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p-Converse to a Theorem of Gross-Zagier, Kolyvagin, and Rubin for Small Primes

Thesis by

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In Partial Fulfillment of the Requirements for the
Degree of
Bachelor of Science

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2021
Defended May 4, 2021

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ACKNOWLEDGEMENTS

The study of this p-converse Theorem first started as a collaborative Summer Undergraduate Research Fellowship (SURF) project with Jacob Ressler under the guidance of Dr. Ashay Burungale in 2020. It was then completed during the course of this senior thesis at Caltech mentored by Dr. Burungale. This project is also partially supported by the Olga Taussky-Todd Prize.

I would like to thank my mentor Dr. Ashay Burungale for his guidance, suggestions, and support during the tough period of the pandemic. I would also like to thank Prof. Dinakar Ramakrishnan and Prof. Elena Mantovan for serving on the thesis committee and providing extremely helpful advice.

ABSTRACT

Let E be a CM elliptic curve over the rationals and p a good, ordinary prime. We show the implication

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1$$

for the p^∞ Selmer group $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})$ and the complex L-function $L(s, E_{/\mathbb{Q}})$ in the case of $p = 2, 3$.

CONTENTS

Chapter 1

INTRODUCTION

Let E be a rational elliptic curve. A fundamental arithmetic invariant of E is the Mordell-Weil rank, which is defined to be the rank of the finitely generated abelian group $E(\mathbb{Q})$, i.e. rational points on the elliptic curve. The Mordell-Weil rank is typically expected to be 0 or 1, but currently there is no systematic approach to find generators of $E(\mathbb{Q})$. An equally mysterious object that governs the arithmetic of E is the conjecturally finite Tate-Shafarevich group $Ш(E_{/\mathbb{Q}})$. The p^∞-Selmer group $\mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}})$ encodes information about the arithmetic of E via the fundamental exact sequence

$$0 \to E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\kappa} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) \xrightarrow{\rho} Ш(E_{/\mathbb{Q}})[p^\infty] \to 0 \qquad (1.0.1)$$

Here κ arises from the Kummer map and ρ is induced by the map in the long exact sequence of cohomologies.

An underlying object on the analytic side is the complex Hasse-Weil L-function $L(s, E_{/\mathbb{Q}})$ corresponding to the elliptic curve $E_{/\mathbb{Q}}$ with $s \in \mathbb{C}$. An important analytic invariant is the analytic rank given by the order of vanishing at $s = 1$, which is denoted by $\mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}})$.

The BSD conjecture predicts a deep relation between the arithmetic and analytic sides. We include here one particular instance of the conjecture:

Conjecture 1 (BSD conjecture)**.** *Let E be an elliptic curve over the rationals. For $r = 0, 1$, the following are equivalent:*

(1) $\mathrm{rank}_{\mathbb{Z}} E(\mathbb{Q}) = r$ *and* $Ш(E_{/\mathbb{Q}})$ *is finite.*

(2) $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = r$ *for a prime* p.

(3) $\mathrm{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = r$.

The implication (1) $\Rightarrow$ (2) follows from the fundamental exact sequence 1.0.1. The implication (3) $\Rightarrow$ (1) is a fundamental result on the BSD conjecture due to Coates-Wiles, Gross-Zagier, Kolyvagin and Rubin. Thus, the implication (2) $\Rightarrow$ (3) is essentially a converse to this fundamental result, and we refer to it as the p-converse theorem.

There has been significant progress made towards the p-converse theorem. When $r = 0$ and E is a CM elliptic curve, the p-converse was first established when $(p, |\mathcal{O}_K^\times|) = 1$ by Rubin in the early 90's. Burungale-Tian then established the case for $p = 2, 3$ in 2019. When $r = 0$ and E is non-CM, Skinner-Urban showed the p-converse for ordinary primes $p > 2$ with certain assumptions on the Galois representation $E[p]$. Wan proved certain cases for supersingular primes $p > 2$. For $r = 1$ and E a non-CM elliptic curve, Skinner and Zhang independently showed the result for good ordinary primes $p > 3$ with certain assumptions on $E[p]$. For $r = 1$ and E a CM elliptic curve, Burungale-Tian (Burungale and Tian, 2020) and Burungale-Skinner-Tian have established the case for good ordinary and good supersingular prime $p > 3$ respectively. The main result of the article is a p-converse in the case of CM elliptic curves.

Theorem 1. *Let E be a CM elliptic curve over the rationals. Let $p \in \{2, 3\}$ be a good ordinary prime for the elliptic curve $E_{/\mathbb{Q}}$. Then,*

$$\operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \operatorname{ord}_{s=1} L(s, E_{/\mathbb{Q}}) = 1$$

In particular, whenever $\operatorname{corank}_{\mathbb{Z}_p} \operatorname{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$, we know $\operatorname{rank}_{\mathbb{Z}} E(\mathbb{Q}) = 1$ and $Ш(E_{/\mathbb{Q}})$ is finite

We restricted ourselves to $p \in \{2, 3\}$ as Burungale-Tian established the same statement in (Burungale and Tian, 2020) except that they assume $p > 3$. Together with their result, the rank 1 p-converse is established for all CM elliptic curves with good ordinary reduction at p.

The approach to prove Theorem 1 closely follows that in (Burungale and Tian, 2020), with modifications to steps that used the assumption $p > 3$. For completeness, we still include the essential components of the argument in (Burungale and Tian, 2020), with emphasis on the modified parts.

In (Burungale and Tian, 2020), the proof first reduces the p-converse implication to an equivalent statement via an auxiliary Rankin-Selberg setup. With the generalized Gross-Zagier formula due to Yuan-Zhang-Zhang (Yuan, S.-W. Zhang, and W. Zhang, 2013), the equivalent statement is proven by showing a certain Heegner point is non-torsion. The proof of the Heegner point being non-torsion relies on the Galois descent of a certain Heegner Main Conjecture, which is Iwasawa-theoretic in nature. When we include $p = 2, 3$, the original approach goes through assuming the Heegner Main Conjecture (HMC). Thus, the major modifications take place in the reformulation and proof of the HMC when $p = 2, 3$.

Notations

Here we introduce notations that will be used throughout unless specified otherwise. Fix an algebraic closure $\overline{\mathbb{Q}}$ of $\mathbb{Q}$. For a subfield $F \subset \overline{\mathbb{Q}}$, we let $G_F = \mathrm{Gal}(\overline{\mathbb{Q}}/F)$. Let Σ denote a set of places of F. Let $F_\Sigma \subset \overline{\mathbb{Q}}$ denote the maximal extension of F unramified outside Σ and $G_{F,\Sigma} = \mathrm{Gal}(F_\Sigma/F)$.

Let $\mathbb{A}_F$ denote the adeles over F. For a finite subset S of places in F, let $\mathbb{A}_F^{(S)}$ denote the adeles outside S amd $\mathbb{A}_{F,S}$ denote the S-part of the adeles.

For a $\mathbb{Z}$-module M, let $M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}$.

Chapter 2

HEEGNER MAIN CONJECTURE

2.1 Setup

This section introduces the objects and hypotheses that will be used in the formulation of the Heegner Main Conjecture. We closely follow §2.1 of (Burungale and Tian, 2020).

Let p be a prime. Fix two embeddings $\iota_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$ and $\iota_p : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$. Let v_p be the p-adic valuation induced via ι_p so that $v_p(p) = 1$.
Let K be an imaginary quadratic field and O_K its ring of integers. We consider K to be a subfield of $\mathbb{C}$ through the embedding ι_∞. Let c be the complex conjugation on $\mathbb{C}$ that induces the unique non-trivial element τ of $\mathrm{Gal}(K/\mathbb{Q})$ via ι_∞.
We assume that p splits in K. Let $\mathfrak{p}$ denote the prime above p in K that was determined via the embedding ι_p. For a positive integer m, let H_m denote the ring class field of K with conductor m and $O_m = \mathbb{Z} + mO_K$ the corresponding order. Let H denote the Hilbert class field.
Let K_∞ be the unique $\mathbb{Z}_p^2$-extension of K and let $\Gamma_K = \mathrm{Gal}(K_\infty/K)$. Note that τ acts on Γ_K. Let K_∞^- denote the anticyclotomic $\mathbb{Z}_p$-extension of K on which τ acts by inversion. Denote $\mathrm{Gal}(K_\infty^-/K)$ by Γ_K^-. Let K_n^- be the subextension of K_∞^- with degree p^n over K. Fix a splitting[1] $\Gamma_K = \Gamma_K^- \times \Gamma_K^+$.

Self-dual pair

Let $g \in S_2(\Gamma_0(N_g), \omega)$ be a weight two elliptic newform with Neben-type ω and let E_g be the corresponding Hecke field. Let $A = A_g$ be an abelian variety over $\mathbb{Q}$ associated to g by Eichler-Shimura such that

$$L(s, A_{/\mathbb{Q}}) = \prod_{\sigma : E_g \hookrightarrow \mathbb{C}} L(s, g^\sigma)$$

Recall that the endomorphism ring for A contains an order O_g in the Hecke field and O_g is generated over $\mathbb{Z}$ by the Hecke eigenvalues of g.
Let χ be an arithmetic Hecke character over K. Let $E_{g,\chi} \subset \mathbb{C}$ be the subfield generated over $\mathbb{Q}$ by the Hecke eigenvalues of g and the image of χ on $\left(\mathbb{A}_K^{(\infty)}\right)^\times$.

[1] When $p \neq 2$, there exists a canonical splitting.

Let $\wp$ be a prime above p in $E_{g,\chi}$ induced via the embedding ι_p. Let L be the completion of $E_{g,\chi}$ at $\wp$ and let O be the integer ring of L. Let $O_{g,\chi} \subset E_{g,\chi}$ be generated over O_g by the values of χ. Let $\wp_0$ be the prime of $O_{g,\chi}$ given by $\wp \cap O_{g,\chi}$. Let O_0 be the localization of $O_{g,\chi}$ at $\wp_0$. We can see that O_0 is a subring of O. Let L_0 be the fraction field of O_0.

We define the integral and the rational Iwasawa algebras:

$$\Lambda^\circ = O[\![\Gamma_K^-]\!], \quad \Lambda = \Lambda^\circ \otimes_O L.$$

We define an involution ι on Λ by $\iota(\gamma) = \gamma^{-1}$ for $\gamma \in \Gamma_K^-$. For any Λ-module M, we let M^ι denote the module twisted by ι.

Now suppose that χ is of finite order and

$$\omega \cdot \chi|_{\mathbb{A}^\times} = 1 \tag{2.1.1}$$

The Rankin-Selberg convolution $L(s, g \times \chi)$ corresponding to the pair (g, χ) is self-dual with functional equation around $s = 1$. Now we are ready to introduce an abelian variety associated to the pair (g, χ) using a variant of Eichler-Shimura construction.

Definition 1. *Let (g, χ) be as above. Let B be the Serre tensor $A \otimes \chi$, an abelian variety defined over K such that*

$$L(s, B_{/K}) = \prod_{\sigma: E_g \hookrightarrow \mathbb{C}} L(s, g^\sigma \times \chi^\sigma)$$

Given 2.1.1, the dual abelian variety $B^\vee$ is isogeneous to B. Fix such an isogeny from now on.

Selmer groups

We introduce Selmer groups associated to B over the anticyclotomic tower K_∞^-. Define

$$\mathcal{S}(B) := \varprojlim_n \varprojlim_m \mathrm{Sel}_{p^m}(B/K_n^-) \otimes_{O_0} L$$

$$\mathcal{X}(B) := (\varinjlim_n \varinjlim_m \mathrm{Sel}_{p^m}(B/K_n^-))^\vee \otimes_{O_0} L$$

Here $(\cdot)^\vee$ denotes taking the Pontryagin dual. Analogously, one can define $\mathcal{S}(B^\vee)$ and $\mathcal{X}(B^\vee)$ for the dual abelian variety $B^\vee$. These Selmer groups carry a natural Λ-module structure through Galois action.

2.2 Heegner Points

The Heegner Main Conjecture concerns Iwasawa theory of Heegner points on the abelian variety B along the anticyclotomic tower.
For the pair (g,χ), we always suppose the following generalized Heegner hypothesis:

- $\omega\cdot\chi|_{\mathbb{A}^\times}=1$.
- $\epsilon(g,\chi)=-1$. Here $\epsilon(g,\chi)$ is the global root number of the Rankin-Selberg convolution.

With these hypothesis, Disegni has constructed in (Disegni, 2017) a norm compatible sequence of Heegner points

$$\big(P(f_{\alpha,n},\chi)\big)_n\in\varprojlim_n B(K_n^-)_{\mathbb{Q}}$$

At $n=0$, we have $P_{g,\chi}$ denoting the Heegner point over K. The construction of the sequence of Heegner points is valid for $p=2,3$. Details can be found in §2.2 of (Burungale and Tian, 2020).
Let T_pB denote the Tate module. The image of $\big(P(f_{\alpha,n},\chi)\big)_n$ under the Kummer map

$$B(K_n^-)\otimes_{\mathcal{O}_0}L\longrightarrow H^1(K_n^-,T_pB)\otimes_{\mathcal{O}_0}L$$

gives the Heegner cohomology class

$$\kappa\in\mathcal{S}(B).$$

2.3 Formulation

Now we are ready to introduce the Heegner Main Conjecture formulated for the CM modular forms. We want to note that such a conjecture is primarily due to Perrin-Riou.
Recall that a Hecke eigenform g is said to be CM if there exists an imaginary quadratic field K and an arithmetic Hecke character ψ over K such that g is the theta series $\theta(\psi)$ associated to ψ. In this case, we say that g has CM by K.

Conjecture 2 (Heegner Main Conjecture (CM-case))**.** *Let (g,χ) be a self-dual pair of a weight two CM modular form (level N_g) with CM by K and a finite order Hecke character over an imaginary quadratic field K with root number -1. Let B the corresponding abelian variety over K. Suppose p is an ordinary prime for the pair and $p\nmid N_g\cdot\mathrm{cond}^r(\chi)$. Then,* *(1)* $\kappa \in \mathcal{S}(B)$ *is* Λ*-non-torsion and*

$$\mathrm{rank}_\Lambda\, \mathcal{S}(B) = \mathrm{rank}_\Lambda\, \mathcal{X}(B) = 1.$$

(2) $\mathrm{char}_\Lambda\, \mathcal{S}(B)/(\kappa) \cdot \mathrm{char}_\Lambda(\mathcal{S}(B)/(\kappa))^\iota = \mathrm{char}_\Lambda\, \mathcal{X}(B)_{tor}$

Here $(\cdot)_{tor}$ denotes the Λ-torsion submodule. Originally, the conjecture was formulated only for odd primes, but we have verified the objects involved, including the Heegner cohomology class κ and the anticyclotomic Selmer groups, are well-defined even when $p = 2$. Although the even prime case is often avoided in Iwasawa-theoretic arguments, it can be included in this formulation as we are only considering a rational statement, where both the Iwasawa algebra Λ and the Selmer groups are tensored with an extension of $\mathbb{Q}_p$.

2.4 Main Result

In (Burungale and Tian, 2020), Conjecture 2 was proven under the assumption $p > 3$. We generalize their approach and obtain the following result that includes $p = 2, 3$.

Theorem 2. *Let ψ be the Hecke character over K with infinity type $(1, 0)$ corresponding to the CM modular form g and suppose $\psi\chi$ corresponds to a CM elliptic curve and has root number -1.*
Then, the CM-case of the Heegner Main Conjecture (Conjecture 2) holds.

Compared to the formulation of Conjecture 2, the only additional assumption in Theorem 2 is the requirement of "$\psi\chi$ corresponds to a CM elliptic curve." The reason behind this assumption will be explained at the end of section 3.4. Moreover, as the reader will see in Chapter 4, the version of Heegner Main Conjecture proven by Theorem 2 suffices for the purpose of our p-converse theorem.

Chapter 3

HEEGNER MAIN CONJECTURE: VERIFICATION AND MODIFICATION

In this section, we explain the main steps taken to generalize Burungale-Tian's approach towards the Heegner Main Conjecture (HMC) and prove Theorem 2. We focus on resolving the key obstacles encountered in the case of $p = 2, 3$ and include other relevant arguments for logical completeness.

There are three natural pieces in the generalization. The first piece is the statement about the Heegner cohomological class κ being Λ-non-torsion. The second piece is concerned with the rank of Selmer groups $\mathcal{S}(B)$ and $\mathcal{X}(B)$ as Λ-modules. We will henceforth refer to it as the "rank statement." The third piece is statement (2) in the HMC, namely, the rational equality of characteristic ideals concerning Selmer groups. We will refer to the third piece as the "equality of characteristic ideals."

3.1 Preliminaries

Before explaining the three pieces in detail, we first introduce some objects that will be useful later.

Selmer Groups

We introduce Selmer groups arising in the process of proving the HMC. We begin with the general definitions and move on to properties that will be used in the rank statement.

Let ψ, g, χ be as in §2.3. Let B be the abelian variety over K associated to the pair (g, χ) in Definition 1. Since we are under the assumption that $p \nmid \mathrm{cond}(\chi)$, the abelian variety B has ordinary reduction at primes above p.
Let V_g denote the p-adic Galois representation $\rho_g : G_{\mathbb{Q}} \to GL_2(L)$ associated to g.
Let $L(\chi)$ denote the one dimensional G_K-representation over L associated with χ.
We can then define a p-adic Galois representation of G_K that is ordinary at p:

$$V := V_g \mid_{G_K} \otimes_L L(\chi)$$

For any Hecke character, we use a superscript $*$ to indicate that it has been pre-composed with the non-trivial element τ of $\mathrm{Gal}(K/\mathbb{Q})$. Define $\lambda = \psi\chi$. There exists an isomorphism of $L[G_K]$ modules:

$$V \cong L(\psi\chi) \oplus L(\psi^*\chi) = L(\lambda) \oplus L(\psi^*\chi)$$

Choose a G_K-stable lattice $T \subset V$ such that

$$T \cong O(\psi\chi) \oplus O(\psi^*\chi) = O(\lambda) \oplus O(\psi^*\chi)$$

Define

$$W = V/T, \quad W(\lambda) = L(\lambda)/O(\lambda), \quad W(\psi^*\chi) = L(\psi^*\chi)/O(\psi^*\chi).$$

$GL_{2/\mathbb{Q}}$ Selmer Groups

We want to define the Bloch-Kato Selmer groups on the $GL_{2/\mathbb{Q}}$-side from the local Bloch-Kato Selmer groups at places v of K.

Let ? represent either $V, T,$ or W.

For $v \nmid p$, the local Bloch-Kato Selmer groups are defined as:

$$H^1_f(K_v, V) := \ker\left(H^1(K_v, V) \to H^1(I_v, V)\right)$$

For $v \mid p$, the local Bloch-Kato Selmer groups are defined as:

$$H^1_f(K_v, V) := \ker\left(H^1(K_v, V) \to H^1(K_v, V^-)\right)$$

Here V^- denotes the maximal unramified quotient of $V|_{G_{K_v}}$.
Define $H^1_f(K_v, T)$ and $H^1_f(K_v, W)$ as the preimage and image resp. of $H^1_f(K_v, V)$ under the natural maps

$$H^1(K_v, T) \to H^1(K_v, V) \to H^1(K_v, W).$$

Using these local conditions, we define the Bloch-Kato Selmer group

$$H^1_f(K, ?) = \ker\left\{H^1(K, ?) \to \prod_v \frac{H^1(K_v, ?)}{H^1_f(K_v, ?)}\right\}$$

We also want to define the Λ-adic Selmer groups using a similar approach. For $v \nmid p$, the Λ-adic local Bloch-Kato Selmer groups are defined as:

$$H^1_f(K_v, V \otimes_L \Lambda) := \ker\left(H^1(K_v, V \otimes_L \Lambda) \to H^1(I_v, V \otimes_L \Lambda)\right)$$

For $v \mid p$, the local Bloch-Kato Selmer groups are defined as:

$$H_f^1\left(K_v, V \otimes_L \Lambda\right) := \ker\left(H^1(K_v, V \otimes_L \Lambda) \to H^1(K_v, V^- \otimes_L \Lambda)\right)$$

Here V^- denotes the maximal unramified quotient of $V\mid_{G_{K_v}}$. One can analogously define $H_f^1\left(K_v, ? \otimes_L \Lambda\right)$.

Definition 2. *The Λ-adic Selmer group $S(V \otimes_L \Lambda)$ corresponding to the pair (g, χ) is given by*

$$S(V \otimes_L \Lambda) := \ker\left\{H^1\left(K, V \otimes_{L_0} \Lambda\right) \to \prod_v H^1\left(K_v, V \otimes_{L_0} \Lambda\right)/H_f^1\left(K_v, V \otimes_{L_0} \Lambda\right)\right\}.$$

The discrete Selmer group $\mathrm{Sel}\left(K, W \otimes_O \Lambda^\circ\right)$ *is given by*

$$\mathrm{Sel}\left(K, W \otimes_O \Lambda^\circ\right) := \ker\left\{H^1\left(K, W \otimes_O \Lambda^\circ\right) \to \prod_v H^1\left(K_v, W \otimes_O \Lambda^\circ\right)/H_f^1\left(K_v, W \otimes_O \Lambda^\circ\right)\right\}.$$

Its Pontryagin dual is given by

$$X(W) = \hom_O\left(\mathrm{Sel}\left(K, W \otimes_O \Lambda^\circ\right), L/O\right)$$

Analogous objects can be defined for the dual Galois representations.

Remark 1. *The Λ-modules $S(V \otimes_L \Lambda)$ and $X(W)_L := X(W) \otimes_O L$ the same as the Λ-modules $\mathcal{S}(B)$ and $\mathcal{X}(B)$.*

$GL_{1/K}$ Selmer Groups

We want to define the Bloch-Kato Selmer groups on the $GL_{1/K}$ side from the local Bloch-Kato Selmer groups corresponding to Hecke characters. For a finite place v of K, the local Bloch-Kato Selmer groups are

$$H_f^1(K_v, L(\lambda)) = \begin{cases} H_{ur}^1(K_v, L(\lambda)), & v \nmid p \\ 0, & v \mid \mathfrak{p}^* \\ H^1(K_v, L(\lambda)), & v \mid \mathfrak{p} \end{cases}$$

and

$$H_f^1(K_v, L(\psi^*\chi)) = \begin{cases} H_{ur}^1(K_v, L(\psi^*\chi)), & v \nmid p \\ H^1(K_v, L(\psi^*\chi)), & v \mid \mathfrak{p}^* \\ 0, & v \mid \mathfrak{p} \end{cases}$$

The unramified local Galoid cohomology is given by

$$H^1_{ur}(K_v, \cdot) = \ker\left(H^1_{ur}(K_v, \cdot) \to H^1_{ur}(I_v, \cdot)\right)$$

Let Σ be the set of primes of K lying above p. Using these local conditions, we define the Selmer groups

$$\mathrm{Sel}^{\Sigma}(K, L(\cdot)) = \ker\left(H^1(K, L(\cdot)) \to \prod_{v \nmid p} H^1(K_v, L(\cdot))/H^1_f(K_v, L(\cdot))\right) \tag{3.1.1}$$

$$H^1_f(K, L(\cdot)) = \mathrm{Sel}(K, L(\cdot)) = \ker\left(\mathrm{Sel}^{\Sigma}(K, L(\cdot)) \to \prod_{v | p} H^1(K_v, L(\cdot))/H^1_f(K_v, L(\cdot))\right) \tag{3.1.2}$$

$$\mathrm{Sel}_{\Sigma}(K, L(\cdot)) = \ker\left(\mathrm{Sel}(K, L(\cdot)) \to \prod_{v | p} H^1(K_v, L(\cdot))\right) \tag{3.1.3}$$

We also want to introduce Λ-adic local conditions as in the $GL_{2/\mathbb{Q}}$ case and subsequently define the Λ-adic Selmer groups associated to the Hecke characters λ and $\psi^*\chi$.

Definition 3. *Let $\cdot$ denote λ or $\psi^*\chi$. The Λ-adic Selmer groups $S(L(\cdot) \otimes_L \Lambda)$ is given by*

$$S(L(\cdot) \otimes_L \Lambda) := \ker\left\{H^1\left(K, L(\cdot) \otimes_L \Lambda\right) \to \prod_{v} H^1\left(K_v, L(\cdot) \otimes_L \Lambda\right)/H^1_f\left(K_v, L(\cdot) \otimes_L \Lambda\right)\right\}.$$

We analogously define the discrete Selmer groups and their Pontryagin duals

$$\mathrm{Sel}\left(K, W(\cdot) \otimes_{\mathcal{O}} \Lambda^{\circ}\right), \quad X(\cdot).$$

Relating $GL_{2/\mathbb{Q}}$ and $GL_{1/K}$ Selmer Groups

Now we introduce some properties relating the Iwasawa-theoretic Selmer groups we have introduced above.

Lemma 1 (Lemma 3.7 in (Burungale and Tian, 2020))**.** *Let ψ (resp. χ) be a Hecke character over an imaginary quadratic field K with infinity type $(1, 0)$ (resp. finite order, unramified at p) and $\lambda = \psi\chi$. Let g be the CM modular form associated to ψ and V the p-adic Galois representation of G_K corresponding to the pair (g, χ). Then we have an isomorphism*

$$S(V \otimes_L \Lambda) \cong S(L(\lambda) \otimes_L \Lambda) \oplus S(L(\psi^*\chi) \otimes_L \Lambda)$$

of Λ-modules and isomorphisms

$$\mathrm{Sel}(K, W \otimes_{O} \Lambda^{\circ}) \cong \mathrm{Sel}(K, W(\lambda) \otimes_{O} \Lambda^{\circ}) \oplus \mathrm{Sel}(K, W(\psi^{*}\chi) \otimes_{O} \Lambda^{\circ})$$

and

$$X(W) \cong X(\lambda) \oplus X(\psi^{*}\chi)$$

of Λ°-modules.

Proof. The decomposition of Selmer groups follows directly from the definition and thus the proof in (Burungale and Tian, 2020) holds for $p = 2, 3$. □

p-adic L-functions

In this part, we introduce p-adic L-functions arising in the process of proving the HMC. It is worth noting that although p-adic L-functions do not appear in the formulation of the HMC, they seem to be inevitable in the proof. Just like the previous parts on Selmer groups, we will introduce underlying p-adic L-functions on both the $GL_{2/\mathbb{Q}}$ and $GL_{1/K}$ sides. We will also relate them through decomposition.

We assumed that the root number of the pair (g, χ) is -1. Thus, without loss of generality, we can suppose the root numbers:

$$\epsilon(\frac{1}{2}, \psi^{*}\chi) = +1, \quad \epsilon(\frac{1}{2}, \lambda) = -1. \tag{3.1.4}$$

$GL_{2/\mathbb{Q}}$-side p-adic L-function

Definition 4 (Definition 3.12 in (Burungale and Tian, 2020))**.** *Recall that g is p-ordinary with U_p-eigenvalue α. Let*

$$L_p(g \times \chi) \in O[\![\Gamma_K]\!] \otimes_{O} L$$

be the two variable Rankin-Selberg p-adic L-function characterized by the interpolation property (for all sufficiently p-ramified finite order characters $\chi' : \Gamma_K \to \mathbb{C}_p^{\times}$)

$$\hat{\chi}'\left(L_p(g \times \chi)\right) = \frac{e_p\left(g \times (\chi\chi')^{-1}\right)}{\alpha^{v(\mathrm{cond}(\chi'))}} \cdot \frac{L^{(p)}(1, g \times (\chi\chi')^{-1})}{\Omega_g}$$

Here

- $\Omega_g := L\left(1, ad(g)\right)$*, where* $ad(g)$ *denotes the adjoint.*

- $e_p\left(g \times (\chi\chi')^{-1}\right) = \varepsilon\left(0, \chi_{\mathfrak{p}}\chi'_{\mathfrak{p}}\right) \cdot \varepsilon\left(0, \chi_{\mathfrak{p}^*}\chi'_{\mathfrak{p}^*}\right)$, *where* $\varepsilon(0, \cdot)$ *is the local* ε*-factor.*
- $L^{(p)}(\cdot)$ *is the* L*-function with Euler factors at primes above* p *removed.*

$GL_{1/K}$-side p-adic L-function

Let W denote a finite flat extension of the Witt ring $W(\mathbf{F})$, where $\mathbf{F}$ is an algebraic closure of $\mathbf{F}_p$. Let $\pi : \Gamma_K^{\sharp} \twoheadrightarrow \Gamma_K$ be a finite cover arising from a finite extension of K_∞ contained in K^{ab} (fixing a tame level prime to p). Let Σ be a p-adic CM type of K corresponding to the embedding ι_∞ defined above. This is not to be confused with the set of places used in the definition of Selmer groups[1]. There exists an element

$$L_\Sigma \in W[\![\Gamma_K^{\sharp}]\!]$$

uniquely characterized by an interpolation property. The domain of interpolation consists of arithmetic Hecke characters $\lambda' : \Gamma_K^{\sharp} \to \mathbb{C}_p^{\times}$ with infinity type $(k+\kappa, -\kappa)$ for $k, \kappa \in \mathbb{Z}$ such that

- $k \geq 1$ and $\kappa \geq 0$, or
- $k \leq 1$ and $k + \kappa > 0$.

Here is the interpolation property.

There exist p-adic CM periods $\Omega_{\Sigma,p} \in \mathbb{C}_p^{\times}$ and complex CM periods $\Omega_{\Sigma,\infty} \in \mathbb{C}^{\times}$ such that for any character λ' in the interpolation domain, we have

$$\frac{L_\Sigma(\lambda')}{\Omega_{\Sigma,p}^{k+2\kappa}} = e_p\left((\lambda')^{-1}\right) \cdot \frac{L^{(p)}(0, (\lambda')^{-1})}{\Omega_{\Sigma,\infty}^{k+2\kappa}} \cdot \frac{\pi^{\kappa}\Gamma(k+\kappa)}{(Im\theta)} \cdot [O_K^{\times} : \mathbb{Z}^{\times}]$$

Here Γ represents the usual Γ-function and $\theta \in K$. For our purpose, we choose a certain tame level and consider the restriction of L_Σ to certain open subsets of $\Gamma_K^{\sharp}$.

Definition 5 (Definition 3.13 in (Burungale and Tian, 2020))**.** *Let* λ_0 *be a* p*-adic Hecke character over* K *with values in* L*. Let*

$$L_\Sigma(\lambda_0) \in W[\![\Gamma_K]\!] \otimes L$$

be the Katz p*-adic* L*-function given by*

$$\hat{\chi}'\left(L_\Sigma(\lambda_0)\right) := L_\Sigma(\lambda_0\chi')$$

The interpolation property of $L_\Sigma(\lambda_0)$ *follows from that of* L_Σ*.*

[1]Future use should be clear from context

Relating $GL_{2/\mathbb{Q}}$ and $GL_{1/K}$ p-adic L-function

Lemma 2 (Lemma 3.14 in (Burungale and Tian, 2020))**.** *Let the notation be as above. Let Σ^* denote the conjugation CM type of Σ. There exists an equality up to a constant in $\overline{\mathbb{Q}}_p^{\times}$:*

$$L_p(g \times \chi) \doteq L_{\Sigma}(\lambda) \cdot L_{\Sigma^*}(\psi^* \chi)$$

3.2 Non-torsion Heegner Cohomological Class

This part of the verification is done in collaboration with Jacob Ressler.
We have verified that the approach used by Burungale-Tian for this piece goes through even when $p = 2, 3$.

3.3 Rank Statement

This part of the modifications is done in collaboration with Jacob Ressler.
Combining Lemma 1 and Remark 1, the Selmer groups $\mathcal{S}(B)$ and $\mathcal{X}(B)$ associated to an abelian variety B can be decomposed into direct sums of Bloch-Kato Selmer groups associated to Hecke characters. The rank statement was then proven by examining the rank of the individual Selmer groups for the characters using Theorem 2.14 in (Arnold, 2007), which assume $p > 3$. Thus, our main modification is to adapt the proof of Theorem 2.14 to $p = 2, 3$.

The original proof of Arnold relies on the non-triviality of an Euler system of elliptic units. The non-triviality was achieved by demonstrating that the Katz's p-adic L-function associated to this Euler system is non-trivial. The restriction $p > 3$ was originally imposed in the construction of the elliptic units, in order to avoid non-integral coefficient in front of the associated p-adic L-function. We fix this issue by adopting the canonical 12th-root of certain elliptic units introduced in (Oukhaba and Viguié, 2013). The p-adic L-function constructed from Oukhaba-Viguié's elliptic units is compatible with the argument in Arnold and is shown to be non-vanishing under desirable circumstances. Upon showing the non-triviality of Euler system, the rest of the argument for proving the statememts in Theorem 2.14 of (Arnold, 2007) follows from properties of Euler system in (Rubin, 2000) and does not assume $p > 3$. To be precise, we have achieved the following:

Theorem 3 (Generalization of Theorem 2.14 in (Arnold, 2007))**.** *Let $\cdot$ denote a Hecke character satisfying conditions in §1.1 of (Arnold, 2007).*
$X_{\Sigma}(K_{\infty}^{-}, \cdot)$ and $Z(K_{\infty}^{-}, \cdot)$ are torsion Λ°-modules and there is a divisibility of charac-

teristic ideals $\mathrm{char}\, X_\Sigma(K_\infty^-, \cdot) \mid \mathrm{char}\, Z(K_\infty^-, \cdot)$. *Moreover,* $\mathrm{Sel}^\Sigma(K_\infty^-, T^*(\cdot))$ *is torsion-free of rank 1 over* Λ°.
If the sign in the functional equation of $\cdot$ *is* -1*, then* $X(K_\infty^-, \cdot)$ *has rank 1 over* Λ° *and* $\mathrm{Sel}(K_\infty^-, T^*(\cdot)) = \mathrm{Sel}^\Sigma(K_\infty^-, T^*(\cdot))$*. Otherwise,* $X(K_\infty^-, \cdot)$ *is a torsion* Λ°*-module.*

Remark 2 (Arnold's Notation)**.** *In the theorem above, we have kept some of the notations for Selmer groups in (Arnold, 2007).*
Here $T^*(\cdot)$ *is the Tate dual*[2] *of* $T(\cdot) = \mathcal{O}(\cdot)$*. The Iwasawa-theoretic Selmer group* $\mathrm{Sel}^\Sigma(K_\infty^-, -)$ *in defined to be a projective limit of the Bloch-Kato Selmer groups in equation 3.1.1, where the limit is taken with respect to corestriction maps between finite extensions of* K *contained in* K_∞^-*. Analogous definitions give* $\mathrm{Sel}(K_\infty^-, -)$ *and* $\mathrm{Sel}_\Sigma(K_\infty^-, -)$*. The divisible Selmer group* $X_\Sigma(K_\infty^-, \cdot)$ *is the Pontryagin dual of* $\mathrm{Sel}_\Sigma(K_\infty^-, W(\cdot))$*. Analogous definition gives* $X^\Sigma(K_\infty^-, \cdot)$ *and* $X(K_\infty^-, \cdot)$*.*
These formulations of Iwasawa-theoretic Selmer groups are equivalent to what we have in Definition 3.
The Λ°*-module* $Z(K_\infty^-)$ *is defined as:*

$$Z(K_\infty^-, \cdot) := \mathrm{Sel}^\Sigma(K_\infty^-, T^*(\cdot))/C(K_\infty^-, \cdot)$$

Here $C(K_\infty^-, \cdot)$ *is the* Λ°*-submodule of* $\mathrm{Sel}^\Sigma(K_\infty^-, T^*(\cdot))$ *generated by* $\{c_\mathfrak{a}(K_\infty^-, \cdot) \mid (\mathfrak{a}, \mathfrak{f}p) = 1\}$*. Note that* $\mathfrak{f}$ *is an ideal of* K *related to the tame level we fixed in the definition of* L_Σ *and* $\mathfrak{a}$ *is an integral ideal of* K*. Most importantly,* $c_\mathfrak{a}$ *denotes an Euler system constructed from our modified elliptic units.*

We will only use Arnold's notations when specified.

3.4 Equality of Characteristic Ideals

In Burungale-Tian, the proof for the equality of characteristic ideals was split into two sides: the $GL_{2/\mathbb{Q}}$-side and the $GL_{1/K}$-side. We examine each side separately for parts that need modification.

On the $GL_{2/\mathbb{Q}}$-side, we desire the following:

Theorem 4. *There is an equality of ideals in* Λ*:*

$$\left(L_p'(g \times \chi)\right) = \mathrm{char}_\Lambda\, \mathcal{S}(B)/(\kappa) \cdot \mathrm{char}_\Lambda(\mathcal{S}(B)/(\kappa))^\iota \cdot R(g \times \chi) \tag{3.4.1}$$

[2] We have flipped the notation for T and T^* compared to Arnold.

Here $R(g \times \chi)$ is the anticyclotomic regulator associated to some Λ-adic height pairing on $S(V \otimes L)$ and $L'_p(g \times \chi)$ denotes the cyclotomic derivative of the two-variable Rankin-Selberg p-adic L-function $L_p(g \times \chi)$.

Proof. The proof by Burungale-Tian relies on a Λ-adic Gross-Zagier formula established in Theorem C of (Disegni, 2017):

$$\langle \kappa, \kappa^{\iota} \rangle \doteq L'_p(g \times \chi)$$

Here, "$\doteq$" denotes equality up to some constant in $\overline{\mathbb{Q}}_p^{\times}$. In particular, this equality holds as ideals in Λ arising from both sides. From our current understanding, the pairing and formula includes the case of $p = 2, 3$ and there are no obstacles for adopting the same approach. □

On the $GL_{1/K}$-side, we want the following identities:

Theorem 5. *There are equalities of ideals in Λ:*

$$(L^-_{\Sigma^*}(\psi^*\chi)) = \mathrm{char}_\Lambda X(\psi^*\chi) \tag{3.4.2}$$

$$(L'_\Sigma(\lambda)) = \mathrm{char}_\Lambda X(\lambda)_{tor} \cdot R(\lambda) \tag{3.4.3}$$

Combining the two we get:

$$\left(L'_\Sigma(\lambda) \cdot L^-_{\Sigma^*}(\psi^*\chi)\right) = \mathrm{char}_\Lambda X(\lambda)_{tor} \cdot \mathrm{char}_\Lambda X(\psi^*\chi) \cdot R(\lambda) \tag{3.4.4}$$

Recall, $L_\Sigma(\lambda)$ and $L^*_\Sigma(\psi^*\chi))$ are two variable Katz's p-adic L-function defined in §3.1. The superscript $'$ denotes taking cyclotomic derivative and the superscript $^-$ denotes the projection to the anticyclotomic component by setting the cyclotomic variable to 0. The anticyclotomic regulator $R(\lambda)$ is associated to height pairing on anticyclotomic Bloch-Kato Selmer groups.

In Burungale-Tian, the two equalities were proven using anticyclotomic results in Theorem 2.14, Theorem 3.9, and Theorem 4.17 of (Arnold, 2007). Modifications of Arnold's theorems and their proofs are needed as they assume $p > 3$.

The case of $\psi^*\chi$

We will first discuss the equality of ideals involving $\psi^*\chi$, whose root number satisfies

$$\epsilon(\frac{1}{2}, \psi^*\chi) = +1.$$

The original proof of equation 3.4.2 by Arnold relies on Rubin's Iwasawa Main Conjecture (IMC) established in (Rubin, 1991), which is an integral statement and assumes $p > 3$ in an essential manner. Instead of trying to directly generalize Rubin's result, the key modification in this paper is to use a rational IMC, which includes the case of $p = 2, 3$. This rational IMC is a corollary of the main result by Johnson-Leung and Kings in (Johnson-Leung and Kings, 2011).

Theorem 6 (Rational Iwasawa Main Conjecture)**.** *Suppose p splits into two distinct primes in K, then there is an equality of ideals in $\mathbb{Z}_p[\![\Gamma_K]\!] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:*

$$\mathrm{char}\left(\overline{\mathscr{E}}_\infty/\overline{\mathscr{C}}_\infty \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right) = \mathrm{char}\left(A_\infty \otimes_{\mathbb{Z}_p} \mathbb{Q}_p\right)$$

and

$$\mathrm{char}(X_\infty \otimes_{\mathbb{Z}_p} \mathbb{Q}_p) = \mathrm{char}(U_\infty/\overline{\mathscr{C}}_\infty \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

Proof. This follows from the proof of Theorem 5.7 in (Johnson-Leung and Kings, 2011) and the relation to Rubin's classical formulation of IMC given in §5.4 of (Johnson-Leung and Kings, 2011). □

Here the notation for the Iwasawa modules are consistent with Rubin's original notations in §4 of (Rubin, 1991). This result is rational in the sense that both the Iwasawa algebra and the modules are tensored with $\mathbb{Q}_p$. In a sense, they are "locally inverted" at p.

For the inclusion

$$(L^-_{\Sigma^*}(\psi^*\chi)) \supset \mathrm{char}_\Lambda X(\psi^*\chi),$$

we follow Arnold's idea of proof but make use of the rational IMC instead of Rubin's IMC. The rational IMC was first used to generalize a two variable main conjecture (Theorem 3.2 in (Arnold, 2007)) for the representation $W(\psi^*\chi)$. Arnold's notations in Remark 2 are used in the following theorem.

Theorem 7 (Generalization of Theorem 3.2 in Arnold, 2007)**.** *$X(K_\infty, \psi^*\chi)$ is a torsion $\mathcal{O}[\![\Gamma_K]\!]$-module.* $\mathrm{Sel}^\Sigma(K_\infty, T^*(\psi^*\chi))$ *has rank 1 over $\mathcal{O}[\![\Gamma_K]\!]$, and there are equalities of ideals in $\mathcal{O}[\![\Gamma_K]\!] \otimes_{\mathcal{O}} L$*

$$\mathrm{char}\, X(K_\infty, \psi^*\chi) \otimes_{\mathcal{O}} L = \mathrm{char}\left(H^1(K_{\infty,\mathfrak{p}}, T(\psi^*\chi))/\mathrm{loc}_{\mathfrak{p}}\, C(K_\infty, \psi^*\chi)\right) \otimes_{\mathcal{O}} L$$

and

$$\mathrm{char}\, X_\Sigma(K_\infty, \psi^*\chi) \otimes_{\mathcal{O}} L = \mathrm{char}\, Z(K_\infty, \psi^*\chi) \otimes_{\mathcal{O}} L$$

Proof. Here we give a sketch of the proof to this theorem.
Let L_∞ be an abelian extension of K which is finite over K_∞ and splits both $V = L(\psi^*\chi)$ and V^*. By splitting over V we mean that the character associated to the Galois representation is trivial when restricted to G_{L_∞}. There is a decomposition $\mathrm{Gal}(L_\infty/K) \cong \mathrm{Gal}(K_\infty/K) \times \Delta$. When $p \mid [L_\infty : K_\infty]$, such a decomposition is not canonical.
Similar to Rubin's notation, for any subfield F of L_∞, define $M^\Sigma(F)$, $M(F)$, $M_\Sigma(F)$ to be the maximal abelian p-extension of F unramified outside of p, $\mathfrak{p}$, and everywhere respectively. When F is finite over K, we let $\mathscr{E}(F)$ denote the completion of the global units $\mathcal{O}_F^\times \otimes \mathcal{O}$. If F/K is infinite, let $\mathscr{E}(F) = \varprojlim \mathscr{E}(F')$. If $\mathfrak{a} \subset K$ is an integral ideal prime to $\mathfrak{f}p$, define $\mathscr{U}_\mathfrak{a}(F) \subset \mathscr{E}(F)$ to be the submodule generated by the elliptic unit Euler system. Define $\mathscr{U}(F)$ to be the submodule generated by all such $\mathscr{U}_\mathfrak{a}(F) \subset \mathscr{E}(F)$.

There is an $\mathcal{O}$-module isomorphism

$$\alpha : \mathrm{Hom}\left(\mathrm{Gal}(M^\Sigma(L_\infty)/L_\infty), \Phi/\mathcal{O}\right) \to \mathrm{Sel}^\Sigma(L_\infty, W^*)$$

The analogous statement is true for $M(L_\infty)$ and $M_\Sigma(L_\infty)$.
There also exist isomorphisms

$$\beta : \mathscr{E}(L_\infty) \otimes \mathcal{O} \to \mathrm{Sel}^\Sigma(L_\infty, T)$$
$$\beta : \mathscr{U}(L_\infty) \otimes \mathcal{O} \to C(L_\infty)$$

By the assumption that L_∞ splits V^*, we can view $\psi^*\chi$ as a character of $\mathrm{Gal}(L_\infty/K) \cong \mathrm{Gal}(K_\infty/K) \times \Delta$ and write $\psi^*\chi = \kappa^*\nu^*$. Here ν^* is a character of the finite group Δ. After descenting Selmer groups from L_∞ to K_∞, we take Δ-invariant of the $\mathcal{O}[\![\mathrm{Gal}(L_\infty/K)]\!]$-modules present in the isomorphisms α and β. Applying Theorem 6 and twisting theorems in Ch.6. of (Rubin, 2000) gives the desired result. □

This two variable main conjecture is then descended to the anticyclotomic tower to give the inclusion. The descent argument does not assume $p > 3$.

The reverse inclusion

$$(L^-_{\Sigma^*}(\psi^*\chi)) \subset \mathrm{char}_\Lambda X(\psi^*\chi)$$

follows from Theorem 3. The modification of Euler systems for $p = 2, 3$ is needed here.

The case of λ

Now we move on to the equality of ideals involving $\lambda = \psi\chi$, whose root number satisfies

$$\epsilon(\frac{1}{2}, \lambda) = -1.$$

This equality is shown in Theorem 4.17 of (Arnold, 2007) under the assumption $p > 3$. One essential component of Arnold's proof is a p-adic height pairing on Selmer groups associated to Hecke characters. This height pairing, first introduced by Perrin-Riou in (Perrin-Riou, 1992), exists for Selmer groups associated to any suitablely nice p-adic Galois representation of a number field. The height pairing carries good properties such as having a bounded image, being Galois-equivariant and compatible with restrictions (corestrictions). Moreover, Rubin has proven a height formula when the pairing was evaluated on classes associated to the elliptic units. To summarize, we have the following:

Theorem 8 (Theorem 4.7 in (Arnold, 2007))**.** *Recall that for each finite extension F of K, local Tate duality gives a pairing:*

$$\langle\ ,\ \rangle_{F_\mathfrak{p}} : H^1(F_\mathfrak{p}, T^*) \times H^1(F_\mathfrak{p}, T) \to \mathcal{O}$$

Set $\langle\ ,\ \rangle_n := \langle\ ,\ \rangle_{K^-_{n,\mathfrak{p}}}$.

For all n, there is a p-adic height pairing

$$h_n : \mathrm{Sel}(K_n^-, T^*) \times \mathrm{Sel}(K_n^-, T) \to L$$

canonically determined by a chosen isomorphism[3] $\rho : \Gamma_K^+ \simeq \mathbb{Z}_p$ *up to sign, satisfying:*

1. *(bounded image) There is an integer k, independent of n, such that the image of h_n lies in $p^{-k}\mathcal{O}$.*
2. *(Galois-equivariance) For any $a \in \mathrm{Sel}(K_n^-, T^*)$, $b \in \mathrm{Sel}(K_n^-, T)$, and $\sigma \in \mathrm{Gal}(K_n^-/K)$, we have*
$$h_n(a^\sigma, b^\sigma) = h_n(a, b)$$
3. *(compatibility) For $a_n \in \mathrm{Sel}(K_n^-, T^*)$ and $b_{n+1} \in \mathrm{Sel}(K_{n+1}^-, T)$, we have*
$$h_n(a_n, \mathrm{cor}(b_{n+1})) = h_{n+1}(\mathrm{res}(a_n), b_{n+1})$$

[3] sending a chosen topological generator γ of $\mathrm{Gal}(K_\infty^+/K)$ to $1 \in \mathbb{Z}_p$

4. *(height formula) If $b \in \mathrm{Sel}(K_n^-, T)$, then (after extending scalars to $\tilde{L}$, the completion of $\mathbb{Q}_p^{ur} \cdot L$),*

$$h_n(c_{\mathfrak{a}}(K_n^-), b) = \langle \alpha_n, \mathrm{loc}_{\mathfrak{p}}\, b \rangle_n$$

The construction of the pairing and the proof of height formula is essentially the same for $p = 2, 3$. This height pairing was further generalized into an Iwasawa-theoretic pairing and inherits many properties, including the height formula, from the original height pairing by definition.

Definition 6 (Iwasawa-theoretic Height Pairing)**.** *Let $\tilde{\Lambda}^\circ := \tilde{O}[\![\Gamma_K^-]\!]$ and $\tilde{\Lambda} := \tilde{\Lambda}^\circ \otimes_{\tilde{O}} \tilde{L}$, where $\tilde{O}$ the integer ring of $\tilde{L}$. We first define the Iwasawa-theoretic Tate pairing*

$$\langle\ ,\ \rangle_\infty : \left(H^1(K_{\infty,\mathfrak{p}}^-, T^*)\hat{\otimes}\tilde{L}\right) \otimes_{\tilde{\Lambda}^\circ} \left(H^1(K_{\infty,\mathfrak{p}}^-, T)^\iota\hat{\otimes}\tilde{L}\right) \to \tilde{\Lambda}^\circ$$

by setting

$$\langle a_\infty, b_\infty \rangle_\infty = \varprojlim_n \sum_{\sigma \in \mathrm{Gal}(K_n^-/K)} \langle a_n^\sigma, b_n \rangle_n \sigma^{-1}$$

We then define the Iwasawa-theoretic p-adic height pairing

$$h_\infty : \left(\mathrm{Sel}(K_\infty^-, T^*)\hat{\otimes}\tilde{L}\right) \otimes_{\tilde{\Lambda}^\circ} \left(\mathrm{Sel}(K_\infty^-, T)^\iota\hat{\otimes}\tilde{L}\right) \to \tilde{\Lambda}^\circ$$

by setting

$$h_\infty(a_\infty, b_\infty) = \varprojlim_n \sum_{\sigma \in \mathrm{Gal}(K_n^-/K)} h_n(a_n^\sigma, b_n)\sigma^{-1}$$

Remark 3 (Iwasawa Height Formula)**.** *For every $b_\infty \in \mathrm{Sel}(K_\infty^-, T)$, there is a height formula*

$$h_\infty(c_{\mathfrak{a}}(K_\infty^-), b_\infty) = \langle \alpha_\infty, \mathrm{loc}_{\mathfrak{p}}\, b_\infty \rangle_\infty$$

for $\alpha_\infty \in H^1(K_{\infty,\mathfrak{p}}^-, T^*)\hat{\otimes}\tilde{L}$ defined in Lemma 4.6 of (Arnold, 2007).

Another important piece in Arnold's argument towards equation 3.4.3 is the duality result about characteristic ideals of the divisible Selmer groups $X(\cdot)$ (see Proposition 4.1 and Proposition 4.2 of (Arnold, 2007)). The duality result does not assume $p > 3$. The Iwasawa height formula, combined with the duality result and results derived from the rational IMC, gives the equality of the root number -1 case.

To summarize, in the root number -1 case, apart from employing the rational IMC, no additional modifications are needed for $p = 2, 3$.

We are now ready to combine results from the $GL_{2/\mathbb{Q}}$-side and the $GL_{1/K}$-side, namely, equation 3.4.1 and 3.4.4, to prove (2) of Conjecture 2 under the assumptions in Theorem 2. The following strategy closely follows that discussed in Burungale-Tian but uses our generalized results above.

Proof of Equality of Characteristic Ideals.
The LHS of equation 3.4.1 and 3.4.4 are equal due to a corollary of Lemma 2, which says:

$$L_p^{'}(g \times \chi) = L_\Sigma'(\lambda) \cdot L_{\Sigma^*}^{-}(\psi^* \chi)$$

up to some constant in $\overline{\mathbb{Q}}_p^\times$. Thus, combining equation 3.4.1 and 3.4.4 gives the following equality in Λ:

$$\mathrm{char}_\Lambda \mathcal{S}(B)/(\kappa) \cdot \mathrm{char}_\Lambda(\mathcal{S}(B)/(\kappa))^\iota \cdot R(g \times \chi) = \mathrm{char}_\Lambda X(\lambda)_{tor} \cdot \mathrm{char}_\Lambda X(\psi^* \chi) \cdot R(\lambda) \tag{3.4.5}$$

From the decomposition of the Selmer group $\mathcal{X}(B)$ given by Lemma 1 and Remark 1, we have

$$\mathrm{char}_\Lambda \mathcal{X}(B)_{\mathrm{tor}} = \mathrm{char}_\Lambda X(\lambda)_{tor} \cdot \mathrm{char}_\Lambda X(\psi^* \chi)_{tor}$$

When proving the rank statement, $X(\psi^* \chi)$ was determined to be a torsion Λ-module. Thus, the equality becomes

$$\mathrm{char}_\Lambda \mathcal{X}(B)_{\mathrm{tor}} = \mathrm{char}_\Lambda X(\lambda)_{tor} \cdot \mathrm{char}_\Lambda X(\psi^* \chi) \tag{3.4.6}$$

Employing again the decomposition of Selmer groups and the rank statement, the definition of Λ-adic height pairing on Selmer groups gives the following relation between regulators:

$$R(g \times \chi) = R(\lambda) \tag{3.4.7}$$

With equation 3.4.5, 3.4.6, and 3.4.7, the equality of characteristic ideals will be complete if we can show that the regulator $R(\lambda)$ does not vanish in the case of $p = 2, 3$.

When $p > 3$, Rubin has demonstrated $R(\lambda) \neq 0$ in the appendix of (Agboola and Howard, 2006), when λ is associated to a CM elliptic curve.
Let F be a finite extension of K contained in K_∞. Rubin first defined the following

more generalized versions of the p-adic height pairing h_n:

$$\begin{aligned} h_F &: \mathrm{Sel}(F,T) \otimes \mathrm{Sel}(F,T^*) \to \mathrm{Gal}(K_\infty/F) \otimes \mathbb{Q}_p \\ h_{F,cycl} &: \mathrm{Sel}(F,T) \otimes \mathrm{Sel}(F,T^*) \to \mathrm{Gal}(FK_\infty^+/F) \otimes \mathbb{Q}_p \\ h_{F,anti} &: \mathrm{Sel}(F,T) \otimes \mathrm{Sel}(F,T^*) \to \mathrm{Gal}(FK_\infty^-/F) \otimes \mathbb{Q}_p \\ h_{F,\mathfrak{p}} &: \mathrm{Sel}(F,T) \otimes \mathrm{Sel}(F,T^*) \to \mathrm{Gal}(FL_\infty/F) \otimes \mathbb{Q}_p \end{aligned}$$

Here L_∞ is the unique $\mathbb{Z}_p$-extension of K unramified outside $\mathfrak{p}$.
The first piece in the proof of non-vanishing is Theorem A.1 in (Agboola and Howard, 2006), which states that on a CM elliptic curve, the $\mathfrak{p}$-adic height $h_{F,\mathfrak{p}}$ of a point of infinite order is nonzero. This result is due to Bertrand and does not use the assumption $p > 3$.
The second piece is showing that there are K_n^--rational points of infinite order which are universal norms in the Selmer group $\mathrm{Sel}(K_n^-, T)$ and $\mathrm{Sel}(K_n^-, T^*)$. This part uses results about the rank of $\mathrm{Sel}(K_n^-, T)$ and $X(\lambda)$ as Λ-modules when λ is of root number -1. They assume $p > 3$. Luckily, we have already generalized these statements in Theorem 3 to include $p = 2, 3$ using a modified Euler system.
The third piece is showing that if $R(\lambda)$ is zero, then the p-adic height pairing h_n is zero. This piece does not use the assumption $p > 3$.
Combining these three pieces above, we have demonstrated that when $p = 2, 3$, the regulator $R(\lambda)$ does not vanish for λ associated to a CM elliptic curve.

When λ is more general, for example, associated to a CM abelian variety, we have not yet investigated the possibility of proving the non-vanishing of $R(\lambda)$ for $p = 2, 3$. This is the reason for imposing the additional assumption in Theorem 2 compared to Conjecture 2 proven in Burungale-Tian. Collecting the steps above, we have arrived at the equality of characteristic ideals and Theorem 2 is complete. $\square$

Chapter 4

p-CONVERSE THEOREM

In this section, we briefly outline how Theorem 2 helps to prove the CM rank 1 p-converse theorem, namely Theorem 1, when $p = 2, 3$. We would like to reemphasize that the main modification for $p = 2, 3$ takes place in Theorem 2. With Theorem 2 established, the steps discussed below are only a restatement of that in Burungale-Tian and not original to this project. The purpose of restating their arguments is to highlight the fact that the modified version of HMC, namely Theorem 2, still suffices to prove the p-converse.

Recall that E be a CM elliptic curve over an imaginary quadratic field K and let λ be the algebraic Hecke character associated to E. According to Rohrlich in (Rohrlich, 1984), there exists a finite order Hecke character χ unramified at p, such that

$$\mathrm{ord}_{s=1} L(s, \lambda^* \cdot \frac{\chi}{\chi^*}) = 0 \tag{4.0.1}$$

When $\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1$, the Parity Conjecture by Nekovář shows that the root number of λ is -1. Let g be the CM modular form associated to $\lambda\chi^{-1}$, then the Rankin-Selberg L-function associated to the pair (g, χ) decomposes in the following form:

$$L(s, g \times \chi) = L(s, \lambda) L(s, \lambda^* \cdot \frac{\chi}{\chi^*})$$

Thus, by the choice of the twist χ, showing the p-converse is equivalent to showing

$$\mathrm{corank}_{\mathbb{Z}_p} \mathrm{Sel}_{p^\infty}(E_{/\mathbb{Q}}) = 1 \implies \mathrm{ord}_{s=1} L(s, g \times \chi) = 1$$

The properties of λ and χ ensure that the generalized Heegner hypothesis is satisfied for the pair (g, χ). Thus, there exists a Heegner point

$$P_{g,\chi} \in B(K) \otimes_{\mathbb{Z}} \mathbb{Q}$$

with $B_{/K}$ a CM abelian variety associated to the pair (g, χ). According to the generalized Gross-Zagier formula by Yuan-Zhang-Zhang (Yuan, S.-W. Zhang, and W. Zhang, 2013), there is an equivalence:

$$\mathrm{ord}_{s=1} L(s, g \times \chi) = 1 \Leftrightarrow P_{g,\chi} \neq 0$$

Due to the work of Kolyvagin/Rubin, 4.0.1 implies

$$\operatorname{corank}_{\mathcal{O}_\wp} \operatorname{Sel}_{\wp^\infty}(\lambda^* \cdot \frac{\chi}{\chi^*}) = 0$$

Combining all of the above, the p-converse theorem becomes equivalent to the implication

$$\operatorname{corank}_{\mathcal{O}_\wp} \operatorname{Sel}_{\wp^\infty}(B_{/K}) = 1 \implies P_{g,\chi} \neq 0$$

This implication is proven from the Galois decent of Theorem 2, as $P_{g,\chi}$ is at the lowest level of the norm-compatible system of Heegner points. The use of Theorem 2 is justified as the pair (g,χ) satisfies the requirement of Conjecture 2 by construction and the Hecke character λ with root number -1 is associated to the CM elliptic curve E by definition.

Chapter 5

CONCLUSION AND FUTURE WORK

The specific CM case of the Heegner Main Conjecture discussed in the proof of our CM rank 1 p-converse theorem has some independent interest, especially the part about anticyclotomic main conjectures. One possible future direction of research is to establish a more general version of the CM Heegner Main Conjecture for $p = 2, 3$.

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