ON THE TRANSONIC FLOW PAST THIN AIRFOILS

Thesis by

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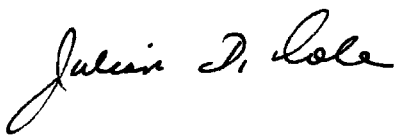

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INTRODUCTION

This research was undertaken as part of a general program of the investigation of the formation of shock waves on airfoils. Consequently, it was thought advisable to study potential solutions for transonic flow to obtain information concerning the occurrence, size, and shape of supersonic zones. From this point of view it is especially desirable to include the effect of wind tunnel walls in modifying the flow field since the most reliable data can be obtained in wind tunnels.

Until recently the main methods of investigating transonic flow were:

(1) Expansion of the velocity potential in a double power series of space coordinates. This method, as applied by Taylor (Ref. 1) and Gortler (Ref. 2), solves an exact equation for the flow with approximate boundary conditions. Since the area of convergence is very limited the use of this method is restricted to local problems such as the flow in the throat of a nozzle, etc.

(2) Solution of the Hodograph Equation. The equation for the two-dimensional flow of a compressible fluid may be linearized by a transformation of the independent variables. (See Ref. 3) Then, in general, the simplified equation must be solved with correspondingly more difficult boundary conditions. Tsien and Kuo (Ref. 4) have applied this method to the flow past bodies of nearly circular shape, but even here the computations were lengthy.

(3) Development of velocity potential in power series of thickness ratio. In the first approximation this is the so-called "small perturbation" or Prandtl-Glauert method and has been used by many authors to obtain information about compressible flows. The first approximation is relatively simple and where carefully considered yields much information, even in transonic flow. Gortler obtained supersonic regions in the flow past a wavy wall in Ref. 8. The second approximation is more complex and has only been worked out for special cases, e.g. Hantzche and Wendt (Ref. 5). (For a revision of these methods see Ref. 3)

(4) Solution of an Approximate Transonic Equation. If in the two-dimensional, irrotational, isentropic flow of a compressible fluid,

a = velocity of sound at a point x,y in space

u = velocity of flow in x-direction at that point

v = velocity of flow in y-direction at that point,

$$(a^2-u^2)\frac{\partial u}{\partial x} - 2uv\frac{\partial u}{\partial y} + (a^2-v^2)\frac{\partial v}{\partial y} = 0 \qquad (1)$$

Then, u is considered to be only slightly different from a velocity of sound and v is considered to be small, i.e.,

$$u = a^* + \frac{\partial \phi}{\partial x} \quad \text{say}$$

$$v = \frac{\partial \phi}{\partial y}$$

where $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ are much less than u, so that squares of their ratio may be neglected and where a* = velocity of sound when it equals the flow velocity on $a^{*2} = u^2 + v^2$, Eq. (1) may be written

$$\frac{\gamma+1}{q^*}\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x^2} + \frac{2}{q^*}\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad (2)$$

Solution of this equation should apply to transonic flow. Recently, E. Graham and Th. von Karman have used Eq. (2) as a starting point and have, by similarity methods, obtained some results for the drag of these bodies in transonic flow.

A consideration of the methods available made a logical first choice the application of the small perturbation theory to the transonic flow past airfoils. The small perturbation theory makes the solution of the flow of a compressible fluid an affine transformation of the solution of the flow of an incompressible fluid. The resulting problem in potential theory is solved by means of a Fourier integral. It can then be expected that as long as the assumptions made in the theory are not violated, the results will approximate the true flow. In this way, supersonic regions over an airfoil and wind tunnel wall effects can be considered.

When exact equations, like (1), are used, supersonic flow is governed by a hyperbolic equation and subsonic flow by an elliptic equation, or one can say that a supersonic zone is an hyperbolic region and a subsonic zone is an elliptic region. The linearized equation (see Eq. 5) used in the solution is, however, elliptic for all space. This implies the analytic continuation of an elliptic solution through a hyperbolic region. (The problem in general was devised by H. Bateman, Ref. 7). However, this is not too strange, for the limited hyperbolic region of a finite supersonic zone behaves elliptically, i.e., every point on it influences every other point and in fact, every point in space. This occurs in the following way:

Fig. 1, the influence of a disturbance or pressure pulse at B is felt in the domain downstream through B. Thus, the influence of B is flet in the boundary of the elliptic region and affects the rest of the elliptic region and especially that part of the boundary in whose domain of influence C lies.

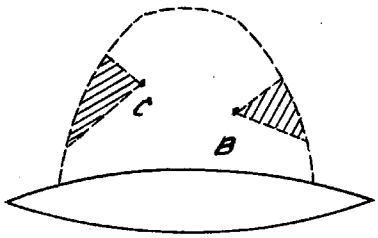


Fig. 1

Physically, the small perturbation approximations are interesting for it will be noticed that stream tubes have no throat at sonic speed nor do they diverge when the velocity becomes supersonic. However, this is a natural consequence of replacing the exact equation for a stream tube, $(1 - M^2) \frac{du}{u} = -\frac{\partial A}{A}$ where M = Mach number = $\frac{u}{a}$

A = area of stream tube

by an approximate one

$(1 - M_\infty^2) \frac{\Delta u}{u} = -\frac{\Delta A}{A}$ where M_∞ = Mach number far from disturbance, as, in affect, the theory does.

II APPLICATION OF SMALL PERTURBATION METHOD TO TRANSONIC FLOW

By considering the conservation of mass, energy, and momentum in a perfect compressible fluid which is disturbed only slightly from steady rectilinear flow, it is possible to describe the motion by the following linearized equation:

$$\left(1-M_\infty^2\right)\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}=0 \qquad (3)$$

where M = Mach number in the undisturbed stream $= \frac{V}{a_\infty}$

if V = velocity of flow in free stream, taken parallel to x-axis

and a_∞ = varied velocity in free stream

Alos, if u, v, w are the velocity components at a point x, y, z in the x, y, z direction respectively, then the induced velocities

$$u' = u - V$$

$$v' = v$$

$$w' = w.$$

In the diameter of B it is assumed that the induced velocities are small compared to the free stream velocity so that squares of this ratio may be neglected and that $\frac{t}{V}\frac{\partial u_i'}{\partial x_k}$ is also small, where t = a characteristic length of the body causing the disturbance, u_i' is any induced velocity, and x_k is any coordinate. (See Ref. 1) These assumptions must be later checked. It can also be shown, using the same approximation that the pressure coefficient

$$C_p = \frac{p-p_\infty}{\frac{1}{2}\rho_\infty V^2} = -\frac{2u'}{V} \qquad (4)$$

when p denotes pressure and ρ density.

If the free stream has subsonic velocity and the entire motion can be considered two dimensional,

$$\beta^2 \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \qquad \text{where } \phi \text{ is a perturbation} \qquad (5)$$

potential such that $u' = \frac{\partial \phi}{\partial x}$

$v' = \frac{\partial \phi}{\partial y}$ $\qquad \beta^2 = 1 - M_\infty^2 > 0$

Some solutions of (3) in special cases are now given.

A. Flow past a wall of almost arbitrary shape, the medium extending to infinity y, as in Fig. 1, x,y are the coordinates in a physical plane and y = f(x) gives the shape of the wall, then (3) must be solved with certain boundary conditions. These are:

1. No disturbance in y-direction far from the wall on

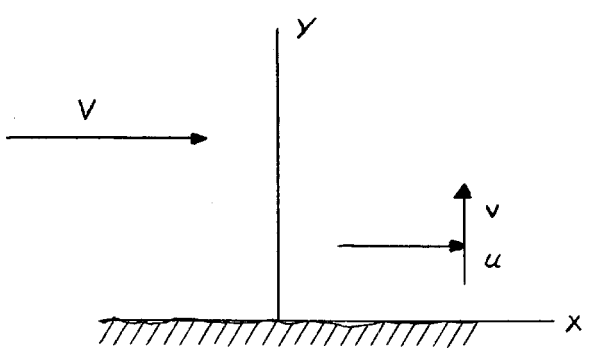
(6) $\qquad v' = \frac{\partial \phi}{\partial y} \to 0 \qquad y \to \infty$

and

2. Flow tangent to the wall.

Within the approximation of the theory this can be expressed as:

(7) $\qquad v(x,0) = V f'(x)$, i.e., the boundary condition is applied on the x-axis.



Now let $\xi = x$, $\eta = \beta y$ so that

(8) $\phi_{\xi\xi} + \phi_{\eta\eta} = 0$ with these boundary conditions

1a. $\frac{\partial \phi}{\partial \eta} \to 0 \qquad \eta \to \infty$

2a. $\frac{\partial \phi}{\partial \eta} = \frac{V}{\beta} f'(\xi) \qquad \eta = 0$

Then the solution may be written (if f'(t) is suitably restricted).

(9) $$\phi = -\frac{V}{\beta\pi} \int_0^\infty \frac{d\lambda}{\lambda} \int_{-\infty}^{\infty} e^{-\lambda\eta} f'(t) \cos \lambda(t-\xi)\, dt$$

for $\frac{\partial^2 \phi}{\partial \xi^2} + \frac{\partial^2 \phi}{\partial \eta^2} = 0$, $\frac{\partial \phi}{\partial \eta} = \frac{V}{\beta\pi} \int_0^\infty d\lambda \int_{-\infty}^{\infty} e^{-\lambda y} f'(t) \cos \lambda(t-\xi)\, dt$

that $\frac{\partial \phi}{\partial \eta} \to 0$ as $y \to \infty$ and at $\eta = 0$

$$\frac{\partial \phi}{\partial \eta} = \frac{V}{\beta\pi} \int_0^\infty d\lambda \int_{-\infty}^{\infty} f'(t) \cos \lambda(t-\xi)\, dt = \frac{V}{\beta} f'(\xi)$$

The pressure distribution may then be found from

$$c_p = -\frac{2\frac{\partial \phi}{\partial \xi}}{V} = \frac{2}{\beta\pi} \int_0^\infty d\lambda \int_{-\infty}^{\infty} e^{-\lambda\eta} f'(t) \sin \lambda(t-\xi)\, dt$$

changing the order of integration yields,

$$c_p = \frac{2}{\beta\pi} \int_{-\infty}^{\infty} f'(t)\, dt \int_0^\infty e^{-\lambda\eta} \sin \lambda(t-\xi)\, d\lambda$$

and integrating,

$$c_p(\xi, \eta) = \frac{2}{\beta\pi} \int_{-\infty}^{\infty} \frac{f'(t)(t-\xi)}{(t-\xi)^2 + \eta^2}\, dt$$, or in physical coordinates

$$c_p = \frac{2}{\beta\pi} \int_{-\infty}^{\infty} \frac{(t-x)\, f'(t)}{(t-x)^2 + \beta^2 y^2}\, dt$$

The result can now be applied to the flow past a thin airfoil at zero angle of attack.

1. Circular arc airfoil

If the flow is past the circular arc airfoil shown in Fig. 3,

$$f'(x) = \frac{-x}{R} \quad \text{for} \quad |x| < a$$

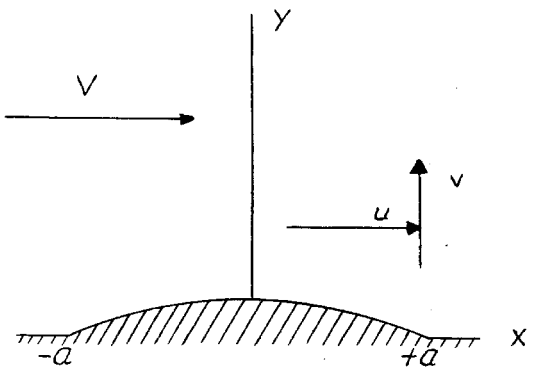
(11)

$$= 0 \quad \text{for} \quad |x| > a$$

to the order of the theory. Then, from (9),

$$c_p = -\frac{2}{\beta\pi R} \int_{-a}^{a} \frac{(t-\xi)\, t}{(t-\xi)^2 + y^2}\, dt$$

(12)



Let a = 1, so that integrating by parts

$$c_p = \frac{-1}{\beta\pi R} \left[t \log \left(y^2 + [t-\xi]^2 \right) \Big|_{-1}^{1} - \int_{-1}^{1} \left(y^2 + [t-\xi]^2 \right) dt \right]$$

$$= \frac{-1}{\beta\pi R} \left[\xi \log \frac{y^2 + (1-\xi)^2}{y^2 + (1+\xi)^2} + 4 \right.$$

$$\left. - 2y \tan^{-1} \frac{1-\xi}{y} - 2y \tan^{-1} \frac{1+\xi}{y} \right]$$

or in physical coordinates

$$(13) \quad c_p(x,y) = \frac{-4}{\beta\pi R} \left[1 - \frac{x}{4} \log \frac{\beta^2 y^2 + (1+x)^2}{\beta^2 y^2 + (1-x)^2} \right.$$

$$\left. - \frac{\beta y}{2} \left(\tan^{-1} \frac{1+x}{\beta y} + \tan^{-1} \frac{1-x}{\beta y} \right) \right]$$

Then

$$(14) \quad c_p(0,0) = c_{p_{max}} = \frac{-4}{\beta\pi R}$$

approximating the surface by y = 0,

and above the maximum thickness

$$(15) \quad c_p(0,y) = c_{p_{max}} \left[1 - \beta y \tan^{-1} \frac{1}{\beta y} \right]$$

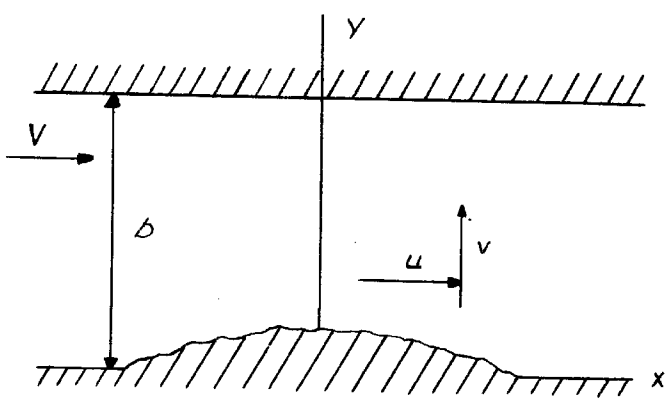
On the surface of the airfoil, or approximately at y = 0

$$c_p(x,0) = c_{p_0}{}_{max}\left[1 - \frac{x}{2}\log\frac{1+x}{1-x}\right] \qquad |x|<1$$

The method of solution can now be extended to flow between boundaries.

A. Flow in a channel past a wall of arbitrary shape.

For closed channels, only the case of flow between a smooth wall and a disturbing wall will be considered but it will be obvious how flow between two disturbing walls can be built up. For the case shown in Fig. 4 the differential Eq. (8) still holds as does boundary condition 2a of (7) but boundary condition 1a must be modified so that



$$(2'a). \quad V = \frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial\eta}\frac{1}{\beta} = 0 \text{ at}$$

$$y = b \qquad \eta = \beta b = h \text{ say}$$

Then the solution is

$$\phi = \frac{-V}{\beta\pi}\int_0^\infty \frac{d\lambda}{\lambda}\int_{-\infty}^{\infty}\frac{\cosh\lambda(h-\eta)}{\sinh\lambda h} f'(t)\cos\lambda(t-\xi)\,dt$$

for

$$\nabla^2\phi = 0 \quad \text{and}$$

$$\frac{\partial\phi}{\partial\eta} = \frac{V}{\beta\pi}\int_0^\infty d\lambda\int_{-\infty}^{\infty}\frac{\sinh\lambda(h-\eta)}{\sinh\lambda h} f'(t)\cos\lambda(t-\xi)\,dt \quad \text{so that}$$

$$\frac{\partial\phi}{\partial\eta} = 0 \text{ at} \qquad \eta = h \quad \text{and at} \quad \eta = 0$$

$$\frac{\partial\phi}{\partial\eta} = \frac{V}{\beta\pi}\int_0^\infty d\lambda\int_{-\infty}^{\infty} f'(t)\cos\lambda(t-\xi)\,dt = \frac{V}{\beta}f'(\xi)$$

Then

$$(18)\qquad c_p = \frac{2}{\beta\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} \frac{\cosh \lambda(h-y)}{\sinh \lambda h} f'(t) \sin \lambda(t-\xi)\, dt$$

changing the order of integration.

$$c_p = \frac{2}{\beta\pi}\int_{-\infty}^{\infty} f'(t)\, dt \int_0^\infty \frac{\cosh \lambda(h-y)}{\sinh \lambda h} \sin \lambda(t-\xi)\, d\lambda$$

This may be integrated with respect to for

$$\int_0^\infty \frac{\cosh \beta\xi}{\sinh \beta\xi} \sin u\xi\, d\xi = \frac{\pi}{2\beta} \frac{\sinh \frac{u\pi}{\beta}}{\cosh \frac{u\pi}{\beta} + \cos \frac{v\pi}{\beta}} \qquad \beta^2 \le \beta^2$$

(See Ref. 9)

Hence $$c_p = \frac{2}{\beta\pi}\,\frac{\pi}{2b}\int_{-\infty}^{\infty} f'(t)\, dt \; \frac{\sinh\left(\frac{t-\xi}{b}\pi\right)}{\cosh\left(\frac{t-\xi}{b}\pi\right) + \cos\left(\frac{h-y}{b}\pi\right)}$$

or

$$(19)\qquad c_p(x,y) = \frac{1}{\beta^2 b}\int_{-\infty}^{\infty} \frac{\sinh\left(\frac{t-x}{\beta b}\pi\right) f'(t)}{\cosh\left(\frac{t-x}{\beta b}\pi\right) - \cos\left(\frac{y}{b}\pi\right)}$$

For the case where the disturbance is restricted to -a x a, the pressure coefficient at the surface (y = 0) is given by

$$(20)\qquad c_p(x,0) = \frac{1}{\beta^2 b}\int_{-a}^{a} f'(t) \coth\left(\frac{t-x}{\beta b}\pi\right) dt$$

This is identically the result obtained by Tsien and Lees in a different way (See Ref. 10). The results can now be applied to a special case.

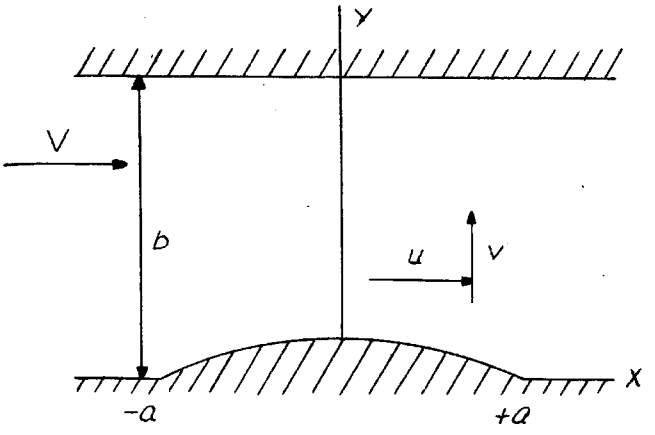
1. Circular arc airfoil at 0° angle of attack in a wind tunnel

For this case, as before,

$$f'(x) = \frac{-x}{R} \quad \text{for } |x| < a$$

$$= 0 \quad \text{for } |x| > a$$

(See Fig. 5) Then from (18)



$$(21) \qquad \varphi(x,y) = \frac{-1}{\beta^2 b R} \int_{-a}^{a} \frac{t \sinh\left(\frac{t-x}{\beta b}\pi\right)}{\cosh\left(\frac{t-x}{\beta b}\pi\right) - \cos\left(\frac{y}{b}\pi\right)}\, dt$$

Now let $a = 1$

$$\frac{t\pi}{h} = \frac{t\pi}{\beta b} = \rho$$

$$\frac{x\pi}{h} = \frac{x\pi}{\beta b} = \sigma$$

$$\bar{y} = \frac{y}{b}$$

$$\frac{\pi}{h} = \frac{\pi}{\beta b} = \mu = \frac{\alpha}{\beta} \; ; \quad \alpha = \frac{\pi}{b}$$

Then

$$\varphi = \frac{-1}{\beta^2 b R} \frac{h^2}{\pi^2} \int_{-\mu}^{\mu} \frac{\rho \sinh(\rho - \sigma)}{\cosh(\rho - \sigma) - \cos \pi \bar{y}}\, d\rho$$

or

$$(22) \qquad \varphi = \frac{-1}{\pi R \alpha} \int_{-\mu}^{\mu} \frac{\rho \sinh(\rho - \sigma)}{\cosh(\rho - \sigma) - \cos \pi \bar{y}}\, d\rho$$

This can easily be evaluated at the surface of the airfoil ($\bar{y} = 0$), for

$$(23) \qquad \varphi(x,0) = \frac{-1}{\pi R \alpha} \int_{-\mu}^{\mu} \rho \coth\left(\frac{\rho - \sigma}{2}\right) d\rho.$$

Now $\coth\left(\frac{p-q}{2}\right) = \frac{2}{p-q}\left\{1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}B_{2n-1}}{(2n)!}\left(\frac{p-q}{2b}\right)^{2n}\right\}$ $\left(\frac{p-q}{2}\right)^2 < \pi^2$

where the B_K are Bernoulli numbers.

This converges everywhere for a worst case of $b = \frac{1}{2\beta}$. For a narrower channel other forms of expansion may be used, as for example in Ref. 3, but this is sufficient for very practical cases.

Hence $$c_p(x,0) = \frac{-2}{\pi R\alpha}\int_{-\mu}^{\mu}\frac{p}{p-q}\,dp - \frac{2}{\pi R\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n-1}}{(2n)!}\int_{-\mu}^{\mu}(p-q)^{2n-1}p\,dp$$

or

$$c_p(x,0) = \frac{-4}{\pi R\beta}\left[1 - \frac{x}{2}\log\left|\frac{1+x}{1-x}\right|\right] - \frac{2}{\pi R\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n-1}}{(2n)!}\int_{-\mu}^{\mu}(p-q)^{2n-1}p\,dp \qquad (24)$$

It is seen that the first term of (24) corresponds to a surface distribution of pressure (c_{p_o}) without a tunnel wall (Eq. 16), and the second term gives the interference effects.

i.e. $(24)\ c_p(x,0) = c_{p0} + \Delta c_p(x,0)$

where $$\Delta c_p(x,0) = \frac{-2}{\pi R\alpha}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n-1}}{(2n)!}\int_{-\mu}^{\mu}(p-q)^{2n-1}p\,dp$$

But $$\int_{-\mu}^{\mu}p(p-q)^{2n-1}dp = \frac{p(p-q)^{2n}}{2n}\Big|_{-\mu}^{\mu} - \int_{-\mu}^{\mu}\frac{(p-q)^{2n}}{2n}dp$$

$$= \frac{(q+2n\mu)(\mu-q)^{2n} - (q-2n\mu)(\mu+q)^{2n}}{2n(2n+1)}$$

Hence,

$$\Delta c_p(x,0) = \frac{-2}{\pi R\beta}\sum_{n=1}^{\infty}(-1)^{n-1}\frac{B_{2n-1}}{(2n+1)!(2n)}\left[(x+2n)(1-x)^{2n} - (x-2n)(1+x)^{2n}\right] \qquad (26)$$

and for x = 0

$$\Delta C_{p_{max}} = C_{p'_{max}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n-1}}{(2n+1)!} \left(\frac{\alpha}{\beta}\right)^{2n} \quad \text{where} \quad C_{p'_{max}} = \frac{-4}{\beta \pi R}$$

or

$$(27) \quad C_p(0,\omega) = C_{p_{max}} \left[1 + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n-1}}{(2n+1)!} \left(\frac{\alpha}{\beta}\right)^{2n} \right]$$

The c_p can also be easily obtained at the tunnel wall $\bar{y} = 1$. From (22)

$$(28) \quad C_p(x,b) = \frac{-1}{\pi R \alpha} \int_{-\mu}^{\mu} p \tanh\left(\frac{p-q}{2}\right) dp.$$

But

$$\tanh\left(\frac{p-q}{2}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n)!} \left(\frac{p-q}{2}\right)^{2n-1} \quad \text{for} \quad \left(\frac{p-q}{2}\right)^2 < \frac{\pi^2}{4}$$

Then proceeding as before we get

$$(29) \quad C_p(x,b) = \frac{-1}{\pi R \beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2^{2n}-1) B_{2n-1}}{n(2n+1)!} \left[(x+2n)(1-x)^{2n} - (x-2n)(1+x)^{2n} \right] \left(\frac{\alpha^2}{\beta}\right)^{2n}$$

and at x = 0 or above the maximum thickness

$$(30) \quad C_p(0,b) = C_{p'_{max}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(2^{2n}-1) B_{2n-1}}{(2n+1)!} \left(\frac{\alpha}{\beta}\right)^{2n}$$

Another place where c_p may be easily found is $\bar{y} = 1/2$ or half way between the wall and airfoil. Then from (22)

$$(31) \quad C_p(x, b/2) = \frac{-1}{\pi R \alpha} \int_{-\mu}^{\mu} p \tanh(p-q)\, dp$$

so that

$$C_p(x, b/2) = \frac{-1}{\pi R \beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1) B_{2n-1}}{n(2n+1)!} \left[(x+2n)(1-x)^{2n} - (x-2n)(1+x)^{2n} \right] \left(\frac{\alpha^2}{\beta}\right)^{2n}$$

or

$$\varphi_p(0, b/2) = \frac{-2}{\pi R \beta} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n}(2^{2n}-1)}{(2n+1)!} B_{2n-1} \left(\frac{\alpha}{\beta}\right)^{2n} \tag{32}$$

$$\varphi_p(0, b/2) = \varphi_{p_{max}} \sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^{2n-1}(2^{2n}-1)}{(2n+1)!} B_{2n-1} \left(\frac{\alpha}{\beta}\right)^{2n}$$

This last solution may be extended by a Taylor expansion, by putting

$$\varphi_p(0,y) = \varphi_p(0, b/2) + (y - b/2)\varphi_p'(0, b/2) + \frac{(y-b/2)^2}{2!}\varphi_p''(0, b/2) + \cdots \tag{33}$$

when $\varphi_p' = \frac{\partial \varphi_p}{\partial y}$ etc.

From (21) we have,

$$\varphi_p(0,y) = \frac{-2}{\pi R \alpha} \int_0^{\mu} \frac{p \sinh p}{\cosh p - \cos \pi \bar{y}} \, dp \tag{34}$$

Hence,

$$\varphi_p'(0,y) = \frac{2}{\pi R \alpha} \int_0^{\mu} \frac{p \sinh p}{(\cosh p - \cos \pi \bar{y})^2} \left(\frac{\pi}{b}\right) \sin \pi \bar{y} \, dp$$

$$\varphi_p'(0, b/2) = \frac{2}{\pi R \alpha} \int_0^{\mu} \left(\frac{\pi}{b}\right) \frac{p \sinh p}{\cosh^2 p} \, dp$$

$$= \frac{2}{\pi R \alpha} \left(\frac{\pi}{b}\right) \left[-\frac{p}{\cosh p} \Big|_0^{\mu} + \int_0^{\mu} \frac{dp}{\cosh p} \right] \tag{35}$$

so

$$\varphi_p'(0, b/2) = \frac{2}{\pi R \alpha} \left[\tan^{-1}(\sinh \mu) - \frac{\mu}{\cosh \mu} \right] \left(\frac{\pi}{b}\right) \tag{36}$$

Then

$$\varphi_p''(0,y) = \frac{2}{\pi R \alpha} \int_0^{\mu} \left(\frac{\pi}{b}\right)^2 p \sinh p \left[\frac{\cos \pi \bar{y}}{(\cosh p - \cos \pi \bar{y})^2} - \frac{2 \sin^2 \pi \bar{y}}{(\cosh p - \cos \pi \bar{y})^2} \right] dp$$

and $$c_p''(0,b/2) = \frac{2}{\pi R\alpha}\left(\frac{\pi}{b}\right)^2 \int_0^\mu \left(\frac{-2\sinh\phi}{\cosh^2\phi}\right)\phi\, d\phi.$$

$$= \frac{2}{\pi R\alpha}\left(\frac{\pi}{b}\right)^2 \left[\frac{\phi}{\cosh^2\phi}\Big|_0^\mu - \int_0^\mu \frac{d\phi}{\cosh^2\phi}\right]$$

or

(37) $$c_p''(0,b/2) = \frac{2}{\pi R\alpha}\left(\frac{\pi}{b}\right)^2 \left[\tanh\mu - \frac{\mu}{\cosh^2\mu}\right]$$

Then from (32) and (33)

$$c_p(0,y) = \frac{-2}{\pi R\alpha}\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n}(2^{2n}-1)}{(2n+1)!}B_{2n-1}\left(\frac{\nu}{\beta}\right)^{2n+1}\right.$$

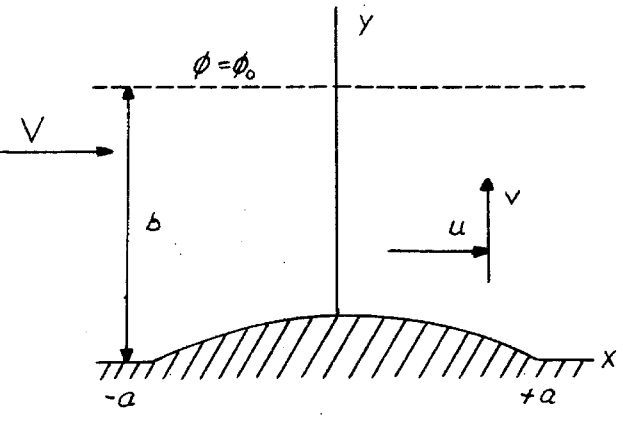
$$- \left(\frac{\pi}{b}\right)(y-b/2)\left[\tan^{-1}(\operatorname{asinh}\frac{\alpha}{\beta}) - \frac{\alpha}{\beta\cosh\frac{\alpha}{\beta}}\right]$$

$$\left. - \frac{1}{2}\left(\frac{\pi}{b}\right)^2(y-\tfrac{b}{2})^2\left[\tanh\frac{\alpha}{\beta} - \frac{\alpha}{\beta\cosh^2\frac{\alpha}{\beta}}\right] + \cdots\right\}$$

(38) $$c_p(0,y) = c_{p_{0max}}\left\{\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{2n-1}(2^{2n}-1)}{(2n+1)!}B_{2n-1}\left(\frac{\alpha}{\beta}\right)^{2n} + \frac{\pi}{2}\frac{\beta}{\alpha}(\tfrac{1}{2}-\bar{y})\left[\tan^{-1}(\operatorname{asinh}\frac{\alpha}{\beta}) - \frac{\alpha}{\beta\cosh\frac{\alpha}{\beta}}\right]\right.$$

$$\left. - \frac{\pi^2}{2}\left(\frac{\beta}{2\alpha}\right)(\tfrac{1}{2}-\bar{y})^2\left[\tanh\frac{\alpha}{\beta} - \frac{\alpha}{\beta\cosh^2\frac{\alpha}{\beta}}\right] + \cdots\right\}$$

For the case when the disturbing wall is in a jet, or the airfoil is in an open wind tunnel, the same method can be applied. For the flow as in Fig. 5a the boundary condition 2'a must be changed so that (2" a $\phi = \phi_o$ a constant at y = b or y = h), but otherwise the problem is the same. Hence, the solution can be written.



(39) $$\phi = \frac{-V}{\beta\pi}\int_0^\infty \frac{d\lambda}{\lambda}\int_{-\infty}^{\infty} \frac{\sinh \lambda(h-\eta)}{\cosh \lambda h} f(t)\cos\lambda(t-\xi)\,dt + \phi_0$$

for then

$$\frac{\partial^2\phi}{\partial\xi^2} + \frac{\partial^2\phi}{\partial\eta^2} = 0 \quad , \quad \phi = \phi_0 \text{ at } \eta = h \quad , \qquad \eta = 0$$

$$\frac{\partial\phi}{\partial\eta} = \frac{V}{\beta\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} f(t)\cos\lambda(t-\xi)\,dt = \frac{V}{\beta} f'(\xi).$$

Then, in the same way as before,

(40) $$c_p = \frac{2}{\beta\pi}\int_0^\infty d\lambda \int_{-\infty}^{\infty} \frac{\sinh\lambda(h-\eta)}{\cosh\lambda h} f'(t)\sin\lambda(t-\xi)\,dt$$

or changing the order of integration

$$c_p = \frac{2}{\beta\pi}\int_{-\infty}^{\infty} f'(t)\,dt \int_0^\infty \frac{\sinh\lambda(h-\eta)}{\cosh\lambda h}\sin\lambda(t-\xi)\,d\lambda$$

But $$\int_0^\infty \frac{\sinh \frac{p}{q}w}{\cosh \frac{q}{q}w}\sin hw\,dw = \frac{\pi}{b}\sin\frac{\frac{p}{2q}\pi}{\cosh\frac{h}{b}\pi + \cos\frac{p}{q}\pi} \qquad p < 2q$$

See Ref. 2

so that

(41) $$c_p(x,y) = \frac{2}{\beta^2 b}\sin\left[\frac{\pi}{2}\left(1-\frac{y}{b}\right)\right]\int_{-\infty}^{\infty} \frac{f'(t)\sinh\left(\frac{t-x}{\beta b}\frac{\pi}{2}\right)}{\cosh\left(\frac{t-x}{\beta b}\pi\right) - \cos\frac{y}{b}\pi}\,dt$$

These results can be applied to the same disturbance as before.

2. Circular arc airfoil in open wind tunnel.

Then in the same notation as before

(42) $$c_p = \frac{-2}{\beta^2 bR}\sin\left[\frac{\pi}{2}(1-\bar{y})\right]\int_{-1}^{1} \frac{t\sinh\left(\frac{t-x}{\beta b}\frac{\pi}{2}\right)}{\cosh\left(\frac{t-x}{\beta b}\pi\right) - \cos\pi\bar{y}}\,dt$$

$$C_p = \frac{-2}{\beta^2 bR} \sin\left[\frac{\pi}{2}(1-\bar{y})\right] \frac{h^2}{\pi^2} \int_{-\mu}^{\mu} \frac{p \sinh\left(\frac{p-q}{2}\right)}{\cosh(p-q) - \cos\pi\bar{y}}\, dp. \qquad 17.$$

and a special case of some interest is x = y = 0. Then

$$C_p(0,0) = -\frac{4}{\pi R\beta} \int_0^{\mu} \frac{p \sinh\frac{p}{2}}{\cosh p - 1}\, dp$$

(43)

$$= C_{p_{max}} \int_0^{\mu} \frac{p}{2} \operatorname{csch}\frac{p}{2}\, dp$$

Expanding

$$\operatorname{csch}\frac{p}{2} = \frac{2}{p}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{(2^{2n-1}-1)B_{2n-1}}{(2n)!}\left(\frac{p}{2}\right)^{2n}\right]$$

and integrating yields

$$C_p = C_{p_{max}}\left[1 + 2\sum_{n=1}^{\infty}(-1)^n \frac{(2^{2n-1}-1)B_{2n-1}}{2^{2n}(2n+1)!}\left(\frac{\alpha}{\beta}\right)^{2n}\right]$$

so that

$$(44)\quad C_p(0,0) = C_{p_{max}}\left[1 + \sum_{n=1}^{\infty}(-1)^n \frac{(2^{2n-1}-1)B_{2n-1}}{2^{2n-1}(2n+1)!}\left(\frac{\alpha}{\beta}\right)^{2n}\right].$$

III. RESULTS AND COMPARISON WITH EXPERIMENTS

Some results and comparison with experiments are shown in Figs. 6-13. The experiments were made in the 2" x 20" transonic wind tunnel of the GALCIT (see Ref. 11).

Figs. 6 and 7 show the pressure distribution at the airfoil surface. $C_{p\,critical}$ denotes pressure at which M = 1. As would be expected, the agreement is better in the leading half of the airfoil and at lower Mach numbers. There is intrinsically less error in the theory at lower Mach numbers. Also, boundary layer effects in distorting the shape of the surface are less at smaller Mach numbers and on the leading part of the airfoil. The effect of boundary layer and occurrence of shock waves on the aft part of the airfoil have been noted in schlieren photographs (see Ref. 13). The parameters used in the computation were

a = half chord of airfoil = 1.5" = 1

R = radius of curvature of airfoil surface = 6.785 = 4.51

b = 1/2 height of tunnel = 10" = 6.67

$\alpha = \frac{\pi}{b}$ tunnel height parameter = $\frac{\pi}{6.67}$ = .471.

It was interesting to note that the wall interference effect was approximately constant along the surface of the airfoil when computed with these parameters.

Fig. 8 makes some comparison of the wind tunnel wall effects for this tunnel and shows especially the increasing effect at higher M . It is noted that the interference effect of an open wind tunnel is less than that of a closed one. Fig. 9 shows how strongly

the ratio of tunnel height to model chord affects the interference. Fig. 10 gives some indication of the decay of airfoil disturbance above the surface. It also indicates that the supersonic region predicted would be too small.

In order to get the shape of some typical supersonic regions Figs. 11, 12, 13 were prepared and for simplicity wind tunnel wall effects were neglected. To show how much the local supersonic zone computed differs from the supersonic flow, the distribution of velocity on the supersonic zone at the surface was used as a starting point and the flow field extended by the method of characteristics (See Ref. 3). The result is shown as the dotted line in Fig. 11. Actually, the supersonic zones in this airfoil become more asymmetric at higher M and hence the agreement with potential theory would be worse.

An interesting theoretical method to determine conditions within the supersonic zone would be to successively alter the surface distribution of pressure in the supersonic zone and make the $M = 1$ boundary of the characteristic extension agree with that predicted by the small perturbation theory.

The small perturbation theory has then yielded some interesting results for transonic flow over airfoils, but for more information more accurate theories are needed. It is felt that the best attacks in the future will be through the second approximation of this theory or through the approximate transonic equation.

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FIG. 6

COMPARISON OF EXPERIMENTAL DATA
AND COMPUTED PRESSURE DISTRIBUTION
INCLUDING WALL EFFECT

3" x 12% CIRCULAR ARC AIRFOIL
$M_\infty = 0.795$

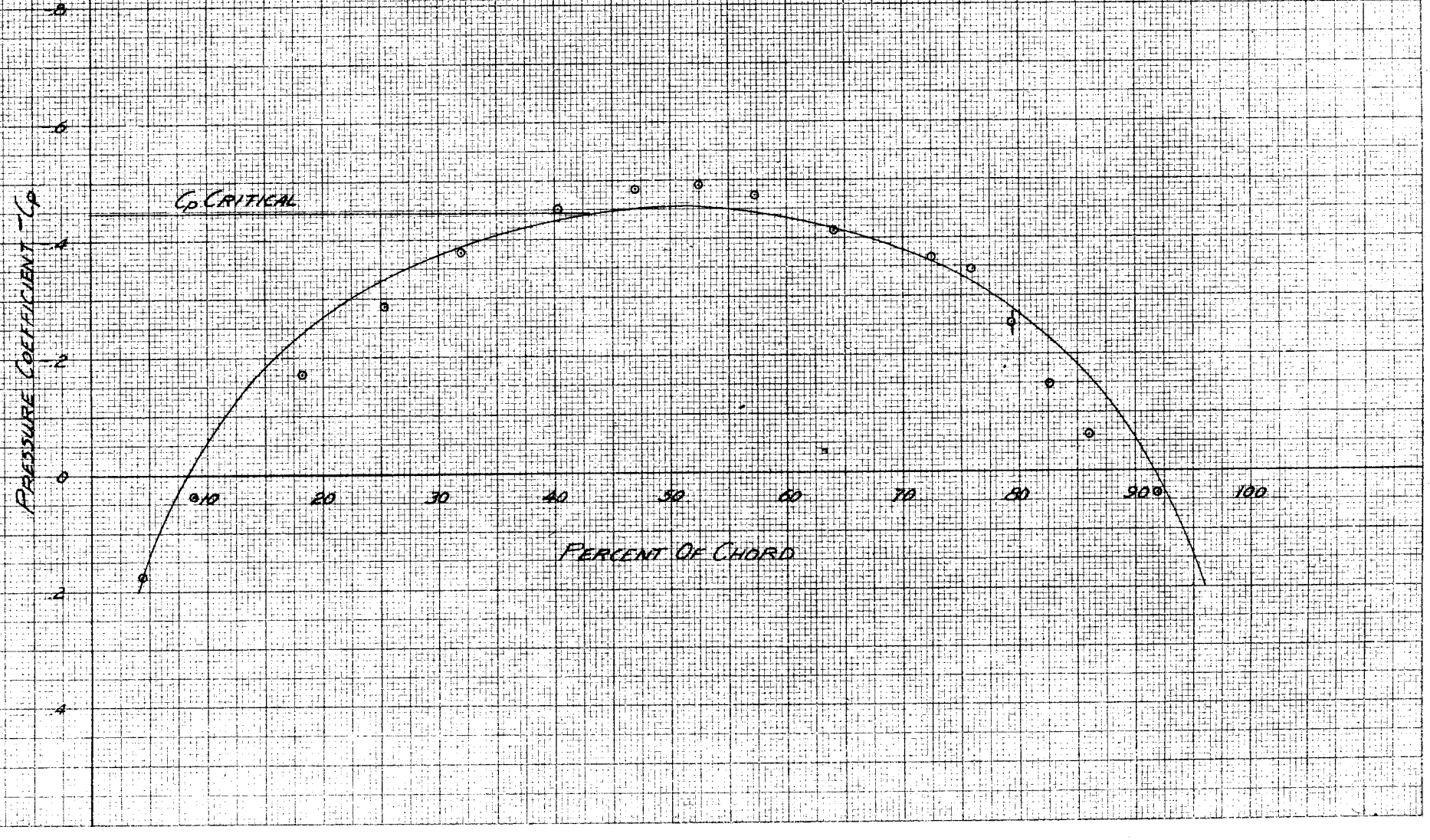


FIG. 7

COMPARISON OF EXPERIMENTAL DATA AND COMPUTED PRESSURE DISTRIBUTION INCLUDING WALL EFFECT

3"x12% CIRCULAR ARC AIRFOIL
$M_\infty = 0.846$

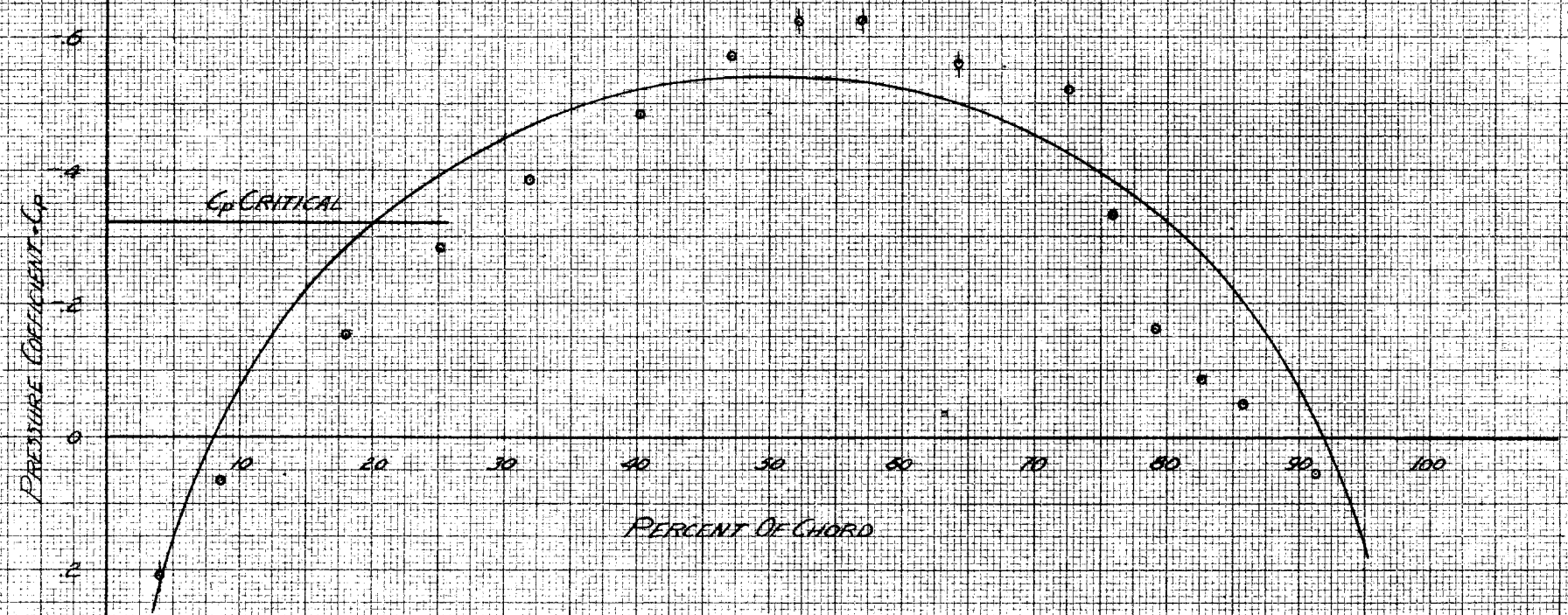


FIG. 8

COMPARISON OF EXPERIMENTAL DATA AND COMPUTED VALUES FOR THE MAX. C_p OF A 3"X12% BICONVEX CIRCULAR ARC AIRFOIL IN A 2"X20" TRANSONIC WIND TUNNEL (CLOSED)

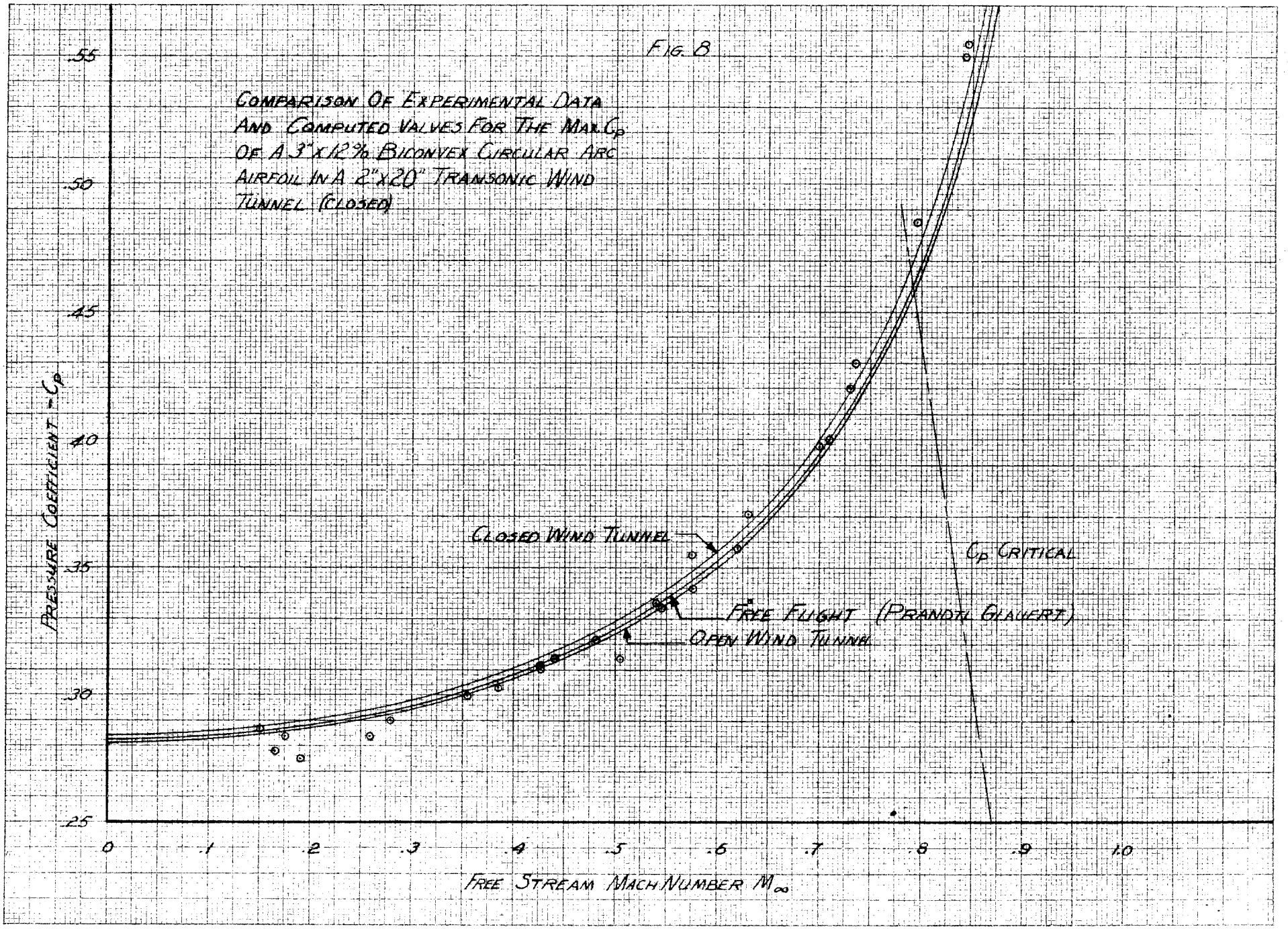


FIG. 9

EFFECT OF WIND TUNNEL INTERFERENCE ON THE MAX. C_p OF A 3" x 12 2/3 CIRCULAR ARC AIRFOIL IN A 2" x 20" WIND TUNNEL

RATIO OF TUNNEL HEIGHT TO MODEL CHORD:

——— 6.67

- - - - 3.33

CLOSED TUNNELS

FREE FLIGHT

OPEN TUNNELS

$-C_p$ PRESSURE COEFFICIENT

.55 .50 .45 .40 .35 .30 .25

1.0 1.1 1.2 1.3 1.4 1.5 1.6 1.7 1.8 1.9 2.0

$\frac{1}{\sqrt{1-M_\infty^2}} = \frac{1}{\beta}$ COMPRESSIBILITY PARAMETER

FIG. 10

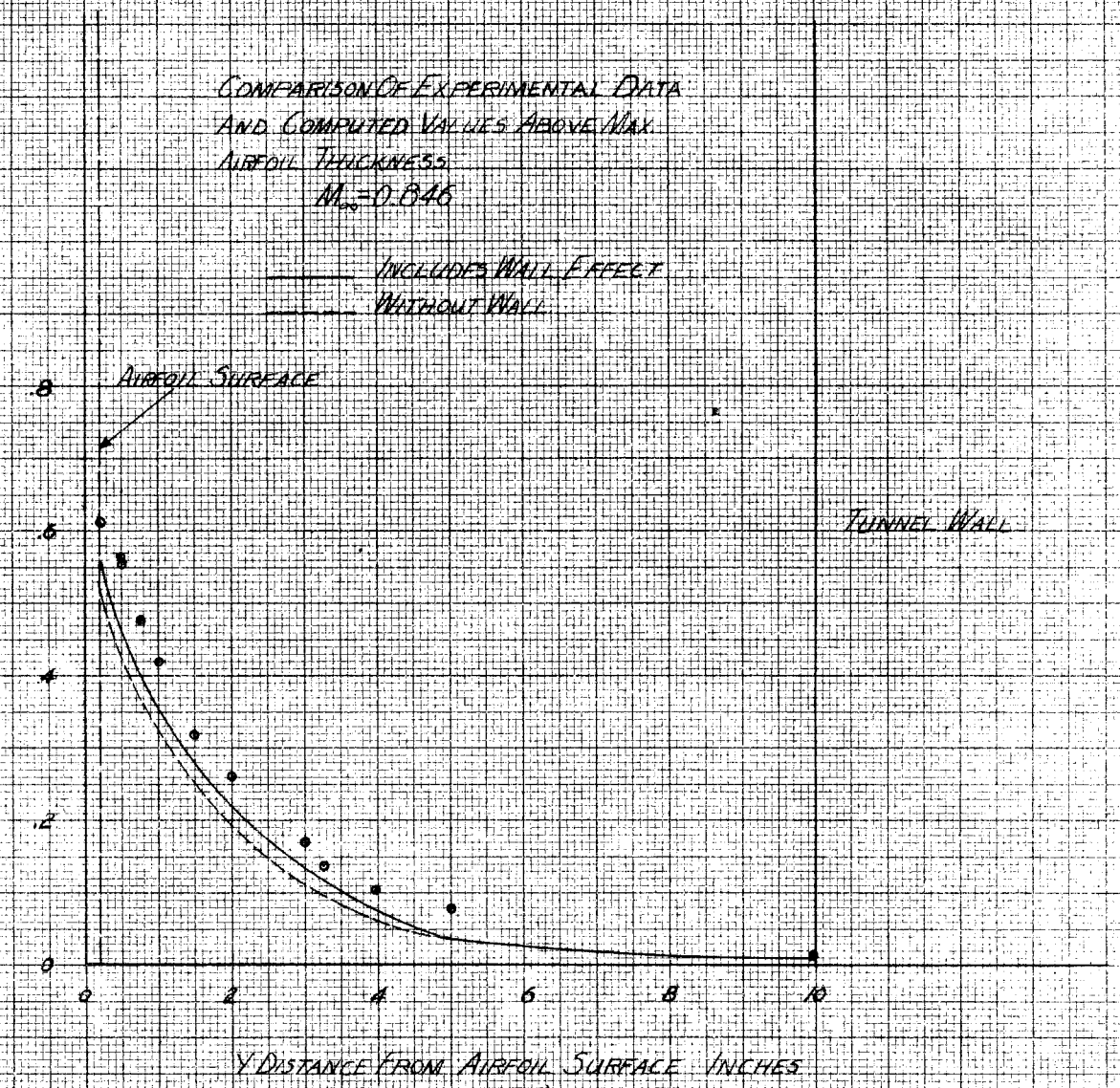

COMPARISON OF EXPERIMENTAL DATA
AND COMPUTED VALUES ABOVE MAX.
AIRFOIL THICKNESS.
$M_\infty = 0.846$
INCLUDES WALL EFFECT
WITHOUT WALL
AIRFOIL SURFACE
TUNNEL WALL
$-C_p$
.8
.6
.4
.2
0
0
2
4
6
8
10
Y DISTANCE FROM AIRFOIL SURFACE INCHES

FIG. 11

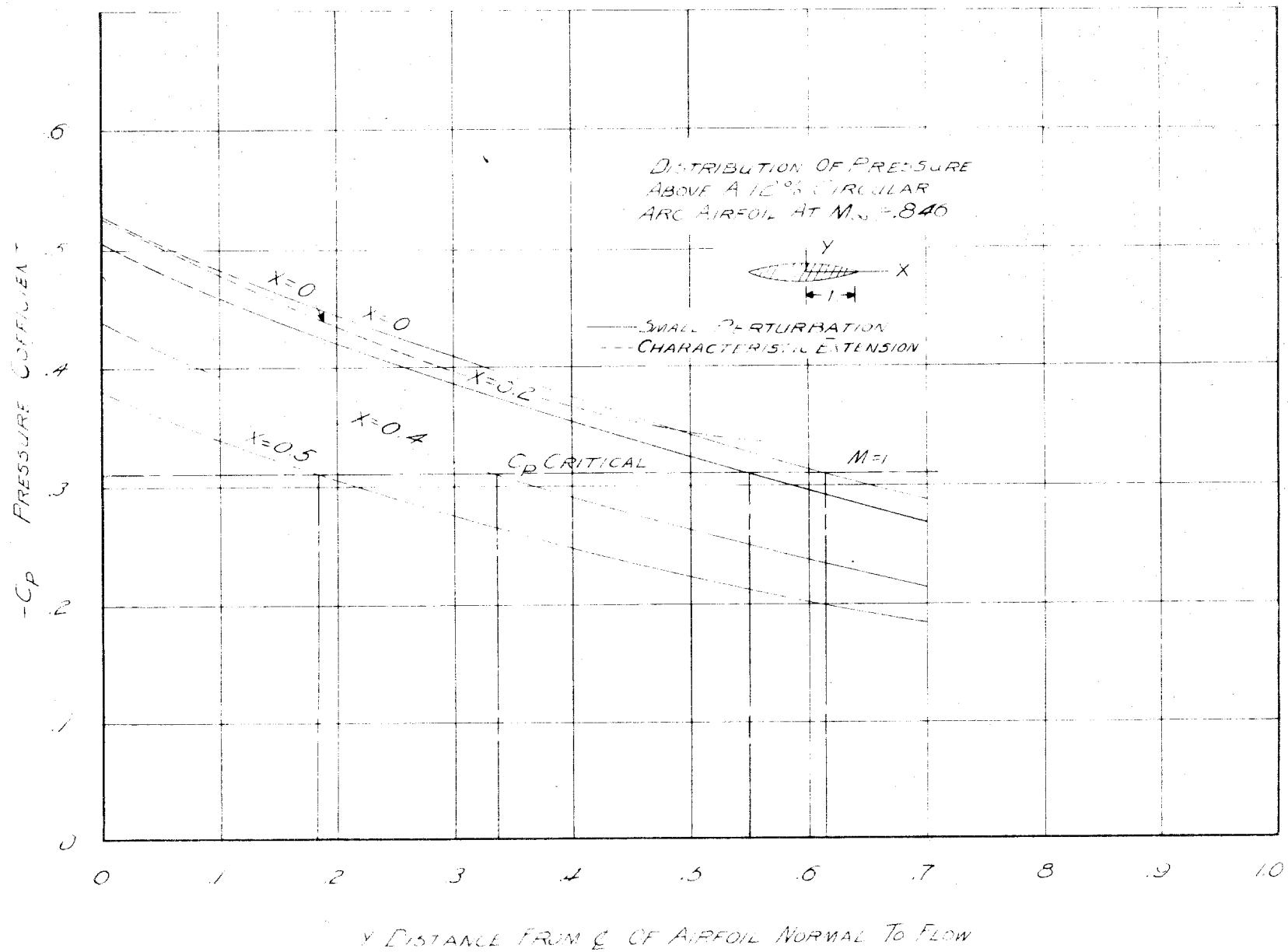

DISTRIBUTION OF PRESSURE ABOVE A 12% CIRCULAR ARC AIRFOIL AT M∞ = .846
Y
X
1
SMALL PERTURBATION
CHARACTERISTIC EXTENSION
X=0
X=0
X=0.2
X=0.4
X=0.5
CP CRITICAL
M=1
-CP PRESSURE COEFFICIENT
0
.1
.2
.3
.4
.5
.6
.7
.8
.9
1.0
Y DISTANCE FROM ℄ OF AIRFOIL NORMAL TO FLOW

FIG 12

SHAPE OF SUPERSONIC ZONE FOR $M_\infty = .846$ AND SOME ILLUSTRATION OF GROWTH OF SUPERSONIC ZONE FOR VARIOUS M_∞

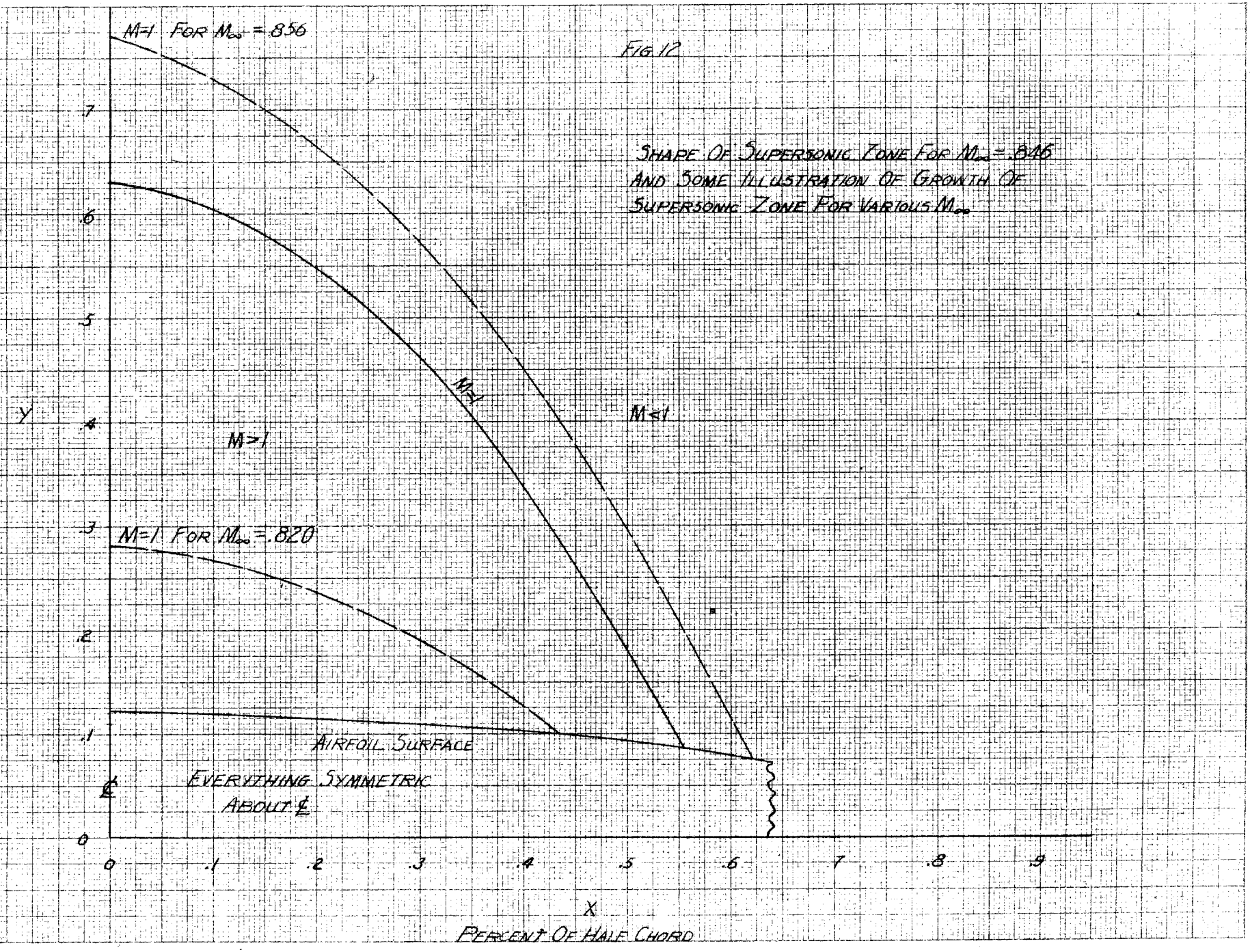
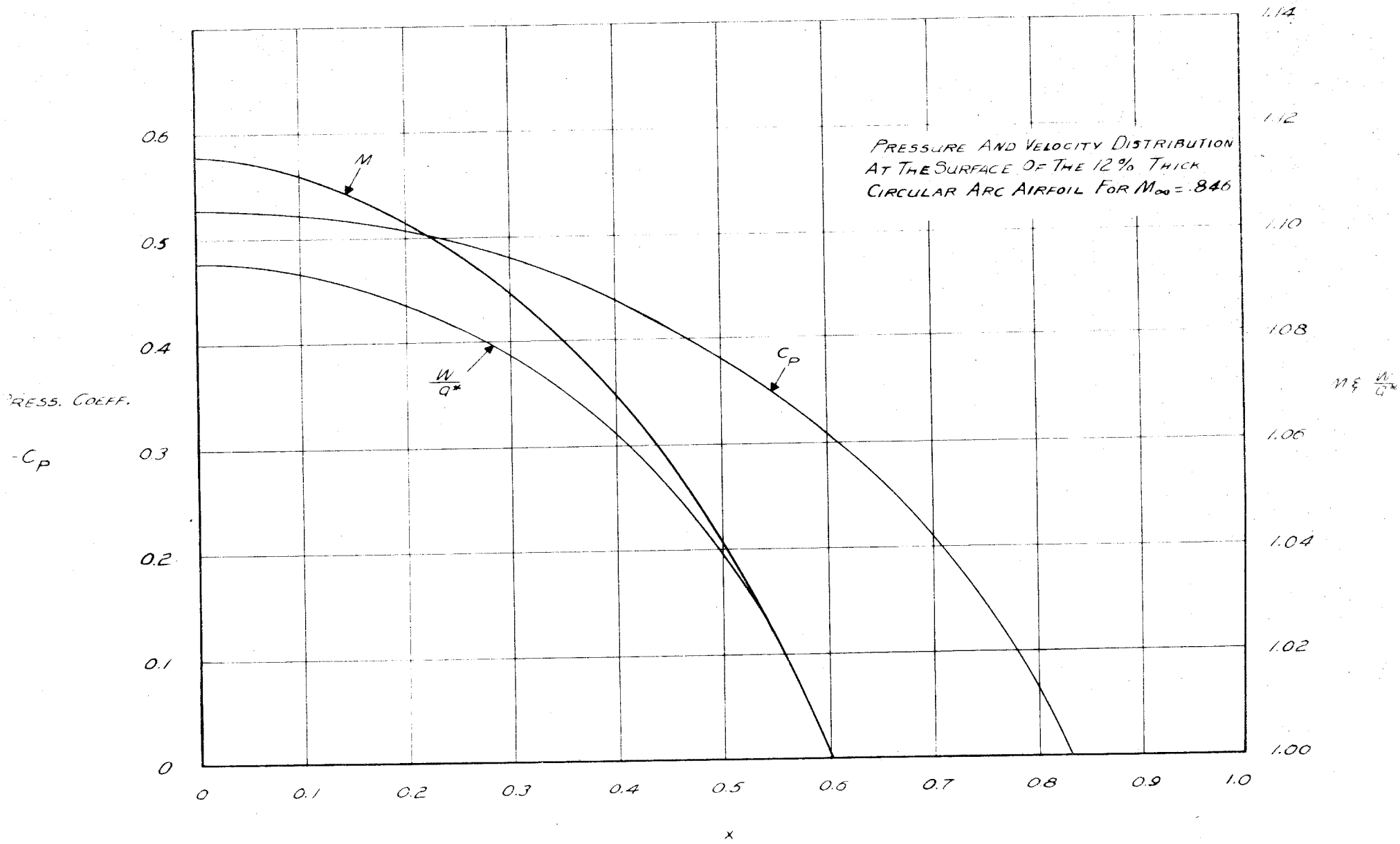


FIG. 13


PRESSURE AND VELOCITY DISTRIBUTION AT THE SURFACE OF THE 12% THICK CIRCULAR ARC AIRFOIL FOR $M_\infty = .846$
M
$\frac{W}{a^*}$
C_p
$-C_p$ PRESS. COEFF.
$M \& \frac{W}{a^*}$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
1.00
1.02
1.04
1.06
1.08
1.10
1.12
1.14
x
DISTANCE FROM MAX. THICKNESS C = 2