Analogues of Amenability

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ABSTRACT

In this thesis I study multiple notions related to and inspired by amenability coming from the points of the view of random walks on groups, dynamical systems, Borel equivalence relations, and descriptive linear algebra. In particular I study notions related to harmonic functions, invariant measures, hyperfiniteness, and dichotomy theorems.

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Chapter 1

INTRODUCTION

In this thesis I analyze several strengthenings, variants, and analogues of amenability in its various guises. A countable group G is called *amenable* if there exist an infinite sequence F_n of subsets such that $\frac{|gF_n \Delta F_n|}{|F_N|}$ converges to 0 for every $g \in G$. (We will use $|*|$ for the cardinality of a finite set throughout this Thesis) Amenability is famously equivalent to a swath of alternate conditions, One of the most notable among these alternate conditions is the existence of a left invariant mean. That is a (countable discrete) group G is amenable if and only if there exist a nontrivial continuous linear map from $l^\infty(G) \to \mathbb{R}$ which is invariant under translation by G. Because of the bevy of equivalent definitions it is completely impossible to give any more than the most perfunctory survey of them in this introduction, we direct interested readers to [Pat00] for more details and proofs of some of these equivalences. I will now discuss in more detail the results of the various chapters of this thesis.

Chapter 2

In Chapter 2 we will analyze a strengthening of amenability coming from the point of view of random walks and harmonic functions. Given a countable group G and a probability measure μ supported on G we say a function f in $l^\infty(G)$ is μ *harmonic* if $f(k) = \sum_{g \in G} \mu(g) f(kg)$. This can be generalized straightforwardly to locally compact groups H where μ is now assumed to be a probability measure on the Borel σ-algebra of H by the formula $f(k) = \int_{g \in G} f(kg) d\mu(g)$. On $\mathbb{R}^n$ the mean value property of harmonic functions shows this notion of harmonicity when specialized to the uniform probability measure on unit ball is equivalent to the classical differential equations definition.

The bounded harmonic functions on a group are, in many cases, surprisingly rigid. Specialized to the case of $\mathbb{R}^n$ (with the aforementioned uniform probability measure on unit ball) an example of this is Liouville's theorem which states that any bounded harmonic function is in fact constant. A groundbreaking example of this rigidity in the setting of discrete groups comes from a theorem of Blackwell [Bla55] who observed that for $\mathbb{Z}^d$ and any choice of measure μ a bounded harmonic function must also be constant. Furstenberg [Fur73] proved that for a countable non-amenable group G any measure μ whose support generates G as a semigroup has non-constant G, μ bounded harmonic functions. Conversely in a groundbreaking paper it was [KV83] proved that any amenable group H has a measure ν (whose support generates H as a semigroup) where the only H, ν harmonic functions are constant. This naturally left open a generalization of Blackwell's Theorem. For which countable groups are there no measures μ and no bounded non-constant functions f such that f, μ is harmonic? Groups with this property are called *Choquet- Deny*

To answer this question fully we must define the notion of an infinite conjugacy class group (abbreviated simply as I.C.C.). A group G has the infinite conjugacy class property if for every non-identity element $g \in G$ there are infinitely many distinct conjugates hgh^{-1}

Jaworski [Jaw04] proved that every group without an I.C.C. quotient is *Choquet-Deny*. In Chapter 2 we prove the converse that every group with an I.C.C. quotient is not *Choquet-Deny*

Chapter 3

Amenable groups have a dynamical interpretation. To explain this we must explain the dynamical notion of a topological dynamical system. To avoid unnecessary technicalities we will assume in this section that all groups are countable and discrete.

A topological dynamical system is a group G, a compact Hausdorff space X and a homomorphism τ from G to the group of homeomorphisms of X . Given a topological dynamical system G, τ, X a probability measure μ on X is *invariant* if $g(\mu) = \mu$ for all $g \in G$. A (countable) group is amenable if and only if every dynamical system with acting group G has an invariant probability measure.

The *Full Shift* on $\mathbb{Z}$ with finite alphabet A, denoted by $A^{\mathbb{Z}}$ is the set of all functions from $\mathbb{Z}$ to A equipped with the pointwise convergence (i.e. product) topology. It has a natural $\mathbb{Z}$ action associated with it where the generator T of $\mathbb{Z}$ acts by $T(f(x)) = f(x+1)$. This action is continuous and the space is compact by Tychonoff's theorem so this is an example of a topological dynamical system. A *subshift* is a closed subset of $A^{\mathbb{Z}}$ which is invariant under T and T^{-1}. These are also dynamical systems. Given a subshift S, let S_n be the restriction of the functions in S to the domain $(1, 2, \ldots, n)$. A subshift has *0 entropy* if $lim_{n\to\infty}\frac{log|S_n|}{n} = 0$.

Given a topological dynamical system, a continuous map ϕ from $X \to X$ is an *automorphism* if it is a homeomorphism ϕ of X such that $\phi(g(x)) = g(\phi(x))$ for all $x \in X$. It is easy to see that the automorphisms of a dynamical system form a group. In this chapter I show that, for a 0 entropy subshift S, there exist a probability measure μ which is invariant under all automorphisms of S, i.e., $\phi(\mu) = \mu$ for all ϕ which are automorphisms of S. This gives some evidence to suggest that such automorphism groups are amenable.

1.1 Chapter 4

Amenability in descriptive set theory is deeply related to the notion of hyperfiniteness. To define hyperfiniteness and clarify the connection we must, however, first define some preliminary notions.

A *Polish Space* is a second countable completely metrizable topological space. Given a Polish space X a *countable Borel equivalence relation* is a Borel subset of X^2 which is an equivalence relation such that all equivalence classes are countable. Similarly a *finite* Borel equivalence relation is a subset of X^2 which is an equivalence relation all of whose equivalence classes are finite.

A countable Borel equivalence E relation is hyperfinite if there is an increasing sequence $E_1 \subset E_2 \subset E_3, \ldots$ of finite Borel equivalence relations whose union $\bigcup E_n$ is E.

Given a countable group G and a map ϕ from G to Borel bijections of a polish space X the *orbit equivalence relation* of ϕ is the countable Borel equivalence relation where two elements $x_1, x_2 \in X$ are equivalent if and only if there exist some $g \in G$ so that $g(x_1) = x_2$. This is always a Borel equivalence relation. If the space X has a Borel σ finite measure μ on it and if the group G is amenable then, except possibly on a Borel set with μ measure 0, the equivalence relation is hyperfinite. (See [CFW81] for more details). Thus Borel hyperfiniteness can be considered an analogue of amenability.

A *Polish group* is a topological group with a Polish topology. In this chapter I prove that that if a Polish group G has a countable normal subgroup N then the left (equivalently right) coset equivalence relation given by multiplication on the left by N is hyperfinite.

1.2 Chapter 5

The importance of hyperfiniteness was reinforced when Harrington, Kechris, and Louveau proved that hyperfinite equivalence relations are minimal among the non-smooth ones [HKL90]. In this chapter I will discuss an analogous minimality that holds in the case of Polish modules.

A *Polish ring* is a ring R with a Polish topology where both addition and multiplication are jointly continuous maps. A *Polish module* is a module over a Polish ring R where both addition and multiplication by elements of R are jointly continuous. Given two Polish modules V, W over the same Polish ring R we say V *embeds* into W if there exists a continuous map ϕ from V to W such that ϕ is injective. In this chapter we show that for countable discrete fields F there exist an uncountable Polish module V_F which embeds into any uncountable Polish F module.

We moreover show that for a discrete Noetherian ring R there is a countable family of uncountable polish R modules which we denote $V_{R,I}$ each associated with an ideal I in R such that for any uncountable Polish R module M at least one of of $V_{R,I}$ embeds into M. In analogy with the above, we also find a minimal Polish module amongst the uncountable dimensional Polish modules over each of $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$.

Chapter 2

TWO

2.1 Introduction

Let G be a countable discrete group. A probability measure μ on G is *non-degenerate* if its support generates G as a semigroup.[1] A function $f\colon G \to \mathbb{R}$ is μ-*harmonic* if $f(k) = \sum_{g \in G} \mu(g) f(kg)$ for all $k \in G$. We say that the *measured group* (G, μ) is *Liouville* if all the bounded μ-harmonic functions are constant; this is equivalent to the triviality of the Poisson boundary $\Pi(G, \mu)$ [Fur63b; Fur71; Fur73] (also called the Furstenberg-Poisson boundary; for formal definitions see also, e.g., Furstenberg and Glasner [FG10], Bader and Shalom [BS06], or a survey by Furman [Fur02]).

When G is non-amenable, (G, μ) is not Liouville for every non-degenerate μ [Fur73]. Conversely, when G is amenable, then there exists some non-degenerate μ such that (G, μ) is Liouville, as shown by Kaimanovich and Vershik [KV83] and Rosenblatt [Ros81]. It is natural to ask for which groups G it holds that (G, μ) is Liouville for *every* non-degenerate μ. We call such groups *Choquet-Deny* groups; as we discuss in §2.1, there are a few variants of this definition (see, e.g., [Gla76a; Gla76b; Gui73], or [JR07]), which, however, we show to be equivalent.

The classical Choquet-Deny Theorem (which was first proved for $\mathbb{Z}^d$ by Blackwell [Bla55]) states that abelian groups are Choquet-Deny [CD60]; the same holds for virtually nilpotent groups [DM61]. There are many examples of amenable groups that are not Choquet-Deny: first examples of such groups[2] are due to Kaimanovich [Kai83] and Kaimanovich and Vershik [KV83], and include locally finite groups; Erschler shows that finitely generated solvable groups that are not virtually nilpotent are not Choquet-Deny [Ers04b], and that even some groups of intermediate growth are not Choquet-Deny [Ers04a]. Kaimanovich and Vershik [KV83, p. 466] conjecture that "Given an exponential group G, there exists a symmetric (nonfinitary, in general) measure with non-trivial boundary." See Bartholdi and Erschler [BE17] for additional related results and further references and discussion.

Our main result is a characterization of Choquet-Deny groups. We say that G has

[1] In the context of Markov chains such measures are called *irreducible*.

[2] In the Lie group setting, an example of an amenable group that is not Choquet-Deny was already known to Furstenberg [Fur63b].

the *infinite conjugacy class* property (ICC) if it is non-trivial, and if each of its non-trivial elements has an infinite conjugacy class. We say that μ is *fully supported* if $\operatorname{supp}\mu = G$; obviously this implies that μ is non-degenerate.

Theorem 1. *A countable discrete group G is Choquet-Deny if and only if it has no ICC quotients. Moreover, when G does have an ICC quotient, then there exists a fully supported, symmetric, finite entropy probability measure μ on G such that (G, μ) is not Liouville. In particular, if G is finitely generated, then it is Choquet-Deny if and only if it is virtually nilpotent.*

That a group with no ICC quotients is Choquet-Deny was shown by Jaworski [Jaw04, Theorem 4.8].[3] Our contribution is therefore in the proof of the converse, which appears in §2.2.

Groups with no ICC quotients are known as FC-hypercentral (see, e.g., [DM56; McL56], or [Rob72, §4.3]). This class is closed under forming subgroups, quotients, direct products and finite index extensions, and includes all virtually nilpotent groups. Among finitely generated groups, virtually nilpotent groups are precisely those with no ICC quotients (see [McL56, Theorem 2] and [DM56, Theorem 2]); this implies the result in Theorem 1 for finitely generated groups. Since finitely generated groups of exponential growth are not virtually nilpotent, Theorem 1 implies that the above mentioned conjecture of Kaimanovich and Vershik [KV83] is correct.

A very recent result by three of the authors of this paper shows that a countable discrete group is strongly amenable if and only if it has no ICC quotients [FTF18]. This implies that G is strongly amenable if and only if (G, μ) is Liouville for every non-degenerate μ, paralleling the above mentioned characterization of amenability as equivalent to the existence of a non-degenerate μ such that (G, μ) is Liouville. While the proofs of these two similar results are different, it is natural to ask whether there is some deeper connection between strong amenability and the Choquet-Deny property.

Different possible definitions of Choquet-Deny groups

Our definition of Choquet-Deny groups is not the usual one, which states that a group is Choquet-Deny if (G, μ) is Liouville for every *adapted* measure μ, where μ is called adapted if its support generates G as a *group* (rather than as a semigroup, as in the non-degenerate case) [Gla76a; Gla76b; Gui73]. Yet another definition

[3] In fact, Jaworski proves there a stronger statement; see the discussion in §2.1.

used in the literature requires that for *every* μ, every bounded μ-harmonic function is constant on the left cosets of G_μ, where G_μ is the subgroup of G generated by the support of μ [JR07].

While a priori these are different definitions, they are equivalent, as demonstrated by our result and by Jaworski's Theorem 4.8 in [Jaw04]. Jaworski's result shows that groups with no ICC quotients are Choquet-Deny according to any of these definitions. Since our construction of μ with a non-trivial boundary yields measures that are supported on all of G (hence non-degenerate, hence adapted), it shows that groups with ICC quotients are not Choquet-Deny according to any of these definitions. Moreover, our result shows that the class of Choquet-Deny groups (whether defined with adapted or with non-degenerate measures) is closed under taking subgroups, which, to the best of our knowledge, was also not previously known.

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2.2 Proofs

In this section we prove the main result of our paper, Theorem 1. Unless stated otherwise, we will assume that all groups are countable and discrete.

Recall that a probability measure μ on G is symmetric if $\mu(g) = \mu(g^{-1})$ for all $g \in G$. Its Shannon entropy (or just entropy) is $H(\mu) = -\sum_{g \in G} \mu(g) \log \mu(g)$.

Our Theorem 1 is a direct consequence of [Jaw04, Theorem 4.8], which proves it for the case of groups with no ICC quotients, and of the following proposition, which handles the case of groups with ICC quotients.

Proposition 2.2.1. *Let G be a group with an ICC quotient. Then there exists a fully-supported, symmetric, finite entropy probability measure μ on G such that $\Pi(G, \mu)$ is non-trivial.*

The main technical effort in the proof of Proposition 2.2.1 is in the proof of the following proposition.

Proposition 2.2.2. *Let G be an amenable ICC group. For every $h \in G \setminus \{e\}$ there exists a fully supported, symmetric, finite entropy probability measure μ such that*

$$\lim_{m\to\infty} \|h\mu^{*m} - \mu^{*m}\| > 0. \tag{2.2.1}$$

Here μ^{*m} is the m-fold convolution $\mu * \cdots * \mu$. We will prove this Proposition later, and now turn to the proof of Proposition 2.2.1.

Proof of Proposition 2.2.1. The case of non-amenable G is known, so assume that G is amenable and has an ICC quotient Q. Let h be a non-identity element of Q. Applying Proposition 2.2.2 to Q and h yields a finite entropy, symmetric measure $\bar{\mu}$ on Q that is fully supported, and satisfies (2.2.1).

Since $\bar{\mu}$ has full support and satisfies (2.2.1), it follows from [Gla76a, Theorem 2] that $(Q, \bar{\mu})$ has a non-trivial Poisson boundary. Let μ be any symmetric, finite entropy non-degenerate probability measure on G that is projected to $\bar{\mu}$; the existence of such a μ is straightforward. Then (G, μ) has a non-trivial Poisson boundary. □

Switching Elements

Here we introduce two notions: switching elements and super-switching elements. We will use these notions in the proof of Proposition 2.2.2.

Definition 2.2.3. Let X be a finite symmetric subset of a group G.

- We call $g \in G$ a *switching element for X* if

$$X \cap gXg^{-1} \subseteq \{e\}.$$

- We call $g \in G$ a *super-switching element for X* if

$$X \cap \left(gXg \cup gXg^{-1} \cup g^{-1}Xg \cup g^{-1}Xg^{-1}\right) \subseteq \{e\}.$$

Note that since X is symmetric, $g \in G$ is a switching element for X if and only if g^{-1} is a switching element for X.

Claim 2.2.4. *Let X be a finite symmetric subset of a group G and let $g \in G$ be a super-switching element for X. If $g^{w_1}xg^{w_2} = y$ for $x, y \in X$ and $w_1, w_2 \in \{-1, +1\}$, then $x = y = e$.*

Proof. Let $g^{w_1}xg^{w_2} = y$ for $x, y \in X$ and $w_1, w_2 \in \{-1, +1\}$. Since

$$y = g^{w_1}xg^{w_2} \in \left(gXg \cup gXg^{-1} \cup g^{-1}Xg \cup g^{-1}Xg^{-1}\right)$$

and $y \in X$, it follows from the definition of a super-switching element for X that $y = e$.

From $g^{w_1}xg^{w_2} = y$, we get $g^{-w_1}yg^{-w_2} = x$. So, by symmetry, the same argument shows $x = e$. □

Proposition 2.2.5. *Let G be a discrete (not necessarily countable) amenable ICC group, and let X be a finite symmetric subset of G. The set of super-switching elements for X is infinite.*

Proof of Proposition 2.2.5. Fix an invariant finitely additive probability measure d on G. For $A \subseteq G$, we call $d(A)$ the density of A. We will need the fact that infinite index subgroups have zero density, and that $d(A) = 0$ for every finite subset $A \subset G$.

Let $C_G(x)$ be the centralizer of a non-identity $x \in X$. Then, since X is finite, there is a finite set of cosets of $C_G(x)$ that includes all $g \in G$ such that $g^{-1}xg \in X$. So, non-switching elements for X are in the union of finitely many cosets of subgroups with infinite index, since G is ICC. This means that the set of non-switching elements for X has zero density, and so the set S of switching elements for X has density one.

Let T be the set of all super-switching elements for X. Let $A \subseteq G$ be the set of involutions $\{g \in G \mid g^2 = e\}$.

If $d(A) > 0$, then $d(A \cap S) > 0$. On the other hand, for any $g \in A \cap S$, since g is switching for X and $g^{-1} = g$, g is super-switching for X. Hence $A \cap S \subseteq T$. This shows that if $d(A) > 0$, then $d(T) \geq d(A \cap S) > 0$, and so we are done.

So, we can assume that $d(A) = 0$. For any $x, y \in X$, let $S_{x,y} = \{g \in S \mid gxg = y\}$. Note that

$$T = S \setminus \bigcup_{\substack{x,y \in X \\ (x,y) \neq (e,e)}} S_{x,y}.$$

It is thus enough to be shown that each $S_{x,y}$ has zero density when $(x, y) \neq (e, e)$. So assume for the sake of contradiction that $d(S_{x,y}) > 0$. Fix $g \in S_{x,y}$. We have the following for all $h \in g^{-1}S_{x,y}$.

$$\begin{aligned} gxg = y = ghxgh &\implies (xg) = h(xg)h \\ &\implies (xg)^{-1}h^{-1}(xg) = h \\ &\implies h = (xg)^{-1}h^{-1}(xg) \\ &\qquad = (xg)^{-1}[(xg)^{-1}h^{-1}(xg)]^{-1}(xg) \\ &\qquad = (xg)^{-2}h(xg)^2 \\ &\implies h \text{ is in the centralizer of } (xg)^2. \end{aligned}$$

So, the centralizer of $(xg)^2$ includes $g^{-1}S_{x,y}$, which has a positive density. So, the centralizer of $(xg)^2$ has finite index. This implies that $(xg)^2 = e$, because in an ICC group only the identity can have a finite index centralizer. Hence $xg \in A$ for all $g \in S_{x,y}$. So $xS_{x,y} \subseteq A$. Hence $S_{x,y}$ also has zero density, which is a contradiction. □

A Heavy-Tailed Probability Distribution on $\mathbb{N}$.

Here we state and prove a lemma about the existence of a probability distribution on $\mathbb{N} = \{1, 2, \ldots\}$ such that infinite i.i.d. samples from this measure have certain properties. We will use this distribution in the proof of Proposition 2.2.2.

Lemma 2.2.6. *Let p be the following probability measure on $\mathbb{N}$: $p(n) = cn^{-5/4}$, where $1/c = \sum_{n=1}^{\infty} n^{-5/4}$. Then p has finite entropy and the following property: for any $\varepsilon > 0$ there exist constants $K_\varepsilon, N_\varepsilon \in \mathbb{N}$ such that for any natural number $m \geq K_\varepsilon$ there exists an $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ such that:*

1. *$p^{\times m}(E_{\varepsilon,m}) \geq 1 - \varepsilon$, where $p^{\times m}$ is the m-fold product measure $p \times \cdots \times p$.*
2. *For any $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$, the maximum of $\{s_1, \ldots, s_{K_\varepsilon}\}$ is at most N_ε.*
3. *For any $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$ and for any $K_\varepsilon \leq k \leq m$, the maximum of $\{s_1, \ldots, s_k\}$ is at least k^2.*
4. *For any $s = (s_1, \ldots, s_m) \in E_{\varepsilon,m}$ and for any $K_\varepsilon \leq k \leq m$, the maximum of $\{s_1, \ldots, s_k\}$ appears in $(s_1, \ldots, s_k)$ only once.*

Proof. It is straightforward to see that p has finite entropy.

Let $s = (s_1, s_2, \ldots) \in \mathbb{N}^\infty$ have distribution $p^{\times\infty}$; i.e., s is a sequence of i.d.d. random variables with distribution p. Since each s_i has distribution p, for each

$n \in \mathbb{N}$ we have:

$$\mathbb{P}[s_i \geq n] = \sum_{m=n}^{\infty} p(m) = c \sum_{m=n}^{\infty} m^{-5/4} \geq c \int_n^{\infty} x^{-5/4} \mathrm{d}x = 4cn^{-1/4}. \qquad (2.2.2)$$

For $k \geq 1$, let

$$M_k := \max\{s_1, \ldots, s_k\},$$

and let

$$\text{next}(k) := \min\{i > k \mid s_i \geq M_k\}.$$

In words, $\text{next}(k)$ is the first index $i > k$ for which s_i matches or exceeds M_k.

We first show that with probability one, $M_k \geq k^2$ for all k large enough. To this end, let A_k be the event that $M_k < k^2$. We have:

$$\begin{aligned}
\mathbb{P}[A_k] &= \mathbb{P}\left[s_i < k^2 \ \forall i \in \{1, \ldots, k\}\right] \\
&= (1 - \mathbb{P}\left[s_1 < k^2\right])^k \\
&\leq (1 - 4c(k^2)^{-1/4})^k \\
&\leq e^{-4ck^{1/2}}.
\end{aligned}$$

Since the sum of these probabilities is finite, by Borel-Cantelli we get that

$$\mathbb{P}[A_k \text{ infinitely often}] = 0.$$

Hence $M_k \geq k^2$ for all k large enough, almost surely. Furthermore, the expectation of $1/M_k$ is small:

$$\mathbb{E}\left[\frac{1}{M_k}\right] = \mathbb{E}\left[\frac{1}{M_k}\middle|A_k\right]\mathbb{P}[A_k] + \mathbb{E}\left[\frac{1}{M_k}\middle|\neg A_k\right]\mathbb{P}[\neg A_k] \leq e^{-4ck^{1/2}} + \frac{1}{k^2}. \qquad (2.2.3)$$

Next, we show that, with probability one, $s_{\text{next}(k)} > M_k$ for all k large enough. That is, for large enough k, the first time that M_k is matched or exceeded after index k, it is in fact exceeded.

Let B_k be the event that $s_{\text{next}(k)} = M_k$. We would like to show that this occurs only finitely often. Note that

$$\begin{aligned}
\mathbb{P}[B_k | M_k] &= \mathbb{P}\left[s_{\text{next}(k)} = M_k \middle| M_k\right] \\
&= \sum_{i=k+1}^{\infty} \mathbb{P}[s_i = M_k, \text{next}(k) = i | M_k].
\end{aligned}$$

Applying the definition of $\mathrm{next}(k)$ yields

$$\mathbb{P}\left[B_k|M_k\right] = \sum_{i=k+1}^{\infty} \mathbb{P}\left[s_i = M_k, s_{k+1}, \ldots, s_{i-1} < M_k|M_k\right].$$

By the independence of the s_i's we can write this as

$$\begin{aligned}\mathbb{P}\left[B_k|M_k\right] &= \sum_{i=k+1}^{\infty} \mathbb{P}\left[s_i = M_k|M_k\right] \prod_{n=1}^{i-(k+1)} \mathbb{P}\left[s_{k+n} < M_k|M_k\right] \\ &= \sum_{i=k+1}^{\infty} \frac{c}{M_k^{5/4}} \mathbb{P}\left[s_{k+1} < M_k|M_k\right]^{i-(k+1)}.\end{aligned}$$

By (2.2.2), $\mathbb{P}\left[s_{k+1} < M_k|M_k\right] \leq 1 - 4cM_k^{-1/4}$. Hence

$$\mathbb{P}\left[B_k|M_k\right] \leq \frac{c}{M_k^{5/4}} \cdot \frac{1}{4cM_k^{-1/4}} = \frac{1}{4M_k}.$$

Using (2.2.3) it follows that

$$\mathbb{P}\left[B_k\right] = \mathbb{E}\left[\mathbb{P}\left[B_k|M_k\right]\right] \leq \mathbb{E}\left[\frac{1}{4M_k}\right] \leq \frac{1}{4}e^{-4ck^{1/2}} + \frac{1}{4k^2}.$$

Hence $\sum_k \mathbb{P}\left[B_k\right] < \infty$, and so by Borel-Cantelli B_k occurs only finitely often.

Since A_k and B_k both occur for only finitely many k, the (random) index ind' at which they stop occurring is almost surely finite, and is given by

$$\mathrm{ind}' = \min\{\ell \in \mathbb{N} : s \notin A_k \cup B_k \text{ for all } k \geq \ell\}.$$

Let

$$\mathrm{ind} = \mathrm{next}(\mathrm{ind}').$$

Hence for $k \geq \mathrm{ind}$, $M_k \geq k^2$ and M_k appears in $(s_1, \ldots, s_k)$ only once.

Fix $\varepsilon > 0$. Since ind is almost surely finite, then for large enough constants $K_\varepsilon \in \mathbb{N}$ and $N_\varepsilon \in \mathbb{N}$ the event

$$E_\varepsilon = \{\mathrm{ind} \leq K_\varepsilon \text{ and } M_{K_\varepsilon} \leq N_\varepsilon\}$$

has probability at least $1 - \varepsilon$, and additionally, conditioned on E_ε it holds that $k \geq \mathrm{ind}$ for all $k \geq K_\varepsilon$, and hence $M_k \geq k^2$ and M_k appears in $(s_1, \ldots, s_k)$ only once. Therefore, if for $m \geq K_\varepsilon$ we let $E_{\varepsilon,m}$ be the projection of E_ε to the first m coordinates, then $E_{\varepsilon,m}$ satisfies the desired properties. □

Proof of Proposition 2.2.2

Let $\frac{1}{8} > \varepsilon > 0$. Let $p, K_\varepsilon \in \mathbb{N}, N_\varepsilon \in \mathbb{N}$, and $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ be the probability measure, the constants, and the events from Lemma 2.2.6. To simplify notation let $N = N_\varepsilon$ and $K = K_\varepsilon$.

Let $G = \{a_1, a_2, \dots\}$, where $a_1 = a_2 = \cdots = a_N = e$. We define $(g_n)_n$, $(A_n)_n$, $(B_n)_n$ and $(C_n)_n$ recursively. Given $g_1, \dots, g_n$, let $A_n = \{g_n, g_n^{-1}, a_n, a_n^{-1}\}$ and $B_n = \cup_{i \leq n} A_i$. Denote $C_n = B_n \cup \{h^{-1}, h\}$. Note that A_n, B_n, and C_n are finite and symmetric for any $n \in \mathbb{N}$. Let $g_1 = g_2 = \ldots = g_N = e$. For $n + 1 > N$, given C_n, let $g_{n+1} \in G$ be a super-switching element for $(C_n)^{2n+1}$ which is not in $(C_n)^{8n+1}$. The existence of such a super-switching element is guaranteed by Proposition 2.2.5 and the facts that $(C_n)^{2n+1}$ is a finite symmetric subset of G and that $(C_n)^{8n+1}$ is finite.

For $n \in \mathbb{N}$, define a symmetric probability measure μ_n on A_n by

$$\mu_n = \varepsilon 2^{-n}\left(\frac{1}{2}\delta_{a_n} + \frac{1}{2}\delta_{a_n^{-1}}\right) + (1 - \varepsilon 2^{-n})\left(\frac{1}{2}\delta_{g_n} + \frac{1}{2}\delta_{g_n^{-1}}\right).$$

Here δ_g is the point mass on $g \in G$. Finally, let

$$\mu = \sum_{n=1}^{\infty} p(n)\mu_n.$$

Obviously μ is symmetric and $\operatorname{supp}\mu = G$. Since p has finite entropy and each μ_n has support of size at most 4, it follows easily that μ has finite entropy.

We want to show that

$$\lim_{m\to\infty} \|h\mu^{*m} - \mu^{*m}\| > 0.$$

Fix $m \in \mathbb{N}$ larger than K and N. For each $n \in \mathbb{N}$ define $f_n : \{1, 2, 3, 4\} \to A_n$ by

$$f_n(1) = a_n,\ f_n(2) = a_n^{-1},\ f_n(3) = g_n,\ f_n(4) = g_n^{-1},$$

and define $\nu_n : \{1, 2, 3, 4\} \to [0, 1]$ by

$$\nu_n(1) = \nu_n(2) = \frac{1}{2}\varepsilon 2^{-n},\ \nu_n(3) = \nu_n(4) = \frac{1}{2}(1 - \varepsilon 2^{-n}).$$

Let

$$\Omega = \{(s, w) \mid s \in \mathbb{N}^m,\ w \in \{1, 2, 3, 4\}^m\}.$$

We define the measure η on the countable set Ω by specifying its values on the singletons:

$$\eta(\{(s, w)\}) = p^{\times m}(s)\ \nu_{s_1}(w_1)\ \nu_{s_2}(w_2)\ \ldots\ \nu_{s_m}(w_m).$$

It follows immediately from this definition that η is a probability measure.

Define $r : \Omega \to G$ by

$$r(s, w) = f_{s_1}(w_1) f_{s_2}(w_2) \dots f_{s_m}(w_m).$$

It is not difficult to see that $r_* \eta = \mu^{*m}$, and so we need to show that $\|hr_*\eta - r_*\eta\|$ is uniformly bounded away from zero for m larger than K and N.

Recall that $E_{\varepsilon,m} \subseteq \mathbb{N}^m$ is the event given by Lemma 2.2.6. Fix $s \in E_{\varepsilon,m}$. Define

$$\begin{aligned} i_{s,1} &= \min\{j \in \{1 \dots, m\} \mid s_j > N\}, \\ i_{s,2} &= \min\{j > i_{s,1} \mid s_j \geq s_{i_{s,1}}\}, \\ &\vdots \\ i_{s,l(s)} &= \min\{j > i_{s,l(s)-1} \mid s_j \geq s_{i_{s,l(s)-1}}\}. \end{aligned}$$

Note that by the second property of $E_{\varepsilon,m}$ in Lemma 2.2.6, we know that

$$K < i_{s,1} < i_{s,2} < \dots < i_{s,l(s)},$$

and by the fourth property,

$$N < s_{i_{s,1}} < s_{i_{s,2}} < \dots < s_{i_{s,l(s)}} = \max\{s_1, \dots, s_m\}.$$

Let

$$W^s_\varepsilon = \{w \in \{1, 2, 3, 4\}^m \mid \forall k \leq l(s)\ w_{i_{s,k}} = 3, 4\}.$$

For $s \in \mathbb{N}^m$ let η_s be the measure η, conditioned on the first coordinate equalling s. I.e., let

$$\eta_s(A) = \frac{\eta(A \cap \Omega^s)}{\eta(\Omega^s)},$$

where $\Omega^s = \{s\} \times \{1, 2, 3, 4\}^m \subseteq \Omega$.

Then

$$\begin{aligned} \eta_s(\{s\} \times W^s_\varepsilon) &= 1 - \eta_s(\{w_{i_{s,1}} = 1, 2;\ \text{or } w_{i_{s,2}} = 1, 2;\ \dots;\ \text{or } w_{i_{s,l(s)}} = 1, 2\ \}) \\ &\geq 1 - \sum_{k=1}^{l(s)} \eta_s(\{w_{i_{s,k}} = 1, 2\}) \\ &= 1 - \sum_{k=1}^{l(s)} \varepsilon 2^{-s_{i_{s,k}}} \\ &\geq 1 - \sum_{j=1}^{\infty} \varepsilon 2^{-j} \\ &= 1 - \varepsilon, \end{aligned}$$

where the first inequality follows from the union bound, and the last inequality holds since $s_{i_{s,1}} < s_{i_{s,2}} < \cdots < s_{i_{s,l(s)}}$.

Finally, let

$$\Omega_\varepsilon = \{(s,w) \in \Omega \mid s \in E_{\varepsilon,m},\ w \in W_\varepsilon^s\}.$$

By the above, and since $\eta(E_{\varepsilon,m} \times \{1,2,3,4\}^m) \geq 1-\varepsilon$ by Lemma 2.2.6, we have shown that

$$\eta(\Omega_\varepsilon) \geq \ (1-\varepsilon)(1-\varepsilon) > \ 1-2\varepsilon.$$

Claim 2.2.7. *For any $\alpha, \beta \in \Omega_\varepsilon$, we have $hr(\alpha) \neq r(\beta)$.*

We prove this claim after we finish the proof of the Proposition.

Let η_1 be equal to η conditioned on Ω_ε, and η_2 be equal to η conditioned on the complement of Ω_ε. We have $\eta = \eta(\Omega_\varepsilon)\eta_1 + (1-\eta(\Omega_\varepsilon))\eta_2$, and by the above claim we know $\|hr_*\eta_1 - r_*\eta_1\| = 2$. So for m larger than K and N

$$\begin{aligned}\|h\mu^{*m} - \mu^{*m}\| = \ &\|hr_*\eta - r_*\eta\| \\ = \ &\|\eta(\Omega_\varepsilon)(hr_*\eta_1 - r_*\eta_1) + (1-\eta(\Omega_\varepsilon))(hr_*\eta_2 - r_*\eta_2)\| \\ \geq \ &\eta(\Omega_\varepsilon)\,\|hr_*\eta_1 - r_*\eta_1\| - 2(1-\eta(\Omega_\varepsilon)) \\ \geq \ &2(1-2\varepsilon) - 2(2\varepsilon) = 2-8\varepsilon,\end{aligned}$$

which is uniformly bounded away from zero since $\varepsilon < \frac{1}{8}$. Since $\|h\mu^{*m} - \mu^{*m}\|$ is a decreasing sequence, this completes the proof of Proposition 2.2.2.

Proof of Claim 2.2.7. Let $\alpha = (s,w)$, $\beta = (t,v) \in \Omega_\varepsilon$. Hence $\max\{K,N\} < m$, $s \in E_{\varepsilon,m}$, $t \in E_{\varepsilon,m}$, $w \in W_\varepsilon^s$, and $v \in W_\varepsilon^t$. Assume that $hr(\alpha) = r(\beta)$. So, we have

$$hf_{s_1}(w_1)\cdots f_{s_m}(w_m) = f_{t_1}(v_1)\cdots f_{t_m}(v_m).$$

Let $K < i_1 < i_2 < \cdots < i_{l(s)}$ and $K < j_1 < j_2 < \cdots < j_{l(t)}$ be the indices we defined for s and t in the proof of Proposition 2.2.2. We remind the reader that the unique maximum of $(s_1,\ldots,s_m)$ is attained at $i_{l(s)}$, with a corresponding statement for $(t_1,\ldots,t_m)$ and $j_{l(t)}$. So we have

$$\begin{aligned}&h\,\overbrace{f_{s_1}(w_1)\cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1})}^{b_1}\,f_{s_{i_{l(s)}}}(w_{i_{l(s)}})\,\overbrace{f_{s_{i_{l(s)}+1}}(w_{i_{l(s)}+1})\cdots f_{s_m}(w_m)}^{b_2} \\ &= \underbrace{f_{t_1}(v_1)\cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1})}_{c_1}\,f_{t_{j_{l(t)}}}(v_{j_{l(t)}})\,\underbrace{f_{t_{j_{l(t)}+1}}(v_{j_{l(t)}+1})\cdots f_{t_m}(v_m)}_{c_2}.\end{aligned}$$

Let $p = s_{i_{l(s)}} = \max\{s_1, \ldots, s_m\}$ and $q = t_{j_{l(t)}} = \max\{t_1, \ldots, t_m\}$. Since $w \in W_\varepsilon^s$ and $v \in W_\varepsilon^t$, we know $f_{s_{i_{l(s)}}}(w_{i_{l(s)}}) = g_p^{\pm 1}$ and $f_{t_{j_{l(t)}}}(v_{j_{l(t)}}) = g_q^{\pm 1}$, so

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_q^{\pm 1} c_2. \tag{2.2.4}$$

Since $p = \max\{s_1, \ldots, s_m\}$, and since $m \geq K$, we know that $m \leq m^2 \leq p$. So $b_1, b_2 \in (B_{p-1})^{p-1} \subseteq (C_{p-1})^{p-1}$. Similarly $c_1, c_2 \in (C_{q-1})^{q-1}$.

Consider the case that $p > q$. Then $c_1, c_2, g_q^{\pm 1} \in (C_q)^q \subseteq (C_{p-1})^{p-1}$. Hence $g_p^{\pm 1} = [b_1^{-1}]h^{-1}[c_1 g_q^{\pm 1} c_2 b_2^{-1}]$ by (2.2.4), and so

$$g_p \in (C_{p-1})^{4(p-1)}\{h, h^{-1}\}(C_{p-1})^{4(p-1)} \subseteq (C_{p-1})^{8(p-1)+1},$$

which is a contradiction with our choice of g_p, since $p > N$. Similarly, if $p < q$, we get a contradiction. So we can assume that $p = q$.

If $p = q$, then by (2.2.4) we have

$$hb_1 g_p^{\pm 1} b_2 = c_1 g_p^{\pm 1} c_2,$$

and $c_1, c_2, b_1, b_2 \in (C_{p-1})^{p-1}$. So, for $x = c_1^{-1} h b_1 \in (C_{p-1})^{2(p-1)+1}$ we have $g_p^{\pm 1} x g_p^{\pm 1} = c_2 b_2^{-1} \in (C_{p-1})^{2(p-1)} \subseteq (C_{p-1})^{2(p-1)+1}$. By the fact that g_p is a super-switching element for $(C_{p-1})^{2(p-1)+1}$ and from Claim 2.2.4, we get that x is the identity.

So $hb_1 = c_1$, i.e.

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)}-1}}(w_{i_{l(s)}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)}-1}}(v_{j_{l(t)}-1}).$$

By the exact same argument, we can see this leads to a contradiction unless

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)-1}-1}}(w_{i_{l(s)-1}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)-1}-1}}(v_{j_{l(t)-1}-1}).$$

And again, this leads to a contradiction unless

$$hf_{s_1}(w_1) \cdots f_{s_{i_{l(s)-2}-1}}(w_{i_{l(s)-2}-1}) = f_{t_1}(v_1) \cdots f_{t_{j_{l(t)-2}-1}}(v_{j_{l(t)-2}-1}).$$

Note that if $l(s) \neq l(t)$, at some point in this process we get that either all the s_i's or all the t_i's are at most N while the other string has characters strictly greater than N. This leads to a contradiction similar to the case $p \neq q$, which we explained before. So, by continuing this process, we get a contradiction unless

$$hf_{s_1}(w_1) \cdots f_{s_{i_1-1}}(w_{i_1-1}) = f_{t_1}(v_1) \cdots f_{t_{j_1-1}}(v_{j_1-1}). \tag{2.2.5}$$

Note that $s_1, \ldots, s_{i_1-1} \leq N$, which implies

$$f_{s_1}(w_1) = \cdots = f_{s_{i_1-1}}(w_{i_1-1}) = e.$$

Similarly, $t_1, \ldots, t_{j_1-1} \leq N$ implies that

$$f_{t_1}(v_1) = \cdots = f_{t_{j_1-1}}(v_{j_1-1}) = e.$$

So, from (2.2.5) we get $h = e$, which is a contradiction.

□

Chapter 3

THREE

3.1 Introduction

Let (G, X) be a topological dynamical system: a jointly continuous action of a topological group G on a compact Hausdorff space X. A homeomorphism φ of X is an automorphism of (G, X) if $g \circ \varphi = \varphi \circ g$ for all $g \in G$. We denote by (G, X) the group of automorphisms, equipped with the compact-open topology. A Borel probability measure ν on X is *invariant* if $g_* \nu = \nu$ for all $g \in G$.

Definition 3.1.1. A Borel probability measure ν on X is *characteristic* if $\varphi_* \nu = \nu$ for all $\varphi \in (G, X)$.

Note that characteristic measures are not necessarily invariant, and invariant measures are not necessarily characteristic. However, when G is abelian then G is a subgroup of (G, X), and hence every characteristic measure is G-invariant; this is not true for general G. When G is amenable then (G, X) admits invariant measures, and moreover, if there are characteristic measures, then there are characteristic invariant measures. Likewise, if (G, X) is amenable then there are characteristic measures, and if there are invariant measures then there are characteristic invariant measures. This follows from the fact that G (resp., (G, X)) acts affinely on the compact, convex set of characteristic (resp., invariant) measures.

In this paper we will focus on *symbolic dynamical systems*, or shifts, and restrict our attention to finitely generated G. Let A be a finite alphabet. The *full shift* is the dynamical system (G, A^G), where A^G is equipped with the product topology and the action is by left translations. A *shift* (G, Σ) is a subsystem of (G, A^G), with Σ a closed, G-invariant subset of A^G.

The automorphism groups of shifts are always countable [Hed69]. Even in the simplest case that $G = \mathbb{Z}$, these groups exhibit rich structure; for example $(\mathbb{Z}, 2^{\mathbb{Z}})$ contains the free group on two generators, as well as every finite group (see, e.g., [BLR88]).

Some shifts $(\mathbb{Z}, \Sigma)$ obviously admit characteristic measures: these include uniquely ergodic shifts, shifts with a unique measure of maximal entropy, shifts with periodic points (which include all shifts of finite type), and shifts with amenable automorphism groups. But since $(\mathbb{Z}, \Sigma)$ is in general non-amenable, it is not obvious that every $(\mathbb{Z}, \Sigma)$ admits a characteristic measure. Indeed, we do not know if this holds.

Our main result concerns zero entropy shifts. To define the entropy of a shift, let $N_\Sigma(F)$, the *growth function* of Σ, assign to each finite $F \subset \mathbb{Z}$ the cardinality of the restriction of Σ to F. The entropy of Σ is given by

$$h(\Sigma) = \inf_r \frac{1}{r} \log N_\Sigma(\{1, 2, \ldots, r\}).$$

Theorem 3.1.2. *Let $(\mathbb{Z}, \Sigma)$ be a shift with $h(\Sigma) = 0$. Then $(\mathbb{Z}, \Sigma)$ admits a characteristic measure.*

Our proof techniques critically uses the zero entropy assumption, and thus leaves open the broader question:

Question 3.1.3. *Does every shift $(\mathbb{Z}, \Sigma)$ admit a characteristic measure?*

We more generally do not know of any countable group G and a shift (G, Σ) that does not admit characteristic measures.

Recent work [CQY16; CK15; CK16a; CK16b; Don+16; Sal17; ST15] shows that "small shifts" have "small automorphism groups." For example, minimal shifts with slow stretched exponential growth (that is, shifts with $N_\Sigma(F) = O(e^{|F|^\beta})$ for $\beta < 1/2$) have amenable automorphism groups, as shown by Cyr and Kra [CK16a]. They conjecture that every minimal zero entropy shift has an amenable automorphism group. A proof of this conjecture would imply Theorem 3.1.2 for minimal shifts.

Theorem 3.1.2 is a consequence of the following, more general result that applies to finitely generated groups, and relates the existence of characteristic measures to the growth of the shift. Given a finitely generated group G, we fix a generating set, and denote by $B_r \subset G$ the ball of radius r, according to the corresponding word length metric.

Theorem 3.1.4. *Let G be a finitely generated group. Then every shift (G, Σ) for which*

$$\liminf_r \frac{1}{r} \log N_\Sigma(B_r) = 0$$

admits a characteristic measure.

Theorem 3.1.2 is an immediate specialization of this result to the case $G = \mathbb{Z}$.

Beyond symbolic systems

It is simple to construct a dynamical system $(\mathbb{Z}, C)$, which is not symbolic, and which has no characteristic measures: simply let $\mathbb{Z}$ act trivially on the Cantor set C. This system admits no characteristic measures, since the Cantor set has no measure that is invariant to all of its homeomorphisms.

Recall that a dynamical system (G, X) is said to be *topologically transitive* if for every two non-empty open sets $U, W \subset X$ there is some $g \in G$ such that $gU \cap W \neq \emptyset$. The system (G, X) is *minimal* if X has no closed, G-invariant sets. It is *free* if $gx \neq x$ for every $x \in X$ and every non-trivial $g \in G$; in the important case of $G = \mathbb{Z}$ every non-trivial minimal system is free.

Question 3.1.5. *Does there exist a non-trivial minimal topological dynamical system that does not admit a characteristic measure?*

An example of a topologically transitive $\mathbb{Z}$-system without characteristic measures is the $\mathbb{Z}$ action by shifts on $C^{\mathbb{Z}}$, where C is the Cantor set.

Recall that (G, X) is said to be proximal [Gla76b] if for every $x, y \in X$ there exists a net $(g_i)_i$ in G such that $\lim_i g_i x = \lim_i g_i y$. Many constructions of dynamical systems without invariant measures are proximal (e.g., the Furstenberg boundary of non-amenable groups [Fur63a; Gla76b]). Hence the following claim highlights a tension that needs to be overcome in order to construct minimal systems without characteristic measures.

Claim 3.1.6. *Let (G, X) be a free system. Then $((G, X), X)$ is not proximal.*

Proof. Assume that $((G, X), X)$ is proximal. Then for each $x \in X$ and $g \in G$, there is a net $(\phi_i)_i$ such that $\lim_i \phi_i x = \lim_i \phi_i gx$. Since G and (G, X) commute, and since the action is continuous, we have that $g \lim_i \phi_i x = \lim_i \phi_i x$. Hence (G, X) is not free. □

Soficity of automorphism groups

We show the following result, using techniques that are similar to those used to prove Theorem 3.1.2.

Theorem 3.1.7. *Let $(\mathbb{Z}, \Sigma)$ a minimal shift with $h(\mathbb{Z}, \Sigma) = 0$. Then $(\mathbb{Z}, \Sigma)$ is sofic.*

Soficity, as defined by Gromov [Gro99] (see also Weiss [Wei00]) is a joint weakening of amenability and residual finiteness, and so this result, in a weak sense, supports the aforementioned conjecture that these automorphism groups are amenable.

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3.2 Proofs

Let G be a countable group, A a finite alphabet and (G, Σ) a subshift of (G, A^G). Let F be a finite subset of G. The restriction of $\sigma \in \Sigma$ to F is denoted by $\sigma_F \colon F \to A$. We denote

$$\Sigma_F = \{\sigma_F \ : \ \sigma \in \Sigma\},$$

and denote the growth function of Σ by

$$N_\Sigma(F) = |\Sigma_F|.$$

Proposition 3.2.1. *Let G be a countable group, and let $(F_n)_n$ be an increasing sequence of finite subsets of G with $\cup_n F_n = G$. Let (G, Σ) be a shift with the property that for every finite $K \subset G$ it holds that*

$$\liminf_n \frac{N_\Sigma\left(\cup_{g \in K} g F_n\right)}{N_\Sigma(F_n)} = 1.$$

Then (G, Σ) admits a characteristic measure.

If G is in addition amenable then (G, Σ) admits a characteristic invariant measure. To see this, note that the set of characteristic measures is a compact, convex subset of the Borel measures on Σ. The group G acts on this set, since for any characteristic ν, $g \in G$ and $\varphi \in (G, \Sigma)$ it holds that $\varphi(g\nu) = g\varphi(\nu) = g\nu$. Since G is amenable this action must have a fixed point, which is the desired characteristic invariant measure.

The proof of Proposition 3.2.1 will use the notion of a *memory set*. Given $\varphi \in (G, \Sigma)$, there is some finite $K \subset G$ and a map $\Phi \colon A^K \to A$ such that

$$[\varphi(\sigma)](g) = \Phi\left((g^{-1}\sigma)_K\right).$$

The set K is called a *memory set* of φ; see, e.g., [CC10, p. 6]. We can assume without loss of generality that K contains the identity.

Proof of Proposition 3.2.1. For each n, let $\pi_n\colon \Sigma \to A^{F_n}$ be the restriction map $\sigma \mapsto \sigma_{F_n}$, so that $\pi_n(\Sigma) = \Sigma_{F_n}$. Let $S_n \subset \Sigma$ be a set of representatives of the set $\{\pi_n^{-1}(\sigma_{F_n}) : \sigma \in \Sigma\}$ of preimages of π_n. Hence $\pi_n(S_n) = \Sigma_{F_n}$ and $|S_n| = |\Sigma_{F_n}| = N_\Sigma(F_n)$.

Let ν_n be the uniform measure over S_n, and let ν be any weak limit of a subsequence of $(\nu_n)_n$; such a limit exists by compactness. We will show that ν is characteristic.

Fix $\varphi \in (G, \Sigma)$. Let $K \subset G$ be a memory set of φ, and assume it contains the identity. There is thus $\Phi\colon A^K \to A$ such that $[\varphi(\sigma)](g) = \Phi\left((g^{-1}\sigma)_K\right)$. Denote

$$\tilde{F}_n = \bigcup_{g \in K} F_n g.$$

Let $\tilde{S}_n = \{\sigma_{\tilde{F}_n} : \sigma \in S_n\}$ be the set of projections of the elements of S_n to $\tilde{F}_n$. Since $\tilde{F}_n$ contains F_n it follows that $|S_n| = |\tilde{S}_n|$.

Define $\varphi'\colon \Sigma_{\tilde{F}_n} \to \Sigma_{F_n}$ by

$$[\varphi'(\sigma)](g) = \Phi\left((g^{-1}\sigma)_K\right),$$

for $g \in F_n$.

By the definition of $\tilde{F}_n$ this is well defined, and moreover $\varphi(\sigma)_{F_n} = \varphi'(\sigma_{\tilde{F}_n})$; that is, φ' maps the restriction of σ to $\tilde{F}_n$ to the restriction of $\varphi(\sigma)$ to F_n. Hence $\varphi(S_n)_{F_n} = \varphi'(\tilde{S}_n)$. Also, φ' is onto and so there is a subset $R_n \subseteq \Sigma_{\tilde{F}_n}$ such that the restriction of φ' to R_n is a bijection from R_n to Σ_{F_n}.

For every $\varepsilon > 0$, we can, by the claim hypothesis, take n to be large enough so that $N_\Sigma(F_n) \geq (1-\varepsilon)N_\Sigma(\tilde{F}_n)$. Then R_n and $\tilde{S}_n$ are both of size $N_\Sigma(F_n) \geq (1-\varepsilon)N_\Sigma(\tilde{F}_n)$. Since their union is contained in $\Sigma_{\tilde{F}_n}$ and is thus of size at most $N_\Sigma(\tilde{F}_n)$, their intersection is of size at least $(1-2\varepsilon)N_\Sigma(\tilde{F}_n)$. Since

$$\varphi(S_n)_{F_n} = \varphi'(\tilde{S}_n) \supseteq \varphi'(\tilde{S}_n \cap R_n),$$

and since φ' is a bijection when restricted to R_n, $\varphi(S_n)$ is also of size at least $(1-2\varepsilon)N_\Sigma(\tilde{F}_n)$, which is at least $(1-2\varepsilon)N_\Sigma(F_n)$.

Since ν_n is the uniform distribution on S_n, it follows that the push-forward measures $\pi_n(\nu_n)$ and $\pi_n(\varphi(\nu_n))$ differ by at most 2ε in total variation. Since the sequence $(F_n)_n$ is increasing, this implies that for all $m \leq n$ it also holds that $\pi_m(\nu_n)$ and $\pi_m(\varphi(\nu_n))$ differ by at most 2ε. Thus for each m, $\pi_m(\nu)$ and $\pi_m(\varphi(\nu))$ are identical, and so $\varphi(\nu) = \nu$, since $\cup_n F_n = G$, and so the cylinder sets defined by the restrictions

$(\pi_m)_m$ form a clopen basis for the Borel σ-algebra. We have thus shown that ν is characteristic.

□

Using Proposition 3.2.1, the proof of our main result is straightforward.

Proof of Theorem 3.1.4. Denote $L(r) = \log N_\Sigma(B_r)$. By the claim hypothesis, there is a sequence $(r_k)_k$ such that $\lim_k L(r_k)/r_k = 0$. Thus, and because $L(r)$ is increasing, there is another subsequence r_n such that for every $i > 0$

$$\lim_{\ell} L(r_n + i) - L(r_n) = 0.$$

Hence if we set $F_n = B_{r_n}$ then the conditions of Proposition 3.2.1 are satisfied, and thus the conclusion follows. □

Theorem 3.1.7 is a corollary of the following more general statement.

Theorem 3.2.2. *Let G be a countable group, and let $(F_n)_n$ be an increasing sequence of finite subsets of G with $\cup_n F_n = G$. Let (G, Σ) be a minimal shift with the property that for every finite $K \subset G$ it holds that*

$$\liminf_n \frac{N_\Sigma\left(\cup_{g \in K} gF_n\right)}{N_\Sigma(F_n)} = 1.$$

Then (G, Σ) is sofic.

The following lemma will serve as our working definition of a sofic group; the reduction to the usual definition is straighforward (see, e.g., [Jus21, Lemma 2.1]). A *partially defined map* from a set A to A is a map from a subset of A into A.

Lemma 3.2.3. *Let H be a countable group. Suppose that for all finite subsets $\Phi \subset H$ and all $\varepsilon > 0$ we have a finite set A and a map $g \mapsto \tilde{g}$ that assigns to each $g \in \Phi$ a partially defined map $\tilde{g}$ from A to A which satisfies the following four conditions:*

1. *for every $g \in \Phi$ there is a subset $A_g \subset A$ with $|A \setminus A_g|/|A| < \varepsilon$, such that the map $\tilde{g}$ is defined and injective on A_g.*

2. *For the identity element $e \in G$, $\tilde{e}$ is the identity map wherever it is defined.*

3. *$\widetilde{gh}(a) = \tilde{g}(\tilde{h}(a))$ whenever all three are defined.*

4. *If there is some $a \in A$ such that $\tilde{g}(a) = (a)$, then g is the identity.*

Then H is sofic.

We will need the following compactness lemma.

Lemma 3.2.4. *Let φ be an automorphism of a subshift (G, Σ) such that $\varphi(\sigma) \neq \sigma$ for all $\sigma \in \Sigma$. Then there is some finite set $K \subset G$ such that for all $\sigma \in \Sigma$ the restrictions σ_K and $\varphi(\sigma)_K$ differ.*

Proof. Let $(F_n)_n$ be an increasing sequence of finite subsets of G with $\cup_n F_n = G$. Assume towards a contradiction that for each n there is a $\sigma^n \in \Sigma$ such that $\sigma^n_{F_n} = \varphi(\sigma^n)_{F_n}$. Assume without loss of generality that the sequence $(\sigma^n)_n$ converges to σ. Since the sequence $(F_n)_n$ is increasing, $\varphi(\sigma)_{F_n} = \sigma_{F_n}$ for all n. Hence $\varphi(\sigma) = \sigma$, since $(F_n)_n$ exhausts G. This is in contradiction to our assumption that φ has no fixed points. □

Proof of Theorem 3.2.2. Let Φ be a finite subset of (G, Σ) which includes the identity. Fix $1 > \varepsilon > 0$. Let K be a finite subset of G that contains the memory sets[1] of all $\varphi \in \Phi$.

Since (S, Σ) is minimal, $\varphi(\sigma) \neq \sigma$ for every $\sigma \in \Sigma$ and non-trivial φ. To see this, note that if the set of fixed points of φ is non-empty then it is a subshift, and so, by minimality, must be all of Σ. Accordingly, by Lemma 3.2.4, we can enlarge K (while keeping it finite) so that $\sigma_K \neq \varphi(\sigma)_K$ for all $\sigma \in \Sigma$ and $\varphi \in \Phi$.

To prove the claim we proceed to find partially defined maps which satisfy the assumptions in Lemma 3.2.3 for Φ, ε. Choose k large enough so that $N_\Sigma(\cup_{g \in K} gF_k)/N_\Sigma(F_k) < 1 + \varepsilon$. Denote $F = F_k$ and $\tilde{F} = \cup_{g \in K} gF_k$.

For every $\varphi \in \Phi$, there is a natural map $\varphi' \colon \Sigma_{\tilde{F}} \longrightarrow \Sigma_F$ which, given $\sigma \in \Sigma$, maps the configuration $\sigma_{\tilde{F}}$ to the configuration $\varphi(\sigma)_F$. This is well defined, since K contains the memory set of φ, and hence $\varphi(\sigma)_F$ is determined by $\sigma_{\tilde{F}}$.

Since φ is an automorphism, φ' is surjective. Now we set A to be $\Sigma_{\tilde{F}}$ and let the partially defined map $\tilde{\varphi}$ from A to A be given by $\tilde{\varphi}(a) = b$ whenever there exists a $\sigma \in \Sigma$ such that $a = \sigma_{\tilde{F}}$, and b is the unique element of $A = \Sigma_{\tilde{F}}$ whose projection on Σ_F is $\varphi'(a)$. This map is undefined when uniqueness fails.

We now prove that this map has the four properties required by Lemma 3.2.3.

[1]See the proof of Proposition 3.2.1 for the definition of a memory set.

1. Since the projection map $\pi : \Sigma_{\tilde{F}} \to \Sigma_F$ is surjective, and since $N_\Sigma(\tilde{F})/N_\Sigma(F) < 1+\varepsilon$ there can be at most $\varepsilon N_\Sigma(F)$ many elements in Σ_F with more than one extension to $\Sigma_{\tilde{F}}$. Thus π^{-1} is one-to-one on a $1-\varepsilon$ fraction of Σ_F. Since $\varphi' : \Sigma_{\tilde{F}} \to \Sigma_F$ is surjective, it follows that $\tilde{\varphi}$ is defined on a $(1-\varepsilon)/(1+\varepsilon)$ fraction of $A = \Sigma_{\tilde{F}}$.

2. If $\sigma_F \in \Sigma_F$ has a unique extension to $\Sigma_{\tilde{F}}$ then that extension must be $\sigma_{\tilde{F}}$. Applying this to the identity of (G, Σ) yields the desired condition.

3. Suppose $\tilde{\psi}(a)$, $\tilde{\varphi}(\tilde{\psi}(a))$ and $\widetilde{\varphi\psi}(a)$ are all defined. We show that $\tilde{\varphi}(\tilde{\psi}(a)) = \widetilde{\varphi\psi}(a)$.

   Note that for any $\eta \in \Phi$ and $\sigma_{\tilde{F}} \in A$, if $\tilde{\eta}(\sigma_{\tilde{F}})$ is defined, then $\tilde{\eta}(\sigma_{\tilde{F}}) = \eta(\sigma)_{\tilde{F}}$. Applying this to ψ, φ and $\varphi\psi$ we get that for $a = \sigma_{\tilde{F}}$

$$\tilde{\varphi}(\tilde{\psi}(\sigma_{\tilde{F}})) = \tilde{\varphi}(\psi(\sigma)_{\tilde{F}}) = \varphi\psi(\sigma)_{\tilde{F}} = \widetilde{\varphi\psi}(\sigma_{\tilde{F}}).$$

4. The fourth condition follows from the fact that $K \subseteq \tilde{F}$, and the defining property of K that ensures that σ_K and $\varphi(\sigma)_K$ differ.

We have thus proved that all of the conditions of Lemma 3.2.3 hold, and so and the group is sofic. □

Theorem 3.1.7 is an easy corollary of Theorem 3.2.2, as, by the same argument as in the proof of Theorem 3.1.2, every zero entropy subshift must satisfy

$$\liminf_n \frac{N_\Sigma\left(\cup_{g\in K} gF_n\right)}{N_\Sigma(F_n)} = 1.$$

Chapter 4

FOUR

4.1 Introduction

The purpose of this paper is to study the complexity of quotient groups G/Γ from the point of view of descriptive set theory. In particular, we focus on the case where G is Polish and Γ is a countable normal subgroup. If Γ is a countable group, then the automorphism group of Γ has a natural Polish group structure, and thus the outer automorphism group of Γ is an example, as is any countable subgroup of an abelian group.

A major recent program is the study of complexity of "definable" equivalence relations. Results in this area are often interpreted to be statements about the difficulty of classification of various natural mathematical objects. A particular focus of the theory of definable equivalence relations, and one where much progress has recently been made, is the study of Borel equivalence relations for which every class is countable, the so-called countable Borel equivalence relations. There is a natural preorder on Borel equivalence relations, called Borel reduction, where E reducing to F is interpreted as E being "easier" than F. The theory of countable Borel equivalence relations has been applied in numerous areas of mathematics. For example, the classification of finitely generated groups [TV99], of subshifts [Cle09], and the arithmetic equivalence of subsets of $\mathbb{N}$ [MSS16] are all equally difficult. In fact, they are equivalent to the universal countable Borel equivalence relation E_∞, which is the hardest countable Borel equivalence relation. On the other hand, many other classification problems are easier. For example, classification of torsion-free finite rank abelian groups is substantially below E_∞ [Tho03; Tho09].

Countable Borel equivalence relations can be characterized as those equivalence relations arising from continuous actions of countable groups on Polish spaces, and thus have very strong interplay with dynamics and group theory. By a foundational result of Slaman-Steel and Weiss [SS88; Wei84], the equivalence relations which arise from a continuous (or more generally, Borel) action of $\mathbb{Z}$ are exactly the **hyperfinite** equivalence relations, which are those which can be written as an increasing union of finite Borel equivalence relations. More generally, it has been shown that every Borel action of a countable abelian group [GJ15], and even of a countable locally nilpotent group [SS13], is hyperfinite. It is an open question whether this holds for all countable amenable groups. By a theorem of Harrington-Kechris-Louveau [HKL90], the hyperfinite equivalence relations only occupy the first two levels of the hierarchy of countable Borel equivalence relations on uncountable Polish spaces under Borel reduction, and thus are considered to have low Borel complexity.

In general, if G is a Polish group and $\Gamma \leq G$ is a countable subgroup, then G/Γ can be rather complicated; we will give a non-hyperfinite example in Chapter 4.2. However, perhaps surprisingly, if Γ is a *normal* subgroup of G, then the coset equivalence relation G/Γ must have low Borel complexity: Let G be a Polish group and let Γ be a countable normal subgroup of G. Then G/Γ is hyperfinite. Notably, in contrast to the aforementioned results, we require no hypotheses on the algebraic structure of the acting group. The proof proceeds by showing that the equivalence relation is generated by a Borel action of a countable abelian group, which is sufficient by the aforementioned theorem of Gao and Jackson.

We obtain as a consequence the following result about outer automorphism groups: Let Γ be a countable group. Then $\mathrm{Out}(\Gamma)$ is hyperfinite.

If G is a compact group with a countable normal subgroup $\Gamma \triangleleft G$, then we also show that the algebraic structure of Γ is severely restricted: Let G be a compact group and let Γ be a countable normal subgroup of G. Then Γ is locally virtually abelian, i.e., every finitely generated subgroup of Γ is virtually abelian.

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4.2 Preliminaries and examples

Descriptive set theory

A **Polish space** is a second countable, completely metrizable topological space. A **Borel equivalence relation** on a Polish space X is an equivalence relation E which is Borel as a subset of $X \times X$. A Borel equivalence relation is **countable** (resp. **finite**) if every class is countable (resp. finite). A countable Borel equivalence relation E on X is **smooth** if there is a Borel function $f : X \to \mathbb{R}$ such that xEx' if and only if $f(x) = f(x')$. A Borel equivalence relation E is **hyperfinite** (resp., **hypersmooth**) if $E = \bigcup_n E_n$, where each $E_n \subset E_{n+1}$ (as a subset of $X \times X$) and each E_n is a finite (resp., smooth) Borel equivalence relation. Given a Borel action of a countable group Γ on a Polish space X, we denote by E_Γ^X the **orbit equivalence relation** of $\Gamma \curvearrowright X$, the Borel equivalence relation whose classes are the orbits of the action. We will say that $\Gamma \curvearrowright X$ is hyperfinite (resp., smooth, hypersmooth) if its orbit equivalence relation E_Γ^X is hyperfinite (resp., smooth, hypersmooth).

A **Polish group** is a topological group whose topology is Polish. If G is a Polish group and $H \leq G$ is a closed subgroup, then the quotient topology on the coset space G/H is Polish (see [BK96, p. 1.2.3]).

Countable subgroups of Polish groups

Let G be a Polish group and let $\Gamma \leq G$ be a countable subgroup. When clear from context, we will abuse notation and identify G/Γ with the coset equivalence relation induced by $\Gamma \curvearrowright G$ (technically, G/Γ is induced by the right action $G \curvearrowleft \Gamma$, but this is isomorphic to the left action $\Gamma \curvearrowright G$ via inversion). For example, we will say that G/Γ is hyperfinite if $\Gamma \curvearrowright G$ is hyperfinite.

Note that since the action $\Gamma \curvearrowright G$ is free, G/Γ cannot be universal among countable Borel equivalence relations (see [Tho09, p. 3.10]).

We give below some examples of G/Γ and the associated Borel complexity, for various G and Γ:

1. $\mathbb{R}/\mathbb{Z}$ is smooth, since $\mathbb{Z} \leq \mathbb{R}$ is a discrete subgroup (see [Kan08, 7.2.1(iv)]).

2. $\mathbb{R}/\mathbb{Q}$ is not smooth, since $\mathbb{Q} \leq \mathbb{R}$ is a dense subgroup (see [Gao09, p. 6.1.10]). Similarly, the commensurability relation $\mathbb{R}^+/\mathbb{Q}^+$ is not smooth. Note that both are hyperfinite, since they arise from Borel actions of countable abelian groups (see [GJ15, p. 8.2]).

3. Let $F_2 \leq \mathrm{SO}_3(\mathbb{R})$ be a free subgroup on two generators. Then $\mathrm{SO}_3(\mathbb{R})/F_2$ is not hyperfinite, since the free action $F_2 \curvearrowright \mathrm{SO}_3(\mathbb{R})$ preserves the Haar measure (see [Gao09, p. 7.4.8]).

4. If Γ is a countable group, then $\mathrm{Inn}(\Gamma)$ is a countable subgroup of $\mathrm{Aut}(\Gamma)$, which is a Polish group under the pointwise convergence topology, and we can consider the quotient $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. For example, when $\Gamma = S_{\mathrm{fin}}$ (the group of finitely supported permutations on $\mathbb{N}$), we have $\mathrm{Out}(S_{\mathrm{fin}}) \cong S_\infty/S_{\mathrm{fin}}$, which is hyperfinite and non-smooth.

In the first, second and fourth examples, Γ is a normal subgroup of G.

4.3 Proofs

For any group G and any subset $S \subset G$, let $C_G(S)$ denote the centralizer of S in G:

$$C_G(S) := \{g \in G : \forall s \in S(gs = sg)\}.$$

Note that if G is a topological group, then $C_G(S)$ a closed subgroup of G.

Let G be a Baire group (i.e., a topological group for which the Baire category theorem holds), and let Γ be a finitely generated subgroup of G each of whose elements has countable conjugacy class in G. Then $C_G(\Gamma)$ is open in G.

Proof. Let $\Gamma = \gamma_0, \ldots, \gamma_n$. Since each γ_i has countable conjugacy class in G, we have $[G : C_G(\gamma_i)] \leq \aleph_0$, so by the Baire category theorem, $C_G(\gamma_i)$ is nonmeager. Thus $C_G(\gamma_i)$ is an open subgroup of G, and thus $C_G(\Gamma)$ is also open, since $C_G(\Gamma) = \bigcap_{i \leq n} C_G(\gamma_i)$. □

Note that when $Z(\Gamma)$ is finite, this immediately implies that Γ is a discrete subgroup of G.

When G is a compact group, Chapter 4.3 implies the following algebraic restriction on Γ: Let G be a compact group and let Γ be a countable normal subgroup of G. Then Γ is locally virtually abelian, i.e. every finitely generated subgroup of Γ is virtually abelian.

Proof. Let Δ be a finitely generated subgroup of Γ. Then by Chapter 4.3, $C_G(\Delta)$ is an open subgroup of G, so since G is compact, the index of $C_G(\Delta)$ in G is finite. Thus since $Z(\Delta) = \Delta \cap C_G(\Delta)$, the index of $Z(\Delta)$ in Δ is finite. □

We now prove the main theorem. Let G be a Polish group and let Γ be a countable normal subgroup of G. Then G/Γ is hyperfinite.

Proof. Let $\Gamma = (\gamma_k)_{k<\omega}$ and denote $\Gamma_k := \gamma_0, \ldots, \gamma_k$. Let $C_k := C_G(\Gamma_k) = C_G(\gamma_0, \ldots, \gamma_k)$ and let $Z_k := Z(\Gamma_k) = C_k \cap \Gamma_k$ be the center of Γ_k. By Chapter 4.3, C_k is an open subgroup of G.

Let $A := Z_{k\,k<\omega}$, the subgroup of G generated by the Z_k for all $k < \omega$. Then A is an abelian subgroup of G, since each Z_k is abelian, and since Z_k commutes with Z_l (pointwise) for any $k < l$.

The principal fact we use about A is the following: $\Gamma \curvearrowright G/\bar{A}$ is hyperfinite.

Proof. Since $\bar{A}$ is a closed subgroup of G, the coset space $G/\bar{A}$ is a standard Borel space, and thus $\Gamma \curvearrowright G/\bar{A}$ induces a Borel equivalence relation.

Every hypersmooth countable Borel equivalence relation is hyperfinite (see [DJK94, p. 5.1]), so it suffices to show that $\Gamma \curvearrowright G/\bar{A}$ is hypersmooth. Since Γ is the increasing union of $(\Gamma_n)_n$, it suffices to show that $\Gamma_n \curvearrowright G/\bar{A}$ is smooth. In fact, we will show that every orbit of $\Gamma_n \curvearrowright G/\bar{A}$ is discrete, which implies smoothness (enumerate a basis, then for each orbit, find the first basic open set isolating an element of the orbit, and select that element).

Fix $g \in G$, and fix m large enough such that $g^{-1}\Gamma_n g \subset \Gamma_m$ (for instance, by normality of Γ, take any Γ_m containing $\{g^{-1}\gamma_i g\}_{i<n}$). We claim that the open neighbourhood $gC_m\bar{A}$ of $g\bar{A}$ contains no other elements of the Γ_n-orbit of $g\bar{A}$. Suppose that $\gamma g\bar{A} \subset gC_m\bar{A}$ for some $\gamma \in \Gamma_n$, so that $g^{-1}\gamma g \in C_m\bar{A}$. Since C_m is an open subgroup of G, it follows that $C_m A$ is closed (since its complement is a union of cosets of an open subgroup), so $C_m\bar{A} \subset \overline{C_m A} = C_m A$, and thus $g^{-1}\gamma g \in C_m A$.

Write $g^{-1}\gamma g = ca$ for some $c \in C_m$ and $a \in A$. Now $A = Z_{k\,k<\omega}$, but since A is abelian, we have $A = Z_{k\,k>m} Z_{l\,l\leq m}$, and thus we can write $a = d\beta$, where $d \in Z_{k\,k>m} \subset C_m$ and $\beta \in Z_{l\,l\leq m} \subset \Gamma_m \cap A$. Now we have $g^{-1}\gamma g = cd\beta$, but since $g^{-1}\gamma g$ and β are in Γ_m, we have $cd \in \Gamma_m$. Since c and d are in C_m as well, we get $cd \in Z_m \subset A$. Now since β is also in A, we have $g^{-1}\gamma g = cd\beta \in A \subset \bar{A}$, and thus $g\bar{A} = \gamma g\bar{A}$. □

We now use this lemma to show that E^G_Γ is induced by the action of a countable abelian group. This is sufficient since by a theorem of Gao and Jackson, every orbit equivalence relation of a countable abelian group is hyperfinite ([GJ15, p. 8.2]).

Since $\Gamma \curvearrowright G/\bar{A}$ is hyperfinite, its orbit equivalence relation is generated by a Borel automorphism T of $G/\bar{A}$ (see [DJK94, p. 5.1]). For each left $\bar{A}$-coset C, let $\gamma_{(C)} \in \Gamma$ be minimal such that $T(C) = \gamma_{(C)}C$, and let $U : G \to G$ be the Borel automorphism defined by $U(g) = \gamma_{(g\bar{A})}g$ (the inverse is defined by $g \mapsto (\gamma_{(T^{-1}(g\bar{A}))})^{-1}g$). This induces a Borel action $\mathbb{Z} \curvearrowright G$, denoted $(n, g) \mapsto n \cdot g$, such that

1. $g\bar{A}$ and $h\bar{A}$ are in the same Γ-orbit iff for some n, $(n \cdot g)\bar{A} = h\bar{A}$,
2. $E^G_{\mathbb{Z}} \subset E^G_\Gamma$,

3. and $\mathbb{Z} \curvearrowright G$ commutes with the right multiplication action $G \curvearrowleft (\bar{A} \cap \Gamma)$.

So there is a Borel action of $\mathbb{Z} \times (\bar{A} \cap \Gamma)$ on G such that $E^G_{\mathbb{Z}\times(\bar{A}\cap\Gamma)} \subset E^G_\Gamma$. We claim that in fact, $E^G_{\mathbb{Z}\times(\bar{A}\cap\Gamma)} = E^G_\Gamma$. Suppose that $g, h \in G$ are in the same Γ-coset (note that we don't need to specify left/right since Γ is normal). Then $g\bar{A}$ and $h\bar{A}$ are in the same Γ-orbit, so there is some $n \in \mathbb{Z}$ with $(n \cdot g)\bar{A} = h\bar{A}$. Since $E^G_\mathbb{Z} \subset E^G_\Gamma$, we have that $n \cdot g$ is in the same Γ-coset as g, and thus in the same Γ-coset as h. Since $n \cdot g$ and h are in the same left $\bar{A}$-coset, and also in the same (left) Γ-coset, they are in the same left $\bar{A} \cap \Gamma$-coset. Thus $E^G_{\mathbb{Z}\times(\bar{A}\cap\Gamma)} = E^G_\Gamma$. Since A is abelian, $\bar{A}$ is also abelian, and thus $\mathbb{Z} \times (\bar{A} \cap \Gamma)$ is a countable abelian group. So E^G_Γ is generated by the action of a countable abelian group, and is therefore hyperfinite. □

We can extend this result to a slightly more general class of subgroups: Let G be a Polish group and let $\Gamma \leq G$ be a countable subgroup of G each of whose elements has countable conjugacy class in G. Then G/Γ is hyperfinite.

Proof. Since every element of Γ has countable conjugacy class in G, the subgroup $\Delta := g\Gamma g^{-1}{}_{g\in G}$ is a countable normal subgroup of G, and thus by Chapter 4.3, E^G_Δ is hyperfinite. Since $E^G_\Gamma \subset E^G_\Delta$, we have that G/Γ is also hyperfinite (since hyperfiniteness is closed under subequivalence relations). □

Let Γ be a countable group. Then $\mathrm{Out}(\Gamma)$ is hyperfinite.

Proof. This follows from Chapter 4.3, since $\mathrm{Inn}(\Gamma) \triangleleft \mathrm{Aut}(\Gamma)$. □

We end with some open questions: Let G be a Polish group and let Γ be a countable subgroup. What are the possible Borel complexities of G/Γ? In particular, are they cofinal among orbit equivalence relations arising from free actions?

For a Polish group G, define the subgroup $Z_\omega(G)$ as follows:

$$Z_\omega(G) := \{g \in G : g \text{ has countable conjugacy class}\}.$$

In general, $Z_\omega(G)$ is a characteristic subgroup of G, analogous to the FC-center, and $Z_\omega(G)$ is $\mathbf{\Pi}^1_1$ by Mazurkiewicz-Sierpiński (see [Kec95, p. 29.19]).

Is there a Polish group G such that $Z_\omega(G)$ is $\mathbf{\Pi}^1_1$-complete?

Chapter 5

FIVE

5.1 Introduction

The axiom of choice allows us to construct many abstract algebraic homomorphisms between topological algebraic systems which are incredibly non-constructive. A longstanding theme in descriptive set theory is to study to what extent we can, and to what extent we provably cannot, construct such homomorphisms in a "definable" way. Here the notion of definability is context-dependent but often includes continuous, Borel, or projective maps.

A classical example of such an abstract construction, which provably cannot be constructed with "nice" sets is the existence of a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$. It is well-known that such a basis cannot be Borel, or more generally, analytic. Similar phenomena show up when constructing Hamel bases for topological vector spaces, or constructing an isomorphism of the additive groups of $\mathbb{R}$ and $\mathbb{C}$.

A more recent theme in descriptive set theory is that such undefinability criteria can often be leveraged in order to gain, and hopefully utilize, additional structure. For example, Silver's theorem [Sil80] and the Glimm-Effros dichotomy [HKL90] interpret the non-reducibility of Borel equivalence relations not as a pathology but rather as the first step in the burgeoning theory of invariant descriptive set theory (see [Gao09] for background). Similarly, work starting with [KST99] studies and exploits the difference between abstract chromatic numbers and more reasonably definable (for example, continuous or Borel) chromatic numbers. A key feature in many of these theories (and all of the above examples) is the existence of dichotomy theorems, which state that either an object is simple, or there is a canonical obstruction contained inside of it. This is usually stated in terms of preorders, saying that there is a natural basis for the preorder of objects which are not simple (recall that a **basis** for a preorder P is a subset $B \subseteq P$ such that for every $p \in P$, there is some $b \in B$ with $b \leq p$).

In this paper, we apply a descriptive set-theoretic approach to vector spaces and more generally, modules, over a locally compact Polish ring[1]. For a Polish ring R, a **Polish R-module** is a topological left R-module whose underlying topology

[1] All rings will be assumed to be unital.

is Polish. Given Polish R-modules M and N, we say that M **embeds** into N, denoted $M \sqsubseteq^R N$, if there is a continuous linear injection from M into N. One particularly nice aspect of Polish modules is that the notion of "definable" reduction is much simpler than in the general case. By Pettis's lemma, any Baire-measurable homomorphism between Polish modules is in fact automatically continuous (see [Kec95, p. 9.10]). Thus there is no loss of generality in considering continuous homomorphisms rather than a priori more general Borel homomorphisms.

Our main results give a dichotomy for Polish modules being countably generated. More precisely, we give a countable basis under $\sqsubseteq^R$ for Polish modules which are not countably generated. While these results are stated in a substantial level of generality (they are true for all left-Noetherian countable rings and many Polish division rings), we feel that the most interesting cases are over some of the most concrete rings. For example, over $\mathbb{Q}$, we show the existence of a unique (up to bi-embeddability) minimal uncountable Polish vector space $\ell^1(\mathbb{Q})$. We further show that nothing bi-embeddable with $\ell^1(\mathbb{Q})$ is locally compact, and thus that every uncountable-dimensional locally compact Polish vector space (for example, $\mathbb{R}$) is strictly more complicated than $\ell^1(\mathbb{Q})$.

Another case of particular interest is the case of $\mathbb{Z}$-modules, that is, abelian groups. We show that there is a countable basis of minimal uncountable abelian Polish groups (one for each prime number and one for characteristic 0). Furthermore, there exists a maximal abelian Polish group by [Shk99], as well as many natural but incomparable elements (for example, $\mathbb{Q}_p$ and $\mathbb{R}$ are incomparable under $\sqsubseteq^{\mathbb{Q}}$ as are $\mathbb{Q}_p$ and $\mathbb{Q}_r$ for $p \neq r$).

Our dichotomy theorems will hold for rings equipped with a proper norm. A **(complete, proper) norm** on an abelian group A is a function $\|\cdot\| : A \to [0, \infty)$ such that the map $(a, b) \mapsto \|a - b\|$ is a (complete, proper) metric on A (recall that a metric is proper if every closed ball is compact). A **norm** on a ring R is a norm $|\cdot|$ on $(R, +)$ such that $|rs| \leq |r||s|$ for every $r, s \in R$. A **proper normed ring** is a ring equipped with a proper norm. Every countable ring admits a proper norm (see Chapter 5.3). Given a proper normed ring R, the R-module $\ell^1(R)$ is defined as follows:

$$\ell^1(R) = \left\{ (r_k)_k \in R^{\mathbb{N}} : \sum_k \frac{|r_k|}{k!} < \infty \right\}$$

(here, $\frac{1}{k!}$ can be replaced with any summable sequence). Then $\|(r_k)_k\| := \sum_k \frac{|r_k|}{k!}$ is a complete separable norm on $(\ell^1(R), +)$, turning $\ell^1(R)$ into a Polish R-module.

The following theorems will be obtained as special cases of results in Chapter 5.5.

A **division ring** is a ring R such that every nonzero $r \in R$ has a two-sided inverse. Let R be a proper normed division ring and let M be a Polish R-vector space. Then exactly one of the following holds:

label=(0) $\dim_R(M)$ is countable.

lbbel=(0) $\ell^1(R) \sqsubseteq^R M$.

This seems to be new, even when R is a finite field, in which case $\ell^1(R) = R^{\mathbb{N}}$. This also implies a special case of [Mil12, Theorem 24], which says that if $\dim_R(M)$ is uncountable, then there is a linearly independent perfect set (see Chapter 5.5).

An analogous statement holds for a large class of discrete rings. A ring is **left-Noetherian** if every increasing sequence of left ideals stabilizes. Let R be a left-Noetherian discrete proper normed ring and let M be a Polish R-module. Then exactly one of the following holds:

label=(0) M is countable.

lbbel=(0) $\ell^1(S) \sqsubseteq^R M$ for some nonzero quotient S of R.

Note that this basis is countable since a countable left-Noetherian ring only has countably many left ideals.

For abelian Polish groups, we obtain an irreducible basis (see Chapter 5.4): Let A be an uncountable abelian Polish group. Then one of the following holds:

1. $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.
2. $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ for some prime p.

Related statements have been shown by Solecki, see [Sol99, Proposition 1.3, Theorem 1.7].

The theorems in Chapter 5.5 will be shown for a substantially broader class of modules. In order to contextualize this, we remark that considering even very basic module homomorphisms (for example, the inclusion of $\mathbb{Q}$ into $\mathbb{R}$ as $\mathbb{Q}$-vector spaces) naturally leads us to consider the broader class of quotients of Polish modules by sufficiently definable submodules. Such quotient modules are in general not Polish (they are not necessarily even standard Borel) but are still important objects of descriptive set-theoretic interest. They play a crucial role in [BLP20] in the form of "groups with a Polish cover", and they also form some of the most classical examples of countable Borel equivalence relations (for example, the commensurability relation on the positive reals naturally comes equipped with an abelian group structure). The embedding order on quotient modules will be defined analogously to the homomorphism reductions for Polish groups studied in [Ber14; Ber18].

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5.2 Polish modules

Most Polish modules which cannot be written as direct sums, even over a field: Let R be a Polish ring, let M be a Polish R-module, and let $(N_x)_{x\in\mathbb{R}}$ be a family of submodules of M such that the set $\{(m,x)\in M\times\mathbb{R} : m\in N_x\}$ is analytic. Then there are only countably many $x\in\mathbb{R}$ with N_x nontrivial, and only finitely many $x\in\mathbb{R}$ with N_x uncountable.

Proof. Let A_n be the set of $m\in M$ which can be written in the form $\sum_{i<n} m_i$ with each m_i is in some N_x. Then A_n is analytic, and thus Baire-measurable. Since $M=\bigcup_n A_n$, there is some A_n which is non-meager. By Pettis's lemma, we can replace n with $2n$ and assume that A_n has non-empty interior. Thus M can be covered by countably many translates of A_n.

Let $X\subseteq\mathbb{R}$ be the set of $x\in\mathbb{R}$ with N_x nontrivial.

Suppose that X is uncountable. For each $x\in X$, fix some nonzero $m_x\in M_x$. Fix an equivalence relation E on X with every class of cardinality $n+1$. Then there must be two E-classes C and C' such that $\sum_{x\in C} m_x$ and $\sum_{y\in C'} m_y$ are in the same translate of A_n. But then

$$\sum_{x\in C} m_x - \sum_{y\in C'} m_y \in A_n - A_n = A_{2n},$$

which is a contradiction. Thus X is countable.

Now $M=\bigcup_F \bigoplus_{x\in F} N_x$, where the union is taken over all finite subsets $F\subseteq X$, so since X is countable, there is some F for which $N_F=\bigoplus_{x\in F} N_x$ is non-meager, and thus open, since N_F is analytic. Then M/M_F is countable, so there are only finitely many $x \in \mathbb{R}$ with N_x uncountable. □

In particular, this implies an unpublished result of Ben Miller showing that an uncountable-dimensional Polish vector space does not have an analytic basis.

If $M \sqsubseteq^R N$ and $N \sqsubseteq^R M$, then we say that M and N are **bi-embeddable**. Note that if M and N are R-modules, and S is a subring of R, then $M \sqsubseteq^R N$ implies $M \sqsubseteq^S N$. In particular, if M and N are $\sqsubseteq^S$-incomparable, then they are $\sqsubseteq^R$-incomparable. In general, the preorder $\sqsubseteq^R$ can contain incomparable elements. For example, $\mathbb{R}$ is $\sqsubseteq^{\mathbb{Z}}$-incomparable with the p-adic rationals $\mathbb{Q}_p$, for any prime p. To see this, we have $\mathbb{R} \not\sqsubseteq^{\mathbb{Z}} \mathbb{Q}_p$ since $\mathbb{R}$ is connected, but $\mathbb{Q}_p$ is totally disconnected. On the other hand, $\mathbb{Q}_p \not\sqsubseteq^{\mathbb{Z}} \mathbb{R}$ since $\mathbb{Q}_p$ has a nontrivial compact subgroup, but $\mathbb{R}$ does not. So $\mathbb{R}$ and $\mathbb{Q}_p$ are $\sqsubseteq^{\mathbb{Z}}$-incomparable, and thus also $\sqsubseteq^{\mathbb{Q}}$-incomparable.

For certain rings, no locally compact module embeds into $R^{\mathbb{N}}$, and thus a minimum for $\sqsubseteq^R$ cannot be locally compact: Let R be a Polish ring with no nontrivial compact subgroups, and let M be a locally compact Polish R-module. If $M \sqsubseteq^R R^{\mathbb{N}}$, then M is countably generated.

Proof. Fix a continuous linear injection $f\colon M \hookrightarrow R^{\mathbb{N}}$. Since R has no nontrivial compact subgroups, the same holds for $R^{\mathbb{N}}$, and thus for M. Fix a complete norm $\|\cdot\|$ compatible with $(M,+)$. Let $\pi_n\colon R^{\mathbb{N}} \to R^n$ denote the projection to the first n coordinates, and let $M_n = \ker(\pi_n \circ f)$, which is a closed submodule of M. Fix such that the closed -ball around $0 \in M$ is compact, and let $C = \{m \in M : \frac{}{2} \leq \|m\| \leq\}$. Then $C \cap \bigcap_n M_n = \varnothing$, so since C is compact, there is some n such that $C \cap M_n = \varnothing$. We claim that M_n is discrete. To see this, suppose that the $\frac{}{2}$-ball around $0 \in M$ contained some nonzero $m \in M_n$. Then the subgroup generated by m is not compact, so there is a minimal $k \in \mathbb{N}$ with $\|km\| \geq \frac{}{2}$, and hence $km \in C$, which is not possible. Thus M_n is countable, so if we pick preimages $(m_i)_{i<n}$ in M of the standard basis of R^n, then M is generated by $M_n \cup (m_i)_{i<n}$, and thus countably generated. □

We do not know anything about the preorder $\sqsubseteq^R$ restricted to locally compact modules, including the existence of a minimum or maximum element.

If M_0 and M_1 are Polish R-modules with Baire-measurable submodules N_0 and N_1 respectively, we write $M_0/N_0 \sqsubseteq^R M_1/N_1$ if there is a continuous linear map

$M_0 \to M_1$ which descends to an injection $M_0/N_0 \hookrightarrow M_1/N_1$. This map is a Borel reduction of $E^{M_0}_{N_0}$ to $E^{M_1}_{N_1}$, where $E^{M_i}_{N_i}$ is the coset equivalence relation of N_i in M_i (see [Gao09] for background on Borel reductions). In particular, we have $\mathbb{R}/\mathbb{Q} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}$, since $E^{\mathbb{R}}_{\mathbb{Q}}$ is not smooth. We also have $\mathbb{R} \not\sqsubseteq^{\mathbb{Q}} \mathbb{R}/\mathbb{Q}$, since any nontrivial continuous linear map $\mathbb{R} \to \mathbb{R}$ is surjective, and thus $\mathbb{R}$ and $\mathbb{R}/\mathbb{Q}$ are $\sqsubseteq^{\mathbb{Q}}$-incomparable.

5.3 Proper normed rings

Every proper normed ring is locally compact and Polish. There are many examples of proper normed rings:

- The usual norms on $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ are proper.
- The p-adic norm on $\mathbb{Q}_p$ is proper.
- Every countable ring R admits a proper norm as follows. Let $w \colon R \to \mathbb{N}$ be a function such that $w(0) = 0$, $w(r) \geq 2$ if $r \neq 0$, and $w(r) = w(-r)$. We extend w to every term t in the language $(+, \cdot) \cup R$ by $w(r + s) = w(r) + w(s)$ and $w(r \cdot s) = w(r)w(s)$. Then let $|r|$ be the minimum of $w(t)$ over all terms t representing r.
- Let R be a proper normed ring. If $S \leq R$ is a closed subring, then there is a proper norm on S obtained by restricting the norm on R. If $I \triangleleft R$ is a closed two-sided ideal, then there is a proper norm on R/I given by $|r + I| = \min_{s \in r+I} |s|$.

In general, we do not know if every locally compact Polish ring admits a compatible proper norm.

Given a closed two-sided ideal $I \triangleleft R$, there is a natural quotient map $\ell^1(R) \twoheadrightarrow \ell^1(R/I)$ with kernel $\ell^1(I) := \ell^1(R) \cap I^{\mathbb{N}}$.

If R is finite proper normed ring, then $\ell^1(R) = R^{\mathbb{N}}$, which in particular is homeomorphic to Cantor space. For infinite discrete rings, there is also a unique homeomorphism type. Recall that **complete Erdős space** is the space of square-summable sequences of irrational numbers with the ℓ^2-norm topology. Let R be an infinite discrete proper normed ring. Then $\ell^1(R)$ is homeomorphic to complete Erdős space.

To show this, we will use a characterization due to Dijkstra and van Mill [DM09, Theorem 1.1]. A topological space is **zero-dimensional** if it is nonempty and it has a basis of clopen sets. [Dijkstra-van Mill] Let X be a separable metrizable space.

Then X is homeomorphic to complete Erdős space iff there is a zero-dimensional metrizable topology τ on X coarser than the original topology such that every point in X has a neighbourhood basis (for the original topology) consisting of closed nowhere dense Polish subspaces of (X, τ).

Proof of Chapter 5.3. We check the condition from Chapter 5.3. Let τ be the product topology on $R^{\mathbb{N}}$, which is zero-dimensional and metrizable. It is enough to show that every closed ball is a closed nowhere dense Polish subspace of $(\ell^1(R), \tau)$. By translation, it suffices to consider balls of the form $B = \{m \in \ell^1(R) : \|m\| \leq\}$. Note that B is closed in $R^{\mathbb{N}}$. Thus (B, τ) is Polish, and B is closed in $(\ell^1(R), \tau)$. It remains to show that the complement of B is dense in $(\ell^1(R), \tau)$. Let U be a nonempty open subset of $(\ell^1(R), \tau)$. We can assume that there is a finite sequence $(r_k)_{k<n}$ in R such that U is the set of sequences in $\ell^1(R)$ starting with $(r_k)_{k<n}$. Since R is infinite and the norm is proper, there is some $r \in R$ with $|r| > n!$. Then $(r_0, \ldots, r_{n-1}, r, 0, 0, 0, \ldots) \in U \setminus B$. ◻

5.4 Special cases

For a general Polish ring R, we do not know much about the preorder $\sqsubseteq^R$, including the following: Is there a maximum Polish R-module under $\sqsubseteq^R$?

This is known for some particular rings, which we mention below.

Principal ideal domains

Recall that a **principal ideal domain (PID)** is an integral domain in which every ideal is generated by a single element. There is an irreducible basis for uncountable Polish modules over a PID: Let R be a proper normed discrete PID and let M be a Polish R-module. Then exactly one of the following holds:

1. M is countable.

2. There a prime ideal $\mathfrak{p} \triangleleft R$ such that $\ell^1(R/\mathfrak{p}) \sqsubseteq^R M$.

Moreover, the $\ell^1(R/\mathfrak{p})$ are $\sqsubseteq^R$-incomparable for different $\mathfrak{p}$.

Proof. Suppose that M is not countable. By Chapter 5.1, there is some proper ideal $I \triangleleft R$ such that $\ell^1(R/I) \sqsubseteq^R M$. Then since R is a PID, there is some prime ideal $\mathfrak{p} \triangleleft R$ and some nonzero $s \in R$ such that $I = \mathfrak{p}s$. Then the linear injection $R/\mathfrak{p} \hookrightarrow R/I$ defined by $r \mapsto rs$ induces a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/I)$.

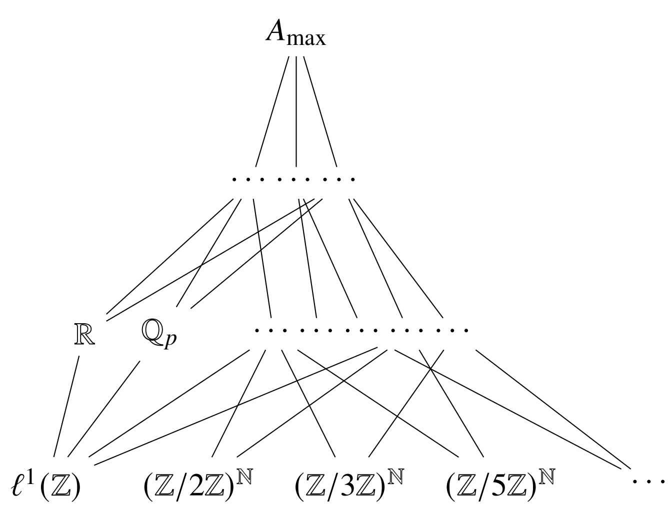
It remains to show that if $\mathfrak{p}$ and ‘’ are prime ideals with $\ell^1(R/\mathfrak{p}) \sqsubseteq^R \ell^1(R/‘)'$, then $\mathfrak{p}$ = ‘’. Fix a continuous linear injection $\ell^1(R/\mathfrak{p}) \hookrightarrow \ell^1(R/‘)'$. Since $R/\mathfrak{p}$ is an integral domain, the annihilator of any nonzero element of $\ell^1(R/\mathfrak{p})$ is $\mathfrak{p}$, and similarly for ‘’. Then for any nonzero $x \in \ell^1(R/\mathfrak{p})$, its image in $\ell^1(R/‘)'$ must have the same annihilator since the map is injective, and thus $\mathfrak{p}$ = ‘’. □

Abelian groups

Applying Chapter 5.4 with $R = \mathbb{Z}$ gives an irreducible basis for uncountable abelian groups: Let A be an uncountable abelian Polish group. Then one of the following holds:

1. $\ell^1(\mathbb{Z}) \sqsubseteq^{\mathbb{Z}} A$.
2. $(\mathbb{Z}/p\mathbb{Z})^{\mathbb{N}} \sqsubseteq^{\mathbb{Z}} A$ for some prime p.

By [Shk99], there is a $\sqsubseteq^{\mathbb{Z}}$-maximum abelian Polish group $A_{\max}$. So the preorder $\sqsubseteq^{\mathbb{Z}}$ on uncountable abelian Polish groups looks like the following:



$\mathbb{Q}$-vector spaces

Fix a proper norm on $\mathbb{Q}$. By Chapter 5.2, a $\sqsubseteq^{\mathbb{Q}}$-minimum uncountable Polish $\mathbb{Q}$-vector space cannot be locally compact. By Chapter 5.1, we have $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} \mathbb{R}$, where the strictness is due to $\ell^1(\mathbb{Q})$ being totally disconnected. However, it is open as to whether there is an intermediate vector space: Is there a Polish $\mathbb{Q}$-vector space V such that $\ell^1(\mathbb{Q}) \sqsubset^{\mathbb{Q}} V \sqsubset^{\mathbb{Q}} \mathbb{R}$?

Real vector spaces

We consider the order $\sqsubseteq^{\mathbb{R}}$ on uncountable-dimensional Polish $\mathbb{R}$-vector spaces. By Chapter 5.1, there is a minimum element $\ell^1(\mathbb{R})$, which is bi-embeddable with the usual space ℓ^1 of absolutely summable sequences. By Chapter 5.2, any uncountable-dimensional locally compact Polish $\mathbb{R}$-vector space must be strictly above ℓ^1. By [Kal77], there is a maximum Polish $\mathbb{R}$-vector space $V_{\max}$.

5.5 Proof of the main theorems

Every abelian Polish group A has a compatible complete norm defined by $\|a\| = d(a, 0)$, where d is an invariant metric on A (see [BK96, pp. 1.1.1, 1.2.2]). If $B \subseteq A$ is a Baire-measurable subgroup, then by Pettis's lemma, B is either open or meager (see [Kec95, p. 9.11]).

Setting $N = 0$ in the following theorem recovers Chapter 5.1. Let R be a proper normed division ring, let M be a Polish R-vector space, and let $N \subseteq M$ be an analytic vector subspace. Then exactly one of the following holds:

label=(0) $\dim_R(M/N)$ is countable.

lbbel=(0) $\ell^1(R) \sqsubseteq^R M/N$.

Proof. Suppose that the dimension of M/N is uncountable. Then N is not open, so N is meager, i.e., we have $N \subseteq \bigcup_k F_k$ for some increasing sequence $(F_k)_k$ of closed nowhere dense sets. Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. For every k, we define ${}_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R/I. We proceed by induction on k. Choose ${}_k > 0$ such that

label=() ${}_k < \frac{1}{2}{}_i$ for every $i < k$,

lbbel=() for every $(r_i)_{i<k}$ such that $\sum_{i<k} \frac{|r_i|}{i!} \leq k$ and there is some $l < k$ with $r_l = 1$ and $r_i = 0$ for $i < l$, the open ${}_k$-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

label=() $m_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$,

lbbel=() $\|rm_k\| < \frac{1}{2}{}_k$ whenever $\frac{|r|}{k!} \leq k$.

We verify that this is possible. When choosing ϵ_k, to satisfy the second condition, note that the set of considered $(r_i)_{i<k}$ is compact, so the set of $\sum_{i<k} r_i m_i$ is also compact, and it is disjoint from N (and hence F_k) by the choice of $(m_i)_{i<k}$. Thus such an ϵ_k must exist. When choosing m_k, note that the first condition holds for a comeager set of m_k, since $N + Rm_0 + Rm_1 \cdots + Rm_{k-1}$ is analytic, and it is not open, since otherwise M/N would have countable dimension. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \leq k$ is compact. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Let $n > l$ be sufficiently large such that $\sum_k \frac{|r_k|}{k!} \leq n$ and $0 \in F_n$. Then

$$_n \leq \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i} m_{n+i}\| < \frac{1}{2}{}_{n+i}$, and thus $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}{}_n$ by inductively using ${}_{k+1} < \frac{1}{2}{}_k$. Thus

$$\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}} \left\| \sum_{k<n} r_k m_k \right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\| \sum_k r_k m_k \right\| < 2 \left\| \sum_{k<n} r_k m_k \right\|.$$

It remains to show that the induced map $\ell^1(R) \to M/N$ is an injection. Let $(r_k)_k \in \ell^1(R)$ be nonzero. By scaling, we can assume that there is some l such that $r_l = 1$ and $r_i = 0$ for $i < l$. Suppose that $n > l$ is sufficiently large such that $\sum_k \frac{|r_k|}{k!} \leq n$. Since $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}{}_n$, we have $\sum_{i\geq 0} \|r_{n+i} m_{n+i}\| <_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. □

We recover [Mil12, Theorem 24] for proper normed division rings: [Miller] Let R be a proper normed division ring, and let M be a Polish R-module. If $\dim_R(M)$ is uncountable, then there is a linearly independent perfect subset of M.

Proof. By Chapter 5.1, we can assume that $M = \ell^1(R)$. Fix an enumeration $(q_n)_{n\in\mathbb{N}}$ of $\mathbb{Q}$. For every $x \in \mathbb{R}$, define $\chi_x \in \ell^1(R)$ by

$$(\chi_z)_n = \begin{cases} 1 & q_n < x \\ 0 & \text{otherwise} \end{cases}$$

Then $(\chi_x)_{x\in\mathbb{R}}$ is an uncountable linearly independent Borel subset of $\ell^1(R)$, so we are done by taking any perfect subset of this. □

There is an analogous generalization of Chapter 5.1.

Let R be a left-Noetherian discrete proper normed ring, let M be a Polish R-module, and let $N \subseteq M$ be a Baire-measurable submodule. Then exactly one of the following holds:

label=(0) M/N is countable.

lbbel=(0) $\ell^1(R)/\ell^1(I) \sqsubseteq^R M/N$ for some proper[2]two-sided ideal $I \triangleleft R$. In particular, there is a linear injection $\ell^1(R/I) \hookrightarrow M/N$.

Proof. Suppose that M/N is not countable. Then N is not open, and thus meager. Let $(U_k)_k$ be a descending neighborhood basis of $0 \in M$, and let $I_k = \{r \in R : rU_k \subseteq N\}$. Then $(I_k)_k$ is an increasing sequence of ideals, so since R is left-Noetherian, this sequence stabilizes at some $I = I_n$. Note that I is a proper ideal, since otherwise $U_n \subseteq N$, a contradiction to N being meager. Note also that I is a two-sided ideal, since if $r \in R$, then there is some $k > n$ with $rU_k \subseteq U_n$, and thus $IrU_k \subseteq IU_n \subseteq N$, and thus $Ir \subseteq I$. By replacing M with the submodule generated by U_n (which is analytic non-meager, and therefore open), we can assume that for every open $V \subseteq M$, we have $\{r \in R : rV \subseteq N\} = I$. Then for every $r \notin I$, the subgroup $\{m \in M : rm \subseteq N\}$ is not open, and therefore meager. Thus more generally, if $m' \in M$, then $\{m \in M : rm \in N + m'\}$ is meager.

Fix a complete norm $\|\cdot\|$ compatible with $(M, +)$. Let $(F_k)_k$ be an increasing sequence of closed nowhere dense sets with $N \subseteq \bigcup_k F_k$. For every k, we define ${}_k > 0$ and $m_k \in M$ such that the image of $(m_k)_k$ in M/N is linearly independent over R/I. We proceed by induction on k. Choose ${}_k > 0$ such that

[2]By proper, we mean a proper subset (no relation to proper norms).

label=() ${}_k < \frac{1}{2}{}_i$ for every $i < k$,

lbbel=() for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i$ nonzero and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, we have ${}_k \leq \|\sum_{i<k} r_i m_i\|$,

lcbel=() for every $(r_i)_{i<k}$ with $\sum_{i<k} r_i m_i \notin N$ and $\sum_{i<k} \frac{|r_i|}{i!} \leq k$, the open ${}_k$-ball centered at $\sum_{i<k} r_i m_i$ is disjoint from F_k.

Then choose $m_k \in M$ such that

label=() $rm_k \notin N + Rm_0 + Rm_1 + \cdots + Rm_{k-1}$ for every $r \notin I$,

lbbel=() $\|rm_k\| < \frac{1}{2}{}_k$ whenever $\frac{|r|}{k!} \leq k$.

We verify that this is possible. When choosing ϵ_k, for the second and third condition, there is only a finite set of $\sum_{i<k} r_i m_i$ to consider, and for the third condition, this set is disjoint from N, and hence from F_k. Thus such an ϵ_k must exist. When choosing m_k, for the first condition, for a fixed $r \notin I$ and $m' \in Rm_0 + \cdots + Rm_{k-1}$, we have shown earlier that $\{rm \notin N + m'\}$ is meager, so by quantifying over the countably many r and m', the set of m_k satisfying the first condition is comeager. The second condition holds for an open set of m_k, since the set of r with $\frac{|r|}{k!} \leq k$ is finite. Thus such an m_k must exist.

We define a map $\ell^1(R) \hookrightarrow M$ by

$$(r_k)_k \mapsto \sum_k r_k m_k.$$

First we show that this is well-defined, from which linearity and continuity are immediate. Let $(r_k)_k \in \ell^1(R)$. We can assume that there is some n such that $\sum_{k<n} r_k m_k$ is nonzero and $\sum_{k<n} \frac{|r_k|}{k!} \leq n$. Then

$${}_n \leq \left\| \sum_{k<n} r_k m_k \right\|.$$

For every i, we have $\|r_{n+i} m_{n+i}\| < \frac{1}{2}{}_{n+i}$, and thus $\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}}{}_n$ by inductively using ${}_{k+1} < \frac{1}{2}{}_k$. Thus

$$\|r_{n+i} m_{n+i}\| < \frac{1}{2^{i+1}} \left\| \sum_{k<n} r_k m_k \right\|.$$

Thus $\sum_k r_k m_k$ is well-defined with

$$\left\|\sum_k r_k m_k\right\| < 2\left\|\sum_{k<n} r_k m_k\right\|.$$

It remains to show that the kernel of the induced map $\ell^1(R) \to M/N$ is $\ell^1(I)$. The kernel clearly contains $\ell^1(I)$, since $IM \subseteq N$. Now let $(r_k)_k \in \ell^1(R) \setminus \ell^1(I)$. Since the image of $(r_k)_k$ in M/N is linearly independent over R/I, if n is sufficiently large, then $\sum_{k<n} r_k m_k \notin N$ and $\sum_k \frac{|r_k|}{k!} \leq n$. Since $\|r_{n+i}m_{n+i}\| < \frac{1}{2^{i+1}}{}_n$, we have $\sum_{i\geq 0} \|r_{n+i}m_{n+i}\| <_n$, and so $\sum_k r_k m_k \notin F_n$. This holds for all sufficiently large n, so $\sum_k r_k m_k \notin N$. □

example

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