Annular links with $\mathfrak{sl}_2$-irreducible annular Khovanov homology

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ABSTRACT

First introduced in [APS04] as a categorification of Kauffman bracket skein modules and recapitulated in [Rob13, GW10], the (sutured) annular Khovanov homology[GN14, GLW18] is a natural generalization of Khovanov homology[Kho00, Kho02, Kho05] to links in the thickened annulus. As in Khovanov homology, annular Khovanov homology is defined in combinatoric manner but is of geometric and representation-theoretic interest and plays a role of powerful link invariant. In this thesis, we explore further on the power of annular Khovanov homology in distinguishing links in the thickened annulus.

Grigsby, Licata, and Wehrli[GLW18] defined an action of the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ over the annular Khovanov homology from the observation that the vector space associated to an annular circle by the Khovanov TQFT can be regarded as a vector representation or (a tensor product of) trivial representations, and the annular Khovanov differential is an intertwiner between such representations. It is a direct consequence of the existence of $\mathfrak{sl}_2$-action that the dimension of the annular Khovanov homology is unimodal with respect to the annular gradings.

We investigate further on the consequences of the $\mathfrak{sl}_2$-action. One of the basic questions regarding a representation of $\mathfrak{sl}_2$ is the irreducibility of the representation. Our main result classifies the annular links whose annular Khovanov homology is irreducible as an $\mathfrak{sl}_2$-representation. This is based on the spectral sequence from the annular Khovanov homology to the annular instanton homology given in [XZ19].

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Chapter 1

INTRODUCTION

The main objects of study in this thesis are annular links, links in a thickened annulus up to ambient isotopies. The study of annular links is often a nice intersection of knot theory and braid theory, as a braid uniquely determines and is determined by its annular closure. In this sense, the study of annular links is more closely related to braid theory than the theory of ordinary knots and links in S^3, which only determines a braid up to Markov equivalences. As in the knot theory in S^3, invariants coming from various other fields of mathematics including algebraic topology, quantum topology, and Floer theory play an important role in the study of annular links. In fact, any invariants of ordinary knots and links can be regarded as an invariant of annular links by embedding the thickened annulus into S^3 in standard, unknotted manner, hence the distinction between ordinary and annular knot invariants comes from the detection of the ambient thickened annulus, or equivalently the axis of the annulus.

First introduced in [APS04] as a categorification of Kauffman bracket skein modules and recapitulated in [Rob13, GW10], there is a natural generalization of Khovanov homology[Kho00, Kho02, Kho05] to a link $\mathbb{L}$ in the thickened annulus. By measuring the winding number with respect to the center of the thickened annulus, Khovanov's chain complex can be endowed with an additional filtration called annular filtration, and the homology $\mathrm{SKh}(\mathbb{L})$ of the associated graded module with respect to this annular filtration turns out to be an invariant of links in the thickened annulus, often called annular Khovanov homology[GLW18, XZ19] or sutured annular Khovanov homology[GN14, BG15] in literature. Just like the Khovanov homology, annular Khovanov homology has proven itself to be a powerful link invariant, e.g. it detects the closed braids[GN14], the closure of trivial braids among closed braids[BG15], and annular links which are contained in a 3-ball in the thickened annulus[XZ19].

Since the earliest stage of the subject, Khovanov homology have drawn much attention not only from topologists but also from algebraists due to its representation theoretic interpretation, and the annular Khovanov homology is no exception. Grigsby, Licaca, and Wehrli[GLW18] defined an action of the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ over

$\mathrm{SKh}(\mathbb{L})$ from the observation that the vector space associated to an annular circle by the Khovanov TQFT can be regarded as a vector representation or (a tensor product of) trivial representations, and the annular Khovanov differential is an intertwiner between such representations. This action of $\mathfrak{sl}_2$ can also be seen from higher category theoretical point of view via trace decategorification of $\mathfrak{sl}_2$ foams, cf. [QR17]. With the action of $\mathfrak{sl}_2$, the annular grading has a clear representation theoretic interpretation as the weight of the $\mathfrak{sl}_2$-representation.

It is a direct consequence of the $\mathfrak{sl}_2$-action on $\mathrm{SKh}(\mathbb{L})$ that the dimension of the annular Khovanov homology is unimodal with respect to the annular gradings.

In this thesis, we investigate further on the consequences of the $\mathfrak{sl}_2$-action on $\mathrm{SKh}(\mathbb{L})$. One of the basic questions regarding a representation of $\mathfrak{sl}_2$ is the irreducibility of the representation. Our main result classifies the annular links whose annular Khovanov homology is irreducible as an $\mathfrak{sl}_2$-representation.

Theorem 1. *The annular Khovanov homology of an annular link is irreducible (as an $\mathfrak{sl}_2$-representation) if and only if it is isotopic to the core of the annulus.*

In a sense, Theorem 1 is comparable to the unknot detection theorem[KM11] of ordinary Khovanov homology if we view the core of the thickened annulus as the simplest annular link. We also note that Xie and Zhang[XZ19, Corollary 1.4] gives another detection theorem of the core of the thickened annulus in terms of the annular Khovanov homology: their result is in terms of the triply graded vector space while Theorem 1 is in terms of the $\mathfrak{sl}_2$-irreducibility, or equivalently annular-graded vector space.

Our proof of Theorem 1 is largely a consequence of the tangle detection theorem[XZ19] and the following theorem, which states that the next-to-top annular grading term of the annular Khovanov homology of a closed braid is strictly larger than the topmost annular grading term.

Theorem 2. *Let σ be a braid of $n \geq 2$ strands and $\widehat{\sigma}$ its closure. Then,*

$$\dim \mathrm{SKh}(\widehat{\sigma}; n-2) > 1 = \dim \mathrm{SKh}(\widehat{\sigma}; n).$$

Theorem 2 is a slight improvement of the unimodality of annular Khovanov homology with respect to the annular grading, cf. [GLW18, Corollary 1]. It is natural to ask whether this strict inequality between the dimensions in consecutive annular gradings continues to hold for smaller annular gradings. While this question turns out to be negative, we observed an interesting patterns in the distribution of the dimension of annular Khovanov homology with respect to the annular gradings experimentally:

Conjecture 3. *For a closed braid* $\mathbb{L}$ *of* n *strands, then there is a positive integer* m *such that the followings hold for every* $k \equiv n \bmod 2$*:*

$$\dim \mathrm{SKh}(\mathbb{L}; k) < \dim \mathrm{SKh}(\mathbb{L}; k-2) \text{ if } m < k \leq n,$$
$$\dim \mathrm{SKh}(\mathbb{L}; k) = \dim \mathrm{SKh}(\mathbb{L}; k-2) \text{ if } 0 < k \leq m.$$

The form of Conjecture 3 resembles the famous Fox's trapezoidal conjecture[Fox62](cf. [Sto14]) stating that a similar family of equalities and inequalities holds for the coefficients of the Alexander polynomial of alternating knots. We have checked Conjecture 3 and a related conjecture affirmatively on more than ~ 1000 random closed braids from 6 to 9 strands.

This thesis is organized as follows. In Chapter 2, we provide the necessary backgrounds on annular links and annular Khovanov homology. In Chapter 3, we prove our main results, Theorem 1 and 2. In Chapter 4, we discuss Conjecture 3 and a related conjecture with the experimental results as a supporting evidence.

Chapter 2

PRELIMINARIES

This chapter is devoted to provide the necessary backgrounds on annular links and annular Khovanov homology. Mainly due to the usage of representation theory over $\mathfrak{sl}_2$, every homology groups considered in this paper is over the complex field $\mathbb{C}$ unless otherwise stated explicitly, though many of the contents in this section allow a straightforward extension to arbitrary coefficient ring.

2.1 Annular links

Let $A = \{z \in \mathbb{C} \mid 1 \leq |z| \leq 2\}$ be a (closed) annulus and $I = [0,1]$. An *annular link* is a (smooth) closed 1-submanifold of $A \times I$, considered up to ambient isotopies of $A \times I$. An *annular knot* is a 1-component annular link. Every annular links in this paper is oriented, though the precise orientation matters little.

Just as in the case of ordinary links in S^3, an annular link $\mathbb{L} \subseteq A \times I$ can be represented as a link diagram on A, obtained by a vertical projection $A \times I \to A$ of a generic representative in the isotopy class of $\mathbb{L} \subseteq A \times I$.

One of the most elementary new feature of annular links compared to ordinary links (in S^3) is the notion of *wrapping* and *winding number*.

Definition 2.1.1. The *wrapping number* of an annular link $\mathbb{L}$ is the minimal (unsigned) intersection number of $\mathbb{L}$ with a meridional disk of $A \times I$. The *winding number* of $\mathbb{L}$ is the signed intersection number of $\mathbb{L}$ with any meridional disk of $A \times I$.

Both the wrapping and winding number are invariants of annular links, and it is clear by definition that the wrapping number of an annular link is always greater than or equal to the winding number.

Remark 2.1.2. Whenever convenient, we choose an orientation for an annular link that maximizes the winding number. In case of braid closures, this coincides with the braid-like orientation, all of whose strands wind positively around the braid axis.

The notion of annular links is particularly well-suited to the study of braids as a closed braid is naturally an annular link and two braids in the same conjugacy class define and are defined by identical annular links.

Definition 2.1.3. Let

$$B_n = \left\langle \sigma_1, \cdots, \sigma_{n-1} \,\middle|\, \begin{array}{ll} \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, & 1 \leq i \leq n-2 \\ \sigma_i\sigma_j = \sigma_j\sigma_i, & |i-j| \geq 2 \end{array} \right\rangle$$

be the braid group on n strands. For a braid $\sigma \in B_n$, there is an associated (n,n)-tangle β in $D^2 \times I$, see Figure 2.1 for a concrete example. The *closure* of σ, denote as $\widehat{\sigma}$, is the annular link obtained by closing up β vertically and embedding it into the thickened annulus $A \times I$ in the way that the braid axis coincides with the center of $A \times I$ and the first strand is placed innermost (equivalently, the last strand is placed outermost) in $A \times I$.

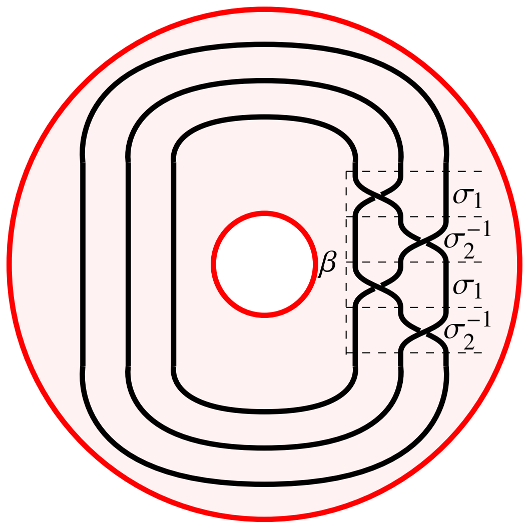

Figure 2.1: Braid closure of $\beta = \sigma_1\sigma_2^{-1}\sigma_1\sigma_2^{-1}$ in the annulus A, presented by an annular diagram.

From the construction, it is clear that both the wrapping number and the winding number of a closed braid of n strands is equal to n.

Definition 2.1.4. Let $\mathbb{L}_1, \mathbb{L}_2$ be annular links. The *annular union* $\mathbb{L}_1 \sqcup \mathbb{L}_2$ of $\mathbb{L}_1$ and $\mathbb{L}_2$ is defined as the annular link embedding $\mathbb{L}_1$ (resp. $\mathbb{L}_2$) into $\{z \in A \mid 1 \leq |z| \leq 3/2\} \times I$(resp. $\{z \in A \mid 3/2 \leq |z| \leq 2\} \times I$) by radial rescaling.

2.2 Annular Khovanov homology

Annular Khovanov homology[APS04, Rob13, GW10] is a natural extension of the Khovanov homology to annular links. Its definition can be best understood in comparison with the construction of ordinary Khovanov homology[Kho00, Kho02, Kho05], which we briefly review below.

At each crossings of a link diagram $\mathcal{D}(L)$ of a link L in $\mathbb{R}^3$, there are two ways to resolve the crossing, depicted as in Figure 2.2.

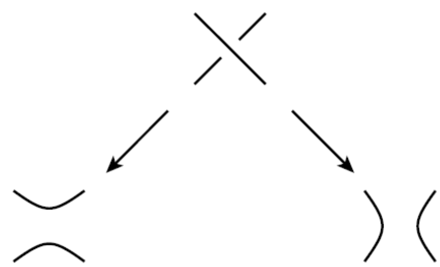

Figure 2.2: Two ways to resolve a crossing.

Applying one of the two resolutions at each crossing, we have a collection of non-intersecting circles on $\mathbb{R}^2$ which is often called a *complete resolution* of $\mathcal{D}(L)$. This process assigns to a tuple in $\{0,1\}^c$ a complete resolution of $\mathcal{D}(L)$ (where c is the number of crossings in $\mathcal{D}(L)$), hence we think of the complete resolutions as vertices in a hypercube, the *cube of resolution* of $\mathcal{D}(L)$. Then the underlying vector space of the Khovanov chain complex $\mathrm{CKh}(\mathcal{D}(L))$ is obtained by applying Khovanov $(1+1)$-dimensional TQFT to the cube of resolution of $\mathcal{D}(L)$, and the differential is defined using the saddle cobordisms associated to the change of resolution at a crossing. The Khovanov homology $\mathrm{Kh}(L)$ is the homology of the chain complex $\mathrm{CKh}(\mathcal{D}(L))$, which can be shown to be invariant under planar isotopies and Reidemeister moves and thus defines an invariant of a link.

Annular Khovanov homology can be constructed in a similar manner by keeping track of the winding number of the complete resolutions. That is, a link diagram $\mathcal{D}(\mathbb{L})$ of an annular link $\mathbb{L}$ can be resolved into a collection of non-intersecting circles in A, but contrary to the planar circles which are all isotopic, a circle in A may be *essential* (isotopic to the core of the annulus) or *inessential* (homotopic to a point). We then assign a grading, often referred as *annular* or k-grading, to the Khovanov TQFT of the annular circles: $\mathbb{C}\{1\} \oplus \mathbb{C}\{-1\}$ if essential, $\mathbb{C}^2\{0\}$ otherwise, where $\{k\}$ denotes the grading shift by k. This additional grading extends linearly with respect to tensor products, thus defines a grading on the Khovanov chain complex $\mathrm{CKh}(\mathcal{D}(\mathbb{L}))$. One can check that the differential on $\mathrm{CKh}(\mathcal{D}(\mathbb{L}))$ is nonincreasing in k-grading, hence $\mathrm{CKh}(\mathcal{D}(\mathbb{L}))$ is filtered with respect to the k-grading. The annular Khovanov homology $\mathrm{SKh}(\mathbb{L})$ is then defined as the homology of the associated graded complex of the k-filtered chain complex $\mathrm{CKh}(\mathcal{D}(\mathbb{L}))$.

Remark 2.2.1. [Rob13] showed that the annular Khovanov homology decategorifies to the Kauffman bracket skein module[HP89].

By definition, annular Khovanov homology admits a decomposition by k-grading:

$$\mathrm{SKh}(\mathbb{L}) = \bigoplus_k \mathrm{SKh}(\mathbb{L}; k).$$

Remark 2.2.2. There is no consensus on which smoothing should be 0- or 1-resolution in literature, thus one can find two different definitions of (annular) Khovanov homology, which compute the mirror of one another. We follow the convention of [GN14], opposed to [Rob13].

Remark 2.2.3. Note that Khovanov homology is bigraded and thus there are two gradings other than the annular grading on annular Khovanov homology. As these two graddings have insignificant roles in this paper, they are omitted in the notation and largely ignored in the followings.

It is clear from definition that for an annular link $\mathbb{L}$ of wrapping number ω, the annular grading of $\mathrm{SKh}(\mathbb{L})$ is supported in $\{-\omega, -\omega+2, \cdots, \omega-2, \omega\}$.

Similar to Khovanov homology, the annular Khovanov homology of the annular union is readily computable from the annular Khovanov homology of factor links:

Proposition 2.2.4. *For annular links $\mathbb{L}_i, i = 1, 2$,*

$$\mathrm{SKh}(\mathbb{L}_1 \sqcup \mathbb{L}_2) = \mathrm{SKh}(\mathbb{L}_1) \otimes \mathrm{SKh}(\mathbb{L}_2)$$

as a k-graded vector space. In particular,

$$\mathrm{SKh}(\mathbb{L}_1 \sqcup \mathbb{L}_2; k) = \bigoplus_{k_1+k_2=k} \mathrm{SKh}(\mathbb{L}_1; k_1) \otimes \mathrm{SKh}(\mathbb{L}_2; k_2).$$

Proof. Given a link diagram $\mathcal{D}(\mathbb{L}_1)$(resp. $\mathcal{D}(\mathbb{L}_2)$ for $\mathbb{L}_1$(resp. $\mathbb{L}_2$), $\mathcal{D}(\mathbb{L}_1) \sqcup \mathcal{D}(\mathbb{L}_2)$ is a link diagram for $\mathbb{L}_1 \sqcup \mathbb{L}_2$. As there is no crossings between $\mathcal{D}(\mathbb{L}_1)$ and $\mathcal{D}(\mathbb{L}_2)$, the annular Khovanov chain complex $\mathrm{CKh}(\mathcal{D}(\mathbb{L}_1) \sqcup \mathcal{D}(\mathbb{L}_2))$ is the tensor product of the chain complexes $\mathrm{CKh}(\mathcal{D}(\mathbb{L}_1))$ and $\mathrm{CKh}(\mathcal{D}(\mathbb{L}_2))$, hence the identity follows from the Künneth formula. □

In [BS15], Batson and Seed constructed the link splitting spectral sequence for Khovanov homology, a spectral sequence from the Khovanov homology of a link to the Khovanov homology of the disjoint union of the link components. We also have an annular analogue:

Theorem 2.2.5. *[Mar21, Theorem 3.1] Let $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$ be a union of sublinks $\mathbb{L}_1$ and $\mathbb{L}_2$. There exists a spectral sequence whose E_1 page is isomorphic to* $\mathrm{SKh}(\mathbb{L})$ *and which converges to* $\mathrm{SKh}(\mathbb{L}_1 \sqcup \mathbb{L}_2)$. *This spectral sequence can be decomposed by k-grading: for each k,*

$$\mathrm{SKh}(\mathbb{L}; k) \Rightarrow \mathrm{SKh}(\mathbb{L}_1 \sqcup \mathbb{L}_2; k)$$

Roughly speaking, Batson-Seed's proof for ordinary Khovanov homology applies equally well to the annular Khovanov case after one realizes that the additional differential d_1 in Batson-Seed's proof consists of saddle cobordisms as in the Khovanov differential, thus is filtered by k-grading and induces the spectral sequence on the associated graded complex.

Closed braids are a particularly nice class of annular links, and annular Khovanov homology can be used to detect them.

Theorem 2.2.6. *[GN14, Corollary 1.2], [XZ19, Theorem 1.3] Let $\mathbb{L}$ be an annular link. Then $\mathbb{L}$ is isotopic to a closed braid if and only if the top k-grading term of* $\mathrm{SKh}(\mathbb{L}; \mathbb{C})$ *is isomorphic to $\mathbb{C}$.*

Remark 2.2.7. The tangle detection theorem, Theorem 2.2.6, is one notable instance where the choice of coefficient ring matters. The proof in [XZ19] relies on the existence of a spectral sequence

$$\mathrm{SKh}(\mathbb{L}) \Rightarrow \mathrm{AKI}(\mathbb{L})$$

to the annular instanton Floer homology $\mathrm{AKI}(\mathbb{L})$. To the best of author's knowledge, extending annular instanton Floer homology over other coefficient rings is a nontrivial task, especially when the coefficient ring is of characteristic two. Nevertheless, Theorem 2.2.6 still holds over fields of characteristic two:

Proposition 2.2.8. *Let $\mathbb{L}$ be an annular link. Then $\mathbb{L}$ is isotopic to a closed braid if and only if the top k-grading term of* $\mathrm{SKh}(\mathbb{L}; \mathbb{Z}/2\mathbb{Z})$ *is isomorphic to $\mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ be the prime field of characteristic 2. The computation of top k-grading term of the annular Khovanov homology of a closed braid is done in [GN14, Corollary 1.2], hence we only prove that the top k-grading term of $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$ is $\mathbb{F}$ only if $\mathbb{L}$ is a closed braid.

By [GW10, Theorem 2.1], there exists a spectral sequence from $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$ to $\mathrm{SFH}(\Sigma(A \times I, \mathbb{L}); \mathbb{F})$, where $\Sigma(A \times I, \mathbb{L})$ is the branched double cover of the product sutured manifold $A \times I$ branched over $\mathbb{L} \subseteq A \times I$. We may view the product sutured manifold $A \times I$ as the knot exterior of z-axis $Z \subseteq S^3$, then the branched double cover $\Sigma(A \times I, \mathbb{L})$ is isomorphic to the link exterior of the preimage $p^{-1}(Z) \subseteq \Sigma(S^3, \mathbb{L})$ with 4 meridional sutures. In particular, the limit page $\mathrm{SFH}(\Sigma(A \times I, \mathbb{L}); \mathbb{F})$ is isomorphic to $\mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \mathbb{F})$ or $\mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \mathbb{F}) \otimes V$ depending on the parity of the wrapping number ω of $\mathbb{L}$, where the knot Floer homology $\mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \mathbb{F})$ is $\frac{1}{2}\mathbb{Z}$-graded by the Alexander grading and $V = \mathbb{F}\left\{\frac{1}{2}\right\} \oplus \mathbb{F}\left\{-\frac{1}{2}\right\}$ is a 2-dimensional vector space with Alexander grading $\pm\frac{1}{2}$. Here we are using symmetrized Alexander grading at the cost of the grading being half-integral.

It is implicit in the proof of [GW10, Theorem 2.1] that this Ozsváth-Szabó spectral sequence above respects the annular and Alexander grading, in the sense that it is decomposed into

$$\mathrm{SKh}(\mathbb{L}; k; \mathbb{F}) \Rightarrow \begin{cases} \mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \frac{k}{2}; \mathbb{F}) & \omega \text{ is even} \\ \bigoplus_{i=\pm 1} \mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \frac{k+i}{2}; \mathbb{F}) & \omega \text{ is odd} \end{cases}$$

Especially, the assumption that the top k-grading term of $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$ is $\mathbb{F}$ implies that the top Alexander grading term of $\mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); \mathbb{F})$ is also isomorphic to $\mathbb{F}$. The fiberedness detection theorem[Ni07] then implies that the preimage $p^{-1}(Z)$ of z-axis is fibered in $\Sigma(S^3, \mathbb{L})$.

When $\mathbb{L}$ is an annular knot, a special case of Goldsmith conjecture[Kir97, Problem 1.28] for branched double cover over a knot, answered affirmatively by [EL83], guarantees that $\mathbb{L}$ must be a closed braid.

When $\mathbb{L}$ has multiple connected components, we may apply the link splitting spectral sequence, Theorem 2.2.5, multiple times to observe that each component of $\mathbb{L}$ is an annular knot of which the top k-grading of the annular Khovanov homology is isomorphic to $\mathbb{F}$, hence a closed braid. This argument also shows that the top k-grading of $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$ is the sum of number of strands of the component braids. We then apply the link splitting spectral sequence again, but now over $\mathbb{C}$. Each component of $\mathbb{L}$ is a braid, hence the top k-grading of their annular Khovanov homology is the number of strands and the top k-grading term isomorphic to $\mathbb{C}$. Hence the top k-grading of $\mathrm{SKh}(\mathbb{L}; \mathbb{C})$ is at least the top k-grading of $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$. (We do not know at this point that there exist no higher k-grading terms of $\mathrm{SKh}(\mathbb{L}; \mathbb{C})$ which degenerate to 0). On the other hand, as the annular Khovanov chain complex can be defined over $\mathbb{Z}$, the universal coefficient theorem for homology, over the principal ideal domain $\mathbb{C}$ and $\mathbb{F}$ respectively, implies that

$$\dim_{\mathbb{C}} \mathrm{SKh}(\mathbb{L}; k; \mathbb{C}) \leq \dim_{\mathbb{F}} \mathrm{SKh}(\mathbb{L}; k; \mathbb{F}),$$

for each k. Hence the top k-grading of $\mathrm{SKh}(\mathbb{L}; \mathbb{C})$ and $\mathrm{SKh}(\mathbb{L}; \mathbb{F})$ coincide and the top k-grading term of $\mathrm{SKh}(\mathbb{L}; \mathbb{C})$ is isomorphic to $\mathbb{C}$, which implies by the tangle detection theorem over $\mathbb{C}$ that $\mathbb{L}$ itself is a closed braid. □

2.3 Wrapping number conjecture

As observed above, annular Khovanov homology gives an immediate lower bound for the wrapping number of the annular link. As the annular Khovanov homology is a categorification of the Kauffman bracket skein module, it is natural to ask if the Kauffman bracket skein module gives a similar lower bound for the wrapping number of the annular link. This is observed in the very beginning of the study of Kauffman bracket skein modules:

Proposition 2.3.1. *[HP11, Lemma 3]For an annular link $\mathbb{L}$ of wrapping number ω and winding number w, let* $\deg(\mathbb{L})$ *be the largest degree appearing in the Kauffman bracket skein module $\langle \mathbb{L} \rangle$ of $\mathbb{L}$. Then,*

$$w \leq \deg(\mathbb{L}) \leq \omega.$$

In the same paper, Hoste and Przytycki conjectured that $\deg(\mathbb{L})$ is exactly the wrapping number ω.

Conjecture 4. *(Wrapping number conjecture[HP11]) For an annular link $\mathbb{L}$, let* $\deg(\mathbb{L})$ *be the largest degree appearing in the Kauffman bracket skein module $\langle \mathbb{L} \rangle$ of $\mathbb{L}$. Then* $\deg(\mathbb{L})$ *is the wrapping number of $\mathbb{L}$.*

An immediate consequence of the wrapping number conjecture is the categorified version of the wrapping number conjecture:

Conjecture 5. *(Categorified wrapping number conjecture[Mar21, Conjecture 1.2]) For an annular link $\mathbb{L}$, the maximal non-zero annular grading of* $\mathrm{SKh}(\mathbb{L})$ *is the wrapping number of $\mathbb{L}$.*

Both of the conjectures above remains unsolved in full generality, but the above spectral sequences give a good lower bound for the maximal non-zero annular grading of $\mathrm{SKh}(\mathbb{L})$ in terms of a variant of Thurston norms of $\mathbb{L}$. This section is devoted to present and compare such arguments, which first was proposed by Eli Grigsby. For the detailed history of the wrapping conjecture, see [Mar21].

The first bound for the maximal non-zero annular grading of $\mathrm{SKh}(\mathbb{L})$ was the winding number, as observed in Proposition 2.3.1. The argument goes as follows:

Both the maximal non-zero annular grading and the winding number are additive under annular unions, hence we may assume without loss of generality that $\mathbb{L}$ is a connected annular link. Up to crossing changes, an annular knot is isotopic to the closure of a twist $\sigma_1\sigma_2\cdots\sigma_{n-1}$, and it is easy to check that the winding number and wrapping number coincides for such annular knots.

A stronger bound can be found using the Ozsváth-Szabó spectral sequence. As before, let $\mathbb{L}$ be an annular knot and ω be the wrapping number of $\mathbb{L}$. Note that the Thurston norm detection property of the knot Floer homology (cf. [OS08, Theorem 1.1], [Ni09, Theorem 1.1]) guarantees the nontriviality of $\mathrm{HFK}(\Sigma(A \times I, \mathbb{L}), p^{-1}(Z); k)$ for k being the Seifert genus $g(p^{-1}(Z))$ of $p^{-1}(Z)$. But the preimage of any Seifert surface of the axis $Z \subseteq S^3$ is a Seifert surface of $p^{-1}(Z)$, thus we have a bound

$$g(p^{-1}(Z)) \leq \begin{cases} 2g + \frac{n-2}{2} & \omega\text{:even}, \\ 2g + \frac{n-1}{2} & \omega\text{:odd}, \end{cases}$$

where n is the geometric intersection number of $\mathbb{L}$ and a Seifert surface of the axis Z, or equivalently a meridional surface in $A \times I$, and g is the genus of the meridional surface. Note that $n \equiv \omega \bmod 2$, thus the fraction term on the right hand side of the inequality above is always an integer. Hence, assuming the best scenario that the equality holds in the inequality above, we have a bound on the maximal non-zero annular grading of $\mathrm{SKh}(\mathbb{L})$ of the form $4g + n$ or $4g + n - 2$, depending on the parity of ω.

A similar bound can be obtained using the annular instanton homology of $\mathbb{L}$. Using the same notation as above for g and n, the adjunction inequality [XZ19, Theorem 8.2] and the spectral sequence $\mathrm{SKh}(\mathbb{L}) \Rightarrow \mathrm{AKI}(\mathbb{L})$ gives a bound $2g + n$. If we can check the equality for $g(p^{-1}(Z))$, the Ozsváth-Szabó spectral sequence will give a better bound. If there is a huge gap between $g(p^{-1}(Z))$ and a generalized Thurston norm of the form $2g + \frac{n-1}{2}$ or $2g + \frac{n-2}{2}$, the annular instanton homology will give a better bound. While such a generalized Thurston norm approach is known not to give a proof of Conjecture 5 by itself, it may be used to check Conjecture 5 for some nice annular links.

2.4 $\mathfrak{sl}_2$-action on Annular Khovanov homology

In [GLW18], Grigsby, Licata, and Wehrli defined an an action of the Lie algebra $\mathfrak{sl}_2 = \mathfrak{sl}_2(\mathbb{C})$ on the annular Khovanov homology. Here we review the basics of the representation theory of $\mathfrak{sl}_2$ and the construction in [GLW18]. Throughout this section, we are working over the complex field $\mathbb{C}$.

The Lie algebra $\mathfrak{sl}_2$ is a 3-dimensional Lie algebra with a standard basis $\{e, f, h\}$, where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

with Lie brackets

$$[e, f] = h, [h, e] = 2e, [h, f] = -2f.$$

A *representation* of the Lie algebra $\mathfrak{sl}_2$, or an $\mathfrak{sl}_2$-representation, is a vector space U together with an action $\rho : \mathfrak{sl}_2 \to \mathrm{End}(U)$ such that $\rho([x, y]) = [\rho(x), \rho(y)]$ holds for each $x, y \in \mathfrak{sl}_2$. An $\mathfrak{sl}_2$-representation is *irreducible* if there are no proper subspaces $W \leqslant U$ such that $\rho(x) \cdot W \leqslant W$ for all $x \in \mathfrak{sl}_2$.

Any finite dimensional representations over $\mathfrak{sl}_2$ can be decomposed into *weight spaces*:

$$U = \bigoplus_{\lambda \in \mathbb{Z}} U_\lambda,$$

where $U_\lambda = \{x \in U \mid h \cdot x = \lambda x\}$ is the weight space of U with weight λ. The theorem of highest weight implies that irreducible representations of $\mathfrak{sl}_2$ are classified by its highest weight, hence for each integer $n \geq 0$, there exists a unique irreducible representation $V_{(n)}$ with highest weight n. Using the Lie bracket relations, the action of e(resp. f) increases the weight by 2(resp. decreases the weight by 2). Hence the irreducible representation $V_{(n)}$ is spanned by

$$\{v, f \cdot v, \cdots, f^n \cdot v\}$$

for a highest weight vector $v \in V_{(n)}$. In particular, the dimension of each weight space of an irreducible representation is 1.

As $\mathfrak{sl}_2$ is (semi)simple, any finite dimensional representation is isomorphic to a direct sum of irreducible representations.

Recall that each circles in a complete resolution of a link diagram is associated to a 2-dimensional vector space, $\mathbb{C}\{1\} \oplus \mathbb{C}\{-1\}$ or $\mathbb{C}^2\{0\}$ depending on its winding number. Grigsby, Licata, and Wehrli observed in [GLW18] that the two vector spaces are isomorphic to the (underlying vector space of) $\mathfrak{sl}_2$-representations $V_{(1)}$ and $V_{(0)}^2$, respectively, and the annular Khovanov differential is an intertwiner between (the tensor products of) such representations. As a result, the action of $\mathfrak{sl}_2$ on the annular Khovanov chain complex, defined by extending the above action on annular circles via tensor products, induces an $\mathfrak{sl}_2$-action on its homology.

A direct consequence of the $\mathfrak{sl}_2$-action on the annular Khovanov homology is the following:

Theorem 2.4.1. *[GLW18, Corollary 1] For $k_1, k_2 \in \mathbb{Z}$ such that $k_1 \equiv k_2 \mod 2$ and $|k_1| \geq |k_2|$,*

$$\dim_{\mathbb{C}} \mathrm{SKh}(\mathbb{L}; k_1) \leq \dim_{\mathbb{C}} \mathrm{SKh}(\mathbb{L}; k_2).$$

In other words, the sequence $\{\dim_{\mathbb{C}} \mathrm{SKh}(\mathbb{L}; k)\}_k$ is unimodal.

Remark 2.4.2. We use the terminology *unimodal* instead of *trapezoidal* chosen by [GLW18], which may cause a confusion with the Fox trapezoidal conjecture[Fox62].

Chapter 3

PROOF OF THE MAIN RESULT

In this chapter, we prove the main results of this thesis, Theorem 1 and Theorem 2.

Proof of Theorem 2. We first show that it suffices to prove the theorem for connected braid closures. Suppose that $\widehat{\sigma}$ is a union of two closed braids $\widehat{\sigma_1} \cup \widehat{\sigma_2}$. Then the splitting spectral sequence applied on $\widehat{\sigma} = \widehat{\sigma_1} \cup \widehat{\sigma_2}$ implies that

$$\begin{aligned}\dim \mathrm{SKh}(\widehat{\sigma}; n-2) &\geq \dim \mathrm{SKh}(\widehat{\sigma_1} \sqcup \widehat{\sigma_2}; n-2)\\ &= \dim \mathrm{SKh}(\widehat{\sigma_1}; n_1 - 2) + \dim \mathrm{SKh}(\widehat{\sigma_2}; n_2 - 2)\end{aligned}$$

when n_i is the number of strands of $\sigma_i, i = 1, 2$. Note that the equality above comes from Proposition 2.2.4. Assuming Theorem 2 for connected braid closures, the right-hand side of the above inequality is greater than 2, hence the proof follows from the induction on the number of connected components of $\widehat{\sigma}$.

Now assume that $\widehat{\sigma}$ is connected (i.e. when it is an annular knot).

Note that the closed braid $\widehat{\sigma}$ is connected if and only if the permutation determined as the image of the braid word representing σ under the natural projection $B_n \to S_n$ is an n-cycle, where S_n is the permutation group of n elements. In fact the number of connected components of the closure is the number of orbits of the permutation. Thus we may conjugate σ so that the resulting permutation is the n-cycle $(1, 2, ..., n-1, n)$. Then the fact that the projection $B_n \to S_n$ is given by imposing the relations $\sigma_i^2 = id$ implies that $\widehat{\sigma}$ transforms to the closure of the positive $\frac{1}{n}$-twist $\beta_n := \sigma_1 \sigma_2 \cdots \sigma_{n-1}$ after a sequence of crossing changes.

But two annular links related by a crossing change share the same set of generators of the Khovanov chain complex; only the differential maps differ. Hence the parity of the annular Khovanov homology at the next-to-top annular grading does not change under a crossing change.

And it is easy to show by a direct computation that $\mathrm{SKh}(\widehat{\beta_n}; n-2)$ is even. For example, taking the 0-(resp. 1-)resolution at the innermost crossing of $\widehat{\beta_n}$ results in the annular link $\widehat{\beta_{n-1} \times 1}$(resp. $\widehat{\beta_{n-2}}$), hence as a k-graded vector space we have an isomorphism

$$\mathrm{CKh}(\widehat{\beta_n}) \cong \mathrm{CKh}(\widehat{\beta_{n-1} \times 1}) \oplus \mathrm{CKh}(\widehat{\beta_{n-2}}).$$

Then we measure the contribution of each summand to $\mathrm{CKh}(\widehat{\beta_n}; n-2)$.

$\widehat{\beta_{n-1} \times 1}$ is the annular union of $\widehat{\beta_{n-1}}$ and the nontrivial annular unknot, hence $\mathrm{CKh}(\widehat{\beta_{n-1} \times 1})$ is isomorphic to two copies of $\mathrm{CKh}(\widehat{\beta_{n-1}})$ with k-gradings shifted by ± 1. Hence the only contribution of this summand to k-grading $n-2$ component is from the topmost component of $\mathrm{CKh}(\widehat{\beta_{n-1}})$ with k-grading shift -1 and the next-to-top component of $\mathrm{CKh}(\widehat{\beta_{n-1}})$ with k-grading shift 1.

And the topmost k-grading of $\mathrm{CKh}(\widehat{\beta_{n-2}})$ is $n-2$, thus the contribution from $\mathrm{CKh}(\widehat{\beta_{n-2}})$ is from the top k-grading. As an annular closure, the topmost k-grading component of $\mathrm{SKh}(\widehat{\beta_{n-1}})$(resp. $\mathrm{SKh}(\widehat{\beta_{n-2}})$ has dimension 1. Hence the contributions from both of the two topmost k-grading components are odd, thus the parity of $\dim \mathrm{CKh}(\widehat{\beta_n}; n-2)$ is the same as that of $\mathrm{CKh}(\widehat{\beta_{n-1}}; n-3)$. Then the claim follows from induction on n and a simple computation that $\mathrm{SKh}(\widehat{\beta_1}) \cong V_{(1)}$.

As the annular Khovanov homology is unimodal with respect to k-grading, evenness of $\dim \mathrm{SKh}(\widehat{\sigma}, n-2)$ implies in particular that $\dim \mathrm{SKh}(\widehat{\sigma}, n-2) \geq 2$, completing the proof for the connected case. □

Remark 3.0.1. [GLW18] found the exact formula for the annular Khovanov homology of $\widehat{\beta_n}$. Using the notations as above (and disregarding the quantum grading), the formula is

$$\mathrm{SKh}(\widehat{\beta_n}) \cong V_{(n)} \oplus V_{(n-2)}.$$

Especially, the next-to-top k-grading summand of $\mathrm{SKh}(\widehat{\beta_n})$ has dimension 2.

Remark 3.0.2. Theorem 2 remains true over fields of arbitrary characteristic, using the universal coefficient theorem argument similar to Proposition 2.2.8. Indeed, for a field $\mathbb{F}$ of positive characteristic,

$$\mathrm{SKh}(\widehat{\sigma}; k; \mathbb{F}) \geq \mathrm{SKh}(\widehat{\sigma}; k; \mathbb{C})$$

for any integer k, and the proof in [GN14] works verbatim to guarantee that the top k-grading term of $\mathrm{SKh}(\widehat{\sigma}; \mathbb{F})$ is isomorphic to $\mathbb{F}$. In characteristic 2, another proof using Ozsváth-Szabó spectral sequence and Baldwin-Vela-Vick's theorem [BV18, Theorem 1.1] on nontriviality of the next-to-top Alexander grading term of the knot Floer homology of fibered knots together with the fiberedness detection theorem is possible.

Combined with the braid detection theorem, Theorem 2.2.6, we can now prove Theorem 1.

Proof of Theorem 1. Let $C = \left\{(z,t) \in A \times I \mid |z| = \frac{3}{2}, t = \frac{1}{2}\right\}$ be the core of the thickened annulus. It is clear from the definition that $\mathrm{SKh}(C) \cong V_{(1)}$ (which is irreducible), hence it suffices to show that an annular link $\mathbb{L}$ is necessarily isotopic to C if $\mathrm{SKh}(\mathbb{L})$ is irreducible.

Note first that irreducibility of the annular Khovanov homology implies that $\dim \mathrm{SKh}(\mathbb{L}; k) = 0$ or $1, \forall k$. In particular, the topmost k-grading term of $\mathrm{SKh}(\mathbb{L})$ is of dimension 1, hence $\mathbb{L}$ must be a closed braid by Theorem 2.2.6.

If $\mathbb{L}$ is isotopic to a closed braid of $n \geq 2$ strands, we may apply Theorem 2 which leads to a contradiction that there exists k such that $\mathrm{SKh}(\mathbb{L}; k) > 1$. Hence $\mathbb{L}$ is isotopic to a closed braid of a single strand, which is simply C. □

Chapter 4

TRAPEZOIDAL AND LOG-CONCAVITY CONJECTURES

While Theorem 2 is true, the strict inequality $\mathrm{SKh}(\widehat{\sigma}; k) < \mathrm{SKh}(\widehat{\sigma}; k-2)$ does not hold in general if k is less than the number of strands of σ. For example, the annular Khovanov homology of the $\frac{1}{n}$-th twist $\widehat{\beta_n}$ is $\mathrm{SKh}(\widehat{\beta_n}) \cong V_{(n)} \oplus V_{(n-2)}$, hence the next-to-top and third-to-top k-grading term of $\mathrm{SKh}(\widehat{\beta_n})$ are both of dimension 2 if n is greater than 4.

Probably the next interesting question is whether the dimension of the annular Khovanov homology with respect to the k-grading continues to be flat once it stops to be strictly larger than its precursor. This is the content of Conjecture 3. A symmetric sequence with the property as above is called *trapezoidal*:

Definition 4.0.1. A sequence $a_{-n}, a_{-n+2}, \cdots, a_{n-2}, a_n$ of nonnegative integers is *symmetric* if $a_i = a_{-i}$ holds for all i. A symmetric sequence is *unimodal* if $a_i \le a_{i-2}$ for all $i > 0$. A symmetric sequence $a_{-n}, a_{-n+2}, \cdots, a_{n-2}, a_n$ is *trapezoidal* if there exists $0 < m \le n$ such that $a_i < a_{i-2}$ if $m < i \le n$ and $a_i = a_{i-2}$ if $0 < i \le m$.

Hence Conjecture 3 claims that the dimension of the annular Khovanov homology of a closed braid forms a trapezoidal sequence.

A general strategy to prove that a sequence is trapezoidal is to show log-concavity.

Definition 4.0.2. A symmetric sequence is *log-concave* if $a_{i+2}a_{i-2} \le a_i^2$ for all $i < n$.

Indeed, for a unimodal sequence, log-concavity implies trapezoidality. As the dimension of annular Khovanov homology is unimodal with respect to k-grading, the following conjecture is stronger than Conjecture 3.

Conjecture 6. *The sequence* $\{\dim \mathrm{SKh}(\widehat{\sigma}; k)\}_{-n \le k \le n}^{k \equiv n \bmod 2}$ *is log-concave.*

Using Morisson's mathematica script[Mor19], we tested the validity of Conjecture 3 and Conjecture 6. In fact, we computed the annular Khovanov homology of slightly more general class of annular links, defined as follows.

Definition 4.0.3. An annular link $\mathbb{L}$ is *minimally wrapping* if the wrapping number and winding number of $\mathbb{L}$ coincides up to changing the orientations of the connected components of $\mathbb{L}$.

Using a similar argument as in the standard proof of Alexander's theorem for links which unwinds the part that winds a candidate braid axis in reverse direction, minimally wrapping links can be equivalently formulated as the annular links admitting a braid closure diagram but the center of the annulus is allowed to lie on the regions between parallel strands closing the braid, not necessarily the braid axis. As a result, we can use the same script as in [Mor19] to compute the annular Khovanov homology of minimally wrapping links only by modifying the k-grading function.

The setup is as follows: We restrict to the braid of at most 9 strands and at most 9 crossings, due to the computational complexity of annular Khovanov homology. Even with the restriction on the number of strands and crossings, there still exists an enormous number of braids to consider; mere count of the braid words in B_9 of length 9 is approximately 300 millions. Hence we take a random approach rather than trying a full computation for such braids and compute the annular Khovanov homology of a random braid word. In case of closed braids, a counterexample to Conjecture 3 or Conjecture 6 can only occur when the number of strands is > 5 due to Theorem 2, hence a random braid word is chosen under this constraint. In case of minimally wrapping link, we have no constraints on the number of strands but additionally choose a random integer which corresponds to the number of strands between the braid axis and the center of the annulus. In both cases, due to Proposition 2.2.4, we only consider the braids which cannot be written as an annular union of

Currently, more than 10000 closed braids of 6 to 7 strands, more than 1000 closed braids of 8 and 9 strands, and about 100 minimally wrapping links have been tested, and every samples tested passed both Conjecture 3 and Conjecture 6. The code and data is available at [Kim21].

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