A Coarse Jacquet-Zagier Trace Formula for GL(n) with Applications

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ABSTRACT

In this thesis we establish a coarse Jacquet-Zagier trace identity for $\mathrm{GL}(n)$. This formula connects adjoint L-functions on $\mathrm{GL}(n)$ with Artin L-functions attached to certain induced Galois representations. We prove the absolute convergence when $\mathrm{Re}(s) > 1$, and obtain holomorphic continuation under almost all character twists. Moreover, as an application, we obtain that holomorphy of certain adjoint L-functions for $\mathrm{GL}(n)$ implies Dedekind conjecture of degree n. Some nonvanishing results are also proved.

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Chapter 1

INTRODUCTION

1.1 Trace Formula: from Arthur-Selberg to Jacquet-Zagier

Let F be a global field, with adele ring $\mathbb{A}_F$. Let $G = \mathrm{GL}(n)$. We consider a smooth function $\varphi : G(\mathbb{A}_F) \to \mathbb{C}$ which is left and right K-finite for a compact subgroup K of $G(\mathbb{A}_F)$, transforms by a unitary character ω of $Z_G(\mathbb{A}_F)$, and has compact support modulo $Z_G(\mathbb{A}_F)$. Denote by $\mathcal{H}(G(\mathbb{A}_F), \omega)$ the set of such functions. Then $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$ defines an integral operator

$$R(\varphi)f(y) = \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x)f(yx)dx, \tag{1.1}$$

on the space $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of functions on $G(F)\backslash G(\mathbb{A}_F)$ which transform under $Z_G(\mathbb{A}_F)$ by ω^{-1} and are square integrable on $G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. This operator can clearly be represented by the kernel function

$$\mathrm{K}^{\varphi}(x, y) = \sum_{\gamma \in Z_G(F)\backslash G(F)} \varphi(x^{-1}\gamma y).$$

We will omit the superscript φ and simply write $\mathrm{K}(x, y)$ for $\mathrm{K}^{\varphi}(x, y)$.

It is well known that $L^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ decomposes into the direct sum of the subspace $L_0^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ of cusp forms and spaces $L_{\mathrm{Eis}}^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ and $L_{\mathrm{Res}}^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$ defined using Eisenstein series and residues of Eisenstein series respectively. Then K splits up as $\mathrm{K} = \mathrm{K}_0 + \mathrm{K}_{\mathrm{Eis}} + \mathrm{K}_{\mathrm{Res}}$. The Selberg trace formula (cf. [Sel56]) gives an expression for the trace of the operator $R(\varphi)$ restricted to the cuspidal spectrum, and is roughly of the form

$$\int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x, x)dx = \Sigma_{\mathrm{Geo}} - \Sigma_{\mathrm{Eis}} - \Sigma_{\mathrm{Res}}, \tag{1.2}$$

where $\Sigma_{\mathrm{Geo}}, \Sigma_{\mathrm{Eis}}$, and Σ_{Res} are contributions from geometric side, continuous spectrum, and residual spectrum, respectively. This formula and its generalizations play important roles in the study of general theory of automorphic representations and Langlands program. Typically, the right hand side of (1.2) has some convergence issue, so a truncation is usually needed.

In [Zag77], Zagier introduced the Rankin-Selberg method into the treatment of (1.2). Precisely, he considered

$$I_0^{\varphi}(s) = \int_{\mathrm{GL}(2,\mathbb{Q})Z(\mathbb{A}_{\mathbb{Q}})\backslash \mathrm{GL}(2,\mathbb{A}_{\mathbb{Q}})} \mathrm{K}_0^{\varphi}(x,x)E(x,s)dx, \tag{1.3}$$

where $E(x,s)$ is an Eisenstein series. Note that $\mathrm{K}_0(x,x)$ is rapidly decreasing and $E(x,s)$ is slowly increasing outside $s=1$, thus the right hand side of (1.3) is well defined as a meromorphic function, which has a simple pole at $s=1$. Zagier obtained a spectral expansion $I_0^{\varphi}(s) = \Sigma_{\mathrm{Geo}}(s) - \Sigma_{\mathrm{Eis}}(s) - \Sigma_{\mathrm{Res}}(s)$ with meromorphic continuation, from which he deduced holomorphy of L-function associated to symmetric square of classical cusp forms.

Zagier's trace identity (1.3) was further developed by Jacquet and Zagier [JZ87] in terms of representation theoretical language to give a new proof of holomorphy of adjoint L-functions on $\mathrm{GL}(2,\mathbb{A}_F)$. They show (after continuation) the contribution from continuous and residual spectrums is a holomorphic multiple of the Dedekind zeta function, and the contribution from elliptic regular conjugacy classes gives certain Artin L-series associated to finitely many quadratic extensions of F. Hence the holomorphy of adjoint L-functions can be deduced from class field theory, or more generally, the (twisted) Dedekind conjecture (see Conjecture 3 below). As another main motivation in loc. cit., studying $I_0^{\varphi}(s)$ provides a new way to derive the Selberg trace formula by taking the reside at $s=1$, avoiding the recourse to Arthur's truncation. See [**Wu19**] for details.

1.2 Statement of the Main Results

Aiming to generalize [JZ87] to higher ranks, we study in this paper a generalization $I_0^{\varphi}(s;\tau)$ of $I_0^{\varphi}(s)$ (defined in (1.3)) for $G=\mathrm{GL}(n)$ over a global field F :

$$I_0^{\varphi}(s,\tau) = \int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0^{\varphi}(x,x)E_P(x,\Phi,\tau;s)dx,$$

where $E_P(x,\Phi,\tau;s)$ is an Eisenstein series. See (1.6) in Sec. 1.3 for the precise definition. Note that the cuspidal part of the Arthur-Selberg trace formula can be realized as the residue of $I_0^{\varphi}(s,\tau)$ (with $\tau=\mathbf{1}$) at $s=1$. In this paper, we obtain a coarse geometric and spectral expansion of $I_0^{\varphi}(s,\tau)$ and verify their absolute convergence when $\mathrm{Re}(s)>1$. We also prove the analytic continuation for almost all character τ's. Some of our main results (Theorem E, F, G and H) may be summarized informally as follows:

Theorem A. *Let notation be as before. Let* $\mathrm{Re}(s) > 1$. *Let* $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$. *Then* $I_0^{\varphi}(s;\tau)$ *admits an expansion:*

$$I_0^{\varphi}(s;\tau) = I_{\mathrm{Geo,Reg}}^{\varphi}(s,\tau) - I_{\infty,\mathrm{Reg}}^{\varphi}(s,\tau) + I_{\mathrm{Sing}}^{\varphi}(s,\tau) - I_{\mathrm{Whi}}^{\varphi}(s,\tau), \tag{1.4}$$

where $I_{\mathrm{Geo,Reg}}(s,\tau)$ *is a finite sum of Tate integrals over certain direct sum of Étale algebras of degree* $\leq n$; $I_{\mathrm{Whi}}^{\varphi}(s,\tau)$ *is an infinite sum of* $I_{\chi}(s,\tau,\lambda)$ *over cuspidal data* χ *associated to proper standard parabolic subgroups of* G, *with each* $I_{\chi}(s,\tau,\lambda)$ *being a multiple of Rankin-Selberg period attached to* χ; $I_{\infty,\mathrm{Reg}}^{\varphi}(s,\tau)$ *is a multiple of*

$$\frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}.$$

Here $\Lambda(s,\cdot)$ *refers to complete Hecke L-functions. Furthermore, if* $\tau^k \neq 1$ *for* $1 \leq k \leq n$, *then* (1.4) *has a meromorphic continuation to* $\mathbb{C}$, *with* $I_{\infty,\mathrm{Reg}}^{\varphi}(s,\tau)/\Lambda(s,\tau)$ *and* $I_{\mathrm{Whi}}^{\varphi}(s,\tau)/\Lambda(s,\tau)$ *being holomorphic in* $\mathrm{Re}(s) \geq 1/2$.

Remark 2. *(1). The expansion* (1.4) *generalizes Jacquet and Zagier's formula for* GL(2) *(see [JZ87]) to* GL(n). *A restricted version was obtained by Flicker [Fli92] under some choice of test functions* φ *so that only elliptic regular part of* $I_{\mathrm{Geo,Reg}}(s,\tau)$ *shows up on the right hand side of* (1.4). *New ideas of our proof are briefly summarized in Section 1.5 below.*

(2). $I_{\mathrm{Sing}}(s,\tau)$ *is defined geometrically, and it appears essentially when* $n \geq 3$. *For certain applications, one can eliminate it by choosing discrete and cuspidal test functions in the sense of [FK88]. Such test functions will be used to deduce Theorem B (see Section 1.4 below), as an application of Theorem A. Also, the analytic continuation of* $I_{\mathrm{Sing}}^{\varphi}(s,\tau)/\Lambda(s,\tau)$ *is given in [***Yan21***] when* $n \leq 4$.

(3). Each individual $I_{\chi}(s,\tau,\lambda)$ *is a period of automorphic forms in the case of* (GL(n) × GL(n), GL(n)) *over the diagonal, in parallel to the* (GL($n+1$) × GL(n), GL(n)) *studied in [IY15].*

The distribution $I_0^{\varphi}(s,\tau)$ and its calculation (4) are interesting for several reasons: $I_0^{\varphi}(s,\tau)$ is the first moment of a family of Rankin-Selberg L-functions; the formula (4) should involve more information than the Arthur-Selberg trace formula, e.g., one may take τ to be of order n and evaluate (1.4) at $s = 1$ to obtain a twisted trace formula for $G = \mathrm{GL}(n)$, which has also been carried out using a different approach in [Kaz83] when n is a prime; and Theorem 2 of [JZ87] reinterpreted the GL(2) case of the twisted trace identity as essentially equivalent to a theorem of Labesse and Langlands [LL79].

On the other hand, the geometric-spectral expansion (4) of $I_0^{\varphi}(s,\tau)$ is quite involved. When $n > 2$, the continuous spectrum has not been investigated before. Nevertheless, the expansion turns out to convey some interesting information connecting L-functions defined analytically and algebraically. In fact, we shall compute the expansion and deduce from it that **holomorphy of certain adjoint L-functions** for $G = \mathrm{GL}(n)$ implies the **Dedekind conjecture** for degree n extensions (see Theorem B on p. 5). The relation between these two problems has been conjectured for a long time, e.g., see [JZ87] and [JR97].

Another consequence of studying $I_0^{\varphi}(s,\tau)$ is holomorphy of adjoint L-functions (and their twists) for all cuspidal representations on $\mathrm{GL}(n)$, $n \leq 4$. This is done in [**Yan21**].

1.3 Basic Notation

Denote by $\mathcal{S}(\mathbb{A}_F^n)$ the space of Schwartz-Bruhat functions on the vector space $\mathbb{A}_F^n$ and by $\mathcal{S}_0(\mathbb{A}_F^n)$ the subspace spanned by products $\Phi = \prod_v \Phi_v$ whose components at real and complex places v have the form

$$\Phi_v(x_v) = e^{-\pi \sum_{j=1}^n x_{v,j}^2} \cdot Q(x_{v,1}, x_{v,2}, \cdots, x_{v,n}),\ x_v = (x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in F_v^n,$$

where $F_v \simeq \mathbb{R}$, and $Q(x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in \mathbb{C}[x_{v,1}, x_{v,2}, \cdots, x_{v,n}]$; and

$$\Phi_v(x_v) = e^{-2\pi \sum_{j=1}^n x_{v,j}\bar{x}_{v,j}} \cdot Q(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}),$$

where $F_v \simeq \mathbb{C}$ and $Q(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n})$ is a polynomial in the ring $\mathbb{C}[x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}]$.

Denote by Ξ_F the set of unitary characters on $F^\times \backslash \mathbb{A}_F^\times$ which are trivial on $\mathbb{R}_+^\times$. For any $\xi \in \Xi_F$, denote by $\Lambda(s,\xi)$ the *complete* Hecke L-function associated to ξ. Let $\Phi \in \mathcal{S}_0(\mathbb{A}_F^n)$. Let $\tau \in \Xi_F$ be fixed. Let $\eta = (0, \cdots, 0, 1) \in F^n$. Set

$$f(x, \Phi, \tau; s) = \tau(\det x)|\det x|^s \int_{\mathbb{A}_F^\times} \Phi(\eta t x)\tau(t)^n |t|^{ns} d^\times t,$$

which is a Tate integral (up to holomorphic factors) for the complete L-function $\Lambda(ns, x.\Phi, \tau^n) = L_\infty(ns, x_\infty.\Phi_\infty, \tau_\infty^n) \cdot L_{\mathrm{fin}}(ns, x_{\mathrm{fin}}.\Phi_{\mathrm{fin}}, \tau_{\mathrm{fin}}^n)$. It converges absolutely uniformly in compact subsets of $\mathrm{Re}(s) > 1/n$. Since the mirabolic subgroup P_0 is the stabilizer of η. Let $P = P_0 Z_G$ be the full $(n-1,1)$ parabolic subgroup of G, then $f(x,s) \in \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\delta_P^{s-1/2}\tau^{-n})$, where δ_P is the modulus character for the parabolic P. Then we can define the Eisenstein series

$$E_P(x,\Phi,\tau;s) = \sum_{\gamma\in P(F)\backslash G(F)} f(x,\Phi,\tau;s), \tag{1.5}$$

which converges absolutely for $\mathrm{Re}(s) > 1$. Also, we define the integral:

$$I_0^{\varphi}(s,\tau) = \int_{G(F)Z(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0^{\varphi}(x,x)E_P(x,\Phi,\tau;s)dx. \tag{1.6}$$

If there is no confusion in the context, we will alway write $I_0(s)$ (resp. $f_\tau(x,s)$ or $f(x,s)$) instead of $I_0^{\varphi}(s;\tau)$ (resp. $f(x,\Phi,\tau;s)$) for simplicity.

1.4 Some Applications

The distribution $I_{\mathrm{Geo,Reg}}(s,\tau)$ in (1.4) turns out to play a role in certain cases of *beyond endoscopy*, see Altuğ's work [Alt15b], [Alt15a], and [Alt17]. In this section, we give other applications of (1.4) to some conjectures on holomorphy of L-functions and nonvanishing problem. First, we recall

Conjecture 3 (τ-twisted Dedekind Conjecture)**.** *Let notation be as before. Let E/F be an extension of global fields. Then $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic when $s\neq 1$, where $N_{E/F}$ is the relative norm.*

When τ is trivial, the above conjecture is conventionally called the Dedekind conjecture, which is known when E/F is Galois by the work of Aramata and Brauer (see Chap. 1 of [Mar77]) or has a solvable Galois closure $\widetilde{E}/F$ by the work of Uchida [Uch75] and van der Waall [Waa75]. Moreover, Dedekind conjecture is the principal case of Artin's holomorphy conjecture. The τ-twisted version of Conjecture 3 has been proved by Murty [MR00] when E/F is either Galois or has a solvable closure. However, the general case (even general degree 5 extensions) is not yet known.

When $n=2$, [JZ87] provides a connection between adjoint L-functions associated to $\pi \in \mathcal{A}_0(\mathrm{GL}(2,F)\backslash \mathrm{GL}(2,\mathbb{A}_F),\omega^{-1})$ and $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ when E/F is quadratic. It was noted in [JR97] that, at least for degree/rank n up to 5, the two families seem to be related on a nuts-and-bolts level in the theory of integral representations, in addition to the relationships suggested by [JZ87].

Let $\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ be the subspace generated by cuspidal representations $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ such that π has a supercuspidal component. Following

[JZ87], Flicker [Fli92] used a simple trace formula to conclude, *modulo* the key Lemma 4 in loc.cit., that Conjecture 3 implies holomorphy of adjoint L-functions

$$\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)=\frac{\Lambda(s,\pi\times\widetilde{\pi}\otimes\tau)}{\Lambda(s,\tau)},\quad \pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1}),$$

when $s\neq 1$. However, this lemma is not correct as pointed out by Flicker himself (ref. [Fli93], P. 202). Consequently, the asserted implication is not complete. In this section we will prove an implication in the opposite direction, obtaining

Theorem B. *Let notation be as before. Assume the twisted adjoint L-functions* $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ *are holomorphic at* $s\neq 1$ *for all* $\pi\in\mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$*. Then the* τ*-twisted Dedekind conjecture holds for all field extensions of* E/F *of degree* n*.*

Remark 5. *(1). This relation provides a new perspective in the study of Dedekind conjecture, which is currently wide open when the degree is larger or equal to* 5*.*

(2). Suppose $\tau^k\neq 1$, $1\leq k\leq n$*. We can conclude from Theorem A, Theorem B and Theorem H (see Sec. 7) that, if* $I_{\mathrm{Sing}}(s,\tau)/\Lambda(s,\tau)$ *admits a holomorphic continuation, then the twisted adjoint L-functions* $L(s,\pi,\mathrm{Ad}\otimes\tau)$ *are holomorphic at* $s\neq 1$ *for all* $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$ *if and only if the* τ*-twisted Dedekind conjecture holds for all fields extensions of* E/F *of degree* n*.*

In Section 9, we will see the proof of Theorem B would provide a result on the nonvanishing of $L(1/2,\pi\times\widetilde{\pi})$:

Theorem C. *Let notation be as before. Let* $n\geq 2$*. Suppose there exists an extension* E/F *with degree* $[E:F]=n$*, and* $\zeta_E(1/2)\neq 0$*. Then there exists a* $\pi=\pi(E)\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$*, such that* $L(1/2,\pi\times\widetilde{\pi})\neq 0$*.*

Remark 7. *Assuming holomorphy of adjoint L-functions, there should be infinitely many number fields F such that* $L(1/2,\pi\times\widetilde{\pi})=0$ *for all* $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$*. Indeed, Fröhlich [Frö72] proved that there are infinitely many number fields F such that* $\zeta_F(1/2)=0$*. Since* $L(s,\pi,\mathrm{Ad})$ *is conjectured to be holomorphic, then for* ***all*** $\pi\in\mathcal{A}_0(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$, $L(1/2,\pi\times\widetilde{\pi})=0$ *conjecturally.*

1.5 Idea of Proof and the Structure of the Thesis

Starting with the spectral decomposition

$$\mathrm{K}_0(x,x)=\mathrm{K}(x,x)-(\mathrm{K}_{\mathrm{Eis}}(x,x)+\mathrm{K}_{\mathrm{Res}}(x,x)),$$

we will further decompose these kernel functions by algebraic and analytic expansion.

Denote by P_0 the mirabolic subgroup of G. Let $\mathfrak{S}$ be the union of $p^{-1}\gamma p$ modulo the center $Z_G(F)$, where γ runs through F-points of standard parabolic subgroups of G, over all $p \in P_0(F)$. Then

$$\mathrm{K}(x,y) = \sum_{\gamma\in Z_G(F)\backslash G(F)-\mathfrak{S}} \varphi(x^{-1}\gamma y) + \sum_{\gamma\in\mathfrak{S}} \varphi(x^{-1}\gamma y). \tag{1.7}$$

By Proposition 12 in Section 2.1, the set $Z_G(F)\backslash G(F) - \mathfrak{S}$ consists of $P_0(F)$-conjugacy classes, giving rise to regular $G(F)$-conjugacy classes.

On the other hand, by Proposition 26 (see Section 3.1), we have the Fourier expansion for ${}_{\infty}(x,y) = \mathrm{K}_{\mathrm{Eis}}(x,x) + \mathrm{K}_{\mathrm{Res}}(x,x)$:

$$\mathrm{K}_{\mathrm{Eis}}(x,y) + \mathrm{K}_{\mathrm{Res}}(x,y) = \int_{[N_P]} \mathrm{K}(ux,y)du + \sum_{k=2}^{n-1} \mathcal{F}_k\, \mathrm{K}(x,y) + \mathrm{K}_{\mathrm{Whi}}(x,y). \tag{1.8}$$

Thus, combining (1.7) and (1.8) together we then obtain

$$\mathrm{K}_0(x,x) = \mathrm{K}_{\mathrm{Reg}}(x) + \mathrm{K}_{\mathrm{Const}}(x,x) + \mathrm{K}_{\mathrm{Sing}}(x,x) + \mathrm{K}_{\mathrm{Whi}}(x,x), \tag{1.9}$$

where

$$\begin{aligned}
\mathrm{K}_{\mathrm{Reg}}(x,x) &= \sum_{\gamma\in Z_G(F)\backslash G(F)-\mathfrak{S}} \varphi(x^{-1}\gamma x),\\
\mathrm{K}_{\mathrm{Const}}(x,x) &= -\int_{[N_P]} \sum_{\gamma\in Z_G(F)\backslash G(F)-\mathfrak{S}} \varphi(x^{-1}u^{-1}\gamma x)du,\\
\mathrm{K}_{\mathrm{Whi}}(x,x) &= -\sum_{\gamma\in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\infty}(u\gamma x,x)\theta(u)du,\\
\mathrm{K}_{\mathrm{Sing}}(x,x) &= \sum_{\gamma\in\mathfrak{S}} \varphi(x^{-1}\gamma y) - \int_{[N_P]} \sum_{\gamma\in\mathfrak{S}} \varphi(x^{-1}u^{-1}\gamma x)du - \sum_{k=2}^{n-1} \mathcal{F}_k\, \mathrm{K}(x,x).
\end{aligned}$$

One then substitutes (1.9) into (1.6) to obtain formally

$$I_0(s,\tau) = I_{\mathrm{Reg}}(s,\tau) + I_{\mathrm{Const}}(s,\tau) + I_{\mathrm{Sing}}(s,\tau) + I_{\mathrm{Whi}}(s,\tau), \tag{1.10}$$

where $I_{\mathrm{Whi}}(s,\tau)$ turns out to be an infinite sum of general Rankin-Selberg periods involving Whittaker functions.

As will be seen in Section 2, stabilizers of elements in $Z_G(F)\backslash G(F)-\mathfrak{S}$ are direct sums of Étale algebras over F of degree less or equal to n. Hence the corresponding distribution $I_{\mathrm{Reg}}(s,\tau)$ would be a sum of certain Artin L-series associated to these Étale algebras. This has been treated in Theorem E in Section 2.2.

In Section 3 we prove Fourier expansion of automorphic forms on $P_0(F)\backslash G(\mathbb{A}_F)$, which implies the decomposition (1.8).

In Section 4.1, we find explicitly representatives of $Z_G(F)\backslash G(F)-\mathfrak{S}$ as $P_0(F)$-conjugacy classes. Then, very roughly, we develop a geometric reduction (in GL(2) case, this is amounts to using Poisson summation, which is not available for GL(n), $n\geq 3$), to relate $I_{\mathrm{Const}}(s,\tau)$ to certain intertwining operators. Hence the convergence and analytic properties follow from theory of intertwining operators. The results are summarized in Theorem F in Section 4.2.

Then the rest of this thesis (Section 5 through 10) is devoted to the distribution $I_{\mathrm{Whi}}(s,\tau)$, which is purely the spectral side. It turns out that Arthur's approach with modified truncation operators is not quite suitable for our situation. The reason is that when we unfold the Eisenstein series and take the Fourier expansion of K_∞, it leads to the loss of $G(F)$-stability. We instead provide a different manipulation, reducing $I_{\mathrm{Whi}}(s,\tau)$ to a Mellin transform of the Kuznetsov relative trace formula, which in turn is majorized by a finite sum of gauges (see Proposition 36). Therefore, we obtain that $I_{\mathrm{Whi}}(s,\tau)$, when $\mathrm{Re}(s)$ is large enough, is an absolute convergent infinite sum of Mellin transforms of certain Rankin-Selberg convolution for *non-discrete* automorphic representations. Concrete statements are given in Theorem G in Section 5.

In Section 6, we prove some properties of Rankin-Selberg periods for non-discrete representations. These results will be used in Section 7 to show absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ in the strip $0<\mathrm{Re}(s)<1$, and thus get a holomorphic function therein, see Theorem H in Section 8 for details. So $I_{\mathrm{Whi}}(s,\tau)$ is holomorphic when $0<\mathrm{Re}(s)<1$ and $\mathrm{Re}(s)>1$. However, for τ such that $\tau^k=1$ for some $1\leq k\leq n$, the function $I_{\mathrm{Whi}}(s,\tau)$ has singularities on the whole boundary $\mathrm{Re}(s)=1$. So we need to find a meromorphic continuation for $I_{\mathrm{Whi}}(s,\tau)$. This is investigated in Section 8, where we obtain continuation of each individual summand of $I_{\mathrm{Whi}}(s,\tau)$ to some zero-free region of Rankin-Selberg L-functions, proving Theorem I, which will be of independent interest, e.g., it will be used in [Yan21]. Further continuation to some open region containing $\mathrm{Re}(s) \geq 1/2$ are obtained for $\mathrm{GL}(n)$, $n \leq 4$, in the Appendix 10.

In Section 9, we gather Theorems E, F, G and H to deduce Theorem A. Furthermore, applying some special test functions φ into Theorem A and dealing with some generalized Tate integral, we then prove Theorems B and C.

1.6 Further Relevant Results

When $n = 2$, $I_{\mathrm{Sing}}(s,\tau)$ has no Fourier part, and has been dealt with in [JZ87]. For general $n \geq 3$, combining Theorems E, F, G, H and the functional equation of Eisenstein series, we conclude that $I_{\mathrm{Sing}}(s,\tau)$ is uniformly convergent when $\mathrm{Re}(s) > 1$, and that it admits a meromorphic continuation to the whole s-plane if $\tau^k \neq 1$ for $1 \leq k \leq n$. Nevertheless, the distribution $I_{\mathrm{Sing}}(s,\tau)$ is rather involved. We will handle it for general τ and $G = \mathrm{GL}(n)$, $n \leq 4$, in [Yan21] by developing different methods from here. The treatment of $I_{\mathrm{Sing}}(s,\tau)$ in [Yan21], together with main results in this thesis, verifies some unknown cases (i.e., $n = 3, 4$) of the Selberg conjecture:

Conjecture 8. *Let notation be as before. Then the complete adjoint L-function* $\Lambda(s,\pi,\mathrm{Ad}) = \Lambda(s,\pi\times\widetilde{\pi})/\Lambda(s,\tau)$ *for* $\mathrm{GL}(n)$ *admits an analytic continuation to the whole complex plane.*

More precisely we prove, in conjunction with Theorem B, that for $n \leq 4$, holomorphy of (twisted) adjoint L-function is equivalent to Dedekind conjecture for degree n. As a consequence, we show in [Yan21] the following:

Theorem D. *Let notation be as before. Let $n \leq 4$. Then the complete L-function* $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ *is entire, unless $\tau \neq 1$ and $\pi\otimes\tau \simeq \pi$, in which case* $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau)$ *is meromorphic with only simple poles at $s = 0, 1$. In particular, Conjecture 8 holds for any cuspidal representation π when $n \leq 4$.*

Remark 10. *If F is a function field over a finite field $\mathbb{F}_q$, by using the cohomology of stacks of shtukas and the Arthur-Selberg trace formula, L. Lafforgue showed the Langlands correspondence between cuspidal automorphic representations π of* $\mathrm{GL}_n(\mathbb{A}_F)$ *and irreducible n-dimensional $\overline{\mathbb{Q}_l}$ representations ρ of the absolute Galois group over F (see [Laf02]), with $l \nmid q$. Then Theorem D follows from the identity* $\Lambda(s,\pi,\mathrm{Ad}\otimes\tau) = \Lambda(s,\mathrm{Ad}\,\rho\otimes\tau)$ *and analytic properties of* $\Lambda(s,\mathrm{Ad}\,\rho\otimes\tau)$, *which is well known by Weil [Wei74]. Our proof works for an arbitrary global field F. So* *it provides a new proof in the function field case. We shall however focus on the case that F is a number field, where such a global Langlands correspondence is not available.*

Remark 11. *If we admit Piatetski-Shapiro's strong conjecture on converse theorem (e.g. see Chap. 10 in [Cog04]), Theorem D would imply that for any cuspidal representation* π *of* $\mathrm{GL}(n, \mathbb{A}_F)$, *there exists an adjoint lifting* $\mathrm{Ad}(\pi)$, *which will be an isobaric automorphic representation of* $\mathrm{GL}(n^2 - 1, \mathbb{A}_F)$, *in the sense of [GJ78]. Hence, in principle, Theorem D will play a role in Langlands functoriality for the adjoint transfer.*

Chapter 2

CONTRIBUTIONS FROM GEOMETRIC SIDES

Let $\mathcal{H}(G(\mathbb{A}_F))$ be the Hecke algebra of $G(\mathbb{A}_F)$ and $\varphi \in \mathcal{H}(G(\mathbb{A}_F))$. For any character ω of $\mathbb{A}_F^\times/F^\times$. Let $\varphi \in C_c^\infty(Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)) \cap \mathcal{H}(G(\mathbb{A}_F))$ be of central character ω. Denote by V_0 the Hilbert space

$$L_0^2\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right) = \bigoplus_{\pi} V_\pi,$$

where $\pi \in \mathcal{A}_0\left(G(F)\backslash G(\mathbb{A}_F), \omega^{-1}\right)$, the set of irreducible cuspidal representation of $G(\mathbb{A}_F)$ with central character ω and V_π is the corresponding isotypical component. By multiplicity one, the representation of $G(\mathbb{A}_F)$ on V_π is equivalent to π. For each π, we choose an orthonormal basis $\mathcal{B}_\pi$ of V_π consisting of K-finite vectors. Let $\mathrm{K}_0(x, y)$ be the kernel function for the right regular representation $R(\varphi)$ on V_0. Then we have the decomposition

$$\mathrm{K}_0(x, y) = \sum_{\pi} \mathrm{K}_\pi(x, y), \text{ where } \mathrm{K}_\pi(x, y) = \sum_{\phi \in \mathcal{B}_\pi} \pi(\varphi)\phi(x)\overline{\phi(y)}. \tag{2.1}$$

All the functions in the summands are of rapid decay in x and y. The sum of $\mathrm{K}_\pi(x, y)$ converges in the space of rapidly decaying functions, by the usual estimates on the growth of cusp forms. The sum over $\mathcal{B}_\pi$ is finitely uniformly in x and y for a given φ because of the K-finiteness of φ.

2.1 Structure of $G(F)$-Conjugacy Classes

Let B be the subgroup of upper triangular matrices of G, and T the Levi component of B. Let N be the unipotent radical of B. Let $W = W_n$ be Weyl group of G with respect to (B, T). Then one can take W to be the subgroup consisting of all $n \times n$ matrices which have exactly 1 in each row and each column, and zeros elsewhere. Let $\Delta = \{\alpha_{1,2}, \alpha_{2,3}, \cdots, \alpha_{n-1,n}\}$ be the set of simple roots, and for each simple root $\alpha_{k,k+1}$, $1 \leq k \leq n-1$, denote by w_k the corresponding reflection. Explicitly, for each $1 \leq k \leq n-1$,

$$w_k = \begin{pmatrix} I_{k-1} & & \\ & S & \\ & & I_{n-k-1} \end{pmatrix}, \text{ where } S = \begin{pmatrix} & 1 \\ 1 & \end{pmatrix}. \tag{2.2}$$

For each $1 \leq k \leq n-1$, let $\mathcal{W}_k$ be the subgroup generated by elements w_i, $i \in \{1, \cdots, n-1\} \setminus \{k\}$. Write $Q_k = B\mathcal{W}_k B$. Then Q_k is a standard maximal parabolic subgroup of G corresponding to the simple root $\alpha_{k,k+1}$. And every maximal parabolic subgroup is conjugate to some Q_k, $1 \leq k \leq n-1$. Clearly, under this notation, one has $P = Q_{n-1}$. Denote by

$$Q_k(F)^{P(F)} = \{pqp^{-1} : \ p \in P(F),\ q \in Q_k(F)\},\ 1 \leq k \leq n-1.$$

The main results in this section is the following two propositions:

Proposition 12. *Let C be a regular $G(F)$-conjugacy classes in $G(F)$. Then there exists a $P(F)$-conjugacy class C_0 such that*

$$C = C_0 \coprod \bigcup_{k=1}^{n-1} C \cap Q_k(F)^{P(F)}. \tag{2.3}$$

Proposition 13. *Let C be an irregular $G(F)$-conjugacy class, then one has*

$$C = \bigcup_{k=1}^{n-1} C \cap Q_k(F)^{P(F)}. \tag{2.4}$$

To prove (2.3) and (2.4), we need rational canonical forms of $g \in G(F)$, which is an analogue of Jordan canonical forms of matrices over $\mathbb{C}$ (of course F is not algebraically closed). The decomposition is given below:

Lemma 14. *Let V be a n-dimensional vector space over F, and $\mathscr{A} \in \mathrm{End}(V)$. Then there exist invariant subspaces $V_l \subseteq V$, $1 \leq l \leq r$, such that*

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_r, \tag{2.5}$$

and for each i, both of the minimal polynomial and characteristic polynomial of $\mathscr{A}_{V_l} = \mathscr{A}\mid_{V_l}$ are of the form $\wp(\lambda)^k$, where $k \in \mathbb{N}_{\geq 1}$ and $\wp(\lambda) \in F[\lambda]$ is a irreducible polynomial over F. Furthermore, for each l, there exists a basis $\alpha_l = \{\alpha_{l_1}, \cdots, \alpha_{l_m}\}$ of V_l such that under α_l, $\mathscr{A}_{V_l}$ has the following quasi-rational canonical form

$$\mathcal{J}\left(\wp(\lambda)^k\right) := \begin{pmatrix} C(\wp) & & & \\ \boldsymbol{N} & C(\wp) & & \\ & \ddots & \ddots & \\ & & \boldsymbol{N} & C(\wp) \end{pmatrix}, \tag{2.6}$$

where $C(\wp)$ is the companion matrix of $\wp(\lambda)$ and $\boldsymbol{N} = \begin{pmatrix} & & 1 \\ & & 0 & \\ & \cdots & & \\ 0 & & & \end{pmatrix}$.

Proof. Let $m(\lambda)$ (resp. $f(\lambda)$) be the minimal polynomial (resp. characteristic polynomial) of $\mathscr{A}$. Consider their primary decompositions over F :

$$m(\lambda) = \prod_i \wp_i(\lambda)^{e_i'} \text{ and } f(\lambda) = \prod_i \wp_i(\lambda)^{e_i},$$

where $\wp(\lambda)_i's$ are distinct irreducible monic polynomials over F, $0 \leq e_i' \leq e_i$, $\forall\ i$. Take $U_i = \ker \wp_i(\mathscr{A})^{e_i'}$. Then U_i is $\mathscr{A}$-invariant. By cyclic decomposition theorem (which holds for general fields), we have

$$U_i = F[\mathscr{A}]\alpha^{i,1} \oplus F[\mathscr{A}]\alpha^{i,2} \oplus \cdots \oplus F[\mathscr{A}]\alpha^{i,r_i},$$

where each $F[\mathscr{A}]\alpha^{i,j}$ is a cyclic subspace of U_i. Then one has the decomposition (2.5) and both of the minimal polynomial and characteristic polynomial of $\mathscr{A}_{V_{i_j}} = \mathscr{A}\mid_{F[\mathscr{A}]\alpha^{i,j}}$ are powers of $\wp_i(\lambda)$.

For any i and $1 \leq j \leq r_i$, we may assume that the minimal polynomial of $\mathscr{A}_{V_{i_j}}$ on $F[\mathscr{A}]\alpha^{i,j}$ is $\wp_i(\lambda)^{e_{i,j}'}$ with some $0 \leq e_{i,j}' \leq e_i'$. Write $\wp_i(\lambda) = \lambda^{d_i} - c_{d_i-1}\lambda^{d_i-1} - \cdots - c_0$. Define

$$\alpha_{sd_i+t} = \mathscr{A}_{V_{i_j}}^{t-1}\wp_i(\mathscr{A}_{V_{i_j}})^s\alpha^{i,j},\ 1 \leq s \leq e_{i,j}', 1 \leq t \leq d_i.$$

Note that for any $1 \leq s \leq e_{i,j}'$,

$$\begin{aligned}
\mathscr{A}_{V_{i_j}}\alpha_{sd_i} &= \mathscr{A}_{V_{i_j}}^{d_i}\wp_i(\mathscr{A}_{V_{i_j}})^{s-1}\alpha^{i,j} \\
&= \left(\mathscr{A}_{V_{i_j}}^{d_i} - \wp_i(\mathscr{A}_{V_{i_j}})^{s-1}\right)\wp_i(\mathscr{A}_{V_{i_j}})^{s-1}\alpha^{i,j} + \wp_i(\mathscr{A}_{V_{i_j}})^s\alpha^{i,j} \\
&= c_0\alpha_{(s-1)d_i+1} + c_1\alpha_{(s-1)d_i+2} + \cdots + c_{d_i-1}\alpha_{sd_i} + \alpha_{sd_i+1}.
\end{aligned}$$

Therefore, under the basis $\{\alpha_{sd_i+t} :\ 1 \leq s \leq e_{i,j}',\ 1 \leq t \leq d_i\}$, $\mathscr{A}_{V_{i_j}}$ is represented by $\mathcal{J}\left(\wp(\lambda)^{e_{i,j}'}\right)$ defined in (2.6). □

To prove (2.3) we need some further preparation. Given arbitrarily an $k \in \mathbb{N}_{\geq 1}$, denote by $H(F) = H_k(F) = GL_k(F)$. Let $\gamma \in H(F)$ be regular and denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_{m_k}(\lambda)^{e_{m_k}}$ its characteristic polynomial, where $e_i \geq 1$, $\wp_i$ is monic and irreducible over F, $1 \leq i \leq m_k$. Let $d_i = \deg \wp_i$. Set $F(\gamma) = F[\lambda]/(f(\lambda))$ be the polynomial algebra generated by γ, and denote by $F(\gamma)^\times$ the set of invertible elements in $F(\gamma)$. Let $P_0^H(F)$ be the mirabolic subgroup of $H(F)$. Also, for any $\delta \in H(F)$, we always write $H_\delta(F)$ for the centralizer of δ in $H(F)$. We will always use these notation henceforth.

Lemma 15. *Let $\gamma \in H(F)$ be regular elliptic, then for any $(a_1, a_2, \cdots, a_k) \in F^k$, there exists a unique element $x \in F(\gamma)$ such that the last row of x is exactly $(a_1, a_2, \cdots, a_k)$.*

Proof. Since γ is regular, $H_\gamma(F) = F(\gamma)$, and $\dim F(\gamma) = k$. Let $\eta = (0, \cdots, 0, 1) \in F^k$. Consider the linear map:

$$\tau : F(\gamma) \longrightarrow F^k, \qquad x \mapsto \tau(x) = \eta x.$$

Since γ is elliptic, $F(\gamma)$ is a field, so any nonzero element is invertible. Consequently, the map τ is injective, and hence surjective. Thus τ is an isomorphism of k-dimensional F-vector spaces. Then the lemma follows. ◻

Remark 16. *Let $\gamma \in H(F)$ be regular elliptic, we have $H(F) = P_0^H(F)F(\gamma)^\times$. In fact, since τ is a bijection, given $g \in H(F)$, there exists $h \in H(F)$ such that $\eta g = \eta h$ which implies that $gh^{-1} \in P_0^H(F)$, the isotropy subgroup of η, i.e., $g \in P_0^H(F)F(\gamma)^\times$.*

Lemma 17. *Let $\gamma \in H(F)$ be regular. Assume further that the characteristic polynomial of γ has only one irreducible factor. Then one can find $\gamma' \in H(F)$ conjugate to γ such that for any $(a_1, a_2, \cdots, a_k) \in F^k$, there exists a unique element $x \in F(\gamma')$ such that the last row of x is exactly $(a_1, a_2, \cdots, a_k)$. In particular, one can take γ' to be the quasi-rational canonical form of γ.*

Proof. Let $f(\lambda) = \wp(\lambda)^e$ be the characteristic polynomial of γ, where $\wp(\lambda) = \lambda^d + c_{d-1}\lambda^{d-1} + \cdots + c_1\lambda + c_0 \in F[\lambda]$ is irreducible. Then $de = k$. By definition, $F[\gamma] = F[\lambda]/(\wp(\lambda)^e)$. Consider the filtration

$$\wp(\lambda)^{i-1}F[\lambda]/\Big(\wp(\lambda)^i\Big) \supseteq \wp(\lambda)^i F[\lambda]/\Big(\wp(\lambda)^{i+1}\Big),\ 1 \le i \le e-1.$$

Pick the basis for $F[\gamma]$ over F as in the proof of Lemma 15, i.e., $\{\lambda^i\wp(\lambda)^j : 0 \le i \le d,\ 0 \le j \le e-1\}$. With respect to this basis, each element of $F[\gamma]$ has the following type

$$\mathcal{S}_\gamma = \left\{ A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix},\ A_i \in M_{d\times d}(F),\ 0 \le i \le e-1 \right\}, \tag{2.7}$$

and under this basis, and the assumption that γ is regular, γ has the quasi-rational canonical form

$$\mathcal{J} = \begin{pmatrix} C & & & \\ \mathbf{N} & C & & \\ & \ddots & \ddots & \\ & & \mathbf{N} & C \end{pmatrix} \in GL_k(F), \tag{2.8}$$

i.e., γ is conjugate to $\mathcal{J}$, where $C = C(\wp)$ be the companion matrix of $\wp(\lambda)$, i.e.,

$$C = \begin{pmatrix} 0 & & & -c_0 \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & -c_{d-1} \end{pmatrix}, \text{ and } \mathbf{N} = \begin{pmatrix} & & 1 \\ & \dots & \\ 0 & & \end{pmatrix} \in GL_d(F).$$

Since elements in the same $G(F)$-conjugacy class have the same characteristic polynomial, we may assume that $\gamma = \mathcal{J}$ is a quasi-rational canonical form.

Then necessarily if $A \in F[\gamma]$ of the form in (2.7), then it commutes with γ. Indeed, since γ is regular, i.e., the minimal polynomial of γ coincides with its characteristic polynomial, any nonsingular matrix commuting with γ must lie in $F[\gamma]^\times$. Thus $F[\gamma]^\times = \{A \in \mathcal{S}_\gamma \cap GL_k(F) : \ A\gamma = \gamma A\}$.

Now we consider the equation $A\gamma = \gamma A$, $A \in \mathcal{S}_\gamma$. Clearly this is equivalent to a system of Sylvester equations

$$\begin{cases} CA_0 = A_0 C \\ \mathbf{N}A_0 + CA_1 = A_1 C + A_0\mathbf{N} \\ \qquad \vdots \\ \mathbf{N}A_{e-2} + CA_{e-1} = A_{e-1}C + A_{e-2}\mathbf{N}. \end{cases} \tag{2.9}$$

Since $A_0 \in F[C]^\times$, and C is regular elliptic, A_0 commuting with C implies that there exists some $h_0(\lambda) \in F[\lambda]$, such that $A_0 = h_0(C)$. We may assume that $d_0 = \deg h_0 \leq d-1$. Let $\eta = (0, \cdots, 0, 1)$ and write $\eta C^i = (b_{d,1}^{(i)}, b_{d,2}^{(i)}, \cdots, b_{d,d}^{(i)})$, $1 \leq i \leq d-1$, for the last row of C^i. Define

$$X_{(i)} = \begin{pmatrix} 0 & b_{d,1}^{(i)} & b_{d,2}^{(i)} & \cdots & b_{d,d-1}^{(i)} \\ 0 & 0 & b_{d,1}^{(i)} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & b_{d,2}^{(i)} \\ \vdots & & \ddots & \ddots & b_{d,1}^{(i)} \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix} \in GL_d(F).$$

Claim 18. *Let notation be as before, then for any* $1 \le i \le d-1$, $X = X_{(i)}$ *is a solution to the Sylvester equation*

$$\mathbf{N}C^i + CX = XC + C^i\mathbf{N}.$$

Write $A_0 = h_0(C) = c'_{d_0}C^{d_0} + c'_{d_0-1}C^{d_0-1} + \cdots + c'_1C + c'_0I_d$, $c'_{d_0} \neq 0$. Define

$$A^{sp}_{h_0} = c'_{d_0}X_{(d_0)} + c'_{d_0-1}X_{(d_0-1)} + \cdots + c'_1X_{(1)}.$$

Clearly $A_1 = A^{sp}_{h_0}$ gives a special solution of the equation $\mathbf{N}A_0 + CA_1 = A_1C + A_0\mathbf{N}$ (the superscript 'sp' refers to 'special'). Given $A_0 = h_0(C)$ as above, one then claims that

$$\mathcal{A}_1 = \{A^{sp}_{h_0} + h_1(C) : \ h_1 \in F[\lambda], \ \deg h_1 \le d-1\}$$

gives all solutions to the equation $\mathbf{N}A_0 + CA_1 = A_1C + A_0\mathbf{N}$. In fact, on the one hand, elements in $\mathcal{A}_1$ obviously satisfies the equation; on the other hand, let A'_1 be any solution to the equation, then $A^{sp}_{h_0} - A'_1$ commutes with C, thus it is a polynomial of C, namely, $A'_1 \in \mathcal{A}_1$. This proves the claim.

Note that $\mathbf{N}A^{sp}_{h_0} = A^{sp}_{h_0}\mathbf{N} = 0$, when substitute $A_1 = A^{sp}_{h_0} + h_1(C)$ into the equation $\mathbf{N}A_1 + CA_2 = A_2C + A_1\mathbf{N}$, to get $\mathbf{N}h_1(C) + CA_2 = A_2C + h_1(C)\mathbf{N}$. Write $h_1(\lambda) = c''_{d_1}\lambda^{d_1} + c''_{d_1-1}\lambda^{d_1-1} + \cdots + c''_1\lambda + c''_0$, and set

$$A^{sp}_{h_1} = c''_{d_1}X_{(d_1)} + c''_{d_1-1}X_{(d_1-1)} + \cdots + c''_1X_{(1)}.$$

Then $\mathcal{A}_2 = \{A^{sp}_{h_1} + h_2(C) : \ h_2 \in F[\lambda], \ \deg h_2 \le d-1\}$ gives all solutions to the equation $\mathbf{N}A_1 + CA_2 = A_2C + A_1\mathbf{N}$. Generally we define $\mathcal{A}_i$, $1 \le i \le e-1$ similarly, and set $\mathcal{A}_0 = \{h_0(C) : \ h_0 \in F[\lambda], \ \deg h_0 \le d-1\}$. These $\mathcal{A}_i$'s describe the structure of $F[\gamma]^\times$.

Therefore, given any $\mathfrak{a} = (a_1, a_2, \cdots, a_k) \in F^k$, by Lemma 15 one can find uniquely an $A_0 \in F[C]$ such that $\eta A_0 = (a_{k-d+1}, a_{k-d+2}, \cdots, a_k)$. Denote the sections of $\mathfrak{a}$ by $\mathfrak{a}_i = (a_{(i-1)d+1}, a_{(i-1)d+2}, \cdots, a_{id})$, $1 \le i \le e-1$. Let $1 \le i_0 \le e-1$, assume that for any $0 \le i < i_0$ one can find uniquely an element $A_i \in M_{d\times d}(F)$ such that the last row of A_i is exactly $\mathfrak{a}_{e-i}$, then let $h_{i_0}(C) \in F[C]^\times$ be the unique element whose last row is $\mathfrak{a}_{e-i_0}$, take $A_{i_0} = A^{sp}_{h_{i_0}} + h_{i_0}(C)$. Then $\eta A_{i_0} = \eta h_{i_0}(C) = \mathfrak{a}_{e-i_0}$. Moreover, such an A_{i_0} is actually unique. Let A'_{i_0} be another matrix satisfying that $\eta A'_{i_0} = \mathfrak{a}_{e-i_0}$. Since A'_{i_0} is a solution of $\mathbf{N}A_{i_0-1} + CX = XC + A_{i_0-1}\mathbf{N}$, $A_{i_0} - A'_{i_0}$ commutes with C. Thus $A_{i_0} - A'_{i_0} \in F[C]$. Note that the last row of $A_{i_0} - A'_{i_0}$ is 0, so by the uniqueness from Lemma 15, $A_{i_0} - A'_{i_0} = 0$. This shows the uniqueness of A_{i_0}.

Therefore, the proof ends with an induction on i_0 and the following proof of Claim 18. □

Proof of Claim 18. We give a proof based on induction, although one might verify the claim by brute force computation (which is pretty complicated). Note that the case $i = 1$ is trivial, since $X_{(1)} = \mathbf{N}$. Now we assume that there exists an i_0 such that $1 < i_0 \leq d-1$, and for any $1 \leq i < i_0$, $X = X_{(i)}$ is a solution to the Sylvester equation $\mathbf{N}C^i + CX = XC + C^i\mathbf{N}$. Write $C^j = \left(b_{s,t}^{(j)}\right)_{1\leq s,t\leq d}$, $1 \leq j \leq d-1$, then a straightforward expansion implies that $X = X_{(i_0)}$ is a solution to the Sylvester equation $\mathbf{N}C^{i_0} + CX = XC + C^{i_0}\mathbf{N}$ if and only if the following system of linear equations holds

$$\begin{cases} b_{d,d}^{(i_0)} = -c_1 b_{d,1}^{(i_0)} - c_2 b_{d,2}^{(i_0)} \cdots - c_{d-1} b_{d,d-1}^{(i_0)}, \\ b_{d,d-1}^{(i_0)} = -c_2 b_{d,1}^{(i_0)} - c_3 b_{d,2}^{(i_0)} - \cdots - c_{d-1} b_{d,d-2}^{(i_0)} + b_{2,1}^{(i_0)}, \\ b_{d,d-2}^{(i_0)} = -c_3 b_{d,1}^{(i_0)} - c_4 b_{d,2}^{(i_0)} - \cdots - c_{d-1} b_{d,d-3}^{(i_0)} + b_{3,1}^{(i_0)}, \\ \qquad \vdots \\ b_{d,2}^{(i_0)} = -c_{d-1} b_{d,1}^{(i_0)} + b_{d-1,1}^{(i_0)}. \end{cases} \tag{2.10}$$

Comparing entries on both sides of $C^{i_0} = C^{i_0-1}C$ leads to the recurrence relations

$$\begin{cases} b_{d,j}^{(i_0)} = b_{d,j+1}^{(i_0-1)}, \ 1 \leq j \leq d-1, \\ b_{d,d}^{(i_0)} = -c_0 b_{d,1}^{(i_0-1)} - c_1 b_{d,2}^{(i_0-1)} - \cdots - c_{d-1} b_{d,d}^{(i_0-1)}. \end{cases} \tag{2.11}$$

Since $i_0 \leq d-1$, $i_0 - 1 < d-1$, then $b_{d,1}^{(i_0-1)} = 0$. Therefore, relations (2.11) implies that

$$b_{d,d}^{(i_0)} = -c_0 b_{d,1}^{(i_0-1)} - c_1 b_{d,1}^{(i_0)} - c_2 b_{d,2}^{(i_0)} \cdots - c_{d-1} b_{d,d-1}^{(i_0)},$$

which is exactly the first equation in (2.10). By other assumption, the system of relations (2.10) holds when i_0 replaced by $i_0 - 1$. Therefore, for any $1 \leq j \leq d-2$, one has

$$b_{d,d-j+1}^{(i_0-1)} = -c_j b_{d,1}^{(i_0-1)} - c_{j+1} b_{d,2}^{(i_0-1)} - \cdots - c_{d-1} b_{d,d-j}^{(i_0-1)} + b_{j,1}^{(i_0-1)}.$$

Note that $b_{j,1}^{(i_0-1)} = b_{j+1,1}^{(i_0)}$, $1 \leq j \leq d-2$, $b_{d,1}^{(i_0-1)} = 0$, and thus (2.11) implies that

$$b_{d,d-j}^{(i_0)} = -c_{j+1} b_{d,1}^{(i_0)} - c_{j+2} b_{d,2}^{(i_0)} - \cdots - c_{d-1} b_{d,d-j-1}^{(i_0)} + b_{j+1,1}^{(i_0)},$$

which is exactly the $(1+j)$-th equation in (2.10). Hence the proof follows from induction. □

Lemma 19. *Let $\gamma \in G(F)$ be regular. Then there exists a finite set of elements $\Gamma_{reg} = \{\gamma_i \in G(F) : \ 0 \le i \le m_0\}$ such that*

1. *$G(F) = \bigcup_{0 \le i \le m_0} P_0(F)\gamma_i F(\gamma)$, where P_0 is the mirabolic subgroup of G;*
2. *There are at most one $\gamma_i \in \Gamma_{reg}$ satisfying that*

$$\gamma_i F(\gamma)\gamma_i^{-1} \not\subseteq \bigcup_{k=1}^{n-1} Q_k(F).$$

Proof. Denote by $f(\lambda) = \wp_1(\lambda)^{e_1} \cdots \wp_m(\lambda)^{e_m}$ the characteristic polynomial of $\gamma \in G(F)$, where $e_i \ge 1$, $\wp_i$'s are distinct monic and irreducible polynomials over F, $1 \le i \le m$. Let $d_i = \deg \wp_i$. Then $d_1 e_1 + d_2 e_2 + \cdots + d_m e_m = \deg f = n$. Set

$$F(\gamma) = F[\lambda]/(f(\lambda)) = \bigoplus_{i=1}^{m} F[\lambda]/(\wp_i(\lambda)^{e_i})$$

be the polynomial algebra generated by γ, and denote by $F(\gamma)^\times$ the set of invertible elements in $F(\gamma)$. Since γ is regular, then by Lemma 14, γ is $G(F)$-conjugate to a matrix of the form

$$\gamma^* = \begin{pmatrix} \mathcal{J}(\wp_1^{e_1}) & & & \\ & \mathcal{J}(\wp_2^{e_2}) & & \\ & & \ddots & \\ & & & \mathcal{J}(\wp_m^{e_m}) \end{pmatrix} \in \begin{pmatrix} G^{(1)}(F) & & & \\ & G^{(2)}(F) & & \\ & & \ddots & \\ & & & G^{(m)}(F) \end{pmatrix},$$

where $G^{(1)}(F) := GL_{d_i e_i}(F)$, $1 \le i \le m$. We may assume $\gamma = \gamma^*$. Write $k_i = d_i e_i$, $1 \le i \le m$. For any $\mathfrak{a} = (a_1, a_2, \cdots, a_n) \in F^n$, let $\eta = (0, 0, \cdots, 0, 1)$ and $\widetilde{\eta_i} : F^n \longrightarrow F^{k_i}$, such that

$$\widetilde{\eta_i}\,(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+1}, \cdots, a_{k_1+\cdots+k_{i-1}+k_i}),\ 1 \le i \le m.$$

Also, for convenience we write $\eta^{(e_i)}\,(\mathfrak{a})$ for the last d_i components of $\widetilde{\eta_i}\,(\mathfrak{a})$, namely $\eta^{(e_i)}\,(\mathfrak{a}) = (a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+1}, a_{k_1+\cdots+k_{i-1}+(e_i-1)d_i+2}, \cdots, a_{k_1+\cdots+k_{i-1}+k_i})$, $1 \le i \le m$. We then split $G(F)$ into a disjoint union of sets following the conditions on the $\mathfrak{a}_i$'s and show that each of the sets is a $P_0(F)\gamma_i F(\gamma)^\times$ for a specific γ_i.

Let $\mathcal{S}_0 = \{\delta \in G(F) :\ \eta\delta = \mathfrak{a} = (a_1, a_2, \cdots, a_n) \in F^n,$ such that for $1 \le i \le m$, $\eta^{(e_i)}\,(\mathfrak{a}) \neq \mathbf{0}\}$. Let $\eta_i = (0, 0, \cdots, 0, 1) \in F^{k_i}$, $1 \le i \le m$. Denote by

$$\gamma_0 = \begin{pmatrix} I_{k_1} & & & \\ \vdots & \ddots & & \\ \vdots & & I_{k_{m-1}} & \\ \eta_1 & \ldots & \eta_{m-1} & I_{k_m} \end{pmatrix}.$$

Then applying Lemma 17 to each $\widetilde{\eta_i}(\mathfrak{a}) \in F^{k_i}$, we find for each $1 \leq i \leq m$, for any $\delta \in \mathcal{S}_0$, a unique $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$, such that $\eta x_i = \widetilde{\eta_i}\delta$. (Write x_i in the form in (2.7), the definition of $\mathcal{S}_0$ implies that $A_0 \neq 0$, thus $A_0 \in F[C]^\times$, so $x_i \in F[\mathcal{J}(\wp_i^{e_i})]^\times$.) Let $x = \mathrm{diag}(x_1, \cdots, x_m)$, then $\eta(\gamma_0 x) = \eta\delta$. Consequently, $\delta(\gamma_0 x)^{-1} \in P_0(F)$, i.e., $\delta \in P_0(F)\gamma_0 F(\gamma)^\times$. Moreover, one has $P_0(F) \cap \gamma_0 F(\gamma)^\times \gamma_0^{-1} = \{I_n\}$. To see this, look at the last row of $\gamma_0 x \gamma_0^{-1}$. A straightforward computation shows that

$$\widetilde{\eta_i}\left(\gamma_0 x \gamma_0^{-1}\right) = \eta_i x_i - \eta_i,\ 1 \leq i \leq m-1,$$

and $\widetilde{\eta_m}\left(\gamma_0 x \gamma_0^{-1}\right) = \eta_m$. Then by uniqueness part of Lemma 17, it follows that $x_i = I_{k_i}$, $1 \leq i \leq m$.

For any $1 \leq l \leq m-1$ and $1 \leq i_1 < \cdots < i_l \leq m$, let $\mathcal{S}^{(l)}_{(i_1,\cdots,i_l)} = \{\delta \in G(F) : \eta\delta = \mathfrak{a} = (a_1, a_2, \cdots, a_n) \in F^n$, such that $\eta^{(e_j)}(\mathfrak{a}) = \mathbf{0}$ iff $j \in \{i_1, \cdots, i_l\}\}$. For any $1 \leq i \leq m$, $1 \leq e \leq e_i - 1$, define $\eta_i^e = (0, \cdots, 0, 1, 0, \cdots, 0) \in F^{k_i}$, where the only nonzero entry (i.e. 1) occurs in the ed_i-position, namely, there are $(e_i - e)d_i$ zeros on the right hand side of the entry 1. Let v_l denote the l-th element of η_i^e, define

$$\eta_i^* = (0, \cdots 0, v_{d_i+1}, v_{d_i+2}, \cdots, v_{k_i-d_i}, v_1, v_2, \cdots, v_{d_i}).$$

For any $1 \leq i \leq m$ and $1 \leq s < t \leq m$, define the Weyl elements $w_i^{(1)}$ and $w_{s,t}^{(2)}$ as

$$w_i^{(1)} = \begin{pmatrix} I_{k_i'} & & & & & \\ & 0 & & \cdots & I_{d_i} & \\ & & I_{d_i} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{d_i} & & \\ & I_{d_i} & & \cdots & 0 & \\ & & & & & I_{k_i''} \end{pmatrix},$$

where $k_i' = k_1 + \cdots k_{i-1}$, $k_i'' = k_{i+1} + \cdots + k_m$, $\forall\ 1 \leq i \leq m$; and

$$w_{s,t}^{(2)} = \begin{pmatrix} I_{k_s'} & & & & & \\ & 0 & & \cdots & I_{k_t} & \\ & & I_{k_{s+1}} & & & \\ & \vdots & & \ddots & \vdots & \\ & & & I_{k_{t-1}} & & \\ & I_{k_s} & & \cdots & 0 & \\ & & & & & I_{k_t''} \end{pmatrix}.$$

We write simply that $w_{s,t}^{(2)} = I_n$ if $s = t$.

Given $\delta \in \mathcal{S}_{(i_1,\cdots,i_l)}^{(l)}$, let $\eta\delta = \mathfrak{a} = (a_1, a_2, \cdots, a_n) \in F^n$. Then $\eta^{(e_j)}(\mathfrak{a}) = \mathbf{0}$ if and only if $j \in \{i_1, \cdots, i_l\}$. If $\widetilde{\eta}_j(\mathfrak{a}) \neq 0$, let $e_j^0 \leq e_j - 1$ be the maximal integral such that $(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \cdots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq 0$. Then by Lemma 17, for each such j, one can find an element x_j of the form in (2.7) such that the $e_j^0 d_j$-row of x_j is exactly

$$(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \cdots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq 0.$$

If $\widetilde{\eta}_j(\mathfrak{a}) = 0$, then take x_j to be an arbitrary element in $F(\mathcal{J}(\wp_j^{e_j}))^\times$. Since $\eta^{(e_i)}(\mathfrak{a}) \neq \mathbf{0}$ for any $i \notin \{i_1, \cdots, i_l\}$, we can pick uniquely elements $x_i \in F(\mathcal{J}(\wp_i^{e_i}))^\times$ such that their last row is $\widetilde{\eta}_i \mathfrak{a}$, $i \in \{1, 2, \cdots, m\} \setminus \{i_1, \cdots, i_l\}$. Let $x = \operatorname{diag}(x_1, x_2, \cdots, x_m)$.

Apply the transposition $\prod_{1 \leq j \leq l}(j, i_j) \in S_n$ on the ordered n-set $(1, 2, \cdots, n)$ to get another ordered n-set $(i_1, \cdots, i_l, i'_{l+1}, \cdots, i'_m)$. Then clearly $\eta^{(e_{i'_m})}(\mathfrak{a}) \neq 0$, hence by our choice, the last row of $x_{i'_m}$ is exactly $\eta_{i'_m}(\mathfrak{a})$. Thus we can define the element $\gamma_{(i_1,\cdots,i_l)} \in G(F)$ as

$$\begin{pmatrix} I_{k_{i_1}} &&&&&&&& \\ & I_{k_{i_2}} &&&&&&& \\ && \ddots &&&&&& \\ &&& I_{k_{i_l}} &&&&& \\ \vdots &&&& I_{k_{i'_{l+1}}} &&&& \\ &&&&& I_{k_{i'_{l+2}}} &&& \\ &&&&&& \ddots && \\ &&&&&&& I_{k_{i'_{m-1}}} & \\ \eta_{i_1}^* & \eta_{i_2}^* & \cdots & \eta_{i_l}^* & \eta_{i'_{l+1}} & \eta_{i'_{l+2}} & \cdots & \eta_{i'_{m-1}} & I_{k_{i'_{m-1}}} \end{pmatrix} \cdot \prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)}.$$

Then by our setting, $\eta\gamma_{(i_1,\cdots,i_l)}^{(l)} x = \mathfrak{a} = \eta\delta$. Then $\delta \in P_0(F)\gamma_{(i_1,\cdots,i_l)} F(\gamma)^\times$. Given any $x \in F(\gamma)^\times$,, one checks directly that

$$\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} x \left(\prod_{j=1}^{l} w_j^{(1)} w_{j,i_j}^{(2)} \right)^{-1} \in Q_{d_{i_1}}(F).$$

Therefore, $\gamma_{(i_1,\cdots,i_l)} x \gamma_{(i_1,\cdots,i_l)}^{-1} \in Q_{d_{i_1}}(F)$, i.e., $\gamma_{(i_1,\cdots,i_l)} F(\gamma)^\times \gamma_{(i_1,\cdots,i_l)}^{-1} \subseteq Q_{d_{i_1}}(F)$.

Now we consider $\mathcal{S}^{(m)} = \{\delta \in G(F) : \ \eta\delta = \mathfrak{a} = (a_1, a_2, \cdots, a_n) \in F^n - \{\mathbf{0}\},\ \eta^{(e_j)}(\mathfrak{a}) = \mathbf{0},\ 1 \leq j \leq m\}$. Let $\delta \in \mathcal{S}^{(m)}$ such that $\eta\delta = \mathfrak{a} = (a_1, a_2, \cdots, a_n) \in$

$F^n - \{\mathbf{0}\}$. For each $1 \le j \le m$ such that $\widetilde{\eta}_j(\mathfrak{a}) \neq 0$, denote by $e_j^0 \le e_j - 1$ the maximal integral such that

$$(a_{k_1+\cdots+(e_j^0-1)d_j+1}, a_{k_1+\cdots+k_{j-1}+(e_j^0-1)d_j+2}, \cdots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j}) \neq 0.$$

Likewise, for each such j, one can find an element x_j of the form in (2.7) such that the $e_j^0 d_j$-row of x_j is exactly $(a_{k_1+\cdots+k_{j-1}+1}, a_{k_1+\cdots+k_{j-1}+1}, \cdots, a_{k_1+\cdots+k_{j-1}+e_j^0 d_j})$. For the remaining j's, take arbitrary $x_j \in F(\mathcal{J}(\wp_j^{e_j}))^\times$. Let $x = \mathrm{diag}(x_1, \cdots, x_m)$.

Now we pick arbitrarily a j_0 such that $\widetilde{\eta}_{j_0}(\mathfrak{a}) \neq 0$. Let $j_0' \neq j_0$ be another integer. Denote by

$$w_{j_0,e_{j_0}}^{(1)} = \begin{pmatrix} I_{k_{j_0}'} & & & \\ & 0 & I_{(e_j-e_{j_0})d_{j_0}} & \\ & I_{e_{j_0}d_{j_0}} & 0 & \\ & & & I_{k_{j_0}''} \end{pmatrix}.$$

Let

$$\gamma_m = \begin{pmatrix} I_{k_{j_0'}} & & & & & & \\ & \ddots & & & & & \\ & & I_{k_1} & & & & \\ \vdots & & & \ddots & & & \\ & & & & I_{k_m} & & \\ & & & & & \ddots & \\ \eta_{i_1}^* & \cdots & \eta_1 & \ldots & \eta_m & \cdots & I_{k_{j_0}} \end{pmatrix} \cdot w_{j_0'}^{(1)} w_{1,j_0'}^{(2)} w_{j_0,e_{j_0}}^{(1)} w_{j_0,m}^{(2)}.$$

Then $\eta\gamma_m x = \eta\delta$. So $\delta \in P_0(F)\gamma_m F(\gamma)^\times$. Moreover, for any $x' \in F(\gamma)^\times$, $\gamma_m x' \gamma_m^{-1} \in Q_{d_{j_0'}}(F)$, the standard maximal parabolic subgroup of type $(j_0', n - j_0')$.

In all, we see that

$$\begin{aligned} G(F) &= \mathcal{S}_0 \coprod \bigcup_{l=1}^{m-1} \bigcup_{1\le i_1<\cdots<i_l\le m} \mathcal{S}_{(i_1,\cdots,i_l)}^{(l)} \coprod \mathcal{S}^{(m)} \\ &= P_0(F)\gamma_0 F(\gamma)^\times \bigcup \bigcup_{\substack{1\le l\le m-1 \\ 1\le i_1<\cdots<i_l\le m}} P_0(F)\gamma_{(i_1,\cdots,i_l)} F(\gamma)^\times \bigcup P_0(F)\gamma_m F(\gamma)^\times, \end{aligned}$$

where $\gamma_m F(\gamma)^\times \gamma_m^{-1}$ and each $\gamma_{(i_1,\cdots,i_l)} F(\gamma)^\times \gamma_{(i_1,\cdots,i_l)}^{-1}$ are contained in some standard maximal parabolic subgroup, and $P_0(F) \cap \gamma_0 F(\gamma)^\times \gamma_0^{-1} = \{I_n\}$. □

Now we prove the result on the structure of conjugacy classes:

Proof of Proposition 12. By Lemma 19 we have $G(F) = \bigcup_{0\leq i\leq m_0} P_0(F)\gamma_i F(\gamma)$, where P_0 is the mirabolic subgroup of G, and $\gamma \in C$. If $\delta \in G(F)$, there exists $p \in P_0(F)$ and $i \in \{0, 2, \cdots, m_0\}$ and $x \in F(\gamma)^{\times}$, such that $\delta = p\gamma_i x$. So one has

$$\delta\gamma\delta^{-1} = p\gamma_i x\gamma x^{-1}\gamma_i^{-1}p^{-1} = p\gamma_i\gamma\gamma_i^{-1}p^{-1}.$$

If $i \geq 1$, then $\delta\gamma\delta^{-1} \in C \cap Q_j(F)^{P(F)}$, for some standard maximal parabolic subgroup Q_j of type $(j, n-j)$, $1 \leq j \leq n-1$. And for $i = 0$, $\delta\gamma\delta^{-1} = p\gamma_0\gamma\gamma_0^{-1}p^{-1}$. Take $\gamma' = \gamma_0\gamma\gamma_0^{-1}$. Then $C_0 = \{p\gamma' p^{-1} : p \in P(F)\}$. This proves the result. □

Note that we have a bijection $\mathcal{W}_{n-1}\backslash W/\mathcal{W}_{n-1} \longleftrightarrow \{1, w_{n-1}\}$. By Bruhat decomposition

$$G(F) = P(F) \bigcup P(F)w_{n-1}P(F). \tag{2.12}$$

Repeating (2.12) we then obtain

$$G(F) = P(F) \bigcup \bigcup_{j=1}^{n-1} P(F)w_{n-1}w_{n-2}\cdots w_j N_j(F), \tag{2.13}$$

where $N_j(F) := (w_j w_{j+1}\cdots w_n N(F) w_n w_{n-1}\cdots w_j \cap N(F))\backslash N(F)$ is of codimension $n-j$ in $N(F)$. Let N^- be the unipotent subgroup of the form $\begin{pmatrix} 1 & & & \\ \vdots & \ddots & & \\ 0 & \ldots & 1 & \\ * & \ldots & * & 1 \end{pmatrix}$, i.e., the lower triangle matrix with entries vanishing outside the diagonal or the bottom.

Proof of Proposition 13. Let $g \in C$ be an representative. Set $m(\lambda)$ (resp. $f(\lambda)$) to be its minimal polynomial (resp. characteristic polynomial) over F. Consider their primary decompositions over F :

$$m(\lambda) = \prod_{i\in I} \wp_i(\lambda)^{e_i'} \text{ and } f(\lambda) = \prod_{i\in I} \wp_i(\lambda)^{e_i},$$

where $\wp(\lambda)_i' s$ are distinct irreducible monic polynomials over F, I is a finite index set such that $e_i > 0$, $\forall\, i \in I$. Write $d_i = \deg \wp_i(\lambda)$, $\forall\, i \in I$. We may assume that $d_1 \leq d_2 \leq \cdots \leq d_{\#I}$. Also, write $d_0 = 0$. Since the conjugacy class C is irregular, $m(\lambda)$ is a proper factor of $f(\lambda)$. Thus we have the following cases:

Case I Suppose $\#I = 1$. Then $m(\lambda) = \wp(\lambda)^{e'}$, $f(\lambda) = \wp(\lambda)^{e}$, and $0 < e' < e = d_1^{-1}n$.

Let C be the companion matrix of $m(\lambda)$. Then by Lemma 14, g is $G(F)$-

conjugate to some element $\tilde{g} = \mathrm{diag}(g_1, \cdots, g_m)$ with

$$g_j = \mathcal{J}_j = \begin{pmatrix} C & & & \\ \mathbf{N} & C & & \\ & \ddots & \ddots & \\ & & \mathbf{N} & C \end{pmatrix}$$

being the quasi-rational canonical form, and $m > 1$. Let $r_j := \mathrm{rank}\, g_j$, $1 \leq j \leq m$. We may assume $r_1 \leq r_2 \leq \cdots \leq r_m$.

For any $h \in G(F)$, if $h \in P(F)$, then clearly $h\tilde{g}h^{-1} \in hQ_{r_1}(F)h^{-1}$; if $h \in G(F) - P(F)$, it can be written as $h = pw_{n-1} \cdots w_k u_k$, where $p \in P(F)$ and u_k is of the form

$$\begin{pmatrix} I_{k-1} & & \\ & 1 & * \\ & & I_{n-k} \end{pmatrix} \in Q_k(F).$$

Suppose $k > r_1$. Then $w_{n-1} \cdots w_k u_k \in \mathrm{diag}(\mathrm{GL}_{r_1}, \mathrm{GL}_{n-r_1})$. So $h\tilde{g}h^{-1} \in Q_{r_1}(F)^{P(F)}$. Hence, we may assume $k \leq r_1$.

Note that there exist a Weyl element $w \in \mathrm{GL}(r_m)$ such that

$$w \begin{pmatrix} C & & & \\ \mathbf{N} & C & & \\ & \ddots & \ddots & \\ & & \mathbf{N} & C \end{pmatrix} w^{-1} = \begin{pmatrix} C & & & & \mathbf{N} & & & \\ \mathbf{N} & C & & & & & & \\ & \ddots & \ddots & & & & & \\ & & \mathbf{N} & C & & & & \\ & & & & C & \mathbf{N} & & \\ & & & & & \ddots & \ddots & \\ & & & & & & C & \mathbf{N} \\ & & & & & & & C \end{pmatrix}, \qquad (2.14)$$

where the left hand side of (2.14) represents wg_mw^{-1} and in the right hand side the upper-left block is a $r_1 \times r_1$-matrix, which is precisely g_1. Namely, one can find a Weyl element $w \in \mathrm{GL}(r_m)$ such that

$$wg_mw^{-1} = \begin{pmatrix} g_1 & B' \\ & A' \end{pmatrix} \in \mathrm{GL}(r_m, F), \qquad (2.15)$$

for some matrices A' and B'. Let $w' = \mathrm{diag}(w, I_{n-r_m})$. Denote by $w'' = w_{r_1-1}w_{r_1-2} \cdots w_k$ if $k < r_1$, and set $w'' = I_n$ if $k = r_1$. Let $g_1'' = w''g_1w''^{-1}$.

Then there exists a Weyl element $w_0 \in P_0(F)$ such that

$$w_0 w_{n-1} w_{n-2} \cdots w_k \tilde{g} w_k \cdots w_{n-2} w_{n-1} w_0^{-1} = \begin{pmatrix} g_m & & & & \\ & g_2 & & & \\ & & \ddots & & \\ & & & g_{m-1} & \\ & & & & g_1'' \end{pmatrix}. \tag{2.16}$$

Let $\tilde{w} = w'' w' w_0 w_{n-1} w_{n-2} \cdots w_k$. Then by (2.15) and (2.16) one has

$$\tilde{w} \tilde{g} \tilde{w}^{-1} = \begin{pmatrix} g_1'' & B'' & \\ & A'' & \\ & & g_1'' \end{pmatrix} \tag{2.17}$$

for some matrices A'' and B''. Note that $w'' w' w_0 \in P_0(F)$. So $N^-(F)$ is stable under the conjugation by $w'' w' w_0$. Also, $w_{n-1} \cdots w_k u_k w_k \cdots w_{n-1} \in N^-(F)$. Then $\tilde{w} u_k \tilde{w}^{-1}$ is of the form $u_k'' \begin{pmatrix} I_{r_1} & & \\ 0 & I_{n-2r_1} & \\ U & 0 & I_{r_1} \end{pmatrix}$, where u_k'' lies inside the intersection of $Q_{r_1}(F)$ and $N^-(F)$, and U is a $r_1 \times r_1$-matrix with the first $r_1 - 1$ rows vanishing. Since g_1'' is regular, by Lemma 17 there exists a unique $r_1 \times r_1$-matrix $\gamma \in F(g_1'')$ such that the last row of γ coincides with the last row of U. So

$$\tilde{w} u_k \tilde{w}^{-1} = u_k'' p' \begin{pmatrix} I_{r_1} & & \\ 0 & I_{n-2r_1} & \\ \gamma & 0 & I_{r_1} \end{pmatrix}, \tag{2.18}$$

for some $p' \in P_0(F)$. Observe that $p'^{-1} u_k'' p' = u_k''$ and

$$\begin{pmatrix} I_{r_1} & & \\ 0 & I_{n-2r_1} & \\ \gamma & 0 & I_{r_1} \end{pmatrix} \begin{pmatrix} g_1'' & B'' & \\ & A'' & \\ & & g_1'' \end{pmatrix} \begin{pmatrix} I_{r_1} & & \\ 0 & I_{n-2r_1} & \\ \gamma & 0 & I_{r_1} \end{pmatrix}^{-1} \in Q_{r_1}(F) \tag{2.19}$$

as $\gamma \in F(g_1'')$. Then we have, by (2.17), (2.18) and (2.19), that

$$\tilde{w} u_k \tilde{g} u_k^{-1} \tilde{w}^{-1} \in Q_{r_1}(F)^{P_0(F)}. \tag{2.20}$$

Recall that $h = p w_{n-1} \cdots w_k u_k$. Note that $p'' := w'' w' w_0 \in P_0(F)$. Then it follows form (2.20) that

$$h \tilde{g} h^{-1} = p p''^{-1} \tilde{w} u_k \tilde{g} u_k^{-1} \tilde{w}^{-1} p'' p^{-1} \in Q_{r_1}(F)^{P(F)}.$$

Case II Suppose $\#I > 1$. Then g is $G(F)$-conjugate to some $\tilde{g} = \mathrm{diag}(\tilde{g}_1, \cdots, \tilde{g}_m)$, where each $\tilde{g}_i$ is of the form $\mathrm{diag}(g_{i,1}, \cdots, g_{i,m_i})$, with

$$g_{i,j} = \begin{pmatrix} C_{i,j} & & & \\ \mathbf{N} & C_{i,j} & & \\ & \ddots & \ddots & \\ & & \mathbf{N} & C_{i,j} \end{pmatrix}$$

and $C_{i,j}$ is elliptic regular; and $\tilde{g}_i$ has characteristic polynomial $\wp_i(\lambda)^{e_i}$. Since g is irregular, so is $\tilde{g}$. Hence there must be some $1 \leq i \leq m$ such that $\tilde{g}_i$ is irregular. We may assume $\tilde{g}_1$ is irregular and $\mathrm{rank}\, g_{1,1} \leq \mathrm{rank}\, g_{1,2} \leq \cdots \mathrm{rank}\, g_{1,m_1}$. Then a similar argument as in the Case I shows that $h\tilde{g}h^{-1} \in Q_{r_1}(F)^{P(F)}$, where $r_1 = \mathrm{rank}\, g_{1,1}$.

Proposition 13 thus follows. □

2.2 Contributions from Nonsingular Conjugacy Classes

Let $s > 1$. Consider the well defined distribution

$$I_0(s,\tau) = \int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x,x)E(x,\Phi;s)dx. \tag{2.21}$$

Unfolding the Eisenstein series (cf. (1.5)) we then obtain

$$\begin{aligned} I_0(s,\tau) &= \int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x,x) \sum_{\gamma\in P(F)\backslash G(F)} f(x,s)dx, \\ &= \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x,x) f(x,s)dx. \end{aligned}$$

Let Q_k be the standard parabolic subgroup of $\mathrm{GL}(n)$ of type $(k, n-k)$. In Proposition 12 we show that for any regular $G(F)$-conjugacy classes C in $G(F)$, there exists a $P(F)$-conjugacy class C_0 such that

$$C = C_0 \coprod \bigcup_{k=1}^{n-1} C \cap Q_k(F)^{P(F)}.$$

Moreover, such a C_0 is uniquely determined by C. When C is a non-regular $G(F)$-conjugacy class, then by Proposition 13, we have

$$C = \bigcup_{k=1}^{n-1} C \cap Q_k(F)^{P(F)}.$$

Take C_0 to be empty set in this case. Denote by

$$\mathfrak{S} = \bigcup_{k=1}^{n-1} (Z_G(F)\backslash Q_k(F))^{P_0(F)}. \tag{2.22}$$

Following the approach in [JZ87], we will treat $I(s)$ via the decomposition

$$\mathrm{K}_0(x,x) = \sum_C \mathrm{K}_C(x,x) + \mathrm{K}_{\mathrm{Geo,Sing}}(x,x) + \mathrm{K}_\infty(x,x),$$

where C runs through all conjugacy classes in $G(F)/Z_G(F)$ and

$$\mathrm{K}_C(x,y) = \sum_{\gamma\in C_0} \varphi(x^{-1}\gamma y) = \sum_{\gamma\in C-\mathfrak{G}} \varphi(x^{-1}\gamma y), \quad \mathrm{K}_{\mathrm{Geo,Reg}}(x,y) = \sum_C \mathrm{K}_C(x,y),$$
$$\mathrm{K}_{\mathrm{Geo,Sing}}(x,y) = \sum_{\gamma\in\mathfrak{G}} \varphi(x^{-1}\gamma y), \qquad \mathrm{K}_\infty(x,y) = \mathrm{K}_{\mathrm{Eis}}(x,y) + \mathrm{K}_{\mathrm{Res}}(x,y).$$

Note that $\mathrm{K}_C(x,x)$ and $\mathrm{K}_{\mathrm{Geo,Sing}}(x,x)$ are not $G(F)$-invariant, but they are $P(F)$-invariant. Then it make sense to integrate them over $Z_G(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)$ against $f(x,s)$. So correspondingly, integrating these partial kernels against the $f(x,s)$ implies that $I_0(s,\tau)$ can be decomposed (at least formally) as

$$I_0(s,\tau) = I_{\mathrm{Geo,Reg}}(s,\tau) + I_{\mathrm{Geo,Sing}}(s,\tau) - I_\infty(s,\tau),\ \mathrm{Re}(s) > 1. \tag{2.23}$$

We will show, under certain geometric restriction of test functions, that $I_{\mathrm{Geo,Reg}}(s,\tau)$, $I_{\mathrm{Geo,Sing}}(s,\tau)$ and $I_\infty(s,\tau)$ all converge absolutely in $\mathrm{Re}(s) > 1$, then the formula (2.23) would be rigorous.

When $G = GL(2)$, Jacquet and Zagier (see [JZ87]) computed the distributions $I_{\mathrm{Geo,Reg}}(s,\tau)$, $I_{\mathrm{Geo,Sing}}(s,\tau)$ and $I_\infty(s,\tau)$ for general test function φ, and verified the convergence. Note that the contribution from $I_{\mathrm{Geo,Reg}}(s,\tau)$ would give Artin L-functions of degree less or equal to n. We shall deal with $I_{\mathrm{Geo,Reg}}(s,\tau)$ in this section, and leaving the computation of $I_{\mathrm{Geo,Sing}}(s,\tau)$ and $I_\infty(s,\tau)$ in the following parts. For each C, let (at least formally)

$$I_C(s,\tau) := \int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \mathrm{K}_C(x,x)f(x,s)dx.$$

Then by definition, $I_C(s,\tau) = 0$ unless C is regular. To describe these conjugacy classes, we introduce the classification of them by factorization of their characteristic polynomials.

Let C be a conjugacy class in $G(F)$. Denote by $P(\lambda; C)$ the characteristic polynomial of C. Factorize it into irreducible ones with multiplicities as

$$P(\lambda; C) = \prod_{i=1}^{g} \wp_i(\lambda; C)^{e_i},$$

where $\wp_i(\lambda; C) \in F[\lambda]$ is an irreducible polynomial of degree f_i. We may assume $f_1 \geq \cdots \geq f_g$. Denote by $\mathbf{f} = (f_1, \cdots, f_g) \in \mathbb{Z}_{\geq 1}^g$ and $\mathbf{e} = (e_1, \cdots, e_g) \in \mathbb{Z}_{\geq 1}^g$. Then $\langle \mathbf{f}, \mathbf{e} \rangle = \sum f_i e_i = n$.

Definition 20. *Let notation be as before. We say C is of type $(\boldsymbol{f}, \boldsymbol{e}; g)$. Let $\Gamma_{\boldsymbol{f},\boldsymbol{e};g}$ be the collection of regular $G(F)$-conjugacy classes of type $(\boldsymbol{f}, \boldsymbol{e}; g)$.*

With the above definition, we have the decomposition:

$$\bigsqcup_{C \text{ regular}} C = \bigsqcup_{\substack{\mathbf{f},\, \mathbf{e} \in \mathbb{Z}_{\geq 1}^g \\ \langle \mathbf{f}, \mathbf{e} \rangle = n}} \Gamma_{\mathbf{f},\mathbf{e};g}. \tag{2.24}$$

A useful observation is that if C is a regular conjugacy class in $G(F)$ of type $(f_1, \cdots, f_g; e_1, \cdots, e_g)$, and $\gamma \in C$, then the centralizer of γ in $G(F)$ can be described by the algebra $\oplus_{1 \leq i \leq g} E_i^{e_i}$, where E_i is a field extension of F with $[E_i; F] = f_i$; and $\oplus_{1 \leq i \leq g} E_i^{e_i}$ denotes the direct sum of e_i copies of E_i.

Let $C \in \Gamma_{\mathbf{f},\mathbf{e};g}$. Let $\gamma_C \in C$ be a fixed element. Let $\lambda_{\mathbf{f},\mathbf{e};g} \in G(F)$ be defined by

$$\lambda_{\mathbf{f},\mathbf{e};g} = \begin{pmatrix} I_{f_1} & & & & & & \\ & \ddots & & & & & \\ & & I_{f_1} & & & & \\ & & & \ddots & & & \\ & & & & I_{f_g} & & \\ & & & & & \ddots & \\ \eta_{f_1} & \cdots & \eta_{f_1} & \cdots & \eta_{f_g} & \cdots & I_{f_g} \end{pmatrix}^{-1}, \tag{2.25}$$

where for each integer m, $\eta_m = (0, \cdots, 1) \in F^m$, the row vector with the last entry being 1 and the rest being 0; and I_m is the identity matrix of rank m.

Then by Proposition 12 and unfolding $E(s, \Phi; s)$, we have, when $\mathrm{Re}(s) > 1$, that

$$I_C(s, \tau) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \sum_{p \in P_0(F)} \varphi(x^{-1}p^{-1}\lambda_{\mathbf{f},\mathbf{e};g}^{-1}\gamma_C\lambda_{\mathbf{f},\mathbf{e};g}px) \sum_{\delta \in P(F)\backslash G(F)} f(\delta x, s)dx.$$

Then switching the sums and changing variables we then obtain

$$I_C(s,\tau)=\int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)}\sum_{p\in P_0(F)}\varphi(x^{-1}p^{-1}\lambda_{\mathbf{f},\mathbf{e};g}^{-1}\gamma_C\lambda_{\mathbf{f},\mathbf{e};g}px)f(x,s)dx$$
$$=\int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\varphi(x^{-1}\gamma_C x)f(\lambda_{\mathbf{f},\mathbf{e};g}^{-1}x,s)dx,$$

supposing the above integrals converge absolutely. Combing this with (2.24) we then deduce (at least formally) that, when $\mathrm{Re}(s)>1$,

$$\sum_C I_C(s,\tau)=\sum_{\substack{\mathbf{f},\ \mathbf{e}\in\mathbb{Z}_{\geq 1}^g\\ \langle\mathbf{f},\mathbf{e}\rangle=n}}\int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{C\in\Gamma_{\mathbf{f},\mathbf{e};g}}\varphi(x^{-1}\gamma_C x)f(\lambda_{\mathbf{f},\mathbf{e};g}^{-1}x,s)dx. \tag{2.26}$$

Moreover, (2.26) would be rigorous if the right hand side converges absolutely, which is indeed the case. To verify, we will consider each type $(\mathbf{f},\mathbf{e};g)$ separately in the following subsections.

Type $(n;1)$

We treat the conjugacy classes of type $(\mathbf{f},\mathbf{e};g)=((n),(1);1)$ first, these are exactly elliptic regular conjugacy classes. Denote by

$$I_{r.e.}(s,\tau)=I_{r.e.}^{\varphi}(s,\tau)=\sum_{C\ \text{regular elliptic}}I_C(s,\tau).$$

Proposition 21. *Let notation be as before. Then for every field extension E/F of degree n, there is an analytic function $Q_E(s)$ such that*

$$I_{r.e.}(s,\tau)=\frac{1}{n}\sum_{[E:F]=n}Q_E(s)\Lambda\left(s,\tau\circ N_{E/F}\right), \tag{2.27}$$

where the summation is taken over only finitely many E's, depending implicitly only on the test function φ.

Proof. Since $\Gamma_{r.e.}(G(F)/Z(F))$ is invariant under $P(F)$-conjugation, we have

$$I_{r.e.}(s,\tau)=\int_{P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{\gamma\in\Gamma_{r.e.}(G(F)/Z(F))}\varphi(x^{-1}\gamma x)\cdot f(x,s)dx.$$

Denote by $\{\Gamma^{r.e.}\}$ a set of representatives for the regular elliptic conjugacy classes in $\Gamma_{r.e.}(G(F)/Z(F))$. For any $\gamma\in\{\Gamma^{r.e.}\}$, the centralizer of γ in $G(F)/Z(F)$ is exactly $F[\gamma]^{\times}$. Then we have

$$\sum_{\gamma\in\Gamma_{r.e.}(G(F)/Z(F))}\varphi(x^{-1}\gamma x)=\sum_{\gamma\in\{\Gamma^{r.e.}\}}\sum_{\delta\in F[\gamma]^{\times}Z_G(F)\backslash G(F)}\varphi(x^{-1}\delta^{-1}\gamma\delta x). \tag{2.28}$$

By Lemma 15 and the Remark after it, one has $G(F) = P(F)F[\gamma]^\times$. Since $P(F) \cap F[\gamma]^\times = Z_G(F)$, every element $\delta \in Z_G(F)\backslash G(F)$ can be written unique as $\delta = p\nu$, where $p \in Z_G(F)\backslash P(F)$ and $\nu \in F[\gamma]^\times$. Hence the inner sum of (2.28) could be taken over $p \in Z_G(F)\backslash P(F)$. Therefore, substituting these into the expression of $I_{r.e.}(s)$ one will obtain

$$I_{r.e.}(s,\tau) = \int_{Y_G} \sum_{\gamma\in\{\Gamma^{r.e.}\}} \sum_{p\in Z_G(F)\backslash P(F)} \varphi(x^{-1}p^{-1}\gamma p x) f(x,s)dx, \tag{2.29}$$

where $Y_G = P(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. For the given F, let E/F be a field extension of degree n. Fix an algebraic closure $\bar{F}$ of F, then E embeds into $\bar{F}$, we look at the contribution from all the regular elliptic conjugacy classes together. We say that a conjugacy class belongs to an extension E of F if it consists of the conjugates of some element $\gamma \in E^\times/F^\times - \{1\}$ with the usual identification. We have to distinguish between two cases:

(a) E/F is Galois.

(b) E/F is not Galois.

The idea is to replace the summation over $\gamma \in \{\Gamma^{r.e.}\}$ by summation over extensions E/F of degree n; and inside, summation over elements of E.

Case (a) When γ varies over $E^\times/F^\times$ we get each conjugacy class belonging to E exactly n times.

Case (b) When γ varies over $E^\times/F^\times$ we get each conjugacy class belonging to E once; but the sets of conjugacy classes belonging to the n embeddings of E in $\bar{F}$ are identical.

So in either case, we can rewrite the integral in (2.29) as

$$I_{r.e.}(s,\tau) = \frac{1}{n}\int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{[E:F]=n} \sum_{\gamma\in E^\times/F^\times-\{1\}} \varphi(x^{-1}\gamma x) f(x,s)dx \tag{2.30}$$

where the right hand summation is over all extensions E/F of degree n. Note that (2.30) is the same as (2.26). Note that the sum over E/F such that $[E : F] = n$ is actually finite, as the summation in (2.28) is finite, as a consequence of the fact that φ is compactly supported.

Moreover, since the coefficients of the characteristic polynomial of every $\gamma \in E^\times/F^\times$ are rational, and lie in a compact set depending on supp φ (and a discrete subset of a compact set is finite), the sum over $\gamma \in E^\times/F^\times - \{1\}$ is a finite sum. Thus we can interchange integrals in (2.30) to get

$$I_{r.e.}(s,\tau) = \frac{1}{n} \sum_{[E:F]=n} \sum_{\substack{\gamma \in E^\times/F^\times \\ \gamma \neq 1}} \int_{G(\mathbb{A}_F)} \varphi(x^{-1}\gamma x)\Phi(\eta x)\tau(\det x)|\det x|^s dx, \qquad (2.31)$$

where $\eta = (0, \cdots, 0, 1) \in \mathbb{A}_F^n$. Let $I_E(s)$ be the inner integral in (2.31), then

$$I_E(s) = \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x^{-1}\gamma x) \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0, \cdots, 0, 1)tx]\tau(\det tx)|\det tx|^s dt dx,$$

where G_γ is the centralizer of γ in G. Hence, $G_\gamma(\mathbb{A}_F) \simeq \mathbb{A}_E^\times$. If we identify $G_\gamma(\mathbb{A}_F)$ with $\mathbb{A}_E^\times$, $\det|_{E^\times}: E^\times \longrightarrow F^\times$ with the norm map $N_{E/F}$, $t \mapsto |\det t|_{\mathbb{A}_F}$ with the idele norm in E, and $\mathcal{S}(\mathbb{A}_F^n)$ with $\mathcal{S}(\mathbb{A}_E)$, we see that the inner integral is just the Tate integral for $\Lambda\left(s, \tau \circ N_{E/F}\right)$. So there is some elementary function $Q(s)$ of s such that $I_E(s) = Q(s)\Lambda\left(s, \tau \circ N_{E/F}\right)$, where $\Lambda\left(s, \tau \circ N_{E/F}\right)$ is the complete L-function attached to E.

Consequently, $I_E(s)$ itself converges normally for $\mathrm{Re}(s) > 1$ and its behavior is given by $Q(s)L_E\left(s, \tau \circ N_{E/F}\right)$. This also given the meromorphic continuation of $I_E(s)$ to the entire s-plane. Since

$$I_{r.e.}(s,\tau) = \sum_{C \text{ of type } ((n),(1))} I_C(s) = \frac{1}{n} \sum_{[E:F]=n} \sum_{\gamma \in E^\times/F^\times - \{1\}} I_E(s),$$

where the sums are finite, then $I_{r.e.}(s,\tau)$ is well defined when $\mathrm{Re}(s) > 1$, admits a meromorphic continuation to $s \in \mathbb{C}$, moreover, (2.27) holds. □

Type $(\mathbf{f}, \mathbf{e}; 1)$

In this subsection, we deal with orbital integrals of general type $(\mathbf{f}, \mathbf{e}; g)$. Note that one of the key ingredients to handle the elliptic regular case is that

$$x \mapsto \sum_{\lambda \in E^\times/F^\times - \{1\}} \varphi(x^{-1}\lambda x) \qquad (2.32)$$

has compact modulo $G_\gamma(\mathbb{A}_F)$. However, the function (2.32) is not compactly supported for general type $(\mathbf{f}, \mathbf{e}; g)$. For example, (2.32) has no compact support for regular unipotent conjugacy classes. So we must proceed differently from the elliptic regular case in Subsection 2.2.

In this section, we handle the case $g = 1$. This will, in conjunction with proof of Proposition 21, play a role in the treatment of general types in the next section.

Let C be of type $(\mathbf{f}, \mathbf{e}; 1)$. We may choose a representative $\gamma \in C_0$ in its quasi-rational canonical form:

$$\gamma = \begin{pmatrix} C & & & & \\ \mathbf{N} & C & & & \\ & \mathbf{N} & C & & \\ & & \ddots & \ddots & \\ & & & \mathbf{N} & C \end{pmatrix},$$

where $C \in \mathrm{GL}(f)$ and there are e such C's in the partitioned matrix above. Then the stabilizer of γ is studied in Lemma 17. In particular, let A be a stabilizer of γ, then A must be of the form:

$$A = \begin{pmatrix} A_0 & & & & \\ A_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ \vdots & \ddots & \ddots & \ddots & \\ A_{e-1} & \dots & A_2 & A_1 & A_0 \end{pmatrix}, \tag{2.33}$$

with $A_i \in M_{f\times f}(F)$, $1 \le i < e$, and $A_0 \in F(C)$. Let $P_{\mathbf{f},\mathbf{e};1}$ be the transpose of standard parabolic subgroup of G of type $(f, f, \cdots, f)$, i.e., it is a lower triangle matrix group. Let $K_{\mathbf{f},\mathbf{e};1}$ be a compact subgroup such that $G(\mathbb{A}_F) = P_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)K_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)$.

Therefore, we can decompose $G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ as: for $x \in G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, write

$$x = \mathbf{B} \begin{pmatrix} I_f & & & & \\ & D_1 & & & \\ & & D_2D_1 & & \\ & & & \ddots & \\ & & & & D_{e-1}\cdots D_1 \end{pmatrix} \begin{pmatrix} T_0 & & & & \\ & T_1 & & & \\ & & T_2 & & \\ & & & \ddots & \\ & & & & T_{e-1} \end{pmatrix} k, \tag{2.34}$$

where each $D_j \in G_C(\mathbb{A}_F)$, which is the stabilizer of C, $1 \le j \le e-1$; and

$$\mathbf{B} = \begin{pmatrix} I_f & & & & \\ & I_f & & & \\ & B_{3,2} & I_f & & \\ & \vdots & \ddots & \ddots & \\ & B_{e,2} & \dots & B_{e,e-1} & I_f \end{pmatrix}; \tag{2.35}$$

and $T_j \in G_C(\mathbb{A}_F)\backslash \mathrm{GL}(f,\mathbb{A}_F)$, $0 \leq j \leq e-1$; and $k \in K_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)$. Denote by $D = \mathrm{diag}(I_f, D_1, \cdots, D_{e-1}\cdots D_1)$ and $\mathbf{T} = \mathrm{diag}(T_0, T_1, \cdots, T_{e-1})$. Note that the decomposition (2.34) follows from Iwasawa decomposition and the unipotent term $\mathbf{B}$ is of the form (2.35) because it's first f-columns can be absorbed by left multiplication of some stabilizer $A \in G_\gamma(\mathbb{A}_F)$ of shape (2.33). Write

$$\mathbf{B}^{-1} = \begin{pmatrix} I_f & & & & \\ & I_f & & & \\ & B_{3,2} & I_f & & \\ & \vdots & \ddots & \ddots & \\ & B_{e,2} & \dots & B_{e,e-1} & I_f \end{pmatrix}^{-1} = \begin{pmatrix} I_f & & & & \\ & I_f & & & \\ & B'_{3,2} & I_f & & \\ & \vdots & \ddots & \ddots & \\ & B'_{e,2} & \dots & B'_{e,e-1} & I_f \end{pmatrix}.$$

For each $B'_{i,j}$, we write $\widetilde{B}'_{i,j} = B'_{i,j}C - CB'_{i,j}$. Let $\mathcal{B}$ be the group of such $\mathbf{B}$'s.

By definition, the contribution from conjugacy classes of type $(\mathbf{f}, \mathbf{e}; 1)$ is

$$I_{\mathbf{f},\mathbf{e};1}(s) = \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{C \in \Gamma_{\mathbf{f},\mathbf{e};1}} \varphi(x^{-1}\gamma_C x) f(\lambda_{\mathbf{f},\mathbf{e};1}^{-1} x, s) dx. \tag{2.36}$$

For two meromorphic functions $h_1(s)$ and $h_2(s)$, we denote by $h_1(s) \sim h_2(s)$ if $h_1(s)/h_2(s)$ admits an analytic continuation to the whole complex plane. We will keep this notation "$\sim$" henceforth. In this subsection, we will show

Proposition 22. *Let notation be as before. Then $I_{f,e;1}(s)$ converges absolutely when* $\mathrm{Re}(s) > 1$ *and*

$$I_{f,e;1}(s) \sim \prod_{j=1}^{e} \sum_{[E_j:F]=f} Q_{E_j}(s) \Lambda_{E_j}(js - j + 1, (\tau \circ N_{E/F})^j),$$

where the sum over number fields E_j's is finite and Q_{E_j} is an entire function of s.

Remark 23. *The function Q_{E_j} is the ratio of the Tate integral and the L-functions. Hence it is entire.*

Proof. Write (2.36) simply as

$$I_{\mathbf{f},\mathbf{e};1}(s) = \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \sum_{\gamma} \varphi(x^{-1}\gamma x) f(x, s) dx,$$

where γ runs over regular elements of type $(\mathbf{f}, \mathbf{e}; 1)$. Then similar as the discussion in Proposition 21, the sum over γ is finite, depending only on the support of φ. We then switch the sum to get

$$I_{\mathbf{f},\mathbf{e};1}(s) = \sum_{\gamma} I_\gamma(s),$$

where

$$I_\gamma(s) := \int_{G_\gamma(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \varphi(x^{-1}\gamma x) \int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\cdots,0,1)tx]\tau(\det tx)|\det tx|^s \, dt dx.$$

Substituting the decomposition (2.34) into the above integral, we obtain

$$I_\gamma(s) = \int_k \int_{(G_C(\mathbb{A}_F))^{e-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e} \int_{\mathcal{B}} \varphi(k^{-1}\mathbf{T}^{-1}D^{-1}\mathbf{B}^{-1}\gamma\mathbf{B}D\mathbf{T}k) \int_{G_\gamma(\mathbb{A}_F)} \cdots,$$

where the integral relative to k is over $K_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)$.

According to the preceding discussion we have

$$I_E(s) = \int_k \int_{(G_C(\mathbb{A}_F))^{e-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e} \int_{\mathcal{B}} \varphi(k^{-1}\mathbf{T}^{-1}D^{-1}MD\mathbf{T}k) \int_{G_\gamma(\mathbb{A}_F)} \cdots,$$

where $M = \mathbf{B}^{-1}\gamma\mathbf{B}$ is of the form

$$\begin{pmatrix} C & & & & \\ \mathbf{N} & C & & & \\ B'_{3,2}\mathbf{N} & \widetilde{B}'_{3,2} + \mathbf{N} & C & & \\ \vdots & \vdots & \ddots & \ddots & \\ B'_{e,2}\mathbf{N} & \widetilde{B}'_{e,2} + B'_{e,3}\mathbf{N} + \widetilde{B}'_{e,3}\mu_{e,2}(B_{i,j},\mathbf{N},C) + \lambda_{e,2}(B_{i,j},\mathbf{N},C) & \dots & \widetilde{B}'_{e,e-1} + \mathbf{N} & C \end{pmatrix},$$

where a typical entry of the above matrix is of the form

$$\widetilde{B}'_{l,k} + B'_{l,k+1}\mathbf{N} + \widetilde{B}'_{l,k+1}\mu_{l,k}(B_{i,j},\mathbf{N},C) + \lambda_{l,k}(B_{i,j},\mathbf{N},C)$$

with $\mu_{l,k}(B_{i,j},\mathbf{N},C)$ and $\lambda_{l,k}(B_{i,j},\mathbf{N},C)$ are polynomials of $B_{i,j}, \mathbf{N}$, and C, with $i \le l-1$ and $j \ge k+2$.

A straightforward computation shows that if the first column of $B'_{i,j}$ is determined, then $\widetilde{B}'_{i,j} = B'_{i,j}C - CB'_{i,j}$ is completely determined by its last $f-1$ columns. Therefore, when $B'_{l,k}\mathbf{N}$ is fixed, then $\widetilde{B}'_{l,k} + B'_{l,k+1}\mathbf{N}$ is an $f \times f$ matrix, linearly determined by the last $f-1$ columns of $B'_{l,k}$ and the first column of $B'_{l,k+1}$, and this correspondence is one to one. So $\mathbf{B} \mapsto M$ gives a one-to-one continuous correspondence. The set of M's which contribute is contained in a compact region by our choice of φ. Then, by a change of variable we see the integral relative to $\mathbf{B}$ has compact support in $\mathcal{B}$. Then

$$I_\gamma(s) = \int_k \int_{(G_C(\mathbb{A}_F))^{e-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e} \int_{\mathcal{B}} \varphi(k^{-1}\mathbf{T}^{-1}D^{-1}MD\mathbf{T}k)\delta_{P_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)}(\mathbf{T})^{-1}$$
$$\int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\cdots,0,1)A\mathbf{B}D\mathbf{T}k]\tau(\det AD\mathbf{T})|\det AD\mathbf{T}|^s dAd\mathbf{B}dDd\mathbf{T}dk,$$

where $\delta_{P_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)}(\mathbf{T})$ is the modular character associated to $P_{\mathbf{f},\mathbf{e};1}(\mathbb{A}_F)$. Change variable $\mathbf{B} \mapsto D\mathbf{T}\mathbf{B}\mathbf{T}^{-1}D^{-1}$ we then obtain

$$I_\gamma(s) = \int_k \int_{(G_C(\mathbb{A}_F))^{e-1}} \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e} \int_{\mathcal{B}} \varphi(k^{-1}\mathbf{B}^{-1}\mathbf{T}^{-1}D^{-1}\gamma D\mathbf{T}\mathbf{B}k)$$
$$\int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\cdots,0,1)AD\mathbf{T}\mathbf{B}k]\tau(\det ADT)|\det AD\mathbf{T}|^s dAd\mathbf{B}dDd\mathbf{T}dk.$$

Then $D^{-1}\gamma D$ is equal to

$$\begin{pmatrix} C &&&&& \\ D_1^{-1}\mathbf{N} & C &&&& \\ 0 & D_1^{-1}D_2^{-1}\mathbf{N}D_1 & C &&& \\ \vdots & \vdots & \ddots && \ddots & \\ 0 & * & \dots & D_1^{-1}\cdots D_{e-2}^{-1}D_{e-1}^{-1}\mathbf{N}D_{e-2}\cdots D_1 && C \end{pmatrix}.$$

We can identify $D_j^{-1}\mathbf{N}$ with an element x_j in $E_j^\times/F^\times - 1$, $1 \le j \le e-1$, where E_j is a field extension of F with $[E_j : F] = f$. Conjugate of $D_j^{-1}\mathbf{N}$ under this identification becomes a Galois action on x_j. Therefore, we have

$$I_\gamma(s) = \frac{1}{f^{e-1}} \prod_{j=1}^{e-1} \sum_{\substack{x_j\in E_j^\times/F^\times-1 \\ [E_j:F]=f}} \int_{\mathbb{A}_{E_j}^\times} \prod_{j=1}^{e-1} \tau^j(N_{E_j/F}(x_j))N_{E_j/F}(x_j)^{js-j+1}$$
$$\int_k \int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e} \int_{\mathcal{B}} \varphi(k^{-1}\mathbf{B}^{-1}\mathbf{T}^{-1}\gamma^D\mathbf{T}\mathbf{B}k)|\det A_0|^{1-e}$$
$$\int_{G_\gamma(\mathbb{A}_F)} \Phi[(0,\cdots,0,1)A\mathbf{T}\mathbf{B}k]\tau(\det A\mathbf{T})|\det A\mathbf{T}|^s dAd\mathbf{B}d\mathbf{T}dkdx_1\cdots dx_{e-1},$$

where the sum over x_j's is finite and

$$\gamma^D = \begin{pmatrix} C &&&& \\ D_1^{-1}\mathbf{N} & C &&& \\ 0 & D_2^{-1}\mathbf{N} & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \dots & D_{e-1}^{-1}\mathbf{N} & C \end{pmatrix} = \begin{pmatrix} C &&&& \\ x_1 & C &&& \\ 0 & x_2 & C && \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & * & \dots & x_{e-1} & C \end{pmatrix}.$$

Similar analysis from the proof of Proposition 21 shows the integral relative to $\mathbf{T}$ actually is over a compact set since φ has compact support. Hence, the function

$$(x_1,\cdots,x_{e-1})\mapsto\int_k\int_{(G_C(\mathbb{A}_F)\backslash G(\mathbb{A}_F))^e}\int_{\mathcal{B}}\varphi(k^{-1}\mathbf{B}^{-1}\mathbf{T}^{-1}\gamma^D\mathbf{T}\mathbf{B}k)|\det A_0|^{1-e}$$
$$\int_{G_\gamma(\mathbb{A}_F)}\Phi[(0,\cdots,0,1)A\mathbf{T}\mathbf{B}k]\tau(\det A\mathbf{T})|\det A\mathbf{T}|^s dAd\mathbf{B}d\mathbf{T}dk$$

is Schwartz; and the function

$$\int_{G_\gamma(\mathbb{A}_F)}\Phi[(0,\cdots,0,1)A\mathbf{T}\mathbf{B}k]\tau(\det A\mathbf{T})|\det A\mathbf{T}|^s|\det A_0|^{1-e}dA$$

is the Tate integral for $\Lambda(es,(\tau\circ N_{E/F})^e)$. Therefore, Proposition 22 follows. □

Orbital Integrals of General Type

Denote by

$$I_{\mathbf{f},\mathbf{e};g}(s)=\int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\sum_{C\in\Gamma_{\mathbf{f},\mathbf{e};g}}\varphi(x^{-1}\gamma_C x)f(\lambda_{\mathbf{f},\mathbf{e};g}^{-1}x,s)dx,$$

where $g\geq 1$, $\mathbf{f},\mathbf{e}\in\mathbb{Z}_{\geq 1}^g$, $\langle\mathbf{f},\mathbf{e}\rangle=n$; and $\mathrm{Re}(s)>1$. We may write $\mathbf{f}=(f_1,\cdots,f_g)$ with $f_1\geq\cdots\geq f_g$; and $\mathbf{e}=(e_1,\cdots,e_g)$.

Let E be a finite extension of F. Let χ be an idele class character of $\mathbb{A}_E^\times$. Let j be a positive integer. Denote by

$$\Lambda_E[j](s,\chi):=\Lambda_E(js-j+1,\chi^j),\tag{2.37}$$

where $\Lambda(s,\chi)$ is the complete Hecke L-function associated to χ.

Proposition 24. *Let notation be as before. Then $I_{\boldsymbol{f},\boldsymbol{e};g}(s)$ converges absolutely when* $\mathrm{Re}(s)>1$ *and*

$$I_{\boldsymbol{f},\boldsymbol{e};g}(s)\sim\prod_{i=1}^g\prod_{j=1}^{e_i}\sum_{[E_{i,j}:F]=f_i}Q_{E_{i,j}}(s)\Lambda_{E_{i,j}}[j]\left(s,\tau\circ N_{E_i/F}\right),\tag{2.38}$$

where for each i, the innermost summation is taken over only finitely many fields $E_{i,j}$'s, depending implicitly only on the test function φ; and each $Q_{E_{i,j}}(s)$ is an entire function.

Proof. Let C be a regular conjugacy class of type $(\mathbf{f}, \mathbf{e}; g)$. By Lemma 14 and proof of Lemma 17, we can write $C = \{\gamma\}$, with a typical element γ given by

$$\gamma = \begin{pmatrix} \alpha_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \alpha_m & & & & & & & \\ & & & \alpha_{m+1} & & & & & & \\ & & & \vdots & \ddots & & & & & \\ & & & * & \cdots & \alpha_{m+1} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \alpha_g & & \\ & & & & & & & \vdots & \ddots & \\ & & & & & & & * & \cdots & \alpha_g \end{pmatrix}.$$

Using this expression and the definition of $\lambda_{\mathbf{f},\mathbf{e};g}^{-1}$ (cf. (2.25)) one then sees that Proposition 24 follows from combining analysis in the proof of Proposition 21 with that in Proposition 22. □

Combining (2.24) and Proposition 24 we then obtain

Theorem E. *Let notation be as before. Let* $\mathrm{Re}(s) > 1$. *Then*

$$I_{\mathrm{Geo,Reg}}(s,\tau) \sim \sum_{g=1}^{n} \sum_{\substack{\boldsymbol{f},\, \boldsymbol{e} \in \mathbb{Z}_{\geq 1}^{g} \\ \langle \boldsymbol{f}, \boldsymbol{e} \rangle = n}} \prod_{i=1}^{g} \frac{1}{f_i^{e_i}} \sum_{[E_i:F]=f_i} Q_{E_i}(s) \prod_{j=1}^{e_i} \Lambda_{E_i}[j] \left(s, \tau \circ N_{E_i/F}\right),$$

where for each i, the innermost summation is taken over only finitely many fields E_i's, depending implicitly only on the test function φ; and each $Q_{E_i}(s)$ is an entire function. Moreover, the right hand side of $I_{\mathrm{Geo,Reg}}(s,\tau)$ in the above formula gives a meromorphic continuation of $I_{\mathrm{Geo,Reg}}(s,\tau)$ to the whole s-plane, with only possible poles at $s \in \{1, 1/2, 2/3, \cdots, 1 - 1/n\}$.

Chapter 3

MIRABOLIC FOURIER EXPANSION OF $\mathrm{K}_\infty(S)$

Take a test function φ as before, then by the definition of $E_P(x,\Phi;s)$ we have

$$I_\infty(s,\tau)=I_\infty^{\varphi}(s,\tau)=-\int_{G(F)Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}\mathrm{K}_\infty(x,x)\sum_{\gamma\in P(F)\backslash G(F)}f(\gamma x,s)dx.$$

where $\mathrm{K}_\infty(x,y)=\mathrm{K}_{\mathrm{Eis}}(x,y)+\mathrm{K}_{\mathrm{Res}}(x,y)$ is left $N(F)$-invariant. Then

$$I_\infty(s,\tau)=-\int_{Z_G(\mathbb{A}_F)P(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_\infty(x,x)f(x,s)dx. \tag{3.1}$$

Now we proceed to compute (3.1) by considering the Fourier expansion of $\mathrm{K}_\infty(x,y)$.

3.1 Mirabolic Fourier Expansions of Automorphic Forms

Fourier expansions of automorphic forms of GL_n are well known (see [Pia75]). Following the idea of Piatetski-Shapiro in [Pia75], we give a new form of Fourier expansions of weak automorphic forms in terms of generalized mirabolic subgroups, via which a further decomposition of $I_\infty(s,\tau)$ is obtained. Here we call a function $f\in\mathcal{C}(G(\mathbb{A}_F))$ a weak automorphic form if it is slowly increasing on $G(\mathbb{A}_F)$, right K-finite and $P_0(F)$-invariant, where P_0 is the mirabolic subgroup of $G=\mathrm{GL}_n$.

Fix an integer $n\geq 2$. The maximal unipotent subgroup of $G(\mathbb{A}_F)$, denoted by $N(\mathbb{A}_F)$, is defined to be the set of all $n\times n$ upper triangular matrices in $G(\mathbb{A}_F)$ with ones on the diagonal and arbitrary entries above the diagonal. Let $\psi_{F/\mathbb{Q}}(\cdot)=e^{2\pi i\,\mathrm{Tr}_{F/\mathbb{Q}}(\cdot)}$ be the standard additive character, then for any $\alpha=(\alpha_1,\cdots,\alpha_{n-1})\in F^{n-1}$, define a character $\psi_\alpha:N(\mathbb{A}_F)\to\mathbb{C}$ by

$$\psi_\alpha(u)=\prod_{i=1}^{n-1}\psi_{F/\mathbb{Q}}\left(\alpha_i u_{i,i+1}\right),\quad\forall\, u=(u_{i,j})_{n\times n}\in N(\mathbb{A}_F).$$

Write $\psi_k=\psi_{(0,\cdots,0,1,\cdots,1)}$ (where the first $n-k$ components are 0 and the remaining k components are 1) and $\theta=\psi_{(1,\cdots,1)}$, the standard generic character used to define Whittaker functions.

For $1\leq k\leq n-1$, let B_{n-k} be the standard Borel subgroup (i.e. the subgroup consisting of nonsingular upper triangular matrices) of GL_{n-k}; let N_{n-k} be the unipotent radical of B_{n-k}. For any $i, j \in \mathbb{N}$, let $M_{i\times j}$ be the additive group scheme of $i \times j$-matrices. Define the unipotent radicals

$$N_{(k,1,\cdots,1)} = \left\{ \begin{pmatrix} I_k & B \\ & D \end{pmatrix} : \ B \in M_{k\times(n-k)},\ D \in N_{n-k} \right\},\ 1 \leq k \leq n-1\ .$$

For $1 \leq k \leq n-1$, set the generalized mirabolic subgroups

$$R_k = \left\{ \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} : \ A \in GL_k,\ C \in M_{k\times(n-k)},\ B \in N_{n-k} \right\}.$$

For $2 \leq k \leq n-1$, define subgroups of R_k by

$$R_k^0 = \left\{ \begin{pmatrix} A & B' & C \\ 0 & a & D \\ 0 & 0 & B \end{pmatrix} : \ \begin{pmatrix} A & B' \\ & a \end{pmatrix} \in GL_k,\ \begin{pmatrix} C \\ D \end{pmatrix} \in M_{k\times(n-k)},\ B \in N_{n-k} \right\}.$$

Also we define $R_0 = R_1^0 = N_{(0,1,\cdots,1)} := N_{(1,1,\cdots,1)}$ to be the unipotent radical of the standard Borel subgroup of GL_n . For simplification, we will denote by $[H] := H(F)\backslash H(\mathbb{A}_F)$ for an algebraic group H over F.

Proposition 26 (Mirabolic Fourier Expansion)**.** *Let h be a continuous function on $P_0(F)\backslash G(\mathbb{A}_F)$. Then we have*

$$h(x) = \sum_{k=1}^{n} \sum_{\delta_k \in R_{k-1}(F)\backslash R_{n-1}(F)} \int_{[N_{(k-1,1,\cdots,1)}]} h(u\delta_k x)\psi_{n-k}(u)du \tag{3.2}$$

if the right hand side converges absolutely and locally uniformly.

Proof. For $1 \leq k \leq n$, we define

$$M_k^0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & I_{n-k} \end{pmatrix} : \ A \in GL(k,F) \right\},$$

$$M_k^\infty = \left\{ \begin{pmatrix} A & b & 0 \\ 0 & c & 0 \\ 0 & 0 & I_{n-k} \end{pmatrix} : \ A \in GL(k-1,F),\ b \in F^{k-1},\ c \in F^\times \right\},$$

where I_{n-k} is the unit matrix of dimension $n-k$. For the sake of simplicity, write

$$J_k h(x) = \sum_{\delta_k \in R_{k-1}\backslash R_{n-1}} \int_{[N_{(k-1,1,\cdots,1)}]} h(n\delta_k x)\psi_{n-k}(n)dn,$$

where $\psi_0 \equiv 1$. Let $N_1 \subset N$ be the subgroup consisting of elements of the form

$$\mathfrak{n}^{(1)}(u_n) = \begin{pmatrix} 1 & & & u_{1,n} \\ & 1 & & \vdots \\ & & \ddots & u_{n-1,n} \\ & & & 1 \end{pmatrix}, \text{ where } u_n = (u_{1,n}, \cdots, u_{n-1,n}) \in \mathbb{A}_F^{n-1}.$$

Since N_1 is abelian, h has the Fourier expansion with respect to N_1 :

$$h(x) = \sum_{\alpha^{(1)}=(\alpha_{1,n},\cdots,\alpha_{n-1,n})\in F^{n-1}} \int_{[N_1]} h(\mathfrak{n}^{(1)}(u_n)x) \prod_{i=1}^{n-1} \psi_{F/\mathbb{Q}}\left(\alpha_{i,n}u_{i,n}\right) du_n.$$

Denote the inner integral by $W^1_{\alpha^{(1)}}h(x)$. Since h is $P_0(F)$-invariant, then

$$\begin{aligned} W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(\gamma x) &= \int_{[N_1]} h(\mathfrak{n}^{(1)}(u_n)\gamma x)\psi_{F/\mathbb{Q}}\left(\alpha_{n-1,n}u_{n-1,n}\right) du_n \\ &= \int_{[N_1]} h\left(\gamma^{-1}\mathfrak{n}^{(1)}(u_n)\gamma x\right)\psi_{F/\mathbb{Q}}\left(\alpha_{n-1,n}u_{n-1,n}\right) du_n, \end{aligned}$$

for any $\gamma = \operatorname{diag}(A, 1)$, where $A \in GL(n-1, F)$. An easy computation shows that $\gamma^{-1}\mathfrak{n}^{(1)}(u_n)\gamma = \mathfrak{n}^{(1)}(u_n')$, where $u_n' = A^{-1}u_n$. Write $A = (a_{i,j})_{(n-1)\times(n-1)}$, then $u_{n-1,n} = a_{n-1,1}u'_{1,n} + \cdots + a_{n-1,n-1}u'_{n-1,n}$. This implies that for any such γ,

$$W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(\gamma x) = W^1_{(a_{n-1,1}\alpha_{n-1,n},a_{n-1,2}\alpha_{n-1,n},\cdots,a_{n-1,n-1}\alpha_{n-1,n})}h(x).$$

Hence one has

$$h(x) = \sum_{\substack{\gamma_{n-1}\in M^{\infty}_{n-1}\backslash M^0_{n-1}\\ \alpha_{n-1,n}\in F^{\times}}} W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(\gamma_{n-1}x) + W^1_{(0,0,\cdots,0)}h(x). \tag{3.3}$$

For any $\alpha_{n-1,n} \in F^{\times}$, let $\mathfrak{a}_{n-1,n} = \operatorname{diag}(1, \cdots, 1, \alpha_{n-1,n}, 1) \in R^0_{n-1}\backslash R_{n-1}$. Since h is left invariant by $\mathfrak{a}_{n-1,n}$, $M^{\infty}_{n-1}\backslash M^0_{n-1} = R^0_{n-1}\backslash R_{n-1}$ and

$$\begin{pmatrix} 1 & & & u_{1,n} \\ & 1 & & \vdots \\ & & \ddots & \alpha_{n-1,n}u_{n-1,n} \\ & & & 1 \end{pmatrix} = \mathfrak{a}_{n-1,n} \begin{pmatrix} 1 & & & u_{1,n} \\ & 1 & & \vdots \\ & & \ddots & u_{n-1,n} \\ & & & 1 \end{pmatrix} \mathfrak{a}^{-1}_{n-1,n},$$

$$W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(\gamma_{n-1}x) = \int_{[N_{(n-1,1)}]} h(n\mathfrak{a}^{-1}_{n-1,n}\gamma_{n-1}x)\psi(u_{n-1,n})dn.$$

Note that $W^1_{(0,0,\cdots,0)}h(x) = J_n h(x)$ and $R^0_{n-1}/R_{n-2} = \mathrm{GL}(1)$, (3.3) then becomes

$$h(x) = \sum_{\delta_{n-1}\in R_{n-2}\backslash R_{n-1}} \int_{N_{(n-1,1)}(F)\backslash N_{(n-1,1)}(\mathbb{A}_F)} h(n\delta_{n-1}x)\psi_1(n)dn + J_n h(x). \tag{3.4}$$

Let $N_2 \subset N$ be the subgroup consisting of elements of the form

$$\mathfrak{n}^{(2)}(u_{n-1}) = \begin{pmatrix} 1 & & & u_{1,n-1} & \\ & 1 & & \vdots & \\ & & \ddots & u_{n-2,n-1} & \\ & & & 1 & \\ & & & & 1 \end{pmatrix}, \ u_{n-1} = (u_{1,n}, \cdots, u_{n-2,n-1}) \in \mathbb{A}_F^{n-2}.$$

Since $N_2(F) \subset M^\infty_{n-1}$, $W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(u_{n-1}x) = W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(x)$, $\forall\ u_{n-1} \in F^{n-1}$. Then we have the Fourier expansion of $W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(x)$ with respect to N_2 :

$$W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(x) = \sum_{\alpha^{(2)}=(\alpha_{1,n-1},\cdots,\alpha_{n-2,n-1})\in F^{n-2}} W^2_{\alpha^{(2)}}h(x),$$

where $W^2_{\alpha^{(2)}}h(x) = W^2_{(\alpha_{1,n-1},\alpha_{2,n-1},\cdots,\alpha_{n-2,n-1})}h(x)$ is defined to be

$$\int_{[N_2]} W^1_{(0,0,\cdots,\alpha_{n-1,n})}h\left(\mathfrak{n}^{(2)}(u_{n-1})x\right)\prod_{i=1}^{n-2}\psi_{F/\mathbb{Q}}\left(\alpha_{i,n-1}u_{i,n-1}\right)du_{n-1}.$$

Likewise, we obtain

$$W^1_{(0,0,\cdots,\alpha_{n-1,n})}h(x) = \sum_{\substack{\gamma_{n-2}\in M^\infty_{n-2}\backslash M^0_{n-2}\\ \alpha_{n-2,n-2}\in F^\times}} W^2_{(0,0,\cdots,\alpha_{n-2,n-1})}h(\gamma_{n-2}x) + W^2_{(0,\cdots,0)}h(x),$$

where, by a direct computation, one has

$$W^2_{(0,\cdots,0)}h(x) = \int_{[N_{(n-2,1,1)}]} h(nx)\psi(u_{n-1,n})dn,$$
$$W^2_{(0,0,\cdots,\alpha_{n-2,n-1})}h(\gamma_{n-2}x) = \int h(n\mathfrak{a}^{-1}_{n-2,n-1}\gamma_{n-2}x)\psi(\alpha_{n-1,n}u_{n-1,n} + u_{n-2,n-1})dn,$$

where the integral is taken over $[N_{(n-2,1,1)}]$. Moreover, noting that $R^0_{n-2}/R_{n-3} = \mathrm{GL}(1)$, then substituting the above computation into (3.3) implies that

$$h(x) = \sum_{\delta_{n-2}\in R_{n-3}\backslash R_{n-1}} \int_{[N_{(n-2,1,1)}]} h(n\delta_{n-2}x)\psi_2(n)dn + J_{n-1}h(x) + J_n h(x).$$

Then clearly the expansion (3.2) follows from repeating this process $n-2$ more times. □

3.2 Decomposition of $I_\infty(s,\tau)$

Applying Proposition 26 to the kernel function $\mathrm{K}(x,y)$ viewed as a function of x, we thus obtain a formal decomposition of the distribution $I_\infty(s)$ when $\mathrm{Re}(s)>1$. In fact, by the spectral decomposition of the kernel function $\mathrm{K}_\infty(x,y)$ (cf. Lemma 2 on p. 263 of [Art79]), one has

$$\mathrm{K}_\infty(x,y)=\sum_{\chi}\sum_{P} n(A)^{-1}\left(\frac{1}{2\pi i}\right)^{\dim(A/Z_G)}\int_{i\mathfrak{a}^G}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathfrak{E}_{P,\chi}(x,y;\phi,\lambda)d\lambda,$$

where χ runs over proper cuspidal datum, P runs over all standard parabolic subgroups that are not equal to G; $\mathfrak{B}_{P,\chi}$ is an orthonormal basis of the automorphic representation induced from P by χ; $\mathfrak{a}^G$ is the root space which can be identified with $\mathbb{R}^{n-1}$; and

$$\mathfrak{E}_{P,\chi}(x,y;\phi,\lambda)=E(x,\mathcal{I}_P(\lambda)\phi,\lambda)\overline{E(y,\phi,\lambda)},$$

with $\mathcal{I}_P$ the operator defined on p. 254 of loc. cit.(cf. line -4); and the integrals on the right hand side converges absolutely.

Since for any $m\in M_P(F)$, we have

$$E(my,\phi,\lambda)=\sum_{\delta\in P(F)\backslash G(F)}\phi(\delta my)e^{(\rho+\lambda)H_P(\delta my)}=E(y,\phi,\lambda),$$

where H_P is the log homomorphism defined by

$$H_P(m)=(n_1^{-1}\log|\det m_1|,\cdots,n_r^{-1}\log|\det m_r|) \tag{3.5}$$

for P of type $(n_1,\cdots,n_r)$ and $\mathrm{diag}(m_1,\ldots,m_r)\in M_P(\mathbb{A}_F)$.

Hence $\mathrm{K}_\infty(x,y)$ is $M_P(F)$-invariant with respect to both variables. Then we can apply Proposition 26 with respect to the first variable of $\mathrm{K}_\infty(x,y)$ to get, at least formally, that

$$I_\infty(s,\tau)=\sum_{k=1}^{n}\int_{X_k}\int_{[N_k^*]}\int_{[N_k']}\mathrm{K}_\infty(n^*n_1x,x)\theta(n_1)dn_1dn^*f(x,s)dx, \tag{3.6}$$

where the generic character $\theta=\psi_{(1,1,\cdots,1)}$ is defined right before Proposition 26, $X_k=Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)$, and $N_k'=N_{(k,1,\cdots,1)}$ and

$$N_k^*=\left\{\begin{pmatrix} I_{k-1} & C & \\ & 1 & \\ & & I_{n-k}\end{pmatrix}:\ C\in\mathbb{G}_a^{k-1}\right\}.$$

Moreover, when both sides of (3.6) converge absolutely, the identity is rigorous.

However, there are usually convergence problem with the decomposition (3.6). In fact, for $1 \leq k \leq n$, if we write $I_\infty^{(k)}(s)$ for the above (formal) integral, namely,

$$I_\infty^{(k)}(s) = \int_{Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)} \int_{[N_k^*]} \int_{[N_k']} \mathrm{K}_\infty(n^* n_1 x, x)\theta(n_1) dn_1 dn^* f(x,s) dx.$$

Then in fact $I_\infty^{(k)}(s)$ might diverge when $2 \leq k \leq n$, if φ does not support in elliptic regular sets. Nevertheless, we can show $I_{\mathrm{Whi}}(s,\tau)$ actually converges absolutely when $\mathrm{Re}(s) > 1$, and thus it defines a holomorphic function therein.

To start with, the first observation is that one can replace K_∞ by K in the definition of $I_\infty^{(k)}(s)$, $2 \leq k \leq n$. Denote by $V_k' = \mathrm{diag}(I_{k-1}, N_{n-k+1})$. Let V_k be the unipotent radical of the standard parabolic subgroup of type $(k-1, n-k+1)$. Then for any function ϕ on $G(\mathbb{A}_F)$ one has, for any $x \in G(\mathbb{A}_F)$, that

$$\int_{[N_k^*]} \int_{[N_k']} \phi(n^* n x)\theta(n) dn dn^* = \int_{[V_k']} \int_{[V_k]} \phi(uu'x) du \theta(u') du'.$$

Since V_k is a unipotent radical, then one has

$$\int_{[N_k^*]} \int_{[N_k']} \phi(n^* n x)\theta(n) dn dn^* = 0, \ \forall\ \phi \in \mathcal{A}_0\left(G(F)\backslash G(\mathbb{A}_F)\right). \tag{3.7}$$

Then by (3.7) and the discrete spectral decomposition (2.1) one has

$$\int_{[N_k^*]} \int_{[N_k']} \mathrm{K}_0(n^* n_1 x, x)\theta(n_1) dn_1 dn_2 dn^* = 0. \tag{3.8}$$

Since $\mathrm{K} = \mathrm{K}_\infty + \mathrm{K}_0$, then by (3.8), for $2 \leq k \leq n$, one sees that (at least formally)

$$\begin{aligned} I_\infty^{(k)}(s) &= \int_{Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)} \int_{[N_k^*]} \int_{[N_k']} \mathrm{K}(n^* n_1 x, x)\theta(n_1) dn_1 dn^* \cdot f(x,s) dx \\ &= \int_{Z_G(\mathbb{A}_F)R_{k-1}(F)\backslash G(\mathbb{A}_F)} \int_{[V_k']} \int_{[V_k]} \mathrm{K}(uu'x, x) du \theta(u') du' \cdot f(x,s) dx. \end{aligned}$$

In fact, since $K_0(x,x)$ is rapidly decreasing, the contribution from K_0 in the above integral is well defined, i.e, it converges absolutely.

Let $n \geq 2$. Recall that $\mathfrak{S}$ is the union of $(Z_G(F)\backslash Q_k(F))^{P_0(F)}$. Let

$$\begin{aligned} \mathrm{K}_{\infty,\mathrm{Sing}}^{(n)}(x,y) &= \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} \sum_{\gamma \in \mathfrak{S}} \varphi(x^{-1}u^{-1}\gamma y) du, \\ \mathrm{K}_{\infty,\mathrm{Reg}}^{(n)}(x,y) &= \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} \mathrm{K}(ux,y) du - \mathrm{K}_{\infty,\mathrm{Sing}}^{(n)}(x,y); \\ \mathrm{K}_\infty^{(k)}(x,y) &= \sum_{\delta_k \in R_{k-1}(F)\backslash P_{n-1}(F)} \int_{[V_k']} \int_{[V_k]} \mathrm{K}(uu'\delta_k x, y) du \theta(u') du', \end{aligned}$$

where $2 \leq k < n$. Define

$$\mathrm{K}_{\mathrm{Sing}}(x,y) := \mathrm{K}_{\mathrm{Geo,Sing}}(x,y) - \mathrm{K}_{\infty,\mathrm{Sing}}^{(n)}(x,y) - \sum_{k=2}^{n-1} \mathrm{K}_{\infty}^{(k)}(x,y). \tag{3.9}$$

Let $\mathrm{Re}(s) > 1$. Correspondingly, we define the distributions by

$$\begin{aligned}
I_{\mathrm{Geo,Reg}}(s,\tau) &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\mathrm{Geo,Reg}}(x,x)\cdot f(x,s)dx;\\
I_{\infty,\mathrm{Reg}}(s,\tau) &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\infty,\mathrm{Reg}}^{(n)}(x,x)\cdot f(x,s)dx;\\
I_{\mathrm{Sing}}(s,\tau) &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\mathrm{Sing}}(x,x)\cdot f(x,s)dx;\\
I_{\mathrm{Whi}}(s,\tau) &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_{\infty}^{(1)}(x,x)\cdot f(x,s)dx;
\end{aligned}$$

where

$$\mathrm{K}_{\infty}^{(1)}(x,y) := \sum_{\delta\in N(F)\backslash P_0(F)} \int_{[N]} \mathrm{K}_{\infty}(n\delta x,\delta y)\theta(n)dn, \tag{3.10}$$

with $N = R_0 = N_1'$ being the unipotent radical of the Borel of G.

Since $\mathrm{K}_0(x,y) = \mathrm{K}(x,y) - \mathrm{K}_\infty(x,y) = \mathrm{K}(x,y) - \sum_{j=1}^{n} \mathrm{K}_{\infty}^{(k)}(x,y)$, then

$$\mathrm{K}_0(x,y) = \mathrm{K}_{\mathrm{Geo,Reg}}(x,y) - \mathrm{K}_{\infty,\mathrm{Reg}}^{(n)}(x,y) + \mathrm{K}_{\mathrm{Sing}}(x,y) - \mathrm{K}_{\infty}^{(1)}(x,y).$$

Therefore, (at least formally) we have

$$I_0(s,\tau) = I_{\mathrm{Geo,Reg}}(s,\tau) - I_{\infty,\mathrm{Reg}}(s,\tau) + I_{\mathrm{Sing}}(s,\tau) - I_{\mathrm{Whi}}(s,\tau). \tag{3.11}$$

The analytic behavior of $I_{\mathrm{Geo,Reg}}(s,\tau)$ has been investigated in Theorem E. In the following sections we will deal with $I_{\infty,\mathrm{Reg}}(s,\tau)$ and $I_{\mathrm{Whi}}(s,\tau)$, and the analytic behavior of $I_{\mathrm{Sing}}(s,\tau)$ would follow from spectral expansion and functional equation. As we will see, $I_{\infty,\mathrm{Reg}}(s,\tau)$ will be handled by Langlands-Shahidi's method after applying some geometric auxiliary results (see Section 2.1); and $I_{\mathrm{Whi}}(s,\tau)$ can be reduced to an infinite sum of Rankin-Selberg convolutions of irreducible generic non-cuspidal representations of $\mathrm{GL}(n,\mathbb{A}_F)$ (see Section 5). We also obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ in Section 6 and Section 7. Hence the expansion (1.4) is well defined on both sides for $\mathrm{Re}(s) > 1$, and can be regarded as an identity between their continuations when $s \in \mathbb{C}$ is arbitrary and τ is such that $\tau^k \neq 1$, $\forall$ $1 \leq k \leq n$.

Chapter 4

CONTRIBUTIONS FROM $I_{\infty,\mathrm{REG}}(S,\tau)$

Now we start with handling the distribution $I_{\infty,\mathrm{Reg}}(s,\tau)$. Our approach is some geometric computation. Recall that, by definition,

$$\begin{aligned} I_{\infty,\mathrm{Reg}}(s,\tau) &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)}\int_{N_P(F)\backslash N_P(\mathbb{A}_F)}\mathrm{K}^{(n)}_{\infty,\mathrm{Reg}}(ux,x)duf(x,s)dx \\ &= \int_{Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)}\int_{[N_P]}\sum_{\gamma\in Z_G(F)\backslash G(F)-\mathfrak{S}}\varphi(x^{-1}u^{-1}\gamma x)duf(x,s)dx. \end{aligned}$$

To simplify $I_{\infty,\mathrm{Reg}}(s,\tau)$, we will write $Z_G(F)\backslash G(F) = \sqcup C$ as a disjoint union of $G(F)$-conjugacy classes modulo $Z_G(F)$, and further decompose each class C into a disjoint union of $P(F)$-conjugacy classes. Then we will find representatives of these $P(F)$-conjugacy classes explicitly. So eventually one can get rid of the factor $P_0(F) = Z_G(F)\backslash P(F)$ in the domain; moreover, one can now apply Iwasawa decomposition to the domain $Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ to compute this integral.

4.1 $P(F)$-conjugacy Classes

For any $G(F)$-conjugacy class C, denote by $C^{P(F)}_{r.e.}$ the component C_0 given in (2.3) if C is regular, and take $C^{P(F)}_{r.e.}$ to be an empty set if C is irregular. Since $C^{P(F)}_{r.e.}$ does not intersect any standard maximal parabolic subgroups and is nontrivial only when C is regular, for convenience, we call $C^{P(F)}_{r.e.}$ the regular elliptic component of C, despite of the fact that it might not be elliptic.

Let $\mathfrak{C}^{P(F)}_{r.e.}$ be the union of regular elliptic components of all $G(F)$-conjugacy classes in $G(F)$. Then $\mathfrak{C}^{P(F)}_{r.e.}$ is a disjoint union of $P(F)$-conjugacy classes in $G(F)$ by Proposition 12. Moreover, Proposition 13 and Proposition 12 give a decomposition of $G(F)$ as $P(F)$-conjugacy classes

$$G(F) = \mathfrak{C}^{P(F)}_{r.e.}\coprod\bigcup_{k=1}^{n-1}Q_k(F)^{P(F)}. \tag{4.1}$$

For any $2 \leq k \leq n$, let P_k be the standard maximal parabolic subgroup of GL_k of type $(k-1,1)$. In the following, we identify P_k with $\mathrm{diag}(P_k, I_{n-k})$ when view it as a subgroup of $G = GL_n$. Write W_k the Weyl group of GL_k with respect to the standard Borel subgroup and its Levi component. Let Δ_k be the set of simple roots. Let S_k be the subgroup of symmetric groups S_n generated by permutations among $\{1, 2, \cdots, k\} \subset \{1, 2, \cdots, n\}$. For any $\alpha \in \Delta_k$, via the isomorphisms and natural inclusion $W_k \xrightarrow{\sim} S_k \hookrightarrow S_n \xrightarrow{\sim} W_n$, we identify it with its natural extension in Δ, the set of simple roots for $G(F) = GL_n(F)$. Henceforth, write $\Delta_k = \{\alpha_{i,i+1} : 1 \leq i \leq k-1\}$, and for each simple root $\alpha_{i,i+1}$, write w_i^k for the corresponding simple reflection and identify it with w_i by the natural embedding.

Denote by $\mathfrak{C}_{r.e.}^{P_k(F)}$ the union of regular elliptic components of all $G(F)$-conjugacy classes in $\mathrm{GL}_k(F), 2 \leq k \leq n$. Let $\mathcal{R}_k$ be a set consisting of exactly all representatives of the $P_k(F)$-conjugacy classes $\mathfrak{C}_{r.e.}^{P_k(F)}$.

To compute $I_{\infty,\mathrm{Reg}}(s, \tau)$, an explicit choice of representatives of $\mathfrak{C}_{r.e.}^{P(F)}$ in Bruhat normal form needs to be taken. We will find at the end of this section that for each $2 \leq k \leq n$, there exists a particular choice of each $\mathcal{R}_k$, such that $\mathcal{R}_k$ is determined by $\mathcal{R}_{k-1}$. Thus a desired $\mathcal{R}_n$ could be obtained by induction. This will be illustrated in Proposition 29, to prove which, we start with the following result to narrow the candidates of representatives for $\mathfrak{C}_{r.e.}^{P(F)}$.

Lemma 27. *Let notation be as before. Set $\mathcal{R}_P = \{w_{n-1}w_{n-2}\cdots w_1 b : b \in B(F)\}$. Denote by $\mathcal{R}_P^{P(F)}$ the union of $P(F)$-conjugacy classes of elements in $\mathcal{R}_P$. Then one has*

$$\mathfrak{C}_{r.e.}^{P(F)} = \mathcal{R}_P^{P(F)}. \tag{4.2}$$

Proof. By Bruhat decomposition, one has

$$G(F) = P(F) \coprod P(F) \cdot w_{n-1} \cdot P(F).$$

For any $g_1 \in P(F)$ and $g_2 \in P(F) \cdot w_{n-1} \cdot P(F)$, since different Bruhat cells do not intersect, the $P(F)$-conjugacy class of g_1 does not intersect with that of g_2. Also note that $P(F)$-conjugacy classes of $P(F)$ lie in $P(F)$, so they are not regular elliptic. Hence we reject all representatives in $P(F)$, and see clearly that $P(F)$-conjugacy classes in $\mathfrak{C}_{r.e.}^{P(F)}$ are represented by elements in $w_{n-1}P(F)$.

For any $g = w_{n-1}\begin{pmatrix} A_{n-1} & b \\ & d_n \end{pmatrix} \in w_{n-1}P(F) \cap \mathfrak{C}_{r.e.}^{P(F)}$, by Bruhat decomposition, either $A_{n-1} \in P_{n-1}(F)$ or $A_{n-1} \in P_{n-1}(F)w_{n-2}P_{n-1}(F)$, where P_{n-1} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-1}(F)$ of type $(n-2, 1)$. If $A_{n-1} \in P_{n-1}(F)$, then $g \in Q_{n-2}(F) \subset \bigcup_{1 \leq k \leq n-1} Q_k(F)^{P(F)}$. Thus $g \notin \mathfrak{C}_{r.e.}^{P(F)}$. Therefore, $A_{n-1} \in$

$P_{n-1}(F)w_{n-2}P_{n-1}(F)$. For any $1 \le k \le n-1$, write R_k^* the standard parabolic subgroup of $G = GL_n$ of type $(k, 1, \cdots, 1)$. So we can write

$$g^{(0)} = g = w_{n-1}\begin{pmatrix} I_{n-2} & c & b_1 \\ & 1 & b_2 \\ & & 1 \end{pmatrix} w_{n-2} \begin{pmatrix} A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n \end{pmatrix} \in w_{n-1}R_{n-1}^*(F),$$

which is conjugate by $w_{n-2}\begin{pmatrix} A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n \end{pmatrix} \in P(F)$ to

$$\begin{aligned} g^{(1)} &= w_{n-2}\begin{pmatrix} A_{n-2} & c_{n-2} & \\ & d_{n-1} & \\ & & d_n \end{pmatrix} w_{n-1}\begin{pmatrix} I_{n-2} & c & b_1 \\ & 1 & b_2 \\ & & 1 \end{pmatrix} \\ &= w_{n-2}w_{n-1}\begin{pmatrix} A_{n-2} & & c_{n-2} \\ & d_n & \\ & & d_{n-1} \end{pmatrix}\begin{pmatrix} I_{n-2} & c & b_1 \\ & 1 & b_2 \\ & & 1 \end{pmatrix} \in w_{n-2}w_{n-1}R_{n-2}^*(F). \end{aligned}$$

Again, apply Bruhat decomposition to $\mathrm{GL}_{n-2}(F) \hookrightarrow GL_n(F)$ to see either $A_{n-2} \in P_{n-2}(F)$ or $A_{n-2} \in P_{n-2}(F)w_{n-3}P_{n-2}(F)$, where P_{n-2} is the standard maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3, 1)$. If $A_{n-2} \in P_{n-2}(F)$, then $g^{(1)} \in Q_{n-3}(F) \subset \bigcup_{1\le k\le n-1} Q_k(F)^{P(F)}$. Thus $g^{(1)} \notin \mathfrak{C}_{r.e.}^{P(F)}$. Therefore,

$$A_{n-2} \in P_{n-2}(F)w_{n-3}P_{n-2}(F).$$

So we can write

$$g^{(1)} = w_{n-2}w_{n-1}\begin{pmatrix} I_{n-3} & c' & c_{n-2}^{(1)} & b_1' \\ & 1 & c_{n-2}^{(2)} & b_2' \\ & & 1 & b_3' \\ & & & 1 \end{pmatrix} w_{n-3}\begin{pmatrix} A_{n-3} & c_{n-3} & & \\ & d_{n-2} & & \\ & & d_n & \\ & & & d_{n-1} \end{pmatrix},$$

which is conjugate by $w_{n-3}\begin{pmatrix} A_{n-3} & c_{n-3} & & \\ & d_{n-2} & & \\ & & d_n & \\ & & & d_{n-1} \end{pmatrix} \in P(F)$ to

$$\begin{aligned} g^{(2)} &= w_{n-3}\begin{pmatrix} A_{n-3} & c_{n-3} & & \\ & d_{n-2} & & \\ & & d_n & \\ & & & d_{n-1} \end{pmatrix} w_{n-2}w_{n-1}\begin{pmatrix} I_{n-3} & c' & c_{n-2}^{(1)} & b_1' \\ & 1 & c_{n-2}^{(2)} & b_2' \\ & & 1 & b_3' \\ & & & 1 \end{pmatrix} \\ &= w_{n-3}w_{n-2}w_{n-1}\begin{pmatrix} A_{n-3} & & & c_{n-3} \\ & d_n & & \\ & & d_{n-1} & \\ & & & d_{n-2} \end{pmatrix}\begin{pmatrix} I_{n-3} & c' & c_{n-2}^{(1)} & b_1' \\ & 1 & c_{n-2}^{(2)} & b_2' \\ & & 1 & b_3' \\ & & & 1 \end{pmatrix}. \end{aligned}$$

Clearly, $g^{(2)} \in w_{n-3}w_{n-2}w_{n-1}R_{n-3}^*(F)$. Continue this process inductively to see that g is $P(F)$-conjugate to some element $g^{(n-2)} \in w_1w_2\cdots w_{n-1}R_1^*(F)$.

Therefore, $\mathfrak{C}_{r.e.}^{P(F)} \subseteq \{\gamma^{P(F)} : \gamma \in w_1w_2\cdots w_{n-1}R_1^*(F)\}$. So we have

$$\begin{aligned} \{g^{-1} : g \in \mathfrak{C}_{r.e.}^{P(F)}\} &\subseteq \{\gamma^{P(F)} : \gamma \in R_1^*(F)w_{n-1}\cdots w_2w_1\} \\ &= \{\gamma^{P(F)} : \gamma \in w_{n-1}w_{n-2}\cdots w_1B(F)\}, \end{aligned}$$

since $R_1^*(F) = B(F) \subseteq P(F)$. Denote by $\iota : G(F) \xrightarrow{\sim} G(F)$, $g \mapsto g^{-1}$, the inversion isomorphism. Then $\mathfrak{C}_{r.e.}^{P(F)}$ is stable under ι, since $\bigcup_{1\leq k\leq n-1} Q_k(F)^{P(F)}$ is stable under ι. Hence,

$$\mathfrak{C}_{r.e.}^{P(F)} = \{g^{-1} : g \in \mathfrak{C}_{r.e.}^{P(F)}\} \subseteq \{\gamma^{P(F)} : \gamma \in w_{n-1}w_{n-2}\cdots w_1B(F)\} = \mathcal{R}_P^{P(F)}.$$

Now we show that $\mathcal{R}_P^{P(F)} \cap \bigcup_{1\leq k\leq n-1} Q_k(F)^{P(F)} = \emptyset$, which implies by (4.1) that $\mathcal{R}_P^{P(F)} \subseteq \mathfrak{C}_{r.e.}^{P(F)}$. Therefore, $\mathfrak{C}_{r.e.}^{P(F)} = \mathcal{R}_P^{P(F)}$.

Assume that $\mathcal{R}_P^{P(F)} \cap Q_k(F)^{P(F)} \neq \emptyset$ for some $1 \leq k \leq n-1$. If $k = n-1$, then $Q_k(F)^{P(F)} = P(F)$. Then the assumption forces that $w_{n-1}w_{n-1}\cdots w_1 \in P(F)$, which is obviously a contradiction. Thus we may assume that $1 \leq k \leq n-2$. Then by Bruhat decomposition, one has

$$P(F) = \coprod_{w\in W_{n-1}} N(F)wB(F), \text{ and } Q_k(F) = \coprod_{w'\in W_k} N(F)w'B(F).$$

For $w \in W_n$, denote by $C(w) = B(F)wB(F)$, the Bruhat cell with respect to w. Then the assumption $\mathcal{R}_P^{P(F)} \cap Q_k(F)^{P(F)} \neq \emptyset$ leads to that

$$C(w)C(w_{n-1}w_{n-2}\cdots w_1)C(w^{-1}) \cap C(w') \neq \emptyset. \tag{4.3}$$

However, Lemma 28 below shows that for any $1 \le k \le n-2$, any $(w, w') \in W_{n-1} \times W_k$, the intersection in the left hand side of (4.3) is always empty, which gives a contradiction and thus ends the proof. □

Lemma 28. *Let notation be as before,* $1 \le k \le n-2$, *then one has*

$$C(w)C(w_{n-1}w_{n-2}\cdots w_1)C(w^{-1}) \cap C(w') = \emptyset,\ \forall\, w \in W_{n-1},\ w' \in W_k.$$

Proof. Recall that for any $w \in W_n$ and $\alpha \in \Delta$, we have (see [Spr09], Lemma 8.3.7)

$$C(s_\alpha)C(w) = \begin{cases} C(s_\alpha w) & \text{if } l(s_\alpha w) = l(w)+1, \\ C(w) \sqcup C(s_\alpha w) & \text{if } l(s_\alpha w) = l(w)-1, \end{cases} \tag{4.4}$$

where $l : W \longrightarrow \mathbb{Z}$ is the length function. Also, a similar computation shows that

$$C(w)C(s_\alpha) = \begin{cases} C(w s_\alpha) & \text{if } l(w s_\alpha) = l(w)+1, \\ C(w) \sqcup C(w s_\alpha) & \text{if } l(w s_\alpha) = l(w)-1. \end{cases} \tag{4.5}$$

Then by (4.4) and (4.5), one obtains that

$$C(w)^{s_\alpha} = \begin{cases} C(s_\alpha w s_\alpha), \text{ if } l(s_\alpha w s_\alpha) = l(w)+2; \\ C(s_\alpha w) \sqcup C(s_\alpha w s_\alpha), \text{ if } l(s_\alpha w) < l(w), l(s_\alpha w s_\alpha) > l(s_\alpha w); \\ C(w s_\alpha) \sqcup C(s_\alpha w s_\alpha), \text{ if } l(w s_\alpha) < l(w), l(s_\alpha w s_\alpha) > l(w s_\alpha); \\ C(w) \sqcup C(s_\alpha w) \sqcup C(w s_\alpha) \sqcup C(s_\alpha w s_\alpha), \text{ otherwise,} \end{cases} \tag{4.6}$$

where we use $C(w)^{s_\alpha}$ to denote by $C(s_\alpha)C(w)C(s_\alpha)$.

Let $w' \in W_k$ and $w \in W_{n-1}$. Let $l(w)$ be the length of w. Then w could be written as a products of $l(w)$ simple reflections s_i, $1 \le i \le n-2$, and each s_i corresponds to the associated reflection of some simple roots in W_{n-1}.

- Assume that $l(w_{n-1}\cdots w_2 w_1 w^{-1}) = l(w_{n-1}\cdots w_2 w_1) + l(w^{-1})$. Take $w = s_{l(w)}\cdots s_2 s_1$ to be a reduced representation by simple reflections and apply (4.4) and (4.5) inductively one then sees that

  $$C(w)C(w_{n-1}w_{n-2}\cdots w_1)C(w^{-1}) = C(w w_{n-1} w_{n-2}\cdots w_1 w^{-1}).$$

  We will simply identify Weyl elements in W_n with translations on the set $\{1, 2, \cdots, n\}$ under the isomorphism $W_n \xrightarrow{\sim} S_n$. Then the cycle type decomposition of $w w_{n-1} w_{n-2}\cdots w_1 w^{-1}$ is the same as that of $w_{n-1}w_{n-2}\cdots w_1$, which is an n-cycle. However, since elements in W_k can never be n-cycles, $C(w w_{n-1} w_{n-2}\cdots w_1 w^{-1}) \cap C(w') = \emptyset, \forall\, w' \in W_k$.

- Assume that $l(w_{n-1}\cdots w_2w_1w^{-1}) < l(w_{n-1}\cdots w_2w_1) + l(w^{-1})$. Denote by $\mathcal{D}(w)$ the set of all possible reduced representations of w' by simple reflections. Then by our assumption, one can take a reduced representation of $w = s'_{l(w)}\cdots s'_2s'_1$ such that $s'_1 = w_1$. Hence one can well define

$$j_w := \max_{1\le j\le l(w)} \left\{s_{l(w)}\cdots s_2s_1 \in \mathcal{D}(w):\ s_i = w_i,\ 1\le i\le j\right\}.$$

Let $w = s_{l(w)}\cdots s_2s_1$ be a reduced representation such that $s_i = w_i, 1\le i\le j_w$. Then $w^{-1} = s_1s_2\cdots s_{l(w)}$. Also, by (4.4) or (4.5) we have

$$C(w) = C(s_{l(w)})\cdots C(s_2)C(s_1), \text{ and } C(w^{-1}) = C(s_1)C(s_2)\cdots C(s_{l(w)}),$$

so

$$C(w)C(\widetilde{w})C(w^{-1}) = C(s_{l(w)})\cdots C(s_2)C(s_1)C(\widetilde{w})C(s_1)C(s_2)\cdots C(s_{l(w)}),$$

where we denote by $\widetilde{w} = w_{n-1}w_{n-2}\cdots w_1$ for convenience. According to (4.6), a brute force computation shows that

$$C(w)C(\widetilde{w})C(w^{-1}) = C(w^*)\coprod\coprod_{1\le i\le j_w} C(w^{(i)}),$$

where $w^* = ww_{n-1}\cdots w_{j_w+2}w_{j_w+1}s_{j_{w+1}}\cdots s_{l(w)}$, and for $1\le i\le j_w$, $w^{(i)} = ww_{n-1}\cdots w_{i+1}w_iw_{i+1}\cdots w_{j_w}s_{j_{w+1}}\cdots s_{l(w)}$.

Let $w_{(j_w)} = s_{l(w)}\cdots s_{j_{w+1}}$, $w^*_{(j_w)} = w_{j_w}\cdots w_1w_{n-1}\cdots w_{j_w+2}w_{j_w+1}$. Then $w^*_{(j_w)}$ is an n-cycle, and thus $w^* = w_{(j_w)}w^*_{(j_w)}w^{-1}_{(j_w)}$ is also an n-cycle. So $w^*\notin W_k$, implying that $C(w^*)\cap C(w') = \emptyset, \forall\ w'\in W_k$.

For each $1\le i\le j_w$, let

$$w^*_{(i)} = w_{i-1}\cdots w_1w_{n-1}w_{n-2}\cdots w_{i+1},$$

$w_{(i)} = s_{l(w)}\cdots s_{j_{w+1}}w_{j_w}\cdots w_i$. Then $w^{(i)} = w_{(i)}w^*_{(i)}w^{-1}_{(i)}$. One can check that $w^*_{(i)} = (1,2,\cdots,i)(i+1,\cdots,n)$, i.e. the cycle type of $w^*_{(i)}$ is $(i,n-i)$. So $w^{(i)}$ also has cycle type $(i,n-i)$. Since elements in W_k can never have type of the form $(i,n-i)$, $w^{(i)}\notin W_k$. Therefore, $C(w^*)\cap C(w^{(i)}) = \emptyset, \forall\ 1\le i\le j_w$, $w'\in W_k$.

This completes the proof. □

Now we consider $P(F)$-conjugation among elements in $\mathcal{R}_P = \{w_{n-1}w_{n-2}\cdots w_1 b : b \in B(F)\}$ to determine representatives of $\mathcal{R}_P^{P(F)}$. Define a relation $\mathscr{R}$ on the set $\left\{w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}\mathfrak{u} :\ \mathfrak{t} \in T(F^\times),\ \mathfrak{u} \in N(F)\right\}$ such that $w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}\mathfrak{u}$ is related to $w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}'\mathfrak{u}'$ if and only if $\mathfrak{u} = \mathfrak{u}'$ and $\det\mathfrak{t} = \det\mathfrak{t}'$, which is equivalent to that there are elements $a_1, \cdots, a_n = a_0 \in F^\times$ such that

$$\begin{cases} a_0 t_1 a_1^{-1} = t_1' \\ a_1 t_2 a_2^{-1} = t_2' \\ \quad\vdots \\ a_{n-1} t_n a_n^{-1} = t_n'. \end{cases} \tag{4.7}$$

One can check easily that $\mathcal{R}_P^{P(F)}$ forms an equivalence relation.

Proposition 29. *Let notation be as before. Set*

$$\widetilde{\mathcal{R}}_P = \left\{w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}\mathfrak{u} :\ \mathfrak{t} \in T\left(F^\times\right),\ \mathfrak{u} \in N_P(F)\right\}/\mathscr{R}.$$

Then $\widetilde{\mathcal{R}}_P$ forms a family of representatives of $\mathcal{R}_P^{P(F)}$.

Proof. Let $w_{n-1}w_{n-2}\cdots w_1 b$ and $w_{n-1}w_{n-2}\cdots w_1 b'$ be two elements in $\mathcal{R}_P$, and write $b = \mathfrak{t}_n\mathfrak{u}$, $b' = \mathfrak{t}_n'\mathfrak{u}'$, the corresponding Levi decomposition. Assume that there exists some $p_n \in P_n(F) = P(F)$ such that

$$p_n w_{n-1}w_{n-2}\cdots w_1 b p_n^{-1} = w_{n-1}w_{n-2}\cdots w_1 b'. \tag{4.8}$$

Then $w_{n-1}p_n w_{n-1} = w_{n-2}\cdots w_1 b' p_n b^{-1} w_1\cdots w_{n-2} \in P(F) = Q_{n-1}(F)$. Since $p_n \in P(F)$, it is necessary of the following form

$$p_n = \begin{pmatrix} A_{n-2} & \mathfrak{c}_{n-1} & \mathfrak{c}_n \\ & a_{n-1} & 0 \\ & & a_n \end{pmatrix} \in \begin{pmatrix} GL_{n-2}(F) & * & * \\ & F^\times & 0 \\ & & F^\times \end{pmatrix} \subset Q_{n-2}(F).$$

Hence, $w_{n-2}w_{n-1}p_n w_{n-1}w_{n-2} = w_{n-3}\cdots w_1 b' p_n b^{-1} w_1\cdots w_{n-3} \in Q_{n-2}(F)$, i.e.,

$$w_{n-2}\begin{pmatrix} A_{n-2} & \mathfrak{c}_n & \mathfrak{c}_{n-1} \\ & a_n & 0 \\ & & a_{n-1} \end{pmatrix} w_{n-2} \in \begin{pmatrix} GL_{n-2}(F) & * & * \\ & F^\times & 0 \\ & & F^\times \end{pmatrix} \subset Q_{n-2}(F).$$

Then A_{n-2} must lie in a maximal parabolic subgroup of $\mathrm{GL}_{n-2}(F)$ of type $(n-3,1)$, and the last component of $\mathfrak{c}_n$ must be vanishing. Thus we can write

$$p_n = \begin{pmatrix} A_{n-3} & \mathfrak{c}_{n-2} & \mathfrak{c}_{n-1}^{(n-3)} & \mathfrak{c}_n^{(n-3)} \\ & a_{n-2} & c_{n-2,n-1} & 0 \\ & & a_{n-1} & 0 \\ & & & a_n \end{pmatrix} \in \begin{pmatrix} GL_{n-3}(F) & * & * & * \\ & F^\times & * & 0 \\ & & F^\times & 0 \\ & & & F^\times \end{pmatrix},$$

where for any column vector $\mathfrak{c}_i = (c_{1,i}, c_{2,i}, \cdots, c_{m,i})^{\mathrm{T}}$, any $1 \leq k \leq m$, write $\mathfrak{c}_i^{(k)} = (c_{1,i}, c_{2,i}, \cdots, c_{k,i})^{\mathrm{T}}$, namely, the first k-entries. Now a similar analysis on the identity

$$w_{n-3}w_{n-2}w_{n-1}p_n w_{n-1}w_{n-2}w_{n-3} = w_{n-4}\cdots w_1 b' p_n b^{-1} w_1 \cdots w_{n-4} \in Q_{n-3}(F)$$

leads to $\mathfrak{c}_n = \mathfrak{c}_n^{(n-4)}$, namely, the last 4 elements of $\mathfrak{c}_n$ are all zeros. Likewise, continue this process $(n-4)$-more times to get $\mathfrak{c}_n = \mathbf{0}$. Now (4.8) becomes

$$\begin{pmatrix} a_n & 0 & 0 & \dots & 0 \\ & a_1 & c_{1,2} & \cdots & c_{1,n-1} \\ & & \ddots & & \vdots \\ & & & a_{n-2} & c_{n-2,n-1} \\ & & & & a_{n-1} \end{pmatrix} b \begin{pmatrix} a_1 & c_{1,2} & \cdots & c_{1,n-1} & 0 \\ & a_2 & \cdots & c_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & a_{n-1} & 0 \\ & & & & a_n \end{pmatrix}^{-1} = b'. \tag{4.9}$$

When expanded, (4.9) becomes (4.13), which will be investigated below. Before seeking for a solution to (4.9), we will simplify it by taking showing that one can actually only consider some special b and b'. This is justified by Claim 30 below.

Write $\mathfrak{t}_n = \mathrm{diag}(t_1, \cdots, t_n)$, and set $t_{i,j} = t_i t_j^{-1}$; for any $n-2 \leq k \leq n-1$, define

$$\mathfrak{n}_{n-k}(F) = \left\{ \begin{pmatrix} I_k & u_k & \\ & 1 & \\ & & I_{n-k-1} \end{pmatrix} : \ u_k \in M_{k\times 1}(F) \right\}.$$

Let $\mathfrak{u}_{n-k} = \mathfrak{u} \cap \mathfrak{n}_{n-k}(F)$, $n-2 \leq k \leq n-1$. Then $\mathfrak{u} = \mathfrak{u}_1\mathfrak{u}_2$.

Claim 30. *For any $b \in B(F)$, there exists a unique $\mathfrak{u} \in \mathfrak{n}_2(F) \subset P(F)$, such that $\mathfrak{u}^{-1} w_{n-1}w_{n-2}\cdots w_1 b \mathfrak{u} \in w_{n-1}w_{n-2}\cdots w_1 T(F)\mathfrak{n}_1(F)$.*

So we only need to consider the $P(F)$-conjugacy of among elements in $\mathcal{R} = \{w_{n-1}w_{n-2}\cdots w_1 \mathfrak{t}\mathfrak{u} :\ \mathfrak{t} \in T(F^\times),\ \mathfrak{u} \in N_P(F)\}$.

Let $w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}_n\mathfrak{u}_1$, $w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}_n'\mathfrak{u}_1' \in \mathfrak{R}$ be $P(F)$-conjugate. Then there exists some $p_n = \mathrm{diag}(b, a_n) \in \mathrm{diag}(B(F), F^\times)$ such that

$$p_n w_{n-1}w_{n-2}\cdots w_1 \mathfrak{t}_n\mathfrak{u}_1 p_n^{-1} = w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}_n'\mathfrak{u}_1'. \tag{4.10}$$

Write $b = \mathrm{diag}(a_1, a_2, \cdots, a_{n-1})\mathfrak{u}$, where we identity $\mathfrak{u}$ with $\mathrm{diag}(\mathfrak{u}, 1) \in \mathfrak{n}_2(F)$. Then comparing the Levi components of both sides in (4.10) leads exactly the system of relations (4.7); while the unipotent radical gives the equation (4.13) with $c^0_{i,j} = 1$, $1 \leq i < j \leq n-1$. By the uniqueness of solution (shown in the proof of Claim 30), $\mathfrak{u} = I_n$. Therefore, $\mathfrak{u}_1 = \mathfrak{u}'_1 \in \mathfrak{n}_1(F) = N_P(F)$. Then the proof follows. □

Proof of Claim 30. Let notation be as in the proof of Proposition 29. Let $\mathfrak{u} = \{u_{i,j}\}_{1\leq i,j\leq n} \in \mathfrak{n}_2(F)$, and $\mathfrak{c} = \{c_{i,j}\}_{1\leq i,j\leq n} = w_1 \cdots w_{n-1}\mathfrak{u}^{-1}w_{n-1} \cdots w_1\mathfrak{t}_n\mathfrak{u}\mathfrak{t}_n^{-1}$. Denote by $\mathfrak{u}^\times = \{u'_{i,j}\}_{1\leq i,j\leq n} \in \mathfrak{n}_2(F)$. Then one has, for any $1 \leq i < j \leq n$, that

$$u_{i,j} + u'_{i,i+1}u_{i+1,j} + u'_{i,i+2}u_{i+2,j} + \cdots + u'_{i,j-1}u_{j-1,j} + u'_{i,j} = 0. \tag{4.11}$$

Also, an elementary computation shows that for any $1 \leq i < j \leq n$, one has

$$c_{i,j} = t_it_j^{-1}u_{i,j} + t_{i+1}t_j^{-1}u'_{i-1,i}u_{i+1,j} + \cdots + t_{j-1}t_j^{-1}u'_{i-1,j-2}u_{j-1,j} + u'_{i-1,j-1}. \tag{4.12}$$

Now fix $c^0_{i,j}$, $1 < i < j < n$, and $\mathfrak{t}_n$, then we show by a double induction that there exists uniquely $u_{i,j}$, $1 < i < j < n$, such that $c_{i,j} = c^0_{i,j}$, $1 < i < j < n$, i.e., we want to solve the system of equations, for fixed $t_1, \cdots, t_n$,

$$t_it_j^{-1}u_{i,j} + t_{i+1}t_j^{-1}u'_{i-1,i}u_{i+1,j} + \cdots + t_{j-1}t_j^{-1}u'_{i-1,j-2}u_{j-1,j} + u'_{i-1,j-1} = c^0_{i,j}. \tag{4.13}$$

When $n \leq 4$, one can check directly by hand that the solution to (4.13) exists and is unique. So from now on we assume that $n > 4$. Let $\mathcal{D}_i = \{D_{i,j} = u_{j+1,j+i+1} : 1 \leq j \leq n-2-i\}$, $1 \leq i \leq n-3$. By (4.12), $c_{1,j} = t_1t_j^{-1}u_{1,j}$, so to make $c^{1,j} = c^0_{1,j}$, one takes $u_{1,j} = t_1^{-1}t_jc^0_{1,j}$, $1 \leq j \leq n-1$. Also, by (4.11), $u'_{i,i+1} = -u_{i,i+1}$, so (4.12) shows that $c_{i,i+1} = u_{i,i+1} + u'_{i-1,i} = u_{i,i+1} - u_{i-1,i}$, $2 \leq i \leq n-2$. Let $c_{i,i+1} = c^0_{i,i+1}$, then $u_{i,i+1} = u_{i-1,i} + c^0_{i,i+1}$, $2 \leq i \leq n-2$. Since $u_{12} = t_1^{-1}t_2c^0_{1,2}$, then a simple induction shows that elements in $\mathcal{D}_1$ are uniquely determined by the equation $c_{i,j} = c^0_{i,j}$, $1 < i < j < n$.

Now, let $1 < i_0 \leq n-3$, assume that $\mathcal{D}_i$ are uniquely solved out by (4.13) for any $1 < i < i_0$. By our assumption and (4.11), $u'_{1,i}$ are now uniquely determined, $1 \leq i \leq i_0 - 1$. Then according to (4.12), $D_{i_0,1} = u_{i_0+1,i_0+2} = c^0_{i_0+1,i_0+2} + u_{1,i_0+1} + u'_{1,2}u_{3,i_0+2} + u'_{1,3}u_{4,i_0+2} + \cdots + u'_{1,i_0}u_{i_0+1,i_0+2}$, where $u_{i,i_0+2} \in \mathcal{D}_{i_0+2-i}$, $3 \leq i \leq i_0+1$. So $D_{i_0,1}$ is uniquely determined. Assume that we have solved out all $D_{i_0,j}$, $j < j_0$, in $\mathcal{D}_{i_0}$. Then by (4.12), $D_{i_0,j_0} = u_{j_0+1,j_0+i_0+1}$ completely depends on $u'_{j_0,j}$, $j_0+1 \leq j \leq j_0+i_0$, and $u_{j_0+k,j_0+i_0+1} \in \mathcal{D}_{i_0+1-k}$, $2 \leq k \leq i_0$. Again, by (4.11), we can inductively compute each $u'_{j_0,j}$ in terms of $u_{i',j'}$, $j_0 \leq i' < j' \leq j$ such that $(i', j') \neq (j_0, j)$.

By our inductive assumption, all these $u_{i',j'}$'s and u_{j_0+k,j_0+i_0+1} have been solved out uniquely. Then D_{i_0,j_0} is thus obtained. By induction, elements in $\mathcal{D}_{i_0}$ are uniquely determined. Therefore, by induction on the index i_0, one verifies that the solution to (4.13) does exist and in fact is unique.

Denote by $\mathfrak{u}_0 = \mathfrak{u}_{\mathfrak{c}^0,\mathfrak{t}_n} \in \mathfrak{n}_2(F)$ the solution to (4.13). Then $\mathfrak{u}_0$ depends only on $\mathfrak{c}^0 = \{c^0_{i,j}\}_{1\leq i,j\leq n}$ and $\mathfrak{t}_n$, where we define $c^0_{i,n} = \delta_{i,n}$, $1 \leq i \leq n$, here δ is the Kronecker symbol. Let $b = \mathfrak{u}_2\mathfrak{u}_1\mathfrak{t}_n$ be an arbitrary element in $B(F)$. Take $\mathfrak{c}^0 = \mathfrak{u}_2$, $\mathfrak{u}_0 \in \mathfrak{n}_2(F)$ the solution to (4.13), and define $\mathfrak{n}_P = \mathfrak{u}_0\mathfrak{t}_n^{-1}\mathfrak{u}_1\mathfrak{t}_n\mathfrak{u}_0^{-1} \in \mathfrak{n}_1(F)$. Then the following conjugacy equation holds:

$$\mathfrak{u}_0^{-1}w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}_n\mathfrak{n}_P\mathfrak{u}_0 = w_{n-1}w_{n-2}\cdots w_1\mathfrak{u}_2\mathfrak{u}_1\mathfrak{t}_n.$$

Therefore, one can take representatives of $P(F)$-conjugacy classes $\mathcal{R}_P^{P(F)}$ in the set $\mathcal{R} = \{w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}\mathfrak{u} : \mathfrak{t} \in T(F^\times), \mathfrak{u} \in N_P(F)\}$. □

Remark 31. *Let* $\gamma = w_{n-1}w_{n-2}\cdots w_1\mathfrak{t}\mathfrak{u} \in GL_n(F)$, $\mathfrak{t} \in T(F^\times)$, $\mathfrak{u} \in N(F)$, *then the* $P(F)$*-conjugacy class of* γ *is thoroughly determined by* $\det\gamma$ *and* $\mathfrak{u} \cap N_P(F)$.

Now we consider for our purpose the decomposition of $Z_G(F)\backslash G(F)$ into $P(F)$-conjugacy classes. By (4.1) one has the following decomposition

$$Z_G(F)\backslash G(F) = Z_G(F)\backslash\mathfrak{C}_{r.e.}^{P(F)} \coprod \bigcup_{k=1}^{n-1} (Z_G(F)\backslash Q_k(F))^{P(F)}. \tag{4.14}$$

Corollary 32. *Let notation be as before. Set* $(F^\times)^n = \{t^n : t \in F^\times\}$, *and let*

$$\widetilde{\mathcal{R}}_P^* = \left\{w_1w_2\cdots w_{n-1}\begin{pmatrix} I_{n-3} & & \\ & t & \\ & & I_2 \end{pmatrix}\mathfrak{u} : t \in F^\times/(F^\times)^n, \ \mathfrak{u} \in N_P(F)\right\}. \tag{4.15}$$

Then $\widetilde{\mathcal{R}}_P^*$ *forms a family of representatives of* $Z_G(F)\backslash\mathfrak{C}_{r.e.}^{P(F)}$.

Proof. By Lemma 27 and Proposition 29,

$$\left\{w_{n-1}w_{n-2}\cdots w_1\begin{pmatrix} I_{n-2} & & \\ & t & 0 \\ & & 1 \end{pmatrix}\mathfrak{u} : t \in F^\times/(F^\times)^n, \ \mathfrak{u} \in N_P(F)\right\} \tag{4.16}$$

forms a family of representatives of $Z_G(F)\backslash\mathfrak{C}_{r.e.}^{P(F)}$. Then the inverse of elements in the set defined in (4.16) also form a family of representatives of $Z_G(F)\backslash\mathfrak{C}_{r.e.}^{P(F)}$. Note that these inverses are bijectively $P_0(F)$-conjugate to $\widetilde{\mathcal{R}}_P^*$, then the proof follows. □

4.2 Holomorphic Continuation

Let $P_0(F)$ be the mirabolic subgroup of $G(F)$, then by definition we have $P_0(F) = R_{n-1}(F)$. For any $\gamma \in G(F)$, write $\gamma^{P_0(F)}$ for the $P_0(F)$-conjugacy class of γ, which is the same as $P(F)$-conjugacy class of γ. Then by Corollary 32 one can decompose $Z_G(F)\backslash G(F)$ as

$$Z_G(F)\backslash G(F) = \coprod_{\gamma \in \widetilde{\mathcal{R}}_P^*} \gamma^{P_0(F)} \coprod \bigcup_{k=1}^{n-1} (Z_G(F)\backslash Q_k(F))^{P_0(F)}. \tag{4.17}$$

By the decomposition (4.17), one can write $\mathrm{K}(x, y) = \mathrm{K}_{\mathrm{Geo,Reg}}(x, y) + \mathrm{K}_{\mathrm{Geo,Sing}}(x, y)$, where

$$\mathrm{K}_{\mathrm{Geo,Reg}}(x, y) = \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \sum_{p \in P_0(F)} \varphi(x^{-1}p^{-1}\gamma p y),$$
$$\mathrm{K}_{\mathrm{Geo,Sing}}(x, y) = \sum_{\gamma^{P_0(F)} \in \mathcal{P}} \sum_{p \in P_0(F)} \varphi(x^{-1}p^{-1}\gamma p y).$$

Hence we have the decomposition

$$I_{\infty,\mathrm{Reg}}(s, \tau) = \int_{X_n} \int_{[N_P]} \mathrm{K}_{\mathrm{Geo,Reg}}(nx, x) dn f(x, s) dx.$$

where $X_n = Z_G(\mathbb{A}_F)R_{n-1}(F)\backslash G(\mathbb{A}_F) = Z_G(\mathbb{A}_F)P_0(F)\backslash G(\mathbb{A}_F)$ and

$$\mathcal{P} = \left\{\gamma^{P(F)} : \ \gamma \in Z_G(F)\backslash Q_k(F) \text{ for some } 1 \leq k \leq n-1\right\}.$$

Note that $f(x, s)$ is $P(F)$-invariant, then by (4.17) and a similar trick of changing variables and interchanging integrals one has formally that

$$\begin{aligned}
I_{\infty,\mathrm{Reg}}(s, \tau) &= \int_{X_n} \int_{[N_P]} \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \sum_{p \in P_0(F)} \varphi(x^{-1}n^{-1}p^{-1}\gamma p x) dn f(x, s) dx \\
&= \int_{Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{N_P(F)\backslash N_P(\mathbb{A}_F)} \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \varphi(x^{-1}n^{-1}\gamma x) dn f(x, s) dx \\
&= \int_{Z_G(\mathbb{A}_F)N_P(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N_P]} \int_{N_P(\mathbb{A}_F)} \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \varphi(x^{-1}u^{-1}n^{-1}\gamma u x) dn du f(x, s) dx \\
&= \int_{Z_G(\mathbb{A}_F)N_P(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N_P]} \int_{N_P(\mathbb{A}_F)} \sum_{\gamma \in \widetilde{\mathcal{R}}_P^*} \varphi(x^{-1}u^{-1}\gamma n x) dn du f(x, s) dx.
\end{aligned}$$

In this section, we will prove the following:

Theorem F. *Let notation be as before, then $I_{\infty,\mathrm{Reg}}(s,\tau)$ converges absolutely and locally normally in the domain* $\mathrm{Re}(s)>1$. *Moreover, $I_{\infty,\mathrm{Reg}}(s,\tau)$ admits a meromorphic continuation. Precisely, one has*

$$I_{\infty,\mathrm{Reg}}(s,\tau)\sim\frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})}. \tag{4.18}$$

Proof. Recall that for any $2\leq k\leq n$, any $v\in\Sigma_F$, we have defined

$$N_k^*=\left\{\begin{pmatrix}I_{k-1}&u_k&\\&1&\\&&I_{n-k}\end{pmatrix}:\ u_k\in M_{(k-1)\times 1}\right\}.$$

Let $N_k^*(\mathbb{A}_F)$ be the restricted product of $N_k^*(F_v)$'s, over $v\in\Sigma_F$. Then

$$N(\mathbb{A}_F)=\prod_{k=2}^{n}N_k^*(\mathbb{A}_F)=N_n^*(\mathbb{A}_F)N_{n-1}^*(\mathbb{A}_F)\cdots N_2^*(\mathbb{A}_F),\ \text{and}\ N(F)=\prod_{k=2}^{n}N_k^*(F).$$

Write $N^P=\prod_{k=2}^{n-1}N_k^*$. Then one can write that $N(\mathbb{A}_F)=N_P(\mathbb{A}_F)N^P(\mathbb{A}_F)$ and $N(F)=N_P(F)N^P(F)$. Apply Iwasawa decomposition:

$$X'(\mathbb{A}_F):=Z_G(\mathbb{A}_F)N_P(\mathbb{A}_F)\backslash G(\mathbb{A}_F)=Z_G(\mathbb{A}_F)\backslash T(\mathbb{A}_F)N^P(\mathbb{A}_F)K,$$

where $T\simeq(\mathbb{G}_m)^n$ is the maximal split torus and K is a maximal compact subgroup. Set $T^*(\mathbb{A}_F)=Z_G(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$ for convenience. For any $\gamma\in\widetilde{\mathcal{R}}_P^*$, write it uniquely as $\gamma=w_1w_2\cdots w_{n-1}\begin{pmatrix}I_{n-2}&&\\&t&0\\&&1\end{pmatrix}\mathfrak{u}$, with $t\in F^\times/(F^\times)^n$, and $\mathfrak{u}\in N(F)$. Set $\widetilde{w}=w_1w_2\cdots w_{n-1}$. There exist unique $\mathfrak{u}_P\in N_P(F)$ and $\mathfrak{u}^P\in N^P(F)$ such that $\mathfrak{u}=\mathfrak{u}_P\mathfrak{u}^P$. Let ρ_{T^*} be the half-sum of positive roots of T^*, and set $\delta_{T^*}(\mathfrak{t})=\mathfrak{t}^{2\rho_{T^*}}$ to be the modular character, explicitly, for any $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\cdots,t_{n-1},1)\in T^*(F)$, $\delta_{T^*}(\mathfrak{t})=$

$\prod_{i=1}^{n-1}|t_i|_{\mathbb{A}_F}^{n-2i+1}$. Substitute these into the expression of $I_{\infty,\mathrm{Reg}}(s)$ to get

$$\begin{aligned} I_{\infty,\mathrm{Reg}}(s,\tau) =& \int_{X'(\mathbb{A}_F)} f(x,s)\int_{N_P(\mathbb{A}_F)}\int_{N_P(F)\backslash N_P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n}\sum_{\mathfrak{u}\in N_P(F)} \\ & \varphi\left(x^{-1}u^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1\end{pmatrix}\mathfrak{u}nx\right)dndudx \\ =& \int_K\int_{N^P(\mathbb{A}_F)}\int_{T^*(\mathbb{A}_F)} f(n^P\mathfrak{t}k,s)\frac{d^\times\mathfrak{t}}{\delta_{T^*}(\mathfrak{t})}dn^Pdk\int_{N_P(\mathbb{A}_F)}\sum_t\int_{[N_P]} \\ & \sum_{\mathfrak{u}_P\in N_P(F)}\varphi\left(k^{-1}\mathfrak{t}^{-1}(n^P)^{-1}u^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1\end{pmatrix}\mathfrak{u}_Pnn^P\mathfrak{t}k\right)dndu \\ =& \int_K\int_{N^P(\mathbb{A}_F)}\int_{T^*(\mathbb{A}_F)} f(\mathfrak{t}n^Pk,s)d^\times\mathfrak{t}dn^Pdk\int_{N_P(\mathbb{A}_F)}\sum_{t\in F^\times/(F^\times)^n} \\ & \times\int_{N_P(\mathbb{A}_F)}\varphi\left(k^{-1}(n^P)^{-1}u^{-1}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1\end{pmatrix}n\mathfrak{t}n^Pk\right)dndu, \end{aligned}$$

where the factor $\delta_{T^*}(\mathfrak{t})$ comes from the Jacobian of change of variables.

Recall that $f(x,s)$ is defined by

$$f(x,s)=\tau(\det x)|\det x|_{\mathbb{A}_F}^s\int_{\mathbb{A}_F^\times}\Phi[(0,\cdots,t)x]\tau^n(t)|t|^{ns}d^\times t, \tag{4.19}$$

which is a Tate integral for the complete L-function $\Lambda(ns,\tau^n)$.

Then $f(\mathfrak{t}n^Pk,s)=\tau(\det\mathfrak{t})|\det\mathfrak{t}|^s f(k,s)$, where we identify $T^*(\mathbb{A}_F)$ with the subgroup $\{\mathrm{diag}(t_1,\cdots,t_{n-1},1):\ t_i\in\mathbb{A}_F^\times,\ 1\le i\le n-1\}$. Therefore one has

$$\begin{aligned} I_{\infty,\mathrm{Reg}}(s,\tau) =& \int_K f(k,s)dk\int_{N^P(\mathbb{A}_F)}dn^P\int_{T^*(\mathbb{A}_F)}\int_{N_P(\mathbb{A}_F)}\sum_t\tau(\det\mathfrak{t})|\det\mathfrak{t}|_{\mathbb{A}_F}^{s+1} \\ & \times\int_{N_P(\mathbb{A}_F)}\varphi\left(k^{-1}(n^P)^{-1}u^{-1}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1\end{pmatrix}\mathfrak{t}nn^Pk\right)dndud^\times\mathfrak{t}, \end{aligned}$$

where t runs through $F^\times/(F^\times)^n$.

Given any $\mathfrak{u}' \in N^P(\mathbb{A}_F)$, and any $\mathfrak{t}' \in T(\mathbb{A}_F)$, we consider the following system of equations with respect to variables $c_{i,j}$, $1 \leq i < j \leq n-1$, and $\mathfrak{u} \in N_P(\mathbb{A}_F)$,

$$\begin{pmatrix} 1 & c_{1,2} & \cdots & c_{1,n-1} & 0 \\ & 1 & \cdots & c_{2,n-1} & 0 \\ & & \ddots & \vdots & \vdots \\ & & & 1 & 0 \\ & & & & 1 \end{pmatrix}^{-1} \cdot \mathfrak{t}' \cdot \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ & 1 & c_{1,2} & \cdots & c_{1,n-1} \\ & & \ddots & & \vdots \\ & & & 1 & c_{n-2,n-1} \\ & & & & 1 \end{pmatrix} = \mathfrak{t}'\mathfrak{u}'\mathfrak{u}. \tag{4.20}$$

One sees easily that equation (4.20) is equivalent to (4.9) or the system of equations (4.13). By the existence of solutions to equation (4.9) (with fixed initial datum), we can find some $\mathfrak{u} = \mathfrak{u}_0 \in N_P(\mathbb{A}_F)$, and $c_{i,j} = c_{i,j}^0 \in \mathbb{A}_F$, $1 \leq i < j \leq n-1$, such that (4.20) holds. Therefore, one can always find some element $\mathfrak{c} \in N^P(\mathbb{A}_F)$ such that

$$\mathfrak{c}^{-1}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1 \end{pmatrix}\mathfrak{t}\mathfrak{c} = \mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1 \end{pmatrix}\mathfrak{t}\mathfrak{u}_0\mathfrak{u}'. \tag{4.21}$$

Hence for any $\mathfrak{u}' \in N^P(\mathbb{A}_F)$, one can rewrite $I_{\infty,\mathrm{Reg}}(s,\tau) = I_{\infty}^{\mathrm{Reg}}(\mathfrak{u}'; s)$, where

$$\begin{aligned} I_{\infty}^{\mathrm{Reg}}(\mathfrak{u}'; s) = &\int_K f(k,s)dk \int_{N^P(\mathbb{A}_F)} dn^P \int_{T^*(\mathbb{A}_F)} \int_{N_P(\mathbb{A}_F)} \sum_t \tau(\det \mathfrak{t}) |\det \mathfrak{t}|_{\mathbb{A}_F}^{s+1} \\ &\times \int_{N_P(\mathbb{A}_F)} \varphi\left(k^{-1}(n^P)^{-1}u^{-1}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1 \end{pmatrix}\mathfrak{t}n\mathfrak{u}'n^P k \right) dndud^{\times}\mathfrak{t}. \end{aligned}$$

Let $c_P = \mathrm{vol}\left(N^P(F)\backslash N^P(\mathbb{A}_F)\right)$. By (4.21), $I_{\infty}^{\mathrm{Reg}}(\mathfrak{u}'; s)$ is $N^P(\mathbb{A}_F)$-invariant, hence one can integrate $I_{\infty}^{\mathrm{Reg}}(\mathfrak{u}'; s)$ over the compact domain $N^P(F)\backslash N^P(\mathbb{A}_F)$ to see that $I_{\infty,\mathrm{Reg}}(s,\tau)$ is equal to

$$\begin{aligned} &\frac{1}{c_P}\int_K f(k,s)dk \int_{N^P(\mathbb{A}_F)} dn^P \int_{[N^P]} \int_{T^*(\mathbb{A}_F)} \int_{N_P(\mathbb{A}_F)} \sum_t \tau(\det \mathfrak{t}) |\det \mathfrak{t}|_{\mathbb{A}_F}^{s+1} \\ &\times \int_{N_P(\mathbb{A}_F)} \varphi\left(k^{-1}(n^P)^{-1}u^{-1}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1 \end{pmatrix}\mathfrak{t}n\mathfrak{u}'n^P k \right) dndud^{\times}\mathfrak{t}d\mathfrak{u}' \\ =&\frac{1}{c_P}\int_K f(k,s)dk \int_{N(\mathbb{A}_F)} d\mathfrak{u} \int_{[N^P]} \int_{T^*(\mathbb{A}_F)} \sum_{t \in F^{\times}/(F^{\times})^n} \tau(\det \mathfrak{t}) |\det \mathfrak{t}|_{\mathbb{A}_F}^{s+1} \\ &\times \int_{N_P(\mathbb{A}_F)} \varphi\left(k^{-1}\mathfrak{u}\mathfrak{t}^{-1}\widetilde{w}\begin{pmatrix} I_{n-2} & & \\ & t & \\ & & 1 \end{pmatrix}\mathfrak{t}\widetilde{w}^{-1} \cdot \widetilde{w}n\mathfrak{u}'k \right) dnd^{\times}\mathfrak{t}d\mathfrak{u}'. \end{aligned}$$

After a changing of variables one obtains

$$I_{\infty,\mathrm{Reg}}(s,\tau)=\frac{1}{c_P}\int_K f(k,s)dk\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{N_P(\mathbb{A}_F)}dn\sum_{t\in F^\times/(F^\times)^n}\Delta_{s,\tau}^{(1)}(\mathfrak{t})$$

$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}\mathfrak{u}\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1^n t\prod_{i=2}^{n-1}t_i\end{pmatrix}\widetilde{w}n\mathfrak{u}'k\right)d^\times\mathfrak{t},$$

where $d^\times\mathfrak{t}=d^\times t_1d^\times t_2\cdots d^\times t_{n-1}$, and for any $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\cdots,t_{n-1},1)\in T^*(\mathbb{A}_F)$,

$$\Delta_{s,\tau}^{(1)}(\mathfrak{t})=\tau(t_1)^{\frac{n(n-1)}{2}}|t_1|_{\mathbb{A}_F}^{\frac{n(n-1)}{2}(s+1)}\prod_{i=2}^{n-1}\tau(t_i^{n-i})|t_i|_{\mathbb{A}_F}^{(n-i)(s+1)}.$$

Depending on the purity of n, we can further simplify $I_{\infty,\mathrm{Reg}}(s,\tau)$. Recall the test function φ has the central character ω, Ξ is the set of idele class characters on $\mathbb{A}_F$, which is trivial on the archimedean places. Denote by $\Xi_{\omega,n}$ the subset $\{\chi\in\Xi:\chi^n=\omega\}\subset\Xi$. Also, let $\Xi_{\tau,2}^n=\{\xi\in\Xi:\ \xi^2=\tau\}$ if n is even, and set $\Xi_{\tau,2}^n$ to be the empty set if n is odd. Then both $\#\Xi_{\tau,2}^n<\infty$ and $\#\Xi_{\tau,2}^n<\infty$.

When n is odd, we have, by the computation above, that $I_{\infty,\mathrm{Reg}}(s,\tau)$ is equal to

$$\frac{1}{c_P}\int_K f(k,s)dk\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{N_P(\mathbb{A}_F)}dn\sum_{\chi\in\Xi_{\omega,n}}\int_{\mathbb{A}_F^\times}\Delta_{s,\tau,\chi}^{od}(\mathfrak{t})d^\times t_1$$

$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}\mathfrak{u}\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\widetilde{w}n\mathfrak{u}'k\right)d^\times t_2\cdots d^\times t_{n-1},$$

where we use the fact that $\left(\mathbb{A}_F^\times\right)^n\cdot F^\times/(F^\times)^n=F^\times\cdot\left(F^\times\backslash\mathbb{A}_F^\times\right)^n$, and $\tau|\cdot|_{\mathbb{A}_F}$ is $F^\times$-invariant, and

$$\Delta_{s,\tau,\chi}^{od}(\mathfrak{t})=\bar{\chi}(t_1)\tau(t_1)^{\frac{n-1}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\tau(t_i^{\frac{n+1}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

When n is even, one has a similar simplification as follows

$$I_{\infty,\mathrm{Reg}}(s,\tau)=\frac{1}{c_P}\int_K f(k,s)dk\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{N_P(\mathbb{A}_F)}dn\sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi^n_{\tau,2}}\Delta^{en}_{s,\tau,\chi,\xi}(\mathfrak{t})$$

$$\times\int_{\mathbb{A}_F^\times}\cdots\int_{\mathbb{A}_F^\times}\varphi\left(k^{-1}\mathfrak{u}\begin{pmatrix}1&&&&\\&t_2^{-1}&&&\\&&\ddots&&\\&&&t_{n-1}^{-1}&\\&&&&t_1\end{pmatrix}\widetilde{w}n\mathfrak{u}'k\right)d^\times t_1\cdots d^\times t_{n-1},$$

where the weighted character $\Delta^{en}_{s,\tau,\chi,\xi}$ is defined to be

$$\Delta^{en}_{s,\tau,\chi,\xi}(\mathfrak{t})=\bar{\chi}(t_1)\xi(t_1)\tau(t_1)^{\frac{n-2}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\xi(t_i)\tau(t_i^{\frac{n}{2}-i})|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

Let $T_*(\mathbb{A}_F^\times)=\{\mathrm{diag}(1,t_1,t_2,\cdots,t_{n-1})\in T(\mathbb{A}_F):\ t_i\in\mathbb{A}_F^\times,\ 1\leq i\leq n-1\}$. Set

$$\iota:\ T^*(\mathbb{A}_F^\times)\longrightarrow T_*(\mathbb{A}_F^\times),\ \mathfrak{t}\mapsto\mathfrak{t}^\iota=\mathrm{diag}(1,t_2^{-1},t_3^{-1},\cdots,t_{n-1}^{-1},t_1).$$

For any $n\in\mathbb{N}_{\geq 2}$, define $\mathfrak{F}_{\chi,\xi}(x;k,s)=\mathfrak{F}_{\chi,\xi}(x;k,s,\varphi,\Phi,\tau)$ by

$$\mathfrak{F}_{\chi,\xi}(x;k,s)=\int_{N(\mathbb{A}_F)}d\mathfrak{u}\int_{[N^P]}d\mathfrak{u}'\int_{T^*(\mathbb{A}_F^\times)}\varphi\left(k^{-1}\mathfrak{u}\mathfrak{t}^\iota x\mathfrak{u}'k\right)\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})d^\times\mathfrak{t},$$

where we write $\delta_n=-\frac{1+(-1)^n}{2}$ and denote by $\Delta_{s,\tau,\chi,\xi,n}(\mathfrak{t})$ the following character

$$\bar{\chi}(t_1)\xi(t_1)^{-\delta_n}\tau(t_1)^{\frac{n-1-\delta_n}{2}}|t_1|_{\mathbb{A}_F}^{\frac{(n-1)(s+1)}{2}}\prod_{i=2}^{n-1}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}.$$

Since $[N^P]=N^P(F)\backslash N^P(\mathbb{A}_F)$ is compact and φ is compactly supported, the integral over $N(\mathbb{A}_F)$ converges absolutely; hence the function $\mathfrak{F}_{\chi,\xi}(x;k,s)$ is well defined for any χ,ξ and $\mathrm{Re}(s)>1$.

Let $b=\mathfrak{u}\mathfrak{t}\in B(\mathbb{A}_F)$, where $\mathfrak{u}\in N(\mathbb{A}_F)$, $\mathfrak{t}=\mathrm{diag}(t_1,t_2,\cdots,t_n)\in T(\mathbb{A}_F)$. Then

$$\mathfrak{F}_{\chi,\xi}(bx;k,s)=\prod_{i=1}^{n}\chi(t_i)\xi(t_i)^{\delta_n}\tau(t_i)^{\frac{n+1-\delta_n}{2}-i}|t_i|_{\mathbb{A}_F}^{[\frac{n+1}{2}-i](s+1)}\cdot\mathfrak{F}_{\chi,\xi}(x;k,s).\qquad(4.22)$$

Since the modular character of $T(\mathbb{A}_F)$ is $\delta_{T(\mathbb{A}_F)}(\mathfrak{t})=\prod_{i=1}^n t_i^{n+1-2i}$, so one has

$$\mathfrak{F}_{\chi,\xi}(x;k,s)\in\mathrm{Ind}_{B(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left(\chi\xi^{\delta_n}\tau^{\lambda_1}|\cdot|_{\mathbb{A}_F}^{\lambda_1 s},\cdots,\chi\xi^{\delta_n}\tau^{\lambda_{n-1}}|\cdot|_{\mathbb{A}_F}^{\lambda_{n-1}s},\chi\xi^{\delta_n}\tau^{\lambda_n}|\cdot|_{\mathbb{A}_F}^{\lambda_n s}\right),$$

where for $1 \leq i \leq n$, $\lambda_i = \frac{n+1-\delta_n}{2} - i$. Denote by

$$G_{\chi,\xi}(x;s) = G_{\chi,\xi}(x;s,\varphi,\Phi,\tau) = \frac{1}{c_P}\int_K f(k,s)\mathfrak{F}_{\chi,\xi}(x;k,s)dk.$$

Then at least formally one can write $I_{\infty,\mathrm{Reg}}(s,\tau)$ as a finite sum:

$$I_{\infty,\mathrm{Reg}}(s,\tau) = \sum_{\chi\in\Xi_{\omega,n}}\sum_{\xi\in\Xi^n_{\tau,2}}\int_{N_P(\mathbb{A}_F)} G_{\chi,\xi}(\widetilde{w}n;s)dn,\ \mathrm{Re}(s) > 1. \tag{4.23}$$

Let $\mathfrak{F}_{1,1,+}(x;k,s) = \mathfrak{F}_{1,1}(x;k,s,|\varphi|,|\Phi|,1)$ and $G_{1,1,+}(x;s) = G_{1,1}(x;s,|\varphi|,|\Phi|,1)$. Then the above interchanging orders of integrals is justified by Fubini's theorem on integral of nonnegative functions. One then has

$$I^+_{\infty,\mathrm{Reg}}(s,\tau) = \sum_{\chi\in\Xi_{1,n}}\sum_{\xi\in\Xi^n_{1,2}}\int_{N_P(\mathbb{A}_F)} G_{1,1,+}(\widetilde{w}n;s)dn,$$

where the sums are finite. Then $\int_{N_P(\mathbb{A}_F)} G_{1,1,+}(\widetilde{w}n;s)dn$ converges absolutely in $\mathrm{Re}(s) > 1$ according to Langlands' theory on intertwining operators. Therefore, by dominant control theorem, $\int_{N_P(\mathbb{A}_F)} G_{\chi,\xi}(\widetilde{w}n;s)dn$ converges absolutely in $\mathrm{Re}(s) > 1$. It is thus a well defined intertwining operator. By Langlands' theory (cf. [Lan71] or [Sha84]) on intertwining operators, we have

$$\int_{N_P(\mathbb{A}_F)} G_{\chi,\xi}(\widetilde{w}n;s)dn \sim \frac{\Lambda(s,\tau)\Lambda(2s,\tau^2)\cdots\Lambda((n-1)s,\tau^{n-1})\Lambda(ns,\tau^n)}{\Lambda(s+1,\tau)\Lambda(2s+1,\tau^2)\cdots\Lambda((n-1)s+1,\tau^{n-1})},$$

where the last factor $\Lambda(ns,\tau^n)$ on the numerator comes from the Tate integral $f(k,s)$ (cf. (4.19)).

So (4.23) is well defined. Then (4.18) follows since the sums in (4.23) is finite. □

Remark 34. *In* GL(2) *case one can also prove Theorem F by Poisson summation (cf. [JZ87]). However, the approach in loc. cit. does not generalize to higher rank case because of a lack of Poisson summation formula. We take advantage of P-conjugacy classes in G to show $I_{\infty,\mathrm{Reg}}(s,\tau)$ can be written as a finite sum of certain auxiliary intertwining operators, and then apply general theory on intertwining operators. This approach steers clear of Poisson summation.*

Chapter 5

CONVERGENCE OF THE SPECTRAL SIDE

We will deal with the spectral side

$$I_{\mathrm{Whi}}(s,\tau)=\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{N(F)\backslash N(\mathbb{A}_F)}\mathrm{K}_\infty(n_1x,x)\theta(n_1)dn_1f(x,s)dx, \tag{5.1}$$

where $\mathrm{K}_\infty(x,y)=\mathrm{K}(x,y)-\mathrm{K}_0(x,y)$ is the non-cuspidal part of the kernel function relative to a *general* test function $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$. The main concern in this section is the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$ when $\mathrm{Re}(s)$ is large.

Typically one needs certain suitable regularization or truncation for K_∞, which is slowly increasing. In the GL(2) case this can be handled by the techniques in [Sel56] or [Zag81]. Also, Arthur (e.g., cf. [Art78], [Art79], [Art80] and [Art81]) develops a truncation approach to regularize the trace formula on general reductive groups successfully. Arthur's truncation operators and their variants (e.g., [**Lap06**]) provides a powerful toolkit to manipulate the convergence problem in the (relative) trace formula.

However, these truncation operators seem to be incompatible with the distribution (5.1). One of the main barriers is that the domain is not the usual automorphic quotient $Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$ but the larger region $Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)$. In particular, the kernel in (5.1) is not $G(F)$-invariant now, which makes the usual truncation operators not work well here. One can appeal to the spectral expansion of K_∞ and apply Arthur's truncation Λ^T to the second Eisenstein series and show it can be integrated over a Siegel domain. With further covering process by Weyl elements conjugation, one can show (5.1) converges absolutely with $\mathrm{K}_\infty(x,y)$ replaced by $\Lambda_2^T\mathrm{K}_\infty(x,y)$, where Λ_2^T means the operator Λ^T is applied to the y-variable. This will be discussed in Section 5.4. Nevertheless, taking Fourier coefficients in the first variable makes the geometric truncation difficult to control, since it is not $G(F)$-invariant. So it is not clear how to compute the spectrally truncated distribution as a polynomial of the parameter T and show ultimately that this polynomial is indeed a constant. (Here the letter 'T' is a conventional notation for the truncation parameter, while we use 'T' to denote the torus elsewhere.)

We will propose an alternative way to show convergence of (5.1). Our strategy is to reduce (5.1) to a Mellin transform of the Kuznetsov relative trace formula, which is majorized by a gauge. So that one obtains convergence of (5.1) for all φ when $\mathrm{Re}(s)$ is large enough.

Then substituting the spectral expansion (5.7) of $\mathrm{K}_\infty(x,y)$ into (5.1), then $I_{\mathrm{Whi}}(s,\tau)$ can be written as

$$\int_{X_G}\sum_{\chi\in\mathfrak{X}}\sum_{\substack{P\in\mathcal{P}\\ P\neq G}}\frac{1}{c_P}\int_{\Lambda^*}\sum_{\phi_1}\sum_{\phi_2}\langle\mathcal{I}_P(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}d\lambda f(x,s)dx, \quad (5.2)$$

where $X_G=Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ and $\mathfrak{X}$ is the (infinite) set of cuspidal data, and W_j's are the Whittaker functions. See (5.10) below for details. We then summarize the final result on the absolute convergence of (5.2) as Theorem G at the end of this section.

5.1 Reduce to the Kuznetsov Relative Trace Formula

Lemma 35. *Let* $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$ *. Let* $\mathrm{K}=\mathrm{K}^{\varphi}$, $\mathrm{K}_0=\mathrm{K}^{\varphi}$ *and* $\mathrm{K}_\infty=\mathrm{K}_\infty^{\varphi}$ *be the corresponding kernel functions. Then*

$$\mathrm{K}_0(x,y)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\left(\mathrm{K}(n\delta x,\delta x)-\mathrm{K}_\infty(n\delta x,\delta x)\right)\theta(n)dn. \quad (5.3)$$

Proof. By the spectral decomposition of $\mathrm{K}_0(x,y)$ we see it is cuspidal as a function of x. Applying Proposition 26 to the first variable of $\mathrm{K}_0(x,y)$ and take $y=x$ we then obtain

$$\mathrm{K}_0(x,y)=\sum_{\delta\in N(F)\backslash P_0(F)}\int_{[N]}\mathrm{K}_0(n\delta x,x)\theta(n)dn.$$

Then (5.3) follows from the spectral decomposition $\mathrm{K}_0(x,y)=\mathrm{K}(x,y)-\mathrm{K}_\infty(x,y)$ and the automorphy of these functions relative to the second variable. □

Let $\mathrm{Re}(s)>1$ in this section. We then plug Lemma 35 into

$$I_0(s,\tau)=\int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)}\mathrm{K}_0(x,x)E(x,s)dx$$

and unfold the Eisenstein series $E(x,s)$ to obtain

$$I_0(s,\tau)=I_{\mathrm{Kl}}(s,\tau)-I_{\mathrm{Whi}}(s,\tau),$$

where

$$I_{\mathrm{Kl}}(s,\tau)=\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)}\int_{[N]}\mathrm{K}(nx,x)\theta(n)dnf(x,s)dx. \quad (5.4)$$

Since K_0 is rapidly decaying, then to show $I_{\mathrm{Whi}}(s,\tau)$ is well defined, it suffices to show $I_{\mathrm{Kl}}(s,\tau)$ converges. We will show $I_{\mathrm{Kl}}(s,\tau)$ converges for all $\varphi \in \mathcal{H}(G(\mathbb{A}_F),\omega)$ and $\mathrm{Re}(s)$ large enough. Then by Cauchy inequality and the convolution decomposition of φ we get the absolute convergence of $I_{\mathrm{Whi}}(s,\tau)$.

By a change of variable one has

$$I_{\mathrm{Kl}}(s,\tau) = \int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} J_{\mathrm{Kuz}}(\varphi,x)f(x,s)dx,$$

where

$$J_{\mathrm{Kuz}}(\varphi,x) = \int_{[N]}\int_{[N]} \mathrm{K}(n_1x,n_2x)\theta(n_1)\overline{\theta}(n_2)dn_1dn_2$$

is a relative trace formula of Kuznetsov type. Hence, $I_{\mathrm{Kl}}(s,\tau)$ is a (multiple) Mellin transform of a Kuznetsov relative trace formula $J_{\mathrm{Kuz}}(\varphi,x)$ since $f(x,s)$ is essentially $|\det x|^s$. Then in principle $I_{\mathrm{Kl}}(s,\tau)$ is a sum of Kloosterman sum zeta functions, which should converge when $\mathrm{Re}(s)$ is large enough. We verify this intuition by showing that $J_{\mathrm{Kuz}}(\varphi,x)$ is majorized by a gauge. Recall that, for $x = \mathrm{diag}(x_1\cdots x_{n-1},\cdots,x_1x_2,x_1,1) \in A(\mathbb{A}_F)$, a gauge $\mathcal{G}$ is a positive function of the form

$$\mathcal{G}(x) = \xi(x_1,x_2,\cdots,x_{n-1})\cdot|x_1x_2\cdots x_{n-1}|^{-M}$$

with $M \geq 0$ and ξ is a Schwartz-Bruhat function on $(\mathbb{A}_F^\times)^{n-1}$.

Proposition 36. *Let notation be as above. Then as a function of $x \in A(\mathbb{A}_F) = Z_G(\mathbb{A}_F)\backslash T(\mathbb{A}_F)$, $J_{\mathrm{Kuz}}(\varphi,x)$ is a majorized by a finite sum of gauges on $A(\mathbb{A}_F)$.*

Proof. By definition of the kernel function $\mathrm{K}(x,y)$ we have

$$J_{\mathrm{Kuz}}(\varphi,x) = \int_{[N]}\int_{[N]} \sum_{\gamma\in Z_G(F)\backslash G(F)} \varphi(x^{-1}n_1^{-1}\gamma n_2x)\theta(n_1)\overline{\theta}(n_2)dn_1dn_2,$$

which converges absolutely since $\mathrm{K}(x,y)$ is continuous and $[N]$ is compact.

Then we consider the double coset $Z_G(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$, whose element is of the form wa, where w is a Weyl element and $a \in Z_G(F)\backslash T(F)$. Let

$$H_{wa} := \left\{(n_1,n_2) \in N\times N:\ n_1^{-1}wan_2a^{-1}w^{-1} \in Z_G\right\}$$

be the stabilizer relative to the representative wa. Then

$$J_{\mathrm{Kuz}}(\varphi,x) = \sum_{wa\in\Phi} \int_{H_{wa}(F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(x^{-1}n_1^{-1}wan_2x)\theta(n_1)\overline{\theta}(n_2)dn_1dn_2,$$

where Φ is a set of complete representatives for $Z_G(\mathbb{A}_F)N(F)\backslash G(F)/N(F)$. Then

$$J_{\mathrm{Kuz}}(\varphi,x)=\sum_{wa\in\Phi}C_{wa}\int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)}\varphi(x^{-1}n_1^{-1}wan_2x)\theta(n_1)\overline{\theta}(n_2)dn_1dn_2,$$

where

$$C_{wa}=\int_{[H_{wa}]}\theta(n_1')\overline{\theta}(n_2')dn_1'dn_2'.$$

Call $wa\in\Phi$ *relevant* if $C_{wa}\neq 0$, i.e., $\theta(n_1')\overline{\theta}(n_2')$ is trivial on $H_{wa}(\mathbb{A}_F)$. Denote by Φ^* the set of relevant elements in Φ. By [**JR92**] (Prop 1 on p. 272) one can take the following realization: Φ^* consists of wa, where w is the longest Weyl element inside a stantard parabolic subgroup $P\subseteq G$ of type $(k_1,\cdots,k_r)$, and $a\in Z_G(F)\backslash\operatorname{diag}(T_{k_1}(F),\cdots,T_{k_r}(F))$ (modulo some further relations), with T_{k_j} being the maximal split torus of $\mathrm{GL}(k_j)$. For instance, when $P=B$ the Borel, then $w=I_n$ and $a=I_n$ and $H_{wa}=N$. Therefore,

$$J_{\mathrm{Kuz}}(\varphi,x)=\sum_{wa\in\Phi^*}\mathrm{vol}([H_{wa}])J_{\mathrm{Kuz}}(\varphi,x;wa),$$

where

$$J_{\mathrm{Kuz}}(\varphi,x;wa)=\int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)}\varphi(x^{-1}n_1^{-1}wan_2x)\theta(n_1)\overline{\theta}(n_2)dn_1dn_2.$$

By definition of Φ^* each w corresponds to a unique (i.e., the minimal one) parabolic subgroup P containing w. Suppose $w\neq I_n$. Then by Levi decomposition it suffices to consider the extreme case where $P=G$ and w is the longest.

Recall that the test function φ is K-finite. Hence there is some compact subgroup $K_0\subset G(\mathbb{A}_{F,\mathrm{fin}})$ such that φ is right K_0-invariant. Let $K_0=\prod_{v<\infty}K_{0,v}$. Note that $J_{\mathrm{Kuz}}(\varphi,x;wa)=\prod_{v\leq\infty}J_{\mathrm{Kuz},v}(\varphi_v,x_v;wa)$, where

$$J_{\mathrm{Kuz},v}(\varphi_v,x_v;wa)=\int_{H_{wa}(F_v)\backslash N(F_v)\times N(F_v)}\varphi_v(x_v^{-1}n_1^{-1}wan_2x)\theta_v(n_1)\overline{\theta}_v(n_2)dn_1dn_2.$$

Then for each finite place v, $J_{\mathrm{Kuz},v}(\varphi_v,x_v;wa)$ is right $K_{0,v}$-invariant. So there exists a compact subgroup $N_{0,v}\subseteq K_{0,v}\cap N(F_v)$, depending only on φ_v, such that

$$J_{\mathrm{Kuz},v}(\varphi_v,x_vu_v;wa)=J_{\mathrm{Kuz},v}(\varphi_v,x_v;wa),\ \text{for all } x_v\in A(F_v)\text{ and }u_v\in N_{0,v}.$$

On the other hand, $J_{\mathrm{Kuz},v}(\varphi_v,x_vu_v;wa)=\theta(x_vu_vx_v^{-1})J_{\mathrm{Kuz},v}(\varphi_v,x_v;wa)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ such that $\theta(x_vu_vx_v^{-1})=1$ if and only if $|\alpha_i(x_v)|_v\leq C_v$, where α_i's are the simple roots of $G(F)$ relative to B.

Note that for all but finitely many $v < \infty$, $K_{0,v} = G(\mathcal{O}_{F,v})$. Thus we can take the corresponding $C_v = 1$. Hence for any $x_v \in A(F_v)$, $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa) \neq 0$ implies that $|\alpha_i(x_v)|_v \leq C_v$, $1 \leq i \leq n-1$, and $C_v = 1$ for all but finitely many finite places v. Denote the compact set by

$$A_{\varphi,\mathrm{fin}} = \Big\{a = (a_v) \in A(\mathbb{A}_{F,\mathrm{fin}}) : \ |\alpha_i(a_v)|_v \leq C_v, \ 1 \leq i \leq n-1\Big\}.$$

Then $\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty})A_{\varphi,\mathrm{fin}}$.

For any $y = \otimes_v(y_{i,j,v}) \in G(\mathbb{A}_F)$. We define $\|y_v\|_v = \max_{i,j} |y_{i,j,v}|_v$ if $v < \infty$; and

$$\|y_v\|_v = \Big[\sum_{i,j} |y_{i,j,v}|_v^2\Big]^{1/2}, \text{ if } v \mid \infty.$$

Then $\|y_v\|_v = 1$ for almost all v. The height function $\|y\| = \prod_v \|y_v\|_v$ is therefore well defined by a finite product. Also, by $\operatorname{supp} J_{\mathrm{Kuz}}(\varphi, x; wa) \subseteq A(\mathbb{A}_{F,\infty})A_{\varphi,\mathrm{fin}}$ and the compactness of $\operatorname{supp}\varphi_v$, we have $\|w^{-1}x_v w x_v a\|_v \leq C_v'$ for some constant C_v' depending only on φ_v, $v < \infty$, and $C_v' = 1$ for almost all v's.

Now we investigate the archimedean $J_{\mathrm{Kuz},v}(\varphi_v, x_v; wa)$, i.e., $v \mid \infty$. Note that φ_v is a compactly supported on $Z_G(F_v)\backslash G(F_v)$. Then $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}^{-1} y_v w n_{2,v} \in \operatorname{supp}\varphi_v$, where $y_v = x_v^{-1} w a x_v w^{-1}$. Hence $\|n_{1,v}^{-1} y_v w n_{2,v} w^{-1}\|_v \leq C_v$ for some constant C_v depending only on φ. A straightforward computation shows that $\|n_{1,v}\|_v + \|n_{2,v}\|_v + \|y_v\|_v \leq C_v'$ for some constant C_v' depending only on φ. So $\varphi_v(n_{1,v} y_v w n_{2,v})$ has compact support relative to $n_{1,v}$ and $n_{2,v}$. Therefore, $J_{\mathrm{Kuz}}(\varphi_v, x_v; wa) = 0$ unless $n_{1,v}, n_{2,v}$ run through a compact set of $N(F_v)$ and $\|y\|_v$ is bounded.

Similar to (**??**) we define an additive character for $x \in A(\mathbb{A}_F)$:

$$\psi_x(u) = \prod_{i=1}^{n-1} \psi_{F/\mathbb{Q}}\left(x_i u_{i,i+1}\right), \quad \forall\, u = (u_{i,j})_{n\times n} \in N(\mathbb{A}_F).$$

Then $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is equal to

$$\delta_w(x)^2 \int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(n_1^{-1} x^{-1} w a x n_2)\theta_x(n_1)\overline{\theta}_x(n_2)dn_1 dn_2,$$

where δ_w is the modular character of the parabolic subgroup associated to w.

Since n_1 and n_2 lie in a compact set determined by $\operatorname{supp}\varphi$, then for a fixed $y \in A(\mathbb{A}_F)$, $v \mid \infty$, the v-th component of

$$\int_{H_{wa}(\mathbb{A}_F)\backslash N(\mathbb{A}_F)\times N(\mathbb{A}_F)} \varphi(n_1^{-1} y w n_2)\theta_x(n_1)\overline{\theta}_x(n_2)dn_1 dn_2$$

is a Schwartz function of x since it is the Fourier transform of a compactly supported smooth function. Hence, $J_{\mathrm{Kuz}}(\varphi, x; wa)$ is majorized by a nonnegative Schwartz-Bruhat function $\Xi(x^{-1}waxw^{-1}, x)$ on $A(\mathbb{A}_F)^2$ with

$$x^{-1}waxw^{-1} \in A^* := \{b \in A(F) : \ \|b\| \leq \prod_v C'_v.\}$$

By properties of the height $\|\cdot\|$ (e.g., see [Art05] p. 70) one has

$$\#\left(w^{-1}x \cdot A^* \cdot wx^{-1}\right) \leq C \cdot (|x_1 \cdots x_{n-1}|^M + |x_1 \cdots x_{n-1}|^{-M}),$$

for some constants C and M depending on supp φ. Therefore,

$$\sum_{a\in A(F)} |J_{\mathrm{Kuz}}(\varphi, x; wa)| \leq \sum_{a\in A^*} \Xi(w^{-1}xwxa, x) = \sum_{a\in w^{-1}x\cdot A^*\cdot wx^{-1}} \Xi(a, x),$$

which is majorized by $|x_1 \cdots x_{n-1}|^{-M} \cdot \xi(x_1, \cdots, x_{n-1})$ for some $M \geq 0$ and Schwartz-Bruhat function ξ.

The remaining case is that $w = I_n$, i.e., $P = B$. In this case

$$\sum_{a\in A(F)} |J_{\mathrm{Kuz}}(\varphi, x; wa) = \delta_w(x) \int_{N(\mathbb{A}_F)} \varphi(an)\overline{\theta}_x(n)dn$$

is the Fourier transform of a Schwartz-Bruhat function. So it is majorized by a gauge. Then Proposition 36 follows. □

As a consequence of Proposition 36 and the Iwasawa decomposition, we have $I_{\mathrm{Kl}}(s, \tau)$ converges absolutely when $\mathrm{Re}(s)$ is large enough. Therefore, $I_{\mathrm{Whi}}(s, \tau)$ converges when $\mathrm{Re}(s)$ is large enough.

To show the absolute convergence of $I_{\mathrm{Whi}}(s, \tau)$ and thus to obtain meromorphic continuation, we need to analyze properties of K_∞ by its spectral expansion.

5.2 Spectral Decomsition of the Kernel Function

In this subsection, we review briefly the spectral theory of automorphic representation of reductive groups, and then apply the results to the non-cuspidal kernel function K_∞. Denote by H a general reductive group and P a standard parabolic subgroup of H. Let M_P (resp. N_P) be the Levi component (resp. unipotent radical) of P.

Let $H^1(\mathbb{A}_F) = \{g \in H(\mathbb{A}_F) : |\lambda(g)|_{\mathbb{A}_F} = 1, \ \forall\ \lambda \in X(H)_F\}$, where $X(H)_F$ is space set of F-rational characters of H. Let $\mathfrak{a}_H = \mathrm{Hom}_{\mathbb{Z}}(X(H)_F, \mathbb{R})$. Let $\mathfrak{a}_H^* = X(H)_F \otimes \mathbb{R}$.

Denote by $\mathfrak{a}_P = \mathfrak{a}_{M_P}$ and $\mathfrak{a}_P^* = \mathfrak{a}_{M_P}^*$. Let P_0 be a fixed minimal parabolic subgroup of H over F. Write $\mathfrak{a}_0$ (resp. $\mathfrak{a}_0^*$) for $\mathfrak{a}_{P_0}$ (resp. $\mathfrak{a}_{P_0}^*$). These notations concur with those used by Arthur, e.g., see p.20-31 of [Art05].

Then by spectral theory (e.g., cf. p. 256 and p. 263 of [Art79]), the decomposition of the Hilbert space $L^2(Z_H(\mathbb{A}_F)N_P(\mathbb{A}_F)M_P(F)\backslash H(\mathbb{A}_F))$ into right $H(\mathbb{A}_F)$-invariant subspaces is determined by the spectral data $\chi = \{(M,\sigma)\}$, where the pair (M,σ) consists of a Levi subgroup M of H and a cuspidal representation $\sigma \in \mathcal{A}_0\left(Z_H(\mathbb{A}_F)\backslash M^1(\mathbb{A}_F)\right)$, where M^1 is defined in a similar way to H^1; the class (M,σ) derives from the equivalence relation $(M,\sigma) \sim (M',\sigma')$ if and only if M is conjugate to M' by a Weyl group element w, and $\sigma' = \sigma^w$ on $Z_H(\mathbb{A}_F)\backslash M^1(\mathbb{A}_F)$. Let $\mathfrak{X}$ be the set of equivalence classes $\chi = \{(M,\sigma)\}$ of these pairs, we thus have

$$L^2(P) := L^2(Z_H(\mathbb{A}_F)N_P(\mathbb{A}_F)M_P(F)\backslash H(\mathbb{A}_F)) = \bigoplus_{\chi\in\mathfrak{X}} L^2(P)_\chi, \tag{5.5}$$

where $L^2(P)_\chi$ consists of functions $\phi \in L^2(Z_H(\mathbb{A}_F)N_P(\mathbb{A}_F)M_P(F)\backslash H(\mathbb{A}_F))$ such that: for each standard parabolic subgroup Q of G, with $Q \subset P$, and almost all $x \in H(\mathbb{A}_F)$, the projection of the function

$$m \mapsto x.\phi_Q(m) = \int_{N_Q(F)\backslash N_Q(\mathbb{A}_F)} \phi(nmx)dn$$

onto the space of cusp forms in $L^2\left(Z_H(\mathbb{A}_F)M_Q(F)\backslash M_Q^1(\mathbb{A}_F)\right)$ transforms under $M_Q^1(\mathbb{A}_F)$ as a sum of representations σ, in which $(M_Q,\sigma) \in \chi$. If there is no such pair in χ, $x.\phi_Q$ will be orthogonal to $\mathcal{A}_0\left(Z_H(\mathbb{A}_F)M_Q(F)\backslash M_Q^1(\mathbb{A}_F)\right)$. Denote by $\mathcal{H}_P$ the space of such ϕ's. Let $\mathcal{H}_{P,\chi}$ be the subspace of $\mathcal{H}_P$ such that for any $(M,\sigma) \notin \chi$, with $M = M_{P_1}$ and $P_1 \subset P$, we have

$$\int_{M(F)\backslash M(\mathbb{A}_F)^1} \int_{N_{P_1}(F)\backslash N_{P_1}(\mathbb{A}_F)} \psi_0(m)\phi(nmx)dn = 0,$$

for any $\psi_0 \in L_0^2\left(M(F)\backslash M(\mathbb{A}_F)^1\right)_\sigma$, and almost all x. This leads us to Langlands' result to decompose $\mathcal{H}_P$ as an orthogonal direst sum $\mathcal{H}_P = \bigoplus_{\chi\in\mathfrak{X}} \mathcal{H}_{P,\chi}$. Let $\mathcal{B}_P$ be an orthonormal basis of $\mathcal{H}_P$, then we can choose $\mathcal{B}_P = \bigcup_{\chi\in\mathfrak{X}} \mathcal{B}_{P,\chi}$, where $\mathcal{B}_{P,\chi}$ is an orthonormal basis of the Hilbert space $\mathcal{H}_{P,\chi}$. We may assume that vectors in each $\mathcal{B}_{P,\chi}$ are K-finite and are pure tensors.

5.3 Spectral Expansion of $I_{\mathrm{Whi}}(s,\tau)$

By definition, we have

$$I_{\mathrm{Whi}}(s,\tau) = \int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} \int_{[N]} \int_{[N]} \mathrm{K}_\infty(n_1x, n_2x)\theta(n_1)\bar{\theta}(n_2)dn_1dn_2 f(x,s)dx.$$

Denote by $\widehat{\mathrm{K}}_\infty(x,y)$ the Fourier expansion of $\mathrm{K}_\infty(x,y)$, namely,

$$\widehat{\mathrm{K}}_\infty(x,y) := \int_{[N]}\int_{[N]} \mathrm{K}_\infty(n_1x,n_2y)\theta(n_1)\bar{\theta}(n_2)dn_1dn_2. \tag{5.6}$$

For our particular purpose here, we take in Section 5.2 that $H = G$. By spectral theory, one can expand $\mathrm{K}_\infty(x,y)$ as

$$\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{k_P!(2\pi)^{k_P}}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x,\mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda, \tag{5.7}$$

where $\mathcal{P}$ is the set of standard parabolic subgroups which are not G; and for any such P, k_P is the number of blocks of the Levi part of P. Also, (5.7) converges absolutely (cf. Lemma 2 on p.263 of [Art79]). Since $[N] = N(F)\backslash N(\mathbb{A}_F)$ is compact, $\widehat{\mathrm{K}}_\infty(x,y)$ is then well defined.

Lemma 37. *Let notation be as before. Then one can interchange the integrals in the definition of* $\widehat{\mathrm{K}}_\infty(x,y)$*, namely, one has*

$$\widehat{\mathrm{K}}_\infty(x,y) = \sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{k_P!(2\pi)^{k_P}}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{P,\chi}} W_{\mathrm{Eis},1}(x;\lambda)\overline{W_{\mathrm{Eis},2}(y;\lambda)}d\lambda, \tag{5.8}$$

where the Fourier coefficient $W_{\mathrm{Eis},1}(x;\lambda) = W_{Eis}(x,\mathcal{I}_P(\lambda,\varphi)\phi,\lambda)$ *is defined by*

$$W_{\mathrm{Eis}}(x,\mathcal{I}_P(\lambda,\varphi)\phi,\lambda) := \int_{N(F)\backslash N(\mathbb{A}_F)} E(n_1x,\mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\theta(n_1)dn_1;$$

and similarly, $W_{\mathrm{Eis},2}(y;\lambda) = W_{\mathrm{Eis}}(y,\phi,\lambda)$ *is given as*

$$W_{\mathrm{Eis}}(y,\phi,\lambda) := \int_{N(F)\backslash N(\mathbb{A}_F)} E(n_2y,\phi,\lambda)\theta(n_2)dn_2.$$

Proof. The main idea of the proof is similar to that in [Art78] (see p. 928-934). For any $P\in\mathcal{P}$, let $c_P = k_P!(2\pi)^{k_P}$. Substitute (5.7) into (5.6) to get a formal expansion of $\widehat{\mathrm{K}}_\infty(x,y)$, which is clearly dominated by the following formal expression

$$\int_{[N]}\int_{[N]}\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{c_P}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\left|E(n_1x,\mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\overline{E(n_2y,\phi,\lambda)}\right|d\lambda dn_1dn_2.$$

Denote by $J_G(\varphi;x,y)$ the above integral. We will show $J_G(\varphi;x,y)$ is finite, hence (5.8) is well defined. One can write the test function φ as a finite linear combination of convolutions $\varphi_1*\varphi_2$ with functions $\varphi_i\in C_c^r(G(\mathbb{A}_F))$, whose archimedean components are differentiable of arbitrarily high order r. Then one applies Hölder inequality to it. Clearly it is enough to deal with the special case that $\varphi = \varphi_j * \varphi_j^*$, where $\varphi_j^*(x) = \overline{\varphi_j(x^{-1})}$, and $x = y$. Define for $g \in G(\mathbb{A}_F)$ that

$$J_G(\varphi_j, g) = \int_{[N]} \mathrm{K}_\infty(ng, ng) dn.$$

Then $J_G(\varphi_j, g)$ is well defined since $[N]$ is compact and K_∞ is continuous. By (5.7) we can expand $J_G(\varphi_j, x)$ as

$$\int_{[N]} \sum_{\chi \in \mathfrak{X}} \sum_{P \in \mathcal{P}} \frac{1}{c_P} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_{\phi \in \mathfrak{B}_{P,\chi}} E(nx, \mathcal{I}_P(\lambda, \varphi_j)\phi, \lambda) \overline{E(nx, \mathcal{I}_P(\lambda, \varphi_j)\phi, \lambda)} d\lambda dn.$$

Note that the summands are nonnegative. In fact, the integral over λ and sum over χ, P and ϕ can be expressed as an increasing limit of nonnegative functions, each of which is the kernel of the restriction of $R(\varphi_j * \varphi_j^*)$, a positive semidefinite operator, to an invariant subspace. Since this limit is bounded by the continuous nonnegative function

$$\mathrm{K}(x, x) = \sum_{\gamma \in Z(F)\backslash G(F)} \varphi_j * \varphi_j^*(x^{-1}\gamma x) = \sum_{\gamma \in Z(F)\backslash G(F)} \varphi(x^{-1}\gamma x),$$

we then obtain

$$J_G(\varphi_j, x) \leq \int_{[N]} \mathrm{K}(nx, nx) dn < \infty$$

since the domain $[N] = N(F)\backslash N(\mathbb{A}_F)$ is compact.

Note that $\mathfrak{B}_{P,\chi}$ is finite due to the K-finiteness assumption, and Eisenstein series holomorphic on λ, hence the integrand becomes

$$\sum_{\phi \in \mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda) \overline{E(y, \phi, \lambda)} = \sum_{\phi \in \mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda, \varphi_j)\phi, \lambda) \overline{E(y, \mathcal{I}_P(\lambda, \varphi_j)\phi, \lambda)}.$$

Then by Cauchy inequality one has $J_G(\varphi; x, y) \leq \sqrt{J_G(\varphi_j, x) J_G(\varphi_j, y)} < \infty$. Hence, $\widehat{\mathrm{K}}_\infty(x, y)$ converges absolutely. Then (5.8) follows from a straightforward computation. □

Let $\phi_2 \in \mathfrak{B}_{P,\chi}$. Then $\mathcal{I}_P(\lambda, \varphi)\phi_2$ can be expanded by a linear combination of vectors in $\mathfrak{B}_{P,\chi}$. As a consequence,

$$E(x, \mathcal{I}_P(\lambda, \varphi)\phi_2, \lambda) = \sum_{\phi_1 \in \mathfrak{B}_{P,\chi}} \langle \mathcal{I}_P(\lambda, \varphi)\phi_2, \phi_1 \rangle E(x, \phi_1, \lambda),$$

where the sum is finite due to the K-finiteness of φ.

For $1 \leq i \leq 2$, define the Whittaker function associated to ϕ_i parameterized by $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ as

$$W_i(x,\lambda) = W_i(x,\phi_\alpha,\lambda) := \int_{N(\mathbb{A}_F)} \phi_i(w_0nx)e^{(\lambda+\rho_P)H_P(w_0nx)}\theta(n)dn, \tag{5.9}$$

where w_0 is the longest element in the Weyl group W_n.

Since the residual spectrum is degenerate, that is, has no Whittaker model, the integral is zero unless the representation is cuspidal. Hence, unfolding the Eisenstein series and by Bruhat decomposition on $G(F)$ one can write the non-constant terms $W_{\mathrm{Eis},i}(x;\lambda)$ in terms of $W_i(x,\lambda)$, e.g., cf. p.123-124 of [Sha10]. Set $X_G = Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$, $c_P = k_P!(2\pi)^{k_P}$, and $\Lambda^* = i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$. Then by (5.8) one can rewrite (at least formally) $I_{\mathrm{Whi}}(s,\tau)$ as

$$\int_{X_G}\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{c_P}\int_{\Lambda^*}\sum_{\phi_1,\phi_2}\langle\mathcal{I}_P(\lambda,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}d\lambda f(x,s)dx, \tag{5.10}$$

where $\phi_i \in \mathfrak{B}_{P,\chi}$, $1 \leq i \leq 2$.

Theorem G. *Let notation be as before. Then there exists a constant c_φ depending only on φ such that $I_{\mathrm{Whi}}(s,\tau)$ converges absolutely for* $\mathrm{Re}(s) > c_\varphi$. *Moreover, when* $\mathrm{Re}(s) > c_\varphi$, *$I_{\mathrm{Whi}}(s,\tau)$ is equal to*

$$\sum_{\chi}\sum_{P\in\mathcal{P}}\frac{1}{c_P}\sum_{\phi_1\in\mathfrak{B}_{P,\chi}}\sum_{\phi_2\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\langle\mathcal{I}_P(\lambda,\varphi)\phi_2,\phi_1\rangle\int_{X_G}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)dxd\lambda,$$

where χ runs over the proper cuspidal data, i.e., χ is not of the form $\{(G,\pi)\}$. Particularly, as a function of s, $I_{\mathrm{Whi}}(s,\tau)$ is analytic in the right half plane $\{z :$ $\mathrm{Re}(z) > c_\varphi\}$.

Proof. For $x \in Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ we write it into the Iwasawa coordinates: $x = ak$, where $a \in A(\mathbb{A}_F)$ and $k \in K$. Then

$$f(x,s) := f(x,\Phi,\tau;s) = \tau(\det a)|\det a|^s\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t.$$

Therefore, $|f(x,s)| = |\det a|^{\mathrm{Re}(s)}h(k,s)$, where

$$h(k,s) := \left|\int_{\mathbb{A}_F^\times}\Phi(\eta tk)\tau(t)^n|t|^{ns}d^\times t\right|$$

is a nonnegative continuous function of k and converges absolutely when $\operatorname{Re}(s) > 1/n$. Let $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$. Then by Proposition 36,

$$\int_{X_G} J_{\mathrm{Kuz}}(\varphi, x) \cdot |f(x, s)| dx = \int_K \int_{A(\mathbb{A}_F)} J_{\mathrm{Kuz}}(\varphi, ak) \cdot |\det a|^{\operatorname{Re}(s)} \delta(a) d^\times a h(k, s) dk$$

converges when $\operatorname{Re}(s)$ is large. By Lemma 35 we have

$$J(\varphi, s) := \int_{X_G} \widehat{\mathrm{K}}_\infty(x, x) |f(x, s)| dx = \int_{X_G} J_{\mathrm{Kuz}}(\varphi, x) \cdot |f(x, s)| dx - G(s),$$

where

$$J_0(\varphi, s) = \int_{Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)} \mathrm{K}_0(x, x) \sum_{\delta \in P(F)\backslash G(F)} |f(\delta x, s)| dx.$$

Since the series $\sum_{\delta \in P(F)\backslash G(F)} |f(\delta x, s)|$ is slowly increasing and $\mathrm{K}_0(x, x)$ decays rapidly on $Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$, then $J_0(\varphi, s)$ converges absolutely. Hence $J(\varphi, s)$ converges and is well defined.

Take test functions of the form $\varphi_0 * \varphi_0^*$, where $\varphi_0^*(x) = \overline{\varphi_0(x^{-1})}$. Plugging the spectral expansion Lemma 37 into $J(\varphi, s)$ to get the convergence of

$$\int_{X_G} \sum_{\chi} \sum_{P \in \mathcal{P}} \frac{1}{c_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \left| W(x; \mathcal{I}_P(\lambda, \varphi_0)\phi, \lambda) \right|^2 |f(x, s)| d\lambda dx \tag{5.11}$$

where χ runs over the proper cuspidal data and

$$W(x; \mathcal{I}_P(\lambda, \varphi_0)\phi, \lambda) = \int_{N(\mathbb{A}_F)} (\mathcal{I}_P(\lambda, \varphi_0)\phi)(w_0 n x) e^{(\lambda + \rho_P) H_P(w_0 n x)} \theta(n) dn,$$

with w_0 being the longest element in the Weyl group W_n. Hence (5.11) is convergent and also nonnegative. So it converges absolutely.

For arbitrary test function $\varphi \in \mathcal{H}(G(\mathbb{A}_F), \omega)$, one can write φ as a finite linear combination of convolutions $\varphi_{j,1} * \varphi_{j,2}$ with functions $\varphi_{j,i} \in C_c^r(G(\mathbb{A}_F))$, whose archimedean components are differentiable of arbitrarily high order r, $1 \leq i \leq 2$, and $j \in J$ is a finite set. Then one applies Hölder inequality to it to see

$$\sum_{\chi \in \mathfrak{X}} \sum_{P \in \mathcal{P}} \sum_{\phi_1 \in \mathfrak{B}_{P,\chi}} \sum_{\phi_2 \in \mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \int_{X_G} \left| \langle \mathcal{I}_P(\lambda, \varphi)\phi_2, \phi_1 \rangle W_1(x; \lambda) \overline{W_2(x; \lambda)} f(x, s) \right| dx d\lambda$$

$$\leq \sum_{j \in J} \prod_{i=1}^{2} \left[\sum_{\chi \in \mathfrak{X}} \sum_{P \in \mathcal{P}} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \int_{X_G} W_{j,i}(x; \lambda) \overline{W_{i,j}(x; \lambda)} \cdot \left| f(x, s) \right| dx d\lambda \right]^{1/2} < \infty,$$

where $W_{j,i}(x; \lambda) = W(x; \mathcal{I}_P(\lambda, \varphi_{j,i})\phi, \lambda)$, for any $1 \leq i \leq 2$, and $j \in J$. This proves the first part of Theorem G. $\square$

Remark 39. *If the base field F is a function field, then it has no archimedean places. Thus* $\operatorname{supp} W_i(x;\lambda)\mid_{A(\mathbb{A}_F)}\subseteq A_{\varphi,\mathrm{fin}}$, $\forall\ \lambda\in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $1\leq i\leq 2$, *namely, the support of* $\widehat{\mathrm{K}}_\infty(x,x)$ *is compact. Also, in the function field case the cuspidal datums have no infinitesimal characters, so the sum over χ's is only finite. Therefore, Theorem G is clear.*

Note that for any χ and P, the space $\mathfrak{B}_{P,\chi}$ depends only on the support and K-finite type of the test function φ. Hence, given any $\lambda_P^\circ=(\lambda_1^\circ,\lambda_2^\circ,\cdots,\lambda_r^\circ)\in\mathfrak{a}_P^*(\mathbb{C})=\mathfrak{a}_P^*\otimes\mathbb{C}$, the function $\varphi(\cdot)\exp\langle\lambda_P^\circ,H_{M_P}(\cdot)\rangle$ shares the same support and K-finite type with the test function φ. Hence, one can replace φ in Theorem G with $\varphi(\cdot)\exp\langle\lambda^\circ,H_{M_P}(\cdot)\rangle$ to get that

Corollary 40. *Let notation be as before. Let $s\in\mathbb{C}$ be such that* $\mathrm{Re}(s)>1$; *and for any standard parabolic subgroup P, let $\lambda_P^\circ=(\lambda_1^\circ,\lambda_2^\circ,\cdots,\lambda_r^\circ)$ be a fixed point in $\mathfrak{a}_P^*(\mathbb{C})$. Let $Y_G=Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. Then the following integral*

$$\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\sum_{\phi_1,\phi_2\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\int_{X_G}\left|\langle\mathcal{I}_P(\lambda+\lambda_P^\circ,\varphi)\phi_2,\phi_1\rangle W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)\right|dxd\lambda$$

is finite, and is uniformly bounded if s lies in some compact subset of the right half plane $\{z:\ \mathrm{Re}(z)>1\}$.

Remark 41. *Let notation be as before, and let $\varphi\in C_0(G(\mathbb{A}_F))$, to apply Theorem G, one still needs to verify that the function $\varphi(\cdot)\exp\langle\lambda^\circ,H_{M_P}(\cdot)\rangle$ lies in $\mathcal{H}(G(\mathbb{A}_F),\omega)$ as well. Noting that they have the same support, one then concludes that for any $\varphi\in\mathcal{H}(G(\mathbb{A}_F),\omega)$, the function $\varphi(\cdot)\exp\langle\lambda^\circ,H_{M_P}(\cdot)\rangle\in\mathcal{H}(G(\mathbb{A}_F),\omega)$. Then Corollary 40 follows from Theorem G.*

To obtain a further holomorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$, we shall study

$$\int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)}W_1(x;\lambda)\overline{W_2(x;\lambda)}f(x,s)dx,\tag{5.12}$$

which is a Rankin-Selberg convolution for (non-cuspidal) automorphic representations of $G(\mathbb{A}_F)$. Note that [IY15] constructed a regularized global integral to compute these Rankin-Selberg periods in the case of $\mathrm{GL}(n+1)\times\mathrm{GL}(n)$. However, such a truncation has not been established in the case of $\mathrm{GL}(n)\times\mathrm{GL}(n)$. Nevertheless, we investigate the analytic behavior of local factors of (5.12) in Section 6, proving (5.12) is a holomorphic multiple of an L-function $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$. See Proposition 60 below for details. Although (5.12) only converges when $\mathrm{Re}(s)>1$, we will obtain its meromorphic continuation to the whole complex plane by analytic properties of $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$.

5.4 Discussion on Arthur's Truncation Operator

In this section we discuss Arthur's truncation operators in our case. We will use the conventional notations on p.24-29 of [Art05]. For any parabolic subgroup P of G, let $\widehat{\tau}_P$ be the characteristic function of the subset $\{t \in \mathfrak{a}_P : \varpi(t) > 0, \forall \varpi \in \widehat{\Delta}_P\}$ of $\mathfrak{a}_P$. Let $\mathfrak{a}_0^+$ be the set of positive coroots. Let $T \in \mathfrak{a}_0^+$, we say T is suitably regular if $\alpha(T)$ is large, for each simple root α. For any suitably regular point $T \in \mathfrak{a}_0^+$ and any function $\phi \in \mathcal{B}_{loc}\left(Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)\right)$, define the truncation function $\Lambda^T\phi$ to be the function in $\mathcal{B}_{loc}\left(Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)\right)$ such that

$$\Lambda^T\phi(x) = \sum_P (-1)^{\dim(A_P/A_G)} \sum_{\delta \in P(F)\backslash G(F)} \widehat{\tau}_P\left(H_P(\delta x) - T\right) \int_{[N_P]} \phi(n\delta x)dn. \quad (5.13)$$

The inner sum may be taken over a finite set depending on x, while the integrand is a bounded function of n.

Before moving on, we still need to choose a nonnegative function $\|\cdot\|$ on $G(\mathbb{A}_F)$ to describe properties of the truncation operator Λ^T quantitatively.

Let $x = (x_v)_v \in G(\mathbb{A}_F)$. Recall that $\|x_v\|_v = \max_{i,j} |x_{i,j,v}|_v$ if $v < \infty$; and

$$\|x_v\|_v = \Big[\sum_{i,j} |x_{i,j,v}|_v^2\Big]^{1/2}, \text{ if } v \mid \infty.$$

Then $\|x_v\|_v = 1$ for almost all v. The height function $\|x\| = \prod_v \|x_v\|_v$ is therefore well defined by a finite product. Then one has $\|xy\| \leq \|x\| \cdot \|y\|, \forall\, x, y \in G(\mathbb{A}_F)$. Also one can check that there is some absolute constants C_0 and N_0 such that for any $x \in G(\mathbb{A}_F)$, we have $\|x\|^{-1} \leq C_0\|x\|^{N_0}$, and

$$\#\{x \in G(F) : \ \|x\| \leq t\} \leq C_0 t^{N_0}, \ t \geq 0.$$

Note that the test function φ is compactly supported. Let

$$\|x \cdot \operatorname{supp}\varphi \cdot y^{-1}\| := \max_{g \in \operatorname{supp}\varphi} \|xgy^{-1}\|.$$

Then one sees that

$$|\operatorname{K}(x, y)| = |\operatorname{K}_\varphi(x, y)| \leq \sum_{\substack{\gamma \in Z_G(F)\backslash G(F) \\ \|\gamma\| \leq \|x \cdot \operatorname{supp}\varphi \cdot y^{-1}\|}} \sup_{g \in Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} |\varphi(g)|.$$

Hence $|\operatorname{K}(x, y)| \ll_\varphi \#\{\gamma \in G(F) : \ \|\gamma\| \leq \|x \cdot \operatorname{supp}\varphi \cdot y^{-1}\|\} \ll_\varphi \|x\|^{N_0'} \cdot \|y\|^{N_0'}$, where N_0' is some absolute constant, i.e., independent of the choice of test function φ. Thus there exists some constant $c(\varphi)$ such that $|\operatorname{K}(x, y)| \leq c(\varphi)\|x\|^{N_0'} \cdot \|y\|^{N_0'}$.

Now we consider derivatives of the kernel $\mathrm{K}(x, y)$. Suppose X and Y are left invariant differential operators on $G(\mathbb{A}_{F,\infty})$ of degrees d_1 and d_2. Suppose also that the test function $\varphi \in C_c^r(G(\mathbb{A}_F))$, for some large positive r. For any cuspidal datum $\chi \in \mathfrak{X}$, define the corresponding kernel function as

$$\mathrm{K}_\chi(x, y) = \sum_{P \in \mathcal{P}} \frac{1}{k_P!(2\pi)^{k_P}} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_{\phi \in \mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda)\overline{E(y, \phi, \lambda)} d\lambda, \quad (5.14)$$

which is convergent absolutely. By [Art79] there exists some function $\varphi_{X,Y} \in C_c^{r-d_1-d_2}(G(\mathbb{A}_F))$ such that its corresponding kernel function $\mathrm{K}_{\chi,\varphi_{X,Y}}(x, y)$ is equal to $XY\,\mathrm{K}_\chi(x, y)$, for all $x, y \in G(\mathbb{A}_F)$. Then one can apply the above estimate for kernel functions to obtain

$$\sum_\chi \left| XY\,\mathrm{K}_\chi(x, y) \right| \leq c(\varphi_{X,Y}) \|x\|^{N_0'} \cdot \|y\|^{N_0'}, \ \forall\ x, y \in G(\mathbb{A}_F). \quad (5.15)$$

Also, for any function $H(x, y)$ in $\mathcal{B}_{loc}\left((G(F)\backslash G(\mathbb{A}_F)^1) \times (G(F)\backslash G(\mathbb{A}_F)^1)\right)$, define the partial Fourier transform of $H(x, y)$ with respect to the x-variable as

$$\mathcal{F}_1 H(x, y) := \int_{N(F)\backslash N(\mathbb{A}_F)} H(n_1 x, y)\theta(n_1) dn_1. \quad (5.16)$$

Let w be a matrix in K_∞, the maximal compact subgroup of $G(\mathbb{A}_{F,\infty})$, representing a Weyl element. Regard w as a matrix in K by setting its components to be the trivial matrix at finite places. Define similarly

$$\mathcal{F}_1^w H(x, y) := \int_{N(F)\backslash N(\mathbb{A}_F)} H(w^{-1} n_1 w x, y)\theta(n_1) dn_1. \quad (5.17)$$

This is well defined since $H(x, y)$ is $G(F)$-invariant on the x-variable, and the quotient $N(F)\backslash N(\mathbb{A}_F)$ is compact.

Let $A_{F,\infty} = Z_G(\mathbb{A}_{F,\infty})\backslash T(\mathbb{A}_{F,\infty})$, where $T(\mathbb{A}_{F,\infty})$ is the $\mathbb{A}_{F,\infty}$-points of the torus isomorphic to $(\mathbb{G}_m)^n$. For any $c > 0$, let $A_{c,\infty}$ be set consisting of all

$$a = \begin{pmatrix} a_1 a_2 \cdots a_{n-1} & & & & \\ & a_1 a_2 \cdots a_{n-2} & & & \\ & & \ddots & & \\ & & & a_1 & \\ & & & & 1 \end{pmatrix} \in A_{F,\infty} \quad (5.18)$$

with $|a_i|_\infty \geq c$ for $1 \leq i \leq n-1$. Clearly $A_{c,\infty} \subsetneq Z_G(\mathbb{A}_{F,\infty})\backslash T(\mathbb{A}_{F,\infty}), \forall\ c > 0$.

A Siegel set $\mathcal{S}_c$ for $c > 0$ is defined to be the set of elements of the form nak, $n \in \Omega \subseteq N(\mathbb{A}_F)$, compact, such that $N(F)\Omega = N(\mathbb{A}_F)$; $k \in K$, and $a \in A_{\geq c} = \{a \in$

$A(\mathbb{A}_F) : \ |\alpha_i(a)| \geq c, \ 1 \leq i \leq n-1\}$. Then from reduction theory (see [Bor69]), there exists some $c_0 > 0$ such that $G(\mathbb{A}_F) = Z_G(\mathbb{A}_F)G(F)\mathcal{S}_{c_0}$. Denote by $S_0 = S_{c_0}$, and we may assume that $0 < c_0 < 1$. Let R be a function on $Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$, we say that R is slowly increasing if there exists some $r > 0$ and $C > 0$ such that $R(x) \leq C\|x\|^r$, $\forall\ x \in G(\mathbb{A}_F)$; we say R is rapidly decreasing if for any positive integer N and any Siegel set $\mathcal{S}_c$ for $G(\mathbb{A}_F)$, there is a positive constant C such that $|R(x)| \leq C\|x\|^{-N}$ for every $x \in \mathcal{S}_c$.

We will apply the truncation operator Λ^T to the *second variable* of kernel functions $\mathrm{K}_\chi(x,y)$ and show that it is absolutely integrable twisted by any slowing increasing functions over $A_{h,\infty}A_{\varphi,\mathrm{fin}} \cdot K$ for some $h > 0$, where $A_{\varphi,\mathrm{fin}}$ be a compact subgroup of $A_{\mathrm{fin}} = Z_G(\mathbb{A}_{F,\mathrm{fin}})\backslash T(\mathbb{A}_{F,\mathrm{fin}})$ depending on φ.

Lemma 42. *Let notation be as above. Then* $\sum_\chi \left|\mathcal{F}_1\Lambda_2^T \mathrm{K}_\chi(x,x)\right|$ *is rapidly decreasing on* $\mathcal{S}_0$*. Moreover, let R be a slowly increasing function on* $Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)$*. Then we have*

$$\int_{\mathcal{S}_0} \sum_\chi \left|\mathcal{F}_1\Lambda_2^T \mathrm{K}_\chi(x,x) \cdot R(x)\right| dx < \infty, \tag{5.19}$$

where χ *runs over all the equivalent classes of cuspidal datum and the truncation operator* Λ^T *acts on the second variable of kernel functions* K_χ *. Moreover,* (5.19) *holds when* $\mathcal{F}_1$ *is replaced by* $\mathcal{F}_1^w$ *(defined in* (5.17)*), where w is a Weyl element.*

Proof. Clearly for any given $x \in G(\mathbb{A}_F)$, $\mathcal{F}_1 \mathrm{K}_\chi(x,y)$ is a well defined function (with respect to y) in $\mathcal{B}_{loc}\left(Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)\right)$. Then according to Lemma 1.4 in [Art80] (or a more explicit version given by Proposition 13.2 in [Art05], p.71) one sees that given a Siegel set $\mathcal{S}$, positive integers M and M_1, and an open compact subgroup K_0 of $G(\mathbb{A}_{F,\mathrm{fin}})$, one can choose a finite set $\{X_i\}$ of left invariant differential operators on $Z_G(\mathbb{A}_{F,\infty})\backslash G(\mathbb{A}_{F,\infty})$ and a positive integer r with the property that if $(\Omega, d\omega)$ is a measure space and $\phi(\omega) \mapsto \phi(\omega, x)$ is any measurable function from Ω to $C^r\left(Z_G(\mathbb{A}_F)G(F)\backslash G(\mathbb{A}_F)/K_0\right)$, then

$$\sup_{x\in\mathcal{S}}\left(\|x\|^M \int_\Omega |\Lambda^T\phi(\omega,x)|d\omega\right) \leq \sup_{x\in G(\mathbb{A}_F)^1}\left(\|x\|^{-M_1}\sum_i\int_\Omega |X_i\phi(\omega,x)|d\omega\right). \tag{5.20}$$

In particular, $\{X_i\}$ is *independent* of $(\Omega, d\omega)$.

Since our test functions lie in the Hecke algebra, we may assume that φ is biinvariant under K_0, where K_0 is an open compact subgroup of $G(\mathbb{A}_{F,\mathrm{fin}})$. Also, set $M_1 = M_0'$ and M large, then we can find a finite set $\{Y_i\}$ of left invariant differential operators on $G(\mathbb{A}_{F,\infty})$ such that (5.20) holds for the measure space $(\Omega, d\omega) = (\mathfrak{X}, d)$, where $\mathfrak{X}$ is the set of cuspidal datum and $d = d_x$ is a discrete measure depending on $x \in G(\mathbb{A}_F)$. Under our current particular choices (5.20) becomes

$$\sup_{y\in\mathcal{S}_0}\left(\|y\|^M \sum_{\chi}\left|\Lambda_2^T \mathrm{K}_\chi(x,y)\right|\right) \le \sup_{y\in G(\mathbb{A})}\left(\sum_{\chi}\|y\|^{-M_1}\left|\sum_i (Y_i)_2 \mathrm{K}_\chi(x,y)\right|\right)$$
$$\le \sup_{y\in G(\mathbb{A})}\left(\sum_{i}\|y\|^{-M_1}\sum_{\chi}\left|(Y_i)_2 \mathrm{K}_\chi(x,y)\right|\right),$$

where $(Y_i)_2$ above indicates that the differential operator acts on the y-variable; and each Y_i is independent of x. By the estimate (5.15), the right hand side of the above inequality is bounded by

$$\sup_{y\in G(\mathbb{A})}\left(\sum_i \|y\|^{-N_1} c(\varphi_{I,Y_i})\|x\|^{N_0'}\cdot\|y\|^{N_0'}\right) \le \sum_i c(\varphi_{I,Y_i})\|x\|^{M_1}, \tag{5.21}$$

for any $x \in \mathcal{S}_0$, where I refers to the trivial identity operator. Denote by $V_0 = \mathrm{vol}([N])$. Then by mean value theorem there exists some $n_0 \in [N]$ such that

$$\sum_\chi \left|\mathcal{F}_1\Lambda_2^T \mathrm{K}_\chi(x,x)\right| \le V_0\sum_\chi\left|\Lambda_2^T \mathrm{K}_\chi(n_0x,x)\right|. \tag{5.22}$$

Substituting (5.21) into the right hand side of (5.22) we then obtain

$$\sum_\chi \left|\mathcal{F}_1\Lambda_2^T \mathrm{K}_\chi(x,x)\right| \le V_0\sum_i c(\varphi_{I,Y_i})\|x\|^{M_1-M},$$

for any $x \in \mathcal{S}_0$, where χ runs over all the equivalent classes of cuspidal datum. Also, since $R(x)$ is slowly increasing, then by taking M to be large enough we conclude that $\sum_\chi \left|\mathcal{F}_1\Lambda_2^T \mathrm{K}_\chi(x,x)\right|\cdot|R(x)|$ is a bounded function on $\mathcal{S}_0$, hence it is integrable. Note that the above argument still works when $\mathcal{F}_1$ is replaced by $\mathcal{F}_1^w$. Then Lemma 42 follows. □

Let S_n be the permutation group on n letters. Let $\sigma \in S_n$. For any $\mathfrak{a} \in A_{F,\infty}$, write it in its Iwasawa normal form given in (5.18). Let $\mathfrak{a}_i' = a_1a_2\cdots a_{n-i}$, $1 \le i \le n-1$; and set $\mathfrak{a}_n' = 1$. For any $1 \le i \le n$, let $\mathfrak{a}_i = a_1^{-1}\cdots a_{\sigma(n)}^{-1}\mathfrak{a}_i'$. Set $\mathfrak{a}_\sigma = \mathrm{diag}(\mathfrak{a}_{\sigma(1)}, \mathfrak{a}_{\sigma(2)}, \cdots, \mathfrak{a}_{\sigma(n)})$. Clearly $\mathfrak{a}_{\sigma(n)} = 1$, and each $\mathfrak{a}_{\sigma(i)}$ is a rational function of $a_1, \cdots, a_{n-1}$. Moreover, each $\mathfrak{a}_{\sigma(i)}$ is a monomial, $1 \le i \le n-1$. Note that $\mathfrak{a}_\sigma$ is of the form in (5.18). So σ induces a well defined map $A_{F,\infty} \longrightarrow A_{F,\infty}$, $\mathfrak{a} \mapsto \mathfrak{a}_\sigma$. Denote by ι_σ this map. Then ι_σ is actually a bijection from $A_{F,\infty}$ to itself.

Write $\mathfrak{a}_{\sigma(i)} = \mathfrak{a}_{\sigma(i)}(a_1, a_2, \cdots, a_{n-1})$ to indicate that $\mathfrak{a}_{\sigma(i)}$ is a fractional function of $a_1, a_2, \cdots a_{n-1}$. For any $c > 0$, let

$$\mathcal{H}_{\sigma}^{c} = \left\{(a_1, a_2, \cdots, a_{n-1}) : \left|\frac{\mathfrak{a}_{\sigma(i)}(a_1, a_2, \cdots, a_{n-1})}{\mathfrak{a}_{\sigma(i+1)}(a_1, a_2, \cdots, a_{n-1})}\right|_{\infty} \geq c,\ 1 \leq i \leq n-1\right\}.$$

Let $S_n^{reg} := \{\sigma \in S_n :\ \sigma(i) \leq \sigma(i+1),\ 3 \leq i \leq n-1\}$. Let $0 < c \leq 1$. Denote by

$$\iota(A_{F,\infty})^{reg} = \left\{(a_1, a_2, \cdots, a_{n-1}) \in \mathbb{G}_m(\mathbb{A}_{F,\infty})^{n-1} :\ \left|a_i\right|_{\infty} \geq c,\ 1 \leq i \leq n-3\right\}.$$

Lemma 43. *Let notation be as above. Let $n \geq 4$. Then one has*

$$\iota(A_{F,\infty})^{reg} \subseteq \bigcup_{\sigma \in S_n^{reg}} \mathcal{H}_{\sigma}^{c}. \tag{5.23}$$

Proof. For any $\sigma \in S_n^{reg}$, if $\sigma(2) < \sigma(3)$, then define $i_\sigma = 2$; if $\sigma(i_0) < \sigma(2) < \sigma(i_0+1)$ for some $3 \leq i_0 \leq n-1$, then define $i_\sigma = i_0$; if $\sigma(2) > \sigma(n)$, then define $i_\sigma = n$. Since such an i_0 (if exists) is unique, then i_σ is well defined. It induces an n to 1 surjection $\iota_{reg} : S_n^{reg} \longrightarrow \{2, 3, \cdots, n\}$, given by $\sigma \mapsto i_\sigma$. For $2 \leq i \leq n$, let $S_{n,i}^{reg}$ be the fibre at $i_\sigma = i$, i.e., $S_{n,i}^{reg} = \iota_{reg}(i)^{-1}$. Then

$$\bigcup_{\sigma \in S_n^{reg}} \mathcal{H}_{\sigma}^{c} = \bigcup_{i=2}^{n} \bigcup_{\sigma \in S_{n,i}^{reg}} \mathcal{H}_{\sigma}^{c}. \tag{5.24}$$

Let $\iota_2(A_{F,\infty})^{reg} = \left\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} :\ \left|a_{n-2}\right|_{\infty} \geq c\right\}$. Denote by $\iota_n(A_{F,\infty})^{reg} = \left\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} :\ \left|a_1 a_2 \cdots a_{n-2}\right|_{\infty} \leq 1/c\right\}$. Define, for any $3 \leq i \leq n-1$, that $\iota_i(A_{F,\infty})^{reg} = \left\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} :\ \left|a_{n-i} a_{n-i+1} \cdots a_{n-2}\right|_{\infty} \geq c,\ \left|a_{n-i+1} a_{n-i+2} \cdots a_{n-2}\right|_{\infty} \leq 1/c\right\}$. Note that, for any $2 \leq i \leq n$, $\iota_i(A_{F,\infty})^{reg}$ is well defined.

Claim 44. *Let notation be as before. Then one has, for any $2 \leq i \leq n$, that*

$$\iota_i(A_{F,\infty})^{reg} \subseteq \bigcup_{\sigma \in S_{n,i}^{reg}} \mathcal{H}_{\sigma}^{c}. \tag{5.25}$$

A straightforward combinatorial analysis shows that $\iota(A_{F,\infty})^{reg}$ is contained in the union of $\iota_i(A_{F,\infty})^{reg}$ over $2 \leq i \leq n$. Hence (5.23) comes from (5.24) and (5.25):

$$\iota(A_{F,\infty})^{reg} \subseteq \bigcup_{i=2}^{n} \iota_i(A_{F,\infty})^{reg} \subseteq \bigcup_{i=2}^{n} \bigcup_{\sigma \in S_{n,i}^{reg}} \mathcal{H}_{\sigma}^{c} = \bigcup_{\sigma \in S_n^{reg}} \mathcal{H}_{\sigma}^{c}.$$

□

Proof of Claim 44. For any $2 \leq i \leq n$ and any $\sigma \in S_{n,i}^{reg}$, recall that $\mathcal{H}_\sigma^c$ is equal to

$$\left\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \left|\frac{\mathfrak{a}_{\sigma(i)}(a_1, a_2, \cdots, a_{n-1})}{\mathfrak{a}_{\sigma(i+1)}(a_1, a_2, \cdots, a_{n-1})}\right|_\infty \geq c, \ i \leq n-1\right\}.$$

Case 1 Let $i = 2$ and $\sigma \in S_{n,i}^{reg}$. If $\sigma(2)+1 < \sigma(1) < n$, then there exists a unique j_σ such that $\sigma(j_\sigma) < \sigma(2) < \sigma(j_\sigma+1)$. In this case $\mathcal{H}_\sigma^c$ is equal to $\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-2}|_\infty \geq c, \ |a_{n-j_\sigma} \cdots a_{n-1}|_\infty \geq c, \ |a_{n-j_\sigma+1} \cdots a_{n-1}|_\infty \leq 1/c\}$. If $\sigma(1) = n$, then $\mathcal{H}_\sigma^c = \{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-i} \cdots a_{n-2}|_\infty \geq c, \ |a_1 \cdots a_{n-1}|_\infty \leq 1/c\}$. If $\sigma(1) = \sigma(2)+1$, then $\mathcal{H}_\sigma^c$ is equal to $\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-2}a_{n-1}|_\infty \geq c, \ |a_{n-1}|_\infty \leq 1/c, \ |a_{n-1}a_{n-2}|_\infty \leq 1/c\}$. If $\sigma(1) = \sigma(2) - 1$, then $\mathcal{H}_\sigma^c$ is equal to $\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-2}|_\infty \geq c, \ |a_{n-1}|_\infty \geq c\}$. Now one sees clearly that the union of $\mathcal{H}_\sigma^c$ over $\sigma \in S_{n,i}^{reg}$ does cover the sets $\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \left|a_{n-2}\right|_\infty \geq c, \ |a_{n-1}|_\infty \geq c\}$ and $\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \left|a_{n-2}\right|_\infty \geq c, \ |a_{n-1}|_\infty \leq 1/c\}$. Therefore it covers $\iota_2(A_{F,\infty})^{reg}$. Hence (5.25) holds in the case where $i = 2$.

Case 2 Let $3 \leq i \leq n-1$ and $\sigma \in S_{n,i}^{reg}$. If $\sigma(2)+1 < \sigma(1) < n$, then there exists a unique j_σ such that $\sigma(j_\sigma) < \sigma(2) < \sigma(j_\sigma+1)$. In this case $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-i} \cdots a_{n-2}|_\infty \geq c, \ |a_{n-i+1} \cdots a_{n-2}|_\infty \leq \frac{1}{c},$$
$$|a_{n-j_\sigma} \cdots a_{n-1}|_\infty \geq c, \| a_{n-j_\sigma+1} \cdots a_{n-1}|_\infty \leq 1/c\Big\}.$$

If $\sigma(1) = n$, then $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-i} \cdots a_{n-2}|_\infty \geq c,$$
$$|a_{n-i+1} \cdots a_{n-2}|_\infty \leq 1/c, \ |a_1 \cdots a_{n-1}|_\infty \leq 1/c\Big\}.$$

If $\sigma(1) = \sigma(2)+1$, then $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-i} \cdots a_{n-2}a_{n-1}|_\infty \geq c,$$
$$|a_{n-1}|_\infty \leq 1/c, \ |a_{n-i+1} \cdots a_{n-2}|_\infty \leq 1/c\Big\}.$$

If $\sigma(1) = \sigma(2) - 1$, then $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_{n-i} \cdots a_{n-2}|_\infty \geq c,$$
$$|a_{n-1}|_\infty \geq c, \ |a_{n-i+1} \cdots a_{n-1}|_\infty \leq 1/c\Big\}.$$

If $1 < \sigma(1) < \sigma(2)-1$, then there exists a unique j_σ such that $\sigma(j_\sigma) < \sigma(2) < \sigma(j_\sigma+1)$. In this case $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ |a_{n-i} \cdots a_{n-2}|_\infty \geq c, |a_{n-i+1} \cdots a_{n-2}|_\infty \leq c^{-1}, \\ |a_{n-j_\sigma} \cdots a_{n-2}a_{n-1}|_\infty \geq c, \ |a_{n-j_\sigma+1} \cdots a_{n-1}|_\infty \leq 1/c\Big\}.$$

If $\sigma(1) = 1$, then

$$\mathcal{H}_\sigma^c = \Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ |a_{n-i}a_{n-i+1} \cdots a_{n-2}|_\infty \geq c, \\ |a_{n-i+1} \cdots a_{n-2}|_\infty \leq c^{-1}, \ c \leq |a_{n-2}a_{n-1}|_\infty\Big\}.$$

Now one sees clearly that the union of $\mathcal{H}_\sigma^c$ over $\sigma \in S_{n,i}^{reg}$ does cover the sets

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ \big|a_{n-i}a_{n-i+1} \cdots a_{n-2}\big|_\infty \geq c, \\ \big|a_{n-i+1}a_{n-i+2} \cdots a_{n-2}\big|_\infty \leq 1/c, \ |a_{n-1}|_\infty \geq c\Big\}$$

and

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ \big|a_{n-i}a_{n-i+1} \cdots a_{n-2}\big|_\infty \geq c, \\ \big|a_{n-i+1}a_{n-i+2} \cdots a_{n-2}\big|_\infty \leq 1/c, \ |a_{n-1}|_\infty \leq 1/c\Big\}.$$

Therefore, it covers $\iota_i(A_{F,\infty})^{reg}$. Hence (5.25) holds.

Case 3 Let $i = n$ and $\sigma \in S_{n,i}^{reg}$. If $\sigma(1) = \sigma(2)+1$, then $\mathcal{H}_\sigma^c = \big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ |a_{n-1}|_\infty \leq 1/c, \ |a_1a_2 \cdots a_{n-2}|_\infty \leq 1/c\big\}$. If $\sigma(1) = \sigma(2)-1$, then $\mathcal{H}_\sigma^c$ is equal to

$$\Big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ |a_{n-1}|_\infty \geq c, \ |a_1a_2 \cdots a_{n-1}|_\infty \leq 1/c\Big\}.$$

If $1 < \sigma(1) < \sigma(2) - 1$, then there exists a unique j_σ such that $\sigma(j_\sigma) < \sigma(2) < \sigma(j_\sigma+1)$. In this case $\mathcal{H}_\sigma^c$ is equal to $\big\{(a_1, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : |a_1 \cdots a_{n-2}|_\infty \leq 1/c, \ |a_{n-j_\sigma} \cdots a_{n-1}|_\infty \geq c, \ |a_{n-j_\sigma+1} \cdots a_{n-1}|_\infty \leq 1/c\big\}$. If $\sigma(1) = 1$, then $\mathcal{H}_\sigma^c = \big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \ |a_{n-i+1} \cdots a_{n-2}|_\infty \leq 1/c, \ |a_1a_2 \cdots a_{n-2}a_{n-1}|_\infty \geq c\big\}$. Now one sees clearly that the union of $\mathcal{H}_\sigma^c$ over $\sigma \in S_{n,i}^{reg}$ does cover the sets $\big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \big|a_1a_2 \cdots a_{n-2}\big|_\infty \leq 1/c, \ |a_{n-1}|_\infty \geq c\big\}$ and $\big\{(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})^{reg} : \big|a_1a_2 \cdots a_{n-2}\big|_\infty \leq 1/c, \ |a_{n-1}|_\infty \leq 1/c\big\}$. Therefore, it covers $\iota_n(A_{F,\infty})^{reg}$. Hence (5.25) holds for $i = n$.

Therefore, Claim 44 follows from the above discussions. □

Proposition 45. *Let notation be as before. Let $\iota : A_{F,\infty} \longrightarrow \mathbb{G}_m(\mathbb{A}_{F,\infty})^{n-1}$ be the isomorphism given by $\mathfrak{a} \mapsto (a_1, a_2, \cdots, a_{n-1})$. Then for any $0 < c \leq 1$, one has*

$$\iota(A_{F,\infty}) \subseteq \bigcup_{\sigma \in S_n} \mathcal{H}_\sigma^c. \tag{5.26}$$

Proof. When $n = 2$, (5.26) is immediate. When $n = 3$, there are six different $\mathcal{H}_\sigma^c$'s. In this case, one can verify by brute force computation that the union of these hyperboloids does cover $\iota(A_{F,\infty})$. Hence (5.26) holds for $n = 3$. From now on, we may assume $n \geq 4$. For any $m \in \mathbb{Z}$, let $\tau_m : \mathbb{Z} \to \mathbb{Z}$ be the shifting map defined by $j \mapsto j + m$, $\forall\ j \in \mathbb{Z}$. Set $S_{n-2}[2] := \{\tau_2 \circ \sigma \circ \tau_{-2} :\ \sigma \in S_{n-2}\}$ as a set of bijections from the set $\{3, 4, \cdots, n\}$ to itself. Regard naturally $S_{n-2}[2]$ as the stabilizer of $\{1, 2\}$ of S_n. Then clearly $S_{n-2}[2]$ is isomorphic to S_{n-2}. Denote by ϱ the natural isomorphism $S_{n-2} \xrightarrow{\sim} S_{n-2}[2]$, $\sigma \mapsto \tau_2 \circ \sigma \circ \tau_{-2}$. Note that $\# S_n^{reg} = n(n-1)$, then we have a bijection:

$$S_{n-2} \times S_n^{reg} \xrightarrow{1:1} S_n, \quad (\sigma, \sigma') \mapsto \varrho(\sigma) \circ \sigma'. \tag{5.27}$$

Assume that (5.26) holds for any $n_0 \leq n - 2$. Let $(a_1, a_2, \cdots, a_{n-1}) \in \iota(A_{F,\infty})$. Let $\mathfrak{a} \in A_{F,\infty}$ be such that $\iota(\mathfrak{a}) = (a_1, a_2, \cdots, a_{n-1})$. Then $(a_1, a_2, \cdots, a_{n-3}) \in \iota^{\leq n-2}(A_{F,\infty})$, where $\iota^{\leq n-2} : A_{F,\infty} \longrightarrow \mathbb{G}_m(\mathbb{A}_{F,\infty})^{n-3}$ is the map given by $\mathfrak{a} \mapsto (a_1, a_2, \cdots, a_{n-3})$. Then by our induction assumption, there exists some $\sigma \in S_{n-2}$ such that $(a_1, a_2, \cdots, a_{n-3}) \in \mathcal{H}_{\sigma_\varrho}^c$, where $\sigma_\varrho := \varrho^{-1}(\sigma)$. Note that for each $1 \leq i \leq n - 3$, $\mathfrak{a}_{\sigma_\varrho(i)}(a_1, a_2, \cdots, a_{n-1})$ is independent of a_{n-2} and a_{n-1}, we may write $\mathfrak{a}_{\sigma_\varrho(i)}(a_1, a_2, \cdots, a_{n-1}) = \mathfrak{a}_{\sigma_\varrho(i)}(a_1, a_2, \cdots, a_{n-3}), 1 \leq i \leq n-3$. Let $\mathfrak{b}_2 = a_1 a_2 \cdots a_{n-2} \cdot \mathfrak{a}_{\sigma_\varrho(1)}(a_1, a_2, \cdots, a_{n-3})^{-1}$, $\mathfrak{b}_1 = a_1 a_2 \cdots a_{n-2} a_{n-1} \cdot \mathfrak{b}_2^{-1}$, and $\mathfrak{b}_n = 1$. Let $\mathfrak{b}_{i+2} = \mathfrak{a}_{\sigma_\varrho(i)}(a_1, a_2, \cdots, a_{n-3})$, $1 \leq i \leq n - 3$. Set $\mathfrak{b} = \operatorname{diag}(\mathfrak{b}_1, \mathfrak{b}_2, \cdots, \mathfrak{b}_n)$. Then clearly $\mathfrak{b} = \mathfrak{a}_{\sigma_\varrho}$. Hence $\mathfrak{b} \in A_{F,\infty}$ and $\iota(\mathfrak{b}) = (b_1, b_2, \cdots, b_{n-1})$, where $b_i = \mathfrak{b}_{n-i} \cdot \mathfrak{b}_{n-i+1}^{-1}$, $1 \leq i \leq n - 1$. By our definition of $\mathfrak{b}$, one sees that $\iota(\mathfrak{b}) \in \iota(A_{F,\infty})^{reg}$. Then by Lemma 43 there exists some $\sigma' \in S_n^{reg}$ such that $\iota(\mathfrak{b}) \in \mathcal{H}_{\sigma'}^c$. Therefore,

$$|\mathfrak{b}_{\sigma'(i)}(b_1, b_2, \cdots, b_{n-1})|_\infty \geq c \cdot |\mathfrak{b}_{\sigma'(i+1)}(b_1, b_2, \cdots, b_{n-1})|_\infty,\ 1 \leq i \leq n - 1.$$

Since $\mathfrak{b} = \mathfrak{a}_{\sigma_\varrho}$, the above system of inequalities becomes, for $1 \leq i \leq n - 1$, that

$$|\mathfrak{a}_{\varrho(\sigma)\circ\sigma'(i)}(a_1, a_2, \cdots, a_{n-1})|_\infty \geq c \cdot |\mathfrak{a}_{\varrho(\sigma)\circ\sigma'(i+1)}(a_1, a_2, \cdots, a_{n-1})|_\infty.$$

Then according to (5.27), $\iota(\mathfrak{a}) \in \mathcal{H}_{\tilde{\sigma}}^c$ for $\tilde{\sigma} = \varrho(\sigma) \circ \sigma' \in S_n$. Hence (5.26) holds for $n_0 = n$. Since it holds for $n_0 = 2$ and $n_0 = 3$, Proposition 45 follows by induction on initial cases. □

Proposition 46. *Let notation be as above. Let $A_{\varphi,\mathrm{fin}}$ be a compact subgroup of $Z_G(\mathbb{A}_{F,\mathrm{fin}})\backslash T(\mathbb{A}_{F,\mathrm{fin}})$ depending only on φ and F. Let $R(x)$ be a slowly increasing function on $\mathcal{S}_0$. Then we have*

$$\int_K\int_{[N]}\int_{A_{F,\infty}A_{\varphi,\mathrm{fin}}}\sum_{\chi}\left|\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_\chi(nak,nak)\cdot R(nak)\right|d^\times a dn dk<\infty, \tag{5.28}$$

where χ runs over all the equivalent classes of cuspidal datum.

Proof. Let $A_{F,\infty}^{\sigma}=\{\mathfrak{a}\in A_{F,\infty}:\ \iota(\mathfrak{a})\in\mathcal{H}_\sigma^c\}$. Then Proposition 45 implies that

$$A_{F,\infty}=\bigcup_{\sigma\in S_n}A_{F,\infty}^{\sigma}. \tag{5.29}$$

The decomposition (5.29) implies that the left hand side of (5.28) is not more than

$$J_c:=\sum_{\sigma\in S_n}\int_K\int_{[N]}\int_{A_{F,\infty}^{\sigma}A_{\varphi,\mathrm{fin}}}\sum_{\chi}\left|\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_\chi(nak,nak)\cdot R(nak)\right|d^\times a dn dk.$$

Note that for any $\sigma\in S_n$, let $\mathfrak{a}\in A_{F,\infty}^{\sigma}$, then $\mathfrak{a}_\sigma\in\mathcal{S}_c$. Let $w\in K_\infty$ be the matrix representation of the Weyl element corresponding to σ so that $\mathfrak{a}_\sigma=w^{-1}\mathfrak{a}w$. Then

$$\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_\chi(nak,nak)=\mathcal{F}_1^w\Lambda_2^T\,\mathrm{K}_\chi^w(w^{-1}nw\mathfrak{a}_\sigma a_{\mathrm{fin}}w^{-1}kw,w^{-1}nw\mathfrak{a}_\sigma a_{\mathrm{fin}}w^{-1}kw),$$

where K_χ^w refers to the kernel function relative to the test function φ^w defined by $\varphi^w(x)=\varphi(wxw^{-1})$.

Let $\iota(\mathfrak{a})=(a_1,a_2,\cdots,a_{n-1})$, and $\iota(\mathfrak{a}_\sigma)=(a_{\sigma,1},a_{\sigma,2},\cdots,a_{\sigma,n-1})$, then a straightforward computation shows that each a_i is a rational monomial P_σ of the variables $a_{\sigma,1},a_{\sigma,2},\cdots,a_{\sigma,n-1}$. Since such a monomial is at most polynomially increasing on $\mathcal{S}_c$, one then changes variables to see

$$\int_K\int_{[N]}\int_{A_{F,\infty}^{\sigma}A_{\varphi,\mathrm{fin}}}\sum_{\chi}\left|\mathcal{F}_1\Lambda_2^T\,\mathrm{K}_\chi(nak,nak)\cdot R(nak)\right|d^\times a dn dk$$

is bounded by the integral over K, $N(F)\backslash N(\mathbb{A}_F)$ of

$$\int_{A_{F,\infty}^{\sigma}A_{\varphi,\mathrm{fin}}}\sum_{\chi}\left|\mathcal{F}_1^w\Lambda_2^T\,\mathrm{K}_\chi^w(X,X)\cdot R(n\mathfrak{a}_\sigma a_{\mathrm{fin}}k)P_\sigma(\mathfrak{a}_\sigma)\right|d^\times\mathfrak{a}_\sigma d^\times a_{\mathrm{fin}}, \tag{5.30}$$

where $X=w^{-1}nw\mathfrak{a}_\sigma a_{\mathrm{fin}}w^{-1}kw,w^{-1}nw\mathfrak{a}_\sigma a_{\mathrm{fin}}w^{-1}kw$. Note that (5.30) is bounded by the integral over $A_{\varphi,\mathrm{fin}}$ of $G_\sigma(n,a_\infty,k)$, where $G_\sigma(n,a_\infty,k)$ is defined to be

$$\int_{A_{F,\infty}\cap\mathcal{S}_c}\sum_{\chi}\left|\mathcal{F}_1^w\Lambda_2^T\,\mathrm{K}_\chi^w(X',X')\cdot R(na_\infty a_{\mathrm{fin}}k)P_\sigma(a_\infty)\right|d^\times a_\infty,$$

where $X' = w^{-1}nwa_\infty a_{\text{fin}}w^{-1}kw$. Since the function $a_\infty \mapsto R(na_\infty a_{\text{fin}}k)P_\sigma(a_\infty)$ is slowly increasing on $A_{F,\infty} \cap \mathcal{S}_c$, Lemma 42 implies that $G_\sigma(n, a_\infty, k)$ is well defined and thus it is continuous. So

$$\int_K \int_{[N]} \int_{A_{\varphi,\text{fin}}} G_\sigma(n, a_\infty, k) d^\times a_{\text{fin}} dndk < \infty, \ \forall\, \sigma \in S_n.$$

Therefore, (5.28) follows from the estimate below:

$$J_c \leq \sum_{\sigma \in S_n} \int_K \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{A_{\varphi,\text{fin}}} G_\sigma(n, a_\infty, k) d^\times a_{\text{fin}} dndk < \infty.$$

Then Proposition 46 follows. □

Corollary 47. *Let notation be as above. Let $R(x)$ be a slowly increasing function on $Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)$. Then we have*

$$\int_{Z_G(\mathbb{A}_F)N(F)\backslash G(\mathbb{A}_F)} \sum_\chi \left| \mathcal{F}_1 \Lambda_2^T \operatorname{K}_\chi(x, x) \cdot R(x) \right| dx < \infty, \tag{5.31}$$

where χ runs over all the equivalent classes of cuspidal data.

Proof. To handle $\mathcal{F}_1 \Lambda_2^T \operatorname{K}_\chi(x, y)$ we can substitute the definitions of $\mathcal{F}_1$ and Λ^T into the expansion of K_χ (cf. (5.14)). Note that (5.14) is convergent absolutely and Λ^T is a finite sum for given x and y. One can thus apply the operators $\mathcal{F}$ and Λ^T inside the integral over $i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ to obtain explicitly that

$$\mathcal{F}_1 \Lambda_2^T \operatorname{K}_\chi(x, y) = \sum_{P \in \mathcal{P}} \frac{1}{c_P} \int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \mathcal{F} E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda) \Lambda^T \overline{E(y, \phi, \lambda)} d\lambda.$$

This an easier analogue of Lemma 37. Then as before, the non-constant Fourier coefficient $\mathcal{F} E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda)$ of $E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda)$ becomes a Whittaker function $W(x; \lambda)$.

Recall that our test function φ is K-finite. Hence there is some compact subgroup $K_0 \subset G(\mathbb{A}_{F,\text{fin}})^1$ such that φ is right K_0-invariant. Then for $\phi \in \mathfrak{B}_{P,\chi}$, $\mathcal{F} E(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda) \Lambda^T \overline{E(y, \phi, \lambda)} = 0$ unless ϕ is right K_0-invariant. Let $K_0 = \prod_{v<\infty} K_{0,v}$. Note that the Whittaker functions are decomposable, i.e.,

$$W(x; \lambda) = \prod_{v \in \Sigma_F} W_v(x; \lambda).$$

Then for each finite place v, $W_v(x; \lambda)$ is right $K_{0,v}$-invariant. So there exists a compact subgroup $N_{0,v} \subseteq K_{0,v} \cap N(F_v)$, depending only on φ, such that

$$W_v(t_v u_v; \lambda) = W_v(t_v; \lambda), \text{ for all } t_v \in T(F_v) \text{ and } u_v \in N_{0,v}.$$

On the other hand, $W_v(t_vu_v;\lambda) = \theta_{t_v}(u_v)W_v(t_v;\lambda)$, where $\theta_{t_v}(n_v) = \theta(t_vn_vt_v^{-1})$, for any $n_v \in N(F_v)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ (hence not on λ) such that $\theta_{t_v}(u_v) = 1$ if and only if $|\alpha_i(t_v)| \leq C_v$, where α_i's are the simple roots of $G(F)$. Note that for all but finitely many $v < \infty$, $K_{0,v} = GL_n(\mathcal{O}_{F,v})$, thus we can take the corresponding $C_v = 1$. Hence for any $t_v \in T(F_v)$, $W_v(t_v;\lambda) \neq 0$ implies that $|\alpha_i(t_v)| \leq C_v$, $1 \leq i \leq n-1$, and $C_v = 1$ for all but finitely many finite places v. Set $A = Z_G\backslash T$, and

$$A_{\varphi,\mathrm{fin}} = \Big\{a = (a_v) \in A(\mathbb{A}_{F,\mathrm{fin}}):\ |\alpha_i(a_v)| \leq C_v,\ 1 \leq i \leq n-1\Big\}.$$

Then $\operatorname{supp} W(x;\lambda)\ |_{A(\mathbb{A}_F)} \subseteq A(\mathbb{A}_{F,\infty})A_{\varphi,\mathrm{fin}}$, $\forall\ \lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $1 \leq i \leq 2$. So after applying Iwasawa decomposition, we see that the integrand in (5.31) are supported in $[N]\cdot A(\mathbb{A}_{F,\infty})A_{\varphi,\mathrm{fin}}\cdot K$, independent of χ. Therefore, (5.31) follows from (5.28). □

Chapter 6

RANKIN-SELBERG CONVOLUTIONS FOR GENERIC REPRESENTATIONS

By Theorem G, we see that when $\mathrm{Re}(s) > 1$, $I_{\mathrm{Whi}}(s, \tau)$ is equal to

$$\sum_{\chi} \sum_{P \in \mathcal{P}} \frac{1}{c_P} \sum_{\phi_1 \in \mathfrak{B}_{P,\chi}} \sum_{\phi_2 \in \mathfrak{B}_{P,\chi}} \int_{\Lambda^*} \langle \mathcal{I}_P(\lambda, \varphi)\phi_2, \phi_1 \rangle \Psi_{P,\chi}(s, W_1, W_2; \lambda)\, d\lambda, \tag{6.1}$$

where $\Psi_{P,\chi}(s, W_1, W_2; \lambda) = \int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x; \lambda)\overline{W_2(x; \lambda)} f(x, s) dx$, and the Whittaker function $W_i(x, \lambda) = \int_{N(\mathbb{A}_F)} \phi_i(w_0 n x) e^{(\lambda+\rho_P)H_P(w_0 n x)} \theta(n) dn$, $1 \leq i \leq 2$, and w_0 is the longest Weyl element.

For our purpose, we need to show that $I_{\mathrm{Whi}}(s, \tau)$ is a holomorphic multiple of $L(s, \tau)$. So we have to compute (6.1) explicitly (up to an entire factor), then continue it to a meromorphic function which is a holomorphic multiple of $L(s, \tau)$ as we desired. To achieve that, we start with computing each $\Psi_{P,\chi}(s, W_1, W_2; \lambda)$ associated to a standard parabolic subgroup P and a cuspidal datum $\chi = (M_P, \sigma) \in \mathfrak{X}$.

Let P be a standard parabolic subgroup of G of type $(n_1, n_2, \cdots, n_r)$, $1 \leq r \leq n$, with $n_1 + n_2 + \cdots n_r = n$. Let $\chi \in \mathfrak{X}$ be represented by (M_P, σ). Let $\mathfrak{B}_{P,\chi}$ be an orthonormal basis of the Hilbert space $\mathcal{H}_{P,\chi}$. For $\phi_i \in \mathfrak{B}_{P,\chi}$, $1 \leq i \leq 2$, define the Whittaker function associated to ϕ_i parameterized by $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ by

$$W_{P,\chi,i}(x, \lambda) = W(x, \phi_i, \lambda) := \int_{N(\mathbb{A}_F)} \phi_i(w_0 n x) e^{(\lambda+\rho_P)H_P(w_0 n x)} \theta(n) dn,$$

where w_0 is the longest element in the Weyl group W_n. Define

$$\Psi_{P,\chi}(s, W_1, W_2; \lambda, \Phi) = \int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_{P,\chi,1}(x; \lambda)\overline{W_{P,\chi,2}(x; \lambda)} f(x, s) dx.$$

From now on, we fix such a standard parabolic subgroup P of type $(n_1, \cdots, n_r)$ and a cuspidal datum $\chi = (M_P, \sigma) \in \mathfrak{X}$, where σ is a unitary representation of M of central character ω. Then there exist r cuspidal representations π_i of $\mathrm{GL}_{n_i}(\mathbb{A}_F)$, $1 \leq i \leq r$, such that $\sigma \simeq \pi_1 \oplus \pi_2 \oplus \cdots \oplus \pi_r$. Let $\pi = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\pi_1, \pi_2 \cdots, \pi_r)$. For any $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, denote by

$$\pi_\lambda = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}\left(\pi_1 \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_1}, \pi_2 \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_2}, \cdots, \pi_r \otimes |\cdot|_{\mathbb{A}_F}^{\lambda_r}\right). \tag{6.2}$$

Then π_λ is also a unitary automorphic representation of $G(\mathbb{A}_F)$. Fix $\phi_1, \phi_2 \in \mathcal{B}_{P,\chi}$ and a point $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$. Write $W_i(x,\lambda) = W_{P,\chi,i}(x,\lambda)$, and $\Psi(s, W_1, W_2; \lambda, \Phi) = \Psi_{P,\chi}(s, W_1, W_2; \lambda, \Phi)$. Since $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $\lambda = -\bar{\lambda}$, one has

$$\Psi(s, W_1, W_2; \lambda, \Phi) = \int_{Z_G(\mathbb{A}_F)N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2\left(x;-\bar{\lambda}\right)} f(x,s)dx. \tag{6.3}$$

Since $W_1(x;\lambda)$ and $W_2\left(x;-\bar{\lambda}\right)$ are dominant by some gauge, and $f(x,s)$ is slowly increasing when $\mathrm{Re}(s) > 1$, then $\Psi(s, W_1, W_2; \lambda, \Phi)$ converges absolutely and normally when $\mathrm{Re}(s) > 1$. Note that $\pi_i = \otimes_v' \pi_{i,v}$, $1 \leq i \leq r$, where, for each $v \in \Sigma_F$, $\pi_{i,v}$ is a unitary irreducible representation of $\mathrm{GL}_{n_i}(F_v)$, of Whittaker tape. Then for each $v \in \Sigma_F$ and $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, denote by $\pi_v = \mathrm{Ind}_{M_P(F_v)}^{G(F_v)}\left(\pi_{1,v}, \pi_{2,v}, \cdots, \pi_{r,v}\right)$ and

$$\pi_{\lambda,v} = \mathrm{Ind}_{M_P(F_v)}^{G(F_v)}\left(\pi_{1,v} \otimes |\cdot|_{F_v}^{\lambda_1}, \pi_{2,v} \otimes |\cdot|_{F_v}^{\lambda_2}, \cdots, \pi_{r,v} \otimes |\cdot|_{F_v}^{\lambda_r}\right).$$

Then $\pi = \otimes_v' \pi_v$ and $\pi_\lambda = \otimes_v' \pi_{\lambda,v}$. Recall that $f(x,s) = \prod_v f_v(x_v, s)$, where

$$f_v(x_v, s) = \tau_v(\det x_v)|\det x_v|_{F_v}^s \int_{Z_G(F_v)} \Phi_v[(0, \cdots, t_v)x_v]\tau_v^n(t_v)|t_v|_{F_v}^{ns} d^\times t_v,$$

if $\Phi = \otimes_v' \Phi_v$. Since ϕ_1 and ϕ_2 both have central character $\omega_\lambda = \omega$, which is unitary. So is $W_1(x;\lambda)$ and $W_2(x;\lambda)$. Hence one can rewrite $\Psi(s, W_1, W_2; \lambda, \Phi)$ as

$$\int_{N(\mathbb{A}_F)\backslash G(\mathbb{A}_F)} W_1(x;\lambda)\overline{W_2\left(x;-\bar{\lambda}\right)}\Phi(\eta x)\tau(\det x)|\det x|_{\mathbb{A}_F}^s dx, \tag{6.4}$$

where $\eta = (0, \cdots, 0, 1) \in F^n$. According to the definition we can write $\phi_i = \otimes_v' \phi_{i,v}$, $1 \leq i \leq 2$. Thus one can factor $W_i(x;\lambda)$ as $\prod_{v \in \Sigma_F} W_{i,v}(x_v;\lambda)$, where

$$W_{i,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{i,v}(w_0 n x_v) e^{(\lambda+\rho_P)H_P(w_0 n x_v)}\theta(n)dn, \ x_v \in F_v, \ 1 \leq i \leq 2.$$

We may assume $\Phi = \otimes_v' \Phi_v$ and $\phi_i = \otimes_v' \phi_{i,v}$, $i = 1, 2$. Then one has

$$\Psi\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi\right) = \prod_{v \in \Sigma_F} \Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right),$$

where each local factor $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is defined to be

$$\int_{N(F_v)\backslash G(F_v)} W_{1,v}(x_v;\lambda)\overline{W_{2,v}\left(x_v;-\bar{\lambda}\right)}\Phi_v(\eta x_v)\tau(\det x_v)|\det x_v|_{F_v}^s dx_v, \tag{6.5}$$

where $W_{i,v}(x_v;\lambda) = \int_{N(F_v)} \phi_{i,v}(w_0 n x) e^{(\lambda+\rho_P)H_{P,v}(w_0 n x)}\theta(n)dn$, $1 \leq i \leq 2$. Since $W_{1,v}(x;\lambda)$ and $W_{2,v}\left(x;-\bar{\lambda}\right)$ are dominant by some local gauge, and $f_v(x_v, s)$ is slowly increasing when $\mathrm{Re}(s) > 1$, then $\Psi\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ converges absolutely and normally when $\mathrm{Re}(s) > 1$, for any $v \in \Sigma_F$.

6.1 Local Theory for $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$

In this section, we shall compute each local integral representation $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ defined via (6.5). Let $v \in \Sigma_F$ be a place of F. Note that v may be archimedean or nonarchimedean. Let $\mathfrak{u} = (u_{j,l})_{1\leq j,l\leq n} \in N(F_v)$, the unipotent of GL_n, we denote by $N_j^0(\mathfrak{u})$ the matrix

$$\begin{pmatrix}
1 & u_{12} & \cdots & \cdots & u_{1,n-j+2} & u_{1,n-j+3} & \cdots & \cdots & u_{1,n}\\
 & 1 & \cdots & \cdots & u_{2,n-j+2} & u_{2,n-j+3} & \cdots & \cdots & u_{2,n}\\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
 & & & 1 & u_{n-j+1,n-j+2} & u_{n-j+1,n-j+3} & \cdots & \cdots & u_{n-j+1,n}\\
 & & & & 1 & 0 & \cdots & \cdots & 0\\
 & & & & & 1 & \cdots & \cdots & u_{n-j+3,n}\\
 & & & & & & \ddots & \vdots & \vdots\\
 & & & & & & & 1 & u_{n-1,n}\\
 & & & & & & & & 1
\end{pmatrix}$$

associated to $\mathfrak{u}$. Denote by $N_j^1(\mathfrak{u})$ the matrix

$$\begin{pmatrix}
1 & u_{12} & \cdots & \cdots & u^1_{1,n-j+2} & u_{1,n-j+3} & \cdots & \cdots & u_{1,n}\\
 & 1 & \cdots & \cdots & u^1_{2,n-j+2} & u_{2,n-j+3} & \cdots & \cdots & u_{2,n}\\
 & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
 & & & 1 & 0 & u_{n-j+1,n-j+3} & \cdots & \cdots & u_{n-j+1,n}\\
 & & & & 1 & 0 & \cdots & \cdots & 0\\
 & & & & & 1 & \cdots & \cdots & u_{n-j+3,n}\\
 & & & & & & \ddots & \vdots & \vdots\\
 & & & & & & & 1 & u_{n-1,n}\\
 & & & & & & & & 1
\end{pmatrix}$$

associated to $\mathfrak{u}$, where $u^1_{l,n-j+2} = u_{l,n-j+2} - u_{l,n-j+1}u_{n-j-1,n-j+2}$, $1 \leq l \leq n-j$. For any $2 \leq j \leq n$, we denote by $N^0_j(\mathfrak{u}^*)$ the matrix

$$\begin{pmatrix} 1 & u_{12} & \cdots & u_{1,n-j+1} & u_{1,n} & u_{1,n-j+2} & \cdots & \cdots & u_{1,n-1} \\ & 1 & \cdots & u_{2,n-j+1} & u_{2,n} & u_{2,n-j+2} & \cdots & \cdots & u_{2,n-1} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & u_{n-j+1,n} & u_{n-j+1,n-j+2} & \cdots & \cdots & u_{n-j+1,n-1} \\ & & & & 1 & 0 & \cdots & \cdots & 0 \\ & & & & & 1 & \cdots & \cdots & u_{n-j+2,n-1} \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & 1 & u_{n-2,n-1} \\ & & & & & & & & 1 \end{pmatrix};$$

and for $\mathfrak{u} = (u_{j,l})_{1\leq j,l\leq n} \in N(F_v)$, we let $N^0_{j+2}(\mathfrak{u}'')$ represent the matrix

$$\begin{pmatrix} 1 & u_{12} & \cdots & u_{1,n-j-1} & u_{1,n} & u'_{1,n-j} & \cdots & \cdots & u_{1,n-1} \\ & 1 & \cdots & u_{2,n-j-1} & u_{2,n} & u'_{2,n-j} & \cdots & \cdots & u_{2,n-1} \\ & & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ & & & 1 & u_{n-j-1,n} & u'_{n-j-1,n-j} & \cdots & \cdots & u_{n-j-1,n-1} \\ & & & & 1 & 0 & \cdots & \cdots & 0 \\ & & & & & 1 & \cdots & \cdots & u_{n-j,n-1} \\ & & & & & & \ddots & \vdots & \vdots \\ & & & & & & & 1 & u_{n-2,n-1} \\ & & & & & & & & 1 \end{pmatrix},$$

with $u'_{k,n-j} = u_{k,n-j} + s_{n-j,n}u_{k,n-j-1}$, $1 \leq k \leq n-j-1$.

Let w_j be the simple root of GL_n corresponding to the permutation $(j, j+1)$, $1 \leq j \leq n-1$. Let τ_n be the longest element in the Weyl group W_n of GL_n, $n \geq 2$. Then $\tau_n = w_{n-1}w_{n-2}w_{n-1}w \cdots w_1w_2 \cdots w_{n-1}$, for any $n \geq 2$. Recall that we write w_0 for the longest element for G; when we highlight the rank n we then use τ_n instead. The τ_n is only used in this section.

Fix a nontrivial additive character $\theta = \theta_v$ on F_v. This notation in only used in this section and should not be thought as a local version of notations in Sec. 3.1. Let $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \in F_v^{n-1}$. Denote by θ_α the character on F_v^{n-1} such that $\theta_\alpha(x_1, \cdots, x_{n-1}) = \theta(\alpha_1x_1 + \cdots + \alpha_{n-1}x_{n-1})$. Extending θ_α to a character on $N(F_v)$ by $\theta_\alpha(\mathfrak{u}) = \theta(\alpha_1u_{12} + \alpha_2u_{23} + \cdots + \alpha_{n-1}u_{n-1,n})$, where $\mathfrak{u} = (u_{k,l})_{1\leq k,l\leq n} \in N(F_v)$. Let

$\phi_v \in \pi_v$. Define the Whittaker function associated to ϕ and α by

$$W_v(\alpha, \lambda) = \int_{N(F_v)} \phi_v\left(\tau_n \mathfrak{u}\right) e^{(\lambda+\rho)H_B(\tau_n\mathfrak{u})}\theta_\alpha(\mathfrak{u})d\mathfrak{u}.$$

Let $\pi_{v,\lambda} = \operatorname{Ind}_{B(F_v)}^{GL_n(F_v)}\left(\chi_{v,1}|\cdot|^{\lambda_1}, \cdots, \chi_{v,n}|\cdot|^{\lambda_n}\right)$ be a principal series. Let $\mathfrak{u}_{n-1} = (u_1, u_2, \cdots, c_{n-1}) \in F_v^{n-1}$. For any $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) \in F_v^{n-1}$, we set

$$W_{v,j}(\widetilde{\alpha}_{n-1}; \widetilde{\lambda}) = \int_{N_{n-1}(F_v)} \phi_v\left(\tau_{n-1}\mathfrak{u}\right) e^{(\lambda+\rho)H_{B_{n-1}}(\tau_{n-1}\mathfrak{u})}\theta_{\widetilde{\alpha}_{n-1}}(\mathfrak{u})d\mathfrak{u},$$

with $\widetilde{\alpha}_{n-1} = (a(u_1)^{-1}a(u_2)\alpha_1, \cdots, a(u_{n-2})^{-1}a(u_{n-1})\alpha_{n-2}) \in F_v^{n-2}$. Hence the function $W_{v,j}(\widetilde{\alpha}_{n-1}; \widetilde{\lambda})$ is a Whittaker function on GL_{n-1} associated to the principal series representation $\operatorname{Ind}_{B_{n-1}(F_v)}^{GL_{n-1}(F_v)}\left(\chi_{v,2}|\cdot|^{\lambda_2}, \cdots, \chi_{v,n}|\cdot|^{\lambda_n}\right)$ and parameter $\widetilde{\alpha}_{n-1} \in F_v^{n-2}$. Let $\chi_{v,l}^{ur} = |\cdot|_v^{i\nu_l}$, $\nu_l \in \mathbb{R}$, $1 \leq l \leq n$. Denote by $z_{k,l} = \lambda_k - \lambda_l + i\nu_k - i\nu_l \in \mathbb{C}$, $1 \leq k < l \leq n$.

Let $x_v \in F_v$. If v is an archimedean place, then define the functions a and s on F_v as follows:

$$a(x_v) = \begin{cases} (1+|x_v|_v^2)^{-1/2},\ s(x_v) = x_v a(x_v),\ \text{if } F_v \simeq \mathbb{R}; \\ (1+|x_v|_v)^{-1/2},\ s(x_v) = \overline{x}_v a(x_v),\ \text{if } F_v \simeq \mathbb{C}. \end{cases}$$

If v is a nonarchimedean place, then define the functions a and s on F_v as follows:

$$a(x_v) = \begin{cases} 1,\ \text{if } x_v \in \mathcal{O}_{F_v}; \\ x_v^{-1},\ \text{otherwise}; \end{cases} \qquad s(x_v) = \begin{cases} 0,\ \text{if } x_v \in \mathcal{O}_{F_v}; \\ 1,\ \text{otherwise}. \end{cases}$$

Lemma 48. *Let notation be as above. Assume that π_v is right K_v-finite. Let $v \in \Sigma_F$ be an arbitrary place and let $\pi_v = \operatorname{Ind}_{B(F_v)}^{GL_n(F_v)}\left(\chi_{v,1}, \cdots, \chi_{v,n}\right)$ be a principal series. Then the Whittaker function $W_v(\alpha; \lambda)$ is equal to*

$$\sum_{j\in \boldsymbol{J}} \int_{F_v^{n-1}} W_{v,j}(\widetilde{\alpha}_{n-1}; \widetilde{\lambda})\widetilde{\theta}_n(u_1, \cdots, u_{n-1}) \prod_{l=2}^{n} |a(u_{n-l+1})|_v^{1+z_{1,l}} \prod_{j=1}^{n-1} du_j, \tag{6.6}$$

where j runs over a finite index set depending only on the K_v-type of ϕ_v; and

$$\begin{aligned}\widetilde{\theta}_n(u_1, \cdots, u_{n-1}) =& \theta(\alpha_{n-1}u_{n-1} - \alpha_{n-2}c(u_{n-1})u_{n-2} - \alpha_{n-3}c(u_{n-2})u_{n-3} \\ & - \cdots - \alpha_{n-j}c(u_{n-j+1})u_{n-j} - \cdots - \alpha_1 c(u_2)u_1).\end{aligned}$$

Proof. Assume that v is an archimedean place. Let $r \in \mathbb{R}$ and $\beta \in [0, 2\pi)$. Then by a straightforward computation we have the Iwasawa decomposition

$$\begin{pmatrix} 1 & re^{i\beta} \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ \frac{re^{-i\beta}}{1+r^2} & 1 \end{pmatrix} \begin{pmatrix} e^{-i\beta}\sqrt{1+r^2} & \\ & \frac{e^{i\beta}}{\sqrt{1+r^2}} \end{pmatrix} \begin{pmatrix} \frac{e^{i\beta}}{\sqrt{1+r^2}} & \frac{re^{2i\beta}}{\sqrt{1+r^2}} \\ -\frac{re^{-2i\beta}}{\sqrt{1+r^2}} & \frac{e^{-i\beta}}{\sqrt{1+r^2}} \end{pmatrix}. \tag{6.7}$$

If v is a nonarchimedean place of F. We then fix an uniformizer ϖ_v of $F_v^\times$. For any $u \in F_v$, one can write $u = u^\circ \varpi_v^m$, for some $m \in \mathbb{Z}$, where $u^\circ \in \mathcal{O}_{F_v}^\times$. If $m \geq 0$, then $u \in \mathcal{O}_{F_v}$, implying that $\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} \in GL(n, \mathcal{O}_{F_v})$. If $m < 0$, then one has that

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix} = \begin{pmatrix} 1 & \\ u^{-1} & 1 \end{pmatrix} \begin{pmatrix} u & \\ & u^{-1} \end{pmatrix} \begin{pmatrix} u^{-1} & 1 \\ 1 & 0 \end{pmatrix}. \tag{6.8}$$

Let v be arbitrary and $u \in F_v$. For any $2 \leq j \leq n$, $1 \leq l \leq 4$, let $M_l(u) = M_{l,j}(u)$ be the matrix defined by

$$M_1(u) = \begin{pmatrix} I_{n-j} & & & \\ & 1 & u & \\ & & 1 & \\ & & & I_{j-2} \end{pmatrix}, \quad M_2(u) = \begin{pmatrix} I_{n-j} & & & \\ & a(u)^{-1} & & \\ & s(u) & a(u) & \\ & & & I_{j-2} \end{pmatrix};$$

$$M_3(u) = \begin{pmatrix} I_{n-j} & & & \\ & a(u)^{-1} & & \\ & & a(u) & \\ & & & I_{j-2} \end{pmatrix}, \quad M_4(u) = \begin{pmatrix} I_{n-j} & & & \\ & 1 & & \\ & c(u) & 1 & \\ & & & I_{j-2} \end{pmatrix},$$

where $c(u) = a(u)s(u)$. Let $\tau_{n,j} = w_{n-1} \cdots w_{n-j+1}$, $2 \leq j \leq n$.

Denote by $w_0 = Id_n$, the identity element. Then one has $N_2^0(\mathfrak{u}) = \mathfrak{u}$, and for any $j \geq 2$, $N_j^0(\mathfrak{u}) = N_j^1(\mathfrak{u}) M_1(u_{n-j+1,n-j+2})$ and $w_{n-j+1} N_j^1(\mathfrak{u}) w_{n-j+1} = N_{j+1}^0(\mathfrak{u}')$, where $\mathfrak{u}' = (u'_{k,l})_{1 \leq k,l \leq n} \in N(F_v)$ is defined by $u'_{k,l} = u_{k,l-1}$ if $l = n-j+2$; $u'_{k,l} = u_{k,l+1}$ if $l = n-j+1$; $u'_{k,l} = u_{k+1,l}$ if $k = n-j+2$; and $u'_{k,l} = u_{k,l}$, otherwise. Now applying (6.8) one then has that $M_1(u_{n+1-j,n-j+2}) = M_2(u_{n+1-j,n-j+2})k = M_4(u_{n+1-j,n-j+2}) M_3(u_{n+1-j,n-j+2})k$, where $k \in K(F_v)$, the maximal compact subgroup of $\mathrm{GL}(n, F_v)$. Consequently, we have $\tau_n N_j^0(\mathfrak{u}) = \tau_n N_j^1(\mathfrak{u}) M_1(u_{n-j+1,n-j+2}) = \tau_n w_{n-j+1} N_{j+1}^0(\mathfrak{u}') w_{n-j+1} M_1(u_{n-j+1,n-j+2})$, which is equal to

$$\tau_n w_{n-j+1} N_{j+1}^0(\mathfrak{u}') w_{n-j+1} M_4(u_{n+1-j,n-j+2}) M_3(u_{n+1-j,n-j+2})k.$$

Note that we have $w_{n-j+1} M_4(u_{n+1-j,n-j+2}) = M_1(u_{n+1-j,n-j+2}) w_{n-j+1}$ and

$$N_{j+1}^0(\mathfrak{u}') M_1(u_{n+1-j,n-j+2}) = M_1(u_{n+1-j,n-j+2}) N_{j+1}^0(\tilde{\mathfrak{u}}),$$

where $\tilde{\mathfrak{u}} = (\tilde{u}_{k,l})_{1 \leq k,l \leq n} \in N(F_v)$ is defined by $\tilde{u}_{k,l} = u'_{k,l} + u_{n+1-j,n-j+2} u'_{k+1,l}$ if $k = n-j+1$; $\tilde{u}_{k,l} = u'_{k,l} + u_{n+1-j,n-j+2} u'_{k,l-1}$ if $l = n-j+2$; and $\tilde{u}_{k,l} = u'_{k,l}$ otherwise.

Therefore,

$$\begin{aligned}\tau_n N_j^0(\mathfrak{u}) &= \tau_n w_{n-j+1} M_1(u_{n+1-j,n-j+2}) N_{j+1}^0(\tilde{\mathfrak{u}}) w_{n-j+1} M_3(u_{n+1-j,n-j+2}) k \\ &= M_1'(u_{n+1-j,n-j+2}) \tau_n w_{n-j+1} N_{j+1}^0(\tilde{\mathfrak{u}}) M_3(u_{n+1-j,n-j+2})^{-1} w_{n-j+1} k,\end{aligned}$$

where $M_1'(u_{n+1-j,n-j+2}) = \tau_n w_{n-j+1} M_1(u_{n+1-j,n-j+2}) w_{n-j+1} \tau_n^{-1}$. Then one has that $M_1'(u_{n+1-j,n-j+2}) \in N(F_v)$. Let $2 \leq j \leq n$ and $\phi_{v,\lambda} = \phi_v e^{(\lambda+\rho)H_B(\cdot)}$. Then

$$\begin{aligned}W_v(\alpha;\lambda) &= \int_{N(F_v)} \phi_v\left(\tau_n N_2^0(\mathfrak{u})\right) e^{(\lambda+\rho)H_B(\tau_n N_2^0(\mathfrak{u}))} \theta_\alpha(\mathfrak{u}) d\mathfrak{u} \\ &= \int_{N(F_v)} \phi_{v,\lambda}\left(M_1'(u_{n-1,n}) \tau_n w_{n-1} N_3^0(\tilde{\mathfrak{u}}) M_3(u_{n-1,n})^{-1} w_{n-1} k\right) \theta_\alpha(\mathfrak{u}) d\mathfrak{u} \\ &= \int_{N(F_v)} \phi_{v,\lambda}\left(M_3^{\tau_n}(u_{n-1,n}) \tau_n w_{n-1} N_3^0(\mathfrak{u}^*) w_{n-1} k\right) \theta_\alpha^{(2)}(\mathfrak{u}) d\mathfrak{u},\end{aligned}$$

where $M_3^{\tau_n}(u_{n-1,n}) = \operatorname{diag}(a(u_{n-1,n}), a(u_{n-1,n})^{-1}, I_{n-2})$; $\theta_\alpha^{(2)}(\mathfrak{u}) = \theta(\alpha_1 u_{12} + \cdots + \alpha_{n-3} u_{n-3,n-2} + \alpha_{n-2} a(u_{n-1,n}) u_{n-2,n-1} + \alpha_{n-1} u_{n-1,n}) \cdot \theta(-\alpha_{n-2} c(u_{n-1,n}) u_{n-2,n})$.

Denote by $M_2^{\tau_n}(\mathfrak{u}) = I_n$. Let $j \geq 3$ and M_l, $3 \leq l \leq j$, be matrices. Denote by $\prod_{l=2}^{j} M_l$ the matrix $M_2 \cdots M_j$. Define the matrix

$$M_j^{\tau_n}(\mathfrak{u}) = \prod_{l=3}^{j} \begin{pmatrix} a(u_{n-l+2,n}) & & & \\ & I_{l-3} & & \\ & & a(u_{n-l+2,n})^{-1} & \\ & & & I_{n-l+1} \end{pmatrix}.$$

Write $a_{k,l}$ for $a(u_{k,l})$; and $c_{k,l}$ for $c(u_{k,l})$. Let $\beta_k(\mathfrak{u}) = a_{k,n}^{-1} a_{k+1,n}$. Denote by $\theta_\alpha^{(j)}(\mathfrak{u})$ the product of

$$\begin{aligned}\theta(\alpha_1 u_{12} + \cdots + &\alpha_{n-j-1} u_{n-j-1,n-j} + \alpha_{n-j} a_{n-j+1,n} u_{n-j,n-j+1} \\ &+ \alpha_{n-j+1} \beta_{n-j+1}(\mathfrak{u}) u_{n-j+1,n-j+2} + \cdots + \alpha_{n-2} \beta_{n-2}(\mathfrak{u}) u_{n-2,n-1})\end{aligned}$$

and $\theta(\alpha_{n-1} u_{n-1,n} - \alpha_{n-2} c_{n-1,n} u_{n-2,n} - \alpha_{n-3} c_{n-2,n} u_{n-3,n} - \cdots - \alpha_{n-j} c_{n-j+1,n} u_{n-j,n})$, for any $2 \leq j \leq n-1$; and $\theta_\alpha^{(n)}(\mathfrak{u}) = \theta(\alpha_1 \beta_1(\mathfrak{u}) u_{12} + \cdots + \alpha_{n-j} \beta_{n-j}(\mathfrak{u}) u_{n-j,n-j+1} + \cdots + \alpha_{n-2} \beta_{n-2}(\mathfrak{u}) u_{n-2,n-1}) \cdot \theta(\alpha_{n-1} u_{n-1,n} - \alpha_{n-2} c_{n-1,n} u_{n-2,n} - \alpha_{n-3} c_{n-2,n} u_{n-3,n} - \cdots - \alpha_{n-j} c_{n-j+1,n} u_{n-j,n} - \cdots - \alpha_1 c_{2,n} u_{1,n})$. Let

$$W_v^{(j)}(\alpha;\lambda) = \int_{N(F_v)} \phi_{v,\lambda}\left(M_{j+1}^{\tau_n}(\mathfrak{u}) \tau_n \tau_{n,j} N_{j+1}^0(\mathfrak{u}^*) k_{j+1}(\mathfrak{u})\right) \theta_\alpha^{(j)}(\mathfrak{u}) \prod_{l=2}^{j} a_{n-l+1,n}^{2-l} d\mathfrak{u}.$$

where $k_{j+1}(\mathfrak{u}) = \tau_{n,j}^{-1} k_j(\mathfrak{u})$ and $k_2(\mathfrak{u}) = k$. Then $W_v(\alpha;\lambda) = W_v^{(2)}(\alpha;\lambda)$. Let $N_{j+1}^*(\mathfrak{u}^*) = (u_{k,l}'')_{1 \leq k,l \leq n}$ such that $u_{k,l}'' = 0$ if $(k,l) = (n-j+1,n)$; and $u_{k,l}'' = u_{k,l}^*$

otherwise. Let $M_4 = M_4(u_{n-j+1,n})$, $M_3 = M_3(u_{n-j+1,n})$. Then a changing of variables leads to that $W_v^{(j)}(\alpha;\lambda)$ is equal to

$$\int_{N(F_v)} \phi_{v,\lambda}\left(M_{j+1}^{\tau_n}(\mathfrak{u})\tau_n\tau_{n,j}N_{j+1}^{*}(\mathfrak{u}^*)M_4M_3k_{j+1}(\mathfrak{u})\right)\theta_\alpha^{(j)}(\mathfrak{u})\prod_{l=2}^{j}a_{n-l+1,n}^{2-l}d\mathfrak{u} =$$

$$\int_{N(F_v)} \phi_{v,\lambda}\left(M_{j+1}^{\tau_n}(\mathfrak{u})M_4'\tau_n\tau_{n,j+1}N_{j+2}^{0}(\mathfrak{u}'')w_{n-j}M_3k_{j+1}(\mathfrak{u})\right)\theta_\alpha^{(j)}(\mathfrak{u})\prod_{l=2}^{j}a_{n-l+1,n}^{2-l}d\mathfrak{u},$$

where $W_4' \in N(F_v)$. Since $\phi_{v,\lambda}$ is left $N(F_v)$-invariant, the right hand side of the above equality is equal to

$$\int_{N(F_v)} \phi_{v,\lambda}\left(M_{j+1}^{\tau_n}(\mathfrak{u})\tau_nM_3\tau_{n,j+1}N_{j+2}^{0}(\mathfrak{u})w_{n-j}k_{j+1}(\mathfrak{u})\right)\theta_\alpha^{(j+1)}(\mathfrak{u})\prod_{l=2}^{j}a_{n-l+1,n}^{2-l}d\mathfrak{u},$$

implying that for any $2 \le j \le n-2$, one has $W_v^{(j)}(\alpha;\lambda) = W_v^{(j+1)}(\alpha;\lambda)$. By our definition of $\theta_\alpha^{(n)}$, a similar computation to the above shows that $W_v^{(n-1)}(\alpha;\lambda) = W_v^{(n)}(\alpha;\lambda)$, namely, one has that $W_v(\alpha;\lambda)$ is equal to

$$\int_{N(F_v)} \phi_{v,\lambda}\left(M_{n+1}^{\tau_n}(\mathfrak{u})\tau_n\tau_{n,n}N_{n+1}^{0}(\mathfrak{u}^*)k_{n+1}(\mathfrak{u})\right)\theta_\alpha^{(n)}(\mathfrak{u})\prod_{l=2}^{n}a_{n-l+1,n}^{2-l}d\mathfrak{u}. \tag{6.9}$$

By definition, one has, for any $\phi_{v,\lambda} \in \pi_{v,\lambda}$, that

$$\phi_v(t_vx_v) = \prod_{j=1}^{n}\chi_{v,j}(t_{v,j})|t_{v,j}|_v^{\frac{n+1}{2}-j+\lambda_j}\cdot\phi_v(x_v),\ x_v \in GL(n,F_v). \tag{6.10}$$

Substituting (6.10) into (6.9) one then sees that $W_v(\alpha;\lambda)$ is equal to

$$\int_{N(F_v)} \phi_{v,\lambda}\left(\tau_n\tau_{n,n}N_{n+1}^{0}(\mathfrak{u}^*)k_{n+1}(\mathfrak{u})\right)\theta_\alpha^{(n)}(\mathfrak{u})\prod_{l=2}^{n}\chi_{1,l}(a_{n-l+1,n})a_{n-l+1,n}^{1+\lambda_1-\lambda_l}d\mathfrak{u}, \tag{6.11}$$

where $\chi_{1,l}(a_{n-l+1,n}) = \chi_{v,1}(a_{n-l+1,n})\chi_{v,l}(a_{n-l+1,n})^{-1}$. Since π_v is right K_v-finite, one then sees, according to (6.11), that $W_v(\alpha;\lambda)$ is equal to

$$\sum_{j\in\boldsymbol{J}}\int_{N(F_v)} \phi_{v,\lambda}^{(j)}\left(\tau_n\tau_{n,n}N_{n+1}^{0}(\mathfrak{u}^*)\right)\theta_\alpha^{(n)}(\mathfrak{u})\prod_{l=2}^{n}\chi_{1,l}(a_{n-l+1,n})a_{n-l+1,n}^{1+\lambda_1-\lambda_l}d\mathfrak{u}, \tag{6.12}$$

where J is a finite set of indexes, whose cardinality depends only on the K_v-finite type of π_v; and each $\phi_{v,\lambda}^{(j)} \in \pi_{v,\lambda}$. Let $W_{v,j}(\alpha;\lambda)$ be the summand of (6.12) corresponding to the index $j \in \boldsymbol{J}$. Let $\widetilde{\lambda}_j = \lambda_j + \lambda_1/(n-1), 2 \le j \le n$. Denote by B_{n-1} the standard Borel subgroup of GL_{n-1} and N_{n-1} the unipotent of B_{n-1}. Then a change of variables implies that $W_{v,j}(\alpha;\lambda)$ is equal to

$$\int_{F_v^{n-1}} W_{v,j}(\widetilde{\alpha}_{n-1};\widetilde{\lambda})\widetilde{\theta}_n(u_1,\cdots,u_{n-1})\prod_{l=2}^{n}\chi_{1,l}(a_{n-l+1})|a_{n-l+1}|_v^{1+\lambda_1-\lambda_l}\prod_{j=1}^{n-1}du_j.$$

Then Lemma 48 follows. □

Let $v \in \Sigma_F$ be a fixed nonarchimedean place, let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v = N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Denote by

$$R_v(s, W_{1,v}, W_{2,v}; \lambda) := \frac{\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1.$$

Proposition 49 (Nonarchimedean Case)**.** *Let notation be as before. Let $s \in \mathbb{C}$ be such that* $\mathrm{Re}(s) > 1$.*Then we have*

(a) *$R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$.*

(b) *We have the local functional equation*

$$\frac{\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})} = \varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta) \cdot \frac{\Psi_v(1-s, \widetilde{W}_{1,v}, \widetilde{W}_{2,v}; -\bar{\lambda}, \widehat{\Phi}_v)}{L_v(1-s, \widetilde{\pi}_{-\bar{\lambda},v} \otimes \bar{\tau}_v \times \pi_{\lambda,v})},$$

where $\varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : 1 \leq i \leq r\}$.

Proof. We shall only prove Part (a), since Part (b) will follow form [JS81].

Let $T(F_v)$ be the maximal torus of $G(F_v)$, and for any $m \in \mathbb{Z}$, let $T^{(m)}(F_v) = \{t \in T(F_v) : |\det t|_{F_v} = q_v^{-m}\}$. Using Iwasawa decomposition and the fact that $W_{i,v}$ and Φ_v are right $G(\mathcal{O}_{F,v})$-finite, we can rewrite $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ as

$$\sum_{j \in J} \int_{T(F_v)} W_{1,v}^{(j)}(a_v; \lambda) \overline{W_{2,v}^{(j)}\left(a_v; -\bar{\lambda}\right)} \Phi_{j,v}(\eta a_v) \tau_v(\det a_v) \delta_T^{-1}(a_v) |\det a_v|_{F_v}^s da_v,$$

where the sum over a finite set J, $W_{i,v}^{(j)}(a_v; \lambda)$ is a Whittaker function associated to some smooth functions in $\mathcal{H}_{P,\chi}$, $1 \leq i \leq 2$, and $\Phi_{j,v}$ is some Schwartz-Bruhat function. Note that for $1 \leq i \leq 2$ and $j \in J$, $W_{i,v}^{(j)}(x_v; \lambda)$ is right $G(\mathcal{O}_{F,v})$-finite. So there exists a compact subgroup $N_{0,v} \subseteq G(\mathcal{O}_{F,v}) \cap N(F_v)$, depending only on φ, such that $W_{i,v}^{(j)}(t_v u_v; \lambda) = W_{i,v}^{(j)}(t_v; \lambda)$, for all $t_v \in T(F_v)$ and $u_v \in N_{0,v}$. On the other hand, $W_{i,v}^{(j)}(t_v u_v; \lambda) = \theta_{t_v}(u_v) W_{i,v}^{(j)}(t_v; \lambda)$, where $\theta_{t_v}(n_v) = \theta(t_v n_v t_v^{-1})$, for any $n_v \in N(F_v)$. But then, there exists a constant C_v depending only on $N_{0,v}$ and θ (hence not on λ) such that $\theta_{t_v}(u_v) = 1$ if and only if $|\alpha_i(t_v)| \leq C_v$, where α_i's are the simple roots of $G(F)$. Thus each $W_{i,v}^{(j)}(x_v; \lambda)$ is compactly supported for a fixed $\lambda \in i\mathfrak{a}_P / i\mathfrak{a}_G$. Therefore, for a fixed λ, $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is a formal Laurent series in q_v^{-s}. Indeed, one can chose some nonnegative integer M independent of λ (but depending on π and φ), such that

$$\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = \sum_{m \geq -M} \Psi_v^{(m)}\left(W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) \cdot q_v^{-ms},$$

where $\Psi_v^{(m)}\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is defined by the integral

$$\sum_{j\in J}\int_{T^{(m)}(F_v)} W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}\left(a_v;-\bar{\lambda}\right)}\Phi_{j,v}(\eta a_v)\tau_v(\det a_v)\delta_T^{-1}(a_v)da_v.$$

Apply the above analysis on $\operatorname{supp} W_{i,v}(a_v;\lambda)$, we see similarly that

$$\operatorname{supp} W_{i,v}^{(j)}(a_v;\lambda)\subseteq\{t\in T^{(m)}(F_v):\ |\alpha_l(t)|_{F_v}\leq C_v^{(j)},\ 1\leq l\leq n-1\}$$

for some constants $C_v^{(j)}$. Hence, for each $j\in J$, $m\geq -N$ and $a_v\in T^{(m)}(F_v)$, the function $a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}\left(a_v;-\bar{\lambda}\right)}$ is analytic and is a formal Laurent series in $\{q_v^{-\lambda_i}\ :\ 1\leq i\leq r\}$ by (2.5.2) of [JS81], and the function

$$a_v\mapsto W_{1,v}^{(j)}(a_v;\lambda)\overline{W_{2,v}^{(j)}\left(a_v;-\bar{\lambda}\right)}\Phi_{j,v}(\eta a_v)\tau(\det a_v)\delta_T^{-1}(a_v)$$

is locally constant. Therefore, $\Psi_v^{(m)}\left(W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is an analytic function of λ and is a formal Laurent series in $\{q_v^{-\lambda_i}\ :\ 1\leq i\leq r\}$.

Since $\pi_{\lambda,v}$ is of Whittaker type, we can use Theorem 2.7 of [JS81] to see that, for fixed $\lambda\in i\mathfrak{a}_P/i\mathfrak{a}_G$, $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)\cdot L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}\}$ with coefficients functions of λ. Moreover, $L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})^{-1}$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i}\ :\ 1\leq i\leq r\}$. So we can write

$$L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{\lambda,v})^{-1}=\sum_{|l|\leq N}Q_l(\lambda)q_v^{-ls},$$

where N is a positive integer and $Q_l(\lambda)$ are polynomials in $\{q_v^{\lambda_i}, q_v^{-\lambda_i}\ :\ 1\leq i\leq r\}$. Then for $\lambda\in i\mathfrak{a}_P/i\mathfrak{a}_G$, $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)\cdot L(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})^{-1}$ is equal to the sum over $m\geq -N-M$ of $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)q_v^{-ms}$, where

$$R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)=\sum_{\substack{i+j=m\\ |i|\leq N, j\geq -M}}Q_i(\lambda)\Psi_v^{(m)}\left(W_{1,v}, W_{2,v}; \lambda, \Phi_v\right).$$

Since the sum on the right hand side is finite, $R_l\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is analytic in λ. Moreover, it is a formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i}\ :\ 1\leq i\leq r\}$. Therefore, part (a) of Proposition 49 follows from Claim 50 below. □

Claim 50. *There exists some $M_0\in\mathbb{Z}$, independent of $\lambda\in i\mathfrak{a}_P/i\mathfrak{a}_G$, such that $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)=0$ for all $m\geq M_0$ and for all $\lambda\in i\mathfrak{a}_P/i\mathfrak{a}_G$. Moreover, for each $m\in\mathbb{Z}$, $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is a polynomial in $\{q_v^{\pm s}, q_v^{\pm\lambda_i}\ :\ 1\leq i\leq r\}$.*

Proof of Claim 50. Let $l \in \mathbb{Z}$. One then defines

$$\Lambda_l = \left\{\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G : \ R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = 0 \text{ for all } m \geq l\right\}.$$

Then each Λ_l is closed since $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = 0$ are analytic (hence continuous) in λ. Since $R_v(s,\lambda) = \sum_m R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) q_v^{-ms} \in \mathbb{C}[q_v^s, q_v^{-s}]$, for fixed $\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G$, there exists some $M(\lambda)$ such that $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = 0$ as long as $m \geq M(\lambda)$. Therefore, $i\mathfrak{a}_P/i\mathfrak{a}_G$ is covered by the union of all Λ_l. Noting that $i\mathfrak{a}_P/i\mathfrak{a}_G \simeq R^{r-1}$ is a Banach space, by Baire category theorem there exists some Λ_{l_0} having nonempty interior, $\mathrm{Int}(\Lambda_{l_0})$, say. Then for any $\lambda \in \mathrm{Int}(\Lambda_{l_0})$, $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = 0$ for any $m \geq l_0$. Since $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is analytic for any $l \in \mathbb{Z}$, $R_m\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = 0$ for all $\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G$, proving the first part. For the remaining part, we consider the functional equation (see [JS81]):

$$\frac{\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})} = \varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta) \cdot \frac{\Psi_v(1-s, \widetilde{W}_{1,v}, \widetilde{W}_{2,v}; -\bar{\lambda}, \widehat{\Phi}_v)}{L_v(1-s, \widetilde{\pi}_{-\bar{\lambda},v} \otimes \bar{\tau}_v \times \pi_{\lambda,v})},$$

where $\varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta)$ is a polynomial in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : \ 1 \leq i \leq r\}$.

We can interpret the functional as an identity between formal Laurent series in $\{q_v^{\lambda_i}, q_v^{-\lambda_i} \ : \ 1 \leq i \leq r\}$. The left hand side are formal Laurent series of the form $\sum_{m_1 \geq -M_1} q_v^{m_1\lambda_i}$, while the right hand side are formal Laurent series of the form $\sum_{m_2 \geq -M_2} q_v^{-m_2\lambda_i}$. Since they are equal, they must be both polynomials in $\{q_v^s, q_v^{-s}, q_v^{\lambda_i}, q_v^{-\lambda_i} : \ 1 \leq i \leq r\}$. Then the proof of Claim 50 follows. ◻

One will see that Proposition 49 is insufficient for our continuation in next few sections. Hence we need to compute $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ more explicitly. We will do principal series case below since this is the only case we need for the particular purpose of this thesis.

Lemma 51. *Let v be a nonarchimedean place of F. Let π_v be a principal series characters $\chi_{v,1}, \chi_{v,2}, \cdots, \chi_{v,n}$. Assume that π_v is right K_v-finite. Let $\alpha \in \mathbb{G}_m(F_v)^{n-1}$ and let $W_v(\alpha,\lambda)$ be a Whittaker function associated to $\pi_{v,\lambda}$ and α. Then $W_v(\alpha,\lambda)$ is of the form $\mathcal{B}_v(\alpha,\lambda)\mathcal{L}_v(\lambda)$, where $\mathcal{B}_v(\alpha,\lambda)$ is a holomorphic function, and*

$$\mathcal{L}_v(\lambda) = \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{v,i}\overline{\chi}_{v,j})^{-1}.$$

Proof. It follows from Lemma 48 and induction that Lemma 51 holds for any n if it holds for $n = 2$ case. Now we show that Lemma 51 holds for $n = 2$.

We may assume that $\chi_{1,2} = \chi_{v,1}\chi_{v,2}^{-1}$ is unramified. Otherwise, the local L-function $L(s, \chi_1\overline{\chi}_2)$ is trivial, and Lemma 51 follows from Part (a) of Proposition 49. According to (6.8) and the K_v-finiteness condition, one has

$$W_v(\alpha, \lambda) = \sum_{j\in \boldsymbol{J}} \sum_{l=1}^{\infty} c_j \int_{\varpi_v^{-l} O_v^{\times}} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)du + W_v^{\circ}(\alpha, \lambda),$$
$$W_v^{\circ}(\alpha, \lambda) = \sum_{j\in \boldsymbol{J}} c_j \int_{O_v} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)du = \sum_{j\in \boldsymbol{J}} c_j \int_{O_v} \theta(\alpha u)du,$$

where j runs over a finite set $\boldsymbol{J}$ and c_j's are constants; moreover, $\boldsymbol{J}$ and c_j's relay only on the K_v-type of π_v. For $u \in F_v^{\times}$, write $u = u^{\circ}\varpi_v^{l}$, where $u^{\circ} \in O_v^{\times} = O_{F_v}^{\times}$, and $l \in \mathbb{Z}$. Write $\alpha = \alpha^{\circ}\varpi_v^{k}$, where $\alpha^{\circ} \in O_v^{\times}$. Recall that by definition the one sees that the conductor of θ is precisely the inverse different of F_v, which is $\mathfrak{D}_{F_v}^{-1} = \{x_v \in F_v : \mathrm{tr}_{F_v/\mathbb{Q}_p}(x_v) \in \mathbb{Z}_p\}$, where p is the characteristic of residue field of O_v. Note that $\mathfrak{D}_{F_v}^{-1}$ is a $\mathbb{Z}_p$-module of F_v and thus has the representation $\mathfrak{D}_{F_v}^{-1} = \varpi_v^{-d}O_v$, where $d \in \mathbb{N}_{\geq 0}$. Hence one sees that

$$I = \int_{O_v} \theta(\alpha u)du = \int_{O_v} \theta(\alpha^{\circ} u \varphi_v^{k})du = \int_{O_v} \theta(u\varpi_v^{k})du$$

is vanishing if $k \leq -d-1$. Clearly $I = 1$ if $k \geq -d$. Note that

$$\int_{\varpi_v^{-l} O_v^{\times}} \chi_{12}(a(u))|a(u)|_v^{1+\lambda_1-\lambda_2}\theta(\alpha u)du = \chi_{12}(\varpi_v)^{l}|\varpi_v|_v^{(1+\lambda_1-\lambda_2)l} \int_{\varpi_v^{-l} O_v^{\times}} \theta(\alpha u)du$$

is vanishing if $l \geq k+d+2$. Let $q_v = |\varpi_v|_v^{-1}$. Then one sees that

$$W_v(\alpha, \lambda) = C + C\sum_{l=1}^{k+d}(1-q_v^{-1})\chi_{12}(\varpi_v)^{l} q_v^{-(\lambda_1-\lambda_2)l} + C \cdot W_{re}, \tag{6.13}$$

where C is a constant depending only on F and K_v-type of ϕ_v and

$$W_{re} = \chi_{12}(\varpi_v)^{k+d+1} q_v^{-(k+d+1)(1+\lambda_1-\lambda_2)} \int_{\varpi_v^{-k-d-1} O_v^{\times}} \theta(u\varpi_v^{k})du. \tag{6.14}$$

Since θ is nontrivial on $\varpi_v^{-d-1}O_v$, then $\int_{\varpi_v^{-k-d-1} O_v} \theta(u\varpi_v^{k})du = 0$. Note that $\varpi_v^{-k-d-1}O_v^{\times} = \varpi_v^{-k-d-1}O_v \setminus \varpi_v^{-k-d}O_v$. Then one has that

$$\int_{\varpi_v^{-k-d-1} O_v^{\times}} \theta(u\varpi_v^{k})du = \int_{\varpi_v^{-k-d-1} O_v} \theta(u\varpi_v^{k})du - \int_{\varpi_v^{-k-d} O_v} \theta(u\varpi_v^{k})du$$
$$= -\int_{\varpi_v^{-k-d} O_v} \theta(u\varpi_v^{k})du = -\operatorname{vol}(\varpi_v^{-k-d}O_v) = -q_v^{k+d}.$$

Then it follows form (6.13) and (6.14) that $W_v(\alpha,\lambda)$ is equal to C multiplying

$$L = 1 + \sum_{l=1}^{k+d}(1-q_v^{-1})\chi_{12}(\varpi_v)^l q_v^{-(\lambda_1-\lambda_2)l} - \chi_{12}(\varpi_v)^{k+d+1}q_v^{-(k+d+1)(\lambda_1-\lambda_2)-1}.$$

An elementary computation leads to the identity

$$L = (1-\chi_{12}(\varpi_v)q_v^{-(1+\lambda_1-\lambda_2)}) \cdot P(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)}), \tag{6.15}$$

where $P(z) = (1-z^{k+d+1})\cdot(1-z)^{-1} = 1 + z + \cdots + z^{k+d} \in \mathbb{C}[z]$.

Therefore, one has that $W_v(\alpha,\lambda) = CQ(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)})\cdot L_v(1+\lambda_1-\lambda_2,\chi_{12})$, where $Q(z) = P(z)$ if $k \geq -d$; $Q(z) \equiv 0$, otherwise. Taking $\mathcal{B}_v(\alpha,\lambda)$ to be the function $CQ(\chi_{12}(\varpi_v)q_v^{-(\lambda_1-\lambda_2)})$ we then obtain Lemma 51 in $n = 2$ case. The general case follows from this and induction, since integral with respect to $\chi_{l,j}$ is exactly the same as above, $1 \leq l < j \leq n$. □

Proposition 52 (Principal Series Case: nonarchimedean)**.** *Let v be a nonarchimedean place of F. Let π_v be a principal series characters $\chi_{1,v},\chi_{2,v},\cdots,\chi_{n,v}$. Assume that π_v is right K_v-finite. Then the function $R_v(s,W_{1,v},W_{2,v};\lambda)$ is of the form $Q_v(s,\lambda)\mathcal{L}_v(\lambda)$, where the function $Q_v(s,\lambda) \in \mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$; and $\mathcal{L}_v(\lambda)$ is defined to be*

$$\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j,\chi_{i,v}\overline{\chi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j,\overline{\chi}_{i,v}\chi_{j,v})^{-1}.$$

Proof. By Lemma 51 the function

$$W_v(x_v;\phi_{1,v},\lambda)\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j,\chi_{i,v}\overline{\chi}_{j,v}) \in \mathbb{C}[q_v^{\pm\lambda_j} : 1 \leq j \leq n].$$

Then applying expansions in [JS81] and changing orders of summations we see that

$$R_v(s,W_{1,v},W_{2,v};\lambda)\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j,\chi_{i,v}\overline{\chi}_{j,v})\cdot L_v(1-\lambda_i+\lambda_j,\overline{\chi}_{i,v}\chi_{j,v})$$

lies in $\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq n]$. Done. □

Corollary 53. *Let $v \in \Sigma_{F,\mathrm{fin}}$ be a finite place such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$. Assume that $\phi_{1,v} = \phi_{2,v} = \phi_v^\circ$ be the unique $G(\mathcal{O}_{F,v})$-fixed vector in the space of π_v such that $\phi_v^0(e) = 1$. Then $R_v(s,W_{1,v},W_{2,v};\lambda)$ is equal to*

$$\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j,\pi_{i,v}\times\widetilde{\pi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j,\widetilde{\pi}_{i,v}\times\pi_{j,v})^{-1}.$$

In particular, $R_v(s,\lambda)$ is independent of s.

Proof. Fix $\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G$. Let $W_{i,v}^{\circ}$ be the $G(O_{F,v})$-invariant vectors such that $W_{i,v}^{\circ}(e) = 1$, $1 \leq i \leq 2$. Then by the computation from [JS81], we know that

$$\Psi_v\left(s, W_{1,v}^{\circ}, W_{2,v}^{\circ}; \lambda, \Phi_v^{\circ}\right)/L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v}) = 1,$$

where Φ_v° is the characteristic function of $O_{F,v}^n$. Then Corollary (53) follows from induction and unramified computations of nonconstant Fourier coefficients of Eisenstein series (see Chap. 7 of [Sha10]). $\square$

Now we move to the archimedean case. In the current state of affairs the local L-functions $L_\infty(s, \pi_\lambda \times \tau \times \widetilde{\pi}_{-\lambda}) = \prod_{v|\infty} L_v(s, \pi_{\lambda,v} \times \tau_v \times \widetilde{\pi}_{-\lambda,v})$ are not defined intrinsically through the integrals as nonarchimedean case, but rather extrinsically through the Langlands correspondence and then related to the integrals. Let $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) = 2^{1-s}\pi^{-s}\Gamma(s)$. Then by Langlands classification (e.g. see [Kna94]), each archimedan L-function $L_v(s, \pi_{\lambda,v} \times \tau_v \times \pi_{-\lambda,v})$ is of the form

$$\prod_{i \in I} \Gamma_{\mathbb{R}}(s + \mu_i) \prod_{j \in J} \Gamma_{\mathbb{C}}(s + \mu_j'), \tag{6.16}$$

where I and J are finite set of inters satisfying $\#I + \#J \leq n$; $\mu_i, \mu_j' \in \mathbb{C}$.

Combining results from [Jac09] and well known estimates on archimedean Satake parameters one concludes the following result.

Proposition 54 (Archimedean Case)**.** *Let notation be as before. Let $v \in \Sigma_{F,\infty}$ be an archimedean place. Then we have*

(a) $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ *converges absolutely and normally in the right half plane* $\{s \in \mathbb{C} : \operatorname{Re}(s) > 1 - 2/(n^2+1)\}$, *uniformly in $\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G$. Moreover, it is bounded at infinity in any strip of finite width.*

(b) *The function $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is a holomorphic function of s and λ. Hence,* $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right) = R_v(s, W_{1,v}, W_{2,v}; \lambda) L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})$ *admits a meromorphic continuation to the whole complex plane.*

(d) *We have the local functional equation*

$$\frac{\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})} = \varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta) \cdot \frac{\Psi_v(1-s, \widetilde{W}_{1,v}, \widetilde{W}_{2,v}; -\bar{\lambda}, \widehat{\Phi}_v)}{L_v(1-s, \widetilde{\pi}_{-\bar{\lambda},v} \otimes \bar{\tau}_v \times \pi_{\lambda,v})},$$

where $\varepsilon(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}, \theta)$ is a holomoprhic function.

Remark 55. *It follows from Lemma 5.4 in [Jac09] that if both* π *is tempered, then the Rankin-Selberg convolution* $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ *converges absolutely and normally in the right half plane* $\{s \in \mathbb{C} : \operatorname{Re}(s) > 0\}$, *uniformly in* $\lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G$.

We need a more explicit description of the polynomial $Q_v(s, \lambda)$ in Proposition 54 when π_v is a principal series. To start with, we recall the definition of archimedean L-function associated to a unitary Hecke character. If $F_v \simeq \mathbb{R}$, then the only possible choices for a unitary Grössencharacter are $x_v \mapsto \operatorname{sgn}(x_v)^k |x_v|_v^{i\nu}$ for $k \in \{0, 1\}$ and $\nu \in \mathbb{R}$. If $F_v \simeq \mathbb{C}$, then the only possible choices for a unitary Grössencharacter are $x_v \mapsto (x_v \cdot |x_v|_v^{-1/2})^k |x_v|_v^{i\nu}$ for $k \in \mathbb{Z}$ and $\nu \in \mathbb{R}$. Furthermore, since the units are killed by such a character, then the sum of those ν's must be 0. The Gamma factors at the real infinite places are $\Gamma((s + i\nu + k)/2)$ and at the complex places are $\Gamma(s + i\nu + |k|/2)$. To prove Proposition 58, we need some preparation.

Lemma 56. *Let* v *be an archimedean place of* F. *Let* π_v *be a principal series characters* $\chi_{v,1}, \chi_{v,2}, \cdots, \chi_{v,n}$. *Assume that* π_v *is right* K_v*-finite. Let* $\alpha \in \mathbb{G}_m(F_v)^{n-1}$ *and let* $W_v(\alpha, \lambda)$ *be a Whittaker function associated to* $\pi_{v,\lambda}$ *and* α. *Then* $W_v(\alpha, \lambda)$ *is of the form* $\mathcal{B}_v(\alpha, \lambda)\mathcal{L}_v(\lambda)$, *where* $\mathcal{B}_v(\alpha, \lambda)$ *is a holomorphic function, and*

$$\mathcal{L}_v(\lambda) = \prod_{1 \leq i < r} \prod_{i < j \leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{v,i}^{ur} \overline{\chi}_{v,j}^{ur})^{-1},$$

where for any $\chi_{v,l}$, $1 \leq l \leq n$, $\chi_{v,l}^{ur} = \chi_{v,l} \circ |\cdot|_v^{1/[F_v:\mathbb{R}]}$ *is the unramified part of* $\chi_{v,l}$.

Proof. It follows from (6.6) and induction that Lemma 56 holds for any n if it holds for $n = 2$ case. Now we show that Lemma 56 holds for $n = 2$.

By (6.7) it suffices to show that for any $\alpha, z \in \mathbb{C}$, one has

$$\int_{F_v} |a(u)|_v^z \theta_v(\alpha u) du \sim \Gamma_{F_v}(z + 1)^{-1}. \tag{6.17}$$

Since the proof is similar, we only consider real places. Let $F_v \simeq \mathbb{R}$. Then

$$\begin{aligned}
\Gamma_{\mathbb{R}}(z + 1) \int_{F_v} |a(u)|_v^z \theta_v(\alpha u) du &= \int_0^\infty \int_{\mathbb{R}} e^{-t} t^{\frac{z+1}{2}} \cdot \frac{e^{2\pi i \alpha u}}{(1 + u^2)^{\frac{z+1}{2}}} du \frac{dt}{t} \\
&= \int_0^\infty \int_{\mathbb{R}} e^{-t(1+u^2)+2\pi i \alpha u} t^{\frac{z+1}{2}} du \frac{dt}{t} \\
&= \pi^{\frac{z+1}{2}} |\alpha|^{z/2} \int_0^\infty e^{-\pi|\alpha|(t+t^{-1})} t^{z/2} \frac{dt}{t}.
\end{aligned}$$

Since the function $g(t) = e^{-\pi|\alpha|(t+t^{-1})}$ is Schwartz, then its Mellin transform

$$\int_0^\infty g(t)t^{z/2}d^\times t$$

is entire. Hence one has a continuation of Whittaker functions and proves (6.17). □

Remark 57. *Note that in the proof of Lemma 56, $\int_0^\infty g(t)t^{z/2}d^\times t \neq 0$ for any $\alpha, z \in \mathbb{C}$. Hence $\int_{F_v} |a(u)|_v^z \theta_v(\alpha u)du$ never vanishes. Then by induction one concludes that $W_v(\alpha, \lambda)/\mathcal{L}_v(\lambda) \neq 0$, for any α and λ.*

Combining Lemma 56 and [JS81] one then concludes the following result.

Proposition 58 (Principal Series Case: archimedean)**.** *Let v be an archimedean place of F. Let π_v be a principal series characters $\chi_{v,1}, \chi_{v,2}, \cdots, \chi_{v,n}$. Then the function $R_v(s, W_{1,v}, W_{2,v}; \lambda)$ is of the form $Q_v(s, \lambda)\mathcal{B}_v(\lambda)\mathcal{L}_v(\lambda)$, where $Q_v(s, \lambda)$ is a polynomial in s and λ_j, $1 \leq j \leq n$; $\mathcal{B}_v(\lambda)$ is a holomorphic function, and*

$$\mathcal{L}_v(\lambda) = \prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1 + \lambda_i - \lambda_j, \chi_{v,i}^{ur}\overline{\chi}_{v,j}^{ur})^{-1} \cdot L_v(1 - \lambda_i + \lambda_j, \overline{\chi}_{v,i}^{ur}\chi_{v,j}^{ur})^{-1}.$$

Remark 59. *Let $v \in \Sigma_{F,\infty}$ be an archimedean place such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$. Assume that $\phi_{1,v} = \phi_{2,v} = \phi_v^\circ$ be the unique $G(\mathcal{O}_{F,v})$-fixed vector in the space of π_v such that $\phi_v^0(e) = 1$. Applying the result in [Sta02] we then have that $\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)$ is equal to*

$$\prod_{1\leq i<r}\prod_{i<j\leq r} \frac{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}{L_v(1 + \lambda_i - \lambda_j, \pi_{i,v} \times \widetilde{\pi}_{j,v}) \cdot L_v(1 - \lambda_i + \lambda_j, \widetilde{\pi}_{i,v} \times \pi_{j,v})}.$$

In particular, $R_v(s, \lambda)$ is independent of s.

6.2 Global Theory for $\Psi(s, W_1, W_2; \lambda)$

In this section, we shall compute the global integral representation $\Psi(s, W_1, W_2; \lambda, \Phi)$ defined via (6.4).

Let $\widetilde{\pi}_{\lambda,v}$ be the contragredient of $\pi_{\lambda,v}$. Let ϖ_v be a uniformizer of $\mathcal{O}_{F,v}$, the ring of integers of F_v. Let $q_v = N_{F_v/\mathbb{Q}_p}(\varpi_v)$, where p is the rational prime such that v is above p. Denote by

$$R(s, W_1, W_2; \lambda) := \prod_{v\in\Sigma_F} \frac{\Psi_v\left(s, W_{1,v}, W_{2,v}; \lambda, \Phi_v\right)}{L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v})}, \quad \mathrm{Re}(s) > 1. \tag{6.18}$$

Then $R(s, W_1, W_2; \lambda)$ is holomorphic for any $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$. Putting the local computations together in the last section, we get

Proposition 60 (Global Case)**.** *Let notation be as before. Let $s \in \mathbb{C}$ be such that* $\mathrm{Re}(s) > 1$.*Then*

(a) *The integral* $\Psi(s, W_1, W_2; \lambda, \Phi)$ *converges absolutely in* $\mathrm{Re}(s) > 1$.

(b) *We have the global functional equation for* $\mathrm{Re}(s) > 1$:

$$\Psi\left(1 - s, \widetilde{W}_1, \widetilde{W}_2; \lambda, \tau^{-1}, \widehat{\Phi}\right) = \Psi(s, W_1, W_2; \lambda, \tau, \Phi).$$

(c) *For any fixed* $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, $R(s, W_1, W_2; \lambda)$ *can be continued to an entire function.*

Remark 61. *By Proposition 52, we see that if the irreducible representation* $\pi = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\pi_1, \cdots, \pi_n)$ *is a principal series which is K-finite, then*

$$\frac{\Psi_f(s, W_1, W_2; \lambda, \Phi)}{L(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})} = \mathcal{H}_f(s, \lambda) \prod_{\substack{1 \le i,j \le n \\ i \neq j}} \prod \frac{1}{L(1 + \lambda_i - \lambda_j, \pi_i \times \widetilde{\pi}_j)}, \tag{6.19}$$

where $\Psi_f(s, W_1, W_2; \lambda, \Phi)$ *is the finite component of* $\Psi(s, W_1, W_2; \lambda, \Phi)$ *and* $\mathcal{H}_f(s, \lambda)$ *is a finite product of polynomials, depending on the K-type of* π.

According to Proposition 58 we have, for each irreducible representation $\pi = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\pi_1, \cdots, \pi_n)$ *is a principal series which is K-finite, that*

$$\frac{\Psi_\infty(s, W_1, W_2; \lambda, \Phi)}{L_\infty(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})} = \mathcal{H}_\infty^*(s, \lambda) \prod_{\substack{1 \le i,j \le n \\ i \neq j}} \prod \frac{1}{L_\infty(1 + \lambda_i - \lambda_j, \pi_i^{ur} \times \widetilde{\pi}_j^{ur})}, \tag{6.20}$$

where $\mathcal{H}_\infty^*(s, \lambda)$ *is a product of polynomials and Mellin transform of Schwartz functions. Moreover,* $\mathcal{H}_\infty^*(s, \lambda)$ *is nonvanishing when* $\mathrm{Re}(s) \ge 1 - 2/(n^2 + 1)$. *Let v be an archimedean place. Let Σ_1 be the set of archimedean places such that $F_v \simeq \mathbb{R}$ and $\pi_{v,i}\widetilde{\pi}_{v,j}$ is ramified. Then for any $v \in \Sigma_1$, one has* $L_\infty(1 + \lambda_i - \lambda_j, \pi_i^{ur} \times \widetilde{\pi}_j^{ur}) L_\infty(1 + \lambda_i - \lambda_j, \pi_i \times \widetilde{\pi}_j)^{-1} = \Gamma_{\mathbb{R}}(1 + \lambda_i - \lambda_j) \cdot \Gamma_{\mathbb{R}}(2 + \lambda_i - \lambda_j)^{-1}$. *Let Σ_2 be the set of archimedean places such that $F_v \simeq \mathbb{C}$ and $\pi_{v,i}\widetilde{\pi_{v,j}}$ is ramified, then for any $v \in \Sigma_2$, one has* $L_\infty(1 + \lambda_i - \lambda_j, \pi_i^{ur} \times \widetilde{\pi}_j^{ur}) L_\infty(1 + \lambda_i - \lambda_j, \pi_i \times \widetilde{\pi}_j)^{-1} = \Gamma_{\mathbb{C}}(1 + \lambda_i - \lambda_j) \cdot \Gamma_{\mathbb{C}}(k_v + 1 + \lambda_i - \lambda_j)^{-1} = \prod_{l=0}^{k_v - 1}(l + 1 + \lambda_i - \lambda_j)^{-1}$, *where* $k_v \in \mathbb{N}_{\ge 1}$.

Let $\mathcal{H}_\infty(s, \lambda)$ *be the product of* $\mathcal{H}_\infty^*(s, \lambda)$ *and the function*

$$\prod_{\substack{1 \le i,j \le n \\ i \neq j}} \prod \prod_{v_2 \in \Sigma_2} \prod_{l=0}^{k_{v_2} - 1} (l + 1 + \lambda_i - \lambda_j) \prod_{v_1 \in \Sigma_1} \Gamma\left(\frac{2 + \lambda_i - \lambda_j}{2}\right) \Gamma\left(\frac{1 + \lambda_i - \lambda_j}{2}\right)^{-1}.$$

Then $\mathcal{H}_\infty(s,\lambda)$ is holomorphic with respect to $s \in \mathbb{C}$ and with respect to $\lambda = (\lambda_1, \cdots, \lambda_n)$ in the domain $|\lambda_i - \lambda_j| < 2$, $1 \le i,j \le n$.

Let $\mathcal{H}(s,\lambda) = \mathcal{H}_\infty(s,\lambda)\mathcal{H}_f(s,\lambda)$. Then by (6.19) *and* (6.20) *we have*

$$\frac{\Psi(s,W_1,W_2;\lambda,\Phi)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})} = \mathcal{H}(s,\lambda)\prod_{\substack{1\le i,j\le n\\ i\neq j}}\prod\frac{1}{\Lambda(1+\lambda_i-\lambda_j,\pi_i\times\widetilde{\pi}_j)}. \tag{6.21}$$

Let notation be as before, we then define, for $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$ and $\phi_2 \in \mathfrak{B}_{P,\chi}$, that

$$R_\varphi(s,\lambda;\phi_2) = \sum_{\phi_1\in\mathfrak{B}_{P,\chi}}\langle\mathcal{I}_P(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\frac{\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})},\ \mathrm{Re}(s)>1, \tag{6.22}$$

where $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$ is the complete L-function, defined by $\prod_{v\in\Sigma_F}L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$. Write φ as a finite sum of convolutions $\varphi_\alpha * \varphi_\beta$. Since $\mathfrak{B}_{P,\chi}$ is finite dimensional, we have, when $\mathrm{Re}(s)>1$, that

$$\sum_{\phi_1\in\mathfrak{B}_{P,\chi}}\langle\mathcal{I}_P(\lambda,\varphi)\phi_1,\phi\rangle\cdot\frac{\Psi(s,W_1,W_2;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})} = \sum_\alpha\sum_\beta\frac{\Psi(s,W_\alpha,W_\beta;\lambda)}{\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})}, \tag{6.23}$$

where $W_\beta(x;\lambda) = W(x,\mathcal{I}_P(\lambda,\varphi_\beta)\phi;\lambda)$ and $W_\alpha(x;\lambda)$ is the Whittaker function defined by $W_\alpha(x;\lambda) = W(x,\mathcal{I}_P(\lambda,\varphi_\alpha)\phi;\lambda)$. Then we have $\Psi(s,W_\alpha,W_\beta;\lambda)$ equal to $\prod\Psi_v(s,W_{\alpha,v},W_{\beta,v};\lambda)$, $\mathrm{Re}(s)>1$; and each $\Psi_v(s,W_{\alpha,v},W_{\beta,v};\lambda)$ is a finite sum of $\Psi_v(s,W_{1,v},W_{2,v};\lambda)$. Then according to Proposition 49 and Proposition 54 we see that, when $\mathrm{Re}(s)>1$, $\Psi_v(s,W_{\alpha,v},W_{\beta,v};\lambda)\Lambda_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda},v)^{-1}$ are independent of s for all but finitely many places v, and as a function of s, is a finite product of holomorphic function in $\mathrm{Re}(s)>0$. Hence both sides of (6.23) are well defined and is meromorphic in $\mathrm{Re}(s)$. Then after continuation we have, for $\mathrm{Re}(s)>0$, that

$$R_\varphi(s,\lambda;\phi) = \sum_{\phi_1\in\mathfrak{B}_{P,\chi}}\langle\mathcal{I}_P(\lambda,\varphi)\phi_1,\phi\rangle R(s,W_1,W_2;\lambda) = \sum_{\alpha,\beta}R(s,W_\alpha,W_\beta;\lambda). \tag{6.24}$$

Then clearly by Theorem G we have that, for $\mathrm{Re}(s)>1$, $I_{\mathrm{Whi}}(s,\tau)$ is equal to

$$\sum_\chi\sum_{P\in\mathcal{P}}\frac{1}{c_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})d\lambda.$$

Note that the integrands make sense in the critical strip $0<\mathrm{Re}(s)<1$. To continue $I_{\mathrm{Whi}}(s,\tau)$ to a meromorphic function in the right half plane $\mathrm{Re}(s)>0$, we need to show that the summation expressing $I_{\mathrm{Whi}}(s,\tau)$ in Theorem G in fact converges absolutely in the strip $S_{(0,1)} = \{s\in\mathbb{C}:\ 0<\mathrm{Re}(s)<1\}$, which we call the critical strip.

Chapter 7

ABSOLUTE CONVERGENCE IN THE CRITICAL STRIP $S_{(0,1)}$

Let $1 \leq m \leq n$ be an integer and $\pi \in \mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ be a cuspidal representation of GL_m over F. For $v \in \Sigma_{F,\mathrm{fin}}$, let $f(\pi_v)$ be the conductor of π_v, set $C(\pi_v) = q_v^{f(\pi_v)}$, where q_v is the cardinality of the residual field of F_v, then $C(\pi_v) = 1$ for all but finitely many finite places v. For $v \in \Sigma_{F,\infty}$, then $F_v \simeq \mathbb{R}$ or $F_v \simeq \mathbb{C}$. Let $L_v(s, \pi_v) = \prod_j \Gamma_{F_v}(s + \mu_{\pi_v,j})$ be the associated L-factor of π_v. Denote in this case by $C(\pi_v; t) = \prod_j (2 + |it + \mu_{\pi,j}|_{F_v})^{[F_v:\mathbb{R}]}$, $t \in \mathbb{R}$, and set $C(\pi_v) = C(\pi_v; 0)$.

Definition 62 (Analytic Conductor)**.** *Let notation be as above, denote by* $C(\pi; t) = \prod_{v \in \Sigma_F} C(\pi_v; t)$, $t \in \mathbb{R}$. *We call* $C(\pi) = C(\pi; 0)$ *the analytic conductor of* π. *Note that it is well defined.*

To prove our Theorem H, we need an explicit upper bound for Rankin-Selberg L-functions in the critical strip in terms of the corresponding analytic conductors. Nevertheless, the standard convexity bound $L(1/2, \sigma \otimes \tau \times \sigma') \ll_\epsilon C(\sigma \otimes \tau \times \sigma')^{1/2+\epsilon}$ is unknown unconditionally for general cuspidal representations $\sigma \in \mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ and $\sigma' \in \mathcal{A}_0(GL_{m'}(F)\backslash GL_{m'}(\mathbb{A}_F))$. To remedy this, we prove a preconvex estimate (which is sufficient for our purpose) for $L(s, \sigma \otimes \tau \times \sigma')$ in the critical strip $0 < \mathrm{Re}(s) < 1$.

Lemma 63 (Preconvex bound)**.** *Let* $1 \leq m, m' \leq n$ *be two integers. Let* $\sigma \in \mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ *and* $\sigma' \in \mathcal{A}_0(GL_{m'}(F)\backslash GL_{m'}(\mathbb{A}_F))$. *Let* $\beta_{m,m'} = 1 - 1/(m^2 + 1) - 1/(m'^2 + 1)$. *Then for* $s \in \mathbb{C}$ *such that* $0 < \mathrm{Re}(s) < 1$, *we have*

$$L(s, \sigma \otimes \tau \times \sigma') \ll_{F,\epsilon} \left(1 + |s(s-1)|^{-1}\right) C(\sigma \otimes \tau \times \sigma'; s)^{\frac{1+\beta_{m,m'} - \mathrm{Re}(s)}{2} + \epsilon}, \qquad (7.1)$$

where the implies constant is absolute, depending only on ϵ *and the base field* F.

Proof. By definition, τ extends to a character on $G(\mathbb{A}_F)$ via composing with the determinant map, i.e., by setting $\tau(x) = \tau(|\det x|_{\mathbb{A}_F})$, for any $x \in G(\mathbb{A}_F)$. Thus τ is automorphic and invariant on $N(\mathbb{A}_F)$. Hence $\sigma \otimes \tau$ is also cuspidal. We may write the cuspidal representations as $\sigma \otimes \tau = \otimes_v (\sigma_v \otimes \tau_v)$ and $\sigma' = \otimes'_v \sigma'_v$. For prime ideals $\mathfrak{p}$ at which neither $\sigma_\mathfrak{p}$ or $\sigma'_\mathfrak{p}$ is ramified, let $\{St_{\sigma \otimes \tau, i}(\mathfrak{p})\}_{i=1}^m$ and $\{St_{\sigma', j}(\mathfrak{p})\}_{j=1}^{m'}$ be the respective Satake parameters of $\sigma \otimes \tau$ and σ'. The Rankin-Selberg L-function at such a $\mathfrak{p}$ (there are all but finitely many such primes) is defined to be

$$L_{\mathfrak{p}}(s, \sigma_{\mathfrak{p}} \otimes \tau_{\mathfrak{p}} \times \sigma'_{\mathfrak{p}}) = \prod_{i=1}^{m} \prod_{j=1}^{m'} \left(1 - St_{\sigma\otimes\tau,i}(\mathfrak{p}) St_{\sigma',j}(\mathfrak{p}) N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1}.$$

Since τ is unitary, by [LRS99] we have $\left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})} |St_{\sigma\otimes\tau,i}(\mathfrak{p})| \leq 1/2 - 1/(m^2+1)\right.$, and $\left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})} |St_{\sigma',j}(\mathfrak{p})| \leq 1/2 - 1/(m'^2+1)\right.$. For the remaining places $\mathfrak{p}$, The Rankin-Selberg L-function at such a $\mathfrak{p}$ can be written as

$$L_{\mathfrak{p}}(s, \sigma_{\mathfrak{p}} \otimes \tau_{\mathfrak{p}} \times \sigma'_{\mathfrak{p}}) = \prod_{i=1}^{m} \prod_{j=1}^{m'} \left(1 - St_{\sigma\otimes\tau\times\sigma',i,j}(\mathfrak{p}) N_{F/\mathbb{Q}}(\mathfrak{p})^{-s}\right)^{-1},$$

with $\left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})} |St_{\sigma\otimes\tau\times\sigma',i,j}(\mathfrak{p})|\right| \leq \left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})} |St_{\sigma\otimes\tau,i}(\mathfrak{p})| + \left|\log_{N_{F/\mathbb{Q}}(\mathfrak{p})} |St_{\sigma',j}(\mathfrak{p})|\right.\right.$, which is bounded by $\beta_{m,m'} = 1 - 1/(m^2+1) - 1/(m'^2+1)$. Then an easy estimate implies that for any s such that $\beta = \mathrm{Re}(s) > 1 + \beta_{m,m'}$, we have

$$\begin{aligned}\left|L(s, \sigma \otimes \tau \times \sigma')\right| &= \prod_{\mathfrak{p}} \left|L_{\mathfrak{p}}(s, \sigma_{\mathfrak{p}} \otimes \tau_{\mathfrak{p}} \times \sigma'_{\mathfrak{p}})\right| \leq \prod_{\mathfrak{p}} \prod_{i=1}^{m} \prod_{j=1}^{m'} \left|1 - N_{F/\mathbb{Q}}(\mathfrak{p})^{\beta_{m,m'}-\beta}\right|^{-1} \\ &= \prod_{\mathfrak{p}} \left|1 - N_{F/\mathbb{Q}}(\mathfrak{p})^{\beta_{m,m'}-\beta}\right|^{-mm'} = \zeta_F(\beta - \beta_{m,m'})^{mm'},\end{aligned}$$

where $\zeta_F(s)$ is the Dedekind zeta function associated to $F/\mathbb{Q}$. In particular,

$$\left|L(\beta + i\gamma, \sigma \otimes \tau \times \sigma')\right| \leq \zeta_F(\beta - \beta_{m,m'})^{mm'} = O_{\beta_0}(1), \ \beta \geq \beta_0 > 1 + \beta_{m,m'}. \quad (7.2)$$

Also, at each infinite place $v \mid \infty$, there exists a set of mm' complex parameters $\{\mu_{\sigma\otimes\tau\times\sigma';v,i,j} : 1 \leq i \leq m, 1 \leq j \leq m'\}$ such that each local L-factor at v is

$$L_v(s, \sigma_v \otimes \tau_v \times \sigma'_v) = Q_v(s) \prod_{i=1}^{m} \prod_{j=1}^{m'} \Gamma_{F_v}\left(s + \mu_{\sigma\otimes\tau\times\sigma';v,i,j}\right),$$

where $Q_v(s)$ is entire. Likewise, we have $\left|\mu_{\sigma\otimes\tau\times\sigma';v,i,j}\right| \leq \beta_{m,m'}$, according to loc. cit. Moreover, since $\widetilde{\sigma_v \otimes \tau_v} = \widetilde{\sigma}_v \otimes \widetilde{\tau}_v$, the finite set $\{\overline{\mu_{\sigma\otimes\tau\times\sigma';v,i,j}} : 1 \leq i \leq m, 1 \leq j \leq m'\}$ is equal to $\{\mu_{\widetilde{\sigma}\otimes\widetilde{\tau}\times\widetilde{\sigma}';v,i,j} : 1 \leq i \leq m, 1 \leq j \leq m'\}$ for any $v \in \Sigma_{F,\infty}$. Note that by Stirling's formula one has, for $s = \beta + i\gamma$, where $\beta < 1 - \beta_{m,m'}$, that

$$\Gamma\left(1 - s + \bar{\mu}/2\right) \cdot \Gamma\left(s + \mu/2\right)^{-1} \ll_{\beta} (1 + |i\gamma + \mu|)^{1/2-\beta},$$

for any $\mu \in \mathbb{C}$ such that $\mathrm{Re}(\mu) > -1 + \beta_{m,m'}$. Then combining these with the duplication formula $\Gamma_{\mathbb{C}}(s) = \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1)$ we have, for $s = \beta + i\gamma$ with $\beta < 1 - \beta_{m,m'}$, that

$$\prod_{v|\infty}\frac{L_v(1-s,\widetilde{\sigma}_v\otimes\widetilde{\tau}_v\times\widetilde{\sigma'}_v)}{L_v(s,\sigma_v\otimes\tau_v\times\sigma'_v)}\ll_\beta\prod_{v|\infty}C(\sigma_v\otimes\tau_v\times\sigma'_v;\gamma)^{1/2-\beta}.$$

Hence together with the functional equation we have

$$L(\beta+i\gamma,\sigma\otimes\tau\times\sigma')=O\left(C(\sigma\otimes\tau\times\sigma';\gamma)^{1/2-\beta}\right),\ \beta\leq\beta_0<-\beta_{m,m'}. \tag{7.3}$$

If $\sigma\otimes\tau\not\simeq\sigma'$, then according to [JS81], $L(s,\sigma\otimes\tau\times\sigma')$ is entire. Then combining the nice analytic properties of $L(s,\sigma\otimes\tau\times\sigma')$ (see loc. cit.) and Phragmén-Lindelöf principle with (7.2) and (7.3) we obtain the following preconvex bound in the interval $-\beta_{m,m'}\leq\beta\leq1+\beta_{m,m'}$:

$$L(\beta+i\gamma,\sigma\otimes\tau\times\sigma')\ll_\epsilon C(\sigma\otimes\tau\times\sigma';\gamma)^{\frac{1+\beta_{m,m'}-\beta}{2}+\epsilon}. \tag{7.4}$$

If $\sigma\otimes\tau\simeq\sigma'$, then according to loc. cit., $L(s,\sigma\otimes\tau\times\sigma')$ has simple poles precisely at $s=1$ and possibly at $s=0$. Consider instead the function $f(s)=s(s-1)(s+2)^{-(5+\beta_{m,m'}-\beta)/2}L(s,\sigma\otimes\tau\times\sigma')$. Then clearly $f(s)$ is holomorphic and of order 1 in the right half plane $\mathrm{Re}(s)>-\beta_{m,m'}$. Hence by (7.2), (7.3) and Phragmén-Lindelöf principle we have that $f(s)$ is bounded by $O_\epsilon\left(C(\sigma\otimes\tau\times\sigma';\gamma)^{(1+\beta_{m,m'}-\beta)/2+\epsilon}\right)$ in the strip $-\beta_{m,m'}\leq\mathrm{Re}(s)\leq1+\beta_{m,m'}$, leading to the estimate

$$L(\beta+i\gamma,\sigma\otimes\tau\times\sigma')\ll_\epsilon|(\beta+i\gamma)(\beta+i\gamma-1)|^{-1}C(\sigma\otimes\tau\times\sigma';\gamma)^{\frac{1+\beta_{m,m'}-\beta}{2}+\epsilon}, \tag{7.5}$$

where $0<\beta<1$. Now (7.1) follows from (7.4) and (7.5). □

Lemma 64. *Let $s=\beta+i\gamma$ such that $\beta>0$ and $\gamma\in\mathbb{R}$. Then one has*

$$|s|^{-1-2|\gamma|}\cdot e^{-C(s)}\leq\left|\Gamma(s)\right|\leq\beta\Gamma(\beta)\cdot|s|^{-1}, \tag{7.6}$$

where $C(s)=\min\left\{\frac{\pi^2|s|^2}{6}+\beta,2|s|\right\}$. Moreover, if $|\gamma|\geq1$, we have uniformly that

$$\Gamma(s)=\sqrt{2\pi}e^{-\frac{\pi}{2}|\gamma|}|\gamma|^{\beta-\frac{1}{2}}e^{i\gamma(\log|\gamma|-1)}e^{\frac{i\pi\delta(s)}{2}\cdot(\beta-\frac{1}{2})}\left(1+\lambda(s)|\gamma|^{-1}\right), \tag{7.7}$$

where $\delta(s)=1$ if $\gamma\geq1$, and $\delta(s)=-1$ if $\gamma\leq-1$; and $|\lambda(s)|\leq e^{1/3+\beta^2/2+\beta^3/3}-1$.

Proof. Consider $1/\Gamma(s)$, which is an entire function. Take the logarithm of its Hadamard decomposition $1/\Gamma(s)=se^{\gamma_0 s}\prod(1+s/n)e^{-s/n}$ (here $\gamma_0=0.57721\cdots$ is the Euler-Mascheroni constant) and take real parts on both sides to get

$$\log\left|\Gamma(s)^{-1}\right|=\log|s|+\gamma_0\beta+\mathrm{Re}\sum_{n<2|s|}\left(\log\left(1+\frac{s}{n}\right)-\frac{s}{n}\right)+S_I,$$

where $S_I = \operatorname{Re} \sum_{n \geq 2|s|} (\log(1+s/n) - s/n)$. Expand the logarithm to see

$$\left|S_I\right| = \left|\operatorname{Re} \sum_{n\geq 2|s|} \sum_{k\geq 2} \frac{(-1)^{k-1} s^k}{k n^k}\right| \leq \frac{1}{2} \sum_{n\geq 2|s|} \frac{|s|^2 n^{-2}}{1-|s|/n} \leq \sum_{n\geq 2|s|} \left(\frac{|s|}{n}\right)^2 \leq C_1(s),$$

where $C_1(s) = \min\{\pi^2/6, |s|^2/(2|s|-1)\}$. Therefore, $\log\left|\frac{1}{\Gamma(s)}\right|$ is no more than

$$\log|s| + \gamma_0 \beta + \sum_{n\leq 2|s|} \left(\frac{|s|}{n} - \frac{\beta}{n}\right) + C_1(s) \leq (1+2|s|-2\beta)\log|s| + C(s),$$

which is further bounded by $(1+2\gamma)\log|s| + C(s)$. This proves the left inequality of (7.6). For the right hand side, consider the integral representation of $\Gamma(s+1) = s\Gamma(s)$, we have $|s\Gamma(s)| = \left|\int_0^\infty t^s e^{-t} dt\right| \leq \int_0^\infty \left|t^s\right| e^{-t} dt = \int_0^\infty t^\beta e^{-t} dt = \beta\Gamma(\beta)$, which proves the right inequality of (7.6). Hence (7.6) holds.

To prove (7.7), we may assume that $\gamma \geq 1$. Write $s = \beta + i\gamma = \rho e^{i\theta}$, then

$$0 < \theta = \frac{\pi}{2} - \arctan\frac{\beta}{\gamma} \leq \begin{cases} \frac{\pi}{2}, \text{ if } \beta \geq 0; \\ \frac{\pi}{2} - \arctan\beta_0, \text{ if } \beta_0 \leq \beta < 0, \end{cases} \tag{7.8}$$

where $\arctan x$ is taken its principal value, i.e., $-\pi/2 < \arctan x < \pi/2, \forall\, x \in \mathbb{R}$.

A standard application of Euler-MacLaurin summation formula leads to that

$$\log\Gamma(s) = (s-1/2)\log s - s + 1/2\log 2\pi + \int_0^\infty \frac{b(u)}{(u+s)^2} du, \tag{7.9}$$

where $b(u) = 1/2\{u\} - 1/2\{u\}^2$, here $\{u\} := u - [u]$ with $[u]$ denoting the Gauss symbol, i.e., $[u]$ is the largest integer no more than u. Then

$$\left|\int_0^\infty \frac{b(u)}{(u+s)^2} du\right| \leq 1/2 \left(\cos\frac{\theta}{2}\right)^{-2} \int_0^\infty \frac{du}{(\rho+u)^2} \leq \frac{1}{6\rho}\left(\cos\frac{\theta}{2}\right)^{-2} \leq \frac{1}{3\gamma}, \tag{7.10}$$

since $0 < \theta/2 \leq \pi/4$ according to (7.8). Substitute (7.10) into (7.9) to get

$$\begin{aligned} \log\Gamma(s) &= (\beta + i\gamma - 1/2)\big[\log\sqrt{\beta^2+\gamma^2} + i\theta\big] - \beta - i\gamma + \log\sqrt{2\pi} + C_1(\gamma) \\ &= (\beta - 1/2)\log\sqrt{\beta^2+\gamma^2} - \gamma\theta - \beta + \frac{1}{2}\log 2\pi + C_1(\gamma) + iC_2(s), \end{aligned}$$

where $C_2(s) = \gamma\log\sqrt{\beta^2+\gamma^2} + (\beta - 1/2)\cdot\theta - \gamma$. Also one has elementary inequalities

$$\begin{cases} \left|\arctan\beta\gamma^{-1} - \beta\gamma^{-1}\right| \leq \left|\beta^3/(3\gamma^3)\right|, \\ \left|\log\sqrt{\beta^2+\gamma^2} - \log\gamma\right| \leq \beta^2/(2\gamma^2). \end{cases} \tag{7.11}$$

Then plugging (7.11) into the expansion of $\log\Gamma(s)$ to get that $\log\Gamma(s)$ is equal to

$$\frac{1}{2}\log 2\pi - \frac{\pi\gamma}{2} + \left(\beta - \frac{1}{2}\right)\log\gamma + i\left\{\gamma\log\gamma - \gamma + \frac{\pi}{2}\left(\beta - \frac{1}{2}\right)\right\} + C_3(s), \quad (7.12)$$

where $\left|C_3(s)\right| \leq \left(1/3 + \beta^2/2 + \beta^3/3\right)\cdot|\gamma|^{-1}$. Then the case $\gamma \geq 1$ of (7.7) follows from (7.12) and the elementary inequality $|e^{cx} - 1| \leq (e^c - 1)x$, for any $c > 0$ and $0 < x \leq 1$. Taking the complex conjugate of both sides gives the case where $\gamma \leq -1$. Hence the lemma follows. □

Corollary 65. *Let $1 \leq m, m' \leq n$ be two integers. Let $\sigma \in \mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ and $\sigma' \in \mathcal{A}_0(GL_{m'}(F)\backslash GL_{m'}(\mathbb{A}_F))$. Let v be an archimedean place. Let $\beta \geq 5$. Then for each $s = \beta_0 + i\gamma \in \mathbb{C}$ such that $1 - 1/(n^2+1) < \beta_0 < 1$ and $\gamma \in \mathbb{R}$, we have*

$$\left|L_v(s, \sigma_v \otimes \tau_v \times \sigma_v')\right| \leq C_\beta \cdot \left|\frac{L_v(\beta + i\gamma, \sigma_v \otimes \tau_v \times \sigma_v')}{C(\sigma_v \otimes \tau_v \times \sigma_v'; \gamma)}\right|, \quad (7.13)$$

where C_β is an absolute constant depending only on β, n and the base field F.

Proof. Let $t_\beta = 2e^{1/3+\beta^2/2+\beta^3/3}$. Then $t_\beta > 2$. Recall that by definition

$$L_v(s, \sigma_v \otimes \tau_v \times \sigma_v') = \prod_{j=1}^{m}\prod_{k=1}^{m'} \Gamma_{F_v}\left(s + \mu_{\sigma\otimes\tau\times\sigma';v,j,k}\right). \quad (7.14)$$

We can write $\mu_{\sigma\otimes\tau\times\sigma';v,j,k} = \beta_{j,k} + i\gamma_{j,k}$. Then $|\beta_{j,k}| \leq 1 - 1/(1+m^2) - 1/(1+m'^2) \leq 1 - 2/(1+n^2)$. Let $t_{j,k} = \gamma + \gamma_{j,k}$, $1 \leq j \leq m$, $1 \leq k \leq m'$. Let $\delta_v = 2/[F_v : \mathbb{R}]$.

Case 1: If $|t_{j,k}| \leq t_\beta$. Then by the estimate (7.6) we see that

$$\left|\frac{\Gamma_{F_v}\left(s + \mu_{\sigma\otimes\tau\times\sigma';v,j,k}\right)\cdot\left(2 + |i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,i,j}|_{F_v}\right)^{[F_v:\mathbb{R}]}}{\Gamma_{F_v}\left(\beta + i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,j,k}\right)}\right|$$

$$\leq\left|\Gamma\left(\frac{\beta_0 + \beta_{j,k}}{\delta_v}\right)\cdot\left(2 + |\beta_{j,k}|_{F_v} + |t_\beta|_{F_v}\right)^2 e^{\frac{2(\beta+|\beta_{j,k}|+t_\beta)}{\delta_v}}\cdot\left[\frac{\beta + |\beta_{j,k}| + t_\beta}{\delta_v}\right]^{1+2t_\beta}\right|,$$

which can be seen clearly to be bounded by

$$C_{1;j,k}(\beta) \triangleq (\beta + |t_\beta|^2 + 1)^{2t_\beta+3} e^{2(\beta+|t_\beta|+1)} \cdot \max_{1/(n^2+1)\leq\beta'\leq 2} \Gamma(\beta'/\delta_v).$$

Case 2: If $|t_{j,k}| \geq t_\beta > 2$. Then by the estimate (7.7) we see that

$$\left|\frac{\Gamma_{F_v}\left(s + \mu_{\sigma\otimes\tau\times\sigma';v,j,k}\right)\cdot\left(2 + |i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,i,j}|_{F_v}\right)^{[F_v:\mathbb{R}]}}{\Gamma_{F_v}\left(\beta + i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,j,k}\right)}\right|$$

$$\leq\left|\frac{\left(2 + |\beta_{j,k}|_{F_v} + |t_\beta|_{F_v}\right)^2\cdot\left(\left(1 + |\lambda(s + \mu_{\sigma\otimes\tau\times\sigma';v,j,k})|\right)\cdot|t_{j,k}|^{-1}\right)}{|\delta_v^{-1}t_{j,k}|^{\beta-\beta_0}\cdot\left(\left(1 - |\lambda(\beta + i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,j,k})|\right)\cdot|t_{j,k}|^{-1}\right)}\right|,$$

which is bounded by $3(3+|t_{j,k}|^2)^2|\delta_v^{-1}t_{j,k}|^{1-\beta} \triangleq C_{3;j,k}(t_{j,k})$. Note that $C_{3;j,k}(t_{j,k})$ is bounded in the interval $[t_\beta, \infty)$. So we can define

$$C_{2;j,k}(\beta) = \sup_{|t_{j,k}|\geq t_\beta} C_{3;j,k}(t_{j,k}).$$

Now let $C_{j,k}(\beta) = \max\{C_{1;j,k}(\beta), C_{2;j,k}(\beta)\}$. Then $C_{j,k}(\beta)$ is well defined, $1 \leq j \leq m$, $1 \leq k \leq m'$. Set $C_\beta = \prod_j \prod_k C_{j,k}(\beta)$. Then by (7.14) and

$$C(\sigma_v \otimes \tau_v \times \sigma_v'; \gamma) = \prod_{i=1}^{m} \prod_{j=1}^{m'} \left(2 + |i\gamma + \mu_{\sigma\otimes\tau\times\sigma';v,i,j}|_{F_v}\right)^{[F_v:\mathbb{R}]},$$

the estimate (7.13) follows. □

Remark 66. *Let* $1 \leq m, m' \leq n$ *be two integers. Let* $\sigma \in \mathcal{A}_0(GL_m(F)\backslash GL_m(\mathbb{A}_F))$ *and* $\sigma' \in \mathcal{A}_0(GL_{m'}(F)\backslash GL_{m'}(\mathbb{A}_F))$. *Let* v *be an archimedean place. Let* $N \geq 1$ *and* $\beta \geq 4N+1$. *Then essentially the same proof of Corollary 65 leads to the result that for each* $s = \beta_0 + i\gamma \in \mathbb{C}$ *such that* $0 < \beta_0 < 1$ *and* $\gamma \in \mathbb{R}$, *we have*

$$\left|L_v(s, \sigma_v \otimes \tau_v \times \sigma_v')\right| \leq C_{N,\beta} \cdot \left|\frac{L_v(\beta + i\gamma, \sigma_v \otimes \tau_v \times \sigma_v')}{C(\sigma_v \otimes \tau_v \times \sigma_v'; \gamma)^N}\right|, \tag{7.15}$$

where $C_{N,\beta}$ *is an absolute constant depending only on* N, β, n *and the base field* F. *This slightly general bound* (7.15) *will be used in [Yan21].*

Let $v \in \Sigma_{F,\mathrm{fin}}$ be a nonarchimedean place of F. Let $\Phi_{v,l}$ be a constant multiplying the characteristic function of some open ball in F_v^n. Then its Fourier transform $\widehat{\Phi_{v,l}}$ is also of the same form, i.e., a constant multiplying the characteristic function of some open ball in F_v^n.

Now we consider integrals $\Psi_v^*\left(s, W_{1,v}, W_{1,v}; \lambda, \Phi_{v,l}\right)$ defined by

$$\int_{N(F_v)\backslash G(F_v)} W_{1,v}(x_v;\lambda)\overline{W_{1,v}\left(x_v;-\bar{\lambda}\right)} \cdot \Phi_{v,l}(\eta x_v)|\det x_v|_{F_v}^s\, dx_v.$$

Let $\widetilde{W}_{1,v}$ be the Whittaker function of $\widetilde{\pi}_{v,-\lambda}$, defined via $\widetilde{W}_{1,v}(x) = W_{1,v}(wx^\iota)$, where $x \in G(F_v)$ and w is the longest element in $W_P\backslash W/W_P$. Define the integral $\Psi_v^*(s, \widetilde{W}_{1,v}, \widetilde{W}_{1,v}; \lambda, \widehat{\Phi_{v,l}})$ by

$$\int_{N(F_v)\backslash G(F_v)} \widetilde{W}_{1,v}\overline{\widetilde{W}_{1,v}\left(x_v;-\bar{\lambda}\right)} \cdot \widehat{\Phi_{v,l}}(\eta x_v)|\det x_v|_{F_v}^s\, dx_v.$$

Lemma 67. *Let notation be as before. Let* $0<\epsilon<1/2$. *Let* q_v *be the cardinality of the residue field of* F_v. *Let* W_1 *be a Whittaker function associated to* $\chi\in\mathfrak{X}_P$. *Then there exists a constant* c_v *depending only on the test function* φ *such that*

$$\left|\Psi_v^*\left(s,W_{1,v},W_{1,v};\lambda,\Phi_{v,l}\right)\right|\leq q_v^{c_v}\left|\Psi_v^*\left(1-\epsilon,W_{1,v},W_{1,v};\lambda,\Phi_{v,l}\right)\right|, \tag{7.16}$$

for any $s\in\mathbb{C}$ *such that* $\operatorname{Re}(s)=\epsilon$.

Proof. We can apply the same argument on the support of $\widetilde{W}_{1,v}$ as that of $W_{1,v}$ in the proof of Corollary 47 to show that there exists a positive integer $m_v=m_v(\varphi_v)$, depending only on the place $v\in S(\pi,\Phi)$ and the K-finite type of the test function φ, such that $\operatorname{supp}\widetilde{W}_{1,v}\mid_{A(F_v)}\subseteq\{x=\operatorname{diag}(x_1,\cdots,x_n)\in A(F_v):\ \max\{|x_i|_v\}\leq q_v^{m_v}\}$. Noting $\widehat{\Phi_{v,l}}$ is an indicator function, then for any s such that $\operatorname{Re}(s)=\epsilon$,

$$\begin{aligned}&q_v^{-nm_v\epsilon}\left|\int_{N(F_v)\backslash G(F_v)}\widetilde{W}_{1,v}(x_v;\lambda)\overline{\widetilde{W}_{1,v}\left(x_v;-\bar{\lambda}\right)}\cdot\widehat{\Phi_{v,l}}(\eta x_v)|\det x_v|_{F_v}^{\epsilon}dx_v\right|\\ \geq&q_v^{-nm_v(1-\epsilon)}\left|\int_{N(F_v)\backslash G(F_v)}\widetilde{W}_{1,v}(x_v;\lambda)\overline{\widetilde{W}_{1,v}\left(x_v;-\bar{\lambda}\right)}\cdot\widehat{\Phi_{v,l}}(\eta x_v)|\det x_v|_{F_v}^{1-\epsilon}dx_v\right|,\end{aligned}$$

from which one easily obtains the inequality that

$$\left|\Psi_v^*(\epsilon,\widetilde{W}_{1,v},\widetilde{W}_{1,v};\lambda,\widehat{\Phi_{v,l}})\right|\geq q_v^{-nm_v}\left|\Psi_v^*(1-\epsilon,\widetilde{W}_{1,v},\widetilde{W}_{1,v};\lambda,\widehat{\Phi_{v,l}})\right|. \tag{7.17}$$

On the other hand, we have the functional equation

$$\left|\frac{\Psi_v^*(1-s,\widetilde{W}_{1,v},\widetilde{W}_{1,v};\lambda,\widehat{\Phi_{v,l}})}{L_v(1-s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})}\right|=\left|\frac{\epsilon(s,\pi_v,\lambda)\Psi_v^*\left(s,W_{1,v},W_{1,v};\lambda,\Phi_{v,l}\right)}{L_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})}\right|, \tag{7.18}$$

where $\epsilon(s,\pi_v,\lambda)=\gamma(s,\pi_v,\lambda)L_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})L_v(1-s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})^{-1}$ is the ϵ-factor, here $\gamma(s,\pi_v,\lambda)$ is the γ-factor. By the stability of γ-factors and [CP17], we have the stability of $\epsilon(s,\pi_v,\lambda)$. Thus $\epsilon(s,\pi_v,\lambda)=\prod\prod\epsilon(s+\lambda_i-\lambda_j,\sigma_{v,i}\times\widetilde{\sigma}_{v,j},\lambda)$. Let $q_v=N_{F/\mathbb{Q}}(\mathfrak{p})$. Then one has that (see [JPS83]) each $\epsilon(s+\lambda_i-\lambda_j,\sigma_{v,i}\times\widetilde{\sigma}_{v,j},\lambda)$ is of the form $cq_v^{-f_vs}$, where $|c|=q_v^{1/2}$ and f_v is the local conductor, which is bounded by an absolute constant depending only on K_v-type of the test function φ. Hence there exists some absolute constant $e_v\in\mathbb{N}_{\geq0}$, relying only on φ, such that

$$\left|\epsilon(s,\pi_v,\lambda)\epsilon(1-s,\pi_v,\lambda)^{-1}\right|\geq q_v^{-e_v\epsilon}. \tag{7.19}$$

Then combine (7.17), (7.18) and (7.19) we have

$$\left|\frac{\Psi_v^*\left(\epsilon,W_{1,v},W_{1,v};\lambda,\Phi_{v,l}\right)}{L_v(\epsilon,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})^2}\right|\leq q^{nm_v+e_v\epsilon}\left|\frac{\Psi_v^*\left(1-\epsilon,W_{1,v},W_{1,v};\lambda,\Phi_{v,l}\right)}{L_v(1-\epsilon,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})^2}\right|. \tag{7.20}$$

Since $\pi_{v,\lambda} \in \mathfrak{X}_P$ is generic, then it is irreducible. Hence, according to [CP17], we have

$$L_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v}) = \prod_{i=1}^{r} \prod_{j=1}^{r} L_v(s + \lambda_i - \lambda_j, \sigma_{v,i} \times \widetilde{\sigma}_{v,j}). \tag{7.21}$$

Let $\mathfrak{p}$ be the prime ideal representing the place $v \in \Sigma_{fin}$. Then for any s,

$$L_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v})^{-1} = \prod_{i=1}^{r} \prod_{j=1}^{r} \prod_{k=1}^{n_i} \prod_{l=1}^{n_j} \left(1 - St_{\sigma\times\sigma',k,l}(\mathfrak{p}) N_{F/\mathbb{Q}}(\mathfrak{p})^{-s-\lambda_i+\lambda_j}\right)$$

is a finite product, hence it is an entire function. Moreover, since $\left|St_{\sigma\times\sigma',k,l}(\mathfrak{p})\right| \leq N_{F/\mathbb{Q}}(\mathfrak{p})^{\beta_{i,j}}$, where $\beta_{i,j} = 1 - 1/(n_i^2 + 1) - 1/(n_j^2 + 1)$, we then have

$$\left|L_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v})^{-1}\right| \leq \prod_{i=1}^{r} \prod_{j=1}^{r} \left(1 + N_{F/\mathbb{Q}}(\mathfrak{p})^{-\operatorname{Re}(s)+\beta_{i,j}}\right)^{n_i+n_j}, \tag{7.22}$$

where n_i and n_j are ranks of components of Levi subgroup of P respectively. Also,

$$\left|L_v(s, \pi_{\lambda,v} \times \widetilde{\pi}_{-\lambda,v})^{-1}\right| \geq \prod_{i=1}^{r} \prod_{j=1}^{r} \left(1 - N_{F/\mathbb{Q}}(\mathfrak{p})^{-\operatorname{Re}(s)+\beta_{i,j}}\right)^{n_i+n_j}. \tag{7.23}$$

Then it follows from (7.20), (7.22) and (7.23) that

$$\left|\Psi_v^*\left(\epsilon, W_{1,v}, W_{1,v}; \lambda, \Phi_{v,l}\right)\right| \leq q_v^{c_v} \left|\Psi_v^*\left(1-\epsilon, W_{1,v}, W_{1,v}; \lambda, \Phi_{v,l}\right)\right|, \tag{7.24}$$

where c_v is a constant depending only on the test function φ. Noting that $\Phi_{v,l}$ is a constant multiplying the characteristic function of some connected compact subset of F_v^n, so $\left|\Psi_v^*\left(s, W_{1,v}, W_{1,v}; \lambda, \Phi_{v,l}\right)\right| \leq \left|\Psi_v^*\left(\operatorname{Re}(s), W_{1,v}, W_{1,v}; \lambda, \Phi_{v,l}\right)\right|$. Then (7.16) follows from this inequality and (7.24). □

With the preparation above, now we can prove the following result:

Theorem H. *Let $s \in \mathbb{C}$ be such that $0 < \operatorname{Re}(s) < 1$, then*

$$\sum_{\chi} \sum_{P\in\mathcal{P}} \frac{1}{c_P} \sum_{\phi\in\mathfrak{B}_{P,\chi}} \int_{\Lambda^*} R_\varphi(s, \lambda; \phi) \Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) d\lambda, \tag{7.25}$$

converges absolutely, normally with respect to s, where $R_\varphi(s, \lambda; \phi)$ is defined in (6.22) *and $\Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is the complete L-function.*

Proof. Fix a proper parabolic subgroup $P \in \mathcal{P}$ of type $(n_1, n_2, \cdots, n_r)$. Let $\mathfrak{X}_P$ be the subset of cuspidal data $\chi = \{(M, \sigma)\}$ such that $M = M_P$. Denote by

$$J_P(s) = \sum_{\chi\in\mathfrak{X}_P} \sum_{\phi\in\mathfrak{B}_{P,\chi}} \int_{\Lambda^*} R_\varphi(s, \lambda; \phi) \Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) d\lambda.$$

Let $M_P = \mathrm{diag}(M_1, M_2, \cdots, M_r)$, where M_i is n_i by n_i matrix, $1 \leq i \leq r$. We may write $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_r)$, where $\sigma_i \in \mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. By the K-finiteness of φ, each σ_i has a fixed finite type, so its conductor is bounded uniformly (depending only on φ). Let $C_\infty(\sigma_i \otimes \tau \times \sigma_j; t) = \prod_{v \in \Sigma_{F,\infty}} C(\sigma_{i,v} \otimes \tau_v \times \sigma_{j,v}; t)$. Then one has $C_\infty(\sigma_i \otimes \tau \times \sigma_j; t) \asymp_\varphi C(\sigma_i \otimes \tau \times \sigma_j; t)$, where the implies constant depends only on $\mathrm{supp}\,\varphi$. For any $\Phi = \Phi_\infty \cdot \prod_{v<\infty} \Phi_v \in \mathcal{S}_0(\mathbb{A}_F^n)$, where $\Phi_\infty = \prod_{v|\infty} \Phi_v$. Let $x_v = (x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in F_v^n$, then by definition, Φ_v is of the form

$$\Phi_v(x_v) = e^{-\pi \sum_{j=1}^n x_{v,j}^2} \cdot \sum_{k=1}^m Q_k(x_{v,1}, x_{v,2}, \cdots, x_{v,n}), \tag{7.26}$$

where $F_v \simeq \mathbb{R}$, $Q_k(x_{v,1}, x_{v,2}, \cdots, x_{v,n}) \in \mathbb{C}[x_{v,1}, x_{v,2}, \cdots, x_{v,n}]$ are monomials; and

$$\Phi_v(x_v) = e^{-2\pi \sum_{j=1}^n x_{v,j}\bar{x}_{v,j}} \cdot \sum_{k=1}^m Q_k(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}), \tag{7.27}$$

where $F_v \simeq \mathbb{C}$ and $Q_k(x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n})$ are monomials in the ring $\mathbb{C}[x_{v,1}, \bar{x}_{v,1}, x_{v,2}, \bar{x}_{v,2}, \cdots, x_{v,n}, \bar{x}_{v,n}]$. Thus there exists a finite index set J such that

$$\Phi_\infty(x_\infty) = \sum_{\boldsymbol{j}=(j_v)_{v|\infty} \in J} \prod_{v|\infty} \Phi_{v,j_v}(x_v), \quad x_\infty = \prod_{v|\infty} x_v \in G(\mathbb{A}_{F,\infty}),$$

where each Φ_{v,j_v} is of the form in (7.26) or (7.27) with $m = 1$. Let $\Phi_{\infty,\boldsymbol{j}} = \prod_{v|\infty} \Phi_{v,j_v}$, $\boldsymbol{j} = (j_v)_{v|\infty} \in J$. Then Φ is equal to the sum over $\boldsymbol{j} \in J$ of each $\Phi_{\boldsymbol{j}} = \Phi_{\infty,\boldsymbol{j}} \prod_{v<\infty} \Phi_v \in \mathcal{S}_0(\mathbb{A}_F^n)$.

Since for each $v \mid \infty$ and $\boldsymbol{j} \in J$, $\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j}\right)$ converges absolutely in $\mathrm{Re}(s) > 0$ (see [Jac09]), one has

$$\left| \prod_{v|\infty} \Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_v\right) \right| \leq \sum_{\boldsymbol{j} \in J} \left| \prod_{v|\infty} \Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right) \right|. \tag{7.28}$$

Since each Φ_{v,j_v} is a monomial multiplying an exponential function with negative exponent, $\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right)$ is in fact of the form $c_1 \pi^{c_2 s} \prod_i \prod_j \Gamma(s + \nu_{i,j})$, where c_1, c_2 and ν_i are some constants and the product is finite. Although these parameters depend on the representations σ and τ, the local Rankin-Selberg integral $\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right)$ is either nonvanishing in $\mathrm{Re}(s) > 0$ or vanishing identically (i.e. $c_1 = 0$). Note that for each archimedean place v, there exists a polynomial $Q(s) = Q(s, \lambda)$ (see loc. cit.) depending on π_∞ and λ, such that

$$L_v(s, \pi_{\lambda,v} \otimes \tau_v \times \widetilde{\pi}_{-\lambda,v}) = Q(s, \lambda) \prod_{i=1}^r \prod_{j=1}^r L_v(s + \lambda_i - \lambda_j, \sigma_{v,i} \otimes \tau_v \times \widetilde{\sigma}_{v,j}),$$

where $\mathrm{Re}(s) > \beta_{n,n} = 1 - 2/(n^2+1)$. Clearly $Q(s,\lambda)$ is nonvanishing in $\mathrm{Re}(s) > \beta_{n,n}$. Combining this with the preceding discussion we conclude that there exists a polynomial $Q_{v,j}(s;\lambda)$ (depending on π) for each λ such that

$$\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right) = Q_{v,j_v}(s;\lambda)\prod_{i=1}^{r}\prod_{j=1}^{r} L_v(s+\lambda_i-\lambda_j, \sigma_{v,i}\otimes\tau_v\times\widetilde{\sigma}_{v,j}).$$

Then the above analysis leads to that each $Q_{v,j_v}(s;\lambda)$ is either nonvanishing in $\mathrm{Re}(s) > \beta_{n,n}$ or vanishing identically (i.e. $c_1 = 0$). Write $Q_{v,j_v}(s;\lambda) = c_{c,j_v}\prod(s-\varrho_\lambda)$, with $\mathrm{Re}(\varrho_\lambda) \leq \beta_{n,n}$. Let $s_0 = \beta_0 + i\gamma_0$ such that $\beta_0, \gamma_0 \in \mathbb{R}$ and $1-1/(n^2+1) \leq \beta_0 < 1$. Let $U_\epsilon(s_0) \subsetneq S[0,1]$ (with $0 < \epsilon < (1-\mathrm{Re}(s_0))/10$) be a neighborhood of s_0. Then $|s-\varrho_\lambda| \leq |s'-\varrho_\lambda|$, for any $s \in U_\epsilon(s_0)$ and $s' = \beta + i\,\mathrm{Im}(s)$, $\beta \geq 5$. Therefore,

$$|Q_{v,j_v}(s;\lambda)| \leq |Q_{v,j_v}(s';\lambda)|,\ \forall\ v \mid \infty,\ \boldsymbol{j} = (j_v)_{v|\infty} \in J,\ \lambda \in i\mathfrak{a}_P/i\mathfrak{a}_G^*, \tag{7.29}$$

where $s = \beta + i\gamma \in U_\epsilon(s_0)$ and $s' = 5 + i\gamma$. Combining (7.29) with (7.13) leads to

$$\left|\prod_{v|\infty}\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right)\right| \leq C_5^{d_F}\cdot\left|\prod_{v|\infty}\frac{\Psi_v\left(s', W_{\alpha,v}, W_{\beta,v}; \lambda, \Phi_{v,j_v}\right)}{C_v(\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v};\gamma)}\right|, \tag{7.30}$$

where $d_F = [F:\mathbb{Q}]$. Let $S(\pi,\Phi)$ be the finite set of nonarchimedean places such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$ outside $\Sigma_{F,\infty}\cup S(\pi,\Phi)$. Then by Proposition 49 we have

$$R_{S(\pi,\Phi)}(s,\lambda) = \prod_{v\in S(\pi,\Phi)} R_v(s, W_{\alpha,v}, W_{\beta,v};\lambda) \in \bigotimes_{v\in S(\pi,\Phi)}\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i}\ :\ 1\leq i\leq r].$$

Let $v \in S(\pi,\Phi)$. Write $R_{S(\pi,\Phi)}(s, W_\alpha;\lambda) = \prod_{v\in S(\pi,\Phi)} R_v(s, W_{\alpha,v}, W_{\alpha,v};\lambda)$; write $R_{S(\pi,\Phi)}(s, W_\beta;\lambda) = \prod_{v\in S(\pi,\Phi)} R_v(s, W_{\beta,v}, W_{\beta,v};\lambda)$. Then they both lie in the ring $\bigotimes_{v\in S(\pi,\Phi)}\mathbb{C}[q_v^{\pm s}, q_v^{\pm\lambda_i} : 1 \leq i \leq r]$. Write $R_v(s;\lambda) = R_v(s, W_{\alpha,v}, W_{\beta,v};\lambda)$ in this proof. By definition we have $R_v(s;\lambda) = \Psi_v\left(s, W_{\alpha,v}, W_{\beta,v};\lambda,\Phi_{v,j}\right)\cdot L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})^{-1}$, when $\mathrm{Re}(s) > 1$. Recall that $\Psi_v\left(s, W_{1,v}, W_{2,v};\lambda,\Phi_v\right)$ is equal to

$$\int_{N(F_v)\backslash G(F_v)} W_{\alpha,v}(x_v;\lambda)\overline{W_{\beta,v}\left(x_v;-\bar{\lambda}\right)}\Phi_v(\eta x_v)\tau(\det x_v)|\det x_v|_{F_v}^s\,dx_v,$$

which converges normally in $\mathrm{Re}(s) > 0$, uniformly in $\lambda \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^*$, due to the standard estimate on Whittaker functions (they are bounded by compactly supported functions in this case). Thus it defines an holomorphic function with respect to s in the region $\mathrm{Re}(s) > 0$. By definition and gauge argument we see that the integral $\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v};\lambda,\Phi_v\right)$ converges normally in $\mathrm{Re}(s) > 0$. Therefore, $\Psi_v\left(s, W_{\alpha,v}, W_{\beta,v};\lambda,\Phi_v\right)\cdot L_v(s,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})^{-1}$ is exactly $R_v(s,\lambda)$ for any

$\mathrm{Re}(s)>0$. Since Φ_v is a Schwartz-Bruhat function, we can write Φ_v as a finite sum of $\Phi_{v,l}$, where each $\Phi_{v,l}$ is a constant multiplying a characteristic function of some open ball in F_v^n. Then the Fourier transform of $\Phi_{v,l}$ is of the same form. Recall that the integral $\Psi_v^*\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)$ is defined by

$$\int_{N(F_v)\backslash G(F_v)} W_{\alpha,v}(x_v;\lambda)\overline{W_{\alpha,v}\left(x_v;-\bar{\lambda}\right)}\cdot\Phi_{v,l}(\eta x_v)|\det x_v|_{F_v}^s\,dx_v.$$

Hence $\left|\Psi_v^*\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|\leq\left|\Psi_v^*\left(\mathrm{Re}(s),W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|$. Likewise, one has $\left|\Psi_v^*(s,\widetilde{W}_{\alpha,v},\widetilde{W}_{\alpha,v};\lambda,\widehat{\Phi_{v,l}})\right|\leq\left|\Psi_v^*(\mathrm{Re}(s),\widetilde{W}_{\alpha,v},\widetilde{W}_{\alpha,v};\lambda,\widehat{\Phi_{v,l}})\right|$. Define

$$H_{v,l}(s,c)=\frac{q_v^{cs}\Psi_v^*\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)}{(6-s)(s+5)L_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})},\ 1/2\leq\mathrm{Re}(s)\leq 5,$$

where $v\in S(\pi,\Phi)$ and $c>0$ is a parameter to be determined, depending only on the test function φ. Clearly for any c, $H_{v,l}(s,c)$ is bounded in the strip $1/2\leq\mathrm{Re}(s)\leq 5$, tends to zero as $\mathrm{Im}(s)$ tends to infinity. Let $0<\epsilon<2/(n^2+1)$ and $s_0'\in(1-\epsilon,1)$. Then by maximal principle, there exists an s_1 such that $\mathrm{Re}(s_1)=5$ or $\mathrm{Re}(s_1)=1/2$ such that $\left|H_{v,l}(s_0',c)\right|\leq\left|H_{v,l}(s_1,c)\right|$. Now we assume $\mathrm{Re}(s_1)=1/2$. Consider the functional equation (7.18). Let $q_v=N_{F/\mathbb{Q}}(\mathfrak{p})$. By the stability of $\epsilon(s,\pi_v,\lambda)$, one has that (see [JPS83]) $\epsilon(s,\pi_v,\lambda)=\prod\prod\epsilon(s+\lambda_i-\lambda_j,\sigma_{v,i}\times\widetilde{\sigma}_{v,j},\lambda)$ and each $\epsilon(s+\lambda_i-\lambda_j,\sigma_{v,i}\times\widetilde{\sigma}_{v,j},\lambda)$ is of the form $cq_v^{-f_vs}$, where $|c|=q_v^{1/2}$ and f_v is the local conductor, which is bounded by an absolute constant depending only on K_v-type of the test function φ. Hence there exists some absolute constant $e_v\in\mathbb{N}_{\geq 0}$, relying only on φ, such that

$$\left|\epsilon(s,\pi_v,\lambda)\epsilon(1/2,\pi_v,\lambda)^{-1}\right|\geq q_v^{-e_v\,\mathrm{Re}(s)}. \tag{7.31}$$

The same argument on the support of $\widetilde{W}_{1,v}$ as that of $W_{1,v}$ in the proof of Corollary 47 shows that there exists a positive integer $m_v=m_v(\varphi_v)$, depending only on the place $v\in S(\pi,\Phi)$ and the K-finite type of the test function φ, such that $\mathrm{supp}\,\widetilde{W}_{1,v}\mid_{A(F_v)}\subseteq\{x=\mathrm{diag}(x_1,\cdots,x_n)\in A(F_v):\ \max\{|x_i|_v\}\leq q_v^{m_v}\}$. Then one has, for any s in the strip $0<\mathrm{Re}(s)\leq 1/2$, that

$$\begin{aligned}&q_v^{-nm_v\,\mathrm{Re}(s)}\left|\int_{N(F_v)\backslash G(F_v)}\widetilde{W}_{\alpha,v}(x_v;\lambda)\overline{\widetilde{W}_{\alpha,v}\left(x_v;-\bar{\lambda}\right)}\cdot\widehat{\Phi_{v,l}}(\eta x_v)|\det x_v|_{F_v}^{\mathrm{Re}(s)}dx_v\right|\\ \geq&q_v^{-nm_v/2}\left|\int_{N(F_v)\backslash G(F_v)}\widetilde{W}_{\alpha,v}(x_v;\lambda)\overline{\widetilde{W}_{\alpha,v}\left(x_v;-\bar{\lambda}\right)}\cdot\widehat{\Phi_{v,l}}(\eta x_v)|\det x_v|_{F_v}^{1/2}dx_v\right|.\end{aligned}$$

Therefore we can substitute $s=1-s_0'$ into the above inequality to get

$$\left|\Psi_v^*(1-s_0',\widetilde{W}_{\alpha,v},\widetilde{W}_{\alpha,v};\lambda,\widehat{\Phi_{v,l}})\right|\geq q_v^{nm_v(\frac{1}{2}-s_0')}\left|\Psi_v^*(1/2,\widetilde{W}_{\alpha,v},\widetilde{W}_{\alpha,v};\lambda,\widehat{\Phi_{v,l}})\right|. \tag{7.32}$$

Then combining this with (7.18), (7.31) and (7.32) one has

$$\left|\frac{\Psi_v^*\left(s_0',W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)}{L_v(s_0',\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})}\right|\geq q_v^{\nu(s_0')}\cdot\left|\frac{\Psi_v^*\left(1/2,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)}{L_v(1-s_0',\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})}\right|, \tag{7.33}$$

where $\nu(s_0')=nm_v(1/2-s_0')+e_vs_0'$ is a constant depending only on the test function φ. Denote by $R_{v,l}^*(s,\lambda)$ the function $\Psi_v^*\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)/L_v(s,\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})$, $\mathrm{Re}(s)>0$. Then one combines (7.33) with (7.22) and (7.23) to get

$$\left|R_{v,l}^*(s_0,\lambda)\right|\geq q_v^{\nu(s_0)}\cdot\prod_{i=1}^r\prod_{j=1}^r\left|\frac{1-q_v^{s_0-1+\beta_{i,j}}}{1+q_v^{-1/2+\beta_{i,j}}}\right|^{n_i+n_j}\cdot\left|R_{v,l}^*(1/2,\lambda)\right|. \tag{7.34}$$

Let $c_{v,0}$ be a positive constant such that

$$q_v^{(s_0'-1/2)c_{v,0}}>q_v^{-\nu(s_0')}\left|\frac{(6-s_0')(s_0'+5)}{(6-1/2)(1/2+5)}\right|\cdot\prod_{i=1}^r\prod_{j=1}^r\left|\frac{1-q_v^{-1/2+\beta_{i,j}}}{1-q_v^{s_0'-1+\beta_{i,j}}}\right|^{n_i+n_j}.$$

Note that such a $c_{v,0}$ always exists since $s_0'>1/2$. Then it follows from (7.22), (7.23) and (7.34) that $\left|H_{v,l}(s_0',c)\right|\leq\left|H_{v,l}(s_1,c_{v,0})\right|$. Therefore, we have a contradiction by assuming that $\mathrm{Re}(s_1)=1/2$. Hence, we have $\mathrm{Re}(s_1)=5$ if $c=c_{v,0}$. Then for any s such that $\mathrm{Re}(s)=s_0'$, $\left|\Psi_v\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|\leq\left|\Psi_v^*\left(s_0',W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|$, which bounded, since $\left|H_{v,l}(s_0',c_{v,0})\right|\leq\left|H_{v,l}(s_1,c_{v,0})\right|$, by

$$\left|\frac{q_v^{5c_{v,0}}(6-s_0')(s_0'+5)L_v(s_0',\pi_{\lambda,v}\times\widetilde{\pi}_{-\lambda,v})}{10q_v^{c_{v,0}s_0'}L_v(s_1,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})}\right|\cdot\left|\Psi_v^*\left(5,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|.$$

Then by (7.23) and trivial estimate on $L_v(s_1,\pi_{\lambda,v}\otimes\tau_v\times\widetilde{\pi}_{-\lambda,v})$ one concludes that there exists some constant c_v'', depending only on φ, such that

$$\left|\Psi_v^*\left(s_0',W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_v\right)\right|\leq q_v^{c_v''}\sum_{l=1}^{L_v}\left|\Psi_v^*\left(5,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|. \tag{7.35}$$

Then combining (7.35) and Lemma 67 we have, for any s with $\epsilon\leq\mathrm{Re}(s)\leq1-\epsilon$,

$$\left|\Psi_v\left(s,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_v\right)\right|\leq q_v^{c_v'}\sum_{l=1}^{L_v}\left|\Psi_v^*\left(5,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l}\right)\right|, \tag{7.36}$$

where c_v' is a constant depending only on the test function φ.

Note that when φ_v is $G(\mathcal{O}_{F,v})$-invariant, then $\pi_{v,\lambda}$ is unramified. So the cardinality of the finite set $S(\pi,\Phi)$ is bounded in terms of τ, Φ and the K-finite type of the test function φ. Namely, there exists a finite set $S_{\varphi,\tau,\Phi}$ of prime ideals such that for any π from some cuspidal datum $\chi \in \mathfrak{X}_P$, one has $S(\pi,\Phi) \subseteq S_{\varphi,\tau,\Phi}$. Therefore, we conclude that

$$\left|R_{S(\pi,\Phi)}(s,W_\alpha;\lambda)\right| \leq \sum_{\boldsymbol{l}=(l_v)_{v\in S(\pi,\Phi)}} \prod_{v\in S(\pi,\Phi)} q_v^{c_v'}\left|\Psi_v^*\left(5,W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l_v}\right)\right|, \tag{7.37}$$

where the sum over multi-index $\boldsymbol{l}$ is finite in terms of φ, τ and Φ. Similarly,

$$\left|R_{S(\pi,\Phi)}(s,W_\beta;\lambda)\right| \leq \sum_{\boldsymbol{l}=(l_v)_{v\in S(\pi,\Phi)}} \prod_{v\in S(\pi,\Phi)} q_v^{c_v'}\left|\Psi_v^*\left(5,W_{\beta,v},W_{\beta,v};\lambda,\Phi_{v,l_v}\right)\right|. \tag{7.38}$$

By Proposition 49 we have, when $v \in \Sigma_{F,\text{fin}} - S(\pi,\Phi) \triangleq S_{\pi,\Phi}^{u.r.}$, that

$$R_v(s,\lambda) = \prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\lambda_i-\lambda_j,\pi_{i,v}\times\widetilde{\pi}_{j,v})^{-1}\cdot L_v(1-\lambda_i+\lambda_j,\widetilde{\pi}_{i,v}\times\pi_{j,v})^{-1}$$

is independent of s. So we write $R_v(\lambda)$ for $R_v(s,\lambda)$ in this case.

Let $s \in U_\epsilon(s_0)$ and $s' = 5 + i\,\mathrm{Im}(s)$. Then by (7.28) and (7.37) we see that when $\phi_1 = \phi_2 \in \mathfrak{B}_{P,\chi}$, $\left|R(s,\lambda)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right|$ is bounded by $\left|R_{S(\pi,\Phi)}(s,\lambda)\right|$ multiplying

$$\left|L(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right|\prod_{v\in S_{\pi,\Phi}^{u.r.}}\left|R_v(\lambda)\right|\cdot\sum_{\boldsymbol{j}\in J}\prod_{v\mid\infty}\left|\Psi_v\left(s,W_{\alpha,v},W_{\beta,v};\lambda,\Phi_{v,j_v}\right)\right|.$$

By (7.1) and (7.21) we have the preconvex bound $L(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda}) \ll_{F,\epsilon} C_\infty(\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda};\mathrm{Im}(s))$. Then combining this bound with (7.30) we have

$$\prod_{v\mid\infty}\left|\Psi_v\left(s,W_{\alpha,v},W_{\beta,v};\lambda,\Phi_{v,j_v}\right)\right| \ll \prod_{v\mid\infty}\left|\frac{\Psi_v\left(s',W_{\alpha,v},W_{\beta,v};\lambda,\Phi_{v,j_v}\right)}{L(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})}\right|, \tag{7.39}$$

where the implied constant is absolute. Let

$$\Psi_\infty\left(s',W_\alpha,W_\beta;\lambda,\Phi_{\boldsymbol{j}}\right) = \prod_{v\mid\infty}\Psi_v\left(s',W_{\alpha,v},W_{\beta,v};\lambda,\Phi_{v,j_v}\right)$$

if $\boldsymbol{j} = (j_v)_{v\mid\infty}$. Similarly, for any $\boldsymbol{l} = (l_v)_{v\in S(\pi,\Phi)}$, we denote by

$$\Psi_{ra}^*\left(s',W_\alpha;\lambda,\Phi_{\boldsymbol{l}}\right) = \prod_{v\in S(\pi,\Phi)}\Psi_v^*\left(s',W_{\alpha,v},W_{\alpha,v};\lambda,\Phi_{v,l_v}\right).$$

Let $C = \prod_{v\in S(\pi,\Phi)} q_v^{c_v'} < \infty$. Then combine (7.37), (7.38) and (7.39) to conclude that

$$\begin{aligned}\left|J_P(s)\right| \leq &\sum_\alpha\sum_\beta\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\left|R(s,W_\alpha,W_\beta;\lambda)\cdot\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right||d\lambda|\\ \leq &C\sum_\alpha\sum_\beta\left[\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}J_\alpha(\lambda)|d\lambda|\right]^{\frac{1}{2}}\left[\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}J_\beta(\lambda)|d\lambda|\right]^{\frac{1}{2}},\end{aligned}$$

where $J_\alpha(\lambda) = J_\alpha(\lambda; \chi, \phi)$ is defined by

$$\sum_{\boldsymbol{j}\in J}\sum_{\boldsymbol{l}}\left|\Psi_\infty\left(s', |W_\alpha|, |W_\alpha|; \lambda, |\Phi_{\boldsymbol{j}}|\right)\Psi_{ra}^*\left(5, W_\alpha; \lambda, \Phi_{\boldsymbol{l}}\right)\right|\cdot\prod_{v\in S_{\pi,\Phi}^{u.r.}}\left|R_v(\lambda)\right|.$$

Likewise, we have definition of $J_\beta(\lambda) = J_\beta(\lambda; \chi, \phi)$ of the same form. Note that

$$\begin{aligned}
&\left|\Psi_\infty\left(s', |W_\alpha|, |W_\alpha|; \lambda, |\Phi_{\boldsymbol{j}}|\right)\Psi_{ra}^*\left(5, W_\alpha; \lambda, \Phi_{\boldsymbol{l}}\right)\right|\cdot\prod_{v\in S_{\pi,\Phi}^{u.r.}}\left|R_v(\lambda)\right|\\
\leq&\left|\Psi_\infty\left(s', |W_\alpha|, |W_\alpha|; \lambda, |\Phi_{\boldsymbol{j}}|\right)\Psi_{ra}^*\left(5, W_\alpha; \lambda, \Phi_{\boldsymbol{l}}\right)\right|\cdot\prod_{v\in S_{\pi,\Phi}^{u.r.}}\left|\frac{\Psi_v^*\left(s', W_{\alpha,v}; \lambda, |\Phi_v|\right)}{L(s', \pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})}\right|\\
\leq&C_0\prod_{v\in\Sigma_F}\int_{N(F_v)\backslash G(F_v)}\left|W_\alpha(x_v;\lambda)\overline{W_\alpha\left(x_v;\lambda\right)}\Phi_{\boldsymbol{j},\boldsymbol{l},v}(\eta x_v)\right|\cdot|\det x_v|_{F_v}^5\,dx_v,
\end{aligned}$$

where $\Phi_{\boldsymbol{j},\boldsymbol{l},v}$ is certain positive Schwartz function of form (7.26) and (7.27) if v is archimedean; $\Phi_{\boldsymbol{j},\boldsymbol{l},v} = |\Phi_{v,l_v}|$, if $v \in S_{\varphi,\tau,\Phi}$; and $\Phi_{\boldsymbol{j},\boldsymbol{l},v} = |\Phi_v|$ otherwise; and C_0 is an absolute constant, independent of π and λ. Note that $\Phi_{\boldsymbol{j},\boldsymbol{l}} \in \mathcal{S}_0(\mathbb{A}_F^n)$. Denote by $\Psi^*\left(5, W_\alpha; \lambda, \Phi_{\boldsymbol{j},\boldsymbol{l}}\right)$ the last integral in the above inequalities. Then we have by the first part of Theorem G that

$$\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}J_\alpha(\lambda)d\lambda \leq \sum_{\boldsymbol{j}\in J}\sum_{\boldsymbol{l}}\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\Psi^*\left(5, W_\alpha; \lambda, \Phi_{\boldsymbol{j},\boldsymbol{l}}\right)d\lambda < \infty,$$

since the sums over $\boldsymbol{j}$ and $\boldsymbol{l}$ are finite. Similarly, one has

$$\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}J_\beta(\lambda)d\lambda \leq \sum_{\boldsymbol{j}\in J}\sum_{\boldsymbol{l}}\sum_{\chi\in\mathfrak{X}_P}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\Psi^*\left(5, W_\beta; \lambda, \Phi_{\boldsymbol{j},\boldsymbol{l}}\right)d\lambda < \infty.$$

Since the sums over α and β are finite, and since there are only finitely many standard parabolic subgroups P of G, we have shown that

$$\sum_{\chi}\sum_{P\in\mathcal{P}}\frac{1}{|c_P|}\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{\Lambda^*}\max_{\mathrm{Re}(s)=s_0}\left|R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right|d\lambda < \infty, \tag{7.40}$$

for any $1-1/(n^2+1) \leq s_0 < 1$. Now we apply Proposition 60 to this result to see that (7.40) holds for any $0 < \mathrm{Re}(s) \leq 1/(n^2+1)$. Note that $R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$ is analytic inside the strip $1/(n^2+1) \leq \mathrm{Re}(s) \leq 1-1/(n^2+1)$. Then by Phragmén-Lindelöf principle we have that $\left|R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right|$ is bounded by

$$\max_{s_0\in\{1/(n^2+1),1-1/(n^2+1)\}}\max_{\mathrm{Re}(s)=s_0}\left|R_\varphi(s,\lambda;\phi)\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})\right|.$$

Therefore, (7.25) holds for all $s \in S_{(0,1)}$. □

Corollary 69. *Let notation be as before. Assume τ is such that $\tau^k \neq 1$ for all $1 \leq k \leq n$. Then*

$$\sum_{\chi} \sum_{P \in \mathcal{P}} \frac{1}{c_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{\Lambda^*} R_{\varphi}(s, \lambda; \phi) \Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda}) d\lambda$$

admits a holomorphic continuation to the whole s-plane.

Proof. Since $\tau^k \neq 1$ for all $1 \leq k \leq n$, then $\pi_\lambda \otimes \tau \not\simeq \pi_\lambda$ for all λ. Then $\Lambda(s, \pi_\lambda \otimes \tau \times \widetilde{\pi}_{-\lambda})$ is entire. Hence the arguments in the proof of Theorem H (with $V(s, \lambda)$ removed) works here for all $\mathrm{Re}(s) > 0$. Then Corollary 69 follows from the functional equation Proposition 60. □

We note that (7.25) converges absolutely when $0 < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(s) > 1$, and Corollary 69 gives a special case where (7.25) converges for all $\mathrm{Re}(s) > 0$. However, for *general* τ, the holomorphic functions defined by (7.25) in $0 < \mathrm{Re}(s) < 1$ and $\mathrm{Re}(s) > 1$ are not compatible, i.e., they do not give a natural continuation. The reason is that in this case (7.25) may diverge for *all* s with $\mathrm{Re}(s) = 1$, e.g., this happens when τ is trivial.

To handle these (finitely many) cases, we will consider the $I_{\mathrm{Whi}}(s, \tau)$ in $\mathrm{Re}(s) > 1$ and obtain its continuation to the half plane by analyzing residues of certain functions of several complex variables in Section 8.

Chapter 8

HOLOMORPHIC CONTINUATION VIA MULTIDIMENSIONAL RESIDUES

From preceding estimates, we see that when $\mathrm{Re}(s) > 1$, $I_{\mathrm{Whi}}(s,\tau)$ is a combination of Rankin-Selberg convolutions for automorphic functions which are not of rapid decay. Zagier [Zag81] computed the Rankin-Selberg transform of some type of automorphic functions and derived the desired holomorphic continuation for $n = 2$ and $F = \mathbb{Q}$ case. However, general Eisenstein series for $\mathrm{GL}(n)$ do not have the asymptotic properties as Zagier considered, since there are mixed terms in the Fourier expansion (see Proposition 26). Thus one needs to develop a different approach to obtain the continuation.

$I_{\mathrm{Whi}}(s,\tau)$ can be written as a sum of functions $\int_\Lambda \mathcal{F}(s,\lambda)d\lambda$, which is well defined when $\mathrm{Re}(s) > 1$. Moreover, for each s_0 with $\mathrm{Re}(s_0) = 1$, there exists some $\lambda_0 \in \Lambda$ such that $F(s,\lambda_0)$ is singular at $s = s_0$. Hence the original integral representations for $I_{\mathrm{Whi}}(s,\tau)$ have singularities at all points on the line $\mathrm{Re}(s) = 1$. We shall use contour-shifting and Cauchy's theorem to continue $I_{\mathrm{Whi}}(s,\tau)$. To illustrate the underlying idea, we simply "think" $\mathcal{F}(s,\lambda) = (s-1-\lambda)^{-1}\cdot(s-1+\lambda)^{-1}$ and $\Lambda = i\mathbb{R}$, namely,

$$I_{\mathrm{Whi}}(s,\tau) = \int_{-i\infty}^{i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}d\lambda, \quad \mathrm{Re}(s) > 1.$$

Now we fix s such that $1 < \mathrm{Re}(s) < 1 + \epsilon/2$, for some small $\epsilon > 0$. Then shift contour to see

$$I_{\mathrm{Whi}}(s,\tau) = \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{1}{(s-1-\lambda)(s-1+\lambda)}d\lambda - \frac{1}{2(s-1)}. \tag{8.1}$$

Note that the right hand side of (8.1) defines a meromorphic function in the region $1-\epsilon/2 < \mathrm{Re}(s) < 1+\epsilon/2$, with a simple at $s = 1$. Hence we obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the region $\mathrm{Re}(s) > 1-\epsilon/2$. Do this process one more time one then gets a meromorphic continuation to the whole complex plane, with explicit description on poles.

Just as the above prototype, the genuine situation admits the same idea of continuation, but with more delicate techniques required, since $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ is typically infinitely many sums of such integrals. Details will be provide in the following sections. Moreover, we find all possible explicit poles of the continuation of each such integral as well, and show they cancel with each other except for $s = 1/2$, where $I_{\mathrm{Whi}}(s,\tau)/\Lambda(s,\tau)$ has at most a simple pole if $\tau^2 = 1$.

8.1 Continuation via a Zero-free Region

Recall that we fix the unitary character τ. Let $\mathcal{D}_\tau$ be a standard (open) zero-free region of $L(s,\tau)$ (e.g. see [Bru06]). We fix such a $\mathcal{D}_\tau$ once for all. We thus can form a domain

$$\mathcal{R}(1/2;\tau)^- := \{s \in \mathbb{C} :\ 2s \in \mathcal{D}_\tau\} \supsetneq \{s \in \mathbb{C} :\ \mathrm{Re}(s) \geq 1/2\}. \tag{8.2}$$

In Section 8.3, we will continue $I_{\mathrm{Whi}}(s,\tau)$ to the open set $\mathcal{R}(1/2;\tau)^-$. Invoking (8.2) with functional equation we then obtain a meromorphic continuation of $I_{\mathrm{Whi}}(s,\tau)$ to the whole complex plane.

Let P be a standard parabolic subgroup of G of type $(n_1, n_2, \cdots, n_r)$. Let $\mathfrak{X}_P$ be the subset of cuspidal data $\chi = \{(M,\sigma)\}$ such that $M = M_P = \mathrm{diag}(M_1, M_2, \cdots, M_r)$, where M_i is n_i by n_i matrix, $1 \leq i \leq r$. We may write $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_r)$, where $\sigma_i \in \mathcal{A}_0(M_i(F)\backslash M_i(\mathbb{A}_F))$. Let π be a representation induced from $\chi = \{(M,\sigma)\}$.

For any $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_r) \in i\mathfrak{a}_P^*/i\mathfrak{a}_G^* \simeq (i\mathbb{R})^{r-1}$, satisfying that $\lambda_1+\lambda_2+\cdots+\lambda_r = 0$, we let $\boldsymbol{\kappa} = (\kappa_1, \kappa_2, \cdots, \kappa_r) \in \mathbb{C}^{r-1}$ be such that

$$\begin{cases} \kappa_j = \lambda_j - \lambda_{j+1},\ 1 \leq j \leq r-1, \\ \kappa_r = \lambda_1 - \lambda_r = \kappa_1 + \kappa_2 + \cdots + \kappa_{r-1}. \end{cases} \tag{8.3}$$

Then we have a bijection $i\mathfrak{a}_P^*/i\mathfrak{a}_G^* \overset{1:1}{\longleftrightarrow} i\mathfrak{a}_P^*/i\mathfrak{a}_G^*, \boldsymbol{\lambda} \mapsto \boldsymbol{\kappa}$ given by (8.3), which induces a change of coordinates with $d\boldsymbol{\lambda} = m_P d\boldsymbol{\kappa}$, where m_P is an absolute constant (the determinant of the transform (8.3)). So that we can write $\boldsymbol{\lambda} = \boldsymbol{\lambda}(\boldsymbol{\kappa})$. Let $R_\varphi(s,\boldsymbol{\lambda};\phi)$ be defined by (6.22) and $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda})$ be the complete L-function. Then we can write $R_\varphi(s,\boldsymbol{\lambda};\phi) = R_\varphi(s,\boldsymbol{\kappa};\phi)$ and $\Lambda(s,\pi_\lambda\otimes\tau\times\widetilde{\pi}_{-\lambda}) = \Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}})$. Recall that if $v \in \Sigma_{F,\mathrm{fin}}$ is a finite place such that π_v is unramified and $\Phi_v = \Phi_v^\circ$ is the characteristic function of $G(\mathcal{O}_{F,v})$. Assume further that $\phi_{1,v} = \phi_{2,v} = \phi_v^\circ$ be the unique $G(\mathcal{O}_{F,v})$-fixed vector in the space of π_v such that $\phi_v^0(e) = 1$. Then $R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\lambda}) = R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\kappa})$ is equal to (53), which is, in the $\boldsymbol{\kappa}$-coordinate, that

$$\prod_{1\leq i<r}\prod_{i<j\leq r} L_v(1+\boldsymbol{\kappa}_{i,j}, \sigma_{i,v}\times\widetilde{\sigma}_{j,v})^{-1}\cdot L_v(1-\boldsymbol{\kappa}_{i,j}, \widetilde{\sigma}_{i,v}\times\sigma_{j,v})^{-1}, \tag{8.4}$$

where $\boldsymbol{\kappa}_{i,j} = \boldsymbol{\kappa}_i + \cdots + \boldsymbol{\kappa}_{j-1}$. By the K-finiteness of φ, there exists a finite set $S_{\varphi,\tau,\Phi}$ of nonarchimedean places such that for any π from some cuspidal datum

$\chi \in \mathfrak{X}_P$, $R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\kappa})$ is equal to the formula in (8.4). Then according to Proposition 49 and Proposition 54 we see that, when $\mathrm{Re}(s) > 0$, $R_v(s, W_{1,v}, W_{2,v}; \boldsymbol{\kappa})$ are independent of s for all but finitely many places v. Therefore, as a function of s, $R_\varphi(s, \boldsymbol{\kappa}; \phi)$ is a finite product of holomorphic function in $\mathrm{Re}(s) > 0$; for any given s such that $\mathrm{Re}(s) > 0$, as a complex function of multiple variables with respect to $\boldsymbol{\kappa}$, $R_\varphi(s, \boldsymbol{\kappa}; \phi)$ has the property that $R_\varphi(s, \boldsymbol{\kappa}; \phi) L_S(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is holomorphic, where $L_S(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is denoted by the meromorphic function

$$\prod_{1 \leq i < r} \prod_{i < j \leq r} \prod_{v \in S_{\varphi,\tau,\Phi}} L_v(1 + \boldsymbol{\kappa}_{i,j}, \sigma_{i,v} \times \widetilde{\sigma}_{j,v}) \cdot L_v(1 - \boldsymbol{\kappa}_{i,j}, \widetilde{\sigma}_{i,v} \times \sigma_{j,v}).$$

Hence $R_\varphi(s, \boldsymbol{\kappa}; \phi)$ is holomorphic in some domain $\mathcal{D}$ if $L_S(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is nonvanishing in $\mathcal{D}$. Now we are picking up such a zero-free region $\mathcal{D}$ explicitly.

Let $1 \leq m, m' \leq n$ be two integers. Let $\sigma \in \mathcal{A}_0(GL_m(F) \backslash GL_m(\mathbb{A}_F))$ and $\sigma' \in \mathcal{A}_0(GL_{m'}(F) \backslash GL_{m'}(\mathbb{A}_F))$. Fix $\epsilon_0 > 0$. For any $c' > 0$, let $\mathcal{D}_{c'}(\sigma, \sigma')$ be

$$\left\{ \kappa = \beta + i\gamma : \ \beta \geq 1 - c' \cdot \left[\frac{(C(\sigma)C(\sigma'))^{-2(m+m')}}{(|\gamma| + 3)^{2mm'[F:\mathbb{Q}]}} \right]^{\frac{1}{2} + \frac{1}{2(m+m')} - \epsilon_0} \right\}, \tag{8.5}$$

if $\sigma' \not\simeq \widetilde{\sigma}$; and let $\mathcal{D}_{c'}(\sigma, \sigma')$ denote by the region

$$\left\{ \kappa = \beta + i\gamma : \ \beta \geq 1 - c' \cdot \left[\frac{(C(\sigma))^{-8m}}{(|\gamma| + 3)^{2mm^2[F:\mathbb{Q}]}} \right]^{-\frac{7}{8} + \frac{5}{8m} - \epsilon_0} \right\}, \tag{8.6}$$

if $\sigma' \simeq \widetilde{\sigma}$. According to [Bru06] and the Appendix of [Lap13], there exists a constant $c_{m,m'} > 0$ depending only on m and m', such that $L(\boldsymbol{\kappa}, \sigma \times \sigma')$ does not vanish in $\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_r) \in \mathcal{D}_{c_{m,m'}}(\sigma, \sigma') \times \cdots \times \mathcal{D}_{c_{m,m'}}(\sigma, \sigma')$. Let $c = \min_{1 \leq m, m' \leq n} c_{m,m'}$ and $C(\sigma, \sigma')$ be the boundary of $\mathcal{D}_c(\sigma, \sigma')$. We may assume that c is small such that the curve $C(\sigma, \sigma')$ lies in the strip $1 - 1/(n+4) < \mathrm{Re}(\kappa_j) < 1$, $1 \leq j \leq r$. Fix such a c henceforth. Note that by our choice of c, $L(\boldsymbol{\kappa}, \sigma \times \sigma')$ is nonvanishing in $\mathcal{D}_c(\sigma, \sigma') \times \cdots \times \mathcal{D}_c(\sigma, \sigma')$ for any $1 \leq m, m' \leq n$. For $v \in S_{\varphi,\tau,\Phi}$, we have that

$$\left| L_v(\boldsymbol{\kappa}, \sigma_v \times \sigma'_v)^{-1} \right| \leq \prod_{i=1}^{r} \prod_{j=1}^{r} \left(1 + q_v^{1 - \frac{1}{m^2+1} - \frac{1}{m'^2+1}} \right)^{n_i + n_j} < \infty,$$

for any $\boldsymbol{\kappa}$ such that each $\mathrm{Re}(\kappa_j) \geq 0$, $1 \leq j \leq r$. Let $L_S(\boldsymbol{\kappa}, \sigma \times \sigma') = L(\boldsymbol{\kappa}, \sigma \times \sigma') \prod_{v \in S_{\varphi,\tau,\Phi}} L_v(\boldsymbol{\kappa}, \sigma_v \times \sigma'_v)^{-1}$. Then $L_S(\boldsymbol{\kappa}, \sigma \times \sigma')$ is nonvanishing in $\mathcal{D}_c(\sigma, \sigma') \times \cdots \times \mathcal{D}_c(\sigma, \sigma')$ for any $1 \leq m, m' \leq n$.

Let $\chi \in \mathfrak{X}_P$ and $\pi = \mathrm{Ind}_{P(\mathbb{A}_F)}^{G(\mathbb{A}_F)}(\sigma_1, \sigma_2, \cdots, \sigma_r) \in \chi$. For any $\epsilon \in (0, 1]$ we set

$$\mathcal{D}_\chi(\epsilon) = \bigcap_{1 \leq i \leq r} \bigcap_{i < j \leq r} \left\{ \kappa \in \mathbb{C} : \ \mathrm{Re}(\kappa) \geq 0, \ 1 - \kappa \in \mathcal{D}_{c\epsilon}(\sigma_i, \sigma_j) \right\}. \tag{8.7}$$

Also, for $\epsilon = 0$, we set $\mathcal{D}_\chi(0) = \{\kappa \in \mathbb{C} : \operatorname{Re}(\kappa) \geq 0\}$. Then by the above discussion, as a function of $\boldsymbol{\kappa}$, $L_S(\boldsymbol{\kappa}, \pi, \widetilde{\pi})$ is nonzero in the region $\mathcal{D}_\chi(\boldsymbol{\epsilon}) = \{\boldsymbol{\kappa} = (\kappa_1, \cdots, \kappa_r) \in \mathbb{C}^r : \kappa_l \in \mathcal{D}_\chi(\epsilon_l)\}$, where $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_r) \in [0,1]^r$. We can write $\mathcal{D}_\chi(\boldsymbol{\epsilon})$ as a product space $\mathcal{D}_\chi(\boldsymbol{\epsilon}) = \prod_{l=1}^r \mathcal{D}_\chi(\epsilon_l)$, and let $\partial\mathcal{D}_\chi(\epsilon_l)$ be the boundary of $\mathcal{D}_\chi(\epsilon_l)$. Then when $\epsilon_l > 0$, $\partial\mathcal{D}_\chi(\epsilon_l)$ has two connected components and one of which is exactly the imaginary axis. Let $C_\chi(\epsilon_l)$ be the other component, which is a continuous curve, where $0 \leq \epsilon_l \leq 1$. When $\epsilon_l = 0$, let $C_\chi(\epsilon_l)$ be the maginary axis. Set $C_\chi(\boldsymbol{\epsilon}) = C_\chi(\epsilon_1) \times \cdots \times C_\chi(\epsilon_{r-1})$, $0 \leq \epsilon_l \leq 1$, $1 \leq l \leq r-1$.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_{r-1}) \in [0,1]^{r-1}$. Then by the above construction, $R_\varphi(s, \boldsymbol{\kappa}; \phi)$ is holomorphic in $\mathcal{D}_\chi(\boldsymbol{\epsilon})$. Hence $R_\varphi(s, \boldsymbol{\kappa}; \phi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}})$ is holomorphic in $\mathcal{D}_\chi(\boldsymbol{\epsilon})$. Moreover, $L_S(\boldsymbol{\kappa}, \pi, \widetilde{\pi}) \neq 0$ on $C_\chi(\boldsymbol{\epsilon})$, for any $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_{r-1}) \in [0,1]^{r-1}$ and any cuspidal datum $\chi \in \mathfrak{X}_P$. Let $\operatorname{Re}(s) > 1$. For any $\phi \in \mathfrak{B}_{P,\chi}$ and $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_{r-1}) \in [0,1]^{r-1}$, let

$$J_{P,\chi}(s; \phi, C_\chi(\boldsymbol{\epsilon})) = \int_{C_\chi(\boldsymbol{\epsilon})} R_\varphi(s, \boldsymbol{\kappa}; \phi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}) d\boldsymbol{\kappa}. \tag{8.8}$$

which is well defined because $J_{P,\chi}(s; \phi, C_\chi(\boldsymbol{\epsilon})) = J_{P,\chi}(s; \phi, C_\chi(\mathbf{0}))$ by Cauchy integral formula. Therefore, according to Theorem G,

$$\sum_P \frac{1}{c_P} \sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{C_\chi(\boldsymbol{\epsilon})} \left| R_\varphi(s, \boldsymbol{\kappa}; \phi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}) \right| d\boldsymbol{\kappa} < \infty$$

for any $\operatorname{Re}(s) > 1$, $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_{r-1}) \in [0,1]^{r-1}$.

Let $\boldsymbol{\epsilon} = (\epsilon_1, \cdots, \epsilon_{r-1}) \in [0,1]^{r-1}$. For any $\beta \geq 1/2$, we denote by

$$\mathcal{R}(\beta; \chi, \boldsymbol{\epsilon}) = \left\{s \in 1 + \mathcal{D}_\chi(\boldsymbol{\epsilon})\right\} \bigcup \left\{s \in 1 - \mathcal{D}_\chi(\boldsymbol{\epsilon})\right\}. \tag{8.9}$$

Lemma 70. *Let notation be as before. Let $P \in \mathcal{P}$ and let $\boldsymbol{\epsilon} = (1/n, 1/n, \cdots, 1/n) \in \mathbb{R}^{n-1}$. Then for any $s \in \mathcal{R}(1; \chi, \boldsymbol{\epsilon}) \setminus \{1\}$, we have*

$$\sum_{\chi \in \mathfrak{X}_P} \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{C_\chi(\boldsymbol{\epsilon})} \left| R_\varphi(s, \boldsymbol{\kappa}; \phi)\Lambda(s, \pi_{\boldsymbol{\kappa}} \otimes \tau \times \widetilde{\pi}_{-\boldsymbol{\kappa}}) \right| d\boldsymbol{\kappa} < \infty. \tag{8.10}$$

Proof. We start with a variant of Lemma 37:

Claim 71. *Let notation be as before. Let*

$$\widehat{\mathrm{K}}_\infty(x, y) := \int_{N(F)\backslash N(\mathbb{A}_F)} \int_{N(F)\backslash N(\mathbb{A}_F)} \mathrm{K}_\infty(n_1 x, n_2 y)\theta(n_1)\bar{\theta}(n_2) dn_1 dn_2. \tag{8.11}$$

Then $\widehat{\mathrm{K}}_\infty(x, y)$ is equal to

$$\sum_{\chi \in \mathfrak{X}} \sum_{P \in \mathcal{P}} \frac{1}{k_P!(2\pi)^{k_P}} \int_{C_\chi(\boldsymbol{\epsilon})} \sum_{\phi \in \mathfrak{B}_{P,\chi}} W(x, \mathcal{I}_P(\lambda, \varphi)\phi, \lambda)\overline{W(y, \phi, \lambda)} d\lambda. \tag{8.12}$$

Then the proof is similar as that of Theorem H except that Lemma 37 should be replaced with Claim 71 and the constant e_v in (7.31) is replaced with $e_v + 1$. □

Proof of Claim 71. The main idea of the proof is similar to Lemma 37. For any $P \in \mathcal{P}$, let $c_P = k_P!(2\pi)^{k_P}$. Applying Cauchy's integral formula we see that $\mathrm{K}_\infty(x, y)$ is equal to

$$\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{k_P!(2\pi)^{k_P}}\int_{C_\chi(\epsilon)}\sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda, \tag{8.13}$$

the absolute convergence of (8.13) is justified in [Art79] invoking Langlands' work on Eisenstein theory (see [Lan76]).

Substitute (8.13) into (8.11) to get an at least formal expansion of $\widehat{\mathrm{K}}_\infty(x, y)$, which is clearly dominated by the following formal expression

$$\int_{[N]}\int_{[N]}\sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{c_P}\left|\int_{C_\chi(\epsilon)}\sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\overline{E(y,\phi,\lambda)}d\lambda\right|dn_1dn_2.$$

Denote by J_G the above integral. We will show J_G is finite, hence

$$\widehat{\mathrm{K}}_\infty(x, y) = \sum_{\chi\in\mathfrak{X}}\sum_{P\in\mathcal{P}}\frac{1}{k_P!(2\pi)^{k_P}}\int_{i\mathfrak{a}_P^*/i\mathfrak{a}_G^*}\sum_{\phi\in\mathfrak{B}_{P,\chi}} W_{\mathrm{Eis},1}(x;\lambda)\overline{W_{\mathrm{Eis},2}(y;\lambda)}d\lambda,$$

is well defined. One can write the test function φ as a finite linear combination of convolutions $\varphi_1 * \varphi_2$ with functions $\varphi_i \in C_c^r(G(\mathbb{A}_F))$, whose archimedean components are differentiable of arbitrarily high order r. Then one applies Hölder inequality to it. Clearly it is enough to deal with the special case that $\varphi = \varphi_j * \varphi_j^*$, where $\varphi_j^*(x) = \overline{\varphi_j(x^{-1})}$, and $x = y$. Note that $\mathfrak{B}_{P,\chi}$ is finite due to the K-finiteness assumption, and Eisenstein series converge absolutely for our λ, hence the integrand

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda,\varphi)\phi,\lambda)\overline{E(x,\phi,\lambda)} = \sum_{\phi\in\mathfrak{B}_{P,\chi}} E(x, \mathcal{I}_P(\lambda,\varphi_j)\phi,\lambda)\overline{E(x,\mathcal{I}_P(\lambda,\varphi_j)\phi,\lambda)}$$

is well defined and obviously nonnegative. In fact, the double integral over λ and ϕ can be expressed as an increasing limit of nonnegative functions, each of which is the kernel of the restriction of $R(\varphi_j * \varphi_j^*)$, a positive semidefinite operator, to an invariant subspace. Since this limit is bounded by the nonnegative function

$$\mathrm{K}_j(x, x) = \sum_{\gamma\in Z(F)\backslash G(F)} \varphi_j * \varphi_j^*(x^{-1}\gamma x),$$

and the domain $[N] = N(F)\backslash N(\mathbb{A}_F)$ is compact, the integral J_G converges. □

8.2 Meromorphic continuation of $J_{P,\chi}(s;\phi,C_\chi(\boldsymbol{\epsilon}))$ across the critical line $\mathrm{Re}(s)=1$

Let $\boldsymbol{\epsilon}=(1/n,1/n,\cdots,1/n)\in\mathbb{R}^{n-1}$ and $s\in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon})$ and $\mathrm{Re}(s)>1$. Then by (6.19) we see that $R(s,W_1,W_2;\boldsymbol{\kappa},\phi)\Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}})$ is equal to a holomorphic function multiplying

$$\prod_{k=1}^{r}\Lambda(s,\sigma_k\otimes\tau\times\widetilde{\sigma}_k)\prod_{j=1}^{r-1}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\sigma_i\otimes\tau\times\widetilde{\sigma}_{j+1})\Lambda(s-\kappa_{i,j},\sigma_{j+1}\otimes\tau\times\widetilde{\sigma}_i)}{\Lambda(1+\kappa_{i,j},\sigma_i\times\widetilde{\sigma}_{j+1})\Lambda(1-\kappa_{i,j},\sigma_{j+1}\times\widetilde{\sigma}_i)}.$$

Let $\mathcal{G}(\boldsymbol{\kappa};s)=\mathcal{G}(\boldsymbol{\kappa};s,P,\chi)$ denotes the above product. Also, for simplicity, we denote by $\mathcal{F}(\boldsymbol{\kappa};s)=\mathcal{F}(\boldsymbol{\kappa};s,P,\chi)$ the function $R_\varphi(s,\boldsymbol{\kappa};\phi)\Lambda(s,\pi_{\boldsymbol{\kappa}}\otimes\tau\times\widetilde{\pi}_{-\boldsymbol{\kappa}})$ if χ is fixed in the context. Then the Rankin-Selberg theory implies that $\mathcal{F}(\boldsymbol{\kappa};s)/\mathcal{G}(\boldsymbol{\kappa};s)$ can be continued to an entire function. We will write C for the boundary $C_\chi(1)$, and (0) for the imaginary axis. Then an analysis on the potential poles of $\mathcal{G}(\boldsymbol{\kappa};s)$ leads to an expression for the integral $J_{P,\chi}(s;\phi,C_\chi(\mathbf{0}))=J_{P,\chi}(s;\phi,C)-\mathcal{J}_\chi(s)$, where

$$\mathcal{J}_\chi(s)=\sum_{j=1}^{r-1}\sum_{i=1}^{j}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j+1},$$

where $\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)$ is identically vanishing unless $\sigma_i\otimes\tau\simeq\sigma_{j+1}$, in which case one must have $n_i=n_{j+1}$. Let $S(r)$ be the symmetric group acting on $\{1,2,\cdots,r\}$. To obtain meromorphic continuation of $J_{P,\chi}(s;\phi,C_\chi(\boldsymbol{\epsilon}))$ to the critical strip $0<\mathrm{Re}(s)<1$, we start with the following initial step:

Proposition 72. *Let notation be as before. Let $\chi\in\mathfrak{X}_P$. Let $\boldsymbol{\epsilon}=(1/n,1/n,\cdots,1/n)$ and $s\in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon})$ and $\mathrm{Re}(s)>1$. Then*

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C_\chi(\mathbf{0}))=\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C)-\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{J}(s;\phi,C),\qquad(8.14)$$

where $C=C_\chi$, and the summand $\mathcal{J}(s;\phi,C)$ is defined to be

$$\sum_{m=1}^{r-1}\sum_{\substack{j_m,j_{m-1},\cdots,j_1\\1\leq j_m<\cdots<j_1\leq r-1}}\cdots\sum c_{j_1,\cdots,j_m}\int_C\cdots\cdots\int_C\operatorname*{Res}_{\kappa_{j_m}=s-1}\cdots\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}},$$

where $c_{j_1,\cdots,j_m}$'s are some explicit integers, and $d\kappa_{r-1}\cdots d\kappa_1/(d\kappa_{j_m}\cdots d\kappa_{j_1})$ means $d\kappa_{r-1}\cdots\widehat{d\kappa_{j_m}}\cdots\widehat{d\kappa_{j_1}}\cdots d\kappa_1$; namely, omitting $d\kappa_{j_m},\cdots,d\kappa_{j_1}$. Moreover, the terms in (8.14) *converges absolutely and normally inside $\mathcal{R}(1;\chi,\boldsymbol{\epsilon})\setminus\{1\}$, where $\mathcal{R}(1;\chi,\boldsymbol{\epsilon})$ is defined in* (8.9). *Hence* (8.14) *gives a meromorphic continuation of the function $\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C_\chi(\mathbf{0}))$ to $\mathcal{R}(1;\chi,\boldsymbol{\epsilon})$, with a potential pole at $s=1$.*

Proof. For any $1 \leq j \leq r-1$ and $1 \leq i \leq j$, if $n_i = n_{j+1}$, we can take the following change of variables to simplify the integral of $\underset{\kappa_{i,j}=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}; s)$:

$$\begin{cases} \lambda_l' = \lambda_l, \ l \neq i, j; \\ \lambda_i' = \lambda_j, \ \lambda_j' = \lambda_i, \end{cases} \quad \text{and} \quad \begin{cases} \sigma_l' = \sigma_l, \ l \neq i, j; \\ \sigma_i' = \sigma_j, \ \sigma_j' = \sigma_i. \end{cases}$$

Let $\kappa_l' = \lambda_l' - \lambda_{l+1}'$, $1 \leq l \leq r-1$; and $\kappa_{l,m}' = \kappa_l' + \cdots + \kappa_m'$, $1 \leq l \leq m \leq r-1$. To describe the relation between $\{\kappa_l : \ 1 \leq l \leq r-1\}$ and $\{\kappa_l' : \ 1 \leq l \leq r-1\}$, we need to consider separately as follows:

Case 1 If $i = j-1$. Then a direct computation shows that

$$\begin{cases} \kappa_l = \kappa_l', \ 1 \leq l \leq r-1, l \neq i-1, i, i+1; \\ \kappa_{i-1} = \kappa_{i-1,i}', \ \kappa_i = -\kappa_i', \ \kappa_{i+1} = \kappa_{i,i+1}'. \end{cases}$$

Hence, the domains $\operatorname{Re}(\kappa_l) = 0, 1 \leq l \leq i = j-1$ are equivalent to $\operatorname{Re}(\kappa_l') = 0$, $1 \leq l \leq i = j-1$. Note that $\det\{\partial\kappa_l/\partial\kappa_m'\}_{1\leq l,m\leq r-1} = -1$, and $\kappa_j' = \kappa_{i,j} = s-1$, then one has

$$\begin{aligned} &\int_{(0)} \cdots \int_{(0)} d\kappa_{j-1} \cdots d\kappa_1 \int_C \cdots \int_C \underset{\kappa_{i,j}=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}; s) d\kappa_{r-1} \cdots d\kappa_{j+1} \\ = &-\int_{(0)} \cdots \int_{(0)} d\kappa_{j-1}' \cdots d\kappa_1' \int_C \cdots \int_C \underset{\kappa_j=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}; s, P, \chi') d\kappa_{r-1}' \cdots d\kappa_{j+1}', \end{aligned}$$

where χ' is the cuspidal datum attached to representations $(\sigma_1', \cdots, \sigma_r')$. Hence $\chi' = \chi$ as an equivalent class.

Case 2 If $i \leq j-2$. Then a direct computation leads to that

$$\begin{cases} \kappa_l = \kappa_l', \ 1 \leq l \leq r-1, l \neq i-1, i, j-1, j; \\ \kappa_{i-1} = \kappa_{i-1,j-1}', \ \kappa_i = -\kappa_{i+1,j-1}', \ \kappa_{j-1} = -\kappa_{i,j-2}', \ \kappa_j = \kappa_{i,j}'. \end{cases}$$

One can show inductively that the domains $\operatorname{Re}(\kappa_l) = 0$, $1 \leq l \leq i = j-1$ are equivalent to $\operatorname{Re}(\kappa_l') = 0$, $1 \leq l \leq i = j-1$. Note that the determinant of transition matrix $\det\{\partial\kappa_l/\partial\kappa_m'\}_{1\leq l,m\leq r-1} = -1$, and $\kappa_j' = \kappa_{i,j}$, so again

$$\begin{aligned} &\int_{(0)} \cdots \int_{(0)} d\kappa_{j-1} \cdots d\kappa_1 \int_C \cdots \int_C \underset{\kappa_{i,j}=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}; s) d\kappa_{r-1} \cdots d\kappa_{j+1} \\ = &-\int_{(0)} \cdots \int_{(0)} d\kappa_{j-1}' \cdots d\kappa_1' \int_C \cdots \int_C \underset{\kappa_j=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}; s, P, \chi') d\kappa_{r-1}' \cdots d\kappa_{j+1}'. \end{aligned}$$

While if $n_i \neq n_{j+1}$, then $\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)=0$. In all, we have

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{i,j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j+1}$$
$$=-\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j+1}.$$

Therefore, we see that $\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C_\chi(\mathbf{0}))-\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C)$ equals

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}}\sum_{j=1}^{r-1}c_j'\int_{(0)}\cdots\int_{(0)}d\kappa_{j-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{j}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j+1},$$

where c_j''s are some explicit constants, depending only on the type of P. Consider

$$\int_{(0)}\cdots\int_{(0)}d\kappa_{j_1-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j_1+1},\ 1\leq j_1\leq r-1.$$

Then by Cauchy integral formula we can write it as the sum of

$$\int_C\cdots\int_C d\kappa_{j_1-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)d\kappa_{r-1}\cdots d\kappa_{j_1+1}\text{ and}$$

$$\sum_{j_2=1}^{j_1-1}\sum_{i_2=1}^{j_2}c_{i_2,j_2}'\int_{(0)}\cdots\int_{(0)}d\kappa_{j_2-1}\cdots d\kappa_1\int_C\cdots\int_C\operatorname{Res}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_{j_2+1}}{d\kappa_{j_1}},$$

where c_{i_2,j_2}' are some explicit integers depending only on the type of P. $\operatorname{Res}\mathcal{F}(\boldsymbol{\kappa};s)=\operatorname*{Res}_{\kappa_{i_2,j_2}=s-1}\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)$. Then one can do the similar analysis to replace $\kappa_{i_2,j_2}=s-1$ with $\kappa_{j_2}=s-1$. Then by induction (or simply continue this process until $m=r-1$) we obtain the expression (8.14). Recall that by definition

$$\mathcal{F}(\boldsymbol{\kappa};s)=\sum_{\phi_1\in\mathfrak{B}_{P,\chi}}\langle\mathcal{I}_P(\lambda,\varphi)\phi_1,\phi_2\rangle\cdot\Psi(s,W_1,W_2;\lambda).\tag{8.15}$$

Then $\mathcal{F}(\boldsymbol{\kappa};s)$ is a Schwartz function of $\boldsymbol{\kappa}$ by Claim **??**. Hence all the above integrals converge absolutely. Then the proof is completed. □

Let notation be as in Proposition 72. Denote by $\mathcal{I}_0(s;\chi)$ the summand of the first term of the right hand side of (8.14), i.e.,

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{P,\chi}}J_{P,\chi}(s;\phi,C),\quad s\in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon}),\ \operatorname{Re}(s)>1.$$

Proposition 73. *Let notation be as before. Let $s \in 1+\mathcal{D}_\chi(\boldsymbol{\epsilon})$ and* $\mathrm{Re}(s)>1$. *Then*

$$\mathcal{I}_{0,\chi}(s)=\sum_{\phi\in\mathfrak{B}_{P,\chi}} J_{P,\chi}(s;\phi,C_\chi(\mathbf{0}))+\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{J}_\chi^0(s), \tag{8.16}$$

where $C=C_\chi$; and the summand $\mathcal{J}_\chi^0(s)$ is defined to be

$$\sum_{m=1}^{r-1}\sum_{\substack{j_m,j_{m-1},\cdots,j_1\\1\le j_m<\cdots<j_1\le r-1}}\cdots\sum\tilde{c}_{j_1,\cdots,j_m}\int_{(0)}\cdots\cdots\int_{(0)}\operatorname*{Res}_{\kappa_{j_m}=1-s}\cdots\operatorname*{Res}_{\kappa_{j_1}=1-s}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}},$$

where $\tilde{c}_{j_1,\cdots,j_m}$'s are some explicit integers, depending only on P; and the measure $d\kappa_{r-1}\cdots d\kappa_1/(d\kappa_{j_m}\cdots d\kappa_{j_1})$ means $d\kappa_{r-1}\cdots\widehat{d\kappa_{j_m}}\cdots\widehat{d\kappa_{j_1}}\cdots d\kappa_1$. Moreover, the terms in (8.16) *converges absolutely and normally inside any bounded strip.*

Proof. The proof is pretty similar to that of Proposition 72. Hence we will omit it. □

8.3 Meromorphic Continuation Inside the Critical Strip

Let $s\in\mathcal{R}(1;\chi,\boldsymbol{\epsilon})$ and $1\le m\le r-1$. Let $j_m,j_{m-1},\cdots,j_1$ be m integers such that $1\le j_m<\cdots<j_1\le r-1$. Consider the summand in the second term of (8.14):

$$\mathcal{I}_{m,\chi}(s):=\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_C\cdots\cdots\int_C\operatorname*{Res}_{\kappa_{j_m}=s-1}\cdots\operatorname*{Res}_{\kappa_{j_1}=s-1}\mathcal{F}(\boldsymbol{\kappa};s)\frac{d\kappa_{r-1}\cdots d\kappa_1}{d\kappa_{j_m}\cdots d\kappa_{j_1}}.$$

Then each $\mathcal{I}_{m,\chi}(s)$ is naturally meromorphic in $\mathcal{R}(1;\chi,\boldsymbol{\epsilon})$ with a possible at $s=1$.

Theorem I. *Let notation be as before. Let $n\le 4$. Let $\chi\in\mathfrak{X}_P$. Assume that the adjoint L-function $L(s,\sigma,\mathrm{Ad}\otimes\tau)$ is holomorphic inside the strip $S_{(0,1)}$ for any cuspidal representation $\sigma\in\mathcal{A}_0(GL(k,\mathbb{A}_F))$, and any $k\le n-1$. Then for any $0\le m\le r-1$, the function*

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{I}_{m,\chi}(s),\qquad s\in\mathcal{R}(1;\chi,\boldsymbol{\epsilon}),$$

admits a meromorphic continuation to the area $\mathcal{R}(1/2;\tau)^-$, with possible simple poles at $s\in\{1/2,2/3,\cdots,(n-1)/n,1\}$, where $\mathcal{R}(1/2;\tau)^-$ is defined in (8.2). Moreover, for any $3\le k\le n$, if $L((k-1)/k,\tau)=0$, then $s=(k-1)/k$ is not a pole.

Remark 75. *In can be seen from the proof that when $n\le 3$, we can continue the functions $\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{I}_{m,\chi}(s)$ to* $\mathrm{Re}(s)>1/3$. *When $n=4$, we can only continue $\sum_{\phi\in\mathfrak{B}_{P,\chi}}\mathcal{I}_{m,\chi}(s)$ to $\mathcal{R}(1/2;\tau)^-$, an open set just containing the right half plane* $\mathrm{Re}(s)\ge 1/2$. *This is because some of its components involve $\Lambda(2s,\tau^2)^{-1}$ as a factor. The key ingredient is that $\mathcal{R}(1/2;\tau)^-$ is uniform with respect to $\chi\in\mathfrak{X}_P$.*

Remark 76. *We restrict ourselves to the case $n \leq 4$ for the following two reasons. On the one hand, we actually need to assume Dedekind Conjecture of degree n to handle the contribution from geometric side. This conjecture has been confirmed when $n \leq 4$, so we will get unconditional results if $n \leq 4$. On the other hand, when $n \geq 5$, the procedure of meromorphic continuation is even more complicated, since we are lack of a symmetrical description of this process. Thus, we will focus on $n \leq 4$ case in this thesis.*

Since the case $n = 2$ has been done in [GJ78], we only need to care about the situation where $3 \leq n \leq 4$. To prove Theorem I in these cases, we deal with $n = 3$ and $n = 4$ separately, since we want to give explicit descriptions.

Let notation be as before. To simplify our computations below, we shall write, for any $\beta \in \mathbb{R}$, that $\mathcal{R}(\beta) = \mathcal{R}(\beta; \chi, \boldsymbol{\epsilon})$, $\mathcal{R}(\beta)^- = \mathcal{R}(\beta; \chi, \boldsymbol{\epsilon}) \cap \{s : \operatorname{Re}(s) < \beta\}$, and $\mathcal{R}(\beta)^+ = \mathcal{R}(\beta; \chi, \boldsymbol{\epsilon}) \cap \{s : \operatorname{Re}(s) > \beta\}$. Recall also that we use $S_{(a,b)}$ to denote the strip $a < \operatorname{Re}(s) < b$, for any $a < b$.

8.4 Proof of Theorem I when $n = 3$

Proof. Let $n = 3$. Then there are two possibilities for $r : r = 2$ or $r = 3$. If $r = 2$, then the parabolic subgroup P is maximal, and any associated cuspidal datum is of the form $\chi \simeq (\sigma_1, \sigma_2)$, where σ_1 is a cuspidal representation of $\mathrm{GL}(2, \mathbb{A}_F)$ and σ_2 is a Hecke character on $\mathbb{A}_F^\times$. In this case, $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s+\kappa_1, \sigma_1 \otimes \tau \times \widetilde{\sigma}_2)\Lambda(s-\kappa_1, \sigma_2 \otimes \tau \times \widetilde{\sigma}_1)}{\Lambda(1+\kappa_1, \sigma_1 \times \widetilde{\sigma}_2)\Lambda(1-\kappa_1, \sigma_2 \times \widetilde{\sigma}_1)} \cdot \prod_{k=1}^{2} \Lambda(s, \sigma_k \otimes \tau \times \widetilde{\sigma}_k). \qquad (8.17)$$

Since each completed L-functions in (8.17) is entire inside $S_{(0,1)}$, then $\mathcal{F}(\boldsymbol{\kappa}, s)$ is holomorphic (after continuation) when $0 < \operatorname{Re}(s) < 1$. On the other hand, $\mathcal{F}(\boldsymbol{\kappa}, s)$ vanishes when $\operatorname{Im}(\kappa_1) \to \infty$. Let $\operatorname{Re}(s) > 1$. By Cauchy integral formula,

$$J_{P,\chi}(s; \phi, C(\mathbf{0})) = \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 = \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_{C} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1,$$

which gives holomorphic continuation to an area $\operatorname{Re}(s) > 1 - \epsilon_1$, for some $\epsilon_1 > 0$. Hence we obtain holomorphic continuation of $J_{P,\chi}(s; \phi, C(\mathbf{0}))$ to $\operatorname{Re}(s) > 0$.

Now we handle the more complicated case where $r = 3$. In this case, cuspidal data χ correspond to (χ_1, χ_2, χ_3), where χ_i's are unitary Hecke characters such that

$\chi_1\chi_2\chi_3 = \omega$, the fixed central character. Then $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\Lambda(s,\tau)^3 \prod_{j=1}^{2}\prod_{i=1}^{j} \frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\overline{\chi}_{j+1})\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\overline{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\overline{\chi}_{j+1})\Lambda(1-\kappa_{i,j},\chi_{j+1}\overline{\chi}_i)}, \tag{8.18}$$

where $\mathcal{H}(s,\boldsymbol{\kappa})$ is an entire function and $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Let $\sum_\phi$ denote the double summation over $\phi \in \mathfrak{B}_{P,\chi}$. Then by Proposition 72,

$$\begin{aligned} J_{P,\chi}(s;\phi,C(\mathbf{0})) = &\sum_\phi \int_C\int_C \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 - c_{1,2}\sum_\phi \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\ &- c_1\sum_\phi\int_C \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - c_2\sum_\phi\int_C \operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1, \end{aligned}$$

for some integers c_1, c_2 and $c_{1,2}$; and $s \in 1+\mathcal{D}(\boldsymbol{\epsilon})$. Denote by $J^1_{P,\chi}(s;\phi,C(\mathbf{0}))$ the right hand side of the above equality. Then $J^1_{P,\chi}(s;\phi,C(\mathbf{0}))$ is meromorphic in the domain $s \in \mathcal{R}(1)$. Then we get a meromorphic continuation inside $\mathcal{R}(1)^-$ with a possible pole at $s=1$. We will handle these integrals respectively.

Recall that, for meromorphic functions $A(s)$ and $B(s)$, by $A(s) \sim B(s)$ if there exists some holomorphic function $C(s)$ such that $A(s) = C(s)B(s)$. Then by (8.18),

$$\begin{aligned} &\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(s-\kappa_2,\chi_1\overline{\chi}_2\tau)\Lambda(2s-1+\kappa_2,\chi_2\overline{\chi}_1\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_1)\Lambda(2-s-\kappa_2,\chi_1\overline{\chi}_2\tau^{-1})\Lambda(2-s,\tau^{-1})};\\ &\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(s-\kappa_1,\chi_2\overline{\chi}_1\tau)\Lambda(2s-1+\kappa_1,\chi_1\overline{\chi}_2\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_1\overline{\chi}_2)\Lambda(2-s-\kappa_1,\chi_2\overline{\chi}_1\tau^{-1})\Lambda(2-s,\tau^{-1})};\\ &\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \end{aligned}$$

Hence by Cauchy integral formula we have, for $s \in \mathcal{R}(1)^-$, that

$$\int_C \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 = \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{8.19}$$

where the right hand side is holomorphic inside $1/2 < \mathrm{Re}(s) < 1$, since

$$\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{8.20}$$

From (8.20) we see $\int_C \mathrm{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ has a potential pole at $s=2/3$ when $\tau^3=1$. Likewise, we have the continuation for $\int_C \mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$:

$$\int_C \operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 = \int_{(0)} \operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 - \operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{8.21}$$

where the right hand side is holomorphic inside $1/2 < \mathrm{Re}(s) < 1$, since

$$\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{8.22}$$

From (8.22) we see $\int_C \mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ has a potential pole at $s=2/3$ when $\tau^3=1$. Now we deal with the remaining term $\int_C\int_C \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1$. By Proposition 73, for $s\in\mathcal{R}(1)^-$, there are integers c_1, c_2 and $c_{1,2}$, such that

$$\begin{aligned}\int_C\int_C \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 &= \sum_{\phi}\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 - c'_{1,2}\sum_{\phi}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\\ &\quad -c'_1\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - c'_2\sum_{\phi}\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1.\end{aligned}$$

According to (8.18), one can compute the partial residues of $\mathcal{F}(\boldsymbol{\kappa},s)$:

$$\begin{aligned}&\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s+\kappa_1,\chi_1\overline{\chi}_2\tau)\Lambda(2s-1-\kappa_1,\chi_2\overline{\chi}_1\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_1,\chi_2\overline{\chi}_1)\Lambda(2-s+\kappa_1,\chi_1\overline{\chi}_2\tau^{-1})\Lambda(2-s,\tau^{-1})};\\ &\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_1\tau)\Lambda(2s-1-\kappa_2,\chi_1\overline{\chi}_2\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_2,\chi_1\overline{\chi}_2)\Lambda(2-s+\kappa_2,\chi_2\overline{\chi}_1\tau^{-1})\Lambda(2-s,\tau^{-1})};\\ &\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.\end{aligned}$$

From the above formulas and combining with the analytic behavior of the function $\mathrm{Res}_{\kappa_1=s-1}\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ we conclude that $\int_C\int_C \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1$ admits a meromorphic continuation to $1/2<\mathrm{Re}(s)<1$, with a possible pole at $s=2/3$ when $\tau^3=1$. Denote by $J_{P,\chi}^{(1/2,1)}(s;\phi,C(\mathbf{0}))$ this continuation. Now we continue our meromorphic continuation to some open set containing $\mathrm{Re}(s)\geq 1/2$. Let $s\in\mathcal{R}(1/2)^+$. Then one can plug (8.19) and (8.21) into formulas for $\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1$ and $\int_C\int_C \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1$ and shift contours to see that $J_{P,\chi}^{(1/2,1)}(s;\phi,C(\mathbf{0}))$ is equal to the sum over and $\phi \in \mathfrak{B}_{P,\chi}$ of

$$\begin{aligned}
&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 - c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s) - c'_1\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2\\
&- c'_2\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 - c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1\\
&+ c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) + c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
=&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 - c'_{1,2}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s) - c'_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2\\
&- c'_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 - c_1\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - c_2\int_{\mathcal{C}}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1\\
&+ c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) + c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\
&+ c'_1\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s) + c'_2\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s),
\end{aligned}$$

where the right hand side of the above equality has a natural meromorphic continuation to the domain $\mathcal{R}(1/2)$. Denote by $J_{P,\chi}^{1/2}(s;\phi,\mathcal{C}(\mathbf{0}))$ the last expression. Note that a direct computation leads to that

$$\begin{aligned}
\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)};\\
\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s) \sim \frac{\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}.
\end{aligned}$$

Also, when $s \in \mathcal{R}(1/2)$, $2-2s$ lies in the zero-free region of $L(s,\tau^{-2})$ and $L_\infty(2-2s,\tau^{-2})$ is holomorphic (hence nonvanishing), then $\Lambda(2-2s,\tau^{-2}) \neq 0$. So the last two terms of $J_{P,\chi}^{1/2}(s;\phi,\mathcal{C}(\mathbf{0}))$ is meromorphic in $\mathcal{R}(1/2)$ with a possible simple pole at $s = 1/2$ when $\tau^2 = 1$. Hence, we have a meromorphic continuation of $J_{P,\chi}(s;\phi,\mathcal{C}(\mathbf{0})) = J_{P,\chi}^{1/2}(s;\phi,\mathcal{C}(\mathbf{0}))$ to the region $\mathcal{R}(1/2)$ with a possible simple pole at $s = 1/2$ when $\tau^2 = 1$.

Now consider $J_{P,\chi}^{1/2}(s;\phi,\mathcal{C}(\mathbf{0}))$, where $s \in \mathcal{R}(1/2)^-$. Invoking the analytic behaviors of $\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ and $\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ with Cauchy's formula we obtain that $J_{P,\chi}^{(1/3,1/2)}(s;\phi,C(\mathbf{0}))$ is equal to the sum over and $\phi\in\mathfrak{B}_{P,\chi}$ of

$$\begin{aligned}&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_1-c_{1,2}'\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)-c_1'\int_C\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2\\&-c_2'\int_C\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-c_1\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2-c_2\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1\\&+c_1\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+c_2\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-c_{1,2}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\&+c_1'\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c_2'\operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+c_1\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\&+c_2\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).\end{aligned}$$

Denote by $J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0}))$ the last expression. Note that we have

$$\begin{aligned}\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\sim\frac{\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)};\\\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)&\sim\frac{\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}.\end{aligned}$$

Note also that the integrals in $J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0}))$ converges locally normally when $1/3<\operatorname{Re}(s)<1/2$. Hence $J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0}))$ has a natural continuation to $1/3<\operatorname{Re}(s)<1/2$, where it is holomorphic. In all, we obtain the meromorphic continuation of $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ to $S_{(1/3,1)}\cup\mathcal{R}(1)$ as follows:

$$J_{P,\chi}(s;\phi,C(\mathbf{0}))=\begin{cases}J_{P,\chi}^{1}(s;\phi,C(\mathbf{0})),\ s\in\mathcal{R}(1);\\J_{P,\chi}^{(1/2,1)}(s;\phi,C(\mathbf{0})),\ s\in S_{(1/2,1)};\\J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0})),\ s\in\mathcal{R}(1/2);\\J_{P,\chi}^{(1/3,1/2)}(s;\phi,C(\mathbf{0})),\ s\in\mathcal{S}(1/3,1/2).\end{cases}$$

Moreover, the continued function $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ is meromorphic inside $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=1/2$ and $s=2/3$ when $\tau^2=1$ and $\tau^3=1$, respectively. □

8.5 Proof of Theorem I when $n=4$

The case $n=4$ seems to be much more complicated than $n=3$, but they share the same underlying idea. The proof is similar, but does not quite follow from GL(3) case. In fact, the essential difficulty as n increases is the determination of partial residues of each continuation: there are roughly $O(n^2)$ such multiple residues, and there is not likely a simple systematical description of them, so we give a proof by explicitly dealing with all possible cases. Some careful computation and continuation are carried out in the appendix (see Section 10 for details).

Proof. Let $n = 4$. Then there are three possibilities for $r : r = 2, r = 3$ or $r = 4$. We will deal with these cases separately.

$r = 2:$ In this case, the parabolic subgroup P is of type $(2, 2)$, and any associated cuspidal datum is of the form $\chi \simeq (\sigma_1, \sigma_2)$, where σ_1 and σ_2 are cuspidal representations of $\mathrm{GL}(2, \mathbb{A}_F)$. In this case, $\mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to an entire function multiplying

$$\frac{\Lambda(s+\kappa_1, \sigma_1 \otimes \tau \times \widetilde{\sigma}_2)\Lambda(s-\kappa_1, \sigma_2 \otimes \tau \times \widetilde{\sigma}_1)}{\Lambda(1+\kappa_1, \sigma_1 \times \widetilde{\sigma}_2)\Lambda(1-\kappa_1, \sigma_2 \times \widetilde{\sigma}_1)} \cdot \prod_{k=1}^{2} \Lambda(s, \sigma_k \otimes \tau \times \widetilde{\sigma}_k). \quad (8.23)$$

Let $s \in \mathcal{R}(1)^+$. Since $\mathcal{F}(\boldsymbol{\kappa}, s)$ vanishes when $\mathrm{Im}(\kappa_1) \to \infty$, then by Cauchy integral formula, we have that

$$J_{P,\chi}(s; \phi, C(\mathbf{0})) = \sum_{\phi \in \mathfrak{B}_{P,\chi}} \int_C \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \sum_{\phi \in \mathfrak{B}_{P,\chi}} \operatorname*{Res}_{\kappa_1 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s). \quad (8.24)$$

The term $\mathrm{Res}_{\kappa_1 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is nonvanishing unless $\sigma_1 \simeq \sigma_2 \otimes \tau$. Hence

$$\operatorname*{Res}_{\kappa_1 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(2s-1, \sigma_1 \otimes \tau^2 \times \widetilde{\sigma}_1)\Lambda(s, \sigma_1 \otimes \tau \times \widetilde{\sigma}_1)}{\Lambda(2-s, \sigma_1 \otimes \tau^{-1} \times \widetilde{\sigma}_1)}.$$

So $\mathrm{Res}_{\kappa_1 = s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ admits a meromorphic continuation inside the domain $\mathcal{R}(1/2) \cup S_{(1/2,1)}$, with possible simple poles at $s = 1/2$. Now the right hand side of (8.24) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s = 1$. Denote by $J^1_{P,\chi}(s; \phi, C(\mathbf{0}))$ the continuation of $J_{P,\chi}(s; \phi, C(\mathbf{0}))$ in $\mathcal{R}(1)$. Apply Cauchy formula again to get

$$\int_C \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 = \int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1 = 1-s} \mathcal{F}(\boldsymbol{\kappa}, s), \quad (8.25)$$

where $s \in \mathcal{R}(1)^-$. By (8.23), $\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$ is holomorphic inside $S_{(1/2,1)}$; also, $\mathrm{Res}_{\kappa_1 = 1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ is nonvanishing unless $\sigma_2 \simeq \sigma_1 \otimes \tau$, in which case one has

$$\operatorname*{Res}_{\kappa_1 = 1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(2s-1, \sigma_2 \otimes \tau^2 \times \widetilde{\sigma}_2)\Lambda(s, \sigma_2 \otimes \tau \times \widetilde{\sigma}_2)}{\Lambda(2-s, \sigma_2 \otimes \tau^{-1} \times \widetilde{\sigma}_2)}.$$

So $\mathrm{Res}_{\kappa_1 = 1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ admits a meromorphic continuation to $\mathcal{R}(1/2) \cup S_{(1/2,1)}$, with possible simple poles at $s = 1/2$. Substituting this with (8.25) into (8.24) we conclude that $J^1_{P,\chi}(s; \phi, C(\mathbf{0}))$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2)\cup S_{(1/2,1)}$. Denote by $J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0}))$ this continuation. Hence we have

$$J_{P,\chi}(s;\phi,C(\mathbf{0}))=\begin{cases}J_{P,\chi}^{1}(s;\phi,C(\mathbf{0})),\ s\in\mathcal{R}(1);\\ J_{P,\chi}^{1/2}(s;\phi,C(\mathbf{0})),\ s\in S_{(0,1)}.\end{cases}$$

Moreover, by assumption $\Lambda(s,\sigma_2\otimes\tau\times\widetilde{\sigma}_2)L(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, then from the expressions above we see that $J_{P,\chi}(s;\phi,C(\mathbf{0}))L(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

$r=3:$ In this case, the parabolic subgroup P is of type $(2,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\sigma_1,\chi_2,\chi_3)$, where σ_1 is a cuspidal representations of $\mathrm{GL}(2,\mathbb{A}_F)$; and χ_2, χ_3 are unitary Hecke characters on $\mathbb{A}_F^{\times}$. Since $\Lambda(s,\sigma_1\otimes\tau\times\chi_i)$ is entire, $2\leq i\leq 3$, then $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to an entire function $\mathcal{H}(\boldsymbol{\kappa},s)$ multiplying

$$\frac{\Lambda(s+\kappa_2,\chi_2\overline{\chi}_3\tau)\Lambda(s-\kappa_2,\chi_3\overline{\chi}_2\tau)\Lambda(s,\sigma_1\otimes\tau\times\widetilde{\sigma}_1)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_2,\chi_2\overline{\chi}_3\tau)\Lambda(1-\kappa_2,\chi_3\overline{\chi}_2\tau)}. \tag{8.26}$$

Let $s\in\mathcal{R}(1)^+$. Since $\mathcal{F}(\boldsymbol{\kappa},s)$ vanishes when $\mathrm{Im}(\kappa_1)\to\infty$, then by Cauchy integral formula, we have that $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ is equal to

$$\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_C\int_C\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1d\kappa_2-\sum_{\phi\in\mathfrak{B}_{P,\chi}}\int_C\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1. \tag{8.27}$$

The term $\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\chi_1\simeq\chi_2\otimes\tau$. Hence

$$\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)=\mathcal{H}(s,\kappa_1)\frac{\Lambda(2s-1,\tau^2)\Lambda(s,\tau)\Lambda(s,\sigma_1\otimes\tau\times\widetilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})},$$

where $\mathcal{H}(s,\kappa_1)$ is an holomorphic function. So $\mathrm{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation inside the domain $S_{(0,1)}$, with possible simple poles at $s=1/2$. Now the right hand side of (8.27) is meromorphic inside $\mathcal{R}(1)$, with a possible pole at $s=1$. Denote by $J_{P,\chi}^1(s;\phi,C(\mathbf{0}))$ the continuation of $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ in $\mathcal{R}(1)$. Apply Cauchy formula again to get

$$\int_C\int_C\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1d\kappa_2=\int_C\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_1+\int_C\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1, \tag{8.28}$$

where $s\in\mathcal{R}(1)^-$. By (8.26), $\int_C\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_1$ is holomorphic inside $S_{(1/3,1)}$; also, $\mathrm{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is nonvanishing unless $\sigma_2\simeq\sigma_1\otimes\tau$, in which case one has

$$\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(2s-1,\tau^2)\Lambda(s,\tau)\Lambda(s,\sigma_1\otimes\tau\times\widetilde{\sigma}_1)}{\Lambda(2-s,\tau^{-1})}.$$

So $\int_C \operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ admits a meromorphic continuation to $S_{(1/3,1)}$, with possible simple poles at $s=1/2$. Substituting this and (8.28) into (8.24) we conclude that $J^1_{P,\chi}(s;\phi,C(\mathbf{0}))$ admits a meromorphic continuation to the domain $S_{(1/3,1)}$. Denote by $J^{(1/3,1)}_{P,\chi}(s;\phi,C(\mathbf{0}))$ this continuation. Hence invoking the above discussion we have

$$J_{P,\chi}(s;\phi,C(\mathbf{0}))=\begin{cases}J^1_{P,\chi}(s;\phi,C(\mathbf{0})),\ s\in\mathcal{R}(1);\\ J^{(1/3,1)}_{P,\chi}(s;\phi,C(\mathbf{0})),\ s\in S_{(1/3,1)}.\end{cases}$$

Moreover, by assumption $\Lambda(s,\sigma_2\otimes\tau\times\widetilde{\sigma}_2)L(s,\tau)^{-1}$ is holomorphic in $S_{(0,1)}$, then from the expressions above we see that $J_{P,\chi}(s;\phi,C(\mathbf{0}))L(s,\tau)^{-1}$ admits a meromorphic continuation in $s\in S_{(1/3,1)}$ with a possible simple pole at $s=1/2$ when $\tau^2=1$.

$r=4:$ In this case, the parabolic subgroup P is of type $(1,1,1,1)$, and any associated cuspidal datum is of the form $\chi\simeq(\chi_1,\chi_2,\chi_3,\chi_4)$, where χ_i's are unitary Hecke characters on $\mathbb{A}_F^\times$ such that $\chi_1\chi_2\chi_3\chi_4=\omega$. Then there exists an entire function $\mathcal{H}(s,\boldsymbol{\kappa})$ such that $\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to

$$\mathcal{H}(s,\boldsymbol{\kappa})\Lambda(s,\tau)^4\prod_{j=1}^{3}\prod_{i=1}^{j}\frac{\Lambda(s+\kappa_{i,j},\tau\chi_i\overline{\chi}_{j+1})\Lambda(s-\kappa_{i,j},\tau\chi_{j+1}\overline{\chi}_i)}{\Lambda(1+\kappa_{i,j},\chi_i\overline{\chi}_{j+1})\Lambda(1-\kappa_{i,j},\chi_{j+1}\overline{\chi}_i)},\qquad(8.29)$$

where $\Lambda(s,\chi')$ is the completed Hecke L-function associated to the unitary Hecke character χ' over F. Then by Proposition 72, when $s\in\mathcal{R}(1)^+$, $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ is equal to

$$\sum_\phi\int_C\int_C\int_C\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_2d\kappa_1-c_1\sum_\phi\int_C\int_C\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_2-$$
$$c_2\sum_\phi\int_C\int_C\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_1-c_3\sum_\phi\int_C\int_C\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_1-$$
$$c_{1,2}\sum_\phi\int_C\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3-c_{1,3}\sum_\phi\int_C\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2-$$
$$c_{2,3}\sum_\phi\int_C\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-c_{1,2,3}\sum_\phi\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s),$$

where the coefficients c_1, c_2, c_3, $c_{1,2}$, $c_{1,3}$, $c_{2,3}$ and $c_{1,2,3}$ are some absolute integers; and the sum with respect to ϕ in taken over $\phi\in\mathfrak{B}_{P,\chi}$.

Due to the finiteness of $\mathfrak{B}_{P,\chi}$ and rapidly decay of $\mathcal{F}(\boldsymbol{\kappa},s)$ as a function of $\boldsymbol{\kappa}$ (see Claim **??**), each term in the above expression converges absolutely and locally normally. Hence we only need to consider each summand in the above expression. Denote by $\chi_{ij}=\chi_i\overline{\chi}_j$, $1\leq i,j\leq 4$. By (8.29) we have

$$\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{12},\chi_{31}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1-\kappa_1,\chi_{21})\Lambda(1+\kappa_2,\chi_{23})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\times$$
$$\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{12},\chi_{13}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{12},\chi_{13})\Lambda(2-s-\kappa_{12},\chi_{31}\tau^{-1})\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s+\kappa_{13},\chi_{14}\tau)\Lambda(s-\kappa_{13},\chi_{41}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1+\kappa_3,\chi_{34})\Lambda(1+\kappa_{13},\chi_{14})\Lambda(2-s-\kappa_1,\chi_{21}\tau^{-1})}$$
$$\cdot\frac{\Lambda(2s-1+\kappa_3,\chi_{34}\tau^2)\Lambda(2s-1+\kappa_1,\chi_{12}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{13},\chi_{41})\Lambda(2-s-\kappa_3,\chi_{43}\tau^{-1})\Lambda(2-s,\tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(s-\kappa_{23},\chi_{42}\tau)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})}\times$$
$$\frac{\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1+\kappa_{23},\chi_{24}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1+\kappa_{23},\chi_{24})\Lambda(2-s-\kappa_{23},\chi_{42}\tau^{-1})\Lambda(2-s,\tau^{-1})}.$$

Hence from the above expressions we see that $\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-2+\kappa_1,\chi_{12}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(3-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{8.30}$$

Likewise, $\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ and

$$\frac{\Lambda(1-\kappa_2,\chi_{31})\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(3s-2+\kappa_2,\chi_{23}\tau^3)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})\Lambda(3-2s-\kappa_2,\chi_{32}\tau^{-2})}. \tag{8.31}$$

Also, the function $\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the following function

$$\frac{\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(3s-2+\kappa_3,\chi_{34}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_3,\chi_{34})\Lambda(3-2s-\kappa_3,\chi_{43}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{8.32}$$

Moreover, one can continue the computation to see that

$$\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\operatorname*{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Therefore, we have, from the above expressions, that $J_{P,\chi}(s;\phi,C(\mathbf{0}))$ admits a meromorphic continuation to $s\in\mathcal{S}(1)$. Denote by $J_{P,\chi}^1(s;\phi,C(\mathbf{0}))$ the continuation. Then clearly $J_{P,\chi}^1(s;\phi,C(\mathbf{0}))$ is holomorphic when $s\in\mathcal{R}(1)^-$.

Let $s\in\mathcal{R}(1)^-$. Let $L(s,\tau)$ be the finite part of Hecke L-function with respect to τ. Then by Cauchy integral formula we have that

Claim 77. $\int_C \int_C \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_2$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

Claim 78. $\int_C \int_C \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_1$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

Claim 79. $\int_C \int_C \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 d\kappa_1$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

Claim 80. $\int_C \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

Claim 81. $\int_C \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

Claim 82. $\int_C \operatorname*{Res}_{\kappa_2=s-1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$ *admits a meromorphic continuation to the domain* $S_{(1/3,\infty)}$. *When restricted to* $\mathcal{R}(1/2;\tau)^- \cup S_{(1/2,1)}$, *it only has possible simple poles at* $s = 3/4$, $s = 2/3$ *and* $s = 1/2$. *Moreover, if* $L(3/4, \tau) = 0$, *then* $s = 3/4$ *is not a pole; if* $L(2/3, \tau) = 0$, *then* $s = 2/3$ *is not a pole.*

By Proposition 73, for $s \in \mathcal{R}(1)^-$, there are integers $c'_1, c'_2, c'_3, c'_{1,2}, c'_{1,3}, c'_{2,3}$ and

$c'_{1,2,3}$, such that $\sum_\phi \int_C \int_C \int_C \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_2 d\kappa_1$ is equal to

$$\sum_\phi \int_{(0)} \int_{(0)} \int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_2 d\kappa_1 - c'_1 \sum_\phi \int_{(0)} \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_2 -$$
$$c'_2 \sum_\phi \int_{(0)} \int_{(0)} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 d\kappa_1 - c'_3 \sum_\phi \int_{(0)} \int_{(0)} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 d\kappa_1 -$$
$$c'_{1,2} \sum_\phi \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 - c'_{1,3} \sum_\phi \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 -$$
$$c'_{2,3} \sum_\phi \int_{(0)} \operatorname*{Res}_{\kappa_2=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - c'_{1,2,3} \sum_\phi \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_2=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s),$$

where the coefficients c'_1, c'_2, c'_3, $c'_{1,2}$, $c'_{1,3}$, $c'_{2,3}$ and $c'_{1,2,3}$ are some absolute integers; and the sum with respect to ϕ in taken over $\phi \in \mathfrak{B}_{P,\chi}$.

Due to the finiteness of $\mathfrak{B}_{P,\chi}$ and rapidly decay of $\mathcal{F}(\boldsymbol{\kappa}, s)$ as a function of $\boldsymbol{\kappa}$ (see Claim **??**), each term in the above expression converges absolutely and locally normally. Hence we only need to consider each summand in the above expression. According to (8.29), we have that

$$\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(s-\kappa_1, \chi_{21}\tau)\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(s+\kappa_{12}, \chi_{13}\tau)}{\Lambda(1+\kappa_1, \chi_{12})\Lambda(1-\kappa_1, \chi_{21})\Lambda(1-\kappa_2, \chi_{32})\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(2s-1-\kappa_{12}, \chi_{31}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{12}, \chi_{31})\Lambda(2-s+\kappa_{12}, \chi_{13}\tau^{-1})\Lambda(2-s, \tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(s+\kappa_{13}, \chi_{14}\tau)\Lambda(s-\kappa_{13}, \chi_{41}\tau)}{\Lambda(1-\kappa_1, \chi_{21})\Lambda(1-\kappa_3, \chi_{43})\Lambda(1+\kappa_{13}, \chi_{14})\Lambda(2-s+\kappa_1, \chi_{12}\tau^{-1})} \cdot \frac{\Lambda(2s-1-\kappa_3, \chi_{43}\tau^2)\Lambda(2s-1-\kappa_1, \chi_{21}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{13}, \chi_{41})\Lambda(2-s+\kappa_3, \chi_{34}\tau^{-1})\Lambda(2-s, \tau^{-1})};$$
$$\operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(s-\kappa_3, \chi_{43}\tau)\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(s+\kappa_{23}, \chi_{24}\tau)}{\Lambda(1-\kappa_2, \chi_{32})\Lambda(1-\kappa_3, \chi_{43})\Lambda(1+\kappa_3, \chi_{34})\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(2s-1-\kappa_{23}, \chi_{42}\tau^2)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^3}{\Lambda(1-\kappa_{23}, \chi_{42})\Lambda(2-s+\kappa_{23}, \chi_{24}\tau^{-1})\Lambda(2-s, \tau^{-1})}.$$

Hence from the above expressions we see that $\operatorname*{Res}_{\kappa_2=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s+\kappa_1, \chi_{12}\tau)\Lambda(3s-2-\kappa_1, \chi_{21}\tau^3)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(1-\kappa_1, \chi_{21})\Lambda(3-2s+\kappa_1, \chi_{12}\tau^{-2})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}. \tag{8.33}$$

Likewise, $\operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying the product of $\Lambda(2s-1, \tau^2)^2 \Lambda(s, \tau)^2 \Lambda(2-s, \tau^{-1})^{-2}$ and

$$\frac{\Lambda(1+\kappa_2, \chi_{13})\Lambda(s+\kappa_2, \chi_{23}\tau)\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(3s-2-\kappa_2, \chi_{32}\tau^3)}{\Lambda(1-\kappa_2, \chi_{32})\Lambda(s-\kappa_2, \chi_{32}\tau)\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})\Lambda(3-2s+\kappa_2, \chi_{23}\tau^{-2})}. \tag{8.34}$$

Also, the function $\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the following function

$$\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(3s-2-\kappa_3,\chi_{43}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1-\kappa_3,\chi_{43})\Lambda(3-2s+\kappa_3,\chi_{34}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{8.35}$$

Moreover, one can continue the computation to see that

$$\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}.$$

Let $s\in\mathcal{R}(1)^-$. Then by Cauchy integral formula we have that

Claim 83. *$\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_2$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

Claim 84. *$\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_1$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

Claim 85. *$\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_1$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

Claim 86. *$\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

Claim 87. *$\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

Claim 88. *$\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_3=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ admits a meromorphic continuation to the domain $S_{(1/3,\infty)}$. When restricted to $\mathcal{R}(1/2;\tau)^-\cup S_{(1/2,1)}$, it only has possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$. Moreover, if $L(3/4,\tau)=0$, then $s=3/4$ is not a pole; if $L(2/3,\tau)=0$, then $s=2/3$ is not a pole.*

The proof of these claims are given in the Appendix 10. Then Theorem I follows. □

Chapter 9

PROOF OF THEOREMS IN APPLICATIONS

By Theorem E, F, G and H, we conclude the first part of Theorem A, obtaining (1.4), namely, for $\mathrm{Re}(s) > 1$,

$$I_0^{\varphi}(s,\tau) = I^{\varphi}_{\mathrm{Geo,Reg}}(s,\tau) + I^{\varphi}_{\infty,\mathrm{Reg}}(s,\tau) + I^{\varphi}_{\mathrm{Sing}}(s,\tau) + I_{\mathrm{Whi}}(s,\tau).$$

Moreover,$I_0^{\varphi}(s,\tau)$, $I^{\varphi}_{\mathrm{Geo,Reg}}(s,\tau)$, and $I^{\varphi}_{\infty,\mathrm{Reg}}(s,\tau)$ admit meromorphic continuation to the whole s-plane. Consequently, $I^{\varphi}_{\mathrm{Sing}}(s,\tau)$ can be continued to a meromorphic function on $\mathbb{C}$.

Assume τ is such that $\tau^k \neq 1$, $1 \leq k \leq n$. Then by Corollary 69 we conclude that $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to $\mathrm{Re}(s) > 0$. Then by functional equation of Eisenstein series, we conclude that $I_{\mathrm{Whi}}(s,\tau)$ has a meromorphic continuation to the whole s-plane. Then Theorem A follows.

Let $\Sigma = \Sigma_\infty \coprod \Sigma_f$ be the set of places of F, where Σ_∞ denotes the subset of archimedean places, and Σ_f denotes the subset of nonarchimedean places.

For a place $v \in \Sigma_f$, we say that a test function $\varphi = \otimes_v \varphi_v \in \mathcal{H}(G(\mathbb{A}_F))$ is *discrete at* v if φ_v is supported on the intersection of $G(\mathcal{O}_{F_v})$ and the regular elliptic subset of $G(F_v)$. Let $\mathcal{F}^*(\omega)$ be the set of smooth functions $\varphi = \otimes'_v \varphi_v : G(\mathbb{A}_F) \to \mathbb{C}$ which is left and right K-finite, is discrete at some $v \in \Sigma_f$, transforms by the character ω of $Z_G(\mathbb{A}_F)$, and has compact support modulo $Z_G(\mathbb{A}_F)$. Let $\mathcal{F}(\omega)$ be the space spanned linearly by functions in $\mathcal{F}^*(\omega)$.

Proof of Theorem B. Fix a field extension E/F of degree n. Let $s_0 \in \mathbb{C} - \{0,1\}$. Let $\gamma_0 \in G(F)$ be such that $F[\gamma_0]^\times = E$. Although such γ_0's are not unique, we fix one γ_0.

Consider the continuous map

$$\sigma : G(F) \longrightarrow F^n, \quad \gamma \mapsto (a_{n-1}(\gamma), \cdots, a_1(\gamma), a_0(\gamma)),$$

where $a_i(\gamma)$'s are the coefficients of characteristic polynomial f_γ of γ, namely, $f_\gamma(t) = \det(tI_n - \gamma) = t^n + a_{n-1}(\gamma)t^{n-1} + \cdots + a_1(\gamma)t + a_0(\gamma)$. Then σ extends to a continuous function $G(\mathbb{A}_F) \longrightarrow \mathbb{A}_F^n$.

Note that when γ runs through $G(F)$, the image $\sigma(\gamma)$ is discrete in $\mathbb{A}_F^n$. Take a compact neighborhood U_{γ_0} of $\sigma(\gamma_0)$ in $\mathbb{A}_F^n$, such that U_{γ_0} does not intersect with other $\sigma(\gamma)$ when $\sigma(\gamma) \neq \sigma(\gamma_0)$ and $\gamma \in G(F)$. Let $C_0 \subset G_{\gamma_0}(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$ be a small compact neighborhood of the identity such that $\sigma(c_0^{-1}\gamma_0 c_0) \in U_0$, and $\sigma(c_0^{-1}\gamma c_0) \notin U_0$ for all $c_0 \in C_0$, where $\gamma \in G(F)$ satisfies $\sigma(\gamma) \neq \sigma(\gamma_0)$. Shrink C_0 suitably if necessary so that we may assume $\tau \circ \det$ is trivial on C_0. Let $x \in C_0$. Denote by

$$T(s,x) = \int_{\mathbb{A}_E^\times} \Phi(\eta t x)\tau(\det tx)|\det tx|^s d^\times t.$$

Then by Tate's thesis, $T(s,x)$ is an integral representation for $\Lambda(s,\tau \circ N_{E/F})$. So $T(s,x) = Q(s,x)\Lambda(s,\tau \circ N_{E/F})$, where $Q(s,x)$ is a function holomorphic in s and smooth in x, depending on Φ, τ, and E. Moreover, one can choose Φ such that $Q(s,x) \equiv 1$ when $x = 1$. Fix the choice of Φ henceforth. Then $Q(s,x) = \prod_v Q_v(s,x_v)$ with $Q_v(s,x_v) \equiv 1$ for $v \notin S_{E/F,\tau}$ and $x_v \in G(\mathcal{O}_{F_v})$, where $S_{E/F,\tau}$ is a finite set of places including the archimedean ones determined by E/F and τ.

Let $C = \prod_v C_v$ be a compact subset of $C_0 = \prod_v C_{0,v}$. Let $\widetilde{C} = \cup_{c\in C} c^{-1}\gamma_0 c$. Then $\widetilde{C}$ is a compact set in $Z_G(\mathbb{A}_F)\backslash G(\mathbb{A}_F)$. Shrink C suitably if necessary such that there exists a nonzero $\widetilde{\varphi} \in \mathcal{H}(G(\mathbb{A}_F),\omega)$ such that $\widetilde{\varphi}_v \geq 0$ for all $v \in \Sigma_F$ and the support $\operatorname{supp}\widetilde{\varphi} \subseteq \widetilde{C}$, and

$$\int_{\widetilde{C}} \widetilde{\varphi}(x^{-1}\gamma_0 x)Q(s_0,x)dx = \prod_v \int_{\widetilde{C}_v} \widetilde{\varphi}_v(x_v^{-1}\gamma_0 x_v)Q_v(s_0,x_v)dx_v \neq 0, \tag{9.1}$$

where the product only takes over finitely many $v \in \Sigma_F$. The existence of such a C comes from the fact that $Q(s_0,x)$ is continuous.

Let u be a place of F such that u splits in E, τ_u is unramified, and $\widetilde{\varphi}_u$ is the characteristic function of $G(\mathcal{O}_{F_u})$ and $\gamma_{0,u} \in G(\mathcal{O}_{F_u})$. Let ρ be a finite dimensional admissible representation of $G(\mathcal{O}_{F_u})$. Let $\rho^\vee$ be the contragredient of ρ. Denote by $\Theta_{\rho^\vee}$ the character of $\rho^\vee$. Since γ_0 is elliptic regular, we can take such a ρ with properties that $\Theta_{\rho^\vee}(\gamma_{0,u}) \neq 0$ and the compact induction $\pi_u = c - \operatorname{Ind}_{G(\mathcal{O}_{F_u})}^{G(F_u)} \rho$ is irreducible. Hence π_u is supercuspidal. Let

$$m_{\pi_u}(x) = \begin{cases} \Theta_{\rho^\vee}(x), & \text{if } x \in G(\mathcal{O}_{F_u}); \\ 0, & \text{otherwise.} \end{cases}$$

Now take $\varphi(x) = \otimes_{v\neq u}\widetilde{\varphi}_v(x_v)\otimes m_{\pi_u}(x_u)$. Then $\varphi \in \mathcal{F}(\omega)$ and supp $\varphi \subseteq \widetilde{C}$. Moreover, at $v = u$ we have $Q(s_0, x_u) = 1$. Hence,

$$\int_{C_u} \varphi_v(x_u^{-1}\gamma_{0,u}x_u)Q(s_0, x_u)dx_u = \Theta_{\rho^\vee}(\gamma_{0,u}) \neq 0.$$

In conjunction with (9.1) at $v \neq u$ we then obtain that

$$\int_{\widetilde{C}} \varphi(x^{-1}\gamma_0 x)Q(s_0, x)dx \neq 0. \tag{9.2}$$

Substituting this choice of φ into Theorem A we obtain

$$\frac{I_0^{\varphi}(s,\tau)}{\Lambda(s,\tau)} = \frac{\Lambda(s,\tau\circ N_{E/F})}{n\Lambda(s,\tau)} \int_{\widetilde{C}} \varphi(x^{-1}\gamma_0 x)Q(s,x)dx. \tag{9.3}$$

Assume that the twisted adjoint L-function $L(s,\pi,\mathrm{Ad}\otimes\tau)$ is holomorphic outside $s = 1$ for all $\pi \in \mathcal{A}_0^{\mathrm{simp}}(G(F)\backslash G(\mathbb{A}_F),\omega^{-1})$. Then by spectral expansion (2.1), the function $I_0^{\varphi}(s,\tau)/\Lambda(s,\tau)$ is holomorphic at $s = s_0$. Therefore, it follows from (9.2) and (9.3) that $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is regular at $s = s_0$. Then the meromorphic function $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic at $s = s_0$.

Since s_0 is arbitrary, then the meromorphic function $\Lambda(s,\tau\circ N_{E/F})/\Lambda(s,\tau)$ is holomorphic outside $s = 0, 1$. So the τ-twisted Dedekind conjecture holds. Then Theorem B follows. □

Remark 89. *It is conjectured (cf. [JZ87], [JR97]) that the reverse direction also holds, namely, the τ-twisted Dedekind conjecture for all field extensions E/F of degree n should imply holomorphy of the τ-twisted adjoint L-functions. This is proved in [***Yan21***] for $n \leq 4$.*

Proof of Theorem C. Let E be a field extension of F of degree n, such that $\zeta_E(1/2) \neq 0$. By the proof of Theorem B, one can choose some test function $\varphi \in \mathcal{F}(\omega)$, such that

$$\int_{\widetilde{C}} \varphi(x^{-1}\gamma_0 x)Q(1/2, x)dx \neq 0.$$

It then follows from (9.3) that

$$I_0^{\varphi}(1/2,\tau) \neq 0. \tag{9.4}$$

Theorem C then follows from (9.4) and the spectral expansion (2.1) of the cuspidal kernel function $\mathrm{K}_0(x,x)$. □

Chapter 10

APPENDIX: CONTINUATION ACROSS THE CRITICAL LINE FOR GL(4)

In this appendix, we shall prove the claims in our proceeding proof of Theorem I when $n = 4$ in subsection 8.5. The processes here are in the same flavor of those in the $n = 3$ case in subsection 8.4, but they are typically much more complicated.

Proof of Claim 77. Let $s \in \mathcal{R}(1)^+$. Let $J_1(s) := \int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3 d\kappa_2$, and $J_1^1(s) := \int_C \int_C \operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3 d\kappa_2$. By the analytic property of $\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ we see that $J_1^1(s)$ is meromorphic in the domain $\mathcal{R}(1)$, with a possible pole at $s = 1$. Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then see that

$$J_1^1(s) = \int_C \int_{(0)} \operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 d\kappa_3 + \int_C \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3, \tag{10.1}$$

where $\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s+\kappa_3, \chi_{34}\tau)\Lambda(3s-2-\kappa_3, \chi_{43}\tau^3)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(1-\kappa_3, \chi_{43})\Lambda(3-2s+\kappa_3, \chi_{34}\tau^{-2})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}. \tag{10.2}$$

Then $J_1^1(s)$ is equal to, after applications of Cauchy integral formula to (10.1),

$$\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 d\kappa_3 + \int_{(0)} \operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 + \int_{(0)} \operatorname{Res}_{\kappa_3=2-2s-\kappa_2}$$
$$\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 + \int_{(0)} \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3 + \operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s),$$

where $\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1, \tau^2)^2\Lambda(s, \tau)^2 \cdot \Lambda(2-s, \tau^{-1})^{-2}$ and

$$\frac{\Lambda(s-\kappa_2, \chi_{32}\tau)\Lambda(2s-1-\kappa_2, \chi_{32}\tau^2)\Lambda(2s-1+\kappa_2, \chi_{23}\tau^2)\Lambda(s+\kappa_2, \chi_{23}\tau)}{\Lambda(1+\kappa_2, \chi_{23})\Lambda(1-\kappa_2, \chi_{32})\Lambda(2-s-\kappa_2, \chi_{32}\tau^{-1})\Lambda(2-s+\kappa_2, \chi_{23}\tau^{-1})}; \tag{10.3}$$

and $\operatorname{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s-\kappa_2, \chi_{32}\tau)\Lambda(3s-2+\kappa_2, \chi_{23}\tau^3)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(1+\kappa_2, \chi_{23})\Lambda(3-2s-\kappa_2, \chi_{32}\tau^{-2})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}. \tag{10.4}$$

From the formula (10.2), we see that $\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.5}$$

We thus see from the proceeding computations of analytic behaviors of the functions $\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ and $\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, that $\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3$ and $\int_{(0)}\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ admit meromorphic continuation to the domain $1/2<\operatorname{Re}(s)<1$, with a possible pole at $s=2/3$ if $\tau^3=1$; and $\operatorname{Res}_{\kappa_3=1-s}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2)^-\cup S_{[1/2,1)}$, with possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$, when $\tau^4=1$, $\tau^3=1$ and $\tau^2=1$, respectively, according to (10.5).

From (10.4) we see that the function $\int_{(0)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ admits holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. From (10.2) we see that the function $\int_{(0)}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ admits holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. Then combining these with (10.3) and (10.5) one sees that $J_1^1(s)$ admits a holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. Denote by $J_1^{(2/3,1)}(s)$ this continuation, where $2/3<\operatorname{Re}(s)<1$.

Let $s\in\mathcal{R}(2/3)^+$, then by Cauchy integral formula we have

$$\int_{(0)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2=\int_{\mathcal{C}}\operatorname{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2. \tag{10.6}$$

Likewise, for $s\in\mathcal{R}(2/3)^+$, $\int_{(0)}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ is equal to

$$\int_{\mathcal{C}}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3-\operatorname{Res}_{\kappa_3=3s-2}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s). \tag{10.7}$$

Then according to (10.2), (10.3), (10.4), (10.6), (10.7), and the computation that the function $\operatorname{Res}_{\kappa_3=3s-2}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}, \tag{10.8}$$

we see that $J_1^{(2/3,1)}(s)$ admits a meromorphic continuation to the domain $\mathcal{R}(2/3)$, with a possible pole at $s=2/3$ when $\tau^3=1$. Denote by $J_1^{2/3}(s)$ this continuation,

$s \in \mathcal{R}(2/3)$. Now let $s \in \mathcal{R}(2/3)^-$. Then we have

$$J_1^{2/3}(s) = \int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$$
$$+ \int_C \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \int_C \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$$
$$+ \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_3=3s-2}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s).$$

According to (10.2), (10.4), (10.6) and (10.7), the terms in the right hand side of the above formula are holomorphic in $1/2 < \mathrm{Re}(s) < 2/3$ except the term $\int_C \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$, which is equal to, by Cauchy integral formula, that

$$\int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \operatorname*{Res}_{\kappa_2=2-3s}\operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s). \tag{10.9}$$

By (10.4), one sees that $\operatorname*{Res}_{\kappa_2=2-3s}\operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.10}$$

By (10.9) and (10.10) one sees that $\int_C \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ admits a meromorphic continuation to $S_{(1/3,2/3)}$ with a at most double pole at $s = 1/2$ when $\tau^2 = 1$. Hence we obtain a meromorphic continuation of $J_1^{2/3}(s)$ to the strip $1/2 < \mathrm{Re}(s) < 2/3$. Denote by $J_1^{(1/2,2/3)}$ this continuation. Then

$$J_1^{(1/2,2/3)}(s) = \int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$$
$$+ \int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$$
$$+ \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_3=3s-2}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$$
$$+ \operatorname*{Res}_{\kappa_2=2-3s}\operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s).$$

One sees clearly that the terms in the right hand side of the above expression are meromorphic in $\mathcal{R}(1/2)$, except the terms $\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3$ and $\int_{(0)} \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$, which by Cauchy integral formula and (10.3), is equal to

$$\int_C \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s), \tag{10.11}$$

where $s \in \mathcal{R}(1/2)^+$. From the formula (10.3), we see that $\operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(1-s,\tau^{-1})}{\Lambda(3-3s,\tau^{-3})\Lambda(2s,\tau^2)\Lambda(1+s,\tau)\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})^2}. \tag{10.12}$$

We then apply the functional equation $\Lambda(2-2s,\tau^{-2}) \sim \Lambda(2s-1,\tau^2)$ to (10.12) to see that $\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(3-3s,\tau^{-3})\Lambda(2s,\tau^2)\Lambda(1+s,\tau)\Lambda(2-s,\tau^{-1})^2}. \tag{10.13}$$

Note that when $s \in \mathcal{R}(1/2)^-$, $2s$ lies in a zero-free region of $\Lambda(s,\tau^2)$. Also, Note that $\int_{(0)}\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 = \int_C\int_C \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3$ when $s \in \mathcal{R}(1/2)^+$. Then by (10.11) and (10.12) we conclude that $J_1^{(1/2,2/3)}(s)$ admits a meromorphic continuation to the area $\mathcal{R}(1/2)$. Denote by $J_1^{1/2}(s)$ this continuation, then

$$\begin{aligned} J_1^{1/2}(s) = &\int_C\int_C \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 + \int_C \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\ &+ \int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 \\ &+ \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_3=3s-2}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \\ &+ \operatorname*{Res}_{\kappa_2=2-3s}\operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s). \end{aligned}$$

Let $s \in \mathcal{R}(1/2)^-$. By Cauchy's formula we have

$$\begin{aligned} \int_C\int_C \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 = &\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 \\ &+ \int_C \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=1-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2; \end{aligned}$$

and the function $\int_C \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ is equal to

$$\int_{(0)} \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s),$$

where we have $\operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \sim \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$. Now we have a continuation of $J_1^{(1/2)}(s)$ to the region $1/3 < \mathrm{Re}(s) < 1/2$. Denote by $J_1^{(1/3,1/2)}(s)$

this continuation, then

$$
\begin{aligned}
J_1^{(1/3,1/2)}(s) = & \int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\
& + \int_{C} \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=1-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\
& + \int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 \\
& + \operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_3=3s-2}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
& + \operatorname*{Res}_{\kappa_2=2-3s}\operatorname*{Res}_{\kappa_3=2-2s-\kappa_2}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
& + \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_3=1-s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa},s).
\end{aligned}
$$

Thus we obtain a meromorphic continuation of $J_1(s)$ to the area $S_{(1/3,\infty)}$:

$$
\widetilde{J}_1(s) = \begin{cases} J_1(s),\ s \in S_{(1,+\infty)}; \\ J_1^1(s),\ s \in \mathcal{R}(1); \\ J_1^{(2/3,1)}(s),\ s \in S_{(2/3,1)}; \\ J_1^{2/3}(s),\ s \in \mathcal{R}(2/3); \\ J_1^{(1/2,2/3)}(s),\ s \in S_{(1/2,2/3)}; \\ J_1^{1/2}(s),\ s \in \mathcal{R}(1/2); \\ J_1^{(1/3,1/2)}(s),\ s \in S_{(1/3,1/2)}; \end{cases} \tag{10.14}
$$

From the above formulas one sees that $\widetilde{J}_1(s)$ has possible poles at $s = 3/4$, $s = 2/3$ and $s = 1/2$; and the potential poles at $s = 3/4$, $s = 2/3$ are at most simple, the possible pole at $s = 1/2$ has order at most 2. Moreover, from the above explicit expressions of $\widetilde{J}_1(s)$, we see that $\widetilde{J}_1(s) \cdot \Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2,\tau) = 0$.

Case 1: If $L(3/4,\tau) = 0$, then by functional equation we have that $\Lambda(1/4,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_1(s)$ has a pole at $s = 3/4$, then from the proceeding explicit expressions, we must have that $\tau^4 = 1$, and the singular part of $\widetilde{J}_1(s)$ around $s = 3/4$ is a holomorphic function multiplying $\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)$. Note that $\Lambda(3s-2,\tau^3)\mid_{s=3/4} = \Lambda(1/4,\tau^3) = \Lambda(1/4,\tau^{-1}) = 0$. Hence, when $L(3/4,\tau) = 0$, $\widetilde{J}_1(s)$ is holomorphic at $s = 3/4$.

Case 2: If $L(2/3,\tau) = 0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_1(s)$ has a pole at $s = 2/3$, then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{J_1}(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\mid_{s=2/3}= \Lambda(1/3,\tau^2) = \Lambda(1/3,\tau^{-1}) = 0$. Hence, when $L(2/3,\tau) = 0$, $\widetilde{J_1}(s)$ is holomorphic at $s = 2/3$.

Now the proof of Claim 77 is complete. □

Proof of Claim 78. Let $s \in \mathcal{R}(1)^+$. Let $J_2(s) := \int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_3$, and $J_2^1(s) := \int_C\int_C \operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_3$. By the analytic property of $\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ we see that $J_2^1(s)$ is meromorphic in the domain $\mathcal{R}(1)$, with a possible pole at $s = 1$. Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then see that

$$J_2^1(s) = \int_C\int_{(0)} \operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_3 + \int_C \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3, \tag{10.15}$$

where $\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying the product of $\Lambda(s,\tau)^2$ and the meromorphic function

$$\frac{\Lambda(2s-1-\kappa_3,\chi_{43}\tau^2)\Lambda(2s-1+\kappa_3,\chi_{34}\tau^2)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)}{\Lambda(2-s+\kappa_3,\chi_{34})\Lambda(2-s-\kappa_3,\chi_{43})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.16}$$

Then $J_2^1(s)$ is equal to, after applications of Cauchy integral formula to (10.15),

$$\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_3 + \int_{(0)} \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 + \int_{(0)} \operatorname{Res}_{\kappa_3=2-2s-\kappa_1}$$
$$\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 + \int_{(0)} \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3 + \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s),$$

where $\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying the product of the meromorphic function $\Lambda(s,\tau)^2$ and

$$\frac{\Lambda(2s-1+\kappa_1,\chi_{12}\tau^2)\Lambda(2s-1-\kappa_1,\chi_{21}\tau^2)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)}{\Lambda(2-s+\kappa_1,\chi_{12})\Lambda(2-s-\kappa_1,\chi_{21})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}; \tag{10.17}$$

also, $\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2 \cdot \Lambda(2-s,\tau^{-1})^{-2}$ and

$$\frac{\Lambda(1-\kappa_1,\chi_{21})\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-2+\kappa_1,\chi_{12}\tau^3)\Lambda(2s-1+\kappa_1,\chi_{12}\tau^2)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(3-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(2-s-\kappa_1,\tau^{-1})}. \tag{10.18}$$

From the formula (10.16), we see that $\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.19}$$

We thus see from the proceeding computations of analytic behaviors of the functions $\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, $\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$, and

$$\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s),$$

that the functions

$$\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3,\quad \int_{(0)}\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$$

and $\int_{(0)}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)f\kappa_3$ admit meromorphic continuation to the domain $1/2<\operatorname{Re}(s)<1$, with a possible pole at $s=2/3$ if $\tau^3=1$; and

$$\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$$

admits a meromorphic continuation to the domain $\mathcal{R}(1/2)^-\cup S_{[1/2,1)}$, with possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$, when $\tau^4=1$, $\tau^3=1$ and $\tau^2=1$, respectively.

From (10.18) we see that the function $\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ might have infinitely many poles in the strip $1/2<\operatorname{Re}(s)<1$. These poles come from nontrivial zeros of $L(s,\chi_{12}\tau)$ is this strip. Hence we may have a problem shifting contours if we try to continue $\int_{(0)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ directly. To remedy this, we need to first deal with the factor $\Lambda(s+\kappa_1,\chi_{12}\tau)$. Thanks to the uniform zero-free region of Rankin-Selberg L-functions defined in Section 8.1, the function $\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is holomorphic in the domain $\mathcal{R}(1-s)$. Then we can apply Cauchy integral formula to obtain that

$$\int_{(0)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1=\int_{(1-s)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1,\tag{10.20}$$

where the integral on the right hand side is taken over $(1-s):=\{z\in\mathbb{C}:\ \operatorname{Re}(z)=1-\operatorname{Re}(s)\}$. Let $\kappa_1'=\kappa_1+s-1$, $\kappa_2'=\kappa_2$ and $\kappa_3'=\kappa_3$. Denote by $\boldsymbol{\kappa}'=(\kappa_1',\kappa_2',\kappa_3')$. Then $d\kappa_j'=d\kappa_j$, $1\leq j\leq 3$. Hence we have

$$\int_{(0)}\operatorname{Res}_{\kappa_3=2-2s-\kappa_1}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1=\int_{(0)}\operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1',\tag{10.21}$$

where by (10.18), $\operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\cdot\Lambda(2-s,\tau^{-1})^{-2}$ and

$$\frac{\Lambda(s+\kappa_1',\chi_{12}\tau)\Lambda(s-\kappa_1',\chi_{21}\tau)\Lambda(2s-1+\kappa_1',\chi_{12}\tau^2)\Lambda(2s-1-\kappa_1',\chi_{21}\tau^2)}{\Lambda(1+\kappa_1',\chi_{12})\Lambda(1-\kappa_1',\chi_{21})\Lambda(2-s+\kappa_1',\chi_{12}\tau^{-1})\Lambda(2-s-\kappa_1',\chi_{21}\tau^{-1})}.\tag{10.22}$$

Then from (10.20), (10.21) and (10.22) we conclude that

$$\int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$$

admits a meromorphic continuation to the strip $1/2 < \mathrm{Re}(s) < 1$. We then have a meromorphic continuation of $J_2^1(s)$ to the area $S_{(1/2,1)}$. Denote by $J_2^{(1/2,1)}$ this continuation. Then $J_2^{(1/2,1)}(s)$ is equal to

$$\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \int_{(0)} \operatorname*{Res}_{\kappa_3'=1-s-\kappa_1'}$$
$$\operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}', s) d\kappa_1' + \int_{(0)} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 + \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s).$$

Let $s \in \mathcal{R}(1/2)^+$. Then by (10.17) and Cauchy integral formula we see that the function $\int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$ is equal to

$$\int_{C} \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.23}$$

and $\operatorname*{Res}_{\kappa_1=2s-1} \operatorname*{Res}_{\kappa_3=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.24}$$

By (10.22) and Cauchy formula we see that $\int_{(0)} \operatorname*{Res}_{\kappa_3'=1-s-\kappa_1'} \operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}', s) d\kappa_1'$ equals

$$\int_{C} \operatorname*{Res}_{\kappa_3'=1-s-\kappa_1'} \operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}', s) d\kappa_1' - \operatorname*{Res}_{\kappa_1'=2s-1} \operatorname*{Res}_{\kappa_3'=2-2s} \operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.25}$$

and $\operatorname*{Res}_{\kappa_1'=2s-1} \operatorname*{Res}_{\kappa_3'=2-2s} \operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)^2\Lambda(1-s,\tau-1)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})^2\Lambda(2s,\tau^2)\Lambda(1+s,\tau)}. \tag{10.26}$$

By (10.16) and Cauchy formula we see that $\int_{(0)} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3$ equals

$$\int_{C} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 - \operatorname*{Res}_{\kappa_3=2s-1} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.27}$$

and $\operatorname*{Res}_{\kappa_3=2s-1} \operatorname*{Res}_{\kappa_1=2-2s} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.28}$$

Note that $\int_C \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$, $\int_C \operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1'$ and the function $\int_C \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ are meromorphic inside $\mathcal{R}(1/2)$, with a potential pole of order less or equal to 2 at $s = 1/2$. Moreover, it follows from (10.20), (10.21) and (10.22) that if $L(1/2,\tau) = 0$, then these three integrals are holomorphic at $s = 1/2$; and the ratio of these integrals and $\Lambda(s,\tau)$ have at most a simple pole at $s = 1/2$. In particular, combining equations (10.23), (10.24), (10.25), (10.26), (10.27) and (10.28), one thus has a meromorphic continuation of $J_2^{(1/2,1)}(s)$ to the domain $\mathcal{R}(1/2)$, with a potential pole of order less or equal to 2 at $s = 1/2$. Denote by $J_2^{1/2}(s)$ this continuation. Then $J_2^{1/2}(s)\cdot\Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2,\tau) = 0$. Explicitly, by Cauchy's formula we have

$$\begin{aligned}J_2^{1/2}(s) = &\int_C\int_C \operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3 + \int_C \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1\\ &+ \int_C \operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1' + \int_C \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3\\ &+ \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname{Res}_{\kappa_1=2s-1}\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\ &- \operatorname{Res}_{\kappa_1'=2s-1}\operatorname{Res}_{\kappa_3'=2-2s}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname{Res}_{\kappa_3=2s-1}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).\end{aligned}$$

Let $s \in \mathcal{R}(1/2)^-$. Then $\int_C\int_C \operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3$ is equal to

$$\begin{aligned}&\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3 + \int_{(0)} \operatorname{Res}_{\kappa_1=1-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3\\ &\quad+ \int_{(0)} \operatorname{Res}_{\kappa_3=1-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \operatorname{Res}_{\kappa_3=1-2s}\operatorname{Res}_{\kappa_1=1-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).\end{aligned}$$

Likewise, the function $\int_C \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ is equal to

$$\int_{(0)} \operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \operatorname{Res}_{\kappa_1=1-2s}\operatorname{Res}_{\kappa_3=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s);$$

the function $\int_C \operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1'$ is equal to

$$\int_{(0)} \operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1' + \operatorname{Res}_{\kappa_1'=1-2s}\operatorname{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname{Res}_{\kappa_2'=s-1}\mathcal{F}(\boldsymbol{\kappa}',s);$$

and the function $\int_C \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ is equal to

$$\int_{(0)} \operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 + \operatorname{Res}_{\kappa_3=1-2s}\operatorname{Res}_{\kappa_1=2-2s}\operatorname{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s).$$

As before, a computation of the above integrals leads to a meromorphic continuation of $J_2^{1/2}(s)$ to the region $1/3 < \mathrm{Re}(s) < 1/2$. Denote by this continuation $J_2^{(1/3,1/2)}(s)$, then

$$\begin{aligned}
J_2^{(1/3,1/2)}(s) = &\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3 + \int_{(0)} \operatorname*{Res}_{\kappa_3=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 \\
&+ \int_{(0)} \operatorname*{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}',s)d\kappa_1' + \int_{(0)} \operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 \\
&+ \operatorname*{Res}_{\kappa_3=2-2s}\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_3=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
&- \operatorname*{Res}_{\kappa_1'=2s-1}\operatorname*{Res}_{\kappa_3'=2-2s}\operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_3=2s-1}\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
&+ \int_{(0)} \operatorname*{Res}_{\kappa_3=1-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \operatorname*{Res}_{\kappa_3=1-2s}\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
&+ \int_{(0)} \operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 + \operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_3=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
&+ \operatorname*{Res}_{\kappa_1'=1-2s}\operatorname*{Res}_{\kappa_3'=1-s-\kappa_1'}\operatorname*{Res}_{\kappa_2'=s-1} \mathcal{F}(\boldsymbol{\kappa}',s) + \operatorname*{Res}_{\kappa_3=1-2s}\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa},s).
\end{aligned}$$

Thus we obtain a meromorphic continuation of $J_2(s)$ to $S_{(1/3,\infty)}$:

$$\widetilde{J_2}(s) = \begin{cases} J_2(s), \ s \in S_{(1,+\infty)}; \\ J_2^1(s), \ s \in \mathcal{R}(1); \\ J_2^{(1/2,1)}(s), \ s \in S_{(1/2,1)}; \\ J_2^{1/2}(s), \ s \in \mathcal{R}(1/2); \\ J_2^{(1/3,1/2)}(s), \ s \in S_{(1/3,1/2)}. \end{cases} \tag{10.29}$$

From the above formulas one sees that $\widetilde{J_2}(s)$ has possible poles at $s = 3/4$, $s = 2/3$ and $s = 1/2$; and the potential poles at $s = 3/4$, $s = 2/3$ are at most simple, the possible pole at $s = 1/2$ has order at most 2. Moreover, from the above explicit expressions of $\widetilde{J_2}(s)$, we see that $\widetilde{J_2}(s) \cdot \Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2,\tau) = 0$.

Case 1: If $L(3/4,\tau) = 0$, then by functional equation we have that $\Lambda(1/4,\tau^{-1}) = 0$. Suppose that $\widetilde{J_2}(s)$ has a pole at $s = 3/4$, then from the proceeding explicit expressions, we must have that $\tau^4 = 1$, and the singular part of $\widetilde{J_2}(s)$ around $s = 3/4$ is a holomorphic function multiplying $\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)$. Note that $\Lambda(3s-2,\tau^3)\mid_{s=3/4} = \Lambda(1/4,\tau^3) = \Lambda(1/4,\tau^{-1}) = 0$. Hence, when $L(3/4,\tau) = 0$, $\widetilde{J_2}(s)$ is holomorphic at $s = 3/4$.

Case 2: If $L(2/3,\tau) = 0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_2(s)$ has a pole at $s = 2/3$, then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{J}_2(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\mid_{s=2/3} = \Lambda(1/3,\tau^2) = \Lambda(1/3,\tau^{-1}) = 0$. Hence, when $L(2/3,\tau) = 0$, $\widetilde{J}_2(s)$ is holomorphic at $s = 2/3$.

Now the proof of Claim 78 is complete. □

Proof of Claim 79. Let $s \in \mathcal{R}(1)^+$. Let $J_3(s) := \int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2$, and $J_3^1(s) := \int_{\mathcal{C}}\int_{\mathcal{C}} \operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2$. By the analytic property of $\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ we see that $J_3^1(s)$ is meromorphic in the domain $\mathcal{R}(1)$, with a possible pole at $s = 1$. Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then see that

$$J_3^1(s) = \int_{\mathcal{C}}\int_{(0)} \operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2 + \int_{\mathcal{C}} \operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2, \tag{10.30}$$

where $\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2 \cdot \Lambda(2-s,\tau^{-1})^{-2}$ and

$$\frac{\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2s-1+\kappa_2,\chi_{23}\tau^2)\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)}{\Lambda(1+\kappa_2,\chi_{23})\Lambda(1-\kappa_2,\chi_{32})\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})\Lambda(2-s-\kappa_2,\tau^{-1})}. \tag{10.31}$$

Then after applications of Cauchy integral formula to (10.30), we obtain that

$$\begin{aligned} J_1^1(s) = &\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2 + \int_{(0)} \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 \\ &+ \int_{(0)} \operatorname{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \int_{(0)} \operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\ &+ \operatorname{Res}_{\kappa_2=1-s}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s) + \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s), \end{aligned}$$

where $s \in \mathcal{R}(1)^-$ and $\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying the product of the meromorphic function $\Lambda(s,\tau)^2$ and

$$\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(3s-2-\kappa_1,\chi_{21}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(3-2s+\kappa_1,\chi_{12})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}; \tag{10.32}$$

and $\operatorname{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-2+\kappa_1,\chi_{12}\tau^3)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(3-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.33}$$

From the formula (10.31), we see that $\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.34}$$

Moreover, by (10.31) and the fact that $\Lambda(s+\kappa_2,\chi_{23}\tau)\cdot\Lambda(2-s-\kappa_2,\chi_{32}\tau^{-1})$ is holomorphic at $\kappa_2=1-s$ when $\chi_{23}=\tau^{-1}$, we deduce that $\underset{\kappa_2=1-s}{\operatorname{Res}}\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)\equiv 0$.

We thus see from the proceeding computations of analytic behaviors of the functions $\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$, $\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$ and $\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$, that

$$\int_{(0)}\int_{(0)}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2$$

and $\int_{(0)}\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ admit meromorphic continuation to the domain $1/2<\operatorname{Re}(s)<1$; and $\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_1=1-s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation to the domain $\mathcal{R}(1/2)\cup S_{(1/2,1)}$, with possible simple poles at $s=3/4$, $s=2/3$ and $s=1/2$, when $\tau^4=1$, $\tau^3=1$ and $\tau^2=1$, respectively, according to (10.34).

From (10.32) we see that the function $\int_{(0)}\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ admits holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. From (10.33) we see that the function $\int_{(0)}\underset{\kappa_2=2-2s-\kappa_1}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ admits holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. Then combining these with (10.32) and (10.34) one sees that $J_3^1(s)$ admits a holomorphic continuation to the domain $2/3<\operatorname{Re}(s)<1$. Denote by $J_3^{(2/3,1)}(s)$ this continuation, where $2/3<\operatorname{Re}(s)<1$.

Let $s\in\mathcal{R}(2/3)^+$, then by Cauchy integral formula we have

$$\int_{(0)}\underset{\kappa_2=2-2s-\kappa_1}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1=\int_{\mathcal{C}}\underset{\kappa_2=2-2s-\kappa_1}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1. \tag{10.35}$$

Likewise, for $s\in\mathcal{R}(2/3)^+$, by (10.32), $\int_{(0)}\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ is equal to

$$\int_{\mathcal{C}}\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-\underset{\kappa_1=3s-2}{\operatorname{Res}}\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_3=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s), \tag{10.36}$$

where $\underset{\kappa_3=3s-2}{\operatorname{Res}}\underset{\kappa_2=2-2s}{\operatorname{Res}}\underset{\kappa_1=s-1}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.37}$$

Then according to (10.32), (10.33), (10.35), (10.36) and (10.37), we see that $J_1^{(2/3,1)}(s)$ admits a meromorphic continuation to the domain $\mathcal{R}(2/3)$, with a possible pole at $s = 2/3$ when $\tau^3 = 1$. Denote by $J_3^{2/3}(s)$ this continuation, $s \in \mathcal{R}(2/3)$. Now let $s \in \mathcal{R}(2/3)^-$. Then we have that

$$\begin{aligned} J_3^{2/3}(s) = &\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 + \int_C \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &+ \int_C \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 \\ &+ \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=3s-2} \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s). \end{aligned}$$

According to (10.33), (10.34), (10.35), (10.36) and (10.37), the terms in the right hand side of the above formula are holomorphic in $1/2 < \operatorname{Re}(s) < 2/3$ except the term $\int_C \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$, which is equal to, by Cauchy integral formula, that

$$\int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \operatorname*{Res}_{\kappa_1=2-3s} \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.38}$$

where $s \in \mathcal{R}(2/3)^-$. By (10.33), one sees that $\operatorname*{Res}_{\kappa_2=2-3s} \operatorname*{Res}_{\kappa_3=2-2s-\kappa_2} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2, \tau^4)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)^2}{\Lambda(3-3s, \tau^{-3})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})\Lambda(1+s, \tau)}. \tag{10.39}$$

By (10.33), (10.38) and (10.39) one sees that $\int_C \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$ admits a meromorphic continuation to $S_{(1/3,2/3)}$ with a at most double pole at $s = 1/2$ when $\tau^2 = 1$. Hence we obtain a meromorphic continuation of $J_1^{2/3}(s)$ to the strip $1/2 < \operatorname{Re}(s) < 2/3$. Denote by $J_3^{(1/2,2/3)}$ this continuation, namely,

$$\begin{aligned} J_3^{(1/2,2/3)}(s) = &\int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 \\ &+ \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 + \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 \\ &+ \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=3s-2} \operatorname*{Res}_{\kappa_2=2-2s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2-3s} \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s). \end{aligned}$$

One sees clearly that the terms in the right hand side of the above expression are meromorphic in $\mathcal{R}(1/2)$, except the term $\int_{(0)} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2$, which by Cauchy integral formula and (10.31), is equal to

$$\int_C \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2 - \operatorname*{Res}_{\kappa_2=2s-1} \operatorname*{Res}_{\kappa_1=1-s} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.40}$$

where $s \in \mathcal{R}(1/2)^+$. By formula (10.31), we see that $\operatorname{Res}_{\kappa_2=2s-1}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(1-s,\tau^{-1})}{\Lambda(3-3s,\tau^{-3})\Lambda(2s,\tau^2)\Lambda(1+s,\tau)\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})^2}. \tag{10.41}$$

We then apply the functional equation $\Lambda(2-2s,\tau^{-2}) \sim \Lambda(2s-1,\tau^2)$ to (10.41) to see that $\operatorname{Res}_{\kappa_2=2s-1}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2\Lambda(1-s,\tau^{-1})}{\Lambda(3-3s,\tau^{-3})\Lambda(2s,\tau^2)\Lambda(1+s,\tau)\Lambda(2-s,\tau^{-1})^2}. \tag{10.42}$$

Note that when $s \in \mathcal{R}(1/2)^-$, $2s$ lies in a zero-free region of $\Lambda(s,\tau^2)$. Then by (10.40) and (10.42) we conclude that $J_1^{(1/2,2/3)}(s)$ admits a meromorphic continuation to the region $\mathcal{R}(1/2)$. Denote by $J_3^{1/2}(s)$ this continuation, then

$$\begin{aligned} J_1^{1/2}(s) = &\int_C\int_C \operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2 + \int_{(0)} \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 \\ &+ \int_{(0)} \operatorname{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \int_C \operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\ &+ \operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname{Res}_{\kappa_1=3s-2}\operatorname{Res}_{\kappa_2=2-2s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s) \\ &+ \operatorname{Res}_{\kappa_1=2-3s}\operatorname{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s) - \operatorname{Res}_{\kappa_2=2s-1}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s). \end{aligned}$$

Let $s \in \mathcal{R}(1/2)^-$. Then by Cauchy's integral formula we have

$$\begin{aligned} \int_C\int_C \operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2 = &\int_{(0)}\int_{(0)} \operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_1 \\ &+ \int_C \operatorname{Res}_{\kappa_2=1-2s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \int_{(0)} \operatorname{Res}_{\kappa_1=1-2s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2. \end{aligned}$$

Also, by (8.31) we have $\int_C \operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2$ equal to

$$\int_{(0)} \operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 + \operatorname{Res}_{\kappa_2=1-2s}\operatorname{Res}_{\kappa_1=1-s}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s).$$

Substituting the above equalities into the expression of $J_3^{1/2}(s)$ we then obtain a continuation of $J_3^{1/2}(s)$ into $1/3 < \operatorname{Re}(s) < 1/2$. Denote this continuation by

$J_3^{(1/2,2/3)}(s)$, then

$$\begin{aligned}
J_3^{(1/2,2/3)}(s) = & \int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_2 + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 \\
& + \int_{(0)} \operatorname*{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \int_{(0)} \operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\
& + \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_1=3s-2}\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
& + \operatorname*{Res}_{\kappa_1=2-3s}\operatorname*{Res}_{\kappa_2=2-2s-\kappa_1}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s) - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s) \\
& + \int_{C} \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \int_{(0)} \operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 \\
& + \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_3=s-1} \mathcal{F}(\boldsymbol{\kappa},s).
\end{aligned}$$

Thus we obtain a meromorphic continuation of $J_3(s)$ to the area $S_{(1/3,\infty)}$:

$$\widetilde{J}_3(s) = \begin{cases} J_3(s), \ s \in S_{(1,+\infty)}; \\ J_3^1(s), \ s \in \mathcal{R}(1); \\ J_3^{(2/3,1)}(s), \ s \in S_{(2/3,1)}; \\ J_3^{2/3}(s), \ s \in \mathcal{R}(2/3); \\ J_3^{(1/2,2/3)}(s), \ s \in S_{(1/2,2/3)}; \\ J_3^{1/2}(s), \ s \in \mathcal{R}(1/2); \\ J_3^{(1/3,1/2)}(s), \ s \in \mathcal{S}(1/3,1/2). \end{cases} \tag{10.43}$$

From the above formulas one sees that $\widetilde{J}_3(s)$ has possible poles at $s = 3/4$, $s = 2/3$ and $s = 1/2$; and the potential poles at $s = 3/4$, $s = 2/3$ are at most simple, the possible pole at $s = 1/2$ has order at most 2. Moreover, from the above explicit expressions of $\widetilde{J}_3(s)$, we see that $\widetilde{J}_3(s) \cdot \Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2,\tau) = 0$.

Case 1: If $L(3/4,\tau) = 0$, then by functional equation we have that $\Lambda(1/4,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_3(s)$ has a pole at $s = 3/4$, then from the proceeding explicit expressions, we must have that $\tau^4 = 1$, and the singular part of $\widetilde{J}_3(s)$ around $s = 3/4$ is a holomorphic function multiplying $\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)$. Note that $\Lambda(3s-2,\tau^3)\mid_{s=3/4} = \Lambda(1/4,\tau^3) = \Lambda(1/4,\tau^{-1}) = 0$. Hence, when $L(3/4,\tau) = 0$, $\widetilde{J}_3(s)$ is holomorphic at $s = 3/4$.

Case 2: If $L(2/3,\tau) = 0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_3(s)$ has a pole at $s = 2/3$, then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{J}_3(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\mid_{s=2/3}= \Lambda(1/3,\tau^2) = \Lambda(1/3,\tau^{-1}) = 0$. Hence, when $L(2/3,\tau) = 0$, $\widetilde{J}_3(s)$ is holomorphic at $s = 2/3$.

Now the proof of Claim 79 is complete. □

Proof of Claim 80. Let $s \in \mathcal{R}(1)^+$. Let $J_{12}(s) = \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$, and $J_{12}^1(s) = \int_C \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$. Then by (8.32) one sees that $J_{12}^1(s)$ is meromorphic in the region $\mathcal{R}(1)$, with a possible pole at $s = 1$.

Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then have that

$$J_{12}^1(s) = \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 + \operatorname*{Res}_{\kappa_3=3-3s}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{10.44}$$

where $\operatorname*{Res}_{\kappa_3=3-3s}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.45}$$

Then one sees, by (8.32) and (10.45), that $\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ and the function $\operatorname*{Res}_{\kappa_3=3-3s}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ are meromorphic in the strip $2/3 < \operatorname{Re}(s) < 1$, with possible simple poles at $s = 3/4$ if $\tau^4 = 1$. Hence, by (10.44), we obtain a meromorphic continuation of $J_{12}^1(s)$ to the strip $2/3 < \operatorname{Re}(s) < 1$, with possible simple poles at $s = 3/4$ if $\tau^4 = 1$. Denote by $J_{12}^{(2/3,1)}(s)$ this continuation.

Let $s \in \mathcal{R}(2/3)^+$. Applying Cauchy integral formula to (8.32) to see that the function $\int_{(0)} \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ is equal to

$$\int_C \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 - \operatorname*{Res}_{\kappa_3=2-3s}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{10.46}$$

and $\operatorname*{Res}_{\kappa_3=2-3s}\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(1+s,\tau)\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.47}$$

Then by (10.46), (10.47) and the fact that $\int_C \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3$ is holomorphic in $\mathcal{R}(2/3)$, we obtain a meromorphic continuation of $J_{12}^{(2/3,1)}(s)$ to the region $\mathcal{R}(2/3)$, with a possible simple pole at $s = 2/3$, if $\tau^3 = 1$. Denote by $J_{12}^{2/3}(s)$ the continuation.

Let $s \in \mathcal{R}(2/3)^{-}$. Then by (10.44) and (10.46) one has

$$J_{12}^{2/3}(s) = \int_{C} \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_3 - \operatorname*{Res}_{\kappa_3=2-3s} \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_3=3-3s} \operatorname*{Res}_{\kappa_1=s-1} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s).$$

Since the right hand side is meromorphic in the strip $S_{(0,2/3)}$, with a possible simple pole at $s = 1/2$ if $\tau^2 = 1$. We thus obtain a meromorphic continuation of $J_{12}^{2/3}(s)$ to the region $0 < \operatorname{Re}(s) < 2/3$, with a possible simple pole at $s = 1/2$ if $\tau^2 = 1$. Denote by $J_{12}^{(1/3,2/3)}(s)$ this continuation. Thus we obtain a meromorphic continuation of $J_{12}(s)$ to the area $S_{(1/3,\infty)}$:

$$\widetilde{J}_{12}(s) = \begin{cases} J_{12}(s), \ s \in S_{(1,+\infty)}; \\ J_{12}^{1}(s), \ s \in \mathcal{R}(1); \\ J_{12}^{(2/3,1)}(s), \ s \in S_{(2/3,1)}; \\ J_{12}^{2/3}(s), \ s \in \mathcal{R}(2/3); \\ J_{12}^{(1/3,2/3)}(s), \ s \in \mathcal{R}(1/2) \cup S_{(1/3,2/3)}. \end{cases} \tag{10.48}$$

From the above formulas one sees that $\widetilde{J}_{12}(s)$ has possible poles at $s = 3/4$, $s = 2/3$ and $s = 1/2$; and these potential poles are all at most simple. Moreover, from the above explicit expressions of $\widetilde{J}_{12}(s)$, we see that $\widetilde{J}_{12}(s) \cdot \Lambda(s, \tau)^{-1}$ has at most a simple pole at $s = 1/2$. We discuss the other two possible poles separately.

Case 1: If $L(3/4, \tau) = 0$, then by functional equation we have that $\Lambda(1/4, \tau^{-1}) = 0$. Suppose that $\widetilde{J}_{12}(s)$ has a pole at $s = 3/4$, then from the proceeding explicit expressions, we must have that $\tau^4 = 1$, and the singular part of $\widetilde{J}_{12}(s)$ around $s = 3/4$ is a holomorphic function multiplying $\Lambda(4s - 3, \tau^4)\Lambda(3s - 2, \tau^3)$. Note that $\Lambda(3s - 2, \tau^3) \mid_{s=3/4} = \Lambda(1/4, \tau^3) = \Lambda(1/4, \tau^{-1}) = 0$. Hence, when $L(3/4, \tau) = 0$, $\widetilde{J}_{12}(s)$ is holomorphic at $s = 3/4$.

Case 2: If $L(2/3, \tau) = 0$, then by functional equation we have that $\Lambda(1/3, \tau^{-1}) = 0$. Suppose that $\widetilde{J}_{12}(s)$ has a pole at $s = 2/3$, then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{J}_{12}(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s - 2, \tau^3)\Lambda(2s - 1, \tau^2)$. Note that $\Lambda(2s - 1, \tau^2) \mid_{s=2/3} = \Lambda(1/3, \tau^2) = \Lambda(1/3, \tau^{-1}) = 0$. Hence, when $L(2/3, \tau) = 0$, $\widetilde{J}_{12}(s)$ is holomorphic at $s = 2/3$.

Now the proof of Claim 80 is complete. □

Proof of Claim 81. Let $s \in \mathcal{R}(1)^+$. Let $J_{13}(s) = \int_{(0)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2$, and $J_{13}^1(s) = \int_C \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2$. Then by (8.31) one sees that $J_{13}^1(s)$ is meromorphic in the region $\mathcal{R}(1)$, with a possible pole at $s = 1$.

Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then have that

$$J_{13}^1(s) = \int_{(0)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 + R_1(s) + R_2(s), \tag{10.49}$$

where $R_1(s) := \underset{\kappa_2=2-2s}{\operatorname{Res}} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)$, and $R_2(s)$ denotes the meromorphic function $\underset{\kappa_2=3-3s}{\operatorname{Res}} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)$. Then by (8.31) $R_1(s) \equiv 0$ if $\chi_{23}\tau^2 \neq 1$. Let $\chi_{23}\tau^2 = 1$. Then the function $G(\kappa_2) = \Lambda(2s-1+\kappa_2) \cdot \Lambda(3-2s-\kappa_2)^{-1}$ is holomorphic at $\kappa_2 = 2-2s$. Hence $R_1(s) \equiv 0$. Also, according to (8.31),

$$R_2(s) \sim \frac{\Lambda(4s-3, \tau^4)\Lambda(3s-2, \tau^3)\Lambda(2s-1, \tau^2)\Lambda(s, \tau)}{\Lambda(4-3s, \tau^{-3})\Lambda(3-2s, \tau^{-2})\Lambda(2-s, \tau^{-1})}. \tag{10.50}$$

Thanks to the uniform zero-free region of Rankin-Selberg L-functions defined in Section 8.1, the function $\underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)$ is holomorphic in the domain $\mathcal{R}(1-s)$. Then we can apply Cauchy integral formula to obtain that

$$\int_{(0)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 = \int_{(1-s)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2, \tag{10.51}$$

where the integral on the right hand side is taken over $(1-s) := \{z \in \mathbb{C} : \operatorname{Re}(z) = 1 - \operatorname{Re}(s)\}$. Let $\kappa_2' = \kappa_2 + s - 1$, $\kappa_1' = \kappa_1$ and $\kappa_3' = \kappa_3$. Denote by $\boldsymbol{\kappa}' = (\kappa_1', \kappa_2', \kappa_3')$. Then $d\kappa_j' = d\kappa_j$, $1 \leq j \leq 3$. Hence we have

$$\int_{(0)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 = \int_{(0)} \underset{\kappa_1'=s-1}{\operatorname{Res}} \underset{\kappa_3'=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2', \tag{10.52}$$

where by (8.31), $\underset{\kappa_1'=s-1}{\operatorname{Res}} \underset{\kappa_3'=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}', s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1, \tau^2)^2\Lambda(s, \tau)^2 \cdot \Lambda(2-s, \tau^{-1})^{-2}$ and

$$\frac{\Lambda(s+\kappa_2', \chi_{23}\tau)\Lambda(s-\kappa_2', \chi_{32}\tau)\Lambda(2s-1+\kappa_2', \chi_{23}\tau^2)\Lambda(2s-1-\kappa_2', \chi_{32}\tau^2)}{\Lambda(1+\kappa_2', \chi_{23})\Lambda(1-\kappa_2', \chi_{32})\Lambda(2-s+\kappa_2', \chi_{23}\tau^{-1})\Lambda(2-s-\kappa_2', \chi_{32}\tau^{-1})}. \tag{10.53}$$

Then from (10.52) and (10.53), we conclude that $\int_{(0)} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2$ admits a meromorphic continuation to the strip $1/2 < \operatorname{Re}(s) < 1$. Combining this with equations (10.49) and (10.50), we then obtain a meromorphic continuation of $J_{13}^1(s)$ to the area $S_{(1/2,1)}$. Denote by $J_{13}^{(1/2,1)}$ this continuation. Then

$$J_{13}^{(1/2,1)}(s) = \int_{(0)} \underset{\kappa_1'=s-1}{\operatorname{Res}} \underset{\kappa_3'=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2' + \underset{\kappa_2=3-3s}{\operatorname{Res}} \underset{\kappa_1=s-1}{\operatorname{Res}} \underset{\kappa_3=s-1}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s). \tag{10.54}$$

Let $s \in \mathcal{R}(1/2)^+$. Applying Cauchy integral formula to (10.54) to obtain that

$$J_{13}^{(1/2,1)}(s) = \int_C \underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2' + R_2(s) - R_3(s), \tag{10.55}$$

where $R_3(s) := \underset{\kappa_2'=2s-1}{\mathrm{Res}}\,\underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s)$. By (8.31), we have that

$$R_3(s) \sim \frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)^2\Lambda(1-s,\tau-1)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})^2\Lambda(2s,\tau^2)\Lambda(1+s,\tau)}. \tag{10.56}$$

By (10.56), the right hand side is meromorphic in $\mathcal{R}(1/2)$, with a possible pole at $s = 1/2$ of order at most 1 according to the functional equation $\Lambda(2s-1,\tau^2) \sim \Lambda(2-2s,\tau^{-2})$. Hence we obtain a meromorphic continuation of $J_{13}^{(1/2,1)}(s)$ to the domain $\mathcal{R}(1/2)$. Denote by $J_{13}^{1/2}(s)$ this continuation.

Let $s \in \mathcal{R}(1/2)^-$. Then by (10.53), $\int_C \underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2'$ is equal to

$$\int_{(0)} \underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2' + \underset{\kappa_2'=1-2s}{\mathrm{Res}}\,\underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s),$$

where $\underset{\kappa_2'=1-2s}{\mathrm{Res}}\,\underset{\kappa_1'=s-1}{\mathrm{Res}}\,\underset{\kappa_3'=s-1}{\mathrm{Res}}\,\mathcal{F}(\boldsymbol{\kappa}', s)$ is equal to a holomorphic function multiplying

$$\frac{\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(3s-1,\tau^3)\Lambda(1-s,\tau^{-1})\Lambda(4s-2,\tau^4)}{\Lambda(2-s,\tau^{-1})^2\Lambda(2-2s,\tau^{-2})\Lambda(2s,\tau^2)\Lambda(3-3s,\tau^{-3})\Lambda(s+1,\tau)}.$$

Thus we obtain a meromorphic continuation of $J_{12}(s)$ to the area $S_{(1/3,\infty)}$:

$$\widetilde{J}_{13}(s) = \begin{cases} J_{13}(s), \ s \in S_{(1,+\infty)}; \\ J_{13}^1(s), \ s \in \mathcal{R}(1); \\ J_{13}^{(1/2,1)}(s), \ s \in S_{(1/2,1)}; \\ J_{13}^{1/2}(s), \ s \in \mathcal{R}(1/2); \\ J_{13}^{(1/3,1/2)}(s), \ s \in S_{(1/3,1/2)}. \end{cases} \tag{10.57}$$

From the above formulas one sees that $\widetilde{J}_{13}(s)$ has possible poles at $s = 3/4$, $s = 2/3$ and $s = 1/2$; and these potential poles are all at most simple. Moreover, from the above explicit expressions of $\widetilde{J}_{13}(s)$, we see that $\widetilde{J}_{13}(s)\cdot\Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$. We discuss the other two possible poles separately.

Case 1: If $L(3/4,\tau) = 0$, then by functional equation we have that $\Lambda(1/4,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_{13}(s)$ has a pole at $s = 3/4$, then from the proceeding explicit expressions, we must have that $\tau^4 = 1$, and the singular part of $\widetilde{J}_{13}(s)$ around $s = 3/4$ is a holomorphic function multiplying $\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)$. Note that $\Lambda(3s-2,\tau^3)\ |_{s=3/4}= \Lambda(1/4,\tau^3) = \Lambda(1/4,\tau^{-1}) = 0$. Hence, when $L(3/4,\tau) = 0$, $\widetilde{J}_{13}(s)$ is holomorphic at $s = 3/4$.

Case 2: If $L(2/3,\tau) = 0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1}) = 0$. Suppose that $\widetilde{J}_{13}(s)$ has a pole at $s = 2/3$, then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{J}_{13}(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\ |_{s=2/3}= \Lambda(1/3,\tau^2) = \Lambda(1/3,\tau^{-1}) = 0$. Hence, when $L(2/3,\tau) = 0$, $\widetilde{J}_{13}(s)$ is holomorphic at $s = 2/3$.

Now the proof of Claim 81 is complete. □

Proof of Claim 82. Let $s \in \mathcal{R}(1)^+$. Let $J_{23}(s) = \int_{(0)} \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$, and $J_{23}^1(s) = \int_C \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$. Then by (8.30) one sees that $J_{23}^1(s)$ is meromorphic in the region $\mathcal{R}(1)$, with a possible pole at $s = 1$.

Let $s \in \mathcal{R}(1)^-$. Applying Cauchy integral formula we then have that

$$J_{23}^1(s) = \int_{(0)} \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 + \operatorname{Res}_{\kappa_1=3-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{10.58}$$

where $\operatorname{Res}_{\kappa_1=3-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)}{\Lambda(4-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.59}$$

Then one sees, by (8.30) and (10.59), that $\int_{(0)} \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ and the function $\operatorname{Res}_{\kappa_1=3-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ are meromorphic in the strip $2/3 < \mathrm{Re}(s) < 1$, with possible simple poles at $s = 3/4$ if $\tau^4 = 1$. Hence, by (10.58), we obtain a meromorphic continuation of $J_{23}^1(s)$ to the strip $2/3 < \mathrm{Re}(s) < 1$, with possible simple poles at $s = 3/4$ if $\tau^4 = 1$. Denote by $J_{23}^{(2/3,1)}(s)$ this continuation.

Let $s \in \mathcal{R}(2/3)^+$. Applying Cauchy integral formula to (8.32) to see that the function $\int_{(0)} \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ is equal to

$$\int_C \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 - \operatorname{Res}_{\kappa_1=2-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s), \tag{10.60}$$

and $\operatorname{Res}_{\kappa_1=2-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(1+s,\tau)\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})}. \tag{10.61}$$

Then by (10.60), (10.61) and the fact that $\int_C \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$ is holomorphic in $\mathcal{R}(2/3)$, we obtain a meromorphic continuation of $J_{23}^{(2/3,1)}(s)$ to the region $\mathcal{R}(2/3)$, with a possible simple pole at $s=2/3$, if $\tau^3=1$. Denote by $J_{23}^{2/3}(s)$ the continuation. Let $s\in\mathcal{R}(2/3)^-$. Then by (10.58) and (10.60) one has

$$J_{23}^{2/3}(s)=\int_C \operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-R_1(s)+R_2(s),$$

where $R_1(s)=\operatorname{Res}_{\kappa_1=2-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$; $R_2(s)=\operatorname{Res}_{\kappa_1=3-3s}\operatorname{Res}_{\kappa_2=s-1}\operatorname{Res}_{\kappa_3=s-1}\mathcal{F}(\boldsymbol{\kappa},s)$.

Since the right hand side is meromorphic in the strip $S_{(0,2/3)}$, with a possible simple pole at $s=1/2$ if $\tau^2=1$. We thus obtain a meromorphic continuation of $J_{23}^{2/3}(s)$ to the region $0<\operatorname{Re}(s)<2/3$, with a possible simple pole at $s=1/2$ if $\tau^2=1$. Denote by $J_{23}^{(1/3,2/3)}(s)$ this continuation. Thus we obtain a meromorphic continuation of $J_{23}(s)$ to the area $S_{(1/3,\infty)}$:

$$\widetilde{J}_{23}(s)=\begin{cases} J_{23}(s), \ s\in S_{(1,+\infty)};\\ J_{23}^{1}(s), \ s\in\mathcal{R}(1);\\ J_{23}^{(2/3,1)}(s), \ s\in S_{(2/3,1)};\\ J_{23}^{2/3}(s), \ s\in\mathcal{R}(2/3);\\ J_{23}^{(1/3,2/3)}(s), \ s\in S_{(1/3,2/3)}.\end{cases} \tag{10.62}$$

From the above formulas one sees that $\widetilde{J}_{23}(s)$ has possible poles at $s=3/4$, $s=2/3$ and $s=1/2$; and these potential poles are all at most simple. Moreover, from the above explicit expressions of $\widetilde{J}_{23}(s)$, we see that $\widetilde{J}_{23}(s)\cdot\Lambda(s,\tau)^{-1}$ has at most a simple pole at $s=1/2$. We discuss the other two possible poles separately.

Case 1: If $L(3/4,\tau)=0$, then by functional equation we have that $\Lambda(1/4,\tau^{-1})=0$. Suppose that $\widetilde{J}_{23}(s)$ has a pole at $s=3/4$, then from the proceeding explicit expressions, we must have that $\tau^4=1$, and the singular part of $\widetilde{J}_{23}(s)$ around $s=3/4$ is a holomorphic function multiplying $\Lambda(4s-3,\tau^4)\Lambda(3s-2,\tau^3)$. Note that $\Lambda(3s-2,\tau^3)\mid_{s=3/4}=\Lambda(1/4,\tau^3)=\Lambda(1/4,\tau^{-1})=0$. Hence, when $L(3/4,\tau)=0$, $\widetilde{J}_{23}(s)$ is holomorphic at $s=3/4$.

Case 2: If $L(2/3,\tau)=0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1})=0$. Suppose that $\widetilde{J}_{23}(s)$ has a pole at $s=2/3$, then from the proceeding explicit expressions, we must have that $\tau^3=1$, and the singular part of $\widetilde{J}_{23}(s)$ around $s=2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\mid_{s=2/3}=\Lambda(1/3,\tau^2)=\Lambda(1/3,\tau^{-1})=0$. Hence, when $L(2/3,\tau)=0$, $\widetilde{J}_{23}(s)$ is holomorphic at $s=2/3$.

Now the proof of Claim 82 is complete. □

Proof of Claim 83. Let $s\in\mathcal{R}(1)^-$. Let $H_1^{(1/2,1)}(s):=\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3d\kappa_2$. Recall that we have computed the analytic property of $\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$:

$$\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(s+\kappa_{23},\chi_{24}\tau)}{\Lambda(1-\kappa_2,\chi_{32})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})}\times$$
$$\frac{\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(2s-1-\kappa_{23},\chi_{42}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{23},\chi_{42})\Lambda(2-s+\kappa_{23},\chi_{24}\tau^{-1})\Lambda(2-s,\tau^{-1})}.$$

Therefore, we see that $H_1^{(1/2,1)}(s)$ is holomorphic in the strip $1/2<\operatorname{Re}(s)<1$. Let $s\in\mathcal{R}(1/2)^+$. By Cauchy integral formula we have

$$H_1^{(1/2,1)}(s)=\int_{(0)}\int_{\mathcal{C}}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2d\kappa_3-\int_{(0)}\left[R_1(\kappa_3)+R_2(\kappa_3)\right]d\kappa_3,\qquad(10.63)$$

where $R_1(\kappa_3)=R_1(\kappa_3;s)=\operatorname{Res}_{\kappa_2=2s-1}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$, and

$$R_2(\kappa_3)=R_2(\kappa_3;s)=\operatorname{Res}_{\kappa_2=2s-1-\kappa_3}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s).$$

By functional equation of Hecke L-functions over F we see that $R_1(\kappa_3)$ is equal to some holomorphic function multiplying the product of $\Lambda(3s-1,\tau^3)\Lambda(s,\tau)^3\cdot\Lambda(2-s,\tau^{-1})^{-1}$ and

$$\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(3s-1+\kappa_3,\chi_{34}\tau^3)}{\Lambda(1-\kappa_3,\chi_{43})\Lambda(2-2s-\kappa_3,\chi_{43}\tau^{-2})\Lambda(1+s+\kappa_3,\chi_{34}\tau)\Lambda(1+s,\tau)}.\qquad(10.64)$$

Also, applying functional equation of Hecke L-functions to $\operatorname{Res}_{\kappa_2=2s-1-\kappa_3}\operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ leads to that $R_2(\kappa_3)$ is equal to some holomorphic function multiplying the product of $\Lambda(3s-1,\tau^3)\Lambda(s,\tau)^3\cdot\Lambda(2-s,\tau^{-1})^{-1}$ and

$$\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s-\kappa_3,\chi_{43}\tau)\Lambda(3s-1-\kappa_3,\chi_{43}\tau^3)}{\Lambda(1+\kappa_3,\chi_{34})\Lambda(2-2s+\kappa_3,\chi_{34}\tau^{-2})\Lambda(1+s-\kappa_3,\chi_{43}\tau)\Lambda(1+s,\tau)}.\qquad(10.65)$$

Due to the uniform zero-free region discussed in Section 8.1, one sees that both $\int_{(0)} R_1(\kappa_3)d\kappa_3$ and $\int_{(0)} R_2(\kappa_3)d\kappa_3$ converges normally in the region $\mathcal{R}(1/2)$. Hence they are holomorphic in this are. Also, note that $\int_{(0)}\int_C \operatorname{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2 d\kappa_3$ is meromorphic in the region $\mathcal{R}(1/2)$, with a possible simple pole at $s=1/2$ if $\tau^2=1$. Denote by $H_1^{1/2}(s)$ this continuation. It's clear that $H_1^{1/2}(s)$ admits a natural meromorphic continuation to the region $1/3<\mathrm{Re}(s)<1/2$. Denote by $H_1^{(1/3,1/2)}(s)$ this continuation. Then we obtain $\widetilde{H}_1(s)$, a meromorphic continuation of $H_1^{(1/2,1)}(s)$ to the domain $S_{(1/3,1)}$, by (10.63), (10.64) and (10.65). Explicitly, we have that

$$\widetilde{H}_1(s)=\begin{cases} H_1^{(1/2,1)}(s), \ s\in S_{(1/2,1)};\\ H_1^{1/2}(s), \ s\in\mathcal{R}(1/2);\\ H_1^{(1/3,1/2)}(s).\end{cases} \tag{10.66}$$

Moreover, $\widetilde{H}_1(s)$ has a possible simple pole at $s=1/2$ if $\tau^2=1$. Now the proof of Claim 83 is complete. □

Proof of Claim 84. Let $s\in\mathcal{R}(1)^-$. Let $H_2^{(1/2,1)}(s):=\int_{(0)}\int_{(0)}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_3 d\kappa_1$. Recall that we have computed the analytic property of $\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$:

$$\begin{aligned}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)\sim&\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(s+\kappa_{13},\chi_{14}\tau)\Lambda(s-\kappa_{13},\chi_{41}\tau)}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+\kappa_{13},\chi_{14})\Lambda(2-s+\kappa_1,\chi_{12}\tau^{-1})}\\ &\cdot\frac{\Lambda(2s-1-\kappa_3,\chi_{43}\tau^2)\Lambda(2s-1-\kappa_1,\chi_{21}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{13},\chi_{41})\Lambda(2-s+\kappa_3,\chi_{34}\tau^{-1})\Lambda(2-s,\tau^{-1})}.\end{aligned}$$

Therefore, we see that $H_2^{(1/2,1)}(s)$ is holomorphic in the strip $1/2<\mathrm{Re}(s)<1$. Let $s\in\mathcal{R}(1/2)^+$. By Cauchy integral formula we have

$$\begin{aligned}H_2^{(1/2,1)}(s)&=\int_{(0)}\int_C\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3-\int_{(0)}\left[R_1(\kappa_3)+R_2(\kappa_3)\right]d\kappa_3\\ &=\int_C\int_C\mathrm{Res}_2(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3-\int_{(0)}\left[R_1(\kappa_3)+R_2(\kappa_3)\right]d\kappa_3-\int_C R(\kappa_1)d\kappa_1,\end{aligned}$$

where $\mathrm{Res}_2(\boldsymbol{\kappa},s)=\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$; $R_1(\kappa_3)=R_1(\kappa_3;s)=\operatorname{Res}_{\kappa_1=2s-1}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$, $R_2(\kappa_3)=R_2(\kappa_3;s)=\operatorname{Res}_{\kappa_1=2s-1-\kappa_3}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$, and the meromorphic function $R(\kappa_1)=R(\kappa_1;s)=\operatorname{Res}_{\kappa_3=2s-1}\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$. By analytic properties of $\operatorname{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)$ and functional equation of Hecke L-functions over F we see that $R_1(\kappa_3)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3\cdot\Lambda(2-s,\tau^{-1})^{-1}\cdot\Lambda(2-2s,\tau^{-2})^{-1}$ and

$$\frac{\Lambda(2s+\kappa_3,\chi_{34}\tau^2)\Lambda(2s-1-\kappa_3,\chi_{43}\tau^2)\Lambda(1+\kappa_3,\chi_{34})\Lambda(3s-1,\tau^3)}{\Lambda(1-\kappa_3,\chi_{43})\Lambda(1+s+\kappa_3,\chi_{34}\tau)\Lambda(2-s+\kappa_3,\chi_{34}\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.67}$$

Also, applying functional equation of Hecke L-functions to $\underset{\kappa_1=2s-1-\kappa_3}{\operatorname{Res}}\ \underset{\kappa_2=1-s}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa}, s)$ leads to that $R_2(\kappa_3)$ is equal to some holomorphic function multiplying

$$\frac{\Lambda(s+\kappa_3,\chi_{34}\tau)\Lambda(3s-1-\kappa_3,\chi_{43}\tau^3)\Lambda(2s,\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(2-s+\kappa_3,\chi_{34}\tau^{-1})\Lambda(1+s-\kappa_3,\chi_{43}\tau)\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.68}$$

Again, by functional equation of Hecke L-functions we see that $R(\kappa_1)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2\cdot\Lambda(2-s,\tau^{-1})^{-1}\cdot\Lambda(1+s,\tau)^{-1}$ and the meromorphic function

$$\frac{\Lambda(1+\kappa_1,\chi_{12})\Lambda(2s-1-\kappa_1,\chi_{21}\tau^2)\Lambda(2s+\kappa_1,\chi_{12}\tau^2)\Lambda(3s-1,\tau^3)}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(2-s+\kappa_1,\chi_{12}\tau^{-1})\Lambda(1+s+\kappa_1,\chi_{12}\tau)\Lambda(2-2s,\tau^{-2})}. \tag{10.69}$$

Due to the uniform zero-free region discussed in Section 8.1, one sees from (10.68) and (10.69) that both $\Lambda(2s,\tau^2)^{-1}\cdot\Lambda(2s-1,\tau^2)^{-1}\cdot\int_{(0)}R(\kappa_1;s)d\kappa_3$ and $\Lambda(2s-1,\tau^2)^{-1}\cdot\int_C R(\kappa_1;s)d\kappa_1$ converge normally for any $s\in\mathcal{R}(1/2)$. Hence they are holomorphic in this area. Then we obtain a meromorphic continuation of $\int_{(0)}R(\kappa_1;s)d\kappa_3$ to $\mathcal{R}(1/2)$, with a possible pole of order at most 2 at $s=1/2$ if $\tau^2=1$; and a meromorphic continuation of $\int_C R(\kappa_1;s)d\kappa_1$ to $\mathcal{R}(1/2)$, with a possible simple pole at $s=1/2$ if $\tau^2=1$. Moreover, if $L(1/2,\tau)=0$, then both $\int_{(0)}R(\kappa_1;s)d\kappa_3$ and $\int_C R(\kappa_1;s)d\kappa_1$ are holomorphic at $s=1/2$.

By (10.67), one can apply Cauchy integral formula to deduce that

$$\begin{aligned} H_2^{(1/2,1)}(s) = &\int_C\int_C \operatorname{Res}_2(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3 - \int_C \left[R_1(\kappa_3)+R_2(\kappa_3)\right]d\kappa_3 - \int_C R(\kappa_1)d\kappa_1 \\ &+ \underset{\kappa_3=2s-1}{\operatorname{Res}}\ \underset{\kappa_1=2s-1}{\operatorname{Res}}\ \underset{\kappa_2=1-s}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa},s), \end{aligned}$$

where $\underset{\kappa_3=2s-1}{\operatorname{Res}}\ \underset{\kappa_1=2s-1}{\operatorname{Res}}\ \underset{\kappa_2=1-s}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa},s)$ is equal to, according to (10.67), some holomoprhic function multiplying the meromorphic function

$$\frac{\Lambda(4s-1,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(2s,\tau^2)\Lambda(s,\tau)^3}{\Lambda(2-2s,\tau^{-2})^2\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)^2\Lambda(3s,\tau^3)}. \tag{10.70}$$

Hence $\underset{\kappa_3=2s-1}{\operatorname{Res}}\ \underset{\kappa_1=2s-1}{\operatorname{Res}}\ \underset{\kappa_2=1-s}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa},s)$ admits a meromorphic continuation to $\mathcal{R}(1/2)$, with a possible pole of order at most 2 at $s=1/2$ if $\tau^2=1$. Moreover, if $L(1/2,\tau)=0$, then $\underset{\kappa_3=2s-1}{\operatorname{Res}}\ \underset{\kappa_1=2s-1}{\operatorname{Res}}\ \underset{\kappa_2=1-s}{\operatorname{Res}} \mathcal{F}(\boldsymbol{\kappa},s)$ is holomorphic at $s=1/2$.

Also, note that $\int_C\int_C \operatorname{Res}_2(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3$, $\int_C R_1(\kappa_3)d\kappa_3$ and $\int_C R_2(\kappa_3)d\kappa_3$ are meromorphic in $S_{(1/3,1/2)}\cup\mathcal{R}(1/2)$, with a possible pole of order at most 2 at $s=1/2$ if $\tau^2=1$. Moreover, $L(1/2,\tau)^{-1}\cdot\int_C\int_C \operatorname{Res}_2(\boldsymbol{\kappa},s)d\kappa_1 d\kappa_3$, $L(1/2,\tau)^{-1}\cdot\int_C R_1(\kappa_3)d\kappa_3$ and $L(1/2,\tau)^{-1}\cdot\int_C R_2(\kappa_3)d\kappa_3$ all have at most a simple pole at $s=1/2$. Denote by $H_2^{(1/3,1/2]}(s)$ this continuation of $H_2^{(1/2,1)}(s)$ to $\mathcal{R}(1/2)$.

Thus, we obtain $\widetilde{H}_2(s)$, a meromorphic continuation of $H_2^{(1/2,1)}(s)$ to the domain $S_{(1/3,\infty)}$, by (10.67), (10.68), (10.69) and (10.70). Explicitly, we have that

$$\widetilde{H}_2(s) = \begin{cases} H_2^{(1/2,1)}(s), \ s \in S_{(1/2,1)}; \\ H_2^{(1/3,1/2]}(s), \ s \in S_{(1/3,1/2)} \cup \mathcal{R}(1/2). \end{cases} \tag{10.71}$$

Moreover, $\widetilde{H}_2(s) \cdot \Lambda(1/2,\tau)^{-1}$ has a possible pole of order at most 1 at $s = 1/2$ if $\tau^2 = 1$. Moreover, if $L(1/2,\tau) = 0$, then $\widetilde{H}_2(s)$ is holomorphic at $s = 1/2$. Now the proof of Claim 84 is complete. □

Proof of Claim 85. Let $s \in \mathcal{R}(1)^-$. Let $H_3^{(1/2,1)}(s) := \int_{(0)}\int_{(0)} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 d\kappa_1$. Recall that we have computed the analytic property of $\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$:

$$\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(s+\kappa_{12},\chi_{13}\tau)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(1-\kappa_1,\chi_{21})\Lambda(1-\kappa_2,\chi_{32})\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})} \times \frac{\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(2s-1-\kappa_{12},\chi_{31}\tau^2)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(1-\kappa_{12},\chi_{31})\Lambda(2-s+\kappa_{12},\chi_{13}\tau^{-1})\Lambda(2-s,\tau^{-1})}.$$

Therefore, we see that $H_3^{(1/2,1)}(s)$ is holomorphic in the strip $1/2 < \operatorname{Re}(s) < 1$. Let $s \in \mathcal{R}(1/2)^+$. By Cauchy integral formula we have

$$H_3^{(1/2,1)}(s) = \int_{(0)}\int_{\mathcal{C}} \operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 d\kappa_1 - \int_{(0)} \left[R_1(\kappa_1) + R_2(\kappa_1)\right] d\kappa_1, \tag{10.72}$$

where $R_1(\kappa_1) = R_1(\kappa_1; s) = \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$, and

$$R_2(\kappa_1) = R_2(\kappa_1; s) = \operatorname*{Res}_{\kappa_2=2s-1-\kappa_1}\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s).$$

By functional equation of Hecke L-functions over F we see that $R_1(\kappa_1)$ is equal to some holomorphic function multiplying the product of $\Lambda(3s-1,\tau^3)\Lambda(s,\tau)^3 \cdot \Lambda(2-s,\tau^{-1})^{-1}$ and

$$\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-1+\kappa_1,\chi_{12}\tau^3)}{\Lambda(1-\kappa_1,\chi_{21})\Lambda(2-2s-\kappa_1,\chi_{21}\tau^{-2})\Lambda(1+s+\kappa_1,\chi_{12}\tau)\Lambda(1+s,\tau)}. \tag{10.73}$$

Also, applying functional equation of Hecke L-functions to $\operatorname*{Res}_{\kappa_2=2s-1-\kappa_1}\operatorname*{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ leads to that $R_2(\kappa_1)$ is equal to some holomorphic function multiplying the product of $\Lambda(3s-1,\tau^3)\Lambda(s,\tau)^3 \cdot \Lambda(2-s,\tau^{-1})^{-1}$ and

$$\frac{\Lambda(s+\kappa_1,\chi_{12}\tau)\Lambda(s-\kappa_1,\chi_{21}\tau)\Lambda(3s-1-\kappa_1,\chi_{21}\tau^3)}{\Lambda(1+\kappa_1,\chi_{12})\Lambda(2-2s+\kappa_1,\chi_{12}\tau^{-2})\Lambda(1+s-\kappa_1,\chi_{21}\tau)\Lambda(1+s,\tau)}. \tag{10.74}$$

Due to the uniform zero-free region discussed in Section 8.1, one sees that both $\int_{(0)} R_1(\kappa_1)d\kappa_1$ and $\int_{(0)} R_2(\kappa_1)d\kappa_1$ converges normally in the region $S_{(1/3,1/2)} \cup \mathcal{R}(1/2)$. Hence they are holomorphic in this are. Also, note that $\int_{(0)} \int_{C} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 d\kappa_1$ is meromorphic in the region $S_{(1/3,1/2)} \cup \mathcal{R}(1/2)$, with a possible simple pole at $s = 1/2$ if $\tau^2 = 1$. Denote by $H_3^{(1/3,1/2]}(s)$ this continuation. Then we obtain $\widetilde{H}_3(s)$, a meromorphic continuation of $H_3^{(1/2,1)}(s)$ to the domain $S_{(1/3,\infty)}$, by (10.72), (10.73) and (10.74). Explicitly, we have that

$$\widetilde{H}_3(s) = \begin{cases} H_3^{(1/2,1)}(s), \ s \in S_{(1/2,1)}; \\ H_3^{(1/3,1/2]}(s), \ s \in S_{(1/3,1/2)} \cup \mathcal{R}(1/2). \end{cases} \tag{10.75}$$

Moreover, $\widetilde{H}_3(s)$ has a possible simple pole at $s = 1/2$ if $\tau^2 = 1$. Now the proof of Claim 85 is complete. □

Proof of Claim 86. Let $s \in \mathcal{R}(1)^-$. Let $H_{12}^{(2/3,1)}(s) := \int_{(0)} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3$. Recall that by (8.35) one sees that $H_{12}^{(2/3,1)}(s)$ admits a natural holomorphic continuation to the strip $2/3 < \operatorname{Re}(s) < 1$. Now let $s \in \mathcal{R}(2/3)^+$. Then we have

$$H_{12}^{(2/3,1)}(s) = \int_{C} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3 - \operatorname{Res}_{\kappa_3=3s-2} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.76}$$

where $\operatorname{Res}_{\kappa_3=3s-2} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}.$$

Then by functional equation $\Lambda(3s-2,\tau^3) \sim \Lambda(3-3s,\tau^{-3})$, we have that

$$\operatorname{Res}_{\kappa_3=3s-2} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(4s-2,\tau^4)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.77}$$

Hence $\operatorname{Res}_{\kappa_3=3s-2} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ admits a meromorphic continuation to the region $S_{(1/3,1)}$, with a possible pole of order at most 2 at $1/2$ if $\tau^2 = 1$.

Moreover, due to (8.35), the function $\int_{C} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_3$ is meromorphic in the region $S_{(1/3,2/3)} \cup \mathcal{R}(2/3)$, with a possible simple pole at $s = 2/3$ if $\tau^3 = 1$; and a possible simple pole at $1/2$ if $\tau^2 = 1$. Thus we get a meromorphic continuation of $H_{12}^{(2/3,1)}(s)$ to the region $S_{(1/3,2/3)} \cup \mathcal{R}(2/3)$. Denote by $H_{12}^{(1/3,2/3]}(s)$ this continuation. Now we obtain from (10.76) and (10.77) a meromorphic continuation of $H_{12}^{(2/3,1)}(s)$

to the region $S_{(1/3,1)}$, namely,

$$\widetilde{H}_{12}(s) = \begin{cases} H_{12}^{(2/3,1)}(s), \ s \in S_{(2/3,1)}; \\ H_{12}^{(1/3,2/3]}(s), \ s \in S_{(1/3,2/3)} \cup \mathcal{R}(2/3). \end{cases} \tag{10.78}$$

From (8.35) and the above formulas one sees that $\widetilde{H}_{12}(s)$ has possible poles at $s = 2/3$ and $s = 1/2$; and these potential pole at $s = 2/3$ is at most simple, the possible pole at $s = 1/2$ has order at most 2. Moreover, from the above explicit expressions of $\widetilde{H}_{12}(s)$, we see that $\widetilde{H}_{12}(s) \cdot \Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2,\tau) = 0$. In additional, if $L(2/3,\tau) = 0$, then by functional equation we have that $\Lambda(1/3,\tau^{-1}) = 0$. Suppose that $\widetilde{H}_{12}(s)$ has a pole at $s = 2/3$. Then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{H}_{12}(s)$ around $s = 2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2) \mid_{s=2/3} = \Lambda(1/3,\tau^2) = \Lambda(1/3,\tau^{-1}) = 0$. Hence, when $L(2/3,\tau) = 0$, $\widetilde{H}_{12}(s)$ is holomorphic at $s = 2/3$. Now the proof of Claim 86 is complete. □

Proof of Claim 87. Let $s \in \mathcal{R}(1)^-$. Let $H_{12}^{(1/2,1)}(s) := \int_{(0)} \operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_2$. Let $\kappa_2' = 1 - s + \kappa_2$, $\kappa_1' = \kappa_1$ and $\kappa_3' = \kappa_3$. Denote by $\boldsymbol{\kappa}' = (\kappa_1', \kappa_2', \kappa_3')$. Recall that $\operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ and the function

$$\frac{\Lambda(1+\kappa_2,\chi_{13})\Lambda(s+\kappa_2,\chi_{23}\tau)\Lambda(2s-1-\kappa_2,\chi_{32}\tau^2)\Lambda(3s-2-\kappa_2,\chi_{32}\tau^3)}{\Lambda(1-\kappa_2,\chi_{32})\Lambda(s-\kappa_2,\chi_{32}\tau)\Lambda(2-s+\kappa_2,\chi_{23}\tau^{-1})\Lambda(3-2s+\kappa_2,\chi_{23}\tau^{-2})}.$$

Then after the above changing of variables, we have that $\operatorname{Res}_{\kappa_1=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) = \operatorname{Res}_{\kappa_1'=1-s} \operatorname{Res}_{\kappa_3'=1-s} \mathcal{F}(\boldsymbol{\kappa}', s)$ is equal to some holomorphic function multiplying the product of $\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(2-s,\tau^{-1})^{-2}$ and the function

$$\frac{\Lambda(s+\kappa_2',\chi_{23}\tau)\Lambda(s-\kappa_2',\chi_{32}\tau)\Lambda(2s-1+\kappa_2',\chi_{23}\tau^2)\Lambda(2s-1-\kappa_2',\chi_{32}\tau^2)}{\Lambda(1+\kappa_2',\chi_{23})\Lambda(1-\kappa_2',\chi_{32})\Lambda(2-s+\kappa_2',\chi_{23}\tau^{-1})\Lambda(2-s-\kappa_2',\chi_{32}\tau^{-2})}. \tag{10.79}$$

One then sees that $H_{13}^{(2/3,1)}(s)$ admits a natural holomorphic continuation to the strip $1/2 < \operatorname{Re}(s) < 1$. Now let $s \in \mathcal{R}(1/2)^+$. Then we have

$$H_{13}^{(1/2,1)}(s) = \int_{\mathcal{C}} \operatorname{Res}_{\kappa_1'=1-s} \operatorname{Res}_{\kappa_3'=1-s} \mathcal{F}(\boldsymbol{\kappa}', s) d\kappa_2' - \operatorname{Res}_{\kappa_2'=2s-1} \operatorname{Res}_{\kappa_1'=1-s} \operatorname{Res}_{\kappa_3'=1-s} \mathcal{F}(\boldsymbol{\kappa}', s), \tag{10.80}$$

where $\operatorname{Res}_{\kappa_2'=2s-1} \operatorname{Res}_{\kappa_1'=1-s} \operatorname{Res}_{\kappa_3'=1-s} \mathcal{F}(\boldsymbol{\kappa}', s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(1-s,\tau^{-1})}{\Lambda(3-3s,\tau^{-3})\Lambda(2-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})^2\Lambda(2s,\tau^2)\Lambda(1+s,\tau)}. \tag{10.81}$$

Denote by $R_{213}\mathcal{F}(\boldsymbol{\kappa}', s) = \underset{\kappa_2'=2s-1}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)$. Applying the functional equation $\Lambda(2s-1,\tau^2) \sim \Lambda(2-2s,\tau^{-2})$ and $\Lambda(1-s,\tau^{-1}) \sim \Lambda(s,\tau)$, one then has

$$R_{213}\mathcal{F}(\boldsymbol{\kappa}', s) \sim \frac{\Lambda(4s-2,\tau^4)\Lambda(3s-1,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^3}{\Lambda(3-3s,\tau^{-3})\Lambda(2-s,\tau^{-1})^2\Lambda(2s,\tau^2)\Lambda(1+s,\tau)}. \tag{10.82}$$

Note that for $s \in \mathcal{R}(1/2)$, $\Lambda(3-3s,\tau^{-3})^{-1}\cdot\Lambda(2s,\tau^2)^{-1}$ is holomophic, since $3-3s$ and $2s$ lie in a zero-free region (see Section 8.1). Hence $\underset{\kappa_2'=2s-1}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)$ admits a meromorphic continuation to the region $S_{(1/3,1/2)}\cup\mathcal{R}(1/2)$, with a possible pole of order at most 2 at $s=1/2$ if $\tau^2=1$. Moreover, if $L(1/2,\tau)=0$, then $\Lambda(s,\tau)^{-1}\cdot\underset{\kappa_2'=2s-1}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)$ is holomorphic at $s=1/2$.

On the other hand, the function $\int_{\mathcal{C}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2'$ is clearly meromorphic in $\mathcal{R}(1/2)$, with a possible pole of order at most 2 at $s=1/2$ if $\tau^2=1$. Moreover, if $L(1/2,\tau)=0$, then $\Lambda(s,\tau)^{-1}\cdot\int_{\mathcal{C}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2'$ is holomorphic at $s=1/2$. Then we obtain a meromorphic continuation of $H_{13}^{(1/2,1)}(s)$ to the region $\mathcal{R}(1/2)$. Denote by $H_{13}^{1/2}(s)$ this continuation.

Let $s\in\mathcal{R}(1/2)^-$. Then $\int_{\mathcal{C}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2'$ is equal to

$$\int_{(0)}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2' + \underset{\kappa_2'=1-2s}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s), \tag{10.83}$$

where $\underset{\kappa_2'=1-2s}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)$ is equal to a holomorphic function multiplying

$$\frac{\Lambda(2s-1,\tau^2)^2\Lambda(s,\tau)^2\Lambda(1-s,\tau^{-1})\Lambda(3s-1,\tau^3)\Lambda(4s-2,\tau^4)}{\Lambda(2-s,\tau^{-1})^2\Lambda(2-2s,\tau^{-2})\Lambda(2s,\tau^2)\Lambda(3-3s,\tau^{-3})\Lambda(s+1,\tau)}.$$

Now we obtain from (10.79), (10.80), (10.82) and (10.83) a meromorphic continuation of $H_{13}^{1/2}(s)$ to the region $S_{(1/3,1/2)}$. Denote by $H_{13}^{(1/3,1/2)}(s)$ this continuation, then $H_{13}^{(1/3,1/2)}(s)$ can be expressed as

$$\int_{(0)}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s)d\kappa_2' + \underset{\kappa_2'=1-2s}{\operatorname{Res}}\underset{\kappa_1'=1-s}{\operatorname{Res}}\underset{\kappa_3'=1-s}{\operatorname{Res}}\mathcal{F}(\boldsymbol{\kappa}', s) - R_{213}\mathcal{F}(\boldsymbol{\kappa}', s).$$

In all, we obtain a meromorphic continuation of $H_{13}^{(1/2,1)}(s)$ to the region $S_{(1/3,1)}$:

$$\widetilde{H}_{13}(s) = \begin{cases} H_{13}^{(1/2,1)}(s), \ s\in S_{(1/2,1)}; \\ H_{13}^{1/2}(s), \ s\in\mathcal{R}(1/2); \\ H_{13}^{(1/3,1/2)}(s), \ s\in S_{(1/3,1/2)}. \end{cases} \tag{10.84}$$

From the above discussions one sees that $\widetilde{H}_{13}(s)$ has a possible pole of order at most 2 at $s = 1/2$ if $\tau^2 = 1$. Moreover, if $L(1/2, \tau) = 0$, then $\Lambda(s,\tau)^{-1} \cdot \widetilde{H}_{13}(s)$ is holomorphic at $s = 1/2$. Now the proof of Claim 87 is complete. □

Proof of Claim 88. Let $s \in \mathcal{R}(1)^-$. Let $H_{23}^{(2/3,1)}(s) := \int_{(0)} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$. Recall that by (8.33) one sees that $H_{23}^{(2/3,1)}(s)$ admits a natural holomorphic continuation to the strip $2/3 < \operatorname{Re}(s) < 1$. Now let $s \in \mathcal{R}(2/3)^+$. Then we have

$$H_{23}^{(2/3,1)}(s) = \int_{C} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1 - \operatorname{Res}_{\kappa_1=3s-2} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s), \tag{10.85}$$

where $\operatorname{Res}_{\kappa_1=3s-2} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ equals some holomorphic function multiplying

$$\frac{\Lambda(4s-2,\tau^4)\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-3s,\tau^{-3})\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}.$$

Then by functional equation $\Lambda(3s-2,\tau^3) \sim \Lambda(3-3s,\tau^{-3})$, we have that

$$\operatorname{Res}_{\kappa_1=3s-2} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \sim \frac{\Lambda(4s-2,\tau^4)\Lambda(2s-1,\tau^2)\Lambda(s,\tau)^2}{\Lambda(3-2s,\tau^{-2})\Lambda(2-s,\tau^{-1})\Lambda(1+s,\tau)}. \tag{10.86}$$

Hence $\operatorname{Res}_{\kappa_1=3s-2} \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)$ admits a meromorphic continuation to the region $\mathcal{R}(1/2) \cup S_{(1/2,1)}$, with a possible pole of order at most 2 at $1/2$ if $\tau^2 = 1$.

Moreover, $\int_C \operatorname{Res}_{\kappa_2=1-s} \operatorname{Res}_{\kappa_3=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) d\kappa_1$ is meromorphic in the region $S_{(1/3,2/3)} \cup \mathcal{R}(2/3)$, with a possible simple pole at $s = 2/3$ if $\tau^3 = 1$; and a possible simple pole at $1/2$ if $\tau^2 = 1$. Thus we get a meromorphic continuation of $H_{23}^{(2/3,1)}(s)$ to the region $S_{[1/3,2/3)} \cup \mathcal{R}(2/3)$. Denote by $H_{23}^{(1/3,2/3]}(s)$ this continuation. Now we obtain from (10.85) and (10.86) a meromorphic continuation of $H_{23}^{(2/3,1)}(s)$ to the region $S_{(1/3,1)}$, namely,

$$\widetilde{H}_{23}(s) = \begin{cases} H_{23}^{(2/3,1)}(s), \ s \in S_{(2/3,1)}; \\ H_{23}^{(1/3,2/3]}(s), \ s \in S_{(1/3,2/3)} \cup \mathcal{R}(2/3). \end{cases} \tag{10.87}$$

From (8.33) and the above formulas one sees that $\widetilde{H}_{23}(s)$ has possible poles at $s = 2/3$ and $s = 1/2$; and these potential pole at $s = 2/3$ is at most simple, the possible pole at $s = 1/2$ has order at most 2. Moreover, from the above explicit expressions of $\widetilde{H}_{23}(s)$, we see that $\widetilde{H}_{23}(s) \cdot \Lambda(s,\tau)^{-1}$ has at most a simple pole at $s = 1/2$ if $L(1/2, \tau) = 0$. In additional, if $L(2/3, \tau) = 0$, then by functional equation we have that $\Lambda(1/3, \tau^{-1}) = 0$. Suppose that $\widetilde{H}_{23}(s)$ has a pole at $s = 2/3$. Then from the proceeding explicit expressions, we must have that $\tau^3 = 1$, and the singular part of $\widetilde{H}_{23}(s)$ around

$s=2/3$ is a holomorphic function multiplying $\Lambda(3s-2,\tau^3)\Lambda(2s-1,\tau^2)$. Note that $\Lambda(2s-1,\tau^2)\ |_{s=2/3}=\Lambda(1/3,\tau^2)=\Lambda(1/3,\tau^{-1})=0$. Hence, when $L(2/3,\tau)=0$, $\widetilde{H}_{23}(s)$ is holomorphic at $s=2/3$. Now the proof of Claim 88 is complete. □

Remark 90. *One can of course deal with each individual* $\iint\mathcal{F}(\boldsymbol{\kappa},s)$ *instead of the infinite sum* $\sum_\chi\sum_\phi\iint\mathcal{F}(\boldsymbol{\kappa},s)$. *However, without Proposition 72 or Proposition 73, the expression of each single* $\iint\mathcal{F}(\boldsymbol{\kappa},s)$ *would be super complicated. For example, one needs to consider residues with respect to* κ_{12}. *We give meromorphic continuation of* $\iint\mathcal{F}(\boldsymbol{\kappa},s)$ *as follows, which involves 56 terms in total for* GL(3) *case (also some of them are same but locate in different regions). Let* $J(s)=\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1d\kappa_2$. *When* $s\in\mathcal{R}(1)^+$, *we have, by Cauchy integral formula, that* $J(s)$ *is equal to*

$$\int_C\int_C\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1d\kappa_2-\int_C\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2-\int_C\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1$$
$$-\int_C\operatorname*{Res}_{\kappa_2=s-1-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1+\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)+\operatorname*{Res}_{\kappa_1=2s-2}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s).$$

Since the right hand side is meromorphic in $\mathcal{R}(1)$, *we get meromorphic continuation of* $J(s)$ *in* $\mathcal{R}(1)^-$. *Denote by* $J_1(s)$ *this continuation. Let* $s\in\mathcal{R}(1)^-$. *Then*

$$\begin{aligned}J_1(s)=&\int_{(0)}\int_{(0)}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1d\kappa_2+\int_{(0)}\operatorname*{Res}_{\kappa_1=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2+\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1\\&+\int_{(0)}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1-\int_{(0)}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_2-\\&\int_{(0)}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s)d\kappa_1+\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s}\mathcal{F}(\boldsymbol{\kappa},s)+\operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-\\&\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1}\mathcal{F}(\boldsymbol{\kappa},s)-\operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s)-\operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1}\mathcal{F}(\boldsymbol{\kappa},s)\\&+\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1}\mathcal{F}(\boldsymbol{\kappa},s),\end{aligned}$$

where the right hand side is meromorphic in $1/2<\mathrm{Re}(s)<1$. *Hence we obtain a meromorphic of* $J_1(s)$ *to the domain* $S_{(1/2,1)}$. *Denote by* $J_2(s)$ *this continuation. Let*

$s \in \mathcal{R}(1/2)^+$. Then we have, again, by Cauchy integral formula, that

$$\begin{aligned}J_2(s) = &\int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_2 + \int_{C} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 + \int_{C} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 \\ &+ \int_{C} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 - \int_{C} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 - \int_{C} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 - \\ &\int_{C} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 + \operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \\ &\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=s-1-\kappa-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s),\end{aligned}$$

where the right hand side is meromorphic in $\mathcal{R}(1/2)$. Hence we obtain a meromorphic continuation of $J_2(s)$ in $s \in \mathcal{R}(1/2)$. Let $s \in \mathcal{R}(1/2)^-$. Then we have, again, by Cauchy integral formula, that

$$\begin{aligned}J_2(s) = &\int_{(0)}\int_{(0)} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 d\kappa_2 + \int_{C} \operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 + \int_{C} \operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 \\ &+ \int_{C} \operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 - \int_{(0)} \operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_2 - \\ &\int_{(0)} \operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s)d\kappa_1 + \operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=s-1}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \\ &\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=1-s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2s-1}\operatorname*{Res}_{\kappa_1=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=2-2s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_2=2-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s} \mathcal{F}(\boldsymbol{\kappa}, s) \\ &+ \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=s-1-\kappa-1} \mathcal{F}(\boldsymbol{\kappa}, s) - \operatorname*{Res}_{\kappa_1=2s-1}\operatorname*{Res}_{\kappa_2=1-s-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s) + \\ &\operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_2=1-2s}\operatorname*{Res}_{\kappa_1=s-1} \mathcal{F}(\boldsymbol{\kappa}, s) + \operatorname*{Res}_{\kappa_1=1-2s}\operatorname*{Res}_{\kappa_2=s-1-\kappa_1} \mathcal{F}(\boldsymbol{\kappa}, s),\end{aligned}$$

where the right hand side is meromorphic in $1/3 < \mathrm{Re}(s) < 1/2$. Hence we obtain a meromorphic continuation of $J_2(s)$ in $s \in \mathcal{S}(1/3, 1/2)$. Therefore, putting the above computation together, we get a meromorphic continuation of $J(s)$ to the domain $s \in S_{(1/3,1)}$.

Then one needs to investigate these terms individually. What is worse, the situation would be much more complicated in GL(4) *case.*

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