Representation Theory of Real Reductive Groups

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ABSTRACT

The representation theory of Lie groups has connections to various fields in mathematics and physics. In this thesis, we are interested in the classification of irreducible admissible complex representations of real reductive groups. We introduce two approaches through the local Langlands correspondence and the Beilinson-Bernstein localization respectively. Then we investigate the connection between these two classifications for the group $\mathrm{GL}(2, \mathbb{R})$.

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Chapter 1

INTRODUCTION

1.1 Background

In the late nineteenth century, motivated by the view of Klein that the geometry of space is determined by its group of symmetries, Lie started to investigate group actions on manifolds, laying the foundation of the theory of Lie groups. Around the same time, while working on algebraic number theory, Dedekind discovered a formula involving multiplicative characters of finite abelian groups. Built on this idea, Frobenius and Schur developed the representation theory of finite groups. With the introduction of invariant integration, analogous results for compact groups were also established soon after. Since then, representation of Lie groups has found applications to numerous fields in mathematics and physics, notably number theory (Tate thesis) and quantum mechanics (Heisenberg and Lorentz groups). Progress was made to expand the theory to locally compact groups, nilpotent/solvable Lie groups, and semisimple Lie groups. These efforts culminated in Langlands classification [9] of all irreducible admissible representations of reductive groups as quotients of induced representations. When the field is $\mathbb{R}$ or $\mathbb{C}$, this classification takes an alternative, more concrete form in terms of representation of Langlands groups. From an entirely different viewpoint and as a generalization of the Borel-Weil theorem for compact groups, Beilinson and Bernstein [1] identified representations of Lie algebras with global sections of certain $\mathcal{D}$-modules on the flag variety, paving the way for a geometric classfication. In this thesis, we first introduce the details of these two distinct approaches, following the expositions in [7], [3], and [4]. Then we apply them explicitly to the group $\mathrm{GL}(2,\mathbb{R})$ and investigate the connection between the results.

1.2 Notation

Throughout the thesis, unless indicated otherwise, we adopt the following notations:

G_0 is a connected real reductive Lie group
G is the complexification of G_0 with Lie algebra $\mathfrak{g}$
B, H are Borel and Cartan subgroups of G
K_0 is the maximal compact subgroup of G_0 with complexification K
$U(\mathfrak{g})$ is the universal enveloping algebra of $\mathfrak{g}$ with center $Z(\mathfrak{g})$

1.3 Preliminaries

Definition 1.3.1 *A connected linear algebraic group L over $\mathbb{R}$ is reductive if the base change $L_{\mathbb{C}}$ is reductive, which is equivalent to the largest smooth connected unipotent normal subgroup of $L_{\mathbb{C}}$ being trivial.*

Definition 1.3.2 *A real Lie group G_0 is reductive if there exists a linear algebraic group L over $\mathbb{R}$ whose identity component (in the Zariski topology) is reductive and a homomorphism $\varphi : G_0 \to L(\mathbb{R})$ with $\ker\varphi$ finite and $\operatorname{im}\varphi$ open in $L(\mathbb{R})$.*

Example 1.3.3 *Every connected semisimple real Lie group (i.e. with semisimple Lie algebra) with finite center is reductive.*

Definition 1.3.4 *A subgroup B of an algebraic group G is Borel if it is maximal among all Zariski closed connected solvable subgroups. A subgroup $H \subset G$ is Cartan if it is the centralizer of a maximal torus.*

Example 1.3.5 *For $\mathrm{GL}(n, \mathbb{C})$, the subgroup of upper triangular matrices is a Borel subgroup, while the subgroup of diagonal matrices is a Cartan subgroup.*

Definition 1.3.6 *A continuous representation (ρ, V) of G_0 on a complex Hilbert space V is admissible if $\rho|_{K_0}$ is unitary and each irreducible representation of K_0 occurs with at most finite multiplicity in $\rho|_{K_0}$. If V is admissible, a vector $v \in V$ is called K_0-finite if it lies in a finite-dimensional space under the action of K_0.*

Definition 1.3.7 *A $(\mathfrak{g}, K)$-module is a vector space V with actions of $\mathfrak{g}$ and K such that for any $v \in V$, $k \in K$, $X \in \mathfrak{g}$, and $Y \in \mathfrak{k}$ (the Lie algebra of K), we have*

$$k \cdot (X \cdot v) = (\mathrm{Ad}(k)X) \cdot (k \cdot v)$$

$Kv \subset V$ *spans a finite-dimensional subspace on which K acts continuously*

$$\left(\tfrac{d}{dt} \exp(tY) \cdot v\right)\Big|_{t=0} = Y \cdot v$$

Given an admissible representation (ρ, V) of G_0, we can consider the space V_K of K_0-finite vectors. Since the action of K_0 on V_K is locally finite, it extends to an action of K. Moreover, it is shown that V_K is dense in V and inherits a $\mathfrak{g}$ action. The actions of K and $\mathfrak{g}$ are compatible in the sense that (ρ, V_K) is a $(\mathfrak{g}, K)$-module. [6, Chapter III]. We also have the following result [6, Chapter VIII].

Proposition 1.3.8 *(ρ, V) is irreducible if and only if (ρ, V_K) is irreducible.*

Another result by [10] shows that:

Proposition 1.3.9 *Every irreducible* $(\mathfrak{g}, K)$*-module is isomorphic to* (ρ, V_K) *for some irreducible admissible representation* (ρ, V) *of* G_0.

Definition 1.3.10 *Two admissible representations of* G_0 *are infinitesimally equivalent if their underlying* $(\mathfrak{g}, K)$*-modules are isomorphic.*

We would like to classify irreducible admissible representations of G_0 up to infinitesimal equivalence, or equivalently, irreducible $(\mathfrak{g}, K)$-modules up to isomorphism.

Chapter 2

LOCAL LANGLANDS CORRESPONDENCE

2.1 Parabolic Induction

Definition 2.1.1 *A unitary representation of G_0 is a continuous norm-preserving group action of G_0 by linear transformations on a Hilbert space.*

Definition 2.1.2 *Let (ρ, V) be an irreducible admissible representation of G_0 and v, v' be K_0-finite vectors. Then a K_0-finite matrix coefficient of ρ is a function $x \mapsto \langle \rho(x)v, v'\rangle$. If all K_0-finite matrix coefficients of ρ are in $L^{2+\epsilon}(G)$ for $\forall \epsilon > 0$, we say ρ is a tempered representation.*

Proposition 2.1.3 (Iwasawa) *Let $\mathfrak{g}_0$ be the Lie algebra of G_0 and $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{p}_0$ be the Cartan decomposition. Then there is moreover a decomposition $\mathfrak{g}_0 = \mathfrak{k}_0 + \mathfrak{a}_0 + \mathfrak{n}_0$ such that $\mathfrak{a}_0$ is abelian, $\mathfrak{n}_0$ is nilpotent, and $[\mathfrak{a}_0 \oplus \mathfrak{n}_0, \mathfrak{a}_0 \oplus \mathfrak{n}_0] = \mathfrak{n}_0$.*

Definition 2.1.4 *Let $A_\mathfrak{p}, N_\mathfrak{p}$ be the analytic subgroups of G_0 with Lie algebras $\mathfrak{a}_0$ and $\mathfrak{n}_0$ respectively. Let $M_\mathfrak{p}$ be the centralizer of $\mathfrak{a}_0$ in K_0. The subgroup $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ is call a minimal parabolic subgroup of G_0. Any closed subgroup Q_0 containing $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$ is called a standard parabolic subgroup.*

Proposition 2.1.5 *Any standard parabolic subgroup Q_0 has a Langlands decomposition $Q_0 = M_0A_0N_0$, where M_0 is reductive, A_0 is abelian, and N_0 is nilpotent.*

Example 2.1.6 *For $\mathrm{GL}(2, \mathbb{R})$, the only proper parabolic is the subgroup of upper triangular matrices with*

$$M_0 = \left\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}\right\} \quad A_0 = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x, y > 0\right\} \quad N_0 = \left\{\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right\}$$

Theorem 2.1.7 (Langlands) *[8] The equivalence classes (up to infinitesimal equivalence) of irreducible admissible representations of G_0 are in bijection with all triples $(M_0A_0N_0, [\sigma_0], \nu_0)$ satisfying*

a) *$M_0A_0N_0$ is a parabolic subgroup of G_0 containing $M_\mathfrak{p}A_\mathfrak{p}N_\mathfrak{p}$*
b) *σ_0 is an irreducible tempered (unitary) representation of M_0 and $[\sigma_0]$ its equivalence class*

c) ν_0 is a member of $\mathfrak{a}_0'$ such that $\mathrm{Re}\langle\nu_0, \alpha\rangle > 0$ *for every positive restricted root not vanishing on $\mathfrak{a}_0$*

In particular, the correspondence is that $(M_0A_0N_0, [\sigma_0], \nu_0)$ corresponds to the class of the unique irreducible quotient of $\mathrm{Ind}_{M_0A_0N_0}^{G_0}(\sigma_0 \otimes \nu_0 \otimes 1)$.

Specifically, the induced representation $\mathrm{Ind}_{M_0A_0N_0}^{G_0}(\sigma_0 \otimes \nu_0 \otimes 1)$ is given by first considering all $F \in C(G_0, V)$ with

$$F(xman) = e^{-(\nu_0+\rho)\log a}\sigma_0(m)^{-1}F(x)$$

and defining $\mathrm{Ind}_{M_0A_0N_0}^{G_0}(\sigma_0 \otimes \nu_0 \otimes 1)(g)F(x) = F(g^{-1}x)$.

Definition 2.1.8 *The parameters $(M_0A_0N_0, [\sigma_0], \nu_0)$ are called the Langlands parameters of the given irreducible admissible representation.*

2.2 Representations of the Weil Group $W_{\mathbb{R}}$

Definition 2.2.1 *The Weil group of $\mathbb{R}$, denoted $W_{\mathbb{R}}$, is the nonsplit extension of $\mathbb{C}^\times$ by $\mathbb{Z}/2\mathbb{Z}$ given by $W_R = \mathbb{C}^\times \cup j\mathbb{C}^\times$, where $j^2 = -1$ and $jcj^{-1} = \overline{c}$ for $\forall c \in \mathbb{C}^\times$.*

For reasons that will become clear, we are interested in classifying n-dimensional complex representations of $W_{\mathbb{R}}$ whose images consist of semisimple elements (i.e. semisimple matrices in $\mathrm{GL}(n, \mathbb{C})$). We refer to them as semisimple representations.

Lemma 2.2.2 *One-dimensional representations of $\mathbb{C}^\times$ are of the form $z \mapsto z^\mu \overline{z}^\nu$ with $\mu, \nu \in \mathbb{C}$ and $\mu - \nu \in \mathbb{Z}$. (Note that $\mathbb{C}^\times \cong S^1 \times \mathbb{R}^+$, and representations of the components are known.)*

Proposition 2.2.3 *One-dimensional representations φ of $W_{\mathbb{R}}$ are of the form*

$$\begin{aligned} \langle +, t\rangle: \quad & \varphi(z) = |z|_{\mathbb{R}}^t \quad \textit{and} \quad \varphi(j) = 1 \\ \langle -, t\rangle: \quad & \varphi(z) = |z|_{\mathbb{R}}^t \quad \textit{and} \quad \varphi(j) = -1 \end{aligned}$$

where t can be any complex number.

Proof: Suppose $\varphi(j) = w \in \mathbb{C}^\times$ and $\forall z \in \mathbb{C}^\times$, $\varphi(z) = z^\mu \overline{z}^\nu$. Then since $\varphi(z) = w\varphi(z)w^{-1} = \varphi(jzj^{-1}) = \varphi(\overline{z})$, it follows that $\mu = \nu$, so $\varphi(re^{i\theta}) = r^{2\mu}$. Since $w^2 = \varphi(j^2) = \varphi(-1) = 1$, $w = \pm 1$. Let $t = 2\mu$. It is easy to check that the formulas above indeed give valid representations.

Proposition 2.2.4 *Equivalent classes of irreducible two-dimensional semisimple representations φ of $W_{\mathbb{R}}$ are of the form*

$$\langle l, t\rangle : \quad \varphi(re^{i\theta})\alpha = r^{2t}e^{il\theta}\alpha \quad \textit{and} \quad \varphi(j)\alpha = \beta$$
$$\varphi(re^{i\theta})\beta = r^{2t}e^{-il\theta}\beta \quad \textit{and} \quad \varphi(j)\beta = (-1)^l\alpha$$

where α, β form a basis of the two dimensional vector space and $l \in \mathbb{Z}^+, t \in \mathbb{C}$.

Proof: Given such a representation φ, pick a basis a, b such that $\varphi(\mathbb{C}^\times)$ consist of diagonal matrices. By the lemma, we can suppose $\varphi(z)a = z^{\mu_1}\overline{z}^{\nu_1}a$ and $\varphi(z)b = z^{\mu_2}\overline{z}^{\nu_2}b$. Since φ is irreducible, either $\mu_1 \neq \mu_2$ or $\nu_1 \neq \nu_2$. Let $a' := \varphi(j)a$. Then

$$\varphi(z)a' = \varphi(j\overline{z}j^{-1})a' = \varphi(j\overline{z})a = z^{\nu_1}\overline{z}^{\mu_1}a'$$

If $\mu_1 = \nu_1$, then $a' \in \mathbb{C}a$ and $\mathbb{C}a$ is an invariant subspace. Contradiction! Thus $a' \in \mathbb{C}b$ by the equation above, which means $\nu := \nu_1 = \mu_2$ and $\mu := \mu_1 = \nu_2$. Since $\varphi(j)^{-1} = \varphi(j)\varphi(-1) = (-1)^{\mu-\nu}\varphi(j)$, we have

$$\varphi(z)a = z^{\mu}\overline{z}^{\nu}a \quad \text{and} \quad \varphi(j)a = a'$$
$$\varphi(z)a' = z^{\nu}\overline{z}^{\mu}a' \quad \text{and} \quad \varphi(j)a' = (-1)^{\mu-\nu}a$$

Considering instead the basis $a', (-1)^{\mu-\nu}a$ and in view of the symmetry, we can suppose without the loss of generality that the integer $l := \mu - \nu$ is positive. Let $t = \frac{1}{2}(\mu + \nu)$, $\alpha = a$, and $\beta = a'$. Then the result follows immediately.

Proposition 2.2.5 *Every finite-dimensional semisimple representation φ of $W_{\mathbb{R}}$ is fully reducible, and each irreducible representation has dimension one or two.*

Proof: Given a representation φ acting on V, since $\varphi(\mathbb{C}^\times)$ consists of commuting diagonalizable matrices, V is the direct sum of spaces $V_{\mu,\nu}$ on which $\varphi(z)$ acts by $z^{\mu}\overline{z}^{\nu}$. Meanwhile, $\varphi(j)V_{\mu,\nu} = V_{\nu,\mu}$. If $\mu = \nu$, we can choose a basis of eigenvectors for $\varphi(j)$ in $V_{\mu,\nu}$, so the span of each vector is an one-dimensional invariant subspace under $\varphi(W_{\mathbb{R}})$. If $\mu \neq \nu$, pick a basis $u_1, u_2, \cdots u_r$ of $V_{\mu,\nu}$. Then $\mathbb{C}u_i \oplus \mathbb{C}\varphi(j)u_i$ is a two-dimensional invariant subspace. Moreover, $V_{\mu,\nu} \oplus V_{\nu,\mu} = \bigoplus_{i=1}^{r} \mathbb{C}u_i \oplus \mathbb{C}\varphi(j)u_i$.

2.3 L-function and ϵ-factor

Given any irreducible admissible representations ρ of G_0, we can attach two invariants, called the L-function and ϵ-factor. For simplicity, we define them here for the special case of $G_0 = \mathrm{GL}(n, \mathbb{R})$. Fix an additive character ψ of $\mathbb{R}$ given by $x \mapsto e^{2\pi i x}$.

Let $M_n(\mathbb{R})$ be the collection of $n \times n$ matrices and $\mathcal{I}_0$ be the space of functions $M_n(K) \to \mathbb{R}$ of the form $P(x_{ij})\exp(-\pi \sum x_{ij}^2)$ for any polynomial P. For any K_0-finite matrix coefficient c of ρ, $\forall f \in \mathcal{I}_0$, and $s \in \mathbb{C}$, we define

$$\zeta(f, c, s) = \int_{M_n(\mathbb{R})} f(x)c(x)|\det x|^s d^\times x$$

Here $d^\times x = |\det x|^{-n} dx$, where dx is a fixed invariant measure for $M_n(\mathbb{R})$.

Proposition 2.3.1 *If ρ is irreducible, then all $\zeta(f, c, s)$ converge in the right half plane and extend to a meromorphic function (in terms of s) on $\mathbb{C}$. Moreover, there is a finite collection $\{(c_i, f_i)\}$ such that*

$$L(s, \rho) = \sum_i \zeta(f_i, c_i, s)$$

satisfies $\forall (c, f), \zeta(f, c, s + \frac{1}{2}(n-1)) = P(f, c, s)L(s, \rho)$ with a polynomial P in s. In particular $L(s, \rho)$ is uniquely defined this way (up to a scalar).

Now for $\forall f \in \mathcal{I}_0$, we define

$$\hat{f} = \int_{M_n(\mathbb{R})} f(y)\psi(\mathrm{Tr}(xy))dy$$

where dy is the self-dual Haar measure on $M_n(\mathbb{R})$. Then $\hat{f} \in \mathcal{I}_0$.

Proposition 2.3.2 *If ρ is irreducible, there exists a meromorphic function $\gamma(s, \rho, \psi)$ independent of f and c with $\zeta(\hat{f}, \check{c}, 1-s+\frac{1}{2}(n-1)) = \gamma(s, \rho, \psi)\zeta(f, c, s+\frac{1}{2}(n-1))$ for $\forall f \in \mathcal{I}_0$ and any K_0 finite matrix coefficient c of ρ. Here $\check{c}(x) = c(x^{-1})$.*

Definition 2.3.3 *We refer to $L(\rho, s)$ as the L-function of ρ. The ϵ-factor is defined as*

$$\epsilon(s, \rho, \psi) = \frac{\gamma(s, \rho, \psi)L(s, \rho)}{L(1-s, \tilde{\rho})}$$

where $\tilde{\rho}$ is the admissible dual of ρ.

We can also attach an L-function and an ϵ-factor to a representation of $W_{\mathbb{R}}$ as follows.

Definition 2.3.4 *Given an irreducible finite-dimensional semisimple representation φ of $W_{\mathbb{R}}$, we can attach an L-function*

$$L(s, \varphi) = \begin{cases} \pi^{-\frac{s+t}{2}}\Gamma(\frac{s+t}{2}) & \text{for } \varphi \text{ given by } \langle +, t\rangle \\ \pi^{-\frac{s+t+1}{2}}\Gamma(\frac{s+t+1}{2}) & \text{for } \varphi \text{ given by } \langle -, t\rangle \\ 2(2\pi)^{-(s+t+\frac{l}{2})}\Gamma(s+t+\frac{l}{2}) & \text{for } \varphi \text{ given by } \langle l, t\rangle \end{cases}$$

and an ϵ-factor:

$$\epsilon(s,\varphi,\psi) = \begin{cases} 1 & \textit{for } \varphi \textit{ given by } \langle +, t\rangle \\ i & \textit{for } \varphi \textit{ given by } \langle -, t\rangle \\ i^{l+1} & \textit{for } \varphi \textit{ given by } \langle l, t\rangle \end{cases}$$

Now to state the local Langlands correspondence, we need one last ingredient. Roughly speaking, the L-group of a connected reductive group G_0 over $\mathbb{R}$ is given by the semi-direct product of a complex reductive group with dual root datum to G_0 and the Galois group $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. We denote it by ${}^L G_0$. Concretely,

Definition 2.3.5 *The L-groups of* $\mathrm{GL}(n,\mathbb{R}), \mathrm{SL}(n,\mathbb{R}), \mathrm{SO}(2n{+}1,\mathbb{R})$*, and* $\mathrm{SO}(2n,\mathbb{R})$ *are* $\mathrm{GL}(n,\mathbb{C}), \mathrm{PGL}(n,\mathbb{C}), \mathrm{Sp}(n,\mathbb{C})$*, and* $\mathrm{SO}(2n,\mathbb{C})$ *respectively.*

2.4 Main Results

Theorem 2.4.1 (Local Langlands Correspondence) *There is a bijection between semisimple representations of $W_{\mathbb{R}}$ into ${}^L G_0$ and L-packets of irreducible admissible representations of G_0.*

Remark 2.4.2 *This theorem means we have a partition of all the irreducible admissible representations of G_0, indexed by semisimple representations of $W_{\mathbb{R}}$ into ${}^L G_0$. An L-packet is a collection of representations of G_0 that correspond to the same representation of $W_{\mathbb{R}}$, which is called the L-parameter of that packet.*

Proposition 2.4.3 *The L-parameter and representations in its corresponding L-packet have the same L-function and ϵ-factor.*

A priori, there can be multiple representations in a single L-packet. However, when $G_0 = \mathrm{GL}(n,\mathbb{R})$, each L-packet has size one. In particular,

Theorem 2.4.4 *There is a bijection between the set of equivalence classes of n-dimensional semisimple complex representations of $W_{\mathbb{R}}$ and the set of equivalence classes of irreducible admissible representations of* $\mathrm{GL}(n,\mathbb{R})$.

To make the correspondence explicit, we first define some classes of representations of $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$. Note that for $j = 1,2$, any matrix $X \in \mathrm{GL}(j,\mathbb{R})$ admits a decomposition $X = YZ$ where $Y \in \mathrm{SL}^{\pm}(j,\mathbb{R})$ and Z is positive scalar. Denote the two characters of $\mathrm{SL}^{\pm}(1,\mathbb{R})$ as 1 and sgn. For $l \in \mathbb{Z}^+$, let D_l^+ be the representation of $\mathrm{SL}(2,\mathbb{R})$ on the space of analytic functions f on the upper half plane satisfying $\|f\|^2 = \int\int |f(z)|^2 y^{l-1} dxdy < \infty$ given by

$$D_l^+ \begin{pmatrix} a & b \\ c & d \end{pmatrix} f(z) = (bz+d)^{-l-1} f(\frac{az+c}{bz+d})$$

Consider $D_l = \mathrm{ind}_{\mathrm{SL}(2,\mathbb{R})}^{\mathrm{SL}^\pm(2,\mathbb{R})}(D_l^+)$. For $\mathrm{GL}(1,\mathbb{R})$, $\forall t \in \mathbb{C}$, we have representations

$$\begin{aligned} 1 \otimes |\cdot|^t : &\quad X \mapsto |Z|^t \\ \mathrm{sgn} \otimes |\cdot|^t : &\quad X \mapsto \mathrm{sgn}(Y)|Z|^t \end{aligned}$$

Similarly for $\mathrm{GL}(2,\mathbb{R})$, we have

$$D_l \otimes |\det(\cdot)|^t : \quad X \mapsto D_l(Y)|\det(Z)|^t$$

Now given a representation of φ of $W_{\mathbb{R}}$, suppose $\varphi = \bigoplus_{i=1}^j \varphi_j$, where φ_j is a n_j-dimensional representation with $n_j = 1$ or 2. We associate to φ_j to a representation ψ_j of $\mathrm{GL}(n_j,\mathbb{R})$ in the following way

$$\begin{aligned} \langle +, t\rangle &\quad\to\quad 1 \otimes |\cdot|^t \\ \langle -, t\rangle &\quad\to\quad \mathrm{sgn} \otimes |\cdot|^t \\ \langle l, t\rangle &\quad\to\quad D_l \otimes |\det(\cdot)|^t \end{aligned}$$

Let t_j be the complex number corresponding to φ_j. Suppose wlog that $n_1^{-1}\,\mathrm{Re}\, t_1 \geq n_2^{-1}\,\mathrm{Re}\, t_2 \geq \cdots \geq n_j^{-1}\,\mathrm{Re}\, t_j$. Let $D = \prod_{i=1}^j \mathrm{GL}(n_j,\mathbb{R})$ and U the block strictly upper triangular subgroup of $\mathrm{GL}(n,\mathbb{R})$. We can extend $\bigoplus_{i=1}^j \psi_j$ to a representation ψ' on DU by making it trivial on U.

Proposition 2.4.5 *The induced representation* $\mathrm{ind}_{DU}^{\mathrm{GL}(n,\mathbb{R})}$ *of* $\mathrm{GL}(n,\mathbb{R})$ *has a unique irreducible quotient* ψ*, which is admissible.*

Remark 2.4.6 *The bijection in Theorem 2.4.4 is given by* $\varphi \mapsto \psi$ *described above.*

Chapter 3

BEILINSON–BERNSTEIN LOCALIZATION

3.1 Representation of Lie Algebras

If we choose a positive root system Δ^+ for $\mathfrak{g}$ then the set of roots of $\mathfrak{g}$ is given by $\Delta^+ \cup \{0\} \cup \Delta^-$ and we have a decomposition

$$\mathfrak{g} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Cartan subalgebra. Let $\mathfrak{n} = \bigoplus_{\alpha\in\Delta^+} \mathfrak{g}_\alpha$ and $\mathfrak{n}^- = \bigoplus_{\alpha\in\Delta^-} \mathfrak{g}_\alpha$. Then $\mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}$ is a Borel subalgebra. Let $\Pi = \{\alpha_j\}_{1\le j\le n}$ be a collection of simple roots.

Proposition 3.1.1 *For each $\alpha_i \in \Pi$, there exists $\alpha_i^\vee \in \mathfrak{h}$ such that $\alpha_i^\vee(\alpha_i) = 2$ and $s_{\alpha_i^\vee,\alpha_i}(\Delta) = \Delta$, where the map $s_{\alpha_i^\vee,\alpha_i} : h^* \to h^*$ is given by $v \mapsto v - \alpha^\vee(v)\alpha$.*

Definition 3.1.2 *The subgroup of* $\mathrm{GL}(\mathfrak{h}^*)$ *generated by $\{s_{\alpha_i^\vee,\alpha_i}\}_{1\le i\le n}$ is called the Weyl group, denoted W. We also define the following subsets of $\mathfrak{h}^*$:*

The fundamental weights π_i are defined by $\alpha_i(\pi_j) = \delta_{ij}$
$\rho = \sum_{i=1}^n \pi_i = \frac{1}{2}\sum_{i=1}^n \alpha_i$
Q^+ is the nonnegative integral span of $\{\alpha_i\}$, i.e. $\bigoplus_{i=1}^n \mathbb{Z}_{\ge 0}\alpha_i$
$P^+ = \bigoplus_{i=1}^n \mathbb{Z}_{\ge 0}\pi_i$, with elements referred to as dominant weights
Elements in $-P^+$ are referred to as anti-dominant weights
Elements in $\{\lambda \in \mathfrak{h}^* : a_i^\vee(\lambda) < 0\} \subset -P^+$ are called regular weights

Given a representation V of $\mathfrak{g}$, we can write $V = \bigoplus V_\lambda$, where all $\lambda \in \mathfrak{h}^*$ appearing in the decomposition are called weights of V.

Theorem 3.1.3 *Any irreducible representation V of $\mathfrak{g}$ has a unique maximal weight λ and a unique minimal weight μ with respect to the partial ordering given by Q^+, with λ dominant and μ anti-dominant. Moreover, for $\forall\lambda \in P^+$ there is a unique representation of $\mathfrak{g}$, denoted $L^+(\lambda)$ with maximal weight λ.*

Definition 3.1.4 *It is known that the center $Z(\mathfrak{g}) \subset U(\mathfrak{g})$ acts on $L^+(\lambda)$ by scalars, which defines a map $\chi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$, called the central character associated to λ.*

Consider the map $f : U(\mathfrak{h}) \to U(\mathfrak{h})$ induced by the linear map $\mathfrak{h} \to U(\mathfrak{h})$ with $h \mapsto h + \rho(h)1$. Since $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-)$, $\forall z \in Z(\mathfrak{g})$ can be written as $z = h + n$ for some $h \in U(\mathfrak{h})$ and $n \in (\mathfrak{n}U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}^-)$. The Harish-Chandra homomorphism is the map $\gamma : Z(\mathfrak{g}) \to U(\mathfrak{h})$ given by $\gamma(z) = f(h)$. Moreover, for the central character, we have $\chi_\lambda(z) = \lambda(\gamma(z))$.

Theorem 3.1.5 (Harish-Chandra) *The Harish-Chandra homomorphism is injective and an isomorphism onto $U(\mathfrak{h})^{W\cdot}$, with the dotted action of W on $\mathfrak{h}^*$ given by $w \cdot \lambda = w(\lambda + \rho) - \rho$.*

3.2 Algebraic $\mathcal{D}$-modules

In this subsection, X denotes a smooth algebraic variety over $\mathbb{C}$, with structure sheaf $\mathcal{O}_X$ and tangent sheaf $\mathcal{T}_X$. For simplicity, we only discuss the affine case here, but the definitions can be extended to general cases and the propositions hold as well.

Definition 3.2.1 *A linear map $D : \mathcal{O}_X \to \mathcal{O}_X$ is a differential operator of order 0 if it is multiplication by a function. For $k \in \mathbb{Z}^+$, D is a differential operator of order k if for any function $f \in \mathcal{O}_X$, the commutator $[D, f]$ is a differential operator f order $k - 1$. Let $D_{X,k}$ be the collection of differential operators of order k and $D_X = \bigcup_{k=0}^{\infty} D_{X,k}$*

Concretely, we have the following description of D_X.

Proposition 3.2.2 *The graded algebra $\mathrm{gr}(D_X)$ is isomorphic to $\mathrm{Sym}_{\mathcal{O}_X}(\mathcal{T}_X)$.*

Definition 3.2.3 *A (left) D-module is an $\mathcal{O}_X$-quasicoherent sheaf of (left) modules over D_X*

Now consider any line bundle $\mathcal{L}$ on X with dual sheaf $\mathcal{L}^\vee$.

Definition 3.2.4 *A map $D : \mathcal{L} \to \mathcal{L}$ is a differential operator of order 0 on $\mathcal{L}$ if it is multiplication by a function. For $k \in \mathbb{Z}^+$, D is a differential operator of order k on $\mathcal{L}$ if for any function $f \in \mathcal{O}_X$, the commutator $[D, f]$ is a differential operator of order $k - 1$. Denote the collection of all such differential operators by $D_{\mathcal{L}}$.*

Proposition 3.2.5 *Concretely, we have $D_{\mathcal{L}} = \mathcal{L} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} \mathcal{L}^\vee$.*

3.3 Main Results

Definition 3.3.1 *The quotient G/B is called the flag variety of G.*

Proposition 3.3.2 *The G-equivariant vector bundles on G/B are in bijection with representations of B.*

Proposition 3.3.3 *The one-dimensional representations of B are in bijection with characters of H (i.e. integral characters of $\mathfrak{h}$).*

Hence given an integral weight $\lambda \in \mathfrak{h}^*$, we can attach an one dimensional representation $\mathbb{C}_\lambda$ of B and a G-equivariant line bundle $\mathcal{L}(\lambda)$

Definition 3.3.4 *For an integral weight $\lambda \in \mathfrak{h}^*$, we define D_λ as the sheaf $D_{\mathcal{L}(\lambda+\rho)}$.*

We denote by $D_\lambda(G/B)$ the category of D_λ-modules that are quasi-coherent as $O_{G/B}$-modules, and by $\mathfrak{g}\text{-mod}_\lambda$ the category of $U(\mathfrak{g})$-modules on which $Z(\mathfrak{g})$ acts by the character χ_λ. Then we have the following crucial result:

Theorem 3.3.5 (Beilinson–Bernstein Localization) *If λ is regular, then the global section functor $\Gamma : D_\lambda(G/B) \longrightarrow \mathfrak{g}\text{-mod}_\lambda$ is an equivalence of categories with quasi-inverse given by $D_\lambda \otimes_{U(\mathfrak{g})} (\cdot)$*

This is still not exactly what's needed for our purpose, as we are intersted in classifying $(\mathfrak{g}, K)$-modules. As it turns out, the appropriate notion corresponding to $(\mathfrak{g}, K)$-modules with central character χ_λ on the geometric side is the category $D_\lambda(G/B)^K$ of "K-equivariant" D_λ-modules on G/B. In other words, we have an equivalence of categories $D_\lambda(G/B)^K \cong (\mathfrak{g}, K)\text{-mod}_\lambda$. Thus to classify irreducible $(\mathfrak{g}, K)$-modules, it suffices to classify simple objects in $D_\lambda(G/B)^K$, for which the following two results are necessary.

Proposition 3.3.6 *For every simple $\mathcal{M} \in D_\lambda(G/B)^K$, there is a K-orbit Q of G/B and a simple $\mathcal{N} \in D_\lambda(Q)^K$ such that $\mathcal{M} \cong j_{!*}(\mathcal{N})$. (For a detailed discussion of the functor $j_{!*}$, see [3, Section 5].)*

Proposition 3.3.7 *Fix a point x in a K-orbit Q. Let K_x be the stabilizer of x in K. Then $D_\lambda(Q)^K = 0$ unless λ integrates to a character of the identity component K_x^0 of K_x, in which case $D_\lambda(Q)^K \cong \mathrm{Rep}(K_x/K_x^0)$, the category of representations.*

Remark 3.3.8 *The integrality condition means we can first conjugate K_x so that it is contained in H and then find a character of K_x^0 whose derivative coincides with the restriction of λ.*

3.4 Induction in the Geometric Context

Parallel to the procedure of parabolic inductions, there is also a notion of induction for $(\mathfrak{g}, K)$-modules. In particular, let $K_H = K \cap B$.

Definition 3.4.1 *The Jacquet functor* r_H^G *is given by* $(\mathfrak{g}, K)$ -mod $\to$ $(\mathfrak{h}, K_H)$ -mod *with* $M \mapsto U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} M$*, in other words, taking* $\mathfrak{n}$ *coinvariants. Notably, it admits a right adjoint, denoted* i_B^G*.*

Remark 3.4.2 *On the level of* K*-modules,* i_B^G *is equivalent to the functor* $\mathrm{ind}_{K_H}^K$ *[5].*

Chapter 4

AN EXAMPLE OF GL$(2,\mathbb{R})$

In this section, we investigate the connection between the classifications of irreducible admissible representations of GL$(2,\mathbb{R})$ through the "Langlands" approach and the "geometric" approach.

4.1 Notation

$G_0 = \mathrm{GL}(2,\mathbb{R})$
$G = \mathrm{GL}(2,\mathbb{C})$
$K = \mathrm{O}(2,\mathbb{C})$
B is the subgroup of upper triangular matrices
H is the subgroup of diagonal matrices

4.2 Langlands Approach

By Remark 2.4.6, all irreducible admissible representations of GL$(2,\mathbb{R})$ are parameterized in the following way, with $l \in \mathbb{Z}^+$, $t, t_1, t_2 \in \mathbb{C}$ satisfying $\mathrm{Re}\, t_1 \geq \mathrm{Re}\, t_2$:

$\langle 1, t_1, 1, t_2\rangle : (1 \otimes |\cdot|^{t_1}) \underline{\otimes} (1 \otimes |\cdot|^{t_2})$
$\langle 1, t_1, \mathrm{sgn}, t_2\rangle : (1 \otimes |\cdot|^{t_1}) \underline{\otimes} (\mathrm{sgn} \otimes |\cdot|^{t_2})$
$\langle \mathrm{sgn}, t_1, 1, t_2\rangle : (\mathrm{sgn} \otimes |\cdot|^{t_1}) \underline{\otimes} (1 \otimes |\cdot|^{t_2})$
$\langle \mathrm{sgn}, t_1, \mathrm{sgn}, t_2\rangle : (\mathrm{sgn} \otimes |\cdot|^{t_1}) \underline{\otimes} (\mathrm{sgn} \otimes |\cdot|^{t_2})$
$\langle l, t\rangle : D_l \otimes |\det(\cdot)|^t$

Here $\underline{\otimes}$ refers to the procedure of taking induced representations and irreducible quotient as described in Remark 2.4.6.

Proposition 4.2.1 *The center $Z(\mathfrak{gl}(2,\mathbb{C}))$ is generated by I and the Casimir $\Delta = -\frac{1}{4}(H^2 - 2H + 4EF)$, where*

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad E = \begin{pmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{i}{2} \end{pmatrix} \quad F = \begin{pmatrix} \frac{i}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{i}{2} \end{pmatrix}$$

Proposition 4.2.2 *[2] For the representations $(\mathrm{sgn}^{\epsilon_1} \otimes |\cdot|^{t_1}) \otimes (\mathrm{sgn}^{\epsilon_2} \otimes |\cdot|^{t_2})$ of* GL$(2,\mathbb{R})$ *with $\epsilon_1, \epsilon_2 \in \{1, 2\}$, the central character χ_λ is given by*

$$I \mapsto t_1 + t_2, \quad \Delta \mapsto \frac{1}{4}[1 - (t_1 - t_2)^2]$$

Let $t = \frac{1}{2}(t_1 - t_2 + 1)$ *and* $\epsilon = \epsilon_1 + \epsilon_2$*. Then moreover*

> *If* $2t \not\equiv \epsilon$ *(mod 2), the representation above is irreducible.*
> *Otherwise, it has two irreducible factors: a finite representation (trivial if and only if* $t = \frac{1}{2}$*) and a discrete series representation (a limit of discrete series if and only if* $t = \frac{1}{2}$*).*

Remark 4.2.3 *As all irreducible representations come from parabolically induced ones, we will focus on finding a geometric interpretation of parabolic induction.*

4.3 Geometric Approach

By explicit computation, there are two K-orbits on G/B, represented by

$$\left\{p := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right\} \quad \text{and} \quad \left\{q := \begin{pmatrix} 1 & 0 \\ i & 1 \end{pmatrix}\right\}$$

respectively. Moreover, the stabilizer groups are given by

$$K_p = \left\{\begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x^2 = y^2 = 1\right\}$$
$$K_q = \left\{\begin{pmatrix} x & y \\ -y & x \end{pmatrix} : x^2 + y^2 = 1\right\}$$

Note that K_p is discrete with size four, so $\operatorname{Rep}(K_p/K_p^0)$ has four simple objects. Moreover, the condition from Proposition 3.3.7 that λ integrates to a character of K_p^0 is always satisfied. These $\mathcal{D}$-modules correspond to the irreducible representations $\langle \operatorname{sgn}^{\epsilon_1}, t_1, \operatorname{sgn}^{\epsilon_2}, t_2\rangle$, where $\langle t_1, t_2\rangle$ parametrize the weight λ and $\langle \epsilon_1, \epsilon_2\rangle$ the simple object in $\operatorname{Rep}(K_p/K_p^0)$.

On the other hand, K_q is connected, so $\operatorname{Rep}(K_q/K_q^0)$ has exactly one simple object. Note that characters of K_q are parametrized by integers $n \in \mathbb{Z}$.

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \mapsto (x + yi)^n$$

These $\mathcal{D}$-modules correspond to the discrete series representations $D_l \otimes |\det(\cdot)|^t$ with $l \in \mathbb{Z}^+$ and $t \in \mathbb{C}$ that we described in Theorem 2.4.4.

Hence as expected, the Langlands approach and the geometric approach both yield the same classification for $\mathrm{GL}(2, \mathbb{R})$.

4.4 Parabolic Induction

For a dominant regular weight λ, let $\Delta_\lambda : (\mathfrak{g}, K)\text{-mod}_\lambda \to D_\lambda(G/B)^K$ be the localization functor from Theorem 3.3.5. Let Q be the orbit of p in G/B and j the inclusion $Q \hookrightarrow G/B$. Finally, let $T_p(-)$ be the functor of taking fiber at p.

Proposition 4.4.1 *We have a commutative diagram:*

$$
\begin{array}{ccc}
(\mathfrak{g}, K)\text{-}mod_\lambda & \xrightarrow{\ \Delta_\lambda\ } & D^\lambda(G/B)^K \\
\big\downarrow & & \big\downarrow j^* \\
(\mathfrak{g}, K)\text{-}mod & & D^\lambda(Q)^K = D(Q)^K \\
r_H^G \big\downarrow & & \big\downarrow T_p(-) \\
(\mathfrak{h}, K_H)\text{-}mod & \xrightarrow[\mathbb{C}_\lambda \otimes_{V_\theta} (-)]{} & \mathrm{Rep}(K_p/K_p^0)
\end{array}
$$

Here $\theta = W \cdot \lambda$ and $V_\theta = U(\mathfrak{h})/J_\theta U(\mathfrak{h})$ where J_θ is the kernel of $U(\mathfrak{h}) \to \mathbb{C}$ determined by $\lambda + \rho$. Notably, on the K-orbit Q, any λ-twisting is trivialized.

Proof: By [11, Theorem 2.5], we have an isomorphism between the left derived functors $LT_p \circ L\Delta_\lambda$ and $\mathbb{C}_\lambda \otimes^L_{V_\theta} (U(\mathfrak{h}) \otimes^L_{U(\mathfrak{b})} -)$. As j^* doesn't change the fiber and r_H^G is exactly $U(\mathfrak{h}) \otimes_{U(\mathfrak{b})} -$, the diagram follows.

Remark 4.4.2 *This gives two interpretations of the localization of principal series representations. Now take right adjoints.*

$$
\begin{array}{ccc}
(\mathfrak{g}, K)\text{-}mod_\lambda & \xleftarrow{\ \Gamma_\lambda(-)\ } & D^\lambda(G/B)^K \\
\big\uparrow & & \big\uparrow j_* \\
(\mathfrak{g}, K)\text{-}mod & & D^\lambda(Q)^K = D(Q)^K \\
i_B^G \big\uparrow & & \big\uparrow \\
(\mathfrak{h}, K_H)\text{-}mod & \longleftarrow & \mathrm{Rep}(K_p/K_p^0)
\end{array}
$$

We obtain two parametrizations of the principal series representations by the four simple objects in $\mathrm{Rep}(K_p/K_p^0)$*. On the right hand side, this is exactly the geometric approach, while on the left hand side, note that i_B^G is the same as* $\mathrm{ind}_{DU}^{\mathrm{GL}(n,\mathbb{R})}$ *from Proposition 2.4.5, so it is essentially the Langlands approach.*

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