Effective Field Theory Topics in the Modern S-Matrix Program

Thesis by

James Mangan

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James Mangan
ORCID: 0000-0002-9713-7446

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ABSTRACT

Quantum field theory is the most predictive theory of nature ever tested, yet the scattering amplitudes produced from the standard application of Lagrangians and Feynman rules belie the simplicity of the underlying physics, obscuring the physical answers behind off-shell actions and gauge redundant descriptions. The aim of the modern S-matrix program (or the “amplitudes” subfield) is to reformulate specific field theories and manifest underlying structures in order to make high multiplicity and/or high loop scattering calculations tractable.

Many of the systems amenable to amplitudes techniques are actually intimately related to each other through the double-copy relations. We argue that conformal invariance is common thread linking several of the scalar effective field theories appearing in the double copy. For a derivatively coupled scalar with a quartic $\mathcal{O}(p^4)$ vertex, classical conformal invariance dictates an infinite tower of additional interactions that coincide exactly with Dirac-Born-Infeld theory analytically continued to spacetime dimension $D = 0$. For the case of a quartic $\mathcal{O}(p^6)$ vertex, classical conformal invariance constrains the theory to be the special Galileon in $D = -2$ dimensions. We also verify the conformal invariance of these theories by showing that their amplitudes are uniquely fixed by the conformal Ward identities. In these theories, conformal invariance is a much more stringent constraint than scale invariance.

Although many of the theories in the double-copy admit a high degree of space-time symmetry, amplitudes tools can be applied to non-relativistic theories as well. We explore the scattering amplitudes of fluid quanta described by the Navier-Stokes equation and its non-Abelian generalization. These amplitudes exhibit universal infrared structures analogous to the Weinberg soft theorem and the Adler zero. Furthermore, they satisfy on-shell recursion relations which together with the three-point scattering amplitude furnish a pure S-matrix formulation of incompressible fluid mechanics. Remarkably, the amplitudes of the non-Abelian Navier-Stokes equation also exhibit color-kinematics duality as an off-shell symmetry, for which the associated kinematic algebra is literally the algebra of spatial diffeomorphisms. Applying the double copy prescription, we then arrive at a new theory of a tensor bi-fluid. Finally, we present monopole solutions of the non-Abelian and tensor Navier-Stokes equations and observe a classical double copy structure.

PUBLISHED CONTENT AND CONTRIBUTIONS

Chapters 2 and 3 of this thesis are based upon the publications listed below. Along with my collaborators, I was a primary author of these papers in that I helped calculate the results and prepare the manuscripts.

C. Cheung and J. Mangan, "Scattering Amplitudes and the Navier-Stokes Equation," arXiv:2010.15970 [hep-th].

C. Cheung, J. Mangan, and C.-H. Shen, "Hidden Conformal Invariance of Scalar Effective Field Theories," *Phs. Rev. D* **102** no. 12, (2020) 125009, arXiv:2005.13027 [hep-th].

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Chapter 1

INTRODUCTION

Quantum field theory (QFT) is the most accurately tested theory of the world, but this accuracy comes at the cost of incredibly complex calculations. The complications are compounded by our usual approach to QFT, in which we start from a Lagrangian, generate Feynman rules, and calculate observables from those rules. By the time we get to the on-shell scattering amplitudes at the end, all of the off-shell, gauge dependent, and field basis dependent redundancies of the Lagrangian have evaporated. This plethora of redundancies has spurred on the development of the modern S-matrix program, which aims to calculate observables in a more direct way by unearthing structures invisible at the action level or by reformulating theories in more on-shell friendly frameworks.

One of the initial successes of the modern S-matrix program was the discovery of the Parke-Taylor formula in the 1980s [1]. The five gluon amplitude generated by Feynman diagrams involves some 10,000 terms but, when taken on-shell and expressed in the correct variables, all of these terms combine to leave a *single* term. This pattern extends to any number of particles; a certain sector of the gluon amplitude always simplifies to just a single term.

Since the '80s there have been many exciting developments in the field. A pedagogical review of many of these discoveries can be found in Refs. [2–8] but we will briefly outline some of them here. High-order loop calculations in Yang-Mills (YM) gauge theory and gravity (GR) were made possible through the generalized unitarity method, which is essentially the optical theorem relating tree and 1-loop amplitudes on steroids (see Ref. [8] and references therein). Although gauge theory is usually characterized by an off-shell action, purely on-shell formulations were found at tree level by making use of on-shell recursion relations [9]. These recursion relations work by complexifying the kinematics while maintaining the on-shell conditions. The amplitude of interest is then expressed via the residue theorem in terms of lower point on-shell amplitudes. Although on-shell recursion relations were first found for gauge theory in spinor helicity, they have been adapted to general kinematics and to other theories including gravity and several scalar effective field theories (EFT's) [10–12].

Via the KLT and BCJ double-copy relations, the problem of graviton scattering was reduced to the far simpler problem of scattering gluons [13–15]. The (field theortic) KLT relations write a gravity tree amplitude in terms of sums of two YM amplitudes along with some inverse propagator factors. The BCJ double-copy is, at its core, a generalization of the KLT relations that extends to the loop (integrand) level. Gauge theory amplitudes exhibit a color-kinematics duality where the kinematic numerators of the diagrams can be shuffled around to obey Jacobi identities in exact parallel with the color factors of the diagrams. Gravity is then obtained from gauge theory by keeping all of the same diagrams as gauge theory but "squaring" (or "double-copying") the kinematic numerators and dropping the color factors. Much like the recursion relations mentioned above, KLT and BCJ were first discovered for gauge theory and gravity and then extended to other theories. While the double copy constructs more complex theories (GR) from simpler ones (YM), the inverse process is possible in many cases as well. The simpler theory in the double-copy can often be obtained from the more complex one by dimensional reduction or "transmutation" [16].

Several novel reformulations of certain QFT's have emerged over the last few decades of developing the modern S-matrix program. Amplitudes from gauge theory and ϕ^3 theory (with a bi-adjoing color structure) can be expressed as volumes over kinematic polytopes known as the amplituhedron and associahedron [17–19]. The amplitudes of gauge theory can also be understood in the CHY formalism as integrals of two "determinants" over moduli space $A_n \sim \int d\mu\, \mathcal{I}_1 \mathcal{I}_2$ [20–23]. By swapping out these determinant factors for slightly different ones, gravity can be written in exactly the same way. This is simply a reflection of the fact that the BCJ product acts in an extremely natural and simple way in this space of determinant factors.

While all of these developments have either enabled higher multiplicity or higher loop calculations or elucidated some structure underlying QFT, almost all of these tools come with tradeoffs. Spinor helicity, which underpins the Parke-Taylor formula, clarifies gauge invariance and the physical states at the cost of making momentum conservation a non-linear constraint. On-shell recursion relations alleviate the off-shell redundancies of Feynman diagrams but only by introducing spurious poles that cancel in the final answers. The KLT relations dramatically simplify gravity amplitudes but they obscure permutation invariance and introduce non-trivial cancellations between propagators.

BCJ and KLT can be seen as connecting all of the major theories in the modern

S-matrix program. Although BCJ and KLT (and on-shell recursion relations) were first developed for gauge theory, they were subsequently adapted to several effective field theories (EFT's) including the non-linear sigma model (NLSM), Born-Infeld (BI), Dirac-Born-Infeld (DBI), and the special Galileon (sGal) [11]. All of these theories form an intricate "web" under the double-copy and dimensional reduction [16]. Much of this thesis is devoted to understanding and expanding this web.

With the benefit of hindsight we can organize the modern S-matrix program in terms of a few overarching questions that we have loosely grouped into three categories. One, what principles fix a QFT? These principles are more or less the "pillars" of amplitudes constructions, showing up time and again. These principles include locality (simple, non-overlapping poles), factorization on poles into lower-point amplitudes, and gauge invariance. In scalar theories, the Ward identity is often replaced by a soft theorem. Other guiding principles include color structure and dimensional reduction.

Two, how can we describe a QFT? The most widely applicable approach is, of course, by writing a Lagrangian for the QFT. For more specialized theories we can characterize them by recursion relations, through the CHY formalism, by "squaring" a simpler theory, or by dimensionally reducing from a more complex theory.

Third, what are the "nice" theories that the amplitudes techniques will work for? These are typically massless, bosonic theories with a single coupling constant. If they involve fermions, then it is usually through supersymmetry. Almost all of the theories are either the input to or output from a BCJ product. These theories include those mentioned above and their supersymmetric generalizations including YM, GR, BI, DBI, the bi-adjoing sclar (BS) theory, NLSM, and sGal. We will describe each of these theories in more detail when they come up, so we will spend the rest of this introduction giving a preview of the results presented in this thesis.

As with any successful toolset, the goal is always to broaden its scope as much as possible. What theories can you apply the modern S-matrix approach to? Although amplitudes gives alternative formulations of QFT's without the use of Lagrangians, almost all of the underlying theories have actions nonetheless. Furthermore, all of the theories are Lorentz invariant. Can we apply amplitudes ideas to theories that lack these properties? This is the subject of Chapter 2, based on [24], where we show that almost all of the on-shell technology mentioned above carries over to a non-Abelian generalization of Navier-Stokes. While a colored fluid shares many structural similarities to gauge theory, a fluid is dissipative so it is impossible to generate an action containing *only* the fluid degrees of freedom. Armed only with the non-relativistic equation of motion for the fluid, we calculate scattering amplitudes, develop on-shell recursion relations which characterize the theory purely in terms of on-shell data, explore a spinor-helicity formalism, and, most amazingly, demonstrate that this fluid is BCJ complaint *off-shell*.

The theories in amplitudes are tied together by the double copy, but it remains unclear what physical principles unite these strange bedfellows. What properties do these theories have in common? Almost all of the theories either have a gauge symmetry or a soft theorem that can be used to reconstruct the theory in question. But there is another thread linking many of the theories, namely, conformal invariance. Since gauge theory and gravity (in any dimension [25]) are known to be classically conformal, the interesting question is if the EFT's in the double copy are conformal. This is the topic of Chapter 3, based on [26]. Each of the theories in the double copy has a single coupling constant so classical (tree-level) scale invariance is trivially ensured in the critical dimension. However in this context scale invariance does not guarantee full conformal invariance so conformal invariance is key in determining the full tower of EFT interactions. We establish the conformal invariance of these theories from both a Lagrangian and an amplitudes perspective.

Because the modern S-matrix program deals with high multiplicity and/or high loop amplitudes, the expressions quickly become too cumbersome to deal with by pen and paper alone. We discuss some of the computer techniques essential for amplitudes in Appendix A, including the role of sparse matrix solvers over finite fields $\mathbb{Z}_p$. These row reduction algorithms show up when solving large ansatze or when doing the integration by parts reduction of generalized unitarity.

In Appendix B we present a "color bootstrap" for amplitudes, independently developed in [27]. Since all of the theories in the web can be uniquely characterized by simple principles like gauge invariance, soft theorems, locality, factorization, and so on, it is natural to ask if any of the other properties of these theories can be used as defining principles. All of the theories in the web are built on the double-copy, which is intimately connected to the color structure of the theories, so it seems reasonable to think that the color structure of theories might be a defining attribute. For NLSM this turns out to be true in that color, along with a few other principles like locality and factorization, is enough to completely define the theory at tree level. This appendix also serves as a practical application of the computational tools discussed in Appendix A.

Finally, we discuss an on-shell recursion relation for NLSM in Appendix C. The known recursion relations for NLSM are either computationally somewhat cumbersome (involving square roots that appear in intermediary expressions but cancel in the final results) or they rely on embedding NLSM in a larger more complex theory [11, 28]. In the appendix, we take a somewhat different approach. Rather than complexifying the kinematics over a single complex variable, we introduce multiple complex variables [29]. The kinematics are engineered so as to avoid square roots while remaining in a theory of pure pions.

Chapter 2

NAVIER-STOKES

2.1 Introduction.

The Navier-Stokes equation (NSE) is remarkably simple and follows trivially from the laws of classical mechanics. Still, its unassuming form and humble origins belie a daunting complexity: the problem of turbulence, which has confounded physicists for generations. The root of this difficulty is that the turbulent regime is essentially a strong coupling limit of the theory.

Of course, non-perturbative dynamics are not intractable per se. But in prominent examples such as quantum chromodynamics, progress has hinged crucially on an action formulation. Because the NSE is dissipative it does not follow trivially from a least action principle, so work in this area has focused on perfect fluids [30] and approaches utilizing auxiliary degrees of freedom [31–34].

Notably, the very premise of the modern S-matrix program (see [2, 3, 7] for reviews) is to bootstrap scattering dynamics from first principles *without* the aid of an action. These efforts have centered primarily on gauge theory and gravity, which are stringently constrained by fundamental properties like Poincare invariance, unitarity, and locality. These theories are "on-shell constructible" since their S-matrices are fully dictated at tree level by on-shell recursion [9, 35] and at loop level by generalized unitarity [8]. Remarkably, the modern S-matrix approach has also uncovered genuinely new structures within quantum field theory such as color-kinematics duality [14, 15], the scattering equations [20–23], and reformulations of amplitudes as volumes of abstract polytopes [17–19].

The NSE does not originate from an action but it nevertheless encodes an S-matrix characterizing the scattering of fluid quanta. In particular, by solving the NSE in the presence of an arbitrary source one obtains the generating functional for all tree-level scattering amplitudes [36]. The turbulent regime then corresponds to the S-matrix at strong coupling, which here is unrelated to a breakdown of the $\hbar$ expansion because the NSE is intrinsically classical and hence devoid of any a priori notion of loops.[1] Instead, turbulence is encoded in tree-level scattering processes

[1] A notion of loops emerges if we introduce stochastic correlations between sources but we will not consider this here.

at arbitrarily high multiplicity, where traditional perturbative methods are rather limited. Nevertheless, there are reasons for optimism in light of the modern S-matrix program, whose tools have uncovered analytic formulae for precisely this kind of arbitrary-multiplicity process involving maximally helicity violating gluons [1] and gravitons [37].

In this paper we initiate a study of the perturbative scattering amplitudes of the NSE and its natural non-Abelian generalization, which we dub the non-Abelian Navier-Stokes equation (NNSE). To begin, we recapitulate the explicit connection between equations of motion and S-matrices [36], drawing on the close analogy between the incompressibility of a fluid and the transverse conditions of a gauge theory. We present the Feynman rules for these theories and compute their three- and four-point scattering amplitudes. Next, we examine the infrared properties of these theories, proving that they exhibit a leading soft theorem essentially identical to that of gauge theory [38] as well as a soft Adler zero [39] reminiscent of the non-linear sigma model. Exploiting these properties, we then derive on-shell recursion relations that express all higher-point amplitudes as sums of products of three-point amplitudes, thus establishing that the NSE and the NNSE are on-shell constructible.

Remarkably, we discover that the *off-shell* Feynman diagrams of the NNSE automatically satisfy the kinematic Jacobi identities required for color-kinematics duality [14, 15]. This implies the existence of an off-shell color-kinematic symmetry and a corresponding conservation law, which we derive explicitly. Applying the double copy prescription, we then square the NNSE to obtain a tensor Navier-Stokes equation (TNSE) describing the dynamics of a bi-fluid degree of freedom. Last but not least, we derive monopole solutions to the NNSE and the TNSE and discuss the classical double copy.

2.2 Setup.

To begin, let us consider an incompressible fluid described by a velocity field u_i.[2] Incompressibility implies that velocity field is solenoidal, so $\partial_i u_i = 0$. The dynamics of the fluid are governed by the NSE,

$$\left(\partial_0 - \nu\partial^2\right) u_i + u_j \partial_j u_i + \partial_i \left(\frac{p}{\rho}\right) = J_i\,, \tag{2.1}$$

[2] Late lower-case Latin indices $i, j, k, \ldots$ run over spatial dimensions, early lower-case Latin indices $a, b, c, \ldots$ run over colors, and upper-case Latin indices $A, B, C, \ldots$ run over external legs. Dot products are denoted by $v_i w_i = vw$ and $v_i v_i = v^2$.

where ρ is the constant energy density, p is the pressure, ν is the viscosity, and J_i is a source term which we also assume to be solenoidal. Taking the divergence of Eq. (2.1), we obtain $\partial^2(p/\rho) = -\partial_i u_j \partial_j u_i$, from which we then solve for p/ρ and insert back into Eq. (2.1) to obtain

$$\left(\partial_0 - \nu\partial^2\right) u_i + \left(\delta_{ij} - \frac{\partial_i \partial_j}{\partial^2}\right) u_k \partial_k u_j = J_i \,. \tag{2.2}$$

Hence, the pressure has the sole purpose of projecting out all but the solenoidal modes.

We can generalize this setup to an incompressible *non-Abelian* fluid described by a velocity field u_i^a satisfying the solenoidal condition $\partial_i u_i^a = 0$ and the NNSE,[3]

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + f^{abc} u_j^b \partial_j u_i^c + \partial_i \left(\frac{p^a}{\rho}\right) = J_i^a \,. \tag{2.3}$$

Here f^{abc} is a fully antisymmetric structure constant and we have introduced non-Abelian versions of the pressure p^a and the solenoidal source term J_i^a. The divergence of Eq. (2.3), $\partial^2(p^a/\rho) = -f^{abc}\partial_i u_j^b \partial_j u_i^c = 0$ is identically zero due to antisymmetry of the structure constants. We thus drop the pressure altogether to obtain

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + f^{abc} u_j^b \partial_j u_i^c = J_i^a \,, \tag{2.4}$$

The absence of a projector in Eq. (2.4) as compared to Eq. (2.2) results in substantial simplifications.

The NSE is simply conservation of energy-momentum, $\partial_0 T_{0j} = \partial_i T_{ij}$, in the Newtonian limit where $T_{0i} = -\rho u_i$ and $T_{ij} = \rho u_i u_j + p\delta_{ij} - \rho\nu\partial_{(i}u_{j)}$. Analogously, the NNSE can be recast as conservation of a peculiar non-Abelian tensor, $\partial_0 T_{0j}^a = \partial_i T_{ij}^a$, where $T_{0i}^a = -\rho u_i^a$ and $T_{ij}^a = \rho f^{abc} u_i^b u_j^c + p^a \delta_{ij} - \rho\nu\partial_{(i}u_{j)}^a$.

2.3 Amplitudes.

As is well-known, the tree-level S-matrix can be computed by solving the classical equations of motion for a field in the presence of arbitrary sources. The field itself is the generating functional of all tree-level scattering amplitudes. Hence, the Berends-Giele recursion relations for gauge theory [41] and gravity [42] are literally the classical equations of motion. Applying identical logic to the NSE, one obtains the Wyld formulation of fluid dynamics [36], which we summarize below.

[3] The quark-gluon plasma is also described by a colored fluid [40], though crucially with equations of motion different from ours.

To begin, we define the notion of an "asymptotic" quantum of fluid. Inserting a plane wave ansatz $u_i \sim e_i e^{-i\omega t} e^{ipx}$ into the linearized NSE and the solenoidal condition, we obtain the on-shell conditions, $i\omega - \nu p^2 = 0$ and $pe = 0$, which exactly mirror those of gauge theory. The solenoidal condition eliminates the longitudinal mode, leaving $-$ and $+$ helicity modes corresponding to left and right circularly polarized fluid quanta. Since the on-shell energy is imaginary, the on-shell solution, $u_i \sim e_i e^{-\nu p^2 t} e^{ipx}$, is a diffusing wavepacket, as expected for a fluid velocity field undergoing viscous dissipation.

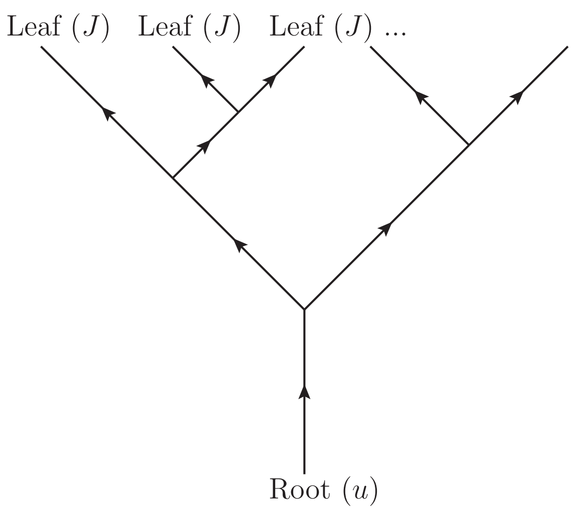


Figure 2.1: The perturbative solution Diagrammatic representation of the perturbative solution for u.

Next, we solve the NSE perturbatively in the source to obtain the one-point function of the velocity field $u_i(t, x, J)$ as a function of spacetime and a functional of on-shell sources J. In other words, we solve the NSE for u in terms of J and iteratively build higher and higher order solutions. Schematically this looks like

$$u \sim \frac{1}{\Box}(J + u\partial u) \tag{2.5}$$

$$\sim \frac{1}{\Box}J + \frac{1}{\Box}(J + u\partial u)\partial\frac{1}{\Box}(J + u\partial u) \tag{2.6}$$

$$\sim \frac{1}{\Box}J + \frac{1}{\Box^2}J\partial J + \ldots \tag{2.7}$$

More precisely, Fourier transforming to energy and momentum space yields $u_i(\omega, p, J)$, whose functional derivative

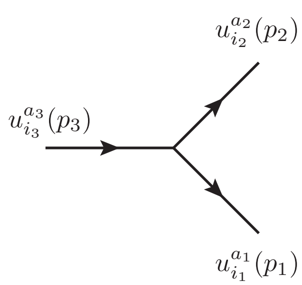
$$G_{n+1} = \left[\prod_{A=1}^{n} e_{Ai_A} \frac{\delta}{\delta J_{i_A}(\omega_A, p_A)}\right] u_i(\omega, p, J)\Big|_{J=0} \tag{2.8}$$

is the correlation function for n fluid quanta which are emitted by J and subsequently absorbed by the one-point function, $u_i(\omega, p, J)$. Here (ω_A, p_A) are the energy and momentum flowing from each "leaf" leg originating from an emission and $(\omega, p) = (\sum_{A=1}^n \omega_A, \sum_{A=1}^n p_A)$ are the total energy and momentum flowing into the "root" leg upon absorption. This recursive construction is associated with the trivalent graph in Fig. 2.1 where the root u branches into two more u's (as indicated by the "interaction" term in the EOM $u\partial u$) and the branching continues until the lines terminate on on-shell leaf leg sources J. The root and leaf monikers were so chosen because of the resemblance of the graph to an actual tree in nature. The arrows in the diagram are needed to keep track of the direction of energy flow. The scattering amplitude A_{n+1} is then obtained from G_{n+1} by amputating the external legs and stripping off the delta functions for energy and momentum conservation. For the remainder of this paper we assume that the leaf legs are on-shell but the root leg is not, so A_{n+1} is in actuality a *semi-on-shell* amplitude.

The Feynman rules for the NSE can be found in [36] so we do not present them again here. Instead we focus on the NNSE. The propagator in this theory is

$$= \frac{\delta^{a_1 a_2}\delta_{i_1 i_2}}{i\omega_1 - \nu p_1^2} \,. \tag{2.9}$$

where the energy flow direction is important for the sign of ω_1. The only interaction is the three-point vertex,



$$\begin{aligned} &= f^{a_1 a_2 a_3}\left(p_{1i_2}\delta_{i_1 i_3} - p_{2i_1}\delta_{i_2 i_3}\right) \\ &\sim f^{a_1 a_2 a_3}(p_{3i_1}\delta_{i_2 i_3} - p_{3i_2}\delta_{i_1 i_3}) \end{aligned} \,, \tag{2.10}$$

where in the second line we have used momentum conservation together with the fact that all terms proportional to p_{1i_1} or p_{2i_2} vanish when dotted into sources or interaction vertices due to the solenoidal condition. Note that the kinematic factors in Eq. (2.10) are not fully antisymmetric since the root leg and the leaf legs are distinguishable. The Feynman rules for NSE are identical except with color structures dropped and plus signs in Eq. (2.10).

Remarkably, the above Feynman rules imply that all amplitudes are manifestly *energy independent* in the sense they they depend only on dot products of momenta and polarizations. This is property is obvious for the three-point interaction vertex but slightly more subtle for the propagators. Regarding the latter, consider that the energy and momentum of each leaf leg is (ω_A, p_A), so the energy and momentum flowing through any intermediate leg is $(\sum_A \omega_A, \sum_A p_A)$, where the sum runs over a subset of the legs. The corresponding propagator is then

$$\frac{\delta^{a_1 a_2}\delta_{i_1 i_2}}{i\sum\limits_A \omega_A - \nu\big(\sum\limits_A p_A\big)^2} = -\frac{1}{\nu}\frac{\delta^{a_1 a_2}\delta_{i_1 i_2}}{\sum\limits_{A\neq B} p_A p_B}, \tag{2.11}$$

which is independent of energy. Here we have made use of the on-shell conditions for the leaf legs.

From Eq. (2.11) we realize that each propagator appears with the effective coupling constant $1/\nu$, in perfect analogy with graviton perturbation theory, where each propagator appears with the gravitational constant G. Hence, the turbulent regime of high Reynolds number, i.e. low viscosity, corresponds to strong coupling.

Let us consider a few examples. The three-point scattering amplitude of the NNSE is

$$A(123) = f^{a_1 a_2 a_3} \times \Big[(p_1 e_2)(e_1 e_3) - \{1 \leftrightarrow 2\}\Big], \tag{2.12}$$

while the four-point scattering amplitude is

$$\begin{aligned} A(1234) = & f^{a_1 a_2 b} f^{b a_3 a_4} \times \frac{1}{p_1 p_2}\Big[(p_1 e_2)(p_3 e_1)(e_3 e_4) + (p_1 e_2)(p_4 e_3)(e_1 e_4) - \{1 \leftrightarrow 2\}\Big] \\ & + t\text{-channel} + u\text{-channel}, \end{aligned} \tag{2.13}$$

dropping all coupling constant prefactors ν throughout. As advertised there is no explicit energy dependence.

2.4 Relativistic Spinor Helicity.

One of the great triumphs of the modern S-matrix program is the Parke-Taylor formula which is a closed formula for specific tree-level gluon amplitudes *at any multiplicity* in 4D [1]. Spinor helicity is at the heart of the Parke-Taylor formula, in part because it makes the physical states manifest. Spinor helicity is more than a trivial change of kinematic variables because it exposes certain structures that are invisible in terms of normal 4-momenta. Since YM simplifies so dramatically in this formalism and YM and NSE share deep structural similarities, it is natural to explore NSE in these variables. However, before describing the non-relativistic setup for NSE we will need to review the salient features of the relativistic formalism. For a pedagogical treatment of 4D spinor helicity see [2, 3]. For an extension to 6D see [43].

The main idea in 4D spinor helicity is to convert all kinematic variables like p^μ into 2D spinors using the Pauli matrices. We will trade momentum vectors for bi-spinors using

$$p_{a\dot{b}} = p_\mu(\sigma^\mu)_{a\dot{b}} = \begin{pmatrix} -p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & -p^0 - p^3 \end{pmatrix}. \tag{2.14}$$

When regarded as a matrix, the determinant of $p_{a\dot{b}}$ is just

$$\det p = -p_\mu p_\mu = 0 \tag{2.15}$$

for massless particles like the gluons we will be focusing on. Since this determinant vanishes, we can write the $p_{a\dot{b}}$ as the outer product of two vectors

$$p_{a\dot{b}} = -|p]_a\langle p|_{\dot{b}} \tag{2.16}$$

where the angle and square spinors ($\langle p|_{\dot{b}}$ and $|p]_a$) are just some 2D spinors that can be manipulated in the standard way with the Levi-Civita tensors ϵ^{ab} and $\epsilon^{\dot{a}\dot{b}}$. With a slight abuse of notation we can write Eq. (2.16) in terms of γ matrices as $-\not{p} = |p\rangle[p| + |p]\langle p|$. When two spinors of the same type are contracted together we drop the ϵ's so that, for example, the Mandelstams look like

$$\langle pq\rangle[pq] = 2p_\mu q_\mu = (p+q)^2 \tag{2.17}$$

after a little algebra. Note that it is impossible to contract an angle with a square bracket because there is no $\epsilon^{a\dot{b}}$ tensor. This will change for the non-relativistic case.

Now that we know how to translate dot products into spinor helicity, we are one step closer to working with on-shell amplitudes in this formalism. We will also need to impose momentum conservation which, in this language, looks like

$$\sum_{i=1}^{n} |i\rangle[i| = 0\,. \tag{2.18}$$

In order to simplify expressions we will also need the Schouten identity. A generic set of three angle spinors will be linearly dependent on each other because they live in just two dimensions. Working out the linear relation gives the identity

$$|i\rangle\langle jk\rangle + |j\rangle\langle ki\rangle + |k\rangle\langle ij\rangle = 0 \tag{2.19}$$

where a similar identity holds for square spinors. We will also need the polarization vectors in spinor helicity in order to work with gluon amplitudes. In 4D there are two independent polarization states of a massless spin-1 particle

$$e_+^\mu = -\frac{1}{\sqrt{2}}\frac{|p]\langle\eta|}{\langle\eta p\rangle} \tag{2.20}$$

$$e_-^\mu = -\frac{1}{\sqrt{2}}\frac{|\eta]\langle p|}{[\eta p]} \tag{2.21}$$

where η is a reference spinor. This reference dependence is due to gauge invariance – we are free to shift $e \longrightarrow e + \alpha p$ since $p_\mu A_n^\mu = 0$ by the Ward identity.

One striking feature of the polarization vectors is that they involve different numbers of angle and square spinors. This is a reflection of the fact that plus and minus helicity polarizations carry different little group weights. Recall that a massless momentum vector has three (complex) degrees of freedom in 4D. This might seem at odds with the fact that we're using two 2D spinors (with a total of four degrees of freedom) to represent the momenta. However, the definition of the spinors Eq. (2.16) is invariant under a mutual rescaling

$$|p\rangle \longrightarrow w^{+1}|p\rangle \tag{2.22}$$

$$|p] \longrightarrow w^{-1}|p] \tag{2.23}$$

and so there really are only three degrees of freedom in the spinors after all. The power of w defines the little group weight. Looking at the polarization vectors, we see that $e_- \sim w^{+2}$ (a negative helicity gluon has little group weight +2) and $e_+ \sim w^{-2}$ so that helicity and little group weight are the same concept up to a factor of -2.

The concept of helicity weight naturally leads to a classification of amplitudes known as the N^{k}MHV classification based on the number of negative helicity gluons in an amplitude. Rather conveniently, the all plus and one minus gluon amplitudes vanish at tree level, i.e., $A(+++...+) = 0$ and $A(-++...+) = 0$. This can be shown by looking at the helicity weights of the amplitude, picking the correct reference vectors η, and leveraging the power counting of the numerators and denominators of Feynman diagrams. So the first non-trivial gluon tree amplitude involves two negative helicity states. This is known as the maximally helicity violating (MHV) amplitude. The nomenclature comes from thinking about the $2 \longrightarrow (n-2)$ process, which is related to the one we are working with by crossing symmetry. The next to maximally helicity violating (NMHV) amplitude involves three negative helicity states and generally the N^{k}MHV amplitude has $k+2$ negative helicity gluons. The remarkable fact found by Parke and Taylor is that the MHV n-pt amplitude has a closed form description in terms of spinor helicity variables [1]. In terms of 4-vectors and Feynman diagrams this amplitude looks roughly like

$$A_n(1^+2^+ ... i^- ... j^- ... n^+) \sim \sum \frac{(ee)(ep)^{n-2}}{\prod p^2} \tag{2.24}$$

but in terms of spinors it is *exactly*

$$A_n[1^+2^+ ... i^- ... j^- ... n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle} \,. \tag{2.25}$$

In this last line the angle brackets surrounding the arguments of the amplitude $A_n[...]$ signify that we have color-ordered the amplitude by stripping off a trace of the $SU(N)$ generators

$$A_n(123...n) = \sum_{\text{perms } \sigma} \text{Tr}(T^1 T^{\sigma_2} T^{\sigma_3} ... T^{\sigma_n}) A_n[1\sigma_2\sigma_3 ... \sigma_n] \,. \tag{2.26}$$

These color ordered partial amplitudes $A_n[..]$ will play an important role in the rest of this thesis.

2.5 Non-Relativistic Spinor Helicity.

Given how graceful YM looks in spinor helicity variables, it is natural to translate the NNSE amplitudes Eq. (2.13) into the non-relativistic spinor helicity formalism of [44]. Despite the non-relativistic setting, the energy and 3-momenta are embedded in a 4-vector by *defining* $p_0 = \sqrt{\omega}$ so that the on-shell condition looks relativistic $p_\mu p^\mu = 0$. In the non-relativistic case, there is one more Levi-Civita tensor ϵ_{ab} associated with spatial rotations. Since $|p_i|$ is invariant under a spatial rotation, this Levi-Civita tensor picks out the the energy (or time direction) as special, just like one would expect for a non-relativistic theory. The new ϵ tensor allows one to take the inner product between angle and square spinors, the result of which is essentially an energy. In particular energy and 3-momentum conservation look like

$$\sum_A \langle AA]^2 = 0 \tag{2.27}$$

$$\sum_A |A\rangle[A| + \frac{1}{2}\epsilon\langle AA] = 0 \,. \tag{2.28}$$

In this formalism, 3-momenta and their dot products looks like

$$p_A = |A\rangle[A| + \frac{1}{2}\epsilon\langle AA] \tag{2.29}$$

$$p_A^i p_B^i = -\langle AB\rangle[AB] - \frac{1}{2}\langle AA]\langle BB] \,. \tag{2.30}$$

Finally, the polarization vectors are

$$e_A^+ = \frac{|A][A|}{\langle AA]} \tag{2.31}$$

$$e_A^- = \frac{|A\rangle\langle A|}{\langle AA]} \tag{2.32}$$

up to unimportant normalization factors.

When translating the NNSE amplitudes into spinor helicity the usual simplifications enjoyed by relativistic gauge theories do not occur here, for two reasons. First, the inner products of angle and square spinors are rotationally invariant but not Lorentz invariant. For example, the NNSE three-point scattering amplitude in Eq. (2.12) becomes

$$\begin{aligned} A(1^-2^-3^-) &= k\,\langle 12\rangle\langle 23\rangle\langle 31\rangle \\ A(1^+2^-3^-) &= k\,[12\rangle\langle 23\rangle\langle 31] \\ A(1^-2^-3^+) &= k\,[31\rangle\langle 12\rangle\langle 23] \end{aligned} \tag{2.33}$$

where $k = f^{a_1a_2a_3}/\langle 11]\langle 22]$ and with all other helicity configurations obtained by conjugation or permutation. Notably, Eq. (2.33) can be recast into the form of gauge theory amplitudes multiplying a Lorentz violating, purely energy-dependent form factor, e.g.

$$A(1^-2^-3^+) = f^{a_1a_2a_3}\frac{\langle 12\rangle^3}{\langle 13\rangle\langle 32\rangle} \times \left[1 - \frac{\langle 11]}{\langle 22]} - \frac{\langle 22]}{\langle 11]}\right]. \tag{2.34}$$

So little group covariance constrains the amplitudes to an extent but there is substantial freedom left due to the breaking of Lorentz invariance. The second reason spinor helicity formalism does not simplify expressions is that the theory does not exhibit helicity selection rules, as is clear from Eq. (2.33). Hence the helicity violating sectors of the theory are not simpler than others, in contrast with gauge theory.

2.6 Soft Theorems.

We now turn to the infrared properties of the amplitudes for fluids. First, note that the NSE and NNSE amplitudes trivially exhibit an Adler zero,

$$\lim_{p\to 0} A_{n+1}(p_1, \cdots, p_n) = 0, \tag{2.35}$$

when momentum of the root leg, $p = \sum_{A=1}^n p_A$, is taken soft but leaving its energy untouched. This is property is obvious from the Feynman rule for the three-point interaction vertex, e.g. as shown in Eq. (2.10) for the NNSE, which is manifestly proportional to the momentum of the root leg. Physically, the Adler zero arises because the NSE and NNSE are in fact conservation equations, $\partial_0 T_{0j} = \partial_i T_{ij}$ and $\partial_0 T^a_{0j} = \partial_i T^a_{ij}$, for which every term has a manifest derivative.

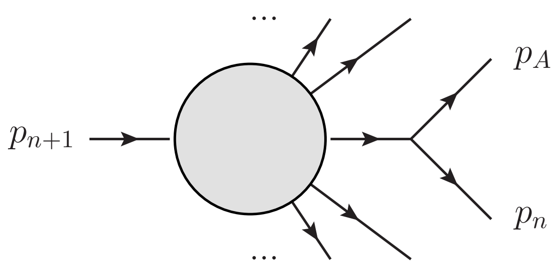


Figure 2.2: As a leaf leg n goes on-shell, the propagators joining n with another leaf leg A become singular. This means that the amplitude is dominated by diagrams of the type shown here.

Second, every leaf leg of a NSE amplitude satisfies a universal leading soft theorem,

$$\lim_{p_n \to 0} A_{n+1}(p_1, \cdots, p_n) = \left[\sum_{A=1}^{n-1} \frac{p_A e_n}{p_A p_n} \right] A_n(p_1, \cdots, p_{n-1}) \,, \tag{2.36}$$

which is highly reminiscent of the Weinberg soft theorem in gauge theory [38]. To derive Eq. (2.36) we realize that the most singular contribution in the $p_n \to 0$ limit arises when leaf leg n fuses with another leaf leg A, resulting in a pole from the merged propagator as shown in Fig. 2.2. The corresponding three-point vertex and propagator is

$$\lim_{p_n \to 0} \frac{1}{p_A p_n} \Big[(p_A e_n) e_{Ai} + (p_n e_A) e_{ni} \Big] = \frac{p_A e_n}{p_A p_n} e_{Ai}, \tag{2.37}$$

where the free index on the polarization dots into a lower-point amplitude, thus establishing Eq. (2.36). This same logic applies trivially to the NNSE as well.

Note that the NSE and NNSE do not have collinear singularities. The two-particle factorization poles go as $1/p_A p_B$, so there are instead "perpendicular" singularities when the momenta are orthogonal.

2.7 Recursion Relations.

Normally we think of a QFT as defined by an action and calculate all of the scattering amplitudes from that. While this approach is extremely flexible, allowing us to describe a plethora of systems, it is computationally inefficient and aesthetically dissatisfying that observables like amplitudes are encoded in off-shell Lagrangians. As first demonstrated by Britto, Cachazo, Feng, and Witten for YM [9], certain theories can be constructed purely from on-shell data via recursion relations (see [2] for a pedagogical review). Although scattering amplitudes are usually thought of as functions of real kinematics, it will be essential to analytically continue them to complex kinematics *while still maintaining the on-shell conditions*. The kinematics are complexified by performing a linear shift in a complex variable z

$$p_i \to p_i(z) = p_i + zq_i \tag{2.38}$$

where the q_i's are carefully chosen reference vectors. (A more intricate shift for pions is discussed in Appendix C.) In order to maintain on-shell kinematics in complex space, the reference vectors are subject to several constraints. For example,

$$\sum_{i=1}^{n} q_i = 0 \tag{2.39}$$

so that total momentum is conserved. Requiring $p(z)^2 = 0$ forces

$$p_i q_i = 0 \tag{2.40}$$

$$q_i^2 = 0. \tag{2.41}$$

Although it is not essential, it is often convenient if

$$q_i q_j = 0 \tag{2.42}$$

so that the (shifted) propagators are linear in z just like the momenta. Linearity in z means that tree amplitudes will only have simple poles in the complex plane, which dramatically simplifies the recursion at a mechanical level. While they are more complicated, shifts with quadratic poles have been put to great effect in reconstructing scalar EFT's [11]. [4]

To produce the actual recursion relation we write the original unshifted amplitude as a contour around the origin

$$A_n(0) = \frac{1}{2\pi i} \oint dz \frac{A_n(z)}{z}. \tag{2.43}$$

The contour is then blown up as in Fig. 2.3 using Cauchy's theorem to include every pole on $\mathbb{CP}^1$ besides the origin. The contour encloses two kinds of poles: the pole

[4] When talking about a linear or quadratic shift, we are referring to how the propagators behave, not how the kinematics in Eq. (2.38) behave. The kinematics will always depend at most linearly on the complex variables.

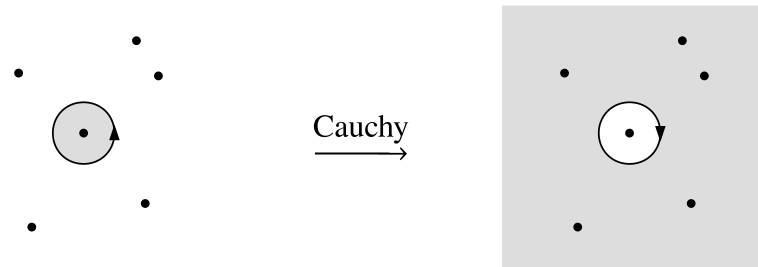


Figure 2.3: On-shell recursion relations for tree amplitudes are obtained by complexifying the kinematics and writing the amplitude as a contour integral around the origin. Using Cauchy's theorem, the contour around the origin can be "flipped" to include all of the poles in the complex plane including the pole at infinity. At the finite poles a propagator has gone on-shell so the amplitude factorizes into lower on-shell amplitudes. The pole at infinity must vanish for the recursion relations to close.

at infinity and the poles at finite z. For now we will assume at the pole at infinity vanishes but we will return to the subject shortly. Because we are looking at tree amplitudes under a linear shift, the amplitude only has simple poles corresponding to when an intermediary propagator $1/P_I(z)^2$ goes on-shell. On these poles the amplitude factorizes into left and right sub-amplitudes A_L and A_R (see Fig. 2.4). Critically, since the kinematics are on-shell everywhere in the complex plane, both A_L and A_R are *on-shell* amplitudes of lower multiplicity and hence this procedure is recursive. Putting this all together gives an analytic expression for the recursion relation

$$A_n = \sum_{z_I} \frac{A_L(z_I)A_R(z_I)}{P_I^2} \tag{2.44}$$

where the sum runs over all diagrams where an internal propagator goes on-shell (at $z = z_I$) and, because of some algebraic cancellations, the propagator is evaluated at its un-shifted location. Using a few low point on-shell seed amplitudes (like 3pt and 4pt), the recursion relation is now able to reconstruct any higher multiplicity on-shell amplitude.

So far we have not discussed the pole at infinity and this is, in fact, the crux of the whole matter. The pole at infinity has to vanish or else the residue theorem will pick up a boundary term at infinity and the recursion relation Eq. (2.44) will not close. The key step in proving any recursion relation is to show that $A_n(z) = \mathcal{O}(1/z)$ for large z so that this boundary term vanishes.

For the case of BCFW, there is only one non-zero reference vector and only two

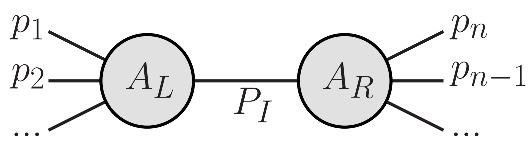


Figure 2.4: The shifted amplitude $A_n(z)$ has poles in the complex plane when a shifted propagator $1/P_I(z)^2$ goes on shell. On this pole the tree amplitude factorizes into lower point left and right sub-amplitudes $A_L(z)$ and $A_R(z)$.

lines are shifted

$$p_i \to p_i + zq \tag{2.45}$$

$$p_j \to p_j - zq \tag{2.46}$$

$$q^2 = p_i q = p_j q = 0. \tag{2.47}$$

For a description of BCFW in spinor helicity see Ref. [9] or for more information on how to described BCFW in a D-dimensional language see [12]. The boundary term can be shown to vanish for certain helicity configurations by analyzing Feynman diagrams [9, 35] or by more general methods [10].

While BCFW is a very simple shift to implement at a mechanical level, other shifts are better suited for certain tasks. The all-line or Risager shift [45] consists of shifting every leg by the same nilpotent reference vector q according to

$$p_i \to p_i + zc_i q \tag{2.48}$$

where the c_i's are chosen to maintain total momentum conservation and we must have $p_i q = 0$ as usual. Under this shift, every N^kMHV amplitude (beyond the MHV sector) vanishes sufficiently quickly at infinity. This shift is a key ingredient in the proof of the CSW rules (or MHV vertex expansion) where arbitrary helicity YM amplitudes are written in terms of sums of products of MHV amplitudes [45–47].

So far we have discussed shifts that work for unitary theories like YM and GR. Under these shifts, EFT's tend to have bad large z behavior. Intuitively this is because a recursion relation defines a whole theory from just a few seed amplitudes like 3pt and 4pt. But a generic EFT has a slew of random couplings so all of the information in the EFT cannot possibly be contained in a finite number of seed amplitudes unless there is some physical principle behind the couplings. For theories with enhanced soft limits (extensions of the Alder zero [39]) it is possible to use the soft information to tame the large z behavior [11]. In particular, the shift

$$p_i \to p_i(1 - c_i z) \tag{2.49}$$

can be used to construct certain theories with degree σ soft theorems

$$A(p_\ell \to 0) = \mathcal{O}(p_\ell^\sigma). \tag{2.50}$$

In order to conserve momentum, the c_i's must satisfy

$$\sum_{i=1}^{n} c_i p_i = 0. \tag{2.51}$$

The trivial solution (all c_i's equal) is insufficient since this will just rescale all of the momenta and will not yield a recursion relation. In order to construct a recursion relation, one notes that $z \to 1/c_i$ probes the soft limit of leg i

$$A_n(z \to 1/c_i) = \mathcal{O}(1 - zc_i)^\sigma. \tag{2.52}$$

Including compensating poles of this form will improve the large z behavior without introducing additional residues because these factors will cancel against the soft limit. The recursion relation derived from

$$\sum_{\text{Res}} \frac{A_n(z)}{z \prod_{i=1}^{n}(1 - c_i z)^\sigma} = 0 \tag{2.53}$$

is able to reconstruct NLSM ($\sigma = 1$), DBI ($\sigma = 2$), and the special Galileon (sGal with $\sigma = 3$) [11]. It is worth noting that unlike BCFW or the Risager shift, the poles in the soft shift are no longer simple poles. In other words, for a nontrivial subset of momenta a propagator $1/P_I(z)^2$ under the soft shift has a z^2 term, not just a term linear in z. This introduces square roots in intermediary stages of the recursion that must cancel in the final amplitude.

2.8 Recursion Relations for Fluids.

Next, let us derive on-shell recursion relations for the NSE and the NNSE. It will suffice to identify a shift of the external kinematics which either has vanishing boundary term or probes an Adler zero of the amplitude. We divide our discussion based on whether the shift modifies the energies in the amplitude or not.

For unshifted energies the momenta must shift so as to maintain the on-shell conditions. A natural choice is

$$p_A \to p_A + z\tau_A e_A, \tag{2.54}$$

for the leaf legs, keeping the energies ω_A and polarizations e_A unchanged. Here the constants τ_A are a priori unconstrained since we implicitly shift the momentum of the rooted leg, which is off-shell, to to conserve momentum. Note that Eq. (2.54) maintains the on-shell conditions since $p_A e_A = e_A^2 = 0$ for circular polarizations.

The boundary term is obtained from the large z behavior of the NSE and NNSE amplitudes. From the Feynman rules it is obvious that these take the schematic form

$$A_{n+1} \sim \sum \frac{(pe)^{n-1}(ee_{n+1})}{(pp)^{n-2}}, \tag{2.55}$$

so every term is proportional to a single dot product of a polarization of a leaf leg with that of the root leg. This of course has the form of the cubic (MHV) Feynman diagrams of gauge theory Eq. (2.24). Now in the large z limit of Eq. (2.54), we find that $pp \sim z^2$, $pe \sim z$, $ee \sim 1$, so Eq. (2.55) implies that $A_{n+1} \sim z^{-n+3}$. The boundary term vanishes when $n \geq 4$, so all amplitudes at five-point and higher are constructible via this shift. A downside of this shift is that the intermediate propagators are quadratic polynomials in z so each factorization channel enters via a pair of residues in the recursion relation [11, 48].

Conveniently, for appropriately chosen τ_A, the momentum shift in Eq. (2.54) will probe the Adler zero of the root leg. For example, at four-point we can write the momentum of the root leg as $p = \tau_1 e_1 + \tau_2 e_2 + \tau_3 e_3$, so $z = 1$ corresponds to the soft limit for which $A_{3+1}(1) = 0$. Hence we can compute the four-point amplitude via $A_{3+1}(0) = \frac{1}{2\pi i}\int \frac{dz}{z}\frac{1}{1-z}A_{3+1}(z)$, where the factor of $(1-z)^{-1}$ improves the large z convergence of the integral.

A more elegant recursion relation can be constructed if we also shift energies. Consider a shift of the leaf legs reminiscent of the Risager deformation [45],

$$p_A \to p_A + z(p_A\eta)\eta, \qquad e_A \to e_A - z(e_A\eta)\eta, \tag{2.56}$$

where $\eta^2 = 0$ is nilpotent and orthogonal to the polarization of the root leg, so $\eta e_{n+1} = 0$. This guarantees that $p_A e_A = 0$ holds after the shift. Here we also implicitly shift the energies of the leaf legs ω_A in whatever way is needed to maintain the on-shell conditions. At large z, the invariants scale as $pp \sim z$, $pe \sim 1$, $ee \sim 1$, so Eq. (2.55) implies $A_{n+1} \sim z^{-n+2}$. The boundary term vanishes for $n \geq 3$, so recursion applies at four point and higher.

However, on closer inspection one realizes that the Feynman diagrammatic numerators are all invariant under Eq. (2.56) and so the only z dependence enters through simple poles in the intermediate propagators. As a result, the factorization diagrams that appear in the recursion relation are literally Feynman diagrams and hence not very useful. Nevertheless it is amusing that the Feynman diagram expansion of the NSE and the NNSE is precisely analogous to the maximally helicity violating vertex expansion of gauge theory [46].

2.9 Color-Kinematics Duality.

A central aim of the modern S-matrix program is to elucidate structures hidden by Lagrangians and nowhere has this seen greater success than with gravity. From a particle physics perspective, GR is a dauntingly complicated theory. The action has infinitely many interactions each with a sea of indices. The cubic vertex alone takes up about half a page of text [5, 49]. In sharp contrast, the 3pt vertex for YM can be written in a single line. Despite the apparently disparate information content of these theories, Kawai, Lewellen, and Tye discovered that YM is actually enough to reconstruct gravity [5] at tree level [13]. Schematically they found that GR comes from two copies of YM

$$M_{\text{GR}}(1,2...n) \sim \sum_{\text{perms } \sigma_L,\sigma_R} s^{n-3} A_{\text{YM}}[1,\sigma_L] A_{\text{YM}}[1,\sigma_R] \tag{2.57}$$

where s is a generalized Mandelstam invariant, σ_L and σ_R run over permutations of $2,3...n-1$, and the YM amplitudes are color ordered as in Eq. (2.26). For more details on the field theoretic KLT relations see Ref. [50] because we will move directly onto the double copy.

The BCJ double copy is a generalization of KLT that works directly at the tree and loop integrand level [14, 15]. For a pedagogical review see [6]. The basis for the double copy is color-kinematics duality. A generic gluon tree amplitude can be written as

$$A_n = \sum_{\text{cubic graphs } i} \frac{c_i n_i}{d_i} \tag{2.58}$$

where c_i is the color factor (some polynomial in structure constants), n_i is the kinematic numerator (containing all of the momentum and polarization vectors), and d_i is the collection of propagators of the graph. Crucially, this sum runs over only cubic graphs; all of the quartic contact terms that appear in the normal YM Feynman rules have to be blown up into products of cubic vertices. This way of writing YM is far from unique since Jacobi identities may relate different color

[5] Technically, the theory of gravity in the KLT relations isn't pure gravity, but instead the theory of a graviton coupled to a 2-form and dilaton.

factors and there is no preferred way to blow up the 4pt vertices. Even though there are many ways to shuffle terms around in Eq. (2.58), color-kinematics (CK) duality says that *there exists* a way of writing YM in the form Eq. (2.58) such that color and kinematics mirror each other. Specifically, the duality states that if three color factors are related to each other by a Jacobi identity $c_i + c_j + c_k = 0$ as in Fig. 2.5, then there is a way of writing the kinematic numerators such that they also obey the same Jacobi identity $n_i + n_j + n_k = 0$. Another, slightly more trivial, requirement is that if two color factors are related by a sign, $c_i = -c_j$, like what happens when two legs in an internal 3pt vertex are swapped, then the numerators must also obey the same relation, $n_i = -n_j$. Again, it is worth noting that the CK dual n_i's are not unique due to a "generalized gauge invariance " [14]. Once the CK numerators are found, tree-level gravity is given by the remarkably simple expression

$$M_n = \sum_{\text{cubic graphs } i} \frac{n_i n_i}{d_i} \tag{2.59}$$

where an analogous statement holds at the loop integrand level. The shorthand for this squaring relation for GR is written as GR = YM ⊗ YM.

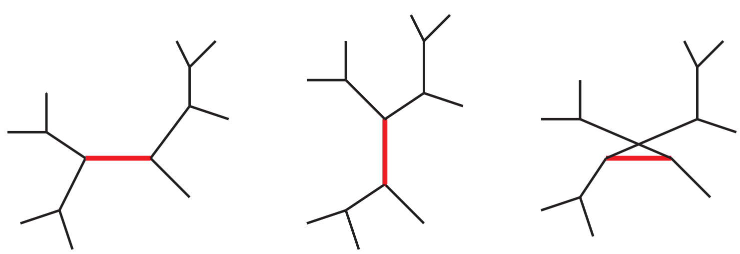


Figure 2.5: Color-kinematics duality states that if the color factors of a triplet of s, t, u diagrams are related by a Jacobi identity, $c_s + c_t + c_u = 0$, then there exists a form of the kinematic numerators such that they obey the same Jacobi identity $n_s + n_t + n_u = 0$. The propagator that is not shared between the diagrams is highlighted in red for clarity.

Since gravity is given so simply in terms of CK numerators from YM, this means that the extra complexity of GR is buried in actually finding those CK numerators. While constructing CK numerators is substantially easier than generating GR from Feynman diagrams, it is worth going over how the numerators can be constructed so we will describe one method that works at tree level. The first step is to carefully choose the right basis for color factors. The $(n-1)!$ trace basis for color ordered partial amplitudes in Eq. (2.26) is overcomplete due to the $U(1)$ decoupling and

Kleiss-Kuijf (KK) relations [51]. The $U(1)$ decoupling identity reads

$$A_n[123...n] + A_n[213...n] + A_n[2314...n] + A_n[23...1n] = 0 \tag{2.60}$$

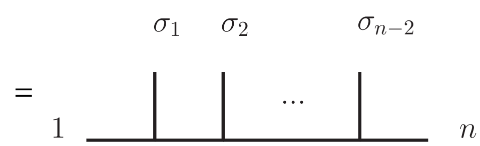
and can be derived by setting one of the group generators T^a to the identity matrix. Similar to the $U(1)$ decoupling identity, the KK relations are linear relations amongst color-ordered partial amplitudes

$$A_n[1, \alpha, n, \beta] = (-1)^{|\beta|} \sum_{\sigma \in \alpha \sqcup\!\sqcup \beta^T} A_n[1, \sigma, n] \tag{2.61}$$

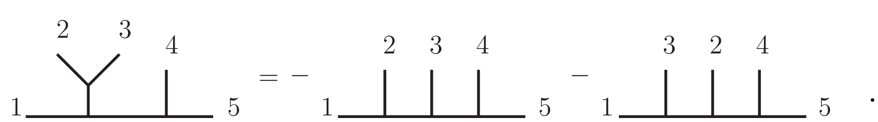
where α and β are lists of legs, $|\beta|$ means the length of the set, β^T is the set β but with reversed ordering, and $X \sqcup\!\sqcup Y$ is the set of all "shuffle" products formed by interleaving elements of X and Y while maintaining their relative orderings.

After imposing the $U(1)$ and KK relations, the basis of amplitudes reduces to the Del Duca-Dixon-Maltoni (DDM) half ladder basis [52]. The DDM basis consists of the $(n-2)!$ color factors

$$\mathrm{DDM}[1\sigma_2\sigma_3...\sigma_{n-2}n] = f^{1\sigma_1 c_1} f^{c_1\sigma_2 c_2} f^{c_2\sigma_3 c_3} ... f^{c_{n-3}\sigma_{n-2}n} \tag{2.62}$$



which runs over the permutations of $\sigma_1, \sigma_2, ...\sigma_{n-2}$ and the "encaps" are usually chosen to be 1 and n. (This graph is nicknamed a half ladder because it looks like a ladder cut down the middle.) By repeated application of the Jacobi identity, every color factor can be brought into the form of a DDM half ladder, for example



(2.63)

At tree level, a spanning set of color identities is given by:

1. Decomposing every DDM with left endcap 1 and right endcap $i = 2, 3...n-1$ in terms of DDM's with endcaps 1 and n

2. $\mathrm{DDM}[1, 2...n] = (-1)^n\ \mathrm{DDM}[n, ...2, 1]$

With the combination of these two requirements, any DDM with any endcaps can be written in terms of the DDM's with the canonical endcaps. Finding the CK numerators then boils down to generating an ansatz for one numerator with the canonical ordering $n[1, 2...n]$ and then enforcing the above DDM color relations with each DDM replaced by a numerator. The numerators with non-canonical ordering like $n[2, 1...n]$ are obtained from the canonical numerator by relabeling particles. The last constraint on n is that it must generate the correct YM amplitude. For constructing and solving such an ansatz, see Appendix A.

Usually the kinematic Jacobi identities only hold on-shell so each one has to be checked one at a time (typically on a computer). In very rare instances, it is possible to show that CK duality holds off-shell, in which case it is sufficient to check just the single triplet of diagrams in Fig. 2.5. We will encounter an off-shell CK duality for fluids in the next section.

Before moving on to the case of fluids, it will be important later in this thesis to know that the double copy is not restricted to just YM and GR. Identical squaring relations hold for a much larger "web" of theories. Both the non-linear sigma model (NLSM) and bi-adjoint scalars (BS)

$$\mathcal{L}_{\rm BS} = -\frac{1}{2}\partial_\mu\phi^{a\bar{a}}\partial_\mu\phi^{a\bar{a}} + f^{abc}f^{\bar{a}\bar{b}\bar{c}}\phi^{a\bar{a}}\phi^{b\bar{b}}\phi^{c\bar{c}} \tag{2.64}$$

obey color-kinematics duality. Thus each n_i in Eq. (2.59) can come from any of the three theories (BS, NLSM, and YM) to produce a valid product theory. Any theory double copied with BS produces the same theory again so BS plays the role of the identity in the BCJ product. This is actually a reflection of the fundamental BCJ identities presented in Ref. [14]. Besides GR there are two other product theories. The first is YM $\otimes$ NLSM which produces Born-Infeld (BI), the EFT of a self-interacting photon originally proposed by Born and Infeld to cancel the divergence of the electron self energy. The second product theory comes from NLSM $\otimes$ NLSM which is the sGal theory mentioned briefly in 2.7. sGal is a scalar EFT with an enhanced soft theorem similar to the Adler zero of pions [53]. All of these double copy relations can be summarized in the "multiplication table"

Theory A	Theory B	Product theory $C = A \otimes B$
YM	YM	GR
YM	NLSM	BI
NLSM	NLSM	sGal
BS	X	X

where X stands for any of the three input theories (BS, NLSM, and YM). This multiplication table is extremely obvious in the Cachazo-He-Yuan (CHY) formalism

[20–23] where amplitudes are written as integrals over moduli space, much like the string tree amplitudes. Although YM, NLSM, sGal, etc., all appear in the double copy, it is unclear what physical principle unites this web of theories. In the next chapter we will look at how classical conformal invariance ties many of these theories together, but before doing that, we will look at the CK properties of fluids.

2.10 Manifest Color-Kinematics Duality for Fluids.

The NNSE is purely trivalent and has a strong resemblance to gauge theory. It is then perhaps unsurprising that it also exhibits color-kinematics duality. To see why, consider a triplet of *off-shell* Feynman diagrams describing the s, t, and u channel exchange of a quantum of fluid within some larger arbitrary scattering process. The sum of these contributions is $\frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u}$, where c_s, c_t, c_u and n_s, n_t, n_u are the color factor and kinematic numerator of each Feynman diagram and $c_s + c_t + c_u = 0$. In the s-channel we find that $c_s = f^{a_1 a_2 b} f^{b a_3 a_4}$ while

$$n_s = \left[p_{1i_2}(p_{1i_3} + p_{2i_3})\delta_{i_1 i_4} + p_{2i_1} p_{3i_2} \delta_{i_3 i_4} - \{1 \leftrightarrow 2\}\right] , \tag{2.65}$$

with the t- and u-channel contributions related by permuting legs 1,2,3. To obtain Eq. (2.65) we have set $p_{1i_1} = p_{2i_2} = p_{3i_3} = 0$ due to the solenoidal condition for the one-point function of the velocity field. Remarkably, Eq. (2.65) implies that $n_s + n_t + n_u = 0$, establishing an off-shell duality between color and kinematics for the NNSE. With this in mind we recast the NNSE in Eq. (2.4) into a more suggestive form

$$\left(\partial_0 - \nu\partial^2\right) u_i^a + \frac{1}{2} f^{abc} f_{ijk} u_j^b u_k^c = J_i^a , \tag{2.66}$$

where we have defined a kinematic structure constant f_{ijk} which acts as a differential operator

$$f_{ijk} v_j w_k = v_j \partial_j w_i - w_j \partial_j v_i , \tag{2.67}$$

and by construction coincides with the Feynman rule for the three-point interaction vertex. Observing that

$$\left[v_j \partial_j, w_k \partial_k\right] = f_{ijk} v_j w_k \partial_i , \tag{2.68}$$

we see that f_{ijk} are the structure constants of the diffeomorphism algebra. Note the similarity of this kinematic algebra to that of self-dual gauge theory [54] and the non-linear sigma model [55].

Color-kinematics duality implies that the NNSE is invariant under the independent global symmetries,

$$\begin{aligned} \text{color:} \quad & u_i^a \rightarrow u_i^a + f^{abc}\theta^b u_i^c \\ \text{kinematic:} \quad & u_i^a \rightarrow u_i^a + f_{ijk}\theta_j u_k^a , \end{aligned} \tag{2.69}$$

where the θ parameters are constant. Note that the kinematic transformation is the global subgroup of diffeomorphisms, i.e. it is *literally* a translation. Without an action for the NNSE we cannot use Noether's theorem to derive the associated conserved currents. However, it is natural to define a vector current $J_{li} = f_{ijk}u_j^a \overleftrightarrow{\partial}_l u_k^a$ whose divergence is $\partial_l J_{li} = \frac{1}{\nu} f_{ijk} u_j^a \overleftrightarrow{\partial}_0 u_k^a$ after plugging in the NNSE. The volume integral of this quantity is the dissipation rate of kinematic charge,

$$\partial_0 Q_i = \int d^3x \; \partial_l J_{li} = \frac{1}{\nu} \int d^3x \; f_{ijk} u_j^a \overleftrightarrow{\partial}_0 u_k^a \, . \tag{2.70}$$

Since the integrand is a total derivative the kinematic charge is constant, $\partial_0 Q_i = 0$. To understand this fact diagrammatically, think of $\partial_0 Q_i$ as a three-particle vertex connecting the root leg to two fluid quanta which then cascade decay into the external sources. Due to the space integral in Eq. (2.70) the root leg is soft. Furthermore, the $\overleftrightarrow{\partial}_0$ in Eq. (2.70) implies that the vertex is multiplied by $\omega_1 - \omega_2 = \sum_{A_1 \neq B_1}(p_{A_1} p_{B_1}) - \sum_{A_2 \neq B_2}(p_{A_2} p_{B_2})$, where we have used momentum conservation and the on-shell conditions. Here A_1, B_1 and A_2, B_2 run over the decay products of the first and second fluid quantum at the vertex, respectively. Since $\omega_1 - \omega_2$ is the difference between two inverse propagators, $\partial_0 Q_i$ simply pinches the propagators adjacent to the root leg. Summing over all possible diagrams yields sums of triplets of kinematic numerators which vanish by the Jacobi identity.

To implement the double copy [6, 14, 15] we substitute all color factors with kinematic numerators. At the level of equations of motion this is achieved by squaring the NNSE term by term to obtain the TNSE

$$(\partial_0 - \nu\partial^2)u_{i\bar{i}} + \frac{1}{2}\left(u_{j\bar{j}}\partial_j\partial_{\bar{j}}u_{i\bar{i}} - \partial_j u_{i\bar{j}}\partial_{\bar{j}}u_{j\bar{i}}\right) = J_{i\bar{i}} \, , \tag{2.71}$$

which governs the dynamics of a bi-fluid velocity field $u_{i\bar{i}}$. As with all double copies, the barred and unbarred indices of the TNSE exhibit two independent rotational invariances. Such twofold symmetries are to be expected in any double copy [55, 56]. Note that it is also possible to substitute the kinematic numerators for color factors to obtain the fluid analog of biadjoint scalar theory.

2.11 Classical Solutions.

While the double copy was originally verified at the level of perturbative scattering amplitudes, it actually holds for certain extended solutions as well. In Ref. [57] it was shown that Kerr-Schild metrics (for pure gravity) of the form

$$g_{\mu\nu} = \eta_{\mu\nu} + k_\mu k_\nu \phi \tag{2.72}$$

double-copy to electromagnetic solutions

$$A_\mu = k_\mu \phi \tag{2.73}$$

where ϕ is a scalar field and k_μ is null with respect to both the Minkowski and full metric. In essence the classical double copy asserts that point charge solutions, $\phi \sim 1/r$, to GR are double copies of point charge solutions to E&M.

Much like GR and E&M, the NNSE and the TNSE have monopole solutions which are double copies of each other. In order to equate spatial and color indices we will restrict to 4D and $SU(2)$. For the NNSE we assume a static, spherically symmetric ansatz reminiscent of the 't Hooft-Polyakov monopole [58, 59], $u_i^a = f(r)\epsilon_{aij}x_j$. The NNSE becomes

$$f''(r) + \frac{4f'(r)}{r} - \frac{f(r)^2}{\nu} = 0\,, \tag{2.74}$$

which admits a singular solution, $f(r) = -\frac{2\nu}{r^2}$. For the TNSE we assume a static, spherically symmetric ansatz $u_{i\bar{i}} = g(r)\delta_{i\bar{i}} + h(r)x_i x_{\bar{i}}$. This yields a set of differential equations for g and h, not detailed here, which admit a singular solution, $g(r) = 2\nu$ and $h(r) = \frac{C}{r^4}$, for an arbitrary constant C. Comparing solutions side by side,

$$u_i^a = -\frac{2\nu\epsilon_{aij}x_j}{r^2} \qquad \text{and} \qquad u_{i\bar{i}} = 2\nu\delta_{i\bar{i}} + \frac{Cx_i x_{\bar{i}}}{r^4}\,, \tag{2.75}$$

we find a structure almost identical to the Kerr-Schild double copy for monopoles and black holes in Eq. (2.72) and Eq. (2.73). Note that the classical solutions in Eq. (2.75) depend crucially on the balance between the linear and nonlinear terms in the equations of motion. Also, the equations of motions admit non-singular solutions which can be solved for numerically but which we do not study further here.

2.12 Conclusions.

The present work leaves numerous avenues for future inquiry. First and foremost is the problem of turbulence and whether any insight can be gleaned from the scattering of fluid quanta at arbitrary multiplicity, e.g. with the tools of eikonal resummation or Wilson loops [60]. Related to this is the question of whether the S-matrices for the NNSE and the TNSE exhibit an analog of the Parke-Taylor formula [1].

Second, the miraculous appearance of color-kinematics duality in the NNSE and the TNSE hints at the enticing possibility that these theories might be but a part of a larger unified web of double copy theories. It is then natural to seek supersymmetric or stringy extensions of our results, as well as fluid analogs of the scattering equations [20–23] and transmutation relations [16].

Third, given that the NNSE and TNSE exhibit color-kinematics duality off-shell, it should be possible to draw an explicit connection between the classical and amplitudes double copy. Furthermore, it is likely that there exist other classical double copy solutions, e.g. including spin but perhaps also relating to known solutions in fluid mechanics such as the Taylor-Green vortex.

Chapter 3

CONFORMAL EFT'S

3.1 Introduction.

The modern scattering amplitudes program has exposed an array of extraordinary theoretical structures which include the double copy [6, 13–15], scattering equations [20–23], and novel reformulations of amplitudes as polyhedra [17, 19]. Developing these theoretical structures has also led to important applications. For instance, via the double copy procedure, gravity's highly complex amplitudes can be obtained by "squaring" much simpler amplitudes from gauge theory. This simplification sits at the heart of the recent state-of-art calculation of the black hole binary Hamiltonian at third post-Minkowskian order [61, 62]. Therefore, it cannot be overemphasized how important it is to understand the origins of these novel structures and to carve out the space of theories that enjoys these properties.

Curiously, the same set of theories emerges again and again when studying the double-copy and scattering equations. This set includes well-known theories like gravity and Yang-Mills (YM) in addition to a variety of scalar theories such as the biadjoint scalar (BS), the nonlinear sigma model (NLSM), Dirac-Born-Infeld (DBI) theory[1], and the special Galileon [23, 53, 63]. These scalar theories can be viewed as the cousins of YM and gravity and sometimes serve as simple toy models to decode mysterious properties like the double copy [55]. Gravity, YM, and these scalar theories are also exceptional in that their interactions are fully fixed by economical principles such as Lorentz invariance [2, 7, 64], gauge invariance [65], soft theorems [53, 66–75], color-kinematics duality [6, 13–15, 23, 55, 76–78], unifying relations [16], ultraviolet behavior [12, 27, 79], or symmetry [80–82, 82–85], depending on the theory in question. Although the details of these constructions will not be important to this paper, they motivates us to ask what physical property unites this disparate theories?

We propose that there is an underlying *symmetry* connecting these theories: conformal invariance. For the appropriate critical spacetime dimension D, the coupling constant is dimensionless and classical scale invariance is trivially ensured for BS

[1] In this paper, we consider DBI theory in flat space, rather than the conformal DBI, which describes a brane in an anti-de Sitter background.

theory ($D = 6$), YM theory ($D = 4$), gravity ($D = 2$) and the NLSM ($D = 2$). Notably, YM and the NLSM are curiously similar in their respective critical dimensions, e.g., both exhibit asymptotic freedom and a gapped spectrum. Rather enticingly, versions of these theories which are conformally invariant at the quantum level also expose integrable properties.

While these facts may be incidental, they beg the question of whether DBI and the special Galileon have special conformal properties. Indeed, we will show that these scalar effective field theories (EFTs) are the unique derivatively coupled, classical conformally invariant ($T^{\mu}_{\mu} = 0$) theories in $D = 0$ and $D = -2$, respectively.[2] While these are clearly unphysical choices for the spacetime dimension, our analysis is well-defined provided we work in general D throughout and only analytically continue to these particular values at the very end.[3]

A corollary of our result is that the tree-level scattering amplitudes in these EFTs are annihilated by the generators of the conformal group, $\mathcal{K}A_n = 0$. We then show how the conformal Ward identities—together with Lorentz invariance, locality, factorization, and the leading Adler zero [39]—are sufficient to uniquely bootstrap these amplitudes, confirming via an amplitudes analysis that the corresponding EFTs are fixed by classical conformal invariance.

In addition, our results show that scale invariance does not imply conformal invariance in the peculiar $D = 0$ and $D = -2$ cases we will discuss. Typically, scale invariance implies conformal invariance in numerous contexts [87–94] when principles like unitarity are assumed. It is unclear whether these assumptions hold in the unphysical dimension D here. In fact, our results are concrete examples where conformal invariance imposes further constraints beyond scale invariance.

3.2 Lagrangians from Conformal Invariance.

An obvious necessary condition for conformal invariance is scale invariance. Scale invariance requires that all coupling constants of the theory are dimensionless in

[2] As a reminder, there are two possible choices for the energy-momentum tensor $T_{\mu\nu}$. The first is the canonical energy-momentum tensor that comes from Noether's theorem. However, this tensor does not need to be gauge invariant or even symmetric (see the Belinfante-Rosenfeld energy-momentum tensor), so we will work with the gravitational energy-momentum tensor that comes from coupling a system to gravity.

[3] Note a very interesting recent conjecture of conformal invariance of graviton and YM amplitudes in arbitrary dimension D [25], later proven in Ref. [86].

a given critical dimension D. [4] Following Ref. [53], we define a power counting parameter ρ which characterizes the number of derivatives per interaction for a derivatively coupled scalar field ϕ. A generic vertex takes the form [5]

$$(\partial\phi)^2(g\partial^\rho\phi)^{n-2}, \tag{3.1}$$

where g is the coupling constant and the precise placement of derivatives, i.e., which derivative acts upon which field, is schematic and should be disregarded. Symmetries generally relate interaction vertices of the same ρ, since by dimensional analysis these terms can destructively interfere in scattering amplitudes. As an example to see that ρ is a faithful power counting parameter, observe that the 6pt contact term and the factorization diagram involving two 4pt vertices both scale with momenta as $p^{4\rho+2} = \frac{1}{p^2}(p^{2\rho+2})^2$, see Fig. 3.1 . Additionally, the power counting parameter ρ has a natural role in the BCJ double-copy. If a theory $M = A_L \otimes A_R$ comes from the BCJ product of two other theories A_L and A_R, then the power counting of the product theory is given by $\rho_M = \rho_L + \rho_R + 2$ while the spin is given by $s_M = s_L + s_R$ and the number of adjoint color factors is given by $c_M = c_L + c_R - 2$.

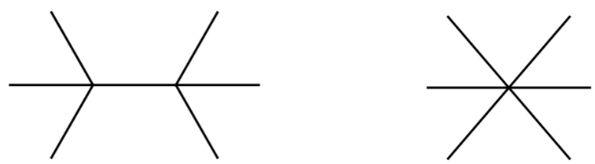


Figure 3.1: For a theory with derivative power counting ρ, the 4pt vertices in the left diagram each scale as $p^{2\rho+2}$ and so the whole diagram, including the propagator, scales as $\frac{1}{p^2}(p^{2\rho+2})^2$. This has exactly the same power counting, $p^{4\rho+2}$, as the contact term shown on the right. Thus ρ is a natural characterization of the interactions in a theory that can interfere destructively by symmetry.

When constructing a Lagrangian ansatz for a conformal theory, scale invariance implies that the coupling constant g is dimensionless. So, in the critical dimension, D and ρ are related to each other by

$$-\rho = \Delta = \frac{D-2}{2}, \tag{3.2}$$

where we have used that the field ϕ has dimension $\Delta = (D-2)/2$. An important feature is that in the critical dimension $D \leq 2$, we have $\rho \geq 0$ and therefore scale

[4] Free theories such as Maxwell (see Ref. [95]) and Klein-Gordon can be scale invariant outside of their naive critical dimensions.

[5] We will assume manifest locality so that no derivatives appear with negative powers in $\mathcal{L}$.

invariance alone still permits an infinite tower of marginal interactions. However, as we will see shortly, the additional assumption of conformal invariance will actually fix this tower uniquely for derivatively coupled scalars. In particular, scale invariance merely implies that $T \equiv T^{\mu}_{\mu} = dJ$ for some virial current J, while conformal invariance imposes the additional constraint that the virial current is conserved, so $T = dJ = 0$.

As is well-known, however, the energy-momentum tensor is only defined modulo improvement terms which are identically conserved, so conformal invariance requires that $T = 0$ up to this ambiguity. A mechanical algorithm to enumerate these improvement terms is to couple the theory to a background metric,

$$\hat{\mathcal{L}} = \sqrt{-g}\,(\mathcal{L} + \Delta\mathcal{L}), \tag{3.3}$$

including all possible minimal and nonminimal gravitational couplings. Since the energy-momentum tensor is the first variation of the background metric, we need to only include nonminimal gravitational interactions which are linear in the Riemann tensor. Higher powers will only contribute to the second variation and higher. Since the linear variation of Riemann has two derivatives in it, the resulting energy-momentum tensor has a trace T which is corrected by some improvement operator of the form $\partial\partial L$ for some local rank two tensor L. Hence, the most general statement of conformal invariance is that $T = \partial\partial L$.[6]

For our analysis, we begin by constructing a general ansatz Lagrangian for a derivatively coupled scalar field ϕ with interactions at a fixed value of ρ. Much like in dimensional regularization, we work in general dimensions such that the variable D only appears at the very end through $\eta^{\mu}{}_{\mu} = D$. We thus ignore all Gram determinant or evanescent effects since these are of course ill-defined for unphysical dimension D anyway. We then constrain the coefficients of the ansatz Lagrangian using conformal invariance.

3.3 Nonlinear Sigma Model.

As a warm up, consider the case of $\rho = 0$, which describes a theory of scalars with at most two derivatives per interaction. This analysis is simple but will serve as a template for more complicated EFTs. The most general two-derivative Lagrangian is[7]

$$\mathcal{L} = -\frac{1}{2}\partial^{\mu}\phi^{i}\partial_{\mu}\phi^{j}G_{ij}, \tag{3.4}$$

[6] See Ref. [96] for a pedagogical review and references therein.

[7] We work in mostly plus signature throughout.

where i, j are internal (target space) indices and $G_{ij}(\phi)$ is field dependent. Although we will not do so here, G can be interpreted as the metric on field space. We will compute the energy-momentum tensor from the coupling to a metric. We couple this theory to a background metric via

$$\hat{\mathcal{L}} = \sqrt{-g}\,(\mathcal{L} + R\,W), \tag{3.5}$$

where $\mathcal{L}$ above is properly covariantized and the arbitrary function $W(\phi)$ parameterizes the improvement terms induced by nonminimal coupling to the Ricci scalar. The energy-momentum tensor is obtained from the first variation of the metric about flat space, $T^{\mu\nu} = 2\frac{\delta S}{\delta g_{\mu\nu}}$, so

$$\begin{aligned} T = &-\frac{1}{2}\partial^\mu\phi^i\partial_\mu\phi^j G_{ij}(D-2) \\ &- 2(D-1)(\partial^\mu\partial_\mu\phi^i W_i + \partial^\mu\phi^i\partial_\mu\phi^j W_{ij}), \end{aligned} \tag{3.6}$$

where $W_i = \frac{dW}{d\phi_i}$ and $W_{ij} = \frac{d^2W}{d\phi_i d\phi_j}$. Thus, in the absence of improvement terms, any two-derivative theory is classically conformal in $D = 2$. In this case, conformal invariance places no restriction on G_{ij} and is identical to scale invariance.

Another well-known example is free theory, where $G_{ij} = \delta_{ij}$. Inserting the equations of motion $\Box\phi^i = 0$ into Eq. (3.6), we obtain

$$T = -\frac{1}{2}\partial^\mu\phi^i\partial_\mu\phi^j\left[(D-2)\delta_{ij} + 4(D-1)W_{ij}\right], \tag{3.7}$$

so for $W_{ij} = -\frac{D-2}{4(D-1)}\delta_{ij}$ we obtain a set of conformally-coupled scalars in any dimension. Note that the first term on the right-hand side of Eq. (3.7) is equal to $-\frac{1}{4}\partial^\mu\partial_\mu\left[(\phi^i)^2(D-2)\right]$ on the support of the free equations of motion. Consequently, in the absence of improvement terms, the trace of the energy-momentum tensor is of the form $T = \partial_\rho\partial_\sigma L^{\rho\sigma}$, as expected for a conformally invariant theory.

3.4 Dirac-Born-Infeld Theory.

We now turn to the case of $\rho = 1$, which is scale-invariant in $D = 0$. The theory we will end up with is DBI which can be obtained from the BI = YM $\otimes$ NLSM theory that appears in the double copy through dimensional reduction. For a derivatively coupled scalar, the Lagrangian is an arbitrary polynomial in $X = (\partial\phi)^2$.[8] Coupling this theory to a background metric, we obtain

$$\hat{\mathcal{L}} = \sqrt{-g}\left(\mathcal{L} + R\,A\,\phi^2 + R^{\mu\nu}B\,\phi^2\nabla_\mu\phi\nabla_\nu\phi\right), \tag{3.8}$$

[8] Working with functions of X is the simplest way to satisfy Lorentz invariance and power counting. However, introducing a scalar multiplet with a more complex derivative structure could lead to more elaborate brane theories such as the multi-field DBI appearing in [66].

where $A(X)$ and $B(X)$ are undetermined functions of X. *A priori*, one can add nonminimal couplings to the Riemann tensor but these all vanish by antisymmetry given the number of derivatives. The trace of the energy-momentum tensor is

$$\begin{aligned} T =& -2\mathcal{L}'X + D\mathcal{L} + (2-D)\partial^\mu\partial^\nu\left(\phi^2\partial_\mu\phi\partial_\nu\phi B\right) \\ &+ 2\partial^\mu\partial_\mu\left[\phi^2\left\{(1-D)A - \frac{1}{2}BX\right\}\right], \end{aligned} \tag{3.9}$$

where the prime denotes differentiation with respect to X.

For classical conformal invariance, $T = 0$ modulo the equations of motion,

$$\Box\phi = -2\frac{\mathcal{L}''}{\mathcal{L}'}Y_\mu Y_\nu Z^{\mu\nu}, \tag{3.10}$$

where $Y_\mu = \partial_\mu\phi$ and $Z_{\mu\nu} = \partial_\mu\partial_\nu\phi$. Plugging this into Eq. (3.9), we find that

$$T = \sum_{i=1}^{6} c_i(X)\mathcal{O}_i \tag{3.11}$$

can be expanded in a basis of six tensor structures,

$$\begin{aligned} \mathcal{O}_i = \{&1,\ \phi Y_\mu Y_\nu Z^{\mu\nu},\ \phi^2(Z^{\mu\nu})^2,\ \phi^2 Y_\mu Y_\nu Y_\rho W^{\mu\nu\rho}, \\ &\phi^2(Y_\mu Z^{\mu\nu})^2,\ \phi^2(Y_\mu Z^{\mu\nu}Y_\nu)^2\}, \end{aligned} \tag{3.12}$$

where $W_{\mu\nu\rho} = \partial_\mu\partial_\nu\partial_\rho\phi$ and the coefficients $c_i(X)$ are

$$c_1 = 2X(2A + BX - \mathcal{L}') \tag{3.13}$$

$$c_2 = 4(4A' + B + 2B'X) - 4(2A + 3BX)\frac{\mathcal{L}''}{\mathcal{L}'} \tag{3.14}$$

$$c_3 = 2(2A' - B'X) \tag{3.15}$$

$$c_4 = 4B' - 4(2A' + B - B'X)\frac{\mathcal{L}''}{\mathcal{L}'} \tag{3.16}$$

$$c_5 = 4(2A'' + B' - B''X) - 8(2A' + B - B'X)\frac{\mathcal{L}''}{\mathcal{L}'} \tag{3.17}$$

$$\begin{aligned} c_6 =& 8B'' - 16B'\frac{\mathcal{L}''}{\mathcal{L}'} - 8(2A' + B - B'X)\frac{\mathcal{L}'''}{\mathcal{L}'} \\ &+ 8(2A' + 2B - B'X)\left(\frac{\mathcal{L}''}{\mathcal{L}'}\right)^2. \end{aligned} \tag{3.18}$$

Treating each $\mathcal{O}_i$ as independent, we find that $c_i = 0$, yielding a system of differential equations for $\mathcal{L}$, A, and B. First, we solve $c_1 = 0$ for A. Plugging A and A' into

$c_2 = 0$ gives an *algebraic* expression for B in terms of derivatives of $\mathcal{L}$. Finally, inserting A and B and their derivatives into $c_3 = 0$ yield

$$\mathcal{L}'\mathcal{L}''' = 3\mathcal{L}''^2, \tag{3.19}$$

from which we obtain the general solution,

$$\begin{aligned} \mathcal{L} &= -\frac{1}{g}\sqrt{1+gX} + \lambda \\ A &= -\frac{gX+2}{8\sqrt{1+gX}} \\ B &= \frac{g}{4\sqrt{1+gX}}, \end{aligned} \tag{3.20}$$

which also solves the remaining equations. Here the decay constant g and cosmological constant λ arise as constants of integration. Remarkably, we narrow down to this particular solution from a class of scale-invariant theories, showing the former is much stronger than the latter in $D = 0$. We thus arrive at a main result of this paper: DBI is the unique conformally invariant, derivatively coupled scalar in $D = 0$.

3.5 Special Galileon.

Next, let us move on to theories with $\rho = 2$, which are scale invariant in $D = -2$. We choose a basis for a derivatively coupled scalar where the n-point interaction vertex takes the form $c_n^{\mu_1 \ldots \mu_{2n-2}} Y_{\mu_1} Y_{\mu_2} Z_{\mu_3\mu_4} \ldots Z_{\mu_{2n-3}\mu_{2n-2}}$, where c_n is an arbitrary constant tensor built from the flat space metric and numerical coefficients. As before, we promote this theory to couple with a background metric and then include all possible improvement terms built from Riemann contracted with derivatives of the scalars, taking the schematic forms $R\phi^2 Z^{n-2}$, $R\phi Y^2 Z^{n-3}$, and $RY^4 Z^{n-4}$.

Setting $T = 0$ on the support of the equations of motion in $D = -2$, we derive constraints on the interaction coefficients though six point. Conformal invariance fixes many but not all of the couplings in the ansatz Lagrangian. Nevertheless, by computing the scattering amplitudes in the resulting theory via Feynman diagrams, we discover that they coincide *exactly* with those of the special Galileon. Hence, the unfixed Lagrangian parameters all evaporate on-shell and can be eliminated by an appropriate field redefinition.

In fact, through a suitable choice of the unfixed parameters, the Lagrangian can be brought to the original representation of the special Galileon [80],

$$\mathcal{L} = -\frac{1}{2}X\Big\{1 - \frac{1}{3!}\left([Z]^2 - [Z^2]\right) + \frac{1}{5!}\big([Z]^4 - 6[Z]^2[Z^2] + 3[Z^2]^2 + 8[Z][Z^3] - 6[Z^4]\big)\Big\} + \ldots, \tag{3.21}$$

where the square brackets denote a trace over spacetime indices $[Z^n] = Z^{\mu_1}{}_{\mu_2}Z^{\mu_2}{}_{\mu_3}\ldots Z^{\mu_n}{}_{\mu_1}$. After rescaling the coupling constant, this action can be rearranged and resummed to obtain

$$\mathcal{L} = -\frac{1}{2}X\frac{\sin\left(\frac{d}{dh}\right)}{\frac{d}{dh}}\det(1+hZ)\Big|_{h=0} \tag{3.22}$$

$$\mathcal{L} = \frac{1}{2}\phi\frac{\sinh\left(\frac{d}{dh}\right)^2}{\frac{d}{dh}}\det(1+hZ)\Big|_{h=0} \tag{3.23}$$

where the trig functions are understood as power series in derivatives of the parameter h and the determinant is expanded as sums of products of traces of $Z_{\mu\nu}$. Similar actions appear in [97].

The freedom of unfixed couplings can also be used to put the improvement terms in a form that depends only on the Ricci tensor,

$$\Delta\mathcal{L} = \phi^2\Big(-\frac{1}{6}[R] - \frac{1}{72}[R][Z^2] + \frac{1}{12}[RZ^2] - \frac{1}{20}[RZ^4] + \frac{1}{40}[RZ^2][Z^2] - \frac{1}{90}[RZ][Z^3] + \ldots\Big), \tag{3.24}$$

which closely mimics those of DBI in Eq. (3.8). While it is computationally difficult to extend these results to higher point, this pattern will almost certainly continue. We leave the question of conformal invariance to all orders for future work.

3.6 Scattering Amplitudes from Conformal Invariance.

Conformal invariance can be enforced at the level of scattering amplitudes rather than the Lagrangian. This has the distinct advantage of trivializing equations of motion and eliminating ambiguities arising from field redefinitions. Here we consider two types of amplitudes constraints which both imply and are implied by conformal invariance.

The first constraint requires coupling the scalar EFT in question to an additional dilaton degree of freedom, τ. Since the dilaton couples via τT and conformal invariance implies that $T = \partial\partial L$, the single-dilaton amplitude exhibits a *double Adler zero* in the soft limit,

$$A_{n+1}(q, p_1, \ldots, p_n)|_{q\to 0} \sim O(q^2), \tag{3.25}$$

where q is the dilaton momentum. To reach this conclusion, one must in general be careful about soft propagator poles spoiling the double Adler zero. However, this is not a problem in a theory of derivatively coupled scalars since the on-shell three-point amplitude vanishes identically due to kinematics.

Notably, the converse proposition is also true: the double Adler zero in Eq. (3.25) implies conformal invariance. To understand this, consider $A_{n_{\min}+1}$ for the smallest possible number of EFT scalars $n_{\min}$ for which the amplitude is nontrivial. By definition, $A_{n_{\min}+1}$ is a local interaction vertex evaluated on-shell with no internal propagators. The $O(q^2)$ soft behavior of the dilaton implies that the lowest order interaction vertex of the dilaton in the off-shell Lagrangian is an operator of the form $\partial\partial\tau L$, where L is a local operator that depends on the EFT scalars. Of course, this operator is ambiguous up to terms which vanish on-shell. Crucially, however, these terms all involve either the on-shell condition for the dilaton, $\Box\tau$ or the on-shell condition for the scalar, $\Box\phi$. The former produces contributions *still* of the form $\partial\partial\tau L$, while the latter can be eliminated via a field redefinition in favor of higher order terms.

Next, we consider A_{n+1} for $n > n_{\min}$. This amplitude has propagator poles, but all the singularities must factorize into lower-point dilaton amplitudes times scalar amplitudes. On these factorization channels, there is always a double Adler zero because the lowest order dilaton interaction vertex is of the form $T = \partial\partial L$ and as discussed before, there are no on-shell three-point amplitudes. Consequently, the residual contact term in the amplitude must independently scale as $O(q^2)$ and should then be added to the definition of L. This argument is then repeated for higher and higher order amplitudes until we obtain $T = \partial\partial L$ to all orders.

The above argument establishes that a double Adler zero for the dilaton implies conformal invariance. However, the dilaton soft theorem is also equivalent to a second type of amplitudes constraint, which is the conformal Ward identity on pure scalar EFT amplitudes. This connection has been shown in the context of gluon and graviton amplitudes [25]. As discussed in Ref. [98], the dilaton soft limit is defined by

$$\begin{aligned} &A_{n+1}(q, p_1, \ldots, p_n)|_{q\to 0} \\ =&(\mathcal{D} + q^\lambda \mathcal{K}_\lambda) A_n(p_1, \ldots, p_n) + O(q^2), \end{aligned} \tag{3.26}$$

where we crucially set $p_n = -\sum_{j=1}^{n-1} p_j$ in order to ensure that the scale and conformal operators commute with momentum conservation [98]. Here $\mathcal{D}$ and $\mathcal{K}_\lambda$ are the scale and conformal boost generators in momentum space,

$$\mathcal{D} = -D + n\Delta + \sum_{i=1}^{n} p_{i\nu} \cdot \partial_{i,\nu} \tag{3.27}$$

$$\mathcal{K}_\lambda = \sum_{i=1}^{n} \left[p_i^\nu \partial_{i,\lambda\nu} - \frac{1}{2} p_{i\lambda} \partial_i^2 + \Delta \partial_{i,\lambda} \right], \tag{3.28}$$

where $\partial_{i,\nu} = \partial/\partial p_i^\nu$, $\partial_{i,\mu\nu} = \partial^2/(\partial p_i^\mu \partial p_i^\nu)$ and $\partial_i^2 = \eta^{\mu\nu}\partial_{i,\mu\nu}$. [9] In the appropriate critical dimension D, all amplitudes are trivially annihilated by $\mathcal{D}$, so the double Adler zero, and hence conformal invariance, hold if and only if

$$\mathcal{K}_\lambda A_n(p_1, \ldots, p_n) = 0. \tag{3.29}$$

For explicit computations, it will be convenient to recast the conformal boost operator in terms of Mandelstam invariants $s_{ij} = -2p_i \cdot p_j$ by dotting $\mathcal{K}_\lambda$ with the momentum p_l^λ of the lth leg [25], so

$$\begin{aligned} p_l \cdot \mathcal{K} = &\sum_{i,j\neq i,k\neq i} \left(s_{ik} s_{lj} - \frac{1}{2} s_{jk} s_{li} \right) \partial_{s_{ij}} \partial_{s_{ik}} \\ &+ \Delta \sum_{i,j\neq i} s_{jl} \partial_{s_{ij}}, \end{aligned} \tag{3.30}$$

where the spacetime dimension D only enters through $\Delta = (D-2)/2$. Note that the above representation is well-defined because the conformal boost commutes with the on-shell condition and we have already fixed p_n to enforce momentum conservation.

We are now equipped to use Eq. (3.29) to "conformally bootstrap" the scattering amplitudes of DBI and the special Galileon. First, let us consider the simplest case of four-point scattering of EFT scalars. The most general ansatz for this amplitude is a linear combination of terms like $s_{12}^a s_{13}^b$ where $a + b = 1 + \rho$. It is straightforward to see that $p_l \cdot \mathcal{K}(s_{12}^a s_{13}^b) = 0$ implies that $\rho = -\Delta$, which is exactly the condition of scale

[9] Although the conformal operators (for acting on particles of any spin) are handled quite nicely in spinor helicity [99], they are unsuitable for the task at hand where the spacetime dimension must be kept general until the end of the calculation.

invariance in Eq. (3.2). Thus, any scale invariant four-point scattering amplitude is automatically conformally invariant. Note that this argument is general and applies to single or multiple scalars which may or may not be derivatively coupled. This result closely mirrors enhanced soft limits [53, 66], which are also automatic at four point.

For higher-point scattering, we construct an ansatz for the amplitude A_n consistent with locality, factorization, Bose symmetry, and a choice of ρ,

$$A_n = A_{n,\text{cont}} + A_{n,\text{fact}} \tag{3.31}$$

where $A_{n,\text{fact}}$ is the factorization contribution obtained by treating all lower point amplitudes as Feynman vertices and summing all Feynman diagrams with at least one internal propagator. For the residual contact contribution, we define a local ansatz function $A_{n,\text{cont}}$ which will be fixed by the conformal Ward identities.[10]

To bootstrap DBI, we consider a general $\rho = 1$ amplitudes ansatz for derivatively coupled scalars. As discussed previously, four-point scattering is automatically conformally invariant. There is no odd-point scattering due to Lorentz invariance so we jump to six point, where the only allowed interaction vertex for a derivatively coupled scalar is

$$A_{6,\text{cont}} = d_6 s_{12} s_{34} s_{56} + \text{perms}. \tag{3.32}$$

for an arbitrary coefficient d_6 and perms stands for the remaining sum over permutations. The condition $\mathcal{K}_\lambda A_6 = 0$ fixes d_6 so that A_6 is precisely the DBI amplitude. The same procedure at eight point then fixes the contact term

$$A_{8,\text{cont}} = d_8 s_{12} s_{34} s_{56} s_{78} + \text{perms}, \tag{3.33}$$

again in such a way that exactly matches DBI. For a discussion of the computer methods needed to solve more complex ansatze see Appendix A.

For the special Galileon, we build an amplitudes ansatz for $\rho = 2$, derivatively coupled scalars. As before, four point is automatic, so we start at five point where there is one independent contact term. Imposing Eq. (3.29) fixes A_5 to zero. Moving on to six point, we perform the same exercise and reproduce the scattering amplitude for the special Galileon. The eight point amplitude is also uniquely fixed to be the special Galileon if we assume each field has at most two derivatives.[11]

[10] A similar approach has been taken to study spontaneously broken conformal symmetry [100].

[11] As a cross check we have used the $O(q^2)$ Adler zero for the dilaton to constrain the the scalar EFT amplitudes. We find again that DBI and the special Galileon are the unique conformally invariant, derivatively coupled amplitudes in $D = 0$ and $D = -2$ up to and including six-point scattering.

It is natural to ask whether there exist other conformally invariant theories in exotic dimensions besides DBI and the special Galileon. We have verified that no such derivatively-coupled scalar theory exists in $D = -4$, at least up to sixth order in the field. This is perfectly analogous to the nonexistence of theories with enhanced Adler zeros at $\rho = 3$ beyond four point [66]. Note that if you relax the assumption of derivative coupling, then there exist additional scalar EFTs which are conformally invariant. An example of such a theory is the six-point contact interaction $\phi^2 \partial_\mu \phi Z^{\mu\nu} \partial_\nu X$, which is conformal all by itself in $D = 0$ but does not exhibit a shift symmetry.

3.7 Conclusions.

Our findings leave a number of avenues for future study. Recently, the Lagrangian form of the conformal symmetry of DBI and the special Galileon was elucidated in [101] and was found to be part of a larger symmetry group. Since DBI and the special Galileon are fixed by conformal invariance, it would be interesting to devise new on-shell recursion relations [9] which exploit this fact. A similar approach was taken in Ref. [11], where enhanced soft limits were leveraged to derive new recursion relations for these very same scalar EFTs.

Second is the question of whether conformal invariance is exhibited by higher-spin theories in the double copy, e.g., the Born-Infeld (BI) photon, whose structure is constrained through soft behavior [74], and the gauge theory constructed in Ref. [102]. It would be interesting to see if the latter can be conformal in $D = 6$. On the other hand, we are actually somewhat pessimistic for BI, simply because a free photon is only conformally invariant in $D = 4$, while scale invariance for interacting BI requires $D = 0$. That said, a more thorough analysis, including other theories with an interacting photon [103], is warranted.

Third, our results suggest an intimate connection between conformal invariance of a derivatively coupled scalar and the enhanced Adler zero condition [53, 66]. Here the underlying symmetry algebras [80–82, 82–85] are likely to shed light, perhaps offering a connection to extended versions of these theories [28, 69, 77, 104, 105].

Last, it would be interesting to see how conformal invariance of DBI and the special Galileon might be extended beyond the classical limit, for instance, by analyzing loops or, more speculatively, through non-perturbative means such as the conformal bootstrap analytically continued to exotic spacetime dimension.

Appendix A

COMPUTATIONAL METHODS

Many of the expressions in amplitudes are too long to deal with by hand, involving thousands or millions of terms. Manipulating these expressions requires the use of computer algebra systems and specialized computational techniques. This appendix provides an overview of some of these methods. The prototypical problem is to construct an ansatz and then solve it subject to some constraints. Hopefully the constraints are linear and the problem reduces to row reducing a matrix. Typical problems of this type include constructing color-kinematics dual representations of YM or bootstrapping a specific amplitude from various criteria. For an example of the latter process, see Appendix B on fixing NLSM from its color structure. For concreteness, I will focus on the problem of fixing an NLSM tree amplitude from the Adler zero, that is, showing that the soft theorem along with a few other properties completely specify NSLM at tree level. Although this property of NLSM has been proven analytically with a recursion relation [11], the point is that the computer is a valuable tool for generating theoretical data when establishing a conjecture or before attempting a proof.

In order to show that an amplitude of n pions is fixed by the Adler zero we need to start off by constructing an ansatz for the amplitude. Although a tree amplitude can be thought of as a function of momentum vectors (and polarization vectors for a higher spin theory), it is vastly superior to think of the Mandelstam invariants $\{p_i p_j, p_i e_j, e_i e_j\}$ as the fundamental variables.[1] Due to momentum conservation, $\sum p_i = 0$, and the on-shell conditions, $p_i^2 = 0$ and $p_i e_i = 0$, some of these variables are linearly dependent. After taking into account all of the kinematical constraints, there are $\frac{n(n-3)}{2}$ of the $p_i p_j$ variables, $n(n-2)$ of the $p_i e_j$ variables, and $\frac{n(n-1)}{2}$ of the $e_i e_j$ variables. One can then construct an ansatz in terms of rational functions of these variables with unknown coefficients c_i sprinkled throughout. Here power counting is obviously very important since it dictates the degree of these rational functions. Locality decides the analytical structure of the poles and bose symmetry constrains how the amplitude behaves when relabeling external particles. The

[1] In the rare event that one needs a null momentum vector in terms of actual components, it is possible to construct one via $p = \{x_1^2 + x_2^2 + x_3^2 + \ldots, -x_1^2 + x_2^2 + x_3^2 + \ldots, 2x_1x_2, 2x_1x_3, \ldots\}$ which is essentially equivalent to going to lightcone coordinates.

critical constraint is that the amplitude should vanish when one of the particles is taken soft. In terms of these generalized Mandelstam invariants, this constraint can be implemented by demanding that the amplitude vanishes as $p_\ell p_i \to 0$ where ℓ is the fixed soft leg. [2]

The constraints are all linear and can be put into the form $\sum c_i s_i = 0$ where s_i is some rational function of Mandelstams. For large systems this must be solved on the computer, so we must convert this to a numerical matrix problem. There are several ways to do this. If we reduce to a Mandelstam basis, then each Mandelstam monomial can be regarded as independent and so the coefficient of each monomial (which is just some linear combination of c_i's) must vanish. This yields a numerical matrix that can be row reduced on a computer to find the c_i's. Of course, if there are poles in $\sum c_i s_i$ then we must either turn the expression into one large fraction and examine the numerator, or we can look at the residues at these poles. Another option for generating a numerical matrix is to simply plug in random integers for each independent Mandelstam. Doing this enough times will generate the desired matrix.

We can now ask the computer to row reduce this numerical matrix. The issue, though, is that we are solving an analytical problem so the row reduction technique must be exact. This means that floating point numbers (or even arbitrary precision floats) are insufficient. In principle, we could row reduce the matrix using arbitrary precision rationals but this is infeasible from a practical standpoint. Although the entries in the initial and row reduced matrices are often quite simple, the intermediary expressions are so complicated that we will run out of either memory or time. The way around this dilemma is to solve the matrix multiple times over different finite fields $\mathbb{Z}_p$ and stitch together the final solution using rational reconstruction (or basically the Chinese Remainder Theorem) [106]. An example of one such solver is `SpaSM` which is available at `https://github.com/cbouilla/spasm` and a package for integrating this into `Mathematica` can be found at `https://gitlab.com/kaelingre/spasmlink`. The advantage of solving the system over $\mathbb{Z}_p$ is that it is exact and so, for example, there is no roundoff error from dividing by small floats. Furthermore, since the field is finite, there is no intermediary expression swell. Thus we have found a way to solve an analytical problem exactly using a numerical technique on the computer. Going back to our original problem

[2] Technically, when setting up the Mandelstam basis some legs are special. For example, momentum conservation eliminates one leg. The leg ℓ cannot be any of the special legs or else, for example, $p_\ell p_i$ might not even appear in the amplitude.

of bootstrapping pions, we would find that all of the c_i's are fixed up to the overall normalization of the coupling constant.

It is also worth noting that this row reduction technique appears outside of ansatz applications. For example, it is used when solving the integration by parts identities that show up in generalized unitarity. The matrices encountered in this process are typically quite sparse so it is essential from a memory standpoint to use a sparse solver (like `SpaSM`) in order to reduce unnecessary fill in.

Appendix B

A COLOR BOOTSTRAP FOR PIONS

Many theories in the amplitudes "web" are fixed by just a few physical principles like some small subset of gauge invariance, soft theorems, power counting, locality, factorization, etc. [11, 12, 65, 67, 68]. The question is, since color is so important to the double copy tying this whole web together, is color a defining property of any the theories in the web? Answering this question (at least experimentally at low point) will involve constructing an ansatz and imposing constraints, so this appendix will serve as a more detailed example of the bootstrapping techniques described in the previous appendix.

We need to find a way to turn some color property into an actual constraint to impose on an amplitude ansatz. The most natural candidate is the fundamental BCJ identity of Ref. [14]

$$\sum_{i=3}^{n}\left(\sum_{j=3}^{i} p_2 p_j\right) A_n[1, 3\ldots i, 2, i+1, \ldots n] = 0\,. \tag{B.1}$$

The reason this is a condition on the color structure is that these relations express the overcompleteness of the DDM basis. The DDM basis has length $(n-2)!$ but under the fundamental BCJ identities the set of independent amplitudes is reduced to just $(n-3)!$ members. From the outset it should be clear that this will only work for theories with at least one color factor like YM or NLSM. Born-Infeld or sGal, for example, are impossible to reconstruct with this method because they have no color structures and so the idea of color-ordering their amplitudes is meaningless.

The fundamental BCJ relations are not enough to fix YM. After constructing an ansatz for the 4pt YM amplitude incorporating locality and power counting, imposing gauge invariance would be enough to completely fix the amplitude [65]. However, replacing the Ward identity with the BCJ constraint is not enough to fix the amplitude because, unlike the case with gauge invariance, the BCJ relations do not relate terms with different polarization structures.

The next theory worth trying to bootstrap is NLSM. This bootstrap for pions was independently found and proven to work in Ref. [27]. In some sense, the bootstrap is much more interesting for pions than for YM because we now have infinitely many interactions to constrain. The miracle is that the fundamental BCJ identities (along with locality, factorization, power counting, and Bose symmetry) uniquely fix NLSM at tree level. In terms of color ordered partial amplitudes, Bose symmetry is just the requirement that the amplitude is invariant under cyclic relabelings of external legs. This means that the bootstrap is trivial at 4pt since there is only cyclically invariant Mandelstam $p_1 p_3$, which is just the pion amplitude up to a normalization factor.

At 6pt the amplitude looks like

$$A_6 = \sum \frac{A_L A_R}{P_I^2} + A_{6,\,\text{contact}} + \text{cyclic} \tag{B.2}$$

where the factorization pieces are fixed by the 4pt amplitude so it is just a matter of fixing the contact term. At 6pt there are $n(n-3)/2 = 9$ independent Mandelstams so the ansatz for the contact term looks like

$$\begin{aligned} A_{6,\,\text{contact}} = & c_{13}s_{13} + c_{14}s_{14} + c_{15}s_{15} + c_{23}s_{23} + c_{24}s_{24} \\ & + c_{25}s_{25} + c_{34}s_{34} + c_{35}s_{35} + c_{45}s_{45} \end{aligned} \tag{B.3}$$

where s_{ij} just means $(p_i + p_j)^2$ and other parameterizations of the contact term are possible. Imposing cyclic invariance reduces the number of independent contact terms to just two and finally imposing the fundamental BCJ identity fixes the ansatz to be NLSM up to a rescaling of the coupling constant. At 8pt there are 20 unknown coefficients which reduces to 3 after imposing cyclicity. [1] Again, imposing the fundamental BCJ identity uniquely fixes the theory to NLSM. This is enough to conjecture that NLSM is fully fixed by BCJ at tree level and, as stated above, this was independently found and proven in [27]. Interestingly, it does not seem to be necessary to impose the $U(1)$ decoupling or KK relations that reduce the $(n-1)!$ elements of the trace basis to the $(n-2)!$ elements of the DDM basis. This is probably because the fundamental BCJ identities are strictly stronger constraints than $U(1)$ and KK.

With the empirical confirmation that the color bootstrap correctly reconstructs pions, we can attempt to search the space of derivatively coupled scalars with one color factor. NLSM corresponds to the power counting $\rho = 0$ where this power counting

[1] A shortcut for finding the contact term is to realize that $c_1 s_{12} + c_2 s_{123} + c_{n/2-1} s_{12\ldots n/2}$ + cyclic furnishes a cyclically invariant basis for the contact term. However, we have chosen to not take this route and instead opted for describing a more general procedure that will work for more complicated theories with higher power counting.

parameter was defined in Eq. (3.1). We can create ansatze for theories with different ρ and see if any of them are compatible with BCJ. It is most reasonable to look for theories with exactly one adjoint color factor, even though there are BCJ complaint theories with two color factors like BS. If there existed a BCJ compliant theory X with two color factors that had a power counting any higher than that of BS ($\rho = -2$), then there would be an infinite tower of derivatively coupled BCJ compliant scalar theories formed by taking $X \otimes X \otimes \ldots$NLSM. This seems a little too good to be true so it is more fruitful to search for theories with just one color factor. In fact, there appears to be one candidate theory with one color factor and the same power counting as sGal ($\rho = 2$). This theory is uniquely fixed at 4pt and 6pt by the color bootstrap but the 8pt amplitude has no BCJ compliant solution. The contact term for the 8pt amplitude has some 80,000 unknown coefficients so it is essential to make good use of the computer techniques described in Appendix A.

Appendix C

AN NLSM RECURSION RELATION

BCFW is the gold standard for recursion because of its simplicity, that is, you only have to shift two legs. However, the simplicity of the shift is also a limitation in that very few theories are 2-line constructible. Many more theories can be recursed by shifting all of the legs and exploiting unitarity [48, 107] or by leveraging soft theorems [53]. These soft recursion relations make it possible to recurse several scalar EFT's (including NLSM, DBI, and sGal) but at the cost of quadratic poles which introduce square roots in intermediary steps that must cancel in the final amplitude. For the special case of NLSM, it can be recursed with a 3-line shift, which is free of quadratic poles, but only by studying and working with the larger theory of NLSM $\times$ BS (pions coupled to bi-adjoint scalars) [28]. In this Appendix we will describe a different route to a linear 3-line recursion relation for pions. The idea is to introduce multiple complex parameters and then use the global residue theorem (a generalization of the normal residue theorem to $\mathbb{CP}^n$) to reconstruct the amplitude. This was first done in Ref. [29] for YM. Normally BCFW or any shift of a single complex parameter only probes single particle exchanges but shifts of multiple complex parameters probe more complex singularities like multi-particle factorization, soft limits, and collinear singularities. Although YM can be constructed perfectly well with just one complex parameter, the additional structure is quite helpful for more elusive theories like pure NLSM.

The recursion process consists of shifting three legs

$$|i] \to |i] + \langle jk\rangle(z_1|i] + z_2|j]) \tag{C.1}$$

$$|j] \to |j] + \langle ki\rangle(z_1|i] + z_2|j]) \tag{C.2}$$

$$|k] \to |k] + \langle ij\rangle(z_1|i] + z_2|j]). \tag{C.3}$$

Complex momentum conservation is guaranteed by the Schouten identity Eq. (2.19) and $p(z)^2 = 0$ holds for any z because we have only shifted square (or angle) brackets, just like in BCFW. Under a linear shift in z, NLSM scales as $\mathcal{O}(z)$ at infinity which means that $A(z)/z$ won't vanish at the boundary and BCFW or similar shifts are insufficient to reconstruct NLSM. This means that we need to include extra factors in the denominator in order to guarantee good large z behavior. We carefully engineer these factors to cancel against the soft behavior of the amplitude. The full amplitude can be on-shell reconstructed from

$$\sum_{\text{Res}} \frac{A_n(z_1, z_2)}{(z_1 f_1)(z_2 f_2)} = 0 \tag{C.4}$$

where $A_n(z_1, z_2)$ scales as $z_1 z_2$ for large z_i and the factors in the denominator are given by

$$f_1 = \left[\frac{2}{[ik]} p_j (p_i + p_k)\right] z_2 + \langle jk \rangle z_1 + 1 \tag{C.5}$$

$$f_2 = \left[\frac{2}{[kj]} p_i (p_j + p_k)\right] z_1 + \langle ki \rangle z_2 + 1. \tag{C.6}$$

The original, unshifted amplitude comes from the $\{z_1, z_2\}$ pole. The rest of the poles cancel against the Adler zero

$$A_n(p_\ell \to 0) = O(p_\ell) \tag{C.7}$$

of the amplitude. Specifically, the $\{z_1, f_2\}$ pole cancels with the soft behavior of leg j, the $\{f_1, z_2\}$ pole cancels with the soft behavior of leg i, and the $\{f_1, f_2\}$ pole cancels with the soft behavior of leg k. There is some freedom in grouping the poles in the global residue theorem so there is no residue at $\{z_1, f_1\}$ for example. The disadvantage of this shift is that spinor helicity and 4D are hardcoded into it while pion tree amplitudes are dimension agnostic and, unlike higher spin theories like YM, reap no benefit from the spinor helicity formalism. The Jacobians from the global residue theorem also present a minor computational inconvenience.

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