= \frac{2\Delta_1\Delta_2}{N\pi\alpha}\left(1-\frac{\theta_1-\theta_2}{2\ \tan\frac{\theta_1-\theta_2}{2}}\right)\left(1-\frac{\theta_3-\theta_4}{2\ \tan\frac{\theta_3-\theta_4}{2}}\right).\end{aligned} \tag{2.46}$$

On the other hand, in the case where $\theta_1 > \theta_3 > \theta_2 > \theta_4$, the delta functions in 2.45 contribute when $\theta, \theta' \in [\theta_2,\ \theta_3]$:

$$\begin{aligned}
&\frac{2\Delta_1\Delta_2}{N\pi\alpha}\int_{\theta_2}^{\theta_3} d\theta\, d\theta' \, \frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}} \Big(-2\pi(\delta''(\theta-\theta') + \delta(\theta-\theta'))\Big) \frac{\sin\frac{\theta_3-\theta'}{2}\sin\frac{\theta'-\theta_4}{2}}{\sin\frac{\theta_3-\theta_4}{2}} \\
&= \frac{-4\Delta_1\Delta_2}{N\alpha}\int_{\theta_2}^{\theta_3} d\theta\, d\theta' \left[\frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}} \frac{\sin\frac{\theta_3-\theta}{2}\sin\frac{\theta-\theta_4}{2}}{\sin\frac{\theta_3-\theta_4}{2}} - \left(\frac{\sin\frac{\theta_1-\theta}{2}\sin\frac{\theta-\theta_2}{2}}{\sin\frac{\theta_1-\theta_2}{2}}\right)' \left(\frac{\sin\frac{\theta_3-\theta}{2}\sin\frac{\theta-\theta_4}{2}}{\sin\frac{\theta_3-\theta_4}{2}}\right)'\right] \\
&= \frac{-\Delta_1\Delta_2}{N\alpha}\frac{\theta_3-\theta_2}{\tan\frac{\theta_1-\theta_2}{2}\tan\frac{\theta_3-\theta_4}{2}} - \frac{2\Delta_1\Delta_2}{N\alpha}\frac{\cos\frac{\theta_1-\theta_4}{2}\cos\frac{\theta_2-\theta_3}{2}}{\sin\frac{\theta_1-\theta_2}{2}\sin\frac{\theta_3-\theta_4}{2}}.
\end{aligned} \tag{2.47}$$

Defining:

$$\theta = \theta_1 - \theta_2, \quad \theta' = \theta_3 - \theta_4, \quad \Delta\theta_+ = \frac{\theta_1+\theta_2}{2} - \frac{\theta_3+\theta_4}{2}. \tag{2.48}$$

The contribution of the Schwarzian modes to the four point function takes the following form [23]:

$$\mathcal{F}\Big(\theta_1,\theta_2,\theta_3,\theta_4\Big) = \frac{4\Delta_1\Delta_2}{S}\begin{cases} \left(1-\frac{\theta}{2\ \tan\frac{\theta}{2}}\right)\left(1-\frac{\theta'}{2\ \tan\frac{\theta'}{2}}\right) & (TO) \\ \frac{-\pi\sin\Delta\theta_+}{2\sin\frac{\theta}{2}\ \sin\frac{\theta'}{2}} - \frac{\pi(\pi-2\Delta\theta_+)}{4\tan\frac{\theta}{2}\tan\frac{\theta'}{2}} + \left(1+\frac{\pi-\theta}{2\ \tan\frac{\theta}{2}}\right)\left(1+\frac{\pi-\theta'}{2\ \tan\frac{\theta'}{2}}\right) & (OTO) \end{cases}. \tag{2.49}$$

Now, the important feature of $\mathcal{F}^{OTO}$ is that in the Lorentzian time where $\theta = \frac{2\pi i t}{\beta}$, it has the exponential growth due to the $\sin\Delta\theta_+$ term at early time, $t \ll \beta \ln S$:

$$\mathcal{F}^{OTO}(\theta_1,\theta_2,\theta_3,\theta_4) \sim \frac{e^{\kappa t}}{2S\sin^2\frac{\epsilon}{2}} + O(1), \quad \kappa = \frac{2\pi}{\beta},\ \ t \ll \frac{\beta}{2\pi}\ln S \tag{2.50}$$

where,

$$\theta_1 = \frac{2\pi i t}{\beta} + \epsilon,\ \ \theta_2 = \frac{2\pi i t}{\beta},\ \ \theta_3 = \epsilon,\ \ \theta_4 = 0. \tag{2.51}$$

The exponential growth in out-of-time-ordered correlators corresponds to scattering amplitude of particles close to the event horizon where the gravitational force becomes dominant [39]. Note that κ, called the Lyapunov exponent, saturates the chaos bound [26].
Note also that from the form of the exponentially growing term in $\mathcal{F}^{OTO}$, we will take the following ansatz for the general case at early time [23]:

$$\mathcal{F}^{OTO}(\theta_1,\theta_2,\theta_3,\theta_4) \approx C^{-1}\, e^{i\tilde{\kappa}(\pi-\theta_1-\theta_2+\theta_3+\theta_4)/2}\, \Upsilon_{\phi_1,\phi_3}(\theta_1-\theta_2)\, \Upsilon_{\phi_2,\phi_4}(\theta_3-\theta_4), \qquad 0<\tilde{\kappa}\le 1. \tag{2.52}$$

We will use this ansatz in Chapter 5.

Chapter 3

THE SACHDEV-YE-KITAEV MODEL

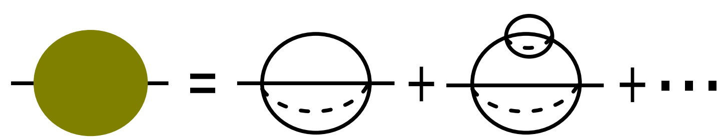
The SYK model[1] [23, 27, 38] is a model of Majorana fermions with all-to-all interaction through a random coupling. The model has dimensionless parameters N, the number of Majorana fermions, and βJ, where β is the inverse temperature and J is the coupling. As we will see at large N the model can be solved. However, the holographic behavior appears at $N \gg \beta J \gg 1$. In this limit the effective action is the Schwarzian theory. The Hamiltonian is given by

$$H = \frac{1}{q!} \sum_{i_1 \cdots i_q} J_{i_1, \cdots i_q} \chi_{i_1} \cdots \chi_{i_q} \qquad \langle J^2_{i_1 \cdots i_q} \rangle = \frac{(q-1)! J^2}{N^{q-1}}. \tag{3.1}$$

where q is a multiple of 4. The Majorana operators satisfy $\{\chi_i, \chi_j\} = \delta_{ij}$. The operators $a_i^\dagger = \frac{\chi_i + i\chi_{i+1}}{\sqrt{2}}$ and $a_i = \frac{\chi_i - i\chi_{i+1}}{\sqrt{2}}$ are the creation and annihilation operators and therefore, the dimension of the associated Hilbert space is $2^{\frac{N}{2}}$. The bare Green's function is :

$$G_0(\tau) = \frac{1}{2}\, \text{sgn}(\tau), \qquad G_0(\omega) = \frac{-1}{i\omega}. \tag{3.2}$$

To write the Schwinger-Dyson equation one can observe the diagrams that contribute to the self energy to the leading order in $\frac{1}{N}$, are of the form,



where the dashed line corresponds to taking the expectation value over Js. For such diagrams the Schwinger-Dyson equation has the following form:

$$\frac{1}{G(\omega)} = -i\omega - \Sigma(\omega), \qquad \Sigma(\tau) = J^2 G^{q-1}(\tau). \tag{3.3}$$

One can also derive 3.3 directly from the partition function:

$$\mathcal{Z} = \int \mathcal{D}\{J\}\, \mathcal{D}\, \{\chi^i\}\, e^{-S} \qquad S = \int d\tau \left(- \sum_i \chi^i \frac{d}{d\tau} \chi^i + H_{SYK} \right). \tag{3.4}$$

The measure $\mathcal{D}\{J\}$ is defined by:

$$\mathcal{D}\{J\} = \exp\left(- \frac{1}{2q!} \sum_{i_1, \cdots, i_q} J^2_{i_1 \cdots i_q} \right) \prod_{i_1 < \cdots < i_q} \frac{dJ_{i_1, \cdots, i_q}}{\sqrt{2\pi}}. \tag{3.5}$$

[1] Here we closely follow [23].

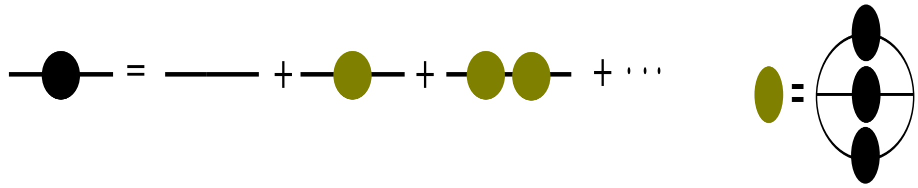


Figure 3.1: The graphical representation of the Schwinger-Dyson equation. The green filled circle is the self energy Σ, and the black filled circle is the two-point function G.

The quantity of interest is the free energy after taking the integral over $J_{i_1\cdots i_q}$,

$$\beta\bar{F} = -\overline{\ln\mathcal{Z}} = -\lim_{M\to 0}\frac{\overline{\mathcal{Z}^M}}{M}. \tag{3.6}$$

For a fixed integer value of M one can define M copies of the Majorana fermions

$$\left\{\chi_i^{\alpha},\quad \alpha = 1,\cdots,N\right\}, \tag{3.7}$$

compute the partition function, and at the end send M to zero.

$$\overline{Z^M} = \int\mathcal{D}\{J\}\int D\chi_\alpha^i \exp\Big(\sum_\alpha\int d\tau\,\Big(-\frac{1}{2}\sum_i\chi_\alpha^i\frac{d}{d\tau}\chi_\alpha^i + \sqrt{\frac{J^2(q-1)!}{N^{q-1}}}\sum J_{i_1\cdots i_q}\,\chi_\alpha^{i_1}\chi_\alpha^{i_2}\cdots\chi_\alpha^{i_q}\Big)\Big) \tag{3.8}$$

Now, defining $G_{\alpha\beta}(\tau,\tau') = \frac{1}{N}\sum_i\chi_\alpha^i(\tau)\chi_\beta^i(\tau')$ and the associated lagrange multiplier $\Sigma_{\alpha\beta}$, they satisfy the following normalization:

$$\int\mathcal{D}G\mathcal{D}\Sigma\;\;\exp\Big(-\frac{N}{2}\sum_{\alpha\beta}\int d\tau d\tau'\Sigma_{\alpha\beta}(\tau,\tau')G_{\alpha\beta}(\tau,\tau')\Big) = 1. \tag{3.9}$$

Inserting

$$\int\mathcal{D}G\mathcal{D}\Sigma\;\;\exp\Big(-\frac{N}{2}\sum_{\alpha\beta}\int d\tau d\tau'\Sigma_{\alpha\beta}(\tau,\tau')\big(G_{\alpha\beta}(\tau,\tau') - \frac{1}{N}\chi_\alpha^i(\tau)\chi_\beta^i(\tau')\big)\Big) = 1 \tag{3.10}$$

in to the path integral, we will get :

$$\begin{aligned}
\overline{Z^M} &= \int\mathcal{D}G\mathcal{D}\Sigma\;\;\exp\Big(-\frac{N}{2}\sum_{\alpha\beta}\int d\tau d\tau'\Sigma_{\alpha\beta}(\tau,\tau')\big(G_{\alpha\beta}(\tau,\tau') - \frac{1}{N}\chi_\alpha^i(\tau)\chi_\beta^i(\tau')\big)\Big)\\
&\times\int\mathcal{D}\{J_{i_1\cdots i_q}\}\int D\chi^i\exp\Big(\sum_\alpha\int d\tau\Big(-\frac{1}{2}\sum_i\chi_\alpha^i\frac{d}{d\tau}\chi_\alpha^i + \sqrt{\frac{J^2(q-1)!}{N^{q-1}}}\frac{1}{q!}\sum J_{i_1\cdots i_q}\,\chi_\alpha^{i_1}\chi_\alpha^{i_2}\cdots\chi_\alpha^{i_q}\Big)\Big)\\
&= \int\mathcal{D}\Sigma\mathcal{D}G\mathcal{D}\chi\exp\Big(\sum_{\alpha\beta}\int d\tau\,d\tau'\Big(-\frac{1}{2}\chi_\alpha^i\big(\delta_{\alpha\beta}\partial_\tau - \Sigma_{\alpha\beta}(\tau,\tau')\big)\chi_\beta^i - \frac{N}{2}\Sigma_{\alpha\beta}(\tau,\tau')G_{\alpha\beta}(\tau,\tau')\Big)\Big)\\
&\int\mathcal{D}\{J_{i_1\cdots i_q}\}\exp\Big(\sum_\alpha\int d\tau\Big(\sqrt{\frac{J^2(q-1)!}{N^{q-1}}}\frac{1}{q!}\sum J_{i_1\cdots i_q}\,\chi_\alpha^{i_1}\chi_\alpha^{i_2}\cdots\chi_\alpha^{i_q}\Big)\Big).
\end{aligned} \tag{3.11}$$

The last integral can be written as :

$$\int \mathcal{D}\{J_{i_1\cdots i_q}\} \sum_n \frac{1}{(2n)!}\Bigg(\sum_\alpha \int d\tau \sqrt{\frac{J^2(q-1)!}{N^{q-1}}}\frac{1}{q!}\sum J_{i_1\cdots i_q}\ \chi_\alpha^{i_1}\chi_\alpha^{i_2}\cdots\chi_\alpha^{i_q}\Bigg)^{2n}. \tag{3.12}$$

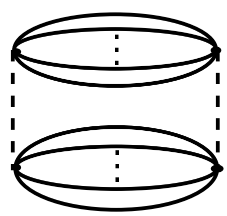
The number of ways to pair up the terms in above is $\frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}}{n!} = \frac{(2n)!}{n!\,2^n}$, and the number of ways to match the indices is $q!$. Taking the integral over J , we will get:

$$\sum_n \frac{\left(\frac{J^2N}{q}\right)^n}{n!}\Big(\int\ d\tau\, d\tau' \sum_{\alpha,\beta} G_{\alpha\beta}(\tau,\tau')\Big)^n = \exp\left(\frac{J^2N}{q}\int\ d\tau\, d\tau'\ \sum_{\alpha\beta} G_{\alpha\beta}(\tau,\tau')\right) \tag{3.13}$$

The partition function will become :

$$-\frac{\beta F}{N} = \ln \mathrm{Pf}\,(\delta_{\alpha\beta}\partial_\tau - \Sigma_{\alpha\beta}(\tau,\tau')) - \frac{1}{2}\int\ d\tau d\tau'\Big(G_{\alpha\beta}(\tau,\tau')\Sigma_{\alpha\beta}(\tau,\tau') - \frac{J^2}{q}G^q_{\alpha\beta}(\tau,\tau')\Big). \tag{3.14}$$

Due to the fact that the bare Green's function is between the Majoranas with the same replica index, the first diagram that contributes to the partition function of Majoranas with different replica index is $O(N^{2-q})$, which is diagrammatically represented as:



Since we are interested in leading order, from now on we consider only the replica diagonal partition function:

$$\Sigma_{\alpha\beta} = \Sigma\ \delta_{\alpha\beta}, \quad G_{\alpha\beta} = G\ \delta_{\alpha\beta}, \tag{3.15}$$

with the free energy given by:

$$-\frac{\beta F}{N} = \ln \mathrm{Pf}\,(-\delta' - \Sigma) - \frac{1}{2}\int\ d\tau d\tau'\Big(G(\tau,\tau')\Sigma(\tau,\tau') - \frac{J^2}{q}G^q(\tau,\tau')\Big). \tag{3.16}$$

The solution to the equation of motion is given by:

$$\partial_\tau G(\tau,\tau') - \int d\tau'' G(\tau,\tau'')\Sigma(\tau'',\tau') = \delta(\tau,\tau'), \quad \Sigma(\tau,\tau') = J^2G^{q-1}(\tau,\tau'). \tag{3.17}$$

At low energy limit, we can ignore the fist term. The rest will have the reparametrization symmetry:

$$\begin{aligned} &G(\tau,\tau') \to G(f(\tau),f(\tau'))\, f'^{\Delta}(\tau)\ f'^{\Delta}(\tau'),\\ &\Sigma(\tau,\tau') \to \Sigma(f(\tau),f(\tau'))\, f'^{(q-1)\Delta}(\tau)\ f'^{(q-1)\Delta}(\tau'), \qquad \Delta = \frac{1}{q}. \end{aligned} \tag{3.18}$$

The conformal solution for G is

$$G_c(\tau) = b\left(\frac{1}{\beta J \sin\frac{\pi\tau}{\beta}}\right)^{2\Delta} sgn(\tau), \qquad \frac{b}{(J\tau)^{2\Delta}} \ sgn(\tau), \qquad b^q = \frac{\pi}{J^2}\left(\frac{1}{2} - \Delta\right) \tan \pi\Delta. \tag{3.19}$$

From now on, we assume that we are in finite temperature and define $G(\tau_1, \tau_2) = G(\theta_1, \theta_2)\left(\frac{2\pi}{\beta J}\right)^{2\Delta}$ where,

$$\theta = \frac{2\pi\tau}{\beta}. \tag{3.20}$$

We also take $\varphi(\theta) = \varphi(\frac{2\pi\tau}{\beta})$ a monotonic reparametrization of θ, $\varphi(\tau+\beta) = \varphi + 2\pi$. Under this, we can rewrite 3.16 as:

$$-\frac{\beta F}{N} = -\ln \mathrm{Pf}\,(-\Sigma) + \frac{1}{2}\int_0^{2\pi} d\varphi d\varphi' \Big(G(\varphi,\varphi')\Sigma(\varphi,\varphi') - \frac{1}{q}G^q(\varphi,\varphi') - \sigma(\varphi,\varphi')G(\varphi,\varphi')\Big), \tag{3.21}$$

where

$$\sigma(\tau_1,\tau_2) = J^2\sigma(\varphi_1,\varphi_2)\left(\frac{2\pi}{\beta J}\right)^{2-2\Delta}\varphi_1'^{1-\Delta}\varphi_2'^{1-\Delta}, \qquad \varphi' = \frac{d\varphi}{d\theta}. \tag{3.22}$$

Note that when $\sigma = 0$, the theory is conformal. We are interested in studying the response of the theory to turning on the perturbation σ. Therefore, we expand the fields around the conformal value i.e. $G = G_c + \delta G, \quad \Sigma = \Sigma_c + \delta\Sigma$ and expand the action to second order:

$$\begin{aligned}-\frac{\beta F}{N} &= \frac{1}{4}\,\mathrm{tr}\,(G_c\delta\Sigma)^2 + \frac{1}{2}\int_0^{2\pi} d\varphi d\varphi' \Big(\delta G(\varphi,\varphi')\delta\Sigma(\varphi,\varphi') - \frac{q-1}{2}G_c^{q-2}(\varphi,\varphi')\delta G(\varphi,\varphi') \\ &\qquad - \sigma(\varphi,\varphi')\Big(G_c(\varphi,\varphi') + \delta G^2(\varphi,\varphi')\Big)\Big) \\ &= -\frac{1}{4}\,\langle\delta f|K_c|\delta f\rangle + \frac{1}{2}\langle\delta g|\delta f\rangle - \frac{1}{4}\langle\delta g|\delta g\rangle - \frac{1}{2}\langle s|g_c + \delta g\rangle,\end{aligned} \tag{3.23}$$

where :

$$\begin{aligned} &g(\varphi_1,\varphi_2) = R_c(\varphi_1,\varphi_2)G(\varphi_1,\varphi_2), \quad s(\varphi_1,\varphi_2) = R_c^{-1}(\varphi_1,\varphi_2),\tilde{\sigma}(\varphi_1,\varphi_2) \\ &R_c(\varphi_1,\varphi_2) = -\sqrt{q-1}\,|G_c(\varphi_1,\varphi_2)|^{\frac{q-2}{2}}.\end{aligned} \tag{3.24}$$

Taking saddle point with respect to δf yields:

$$K_c|\delta f\rangle = |\delta g\rangle, \tag{3.25}$$

where the operator K_c is defined by:

$$\langle\varphi_3,\varphi_4|K_c|\varphi_1,\varphi_2\rangle = (q-1)|\tilde{G}(\varphi_1,\varphi_2)|^{\frac{q-2}{2}}\ \tilde{G}(\varphi_1,\varphi_3)\tilde{G}(\varphi_2,\varphi_4)\ |\tilde{G}(\varphi_3,\varphi_4)|^{\frac{q-2}{2}}. \tag{3.26}$$

Plugging back, we will get:

$$-\frac{\beta F}{N} = \frac{1}{4}\langle\delta g|K_c^{-1} - 1|\delta g\rangle - \frac{1}{2}\langle s|g_c + \delta g.\rangle \tag{3.27}$$

Further saddle point with respect to δg yields:

$$|\delta g\rangle = K_c\Big(1-K_c\Big)^{-1}|s\rangle, \tag{3.28}$$

and so the final answer for the free energy will take the form:

$$-\frac{\beta F}{N} = \frac{-1}{4}\langle s|K_c\Big(1-K_c\Big)^{-1}|s\rangle - \frac{1}{2}\langle s|g_c\rangle. \tag{3.29}$$

Equation 3.28 gives the response δg to the perturbation s. We are interested in the sources that act in the intermediate time $J^{-1} \ll \tau \ll \beta$. A set of eigenfunctions for this equation is $s(\varphi_1,\varphi_2) \propto \frac{\mathrm{sgn}(\varphi_1-\varphi_2)}{|\varphi_1-\varphi_2|^h}$ with eigenvalue $\frac{k_c(h)}{1-k_c(h)}$ where

$$k_c(h) = \frac{u(\Delta-\frac{1-h}{2})u(\Delta-\frac{h}{2})}{u(\Delta+\frac{1}{2})u(\Delta-1)}, \qquad u(x) = \Gamma(2x)\sin(\pi x). \tag{3.30}$$

In particular, the source $|s\rangle$ corresponding to the responses with eigenvalue $k_c(h) = 1$ produces resonances that reach the IR physics. However, the power law eigenfunction does not seem to come from RG. A more careful treatment yields

$$s(\theta_1,\theta_2) = -a_I\ \varepsilon^{h_I-1}\ \frac{\mathrm{sgn}(\theta_1-\theta_2)}{|\theta_1-\theta_2|^{h_I}}\ u(\zeta) \tag{3.31}$$

where $\zeta = \ln\frac{|\theta_{12}|}{\epsilon}$ is the renormalization parameter, and $u(\zeta)$ is a normalized window function, $\int d\zeta\ u(\zeta) = 1$. Transforming back s to σ, we will get:

$$\sigma_I(\tau_1,\tau_2) = -a_I\sqrt{q-1}\ b^{1/2-\Delta}\ J^2\ |J(\tau_1-\tau_2)|^{2\Delta-1-h_I}\,\mathrm{sgn}(\tau_1-\tau_2)\ u(\ln|J(\tau_1-\tau_2)|). \tag{3.32}$$

This clearly implies that $\int d\tau_1\Big(\sigma_I(\tau_1-\tau_2)(\tau_1-\tau_2)\Big)$ is J independent, as is supposed to be. In fact, one can regard it as the widened analog of $\delta'(\tau_1-\tau_2)$. This will justify the coefficient ε^{h_I-1} in 3.31. Now, the leading value is $h_0 = 2$ which corresponds to:

$$\begin{aligned} \sigma_0(\tau_1,\tau_2) &= -a_0\sqrt{q-1}\ b^{\frac{1}{2}-\Delta}\ J^2\ |J(\tau_1-\tau_2)|^{2\Delta-3}\,\mathrm{sgn}(\tau_1-\tau_2)\ u(\ln|J(\tau_1-\tau_2)|) \\ \sigma(\theta_1,\theta_2) &= -a_0\sqrt{q-1}\ b^{\frac{1}{2}-\Delta}\ \varepsilon\ |\theta_1-\theta_2|^{2\Delta-3}\,\mathrm{sgn}(\theta_1-\theta_2)\ u(\ln\frac{|\theta_1-\theta_2|}{\varepsilon}). \end{aligned} \tag{3.33}$$

Plugging back into 3.16 yields,

$$S = \frac{N}{2}\int d\theta_1 d\theta_2\ \tilde{\sigma}_0(\theta_1,\theta_2)\ \tilde{G}_c(\theta_1,\theta_2). \tag{3.34}$$

On the other hand, $G_c(\theta_1,\theta_2) = G_c(\varphi_1,\varphi_2)\ {\varphi'_1}^{\Delta}{\varphi'_2}^{\Delta}$. Taking $\theta_+ = \frac{\theta_1+\theta_2}{2}$, and assuming that $|\theta_1-\theta_2| \ll 1$, we have:

$$\begin{aligned} \varphi'(\theta_1)^{\Delta}\varphi'(\theta_2)^{\Delta} &= \varphi'^{2\Delta}(\theta_+)\Big(1+\frac{\Delta}{4}(-\frac{\varphi''^2}{\varphi'^2}+\frac{\varphi'''}{\varphi'})\,(\theta_1-\theta_2)^2\Big) \\ \Big(2\sin\frac{(\varphi(\theta_1)-\varphi(\theta_2))}{2}\Big)^{-2\Delta} &= \varphi'^{-2\Delta}(\theta_+)(\theta_1-\theta_2)^{-2\Delta}\Big(1-\frac{\Delta}{12}\Big(\frac{\varphi'''}{\varphi'}-\varphi'^2\Big)(\theta_1-\theta_2)^2\Big). \end{aligned} \tag{3.35}$$

Therefore, we will get:

$$
\begin{aligned}
&G(\theta_1,\theta_2) \approx G_{\beta=\infty}(\theta_1-\theta_2)\left(1+\frac{\Delta}{6}\,\mathrm{Sch}\left(e^{i\varphi(\theta_+)},\theta_+\right)(\theta_1-\theta_2)^2\right),\\
&G_{\beta=\infty}(\theta_1-\theta_2) = b^{\Delta}\,|\theta_1-\theta_2|^{-2\Delta}\ \mathrm{sgn}(\theta_1-\theta_2).
\end{aligned}
\tag{3.36}
$$

Hence, 3.34 yields:

$$
\begin{aligned}
&-a_0\sqrt{q-1}\,b^{\frac{1}{2}}\,\frac{N\Delta\varepsilon}{12}\int\frac{d\theta}{|\theta|}\,d\theta_+\ \mathrm{Sch}\left(e^{i\varphi(\theta_+)}\right)u(\ln\frac{|\theta|}{\varepsilon}) = -\alpha_S N\varepsilon\int_0^{2\pi}d\theta\ \mathrm{Sch}\left(e^{i\varphi(\theta)},\theta\right)\\
&\alpha_S = \frac{a_0\sqrt{(q-1)b}\,\Delta}{6}.
\end{aligned}
\tag{3.37}
$$

Next, $k(h_I)=1$ modes will also contribute to the action with a coefficient with higher power of $\frac{1}{\beta J}$. In the limit $N \gg 1, \beta J \gg 1$ where we also have $\frac{N}{\beta J} \gg 1$, the Schwarzian action 3.37 dominates. Thus, one can consider the Majorana fermions in 3.1 as the UV degrees of freedom describing the gravitational theory, the Schwarzian action.

Chapter 4

DISENTANGLING THE THERMOFIELD-DOUBLE STATE

In this section, we will study the states that are produced by evolving the thermofield-double state $|\text{TFD}\rangle \in \mathcal{H}^* \otimes \mathcal{H} \equiv \mathcal{H}_L \otimes \mathcal{H}_R$ by double-traced operators coupling both sides of the thermofield-double:

$$|\widetilde{\text{TFD}}(t)\rangle = \widetilde{U}(t)\,|\text{TFD}\rangle \tag{4.1}$$

where the unitary operator $U(t)$ has the form

$$\begin{aligned} \widetilde{U}(t) &= \mathbf{T}\exp\Big(-i\int_0^t du\,\widetilde{H}(u)\Big) \\ \widetilde{H}(u) &= H_L + H_R + H_{int}(u), \qquad H_{int}(u) = \frac{g(u)}{N}\sum_{i=1}^{N}\phi_L^i\phi_R^i, \qquad N \gg 1 \end{aligned} \tag{4.2}$$

and $H_L = H^* \otimes 1$ and $H_R = 1 \otimes H$. Note also that the time dependence of $\widetilde{H}(u)$ is only through $g(u)$. Here we assume that the fields ϕ^is are bosonic with dimension Δ[1]. Note that the time direction in the left side is the opposite to the one in the right side. In other words, the generator of time evolution is $H = H_R - H_L$ [2] (see appendix B). This means the Hamiltonian $H = H_L + H_R$, while it takes the right fields forward in time, it takes the left fields backward in time. Therefore, in the interaction picture, we will get:

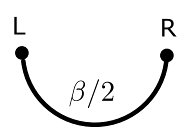
$$U_I(t) = \mathbf{T}\exp\Big(-i\int_0^t du\,H_I(u)\Big), \quad H_I(u) = \frac{1}{N}\sum_{i=1}^{N} g(u)\phi_L^i(-u)\phi_R^i(u). \tag{4.3}$$

We further assume that the unperturbed theory, ($H_{int} = 0$), at low temperature has an approximate conformal symmetry, i.e. the two point function is conformal and in Euclidean time it is: [3]

$$\langle\phi_R^i(\tau)\phi_R^j(\tau')\rangle = \frac{b\delta_{ij}}{\left(\frac{\beta J}{\pi}\sin\frac{\pi(\tau-\tau')}{\beta}\right)^{2\Delta}} \tag{4.4}$$

while higher point functions are described by the Schwarzian modes. To construct the density matrix associated with the right side, we start with the thermofield-double which can be diagrammatically represented as:

$$|\text{TFD}\rangle \qquad \leftrightarrow \tag{4.5}$$

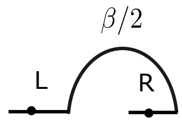


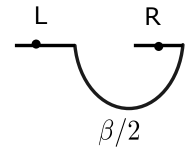
[1] One can also take fermionic fields. In this case, the terms $\phi_L^j\phi_R^j$ in the Hamiltonian should be modified to $i\phi_L^j\phi_R^j$ for the Hamiltonian to be Hermitian. However, the results remain unchanged.

[2] $\phi_R(t) = e^{iH_Rt}\phi_R e^{-iH_Rt}$, while $\phi_L(t) = e^{-iH_Lt}\phi_L e^{iH_Lt}$.

[3] Here, J has dimension of energy, and we extracted it from numerator to make both numerator and denominator manifestly dimensionless.

Then for $0 \leq t_1, t_2 \leq t$, we have the following diagrammatic representations:

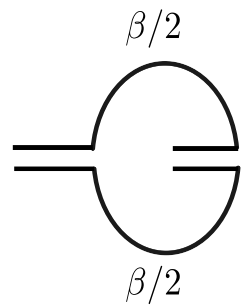
$$\langle \text{TFD}|\phi_L(-t_2)\phi_R(t_1) =$$ 

$$\phi_R(t_1)\phi_L(-t_2)|\text{TFD}\rangle =$$ 

(4.6)

Now, consider two copies of the thermofield-double with the insertions from the unitary operator 4.3

$$\widetilde{U}_I(t)|\text{TFD}\rangle\langle\text{TFD}|\widetilde{U}_I^{-1}(t) \qquad \leftrightarrow \tag{4.7}$$

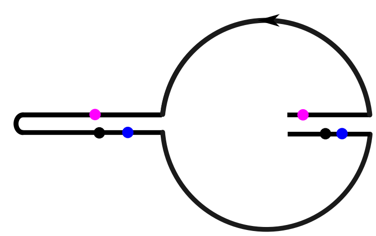


Then we can use the identity [4]

$$\phi_L(-t)|\text{TFD}\rangle = \phi_R(-t + i\frac{\beta}{2})|\text{TFD}\rangle \tag{4.8}$$

to transform all the left fields to the right ones and trace out the left side. This corresponds to gluing the left end points of the contour,

$$\rho_R(t) = \text{Tr}_L\left(\widetilde{U}(t)|\text{TFD}\rangle\langle\text{TFD}|\widetilde{U}^{-1}(t)\right) \qquad \leftrightarrow \tag{4.9}$$



An explicit expression for $\rho_R(t)$ will take the following form:

$$\begin{aligned}\rho_R(t) = &\sum_{n,m} \frac{(i)^m(-i)^n}{N^{n+m} n!\, m!} \int_0^u (du_1 \cdots du_n)(du'_1 \cdots du'_m) g(u_1) \cdots g(u_n) g(u'_1) \cdots g(u'_m) \\ &= \sum_{i_1 j_1, \ldots, i_n j_n = 1}^{N} \frac{e^{-\beta H_R}}{Z(\beta)} \Big[\phi^{i_1}(u_1 - i\beta) \cdots \phi^{i_n}(u_n - i\beta) \phi^{i_n}(-u_n - i\frac{\beta}{2}) \cdots \phi^{i_1}(-u_1 - i\frac{\beta}{2}) \\ &\qquad \phi^{j_1}(-u'_1 - i\frac{\beta}{2}) \cdots \phi^{j_m}(-u'_m - i\frac{\beta}{2}) \phi^{j_m}(u'_m) \cdots \phi^{j_1}(u'_1) \Big], \\ &\qquad (u_1 \geq \cdots \geq u_n \geq 0, \;\; u'_1 \geq \cdots \geq u'_n \geq 0).\end{aligned} \tag{4.10}$$

[4] This is indeed a Euclidean rotation with angle π with the modular operator; see Appendix B. For fermionic fields, we have $\phi_L(-t)|\text{TFD}\rangle = i\phi_R(-t + i\frac{\beta}{2})|\text{TFD}\rangle$.

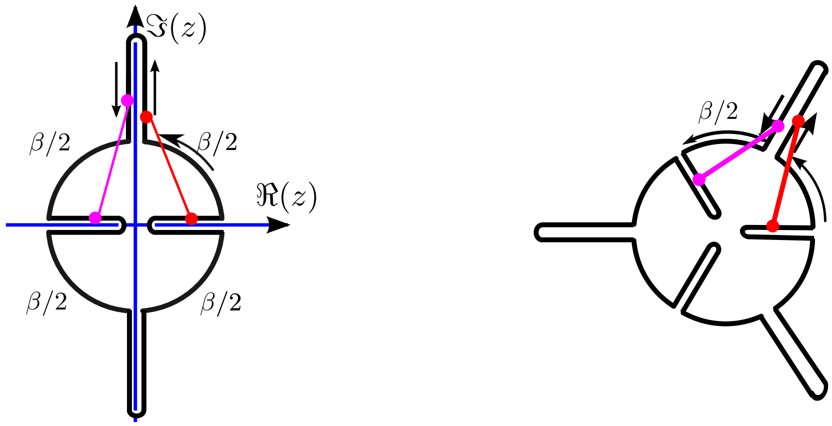


Figure 4.1: The time contour for the second(left) and the third(right) Renyi entropy. The contours can be constructed by gluing two and three contours of 4.9. The insertions are represented by filled circles and the Wick's contractions are represented by solid lines. In the left figure, $z = e^{\frac{\pi(i\tau - t)}{\beta}}$, and in the right figure, $z = e^{\frac{2\pi(i\tau - t)}{3\beta}}$.

One can construct the Renyi entropy by gluing s copies of ρ_R and compute the entanglement entropy:

$$S_{EE} = \lim_{s\to 1} \frac{1}{1-s} \log \operatorname{Tr} \rho_R^s. \tag{4.11}$$

The leading contribution to $\operatorname{Tr}(\rho^s)$ comes from breaking each replica into a product of two-point function fields with the same index at temperature $s\beta$. After some manipulation, one will get:

$$\ln \operatorname{Tr}\left(\rho_R^s(t)\right) = \ln \frac{Z_0(s\beta)}{Z_0^s(\beta)} - is \int_0^t g(u)du\Big(G_{s\beta}(2iu + \frac{\beta}{2}) - G_{s\beta}(-2iu + \frac{\beta}{2})\Big). \tag{4.12}$$

In the limit $s \longrightarrow 1$, the first term in the right hand side is the entanglement entropy of $|\mathrm{TFD}\rangle$. Therefore, ΔS is:

$$\begin{aligned}\Delta S &= i\Big(\frac{\pi}{\beta J}\Big)^{2\Delta} \int_0^u g(u)du \quad \frac{d}{ds}\left[\left(\frac{1}{\sin(\frac{2\pi iu}{s\beta} + \frac{\pi}{2s})}\right)^{2\Delta} - \left(\frac{1}{\sin(\frac{-2\pi iu}{s\beta} + \frac{\pi}{2s})}\right)^{2\Delta}\right]_{s=1} \\ &= 2\pi\Delta\, b\Big(\frac{\pi}{\beta J}\Big)^{2\Delta} \int_0^u g(u)du \quad \frac{\sinh\frac{2\pi u}{\beta}}{\left(\cosh(\frac{2\pi u}{\beta})\right)^{2\Delta+1}}.\end{aligned} \tag{4.13}$$

Considering a quantum quench, $g(u) = g\ \delta(u)$, we can evaluate the integral:

$$\Delta S_{EE}(t) = \frac{\pi b g}{2J}\Big(\frac{\pi}{\beta J}\Big)^{2\Delta-1}\left(1 - \frac{1}{\left(\cosh(\frac{2\pi t}{\beta})\right)^{2\Delta}}\right). \tag{4.14}$$

As is clear, ΔS_{EE} will saturate at $t \sim \frac{2\pi}{\beta}$ with the value:

$$\Delta S^*_{EE} = \frac{\pi b g}{2J}\Big(\frac{\pi}{\beta J}\Big)^{2\Delta-1}. \tag{4.15}$$

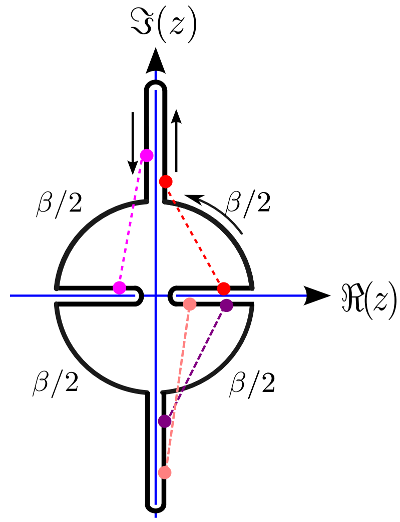


Figure 4.2: The contribution of four-point functions in the second configuration that contributes to the entanglement entropy ($z = e^{\frac{\pi(i\tau - t)}{\beta}}$). There are two types: the first depicted by the pink and red dotted lines where the insertions are disjoint. The second type corresponds to the nested correlator.

One expects at this time the system to thermalize. Since $S \propto T$, the thermalization happens with the new temperature

$$\tilde{\beta} = \beta\left(1 - \frac{\Delta S}{S}\right) = \beta\left(1 - \frac{\pi b g}{2JS}\left(\frac{\pi}{\beta J}\right)^{2\Delta-1} + O(\frac{1}{S^2})\right). \tag{4.16}$$

Second order correction to the entanglement entropy

The second order correction comes from the connected part of the four-point functions in the OPE limit. There are two configurations of the fields. The first configuration corresponds to the pairs of fields with the same index that belong to different replicas:

$$\begin{aligned}\frac{s(s-1)}{2} g^2 \int \, dudu'\, G(2iu + \frac{\beta}{2})\, G(2iu' + \frac{\beta}{2}) \Big[&\mathcal{F}^{TO}(2iu + \frac{\beta}{2}, -2iu' + \frac{\beta}{2}) + \mathcal{F}^{TO}(2iu' + \frac{\beta}{2}, -2iu + \frac{\beta}{2}) \\ &- \mathcal{F}^{TO}(2iu' + \frac{\beta}{2}, 2iu + \frac{\beta}{2}) - \mathcal{F}^{TO}(-2iu' + \frac{\beta}{2}, -2iu + \frac{\beta}{2}) \Big] \\ = \frac{\pi^2 b^2 g^2}{2S}\left(\frac{\beta}{2\pi}\right)^2 \left(\frac{\pi}{\beta J}\right)^{4\Delta} &\left(1 - \frac{1}{\cosh\frac{2\pi t}{\beta}}\right)^2 (s-1).\end{aligned} \tag{4.17}$$

The second configuration consists of the pairs that belong to the same replica, but are located on different branches of the Keldysh contour and the pairs which are located on the same branch of the Keldysh contour, which are nested pairs. The expression associated with the second configuration is equal to:

$$\begin{aligned}&sg^2 \int_0^t \, dudu'\, G(-2iu + \frac{\beta}{2})\, G(+2iu' + \frac{\beta}{2})\, \mathcal{F}(-2iu + \frac{\beta}{2}, 2iu' + \frac{\beta}{2}) \\ &- \frac{s}{2} g^2 \int_0^t \, du \int_0^t \, du' \left[G(-2iu - \frac{\beta}{2} + s\beta) G(2iu' + \frac{\beta}{2}) \mathcal{F}_{\mathbf{T}}(-2iu - \frac{\beta}{2} + s\beta, 2iu' + \frac{\beta}{2}) + c.c. \right],\end{aligned} \tag{4.18}$$

where the second line corresponds to $\left\langle \mathbf{T}\left\{\phi^i(iu+\frac{\beta}{2})\phi^j(iu'+\frac{\beta}{2})\phi^j(-iu')\phi^i(-iu)\right\}\right\rangle + \left\langle \widetilde{\mathbf{T}}\left\{\phi^i(-iu+\frac{\beta}{2})\phi^j(-iu'+\frac{\beta}{2})\phi^j(iu')\phi^i(iu)\right\}\right\rangle$. Both expressions are with respect to the inverse temperature $s\beta$. Note that in the limit $s \to 1$, both 4.17 and 4.18 vanish, which means we can compute their contribution to the entanglement entropy separately. Let us define $X = \frac{iu}{s} + \frac{\pi}{2s}$ and $Y = \frac{-iu}{s} + \frac{\pi}{2s}$, and assume $\beta = 2\pi$. We have $\left\langle \mathbf{T}\left\{\phi^i(iu+\frac{\beta}{2})\phi^j(iu'+\frac{\beta}{2})\phi^j(-iu')\phi^i(-iu)\right\}\right\rangle =$

$$\left(\frac{\pi}{\beta J}\right)^{4\Delta}\frac{4\Delta^2 b^2}{S}\begin{cases}\left(1-\frac{X}{\tan X}+\frac{\pi}{\tan X}\right)\left(1-\frac{X'}{\tan X'}\right)\frac{1}{\sin^{2\Delta}X\ \sin^{2\Delta}X'} & u > u'\\ \left(1-\frac{X}{\tan X}\right)\left(1-\frac{X'}{\tan X'}+\frac{\pi}{\tan X'}\right)\frac{1}{\sin^{2\Delta}X\ \sin^{2\Delta}X'} & u' > u,\end{cases} \tag{4.19}$$

and the anti-time-ordered four-point function is simply the complex conjugate of the above expression. So the expression for 4.18 will take the following form:

$$\begin{aligned}-\frac{sg^2}{2}\frac{4b^2\Delta^2}{S}\left(\frac{\pi}{\beta J}\right)^{4\Delta}\Bigg[\Bigg[\int_0^t du\left(\frac{1}{\sin^{2\Delta}X}\left(1-\frac{X}{\tan X}\right)-\frac{1}{\sin^{2\Delta}Y}\left(1-\frac{Y}{\tan Y}\right)\right)\Bigg]^2\\ +\Bigg[\int_0^t du\ \frac{1}{\sin^{2\Delta}X}\frac{\pi}{\tan X}\int_0^u du'\frac{1}{\sin^{2\Delta}X'}\left(1-\frac{X'}{\tan X'}\right)\\ +\int_0^t du\ \frac{1}{\sin^{2\Delta}X}\left(1-\frac{X}{\tan X}\right)\int_u^t du'\frac{1}{\sin^{2\Delta}X'}\frac{\pi}{\tan X'}+c.c.\Bigg]\Bigg].\end{aligned} \tag{4.20}$$

We can use by-parts and some manipulations to further simplify the second and the third line:

$$\begin{aligned}-\frac{sg^2}{2}\frac{4b^2\Delta^2}{S}\left(\frac{\pi}{\beta J}\right)^{4\Delta}\Bigg[\Bigg[\int_0^t du\left(\frac{1}{\sin^{2\Delta}X}\left(1-\frac{X}{\tan X}\right)-\frac{1}{\sin^{2\Delta}Y}\left(1-\frac{Y}{\tan Y}\right)\right)\Bigg]^2\\ +\Bigg[\frac{\pi^2 s}{8\Delta^2\sin^{4\Delta}\frac{\pi}{2s}}-\frac{\pi^2 s}{4\Delta^2\sin^{2\Delta}\frac{\pi}{2s}}\sin^{-2\Delta}X+\frac{\pi s^2}{4\Delta^2}X\sin^{-4\Delta}X+c.c.\Bigg]\\ -\frac{i\pi s}{2\Delta}(2-\frac{1}{2\Delta})\left(\int_0^t\frac{1}{\sin^{4\Delta}X}-c.c.\right)+\frac{i\pi s}{\Delta}\left(1-\frac{1}{2\Delta}\right)\left(\frac{1}{\sin^{2\Delta}X}\int_0^t\frac{1}{\sin^{2\Delta}X}-c.c.\Bigg]\right.\end{aligned} \tag{4.21}$$

Expanding around $s = 1$ yields:

$$\frac{-\pi^2 b^2 g^2}{2S}\left(\frac{\beta}{2\pi}\right)^2\left(\frac{\pi}{\beta J}\right)^{4\Delta}\left[4\Delta\frac{t\tanh t}{\cosh^{2\Delta}t}-4\Delta\frac{1}{\cosh^{2\Delta}t}+\frac{4\Delta}{\cosh^{4\Delta}t}+4\Delta(2\Delta-1)\frac{\sinh t}{\cosh^{1+2\Delta}t}\int_0^t\frac{du}{\cosh^{2\Delta}u}\right]. \tag{4.22}$$

Therefore, retrieving the β dependence, the second order correction to the entanglement entropy, including 4.17, is

$$\begin{aligned}\Delta S_{EE} = \frac{\pi^2 b^2 g^2}{2S}\left(\frac{\beta}{2\pi}\right)^2\left(\frac{\pi}{\beta J}\right)^{4\Delta}\Bigg[-\left(1-\frac{1}{\cosh^{2\Delta}\frac{2\pi t}{\beta}}\right)^2+\frac{4\Delta}{\cosh^{2\Delta}\frac{2\pi t}{\beta}}\Bigg[\frac{2\pi t}{\beta}\tanh\frac{2\pi t}{\beta}-1+\frac{1}{\cosh^{2\Delta}\frac{2\pi t}{\beta}}\\ +(2\Delta-1)\tanh\frac{2\pi t}{\beta}\int_0^{\frac{2\pi t}{\beta}}\frac{du}{\cosh^{2\Delta}u}\Bigg]\Bigg].\end{aligned} \tag{4.23}$$

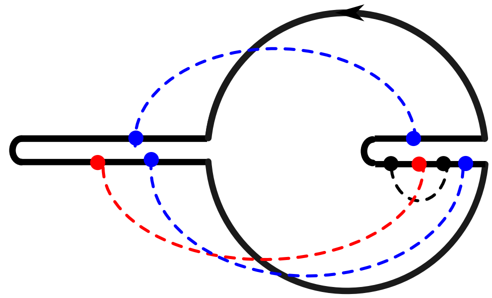


Figure 4.3: The time contour for the four-point function using 4.6. Here, we have $-\frac{\beta}{2} \leq \tau \leq \frac{\beta}{2}$. The figure displays different configurations in the computation of the two-point function. The probing fields are depicted by the black filled circles. The blue/red filled circles represent the field insertions from the unitary evolution. The red insertions make the out-of-time-ordered correlator with the probing fields.

Thermalization

In this section, we will compute the temperature of the deformed thermofield-double state directly by studying the two point function of two probing fields inserted in the right side. In general, for a system out of equilibrium, the two-point function $G(t_1, t_2)$ also depends on $\frac{t_1+t_2}{2}$. In our case, the correction to the two-point function is of two types. The first type is the case where the interaction Hamiltonian makes a time-ordered correlation function with the two probing fields in the two point function, while the second type is the case where they make an out-of-time-ordered correlator; see Figure 4.3. We expect the $\frac{t_1+t_2}{2}$ dependence of the two-point function to come from the second type, from the exponentially growing term in the OTOC configuration [39]. However, as we will see the non-equilibrium part will be suppressed at the time of order $t \sim \frac{\beta}{2\Delta}$ due to exponential decay in the strength of the interaction Hamiltonian, and the system will equilibrate with the new temperature much earlier than the scrambling time. We insert the two probing fields of dimension Δ_1 at time t_1 and t_2:

$$\frac{1}{N}\sum_{i=1}^{N}\Big\langle TFD\Big|U_I(t_0,t_2)\,\phi^i(t_2)\,U_I(t_2,t_1)\,\phi^i(t_1)U_I(t_1,t_0)\Big|TFD\Big\rangle, \tag{4.24}$$

Here, $t_0 = 0$ is the time TFD is prepared. We expand the three evolution operators inserted in the two-point function which to leading order in g yields:

$$\begin{aligned} G_\beta(t_2-t_1)\Bigg(1-i\,\Bigg[&\int_{t_0}^{t_1} du\, g(u)\,\mathcal{F}^{TO}\Big(i(t_2-t_1), 2iu+\frac{\beta}{2}\Big)\;G(2iu+\frac{\beta}{2}) \\ &-\int_{t_0}^{t_2} du\, g(u)\,\mathcal{F}^{TO}\Big(-2iu+\frac{\beta}{2}, i(t_2-t_1)\Big)\;G(2iu+\frac{\beta}{2}) \\ &+\int_{t_1}^{t_2} du\, g(u)\,\mathcal{F}^{OTO}\Big(it_2, it_1, iu, -iu-\frac{\beta}{2}\Big)G(2iu+\frac{\beta}{2})\Bigg]\Bigg). \end{aligned} \tag{4.25}$$

For $t_1, t_2 > t_0$ after manipulation, we get:

$$G((t_2-t_1))\Bigg[1+\frac{\beta bg\Delta_1}{2S(\frac{\beta J}{\pi})^{2\Delta}}\Bigg(\Big(2-\frac{1}{\cosh^{2\Delta}\frac{2\pi t_1}{\beta}}-\frac{1}{\cosh^{2\Delta}\frac{2\pi t_2}{\beta}}\Big)\Big(1-\frac{\frac{\pi(t_2-t_1)}{\beta}}{\tanh\frac{\pi(t_2-t_1)}{\beta}}\Big)$$
$$+\int_{t_1}^{t_2}du\left[\frac{\frac{\pi}{4}\cosh\frac{\pi(t_1+t_2)}{\beta}}{\sinh\frac{\pi(t_2-t_1)}{\beta}}\frac{1}{\cosh^{2\Delta+1}\frac{2\pi u}{\beta}}\right]-\frac{\pi}{2}\coth\frac{\pi(t_2-t_1)}{\beta}\int_{t_1}^{t_2}du\,\frac{1}{(\cosh\frac{2\pi u}{\beta})^{2\Delta}}\Big(1-\frac{\frac{2\pi u}{\beta}}{\coth\frac{2\pi u}{\beta}}\Big)$$
$$+\left[\frac{\frac{\pi(t_1+t_2)}{8\Delta}}{\tanh\frac{\pi(t_2-t_1)}{2}}\Big(\frac{1}{\cosh^{2\Delta}(\frac{2\pi t_2}{\beta})}-\frac{1}{\cosh^{2\Delta}(\frac{2\pi t_1}{\beta})}\Big)\right]\Bigg)\Bigg]. \tag{4.26}$$

At $t_1, t_2 \sim \beta$, one can approximate $\cosh\frac{2\pi t}{\beta} \sim \frac{e^{\frac{2\pi t}{\beta}}}{2}$ and see that all these terms are negligible. Therefore, the terms that survive are:

$$G(t_2-t_1)\left[1+\frac{\beta bg\Delta_1}{S(\frac{\beta J}{\pi})^{2\Delta}}\Big(1-\frac{\frac{\pi(t_2-t_1)}{\beta}}{\tanh\frac{\pi(t_2-t_1)}{\beta}}\Big)\right]=G_{\tilde\beta}(t_2-t_1). \tag{4.27}$$

where

$$\tilde\beta=\beta\Big(1-\frac{\pi bg}{2JS}(\frac{\pi}{\beta J})^{2\Delta-1}\Big). \tag{4.28}$$

This matches the temperature that was predicted from computing the entanglement entropy, i.e. 4.16.

The entanglement entropy from the equation of motion

In this section, we will derive the entanglement entropy from the equation of motion for the boundary. For that, we will use the global AdS coordinate. The relation between coordinate times in Poincare, Schwarzschild, and global time coordinates is:

$$t=\tanh\frac{\pi\tau}{\beta}=\tan\frac{\eta}{2}, \tag{4.29}$$

and so the two-point function between the left and the right boundary in terms the global AdS coordinate is given by:

$$\left\langle\phi_R^i(u_1)\phi_L^j(-u_2)\right\rangle=b\left(\frac{\eta'(u_1)\eta'(u_2)}{4J^2\cos^2\frac{\eta_R(u_1)-\eta_L(u_2)}{2}}\right)^{\Delta}\delta^{ij}. \tag{4.30}$$

Here, η_L and η_R are the restriction of AdS_2 global time η to the left and right boundaries. At low energy limit, we expect the effective action for the boundary to take the following form:

$$S=-\left[\int d\tilde u\,Sch(\tan\frac{\eta_L}{2},\tilde u)+\int d\tilde u\,Sch(\tan\frac{\eta_R}{2},\tilde u)\right]-2\kappa\int d\tilde u\left(\frac{\eta_L'(\tilde u)\,\eta_R'(\tilde u)}{\cos^2\frac{\eta_R-\eta_L}{2}}\right)^{\Delta}$$
$$\kappa=\frac{g}{2}(\frac{b}{(2J)^{2\Delta}})\ (\frac{\phi_r}{8\pi G})^{1-2\Delta},\quad \tilde u=\frac{8\pi G u}{\phi_r}, \tag{4.31}$$

where we approximated the interaction term by $S_{int} \approx \int du\, g(u) \left\langle \phi_L^i(u)\phi_R^i(u)\right\rangle$. We can rewrite the above action as:

$$
\begin{aligned}
S &= \int d\tilde{u} \left(P_{\eta_R}\eta_R' + P_{\eta_L}\eta_L' + P_{\phi_R}\phi_R' + P_{\phi_L}\phi_L' - H\right) \\
H &= \frac{1}{2}\left(P_{\phi_L}^2 + e^{2\phi_L} + 2P_{\eta_L}e^{\phi_L}\right) + \frac{1}{2}\left(P_{\phi_R}^2 + e^{2\phi_R} + 2P_{\eta_R}e^{\phi_R}\right) + 2\kappa\left(\frac{e^{(\phi_R+\phi_L)}}{\cos^2\frac{\eta_R-\eta_L}{2}}\right)^{\Delta}.
\end{aligned}
\tag{4.32}
$$

The equation of motion is given by:

$$
\begin{aligned}
\phi_R' &= P_{\phi_R}, \quad \eta_R' = e^{\phi_R}, \\
P_{\phi_R}' &= -\left(e^{2\phi_R} + P_{\eta_R}e^{\phi_R} + 2\kappa\Delta\left(\frac{e^{(\phi_R+\phi_L)}}{\cos^2\frac{\eta_R-\eta_L}{2}}\right)^{\Delta}\right) \\
P_{\eta_R}' &= -2\kappa\Delta\tan\frac{\eta_R-\eta_L}{2}\left(\frac{e^{(\phi_R+\phi_L)}}{\cos^2\frac{\eta_R-\eta_L}{2}}\right)^{\Delta}.
\end{aligned}
\tag{4.33}
$$

Note that 4.31 still has SL(2,R) symmetry. The corresponding conserved charges are:

$$
Q_3 = Q_3^R + Q_3^L, \quad Q_2 = Q_2^R - Q_2^L, \quad Q_1 = Q_1^R - Q_1^L. \tag{4.34}
$$

where $Q_i^{R(L)}$s are the charges 2.37. Here, the Poisson bracket is with respect to (ϕ_R, P_{ϕ_R}), (ϕ_L, P_{ϕ_L}), (η_L, P_{η_L}), and (η_R, P_{η_R}). Since we are interested in solving 4.33 with initial condition set by B.14 for both sides, and both are symmetric with respect to left and right, $\eta_L(u) = \eta_R(u)$ is always guaranteed, so we can reduce the equation of motion to:

$$
\begin{aligned}
&\phi_R' = P_{\phi_R}, \quad \eta_R' = e^{\phi_R}, \quad P_\eta = 0, \\
&\phi_R'' = \begin{cases} -(e^{2\phi_R} + 2\kappa\Delta\, e^{2\Delta\phi_R}) & u \geq 0 \\ -e^{2\phi_R} & u < 0. \end{cases}
\end{aligned}
\tag{4.35}
$$

One can solve the above equation for $u \geq 0$ perturbatively assuming that $\kappa \ll 1$ with the initial condition

$$
\phi(0) = \ln\frac{S}{2\pi}, \quad \phi'(0) = 0. \tag{4.36}
$$

In particular, we are interested in computing the Casimir function $Q_R = \phi_R'^2 + e^{2\phi_R} = Q_0 + \kappa Q_1 + \kappa^2 Q_2 + O(\kappa^3)$. Equation 4.35 implies:

$$
{\phi_R'}^2 + e^{2\phi_R} + 2\kappa e^{2\Delta\phi_R} = \left(\frac{S}{2\pi}\right)^2 + 2\kappa\left(\frac{S}{2\pi}\right)^{2\Delta}. \tag{4.37}
$$

Taking $\phi = \phi_0 + \kappa\phi_1 + \kappa^2\phi_2 + O(\kappa^3)$ and plugging into 4.35, at leading order we have:

$$
\phi_0(\tilde{u}) = -\ln\frac{2\pi}{S}\cosh\left(\frac{S}{2\pi}\tilde{u}\right), \qquad Q_0 = \left(\frac{S}{2\pi}\right)^2, \tag{4.38}
$$

while in the next order one gets:

$$\phi_1' - \frac{S}{\pi}\frac{1}{\sinh\frac{S\tilde{u}}{\pi}}\,\phi_1 = \Big(\frac{S}{2\pi}\Big)^{2\Delta-1}\Big(1 - \frac{1}{\cosh^{2\Delta}\frac{S\tilde{u}}{2\pi}}\Big)\coth\frac{S\tilde{u}}{2\pi}. \tag{4.39}$$

One can rewrite the above as follows:

$$\Big(\coth\frac{S\tilde{u}}{2\pi}\phi_1\Big)' = \Big(\frac{S}{2\pi}\Big)^{2\Delta-1}\Big(1 - \frac{1}{\cosh^{2\Delta}\frac{Su}{2\pi}}\Big)\coth^2\frac{S\tilde{u}}{2\pi}, \tag{4.40}$$

with the answer:

$$\phi_1 = \Big(\frac{S}{2\pi}\Big)^{2\Delta-2}\Big(-\frac{S\tilde{u}}{2\pi}\tanh\frac{S\tilde{u}}{2\pi} + 1 - \frac{1}{\cosh^{2\Delta}u} + (1-2\Delta)\tanh\frac{S\tilde{u}}{2\pi}\int_0^{\frac{S\tilde{u}}{2\pi}}\frac{1}{\cosh^{2\Delta}x}\Big). \tag{4.41}$$

The leading correction to the Casimir is given by:

$$\begin{aligned} Q_1 &= 2\Big(\Big(\frac{S}{2\pi}\Big)^{2\Delta} - e^{2\Delta\phi_0}\Big)\Big) = 2\Big(\frac{S}{2\pi}\Big)^{2\Delta}\Big(1 - \frac{1}{\cosh^{2\Delta}\frac{S\,\tilde{u}}{2\pi}}\Big), \\ Q_2 &= -4\Delta\, e^{2\Delta\phi_0}\,\phi_1. \end{aligned} \tag{4.42}$$

Then the value of the entropy can be rederived from

$$S(u) = 2\pi\sqrt{Q_R} = 2\pi\sqrt{Q_0}\Big(1 + \frac{\kappa}{2}\frac{Q_1}{Q_0} + \frac{\kappa^2}{2}\Big(\frac{Q_2}{Q_0} - \frac{1}{4}\Big(\frac{Q_1}{Q_0}\Big)^2\Big) + O(\kappa^3)\Big). \tag{4.43}$$

Considering the expansion $S(u) = S_0 + \kappa S_1(u) + \kappa^2 S_2(u)$, we have:

$$\begin{aligned} S_1(u) &= 2\pi\Big(\frac{S}{2\pi}\Big)^{2\Delta-1}\Big(1 - \frac{1}{\cosh^{2\Delta}\frac{S\,\tilde{u}}{2\pi}}\Big), \qquad \frac{S\tilde{u}}{2\pi} = \frac{2\pi u}{\beta}. \\ S_2(u) &= \frac{S}{2}\Big(\frac{S}{2\pi}\Big)^{4\Delta-4}\Big(-\Big(1 - \frac{1}{\cosh^{2\Delta}\frac{S\,\tilde{u}}{2\pi}}\Big)^2 + \frac{4\Delta}{\cosh^{2\Delta}\frac{S\,\tilde{u}}{2\pi}}\Big(\frac{S\tilde{u}}{2\pi}\tanh\frac{S\tilde{u}}{2\pi} - 1 + \frac{1}{\cosh^{2\Delta}\frac{S\tilde{u}}{2\pi}} \\ &\qquad - (1-2\Delta)\tanh\frac{S\tilde{u}}{2\pi}\int_0^{\frac{S\tilde{u}}{2\pi}}\frac{1}{\cosh^{2\Delta}x}\Big)\Big). \end{aligned} \tag{4.44}$$

Note that $\kappa S_1(u)$ and $\kappa^2 S_2(u)$ are exactly equal to 4.14 and 4.23. Notice that the relation of entanglement entropy to the coarsed-grained quantity 4.43, is more than the equality of final answers. In fact, in the second order, the term $\pi\kappa^2\sqrt{Q_0}\frac{Q_2}{Q_0}$ is equal to the second configuration in section 4, and $-\frac{\pi\kappa^2}{4}\sqrt{Q_0}\Big(\frac{Q_1}{Q_0}\Big)^2$ equals the first configuration.

4.1 The explicit solution for $\Delta = \frac{1}{2}$

We can find an exact solution to 4.35 for $\Delta = \frac{1}{2}$:

$$\tan\frac{\eta}{2} = \frac{S}{\tilde{S}}\,\tanh\frac{\pi u}{\tilde{\beta}}, \quad \tilde{\beta} = \beta\left(1 + \frac{4\pi\kappa}{S}\right)^{\frac{-1}{2}}, \tag{4.45}$$

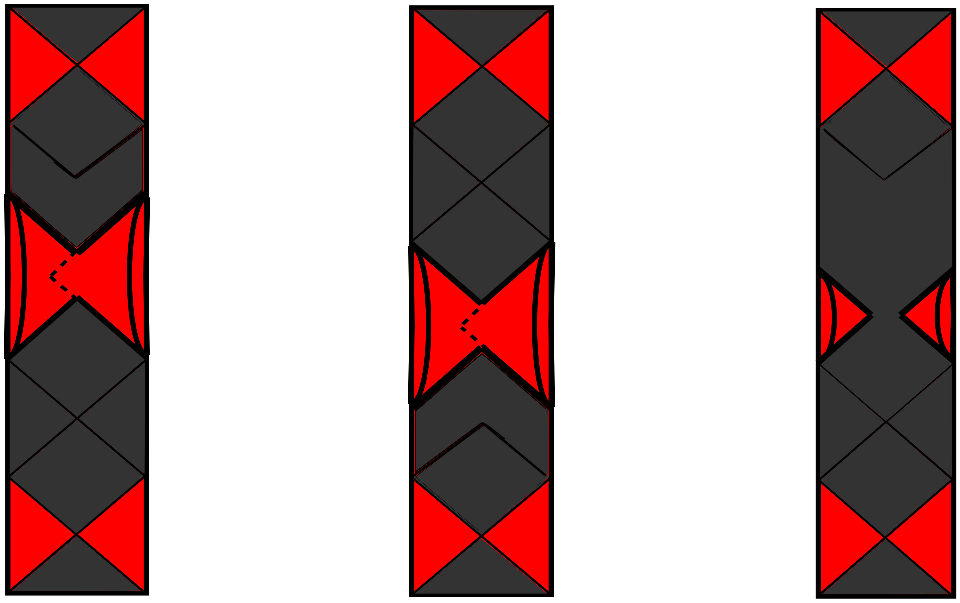


Figure 4.4: The effect of the double trace operator on the black hole in the bulk. (a) represents the case where the interaction is on for $t > 0$. For g<0, the operator will decrease the entropy of the black hole, the black hole's size gets smaller, and so part of the region behind the horizon is now revealed to the outside observer. (b) displays the case where the interaction is on while we go backward in time which shrinks the white hole region. (c) We can repeat (a) but with g>0. This leads to expanding the black hole. The entanglement entropy will increase, the black hole will expand, so the red region will become smaller.

and $\tilde{S}$ is the thermal entropy associated with the new inverse temperature $\tilde{\beta}$, $\left(\frac{S}{\tilde{S}} = \frac{\tilde{\beta}}{\beta}\right)$. Plugging 4.45 into the formula for the Casimir yields:

$$\tilde{S}(u) = 2\pi\sqrt{Q_R} = S\left(\frac{1 + (\frac{\tilde{S}}{S})^2 \tanh^2 \frac{\pi u}{\tilde{\beta}}}{1 + (\frac{S}{\tilde{S}})^2 \tanh^2 \frac{\pi u}{\tilde{\beta}}}\right)^{\frac{1}{2}}. \tag{4.46}$$

Indeed, when $\Delta = \frac{1}{2}$, the effect of the interaction Hamiltonian is to change the black hole's temperature. One can argue that the effect of the double traced operator for $g < 0$ is to shrink the black hole, the Einstein-Rosen bridge, so part of the region that was behind the horizon is now revealed to the outside observer. Therefore, for the black hole interior to be revealed to the outside observer, the observer has to decrease the entanglement entropy of TFD and reach the states $|\widetilde{\text{TFD}}\rangle$ in the Hilbert space. As a consequence, the wormhole between the two sides will shrink and part of the interior can be probed from outside; see Figure 4.4. To quantify this, we define the length of the wormhole at time t as the length of the geodesic that connects the two points at $r = \frac{1}{S}$. Shrinking a given wormhole corresponds to a configuration where the upper bifurcation point will be placed on the wormhole which happens when we decrease the entanglement entropy. For example, for $\Delta = \frac{1}{2}$, the location of the upper bifurcation point

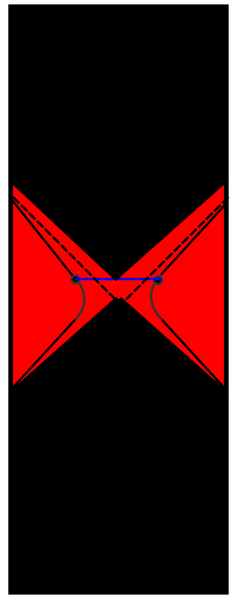


Figure 4.5: The two-sided black hole embedded in the global AdS. The blue segment is the wormhole that connects the two sides whose end points are located on $r = \frac{1}{S}$ (r is the Schwarzschild coordinate). We can shrink such a wormhole by spending the entanglement that exists in the thermofield-double state.

as a function of κ is:

$$\left(\eta_b, \sigma_b\right) = \left(2 \arctan \frac{1}{\sqrt{1 - \frac{4\pi\kappa}{S}}} - \frac{\pi}{2}, 0\right). \tag{4.47}$$

Now, to the leading order in $\frac{1}{S}$, the length of the wormhole satisfies:

$$\cosh d_w = 1 + 2\tan^2 \eta_b, \tag{4.48}$$

or equivalently:

$$\frac{d_w}{2} = -\log\sqrt{1 + \frac{4\pi\kappa}{S}} \approx \frac{2\pi|\kappa|}{S} + O(\frac{\kappa^2}{S^2}) \tag{4.49}$$

where d_w is the length of the wormhole. 4.49 implies that in order to shrink a wormhole of length d_w, one needs to decrease the entanglement entropy to $S - 2\pi|\kappa|$. This may be a manifestation of ER=EPR [30]. To send a message from one side to the other side, we also need to take into account the back reaction of the message on the black hole which leads to the expansion of the wormhole. Such back reactions are due to the shock wave effects [9, 39]. Therefore, to send a message between two parties with a black hole in between, in general, there are two barriers: the wormhole that already exists and the back reaction of the message which leads to its expansion. However, one can use the entanglement as a resource to shrink the wormhole and send the message to the other side.

We can also interpret 4.37 as the conservation of the energy for a particle whose trajectory is the boundary of AdS:

$$\phi_R'^2 + e^{2\phi_R} + 2\kappa e^{2\Delta\phi_R} = E = \left(\frac{S}{2\pi}\right)^2 + 2\kappa\left(\frac{S}{2\pi}\right)^{2\Delta}. \tag{4.50}$$

This implies that at late, time we have:

$$S(u \to \infty) = 2\pi\sqrt{E} = S\,\sqrt{1 + 2\kappa\left(\frac{S}{2\pi}\right)^{2\Delta-2}}. \tag{4.51}$$

This implies a critical value for the coupling κ where the energy of the particle (the entanglement entropy) vanishes:

$$\kappa_* = \frac{-1}{2}\left(\frac{S}{2\pi}\right)^{2-2\Delta} \tag{4.52}$$

Naively, κ_* is the value for which the state $\widetilde{\text{TFD}}$ becomes pure. However, close to this value, our classical computation becomes invalid ($\frac{\phi_r}{\beta J} \sim 1$), and one should consider the quantum Schwarzian [22, 4, 33, 24].

Chapter 5

PERTURBATIVE CALCULATIONS OF THE ENTANGLEMENT ENTROPY

The model and general formulas

Let us consider a quantum system with some Hilbert space $\mathcal{H}_{\mathrm{B}}$ and Hamiltonian H_{B}. For an exact analogy with the evaporation problem, we would have to pick a pure state that looks like thermal to all simple measurements. Instead, we double the system and postulate that its initial state is the thermofield-double, $|\mathrm{TFD}_{\mathrm{B}}\rangle \in \mathcal{H}_{\mathrm{B}}^{*} \otimes \mathcal{H}_{\mathrm{B}}$. Only the right part is coupled to the heat bath, but we are interested in the von Neumann entropy of the double system as its density matrix $\rho_{\mathrm{B^*B}}$ evolves in time. Likewise, the bath is also doubled, so that the initial state of the world is

$$|\Psi_0\rangle = |\mathrm{TFD}_{\mathrm{B}}\rangle \otimes |\mathrm{TFD}_{\mathrm{b}}\rangle \in \mathcal{H}_{\mathrm{B}}^{*} \otimes \mathcal{H}_{\mathrm{B}} \otimes \mathcal{H}_{\mathrm{b}} \otimes \mathcal{H}_{\mathrm{b}}^{*}. \tag{5.1}$$

A similar, but not identical,[1] setting was used in [13, 8], where the s-Renyi entropy for integer $s > 1$ was calculated.

The full Hamiltonian $H = H_{\mathrm{B}} + H_{\mathrm{b}} + H_{\mathrm{Bb}}$ acts only on the two objects in the middle, i.e. on $\mathcal{H}_{\mathrm{B}} \otimes \mathcal{H}_{\mathrm{b}}$. We assume that the interaction term has the form

$$H_{\mathrm{Bb}} = \kappa \sum_{j=1}^{N} O_{\mathrm{B}}^{j} O_{\mathrm{b}}^{j} \tag{5.2}$$

with some bosonic operators O_{B}^{j} and O_{b}^{j}. In the case of fermionic systems like the SYK model [38, 21, 27, 23], we should multiply the coupling parameter κ by i. For simplicity, we will do the computation for a bosonic system, but the final answer will equally be applicable to fermionic systems.

Thus, the evolution of the world in the interaction picture takes the form

$$\rho_{\mathrm{B^*Bbb^*}}(t) = U(t)|\Psi_0\rangle\langle\Psi_0|U^{-1}(t), \qquad U(t) = \mathbf{T}\left(e^{-i\int_0^t H_{\mathrm{Bb}}(u)du}\right), \tag{5.3}$$

where $\mathbf{T}$ stands for time ordering. We also assume that $\langle O_{\mathrm{B}}^{j}\rangle = \langle O_{\mathrm{b}}^{j}\rangle = 0$.[2] In the rest of the section, we will compute the s-Renyi entropy of the system's density matrix $\rho_{\mathrm{B^*B}}(t)$ after tracing out the bath. It is

[1] In Refs. [13, 8], the initial state is taken to be the thermofield-double of two *interacting* subsystems rather than the product of two thermofield-doubles.

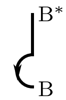
[2] In case that $\langle O^j\rangle \neq 0$, one can work with $O^j - \langle O^j\rangle$.

given by the perturbative expansion

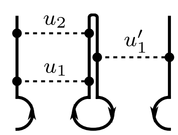
$$\rho_{\mathrm{B^*B}}(t) = \sum_{n,m} \frac{(i\kappa)^n(-i\kappa)^m}{n!\,m!} \int_0^t \Big[\mathbf{T}\{O_{\mathrm{B}}^{j_n}(u_n)\cdots O_{\mathrm{B}}^{j_1}(u_1)\}\big|\mathrm{TFD_B}\big\rangle\big\langle\mathrm{TFD_B}\big|\, \widetilde{\mathbf{T}}\{O_{\mathrm{B}}^{j'_1}(u'_1)\cdots O_{\mathrm{B}}^{j'_m}(u'_m)\} \times \Big\langle \widetilde{\mathbf{T}}\{O_{\mathrm{b}}^{j'_1}(u'_1)\cdots O_{\mathrm{b}}^{j'_m}(u'_m)\}\,\mathbf{T}\{O_{\mathrm{b}}^{j_n}(u_n)\cdots O_{\mathrm{b}}^{j_1}(u_1)\}\Big\rangle\Big]\, du\, du', \tag{5.4}$$

where the expectation value $\langle\cdots\rangle$ is with respect to the bath's thermal state and $\widetilde{\mathbf{T}}$ denotes reverse time ordering. (If $u_1 < \cdots < u_n$ and $u'_1 < \cdots < u'_m$, then the operators are already ordered.) There is also an implicit sum over repeated indices, with each index going from 1 to N. Note that operators with the same indices have the same time argument.

Since the combinatorics might soon get complicated, let us introduce some simplifying graphic notation:

$$|\mathrm{TFD_B}\rangle =$$  , $$\langle\mathrm{TFD_B}| =$$  . (5.5)

Then each term in the expansion (5.4) will look like this (where $\mathbf{T}$, $\widetilde{\mathbf{T}}$, and the indices are omitted):

$$O(u_2)O(u_1)\big|\mathrm{TFD_B}\big\rangle\,\big\langle O(u'_1)O(u_2)O(u_1)\big\rangle_{\mathrm{b}}\big\langle\mathrm{TFD_B}\big|O(u'_1) =$$  for $u_1 < u_2$. (5.6)

The diagram element in the middle is the Keldysh contour for the heat bath. It consists of a circle at the bottom representing imaginary-time evolution and a stem corresponding to the real-time evolution; the time goes up. For integer s, $\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s$ can be represented by gluing such s diagrams (describing different replicas of the density matrix) in the cyclic order — see Figure 5.1, where the replicas are depicted with different colors. The expectation values should be independently computed for each closed contour, whether it corresponds to the system or the bath.

Let us further assume that the system-bath coupling is sufficiently weak. Then the "radiation quanta" emanating from the system are sparse, which means that dominant diagrams have at most two operators with close times per contour. Therefore, the calculation can be done using Wick contraction. Of course, if the fields O_{B}^j, O_{b}^j are Gaussian, then no sparseness condition is necessary.

An example of a (subleading) Wick pairing contributing to $\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s$ is as follows:

$$\big\langle O(v'_1)O(v'_2)\big\rangle_{\mathrm{b}}\big\langle O(v'_1)O(v'_2)O(u_2)O(u_1)\big\rangle_{\mathrm{B}}\big\langle O(u_2)O(u_1)\big\rangle_{\mathrm{b}}, \qquad \text{for } u_1 < u_2, \quad v'_1 < v'_2. \tag{5.7}$$

It corresponds to the black loop at the bottom of Figure 5.1 (a). In general, a Wick contraction diagram is a disjoint union of loops that consists of alternating solid and dotted lines. Solid lines represent contractions of fields on the same contour, whereas dotted lines correspond to interaction terms such as

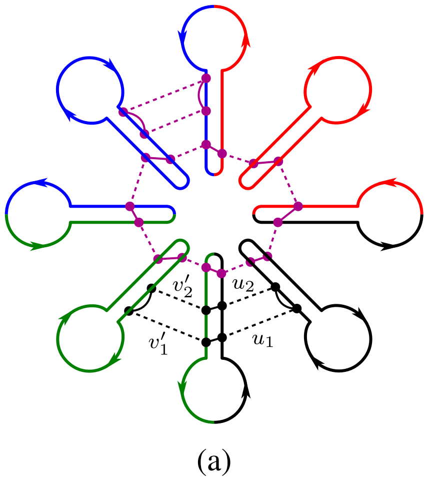


Figure 5.1: The time contour for the system-bath coupling. (a) A diagram for $\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s$, where $\rho_{\mathrm{B^*B}}(t)$ is given by (5.4) and $s = 4$. The thick solid lines make up $2s$ Keldysh contours for the system and the bath in alternating order. Each half of a contour represents a thermofield-double. Lorentzian time is directed toward the center of the diagram. Different replicas are depicted in different colors. Wick contractions of fields are depicted by thin solid lines, while the dotted lines represent system-bath coupling. The leading contributions are due to purple loops; the shorter and the longer loops correspond to equations (5.11) and (5.12), respectively. (b) The simplified diagram obtained by omitting the bath's replicas.

$O^j_{\mathrm{B}}(u)O^j_{\mathrm{b}}(u)$. Each contraction comes with a Kronecker delta identifying the indices of the contracted fields. The result is nonzero if all the indices on each loop are the same. Therefore, each loop with d solid line segments evaluates to $N\kappa^d$ multiplied by the product of two-point functions.

The diagrams for $\ln\bigl(\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s\bigr)$ are connected, i.e. contain a single loop. The factor of N that appears here is usual for extensive thermodynamic quantities; thus, the intensive parameter is κ^d. If κ is small, we should only keep loops with $d = 2$ contractions (in one replica) and $d = 2s$ contractions (traversing all the replicas). Both types of loops are shown in purple in figure 5.1.

There are three types of two-point functions for the system,

$$\bigl\langle \mathbf{T}\{O^j_{\mathrm{B}}(u)O^l_{\mathrm{B}}(v)\}\bigr\rangle = iG_{\mathbf{T}\mathrm{B}}(u-v)\delta_{jl}, \qquad \bigl\langle \widetilde{\mathbf{T}}\{O^j_{\mathrm{B}}(u)O^l_{\mathrm{B}}(v)\}\bigr\rangle = iG_{\widetilde{\mathbf{T}}\mathrm{B}}(u-v)\delta_{jl}, \tag{5.8}$$

$$\bigl\langle O^{j'}_{\mathrm{B}}(u')O^j_{\mathrm{B}}(u)\bigr\rangle = iG_{\mathrm{B}}(u'-u)\delta_{j'j}, \tag{5.9}$$

and similarly for the bath. The expressions for closed paths will be simplified if we think of a two-point function as the matrix element of a bilocal operator $\widehat{G}$,

$$\langle u'|\widehat{G}|u\rangle = f_t(u')G(u'-u)f_t(u), \qquad f_t(u) \equiv \theta(u) - \theta(u-t), \tag{5.10}$$

where $f_t(u)$ is a time window function that vanishes outside the interval $(0,t)$. This way, we can extend the time domain to $(-\infty,\infty)$ and avoid putting limits on the integrals. We also define the transpose of the operator $\widehat{G}$, denoted by $\widehat{G}^{\intercal}$, with the matrix element $\langle t_1|\widehat{G}^{\intercal}|t_2\rangle = G(t_2 - t_1)$. Note that $\widehat{G}_{\mathbf{T}}$ and $\widehat{G}_{\widetilde{\mathbf{T}}}$ for bosons are symmetric, i.e. equal to their transpose. To illustrate this notation, the expression (**??**) involves a loop with four Wick contractions, $v_1' \overset{\mathrm{b}}{\leftarrow} v_2' \overset{\mathrm{B}}{\leftarrow} u_2 \overset{\mathrm{b}}{\leftarrow} u_1 \overset{\mathrm{B}}{\leftarrow} v_1'$; hence, the result takes the form $\mathrm{Tr}\big(\widehat{G}_{\widetilde{\mathbf{T}}\mathrm{b}} \circ \widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathbf{T}\mathrm{b}} \circ \widehat{G}_{\mathrm{B}}^{\intercal}\big)$.

As already mentioned, there are two types of loops that contribute to $\ln\big(\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s\big)$ to leading order. The loops of length $d = 2$ can themselves be of two forms, one of which appears in figure 5.1. They give the following contributions:

$$P_{2,\mathbf{T}}(t) = N\kappa^2\ \mathrm{Tr}\big(\widehat{G}_{\mathbf{T}\mathrm{B}} \circ \widehat{G}_{\mathbf{T}\mathrm{b}}^{\intercal}\big), \qquad P_{2,\widetilde{\mathbf{T}}}(t) = N\kappa^2\ \mathrm{Tr}\big(\widehat{G}_{\widetilde{\mathbf{T}}\mathrm{B}} \circ \widehat{G}_{\widetilde{\mathbf{T}}\mathrm{b}}^{\intercal}\big). \tag{5.11}$$

Here we have used the fact that $(\pm i)^2$ from (5.4) cancels i^2 from (5.8). (The extra factors present in the fermionic case also cancel each other.) There are also loops of length $d = 2s$, which we say to have winding number 1 because they traverse all replicas. The expression for such a loop takes the following form:

$$P_{2s}(t) = N\kappa^{2s}\ \mathrm{Tr}\big(-\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big)^s. \tag{5.12}$$

Since in the end we are interested in the limit $s \to 1$, the quantities (5.12) and (5.11) are of the same order in the coupling.

As an exercise, let us sum up the leading diagrams in $\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s$ — we should get the exponential of a sum of single loops with certain coefficients. It is sufficient to only keep track of the system's replica, leaving the bath implicit as in figure 5.1 (b). Let $m_1, \ldots, m_s$ and $n_1, \ldots, n_s$ be the numbers of fields in the time ordered and anti-time ordered branches of the Keldysh contours. The number of ways to break them into k loops of length $2s$ with winding number 1 and some loops of length 2 (with winding number 0) is given by

$$(k!)^{2s-1} \prod_{r=1}^{s} \binom{m_r}{k}\binom{n_r}{k} \frac{(m_r-k)!}{\left(\frac{m_r-k}{2}\right)!\,2^{\frac{m_r-k}{2}}} \frac{(n_r-k)!}{\left(\frac{n_r-k}{2}\right)!\,2^{\frac{n_r-k}{2}}}. \tag{5.13}$$

Defining $m_r - k = 2p_r$ and $n_r - k = 2q_r$, after manipulation we will get

$$\begin{aligned} \mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s &= \sum_{k=0}^{\infty} \sum_{p_1,\ldots,p_s} \sum_{q_1,\ldots,q_s} \frac{1}{k!} \prod_{r=1}^{s} \frac{1}{(p_r)!(q_r)!\,2^{p_r}2^{q_r}}\ \big(P_{2s}(t)\big)^k\ \big(P_{2,\mathbf{T}}(t)\big)^{p_r} \big(P_{2,\widetilde{\mathbf{T}}}(t)\big)^{q_r} \\ &= \exp\left(\frac{1}{2}\left(P_{2,\mathbf{T}}(t) + P_{2,\widetilde{\mathbf{T}}}(t)\right) + P_{2s}(t)\right). \end{aligned} \tag{5.14}$$

Using the fact that $\mathrm{Tr}\big(\widehat{G}_{\mathbf{T}\mathrm{B}} \circ \widehat{G}_{\mathbf{T}\mathrm{b}}^{\intercal}\big) + \mathrm{Tr}\big(\widehat{G}_{\widetilde{\mathbf{T}}\mathrm{B}} \circ \widehat{G}_{\widetilde{\mathbf{T}}\mathrm{b}}^{\intercal}\big) = 2\,\mathrm{Tr}\big(\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big)$, the final answer is as follows:

$$\boxed{\ln\Big(\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s\Big) = N\Big(s\kappa^2\ \mathrm{Tr}\big(\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big) + \kappa^{2s}\,\mathrm{Tr}\Big(\big(-\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\intercal}\big)^s\Big)\Big).} \tag{5.15}$$

While the above equation was derived for bosonic systems, it is the same for fermionic systems like the SYK model.

Short initial period vs. linear growth

Equation (5.15) allows one to compute the von Neumann entropy $S(\rho_{\mathrm{B^*B}}(t))$ for $t < t_{\mathrm{Page}}$. Although the exact answer is model-dependent, there are two universal regimes: very early times, just after the system-bath coupling is turned on, and intermediate times, when the entropy grows linearly.

Very early times: Let $t \ll t_{\mathrm{UV}}$ such that the effect of the Hamiltonians H_{B} and H_{b} is negligible. For example, $t_{\mathrm{UV}} = J^{-1}$ for the SYK model. More exactly, we assume that the Green functions $G_{\mathrm{B}}(t)$ and $G_{\mathrm{b}}(t)$ may be approximated by some constants. Then the expression (5.15) and the von Neumann entropy take this form:

$$\ln\Big(\mathrm{Tr}\,(\rho_{\mathrm{B^*B}}(t))^s\Big) \approx N\Big(-s\,c\kappa^2t^2+\big(c\kappa^2t^2\big)^s\Big), \tag{5.16}$$

$$\boxed{S\big(\rho_{\mathrm{B^*B}}(t)\big) \approx c\kappa^2t^2\,\big(-\ln(c\kappa^2t^2)+1\big), \qquad \text{where} \quad c = -G_{\mathrm{B}}(0)G_{\mathrm{b}}(0).} \tag{5.17}$$

Intermediate times: For systems with continuous excitation spectrum, connected correlators decay in time. Exponential decay is typical; for example, in a conformal system at finite temperature, the correlator of fields with scaling dimension Δ decays as $\exp\big(-\frac{2\pi\Delta}{\beta}t\big)$ if $t \gg \frac{\beta}{\Delta}$. Let us assume that both $G_{\mathrm{B}}(t)$ and $G_{\mathrm{b}}(t)$ decay exponentially at $t \gg t_*$.

If $t \gg t_*$, then $\mathrm{Tr}\big(\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\mathsf{T}}\big)$ can be approximated as follows. This expression is an integral over $u, u' \in (0,t)$, but the integrand is negligible unless $|u'-u| \lesssim t_*$. Therefore, we may remove the limits on u', and then use the Fourier transform:

$$\mathrm{Tr}\big(\widehat{G}_{\mathrm{B}} \circ \widehat{G}_{\mathrm{b}}^{\mathsf{T}}\big) \approx \int_0^t du \int_{-\infty}^{\infty} du'\, G_{\mathrm{B}}(u',u)G_{\mathrm{b}}(u',u) = t\int \tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\,\frac{d\omega}{2\pi}, \tag{5.18}$$

where $\tilde{G}(\omega) = \int_{-\infty}^{\infty} G(t)e^{i\omega t}\,dt$. The second term in (5.15) can also be approximated in such a way. Thus,

$$\ln\Big(\mathrm{Tr}(\rho(t)_{\mathrm{B^*B}})^s\Big) \approx NA(s)\,t, \tag{5.19}$$

where

$$A(s) = s\kappa^2\int \tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\,\frac{d\omega}{2\pi} + \kappa^{2s}\int \big(-\tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\big)^s\,\frac{d\omega}{2\pi}. \tag{5.20}$$

After analytically continuing to $s = 1$, the von Neumann entropy will be given by

$$S\big(\rho_{\mathrm{B^*B}}(t)\big) \approx -NA'(1)\,t, \tag{5.21}$$

$$A'(1) = \kappa^2\int \tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\Big(-\ln\Big(-\kappa^2\tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)\Big)+1\Big)\frac{d\omega}{2\pi}. \tag{5.22}$$

The integrals in $A(s)$ and $A'(1)$ converge because a possible peak at $\omega = 0$ is broadened to have a width $\omega_{\min} = t_*^{-1}$. There is also a natural UV cutoff at $\omega_{\max} = t_{\mathrm{UV}}^{-1}$. An interesting case is where both the system and the bath are conformal at $\omega \ll \omega_{\max}$. Let us first assume that the temperature is zero; then $\tilde{G}_{\mathrm{B}}(\omega) \propto \omega^{2\Delta_{\mathrm{B}}-1}$ and $\tilde{G}_{\mathrm{b}}(\omega) \propto \omega^{2\Delta_{\mathrm{b}}-1}$ for $\omega > 0$, but both $\tilde{G}_{\mathrm{B}}$ and $\tilde{G}_{\mathrm{b}}$ vanish at $\omega < 0$. (Recall that these are Wightman functions.) Hence, $\tilde{G}_{\mathrm{B}}(\omega)\tilde{G}_{\mathrm{b}}(-\omega)$ is zero for all $\omega \neq 0$. At finite temperature, the integral in (5.22) is dominated by the region $\omega \sim \beta^{-1}$, where $i\tilde{G}_{\mathrm{B}}(\omega) \sim t_{\mathrm{UV}}(t_{\mathrm{UV}}/\beta)^{2\Delta_{\mathrm{B}}-1}$ and $i\tilde{G}_{\mathrm{b}}(\omega) \sim t_{\mathrm{UV}}(t_{\mathrm{UV}}/\beta)^{2\Delta_{\mathrm{b}}-1}$. It follows that

$$\frac{dS(\rho_{\mathrm{B^*B}}(t))}{N\,dt} = -A'(1) \sim \frac{-x\ln x}{\beta}, \qquad \text{where} \quad x = (\beta\kappa)^2\left(\frac{\beta}{t_{\mathrm{UV}}}\right)^{-2(\Delta_{\mathrm{B}}+\Delta_{\mathrm{b}})}. \tag{5.23}$$

A good example is two SYK models at large βJ. (A bath with $\Delta_{\mathrm{b}} = \frac{1}{2}$ can also be realized by a critical Majorana chain.) The Renyi entropies in this case were studied in [8] using the effective action method, which is generally more powerful than perturbation theory. However, the analytic continuation to $s = 1$ was not obtained. The computed growth rate of the s-Renyi entropy for integer $s > 1$ is consistent with our estimate,

$$\frac{dS_s(\rho_{\mathrm{B^*B}}(t))}{N\,dt} = \frac{A(s)}{1-s} \sim \beta\kappa^2(\beta J)^{-2(\Delta_{\mathrm{B}}+\Delta_{\mathrm{b}})}. \tag{5.24}$$

5.1 Perturbations to the saturated phase

We now consider the system at later times, such that its von Neumann entropy has reached the coarse-grained (thermodynamic) entropy. The entanglement entropy in this phase shows interesting behavior under perturbations. For example, a short impulse increasing the system's energy (similar to throwing a rock into a black hole) will cause a resurgence of entropy growth. Indeed, such an action can be described by some unitary operator V. It increases the coarse-grained entropy and effective temperature, though the true microscopic entropy does not change. Letting the system interact with the bath, we should see a behavior similar to the cusp in the Page curve. Specifically, we expect the von Neumann entropy to grow until it becomes equal to the coarse-grained entropy. Since the growth rate is constant while the perturbation can be arbitrarily weak, the resurgence can be short — just slightly longer than the scrambling time. It will be followed by a thermal equilibration period, when both the coarse-grained and microscopic entropies decrease, see Figure 5.2. Thus, the Page curve cusp is accessible in this setting. However, to actually produce a cusp, the perturbation should be sufficiently strong, likely beyond the Taylor expansion. We will study a simpler problem, calculating the effect in the lowest order.

While our formal goal is to compute the von Neumann entropy in a rather general setting, the key result pertains to systems that saturate the chaos bound [26]. The expression obtained in this case admits a holographic interpretation, which will be discussed at the end.

Statement of the problem

For the study of the saturated phase, it is sufficient to consider one copy of the system and the bath rather than the thermofield-double. The bath can be integrated out, giving rise to the interaction function

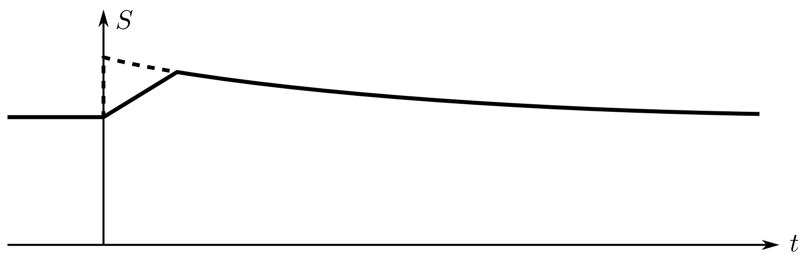


Figure 5.2: Qualitative plot of the system's coarse-grained entropy (dashed line) and the entanglement entropy (solid line) in the presence of an instantaneous perturbation.

$\sigma(\tau_1,\tau_2) = \kappa^2 G_b(\tau_1,\tau_2)$ on the Keldysh contour. Let us simplify the model a bit and replace the interaction σ, which is constantly on, with a superoperator R acting at a specific time. (The results we will obtain in this setting can be easily generalized to the original model.) So, the exact problem involves a quantum system at thermal equilibrium subjected to a sequence of two instantaneous perturbations, V and R.

Let $V = e^{-ixX}$, where X is a Hermitian operator and x is a small parameter. We consider the action of V on the thermal state ρ_0 and expand the resulting density matrix $\rho_1(x)$ to the second order in x:

$$\rho_1(x) = V\rho_0 V^\dagger \approx \rho_0 - ix[X,\rho_0] + x^2\left(X\rho_0 X - \frac{1}{2}\big(X^2\rho_0 + \rho_0 X^2\big)\right). \tag{5.25}$$

In fact, a nontrivial effect will be seen in the second order, and only when combined with a subsequent interaction with the environment. The latter is described by a physically realizable (i.e. completely positive, trace-preserving) superoperator R. Suppose that R is close to the identity such that it can be expanded to the first order in some parameter ϵ:

$$R \approx 1\cdot 1 + \epsilon L, \qquad L = -i(C\cdot 1 - 1\cdot C) + \sum_j \left(A_j\cdot A_j^\dagger - \frac{1}{2}\big(A_j^\dagger A_j\cdot 1 + 1\cdot A_j A_j^\dagger\big)\right), \tag{5.26}$$

where $A\cdot B$ stands for the superoperator that takes ρ to $A\rho B$. The first term in L (which involves a Hermitian operator C and acts as $\rho \mapsto -i[C,\rho]$) may be neglected because it represents an infinitesimal unitary transformation, and thus, does not change the entropy. The sum over j (known as Lindbladian) corresponds to tracing out the environment. As will be justified later, we may replace L with $\sum_j A_j\cdot A_j^\dagger$ so that the final density matrix becomes

$$\begin{aligned}\rho(x,\epsilon) = R(\rho_1(x)) = (1-ixX)\rho_0(1+ixX) + \epsilon\sum_j A_j(1-ixX)\rho_0(1+ixX)A_j^\dagger \\ + \text{unimportant terms.}\end{aligned} \tag{5.27}$$

Our goal is to compute $\frac{\partial^2}{\partial x^2}\frac{\partial}{\partial\epsilon}S(\rho(x,\epsilon))$, where $S(\rho) = -\operatorname{Tr}(\rho\ln\rho)$.

We assume that V acts at time 0, whereas R acts at a later time t. Thus, A_j is understood as $A_j(t) = e^{iH_0 t}A_j(0)e^{-iH_0 t}$, where $A_j(0)$ is some simple (e.g. one- or two-body) operator. The calculation will be done by the replica method for a general large N system in the early time regime, i.e. before the scrambling time. However t is taken to be sufficiently large such that OTOCs are parametrically greater than correlators with non-alternating times. Note that $\rho(x,\epsilon)$ involves only non-alternating operators such as $A_j X \rho_0 X A_j^\dagger$. However, OTOCs appear due to the use of replicas. The "unimportant terms" in (5.27) are exactly those that do not generate any OTOCs.

In the next section, we study partial derivatives of $S(\rho)$, assuming that ρ depends on parameters in some particular way. This setting does not directly include the function $\rho(x,\epsilon)$ given by equation (5.27). To cover this case, we will use a trick called "locking two operators in the same replica", see Section 5.1.

Thermodynamic response theory for the replicated system

Let us recall the standard definition of connected correlators. We begin with the partition function $Z = \operatorname{Tr} W$, where W is the imaginary-time evolution operator:

$$W = \mathbf{T} \exp\left(-\int_0^\beta H(\tau)\, d\tau \right). \tag{5.28}$$

Without perturbation, we have $H(\tau) = H_0$. The insertion of operators $X_1, \dots, X_n$ at times $\tau_1, \dots, \tau_n$ is described by perturbing the Hamiltonian:

$$H(\tau) = H_0 - \sum_{j=1}^{n} x_j \delta(\tau - \tau_j)\, X_j, \qquad \beta \ge \tau_n \ge \cdots \ge \tau_1 \ge 0, \tag{5.29}$$

where x_j are infinitesimal numbers. We generally assume that the operators X_j are bosonic. (If any of them is fermionic, the corresponding variable x_j should be anti-commuting.) Thus,

$$W(\beta, x_n, \dots x_1) = e^{-(\beta-\tau_n)H_0}(1 + x_n X_n) e^{-(\tau_n - \tau_{n-1})H_0} \cdots (1 + x_1 X_1) e^{-\tau_1 H_0} \tag{5.30}$$

and $Z(\beta, x_n, \dots x_1) = \operatorname{Tr} W(\beta, x_n, \dots x_1)$. The full correlator is simply

$$\langle X_n(\tau_n) \cdots X_1(\tau_1) \rangle = Z^{-1} \left. \frac{\partial^n Z}{\partial x_1 \cdots \partial x_n} \right|_{x_1 = \cdots = x_n = 0}. \tag{5.31}$$

The corresponding connected correlator is defined as follows:[3]

$$\langle X_n(\tau_n), \dots, X_1(\tau_1) \rangle = \left. \frac{\partial^n \ln Z}{\partial x_1 \cdots \partial x_n} \right|_{x_1 = \cdots = x_n = 0}. \tag{5.32}$$

For example, $\langle X, Y \rangle = \langle XY \rangle - \langle X \rangle \langle Y \rangle$ and

$$\langle X, Y, Z \rangle = \langle XYZ \rangle - \langle XY \rangle \langle Z \rangle - \langle XZ \rangle \langle Y \rangle - \langle YZ \rangle \langle X \rangle + 2 \langle X \rangle \langle Y \rangle \langle Z \rangle. \tag{5.33}$$

[3] We use commas instead of double brackets because the usual notation (without commas) has some ambiguity.

Now, let us introduce s replicas of the system, such that the partition function becomes

$$Z(s,\beta,x_n,\ldots,x_1) = \mathrm{Tr}\big(W(\beta,x_n,\ldots,x_1)\big)^s. \tag{5.34}$$

We may think of the parameter s as being associated with a *branching operator* $\mathbf{B}$, which commutes with everything. It is not defined by itself but only through its connected correlators:

$$\boxed{\langle \mathbf{B}, X_n(\tau_n),\cdots,X_1(\tau_1)\rangle = \frac{\partial^n}{\partial x_1\cdots\partial x_n}\bigg|_{x_1=\cdots=x_n=0} \frac{\partial \ln Z}{\partial s}\bigg|_{s=1}.} \tag{5.35}$$

The branched correlator (5.35) is related to the entropy $S = S(\rho)$ of the density matrix $\rho = Z^{-1}W$ at $s = 1$ because

$$\big(\partial_s(\ln Z)\big)\big|_{s=1} = \ln Z - S. \tag{5.36}$$

Thus, the entropy derivative with respect to $x_1,\ldots,x_n$ is given by $-\langle \mathbf{B}, X_n(\tau_n),\cdots,X_1(\tau_1)\rangle + \langle X_n(\tau_n),\cdots,X_1(\tau_1)\rangle$. It is usually the easiest to compute the derivative of the relative entropy, $S(\rho||\rho_0) = \mathrm{Tr}(\rho(\ln\rho - \ln\rho_0))$:

$$\begin{aligned}\frac{\partial^n S(\rho||\rho_0)}{\partial x_1\cdots\partial x_n}\bigg|_{x_1=\cdots=x_n=0} &= \big\langle(\mathbf{B}+\beta H_0), X_n(\tau_n),\cdots,X_1(\tau_1)\big\rangle - \big\langle X_n(\tau_n),\cdots,X_1(\tau_1)\big\rangle \\ &= \frac{\partial^n}{\partial x_1\cdots\partial x_n}\bigg|_{x_1=\cdots=x_n=0} \Big(\partial_s\big(s^{-1}\ln\mathrm{Tr}\,(W(\beta/s,\ldots))^s\big)\Big)\Big|_{s=1}.\end{aligned} \tag{5.37}$$

For integer s, the expression $\mathrm{Tr}\,(W(\beta/s,\ldots))^s$ may be interpreted in terms of gluing s intervals of length β/s to make a circle of length β. The operators $X_n(\tau_n),\ldots,X_1(\tau_1)$ are distributed along that circle. Thus, the number in question is, essentially, a correlation function at the given β.

An important caveat is that there is no natural definition of the full correlator $\langle \mathbf{B}\,A\rangle$ as a function of A such that one could compute $\langle \mathbf{B}\,YZ\rangle$ by substituting YZ for A. If such a function (with the usual relation to the connected correlator) existed, we would have this corollary of equation (5.33): $\langle \mathbf{B},Y,Z\rangle = \langle \mathbf{B},YZ\rangle - \langle \mathbf{B},Y\rangle\langle Z\rangle - \langle Y\rangle\langle \mathbf{B},Z\rangle$. But this last identity is false because in the expression for $\langle \mathbf{B},Y,Z\rangle$, the operators Y and Z can occur in different replicas, but in $\langle \mathbf{B},YZ\rangle$, they cannot.

Branched two-point correlator

Suppose that the ordinary correlation function $\langle Y(\tau),X(0)\rangle$ is known on the imaginary axis, $\tau = it$, and let us use its Fourier transform in t. In these terms,

$$\langle Y(\tau),X(0)\rangle = \int_{-\infty}^{\infty} F_{Y,X}(\omega)e^{-\omega\tau}\,\frac{d\omega}{2\pi}. \tag{5.38}$$

The corresponding branched correlator is expected to have a similar form,

$$\big\langle \mathbf{B}+\beta H_0, Y(\tau), X(0)\big\rangle - \big\langle Y(\tau),X(0)\big\rangle = \int_{-\infty}^{\infty} h_{Y,X}(\omega)e^{-\omega\tau}\,\frac{d\omega}{2\pi}. \tag{5.39}$$

The goal of this section is to find the function $h_{Y,X}$.

Let us consider the Fourier modes of the operators Y and X, for example, $Y_\omega = \int Y(it)e^{i\omega t}\,dt$. Their connected correlator is

$$\langle Y_\omega, X_{\omega'}\rangle = F_{Y,X}(\omega)\cdot 2\pi\delta(\omega+\omega'), \tag{5.40}$$

and we also have

$$Y(\tau) = \int \underbrace{Y_\omega e^{-\omega\tau}}_{Y_\omega(\tau)}\,\frac{d\omega}{2\pi}, \qquad X(0) = \int \underbrace{X_\omega}_{X_\omega(0)}\,\frac{d\omega}{2\pi}. \tag{5.41}$$

We now calculate the branched correlator of Y_ω and $X_{\omega'}$, equal to $h_{Y,X}(\omega)\cdot 2\pi\delta(\omega+\omega')$. When the number of replicas s is a positive integer, each of the operators in question can be inserted in any replica, so the calculation involves a double sum. Since each replica's length is β/s, putting Y_ω in the k-th replica is described by $Y_\omega(k\beta/s) = Y_\omega e^{-k\beta\omega/s}$. With this in mind, we get:

$$\big\langle \mathbf{B} + \beta H_0, Y_\omega, X_{\omega'}\big\rangle - \big\langle Y_\omega, X_{\omega'}\big\rangle \tag{5.42}$$

$$= \left(\partial_s\left(s^{-1}\sum_{k=0}^{s-1}\sum_{l=0}^{s-1}\mathbf{T}\Big\langle Y_\omega\Big(\frac{k\beta}{s}\Big),\ X_{-\omega'}\Big(\frac{l\beta}{s}\Big)\Big\rangle\right)\right)\Bigg|_{s=1} \tag{5.43}$$

$$= \left(\partial_s\sum_{k=0}^{s-1}\Big\langle Y_\omega\Big(\frac{k\beta}{s}\Big),\ X_{-\omega'}(0)\Big\rangle\right)\Bigg|_{s=1} \tag{5.44}$$

$$= \big\langle Y_\omega, X_{\omega'}\big\rangle\Bigg(\partial_s\underbrace{\sum_{k=0}^{s-1} e^{-k\beta\omega/s}}_{\frac{1-u}{1-u^{1/s}}\ \text{for}\ u=e^{-\beta\omega}}\Bigg)\Bigg|_{s=1} = 2\pi\delta(\omega+\omega')\cdot F_{Y,X}(\omega)\frac{\beta\omega}{e^{\beta\omega}-1}. \tag{5.45}$$

Thus,

$$\boxed{h_{Y,X}(\omega) = F_{Y,X}(\omega)\frac{\beta\omega}{e^{\beta\omega}-1}.} \tag{5.46}$$

Branched correlator related to early-time OTOCs

Let us recall the original problem of computing $S(\rho(x,\epsilon))$ with $\rho(x,\epsilon)$ given by equation (5.27). In this section, we calculate an analogous branched correlator $\langle \mathbf{B}, A_j(\beta+it), X(\beta), X(0), A_k^\dagger(it)\rangle$ and, more generally,

$$\big\langle \mathbf{B}, X_4(\beta+it_4), X_3(\beta+it_3), X_2(it_2), X_1(it_1)\big\rangle \qquad \text{for}\ t_1,t_4\approx t, \quad t_2,t_3\approx 0. \tag{5.47}$$

One can eliminate β from the time arguments by cyclically permuting $X_4,\dots,X_1$. As already mentioned, the replica calculation involves OTOCs, which are dominant for sufficiently large t. Neglecting all terms with non-alternating times, we get:

$$\begin{aligned}\mathcal{B} := \big\langle \mathbf{B}, X_4(\beta+it_4), X_3(\beta+it_3), X_2(it_2), X_1(it_1)\big\rangle &= \big\langle \mathbf{B}, X_2(it_2), X_1(it_1), X_4(it_4), X_3(it_3)\big\rangle \\ &\approx \Big(\partial_s\big(\mathcal{B}_+(s)+\mathcal{B}_-(s)\big)\Big)\Big|_{s=1},\end{aligned} \tag{5.48}$$

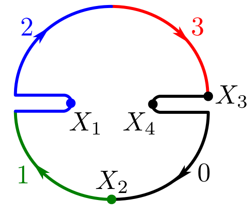


Figure 5.3: Graphic representation of a single term in equation (5.49). In this example, $s = 4$ (with the replicas labeled $0, 1, 2, 3$), $k = 1$, $j = 0$, $l = 2$, and $t_2 = t_3 = 0$.

where

$$\mathcal{B}_+(s) = \sum_{k=0}^{s-1}\sum_{j=0}^{k}\sum_{l=k+1}^{s-1}\left\langle X_1\left(it_1+\frac{l\beta}{s}\right), X_2\left(it_2+\frac{k\beta}{s}\right), X_4\left(it_4+\frac{j\beta}{s}\right), X_3(it_3)\right\rangle, \tag{5.49}$$

$$\mathcal{B}_-(s) = \sum_{k=0}^{s-1}\sum_{j=0}^{k}\sum_{l=k+1}^{s-1}\left\langle X_2(it_2), X_1\left(it_1-\frac{j\beta}{s}\right), X_3\left(it_3-\frac{k\beta}{s}\right), X_4\left(it_4-\frac{l\beta}{s}\right)\right\rangle. \tag{5.50}$$

(Equation (5.49) is illustrated by figure 5.3.) In order to make further progress, we will use the single-mode ansatz for early-time OTOCs,

$$\big\langle X_1(\tau_1), X_2(\tau_2), X_4(\tau_4), X_3(\tau_3)\big\rangle \approx -C^{-1}e^{-i\varkappa(\tau_1+\tau_4-\tau_2-\tau_3-\beta/2)/2}\,\Upsilon^{\mathrm{R}}_{X_1,X_4}(\tau_1-\tau_4)\Upsilon^{\mathrm{A}}_{X_2,X_3}(\tau_2-\tau_3), \tag{5.51}$$

combined with the Fourier representation

$$\Upsilon^{\mathrm{R}}_{X_1,X_4}(\tau) = \int \tilde{\Upsilon}^{\mathrm{R}}_{X_1,X_4}(\omega)e^{-\omega\tau}\,\frac{d\omega}{2\pi}, \qquad \Upsilon^{\mathrm{A}}_{X_2,X_3}(\tau) = \int \tilde{\Upsilon}^{\mathrm{A}}_{X_2,X_3}(\omega)e^{-\omega\tau}\,\frac{d\omega}{2\pi}. \tag{5.52}$$

The result has this general form:

$$\mathcal{B} = -C^{-1}e^{\varkappa(t_1+t_4-t_2-t_3)/2}\int \tilde{\mathcal{B}}(\omega_{14},\omega_{23})\,e^{-i\omega_{14}(t_1-t_4)}e^{-i\omega_{23}(t_2-t_3)}\,\frac{d\omega_{23}}{2\pi}\frac{d\omega_{14}}{2\pi}, \tag{5.53}$$

$$\tilde{\mathcal{B}}(\omega_{14},\omega_{23}) = \tilde{\Upsilon}^{\mathrm{R}}_{X_1,X_4}(\omega_{14})\tilde{\Upsilon}^{\mathrm{A}}_{X_2,X_3}(\omega_{23})\,f(\omega_{14},\omega_{23}). \tag{5.54}$$

Hence, the task is to compute $f(\omega_{14},\omega_{23})$.

First, we find the similar function $f_+(s;\omega_{14},\omega_{23})$ related to $\mathcal{B}_+(s)$. Let

$$u = e^{-\beta\omega_{23}}, \qquad v = e^{-\beta\omega_{14}}, \qquad w = e^{-i\beta\varkappa/2}. \tag{5.55}$$

Then

$$f_+(s;\omega_{14},\omega_{23}) = \sum_{k=0}^{s-1}\sum_{j=0}^{k}\sum_{l=k+1}^{s-1} u^{k/s}v^{(l-j)/s}w^{(j+l-k)/s-1/2} \tag{5.56}$$

$$= \frac{w^{-1/2}}{(1-(w/v)^{1/s})(1-(wv)^{1/s})}\Bigg((vw)^{1/s}\frac{1-u/v}{1-(u/v)^{1/s}} - vw\frac{1-u/w}{1-(u/w)^{1/s}} - w^{2/s}\frac{1-uw}{1-(uw)^{1/s}} + v^{1-1/s}w^{1+1/s}\frac{1-u/v}{1-(u/v)^{1/s}}\Bigg), \tag{5.57}$$

and hence,

$$\left(\partial_s f_+(s;\omega_{14},\omega_{23})\right)\Big|_{s=1} = -\frac{1}{(1-uw)(1-uw^{-1})}\left(\frac{w^{-1/2}}{1-v^{-1}u^{-1}}+\frac{w^{1/2}}{v^{-1}-u^{-1}}\right)\ln u + \frac{1}{(1-v^{-1}w)(1-v^{-1}w^{-1})}\left(\frac{w^{-1/2}}{1-uv}+\frac{w^{1/2}}{u-v}\right)\ln v + \frac{w^{-1/2}(1+uv^{-1})-w^{1/2}(u+v^{-1})}{(1-uw)(1-uw^{-1})(1-v^{-1}w)(1-v^{-1}w^{-1})}\ln w. \tag{5.58}$$

The function f_- is obtained from f_+ by replacing w with w^{-1}. Adding both terms together, we get:

$$f(\omega_{14},\omega_{23}) = \frac{(w^{-1/2}+w^{1/2})(u^{-1}-1)(1+v^{-1})}{(1-uw)(1-uw^{-1})(1-u^{-1}v^{-1})(v^{-1}-u^{-1})}\ln u + \frac{(w^{-1/2}+w^{1/2})(1+u)(1-v)}{(1-v^{-1}w)(1-v^{-1}w^{-1})(1-uv)(u-v)}\ln v + \frac{(w^{-1/2}-w^{1/2})(1+u)(1+v^{-1})}{(1-uw)(1-uw^{-1})(1-v^{-1}w)(1-v^{-1}w^{-1})}\ln w, \tag{5.59}$$

where $u = e^{-\beta\omega_{23}}$, $v = e^{-\beta\omega_{14}}$, and $w = e^{-i\beta\varkappa/2}$.

A great simplification occurs in the maximal chaos case:

$$\boxed{f(\omega_{14},\omega_{23}) = \frac{2\pi}{(1+e^{-\beta\omega_{23}})(1+e^{\beta\omega_{14}})} \qquad \text{if} \quad \varkappa = \frac{2\pi}{\beta}.} \tag{5.60}$$

Importantly, the function $f(\omega_{14},\omega_{23})$ splits into two factors. They may be interpreted in terms of interaction of the fluctuating horizon (which corresponds to **B**) with incoming and outgoing radiation.

Locking two operators in the same replica

We now adapt the obtained result to express the entropy of the density matrix $\rho(x,\epsilon)$. The latter is a normalized version of the operator

$$\overline{W}(\beta,x,\epsilon) = (1-ixX)e^{-\beta H_0}(1+ixX) + \epsilon\sum_j A_j(t)\,(1-ixX)e^{-\beta H_0}(1+ixX)\,A_j^\dagger(t). \tag{5.61}$$

Note that we have made the time explicit and will follow the convention that $A_j = A_j(0)$. To proceed, we replace the set of operators A_j with a single operator Y. This is achieved by extending the physical system with an auxiliary one, comprising a ground state $|0\rangle$ with zero energy and a set of excited states $|j\rangle$ with energy Ω. We denote the Hamiltonian of the extended system by $H(\Omega)$ and set

$$Y = \sum_j A_j \otimes |0\rangle\langle j|, \qquad Y^\dagger = \sum_j A_j^\dagger \otimes |j\rangle\langle 0|. \tag{5.62}$$

Although the transformation just described alters the operator $\overline{W}(\beta, x, \epsilon)$ in a nontrivial way, we will find an agreement in the $\Omega \to \infty$ limit. For the time being, let us construct some operators acting on the extended system that correspond to the two terms in (5.61) as closely as possible:

$$(1 - ixX)e^{-\beta H(\Omega)}(1 + ixX) = (1 - ixX)e^{-\beta H_0}(1 + ixX) \otimes \left(|0\rangle\langle 0| + e^{-\beta\Omega} \sum_j |j\rangle\langle j| \right), \tag{5.63}$$

$$\begin{aligned} &e^{\beta\Omega}\, Y(t)\, (1 - ixX)e^{-\beta H(\Omega)}(1 + ixX)\, Y^\dagger(t) \\ &\qquad = \left(\sum_j A_j(t)\, (1 - ixX)e^{-\beta H_0}(1 + ixX)\, A_j^\dagger(t) \right) \otimes |0\rangle\langle 0|. \end{aligned} \tag{5.64}$$

Now let

$$W(\Omega, \beta, x, y) = (1 + e^{\beta\Omega/2} y Y(t))(1 - ixX)e^{-\beta H(\Omega)}(1 + ixX)(1 + e^{\beta\Omega/2} y Y^\dagger(t)). \tag{5.65}$$

Then

$$\lim_{\Omega\to\infty} W(\Omega, \beta, x, y) = \overline{W}(\beta, x, y^2) \otimes |0\rangle\langle 0|, \tag{5.66}$$

and hence,

$$\lim_{\Omega\to\infty} \mathrm{Tr}\big(W(\Omega, \beta, x, y)\big)^s = \mathrm{Tr}\big(\overline{W}(\beta, x, y^2)\big)^s \tag{5.67}$$

for any s. The last equation can be interpreted as the operators $Y(t)$ and $Y^\dagger(t)$ in the expansion of $\mathrm{Tr}\, W^s$ being locked in the same replica. We now take the s derivative of both sides at $s = 1$ and consider the $x^2 y^2$ term in the Taylor expansion. Thus, equation (5.67) becomes

$$\begin{aligned} &\lim_{\Omega\to\infty} \Big(e^{\beta\Omega} \Big(-\big\langle \mathbf{B}, Y(\beta + it), X(\beta), X(0), Y^\dagger(it) \big\rangle_\Omega + \big\langle Y(\beta + it), X(\beta), X(0), Y^\dagger(it) \big\rangle_\Omega \Big) \Big) \\ &\qquad = \frac{1}{2} \left(\frac{\partial^2}{\partial x^2} \frac{\partial}{\partial \epsilon}\, S(\rho(x, \epsilon)) \right) \bigg|_{x=\epsilon=0}. \end{aligned} \tag{5.68}$$

Finally, let us use the results of the previous section. Part of the correspondence is obvious: $X_1 = Y^\dagger$, $X_2 = X_3 = X$, and $X_4 = Y$. In the OTOC-based approximation, the branched correlator

$$\mathcal{B} = \big\langle \mathbf{B}, Y(\beta + it), X(\beta), X(0), Y^\dagger(it) \big\rangle_\Omega = \big\langle \mathbf{B}, X(0), Y^\dagger(it), Y(it), X(0) \big\rangle_\Omega \tag{5.69}$$

dominates the left-hand side of equation (5.68) and is given by (5.53), (5.54) with $t_1 = t_4 = t$ and $t_2 = t_3 = 0$. Note that if Ω is large, then

$$\Upsilon^{\mathrm{R}}_{Y^\dagger, Y}(\tau) \approx e^{-\Omega(\beta - \tau)} \sum_j \Upsilon^{\mathrm{R}}_{A_j^\dagger, A_j}(\tau) \tag{5.70}$$

by analogy with the rather obvious equation $\langle Y^\dagger(\tau) Y(0)\rangle \approx e^{-\Omega(\beta-\tau)} \sum_j \langle A_j^\dagger(\tau) A_j(0)\rangle$. Hence,

$$\tilde{\Upsilon}^{\mathrm{R}}_{Y^\dagger, Y}(\omega) \approx e^{-\beta\Omega} \sum_j \tilde{\Upsilon}^{\mathrm{R}}_{A_j^\dagger, A_j}(\omega + \Omega). \tag{5.71}$$

On the other hand, the function $\tilde{\mathscr{B}}(\omega_{14},\omega_{23})$ in equation (5.53) may be replaced with $\tilde{\mathscr{B}}(\omega_{14}-\Omega,\omega_{23})$ without affecting the result.[4] Combining (5.54) with (5.71), we get

$$e^{\beta\Omega}\tilde{\mathscr{B}}(\omega_{14}-\Omega,\omega_{23}) = \sum_j \tilde{\Upsilon}^{\mathrm{R}}_{A_j^\dagger,A_j}(\omega_{14})\tilde{\Upsilon}^{\mathrm{A}}_{X,X}(\omega_{23})\, f(\omega_{14}-\Omega,\,\omega_{23}), \tag{5.72}$$

where the prefactor $e^{\beta\Omega}$ is used to match the left-hand side of (5.68).

Thus, we have arrived at the conclusion that the replica locking amounts to replacing the function $f(\omega_{14},\omega_{23})$ in (5.54) with $f(-\infty,\omega_{23})$. Using the explicit formulas (5.59), (5.60), we get:

$$f(-\infty,\omega_{23}) = \frac{\cos\frac{\beta\varkappa}{4}\cdot 2\beta\omega_{23}(1-e^{-\beta\omega_{23}}) + \sin\frac{\beta\varkappa}{4}\cdot\beta\varkappa(1+e^{-\beta\omega_{23}})}{(1-e^{-\beta(\omega_{23}+i\varkappa/2)})(1-e^{-\beta(\omega_{23}-i\varkappa/2)})} \tag{5.73}$$

$$= \frac{2\pi}{1+e^{-\beta\omega_{23}}} \qquad \text{if} \quad \varkappa = \frac{2\pi}{\beta}. \tag{5.74}$$

For comparison, consider a Euclidean black hole in a hyperbolic space (say, in two dimensions). The replica geometry involves an s-fold cover of both the circle and the disk it bounds, with a branching point at the center. Inserting boundary fields slightly deforms the space. In the $s\to 1$ limit, the geometry is given by a smooth metric on the disk and the position of the branching point. The branching point is a special case of a quantum extremal surface [10] (where "surface" means a codimension 2 submanifold). Its position is determined by an extremum of entropy. Instead of the entropy S, we may consider $\ln Z - S$. Indeed, the partition function Z depends only on the space-time metric, which should be fixed before finding the extremal surface.

In the Lorentzian case, the branching point is described by null coordinates (u_+,u_-). (We set aside the ambiguity in the choice of origin due to the deformation of space-time relative to AdS_2.) The entropy can be expanded to the second order in u_+, u_-:

$$S(u_+,u_-) = S_0 + p_+u_+ + p_-u_- - Cu_+u_-, \tag{5.75}$$

where p_+ and p_- depend on the inserted field.[5] Solving the extremum problem, we get

$$S_{\mathrm{ext}} = S_0 + C^{-1}p_+p_-. \tag{5.76}$$

Now, let us forget about geometry. The only property we need is that if there is large time separation, then p_+ and p_- depend only on the fields inserted in the past and the future, respectively. Thus, the change in the entropy should factor into two quantities dependent on the corresponding fields. This is exactly what we observed in the maximal chaos case; see equation (5.60). We leave the interpretation of these quantities to future research.

[4] This is because $t_1 = t_4$. Note, however, that the same condition was implicitly used in (5.64). A more general model of system-bath coupling involves $A_j(t_4)$ and $A_j^\dagger(t_1)$ so that the additional factor $e^{-i\Omega(t_1-t_4)}$ has to be added on the left-hand side of (5.64). To reproduce this factor, one *should* replace $\tilde{\mathscr{B}}(\omega_{14},\omega_{23})$ with $\tilde{\mathscr{B}}(\omega_{14}-\Omega,\omega_{23})$ in (5.53).

[5] This expression is similar to 't Hooft's effective action [1] for the fluctuating horizon, where p_+ and p_- are null energies. In our case, they are just abstract coefficients.

Appendix A

THE ADS$_2$ SPACE IN DIFFERENT COORDINATES

The AdS$_2$ space is the maximally symmetric space with constant curvature $R = -2$ which is defined by:

$$Y_{-1}^2 + Y_0^2 - Y_1^2 = 1, \quad ds^2 = -dY_{-1}^2 - dY_0^2 + dY_1^2 \tag{A.1}$$

with the symmetry $SO(1,2)$. The global coordinate is defined by

$$Y_{-1} = \frac{\cos\eta}{\cos\theta}, \quad Y_0 = \frac{\sin\eta}{\cos\theta}, \quad Y_1 = \tan\theta, \qquad ds^2 = \frac{-d\eta^2 + d\theta^2}{\cos^2\theta}, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}, \quad -\infty < \eta < \infty \tag{A.2}$$

which has the topology of a strip with the boundaries located at $\theta = -\frac{\pi}{2}, \frac{\pi}{2}$. The Poincare patch is defined by

$$z = \frac{1}{Y_{-1} + Y_1}, \quad t = \frac{Y_0}{Y_{-1} + Y_1}, \quad ds^2 = \frac{-dt^2 + dz^2}{z^2}, \qquad 0 < z < \infty \tag{A.3}$$

with the boundary located at $z = 0$ $(\theta = \frac{\pi}{2})$. The Schwarzschild coordinate is defined as

$$\begin{aligned} &Y_{-1} = r, \quad Y_0 = (r^2 - 1)^{\frac{1}{2}} \sinh t, \quad Y_1 = (r^2 - 1)^{\frac{1}{2}} \cosh t \\ &ds^2 = -(r^2 - 1)dt^2 + \frac{dr^2}{r^2 - 1}. \end{aligned} \tag{A.4}$$

Note that the event horizon is located at $r = 1$. One can also define the Kruskal-Szekeres coordinate as the maximal extension of the black hole as follows :

$$Y_{-1} = \frac{1 - uv}{1 + uv}, \quad Y_0 = \frac{u + v}{1 + uv}, \quad Y_1 = \frac{v - u}{1 + uv}, \qquad ds^2 = \frac{-4du\, dv}{(1 + uv)^2}. \tag{A.5}$$

The dilaton profile in the bulk

As was derived in the first chapter the dilaton field in the bulk with no matter satisfies:

$$\nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^2 \phi + g_{\mu\nu} \phi = 0. \tag{A.6}$$

In global coordinate, the equation implies:

$$\phi = \alpha \frac{\sin\eta}{\cos\theta} + \beta \frac{\cos\eta}{\cos\theta} + \gamma \tan\theta = A.Y \tag{A.7}$$

where the last expression is in the embedding coordinate for an arbitrary vector A. The value of the parameters is fixed by the boundary condition $\phi_b = \frac{\phi_r}{\epsilon}$. For the Poincare patch where the boundary is located at $z = \epsilon$, $\boxed{\phi(z,t) = \frac{\phi_r}{z}}$. It is clear that at $z = \infty$, the Poincare horizon, the dilaton vanishes. The profile of dilaton in Schwarzschild coordinate is $\phi(r,t) = \phi_0\, r = \phi_0\, \frac{\cos\eta}{\cos\theta}$. To compute ϕ_0, the location of the boundary is at $r = \frac{\dot{\tau}}{\epsilon}$ where τ is the Euclidean times. The solution is $\tau = \frac{2\pi u}{\beta}$. Therefore,

$$\phi_0 = \frac{2\pi}{\beta}\, \phi_r = 4G\, S. \tag{A.8}$$

Appendix B

A BRIEF REVIEW OF THE THERMOFIELD-DOUBLE STATE AND ITS PROPERTIES

By definition, $|\mathrm{TFD}\rangle$ is the purification of the canonical ensemble $\rho = \frac{e^{-\beta H}}{\mathcal{Z}}$, where the Hamiltonian is a Hermitian operator. Given a separable Hilbert space, if we assume that the eigenstates of the Hamiltonian are $\{|E_n\rangle, n \in \mathbb{N}\}$, then the $|TFD\rangle$ can be represented as

$$|\mathrm{TFD}\rangle = \frac{1}{\mathcal{Z}^{\frac{1}{2}}} \sum_n e^{-\frac{\beta E_n}{2}} \, |E_n^*\rangle_L \otimes |E_n\rangle_R. \tag{B.1}$$

However, to understand the relation between the left and right kets, it is more illuminating to start with an operator algebra, and then construct the associated Hilbert space [1]. Therefore, assume that we start with some Hilbert space $\mathcal{H}$ and consider the algebra of the bounded operators, denoted by $\mathcal{B}(\mathcal{H})$, as the set of operators which is a vector space over the complex field and also closed under multiplication where

$$\|A\| = \sup_{\psi \in \mathcal{H}} \frac{\|A\psi\|}{\|\psi\|} < \infty. \tag{B.2}$$

Throughout this article, we assume that $\mathcal{B}(\mathcal{H})$ is in fact a von Nuemann Algebra, meaning it is an $*-$algebra that includes unit which is closed w.r.t. the weak operator topology.

The GNS construction: Given a $*-$ algebra as above, associated to each positive bilinear form $\omega : \mathcal{B}(\mathcal{H}) \otimes \mathcal{B}(\mathcal{H}) \to \mathbb{C}$, there is a Hilbert space $\mathcal{H}_\omega$.

Now, associated to each operator $A \in \mathcal{B}(\mathcal{H})$, we define a vector $|A\rangle_p$.

$$A \leftrightarrow |A\rangle_p. \tag{B.3}$$

This way, we can naturally define the action operators on the vectors:

$$B|A\rangle = |BA\rangle, \tag{B.4}$$

where $|BA\rangle \leftrightarrow BA$. The inner product on the vectors is defined by ω, namely,

$$\left\langle A|B \right\rangle = \omega(A^* B). \tag{B.5}$$

Such a construction defines a pre-Hilbert space. In our construction, vectors of zero norm may be produced. Using the Cauchy-Schwarz inequality

$$\left\langle A|B \right\rangle^2 \leq \|A\| \, \|B\| \,. \tag{B.6}$$

[1] some of the standard references for this section are [14, 43, 6].

It is clear that such states are in fact orthogonal to all other states of the pre-Hilbert space. CS inequality also implies that if $|A\rangle_p$ has a zero norm, $|BA\rangle_p$ also has a zero norm. This means, $\mathcal{J}$ the set of zero norm states, is a left ideal. Therefore, we can take the quotient of our pre-Hilbert space w.r.t. $\mathcal{J}$, or equivalently, define the state $|A\rangle$ as follows:

$$|A\rangle \equiv \{|A\rangle_p + |X\rangle_p,\ s.t.\ |X\rangle_p \in \mathcal{J}\}. \tag{B.7}$$

The last part of our construction is to consider the completion of the above space. The result will be denoted by $\mathcal{H}_\omega$. Here the vacuum state $|\Omega\rangle$ is associated to the unit operator 1. We have:

$$\omega(A) = \langle\Omega|A|\Omega\rangle. \tag{B.8}$$

Associated to any other vector $|\psi\rangle \in \mathcal{H}$, one can define the state ω_ψ:

$$\omega_\psi(A) = \langle\psi|A|\psi\rangle. \tag{B.9}$$

We will extend our states to include the density matrices ρ so that:

$$\omega_\rho(A) = \mathrm{Tr}(\rho A), \quad Tr(\rho) = 1, \tag{B.10}$$

where $\rho \in \mathcal{B}(\mathcal{H})$ is a positive trace class operator. These are called the normal states.
We call the state ω_ρ has a one parameter symmetry group $\alpha_t,\ t \in \mathbb{R}$, generated by the operator H if

$$\omega_\rho(\alpha_t(A)) = \omega_\rho(A). \tag{B.11}$$

The state ω_ρ is **KMS** if it satisfies:

$$\omega_\rho(A\alpha_t(B)) = \omega_\rho(\alpha_{t+i\beta}(B)A). \tag{B.12}$$

It is easy to prove that if the state ω_ρ satisfies the KMS condition, then [2]

$$\rho \propto e^{-\beta H}. \tag{B.13}$$

Now, consider the operator $\rho^{\frac{1}{2}}$. The claim is:

$$\rho^{\frac{1}{2}} \leftrightarrow |\rho^{\frac{1}{2}}\rangle \equiv |\mathrm{TFD}\rangle. \tag{B.14}$$

From our definition:

$$\omega_\rho(A) = \langle\mathrm{TFD}|A|\mathrm{TFD}\rangle, \tag{B.15}$$

where, in the r.h.s, the inner product is with respect to the trace.

[2]Consider $\rho e^{\beta H}$. Using the KMS condition, one can prove that $[\rho e^{\beta H}, A] = 0 \quad \forall A \in \mathcal{B}(\mathcal{H}_\omega)$. Since our representation is irreducible, this means that $\rho e^{\beta H} \propto 1$.

Before studying the representation of the operators on this state, we provide some definitions: we call a state to be *cyclic* for an operator algebra $\mathcal{A}$, if the action of the operators on this state will give a dense subset of the Hilbert space. The state $|\Psi\rangle$ is called *separating* if

$$A|\Psi\rangle = 0 \Rightarrow A = 0. \tag{B.16}$$

The operator algebra $\mathcal{A}'$ is called the commutant of $\mathcal{A}$ if:

$$[A, B] = 0, \quad A \in \mathcal{A},\ B \in \mathcal{A}'. \tag{B.17}$$

Assume that the state $|\Omega\rangle$ is cyclic and separating for $\mathcal{A}$ and $\mathcal{A}'$.[3] We define the *Tomita operator*:

$$S_\Omega\, A|\Omega\rangle = A^\dagger|\Omega\rangle, \qquad A \in \mathcal{A} \tag{B.18}$$

From the definition, it follows that $S_\Omega^2 = 1$, and so S_Ω is invertible and unbounded. One can check that the Tomita operator for $\mathcal{A}'$ is $S'_\Omega = S_\Omega^\dagger$. Since S_Ω is invertible, it has a unique polar decomposition:

$$S_\Omega = J\, \Delta^{\frac{1}{2}}, \tag{B.19}$$

where $\Delta^{\frac{1}{2}}$ is a positive operator and J is anti-unitary with the following properties:

$$J^2 = 1, \quad J' = J, \quad \Delta' = \Delta^{-1}, \quad J\,\Delta^{\frac{1}{2}} J = \Delta^{\frac{-1}{2}}, \quad J\,\Delta^{is} J = \Delta^{is}, \quad s \in \mathbb{R} \tag{B.20}$$

Where the polar decomposition of S'_Ω is given by $S'_\Omega = J'\, \Delta'^{\frac{1}{2}}$. The above properties can be proven easily. The rather nontrivial consequence of the above definitions is the following:

$$J\, \mathcal{A}\, J = \mathcal{A}'. \qquad\qquad \text{(Tomita-Takesaki)} \tag{B.21}$$

Therefore, from the definition, we have:

$$\Delta^{\frac{1}{2}} A|\Omega\rangle = J A^\dagger|\Omega\rangle. \tag{B.22}$$

Note that the r.h.s and so the l.h.s belong to $\mathcal{A}'$.

Now, consider the thermofield-double $|\rho^{\frac{1}{2}}\rangle$ defined in B.14. We define the two representations of the operator algebra on this state as follows:

$$\pi_R(A)|\rho^{\frac{1}{2}}\rangle \equiv |A\rho^{\frac{1}{2}}\rangle = A_R|\rho^{\frac{1}{2}}\rangle =, \qquad \pi_L(A)|\rho^{\frac{1}{2}}\rangle \equiv |\rho^{\frac{1}{2}} A^\dagger\rangle = A_L^*|\rho^{\frac{1}{2}}\rangle. \tag{B.23}$$

where in the last equation $*$ is the complex conjugate. It is clear that the operator algebra $[\mathcal{A}_R, \mathcal{A}_L] = 0$ and they are in fact each other's commutant. The Hamiltonian is defined by:

$$H|\text{TFD}\rangle = 0. \tag{B.24}$$

[3]The state $|\Omega\rangle$ is cyclic for $\mathcal{A}$ iff it is separating for $\mathcal{A}'$.

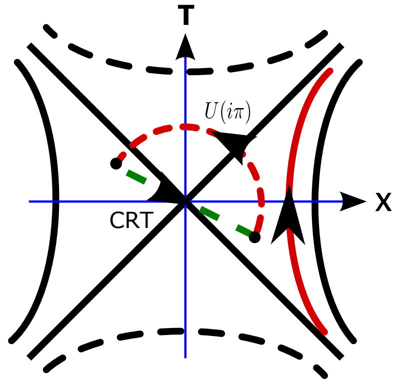


Figure B.1: Representation of the Tomita operator in the two-sided black hole.

The associated unitary operator is:

$$e^{-iHt} \equiv \pi_R(e^{-iHt})\pi_L(e^{-iHt}) = e^{-i(1\otimes H_R - H_L\otimes 1)t} \Rightarrow H = H_R - H_L, \tag{B.25}$$

where in the last equation, we dropped the tensor product notation for simplicity. Therefore, under the evolution with U, the time directions in the left and right sides are opposite. Now, one can define the operators Δ, J and study their action on this state.

In AdS/CFT correspondence, it was proposed by Maldacena that given a holographic CFT, the thermofield-double state represents a two-sided black hole in the bulk. To construct the Tomita operator, it should be noted that such black holes have a time-like Killing vector which is ∂_t in the Schwarzschild coordinate with the generator H. Close to the horizon, the geometry of the black hole to the first order is similar to the geometry of the Rindler space and H is the boost generator. On the boundary, the generator H will coincide with the modular Hamiltonian B.13. To construct the Tomita operator we consider $U(i\pi)$ generated by the modular hamiltonian which is the Euclidean rotation in the (T, X) plane, and use the CRT[4] operator to bring it back to the same point, B.1. Defining

$$J = \text{CRT}, \qquad \Delta^{\frac{1}{2}} = e^{-\frac{\beta}{2}H}, \qquad H = H_R - H_L \tag{B.26}$$

, for real scalar fields we have:

$$\phi_L(t, r, \vec{x})|\text{TFD}\rangle = \Delta^{\frac{1}{2}}\, \phi_R(t, r, \vec{x})|\text{TFD}\rangle = \phi_R(t + i\frac{\beta}{2}, r, \vec{x})|\text{TFD}\rangle \tag{B.27}$$

, while for fermions:

$$\psi_L(t, r, \vec{x})|\text{TFD}\rangle = \Delta^{\frac{1}{2}}\, \psi_R(t, r, \vec{x})|\text{TFD}\rangle = i\psi_R(t + i\frac{\beta}{2}, r, \vec{x})|\text{TFD}\rangle \tag{B.28}$$

where $(t, r, \vec{x})$ is the Schwarzschild coordinate, $(\vec{x})$ is the transverse coordinate.

In AdS_2, the generator of rotation in the Euclidean (T, X) plane is given by $\Lambda_0 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$.

[4]C, R, T are the charge conjugation operator, reflection operator, and time reversal, respectively.

Appendix C

THE MAXIMAL EXTENSION OF THE BLACK HOLE FINAL STATES

In this section, we will review the Racz-Wald [36, 37] construction of a bifurcate horizon[1]. We first fix our conventions: the Killing vectors are the symmetry generators of the spacetime. Their components satisfy:

$$\nabla_\mu \zeta_\nu + \nabla_\nu \zeta_\mu = 0, \quad \nabla_\mu \nabla_\nu \zeta_\rho = -R^\sigma{}_{\mu\rho\nu}\, \zeta_\sigma. \tag{C.1}$$

A Killing hypersurface is a null hypersurface whose normal vector is Killing. This implies the definition of the surface gravity κ as

$$\zeta^\mu \nabla_\mu \zeta_\nu = \kappa \zeta_\nu. \tag{C.2}$$

Note that the definition satisfies that κ remains constant on the orbits of the Killing vector i.e. $\zeta^\mu \nabla_\mu \kappa = 0$ which can be derived by acting with $\zeta^\rho \nabla_\rho$ on both sides of C.2. This implies that we can easily normalize the Killing vector and get $\ell^\mu = e^{-\kappa v} \zeta^\mu$ which is affine ($\zeta^\mu \nabla_\mu v = 1$):

$$\ell^\mu \nabla_\mu \ell_\nu = 0. \tag{C.3}$$

Note that κ can be the function of the transverse coordinate. We can choose a basis on $\mathcal{N}$ as follows. We assume that the cross sections of $\mathcal{N}$ admit a basis e^A_μ, where $A = 1, \cdots, d-2$. We also define the null vector n^μ which is orthonormal to e^A_μ and satisfies $n.\ell = 1$. Then the vectors $\{\ell^\mu, n^\mu, e^\mu_A\}$ are a basis for the spacetime.

Theorem: Consider a stationary spacetime $\mathcal{M}$ which admits a one-sided black hole but no white hole. Assuming that the event horizon $\mathcal{N}$ of the black hole is also a Killing horizon with a compact cross section whose Killing orbits are diffeomorphic to $\mathbb{R}$, it is possible to embed the spacetime into a bigger spacetime $\mathcal{M}'$ where $\mathcal{N}$ will become a proper subset of a bifurcate horizon iff the surface gravity κ associated to $\mathcal{N}$ is a nonzero constant.

Proof: We first prove if $\nabla_\mu \kappa$ does not vanish along the orbits of the Killing vector, γ, then at least one of the components of the curvature tensor will blow up along γ. Note that $e^\mu_A \nabla_\mu \kappa$ is constant along γ :

$$\begin{aligned} \zeta^\alpha \nabla_\alpha \left(e^\beta_A \kappa_{,\beta} \right) &= \zeta^\alpha \nabla_\alpha e^\beta_A \nabla_\beta \kappa + e^\beta_A \zeta^\alpha \nabla_\alpha \nabla_\beta \kappa = \zeta^\alpha \nabla_\alpha e^\beta_A \nabla_\beta \kappa + e^\beta_A \nabla_\beta (\zeta^\alpha \nabla_\alpha \kappa) - e^\beta_A \nabla_\beta \zeta^\alpha \nabla_\alpha \kappa \\ &= \left[\zeta^\alpha \nabla_\alpha, e^\beta_A \nabla_\beta \right] \kappa = 0. \end{aligned} \tag{C.4}$$

[1] Here, we will mostly review the idea and refer the reader to the original papers for the technical details and the mathematical accuracy of the construction.

On the other hand, using C.1 we have:

$$e^{\mu}_{A}\nabla_{\mu}\kappa = e^{\kappa v}\, R_{\sigma\gamma\mu\nu}\, \ell^{\sigma}\, e^{\sigma}_{A}\, \ell^{\mu}\, n^{\nu}. \tag{C.5}$$

Now, the l.h.s is constant as $v \longrightarrow -\infty$, $e^{\kappa v} \longrightarrow 0$. Therefore,

$$R_{\sigma\gamma\mu\nu}\, \ell^{\sigma}\, e^{\gamma}_{A}\, \ell^{\mu}\, n^{\nu} \longrightarrow \infty, \tag{C.6}$$

which, in general, prevents us from further extending the spacetime $\mathcal{M}$.

On the other hand, assume that $\kappa \neq 0$ is a constant on $\mathcal{N}$. We pick the basis $\{\zeta^{\mu}, r^{\mu}, e^{\mu}_{A}\}$ where $r.\,\zeta = 1$ ($r^{\mu} = e^{-\kappa v}n^{\mu}$) and ζ^{μ} belongs to $\mathcal{N}$. Therefore, $\{\zeta^{\mu}, e^{\mu}_{A}\}$ defines the coordinate system (v, x^{A}) on the horizon. Then, in a small neighborhood of $\mathcal{N}$, denoted by $\mathcal{O}$, we can parallel transport the vectors along r^{μ} and so it will be coordinatized by (v, r, x^{A}), called the Eddington-Finkelstein (EF) coordinate, and $\mathcal{N}$ is described by $r = 0$. In $\mathcal{O}$, the metric will take the following form:

$$ds^2 = dv\Big(-F\, dv + 2\, dr\Big) + 2g_{vA}dv\, dx^{A} + g_{AB}dx^{A}dx^{B} \tag{C.7}$$

where $g_{AB} = g(e^{\mu}_{A}\partial_{\mu}, e^{\nu}_{B}\partial_{\nu})$, $g_{vA} = g(\zeta^{\mu}\partial_{\mu}, e^{\nu}_{A}\partial_{\nu})$, and $\zeta^{\mu}\zeta_{\mu} = -F$. From our construction of the coordinate system, the metric components are not a function of v, since on the horizon by going along v geometry does not change, and the u component of the coordinate system does not change as we go along the r direction. Moreover, since $F\Big|_{r=0} = g_{vA}\Big|_{r=0} = 0$, in a small neighbourhood in $\mathcal{O}$, namely $\mathcal{O}'$, one can write:

$$F = r f(r, x^{A}), \qquad g_{vA} = r\tilde{g}_{vA}. \tag{C.8}$$

Now, from the fact that $r^{\mu}\nabla_{\mu}(\zeta.\zeta) = -2\kappa$ on the Killing horizon, we have:

$$\frac{1}{2}\frac{\partial F}{\partial r}\Big|_{r=0} = \kappa \qquad \Rightarrow \qquad f(r = 0, x^{A}) = 2\kappa \neq 0. \tag{C.9}$$

Therefore, in a small neighborhood in $\mathcal{O}'$ denoted by $\mathcal{P}$, we can define $h(r, v, x^{A})$ so that:

$$\frac{1}{f} - \frac{1}{2\kappa} = r h(r, x^{A}). \tag{C.10}$$

Therefore, we can write the term inside the parenthesis in C.7 as:

$$-F dv + 2dr = -F\Big(dv - 2h dr - \frac{1}{\kappa}\frac{dr}{r}\Big). \tag{C.11}$$

Now, define the v coordinate as follows:

$$u = v - \frac{1}{\kappa}\,\ln r - 2\int_{0}^{r} dr'\; h(r', x^{A}). \tag{C.12}$$

Following the Kruskal-Szekeres maximal extension of the Schwarzschild black hole, we define:

$$\begin{gathered} U = -e^{-\kappa u} = -e^{-\kappa v}\, r\, e^{2\kappa \int_0^r dr' h}, \quad V = e^{\kappa v}, \\ UV = r\, e^{2\kappa \int_0^r dr' h}. \end{gathered} \tag{C.13}$$

Then C.7 will take the following form:

$$ds^2 = -\frac{F}{\kappa^2 UV} dUdV + \Big(2g_{VA} + \frac{2F}{\kappa V}\int_0^r dr' \frac{\partial h}{\partial x^A}\Big) dV dx^A + g_{AB}\, dx^A dx^B, \quad V > 0. \tag{C.14}$$

Note that from C.13, we can easily see that $\frac{F}{UV}, \frac{F}{V}$ are analytic near $UV = 0$. Moreover, since $\frac{\partial UV}{\partial r}\Big|_{r=0} \neq 0$, in a local neighborhood of UV=0 we can write:

$$r = r(UV) = UV\phi(UV, x^A). \tag{C.15}$$

Plugging back into the metric, the final form of the metric will be:

$$\begin{aligned} ds^2 &= -\frac{\phi f}{\kappa^2} dUdV + 2U\Big(V\phi\tilde{g}_{VA} + \frac{\phi f}{\kappa}\int_0^r dr' \frac{\partial h}{\partial x^A}\Big) dV dx^A + g_{AB}\, dx^A dx^B \\ &\equiv -2G\, dUdV + 2G_{VA}\, dV\, dx^A + g_{AB}\, dx^A dx^B \quad V > 0. \end{aligned} \tag{C.16}$$

To summarize, we have so far proven that there is a neighborhood $|UV| < \epsilon(x^A)$ of the Killing horizon $\mathcal{N}$ where the metric will take the form C.16 with the restriction that $V > 0$. We can define our manifold in a straightforward way: since we assume that the cross sections of $\mathcal{N}$ are compact, we know that every covering by open sets of the cross section can be reduced to a finite one. For i-th open set, we define the extension as above and call it $\mathcal{M}_i$. Now, we can identify points in $\mathcal{M}_i$ and $\mathcal{M}_j$ if $(U_i, V_i, x_i^A) = (U_j, V_j, x_j^A)$. It is straightforward to check that $\zeta = V\partial_V - U\partial_U$ is a Killing vector on $U = 0$ and $V = 0$ hypersurfaces. This will prove the local construction of the bifurcate horizon. Now, we can also drop the restriction on V which renders the left wedge.

In the static case, we can say more. We will define the new time coordinate $t(v, r, x^A)$ that the Killing vector is orthogonal to the constant time surfaces. First, we must have:

$$\mathcal{L}_\zeta t = 1, \tag{C.17}$$

which implies that :

$$v = t + L(r, x^A), \tag{C.18}$$

for some function L. We can take differential of the above relation and plug it into C.7. Since for $r > 0$, $g_{tr} = g_{tA} = 0$, we have

$$\partial_r L = \frac{1}{F}, \quad \partial_A L = -g_{vA}. \tag{C.19}$$

As a consequence of the first relation we have:

$$t = v - \frac{1}{2\kappa}\ln r - \int_0^r h + H(x^A) = \frac{v+u}{2} + H(x^A). \tag{C.20}$$

In terms of the Kruskal-Szekeres coordinate it becomes:

$$t = \frac{1}{2\kappa}\ln\Big(-\frac{V}{U}\, e^{2\kappa H(x^A)}\Big). \tag{C.21}$$

The stationary case is similar and we refer the reader to the references. It is important to note that nowhere Einstein's equation was used. In fact, the role of Einstein equation and matter condition is to confirm the surface gravity is constant on the horizon.

BIBLIOGRAPHY

[1] Gerard 't Hooft. "The black hole interpretation of string theory". In: *Nucl. Phys.* B335 (1990), pp. 138–154. DOI: 10.1016/0550-3213(90)90174-C.

[2] Ahmed Almheiri and Joseph Polchinski. "Models of AdS_2 backreaction and holography". In: *JHEP* 11 (2015), p. 014. DOI: 10.1007/JHEP11(2015)014. arXiv: 1402.6334 [hep-th].

[3] Ahmed Almheiri et al. "Replica wormholes and the entropy of Hawking radiation". In: *JHEP* 2020 (2020), pp. 1–42. DOI: 10.1007/JHEP05(2020)013. eprint: 1911.12333.

[4] Dmitry Bagrets, Alexander Altland, and Alex Kamenev. "Power-law out of time order correlation functions in the SYK model". In: *Nucl. Phys.* B921 (2017), pp. 727–752. DOI: 10.1016/j.nuclphysb.2017.06.012. arXiv: 1702.08902 [cond-mat.str-el].

[5] J. M. Bardeen, B. Carter, and S. W. Hawking. "The four laws of black hole mechanics". In: *Communications in Mathematical Physics* 31.2 (1973), pp. 161–170. DOI: 10.1007/BF01645742. URL: https://doi.org/10.1007/BF01645742.

[6] J. J Bisognano and E. H. Wichmann. "On the Duality Condition for Quantum Fields". In: *J. Math. Phys.* 17 (1976), pp. 303–321. DOI: 10.1063/1.522898.

[7] B. Carter. "Axisymmetric Black Hole Has Only Two Degrees of Freedom". In: *Phys. Rev. Lett.* 26 (6 Feb. 1971), pp. 331–333. DOI: 10.1103/PhysRevLett.26.331. URL: https://link.aps.org/doi/10.1103/PhysRevLett.26.331.

[8] Yiming Chen, Xiao-Liang Qi, and Pengfei Zhang. "Replica wormhole and information retrieval in the SYK model coupled to Majorana chains". In: (2020). eprint: arXiv:2003.13147.

[9] Tevian Dray and Gerard 't Hooft. "The gravitational shock wave of a massless particle". In: *Nucl. Phys.* B253 (1985), pp. 173–188. DOI: 10.1016/0550-3213(85)90525-5.

[10] Netta Engelhardt and Aron C Wall. "Quantum extremal surfaces: holographic entanglement entropy beyond the classical regime". In: *JHEP* 2015 (2015), p. 73. DOI: 10.1007/JHEP01(2015)073. eprint: 1408.3203.

[11] Alessandro Fabbri and Jose Navarro-Salas. "Modeling Black Hole Evaporation". In: (Jan. 2005).

[12] Ping Gao, Daniel Louis Jafferis, and Aron Wall. "Traversable Wormholes via a Double Trace Deformation". In: *JHEP* 12 (2017), p. 151. DOI: 10.1007/JHEP12(2017)151. arXiv: 1608.05687 [hep-th].

[13] Yingfei Gu, Andrew Lucas, and Xiao-Liang Qi. "Spread of entanglement in a Sachdev-Ye-Kitaev chain". In: *JHEP* 2017 (2017), p. 120. DOI: 10.1007/JHEP09(2017)120. eprint: 1708.00871.

[14] R. Haag. *Local quantum physics: Fields, particles, algebras*. 1992.

[15] S. W. Hawking and G. F. R. Ellis. *The Large Scale Structure of Space-Time*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, Feb. 2011. ISBN: 978-0-521-20016-5, 978-0-521-09906-6, 978-0-511-82630-6, 978-0-521-09906-6. DOI: 10.1017/CBO9780511524646.

[16] Patrick Hayden and John Preskill. “Black holes as mirrors: quantum information in random subsystems”. In: *JHEP* 2007.09 (2007), p. 120. DOI: 10.1088/1126-6708/2007/09/120. eprint: arXiv:0708.4025.

[17] W. Israel. “Thermo-field dynamics of black holes”. In: *Physics Letters A* 57.2 (1976), pp. 107–110. ISSN: 0375-9601. DOI: https://doi.org/10.1016/0375-9601(76)90178-X. URL: https://www.sciencedirect.com/science/article/pii/037596017690178X.

[18] Werner Israel. “Event horizons in static electrovac space-times”. In: *Communications in Mathematical Physics* 8.3 (1968), pp. 245–260. DOI: 10.1007/BF01645859. URL: https://doi.org/10.1007/BF01645859.

[19] Werner Israel. “Event Horizons in Static Vacuum Space-Times”. In: *Phys. Rev.* 164 (5 Dec. 1967), pp. 1776–1779. DOI: 10.1103/PhysRev.164.1776. URL: https://link.aps.org/doi/10.1103/PhysRev.164.1776.

[20] R. Jackiw. “Lower Dimensional Gravity”. In: *Nucl. Phys.* B252 (1985), pp. 343–356. DOI: 10.1016/0550-3213(85)90448-1.

[21] Alexei Kitaev. *Hidden correlations in the Hawking radiation and thermal noise*. Talk at KITP http://online.kitp.ucsb.edu/online/joint98/kitaev/. Feb. 2015.

[22] Alexei Kitaev and S. Josephine Suh. “Statistical mechanics of a two-dimensional black hole”. In: (2018). arXiv: 1808.07032 [hep-th].

[23] Alexei Kitaev and S. Josephine Suh. “The soft mode in the Sachdev-Ye-Kitaev model and its gravity dual”. In: *JHEP* 05 (2018), p. 183. DOI: 10.1007/JHEP05(2018)183. arXiv: 1711.08467 [hep-th].

[24] Ho Tat Lam et al. “Shockwave S-matrix from Schwarzian Quantum Mechanics”. In: *JHEP* 11 (2018), p. 182. DOI: 10.1007/JHEP11(2018)182. arXiv: 1804.09834 [hep-th].

[25] Juan Maldacena and Xiao-Liang Qi. “Eternal traversable wormhole”. In: (2018). arXiv: 1804.00491 [hep-th].

[26] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford. “A bound on chaos”. In: *JHEP* 08 (2016), p. 106. DOI: 10.1007/JHEP08(2016)106. arXiv: 1503.01409 [hep-th].

[27] Juan Maldacena and Douglas Stanford. “Remarks on the Sachdev-Ye-Kitaev model”. In: *Phys. Rev.* D94.10 (2016), p. 106002. DOI: 10.1103/PhysRevD.94.106002. arXiv: 1604.07818 [hep-th].

[28] Juan Maldacena, Douglas Stanford, and Zhenbin Yang. “Conformal symmetry and its breaking in two dimensional Nearly Anti-de-Sitter space”. In: *PTEP* 2016.12 (2016), p. 12C104. DOI: 10.1093/ptep/ptw124. arXiv: 1606.01857 [hep-th].

[29] Juan Maldacena, Douglas Stanford, and Zhenbin Yang. “Diving into traversable wormholes”. In: *Fortsch. Phys.* 65.5 (2017), p. 1700034. DOI: 10.1002/prop.201700034. arXiv: 1704.05333 [hep-th].

[30] Juan Maldacena and Leonard Susskind. “Cool horizons for entangled black holes”. In: *Fortsch. Phys.* 61 (2013), pp. 781–811. DOI: 10.1002/prop.201300020. arXiv: 1306.0533 [hep-th].

[31] Juan M. Maldacena. "The Large N Limit of Superconformal Field Theories and Supergravity". In: *Adv.Theor.Math.Phys.* 2 (1998), pp. 231–252. DOI: 10.1023/A:1026654312961. eprint: hep-th/9711200. URL: https://arxiv.org/pdf/hep-th/9711200.pdf.

[32] Juan Martin Maldacena. "Eternal black holes in anti-de Sitter". In: *JHEP* 04 (2003), p. 021. DOI: 10.1088/1126-6708/2003/04/021. arXiv: hep-th/0106112 [hep-th].

[33] Thomas G. Mertens, Gustavo J. Turiaci, and Herman L. Verlinde. "Solving the Schwarzian via the Conformal Bootstrap". In: *JHEP* 08 (2017), p. 136. DOI: 10.1007/JHEP08(2017)136. arXiv: 1705.08408 [hep-th].

[34] Don N. Page. "Information in black hole radiation". In: *Phys. Rev. Lett.* 71 (1993), pp. 3743–3746. DOI: 10.1103/PhysRevLett.71.3743. eprint: hep-th/9306083. URL: https://link.aps.org/doi/10.1103/PhysRevLett.71.3743.

[35] Geoff Penington et al. "Replica wormholes and the black hole interior". In: (2019). eprint: 1911.11977.

[36] I Racz and R M Wald. "Extensions of spacetimes with Killing horizons". In: *Classical and Quantum Gravity* 9.12 (Dec. 1992), pp. 2643–2656. DOI: 10.1088/0264-9381/9/12/008. URL: https://doi.org/10.1088/0264-9381/9/12/008.

[37] Istvan Racz and Robert M. Wald. "Global Extensions of Spacetimes Describing Asymptotic Final States of Black Holes". In: *Class.Quant.Grav.* 13 (1996), pp. 539–553. DOI: 10.1088/0264-9381/13/3/017. eprint: gr-qc/9507055. URL: https://arxiv.org/pdf/gr-qc/9507055.pdf.

[38] Subir Sachdev and Jinwu Ye. "Gapless spin fluid ground state in a random, quantum Heisenberg magnet". In: *Phys. Rev. Lett.* 70 (1993), p. 3339. DOI: 10.1103/PhysRevLett.70.3339. arXiv: cond-mat/9212030 [cond-mat].

[39] Stephen H. Shenker and Douglas Stanford. "Stringy effects in scrambling". In: *JHEP* 05 (2015), p. 132. DOI: 10.1007/JHEP05(2015)132. arXiv: 1412.6087 [hep-th].

[40] C. Teitelboim. "Gravitation and Hamiltonian Structure in Two Space-Time Dimensions". In: *Phys. Lett.* 126B (1983), pp. 41–45. DOI: 10.1016/0370-2693(83)90012-6.

[41] Robert M. Wald. *General Relativity*. Chicago, USA: Chicago Univ. Pr., 1984. DOI: 10.7208/chicago/9780226870373.001.0001.

[42] Edward Witten. "Anti De Sitter Space And Holography". In: *Adv.Theor.Math.Phys.* 2 (1998), pp. 253–291. eprint: hep-th/9802150. URL: https://arxiv.org/pdf/hep-th/9802150.pdf.

[43] Edward Witten. "Notes on Some Entanglement Properties of Quantum Field Theory". In: *Rev. Mod. Phys.* 90 (2018), p. 45003. DOI: 10.1103/RevModPhys.90.045003. eprint: 1803.04993. URL: https://arxiv.org/pdf/1803.04993.pdf.