EIGENVALUE STRUCTURE IN PRIMITIVE LINEAR GROUPS

Thesis by

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to Mom, Dad, and Mike

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ABSTRACT

One approach to studying finite linear groups over the complex numbers is to classify those groups with an element possessing a certain eigenvalue structure. Let G be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n. Assume $g \in G$ such that X(g) has eigenvalues $\epsilon, \bar{\epsilon}, 1, 1, \ldots, 1$ where ϵ is a primitive r^{th} root of unity. H. F. Blichfeldt and J. H. Lindsey have classified G whenever $r \geq 5$. In this thesis $r = 3$ and 4 are handled. The main results are:

<u>Theorem 1</u>: Let G be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n. Assume there is an element $g \in G$ such that X(g) has eigenvalues $i, -i, 1, 1, \ldots, 1$. Then $n \leq 4$ and G is a known group.

<u>Theorem 2</u>: Let G be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n. Assume there is an element $g \in G$ such that X(g) has eigenvalues $\omega, \bar{\omega}, 1, 1, \ldots, 1$ where $\omega = e^{2\pi i/3}$. Let N be the subgroup of G generated by all such elements. Then either

1. $N \cong A_{n+1}$ and $G/Z(G) \cong A_{n+1}$ or S_{n+1}.
2. $n = 8$, $N = N'$, Z(N) has order 2, and $N/Z(N) \cong O_8^+(2)$; $G/Z(G)$ is a subgroup of the automorphism group of $O_8^+(2)$.
3. $n \leq 7$ and G is a known group.

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INTRODUCTION

Finite linear groups of degree n over the complex numbers can many times be classified according to the eigenvalue structure of an element in the group. As finite linear groups are subdirect products of irreducible linear groups, it is convenient to restrict the study to irreducible groups. For example, Mitchell [21] classified all irreducible linear groups containing an element with eigenvalues $\alpha, \beta, \beta, \ldots, \beta$ where $\alpha \neq \beta$. The next natural step is to consider the case where there exists an element of the group which has an eigenspace of dimension n - 2 corresponding to one of its eigenvalues. In this thesis, we will examine groups which contain an element with eigenvalues $\epsilon\alpha, \bar{\epsilon}\alpha, \alpha, \alpha, \ldots, \alpha$ where $\epsilon \neq 1$.

Let G be a finite group with a faithful, irreducible complex representation X of degree n over the vector space V. The representation X is said to be primitive if there does not exist a set of $m \geq 2$ proper, nontrivial subspaces V_i with $V = V_1 \oplus V_2 \oplus \ldots \oplus V_m$ such that X(g) permutes $\{V_i\}$ for all $g \in G$. An irreducible representation which is not primitive is similar to one induced from a representation of some proper subgroup [5, Theorem 50.2]. Thus it is not very restrictive to consider only primitive representations. The representation X is unimodular if X(g) has determinant 1 for all $g \in G$. Any irreducible representation is projectively equivalent (i.e., as a collineation group) to a unimodular representation. Therefore by classifying unimodular, irreducible, primitive groups containing an element with eigenvalues

$\epsilon, \bar{\epsilon}, 1, 1, \ldots, 1$, we are classifying, up to projective equivalence, those irreducible primitive groups containing an element with eigenvalues $\epsilon\alpha, \bar{\epsilon}\alpha, \alpha, \ldots, \alpha$. Some work has been done on this problem. If X is irreducible and primitive, and if ϵ is a primitive r^{th} root of unity, the results are known for $r \geqslant 5$. A special case of a theorem in Blichfeldt [2, p. 96] proves that if $r \geqslant 6$, then $n \leqslant 2$, $r = 6$, 8, or 10, and the groups are known. If $r = 5$, Lindsey [14, Lemma 2] proves $n \leqslant 4$ and the groups are known. In this thesis the cases $r = 3$ and 4 are handled. The results are:

Theorem 1: Let G be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n. Assume there is an element $g \in G$ such that X(g) has eigenvalues $i, -i, 1, 1, \ldots, 1$. Then $n \leqslant 4$ and G is a known group.

Theorem 2: Let G be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n. Assume there is an element $g \in G$ such that X(g) has eigenvalues $\omega, \bar{\omega}, 1, 1, \ldots, 1$ where $\omega = e^{2\pi i/3}$. Let N be the subgroup of G generated by all such elements. Then either

1. $N \cong A_{n+1}$ and $G/Z(G) \cong A_{n+1}$ or S_{n+1}.
2. $n = 8$, $N = N'$, Z(N) has order 2, and $N/Z(N) \cong O_8^+(2)$; $G/Z(G)$ is a subgroup of the automorphism group of $O_8^+(2)$.
3. $n \leqslant 7$ and G is a known group.

Note that all primitive linear groups of degree 7 or less are known (see Blichfeldt [2], Brauer [3], Lindsey [14, 16, 17], and Wales [27, 28]).

Some authors in recent years have become interested in classifying quasiprimitive groups rather than primitive ones. A representation X of G is quasiprimitive if for every normal subgroup N of G, $X|N$ splits into isomorphic factors. By Clifford's theorem [5], a primitive representation is quasiprimitive. By examining the results of Chapter I, it is easy to see that a quasiprimitive irreducible representation X of G, where X(g) has eigenvalues $i, -i, 1, 1, \ldots, 1$ or $\omega, \bar{\omega}, 1, 1, \ldots, 1$ for some $g \in G$, is indeed primitive. Therefore Theorems 1 and 2 are true if X is quasiprimitive rather than primitive.

Chapters I and II give preliminary material used in the proofs of Theorems 1 and 2. The results help characterize the possible subgroup structure of G; in particular under certain conditions primitive subgroups of codimension 1 or 2 can be constructed. These results are useful for induction purposes to prove the main results. In Chapter III, Theorem 1 is proved. This proof is much easier than that of Theorem 2 because no primitive group of degree 5, 6, or 7 contains an element with eigenvalues $i, -i, 1, \ldots, 1$. In Chapter IV, the alternating groups of Theorem 2 are obtained by exhibiting appropriate generators and relations. The proof of Theorem 2 is completed in Chapter V. The powerful results of Aschbacher-Hall [1] and Stellmacher [26] on groups generated by elements of order 3 are used. The following corollary is also proved:

Corollary: If G is a finite simple group containing A_{n-1} with a nontrivial representation of degree $n \geq 10$, then $G \cong A_{n-1}, A_n$, or A_{n+1}.

The following notation is adopted. If H is a subgroup of a finite group K, $N_K(H)$ is the normalizer in K of H, $C_K(H)$ is the centralizer in K of H, Z(H) is the center of H, and H′ is the derived group of H. If p is a prime, $O_p(H)$ denotes the largest normal p-subgroup of H and $O_\infty(H)$, the largest normal solvable subgroup of H. If $x, y \in H$, $y^{-1}xy$ is denoted by x^y. Also $m 1_H$ is the direct sum of m copies of the trivial representation of H. If H and L are subgroups of K, $[H, L] = \langle h^{-1}\ell^{-1}h\ell \mid h \in H, \ell \in L\rangle$. The symbol Z_k denotes the cyclic group of order k. The order of H is denoted $|H|$; also $H \lhd K$ means H is normal in K.

Let H be a finite group with a faithful, irreducible, primitive complex representation Y of degree m. The term Blichfeldt refers to the result [2, p.96] that H\Z(H) does not contain an element h where Y(h) has an eigenvalue ϵ such that all other eigenvalues are at most 60° away from ϵ. The term Mitchell will refer to two results in [21]: If $h \in H$ and Y(h) has eigenvalues $\alpha, \beta, \ldots, \beta$ such that $h^2 \notin Z(H)$ but $h^4 \in Z(H)$, then $m \leq 2$. If $h \in H$ and Y(h) has eigenvalues $\alpha, \beta, \ldots, \beta$ such that $h \notin Z(H)$ but $h^3 \in Z(H)$, then $m \leq 4$. In the latter case if $m = 3$, H/Z(H) is a split extension of $Z_3 \times Z_3$ by $SL_2(3)$ and if $m = 4$, $H/Z(H) \cong O_5(3)$. Which result the term Mitchell refers to will be clear from the context. If p is a prime, a p-element of H is an element in H of order a power of p. A special 4-element of H is an element $h \in H$ such that Y(h) has eigenvalues i, -i, 1, 1, . . ., 1. A special 3-element of H is an element $h \in H$ such that Y(h) has eigenvalues $\omega, \bar{\omega}, 1, 1, \ldots, 1$ where $\omega = e^{2\pi i/3}$. The group G will be a finite group with a faithful, irreducible, primitive, unimodular complex representation X of degree n over the vector space V. If $v_1, \ldots, v_k \in V$, $\langle v_1, \ldots, v_k\rangle$ denotes the subspace of V generated by $v_1, \ldots, v_k$.

When working on this problem, it was often necessary to consult character tables of various groups. Some of these tables are found in [7], [8], [13], [16], [17], [19], [20], [29], and [30]. General references for group theory and representation theory are [2], [5], [11], [19], [23], and [24].

CHAPTER I

THE CONSTITUENTS OF X WHEN RESTRICTED TO SUBGROUPS

In this chapter, we give properties of the constituents of X when restricted to subgroups of G generated by special 3-elements or special 4-elements. In particular these constituents are shown to be either primitive or monomial. The possible monomial groups are investigated more closely, and conditions are given which guarantee the uniqueness up to scaling and ordering of the basis. These results are useful in constructing large subgroups in later chapters.

<u>Lemma 1.1</u>: Let $N \triangleleft G$ and assume N contains a special 3-element or special 4-element h. Then $X|N$ is irreducible.

<u>Proof</u>: By Clifford's theorem [5], $X|N = X_1 \oplus \ldots \oplus X_t$ where all the X_i's are equivalent irreducible representations of N. In particular the trace of $X_i(h)$ is the trace of $X_1(h)$. By the eigenvalue structure of $X(h)$, $t > 1$ is impossible.

<u>Lemma 1.2</u>: Let H be a subgroup of G generated either by special 3-elements or special 4-elements. Assume $X|H = X_1 \oplus X_2$ where X_1 is irreducible. Then either X_1 is monomial or X_1 is primitive.

<u>Proof</u>: Let $H = \langle h_1, \ldots, h_r \rangle$ where h_i are special 3-elements or special 4-elements. Let X_1 have degree m and act on the subspace V^*. Assume the result is false. Then there exist $\ell > 1$ subspaces

$V_1, \ldots, V_\ell$ of V^* all of dimension $k > 1$ with $m = \ell k$ such that the $X_1(h_i)$ permute $\{V_1, \ldots, V_\ell\}$. We may assume $m \geq 4$. Assume $X_1(h_i)$ fixes exactly t subspaces, say $V_1, \ldots, V_t$ by renumbering if necessary. Let $\chi_j(h_i)$ be the trace of $X_1|_{V_j}(h_i)$ for $j = 1, \ldots, t$. Then

$$m - 3 \leq |\text{trace } X_1(h_i)| = |\sum_{j=1}^{t} \chi_j(h_i)| \leq \sum_{j=1}^{t} |\chi_j(h_i)| \leq kt \quad .$$

So $k\ell - 3 \leq kt$ and hence $t \geq \ell - 1$ as $k > 1$. As $X_1(h_i)$ fixes $\ell - 1$ subspaces, it must fix all ℓ subspaces; so X_1 is reducible, a contradiction.

Lemma 1.1 implies in particular that if V_1 is a proper subspace of V, there exists a special 3-element or special 4-element $g \in G$ such that X(g) does not leave V_1 invariant. It also implies that the subgroup generated by special 3-elements or special 4-elements could not be abelian. These facts along with Lemmas 1.1 and 1.2 are used without reference throughout this paper.

We now want to look more closely at what happens when the hypothesis of Lemma 1.2 holds. In particular we want to investigate H when X_1 is monomial and X_2 is of a special nature.

Lemma 1.3: Let H be a subgroup of G generated by special 4-elements such that $X|H = X_1 \oplus (n-r)1_H$ where X_1 is irreducible of degree $r \geq 3$. Suppose X_1 acts on V_1 and $X|H$ is monomial in the basis $v_1, \ldots, v_n$ of V. Then

1. There exist special 4-elements $h_1, \ldots, h_{r-1} \in H$ such that when $v_1, \ldots, v_n$ are properly scaled and ordered

$$\begin{cases} v_i X(h_i) = -v_{i+1} \\ v_{i+1} X(h_i) = v_i \\ v_\ell X(h_i) = v_\ell \text{ for } \ell \notin \{i, i+1\} \end{cases} .$$

Also $\langle v_1, \ldots, v_r \rangle = V_1$ and $(n-r)1_H$ acts on $\langle v_{r+1}, \ldots, v_n \rangle$.

2. $X | \langle h_1, \ldots, h_j \rangle$ is irreducible on $\langle v_1, \ldots, v_{j+1} \rangle$ for $j \geq 2$.
3. If $X|H$ is monomial in a basis $v_1^*, \ldots, v_n^*$, by ordering and scaling $v_1^*, \ldots, v_n^*$ correctly, $v_1 = v_1^*, \ldots, v_r = v_r^*$ and $\langle v_{r+1}^*, \ldots, v_n^* \rangle = \langle v_{r+1}, \ldots, v_n \rangle$.

<u>Proof</u>: As X_1 is irreducible there is a special 4-element $h_1 \in H$ such that $X(h_1)$ is not diagonal. So for some $j \neq k$, we have

$$\begin{cases} v_j X(h_1) = -\delta v_k \\ v_k X(h_1) = \delta^{-1} v_j \\ v_\ell X(h_1) = v_\ell \text{ for } \ell \notin \{j, k\} \end{cases} .$$

By ordering $v_1, \ldots, v_n$ correctly, we may assume $j = 1$, $k = 2$. Replacing v_2 by δv_2, we may assume $\delta = 1$. So $X(h_1)$ has the desired form. Note that if h is any special 4-element of H with $j \neq k$ and

$$\begin{cases} v_j X(h) = -\epsilon v_k \\ v_k X(h) = \epsilon^{-1} v_j \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{j, k\} \end{cases} ,$$

then the eigenspaces of $X(h)$ corresponding to i and $-i$ span $\langle v_j, v_k \rangle$. Thus $\langle v_j, v_k \rangle \subseteq V_1$, the unique subspace of V on which X_1 acts; in particular $\langle v_1, v_2 \rangle \subseteq V_1$. Assume we have constructed $h_1, \ldots, h_i$ where $i < r-1$. So $\langle v_1, v_2 \rangle, \langle v_2, v_3 \rangle, \ldots, \langle v_i, v_{i+1} \rangle \subseteq V_1$ and as X_1 is irreducible there is a special 4-element $h^*_{i+1} \in H$ such that $X(h^*_{i+1})$ does not leave $\langle v_1, \ldots, v_{i+1}\rangle$ invariant. So there exist j, k with $1 \leq j \leq i+1$ and $i+2 \leq k \leq n$ such that

$$\begin{cases} v_j X(h^*_{i+1}) = -\delta v_k \\ v_k X(h^*_{i+1}) = \delta^{-1} v_j \\ v_\ell X(h^*_{i+1}) = v_\ell \text{ for } \ell \notin \{j, k\} \end{cases} .$$

As $\langle X(h_1), \ldots, X(h_i)\rangle$ acts transitively on $\langle v_1\rangle, \ldots, \langle v_{i+1}\rangle$, we may choose $h \in \langle h_1, \ldots, h_i\rangle$ such that $v_j X(h) = \lambda v_{i+1}$. Letting $h_{i+1} = h^{-1} h^*_{i+1} h$,

$$\begin{cases} v_{i+1} X(h_{i+1}) = -\lambda^{-1}\delta v_k \\ v_k X(h_{i+1}) = \lambda\delta^{-1} v_{i+1} \\ v_\ell X(h_{i+1}) = v_\ell \text{ for } \ell \notin \{i+1, k\} \end{cases} .$$

Replacing v_k by $\lambda^{-1}\delta v_k$ and rearranging $v_{i+2}, \ldots, v_n$, we may assume $\lambda^{-1}\delta = 1$ and $k = i+2$. We have inductively constructed $h_1, \ldots, h_{r-1}$; clearly $\langle v_1, \ldots, v_r\rangle = V_1$ as $\langle v_i, v_{i+1}\rangle \subseteq V_1$. As $(n-r)1_H$ acts on a subspace of $\langle v_{r+1}, \ldots, v_n\rangle$ and $\langle v_{r+1}, \ldots, v_n\rangle$ has dimension $n-r$, $(n-r)1_H$ acts on $\langle v_{r+1}, \ldots, v_n\rangle$, proving 1.

The second assertion is proved by induction. Let $j = 2$. As h_1 and h_2 do not commute, if $X|\langle h_1, h_2\rangle$ is not irreducible on $\langle v_1, v_2, v_3\rangle$, it must contain an eigenvector common to both $X(h_1)$ and $X(h_2)$. This is not true because on $\langle v_1, v_2, v_3\rangle$, $X(h_1)$ has eigenspaces $\langle v_1 + iv_2\rangle$, $\langle v_1 - iv_2\rangle$, $\langle v_3\rangle$ corresponding to $i, -i, 1$, respectively, while $X(h_2)$ has eigenspaces $\langle v_2 + iv_3\rangle$, $\langle v_2 - iv_3\rangle$, $\langle v_1\rangle$ corresponding to $i, -i, 1$. Thus $X|\langle h_1, h_2\rangle$ is irreducible on $\langle v_1, v_2, v_3\rangle$. Assume $X|\langle h_1, \ldots, h_{j-1}\rangle$ is irreducible on $\langle v_1, \ldots, v_j \rangle$ for $j-1 \geq 2$. As $X | \langle h_1, \ldots, h_j \rangle$ is invariant on $\langle v_1, \ldots, v_{j+1} \rangle$ but not on $\langle v_{j+1} \rangle$, it is irreducible on $\langle v_1, \ldots, v_{j+1} \rangle$, proving 2.

To prove 3, we examine $X(h_1), \ldots, X(h_{r-1})$. As a permutation on $\langle v_1^* \rangle, \ldots, \langle v_n^* \rangle$, each $X(h_i)$ acts trivially or as a transposition. As $X | \langle h_1, h_2 \rangle$ has an irreducible constituent of degree 3, the only possibility is that there exist i, j, k distinct with

$$\begin{cases} v_i^* X(h_1) = -\delta v_j^* \\ v_j^* X(h_1) = \delta^{-1} v_i^* \\ v_\ell^* X(h_1) = v_\ell^* \text{ for } \ell \notin \{i, j\} \end{cases} \quad \text{and} \quad \begin{cases} v_j^* X(h_2) = -\epsilon v_k^* \\ v_k^* X(h_2) = \epsilon^{-1} v_j^* \\ v_\ell^* X(h_2) = v_\ell^* \text{ for } \ell \notin \{j, k\} \end{cases}.$$

Replacing v_j^* by δv_j^* and v_k^* by $\epsilon\delta v_k^*$, we may assume $\delta = \epsilon = 1$. By reordering $v_1^*, \ldots, v_n^*$, we may assume $i = 1$, $j = 2$, $k = 3$. So assume under suitable ordering and scaling of $v_1^*, \ldots, v_n^*$ that we have

$$(1) \qquad \begin{cases} v_i^* X(h_i) = -v_{i+1}^* \\ v_{i+1}^* X(h_i) = v_i^* \\ v_\ell^* X(h_i) = v_\ell^* \text{ for } \ell \notin \{i, i+1\} \end{cases}$$

for $1 \leq i \leq j$ and some $j \geq 2$. Clearly $\langle v_1, \ldots, v_{j+1} \rangle = \langle v_1^*, \ldots, v_{j+1}^* \rangle$, as $X | \langle h_1, \ldots, h_j \rangle$ is irreducible on $\langle v_1, \ldots, v_{j+1} \rangle$ and invariant on $\langle v_1^*, \ldots, v_{j+1}^* \rangle$. As $X(h_{j+1})$ does not leave $\langle v_1, \ldots, v_{j+1} \rangle$ invariant, $X(h_{j+1})$ must interchange $\langle v_i^* \rangle$, $\langle v_k^* \rangle$ for some $1 \leq i \leq j+1$ and $j+2 \leq k \leq n$. By renumbering $v_{j+2}^*, \ldots, v_n^*$, we may assume $k = j+2$. As h_{j+1} commutes with $h_1, \ldots, h_{j-1}$, the only possibility is that $i = j+1$. So

$$\begin{cases} v_{j+1}^{*}X(h_{j+1}) = -\delta v_{j+2}^{*} \\ v_{j+2}^{*}X(h_{j+1}) = \delta^{-1} v_{j+1}^{*} \\ v_{\ell}^{*}X(h_{j+1}) = v_{\ell}^{*} \text{ for } \ell \notin \{j+1, j+2\} \end{cases} .$$

By replacing v_{j+2}^{*} by δv_{j+2}^{*}, which doesn't affect the form of $X(h_1), \ldots, X(h_j)$, we may assume $\delta = 1$. By suitably ordering and scaling $v_1^{*}, \ldots, v_n^{*}$, $X(h_i)$ has the form (1) for $1 \leqslant i \leqslant r-1$. Clearly $V_1 = \langle v_1^{*}, \ldots, v_r^{*} \rangle = \langle v_1, \ldots, v_r \rangle$ and $\langle v_{r+1}^{*}, \ldots, v_n^{*} \rangle = \langle v_{r+1}, \ldots, v_n \rangle$. Let $v_i S = v_i^{*}$; clearly S commutes with each h_i. As X_1 is irreducible, S must act as a scalar on V_1. Hence by correctly scaling $v_1^{*}, \ldots, v_r^{*}$, we have 3.

Lemma 1.4: Let H be a subgroup of G generated by special 3-elements such that $X|H = X_1 \oplus \xi \oplus (n-m-1)1_H$ where X_1 is irreducible of degree $m \geqslant 5$. Assume X_1 is monomial on a subspace V_1 of V, ξ is linear and acts on $\langle v \rangle$, and $(n-m-1)1_H$ acts on V_2. Let V_1 have a basis $v_1, \ldots, v_m$ such that X_1 is monomial in that basis. Then there exist special 3-elements $h_1, \ldots, h_{m-2} \in H$ such that by properly scaling and ordering $v_1, \ldots, v_m$, we have

$$\begin{cases} v_i X(h_i) = v_{i+1} \\ v_{i+1} X(h_i) = v_{i+2} \\ v_{i+2} X(h_i) = v_i \\ v_{\ell} X(h_i) = v_{\ell} \text{ for } \ell \notin \{i, i+1, i+2\} \end{cases} .$$

Also $\xi(h_i) = 1$ for $1 \leqslant i \leqslant m-2$. The elements $h \in H$ with $(X_1 \oplus \xi)(h)$ diagonal in the basis $v_1, \ldots, v_m$, v form a normal subgroup F of H with $H = FA$ where $A = \langle h_1, \ldots, h_{m-2} \rangle$. Also $X_1|F$ splits into m distinct linear representations all different from ξ. Furthermore if $v_1^*, \ldots, v_n^*$ is a basis for V in which $X|H$ is monomial and if $v_1^*, \ldots, v_n^*$ are properly scaled and ordered, $v_1 = v_1^*, \ldots, v_m = v_m^*$. If $\xi = 1_H$, $\langle v_{m+1}^*, \ldots, v_n^* \rangle = \langle v \rangle \oplus V_2$. If $\xi \neq 1_H$, by ordering and scaling $v_{m+1}^*, \ldots, v_n^*$ correctly, $v = v_{m+1}^*$ and $V_2 = \langle v_{m+2}^*, \ldots, v_n^* \rangle$.

Proof: As H is not abelian, there is a special 3-element $h_1 \in H$ such that $X_1(h_1)$ is not diagonal. In particular, we must have i, j, k all distinct such that

$$\begin{cases} v_i X(h_1) = a v_j & \\ v_j X(h_1) = b v_k & abc = 1 \\ v_k X(h_1) = c v_i & \\ v_\ell X(h_1) = v_\ell \text{ for } \ell \notin \{i,j,k\} & \end{cases} .$$

By reordering $v_1, \ldots, v_n$ we may assume $i = 1$, $j = 2$, $k = 3$. Replacing v_2 by av_2 and v_3 by abv_3, we may assume $a = b = c = 1$ and we have h_1.

Assume we have constructed $h_1, \ldots, h_i$ for some $i \leq m-4$. Then there is a special 3-element $h \in H$ which does not leave $\langle v_1, \ldots, v_{i+2} \rangle$ invariant, as X_1 is irreducible. So in particular there exist r, s, t all distinct such that

$$\begin{cases} v_r X(h) = a v_s & \\ v_s X(h) = b v_t & abc = 1 \\ v_t X(h) = c v_r & \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{r,s,t\} & \end{cases}$$

where $1 \leq r \leq i+2$ and at least one of s or t is greater than i+2.

Replacing h by h^{-1} if necessary, assume $i + 3 \leq t \leq m$.

Case a) $i + 3 \leq s \leq m$:

By renumbering $v_{i+3}, \ldots, v_m$ we may assume that $s = i+3$, $t = i+4$, which does not affect the form of $X(h_1), \ldots, X(h_i)$. As $X | \langle h_1, \ldots, h_i \rangle$ is the alternating group on $v_1, \ldots, v_{i+2}$, choose $g \in \langle h_1, \ldots, h_i \rangle$ with $v_r X(g) = v_{i+2}$. Letting $h_{i+2} = h^g$,

$$\begin{cases} v_{i+2}X(h_{i+2}) = av_{i+3} & \\ v_{i+3}X(h_{i+2}) = bv_{i+4} & abc = 1 \\ v_{i+4}X(h_{i+2}) = cv_{i+2} & \\ v_\ell X(h_{i+2}) = v_\ell \text{ for } \ell \notin \{i+2, i+3, i+4\} & \end{cases} .$$

Replacing v_{i+3} by av_{i+3}, v_{i+4} by abv_{i+4}, which does not affect the form of $X(h_1), \ldots, X(h_i)$, we may assume $a = b = c = 1$. This is the desired element h_{i+2} and $h_i^{h_{i+2}h_i}$ is the element h_{i+1}.

Case b) $1 \leq s \leq i+2$:

By renumbering $v_{i+3}, \ldots, v_m$, we may assume $t = i+3$. Then clearly $X | \langle h_1, \ldots, h_i, h \rangle$ acts as the alternating group on $\langle v_1 \rangle, \ldots, \langle v_{i+3} \rangle$. As X_1 is irreducible and $i+3 < m$, there is a special 3-element $g \in H$ such that $X_1(g)$ does not leave $\langle v_1, \ldots, v_{i+3} \rangle$ invariant. So there exist ρ, μ, ν all distinct such that

$$\begin{cases} v_\rho X(g) = \alpha v_\mu & \\ v_\mu X(g) = \beta v_\nu & \alpha\beta\gamma = 1 \\ v_\nu X(g) = \gamma v_\rho & \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{\rho, \mu, \nu\} & \end{cases}$$

where not all of ρ, μ, ν are less than or equal to $i+3$, but at least one is. By replacing g by g^{-1} if necessary, we may assume $1 \leqslant \rho \leqslant i+3$ and $i+4 \leqslant \nu \leqslant m$. If $i+4 \leqslant \mu \leqslant m$, then choose $k \in \langle h_1, \ldots, h_i, h\rangle$ such that $v_\rho X(k) = xv_{i+2}$. Then

$$\begin{cases} v_{i+2}X(k^{-1}gk) = x^{-1}\alpha v_\mu \\ v_\mu X(k^{-1}gk) = \beta v_\nu \\ v_\nu X(k^{-1}gk) = x\gamma v_{i+2} \\ v_\ell X(k^{-1}gk) = v_\ell \text{ for } \ell \notin \{i+2, \mu, \nu\} \end{cases} .$$

We now have Case a, with $k^{-1}gk$ in place of h. If $1 \leqslant \mu \leqslant i+3$, let $k \in \langle h_1, \ldots, h_i, h\rangle$ such that $v_\rho X(k) = xv_{i+2}$ and $v_\mu X(k) = yv_{i+3}$. Therefore

$$\begin{cases} v_{i+2}X(k^{-1}gk) = x^{-1}\alpha y v_{i+3} \\ v_{i+3}X(k^{-1}gk) = y^{-1}\beta v_\nu \\ v_\nu X(k^{-1}gk) = x\gamma v_{i+2} \\ v_\ell X(k^{-1}gk) = v_\ell \text{ for } \ell \notin \{i+2, i+3, \nu\} \end{cases} .$$

Again we have Case a, with $k^{-1}gk$ in place of h. So by induction $h_1, \ldots, h_{m-2}$ are found. Clearly $\xi(h_i) = 1$ for $1 \leqslant i \leqslant m-2$ as $X_1(h_i)$ is not diagonal in $v_1, \ldots, v_m$.

Let $F = \{h \in H \mid (X_1 \oplus \xi)(h)$ is diagonal in the basis $v_1, \ldots, v_m, v\}$. If $g \in H$ and $f \in F$, $(X_1 \oplus \xi)(g^{-1}fg)$ is still diagonal and so $F \triangleleft H$. Let g be a special 3-element in H. If $(X_1 \oplus \xi)(g)$ is diagonal, then $g \in F$. Assume $(X_1 \oplus \xi)(g)$ is not diagonal. Then for some i, j, k distinct, $vX(g) = v$ and

$$\begin{cases} v_i X(g) = a v_j & \\ v_j X(g) = b v_k & abc = 1 \\ v_k X(g) = c v_i & \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{i, j, k\} & . \end{cases}$$

As $X_1 | A$ is the alternating group on $v_1, \ldots, v_m$, there is an $h \in A$ with

$$\begin{cases} v_i X(h) = v_j \\ v_j X(h) = v_k \\ v_k X(h) = v_i \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{i, j, k\} \quad . \end{cases}$$

Then $gh^2 \in F$ and so $g \in FA$. Hence $H = FA$ and $F \cap A = 1$.

Consider $X_1 | F = \xi_1 \oplus \ldots \oplus \xi_m$ where ξ_i is linear on $\langle v_i \rangle$ for $1 \leq i \leq m$. Assume first that $\xi_i = \xi_j$ for some $i \neq j$. Let $f \in F$ and by double transitivity, let $g_r \in A$ such that $v_i X(g_r) = v_i$ and $v_r X(g_r) = v_j$. So $\xi_j(g_r^{-1} f g_r) v_j = v_j X(g_r^{-1} f g_r) = \xi_r(f) v_j$ and $\xi_i(g_r^{-1} f g_r) v_i = v_i X(g_r^{-1} f g_r) = \xi_i(f) v_i$. This implies $\xi_r(f) = \xi_j(g_r^{-1} f g_r) = \xi_i(g_r^{-1} f g_r) = \xi_i(f)$. Thus $\xi_1 = \ldots = \xi_m$ contradicting the irreducibility of X_1. So the ξ_i's are distinct. If $\xi \neq 1_H$, for some special 3-element g, $\xi(g) = \omega$ or $\bar{\omega}$. So $(X_1 \oplus \xi)(g)$ is diagonal and $\xi_i(g) \neq \xi(g)$ for $1 \leq i \leq m$. Thus $\xi \neq \xi_i$ for $1 \leq i \leq m$ if $\xi \neq 1_H$. If $\xi = 1_H$, there is an $f \in F$ such that for some i, $\xi_i(f) \neq 1$. Let $g_r \in A$ with $v_i X(g_r) = v_r$. Then $\xi_r(g_r^{-1} f g_r) v_r = v_r X(g_r^{-1} f g_r) = \xi_i(f) v_r$. So $\xi_r(g_r^{-1} f g_r) \neq 1$ and $\xi \neq \xi_i$ for $1 \leq i \leq m$.

Let $v_1^*, \ldots, v_n^*$ be a basis of V in which $X | H$ is monomial. We first consider $X(h_1), \ldots, X(h_{m-2})$ acting on this basis. As $\langle h_1, h_3 \rangle \cong A_5$, clearly the only possibility is that

$$\begin{cases} v_r^* X(h_1) = a v_s^* \\ v_s^* X(h_1) = b v_t^* \qquad abc = 1 \\ v_t^* X(h_1) = c v_r^* \\ v_\ell^* X(h_1) = v_\ell^* \text{ for } \ell \notin \{r, s, t\} \end{cases} \quad \text{and} \quad \begin{cases} v_t^* X(h_3) = \alpha v_u^* \\ v_u^* X(h_3) = \beta v_w^* \qquad \alpha\beta\gamma = 1 \\ v_w^* X(h_3) = \gamma v_t^* \\ v_\ell^* X(h_3) = v_\ell^* \text{ for } \ell \notin \{t, u, w\} \end{cases} .$$

By replacing v_s^* by av_s^*, v_t^* by abv_t^*, v_u^* by αabv_u^*, and v_w^* by $\alpha\beta abv_w^*$, we may assume $a = b = c = \alpha = \beta = \gamma = 1$. Also by renumbering we may assume $r = 1$, $s = 2$, $t = 3$, $u = 4$, $w = 5$. Note that $h_2 = h_1^{h_3 h_1}$. So inductively assume we have reordered and rescaled $v_1^*, \ldots, v_n^*$ such that for some i, with $2 \leq i \leq m-4$,

$$(1) \qquad \begin{cases} v_j^* X(h_j) = v_{j+1}^* \\ v_{j+1}^* X(h_j) = v_{j+2}^* \\ v_{j+2}^* X(h_j) = v_j^* \\ v_\ell^* X(h_j) = v_\ell^* \text{ for } \ell \notin \{j, j+1, j+2\} \end{cases} ,$$

for $1 \leq j \leq i$. As $X(h_{i+2})$ does not commute with $\langle X(h_1), \ldots, X(h_i)\rangle$, $X(h_{i+2})$ must act trivially on at least $n - (i+4)$ of the vectors $v_{i+3}^*, \ldots, v_n^*$. By reordering $v_{i+3}^*, \ldots, v_n^*$, assume $X(h_{i+2})$ acts trivially on $v_{i+5}^*, \ldots, v_n^*$. So $X|\langle h_1, \ldots, h_i, h_{i+2}\rangle$ acts as a permutation group on $\langle v_1^*\rangle, \ldots, \langle v_{i+4}^*\rangle$. As $\langle h_1, \ldots, h_i, h_{i+2}\rangle \cong A_{i+4}$, which is simple, $X|\langle h_1, \ldots, h_i, h_{i+2}\rangle$ cannot act trivially on $\langle v_{i+3}^*\rangle$ or $\langle v_{i+4}^*\rangle$. Thus there is a j with $1 \leq j \leq i+2$ such that by ordering v_{i+3}^*, v_{i+4}^* correctly

$$\begin{cases} v_j^* X(h_{i+2}) = a v_{i+3}^* \\ v_{i+3}^* X(h_{i+2}) = b v_{i+4}^* \qquad abc = 1 \\ v_{i+4}^* X(h_{i+2}) = c v_j^* \\ v_\ell^* X(h_{i+2}) = v_\ell^* \text{ for } \ell \notin \{j, i+3, i+4\} \end{cases} .$$

Replacing v_{i+3}^* by av_{i+3}^* and v_{i+4}^* by abv_{i+4}^*, we may assume $a = b = c = 1$. As h_{i+2} commutes with $h_1, \ldots, h_{i-1}$, $j = i+2$ is the only possibility. Noting that $h_{i+1} = h_i^{h_{i+2}h_i}$, we have by induction and properly scaling and ordering $v_1^*, \ldots, v_n^*$ that $X(h_j)$ has the form (1) for $1 \leq j \leq m-2$.

Let $D = \{h \in H \mid X(h)$ is diagonal in $v_1^*, \ldots, v_n^*\} \triangleleft H$. If g is a special 3-element in H, by looking at $(X_1 \oplus \xi)(g)$ in the basis $v_1, \ldots, v_m, v$, it is clear that g does not commute with A. Thus either $g \in D$ or for i, j, k distinct with $i \leq m$,

$$\begin{cases} v_i^* X(g) = a v_j^* & \\ v_j^* X(g) = b v_k^* & \qquad abc = 1 \\ v_k^* X(g) = c v_i^* & \\ v_\ell^* X(g) = v_\ell^* \text{ for } \ell \notin \{i, j, k\} & . \end{cases}$$

In the latter case if $j, k > m$, by replacing v_j^* by av_j^* and v_k^* by abv_k^*, it is clear that $\langle h_1, \ldots, h_{m-2}, g\rangle \cong A_{m+2}$ and $X \mid \langle h_1, \ldots, h_{m-2}, g\rangle$ has an irreducible constituent of degree $m+1$, a contradiction. If only one of j, k is greater than m, by replacing g by g^{-1} if necessary, we may assume $j \leq m$ and $k > m$. Replacing v_k^* by bv_k^*, we may also assume $c = a^{-1}$ and $b = 1$. If $a \neq 1$, $X \mid \langle h_1, \ldots, h_{m-2}, g\rangle$ is irreducible on $\langle v_1^*, \ldots, v_m^*, v_k^*\rangle$, a contradiction. So $a = 1$ and $K = \langle h_1, \ldots, h_{m-2}, g\rangle \cong A_{m+1}$. As $K \cap F$ is a nontrivial abelian normal subgroup of K, this is a contradiction. So $i, j, k \leq m$ and as done previously, $gh \in D$ for some $h \in A$. Therefore $H = DA$. As DF/F is a normal abelian subgroup of $H/F \cong A_m$, $D \subseteq F$. Hence $D = F$ as $F \cap A = 1$.

Assume first that $\xi = 1_H$. As the only linear constituents of $X \mid H$ are trivial, $X \mid D$ must be trivial on $v_{m+1}^*, \ldots, v_n^*$; so $\langle v\rangle \oplus V_2 =$

$\langle v_{m+1}^*, \ldots, v_n^* \rangle$ and $\langle v_1^*, \ldots, v_m^* \rangle = \langle v_1, \ldots, v_m \rangle$. Suppose $\xi \neq 1_H$. As there is exactly one nontrivial linear constituent of $X|H$, $X|D$ must be trivial on $n - m - 1$ of the vectors $v_{m+1}^*, \ldots, v_n^*$ while ξ acts on the remaining one. By reordering and rescaling, we may assume $v = v_{m+1}^*$; so $\langle v_1^*, \ldots, v_m^* \rangle = \langle v_1, \ldots, v_m \rangle$ and $V_2 = \langle v_{m+2}^*, \ldots, v_n^* \rangle$. As $X_1|F = \xi_1 \oplus \ldots \oplus \xi_m = X_1|D = \mu_1 \oplus \ldots \oplus \mu_m$ where μ_i acts on $\langle v_i^* \rangle$ and each ξ_i is unique, $\{ \langle v_i^* \rangle \mid 1 \leq i \leq m \} = \{ \langle v_j \rangle \mid 1 \leq j \leq m \}$ proving the lemma.

If $n \geq 5$ and H is generated by special 3-elements such that $X|H$ is irreducible, clearly Lemma 1.4 holds when appropriately modified to avoid conclusions concerning ξ or $(n-m-1)1_H$. So when necessary, Lemma 1.4 will include this case.

CHAPTER II

BUILDING UP OF CERTAIN SUBGROUPS

In this chapter lemmas are developed which allow the building up of subgroups of G generated by special 3-elements and special 4-elements. Also under certain conditions it is shown that there are subgroups of G which are primitive of smaller degree. These results are used for induction later.

Lemma 2.1: Let $n \geq 4$ and H be a subgroup of G generated by special 4-elements. Assume $X|H = X_1 \oplus (n-r)1_H$ where X_1 is irreducible such that $3 \leq r < n$. Then there exists a special 4-element $h \in G$ with $X|\langle H,h\rangle = X_2 \oplus (n-s)1_{\langle H, h\rangle}$ where X_2 is irreducible of degree $s = r+1$ or $r+2$.

Proof: Let X_1 act on V_1. Choose a special 4-element h such that X(h) does not leave V_1 invariant. So $X|\langle H,h\rangle = X_2 \oplus (n-r-2)1_{\langle H, h\rangle}$ where either X_2 is irreducible or $X_2 = X_3 \oplus \xi$ such that X_3 is irreducible and ξ is linear. We are done unless $X_2 = X_3 \oplus \xi$ where $\xi \neq 1_{\langle H, h\rangle}$. Assuming this is the case, by Mitchell X_3 is monomial. In particular $X|H$ is monomial in some basis $v_1, \ldots, v_n$. By Lemma 1.3, there exist special 4-elements $h_1, \ldots, h_{r-1} \in H$ such that when $v_1, \ldots, v_n$ are properly scaled and ordered

$$\begin{cases} v_k X(h_k) = -v_{k+1} \\ v_{k+1} X(h_k) = v_k \\ v_\ell X(h_k) = v_\ell \quad \text{for } \ell \notin \{k, k+1\} \end{cases}$$

for $1 \leq k \leq r-1$ and $X|\langle h_1, \ldots, h_{r-1}\rangle$ is irreducible on $\langle v_1, \ldots, v_r\rangle$. Thus $V_1 = \langle v_1, \ldots, v_r\rangle$. As $\xi(u) = 1$ for $u \in H$, $\xi(h) \neq 1$. Since $X(h)$ does not leave V_1 invariant, $X(h)$ is not diagonal in $v_1, \ldots, v_n$. So

$$\begin{cases} v_i X(h) = -\delta v_j \\ v_j X(h) = \delta^{-1} v_i \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{i, j\} \end{cases}$$

where $1 \leq i \leq r$ and $r+1 \leq j \leq n$. Thus $X|\langle H, h\rangle$ is irreducible on $\langle v_1, \ldots, v_r, v_j\rangle$ and trivial on $\langle v_{r+1}, \ldots, v_{j-1}, v_{j+1}, \ldots, v_n\rangle$ contradicting $\xi(h) \neq 1$. So the lemma is proved.

Lemma 2.2: Let H be a subgroup of G generated by special 3-elements such that $X|H = X_1 \oplus X_2 \oplus (n-s)1_H$ where X_1 is a nonunimodular, irreducible, primitive representation of degree 4. Assume X_2 is a representation of degree d with $1 \leq d \leq 4$ and $s = d+4$. Suppose $X_i(H) = H_i$. Then $n = 8$, X_2 is irreducible and primitive of degree 4, and $H_i \cong \widetilde{O_5(3)} \times Z_3$ where $\widetilde{O_5(3)}$ is the nonsplitting central extension of Z_2 by $O_5(3)$. If L_i is the set of all elements of H_i which occur with component the identity of H_j $(j \neq i)$ in the subdirect product, then $L_i \subseteq O_2(Z(H_i))$ for $i = 1$ and 2.

Proof: By Mitchell, as X_1 is not unimodular and X_1 is primitive, $H_1/Z(H_1) \cong O_5(3)$. But $H_1''Z(H_1) = H_1$ and so $H_1' = H_1''$. Thus $H_1'/Z(H_1')$ $\cong O_5(3)$. As $O_5(3)$ does not have a nontrivial representation of degree 4, $Z(H_1') \neq 1$. The Schur multiplier of $O_5(3)$ is Z_2 (see [6]); so $Z(H_1') \cong Z_2$. Because H_1 has elements with determinant ω and only elements with determinants 1, ω, or $\bar{\omega}$, $O_3(Z(H_1)) \cong Z_3$ and $O_2(Z(H_1)) \cong Z_2$ or Z_4. If

$O_2(Z(H_1)) \cong Z_4$, then $K = H_1' O_3(Z(H_1))$ has index 2 in H_1, contradicting the fact that H_1 is generated by 3-elements. Thus $O_2(Z(H_1)) = Z(H_1')$ and $H_1 \cong H_1' \times Z_3$ where $H_1' \cong \widetilde{O_5(3)}$, the nonsplitting central extension of Z_2 by $O_5(3)$. Let L_i be as in the statement of the lemma. By Theorem 5.5.1 of [11], $L_i \trianglelefteq H_i$ and $H_1/L_1 \cong H_2/L_2$. As L_i consists of unimodular matrices, either $L_1 = H_1'$ or $L_1 \subseteq O_2(Z(H_1))$. In the first case L_1 contains an element with eigenvalues ω, $\bar{\omega}$, ω, $\bar{\omega}$ and a central element with eigenvalues -1, -1, -1, -1. But then X(G) contains an element with eigenvalues $-\omega$, $-\bar{\omega}$, $-\omega$, $-\bar{\omega}$, 1, 1, ..., a contradiction to Blichfeldt. So $L_1 \subseteq O_2(Z(H_1))$ and $H_2/L_2 \cong O_5(3) \times Z_3$ or $\widetilde{O_5(3)} \times Z_3$. By examining the possible decompositions of X_2 into its components and consulting Blichfeldt's list [2] carefully, it is easy to see that X_2 is irreducible of degree 4 and is primitive. Carrying out the same analysis as on H_1, we see $H_2 \cong \widetilde{O_5(3)} \times Z_3$ and $L_2 \subseteq O_2(Z(H_2))$. Also in the isomorphism from H_1/L_1 onto H_2/L_2, the central elements are mapped onto central elements. In particular if $n \geq 9$, X(G) contains an element with eigenvalues $-\omega, -\omega, -\omega, -\omega, -\bar{\omega}, -\bar{\omega}, -\bar{\omega}, -\bar{\omega}, 1, \ldots$, contradicting Blichfeldt. So $n = 8$.

Lemma 2.3: Let $n \geq 5$ and H a subgroup of G generated by special 3-elements. Assume $X|H = X_1 \oplus (n-r)1_H$ where X_1 is irreducible of degree r with $3 \leq r < n$. Then there exists a subgroup K generated by special 3-elements such that $X|\langle H, K\rangle = X_2 \oplus (n-s)1_{\langle H,K\rangle}$ where $s = r+1$ or $r+2$ and X_2 is irreducible of degree s.

Proof: Let X_1 act on the subspace V_1 and let h be a special 3-element such that X(h) does not leave V_1 invariant. So

$X \mid \langle H, h\rangle = X_2 \oplus (n-r-2)1_{\langle H, h\rangle}$ where either X_2 is irreducible or $X_2 = X_3 \oplus \xi$ such that X_3 is irreducible and ξ is linear. We are done unless $X_2 = X_3 \oplus \xi$ and $\xi \neq 1_{\langle H, h\rangle}$. Assume this is the case.

Suppose first that $r = 3$. By Lemma 2.2, X_3 is not primitive; so X_3 is monomial. Let X_3 act on $V_3 \supset V_1$ and let v_1, v_2, v_3, v_4 be a basis of V_3 such that X_3 is monomial in that basis. As X_3 is irreducible, there exist special 3-elements $h_1, h_2 \in \langle H, h\rangle$ such that by scaling and ordering $v_1, \ldots, v_4$ correctly,

$$\begin{cases} v_1X(h_1) = v_2 \\ v_2X(h_1) = v_3 \\ v_3X(h_1) = v_1 \\ v_4X(h_1) = v_4 \end{cases} \quad \text{and} \quad \begin{cases} v_1X(h_2) = v_1 \\ v_2X(h_2) = av_3 \\ v_3X(h_2) = v_4 \\ v_4X(h_2) = a^{-1}v_4 \end{cases} .$$

If $a \neq 1$, $X_3 \mid \langle H, h_1, h_2\rangle$ is irreducible and we are done. The case $a = 1$ is handled in the same way as $r \geqslant 4$ is.

Assume $r \geqslant 4$. Let X_3 act on V_3. For $g \in H$, $\xi(g) = 1$; also $\xi(h) \neq 1$. By Mitchell, X_3 is monomial. Let $v_1, \ldots, v_{r+1}$ be a basis of V_3 in which X_3 is monomial.

By Lemma 1.4, there exist $h_1, \ldots, h_{r-1} \in \langle H, h\rangle$ such that by scaling and ordering $v_1, \ldots, v_{r+1}$ properly,

$$\begin{cases} v_iX(h_i) = v_{i+1} \\ v_{i+1}X(h_i) = v_{i+2} \\ v_{i+2}X(h_i) = v_i \\ v_\ell X(h_i) = v_\ell \text{ for } \ell \notin \{i, i+1, i+2\} \end{cases} .$$

Since $\xi(h) \neq 1$, by replacing h by h^{-1} if necessary, $v_jX(h) = \omega v_j$ for some j and $v_i X(h) = v_i$ for $i \neq j$. Let $g \in \langle h_1, \ldots, h_{r-1} \rangle$ with $v_j X(g) = v_k$ for some $k \neq j$. Let $h^* = g^{-1}h^{-1}gh$. Then $\xi(h^*) = 1$ and

$$\begin{cases} v_j X(h^*) = \omega v_j \\ v_k X(h^*) = \bar{\omega} v_k \\ v_\ell X(h^*) = v_\ell \text{ for } \ell \notin \{j, k\} \end{cases} .$$

Letting $K = \langle h_1, \ldots, h_{r-1}, h^* \rangle$, we are done since $X_3 | K$ is irreducible and $\xi | K = 1_K$.

Lemma 2.4: Let H be a subgroup of G generated by special 4-elements (special 3-elements) such that $X | H = X_1 \oplus (n-r)1_H$ where $r \geq 4$ $(r \geq 5)$ and X_1 is irreducible of degree r. Suppose X_1 acts on V_1 and is monomial in a basis $v_1, \ldots, v_r$ of V_1. Let $H_i = \langle h \in H | v_i X(h) = v_i$ and h is a special 4-element (special 3-element)$\rangle$. Then $X | H_i = X_{1,i} \oplus (n-r+1)1_{H_i}$ where $X_{1,i}$ is irreducible and monomial on $\langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_r \rangle$.

Proof: Notice that multiplying v_i by α does not change H_i. Let $(n-r)1_H$ act on V_2. First take the case where H is generated by special 4-elements. By Lemma 1.3 after scaling and ordering $v_1, \ldots, v_r$ correctly, there exist special 4-elements $h_1, \ldots, h_{r-1}$ such that,

$$\begin{cases} v_j X(h_j) = -v_{j+1} \\ v_{j+1} X(h_j) = v_j \\ v_\ell X(h_j) = v_\ell \text{ for } \ell \notin \{j, j+1\} \end{cases} .$$

By transitivity, choose $g_i \in \langle h_1, \ldots, h_{r-1} \rangle$ such that $v_i X(g_i) = \alpha_i v_r$. Then

$H_r = \langle h^{g_r} \mid h \in H_i \rangle$. Clearly H_i has the desired properties if and only if H_r does. But by Lemma 1.3, $X \mid \langle h_1, \ldots, h_{r-2} \rangle$ is irreducible on $\langle v_1, \ldots, v_{r-1} \rangle$. As $X \mid H_r$ clearly acts trivially on $V_2 \oplus \langle v_r \rangle$, we have $X \mid H_r = X_{1,r} \oplus (n-r+1) 1_{H_r}$ where $X_{1,r}$ is irreducible and monomial on $\langle v_1, \ldots, v_{r-1} \rangle$.

Now consider the case where H is generated by special 3-elements. By Lemma 1.4, after scaling and ordering $v_1, \ldots, v_r$ correctly, there exist special 3-elements $h_1, \ldots, h_{r-2}$ such that

$$\begin{cases} v_i X(h_i) = v_{i+1} \\ v_{i+1} X(h_i) = v_{i+2} \\ v_{i+2} X(h_i) = v_i \\ v_\ell X(h_i) = v_\ell \text{ for } \ell \in \{i, i+1, i+2\} \end{cases} .$$

Again as $X \mid \langle h_1, \ldots, h_{r-2} \rangle$ is transitive on $v_1, \ldots, v_r$, it suffices to show the results hold for H_r. Since X_1 is irreducible, there exists $h \in H$ such that

$$\begin{cases} v_i X(h) = a v_j \\ v_j X(h) = b v_k \qquad abc = 1 \\ v_k X(h) = c v_i \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{i, j, k\} \end{cases} \quad \text{or} \quad \begin{cases} v_i X(h) = \omega v_i \\ v_j X(h) = \bar{\omega} v_j \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{i, j\} \end{cases} .$$

In the first case not all of a, b, and c are 1. In either case, for some ℓ, $v_\ell X(h) = v_\ell$. By transitivity, there is a $g \in \langle h_1, \ldots, h_r \rangle$ such that $v_\ell X(g) = v_r$. Replacing h by h^g, we may assume $i, j, k < r$. In particular $X \mid \langle h_1, \ldots, h_{r-3}, h \rangle$ is irreducible on $\langle v_1, \ldots, v_{r-1} \rangle$. Clearly $X \mid H_r$ acts trivially on $V_2 \oplus \langle v_r \rangle$, and so $X|H_r = X_{1,r} \oplus (n-r+1)1_{H_r}$ where $X_{1,r}$ is irreducible and monomial on $\langle v_1, \ldots, v_{r-1} \rangle$.

We are now ready to prove that under certain circumstances there exist primitive subgroups of codimension 1 or 2. These are important for inductive purposes in the proofs of the main results. Lemma 2.7 will also be useful in Chapter V to determine the possible subgroups generated by two special 3-elements.

Lemma 2.5: Let $n \geqslant 6$. Assume there exists a subgroup U of G generated by special 4-elements such that $X|U = X_1 \oplus r1_U$ where $r = 1$ or 2 and X_1 is irreducible. Then there exists a subgroup H of G generated by special 4-elements such that $X|H = Y \oplus y1_H$ where Y is irreducible and primitive, and $y = 1$ or 2.

Proof: Assume the lemma is false. In particular X_1 is monomial. By replacing U by another subgroup, we may assume $r = 1$ as follows. Let X_1 act on V_1 and let $v_1, \ldots, v_{n-2}$ be a basis of V_1 such that X_1 is monomial in that basis. Let $2 \cdot 1_U$ act on V_2. Choose a special 4-element h^* such that $X(h^*)$ does not leave V_1 invariant. Then $X(h^*)$ does not leave both $\langle v_1, \ldots, v_{n-3} \rangle$ and $\langle v_2, \ldots, v_{n-2} \rangle$ invariant. By reordering the v_i's, we may assume $X(h^*)$ does not leave $\langle v_1, \ldots, v_{n-3} \rangle$ invariant. Let $U_{n-2} = \langle h \in U \,|\, v_{n-2}X(h) = v_{n-2}$ and h is a special 4-element$\rangle$. By Lemma 2.4, $X|U_{n-2} = X_{1,n-2} \oplus 3 \cdot 1_{U_{n-2}}$ where $X_{1,n-2}$ is irreducible on $\langle v_1, \ldots, v_{n-3} \rangle$ and monomial in the basis $v_1, \ldots, v_{n-3}$. Also $3 \cdot 1_{U_{n-2}}$ acts on $V' = \langle v_{n-2}, V_2 \rangle$. So $X|\langle U_{n-2}, h^* \rangle = X_2 \oplus 1_{\langle U_{n-2}, h^* \rangle}$ where X_2 is irreducible or $X_2 = X_3 \oplus \xi$ with X_3 irreducible of degree $n-2$. If X_2 is irreducible, replace U by $\langle U_{n-2}, h^* \rangle$. Suppose $X_2 = X_3 \oplus \xi$. If X_3 is primitive, by Mitchell, $\xi = 1_{\langle U_{n-2}, h^* \rangle}$ and the lemma is true, a contradiction. So X_3 is monomial and there is a basis $v_1^*, \ldots, v_n^*$ of V on which $X | \langle U_{n-2}, h^* \rangle$ is monomial. By Lemma 1.3 applied to U_{n-2}, we may reorder and rescale $v_1^*, \ldots, v_n^*$ such that $v_1 = v_1^*, \ldots, v_{n-3} = v_{n-3}^*$ and $V' = \langle v_{n-2}^*, v_{n-1}^*, v_n^* \rangle$. As $\langle v_1, \ldots, v_{n-3} \rangle$ is not left invariant by $X(h^*)$, in the basis $v_1^*, \ldots, v_n^*$, $X(h^*)$ is not diagonal. In particular $\xi = 1_{\langle U_{n-2}, h^* \rangle}$. So $X | \langle U_{n-2}, h^* \rangle$ acts trivially on a subspace V_3 of dimension 2 with $V_3 \subset V'$. As V_2, V_3 have dimension 2 and are in a subspace V' of dimension 3, $V_2 \cap V_3 \neq \{0\}$. Thus $X | \langle U, h^* \rangle$ acts trivially on $V_2 \cap V_3$. Hence $X | \langle U, h^* \rangle = X' \oplus 1_{\langle U, h^* \rangle}$ where X' is irreducible of degree $n-1$, as $X(h^*)$ does not leave V_1 invariant. Replacing U by $\langle U, h^* \rangle$, we may assume $r = 1$.

So $X | U = X_1 \oplus 1_U$ where X_1 is monomial and irreducible on V_1. Let V_1 have a basis $v_1, \ldots, v_{n-1}$ in which X_1 is monomial, and let 1_U act on $\langle v_n \rangle$. First let g be any special 4-element such that $X(g)$ leaves V_1 invariant. Then $X | \langle U, g \rangle = X_1' \oplus \xi$ where X_1' is irreducible on V_1 and ξ acts on $\langle v_n \rangle$. If $\xi \neq 1_{\langle U, g \rangle}$, by Mitchell, X_1' is monomial. If $\xi = 1_{\langle U, g \rangle}$, as we are assuming this lemma is false, X_1' is still monomial. If $v_1^*, \ldots, v_n^*$ is a basis of V in which $X | \langle U, g \rangle$ is monomial, by Lemma 1.3, after suitably scaling and ordering $v_1^*, \ldots, v_n^*$, we have $v_1 = v_1^*, \ldots, v_{n-1} = v_{n-1}^*$ and $\langle v_n^* \rangle = \langle v_n \rangle$. So $X(g)$ is monomial in the basis $v_1, \ldots, v_n$.

Now let g be a special 4-element such that $X(g)$ does not leave V_1 invariant. Either $X(g)$ does not leave $\langle v_1, \ldots, v_{n-3} \rangle$ or $\langle v_3, \ldots, v_{n-1} \rangle$ invariant. By reordering $v_1, \ldots, v_{n-1}$ if necessary, assume $X(g)$ does not leave $\langle v_1, \ldots, v_{n-3}\rangle$ invariant. Let $\widetilde{U} = \langle h \in U \mid h$ is a special 4-element and $v_{n-2}X(h) = v_{n-2}, v_{n-1}X(h) = v_{n-1}\rangle$. By Lemma 2.4 applied twice, $X|\widetilde{U} = \widetilde{X} \oplus 3 \cdot 1_{\widetilde{U}}$ where $\widetilde{X}$ is monomial and irreducible on $\langle v_1, \ldots, v_{n-3}\rangle$ and has a monomial basis $v_1, \ldots, v_{n-3}$ on this subspace. Also $3 \cdot 1_{\widetilde{U}}$ acts on $\langle v_{n-2}, v_{n-1}, v_n\rangle$. So $X|\langle\widetilde{U}, g\rangle = X_2 \oplus 1_{\langle\widetilde{U}, g\rangle}$ where either X_2 is irreducible or $X_2 = X_3 \oplus \xi$ such that X_3 is irreducible and ξ is linear. By assumption that this result is false, X_2 is monomial unless $X_2 = X_3 \oplus \xi$ with $\xi \neq 1_{\langle\widetilde{U}, g\rangle}$. In that case, however, X_3 is monomial by Mitchell. In any case $X|\langle\widetilde{U}, g\rangle$ is monomial in some basis $v_1^*, \ldots, v_n^*$ of V. Applying Lemma 1.3 to $\widetilde{U}$, by ordering and scaling $v_1^*, \ldots, v_n^*$ correctly, $v_1^* = v_1, \ldots, v_{n-3}^* = v_{n-3}$, and $\langle v_{n-2}^*, v_{n-1}^*, v_n^*\rangle = \langle v_{n-2}, v_{n-1}, v_n\rangle$. In particular $v_jX(g) = v_j$ for some $1 \leqslant j \leqslant n-3$. By reordering $v_1, \ldots, v_{n-3}$, we may assume $j = 1$.

Now let $U_1 = \langle h \in U \mid v_1X(h) = v_1$ and h is a special 4-element$\rangle$. By Lemma 2.4, $X|U_1 = Y_1 \oplus 2 \cdot 1_{U_1}$ where Y_1 is irreducible and monomial on $\langle v_2, \ldots, v_{n-1}\rangle$ and Y_1 has a monomial basis $v_2, \ldots, v_{n-1}$. Also $2 \cdot 1_{U_1}$ acts on $\langle v_1, v_n\rangle$. As $X(g)$ does not leave $\langle v_1, \ldots, v_{n-1}\rangle$ invariant, $X(g)$ does not leave $\langle v_2, \ldots, v_{n-1}\rangle$ invariant. So $X|\langle U_1, g\rangle = Y \oplus 1_{\langle U_1, g\rangle}$ where Y acts irreducibly on $\widetilde{V} \supset \langle v_2, \ldots, v_{n-1}\rangle$ and $1_{\langle U_1, g\rangle}$ on $\langle v_1\rangle$. By assumption that the result is false, Y is monomial with a basis $v_2^*, \ldots, v_n^*$. By Lemma 1.3 applied to U_1, after reordering and rescaling $v_2^*, \ldots, v_n^*$, we may assume $v_2^* = v_2, \ldots, v_{n-1}^* = v_{n-1}$ and $v_n^* \in \langle v_1, v_n\rangle$. As $X|U_1$ acts trivially on $\langle v_1, v_n\rangle$, $v_n^*X(u) = v_n^*$ for $u \in U_1$; so as Y is irreducible, we have for some j with $2 \leqslant j \leqslant n-1$,

$$\begin{cases} v_j X(g) = -\delta v_n^* \\ v_n^* X(g) = \delta^{-1} v_j \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{j, n\} \end{cases} .$$

By reordering $v_2, \ldots, v_{n-1}$, we may assume $j = n-1$. In particular, $v_\ell X(g) = v_\ell$ for $1 \leq \ell \leq n-2$.

Let $U_2 = \langle h \in U \mid v_2 X(h) = v_2$ where h is a special 4-element$\rangle$. As in the analysis of $\langle U_1, g\rangle$, there is $v'_n \in \langle v_2, v_n\rangle$ such that

$$\begin{cases} v_j X(g) = -\epsilon v'_n \\ v'_n X(g) = \epsilon^{-1} v_j \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{j, n\} \end{cases} .$$

The only possibility is that $j = n-1$ and $\langle v'_n\rangle = \langle v^*_n\rangle \subseteq \langle v_1, v_n\rangle \cap \langle v_2, v_n\rangle = \langle v_n\rangle$. So in fact $X(g)$ is monomial in the basis $v_1, \ldots, v_n$.

Therefore if h is a special 4-element, $X(h)$ is monomial in the basis $v_1, \ldots, v_n$. Let N be the subgroup of G generated by all special 4-elements. So $N \triangleleft G$. By Lemma 1.3, the set $\{\langle v_i\rangle \mid 1 \leq i \leq n\}$ is the unique set of one dimensional subspaces of V permuted by $X(N)$. For $g \in G$, $h \in N$, $ghg^{-1} = h_1 \in N$. So $(\langle v_i\rangle X(g))X(h) = (\langle v_i\rangle X(h_1))X(g) = \langle v_j\rangle X(g)$ for some j depending on i and h_1. Thus $X(h)$ for all $h \in N$ permutes $\{\langle v_i\rangle X(g) \mid 1 \leq i \leq n\}$. So $\{\langle v_i\rangle X(g) \mid 1 \leq i \leq n\} = \{\langle v_i\rangle \mid 1 \leq i \leq n\}$ and X is monomial, a contradiction.

<u>Lemma 2.6</u>: Let $n \geq 5$. Let h_1 and h_2 be noncommuting special 3-elements of G. Assume $X \mid \langle h_1, h_2\rangle$ is monomial in a basis $v_1, \ldots, v_n$ such that $X \mid \langle h_1, h_2\rangle$ leaves exactly $n-5$ of the subspaces $\langle v_i\rangle$ fixed. Then

$\langle h_1, h_2 \rangle \cong A_5$ and $X | \langle h_1, h_2 \rangle = X_1 \oplus (n-4)1_{\langle h_1, h_2 \rangle}$ where X_1 is irreducible.

Proof: By ordering $v_1, \ldots, v_n$ correctly, we get

$$\begin{cases} v_1X(h_1) = av_2 & \\ v_2X(h_1) = bv_3 & abc = 1 \\ v_3X(h_1) = cv_1 & \\ v_\ell X(h_1) = v_\ell \text{ for } \ell \notin \{1,2,3\} & \end{cases} \quad \text{and} \quad \begin{cases} v_3X(h_2) = \alpha v_4 & \\ v_4X(h_2) = \beta v_5 & \alpha\beta\gamma = 1 \\ v_5X(h_2) = \gamma v_3 & \\ v_\ell X(h_2) = v_\ell \text{ for } \ell \notin \{3,4,5\} & \end{cases}.$$

By replacing v_2 by av_2, v_3 by abv_3, v_4 by αabv_4, and v_5 by $\alpha\beta abv_5$, we may assume $a = b = c = \alpha = \beta = \gamma = 1$. Clearly $\langle h_1, h_2 \rangle \cong A_5$, and $X | \langle h_1, h_2 \rangle = X_1 \oplus (n-4)1_{\langle h_1, h_2 \rangle}$ where X_1 is irreducible.

Lemma 2.7: Let $n \geq 8$ and let K be a subgroup of G satisfying one of the following:

1. $K = \langle g, h \rangle$ where g, h are special 3-elements with $X|K = \widetilde{X} \oplus (n-r)1_K$ such that $\widetilde{X}$ is irreducible of degree $r = 3$ or 4. Assume for $n = 8$ and $r = 4$ that $K \ncong A_5$.
2. $K = \langle g, h, k \rangle$ where g, h, k are special 3-elements with $X|K = \widetilde{X} \oplus (n-4)1_K$. Let $\widetilde{X}$ be monomial on a subspace $\widetilde{V}$ with a basis e_1, e_2, e_3, e_4 such that in this basis

$$\widetilde{X}(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \widetilde{X}(h) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \text{ and } \widetilde{X}(k) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Then there exists a subgroup H of G generated by special 3-elements such that $X|H = Y \oplus y1_H$ where Y is irreducible and primitive with $y = 1$

or 2 and H contains a G-conjugate of K.

<u>Proof</u>: First a few remarks are made about situation 2. The representation $\widetilde{X}$ is irreducible and $\widetilde{X} \mid \langle g, h\rangle = R_1 \oplus R_2$ where R_i are both irreducible with R_1 acting on $\langle e_1 + \overline{\omega} e_2 + \omega e_3\rangle \oplus \langle -e_2 + \omega\, e_3 + \overline{\omega} e_4\rangle$ and R_2 on $\langle e_1 + \omega e_2 + \overline{\omega} e_3\rangle \oplus \langle -e_2 + \overline{\omega} e_3 + \omega e_4\rangle$. Also $\langle g,h\rangle \cong SL_2(3)$, $\langle g,k\rangle \cong A_4$, and $\langle h,k\rangle \cong A_4$. Assume there is a basis $u_1, \ldots, u_n$ on which $X \mid \langle g,h,k\rangle$ is monomial. By Lemma 2.6, after ordering $u_1, \ldots, u_n$ correctly, we may assume $X \mid \langle g,h\rangle$ acts trivially on $u_5, \ldots, u_n$. As $R_1 \oplus R_2$ acts on the same subspace as $\widetilde{X}$ does, $\langle e_1, e_2, e_3, e_4\rangle = \langle u_1, u_2, u_3, u_4\rangle$, and $X \mid \langle g,h,k\rangle$ acts trivially on $u_5, \ldots, u_n$.

Let K satisfy either 1. or 2. By Lemma 2.3 applied inductively, beginning with K, there exists a subgroup U generated by special 3-elements with $K \subseteq U$ such that $X \mid U = X_1 \oplus s \cdot 1_U$ where $s = 1$ or 2 and X_1 is irreducible.

Assume the lemma is false. Then X_1 is monomial. First we want to show that we may assume $s = 1$. So assume $s = 2$. Let X_1 act on V_1 and let $v_1, \ldots, v_{n-2}$ be a basis of V_1 in which X_1 is monomial. Let $2 \cdot 1_U$ act on W. If K satisfies 2., we may assume $X_1 \mid K$ acts trivially on $v_5, \ldots, v_{n-2}$ by ordering $v_1, \ldots, v_{n-2}$ correctly. Suppose K satisfies 1. As g and h do not commute, we may assume by ordering correctly that $X_1 \mid K$ acts trivially on $v_6, \ldots, v_{n-2}$. By Lemma 2.6, if $n = 8$ and $r = 4$, we may also assume that $X \mid \langle g,h\rangle$ acts trivially on v_5. Thus in any case, we may assume $X \mid K$ acts trivially on $v_\sigma, \ldots, v_{n-2}$ where $\sigma = 6$ if $n > 8$ and $\sigma = 5$ if $n = 8$.

Let h^* be a special 3-element such that $X(h^*)$ does not leave V_1

invariant. So $X(h^*)$ does not leave both $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$ and $\langle v_1, \ldots, v_\sigma, v_{\sigma+2}, \ldots, v_{n-2}\rangle$ invariant. By renumbering $v_\sigma, \ldots, v_{n-2}$, we may assume $X(h^*)$ does not leave $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$ invariant. Let $M = \langle u \in U \,|\, v_\sigma X(u) = v_\sigma$ where u is a special 3-element$\rangle$. By Lemma 2.4, $X|M = X_{1,\sigma} \oplus 3 \cdot 1_M$ where $X_{1,\sigma}$ acts irreducibly on $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$ and $3 \cdot 1_M$ on $\langle v_\sigma, W\rangle$. Note $K \subseteq M$. Then $X|\langle M, h^*\rangle = Y \oplus 1_{\langle M, h^*\rangle}$ where either Y is irreducible or $Y = Y_1 \oplus \xi$ with Y_1 irreducible and ξ linear. If Y is irreducible, replace U by $\langle M, h^*\rangle$. Assume then that $Y = Y_1 \oplus \xi$. If Y_1 is primitive, $\xi = 1_{\langle M, h^*\rangle}$ by Mitchell, contradicting the assumption that the result is false. So Y_1 is monomial on the subspace $\widetilde{V}_1 \supset \langle v_1, v_2, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$. If $\widetilde{v}_1, \ldots, \widetilde{v}_{n-2}$ is a basis of $\widetilde{V}_1$ in which Y_1 is monomial, by Lemma 1.4, applied to M, when $\widetilde{v}_1, \ldots, \widetilde{v}_{n-2}$ are ordered and scaled correctly, $\widetilde{v}_1 = v_1, \ldots, \widetilde{v}_{\sigma-1}, = v_{\sigma-1}, \widetilde{v}_{\sigma+1} = v_{\sigma+1}, \ldots, \widetilde{v}_{n-2} = v_{n-2}$ and $\widetilde{v}_\sigma \in \langle v_\sigma, W\rangle$. As $X(h^*)$ does not leave $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$ invariant, $Y_1(h^*)$ is not diagonal. In particular $\xi(h^*) = 1$ and so $X|\langle M, h^*\rangle = Y_1 \oplus 2 \cdot 1_{\langle M, h^*\rangle}$. Let $2 \cdot 1_{\langle M, h^*\rangle}$ act on W^*. Then $W^* \subset \langle v_\sigma, W\rangle$. As W and W^* have dimension 2, $W \cap W^* \neq \{0\}$, and so $X|\langle U, h^*\rangle$ acts trivially on $W \cap W^*$. Hence $X|\langle U, h^*\rangle = \widetilde{X} \oplus 1_{\langle U, h^*\rangle}$ where $\widetilde{X}$ is irreducible of degree $n - 1$, because $X(h^*)$ does not leave V_1 invariant.

So without loss of generality, we may assume $s = 1$ and X_1 acts on V_1. Let $v_1, \ldots, v_{n-1}$ be a basis of V_1 in which X_1 is monomial. Suppose 1_U acts on $\langle v_n\rangle$. Assume first that h is a special 3-element such that $X(h)$ leaves V_1 invariant. Then $X|\langle U, h\rangle = \widetilde{X} \oplus \xi$ where $\widetilde{X}$ acts on V_1 and ξ on $\langle v_n\rangle$. If $\widetilde{X}$ is primitive, by Mitchell, $\xi(h) = 1$, and we have a contradiction to the assumption that the result is false. So $\widetilde{X}$ is monomial and by Lemma 1.4, X(h) is monomial in $v_1, \ldots, v_n$.

Now let h be a special 3-element such that X(h) does not leave V_1 invariant. As earlier, we may order $v_1, \ldots, v_{n-1}$ correctly so that $X|K$ acts trivially on $v_\sigma, \ldots, v_{n-1}$ where $\sigma = 5$ if $n = 8$ and $\sigma = 6$ if $n > 8$. Also X(h) does not leave all three of $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+2}, \ldots, v_{n-1}\rangle$, $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+1}, \ldots, v_{n-2}\rangle$, and $\langle v_1, \ldots, v_\sigma, v_{\sigma+3}, \ldots, v_{n-1}\rangle$ invariant. By numbering $v_\sigma, \ldots, v_{n-1}$ correctly, we may assume X(h) does not leave $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+2}, \ldots, v_{n-1}\rangle$ invariant. Let $M = \langle u \in U \,|\, v_\sigma X(u) = v_\sigma$, $v_{\sigma+1}X(u) = v_{\sigma+1}$ where u is a special 3-element$\rangle$. By Lemma 2.4 applied twice, $X|M = \widetilde{X} \oplus 3 \cdot 1_M$ where $\widetilde{X}$ is irreducible and monomial on $\langle v_1, \ldots, v_{\sigma-1}, v_{\sigma+2}, \ldots, v_{n-1}\rangle$ and $3 \cdot 1_M$ acts on $\langle v_\sigma, v_{\sigma+1}, v_n\rangle$. But $K \subseteq M$ and $X|\langle M, h\rangle = Y \oplus 1_{\langle M, h\rangle}$ where either Y is irreducible or $Y = Y_1 \oplus \xi$ with Y_1 being irreducible and ξ being linear. By assumption that the result is false, if Y is irreducible, Y is monomial. Also if Y is reducible, Y_1 is monomial by Mitchell and the assumption that the result is false. In either case there is an $i < n$ with $v_iX(h) = v_i$ by Lemma 1.4.

If $1 \leq i \leq \sigma-1$, by Lemma 1.4, choose $u \in U$ such that $v_\sigma X(u) = v_i$. If $i \geq \sigma$, let $u = 1$. Then $X|K^u$ acts trivially on v_i. Let $N_i = \langle g \in U \,|\, v_iX(g) = v_i$ and g is a special 3-element$\rangle$. So $K^u \subseteq N_i$ and by Lemma 2.4, $X|N_i = Y_i \oplus 2 \cdot 1_{N_i}$ where Y_i is irreducible and acts with monomial basis $v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}$. Also $2 \cdot 1_{N_i}$ acts on $\langle v_i, v_n\rangle$. As X(h) leaves v_i invariant but not V_1, X(h) does not leave $\langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1}\rangle$ invariant. Thus $X|\langle N_i, h\rangle = R_i \oplus 1_{\langle N_i, h\rangle}$ where R_i acts irreducibly on V_2 and $1_{\langle N_i, h\rangle}$ acts on $\langle v_i\rangle$. As the result is assumed false, R_i is monomial in some basis $\tilde{v}_1, \ldots, \tilde{v}_{n-1}$ of V_2. By Lemma 1.4 applied to N_i, we can renumber and rescale $\tilde{v}_1, \ldots, \tilde{v}_{n-1}$ so that $v_1 = \tilde{v}_1, \ldots, v_{i-1} = = \tilde{v}_{i-1}, v_{i+1} = \tilde{v}_{i+1}, \ldots, v_{n-1} = \tilde{v}_{n-1}$ and $\tilde{v}_i \in \langle v_i, v_n \rangle$. As X(h) does not leave $\langle v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_{n-1} \rangle$ invariant, $\tilde{v}_i R_i(h) \notin \langle \tilde{v}_i \rangle$. So for some $j, k < n$ with i, j, k all distinct,

$$\begin{cases} v_j X(h) = a v_k \\ v_k X(h) = b \tilde{v}_i & \qquad abc = 1 \\ \tilde{v}_i X(h) = c v_j \\ v_\ell X(h) = v_\ell \text{ for } 1 \leq \ell \leq n-1, \ell \neq j, k \end{cases}.$$

Choosing ℓ with $1 \leq \ell \leq n-1$ but distinct from i, j, k, we may apply the same argument to get

$$\begin{cases} v_p X(h) = \alpha v_q \\ v_q X(h) = \beta v_\ell^* & \qquad \alpha\beta\gamma = 1 \\ v_\ell^* X(h) = \gamma v_p \\ v_t X(h) = v_t \text{ for } 1 \leq t \leq n-1, \ t \neq p, q \end{cases}$$

where $p, q, \ell < n$ are distinct and $v_\ell^* \in \langle v_\ell, v_n \rangle$. By comparing the two expressions for X(h), the only possibility is $j = p$ and $k = q$. So $\langle \tilde{v}_i \rangle = \langle v_\ell^* \rangle \subseteq \langle v_i, v_n \rangle \cap \langle v_\ell, v_n \rangle = \langle v_n \rangle$. In particular, X(h) is monomial in the basis $v_1, \ldots, v_n$.

Therefore if N is the normal subgroup of G generated by the special 3-elements of G, $X|N$ is monomial. As in the concluding paragraph of the proof of Lemma 2.5, this is a final contradiction.

Corollary 2.1: Let K be as in Lemma 2.7 and $n \geq 8$. Then there exists a subgroup U of G generated by special 3-elements such that

$X|U = R \oplus r1_U$ where R is primitive and irreducible with $r = 1$ or 2, and $K \subseteq U$.

Proof: Let H be as in the conclusion of Lemma 2.7. Then $K^g \subseteq H$ for some $g \in G$. Letting $U = H^{g^{-1}}$, the desired result follows.

Corollary 2.2: Let H be a subgroup of G generated by special 3-elements such that $X|H = \widetilde{X} \oplus (n-r)1_H$ where $\widetilde{X}$ is irreducible of degree r and $3 \leqslant r < n$. Then there exists a subgroup U of G such that $X|U = Y \oplus y1_U$ where Y is irreducible and primitive with $y = 1$ or 2.

Proof: By Lemma 2.3 and induction, we may assume $r = n-2$ or $n-1$. If $\widetilde{X}$ is primitive, we are done. So assume $\widetilde{X}$ is monomial. By Lemma 1.4, there exist special 3-elements h_1, h_2 such that $X|\langle h_1, h_2\rangle = S \oplus (n-3)1_{\langle h_1, h_2\rangle}$ where S is irreducible of degree 3. We are done by Corollary 2.1.

CHAPTER III
THE PROOF OF THEOREM 1

Theorem 1 will now be proved. The primitive groups of degree 7 or less are known. By examining Brauer [3], Lindsey [14, 16, 17], and Wales [27, 28] very carefully, we see there are no primitive, irreducible, unimodular linear groups of degree 5, 6, or 7 containing special 4-elements. We may assume $n \geqslant 8$.

Let g and h be special 4-elements which do not commute. Then $X | \langle g, h\rangle = Y \oplus (n-4)1_{\langle g, h\rangle}$ where Y has an irreducible constituent of degree at least 2.

<u>Case A</u>: $Y = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 2.

Let $H_i = Y_i(\langle g, h\rangle)$. Then $Y(\langle g, h\rangle) = H$ is a subdirect product of H_1 and H_2. By examining Blichfeldt [2], we check the various possibilities for H_1.

<u>Subcase 1</u>: $H_1/Z(H_1) \cong A_5$. (This proof is as in Lemma 2 of [14]).

Then H' is a subdirect product of H_1' and H_2', which are unimodular groups of 2×2 matrices. Let M_i be the set of all elements in H_i' which occur with component the identity of H_j' $(j \neq i)$ in the subdirect product. By Theorem 5.5.1 of Hall [11], $M_i \triangleleft H_i'$ and $H_1'/M_1 \cong H_2'/M_2$. As $H_1'/Z(H_1') \cong A_5$, either $M_1 = H_1' = H_1''$ or $M_1 \subseteq Z(H_1')$. As A_5 has no representation of degree 2, $|Z(H_1')| = 2$. If $M_1 = H_1'$, there is an element of order 3 in M_1, which must have eigenvalues ω, $\overline{\omega}$. Multiplying by the nontrivial element in $Z(H_1')$ gives an element in $X(G)$ with eigenvalues $-\omega, -\bar{\omega}, 1, 1, 1, \ldots$, contradicting Blichfeldt. If $M_1 \subseteq Z(H_1')$, the only possibility is that $H_2'/Z(H_2') \cong A_5$. In any case there are elements $h_i \in H_i'$ with eigenvalues $-\omega$, $-\bar{\omega}$ which are paired in H. So $X(G)$ contains an element with eigenvalues $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, \ldots$, contradicting Blichfeldt. Subcase 1. is therefore impossible.

<u>Subcase 2</u>: $H_1/Z(H_1) \cong S_4$.

Then H' is isomorphic to a subdirect product of H_1' and H_2', which again are unimodular. Let M_i be as in subcase 1. Then $H_1'/Z(H_1') \cong A_4$. The only element of order 2 in a 2 dimensional unimodular group is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. So $|Z(H_1')| = 2$ and the Sylow 2-subgroup of H_1', which has order 8, is either cyclic or quaternion. As A_4 has no elements of order 4, the Sylow 2-subgroup of H_1' must be quaternion. If M_1 contains the Sylow 2-subgroup of H_1', there exist special 4-elements g_1, g_2 such that $\langle g_1, g_2 \rangle$ is the quaternion group of order 8 and $X | \langle g_1, g_2 \rangle = X^* \oplus (n-2)1_{\langle g_1, g_2 \rangle}$ where $X^*(g_1) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $X^*(g_2) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ in some basis. If M_1 does not contain the Sylow 2-subgroup of H_1', as $H_1'/Z(H_1') \cong A_4$ and $M_1 \triangleleft H_1'$, $M_1 \subseteq Z(H_1')$. As $H_2'/M_2 \cong H_1'/M_1$, the only possibility, by looking at Blichfeldt's list of 2 dimensional groups is $H_2'/Z(H_2') \cong A_4$. As in subcase 1, there are elements $h_i \in H_i'$ with eigenvalues $-\omega, -\bar{\omega}$, which are paired in H. So $X(G)$ contains an element with eigenvalues $-\omega, -\bar{\omega}, -\omega, -\bar{\omega}, 1, 1, \ldots$, contradicting Blichfeldt.

<u>Subcase 3:</u> $H_1/Z(H_1) \cong A_4$.

This subcase is impossible as 2-elements do not generate A_4.

<u>Subcase 4</u>: Y_1 is monomial and unimodular.

As Y_1 is irreducible and unimodular, $Y_1(g)$ and $Y_1(h)$ both have eigenvalues i and -i. So $Y_2 = 2 \cdot 1_{\langle g, h \rangle}$.

<u>Subcase 5</u>: Y_1 is monomial and not unimodular.

Let Y_i act on V_i and let v_1, v_2 be a basis of V_1 in which Y_1 is monomial. As both $Y_1(g)$, $Y_1(h)$ could not be diagonal, only one of $Y_1(g)$, $Y_1(h)$ could have eigenvalue structure 1, i or 1, -i. As Y_1 is not unimodular, we may assume by replacing g by g^{-1} if necessary that $Y_1(g)$ has eigenvalues 1, i. So $Y_2(h)$ is the identity and hence $Y_2 = \xi_1 \oplus \xi_2$. Let ξ_1 act on v_3, ξ_2 on v_4. By ordering v_1, v_2 and v_3, v_4 correctly we get in the basis $v_1, \ldots, v_4$ that

$$(Y_1 \oplus Y_2)(g) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (Y_1 \oplus Y_2)(h) = \begin{pmatrix} 0 & -\delta & 0 & 0 \\ \delta^{-1} & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

But

$$(Y_1 \oplus Y_2)(h^{-1}ghg^{-1}) = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and letting $g_1 = h$, $g_2 = h^{-1}ghg^{-1}$, which are special 4-elements, $X \mid \langle g_1, g_2 \rangle = Y_3 \oplus (n-2)1_{\langle g_1, g_2 \rangle}$ where Y_3 is monomial and irreducible.

Therefore if case A holds, we may choose special 4-elements g_1, g_2 such that $X|\langle g_1, g_2\rangle = X_1 \oplus (n-2)1_{\langle g_1, g_2\rangle}$ where X_1 is irreducible and monomial. Let X_1 act on V_1 and choose a special 4-element g_3 such that $X(g_3)$ does not leave V_1 invariant. So $X|\langle g_1, g_2, g_3\rangle =$ $X_2 \oplus (n-4)1_{\langle g_1, g_2, g_3\rangle}$ where X_2 is irreducible or $X_2 = X_3 \oplus \xi$ such that X_3 is irreducible of degree 3. Assume the latter is the case with $\xi \neq 1_{\langle g_1, g_2, g_3\rangle}$. Then by Mitchell, X_3 is monomial on some basis v_1, v_2, v_3. Consider $X_3(g_i)$ in this basis; $X_3(g_i)$ must act trivially on at least one v_j. Assume first that $X|\langle g_1, g_2\rangle$ does not leave any $\langle v_i\rangle$ invariant. Then by ordering v_1, v_2, v_3 correctly, we get

$$X_3(g_1) = \begin{pmatrix} 0 & -\delta & 0 \\ \delta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad X_3(g_2) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -\epsilon \\ 0 & \epsilon^{-1} & 0 \end{pmatrix} .$$

But then $X_3|\langle g_1, g_2\rangle$ is irreducible, a contradiction. So $X|\langle g_1, g_2\rangle$ leaves some $\langle v_i\rangle$ invariant. But as $\xi(g_1) = \xi(g_2) = 1$, $\xi(g_3) \neq 1$, $X_3(g_3)$ is diagonal and so also leaves $\langle v_i\rangle$ invariant, a contradiction. Thus in any case $X|\langle g_1, g_2, g_3\rangle = \widetilde{X} \oplus (n-s)1_{\langle g_1, g_2, g_3\rangle}$ where $s = 3$ or 4 and $\widetilde{X}$ is irreducible when case A holds.

Case B: $Y = Y_1 \oplus \xi$ where Y_1 is irreducible of degree 3.

If Y_1 is primitive, $\xi = 1_{\langle g, h\rangle}$ by Mitchell. Assume Y_1 is monomial on V_1; let v_1, v_2, v_3 be a basis of V_1 in which Y_1 is monomial. If $Y_1(g)$ or $Y_1(h)$ have eigenvalues $1, 1, -i$ or $1, 1, i$, they are diagonal in this basis. In any case $Y_1(g)$ and $Y_1(h)$ each fix one of the subspaces $\langle v_j\rangle$, j depending on g and h. So neither $Y_1(g)$ nor $Y_1(h)$ is diagonal and hence $\xi = 1_{\langle g, h\rangle}$.

Therefore in any case G contains a subgroup H generated by special 4-elements such that $X|H = X_1 \oplus (n-r)1_H$ where X_1 is irreducible and $r = 3$ or 4. Assume Theorem 1 is false. Let $n \geq 8$ be minimal such that there is a counterexample. Thus, recalling the remark at the beginning of this section, there does not exist irreducible, primitive unimodular groups of degree n-1 or n-2 which contain special 4-elements. Applying Lemma 2.1 inductively, starting with H, there exists a subgroup U generated by special 4-elements such that $X|U = Y \oplus y \cdot 1_U$ where Y is irreducible and $y = 1$ or 2. By Lemma 2.5, we may assume Y is primitive, contradicting the minimality of n. So Theorem 1 holds.

CHAPTER IV

A SPECIAL CASE OF THEOREM 2

We return to the proof of Theorem 2. In this chapter we classify G when $n \geqslant 8$ and a certain hypothesis holds, which will be useful for induction purposes. Define hypothesis (A) as follows:

(A) If U is any subgroup of G generated by special 3-elements such that $X|U = S \oplus s \cdot 1_U$ where S is irreducible and primitive of degree $n-s$ with $s = 1$ or 2, then $U \cong A_{n-s+1}$.

We first prove some results on the irreducible characters of the alternating and symmetric groups.

By the work of Frobenius [9], the characters of S_n are all related to the partitions of n into integers. Let $(\lambda) = \{\lambda_1, \ldots, \lambda_k\}$ be a partition of the integer n into nonnegative integers where $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_k > 0$. Let $\ell_i = \lambda_i + k - i$. So $n \geqslant \ell_1 > \ell_2 > \ldots > \ell_k > 0$. Then the degree $d_{(\lambda)}$ of the irreducible character corresponding to (λ) is

$$d_{(\lambda)} = n! \frac{\prod_{p<q} (\ell_p - \ell_q)}{\ell_1! \ldots \ell_k!} .$$

All irreducible characters come from such partitions.

Lemma 4.1: The group S_n for $n \geqslant 9$ has two irreducible characters of degree 1 due to the partitions $(\lambda) = \{n\}$ and $(\lambda) = \{1, 1, \ldots, 1\}$, two irreducible characters of degree $n-1$ due to the partitions

$(\lambda) = \{n-1, 1\}$ and $(\lambda) = \{2, 1, 1, \ldots, 1\}$. All other irreducible characters have degree greater than $2(n+1)$.

Proof: By checking the character tables, we see that the result holds for $n = 9, \ldots, 14$ (see [13,19,29,30]). So assume $n \geq 15$ and proceed by induction. We split this into several cases.

The case $k = 1$: The only partition is $(\lambda) = \{n\}$. So $d_{(\lambda)} = n! \frac{1}{n!} = 1$.

The case $k = 2$: Then $(\lambda) = \{\lambda_1, \lambda_2\}$ and $\lambda_1 \geq \frac{n}{2}$. If $\lambda_1 = n-1$, then $\ell_1 = n$, $\ell_2 = 1$, and $d_{(\lambda)} = n-1$. If $\lambda_1 = n-2$, then $\ell_1 = n-1$, $\ell_2 = 2$, and $d_{(\lambda)} = \frac{n(n-3)}{2} > 2(n+1)$ for $n \geq 15$.

So assume $\frac{n}{2} \leq \lambda_1 \leq n-3$. Let $\lambda_1 = m$. Then $\ell_1 = m+1$ and $\ell_2 = n-m$. So $d_{(\lambda)} = n! \frac{(m+1+m-n)}{m!(n-m)!} > \frac{n!}{m!(n-m)!}$. Notice that if $m, m' \geq \frac{n}{2}$ with $m > m'$, $\frac{n!}{m'!(n-m')!} > \frac{n!}{m!(n-m)!}$. Also for $m = n-3$ and $n \geq 15$, $\frac{n!}{m!(n-m)!} > 2(n+1)$. So $d_{(\lambda)} > 2(n+1)$.

The case $k = n$: The only partition is $(\lambda) = \{1, 1, \ldots, 1\}$. So $\ell_i = 1 + n - i$ and

$$d_{(\lambda)} = \frac{n! \prod_{p=1}^{n-1} \left(\prod_{q=p+1}^{n} (\ell_p - \ell_q) \right)}{n!(n-1)! \ldots 1!} = \frac{n! \prod_{p=1}^{n-1} (n-p)!}{n!(n-1)! \ldots 1!} = 1 .$$

The case $(\lambda) = \{n-k+1, 1, 1, \ldots, 1\}$ for $3 \leq k \leq n-1$: Then $\ell_1 = n$ and $\ell_i = k - (i-1)$ for $i \geq 2$. So

$$d_{(\lambda)} = n! \frac{\prod_{q=2}^{k} (\ell_1 - \ell_q) \prod_{1<p<q} (\ell_p - \ell_q)}{n!(k-1)!(k-2)!\dots 1!}$$

$$= \frac{\{(n-(k-1))\dots(n-1)\}\{(k-2)!(k-3)!\dots 1!\}}{(k-1)!(k-2)!\dots 1!} = \binom{n-1}{k-1} .$$

If $k = n-1$, $(\lambda) = \{2,1,1,\dots,1\}$ and $d_{(\lambda)} = n-1$. Because $\binom{n-1}{k-1} = \binom{n-1}{n-k}$, to prove $\binom{n-1}{k-1} > 2(n+1)$ for $n \geq 15$ and $3 \leq k \leq n-2$, it suffices to prove it for $3 \leq k \leq \frac{n}{2}$. But if $3 \leq k \leq \frac{n}{2}$, for $n \geq 15$, $2(n+1) < \frac{(n-1)(n-2)}{2} = \binom{n-1}{2} \leq \binom{n-1}{k-1}$.

The remaining cases: In the remaining cases we have $k \geq 3$ and we do not have the partition $(\lambda) = \{n-k+1, 1, \dots, 1\}$. Let $\lambda_k = r$. As $k \geq 3$, $\frac{n}{3} \geq r$. Consider the partition $(\lambda') = \{\lambda_1, \dots, \lambda_{k-1}\}$ of $n-r$. Let $\ell'_i = \lambda_i + (k-1) - i = \ell_i - 1$. Then $\ell'_p - \ell'_q = \ell_p - \ell_q$ and

$$d_{(\lambda)} = n! \frac{\prod_{p<q} (\ell_p - \ell_q)}{\ell_1! \dots \ell_k!}$$

$$= \left\{ (n-r)! \frac{\prod_{p<q<k} (\ell'_p - \ell'_q)}{\ell'_1! \dots \ell'_{k-1}!} \right\} \left\{ \frac{n(n-1)\dots(n-r+1) \prod_{p=1}^{k-1} (\ell_p - r)}{\ell_1 \dots \ell_{k-1}\, r!} \right\} .$$

But

$$d_{(\lambda')} = (n-r)! \frac{\prod_{p<q<k} (\ell'_p - \ell'_q)}{\ell'_1! \dots \ell'_{k-1}!}$$

is the degree of a character of S_{n-r} corresponding to the partition (λ').

By induction, noting that we have enough initial cases (as $r \leqslant \frac{n}{3}$), $d_{(\lambda')} > 2(n-r+1)$ because (λ') is not of the form $\{1,\ldots,1\}$, $\{n-r\}$, $\{n-r-1,1\}$, or $\{2,1,\ldots,1\}$.

Assume first that $k-1 > r$. As $n \geqslant \ell_1 > \ell_2 > \ldots > \ell_k > 0$, $\ell_p - r \geqslant \ell_{p+r}$ for $p \leqslant k-1-r$. So $\ell_p - r/\ell_{p+r} \geqslant 1$ for $p \leqslant k-1-r$. As $\ell_k = r$, $\ell_{k-r} - r > \ldots > \ell_{k-1} - r > 0$ and so

$$\frac{\prod_{p=k-r}^{k-1} (\ell_p - r)}{r!} \geqslant 1 \quad .$$

Thus

$$d_{(\lambda)} = d_{(\lambda')} \frac{n(n-1)\ldots(n-r+1)}{\ell_1 \ell_2 \ldots \ell_r} \prod_{p=1}^{k-1-r} \left(\frac{\ell_p - r}{\ell_{p+r}} \right) \frac{\prod_{p=k-r}^{k-1} (\ell_p - r)}{r!}$$

$$> 2(n-r+1) \frac{n(n-1)\ldots(n-r+1)}{\ell_1 \ell_2 \ldots \ell_r} \quad .$$

As $n \geqslant \ell_1 > \ell_2 > \ldots > \ell_r$, $\frac{n+1-i}{\ell_i} \geqslant 1$ for $2 \leqslant i \leqslant r$. So $d_{(\lambda)} > 2(n-r+1) \frac{n}{\ell_1}$. If $r = 1$, $\lambda_1 \leqslant n-(k-1)$ with equality holding only if $(\lambda) = \{n-k+1, 1, 1, \ldots, 1\}$ which we are excluding. So $\ell_1 = \lambda_1 + k - 1 \leqslant n - 1$ and $d_{(\lambda)} > \frac{n}{n-1} 2n > 2(n+1)$. If $r > 1$, $\lambda_1 \leqslant n - r(k-1)$ and $\ell_1 = \lambda_1 + k - 1$ $\leqslant n - (r-1)(k-1) \leqslant n - 2(r-1)$ as $k \geqslant 3$. So $d_{(\lambda)} > \frac{n}{n-2(r-1)} 2(n-r+1)$. But $\frac{n}{n-2(r-1)} 2(n-r+1) \geqslant 2(n+1)$ is equivalent to $0 \leqslant n(r-2) + 2(r-1)$, which is true for $r \geqslant 2$.

Assume that $k-1 \leqslant r$. Then

$$d_{(\lambda)} = d_{(\lambda')} \left\{ \frac{n(n-1)\ldots(n-(k-2))}{\ell_1 \ell_2 \ldots \ell_{k-1}} \right\} \left\{ \frac{(n-(k-1))(n-k)\ldots(n-r+1)}{r(r-1)\ldots(r-(r-k))} \right\} \times \left\{ \frac{(\ell_1 - r)(\ell_2 - r)\ldots(\ell_{k-1}-r)}{(k-1)(k-2)\ldots 1} \right\} .$$

The middle product is vacuous if $k-1 = r$. As $\ell_1 > \ell_2 > \ldots > \ell_{k-1} > \ell_k = r$, $\ell_i - r \geq k - i$ for $i \leq k-1$. So

$$\prod_{i=1}^{k-1} \left(\frac{\ell_i - r}{k-i} \right) \geq 1 \quad .$$

As $\frac{n}{3} \geq r \geq k-1$, $n-(k-1) \geq r$ and $n-(k-1)-i \geq r-i$ for $0 \leq i \leq r-k$. Thus

$$\left\{ \frac{(n-(k-1))(n-k)\ldots(n-r+1)}{r(r-1)\ldots(r-(r-k))} \right\} \geq 1 \quad .$$

As $n \geq \ell_1 > \ell_2 > \ldots > \ell_{k-1}$, $\frac{n-i}{\ell_{i+1}} \geq 1$. Therefore $d_{(\lambda)} > 2(n-r+1)\frac{n}{\ell_1}$. But $\lambda_1 \leq n-r(k-1)$ and $\ell_1 = \lambda_1 + k - 1 \leq n-(r-1)(k-1) \leq n-2(r-1)$. So $d_{(\lambda)} > \frac{n}{n-2(r-1)} 2(n-r+1)$. Proceeding as above, $d_{(\lambda)} > 2(n+1)$, and the lemma is proved.

Lemma 4.2: If $n \geq 7$, A_n has only one irreducible character of degree 1, which is the trivial character, and only one irreducible character of degree $n-1$, which is the nontrivial constituent of the permutation character. All other irreducible characters have degree greater than $n+1$.

Proof: By Frobenius [10], all irreducible characters of S_n remain irreducible or split into two conjugate irreducible characters, obviously of equal degree, when restricted to A_n. All characters of A_n are obtained in this way. The result is true for $n = 7, 8$ by checking the character tables. Assume $n \geq 9$. Let ρ_1 be the permutation character of S_n on n points and let μ be the nontrivial linear character of S_n. Two irreducible characters of S_n of degree n-1 are $\rho_1 - 1_{S_n}$ and $\mu(\rho_1 - 1_{S_n})$. They are equal and irreducible when restricted to A_n. As all other irreducible characters of S_n have degree greater than $2(n+1)$ by Lemma 4.1, all other irreducible characters of A_n have degree greater than n+1.

Notice that if $H \cong A_m$ for $m \geq 7$ is a subgroup of G such that $X|H = X_1 \oplus (n-m+1)1_H$, X_1 is irreducible and the 3-cycles of H correspond precisely to the special 3-elements of H. Also X_1 is primitive by Lemma 1.4 as A_m is simple. These facts will be used without reference in the rest of the paper. We now give some results on generators and relations of A_n.

<u>Lemma 4.3</u>: Let $U_k = \langle f_1, \ldots, f_{k-2} \rangle$ for $k \geq 5$. Suppose the following relations hold:

(1) $f_1^3 = 1$, $(f_d f_{d-1} \ldots f_1)^2 = 1$ for $d = 2, \ldots, k-2$

(2) $f_{d+1}^3 = 1$ for $d = 1, \ldots, k-3$

(3) $((f_d \ldots f_1)(f_e \ldots f_1))^2 = 1$ for $d = 1, \ldots, k-4$ and $e = d+2, \ldots, k-2$.

Then either $U_k = 1$ or $U_k \cong A_k$. Also in A_k if we let $f_d = (d, d+1, d+2)$, then $f_1, \ldots, f_{k-2}$ satisfy (1), (2), and (3).

Proof: Let $h_d = f_d f_{d-1} \dots f_1$. Then (1) is equivalent to

(1′) $h_1^3 = 1$, $h_d^2 = 1$ for $d = 2, \dots, k-2$

Also (2) is equivalent to

(2′) $(h_d h_{d+1})^3 = 1$

because $(h_d h_{d+1})^3 = (h_d h_{d+1}^{-1})^3 = ((f_d \dots f_1)(f_1^{-1} \dots f_{d+1}^{-1}))^3 = (f_{d+1}^{-1})^3$. In addition, (3) is equivalent to

(3′) $(h_d h_e)^2 = 1$ for $d=1, \dots, k-4$ and $e=d+2, \dots, k-2$.

By Moore [22], $U_k = 1$ or $U_k \cong A_k$ as $U_k = \langle h_1, \dots, h_{k-2} \rangle$.

Now let $f_d = (d, d+1, d+2)$. Then $f_d \dots f_1 = (1,2)(d+1, d+2)$ for $d \geq 2$. Thus (1) and (2) hold. Also $(f_1 f_e \dots f_1)^2 = ((1,2,3)(1,2)(e+1,e+2))^2 = 1$ for $e \geq 3$ and for $d > 1$, $e \geq d+2$, $(f_d \dots f_1 f_e \dots f_1)^2 = ((1,2)(d+1,d+2)(1,2)(e+1,e+2))^2 = 1$. So (3) holds.

Lemma 4.4: Let U be a group with a subgroup $H \cong A_k$ for $k \geq 5$. Let $H = \langle f_1, \dots, f_{k-2} \rangle$ where f_d corresponds to $(d, d+1, d+2)$ on $\{1, \dots, k\}$. Assume $f_{k-1} \in U$ such that $f_{k-1}^3 = 1$, $(f_{k-2} f_{k-1})^2 = 1$, f_{k-1} commutes with $f_1, \dots, f_{k-4}$, and $f_{k-1} f_{k-2} f_{k-3} f_{k-1} = f_{k-2} f_{k-3}$. Then $\langle H, f_{k-1} \rangle \cong A_{k+1}$.

Proof: It suffices to show (1), (2), and (3) hold in Lemma 4.3. By Lemma 4.3, these relations hold for the f_i's with $i \leq k-2$. So for (1) we only need to consider $d = k-1$. But

$$\begin{aligned}(f_{k-1} f_{k-2} \dots f_1)^2 &= (f_{k-1} f_{k-2} f_{k-3} f_{k-1}) f_{k-4} \dots f_1 f_{k-2} \dots f_1 \\ &= f_{k-2} \dots f_1 f_{k-2} \dots f_1 = 1 \quad .\end{aligned}$$

Thus (1) holds. Clearly (2) holds. We only need to consider $e = k-1$ in (3). Assume first that $d \leq k-4$. Then

$(f_d \ldots f_1 f_{k-1} \ldots f_1)^2$

$$= f_d \ldots f_1(f_{k-1}f_{k-2}f_{k-3}f_{k-1})f_{k-4} \ldots f_1 f_d \ldots f_1 f_{k-2} \ldots f_1$$

$$= (f_d \ldots f_1 f_{k-2} \ldots f_1)^2 = 1 \quad \text{as} \quad d+2 \leq k-2.$$

Now assume $d = k-3$. First as $f_{k-1}f_{k-2}f_{k-3}f_{k-1} = f_{k-2}f_{k-3}$ and $(f_{k-2}f_{k-1})^2 = 1$, $f_{k-3}f_{k-1} = f_{k-1}f_{k-2}^2 f_{k-3}$. So

$f_{k-3} \ldots f_1 f_{k-1} f_{k-2} f_{k-3} \ldots f_1 f_{k-3} \ldots f_1 f_{k-1} f_{k-2} \ldots f_1$

$$= f_{k-3} \ldots f_1 f_{k-1} f_{k-2} f_{k-3} \ldots f_1(f_{k-3}f_{k-1})f_{k-4} \ldots f_1 f_{k-2} \ldots f_1$$

$$= f_{k-3} \ldots f_1 f_{k-1} f_{k-2} f_{k-3} \ldots f_1 f_{k-1} f_{k-2}(f_{k-2} f_{k-3} \ldots f_1 f_{k-2} \ldots f_1)$$

$$= f_{k-3} \ldots f_1(f_{k-1}f_{k-2}f_{k-3}f_{k-1})f_{k-4} \ldots f_1 f_{k-2}$$

$$= f_{k-3} \ldots f_1 f_{k-2} \ldots f_1 f_{k-2} = f_{k-2}^{-1}(f_{k-2} \ldots f_1)^2 f_{k-2} = 1 .$$

Lemma 4.5: Let U be a group containing a subgroup $H = \langle h_1, \ldots, h_{k-2}\rangle \cong A_k$ for some $k \geq 6$. Assume H acts on $\{1, \ldots, k\}$ with $h_i = (i, i+1, i+2)$ for $1 \leq i \leq k-2$. Let $g \in U$ such that $\langle H, g\rangle \cong A_{k+s}$ where $s = 1$ or 2. Assume $\langle H, g\rangle$ acts on $\{b_1, \ldots, b_{k+s}\}$ and that $h_1, \ldots, h_{k-2}, g$ are 3-cycles in $\langle H, g\rangle$. Then by numbering $b_1, \ldots, b_{k+s}$ correctly, $h_i = (b_i, b_{i+1}, b_{i+2})$. Also there is an $h \in H$ such that g^h or $(g^{-1})^h$ is $(b_{k+s-2}, b_{k+s-1}, b_{k+s})$; if g commutes with h_1, we can choose $h \in \langle h_4, \ldots, h_{k-2}\rangle$.

Proof: Because $\langle h_1, h_3\rangle \cong A_5$ and h_1, h_3 are 3-cycles in $\langle H, g\rangle$, by correct ordering of $b_1, \ldots, b_{k+s}$, $h_1 = (b_1, b_2, b_3)$ and $h_3 = (b_3, b_4, b_5)$. As $h_2 = h_1^{h_3 h_1}$, $h_2 = (b_2, b_3, b_4)$. Assume we have numbered $b_1, \ldots, b_{k+s}$ correctly so that $h_j = (b_j, b_{j+1}, b_{j+2})$ for $1 \leq j \leq m$ where $m \geq 2$. If $m = k-3$, omit h_m in the latter classification and so assume $m \leq k-4$.

As $\langle h_1, \ldots, h_m, h_{m+2}\rangle \cong A_{m+4}$ and each h_i is a 3-cycle, by numbering $b_{m+3}, \ldots, b_{k+s}$ correctly, $h_{m+2} = (b_k, b_{m+3}, b_{m+4})$ for some $k \leq m+2$. Since h_{m+2} commutes with $h_1, \ldots, h_{m-1}$, the only possibility is $k = m+2$. Also $h_{m+1} = h_m^{h_{m+2}h_m} = (b_{m+1}, b_{m+2}, b_{m+3})$. Therefore by induction $h_1, \ldots, h_{k-2}$ have the desired form. If $s = 1$, $g = (b_i, b_j, b_{k+1})$ where $i, j \leq k$, and if $s = 2$, $g = (b_j, b_{k+1}, b_{k+2})$ where $j \leq k$ upon ordering b_{k+1}, b_{k+2} correctly. By double transitivity, choose $h \in H$ with $b_i h = b_{k-1}$ and $b_j h = b_k$. Then $g^h = (b_{k+s-2}, b_{k+s-1}, b_{k+s})$. Suppose g commutes with h_1. Then $i, j \geq 4$. So we could have chosen $h \in \langle h_4, \ldots, h_{k-2}\rangle$ unless $\langle h_4, \ldots, h_{k-2}\rangle$ is not doubly transitive, which occurs only if $k = 6$. If $k = 6$, however, g^h or $(g^{-1})^h$ where $h \in \langle h_4\rangle$ is of the desired form.

<u>Lemma 4.6</u>: Let U be a group with a subgroup $H = \langle h_1, \ldots, h_{k-2}\rangle \cong A_k$ where $k \geq 7$. Assume H acts on $\{1, \ldots, k\}$ and that $h_i = (i, i+1, i+2)$ for $1 \leq i \leq k-2$. Let $g \in U$ such that g commutes with h_1 and h_2. Let $H_1 = \langle h_2, \ldots, h_{k-2}, g\rangle \cong A_k$ and assume H_1 acts on $\{b_1, \ldots, b_k\}$. Furthermore assume $h_2, \ldots, h_{k-2}, g$ are 3-cycles in H_1. Then $\langle H, g\rangle \cong A_{k+1}$.

<u>Proof</u>: By Lemma 4.5, we may assume $h_i = (b_{i-1}, b_i, b_{i+1})$ for $2 \leq i \leq k-2$ and for some $h \in \langle h_5, \ldots, h_{k-2}\rangle$, g^h or $(g^{-1})^h$ is $g_1 = (b_{k-1}, b_k, b_{k+1})$. Also g_1 commutes with h_1 as g, h do. Thus $g_1^3 = 1$, $(h_{k-2}g_1)^2 = 1$, g_1 commutes with $h_1, \ldots, h_{k-4}$, and $g_1 h_{k-2} h_{k-3} g_1 = h_{k-2} h_{k-3}$. By Lemma 4.4, $\langle H, g\rangle = \langle H, g_1\rangle \cong A_{k+1}$.

<u>Lemma 4.7</u>: Assume hypothesis (A) holds and $n \geq 8$. Let M be a subgroup of G generated by special 3-elements such that $X|M = X_1 \oplus (n-m)1_M$ where X_1 is irreducible of degree $m \geq 5$ (including the

possibility that $m = n$) or $X|M = X_1 \oplus \xi \oplus (n-m-1)1_M$ where X_1 is irreducible of degree $m \geq 5$ and ξ is linear. Then X_1 is primitive and if $X|M$ is of the latter form, $\xi = 1_M$.

Proof: Let K be as in the hypothesis of Lemma 2.7. By Corollary 2.1 and hypothesis (A), there is a subgroup U generated by special 3-elements such that $K \subseteq U \cong A_{n-s+1}$ where U and s are as in hypothesis (A). As two special 3-elements of U, which must be 3-cycles of U, either commute, generate A_4, or generate A_5, K can only satisfy 1. of Lemma 2.7 and $K \cong A_4$ or A_5.

Let X_1 act on the subspace V_1 and assume X_1 is monomial. Let $v_1, \ldots, v_m$ be a basis of V_1 in which X_1 is monomial. By Lemma 1.4, after rescaling and reordering $v_1, \ldots, v_m$, there exist special 3-elements $h_1, \ldots, h_{m-2} \in M$ such that

$$\begin{cases} v_i X(h_i) = v_{i+1} \\ v_{i+1} X(h_i) = v_{i+2} \\ v_{i+2} X(h_i) = v_i \\ v_\ell X(h_i) = v_\ell \quad \text{for } \ell \notin \{i, i+1, i+2\} \end{cases}$$

for $1 \leq i \leq m-2$.

Assume first that $X|M = X_1 \oplus \xi \oplus (n-m-1)1_M$ where $\xi \neq 1_M$. Therefore there is a special 3-element $g \in M$ with $\xi(g) = \bar{\omega}$. So $X_1(g)$ must be diagonal in the basis $v_1, \ldots, v_m$ and for some i, $v_i X(g) = \omega v_i$. Let $g_1, g_2 \in \langle h_1, \ldots, h_{m-2} \rangle$ such that $v_i X(g_1) = v_1$ and $v_i X(g_2) = v_2$. Then $h = g_1^{-1} g g_1 g_2^{-1} g^{-1} g_2$ is a special 3-element and

$$(1)\qquad \begin{cases} v_1X(h) = \omega v_1 \\ v_2X(h) = \bar{\omega} v_2 \\ v_\ell X(h) = v_\ell \text{ for } \ell > 2 \end{cases}.$$

But then $X\,|\,\langle h_1, h\rangle = Y \oplus (n-3)1_{\langle h_1, h\rangle}$ where Y is irreducible and $\langle h_1, h\rangle \not\cong A_4$ or A_5, a contradiction.

Therefore $\xi = 1_M$ and we may assume $X\,|M = X_1 \oplus (n-m)1_M$. As X_1 is irreducible, there is a special 3-element $g \in M$ with $X(g)$ not leaving $\langle v_1+\ldots+v_m\rangle$ invariant. First assume $X_1(g)$ is diagonal in the basis $v_1, \ldots, v_m$. Then $v_iX(g) = \omega v_i$ and $v_jX(g) = \bar{\omega} v_j$ for some i and j. Let $g_1 \in \langle h_1, \ldots, h_{m-2}\rangle$ such that $v_iX(g_1) = v_1$ and $v_jX(g_1) = v_2$, by double transitivity. Letting $h = g_1^{-1}gg_1$, we get $X(h)$ as in (1), which is again a contradiction. Therefore $X_1(g)$ is not diagonal. Hence there exist distinct i, j, and k such that

$$\begin{cases} v_iX(g) = av_j \\ v_jX(g) = bv_k \qquad abc = 1 \\ v_kX(g) = cv_i \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{i, j, k\} \end{cases}$$

where not all of a, b, c are 1. Let $g_1 \in \langle h_1, \ldots, h_{m-2}\rangle$ with $v_iX(g_1) = v_2$, $v_jX(g_1) = v_3$, and $v_kX(g_1) = v_4$. Replacing g by $g_1^{-1}gg_1$, we may assume $i = 2$, $j = 3$, and $k = 4$. If g commutes with h_2, then $a = b = c \in \{\omega, \bar{\omega}\}$; however $X\,|\,\langle h_1, g\rangle = Y \oplus (n-4)1_{\langle h_1, g\rangle}$ where Y is irreducible but $\langle h_1, g\rangle \not\cong A_4$ or A_5, a contradiction. So g does not commute with h_2. Hence $X\,|\,\langle h_2, g\rangle = Y \oplus (n-3)1_{\langle h_2, g\rangle}$ where Y is irreducible. The only possibility is that $\langle h_2, g\rangle \cong A_4$ and $\{a, b, c\} = \{-1, -1, 1\}$. By conjugating g by h_2 or h_2^{-1} if necessary, we may assume $a = -1$, $b = 1$, and $c = -1$. But then situation 2. of Lemma 2.7 occurs, a contradiction.

Therefore we can only conclude X_1 is primitive. If $X|M = X_1 \oplus \xi \oplus (n-m-1)1_M$, by Mitchell $\xi = 1_M$.

Lemma 4.8: Assume hypothesis (A) holds and $n \geq 8$. Let H be a subgroup of G generated by special 3-elements. Assume $X|H = X_1 \oplus (n-m)1_H$ where X_1 is irreducible of degree $m \geq 5$. Let g be a special 3-element which does not commute with H. Then $X|\langle H,g\rangle = Y \oplus (n-s)1_{\langle H,g\rangle}$ where $s = m$, $m+1$, or $m+2$ and Y is irreducible.

Proof: We must have $X|\langle H,g\rangle = R \oplus (n-m-2)1_{\langle H,g\rangle}$ where either

1. R is irreducible.
2. $R = R_1 \oplus \xi$ where R_1 is irreducible and ξ is linear.
3. $R = R_1 \oplus \xi_1 \oplus \xi_2$ where R_1 is irreducible and ξ_1, ξ_2 are both linear.

We are done if 1. holds. If 2. holds, $\xi = 1_{\langle H,g\rangle}$ by Lemma 4.7. If 3. holds, since g does not commute with H, one ξ_i, say ξ_2, is $1_{\langle H,g\rangle}$. By Lemma 4.7, ξ_1 is also $1_{\langle H,g\rangle}$ and the lemma is proved.

Lemma 4.9: Assume hypothesis (A) holds and $n \geq 8$. Let H be a subgroup of G generated by special 3-elements such that $X|H = X_1 \oplus 2\cdot 1_H$ where X_1 is irreducible. Let $g \in G$ be a special 3-element such that $X|\langle H,g\rangle$ is irreducible. Then $\langle H,g\rangle \cong A_{n+1}$.

Proof: By Lemma 4.7, X_1 is primitive; so by hypothesis (A), $H \cong A_{n-1}$. Let H act on the set $\{a_1, \ldots, a_{n-1}\}$. The special 3-elements in H are 3-cycles and so define $h_i = (a_i, a_{i+1}, a_{i+2})$ for $1 \leq i \leq n-3$. Then $H = \langle h_1, \ldots, h_{n-3} \rangle$. As $X | \langle H, g \rangle$ is irreducible, g does not commute with H. Thus g does not commute with both $\langle h_1, \ldots, h_{n-4} \rangle$ and $\langle h_2, \ldots, h_{n-3} \rangle$. Let $g_i = h_{n-i-2}^{-1}$ for $1 \leq i \leq n-3$ and $b_i = a_{n-i}$ for $1 \leq i \leq n-1$. Then $g_i = (b_i, b_{i+1}, b_{i+2})$ and $\langle g_1, \ldots, g_{n-4} \rangle = \langle h_{n-3}, \ldots, h_2 \rangle$. If necessary, replacing h_i by g_i and a_{n-i} by b_i, we may assume notation is chosen so that g does not commute with $\langle h_1, \ldots, h_{n-4} \rangle$.

First consider $X | \langle h_1, \ldots, h_{n-4}, g \rangle$. By Lemma 4.2, $X | \langle h_1, \ldots, h_{n-4} \rangle = X_1 | \langle h_1, \ldots, h_{n-4} \rangle \oplus 2 \cdot 1_{\langle h_1, \ldots, h_{n-4} \rangle} = X_2 \oplus 3 \cdot 1_{\langle h_1, \ldots, h_{n-4} \rangle}$ where X_2 is irreducible. By Lemma 4.8, $X | \langle h_1, \ldots, h_{n-4}, g \rangle = Y \oplus s \cdot 1_{\langle h_1, \ldots, h_{n-4}, g \rangle}$ where $s = 1$, 2, or 3 and Y is irreducible. If $s = 3$, clearly $X | \langle h_1, \ldots, h_{n-4}, g, h_{n-3} \rangle$ is not irreducible, a contradiction. So $s = 1$ or 2. By Lemma 4.7, Y is primitive and hence by hypothesis (A), $\langle h_1, \ldots, h_{n-4}, g \rangle \cong A_{n-s+1}$ and $h_1, \ldots, h_{n-4}, g$ represent 3-cycles. By Lemma 4.5 there is an element $g_1 = g^h$ for some $h \in \langle h_1, \ldots, h_{n-4} \rangle$ such that g_1 commutes with $h_1, \ldots, h_{n-6}$. As $\langle H, g \rangle = \langle H, g_1 \rangle$, by replacing g by g_1, we may assume g commutes with $h_1, \ldots, h_{n-6}$.

As $X | \langle H, g \rangle$ is irreducible, g does not commute with $\langle h_2, \ldots, h_{n-3} \rangle$. By Lemma 4.2, $X | \langle h_2, \ldots, h_{n-3} \rangle = X_1 | \langle h_2, \ldots, h_{n-3} \rangle \oplus 2 \cdot 1_{\langle h_2, \ldots, h_{n-3} \rangle} = X_3 \oplus 3 \cdot 1_{\langle h_2, \ldots, h_{n-3} \rangle}$ where X_3 is irreducible. Therefore by Lemma 4.8, $X | \langle h_2, \ldots, h_{n-3}, g \rangle = R \oplus r \cdot 1_{\langle h_2, \ldots, h_{n-3}, g \rangle}$ where $r = 1$, 2, or 3 and R is irreducible. If $r = 3$, $X | \langle h_2, \ldots, h_{n-3}, g, h_1 \rangle$ is reducible, a contradiction. By Lemma 4.7, R is primitive. If $r = 2$, by hypothesis (A), $\langle h_2, \ldots, h_{n-3}, g \rangle \cong A_{n-1}$. By Lemma 4.6 $\langle H, g \rangle \cong A_n$; however $X | \langle H, g \rangle$ could not be irreducible by Lemma 4.2.

Therefore $r = 1$ and by hypothesis (A), $\langle h_2, \ldots, h_{n-3}, g\rangle \cong A_n$. Let $\langle h_2, \ldots, h_{n-3}, g\rangle$ act on $\{b_1, \ldots, b_n\}$. By Lemma 4.5, we may assume $h_i = (b_{i-1}, b_i, b_{i+1})$ for $2 \leq i \leq n-3$ and $g = (b_j, b_{n-1}, b_n)$. Also there is an element $h \in \langle h_5, \ldots, h_{n-3}\rangle$ such that g^h or $(g^{-1})^h$ is $g_1 = (b_{n-2}, b_{n-1}, b_n)$. As g and h commute with h_1, so does g_1. Let $h_{n-2} = h_{n-3}{}^{g_1 h_{n-3}}$. Then by Lemma 4.6, $\langle H, h_{n-2}\rangle \cong A_n$. But $h_1, \ldots, h_{n-2}$ represent 3-cycles in $\langle H, h_{n-2}\rangle$ and if $\langle H, h_{n-2}\rangle$ acts on $\{c_1, \ldots, c_n\}$, by Lemma 4.5, we may assume $h_i = (c_i, c_{i+1}, c_{i+2})$ for $1 \leq i \leq n-3$ and $h_{n-2} = (c_j, c_k, c_n)$. Looking in $\langle h_2, \ldots, h_{n-2}\rangle$, the only possibility is $h_{n-2} = (c_{n-2}, c_{n-1}, c_n)$. Thus by Lemma 4.6, $\langle H, h_{n-2}, g_1\rangle = \langle H, g\rangle \cong A_{n+1}$.

Lemma 4.10: Assume hypothesis (A) holds and $n \geq 8$. Let H be a subgroup of G generated by special 3-elements such that $X|H = X_1 \oplus 1_H$ where X_1 is irreducible. Let $g \in G$ be a special 3-element such that $X|\langle H, g\rangle$ is irreducible. Then $\langle H, g\rangle \cong A_{n+1}$.

Proof: By Lemma 4.7, X_1 is primitive. Therefore by hypothesis (A), $H \cong A_n$. Assume H acts on $\{a_1, \ldots, a_n\}$. The special 3-elements in H are precisely the 3-cycles and so let $h_i = (a_i, a_{i+1}, a_{i+2})$. Hence $H = \langle h_1, \ldots, h_{n-2}\rangle$ and as $X|\langle H, g\rangle$ is irreducible, g does not commute with H. So g could not commute with both $\langle h_1, \ldots, h_{n-3}\rangle$ and $\langle h_2, \ldots, h_{n-2}\rangle$. As in the proof of Lemma 4.9, we may assume g does not commute with $\langle h_1, \ldots, h_{n-3}\rangle$.

By Lemma 4.2, $X|\langle h_1, \ldots, h_{n-3}\rangle = X_1|\langle h_1, \ldots, h_{n-3}\rangle \oplus 1_{\langle h_1, \ldots, h_{n-3}\rangle} = X_2 \oplus 2 \cdot 1_{\langle h_1, \ldots, h_{n-3}\rangle}$. By Lemma 4.8, $X|\langle h_1, \ldots, h_{n-3}, g\rangle = Y \oplus s \cdot 1_{\langle h_1, \ldots, h_{n-3}, g\rangle}$ where Y is irreducible and $s = 0$, 1, or 2. By Lemma 4.7, Y is primitive. If $s = 2$, by hypothesis

(A), $\langle h_1, \ldots, h_{n-3}, g\rangle \cong A_{n-1} \cong \langle h_1, \ldots, h_{n-3}\rangle$ and $g \in H$, contradicting the irreducibility of $X \mid \langle H, g\rangle$. Thus $s = 0$ or 1 and by Lemma 4.9 or hypothesis (A), respectively, $\langle h_1, \ldots, h_{n-3}, g\rangle \cong A_{n-s+1}$. The elements $h_1, \ldots, h_{n-3}, g$ must all represent 3-cycles and by Lemma 4.5, there is a conjugate g_1 of g by an element in $\langle h_1, \ldots, h_{n-3}\rangle$ such that g_1 commutes with $h_1, \ldots, h_{n-5}$. Without loss of generality, we may replace g by g_1 and hence assume g commutes with $h_1, \ldots, h_{n-5}$.

As $X \mid \langle H, g\rangle$ is irreducible, g does not commute with $\langle h_2, \ldots, h_{n-2}\rangle$. But $X \mid \langle h_2, \ldots, h_{n-2}\rangle = X_1 \mid \langle h_2, \ldots, h_{n-2}\rangle \oplus 1_{\langle h_2, \ldots, h_{n-2}\rangle} = X_3 \oplus 2 \cdot 1_{\langle h_2, \ldots, h_{n-2}\rangle}$ where X_3 is irreducible by Lemma 4.2. So by Lemma 4.8, $X \mid \langle h_2, \ldots, h_{n-2}, g\rangle = R \oplus r \cdot 1_{\langle h_2, \ldots, h_{n-2}, g\rangle}$ where R is irreducible and $r = 0$, 1, or 2. By Lemma 4.7, R is primitive. If $r = 2$, by hypothesis (A), $\langle h_2, \ldots, h_{n-2}, g\rangle \cong A_{n-1} \cong \langle h_2, \ldots, h_{n-2}\rangle$ and so $g \in H$, contradicting the irreducibility of $X \mid \langle H, g\rangle$. If $r = 0$, by Lemma 4.9, $\langle h_2, \ldots, h_{n-2}, g\rangle \cong A_{n+1}$. As $h_2, \ldots, h_{n-2}, g$ are special 3-elements, they represent 3-cycles. Assume $\langle h_2, \ldots, h_{n-2}, g\rangle$ acts on $\{b_1, \ldots, b_{n+1}\}$. By Lemma 4.5, we may assume $h_i = (b_{i-1}, b_i, b_{i+1})$ and $g = (b_j, b_n, b_{n+1})$ for some j. By Lemma 4.5, there is an $h \in \langle h_5, \ldots, h_{n-2}\rangle$ such that g^h or $(g^{-1})^h$ is $g_1 = (b_{n-1}, b_n, b_{n+1})$. As g and h commute with h_1, so does g_1. Let $h_{n-1} = h_{n-2}^{\,g_1 h_{n-2}} = (b_{n-2}, b_{n-1}, b_n)$. This commutes with h_1 and h_2; so by Lemma 4.6, $\langle H, h_{n-1}\rangle \cong A_{n+1}$. Again $h_1, \ldots, h_{n-1}$ represent 3-cycles in $\langle H, h_{n-1}\rangle$ and if $\langle H, h_{n-1}\rangle$ acts on $\{c_1, \ldots, c_{n+1}\}$, by Lemma 4.5, we may assume $h_i = (c_i, c_{i+1}, c_{i+2})$ for $1 \leq i \leq n-2$ and $h_{n-1} = (c_i, c_j, c_{n+1})$. Examining $\langle h_2, \ldots, h_{n-1}\rangle$, the only possibility is $h_{n-1} = (c_{n-1}, c_n, c_{n+1})$. By Lemma 4.6, $\langle H, h_{n-1}, g_1\rangle = \langle H, g\rangle \cong A_{n+2}$, contradicting Lemma 4.2. So $r = 1$

and by hypothesis (A), $\langle h_2, \ldots, h_{n-2}, g\rangle \cong A_n$. By Lemma 4.6, $\langle H, g\rangle \cong A_{n+1}$.

We are now ready to classify G when $n \geq 8$ and hypothesis (A) holds.

Lemma 4.11: Assume hypothesis (A) holds and $n \geq 8$. Assume there is a subgroup H of G generated by special 3-elements such that $X|H = Y \oplus (n-s)1_H$ where Y is irreducible of degree s with $3 \leq s < n$. Then G contains a normal subgroup N generated by special 3-elements such that $N \cong A_{n+1}$ and $G/Z(G) \cong A_{n+1}$ or S_{n+1}.

Proof: By Lemma 2.3 and induction, we may assume $s = n-2$ or $n-1$. Let Y act on the subspace V_1 of dimension s. Let g be a special 3-element such that X(g) does not leave V_1 invariant. If $s = n-1$ $X|\langle H, g\rangle$ is irreducible and by Lemma 4.10, $\langle H, g\rangle \cong A_{n+1}$. If $s = n-2$ and $X|\langle H, g\rangle$ is irreducible, by Lemma 4.9, $\langle H, g\rangle \cong A_{n+1}$. If $s = n-2$ and $X|\langle H, g\rangle = X_1 \oplus \xi$ where X_1 is irreducible of degree n-1, by Lemma 4.7, $\xi = 1_{\langle H, g\rangle}$. In this case, let X_1 act on V_2 and let g_1 be a special 3-element such that $X(g_1)$ does not leave V_2 invariant. So $X|\langle H, g, g_1\rangle$ is irreducible and by Lemma 4.10, $\langle H, g, g_1\rangle \cong A_{n+1}$. Hence in any case, G contains a subgroup N generated by special 3-elements such that $X|N$ is irreducible and $N \cong A_{n+1}$.

Let N act on $\{a_1, \ldots, a_{n+1}\}$ and choose special 3-elements $h_1, \ldots, h_{n-1}$ such that $h_i = (a_i, a_{i+1}, a_{i+2})$. Let h be a special 3-element not in N. By Lemma 4.2, $X|\langle h_1, \ldots, h_{n-2}\rangle = X_1 \oplus 1_{\langle h_1, \ldots, h_{n-2}\rangle}$ where X_1 is irreducible. Let X_1 act on the subspace W. If X(h) leaves W invariant, by Lemma 4.7, $X|\langle h_1, \ldots, h_{n-2}, h\rangle = X_2 \oplus 1_{\langle h_1, \ldots, h_{n-2}, h\rangle}$ where X_2 is primitive. But by hypothesis (A), $h \in \langle h_1, \ldots, h_{n-2}\rangle \cong A_n$, a contradiction. So $X|\langle h_1, \ldots, h_{n-2}, h\rangle$ is irreducible and by Lemma 4.10, $\langle h_1, \ldots, h_{n-2}, h\rangle \cong A_{n+1}$. By Lemma 4.5, there is a $g_1 = h^g$ for some $g \in \langle h_1, \ldots, h_{n-2}\rangle$ such that g_1 commutes with $h_1, \ldots, h_{n-5}$. Also g_1 is not in N. As above $X|\langle h_2, \ldots, h_{n-1}, g_1\rangle$ is irreducible and so by Lemma 4.10, $\langle h_2, \ldots, h_{n-1}, g_1\rangle \cong A_{n+1}$ and $h_2, \ldots, h_{n-1}, g_1$ are 3-cycles. By Lemma 4.6, $\langle h_1, \ldots, h_{n-1}, g_1\rangle \cong A_{n+2}$ which contradicts Lemma 4.2. So h does not exist and N is the subgroup of G generated by all special 3-elements of G.

Therefore $N \lhd G$ and as $X|N$ is irreducible, $C_G(N) = Z(G)$. Thus $G/Z(G)$ is a subgroup of the automorphism group of A_{n+1}. So $G/Z(G) \cong A_{n+1}$ or S_{n+1}. (See [24]).

CHAPTER V

THE PROOF OF THEOREM 2

In this chapter we complete the proof of Theorem 2. We first begin by considering what groups could be generated by two special 3-elements.

Lemma 5.1: Let h_1 and h_2 be special 3-elements such that h_1 and h_2 do not commute. Let $n \geq 6$. Then one of the following holds:

1. $X|\langle h_1, h_2\rangle = Y_1 \oplus Y_2 \oplus (n-4)1_{\langle h_1, h_2\rangle}$ where Y_1, Y_2 are irreducible of degree 2. Let $Y_i(\langle h_1, h_2\rangle) = H_i$ and let M_i be the set of all elements in H_i which occur with component the identity of $H_j (j \neq i)$ in the subdirect product. Then $H_i \cong SL_2(3)$ for $i = 1$ and 2, $|Z(H_i)| = 2$, and either $M_i = Z(H_i)$ for $i = 1$ and 2, or $M_i = 1$ for $i = 1$ and 2. Also $Y_i(h_j)$ are not unimodular for $i = 1,2$ and $j = 1,2$.
2. $X|\langle h_1, h_2\rangle = Y_1 \oplus \xi \oplus (n-4)1_{\langle h_1, h_2\rangle}$ where Y_1 is irreducible of degree 3.
3. $X|\langle h_1, h_2\rangle = Y \oplus (n-4)1_{\langle h_1, h_2\rangle}$ where Y is irreducible of degree 4.

Proof: As h_1 and h_2 do not commute, $X|\langle h_1, h_2\rangle = Y \oplus (n-4)1_{\langle h_1, h_2\rangle}$ where Y has a constituent of degree at least 2. So we have three possibilities:

a. $Y = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 2.

b. $Y = Y_1 \oplus \xi$ where Y_1 is irreducible of degree 3.

c. Y is irreducible.

As b. and c. give 2. and 3., respectively, we only need to show that a. gives 1.

So assume $Y \mid \langle h_1, h_2 \rangle = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree 2. Let $Y_i(\langle h_1, h_2 \rangle) = H_i$. So $Y(\langle h_1, h_2 \rangle)$ is a subdirect product of H_1 and H_2. If Y_1 is monomial in some basis, then $Y_1(h_1)$ and $Y_1(h_2)$ would both be diagonal in that basis as they have odd order. This contradicts the irreducibility of Y_1. So Y_1 is primitive. We now examine the possibilities for Y_1 by examining Blichfeldt's list [2].

Case A: $H_1/Z(H_1) \cong A_5$

This case is impossible as in case A of Chapter 3.

Case B: $H_1/Z(H_1) \cong S_4$

This case is also impossible as S_4 is not generated by its 3-elements.

Case C: $H_1/Z(H_1) \cong A_4$

Assume first that Y_2 is reducible. So $Y_2 = \xi_1 \oplus \xi_2$. Thus H_2 is abelian and so $Y(\langle h_1, h_2 \rangle)' = H_1' \oplus 1 \oplus 1$ and $H_1'/(Z(H_1) \cap H_1')$ is elementary abelian of order 4. As H_1' consists of 2×2 unimodular matrices, the only element of order 2 in H_1' is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$. In particular H_1' contains an element x of order 4. As x is unimodular, x has eigenvalues i and -i. So G contains a special 4-element, a contradiction by Theorem 1.

So Y_2 is irreducible also. As previously, Y_2 is primitive; also $H_2/Z(H_2) \cong A_5$ or S_4 are eliminated as in cases A and B. So

$H_2/Z(H_2) \cong A_4$. Note that if any one of $Y_i(h_j)$ is the identity, then Y_1 or Y_2 is reducible, a contradiction. So each $Y_i(h_j)$ is not unimodular. Let M_i be as defined in the statement of the lemma. By Theorem 5.5.1 of [11], $M_i \triangleleft H_i$ and $H_1/M_1 \cong H_2/M_2$. Let $C_i = Z(H_i)$. Assume first that for some i, M_iC_i/C_i contains the Sylow 2-subgroup of H_i/C_i. Then M_i contains a unimodular subgroup of order 4. As earlier M_i could only have one involution, the central involution, and so has an element with eigenvalues i and -i. Thus G contains a special 4-element, a contradiction to Theorem 1. So for i = 1 and 2, $M_i \subseteq C_i$. As elements of H_1 and H_2 have only determinant 1, ω, or $\bar{\omega}$, C_i could only be 1, Z_2, Z_3, or Z_6. As H_i' contains an involution, which must be $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$, $Z_2 \subseteq C_i$ for i = 1 and 2. Assume $C_i = Z_6$ for some i, say i = 1. As M_1 contains unimodular matrices, $M_1 = Z_2$ or $M_1 = 1$. In either case, as $H_1/M_1 \cong H_2/M_2$, we would have $C_2 = Z_6$ and $M_2 \cong M_1$. The isomorphism between H_1/M_1 and H_2/M_2 maps the centers onto one another; so X(G) contains an element with eigenvalues $-\omega, -\omega, -\bar{\omega}, -\bar{\omega}, 1, 1, \ldots,$ a contradiction to Blichfeldt. So $C_i \cong Z_2$ and $H_i' \supseteq C_i$ for i = 1 and 2. By Schur [25], the only nonsplitting central extension of Z_2 by A_4 is $SL_2(3)$. So $H_1 \cong H_2 \cong SL_2(3)$ and either $M_i = Z(H_i)$ for i = 1 and 2 or $M_i = 1$ for i = 1 and 2.

Lemma 5.2: The only nonabelian linear unimodular group P of degree 3 and order 27 has exponent 3 and is given by $P = \langle g, h \rangle$ where

$$g = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad h = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \bar{\omega} \end{pmatrix}$$

in some basis.

Proof: The representation is monomial (52.1 of [5]). Let v_1, v_2, v_3 be a basis for the space on which the representation is defined. Assume P has an element t of order 9. Assume first that t is not diagonal. By ordering v_1, v_2, v_3 correctly,

$$t = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} .$$

But t has determinant 1. So abc = 1 implying t has order 3, a contradiction. So t is diagonal. Also $P = \langle s, t \mid s^{-1}ts = t^4,\ t^9 = 1,\ s^3 = 1 \rangle$. (See Chapter 4 of [11].) As s could not be diagonal, by scaling and ordering v_1, v_2, v_3 correctly,

$$s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad t = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

where abc = 1. As $t^3 \in Z(P)$, $a^3 = b^3 = c^3 \in \{\omega, \overline{\omega}\}$. Let $a = \epsilon$ where ϵ is a primitive ninth root of unity. If $\epsilon^3 = \omega$ either $b = c = \epsilon\omega$ or $\{b, c\} = \{\epsilon, \epsilon\overline{\omega}\}$. If $\epsilon^3 = \overline{\omega}$ either $b = c = \epsilon\overline{\omega}$ or $\{b, c\} = \{\epsilon, \epsilon\omega\}$. But

$$s^{-1}ts = \begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} = \begin{pmatrix} a^4 & 0 & 0 \\ 0 & b^4 & 0 \\ 0 & 0 & c^4 \end{pmatrix}$$

and none of the combinations of a, b, and c work.

So P has exponent 3. Let s be nondiagonal. By ordering and scaling v_1, v_2, v_3 correctly,

$$s = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

Let t be an element of order 3 not commuting with s. If t is diagonal, by replacing t by t^{-1} if necessary and permuting v_1, v_2, v_3 cyclically, we may assume

$$t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \overline{\omega} \end{pmatrix} .$$

If t is not diagonal, by replacing t by t^{-1} if necessary

$$t = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix}$$

where $abc = 1$. But

$$s^2t = \begin{pmatrix} c & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} .$$

As s and s^2t do not commute and s^2t has order 3, $\{a, b, c\} = \{1, \omega, \overline{\omega}\}$, and the result holds.

<u>Lemma 5.3</u>: Let h_1 and h_2 be special 3-elements. Assume $X | \langle h_1, h_2\rangle = Y_1 \oplus \xi \oplus (n-4)1_{\langle h_1, h_2\rangle}$ where Y_1 is irreducible of degree 3 and $\xi \neq 1_{\langle h_1, h_2\rangle}$. Then there exist special 3-elements h_1' and h_2' contained in $\langle h_1, h_2\rangle$ such that $X | \langle h_1', h_2'\rangle = Y \oplus (n-3)1_{\langle h_1', h_2'\rangle}$ where Y

is irreducible and $\langle h_1', h_2' \rangle$ is the nonabelian group of order 27 and exponent 3.

Proof: Assume first that Y_1 is monomial. Let Y_1 act on V_1 and ξ on $\langle v_4 \rangle$. Let v_1, v_2, v_3 be a basis of V_1 in which Y_1 is monomial; since $Y_1(h_1)$ and $Y_1(h_2)$ cannot both be diagonal, we may assume $Y_1(h_1)$ is not diagonal. By replacing h_2 by h_2^{-1} if necessary and scaling and ordering v_1, v_2, v_3 correctly,

$$(Y_1 \oplus \xi)(h_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad (Y_1 \oplus \xi)(h_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \omega & 0 \\ 0 & 0 & 0 & \bar{\omega} \end{pmatrix} .$$

Let $h^* = h_1 h_2 h_1^{-1} h_2^{-1}$. Then

$$(Y_1 \oplus \xi)(h^*) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega & 0 & 0 \\ 0 & 0 & \bar{\omega} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} .$$

and $h_1' = h_1$, $h_2' = h^*$ gives the desired group by Lemma 5.2.

Assume now that Y_1 is primitive. Let $Y_1(\langle h_1, h_2 \rangle) = H_1$ and $\xi(\langle h_1, h_2 \rangle) = H_2$. Then $H = (Y \oplus \xi)(\langle h_1, h_2 \rangle)$ is a subdirect product of H_1 and H_2. By Mitchell, $H_1/Z(H_1)$ is an extension of $Z_3 \times Z_3$ by $SL_2(3)$. Now H' is a subdirect product of H_1' and $H_2' = 1$; also $|H_1'/(Z(H_1) \cap H_1')| = 72$. If S is the Sylow 3-subgroup of H_1', $S \triangleleft H_1'$ and so S char $H_1' \triangleleft H$. Thus $S \triangleleft H$ and by Clifford's theorem (see [5]), as $S \not\subseteq Z(H_1)$, S is nonabelian. As $Z(H_1) \cap H_1'$ consists of nonsingular unimodular scalar matrices, $|Z(H_1) \cap H_1'| \leq 3$. So as S is nonabelian, $Z(H_1) \cap H_1'$ has order 3 and S has order 27. By Lemma 5.2, S is of exponent 3 and the result follows.

The next three lemmas deal with special 3-elements which satisfy conclusion 1. of Lemma 5.1. The possibility of conclusion 1. occurring is one reason why Theorem 2 is more difficult than Theorem 1. Induction techniques do not work quite as easily in this situation as one would hope.

Lemma 5.4: Let h_1 and h_2 be special 3-elements such that they satisfy 1. of Lemma 5.1. Let $n \geq 6$ and let Y_i act on the subspace V_i. Let h_3 be a special 3-element such that $X(h_3)$ does not leave $V_1 \oplus V_2$ invariant. Then one of the following holds:

1. $X|\langle h_1, h_2, h_3\rangle = U \oplus (n-s)1_{\langle h_1, h_2, h_3\rangle}$ where U is irreducible of degree s with s = 5 or 6.
2. There are special 3-elements $h_1', h_2' \in \langle h_1, h_2, h_3\rangle$ such that $X|\langle h_1', h_2'\rangle = U \oplus (n-3)1_{\langle h_1', h_2'\rangle}$ where U is irreducible of degree 3 and $\langle h_1', h_2'\rangle$ is the nonabelian group of order 27 and exponent 3.
3. $X|\langle h_1, h_2, h_3\rangle = U_1 \oplus U_2 \oplus (n-6)1_{\langle h_1, h_2, h_3\rangle}$ where both U_i are irreducible and primitive of degree 3. Also $U_i(h_j)$ are nonunimodular for i = 1,2 and j = 1,2,3. Let $G_i = U_i(\langle h_1, h_2, h_3\rangle)$ and N_i be the set of all elements in G_i which occur with component the identity of G_j $(i \neq j)$ in the subdirect product. Then $G_i/Z(G_i)$ is an extension of $Z_3 \times Z_3$ by $SL_2(3)$. If $N_i \not\subseteq Z(G_i)$ for some i, 2. also holds.

Proof: We have $X\,|\,\langle h_1, h_2, h_3\rangle = U \oplus (n-6)1_{\langle h_1, h_2, h_3\rangle}$. As $X(h_3)$ does not leave $V_1 \oplus V_2$ invariant, $X(h_3)$ does not leave both V_1 and V_2 invariant. Hence U has an irreducible constituent of degree at least 3. So we have four possibilities for U:

A. U is irreducible.

B. $U = U_1 \oplus \xi$ where U_1 is irreducible of degree 5.

C. $U = U_1 \oplus U_2$ where U_1 is irreducible of degree 4.

D. $U = U_1 \oplus U_2$ where U_1 is irreducible of degree 3.

First, case A gives 1. Assume case B holds; then 1. holds if we show $\xi = 1_{\langle h_1, h_2, h_3\rangle}$. By Mitchell, if U_1 is primitive, $\xi = 1_{\langle h_1, h_2, h_3\rangle}$. Assume U_1 acts on a subspace W and is monomial in a basis $v_1, \ldots, v_5$. As $U_1\,|\,\langle h_1, h_2\rangle = Y_1 \oplus Y_2 \oplus 1_{\langle h_1, h_2\rangle}$, if $\xi \neq 1_{\langle h_1, h_2, h_3\rangle}$, $U_1(h_3)$ must be diagonal in the basis $v_1, \ldots, v_5$. Hence as U_1 is irreducible, $U_1\,|\,\langle h_1, h_2\rangle$ does not leave any $\langle v_i\rangle$ invariant. By Lemma 2.6, $\langle h_1, h_2\rangle \cong A_5$, a contradiction. So $\xi = 1_{\langle h_1, h_2, h_3\rangle}$ and 1. holds.

Assume case C holds. Let U_i act on W_i. If $V_1 \oplus V_2 \subseteq W_1$, $V_1 \oplus V_2 = W_1$ and $X(h_3)$ leaves $V_1 \oplus V_2$ invariant, a contradiction. Hence as V_1, V_2 are unique subspaces, we may assume $V_1 \subseteq W_1$ and $V_2 = W_2$. So $U_1(h_1)$ and $U_1(h_2)$ have eigenvalues $1, 1, 1, \omega$ or $1, 1, 1, \bar{\omega}$. If U_1 is monomial in some basis, $U_1(h_1)$ and $U_1(h_2)$ would have to be diagonal, contradicting $U_1\,|\,\langle h_1, h_2\rangle = Y_1 \oplus 2 \cdot 1_{\langle h_1, h_2\rangle}$. Therefore U_1 is primitive. But this is impossible by Lemma 2.2. So case C does not occur.

Finally assume case D holds. Let $G_i = U_i(\langle h_1, h_2, h_3\rangle)$ and let U_i act on W_i. By ordering Y_1 and Y_2 correctly, we may assume $V_i \subseteq W_i$. Therefore U_2 has a constituent of degree at least 2, and $U_i(h_1)$ and $U_i(h_2)$ are nonunimodular for i = 1 and 2. We have two possibilities:

(i) $U_2 = U_3 \oplus \xi$ where U_3 is irreducible of degree 2 and $\xi \neq 1_{\langle h_1, h_2, h_3\rangle}$.

(ii) U_2 is irreducible or $U_2 = U_3 \oplus 1_{\langle h_1, h_2, h_3\rangle}$ where U_3 is irreducible of degree 2.

If (i) holds, then $\xi(h_1) = \xi(h_2) = 1$ and $\xi(h_3) \neq 1$. As U_1 is irreducible, h_3 does not commute with both h_1 and h_2. Without loss of generality assume h_3 does not commute with h_1. As $U_1(h_3)$ is not trivial, $U_2(h_3)$ is trivial. So $U_1 | \langle h_1, h_3\rangle = R \oplus 1_{\langle h_1, h_3\rangle}$ where R is irreducible of degree 2 and $X | \langle h_1, h_3\rangle = R \oplus \xi_1 \oplus \xi_2 \oplus (n-4)1_{\langle h_1, h_3\rangle}$. But by Lemma 5.1, this is impossible.

So (ii) holds. If either U_i is monomial, in some basis $U_i(h_1)$ and $U_i(h_2)$ are both diagonal as they have eigenvalues $1, 1, \omega$ or $1, 1, \bar{\omega}$. This contradicts the irreducibility of Y_i. So U_1 is primitive; if U_2 is irreducible, it is primitive and if $U_2 = U_3 \oplus 1_{\langle h_1, h_2, h_3\rangle}$, U_3 is primitive. Let $G_i = U_i(\langle h_1, h_2, h_3\rangle)$. So $U(\langle h_1, h_2, h_3\rangle)$ is a subdirect product of G_1 and G_2. By Mitchell, $G_1/Z(G_1)$ is $Z_3 \times Z_3$ extended by $SL_2(3)$. Let N_i be the set of all elements in G_i which occur with the identity of G_j $(j \neq i)$ in the subdirect product. By Theorem 5.5.1 of [11], $N_i \lhd G_i$ and $G_1/N_1 \cong G_2/N_2$. All nontrivial normal subgroups of $G_1/Z(G_1)$ contain $\bar{S}$, a normal subgroup of order 9. Let S be the inverse image of $\bar{S}$ in G_1. Assume first that $N_1 \not\subseteq Z(G_1)$. Then $\bar{S} \subseteq \bar{N_1} = N_1Z(G_1)/Z(G_1)$. Also $S \cap N_1 \lhd G_1$ and $|S| = 9 \cdot |Z(G_1)|$. As N_1 contains only unimodular matrices, $|S \cap N_1| = 9$ or 27. By Clifford's theorem [5] as G_1 is primitive and $S \cap N_1 \not\subseteq Z(G_1)$, $|S \cap N_1| = 27$. So by Lemma 5.2, we have 2. Now assume $N_1 \subseteq Z(G_1)$. As $G_1/N_1 \cong G_2/N_2$, by looking at Blichfeldt's list of primitive groups of degree 2 and 3, U_2 must be irreducible. As it is primitive $G_2/Z(G_2) \cong G_1/Z(G_1)$. If $N_2 \not\subseteq Z(G_2)$, we have 2. as above. We must have $U_i(h_3)$ nonunimodular as both U_1 and U_2 are irreducible, and so 3. holds.

Lemma 5.5: One of the following occurs for $n \geq 6$.

1. There exists a subgroup H of G generated by special 3-elements such that $X|H = Y \oplus (n-r)1_H$ where Y is irreducible of degree r for some r with $3 \leq r \leq 6$.
2. Let h_1 and h_2 be any two special 3-elements. Then either
   a) h_1 and h_2 commute
   b) $\langle h_1, h_2\rangle \cong SL_2(3)$ and $X|\langle h_1, h_2\rangle$ satisfies 1. of Lemma 5.1. For any special 3-element h_3 satisfying the hypothesis of Lemma 5.4, conclusion 3. holds in Lemma 5.4 but 2. doesn't.

Proof: By Lemma 1.1, not all special 3-elements commute. So let h_1 and h_2 be special 3-elements which don't commute. If conclusion 3. of Lemma 5.1 holds, we have 1. If conclusion 2. of Lemma 5.1 holds, we have 1. by Lemma 5.3. So assume conclusion 1. holds of Lemma 5.1. Let Y_i act on V_i. There is a special 3-element h_3 such that $X(h_3)$ does not leave $V_1 \oplus V_2$ invariant. Hence, if for this h_3, conclusion 1. or 2. of Lemma 5.4 hold, we have 1. So assume conclusion 3. holds but 2. doesn't. Then $N_i \subseteq Z(G_i)$. As elements of U_1 have determinant 1, ω, or $\bar{\omega}$, $|Z(G_i)|\,|9$. Also $|\langle h_1, h_2, h_3\rangle| = |G_1| \cdot |N_2| = 216 \cdot |Z(G_1)|\,|N_2| = 2^3 \cdot 3^3 \cdot 3^a$ for some $a \geq 1$. So in particular $48 \nmid |\langle h_1, h_2, h_3\rangle|$ and hence by conclusion 1. of Lemma 5.1, $\langle h_1, h_2\rangle \cong SL_2(3)$.

Lemma 5.6: Assume $n \geq 8$ and that conclusion 2. of Lemma 5.5 holds. Let h_1, h_2, h_3 be special 3-elements of G that satisfy conclusion 3. of Lemma 5.4 where U_i acts on W_i. Let h_4 be a special 3-element of G such that $X(h_4)$ does not leave $W_1 \oplus W_2$ invariant. Then one of the following holds:

1. $X \mid \langle h_1, h_2, h_3, h_4 \rangle = \widetilde{X} \oplus (n-s)1_{\langle h_1, h_2, h_3, h_4 \rangle}$ where $\widetilde{X}$ is irreducible and primitive of degree $s = 7$ or 8.
2. $n = 8$ and $X \mid \langle h_1, h_2, h_3, h_4 \rangle = R_1 \oplus R_2$ where R_1 and R_2 are irreducible and primitive of degree 4. If $F_i = R_i(\langle h_1, h_2, h_3, h_4 \rangle)$, then $F_i \cong \widetilde{O_5(3)} \times Z_3$ where $\widetilde{O_5(3)}$ is the nonsplitting central extension of Z_2 by $O_5(3)$. If L_i is the set of all elements in F_i which occur with component the identity of F_j $(i \neq j)$ in the subdirect product, then $L_i \subseteq O_2(Z(F_i))$ for $i = 1$ and 2.

Proof: As $X(h_4)$ does not leave both W_1 and W_2 invariant, we have the following possibilities:

(i) $X \mid \langle h_1, h_2, h_3, h_4 \rangle = \widetilde{X} \oplus (n-8)1_{\langle h_1, h_2, h_3, h_4 \rangle}$ where $\widetilde{X}$ is irreducible.

(ii) $X \mid \langle h_1, h_2, h_3, h_4 \rangle = \widetilde{X} \oplus Y \oplus (n-8)1_{\langle h_1, h_2, h_3, h_4 \rangle}$ where $\widetilde{X}$ is irreducible of degree r with $4 \leq r \leq 7$ and Y has degree $8-r$.

First assume either (i) holds or (ii) holds with $r = 7$. Let $\widetilde{X}$ act on $\widetilde{V}$. Suppose that $\widetilde{X}$ is monomial on $\widetilde{V}$. Then by Lemma 1.4, there exist special 3-elements generating A_5, a contradiction to the assumption that 2. holds in Lemma 5.5. So $\widetilde{X}$ is primitive and by Mitchell, if (ii) holds with $r = 7$, $Y = 1_{\langle h_1, h_2, h_3, h_4 \rangle}$. So in these cases, 1. holds.

We now assume (ii) holds for $4 \leq r \leq 6$. If $r = 6$, $\tilde{X}$ must be acting on $W_1 \oplus W_2$, a contradiction that $X(h_4)$ does not leave $W_1 \oplus W_2$ invariant. If $r = 5$, the only possibility by correctly ordering U_1 and U_2 is that $\tilde{X} \mid \langle h_1, h_2, h_3 \rangle = U_1 \oplus 2 \cdot 1_{\langle h_1, h_2, h_3 \rangle}$ and $Y \mid \langle h_1, h_2, h_3 \rangle = U_2$. As $\tilde{X}(h_i)$ for $1 \leq i \leq 3$ have eigenvalues $1,1,1,1,\omega$ or $1,1,1,1,\bar{\omega}$, by Mitchell, $\tilde{X}$ is not primitive; so $\tilde{X}$ is monomial. But then in some basis of the space on which $\tilde{X}$ acts, $\tilde{X}(h_i)$ are all diagonal for $1 \leq i \leq 3$, contradicting the fact that U_1 is irreducible. So $r = 4$. Let $\tilde{X} = R_1$ and $Y = R_2$. By correctly ordering U_1 and U_2, the only possibility is $R_i \mid \langle h_1, h_2, h_3 \rangle = U_i \oplus 1_{\langle h_1, h_2, h_3 \rangle}$. If either R_i is monomial, in some basis $R_i(h_1)$, $R_i(h_2)$, and $R_i(h_3)$ are all diagonal because they have eigenvalues $1,1,1,\omega$ or $1,1,1,\bar{\omega}$. This contradicts the irreducibility of U_i. So R_1 is primitive and the result follows by Lemma 2.2.

The next lemma eliminates a possibility which occurs in [15]. Using the powerful results of [1], Lemma 5.8 shows that condition 1. of Lemma 5.5 holds. This allows us to construct primitive subgroups of codimension 1 or 2.

<u>Lemma 5.7</u>: If X_1 is an irreducible representation of a group H containing a special 3-element and X_1 has degree 8, then X_1 is not the tensor product of two representations of smaller degree.

<u>Proof</u>: Assume X_1 is the tensor product of Y_1 and Y_2 each of degree less than 8. Then we may assume Y_1 has degree 4 and Y_2 has degree 2. There exist elements y_1, y_2 such that $Y_1(y_1) \otimes Y_2(y_2)$ has eigenvalues $\omega, \bar{\omega}, 1, 1, 1, 1, 1, 1$. Let $Y_1(y_1)$ have eigenvalues α_1, α_2, α_3, α_4 and $Y_2(y_2)$ have eigenvalues β_1, β_2. So $Y_1(y_1) \otimes Y_2(y_2)$ has

eigenvalues $\alpha_i\beta_j$ for $1 \leq i \leq 4$ and $1 \leq j \leq 2$. We may assume $\alpha_1\beta_1 = \bar{\omega}$. So $\beta_1 \neq \beta_2$ and $\alpha_1 \neq \alpha_2$. Thus $\{\alpha_1\beta_2, \alpha_2\beta_2\} = \{\omega, 1\}$. So $\{\alpha_3\beta_2, \alpha_4\beta_2\} = \{1, 1\}$ and hence $\alpha_3 = \alpha_4$. So $\alpha_3\beta_1 = \alpha_4\beta_1 = 1$ and hence $\beta_1 = \beta_2$, a contradiction.

Lemma 5.8: Let $n \geq 8$. Then 1. of Lemma 5.5 holds.

Proof: Assume 1. of Lemma 5.5 fails. By Lemma 1.1 not all special 3-elements commute. Choose special 3-elements h_1 and h_2 which do not commute. So $\langle h_1, h_2\rangle \cong SL_2(3)$ and by replacing h_2 by h_2^{-1} if necessary, we may assume h_1 and h_2 are conjugate in $\langle h_1, h_2\rangle$. Also $X|\langle h_1, h_2\rangle$ satisfies 1. of Lemma 5.1. There is a special 3-element h_3 satisfying the hypothesis to Lemma 5.4. So $\langle h_i, h_3\rangle = SL_2(3)$ for either $i = 1$ or $i = 2$ and in particular, replacing h_3 by h_3^{-1} if necessary, we may assume h_3 is conjugate to h_1 in $\langle h_1, h_2, h_3\rangle$. Also 3. holds in Lemma 5.4 but 2. doesn't. We may choose h_4 as in the hypothesis of Lemma 5.6, and as above we may assume h_4 is conjugate to h_1 in $\langle h_1, h_2, h_3, h_4\rangle$. Assume 2. holds in Lemma 5.6. Then $n = 8$ and by Lemma 1.1 there is a special 3-element h_5 such that $X|\langle h_1, h_2, h_3, h_4, h_5\rangle$ is irreducible. By Lemma 1.4 and our hypothesis, $X|\langle h_1, \ldots, h_5\rangle$ is primitive. As before we may assume h_5 is conjugate to h_1 in $\langle h_1, \ldots, h_5\rangle$. Therefore in any case we may assume that there is a subgroup H of G generated by an H-conjugate class Ω of special 3-elements such that $X|H = X_1 \oplus (n-r)1_H$ where $r = 7$ or 8 and X_1 is irreducible and primitive. Any two elements in Ω either commute or generate $SL_2(3)$.

By Aschbacher-Hall [1], $H/O_\infty(H) \cong Sp_k(3)$, $U_k(3)$, or $PGU_k(2)$. By examination of Wales [27,28], we see that $r = 7$ is impossible. So assume $r = 8$. First consider the case when $O_\infty(H) > Z(H)$. By Lemma 5.7 and inspection of the theorem and proof in Lindsey [15], there is a 2-group $Q \trianglelefteq H$ with $X_1 | Q$ irreducible. Consider the group $T = \langle h_1, h_2 \rangle Q$ which has order $2^a 3$. Let $X | \langle h_1, h_2 \rangle =$ $Y_1 \oplus Y_2 \oplus (n-4)1_{\langle h_1, h_2 \rangle}$ and let Y_i act on V_i. Assume there is a conjugate $h \in T$ of h_1 by an element in T such that $h \notin \langle h_1, h_2 \rangle$. Then as $3^2 \nmid |T|$, by Lemma 5.5, $X(h)$ must leave $V_1 \oplus V_2$ invariant. If h commutes with $\langle h_1, h_2 \rangle$, $3^2 \mid |T|$, a contradiction. Thus $X | \langle h_1, h_2, h \rangle =$ $Y \oplus \xi \oplus (n-5)1_{\langle h_1, h_2, h \rangle}$ where Y acts on $V_1 \oplus V_2$ and ξ is linear. But for one of h_1 or h_2, say h_1, $X | \langle h_1, h \rangle = U_1 \oplus U_2 \oplus (n-4)1_{\langle h_1, h \rangle}$ where U_1, U_2 are irreducible of degree 2, by Lemma 5.1. So $\xi = 1_{\langle h_1, h_2, h \rangle}$ and by assumption that 1. of Lemma 5.5 fails, $X(h)$ must leave both V_1 and V_2 invariant. Hence $X | \langle h_1, h_2, h \rangle = R_1 \oplus R_2 \oplus (n-4)1_{\langle h_1, h_2, h \rangle}$ where R_i acts on V_i. In a manner analogous to case C of Lemma 5.1, $|\langle h_1, h_2, h \rangle| = 24$ or 48. But $\langle h_1, h_2 \rangle$ has four Sylow 3-subgroups and as $|\langle h_1, h_2, h \rangle| = 24$ or 48, so does $\langle h_1, h_2, h \rangle$, contradicting $h \notin \langle h_1, h_2 \rangle$. Thus all conjugates of h_1 by elements in T lie in $\langle h_1, h_2 \rangle$. In particular Q normalizes $\langle h_1, h_2 \rangle$.

For either $i = 1$ or $i = 2$, $\langle h_i, h_3 \rangle \cong SL_2(3)$. As in the preceding paragraph, Q normalizes $\langle h_i, h_3 \rangle$. So Q normalizes $H_1 = \langle h_1, h_2, h_3 \rangle$. By examining the groups listed in section 3 of Aschbacher-Hall [1], $|H_1| = 648$ and $O_2(H_1) = 1$. As Q and H_1 normalize each other, $[Q, H_1] \subseteq Q \cap H_1 \triangleleft QH_1$. As $O_2(H_1) = 1$, $Q \cap H_1 = 1$ and hence Q and H_1 centralize each other. This is clearly impossible as $X_1 | Q$ is irreducible.

Thus $O_\infty(H) = Z(H)$. By 2D of Brauer [3], we may assume the highest prime dividing $|H|$ is 7. Then $H/Z(H) \cong O_5(3)$, $U_3(3)$, or $U_4(3)$.

As $|Z(H)|\,|8$, if $H/Z(H) \cong U_3(3)$, $3^4 \nmid |H|$. But $3^4 \mid |\langle h_1, h_2, h_3\rangle|$ as $|\langle h_1, h_2, h_3\rangle| = 648$, a contradiction. Let $K \cong O_5(3)$ or $U_4(3)$ and let L be a group with Z(L) a 2-group and $L/Z(L) \cong K$. An element of order 5 in K is self-centralizing, and so $C_L(\langle \pi_5\rangle) = \langle \pi_5\rangle \times Z(L)$ where π_5 is a 5-element of L. All 5-elements of K are conjugate and so all 5-elements of L are conjugate. As $5^2 \nmid |L|$, by Brauer [4, I, Theorem 10], every 5-block of defect 1 has characters of degree $z \equiv \pm 1 \pmod 5$. So L does not have an irreducible character of degree 8, and hence H does not exist. This is a final contradiction and so 1. of Lemma 5.5 holds.

The next lemma allows us to use the results of Stellmacher [26] when $n = 8$. Lemma 5.10 shows that when hypothesis (A) of Chapter IV fails in the case $n = 8$, we get a large primitive subgroup of degree 7. The final two lemmas complete the proof of Theorem 2.

Lemma 5.9: Let $n = 8$. Let h_1 and h_2 be special 3-elements. Then either h_1 and h_2 commute or $\langle h_1, h_2\rangle$ is isomorphic to A_4, $SL_2(3)$, or A_5.

Proof: Assume there is a subgroup H of G generated by special 3-elements such that $X|H = X_1 \oplus (8-s)1_H$ where X_1 is irreducible and primitive of degree $s = 6$ or 7. By Lemma 5.8 and Corollary 2.2, such a subgroup exists. By examination of the primitive groups in Lindsey [14, 16, 17] and Wales [27, 28] and by applying Blichfeldt, we get that if $s = 6$, $H \cong A_7$ or $O_5(3)$ and if $s = 7$, $H \cong A_8$ or $PSp_6(2)$. (If $s = 6$, the group H with $H/Z(H) \cong U_4(3)$ has a special 3-element (see [16]); but $|Z(H)| = 6$ in this case, which gives a contradiction by Blichfeldt.) In particular if $p \mid |G|$ where p is a prime we must have $p \leq 7$ (see Brauer [3]). Also in these cases X_1 has a rational character.

Let h_1 and h_2 be noncommuting special 3-elements. We study the possibilities for $X | \langle h_1, h_2\rangle$ given in Lemma 5.1.

Consider the case $X | \langle h_1, h_2\rangle = Y_1 \oplus \xi \oplus 4 \cdot 1_{\langle h_1, h_2\rangle}$ where Y_1 is irreducible of degree 3. If $\xi \neq 1_{\langle h_1, h_2\rangle}$, by Lemma 5.3, there exist special 3-elements h_1', h_2' such that $X | \langle h_1', h_2'\rangle = Y \oplus 5 \cdot 1_{\langle h_1', h_2'\rangle}$ where $\langle h_1', h_2'\rangle$ is the nonabelian group of order 27 and exponent 3. Then by Corollary 2.1, there is a subgroup H of G generated by special 3-elements such that $\langle h_1', h_2'\rangle \subseteq H$ and $X | H = X_1 \oplus (8-s)1_H$ where X_1 is irreducible and primitive of degree $s = 6$ or 7. By the first paragraph of this proof, X_1 has a rational character, contradicting the existence of an element $z \in Z(\langle h_1', h_2'\rangle)$ such that $X(z)$ has eigenvalues $\omega, \omega, \omega, 1, 1, 1, 1, 1$. So $\xi = 1_{\langle h_1, h_2\rangle}$. Thus by Corollary 2.1, there is a subgroup H of G generated by special 3-elements such that $\langle h_1, h_2\rangle \subseteq H$ and $X | H = X_1 \oplus (8-s)1_H$ where X_1 is irreducible and primitive of degree $s = 6$ or 7. As above $Z(\langle h_1, h_2\rangle) = 1$. If Y_1 is primitive, the only possibility is $\langle h_1, h_2\rangle \cong SL_2(7)$ or A_5 by looking at Blichfeldt's list. If $\langle h_1, h_2\rangle \cong SL_2(7)$, the eigenvalue structure of a 7-element is incorrect in $X_1 | H$. So if Y_1 is primitive, $\langle h_1, h_2\rangle \cong A_5$.

Now assume Y_1 acts on V_1 and is monomial in some basis v_1, v_2, v_3. Not both $Y_1(h_1)$ and $Y_1(h_2)$ are diagonal. So by scaling and ordering v_1, v_2, v_3 correctly and numbering h_1, h_2 correctly, we may assume

$$Y_1(h_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} .$$

As $\langle h_1, h_2\rangle$ cannot be the nonabelian group of order 27 and exponent 3, as above, by replacing h_2 by h_2^{-1} if necessary, we may assume

$$Y_1(h_2) = \begin{pmatrix} 0 & a & 0 \\ 0 & 0 & b \\ c & 0 & 0 \end{pmatrix} .$$

If H is A_7 or A_8, the special 3-elements correspond to 3-cycles and so two of them commute, generate A_4, or generate A_5. So we may assume $H \cong O_5(3)$ or $PSp_6(2)$. But $X_1(h_1^2 h_2)$ has eigenvalues $a, b, c, 1, 1, \ldots$. By examining $X_1|H$, the only possibilities are $\{a, b, c\} = \{1, -1, -1\}$ or $\{1, \omega, \bar{\omega}\}$. The first case gives $\langle h_1, h_2\rangle \cong A_4$ and the second case gives $\langle h_1, h_2\rangle$ the nonabelian group of order 27 and exponent 3, a contradiction. So if case 2. of Lemma 5.1 holds, $\langle h_1, h_2\rangle \cong A_4$ or A_5.

Now assume $X|\langle h_1, h_2\rangle = Y \oplus 4 \cdot 1_{\langle h_1, h_2\rangle}$ where Y is irreducible of degree 4. Let Y act on V_1. Assume first that Y is monomial on V_1. Let v_1, v_2, v_3, v_4 be a basis of V_1 in which Y is monomial. By ordering and scaling v_1, v_2, v_3, v_4 correctly and replacing h_2 by h_2^{-1} if necessary, we may assume

$$Y(h_1) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Y(h_2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & 1 \\ 0 & a^{-1} & 0 & 0 \end{pmatrix}$$

where $a \neq 1$. By Corollary 2.1, there is a subgroup H of G generated by special 3-elements such that $\langle h_1, h_2\rangle \subseteq H$ and $X|H = X_1 \oplus (8-s)1_H$ where X_1 is irreducible and primitive of degree $s = 6$ or 7. By the first paragraph $H \cong A_7, A_8, O_5(3)$, or $PSp_6(2)$. If $H \cong A_7$ or A_8, h_1 and h_2 represent 3-cycles and generate A_4 or A_5 contradicting Y being irreducible and monomial. So $H \cong O_5(3)$ or $PSp_6(2)$. Now $X_1((h_1h_2)^2)$ has eigenvalues $a, a^{-1}, a, a^{-1}, 1, 1, \ldots$. By looking in the character table of $O_5(3)$ or $PSp_6(2)$, the possibilities for a are $-1, \omega, \bar{\omega}, i$, or $-i$.

If $a = -1$, Y has an invariant subspace $\langle v_1 + \bar{\omega}v_2 + \omega v_3\rangle \oplus \langle -v_2 + \omega v_3 + \bar{\omega}v_4\rangle$, a contradiction. Let $g = h_1(h_1h_2)^2h_1^{-1}(h_2^{-1}h_1^{-1})^2$ and $h = h_1gh_1^{-1}g^{-1}$. Then $X_1(h)$ has eigenvalues $a^4, a^{-2}, a^{-2}, 1, 1, 1, \ldots$ which is impossible if $a = \omega$ or $\bar{\omega}$. Let $h = h_1^{-1}(h_2^{-1}h_1^{-1})^2(h_2h_1h_2)^2$. Then $X_1(h)$ has eigenvalues $a^{-1}, a^{-1}, a^3, a^{-1}, 1, 1, \ldots$ which is impossible if $a = \pm i$. Thus Y must be primitive.

There is a special 3-element h_3 such that $X(h_3)$ does not leave V_1 invariant. So $X | \langle h_1, h_2, h_3\rangle = R \oplus 2 \cdot 1_{\langle h_1, h_2, h_3\rangle}$ where either R is irreducible or $R = R_1 \oplus \xi$ where R_1 is irreducible of degree 5. Assume first that R acts on V_2 and is monomial. Then there is a basis $v_1, \ldots, v_6$ of V_2 in which R is monomial. As Y is not monomial, $R | \langle h_1, h_2\rangle$ can fix at most one $\langle v_i\rangle$ trivially. But the only possibility by Lemma 2.6 is that $\langle h_1, h_2\rangle \cong A_5$. So we may assume that if R is irreducible it is primitive, and if $R = R_1 \oplus \xi$, R_1 is primitive and $\xi = 1_{\langle h_1, h_2, h_3\rangle}$ by Mitchell. By Lindsey [18], the Sylow 5-subgroup of G is abelian. Using this fact and the list of primitive groups of degree 5 in Brauer [3], if R is reducible $\langle h_1, h_2, h_3\rangle \cong A_6$ or $O_5(3)$. Now if $\langle h_1, h_2, h_3\rangle \cong O_5(3)$, R_1 doesn't have a rational character. Let R_1 act on V_3 and let h_4 be a special 3-element such that $X(h_4)$ does not leave V_3 invariant. So $X | \langle h_1, h_2, h_3, h_4\rangle = S \oplus 1_{\langle h_1, h_2, h_3, h_4\rangle}$ where either S is irreducible or $S = S_1 \oplus \xi$ with S_1 irreducible of degree 6. By Lemma 1.4, as $O_5(3)$ is simple and not contained in A_7, S cannot be monomial. So S is primitive if it is irreducible, and if $S = S_1 \oplus \xi$, $\xi = 1_{\langle h_1, h_2, h_3, h_4\rangle}$

and S_1 is primitive. By the first paragraph, S has a rational character, a contradiction. So if R is reducible, $\langle h_1, h_2, h_3\rangle \cong A_6$ and the only possibility is that $\langle h_1, h_2\rangle \cong A_5$. So we assume R is irreducible. By the first paragraph, $\langle h_1, h_2, h_3\rangle \cong A_7$ or $O_5(3)$. Noting that in either group a 5-element is self-centralizing, by examining Blichfeldt's list [2] of primitive groups of degree 4, and by applying Blichfeldt, the only possibility is that $\langle h_1, h_2\rangle \cong A_5$. So if 3. holds of Lemma 5.1, $\langle h_1, h_2\rangle \cong A_5$.

Now assume 1. holds in Lemma 5.1. In the notation of conclusion 1. of Lemma 5.1, it suffices to prove $M_i = 1$ for $i = 1$ and 2. Let h_3 be a special 3-element such that $X(h_3)$ does not leave $V_1 \oplus V_2$ invariant by Lemma 1.1. Then the possibilities for $X|\langle h_1, h_2, h_3\rangle$ are listed in Lemma 5.4. Assume $M_i \neq 1$ for both $i = 1$ and $i = 2$, and hence $|\langle h_1, h_2\rangle| = 48$. Assume first that $X|\langle h_1, h_2, h_3\rangle = U \oplus (8-s)1_{\langle h_1, h_2, h_3\rangle}$ where U is irreducible of degree s with $s = 5$ or 6. Suppose U is primitive. By what was done in the previous paragraph, if $s = 5$, $\langle h_1, h_2, h_3\rangle \cong A_6$ and if $s = 6$, $\langle h_1, h_2, h_3\rangle \cong A_7$ or $O_5(3)$. But as $M_i \neq 1$ for $i = 1$ and $i = 2$, there is an element $g \in \langle h_1, h_2\rangle$ with $Y_1(g) = Y_1(h_1)$, $Y_2(g) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} Y_2(h_1)$. Then U(g) does not have a rational trace, a contradiction as U must have a rational character. So U is monomial. Thus in some basis $v_1, \ldots, v_s$ of the subspace on which U acts, by ordering and scaling $v_1, \ldots, v_s$ correctly and replacing h_2 by h_2^{-1} if necessary, we may assume

$$\begin{cases} v_1X(h_1) = v_2 \\ v_2X(h_1) = v_3 \\ v_3X(h_1) = v_1 \\ v_\ell X(h_1) = v_\ell \text{ for } \ell > 3 \end{cases} \quad \text{and} \quad \begin{cases} v_2X(h_2) = av_3 \\ v_3X(h_2) = v_4 \\ v_4X(h_2) = a^{-1}v_2 \\ v_\ell X(h_2) = v_\ell \text{ for } \ell \notin \{2,3,4\} \end{cases} .$$

As $U \mid \langle h_1, h_2\rangle = Y_1 \oplus Y_2 \oplus (s-4)1_{\langle h_1, h_2\rangle}$ and as $U((h_1h_2)^2) = \mathrm{diag}\{a, a^{-1}, a, a^{-1}, 1, \ldots\}$, the only possibility is $a = a^{-1}$. So $a = \pm 1$. If $a = 1$, $\langle h_1, h_2\rangle \cong A_4$, a contradiction. If $a = -1$, $\langle h_1, h_2\rangle \cong SL_2(3)$, a contradiction. Thus if $M_i \neq 1$ for $i = 1$ and $i = 2$, 1. of Lemma 5.4 does not hold. By what we have already done, 2. of Lemma 5.4 could not hold. So assume 3. of Lemma 5.4 holds. In the notation of that lemma, $N_i \subseteq Z(G_i)$ for $i = 1$ and 2 and $|\langle h_1, h_2, h_3\rangle| = 2^3 \cdot 3^a$ for some integer a. But then $|\langle h_1, h_2\rangle| \neq 48$, a contradiction. So in any case, we must conclude $M_i = 1$ for both $i = 1$ and $i = 2$, and hence $\langle h_1, h_2\rangle \cong SL_2(3)$.

Lemma 5.10: Let $n = 8$. Assume the special 3-elements of G do not generate A_9. Then there is a subgroup H of G generated by special 3-elements such that $X|H = R \oplus 1_H$ where R is irreducible and primitive of degree 7 with $H \cong PSp_6(2)$.

Proof: By Lemma 5.8 and Lemma 4.11, hypothesis (A) of Chapter 4 does not hold. Thus there is a subgroup K of G generated by special 3-elements such that $X|K = X_1 \oplus (8-s)1_K$ where X_1 is irreducible and primitive of degree $s = 6$ or 7 and $K \not\cong A_{s+1}$. By the first paragraph of the proof of Lemma 5.9, if $s = 6$, $K \cong O_5(3)$ and if $s = 7$, $K \cong PSp_6(2)$. If $s = 7$, we are done and so assume $s = 6$.

There is a subgroup $K_1 \subset K$ with $K_1 \cong A_6$. This subgroup is the derived group of the stabilizer of a point in the permutation representation of $O_5(3)$ on 36 letters [12]. So $X|K_1 = X_1|K_1 \oplus 2 \cdot 1_{K_1} = X_2 \oplus 3 \cdot 1_{K_1}$

where X_2 is irreducible and primitive, by examining the character table of A_6. Also K_1 is generated by special 3-elements. Let X_i act on V_i for $i = 1$ and 2. So $V_2 \subset V_1$. Let $2 \cdot 1_K$ act on V_1' and $3 \cdot 1_{K_1}$ act on V_2'. Hence $V_1' \subset V_2'$.

Assume the lemma is false. Let g be a special 3-element such that $X(g)$ does not leave V_1 invariant. In particular g does not commute with K. Let $K_1, \ldots, K_k$ be all the K-conjugates of K_1. As K is simple, $K = \langle K_1, \ldots, K_k \rangle$. Thus g does not commute with some K_i and by renumbering, we may assume g does not commute with K_1. Assume first that $X(g)$ leaves V_2 invariant. So $X \mid \langle K_1, g \rangle = X_3 \oplus \xi_1 \oplus \xi_2 \oplus 1_{\langle K_1, g \rangle}$ where X_3 acts on V_2. As g does not commute with K_1, $\xi_1(g) \neq 1$, $\xi_2(g) \neq 1$ is impossible. So one ξ_i, say for $i = 2$, is $1_{\langle K_1, g \rangle}$. If $\xi_1(g) = 1$, $X(g)$ leaves V_1 invariant, a contradiction. So $\xi_1(g) \neq 1$. But this is a contradiction by Mitchell as $X_3 \mid K_1 = X_2$ is primitive. So $X(g)$ does not leave V_2 invariant. We have three possibilities for $X \mid \langle K_1, g \rangle$:

A. $X \mid \langle K_1, g \rangle = R \oplus 1_{\langle K_1, g \rangle}$ where R is irreducible and primitive.

B. $X \mid \langle K_1, g \rangle = R \oplus 1_{\langle K_1, g \rangle}$ where R is irreducible and monomial.

C. $X \mid \langle K_1, g \rangle = R \oplus \xi \oplus 1_{\langle K_1, g \rangle}$ where R is irreducible of degree 6.

We show that in any case there is a special 3-element h such that $X \mid \langle K_1, h \rangle = Y \oplus 2 \cdot 1_{\langle K_1, h \rangle}$ where $X(h)$ does not leave V_1 invariant.

Assume case A holds. As we are assuming the lemma is false, $\langle K_1, g \rangle \cong A_8$. As all special 3-elements of $\langle K_1, g \rangle$ represent 3-cycles, by Lemma 4.5, K_1 is the stabilizer of two points. Choose a special 3-element h such that $\langle K_1, h \rangle$ is the stabilizer of one point, and hence so that $\langle K_1, h \rangle \cong A_7$. So $X \mid \langle K_1, h \rangle = Y \oplus 2 \cdot 1_{\langle K_1, h \rangle}$ where Y is irreducible. If $X(h)$ leaves V_1 invariant, then Y acts on V_1 and

$X|\langle K,h\rangle = S \oplus 2\cdot 1_{\langle K,h\rangle}$ where S is primitive on V_1. So $\langle K,h\rangle \cong O_5(3)$ or A_7. But $\langle K,h\rangle$ contains subgroups isomorphic to both $O_5(3)$ and A_7, which is impossible. So X(h) does not leave V_1 invariant.

Assume case B holds. Let R act on W and $1_{\langle K_1,g\rangle}$ on $\langle v_8\rangle$. Let $v_1,\ldots,v_7$ be a basis of W on which R is monomial. Choose special 3-elements g_1, g_2, g_3, g_4 in K_1 with g_i corresponding to the 3-cycle $(i,i+1,i+2)$. By ordering and scaling $v_1,\ldots,v_7$ correctly we may assume

$$\begin{cases} v_kX(g_k) = v_{k+1} \\ v_{k+1}X(g_k) = v_{k+2} \\ v_{k+2}X(g_k) = v_k \\ v_\ell X(g_k) = v_\ell \text{ for } \ell \notin \{k,k+1,k+2\} \end{cases} \quad \text{and} \quad \begin{cases} v_iX(g) = av_j \\ v_jX(g) = v_7 \\ v_7X(g) = a^{-1}v_i \\ v_\ell X(g) = v_\ell \text{ for } \ell \notin \{i,j,7\} \end{cases} .$$

In the expression for X(g), $a \neq 1$ and $i \neq j$ with $i,j < 7$. By conjugating by an element of K_1, we may assume $i = 5$ and $j = 6$. Let $h = gg_3gg_3^{-1}g^{-1}$. Then

$$\begin{cases} v_4X(h) = v_5 \\ v_5X(h) = av_6 \\ v_6X(h) = a^{-1}v_4 \\ v_\ell X(h) = v_\ell \text{ for } \ell \notin \{4,5,6\} \end{cases} .$$

Now $X|\langle K_1,h\rangle = Y \oplus 2\cdot 1_{\langle K_1,h\rangle}$, where Y is irreducible on $\langle v_1,\ldots,v_6\rangle$. By examining $\langle g_4,h\rangle$, by Lemma 5.9, $a = -1$ is the only possibility. Assume X(h) leaves V_1 invariant. Then $V_1 = \langle v_1,\ldots,v_6\rangle$ and $X|\langle K,h\rangle = S \oplus 2\cdot 1_{\langle K,h\rangle}$. So as $S|K = X_1$, $h \in K$ and $\langle K,h\rangle \cong O_5(3)$. Let $f = g_4^2h$. Then $v_iS(f) = -v_i$ for $i = 4$ and 6 and $v_jS(f) = v_j$ for $j = 1,2,3$, and 5. Conjugating f by elements of K_1, we get elements

h_1, h_2, h_3 with $v_{2i-1}X(h_i) = -v_{2i-1}$, $v_{2i}X(h_i) = -v_{2i}$ and $v_jX(h_i) = v_j$ for $j \neq 2i-1, 2i$. But then $S(h_1h_2h_3)$ is a nontrivial scalar matrix, contradicting $Z(O_5(3)) = 1$. Thus $X(h)$ does not leave V_1 invariant.

Assume case C holds. Suppose $\xi \neq 1_{\langle K_1, g\rangle}$; so $\xi(g) \neq 1$. But there is some special 3-element $g_1 \in K_1$ with g_1 not commuting with g. The only possibility is that $X|\langle g_1, g\rangle = Y_1 \oplus \xi_1 \oplus \xi_2 \oplus 4\cdot 1_{\langle g_1, g\rangle}$ where Y_1 is irreducible of degree 2 or $X|\langle g_1, g\rangle = Y_2 \oplus \mu \oplus 4\cdot 1_{\langle g_1, g\rangle}$ where Y_2 is irreducible of degree 3 and $\mu(g) \neq 1$. By Lemma 5.1, the first case is impossible. By Lemmas 5.3 and 5.9, the second case is impossible. Hence $\xi = 1_{\langle K_1, g\rangle}$ and letting $g = h$, we have the desired result.

So there is a special 3-element h such that $X|\langle K_1, h\rangle = Y \oplus 2\cdot 1_{\langle K_1, h\rangle}$ where $X(h)$ does not leave V_1 invariant. Thus either $X|\langle K, h\rangle$ is irreducible or $X|\langle K, h\rangle = R \oplus \xi$ where R is irreducible of degree 7. Let Y act on W_1 and $2\cdot 1_{\langle K_1, h\rangle}$ on W_1'. Note that $W_1' \subset V_2'$. As V_1' and W_1' are subspaces of dimension 2 contained in the subspace V_2' of dimension 3, $V_1' \cap W_1'$ has dimension at least 1 and $X|\langle K, h\rangle$ acts trivially on $V_1' \cap W_1'$. So $V_1' \cap W_1'$ has dimension 1 and $X|\langle K, h\rangle = R \oplus \xi$ where $\xi = 1_{\langle K, h\rangle}$. As $O_5(3)$ is simple and not contained in A_7, R is primitive by Lemma 1.4. As $O_5(3)$ is not contained in A_8, the only possibility is that $\langle K, h\rangle \cong PSp_6(2)$. This contradicts the assumption that the lemma is false and so the result follows.

<u>Lemma 5.11</u>: Let $n = 8$ and let N be the subgroup of G generated by all special 3-elements. Then one of the following holds:

1. $N \cong A_9$ and $G/Z(G) \cong A_9$ or S_9.

2. $N/Z(N) \cong O_8^+(2)$ where $Z(N)$ has order 2 and $N = N'$. Also $G/Z(G)$ is a subgroup of the automorphism group of $O_8^+(2)$.

<u>Proof</u>: If $N \cong A_9$, then $X|N$ is irreducible and so $C_G(N) = Z(G)$. As $N \trianglelefteq G$, $G/Z(G) = G/C_G(N)$ is contained in the automorphism group of A_9. Thus $G/Z(G) \cong A_9$ or S_9 and 1. holds.

Assume N is not isomorphic to A_9. Then by Lemma 5.10, there is a subgroup H generated by special 3-elements such that $H \cong PSp_6(2)$ and $X|H = R \oplus 1_H$ where R is irreducible and primitive of degree 7. All special 3-elements of H are conjugate in H and the special 3-elements generate H. Also by Aschbacher-Hall [1] and Lemma 5.9, there are two special 3-elements in H which generate A_5. Let R act on V_1 and let g be a special 3-element in G such that X(g) does not leave V_1 invariant. So g does not commute with H; hence g does not commute with some special 3-element $h_1 \in H$. By Lemma 5.9, g is conjugate to h_1 or h_1^{-1} in $\langle g, h_1 \rangle$. Thus g is conjugate to the special 3-elements of H by an element in $\langle H, g \rangle$. Let h be any special 3-element of G. Then h does not commute with $\langle H, g \rangle$ as $X|\langle H, g \rangle$ is irreducible and so h does not commute with some special 3-element $h_2 \in \langle H, g \rangle$. By Lemma 5.9, h is conjugate to h_2 or h_2^{-1} in $\langle g, h_2 \rangle$ and we can conclude that all special 3-elements of G are conjugate in the group N that they generate.

By Lemma 5.9 and Stellmacher [26], $N/O_\infty(N)$ is isomorphic to $PSp_{2k}(2)$ for $k \geq 3$, $O_{2k}^{\pm}(2)$ for $k \geq 3$, A_k for $k \geq 5$, HJ, $G_2(4)$, Sz, or Co_1. As H is simple, $H \cap O_\infty(N) = 1$ and $N/O_\infty(N)$ contains a subgroup isomorphic to $PSp_6(2)$. If p is a prime and $p \mid |G|$, then $p \leq 7$ (see [3]). Therefore using these two facts, we can conclude that $N/O_\infty(N) \cong PSp_6(2)$ or $O_8^+(2)$.

Consider first the case that $O_\infty(N) > Z(N)$. By Lemma 5.7 and inspection of the theorem and proof in Lindsey [15], there is a 2-group $Q \subseteq O_\infty(N)$ such that $Q \triangleleft N$, $X|Q$ is irreducible, and N/Q is a subgroup of $PSp_6(2)$. From the fact that $N/O_\infty(N) \cong PSp_6(2)$ or $O_8^+(2)$, we could only have $O_\infty(N) = Q$ and $N/O_\infty(N) \cong PSp_6(2)$. Therefore as $O_\infty(N) \cap H = 1$, $N = HQ$. By Frame [7], H has a subgroup H_1 with $H_1 \cong O_5(3)$. As $X|H$ has a rational character, the only possibility is that $X|H_1 = R_1 \oplus 2 \cdot 1_{H_1}$ where R_1 is irreducible and primitive of degree 6. H_1 is generated by special 3-elements. As $Q \triangleleft N$, consider the group $N_1 = H_1Q$. Then $|N_1| = |H_1| \cdot 2^a$ for some integer a. Let R_1 act on V_2. Let h_1 be any special 3-element of H_1 and h any conjugate of h_1 by an element of N_1. Assume X(h) does not leave V_2 invariant. Then $X|\langle H_1, h\rangle$ is irreducible or $X|\langle H_1, h\rangle = S \oplus \xi$ where S is irreducible of degree 7. As $7 \nmid |N_1|$, the latter is impossible. So $X|\langle H_1, h\rangle$ is irreducible; as $O_5(3)$ is simple and not a subgroup of A_8, by Lemma 1.4, $X|\langle H_1, h\rangle$ must be primitive. But this contradicts Lemma 5.10. So X(h) leaves V_2 invariant. Hence $X|\langle H_1, h\rangle = S \oplus \xi_1 \oplus \xi_2$ where S is irreducible of degree 6. As $S|H_1$ is primitive, so is S. If $\xi_1(h) \neq 1$ and $\xi_2(h) \neq 1$, h commutes with H_1 and $h \notin H_1$. But then the Sylow 3-subgroup of N_1 is larger than the Sylow 3-subgroup of H_1, a contradiction. So at least one ξ_i is $1_{\langle H_1, h\rangle}$ and by Mitchell, both must be. By the first paragraph of the proof of Lemma 5.9, the only possibility is $h \in H_1$. Therefore $H_1 \triangleleft N_1$ and so $[H_1, Q] \subseteq H_1 \cap Q = 1$. So H_1 centralizes Q, a contradiction as $X|Q$ is irreducible.

Thus $O_\infty(N) = Z(N)$. Hence if $N/O_\infty(N) \cong PSp_6(2)$, $N = HZ(N)$, a contradiction as $X|N$ is irreducible. So $N/Z(N) \cong O_8^+(2)$. By examining Frame [8], it is easy to see that $O_8^+(2)$ has no irreducible representation of degree 8. So Z(N) is cyclic of order 2, 4, or 8. Also $N'Z(N) = N$ as $N/Z(N)$ is simple. Hence N/N' is a 2-group. As N is generated by elements of order 3, $N = N'$. The Schur multiplier of $O_8^+(2)$ by Steinberg (see [6]) is $Z_2 \times Z_2$. As Z(N) is cyclic, we must have $|Z(N)| = 2$. Frame [8] exhibits a group, the Weyl group of E_8, whose derived subgroup has the properties of N described here. As $N \lhd G$ and $X|N$ is irreducible, $C_G(N) = Z(G)$. So $G/Z(G)$ is a subgroup of the automorphism group of N. But the automorphism group of N is clearly isomorphic to a subgroup of the automorphism group of $O_8^+(2)$.

<u>Lemma 5.12</u>: Let $n \geq 9$ and let N be the subgroup of G generated by all special 3-elements. Then $N \cong A_{n+1}$ and $G/Z(G) \cong A_{n+1}$ or S_{n+1}.

<u>Proof</u>: We first consider the case $n = 9$. Assume hypothesis (A) of Chapter 4 does not hold. Then there is a subgroup K of G generated by special 3-elements such that $X|K = X_1 \oplus (9-s)1_K$ where X_1 is primitive and irreducible of degree $s = 7$ or 8 and $K \ncong A_{s+1}$. If $s = 8$, by Lemma 5.11, $K/Z(K) \cong O_8^+(2)$ and $Z(K) \neq 1$. But then X(G) contains an element with eight eigenvalues equal to -1 and the other one equal to 1. By Mitchell [21], $G/Z(G) \cong S_{10}$, clearly a contradiction. So $s = 7$ and by examining Wales [27,28] $K \cong PSp_6(2)$. Let X_1 act on V_1 and $2 \cdot 1_K$ act on V_1'.

By Frame [7], there is a subgroup K_1 of K with $K_1 \cong O_5(3)$ and $X|K_1 = X_2 \oplus 3 \cdot 1_{K_1}$ where X_2 is irreducible and primitive of degree 6. Also K_1 is generated by its special 3-elements. Let g be a special 3-element in G such that X(g) does not leave V_1 invariant. In particular g does not commute with K. As in the proof of Lemma 5.10, by replacing K_1 by an appropriate K-conjugate, we may assume g does not commute with K_1. Let X_2 act on V_2 and $3 \cdot 1_{K_1}$ on V_2'. As in the proof of Lemma 5.10, X(g) does not leave V_2 invariant. Therefore $X \mid \langle K_1, g\rangle = S \oplus 1_{\langle K_1, g\rangle}$ where S is irreducible or $S = Y \oplus \xi$ such that Y is irreducible of degree 7. Assume first that S is irreducible. By Lemma 1.4, as $O_5(3)$ is simple and not a subgroup of A_8, S is primitive. But by Lemma 5.11, $\langle K_1, g\rangle / Z(\langle K_1, g\rangle) \cong O_8^+(2)$ is the only possibility. This is a contradiction, as in the first paragraph. So $S = Y \oplus \xi$. By Lemma 1.4, Y must be primitive. Hence $X \mid \langle K_1, g\rangle = Y \oplus 2 \cdot 1_{\langle K_1, g\rangle}$ by Mitchell. Arguing as in Lemma 5.10, $X \mid \langle K, g\rangle = R \oplus 1_{\langle K, g\rangle}$ where R is irreducible. By Lemma 1.4, since $PSp_6(2)$ is simple and is not contained in A_8, R is primitive. By Lemma 5.11, $\langle K, g\rangle / Z(\langle K, g\rangle) \cong O_8^+(2)$ is the only possibility, a contradiction as in the first paragraph. So hypothesis (A) holds and by Lemmas 5.8 and 4.11, the result holds for $n = 9$.

We now consider $n = 10$. Again assume hypothesis (A) fails. Then there is a subgroup K of G generated by special 3-elements such that $X \mid K = X_1 \oplus (10-s)1_K$ where X_1 is irreducible and primitive of degree $s = 8$ or 9 with $K \not\cong A_{s+1}$. From the case $n = 9$, we must have $s = 8$. By Lemma 5.11, $K/Z(K) \cong O_8^+(2)$. By Lemma 5.10, K contains a subgroup K_1 generated by special 3-elements such that $X \mid K_1 = X_2 \oplus 3 \cdot 1_{K_1}$ where $K_1 \cong PSp_6(2)$ and X_2 is primitive and irreducible. Let X_1 act on V_1 and $2 \cdot 1_K$ act on V_1'. Let X_2 act on V_2 and $3 \cdot 1_{K_1}$ on V_2'.

Let g be a special 3-element such that X(g) does not leave V_1 invariant. As in the proof of Lemma 5.10, X(g) does not leave V_2

invariant. Therefore $X | \langle K_1, g\rangle = S \oplus 1_{\langle K_1, g\rangle}$ where S is irreducible or $S = Y \oplus \xi$ such that Y is irreducible and ξ is linear. By Lemma 1.4, as $PSp_6(2)$ is simple and not a subgroup of A_9, if S is irreducible, it is primitive, and if S is reducible, Y is primitive. From the result for n = 9, S cannot be irreducible as $PSp_6(2)$ is not a subgroup of A_{10}. So $X | \langle K_1, g\rangle = Y \oplus 2 \cdot 1_{\langle K_1, g\rangle}$ by Mitchell. Arguing as in Lemma 5.10, $X | \langle K, g\rangle = R \oplus 1_{\langle K, g\rangle}$ where R is irreducible. But again by Lemma 1.4, R is primitive and so $\langle K, g\rangle \cong A_{10}$. But $K_1 \cong PSp_6(2)$, a contradiction. So hypothesis (A) holds. Therefore by Lemmas 5.8 and 4.11, the result holds for n = 10.

Assume the result fails for some $n \geqslant 11$ and let n be minimal so that a counterexample exists. By Lemmas 5.8 and 4.11, hypothesis (A) fails and so there exists a subgroup H of G generated by special 3-elements such that $X | H = X_1 \oplus (n-s)1_H$ where X_1 is irreducible and primitive of degree s = n-1 or n-2 but $H \not\cong A_{s+1}$. As $n \geqslant 11$, this contradicts the minimality of n and the lemma holds.

Combining Lemmas 5.11 and 5.12, the proof of Theorem 2 is now complete.

The corollary is an easy application of Theorem 2. Let G be simple and X a nontrivial complex representation of G of degree $n \geqslant 10$. Let H be a subgroup of G isomorphic to A_{n-1}. By Lemma 4.2, $X | H = X_1 \oplus 2 \cdot 1_H$ where X_1 is irreducible and primitive of degree n-2. In particular G contains a special 3-element, and as G is simple, it is generated by its special 3-elements. Also $X = Y_1 \oplus Y_2$ where Y_1 is irreducible of degree n-2, n-1, or n. If Y_1 has degree n-2, it is primitive as X_1 is. So special 3-elements $g \in G \backslash H$ have the property that either $Y_1(g)$ or $Y_2(g)$ is trivial; as G is simple, $Y_2(g)$ must be trivial and hence $Y_2 = 2 \cdot 1_G$. By Theorem 2, $G \cong A_{n-1}$. If Y_1 has degree n-1, $Y_2 = 1_G$ as G is simple. By Lemma 1.4, Y_1 is primitive and by Theorem 2, $G \cong A_n$. If X is irreducible, by Lemma 1.3 it is primitive and by Theorem 2, $G \cong A_{n+1}$. The corollary is proved.

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