PROBLEMS IN TRANSONIC FLOW

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1. INTRODUCTION AND SUMMARY

The main problems studied in this thesis deal with transonic flow, flow where the local speed of the gas is close to the local speed of sound. Such problems are difficult because non-linearity of the mathematical problems must be considered if the solutions are to resemble reality. Non-linearity enters the problem in different ways and several of these are considered here. The main aim of the research was a description of the flow past an airfoil through that range of Mach numbers where local supersonic zones and shock waves form on the airfoil. This problem cannot be regarded as solved. However, some progress has been made in that direction.

When gas streams past a curved surface the flow speed may exceed the local speed of sound. The non-linearity which enters here is in the acceleration terms like $u \frac{\partial u}{\partial x}$ and in the variation of the sonic speed and this is balanced by the change of area. When gas flows through a normal shock wave the flow speed also passes through the sonic speed. The non-linearity which is important here is the acceleration term $u \frac{\partial u}{\partial x}$ which tends to steepen the wave front and which, in this case, is balanced by the viscous stresses. When shock waves occur on an airfoil both effects are important, as well as the effect of the viscous layer near the boundary. This boundary layer is omitted in the present work although its effect may, in certain cases, be vital. A proper study of the boundary layer and its interaction with the "outer flow" depends on an understanding of the non-linear phenomena already mentioned.

In the first part of the work some general equations for transonic flow are derived. A law of similarity for these equations is presented and some comparisons made with experiments. Then, omitting shock waves, the first non-linear effect is studied in the boundary value problem for transonic potential flow past an airfoil. All the problems studied deal only with the local supersonic zone itself and some conclusions are reached. However, not much could be done about the important problem of the global flow picture, especially the location of these supersonic zones in arbitrary cases. The main mathematical problems which occur in the first part are questions of the existence and uniqueness of solutions for a non-linear equation of changing type. This has seldom been treated in the literature. The question of the height of the supersonic zone and the drag of a transonic airfoil is also treated briefly in the first part.

In the second part the non-linear effects in shock waves are studied. It is shown approximately how a shock wave approaches a steady state due to the non-linearity. It was planned then to study the presence of both types of non-linear effects for the flow past an airfoil but not too much progress has been made yet along these lines.

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2. FLOW PAST A THIN BODY

In this section some problems in steady flow are considered. The first aim is a derivation of approximate equations suitable for transonic flow and a check on their approximation. Next the properties of these equations are considered in some detail. Some viscous effects are also studied.

2.1 General Equations

The fundamental equations are statements of well-known physical principles. The starting point of the discussion here is the statement of the conservation of momentum in the Navier-Stokes equations. Hence, the assumptions in the derivation of these equations are implicit in the results. We consider, at first, steady, two-dimensional flow and later some remarks will be made about three-dimensional flow. The momentum equations for the x and y directions may be written:

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\frac{1}{3\rho}\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)\right] \quad \text{(a)}$$

$$(2.11)$$

$$u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}=-\frac{1}{\rho}\frac{\partial p}{\partial y}+\frac{1}{3\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}+\mu\frac{\partial v}{\partial y}\right)+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)\right] \quad \text{(b)}$$

where

u = velocity component in x-direction

v = velocity component in y-direction

p = pressure

ρ = density

μ = viscosity

The gas is also assumed to obey the perfect gas law

$$p = \rho R T \tag{2.12}$$

where

R = gas constant = $c_p - c_v$

T = absolute temperature

c_p = specific heat at constant pressure

c_v = specific heat at constant volume

In addition the conservation of matter expressed by the continuity equation

$$\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} = 0 \tag{2.13}$$

The first step will be to derive an equation of motion of a compressible viscous fluid in a form suitable for discussion.

Upon multiplying equation (2.11a) by u and (2.11b) by v and adding, one obtains,

$$\begin{aligned} u^2\frac{\partial u}{\partial x} + uv\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) + v^2\frac{\partial v}{\partial y} = & -\frac{1}{\rho}\left\{u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y}\right\} + \frac{4}{3\rho}\left[u\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right) + v\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)\right] \\ & + \frac{1}{\rho}\left\{u\left[\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) + \frac{1}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial y}\right)\right]\right. \\ & \left. + v\left[\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right) + \frac{1}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}\right)\right]\right\} \end{aligned} \tag{2.14}$$

The expression involving the pressure gradients may be eliminated from (2.14) by considering the equation for the conservation of energy and using the continuity equation. Assuming that no external heat is added to the system consisting of all the fluid, the energy equation is

$$u\frac{\partial E}{\partial x}+v\frac{\partial E}{\partial y}=-\frac{p}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\frac{1}{\rho}\chi+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right] \qquad (2.15)$$

where

E = internal energy = $c_v T$

k = specific thermal conductivity, and

χ = the dissipation function

$$\chi=\mu\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)^2\right]-\frac{2}{3}\mu\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)^2 \qquad (2.16)$$

Equations (2.12), (2.13), and (2.15) imply, assuming c_p and c_v are constant and writing $\gamma=\frac{c_p}{c_v}$,

$$\frac{1}{\rho}\left(u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}\right)=\frac{\gamma-1}{\rho}\chi+\frac{\gamma-1}{\rho}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]-\frac{\gamma p}{\rho}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) \qquad (2.17)$$

Hence, writing $a^2=\frac{\gamma p}{\rho}$, the velocity for an adiabatic disturbance, and using equation (2.17), equation (2.14) may be transformed to

$$\left(1-\frac{u^2}{a^2}\right)\frac{\partial u}{\partial x}-\frac{uv}{a^2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)+\left(1-\frac{v^2}{a^2}\right)\frac{\partial v}{\partial y}=\frac{\gamma-1}{\rho a^2}\chi+\frac{\gamma-1}{\rho a^2}\left[\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right)+\frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right)\right]$$
$$-\frac{4}{3\rho a^2}\left[u\frac{\partial}{\partial x}\left(\mu\frac{\partial u}{\partial x}\right)+v\frac{\partial}{\partial y}\left(\mu\frac{\partial v}{\partial y}\right)\right]$$
$$-\frac{1}{\rho a^2}\left[u\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right)+\frac{u}{3}\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial y}\right)+v\frac{\partial}{\partial x}\left(\mu\frac{\partial v}{\partial x}\right)+\frac{v}{3}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial x}\right)\right]$$

(2.18)

The left hand side of the equation above represents the dynamic terms as usually written while the right hand side represents the effect of viscosity and heat conduction. No problems for (2.18) will even be set up but simplifying assumptions will be made immediately.

2.2 Approximations for Transonic Flow Past a Thin Body

In this section approximate equations applicable to a special type of transonic flow will be developed. Specifically, the theory will be restricted to the flow past a thin airfoil which may cause a local supersonic region in a basically subsonic flow. In general there will exist shock waves in this region.

The most important parameter affecting the nature of the flow field at a point is the local Mach number, or the ratio of the flow speed to the sound speed at the point. For most of the flow field the local sound speed and the flow speed can be simply related by the well-known Bernoulli equation for compressible fluids

$$\frac{u^2+v^2}{2} + \frac{a^2}{\gamma-1} = \text{const.} = \frac{a^{*2}}{2}\,\frac{\gamma+1}{\gamma-1} \quad \text{(say)} \qquad (2.21)$$

In all cases a^* is well defined from conditions given at upstream infinity so that a^* = speed at which the local flow speed and sound speed are equal. Equation (2.21), a statement of the conservation of energy per unit mass, is derived by integrating equation (2.15) along a streamline in a region of the flow field where the dissipative terms are not important. Hence equation (2.21) can be written in another form

$$\frac{u^2+v^2}{2} + c_p T = c_p T_0 = \frac{a^{*2}}{2} + c_p T^* \qquad (2.21a)$$

Transonic flow is defined here as flow in which the local flow speed is everywhere close to a^*. The basic flow is uniform, directed along the x-axis and the body, close to the x-axis, is assumed to cause small disturbances. The assumption of transonic flow and a thin body may be introduced by restricting the velocity components so that

$$u(x,y) = a^* + u'(x,y)$$
$$v(x,y) = v'(x,y) \tag{2.22}$$

where $\frac{u'}{a^*}, \frac{v'}{a^*} \ll 1$. Using equation (2.21) and neglecting terms of higher order the Mach number M can easily be expressed as

$$M^2 - 1 = (\gamma+1)\frac{u'}{a^*} \tag{2.23}$$

The most important terms on the left hand side of equation (2.18) may be isolated by considering the character of flow near M = 1 past a thin body. An essential property is that in general $\frac{\partial u}{\partial x} \gg \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$. This property may be understood by considering the rapid growth of a supersonic zone in the y-direction as the Mach number at infinity is increased. In another way, in the limit as $M_\infty \rightarrow 1$ in supersonic flow, the linearized theory (Ref. 1, p. 142) indicates that the disturbance is propagated practically undamped to infinity in the y-direction but is restricted to a small width in the x-direction.

Furthermore, shock waves in transonic flow are necessarily weak. It is well-known that along any streamline the change in entropy through a normal shock is proportional to $(M_a^2 - 1)^3$ where M_a = Mach number ahead of the shock (Ref. 1, p. 41). Hence for transonic flow the change in entropy depends on $\left(\frac{u'}{a^*}\right)^3$. This means that difficulties due to changes in entropy can be neglected to the second order in $\frac{u'}{a^*}$ and outside of shock waves and boundary layers, the flow can be considered irrotational. Specifically, Crocco's vortex theorem (Ref. 2) implies

$$\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x} \sim \frac{dS}{d\psi} \sim \frac{dS}{dy} \qquad S = \text{entropy}, \quad \psi = \text{STREAM FUNCTION}$$

so that

$$\frac{\partial u'}{\partial y} = \frac{\partial v'}{\partial x} \tag{2.24}$$

Equation (2.24) also implies the existence of a perturbation velocity potential $\varphi(x,y)$, such that

$$u' = \frac{\partial \varphi}{\partial x}, \qquad v' = \frac{\partial \varphi}{\partial y} \tag{2.25}$$

The terms on the right hand side of equation (2.18) are of course unimportant in general except in shock waves or boundary layers. As a further simplification only one type of viscous effect will be considered, namely shock waves. Recent research (Ref. 3) has shown that longitudinal waves (shock waves) and transversal waves in a viscous fluid may to a certain extend be uncoupled and treated separately. The boundary layer represents to a large degree a transversal wave spreading into the fluid. For this work the boundary layer will be omitted entirely but it should be remembered that both theoretical and experimental work (Ref. 4) has shown a strong local interaction. Now make the following assumption about the structure of the shock. Viscosity and heat conduction effects are considered to be important in a region whose dimension Δ in the x-direction is much less than its dimension ℓ in the y-direction. ($\Delta \ll \ell$). Since in this region

$$\rho = \rho^* + O\left(\frac{u'}{a^*}\right) \qquad \mu = \mu^* + O\left(\frac{u'}{a^*}\right)$$

$$a^2 = a^{*2} + O\left(\frac{u'}{a^*}\right) \qquad k = k^* + O\left(\frac{u'}{a^*}\right)$$

Then, in the right hand side of equation (2.18) all the terms of the dissipation function χ and all of the other viscous terms can be seen to be much less than

$$\frac{4u}{3\rho a^2} \frac{\partial}{\partial x}\left(\mu \frac{\partial u}{\partial x}\right) \sim \frac{4}{3} \frac{\mu^*}{\rho^* a^* \Delta}\left(\frac{u'}{\Delta}\right)$$

In addition, since across the shock the change in T (see Eq. (2.21a) $\Delta T \sim \Delta u^2 \sim 2a^* \Delta u'$ a temperature term is seen to be of the same order of magnitude,

$$\frac{\gamma-1}{\rho a^2} \frac{\partial}{\partial x}\left(k \frac{\partial T}{\partial x}\right) \sim \frac{\gamma-1}{\rho^* a^*} \frac{k^*}{\Delta}\left(\frac{u'}{\Delta}\right)$$

Then, under all these assumptions and retaining only the dominant terms, equation (2.18) can be written

$$\frac{\gamma+1}{a^*} u' \frac{\partial u'}{\partial x} - \frac{\partial v'}{\partial y} = \frac{4}{3} \frac{\mu^*}{\rho^* a^*} \frac{\partial^2 u'}{\partial x^2} - \frac{(\gamma-1)k^*}{\rho^* a^{*2}} \frac{\partial^2 T}{\partial x^2} \qquad (2.26)$$

Summarizing, equation (2.26) may be considered a valid approximation under the following assumptions:

i) transonic flow

ii) thin body

iii) Navier-Stokes equation valid

iv) flow outside boundary layer

v) no heat addition

vi) shock wave restricted to a narrow region.

Heat conduction has qualitatively the same effect as viscosity and that is now also omitted for purposes of simplification. Then omitting the primes the basic system, under the assumptions above, is

$$\frac{\gamma+1}{a^*} u \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = \bar{\nu} \frac{\partial^2 u}{\partial x^2} \qquad \text{(a)}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad \text{(b)} \qquad (2.27)$$

where $\bar{\nu} = \frac{4}{3} \frac{\mu^*}{\rho^* a^*}$

In the case of shock free flow the system reduces to

$$\frac{\gamma+1}{a^*} u \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \qquad \text{(a)}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \qquad \text{(b)} \qquad (2.28)$$

or if a potential is introduced

$$\frac{\gamma+1}{a^*} \frac{\partial \varphi}{\partial x} \frac{\partial^2 \varphi}{\partial x^2} - \frac{\partial^2 \varphi}{\partial y^2} = 0 \qquad (2.29)$$

2.3 Discussion of Approximate Equations

In this and the next section it is shown that the approximate equations (2.27 - 2.28) should be adequate for a description of transonic flow phenomena under the assumptions made. This is done in two ways, first by discussing briefly the properties of these equations and secondly by deriving a law of similarity which is compared with experimental results.

The significance of (2.28) can be appreciated better by a comparison with the equation used in linearized theory of supersonic and subsonic motion:

$$(1-\bar{M}^2)\frac{\partial^2\bar{\varphi}}{\partial x^2}+\frac{\partial^2\bar{\varphi}}{\partial y^2}=0 \tag{2.31}$$

where $\bar{M}=\frac{U}{a_\infty}=$ the Mach number of infinity, a constant. It should be noted here that usually $\frac{\partial\bar{\varphi}}{\partial x}$ is taken to vanish at infinity, at least upstream, while $\frac{\partial\varphi}{\partial x}$ does not. The comparison between (2.28) and (2.31) can be made more striking by rewriting (2.29) with the aid of (2.23) as

$$(1-M^2)\frac{\partial^2\varphi}{\partial x^2}+\frac{\partial^2\varphi}{\partial y^2}=0 \tag{2.32}$$

where, of course, $M^2=M^2(x,y)$

Now the Mach lines or characteristics of equation (2.31) are given by the lines whose slope

$$\frac{dy}{dx}=\pm\frac{1}{\sqrt{\bar{M}^2-1}}$$

and are real if the basic flow is supersonic. This is typical of a hyperbolic equation; they are imaginary everywhere typical of an elliptic equation for $\bar{U} < a_\infty$. Partially supersonic and partially subsonic flow cannot be described by the linearized theory. It is clear that an equation for transonic flow must be non-linear like (2.29). The characteristics or Mach lines of such an equation must be real or imaginary depending on the magnitude of the velocity, i.e. the equation must be able to change type, depending on the velocity. The characteristics of equation (2.29) or (2.28) have the differential equation

$$\frac{dy}{dx} = \pm \frac{1}{\sqrt{M^2-1}} = \pm \sqrt{\frac{a^*}{(\gamma+1)\frac{\partial \varphi}{\partial x}}} \tag{2.33}$$

The characteristics are evidently real if $\frac{\partial \varphi}{\partial x} > 0$ and imaginary if $\frac{\partial \varphi}{\partial x} < 0$, as they should be. The non-linear equation (2.32) is thus the simplest potential equation which retains the essential features of transonic flow. Linear theory applies only locally in mixed flow. Of course, the slope of the characteristic at any point is just the Mach angle β for $\tan\beta = \frac{1}{\sqrt{M^2-1}}$, as can be seen from (2.33).

Another interesting contrast with the linearized theory is provided by the fact that any individual stream tube has a throat, as it should at M = 1, in the transonic theory; it cannot have such a throat in the linearized theory. This is essentially a consequence of replacing $1 - M^2$ by $1 - M_\infty^2$ in the one-dimensional continuity equation

$$\frac{1}{M^2-1}\,\frac{dA}{A} = \frac{dw}{w} \tag{2.34}$$

where

A = area of stream tube

w = velocity along stream tube

It is also related, in subsonic flow, to the representation of compressible flow as an affine transformation of incompressible flow in the linearized theory.

Thus we have shown that qualitatively the approximate equation should be valid. Next we can indicate some quantitative results both theoretical and experimental. First we can develop the relationships which are valid for simple waves (Ref. 5, p. 59) in supersonic flow as given by (2.28). These may be found by seeking the solutions of (2.28) for which u is a function of v alone

$$u = f(v) \tag{2.35}$$

(2.28) becomes

$$\frac{\gamma+1}{a^*} f(v) f'(v) v_x - v_y = 0$$

$$f'(v) v_y - v_x = 0$$

so that

$$\frac{\gamma+1}{a^*} f f'^2 = 1$$

and

$$v = C \pm \frac{2}{3} \sqrt{\frac{\gamma+1}{a^*}}\, u^{3/2} \tag{2.36}$$

This agrees with the hodograph epicycloid for supersonic simple waves (Ref. 5) when the Mach number is close to 1 and the deflection angles are small. Since, in a sense, a shock free local supersonic zone can be represented as a construction with these simple waves we conclude that the approximate equation (2.28) should quantitatively describe the flow in such a supersonic zone. This will be discussed in more detail later. Next we consider the system (2.28) in the large and make comparisons with experiments based on definite problems.

2.4 Laws of Similarity

This is done by the transonic similarity law. Laws of similarity are important for several reasons. They contribute to a better understanding of the problem by showing the relation between the various important parameters. They are also needed for comparison of experiments performed under different conditions. Laws of similarity were first given by Karman (Ref. 6) and Guderley (Ref. 7), and some aspects of the problem have been discussed by Graham (Ref. 8).

The derivation of the law of similarity can be regarded as an attempt to see what flows can be related to an already known flow by a class of linear transformations.

Suppose that $\varphi_1(x_1, y_1)$ is the known velocity potential for flow past a given body whose shape is

$$y_1 = \tau_1 f_1\left(\frac{x_1}{c_1}\right) \tag{2.41}$$

where

τ_1 = 1/2 thickness; f (o) = 1

f_1 = shape function

c_1 = 1/2 chord.

The origin is taken at the 1/2 chord point of the airfoil and the subscript 1 is used to indicate quantities connected with the "known" problem. It is assumed that the body is symmetric about y = 0. Then assuming shock free flow $\varphi_1(x_1, y_1)$ satisfies the equation (2.29)

$$\frac{\gamma+1}{q_1^*} \frac{\partial \varphi_1}{\partial x_1} \frac{\partial^2 \varphi_1}{\partial x_1^2} - \frac{\partial^2 \varphi_1}{\partial y_1^2} = 0 \tag{2.42}$$

and the following boundary conditions may be taken:

(i) Linearized boundary conditions of flow tangent to the body

$$\frac{\partial \varphi_1}{\partial y_1} = q_1^* \frac{\tau_1}{c_1} f_1'\left(\frac{x_1}{c_1}\right) \quad at \quad y_1 = 0 \tag{2.43}$$

where the prime denotes differentiation,

(ii) Uniform flow, parallel to the x-axis, at infinity

$$\frac{\partial \varphi_1}{\partial x_1} = -\frac{q_1^*}{\gamma+1}\left(1 - M_{1\infty}^2\right) \tag{2.44}$$

from (2.23).

The potential φ_1 is assumed to be determined uniquely by these conditions.* Now, let us determine the constants A, B, C which will relate a second potential φ_2 to the known potential φ_1 in the following way:

~~$x_2 = Bx_1$~~

~~$y_2 = Cy_1$~~

$$\varphi_2(x_2, y_2) = A\varphi_1(x_1, y_1) \tag{2.45}$$

*This is a touchy point for mixed flows which is discussed later. However, as applied, visual evidence indicates flow of a similar nature.

Corresponding points are defined by $\begin{aligned} x_2 &= Bx_1 \\ y_2 &= Cy_1 \end{aligned}$

Let us see what problem is solved by the new potential φ_2 .

We have

$$\frac{\partial^n \varphi_1}{\partial x_1^n} = \frac{B^n}{A} \frac{\partial^n \varphi_2}{\partial x_2^n} \tag{2.46}$$

and

$$\frac{\partial^n \varphi_1}{\partial y_1^n} = \frac{C^n}{A} \frac{\partial^n \varphi_2}{\partial y_2^n}$$

so that (2.42) becomes

$$\frac{\gamma+1}{a_1^*} \frac{B}{A} \frac{\partial \varphi_2}{\partial x_2} \frac{B^2}{A} \frac{\partial^2 \varphi_2}{\partial x_2^2} - \frac{C^2}{A} \frac{\partial^2 \varphi_2}{\partial y_2^2} = 0 \tag{2.47}$$

In order that φ_2 satisfies an equation similar to that satisfied by φ_1 (2.42) it is necessary that

$$\frac{a_1^*}{a_2^*} \frac{AC^2}{B^3} = 1 \tag{2.48}$$

φ_2 may be considered to describe flow past a second body related to the first by a transformation of the boundary condition at ($y_1 = y_2 = 0$). Then (2.43) transforms to

$$\frac{\partial \varphi_1}{\partial y_1} = \frac{C}{A} \frac{\partial \varphi_2}{\partial y_2} = a_1^* \frac{\tau_1}{c_1} f_1'\left(\frac{x_1}{c_1}\right) = \frac{C}{A} a_2^* \frac{\tau_2}{c_2} f_2'\left(\frac{x_2}{c_2}\right) \quad \text{at } y_1 = y_2 = 0$$

or

$$\frac{A}{C} = \frac{a_2^*}{a_1^*} \frac{\tau_2}{\tau_1} \frac{c_1}{c_2} \frac{f_2'\left(\frac{x_2}{c_2}\right)}{f_1'\left(\frac{x_1}{c_1}\right)} \tag{2.49}$$

if $y_2 = \tau_2 f_2\left(\frac{x_2}{c_2}\right)$ gives the new body shape. In addition the flow at infinity must be altered. Equation (2.44) becomes

$$\frac{\partial \varphi_1}{\partial x_1} = \frac{B}{A}\frac{\partial \varphi_2}{\partial x_2} = -\frac{a_1^*}{(\gamma+1)}(1-M_{1\infty}^2) = -\frac{B}{A}\frac{a_2^*}{\gamma+1}(1-M_{2\infty}^2)$$

or

$$\frac{A}{B} = \frac{a_2^*}{a_1^*}\frac{1-M_{2\infty}^2}{1-M_{1\infty}^2} \tag{2.410}$$

Equations (2.48), (2.49) and (2.410) give the conditions for similarity. They tell how the potential, space coordinate, body shape, Mach number at infinity, and total enthalpy can be altered to produce similar flow; of course comparisons can be made in various ways. These conditions having been satisfied corresponding points (x_1, x_1) and (x_2, y_2) are defined and the properties of the two flows at these points can be related. For example,

$$\frac{a_2^*}{a_1^*}\frac{1-M_2^2(x_2,y_2)}{1-M_1^2(x_1,y_1)} = \frac{A}{B} \tag{2.411}$$

In general, the two flows will have different directions and magnitudes at the corresponding points.

Now, for simplicity, consider the case of $a_2^* = a_1^*$, $B = 1$. The constants may then be determined as

$$A = \frac{1-M_{2\infty}^2}{1-M_{1\infty}^2} \quad ; \quad C = \sqrt{\frac{1-M_{1\infty}^2}{1-M_{2\infty}^2}}$$

and the condition for similar flows then demands

$$\left(\frac{1-M_{2\infty}^2}{1-M_{1\infty}^2}\right)^{3/2} = \frac{\tau_2}{\tau_1}\frac{c_1}{\tau_2}\frac{f_2'\left(\frac{x_2}{c_2}\right)}{f_1'\left(\frac{x_1}{c_1}\right)} \tag{2.412}$$

From this, two important points may be noticed. First, since τ, c

are constants, comparisons can only be made when the slopes of the bodies at corresponding points are in a constant ratio. This can be done when the slopes depend on the same power of x. Second, a simultaneous variation of Mach number and body shape must occur if any comparison is to be made. Picking the same body necessitates the same condition at infinity and the comparison is trivial, i.e. identical flow pattern.

Then a simple set of bodies that can be compared are aifoils of the same shape functions, i.e. $f_2'\left(\frac{x_2}{c_2}\right) \equiv f_1'\left(\frac{x_1}{c_1}\right)$, and of the same chord $c_1 = c_2$. If the condition for similarity (2.412), is satisfied

$$A = \frac{1 - M_{2\infty}^2}{1 - M_{1\infty}^2} = \left(\frac{\tau_2}{\tau_1}\right)^{2/3} \tag{2.413}$$

corresponding points are given by

$$\begin{aligned} x_2 &= x_1 \\ y_2 &= y_1 \sqrt{\frac{1 - M_{1\infty}^2}{1 - M_{2\infty}^2}} \end{aligned} \tag{2.414}$$

and then the Mach numbers at corresponding points are related by

$$\frac{1 - M_1^2(x_1, y_1)}{1 - M_2^2(x_2, y_2)} = \frac{1 - M_{1\infty}^2}{1 - M_{2\infty}^2} \tag{2.415}$$

Summarizing, equations (2.413), (2.414) and (2.415) can be taken as the law of similarity for a case of

(i) flow outside viscous region

(ii) $a_1^* = a_2^*$, same total enthalpy

(iii) $c_1 = c_2$, same chord

(iv) $f_1\left(\frac{x_1}{c_1}\right) = f_2\left(\frac{x_2}{c_2}\right)$, same shape function

The transonic similarity as developed in (2.413 - 2.415) can be checked against experiments in several ways. Experimental measurements (Ref. 4) of the Mach number decay above the maximum thickness of 6% and 12% circular arc airfoils and the points are plotted in Figures 1 and 2. By regarding the flow past the 6% airfoil as known (φ_1), decay curves for the 12% airfoil (φ_2) can be computed. The details of the computation are given below.

The conditions for the similarity law as developed in (2.413-2.415) are met. $a_1^* = a_2^*$, $c_1 = c_2 = 1\ 1/2"$ and in accordance with the linearized boundary conditions $f_1\left(\frac{x_1}{c_1}\right) = f_2\left(\frac{x_2}{c_2}\right) = -2\frac{x}{c}$ for circular arc airfoils. However since M_∞ is not well-defined in the wind tunnel, the two flows are linked by comparing the Mach numbers at the points of maximum thickness. This can be done since the origin remains fixed in the transformation. Then

$$A = \left(\frac{\tau_2}{\tau_1}\right)^{2/3} = \left(\frac{0.12}{0.06}\right)^{2/3} = 1.587 = \frac{1 - M_2^2(x_2, y_2)}{1 - M_1^2(x_1, y_1)} = \frac{1 - M_{1\infty}^2}{1 - M_{2\infty}^2}$$

The corresponding points are given by

$$x_2 = x_1$$

$$y_2 = y_1 \cdot \sqrt{\frac{1 - M_{1\infty}^2}{1 - M_{2\infty}^2}} = y_1 \left(\frac{\tau_1}{\tau_2}\right)^{\frac{1}{3}} = 0.794\, y_1$$

Curves were faired through the experimental points for the 6% airfoil and values of M_1 and y_1 were read off. Values of M_2 and y_2 on the decay for the 12% airfoil were then computed.

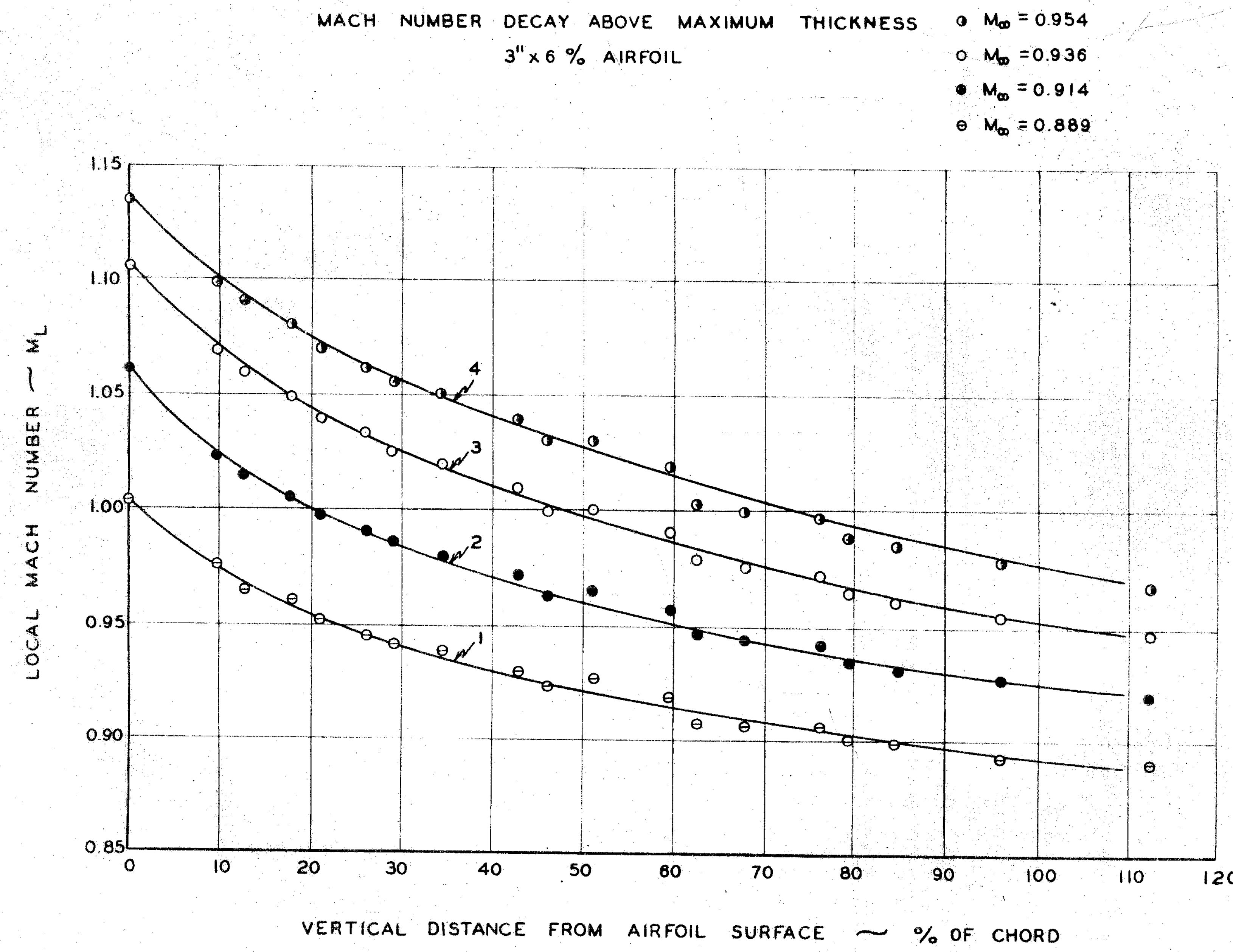

MACH NUMBER DECAY ABOVE MAXIMUM THICKNESS
3" x 6 % AIRFOIL
M∞ = 0.954
M∞ = 0.936
M∞ = 0.914
M∞ = 0.889
LOCAL MACH NUMBER ~ M_L
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0
10
20
30
40
50
60
70
80
90
100
110
120
4
3
2
1
VERTICAL DISTANCE FROM AIRFOIL SURFACE ~ % OF CHORD

FIGURE I

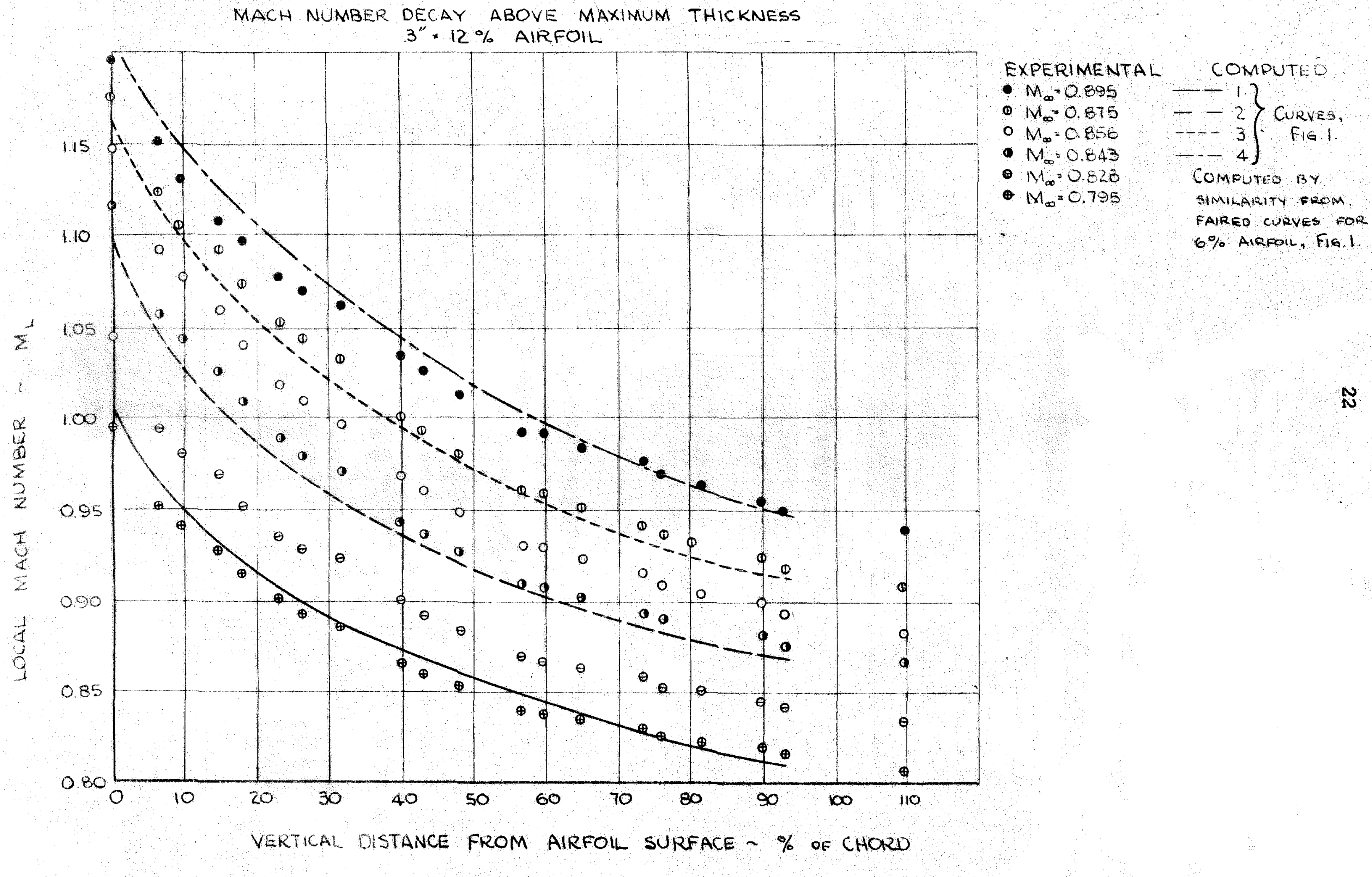

MACH NUMBER DECAY ABOVE MAXIMUM THICKNESS
3" × 12 % AIRFOIL
EXPERIMENTAL
M_∞ = 0.895
M_∞ = 0.875
M_∞ = 0.856
M_∞ = 0.843
M_∞ = 0.828
M_∞ = 0.795
COMPUTED
1
2
3
4
CURVES, FIG. 1.
COMPUTED BY SIMILARITY FROM FAIRED CURVES FOR 6% AIRFOIL, FIG. 1.
LOCAL MACH NUMBER ~ M_L
1.15
1.10
1.05
1.00
0.95
0.90
0.85
0.80
0
10
20
30
40
50
60
70
80
90
100
110
VERTICAL DISTANCE FROM AIRFOIL SURFACE ~ % OF CHORD

FIGURE 2

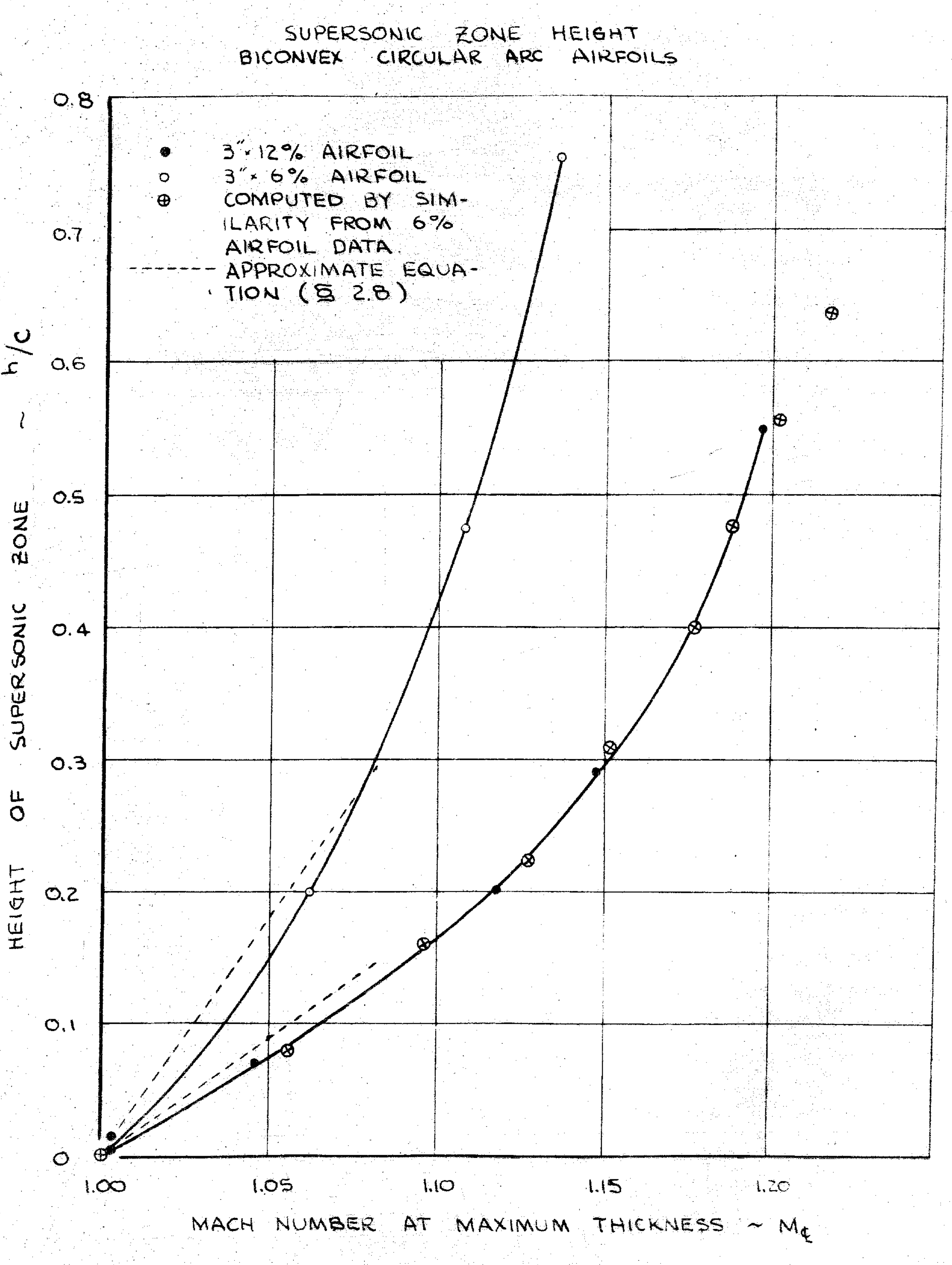

SUPERSONIC ZONE HEIGHT
BICONVEX CIRCULAR ARC AIRFOILS
3" × 12% AIRFOIL
3" × 6% AIRFOIL
COMPUTED BY SIMILARITY FROM 6% AIRFOIL DATA
APPROXIMATE EQUATION (§ 2.8)
HEIGHT OF SUPERSONIC ZONE ~ h/c
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
1.00
1.05
1.10
1.15
1.20
MACH NUMBER AT MAXIMUM THICKNESS ~ M℄

FIGURE 3

The computed curves are plotted in Figure 2 and show good agreement with the experimental decay curves. The height of the supersonic zone over the maximum thickness was checked in the same way and is given in Figure 3.

Extension of Transonic Similarity to Flow with Shock Waves

The similarity law developed in (2.48 - 2.410) can be extended to flow with shock waves, under the assumptions previously mentioned, by considering (2.27). If the potential is introduced we have

$$\frac{\gamma+1}{a^*}\frac{\partial\varphi}{\partial x}\frac{\partial^2\varphi}{\partial x^2} - \frac{\partial^2\varphi}{\partial y^2} = \bar{\nu}\frac{\partial^3\varphi}{\partial x^3} \tag{2.416}$$

Then, following (2.45) we obtain, in addition to (2.48-2.410), the condition that

$$A\frac{q_1^*}{q_2^*}\bar{\nu}_1 = \bar{\nu}_2 \tag{2.417}$$

Thus if the viscosity may be varied comparisons may be made in various ways, as before. However, in the usual case

$$\nu_1^* = \nu_2^* \quad , \quad q_1^* = q_2^* \tag{2.418}$$

so that it is necessary that A = 1. This implies that

$$B = \frac{1-M_{1\infty}^2}{1-M_{2\infty}^2} \quad \text{or} \quad x_2 = \frac{1-M_{1\infty}^2}{1-M_{2\infty}^2}x_1 \tag{2.410}$$

a distortion of the x-direction is necessary for similarity. Also, in the usual case the shape is given by the same function so that the chord must be correspondingly distorted. This is needed since $\frac{\partial\varphi}{\partial y} = 0$ at y = 0 ahead and behind of the airfoil. Thus

$$\frac{c_1}{c_2}\,\frac{1-M_{1\infty}^2}{1-M_{2\infty}^2} = 1 \tag{2.420}$$

so that

$$\frac{x_2}{c_2} = \frac{x_1}{c_1} \tag{2.421}$$

Then

$$C = \frac{\tau_1}{\tau_2}\,\frac{c_2}{c_1} = \frac{\tau_1}{\tau_2}\,\frac{1-M_{1\infty}^2}{1-M_{2\infty}^2} \tag{2.422}$$

Finally the condition for similarity (2.48) shows

$$\frac{\tau_1}{\tau_2} = \sqrt{\frac{1-M_{1\infty}^2}{1-M_{2\infty}^2}} \tag{2.423a}$$

or

$$\frac{\tau_1/c_1}{\tau_2/c_2} = \left(\frac{1-M_{1\infty}^2}{1-M_{2\infty}^2}\right)^{3/2} \tag{2.423b}$$

Corresponding points are now defined and properties at these may be compared. The law of similarity (2.423b) is the same as before (2.413), but it should be remembered that the length scale in the flow direction is altered. This distortion is consistent with the possibility of introducing a characteristic length, e.g. $\frac{\nu^*}{a^*}$, when a real fluid is considered. Some similarity of thé ratio of shock wave thickness to chord is needed. Experiments have not yet been carried out which permit an evaluation of this law. If any comparisons are made, however, the state of the boundary layer will also have to be taken into account (Ref. 4).

Further extension of the similarity law to three-dimensional flows and axially symmetric flows can be made and the results are qualitatively the same. They seem to emphasize the essential non-linearity of the problem. They also indicate that the transonic approximation should give quantitatively correct results for a reasonable range of Mach numbers close to M = 1. It was also seen that the non-viscous behavior should hold, at least up to the maximum thickness of the airfoil.

2.5 Potential Flow

It was shown in the previous section that potential transonic flow should have some significance. Experiments (Ref. 4) have indicated a regime in which shock-free transonic flow exists. Therefore it is desirable to find solutions for transonic potential flow, with local supersonic zones past given bodies and to discuss their meaning. This is difficult for two reasons. First, we are dealing with an equation which is partly elliptic (subsonic) and partly hyperbolic (supersonic) and the proper prescription of conditions to determine a unique solution to the mathematical problem is now known. Also the existence of such a solution is an open question. Secondly, the equations are non-linear so that representations cannot be built up on a superposition principle. Some solutions have been given however (Ref. 9), but these all determine the body shape after starting with a known solution in the hodograph. Hence they can never discover, except perhaps by inspection, what additional conditions (as contrasted to purely subsonic flow) are implicit in the physical problem. An additional difficulty is that it is not known whether potential solutions can depend continuously on the given data. Guderly (Ref. 7) has attempted to show that it is not possible, but the details of his proof are obscure.

For these reasons practically no special problems will be considered here. Instead some very general properties of supersonic gas flows and of gas flows in local supersonic zones will be discussed. The basic concepts used are found in Ref. 5 and some of the results appear in Nikolsky and Tagonoff (Ref. 10).

General Properties of Supersonic Flow

In the following work the transonic approximation will be used exclusively, for the sake of simplicity. Almost all of the results can be derived on the basis of the exact equations of motion. The starting point is thus (2.28)

$$\frac{\gamma+1}{a^*} u \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0 \tag{2.28a}$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0 \tag{2.28b}$$

Now consider a pair of functions u(x,y), v(x,y) in a domain $\mathcal{D}$ of the (x,y) plane. Then to each point P(x,y) there corresponds one point P(u,v) in the hodograph or (u,v) plane. We now study the local mapping from x-y plane to u-v plane in the neighborhood of P. If we consider a neighboring point $P_1(x + dx, y + dy)$ this will map on to a point $P_1(u_1, v_1)$ defined by

$$u_1 = u(x+dx, y+dy) = u(x,y) + (u_x)_P dx + (u_y)_P dy \quad + \text{ smaller terms}$$

$$v_1 = v(x+dx, y+dy) = v(x,y) + (v_x)_P\, dx + (v_y)_P\, dy \quad \text{+ smaller terms}$$

In the limit the smaller terms vanish and the infinitesimal vector PP_1 maps to $\overline{PP_1}$. It has been assumed that u,v are differentiable at P in $\mathcal{D}$. We thus obtain the mapping of infinitesimal vector (dx,dy) into (du,dv)

$$(du) = (u_x)_P\, dx + (u_y)_P\, dy \tag{2.51a}$$

$$(dv) = (v_x)_P\, dx + (v_y)_P\, dy \tag{2.51b}$$

or, the slope of the mapped vector

$$\left(\frac{dv}{du}\right)_{\overline{P}} = \frac{(v_x)_P + (v_y)_P \left(\frac{dy}{dx}\right)_P}{(u_x)_P + (u_y)_P \left(\frac{dy}{dx}\right)_P} \tag{2.52}$$

(2.52) will be used to discuss the mapping of various local vectors under the assumption that: the mapping functions satisfy the equations of motion in $\mathcal{D}$. When the flow is supersonic two distinguished directions at P are the characteristic directions (Mach lines).

$$\frac{dy}{dx} = \pm \sqrt{\frac{a^*}{(\gamma+1)u_P}} = \pm \frac{1}{\sqrt{M^2-1}} \tag{2.53}$$

The streamlines at P are in the first approximation the lines y = constant of the undisturbed flow and these bisect the angle between the 2 characteristics ($\pm$) given by (2.53). It is a general result (Ref. 5) that the streamlines bisect the angle between the characteristics in the physical plane. According to (2.52) the characteristic directions map into

$$\frac{dv}{du} = \frac{v_x \pm v_y \sqrt{\frac{a^*}{(\gamma+1)u}}}{u_x \pm u_y \sqrt{\frac{a^*}{(\gamma+1)u}}}$$

or using (2.28)

$$\frac{dv}{du} = \pm\sqrt{\frac{\gamma+1}{a^*}u} \tag{2.54}$$

Thus a characteristic of one family (+) maps into a direction orthogonal to the characteristic of the other family (-). The characteristic direction in the hodograph may be imbedded in a curve integrating (2.54).

$$V = c \pm \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\, u^{3/2} ; \quad C = \text{const.} \tag{2.55}$$

As will be shown later (2.55) is the equation of the characteristics of the hodograph equations for the flow. Hence we have shown that locally, characteristic directions in the physical plane map to characteristic directions in the hodograph. To show that one characteristic in the physical plane maps into one characteristic in hodograph, i.e. that the constant c is the same, we need to approximate the characteristic in the physical plane by a series of segments and use a covering theorem. We use here the result that: a characteristic in the physical plane maps into a definite characteristic in the hodograph and the equation (2.55) with a fixed constant is valid along both these characteristics.

The mapping of equation (2.55) to the hodograph depends on the possibility of interchanging dependent and independent variables and leads to the linear equations

$$\frac{\gamma+1}{a^*} u y_v - x_u = 0 \tag{2.56a}$$

$$x_v - y_u = 0 \tag{2.56b}$$

This is always possible so long as the Jacobian

$$j = u_x v_y - v_x u_y = \frac{\partial(u,v)}{\partial(x,y)} \tag{2.57}$$

does not vanish, since (Ref. 5)

$$u_x = j y_v \quad , \quad v_x = -j y_u$$
$$u_y = -j x_v \qquad v_y = j x_u \tag{2.58}$$

j represents a directed ratio of the local areas in the hodograph to the local area in the physical plane. This may be seen by considering the map of vector product of $\vec{du}$ and $\vec{dv}$ at P.

$$\vec{du} \times \vec{dv} = (u_x \vec{dx} + u_y \vec{dy}) \times (v_x \vec{dx} + v_y \vec{dy})$$

$$= j \, (\vec{dx} \times \vec{dy})$$

Thus if

$j < 0$ the mapping is order reversing

$j > 0$ the mapping is order preserving

Similarly in hodograph

$$\vec{dx} \times \vec{dy} = J \, \vec{du} \times \vec{dv}$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = x_u y_v - x_v y_u$$

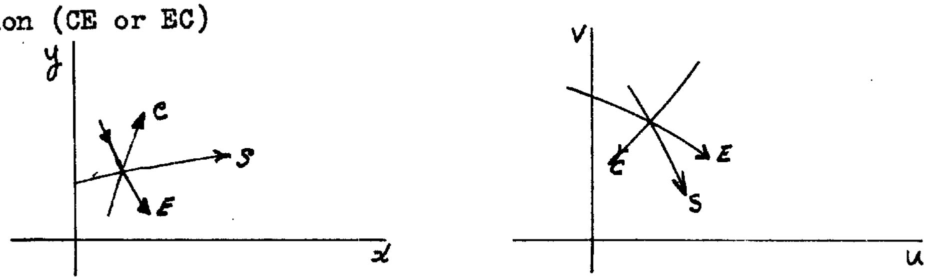
J represents ratio of local area in physical plane to local area in hodograph. It can be shown that J, j < 0 for M < 1 (except for constant flow) but that they may change sign if M > 1. When J = 0 for a solution of (2,56) along a curve in the hodograph the mapping of (u,v) plane may form a fold in (x,y) plane and the edge of the fold is called a limiting line. Limiting lines have been studied in much detail (Ref. 1, 5). All we need to know is that solutions, starting in the hodograph, which show this represent physically impossible flow patterns and that: the critical curve in (u,v) plane which maps into a limiting line can be described by the condition that the images of the streamlines in the hodograph pass through it in a characteristic direction. The characteristic directions in the hodograph can be found from (2.56) and are known in advance since the equations are linear. According to the usual methods the characteristic differential equation for the system (2.54) and the characteristics are given by the semi-cubical parabolas of (2.55).

The direction (of flow) along the streamline in the physical plane is taken as a time-like direction and the direction of the two characteristics at each point in the physical plane is prescribed as downstream. These characteristics bound the region of influence in the initial instants, of a small disturbance introduced into the stationary flow. A time-like direction is thus fixed for the image of the streamline in the hodograph and then the direction of the characteristics in the hodograph is known. Hence, locally, the characteristic in the hodograph can be identified as expansion (E) i.e. increasing velocity or compression (C). This fixes the characteristics in the physical plane by the mapping.

Three types of flow may now be distinguished locally (of course flow of the same type may fill a region):

(a) <u>Source-like Flow:</u> The characteristics leaving a streamline (S) are here both of same type, either both compression (CC) or both expansion (EE). A typical example is shown below.

(b) <u>Vortex-like Flow:</u> The characteristics leaving a streamline are of different types; one carries compression and the other expansion (CE or EC)



(c) <u>Characteristic-like Flow:</u> In the hodograph the streamline S lies on a characteristic; thus S crosses only one set of characteristics, either E or C. This is Prandtl-Meyer Flow or the simple waves of Ref. 5.

Exact solutions of these three types of flow exist showing that each type represents a possibility. In all the discussion so far, except (c), a local bi-uniqueness of the mapping from hodograph to physical plane was assumed. There is in general no bi-uniqueness in the large. For a physical problem the flow in the hodograph may have to be represented on two or an infinite number of sheets. The solutions may have to have certain singularities and folds in the hodograph may also be needed. The problems of the solution in the large will not be studied here and the main attention will be to potential flow in a local supersonic zone near an airfoil.

2.6 Properties of Potential Flow in a Local Supersonic Zone

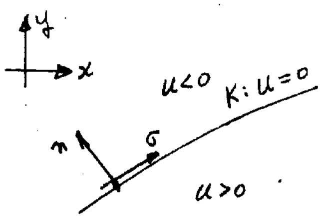
In this section it is assumed that there exists a region of shock-free local supersonic flow and the properties of this flow are studied. In certain cases it will be shown that the assumption of the existence of such a supersonic zone leads to contradictions.

To be specific, assume there are two continuous and piece-wise differentiable functions $u(x,y)$, $v(x,y)$ satisfying (2.28) and the conditions of flow tangent to a given body. Assume also there exists a bounded region $\mathcal{R}$, whose boundary is a simple closed curve, in which $u > 0$, and another region in which $u < 0$. The curve K: $u = 0$ which forms part wall of the boundary of $\mathcal{R}$ is then rectifiable and simple. (The case where $u = 0$ over a region is ruled out.) The body B is assumed to be a thin airfoil lying close to the x-axis. For purposes of the mathematical boundary value problem the airfoil coincides with a segment of the x-axis.

I. <u>$v(x,y)$ is a monotonic decreasing function of the arc length on the sonic line $u = 0$ going in a clockwise direction about the</u>

region of higher velocity.

Consider the sonic line K represented parametrically by $x = x(\sigma)$, $y = y(\sigma)$ where σ is the arc-length along the line. Consider



$$\frac{dv}{d\sigma} = v_x \frac{dx}{d\sigma} + v_y \frac{dy}{d\sigma} = u_y \frac{dx}{d\sigma} = u_y \frac{\partial y}{\partial n} \tag{2.61}$$

by using (2.28). Hence

$$\frac{dv}{d\sigma} \leq 0 \quad \text{on } K. \tag{2.62}$$

In the figure, $u_y < 0$ and $x_\sigma > 0$. It is also possible to have $u_y > 0$ but then $x_\sigma < 0$ since the direction of increasing σ is clockwise about the region of higher velocity. If the equality in (2.62) holds at any point then either $\frac{\partial y}{\partial n} = 0$ or $u_y = 0$ at that point. If $\frac{dy}{dn} \neq 0$ and $u_y = 0$ along a segment of K and if we assume the solution is analytic in a neighborhood of K it follows easily that u_x, v_x, v_y and all higher derivatives vanish in that neighborhood. Under those assumptions $u_y = 0$ leads to a contradiction. However, the assumption of analyticity is hot always plausible so that it is assumed here that $J \neq 0$ on the sonic line.

$$J = u_x v_y - u_y v_x = \frac{\partial H}{\partial a^*} u u_x^2 - u_y^2 \neq 0, \quad \text{on } K \quad J = -u_y^2$$

Hence $u_y \neq 0$. Now if $\frac{dy}{dn} = 0$ we have $u_x \frac{dy}{dn} = u_y \frac{dx}{dn}$ so $u_y = 0$. Hence $\frac{dy}{dn} = 0$ is also ruled out by the assumption $J \neq 0$ and the sonic line cannot be vertical. Thus, in the following work we use

$$\frac{dv}{d\sigma} < 0 \quad \text{on } K \tag{2.62a}$$

It is interesting also to consider a curve K_1, on which $u = u_1$ = constant, which is assumed to separate a region of higher velocity from a region of lower velocity. Then along K_1

$$\frac{dv}{d\sigma} = v_x \frac{dx}{d\sigma} + v_y \frac{dy}{d\sigma} = u_y \frac{dx}{d\sigma} + \left(\frac{\gamma+1}{a^*} u_1\right) u_x \frac{dy}{d\sigma}$$

and

$$\frac{du}{d\sigma} = 0 = u_x \frac{dx}{d\sigma} + u_y \frac{dy}{d\sigma}$$

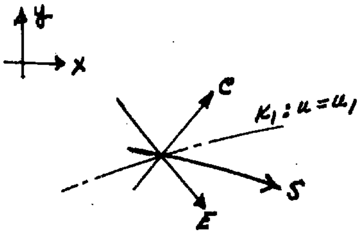
or

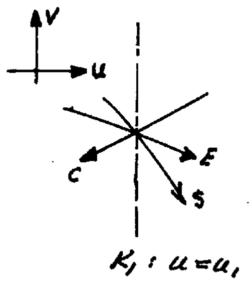
$$\frac{dv}{d\sigma} = \frac{u_y}{\frac{dx}{d\sigma}} \left\{ \left(\frac{dx}{d\sigma}\right)^2 - \left(\frac{\gamma+1}{a^*} u_1\right)\left(\frac{dy}{d\sigma}\right)^2 \right\} \quad \text{on } K_1 \tag{2.63}$$

Hence if $u_1 \le 0$ subsonic or sonic the same conclusion prevails that

$$\frac{dv}{d\sigma} \le 0 \quad \text{along} \quad K_1 : u = u_1 \tag{2.64}$$

If the flow is supersonic (2.64) persists only if the flow is vortex-like. In this case the line K_1: $u = u_1$ and the streamline S both lie in the same quadrant between the characteristics leaving a point when the flow is vortex like (see Fig.). Hence, the slope of K_1





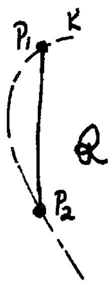
may be bounded in the physical plane as

$$\left|\frac{dy}{dx}\right| = \left|\frac{\frac{dy}{d\sigma}}{\frac{dx}{d\sigma}}\right| < \frac{1}{\sqrt{M^2-1}} = \frac{1}{\sqrt{\frac{\gamma+1}{a^*}u_1}}$$

This inequality combined with (2.63) implies (2.64).

II. <u>The boundary of the region R: $u > 0$ must contain part of body B.</u>

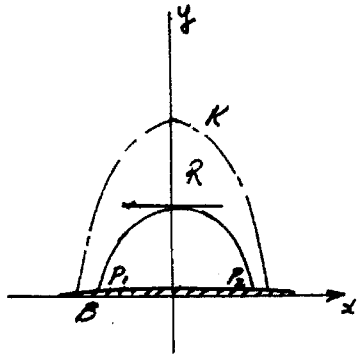
The characteristics (Mach lines) are real at a point where $u > 0$; each real characteristic in the physical plane maps in to a definite characteristic in the hodograph (p. 29). Now assume that the boundary of R does not contain part of the boundary B. Then each characteristic from K must return to K and thus intersect it in two points P_1, P_2. These two points are mapped on the same hodograph characteristic and actually on the same point since u only takes the value zero once on a given hodograph characteristic. This follows from the extension theorem for the solution of ordinary differential equations since the characteristics are defined as the solution of certain differential equations. It can also follow from arguments on the boundedness of the slope of the characteristics. This implies that $v_{P_1} = v_{P_2}$. But by I. this is possible



only if K is vertical at all points, which reduce R to a line. Hence R occurs next to the boundary B. The arguments in I and II apply just as well if there is more than one supersonic region. The monotonicity properties still hold on the sonic line of each region and each must be adjacient to the boundary.

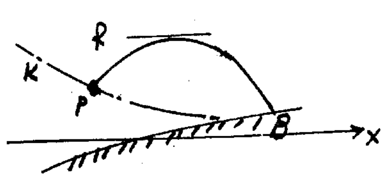
III. <u>The characteristics in R run from the sonic line K to the boundary B.</u>

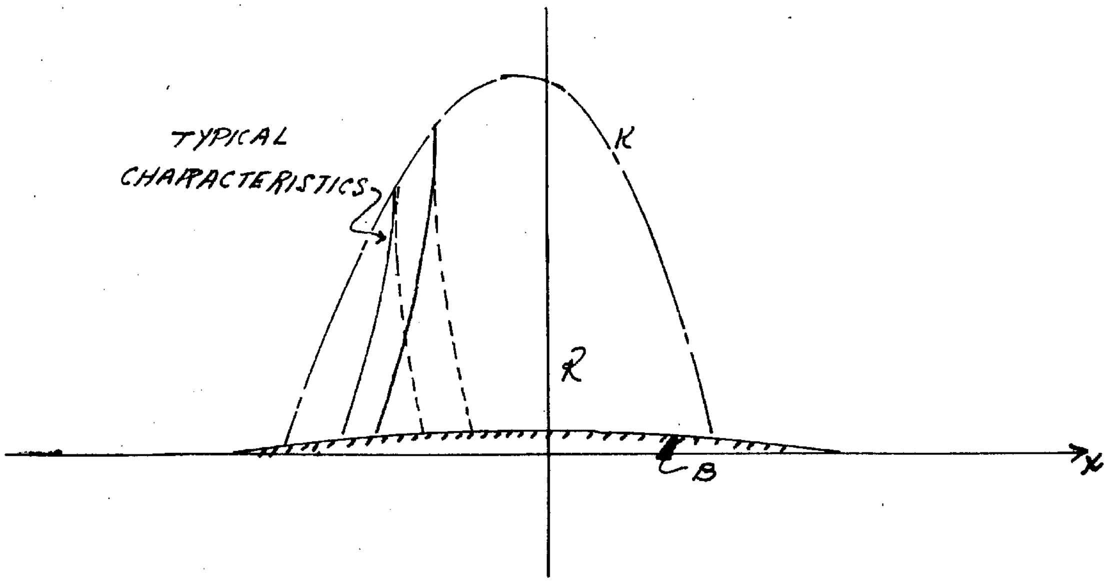
It has already been proved that the characteristics cannot run from the sonic line to the sonic line. Now it is also shown that characteristics cannot run from the boundary B back to the boundary. The slope of the characteristics which is the Mach angle is a continuous function over R since the velocity u is assumed continuous. Hence the characteristics running from P_1 to P_2 must somewhere



attain the horizontal direction. This implies an infinite velocity which is contradiction. Hence III is true. In the above reasoning it is assumed that the body is so thin that the data are given on the x-axis.

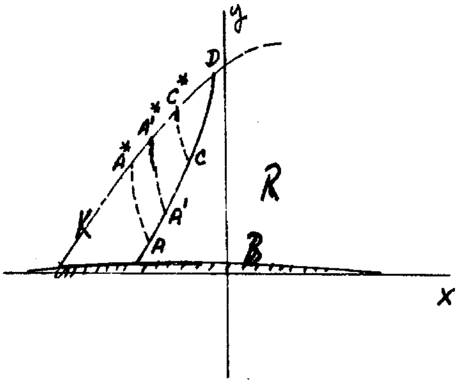
It is interesting to note that the shape of the sonic line K may be fairly well specified. The lines x = constant must run from B through R to the sonic line. If they do not, a contradiction follows. The characteristic leaving P on K must run to the boundary B. If y is not monotone on this characteristic it must assume the horizontal direction, which is forbidden. The supersonic zone must then have the following general appearance.





IV. <u>(u,v) are monotone functions of the distance from body B along a characteristic in a local supersonic zone of potential flow.</u>

Consider a segment AC of a characteristic which hits the sonic line at D and the characteristic of the other family extending from AC to the sonic line K. First, at any point A'



$$\nu_{A'} = \frac{1}{2}\left\{\nu_{A'^*} + \nu_D\right\} \tag{2.65}$$

This follows from the symmetry of the hodograph characteristics onto which A'A* and A'D are mapped or from their equation

$$\nu = \nu^* \mp \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\, u^{3/2}, \text{ resp} \tag{2.66}$$

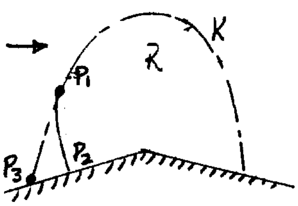
Now along A*C*, ν is decreasing. Hence by (2.65) along AC, ν is decreasing. But (2.66) holds along AC with a fixed constant and sigh. Thus u is monotonic along AC.

It should be noted that this proof depended on the fact that the characteristics from AC extended to the sonic line K and that

A^* to C^* was clockwise. For the other family of characteristics the reverse is true when traveling away from B.

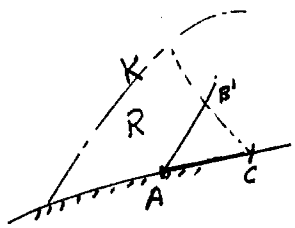
V. <u>Transonic potential flow around a corner formed by straight sides is not possible; more generally, a flat segment inside a local supersonic zone is not possible.</u>

Consider a body with a corner formed by straight sides and consider a local supersonic zone about that corner. Consider a characteristic running from P on the sonic line K to a point P



before the corner on the body. From IV $v_{P_1} > v_{P_2}$ and from I $v_{P_3} > v_{P_1}$. Therefore, for the assumed flow, $v_{P_3} > v_{P_2}$ which contradicts the assumption of straight sides: (v = constant). Therefore the sonic velocity can be reached on the body only at the corner. From symmetry and from discussion of the shape of the supersonic zone under III it then follows the supersonic zone consists of only the corner point. This solution makes sense for a body with infinitely long sides and with the flow at rest at infinity. This problem is solved for incompressible flow. However the infinite velocity which occurs at the corner in the incompressible flow is now replaced by the sonic velocity. When the body dimehsion is infinite there is no characteristic length in terms of which the height of the supersonic zone can be expressed. However, when the body has a finite size, as a double wedge airfoil, shock waves are needed to satisfy the boundary conditions as soon as a sonic region develops.

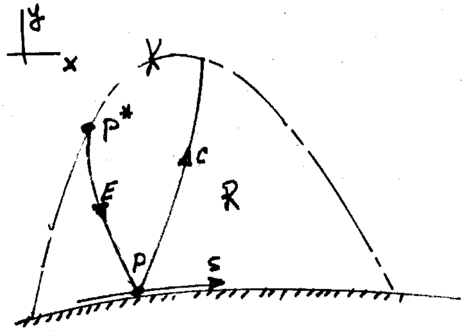
The proof that potential flow cannot exist when there is a flat segment is analogous and depends on monotonicity of along the characteristic. Consider a flat segment joined to the body with continuous slope and the characteristic going downstream from its start. We have $v_A > v_{B'}$ and $v_{B'} > v_C$ which implies $v_A > v_C$ a contradiction. ($v_A = v_{B'} = v_C$ is ruled out as this implies the existence of a simple wave: see next paragraph.)

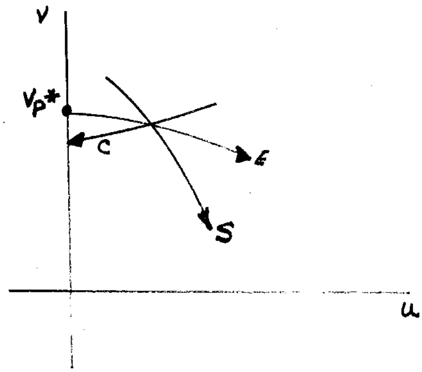


Another interesting result is that the local supersonic zone cannot enclose a body with a corner (v discontinuous). In turning a corner the supersonic potential flow must use a simple wave (Ref. 5). This simple wave is a region which has a degenerate hodograph and which has one family of characteristics which are straight lines. If these straight characteristics extênd to the sonic line they must be vertical and hence the velocity is sonic everywhere, a contradiction. The occurrence of a simple wave or Prandtl-Meyer fan in a local supersonic zone always implies the existence of a shock wave.

VI. <u>The flow along a convex boundary next to a local supersonic zone is vortex-like</u>.

The convex boundary is defined by $\frac{dv}{\partial x} < 0$ along the boundary in the flow direction. The proof of VI is indicated from an inspection of the flow picture and its mapping in the hodograph. In the physical plane one expansion characteristic of E runs from the sonic line into the streamline and one compression characteristic must run from the streamline to the sonic line. In the hodograph a vortex-like streamline is the only one for which this is possible. The result may be demonstrated analytically as





follows. Consider a point P on S and the relationship which holds along an E characteristic

$$v_p = v_{p^*} - \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\, u_p^{3/2} \tag{2.67}$$

and differentiate (2.67) along the streamline. This may be written

$$\frac{dv_p}{dx} = \frac{dv_p^*}{dx} - \sqrt{\frac{\gamma+1}{a^*}}\, u_p^{\frac{1}{2}} \frac{du_p}{dx} \tag{2.68}$$

But by I and since $dv_p < 0$ along S, $\dfrac{dv_p}{dv_p^*} > 0$. Therefore (2.68) gives

$$\frac{dv_p}{du_p} < -\sqrt{\frac{\gamma+1}{a^*}u_p} \qquad (2.69)$$

The slope of E at P in the hodograph is just $-\sqrt{\frac{\gamma+1}{a^*}u_p}$.

Similarly by considering C and the relationships that hold on it

$$\frac{dv_p}{du_p} > \sqrt{\frac{\gamma+1}{a^*}u_p} \qquad (2.610)$$

Hence the flow is vortex-like.

By examining the hodograph picture it is apparent the flow must be vortex-like in the entire region R. Hence the extension of I applies and V must attain its maximum and minimum in the boundary. The lines of constant velocity run from the boundary to the boundary.

If for any reason the inequalities (2.610) must be violated and the image of the streamline in the hodograph is made to appear source-like we have a contradiction and the assumed potential flow cannot exist. For example, it is always possible to do this by introducing a sufficient jump in the curvature of the surface if the velocity is assumed fixed at a point.

<u>Discussion.</u>

The main question so far was the existence of a solution and the consequence of it. It was shown first of all that for certain bodies potential flow could not exist and that shock waves were needed to satisfy the boundary conditions. The necessity of shock waves for this reason is familiar from purely supersonic flow.

Also it was shown that although one body may permit a transonic zone, a body close to it with only a small flat segment cannot. Also assuming a solution for a given body and Mach number at infinity the supersonic potential zone may be made impossible by a change in its curvature.

However, it is not so clear why, for a given smooth body, the potential flow cannot exist as M is increased to one. Friedrichs (Ref. 11) has shown that under these circumstances, starting with the solution in the hodograph, the flow cannot breakdown because of a limiting line. This is of course connected with the fact that a limiting line demands a corner of the streamline and this cannot appear on a given body. The reason for the impossibility of a potential solution might be connected with the violation of equations (2.69) and (2.610) and the impossible flow pattern in the hodograph. However this criterion is not very useful and further research is needed along these lines.

2.7 Drag of Transonic Airfoils

Consider a non-lifting airfoil moving at $M < 1$ through air at rest at ∞ with a local supersonic zone which contains shock waves. In these coordinates it is easy to concentrate the attention on a definite mass of gas, bounded by planes normal to the flight direction at $\pm\infty$. Then the First Law of Thermodynamics applied to this gas states that:

Heat added to gas = Gain in internal energy + work done by gas (2.71)

If the assumption is made, as before, that no external heat is added, all the heat added to the gas comes from the dissipative effects. These effects are mainly in shock waves and the boundary layer. If the flow pattern is steady the temperature distribution is the same so that there is no gain in internal energy. The work done by the gas is all in retarding the airfoil. Thus in a unit time, if

χ = dissipation function

D = drag per unit span (two-dimensional flow)

V = velocity of airfoil

$$\iint_{-\infty}^{\infty} \chi \, dx\, dy = D \cdot V \tag{2.72}$$

For transonic flow $V \doteq a^*$ and χ is approximated assuming a single normal shock of length ℓ. For convenience assume the airfoil is symmetric. The expression for the dissipation, as worked out in (3.2), is

$$\int_{-\infty}^{\infty} \chi \, dx = \frac{4}{3}\mu^* \int_{-\infty}^{\infty} u_x^2 \, dx = \frac{2(\gamma+1)}{3}\rho^* u_a^3 \tag{2.73}$$

replacing $u_{-\infty}$ of (3.2) by u_a = velocity ahead of shock. Hence, if ℓ = height of shock

$$\frac{4(\gamma+1)\rho^*}{3}\int_0^{\ell} u_a^3 \, dy = D \cdot a^*$$

or:

$$D = \frac{4(\gamma+1)}{3}\rho^* \frac{\overline{u_a^3}\,\ell}{a^*} = \frac{4}{3(\gamma+1)^2}\,\overline{(M_a^2-1)^3}\,\ell\,\rho^* a^{*2}$$

bar denotes mean value

$$C_D = \frac{D}{\frac{\rho}{2} a^{*2} C} = \frac{8}{3(\gamma+1)^2} \sqrt{(M_a^2-1)^3}\ \frac{\ell}{c} \tag{2.74}$$

C = chord

(2.74) gives the "wave drag" of a transonic airfoil. It is normally quite small but increases rapidly as M ⟶ 1 for $\frac{\ell}{c}$ becomes large rapidly (see Fig. 3). This drag is, of course, felt in the pressure distribution in the airfoil.

2.8 Determination of the Height of the Supersonic Zone from Data on the Airfoil

In this section it is assumed that we have transonic potential flow adjacent to a convex boundary and that the velocities are known on the boundary. Theoretically it is possible to determine the height and shape of the supersonic zone by the computational methods of characteristics. Practically this is impossible because the slope of the characteristics become vertical near M = 1. It is shown here that a certain functional relationship must exist along the streamline and the determination of the height of the supersonic zone is reduced to the solution of an integral equation. A first approximation is given for a special case.

A solution to the equation of motion must satisfy the following equation in the hodograph

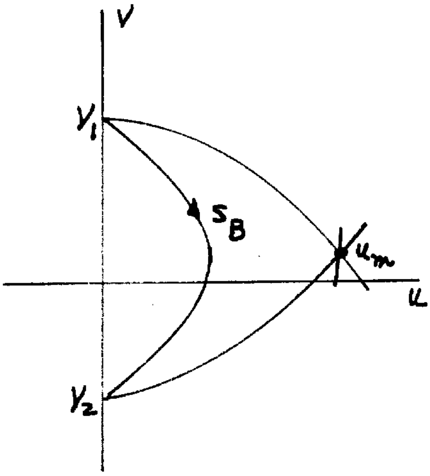
$$\frac{\gamma+1}{a^*} u\, y_{vv} - y_{uu} = 0 \tag{2.81}$$

derived from (2.56). Darboux (Ref. 12) gives the general solution for (2.81) of an initial value problem, with y, y_u given on u = 0 (sonic line), as

$$y(u,v) = \int_0^1 f\left[v + \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\,u^{3/2}(2t-1)\right] t^{-5/6}(1-t)^{-5/6}\,dt$$
$$+\,u\int_0^1 g\left[v + \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\,u^{3/2}(2t-1)\right] t^{-1/6}(1-t)^{-1/6}\,dt \qquad (2.82)$$

which is valid at any point P whose domain of dependence lies on the initial line.

If v_1 and v_2 are the velocities at the entrance and exit of the sonic line the representation (2.82) is valid in the triangle v_1, u_m, v_2 formed by the characteristics through v_1 and v_2. In particular, assuming a convex body, the bounding streamline S_B is vortex-like and hence lies within the triangle so that (2.82) holds in the local supersonic zone. The functions f and g are related to the initial data since at u = 0



$$y(0,v) = f(v)\int_0^1 t^{-5/6}(1-t)^{-5/6}\,dt = I_{5/6}\,f(v)$$

where

$$I_{5/6} = \int_0^1 t^{-5/6}(1-t)^{-5/6}\,dt = \frac{\Gamma^2(5/6)}{\Gamma(5/3)} \qquad (2.83)$$

and

$$y_u(0,v) = I_{\frac{1}{6}}\, g(v) \tag{2.84}$$

Now approximate the airfoil by y = 0 and assume that all data are given on the airfoil. We may express the data as:

$$\text{on } S_B \left\{ \begin{array}{c} y = 0 \\ \frac{2}{3}\sqrt{\frac{\gamma+1}{a^*}}\, u^{3/2} = h(v) \\ x = l(v) \end{array} \right\} \tag{2.85}$$

Since $x_v = y_u = l'(v)$ on S_B we obtain, using (2.85), the following pair of integral equations for f, g which must hold.

$$0 = \int_0^1 f[v+h(2t-1)]\, t^{-5/6}(1-t)^{-5/6}\, dt + \left(\frac{3}{2}\right)^{\frac{2}{3}}\left(\frac{a^*}{\gamma+1}\right)^{\frac{1}{3}} h(v)^{2/3} \int_0^1 g[v+h(2t-1)]\, t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt \tag{2.86}$$

$$\begin{aligned} l'(v) = \sqrt{\frac{\gamma+1}{a^*}}\left(\frac{3}{2}\right)^{\frac{1}{3}}\left(\frac{a^*}{\gamma+1}\right)^{\frac{1}{6}} h(v)^{\frac{1}{3}} \int_0^1 f'[v+h(2t-1)](2t-1)\, t^{-5/6}(1-t)^{-5/6}\, dt \\ + \int_0^1 g[v+h(2t-1)]\, t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt \\ + \frac{3}{2}\, h(v) \int_0^1 g'[v+h(2t-1)](2t-1)\, t^{-\frac{1}{6}}(1-t)^{-\frac{1}{6}}\, dt \end{aligned} \tag{2.87}$$

The general solution of (2.86) and (2.87) has not been worked out. However an approximate answer may be found by putting h = 0 under the integral sign. This gives

$$0 = f(v)\, I_{5/6} + \left(\frac{3}{2}\right)^{2/3}\left(\frac{a^*}{\gamma+1}\right)^{\frac{1}{3}} h(v)^{2/3}\, g(v)\, I_{\frac{1}{6}} \tag{2.88}$$

$$\ell'(v) = \left(\frac{\gamma+1}{a^*}\right)^{\frac{1}{3}} \left(\frac{3}{2}\right)^{\frac{1}{3}} h(v)^{\frac{1}{3}} f'(v)\left[2I_1 - I_{5/6}\right] + g(v)\, I_{1/6} + \frac{3}{2} h(v)\, g'(v)\left[2I_1 - I_{\frac{1}{6}}\right] \qquad (2.89)$$

where

$$I_1 = \int_0^1 t^{\frac{1}{6}}(1-t)^{-5/6}\,dt = \int_0^1 t^{5/6}(1-t)^{-\frac{1}{6}}\,dt$$

For a symmetric solution we have $g'(o) = f'(o)$ so that

$$g(0) \equiv g_0 = \frac{\ell'(0)}{I_{\frac{1}{6}}} \equiv \frac{\ell_0'}{I_{\frac{1}{6}}} \qquad (2.810)$$

$$f(o) \equiv f_0 = \frac{-1}{I_{5/6}} \left(\frac{3}{2}\right)^{2/3} \left(\frac{a^*}{\gamma+1}\right)^{\frac{1}{3}} h_0^{2/3}\, \ell_0' \qquad (2.811)$$

Now $\ell'(v) = \frac{\partial x}{\partial v} = \frac{1}{\frac{\partial v}{\partial x}}$ along S_B. Hence the height of the supersonic zone, on the line $v = 0$, is approximately (see (2.83))

$$y_0 = -\frac{u_{max}}{\left(\frac{dv}{\partial x}\right)_0} \qquad (2.812)$$

Example: Consider a circular arc airfoil so that approximately

$v = -2a^*\tau x$ on S_B ; τ = thickness ratio

$$\frac{dv}{\partial x} = -2a^*\tau$$

Then:

$$y_0 = \frac{1}{2(\gamma+1)} \frac{M_0^2 - 1}{\tau} = \frac{1}{4.8} \frac{M_0^2-1}{\tau} \qquad (2.813)$$

For τ = 0.06, 0.12 (2.813) gives straight lines which are drawn in Fig. 3. In comparison with the one experimental point which is really available the values seem slightly high but the agreement is fair for $M_o < 1.10$.

3. FLOW THROUGH A SHOCK WAVE

A simple example of a non-linear longitudinal wave is a shock wave, which in a real fluid is of course not a discontinuity but a continuous steep variation. The effect of the non-linearity in maintaining the steepness and preventing viscous dispersion of the wave is vital, and is lost in any lihearized theory.

The actual non-linear problems are very difficult and, at present, not capable of solution. However, the main effects can be shown by considering simplified problems. In the following sections an approximate treatment of weak non-stationary waves will be given.

3.1 Derivation of Equations

Basic Equations for Transonic Flow.

Consider one-dimensional longitudinal flow with the velocity $u(x,t)$. The basic assumption is that at $x = -\infty$, the flow is uniform and steady, slightly faster than the speed of sound;

$$u_{-\infty} = a^* + w_{-\infty} \tag{3.11}$$

where the constant a^* is the speed of sound at $M = 1$. Under the conditions assumed the flow is isentropic at $-\infty$ and the Bernoulli equation is

$$\frac{u^2}{2} + \frac{a^2}{\gamma - 1} = \frac{a^{*2}}{2}\,\frac{\gamma+1}{\gamma-1} = \text{constant} \tag{3.12}$$

The equations to be satisfied by the flow are continuity, momentum energy, and the perfect gas law

$$\rho_t + (\rho u)_x = 0 \tag{3.13a}$$

$$\rho u_t + \rho u u_x = -p_x + \frac{4}{3}(\mu u_x)_x \tag{3.13b}$$

$$\rho c_v T_t + \rho u c_v T_x = -\rho\left[p\left(\frac{1}{\rho}\right)_t + u P\left(\frac{1}{\rho}\right)_x\right] + \frac{4}{3}\mu u_x^2 \tag{3.13c}$$

$$P = \rho R T \tag{3.13d}$$

The assumptions of no heat conduction, no heat addition to the system, and a perfect gas have been made. The first step is the derivation of suitable equations from (3.13)

Derivation of a Bernoulli Equation, Including Viscosity.

For convenience introduce a velocity potential $\Phi(x,t)$ such that

$$u(x,t) = \Phi_x \tag{3.14}$$

Then integration of the momentum equation from $-\infty$ to x yields

$$\Phi_t + \frac{1}{2}\Phi_x^2 = -\int_{-\infty}^{x} \frac{1}{\rho}\frac{\partial P}{\partial \xi}d\xi + \frac{4}{3}\int_{-\infty}^{x} \frac{1}{\rho}\frac{\partial}{\partial \xi}\left(\mu \frac{\partial^2 \Phi}{\partial \xi^2}\right)d\xi + \frac{1}{2}u_\infty^2 \tag{3.15}$$

The right hand side of (3.15) can be integrated under certain perturbation assumptions. For transonic flow assume that all the quantities differ slightly from their values at M = 1. Thus

$P = P^*(1+p^*)$ where $a^{*2} = \gamma R T^* = \frac{\gamma P^*}{\rho^*}$ is defined from the condition at $-\infty = x$

$\rho = \rho^*(1+s^*)$

$T = T^*(1+\theta^*)$

$\mu = \mu^*(1+\alpha^*)$

where $\mu^* = \mu(T^*)$

and where all the quantities $p^*, \rho^*, \theta^*, \alpha^* \ll 1$. Further introduce a perturbation potential $\varphi(x,t)$ such that

$$\Phi_x(x,t) = a^* + \varphi_x(x,t) \tag{3.17}$$

where

$$\varphi_x \ll a^*$$

Then using (3.11), (3.16) and (3.17) and neglecting higher order terms we approximate (3.15) by

$$p^*(x,t) = p_\infty + \frac{\gamma}{a^{*2}}\left\{\frac{4}{3}\nu^*\varphi_{xx} - \varphi_t + a^*(u_\infty - \varphi_x)\right\} \tag{3.18}$$

(3.18) is the required relationship and expresses the pressure at any point in terms of the derivatives of the perturbation potential

.

<u>Derivation of an Equation of Motion in Terms of the Perturbation Potential φ.</u>

Multiplying the momentum equation (3.13b) by u yields

$$uu_t + u^2u_x = -\frac{u}{\rho}P_x + \frac{4}{3\rho}u\,(\mu u_x)_x \tag{3.19}$$

The continuity, energy and state equations of (3.13) can be used to eliminate P_x since

$$\frac{u}{\rho}P_x = \frac{4}{3}(\gamma-1)\frac{\mu}{\rho}u_x^2 - \left(\frac{\gamma P}{\rho}\right)u_x - \frac{1}{\rho}P_t \tag{3.110}$$

Then (3.19) and (3.110) are combined as

$$uu_t + \left(u^2 - \frac{\gamma P}{\rho}\right)u_x = \frac{1}{\rho}P_t + \frac{4}{3}\frac{u}{\rho}(\mu u_x)_x - \frac{4}{3}(\gamma-1)\frac{\mu}{\rho}u_x^2 \tag{3.111}$$

Introducing the perturbation assumptions (3.16) and (3.17) in (3.111) and neglecting smaller terms, we have

$$a^* \varphi_{xt} + \{2a^* \varphi_x - a^*(p^* - s^*)\} \varphi_{xx} = \frac{a^{*2}}{\gamma} p^*_t \tag{3.112}$$

In order to obtain an equation for φ, we eliminate p^*_t from (3.113) by using the Bernoulli equation (3.18) and we relate p^* and s^* by using the energy equation (3.13c). Introducing the perturbation assumptions in the energy equations

$$\left(\frac{\partial}{\partial t} + a^* \frac{\partial}{\partial x}\right)\left(\frac{p^*}{\gamma} - s^*\right) = \frac{4}{3} \frac{\nu^*}{a^{*2}} (\gamma-1) \left(\frac{\partial^2 \varphi}{\partial x^2}\right)^2 \tag{3.113a}$$

or integrating

$$\frac{p^*}{\gamma} - s^* = f(x - a^* t) + J(x,t) \tag{3.113b}$$

where J = integral of the dissipation, and $J(-\infty, t) = 0$. The flow is isentropic at $x = -\infty$ so that f = 0 and

$$p^*_{-\infty} - s^*_{-\infty} = -\frac{\gamma+1}{a^*} u_{-\infty} \tag{3.114}$$

Hence

$$p^* - s^* = \frac{\gamma-1}{a^{*2}} \left\{ \frac{4}{3} \nu^* \varphi_{xx} - \varphi_t - a^* \varphi_x \right\} \tag{3.115}$$

Then using (3.18) and (3.115), we obtain from (3.112) the following basic equation for $\varphi(x,t)$.

$$\varphi_{tt} + 2a^* \varphi_{xt} + \{(\gamma+1) a^* \varphi_x + (\gamma-1)\varphi_t - a^{*2} J\} = \frac{4}{3} \nu^* \{a^* \varphi_{xxx} + \varphi_{xxt}\} \tag{3.116}$$

Non-linear terms which allow for both the steepening of the wave $u \frac{\partial u}{\partial x}$ and variations in the local speed of sound are present in this equation. It should be remembered that the equation is expressed in a system of coordinates where the flow is slightly supersonic at $x = -\infty$. Notice also that the dissipation integral does not occur in the right hand side but only in the lower order terms.

3.2 Transonic Flow Through a Shock Wave

Steady-State Solution.

For a shock wave which is already developed $\varphi = \varphi(x)$ and (3.116) becomes

$$(\gamma+1)\,\varphi_x \varphi_{xx} - a^* \mathcal{J}\{\varphi\} = \frac{4}{3} \nu^* \varphi_{xxx} \tag{3.21a}$$

where

$$\mathcal{J}\{\varphi\} = \frac{4}{3} \frac{\nu^*}{a^{*2}} (\gamma-1) \int_{-\infty}^{x} \varphi_{\xi\xi}^2 \, d\xi \tag{3.21b}$$

It is now assumed that the dissipation in (3.21a) is small enough to be neglected and (3.21a) is then integrated from $-\infty$ to x, to give

$$w^2 - w_{-\infty}^2 = \lambda \frac{dw}{dx} \quad \text{where} \quad \lambda = \frac{8\nu^*}{3(\gamma+1)} \tag{3.22}$$

(3.22) may also be integrated and if x = 0 when w = 0 we obtain

$$\frac{w}{w_{-\infty}} = -\tanh\left(\frac{w_{-\infty} x}{\lambda}\right) \tag{3.23}$$

Also we have the potential

$$\varphi(x) = -\lambda \log \cosh\left(\frac{w_{-\infty} x}{\lambda}\right) \tag{3.24}$$

if $\varphi(0) = 0$

(3.24) represents a velocity distribution which is supersonic at $x = -\infty$ and subsonic at $x = +\infty$, with a continuous variation which is most rapid in the neighborhood of the origin $x = 0$. For smaller values of ν^* this region of rapid change becomes increasingly narrower and approaches zero as $\nu^* \to 0$.

The important effect of the non-linearity in allowing a transition from $w > 0$, (supersonic flow), to $w < 0$, (subsonic flow) is thus shown. This transition is missing in any linearized theory.

As a check on the approximations the order of magnitude of the dissipation integral (3.21b) can be found using the solution (3.23). We obtain

$$\mathcal{J}(x) = (\gamma-1)\frac{w_\infty^3}{a^{*3}}\left\{\frac{2}{3} + \tanh\frac{w_\infty x}{\lambda} - \frac{1}{3}\tanh^3\frac{w_\infty x}{\lambda}\right\}$$

so that the total dissipation is

$$\mathcal{J}(\infty) = \frac{4}{3}(\gamma-1)\frac{w_\infty^3}{a^{*3}} \sim \left(M_\infty^2 - 1\right)^3 \qquad (3.24)$$

This indicates that the dissipation can be neglected if the Mach number ahead of the wave is sufficiently close to 1. Actually the steady state problem including the viscosity and dissipation can be solved for waves of arbitrary strength (see Ref. 13). However the viscosity is assumed to be constant in that derivation. For a strong wave the variations of viscosity and heat conduction with temperature, and the dissipation all become important. The weak wave assumption is the only one consistent with assuming constant viscosity.

An Approximate Non-Steady Solution.

In this approximation the integral of the dissipation is again neglected. Further, in order to simplify the problem the assumption is made that

$$\varphi_t \ll a^* \varphi_x \tag{3.25}$$

so that certain terms in (3.116) may be neglected. This is an approximation which becomes better for large t when the solution will be shown to approach the steady state. (3.116) is thus approximated by

$$2\varphi_{xt} + (\gamma+1)\varphi_x \varphi_{xx} = \frac{4}{3}\nu^* \varphi_{xxx} \tag{3.26}$$

Integration of (3.26) from $-\infty$ to x gives

$$\varphi_t + \frac{\gamma+1}{4}(\varphi_x^2 - \underline{w}_\infty^2) = \frac{2\nu^*}{3}\varphi_{xx} \tag{3.27}$$

since $\varphi_x = \underline{w}_\infty$, $\varphi_{xx}, \varphi_t \to 0$ at $x = -\infty$. (3.27) is a non-linear equation for which the general solution of the initial value problem for the domain ($-\infty < x < \infty$, $t \geq 0$) can be given. Only a special example is treated here which is the diffusion by viscosity of an initially sharp wave front. The initial choice is the solution for a weak wave without viscosity and then at t = 0 the viscosity is introduced. The initial conditions are

$$\begin{aligned} \varphi(x,0) &= \underline{w}_\infty x \qquad x<0 \\ &= -\underline{w}_\infty x \qquad x>0 \end{aligned} \tag{3.28}$$

The solution is found by assuming

$$\varphi(x,t) = F\{\theta(x,t)\} \tag{3.29}$$

where θ (x,t) is to satisfy a linear equation and the function F must be determined. Such an assumption about the solution is suggested by similarity solutions of (3.27). Under the assumption (3.29), (3.27) becomes

$$F'(\theta)\,\theta_t + \frac{\gamma+1}{4}\left\{F'(\theta)^2\,\theta_x^2 - w_{-\infty}^2\right\} = \frac{2\nu^*}{3}\left\{F''(\theta)\theta_x^2 + F'(\theta)\,\theta_{xx}\right. \tag{3.210}$$

(3.210) is satisfied if

$$\frac{\gamma+1}{4}F'^2 = \frac{2\nu^*}{3}F'' \tag{3.211}$$

and

$$\theta_t - \frac{1}{F'(\theta)}\,\frac{\gamma+1}{4}w_{-\infty}^2 = \frac{2\nu^*}{3}\,\theta_{xx} \tag{3.212}$$

The general solution of (3.211) is

$$F(\theta) = C_2 - \lambda \log\left\{C_1 - \frac{1}{\lambda}\,\theta(x,t)\right\}$$

The constants C_2, C_1 may be chosen by having $\varphi \to 0$ as $\frac{\gamma+1}{4} \to 0$, and having F = 0 where $\theta = 0$. Then

$$F(\theta) = -\lambda \log\left(1 - \frac{1}{\lambda}\theta(x,t)\right) \tag{3.213}$$

$$F'(\theta) = \frac{1}{1 - \frac{1}{\lambda}\theta(x,t)} \tag{3.214}$$

and the linear equations to be satisfied by θ is a modified heat equation

$$\theta_t - \frac{\gamma+1}{4}\left(1-\frac{1}{\lambda}\theta\right) = \frac{2\nu^*}{3}\theta_{xx} \tag{3.215}$$

The boundary conditions for (3.215) are found from the formulae relating θ, φ through the known function F:

$$\varphi(x,t) = -\lambda \log\left\{1-\frac{1}{\lambda}\theta(x,t)\right\} \tag{3.216a}$$

$$\theta(x,t) = \lambda\left\{1-\exp\left[-\frac{1}{\lambda}\varphi(x,t)\right]\right\} \tag{3.216b}$$

Thus using (3.28) the initial conditions for (3.215) are

$$\theta(x,0) = \lambda\left\{1-e^{-\frac{w_\infty x}{\lambda}}\right\} \quad x<0 \tag{3.217a}$$

$$= \lambda\left\{1-e^{\frac{w_\infty x}{\lambda}}\right\} \quad x>0 \tag{3.217b}$$

The solution to (3,215) satisfying the boundary conditions (3.217) may be obtained by Laplace tbansformation or other methods

$$\theta(x,t) = \lambda\left\{1+\frac{1}{\lambda}e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda}}\int_0^\infty e^{-\tau t}\frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}\,d\tau - 2\cosh\frac{w_\infty x}{\lambda}\right\} \quad x>0 \tag{3.218}$$

It may be verified that (3.218) satisfies the boundary condition (3.217b) by virtue of the formula

$$\frac{w_\infty x}{\lambda}\frac{1}{\pi}\int_0^\infty \frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}\,d\tau = e^{-\frac{w_\infty x}{\lambda}} \quad x>0 \tag{3.219}$$

Thus the potential and velocity distribution for x > 0 are obtained ; and $\varphi_x(-x,t) = -\varphi_x(x,t)$, $\varphi_t(-x,t) = \varphi_t(x,t)$

$$\varphi(x,t) = -\lambda \log\left\{2\cosh\frac{w_\infty x}{\lambda} - \frac{1}{\lambda\pi} e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda}} \int_0^\infty e^{-\tau t} \frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}\,d\tau\right\} \tag{3.220a}$$

$$\varphi_x = -w_\infty \frac{\sinh\frac{w_\infty x}{\lambda} + \frac{1}{\lambda\pi} e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda}} \int_0^\infty e^{-\tau t} \frac{\sin\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}\,d\tau}{\cosh\frac{w_\infty x}{\lambda} - \frac{1}{2\lambda\pi} e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda}} \int_0^\infty e^{-\tau t} \frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}\,d\tau} \tag{3.220b}$$

$$\varphi_t = -2w_\infty^2\sqrt{\frac{2\nu^*}{3\pi t}}\;\frac{e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda} - \frac{x^2}{t\lambda(\gamma+1)}}}{2\cosh\frac{w_\infty x}{\lambda} - \frac{1}{\pi\lambda} e^{-\frac{(\gamma+1)w_\infty^2 t}{2\lambda}} \int_0^\infty e^{-\tau t} \frac{\cos\left(\sqrt{\frac{3\tau}{2\nu^*}}\,x\right)d\tau}{\sqrt{\frac{3\tau}{2\nu^*}}\left(\frac{(\gamma+1)w_\infty^2}{2\lambda}+\tau\right)}} \tag{3.220c}$$

This solution (3.220) approaches the steady-state solution very rapidly because of the exponential damping with time. The ratio $\frac{\varphi_t}{a^*\varphi_x}$ may be computed from the solution (3.220) and it can be seen that for t > 0 this term is very small except when x = 0, where $\varphi_x = 0$. For any fixed x = 0 the ratio becomes very small after some time.

Thus one simple example has been given where the steady state of the shock wave is the limit of special non-steady solution. The problem had to be simplified greatly in order to make an analytical treatment possible. However, the essential features of non-linearity and viscosity were retained.

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