NUMERICAL RANGES AND COMMUTATION PROPERTIES OF HILBERT SPACE OPERATORS

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ABSTRACT

Application of the theory of numerical ranges to the study of commutation properties of operators is the purpose of the thesis.

For a complex, unital Banach algebra $\mathfrak{A}$, $T \in \mathfrak{A}$, the numerical range of T is $V(\mathfrak{A}, T) = \{f(T) : f(1) = 1 = \|f\|, f \in \mathfrak{A}^*\}$. This is a generalization and extension of the notion of the numerical range defined for a bounded operator T on the Hilbert space $\mathfrak{H}$: $W(T) = \{(Tx, x) : x \in \mathfrak{H}, (x,x) = 1\}$. These numerical range concepts are used in studies of multiplicative commutators, derivations, and powers of accretive operators.

An extension of Frobenius' group commutator theorem is obtained: For $T, A, B \in \mathfrak{B}(\mathfrak{H})$, $T = ABA^{-1}B^{-1}$, $AT = TA$, A normal and $0 \notin W(B)^-$ imply $T = 1$. Other extensions of the Frobenius theorem are proved and a special discussion is given about these results in the case $\mathfrak{H}$ is finite dimensional. The sharpness of the results is also reviewed.

For X a Banach space, the numerical range of a derivation acting on $\mathfrak{B}(X)$ is completely characterized. If Δ_T is the derivation induced by $T \in \mathfrak{B}(X)$, then

$$V(\mathfrak{B}(\mathfrak{B}(X)), \Delta_T) = V(\mathfrak{B}(X), T) - V(\mathfrak{B}(X), T) .$$

Normal elements of general Banach algebras are discussed. A consequence of an examination of derivations which are normal is a simple proof of the Fuglede-Putnam Theorem.

A theorem for matrices by C. R. Johnson is generalized to the operator case: for $T \in \mathcal{B}(\mathcal{H})$, $W(T^n) \subset \{\text{Re} z \geq 0\}$, $n = 1, 2, \ldots$ if and only if $T \geq 0$. Examples are given which show neither the necessity nor the sufficiency part of the theorem can be transplanted into the general Banach algebra setting. A containment result for the numerical range of a product is also proved.

TABLE OF CONTENTS

INTRODUCTION

The study of numerical ranges of operators and Banach algebra elements has expanded considerably in recent years (for a survey of these advances, see [10]). It is the purpose of this work to apply newly found numerical range results to study commutativity and multiplicative properties of operators.

A discussion of background material essential to the understanding of this thesis is contained in Chapter 1. Proofs are provided in many cases to make the thesis essentially self-contained. Many of the numerical range techniques used in later chapters are introduced in these proofs.

For $\mathfrak{A}$ a complex unital Banach algebra, $T \in \mathfrak{A}$,

$$V(\mathfrak{A}, T) = \{f(T) : f(1) = 1 = \|f\|, f \in \mathfrak{A}^*\}$$

is the algebra numerical range of T. Chapter 1 delineates properties of this set valued map. How it relates to other concepts of the numerical range and their antecedent in Hilbert space is shown.

Extensions of Frobenius' group commutator theorem are the basis for discussion in Chapter 2. $\mathcal{B}(\mathfrak{H})$ denotes the algebra of bounded operators on the Hilbert space $\mathfrak{H}$. In $\mathcal{B}(\mathfrak{H})$ the following extension of the Frobenius theorem is obtained: for $T, A, B \in \mathcal{B}(\mathfrak{H})$, suppose $T = ABA^{-1}B^{-1}$, $AT = TA$, A normal, and $0 \notin V(\mathcal{B}(\mathfrak{H}), B)$, then $T = 1$.

For $\mathcal{B}(X)$, the algebra of bounded operators on the Banach space X, the numerical range of a derivation on $\mathcal{B}(X)$ is completely characterized. This result is viewed with some surprise because of its applicability to all algebras of the form $\mathcal{B}(X)$; a related characterization of the norm of a derivation by Stampfli does not extend from Hilbert space to the general case. A simple derivation proof of the Fuglede-Putnam theorem is a consequence of our investigations.

Chapter 4 is devoted to an operator proof of a matrix theorem of C. R. Johnson: for $T \in \mathcal{B}(\mathfrak{h})$, $V(\mathcal{B}(\mathfrak{h}), T^n) \subset \{\mathrm{Re}\, z \geq 0\}$, $n = 1, 2, \ldots,$ if and only if $T \geq 0$. Thus, as with complex numbers, operators which have all powers accretive are positive. Discussion is included about extensions of the main result. Examples are presented which show that neither the necessity nor the sufficiency part of the main theorem can be translated into a general Banach algebra setting.

All results are stated for complete (Banach) spaces and algebras. It will be seen that there is no loss of generality in assuming completeness because the algebra numerical range is unaffected by enlargements. The assumed completeness makes the statement of results simpler and hence facilitates the discussion.

CHAPTER 1

PRELIMINARIES: NUMERICAL RANGE PROPERTIES

1. INTRODUCTION

It is the purpose of this work to study multiplicative and commutation properties of Banach algebra elements by means of the numerical range. Investigations are made into multiplicative and additive commutators and powers of operators. The effort is to show that the imposition of numerical range conditions yields useful characteristics of the algebra elements involved. To meet these objectives, therefore, a groundwork is layed in this chapter of the preliminary material needed.

The approach to the subject of numerical ranges is general. While often the attention is focused on the Banach algebra of Hilbert space operators and the well-known numerical range defined for these objects, in mind are thoughts of extending Hilbert space results to a more general Banach algebra case, or else demonstrating the distinctiveness of the Hilbert space case by counterexample.

2. DEFINITIONS AND NOTATION

$\mathfrak{H}$ denotes a complex Hilbert space equipped with the inner product $(\cdot,\cdot)$: $\mathfrak{H} \times \mathfrak{H} \to \mathbb{C}$. If X denotes a complex Banach space, $\mathcal{B}(X)$ symbolizes the algebra of all bounded endomorphisms of X. The script letters $\mathcal{A}$, $\mathcal{B}$, ... are used to denote complex unital Banach algebras, the capitals A, B, ..., S, T, ... are used for Banach algebra elements or operators, and lower case letters f, g, ..., x, y, ... denote Banach space vectors. The unit element of a Banach algebra is written 1.

The norm on a Banach space or algebra is denoted $\|\cdot\|$. Since context will reveal which norm is being applied no further identification is attached to the norm. Recall that for a Banach space X an endomorphism $T : X \to X$ is bounded if the set $\{\|Tx\| : \|x\| = 1, x \in X\}$ is a bounded set of real numbers. The norm induced from the Banach space,

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1, x \in X\} ,$$

to the algebra $\mathcal{B}(X)$ makes $\mathcal{B}(X)$ a Banach algebra.

Associated with a Banach space X is its dual space X*, the Banach space of all continuous linear maps from X to $\mathbb{C}$. Since $\mathcal{H}$ is self-dual no distinction is made between elements of the space and elements of the dual.

For $T \in \mathcal{B}(\mathcal{H})$ T* is the <u>adjoint of T</u>. $T^* \in \mathcal{B}(\mathcal{H})$ and is defined by the relations

$$(T^*x, y) = (x, Ty), \qquad \text{for all } x, y \in \mathcal{H} .$$

$\mathcal{G}(\mathfrak{A})$ is the group of invertible elements in the unital Banach algebra $\mathfrak{A}$. If $T \in \mathfrak{A}$, the <u>spectrum of T in $\mathfrak{A}$</u>, denoted $\sigma_{\mathfrak{A}}(T)$, is the set

$$\sigma_{\mathfrak{A}}(T) = \{\lambda \in \mathbb{C} : \lambda - T \notin \mathcal{G}(\mathfrak{A})\} .$$

The spectrum of a Banach algebra element is a nonempty compact subset of the complex plane. If $\mathfrak{B}$ is the maximal commutative subalgebra of $\mathfrak{A}$ containing T, then

$$\sigma_{\mathfrak{A}}(T) = \sigma_{\mathfrak{B}}(T) = \{\varphi(T) : \varphi : \mathfrak{B} \to \mathbb{C} \text{ is multiplicative}\} \qquad (1)$$

(see, e.g. [32] pages 35 and 111). When the algebra context is clear the spectrum of T is written $\sigma(T)$.

$\Pi_{\mathfrak{A}}(T)$ denotes the approximate point spectrum of T in $\mathfrak{A}$. $\lambda \in \Pi_{\mathfrak{A}}(T)$ implies $\lambda - T$ is a generalized divisor of zero in $\mathfrak{A}$. This means that there exists $\{A_n\}_{n=1}^{\infty}$, $A_n \in \mathfrak{A}$, $\|A_n\| = 1$ such that either $A_n T \to 0$ or $TA_n \to 0$.

r(T) denotes the spectral radius,

$$r(T) = \max\{ |\lambda| : \lambda \in \sigma_{\mathfrak{A}}(T)\} ,$$

and can be computed with the standard formula

$$r(T) = \lim_{n \to \infty} \|T^n\|^{1/n} . \tag{2}$$

For $K \subset \mathbb{C}$, coK denotes the convex hull of K, K^- the closure, $\overset{\circ}{K}$ the interior, and ∂K the set of boundary points of K. $\overline{K} = \{\overline{\lambda} : \lambda \in K\}$ is the set of complex conjugates of the points of K. The real and imaginary parts of elements of K are $\mathrm{Re}K = \{\frac{\lambda + \overline{\lambda}}{2} : \lambda \in K\}$ and $\mathrm{Im}K = \{\frac{\lambda - \overline{\lambda}}{2i} : \lambda \in K\}$, respectively.

3. THE NUMERICAL RANGES

The study of the Hilbert space numerical range dates back to the work of Hilbert, Toeplitz, and others who were largely interested in quadratic forms.

(1.1) DEFINITION. The numerical range of $T \in \mathcal{B}(\mathfrak{H})$ is the collection

$$W(T) = \{(Tx, x) : \|x\| = 1\} .$$

<u>The numerical radius of</u> T <u>is</u>

$$w(T) = \sup\{ |\lambda| : \lambda \in W(T)\} \quad .$$

That W(T) is convex for each $T \in \mathcal{B}(\mathfrak{H})$ is the content of the Toeplitz-Hausdorff Theorem. Several additional properties of this numerical range are properties that will be observed for generalized numerical ranges.

(1.2) THEOREM. <u>For</u> $T \in \mathcal{B}(\mathfrak{H})$

i) W(T) <u>is convex</u>,

ii) $\sigma(T) \subset W(T)^-$,

iii) $w(T) \leq \|T\| \leq 2w(T)$.

To briefly comment on the proof, i) (the Toeplitz-Hausdorff theorem) has many elementary proofs. One of the easiest derives from the observation that the numerical range of a restriction of an operator is contained in its full numerical range (this is made precise in Proposition 1.3 iii). Coupled with a result of Donoghue [14] that numerical ranges of operators on the two dimensional Hilbert space are closed elliptical disks this implies i).

$\lambda \in \partial\sigma(T)$ implies $\lambda \in \Pi_{\mathcal{B}(\mathfrak{H})}(T)$ and in fact that there exists a sequence of unit vectors, $\{x_n\}_{n=1}^{\infty}$, such that $\|(\lambda - T)x_n\| \to 0$ as $n \to \infty$. Hence $(Tx_n, x_n) \to \lambda$ as $n \to \infty$ and $\lambda \in W(T)^-$. Because $\sigma(T)$ is compact and $\partial\sigma(T) \subset W(T)^-$, i) implies ii).

It is clear that $w(T) \leq \|T\|$ by the Schwarz inequality. That $\|T\| \leq 2w(T)$ follows from polarization.

Other numerical range results which pertain only to the Hilbert space case will also be useful in what follows. Some of these results are listed in

(1.3) PROPOSITION. For $T \in \mathcal{B}(\mathfrak{H})$,

i) $\mathrm{Re}T = \frac{T + T^*}{2}$ and $\mathrm{Im}T = \frac{T - T^*}{2i}$, then $\mathrm{Re}W(T) = W(\mathrm{Re}T)$ and $\mathrm{Im}W(T) = W(\mathrm{Im}T)$,

ii) For T normal (i.e., $TT^* = T^*T$), $W(T)^- = \mathrm{co}\,\sigma(T)$,

iii) If P is an orthogonal projection on $\mathfrak{H}$ $(P = P^2 = P^*)$ and $P \neq 0$, then

$$W(PTP) \subset W(T) \quad ,$$

where PTP is considered as an operator on $P\mathfrak{H}$,

iv) $W(T) = W(UTU^*)$, U unitary.

PROOF.

$$\mathrm{Re}(Tx, x) = \frac{(Tx, x) + \overline{(Tx, x)}}{2} = \frac{(Tx, x) + (T^*x, x)}{2} = (\mathrm{Re}Tx, x) \quad .$$

With a similar relation for the imaginary parts i) is proved.

ii) follows from a more general result presented in Chapter 3, Section 4.

PTP considered as an operator on $P\mathfrak{H}$ has the numerical range,

$$W(PTP) = \{(PTPx, x) : x = Px, \ \|x\| = 1\} \quad .$$

But then

$$W(PTP) = \{(Tx, x) : x = Px, \ \|x\| = 1\} \subset W(T) \quad .$$

iv) is valid from the equations

$$\begin{aligned} W(UTU^*) &= \{(UTU^*x, x) : \|x\| = 1\} \\ &= \{(Ty, y) : y = U^*x, \ \|x\| = 1\} \\ &= \{(Ty, y) : \|y\| = 1\} \\ &= W(T) \ . \blacksquare \end{aligned}$$

Additional facts about W will be described as needed. Further general discussion of Hilbert space numerical range properties is found in [20], Chapter 14.

The properties of the Hilbert space numerical range are largely held intact in the generalizations examined in this chapter. Theorem 1.2 has analogous formulations in each new setting.

The modern theory of numerical ranges has its roots in the study of geometrical properties of Banach algebras. One paper of note is that of Bohnenblust and Karlin [5] which studies the geometry of the unit sphere of a unital Banach algebra. The key result of [5], for the purposes of this thesis, is the fact that the unit element is a vertex of the unit sphere. Let $\mathcal{S}(\mathfrak{A})$ denote the set of states for the unital Banach algebra $\mathfrak{A}$:

$$\mathcal{S}(\mathfrak{A}) = \{f \in \mathfrak{A}^* : f(1) = 1 = \|f\|\} \ .$$

(1.4) THEOREM. [5]. For the complex unital Banach algebra $\mathfrak{A}$, $\mathcal{S}(\mathfrak{A})$ separates the points of $\mathfrak{A}$. Furthermore, for $T \in \mathfrak{A}$, if

$$v(T) = \sup\{|f(T)| : f \in \mathcal{S}(\mathfrak{A})\} \ ,$$

then

$$v(T) \leqslant \|T\| \leqslant ev(T) \quad .$$

It is Bonsall [7] who formulates the notation for the set implicitly examined by Bohnenblust and Karlin, Vidav [44], and others. In [7] the following definition is made:

(1.5) DEFINITION. For a unital Banach algebra $\mathfrak{A}$, $T \in \mathfrak{A}$, the algebra numerical range, written $V(\mathfrak{A}, T)$, is the set

$$V(\mathfrak{A}, T) = \{f(T) : f \in \mathcal{S}(\mathfrak{A})\} \quad .$$

The number $v(T)$ described in Theorem 1.4 is the numerical radius of T.

Observe that the numerical range of a Banach algebra element, unlike the spectrum, is not algebra dependent. For if $T \in \mathfrak{B}$, $\mathfrak{B}$ a unital Banach algebra, and $\mathfrak{B} \subset \mathfrak{A}$, then clearly $V(\mathfrak{B}, T) \supset V(\mathfrak{A}, T)$ since a state on $\mathfrak{A}$ restricted to $\mathfrak{B}$ is a state; because a state in $\mathfrak{B}$ can be extended to a state in $\mathfrak{A}$ by the Hahn-Banach theorem, $V(\mathfrak{B}, T) \subset V(\mathfrak{A}, T)$. The numerical range is a norm dependent quantity because it is defined in terms of the states.

It is eventually shown that describing both v and w as the numerical radius is not inconsistent.

It is almost immediate from Definition 1.5 that a theorem analogous to Theorem 1.2 is valid.

(1.6) THEOREM. Let $\mathfrak{A}$ be a complex unital Banach algebra. For $T \in \mathfrak{A}$,

i) $V(\mathfrak{A}, T)$ is closed and convex,

ii) $\sigma(T) \subset V(\mathfrak{A}, T)$,

iii) $v(T) \leqslant \|T\| \leqslant ev(T)$.

PROOF. i) holds because $\mathcal{S}(\mathfrak{A})$ is a compact, convex subset of $\mathfrak{A}^*$ in the weak* topology. Thus, since the map $\varphi : \mathfrak{A}^* \to \mathbb{C}$ defined by

$$\varphi(f) = f(T), \quad f \in \mathfrak{A}^*$$

is weakly continuous and linear, $\varphi(\mathcal{S}(\mathfrak{A})) = V(\mathfrak{A}, T)$ is compact and convex.

$\lambda \in \partial\sigma_{\mathfrak{A}}(T)$ implies $\lambda \in \Pi_{\mathfrak{A}}(T)$ and even that there exists a sequence of unit elements, $\{A_n\}_{n=1}^{\infty}$ such that $(\lambda - T)A_n \to 0$ as $n \to \infty$. From the Hahn-Banach theorem there exist functionals $f_n \in \mathfrak{A}^*$ with $f_n(A_n) = \|f_n\| = 1$. But then $g_n(\cdot) = f_n(\cdot A_n)$ is a state and $g_n(T) \to \lambda$ as $n \to \infty$. Thus $\lambda \in V(\mathfrak{A}, T)$. The compactness of $\sigma_{\mathfrak{A}}(T)$ and convexity of V imply ii).

iii) is Theorem 1.4.■

Before the algebra numerical range was formulated Lumer [27] introduced the concept of the semi-inner-product space (s.i.p.s.). Independently, Bauer [2] introduced a related notion for finite dimensional spaces. Both sought to explore operators on spaces other than $\mathfrak{H}$ by imitating Hilbert space structure. To do this to each element x of a Banach space X we associate a functional f_x such that

$$f_x(x) = \|x\|^2 \quad \text{and} \quad \|f_x\| = \|x\|.$$

With such an association we define a semi-inner-product on X, $[\cdot,\cdot] : X \times X^* \to \mathbb{C}$, which satisfies the relations

$$[x, y] = f_y(x), \quad x, y \in X \quad .$$

$[x,x] = \|x\|^2$ and $[\cdot,\cdot]$ is linear in the first variable. Otherwise the semi-inner-product does not satisfy any of the usual Hilbert space inner product relations. Note that the selection of the functional associated with the vector x is not (in general) unique. However, if $X = \mathfrak{H}$ then the selection is unique and the semi-inner-product coincides with the established inner product on $\mathfrak{H}$.

(1.7) DEFINITION. For X a Banach space, $T \in \mathcal{B}(X)$, and $[\cdot,\cdot]$, a fixed semi-inner-product defined on X, put

$$W_{[,]}(T) = \{[Tx,x] : \|x\| = 1\} .$$

$W_{[,]}(T)$ is the Lumer numerical range relative to the semi-inner-product $[\cdot,\cdot]$.

(1.8) DEFINITION. Let $\mathcal{S}(X)$ denote the family of all semi-inner-products on the Banach space X. Then for $T \in \mathcal{B}(X)$, the spatial numerical range of T is

$$W(T) = \bigcup_{\mathcal{S}(X)} W_{[,]}(T) .$$

$$w(T) = \sup\{|\lambda| : \lambda \in W(T)\}$$

is the numerical radius of T.

By the remark above there is compatibility between Definitions 1.8 and 1.1. These definitions also imply

$$W_{[,]}(T) \subset W(T) \subset V(\mathcal{B}(X),T), \quad T \in \mathcal{B}(X) . \tag{3}$$

It is also possible to produce a theorem analogous to Theorems 1.2 and 1.6 for the spatial numerical range. Two intermediate results are needed, however.

(1.9) LEMMA. ([10], page 83). <u>For</u> T $\in \mathcal{B}(X)$ <u>and</u> $[,] \in \mathcal{J}(X)$,

$$\inf\left\{\frac{\|1+aT\|-1}{a} : a>0\right\} = \sup\{\mathrm{Re}[Tx,x] : \|x\|=1\}$$
$$= \sup\{\mathrm{Re}\lambda : \lambda \in V(\mathcal{B}(X),T)\} .$$

PROOF. Put $\alpha = \inf\{\frac{\|1+aT\|-1}{a} : a>0\}$, $\beta = \sup\{\mathrm{Re}[Tx,x] : \|x\|=1\}$, and $\gamma = \sup\{\mathrm{Re}\lambda : \lambda \in V(\mathcal{B}(X),T)\}$. Since $W_{[,]}(T) \subset V(\mathcal{B}(X),T)$, $\beta \leqslant \gamma$ is clear.

For f $\in \mathcal{S}(\mathcal{B}(X))$

$$\mathrm{Ref}(T) = \frac{\mathrm{Ref}(1+aT)-1}{a} \leqslant \frac{\|1+aT\|-1}{a}, \quad a>0 ;$$

$\gamma \leqslant \alpha$ is also clear.

For $a > 0$ sufficiently small,

$$\|(I-aT)x\| \geqslant \mathrm{Re}[(1-aT)x,x]$$
$$= 1 - a\mathrm{Re}[Tx,x] \geqslant 1-a\beta > 0 , \quad \|x\|=1 . \qquad (4)$$

With x replaced by $(I+aT)x$ (4) becomes

$$\|(1-a^2T^2)x\| \geqslant (1-a\beta)\|(1+aT)x\| , \quad x \subset X .$$

Then

$$\frac{\|1+aT\|-1}{a} \leqslant \frac{\beta+a\|T^2\|}{1-a\beta}, \quad a>0 .$$

Hence $\alpha \leqslant \beta$. ∎

(1.10) THEOREM. [10]. <u>For</u> $T \in \mathcal{B}(X)$ <u>and</u> $[,] \in \mathcal{L}(X)$, $\mathrm{co}W_{[,]}(T)^- = \mathrm{co}W(T)^- = V(\mathcal{B}(X), T)$, <u>and</u> $w(T) = v(T)$.

PROOF. Because of the inclusion (3) the theorem will be proved once it is verified that $\mathrm{co}W_{[,]}(T)^- = V(\mathcal{B}(X), T)$. Since both sets in question are compact and convex it suffices to show $\sup\{\mathrm{Re}\lambda : \lambda \in e^{i\theta}W_{[,]}(T)\} = \sup\{\mathrm{Re}\lambda : \lambda \in e^{i\theta}V(\mathcal{B}(X), T)\}$, for all $\theta \in R$. Because $e^{i\theta}W_{[,]}(T) = W_{[,]}(e^{i\theta}T)$ and $e^{i\theta}V(\mathcal{B}(X), T) = V(\mathcal{B}(X), e^{i\theta}T)$, Lemma 1.9 can be applied to yield the result.■

The spatial numerical range analog to Theorems 1.2 and 1.6 is now available.

(1.11) THEOREM. <u>For</u> $T \in \mathcal{B}(X)$,

i) $W(T)$ <u>is connected,</u>

ii) $\sigma_{\mathcal{B}(X)}(T) \subset W(T)^-$

iii) $w(T) \leq \|T\| \leq ew(T)$.

That W(T) is connected was first shown by Bonsall, Cain, and Schneider [8]. They show that the set

$$Z = \{(x,f) : x \in X,\ f \in X^*,\ f(x) = 1 = \|f\| = \|x\|\}$$

is connected in the norm × weak* topology on $X \times X^*$. It is then a simple matter to show that W(T) is a continuous image of this set.

Williams [46] gives an elegant proof that $\sigma_{\mathcal{B}(X)}(T) \subset W(T)^-$. He uses the result of Bishop and Phelps [3]: for X a Banach space,

$$D = \{f \in X^* : \|f\| = 1 = f(x) = \|x\| \quad \text{for some} \quad x \in X\}$$

is norm dense in the unit sphere of X^*.

iii) follows from Theorems 1.4 and 1.10.

One additional lemma is helpful in the sequel. The proof is similar to that of Lemma 1.9 and is omitted.

(1.12) LEMMA. [10]. <u>For the complex unital Banach algebra</u> $\mathcal{A}$, $T \in \mathcal{A}$,

$$\max\{\operatorname{Re}\lambda : \lambda \in V(\mathcal{A}, T)\} = \lim_{a \to 0^+} \frac{1}{a} \log \|\exp aT\| = \sup_{a>0} \frac{1}{a} \log \|\exp aT\| , \tag{5}$$

<u>and</u>

$$\max\{\operatorname{Re}\lambda : \lambda \in \sigma_{\mathcal{A}}(T)\} = \lim_{a \to +\infty} \frac{1}{a} \log \|\exp aT\| = \inf_{a>0} \frac{1}{a} \log \|\exp aT\| . \tag{6}$$

The limits in (5) and (6) exist and equal the sup and inf respectively by the subadditivity of the function $\frac{1}{a} \log \|\exp aT\|$ (see [22], pages 135-145).

For (6) note that $\lim_{a \to \infty} \log \|\exp aT\|^{1/a} = \log r(\exp T) = \max\{\operatorname{Re}\lambda : \lambda \in \sigma_{\mathcal{A}}(T)\}$.

4. HERMITIAN ELEMENTS

Vidav [44] introduced a norm characterization of hermitian elements in Banach algebras. The study of such elements has been an important aspect of the investigation of numerical range properties.

(1.13) DEFINITION. <u>For the unital Banach algebra</u> $\mathfrak{A}$, $H \in \mathfrak{A}$ <u>is called hermitian if</u> $\|\exp iaH\| = 1$, <u>for all</u> $a \in \mathbb{R}$.

From this definition a numerical range characterization of hermitian elements is available.

(1.14) THEOREM. [27], [10]. $H \in \mathfrak{A}$ <u>is hermitian if and only if</u> $V(\mathfrak{A}, H) \subset \mathbb{R}$.

PROOF. $\|\exp iaH\| = 1$, $a \in \mathbb{R}$ implies both

$$\sup_{a>0} \frac{1}{a} \log \|\exp iaH\| = 0$$

and

$$\sup_{a>0} \frac{1}{a} \log \|\exp - iaH\| = 0 \quad .$$

By Lemma 1.12, $\mathrm{Im}\lambda = 0$ for $\lambda \in V(\mathfrak{A}, H)$.

If $V(\mathfrak{A}, H) \subset \mathbb{R}$ then (5) and (6) imply $\log \|\exp iaH\| = 0$ for all $a \in \mathbb{R}$. Thus $\|\exp iaH\| = 1$, $a \in \mathbb{R}$. ∎

In $\mathcal{B}(\mathfrak{h})$, therefore, the usual definition of hermiticity corresponds to that given by Vidav. One other property which extends from the Hilbert space to the general case is that the norm and the spectral radius of a hermitian element are equal. This is the content of Sinclair's Theorem [37] (an elementary proof is given in [9]):

(1.15) THEOREM. <u>For a unital Banach algebra</u> $\mathfrak{A}$, <u>if</u> $H \in \mathfrak{A}$ <u>is hermitian, then</u> $r(H) = v(H) = \|H\|$.

An immediate corollary of the theorem is that $V(\mathfrak{A}, H) = \mathrm{co}\,\sigma(H)$ whenever H is hermitian.

It is essential to note, however, that few other properties of Hilbert space hermitians carry over to the general setting. For example, it need not follow that H^2 is hermitian if H is hermitian. A discussion of some of these differences is given in Chapter 4, Section 6.

In the interest of previewing theorems which appear later in this thesis we combine two elementary facts to obtain an additive commutator theorm for a general Banach algebra.

Recall the Kleinecke-Shirokov Theorem for Banach algebra elements (see [20], page 128):

(1.16) THEOREM. Suppose that $A, B \in \mathfrak{A}$ and that A commutes with $D = AB - BA$. Then $\sigma_{\mathfrak{A}}(D) = \{0\}$.

This algebraic theorem has immediate application in an additive commutator theorem.

(1.17) THEOREM. Suppose $\mathfrak{A}$ is a unital Banach algebra and that both H and K are hermitian in $\mathfrak{A}$. If H commutes with $D = HK - KH$, then $D = 0$.

PROOF. It suffices to show that iD is hermitian, for by Theorem 1.15 $\sigma(D) = \{0\}$ implies $\|D\| = 0$.

To sketch that iD is hermitian (see [10], page 48 for details), the hermiticity of H and K is used to obtain

$$\|\exp(iaH)\ \exp(iaK)\ \exp(-iaH)\ \exp(-iaK)\| = 1\ . \qquad (7)$$

Expanding the term on the left one finds

$$\|1 - a^2 D\| + o(a^2) = 1 \quad . \tag{8}$$

Lemma 1.9 implies that

$$\inf\{\mathrm{Re}\lambda : \lambda \in V(\mathfrak{A}, D)\} = -\inf_{a>0} \frac{\|1 - a^2 D\| - 1}{a^2} = -\lim_{a^2 \to 0+} \frac{\|1 - a^2 D\| - 1}{a^2} = 0 \quad .$$

Changing the order of the signs in the left-hand side of (7) one obtains the variation of (8)

$$\|1 + a^2 D\| + o(a^2) = 1 \quad .$$

Thus $\sup\{\mathrm{Re}\lambda : \lambda \in V(\mathfrak{A}, D)\} = 0$, as well, and iD is hermitian. ∎

5. WILLIAMS' THEOREM

Numerical ranges possess several useful manipulative properties which will be in constant use in the sequel. One result due to Williams will prove particularly valuable. This result will be discussed after several general results are collected in

(1.18) PROPOSITION. <u>Let $\mathfrak{A}$ be a unital Banach algebra.</u>

i) <u>If</u> $T \in \mathfrak{A}$, <u>then</u> $V(\mathfrak{A}, \lambda T) = \lambda V(\mathfrak{A}, T)$, $\lambda \in \mathbb{C}$,

ii) <u>if</u> $S, T \in \mathfrak{A}$, <u>then</u> $V(\mathfrak{A}, S+T) \subset V(\mathfrak{A}, S) + V(\mathfrak{A}, T)$,

$V(\mathfrak{A}, \lambda+T) = \lambda + V(\mathfrak{A}, T)$, $\lambda \in \mathbb{C}$,

iii) <u>for</u> $\Delta_\epsilon = \{z \in \mathbb{C} : |z| \leq \epsilon\}$, <u>if</u> $\|S - T\| \leq \epsilon$ <u>then</u>

$V(\mathfrak{A}, T) \subset V(\mathfrak{A}, S) + \Delta_\epsilon$.

PROOF. i) is trivial from the definition; ii) follows from the linearity of the states.

If $\|S - T\| \leq \epsilon$ then $|f(S) - f(T)| \leq \epsilon$ for each $f \in \mathcal{S}(\mathfrak{A})$.

Thus each element of $V(\mathcal{Q}, S)$ is within a distance ϵ from an element of $V(\mathcal{Q}, T)$. ■

It is important to observe that each part of this proposition also applies to the spatial and semi-inner-product numerical ranges. Part iii) of the proposition is the continuity which plays an important role in Chapter 4.

Williams' Theorem gives a containment for the spectrum of a product of two Banach algebra elements. Despite the apparent roughness of the approximation to the spectrum, the result is widely used in what follows.

(1.19) THEOREM. [46]. $\mathcal{Q}$ is a unital Banach algebra, $S, T \in \mathcal{Q}$. If $0 \notin V(\mathcal{Q}, T)$, then $\sigma_{\mathcal{Q}}(ST^{-1}) \subset V(\mathcal{Q}, S)/V(\mathcal{Q}, T)$.

PROOF. $0 \notin V(\mathcal{Q}, T)$ implies that T is invertible. $\lambda \in \sigma_{\mathcal{Q}}(ST^{-1})$ implies $0 \in \sigma_{\mathcal{Q}}(\lambda T - S)$. From Theorem 1.6 and Proposition 1.18 i) and ii)

$$0 \in V(\mathcal{Q}, \lambda T - S) \subset \lambda V(\mathcal{Q}, T) - V(\mathcal{Q}, S) \ .$$

But this means $\lambda \in V(\mathcal{Q}, S)/V(\mathcal{Q}, T)$. ■

If S and T commute more can be said. Let $\mathcal{B}$ be a maximal commutative subalgebra of $\mathcal{Q}$ containing S and T. By (1)

$$\begin{aligned}\sigma_{\mathcal{Q}}(ST) = \sigma_{\mathcal{B}}(ST) &= \{\varphi(ST) : \varphi \text{ multiplicative on } \mathcal{B}\} \\ &\subset \sigma_{\mathcal{B}}(S) \cdot \sigma_{\mathcal{B}}(T) = \sigma_{\mathcal{Q}}(S) \cdot \sigma_{\mathcal{Q}}(T) \ .\end{aligned}$$

This fact is listed for reference as

(1.20) THEOREM. For $S, T \subset \mathcal{Q}$. If S and T commute then

$$\sigma_{\mathcal{Q}}(ST) \subset \sigma_{\mathcal{Q}}(S)\sigma_{\mathcal{Q}}(T) .$$

6. CORNERS AND BOUNDARIES OF NUMERICAL RANGES

Hildebrandt [21] discovered that the corner of a Hilbert space numerical range is a point of the spectrum of the operator in question. This and other geometrical properties of numerical range sets will be studied in this section.

(1.21) THEOREM. [21]. For $T \in \mathcal{B}(\mathcal{H})$, a corner of $W(T)$ (a point of $\partial W(T)$ at which nonunique tangents to $W(T)$ exist) is a spectral point of T.

PROOF. It can be supposed (by translation and rotation of T) that 0 is the corner point of W(T) and that $W(T) \subset \{Z \in \mathbb{C} : \mathrm{Re}z \geq 0\}$. By the nonuniqueness of tangents at 0, there exists an angle $\theta \neq 0$ such that also $W(e^{i\theta}T) \subset \{\mathrm{Re}z \geq 0\}$. There exists a sequence of unit vectors $\{x_n\}_{n=1}^{\infty}$ such that $(Tx_n, x_n) \to 0$ as $n \to \infty$. But then both $(\mathrm{Re}Tx_n, x_n)$ and $(\mathrm{Re}e^{i\theta}Tx_n, x_n) \to 0$ as $n \to \infty$. Since both ReT and $\mathrm{Re}e^{i\theta}T$ are positive, this implies $\mathrm{Re}Tx_n \to 0$ and $\mathrm{Re}e^{i\theta}Tx_n \to 0$ as $n \to \infty$. Hence $Tx_n \to 0$ and $n \to \infty$ and $0 \in \sigma(T)$.∎

Schmidt [36] has extended this theorem with restrictions to the general Banach algebra case. He proves that if λ is a corner of $V(\mathcal{Q}, T)$ such that $V(\mathcal{Q}, T)$ is contained in a sector with vertex at λ of angular opening less than $\pi/2$ then $\lambda \in \sigma_{\mathcal{Q}}(T)$. These results are also shown to be sharp [36].

Sinclair and Crabb [38] investigate properties of points which belong to the boundary of the numerical range of an operator and some or all of its powers. One of their results will be of use in Chapter 4.

(1.22) THEOREM. [38]. Let $\mathfrak{A}$ be a complex unital Banach algebra. For $T \in \mathfrak{A}$ suppose $0 \in \partial V(\mathfrak{A}, T^n)$, $n = 1, 2, \ldots$, then $8r(T) \geq \|T\|$.

CHAPTER 2

MULTIPLICATIVE COMMUTATORS

1. INTRODUCTION

This chapter is devoted to a study of multiplicative properties of operators using the multiplicative commutator as the main tool. The use of such a commutator as a tool is demonstrated in a theorem of Frobenius (see [29], [42] for background details).

(2.1) THEOREM. *For* $T, A, B \in \mathcal{B}(\mathfrak{h})$, $\dim \mathfrak{h} < \infty$, *suppose* $A, B \in \mathcal{B}(\mathfrak{h})$ *such that*

$$T = ABA^{-1}B^{-1} \quad , \qquad (1)$$

$$AT - TA = [A, T] = 0 \quad , \qquad (2)$$

and both A, B *are unitary*. *Then* $0 \in W(B)$ *or* $[A, B] = 0$.

The central result of this chapter is an extension of the Frobenius Theorem which contains all known improvements of this theorem. The directions in which the central result is the best possible theorem are also discussed.

2. EXTENSIONS OF THE FROBENIUS THEOREM

A detail which plays a role in the theorems of this and the next section is the closedness of the numerical range. In general, W(T) is not closed. However, if $\dim \mathfrak{h} < \infty$ then, as a continuous image of a compact set, W(T) is closed. Normally the extension of a theorem with a numerical range condition from the finite dimensional to the infinite dimensional case requires the closure of the numerical range as the condition. That Theorem 2.4, the extension to the operator case of Theorem 2.3, does not require this modification is a point of interest of this chapter.

That Theorem 2.1 is not the best possible commutator theorem with hypotheses (1) and (2) is demonstrated in Putnam [30]. There the following is shown to be true:

(2.2) THEOREM. For $T, A, B \in \mathcal{B}(\mathfrak{h})$, suppose $A, B \in \mathcal{B}(\mathfrak{h})$, such that (1) and (2) hold, and that A is unitary. Then $0 \in W(B)^-$ or $\sigma(T) = \{1\}$.

Of course this implies Theorem 2.1 and shows that Theorem 2.1 can be extended to the infinite dimensional case. That this result contains Theorem 2.1 derives from the fact that $T = ABA^{-1}B^{-1}$ and both A and B unitary imply T unitary. The only unitary T with $\sigma(T) = \{1\}$ is $T = 1$. A proof of Theorem 2.2 will be given in Chapter 4 (page 57) by techniques which are fundamentally different from those of this chapter.

In a somewhat different situation Marcus and Thompson [29] have also obtained an extension of Theorem 2.1. In this case the condition that A, B (hence T) be unitary is changed to the weaker condition that only A, T be normal.

(2.3) THEOREM. [29]. Let $\dim \mathfrak{h} < \infty$. For $T, A, B \in \mathcal{B}(\mathfrak{h})$ suppose (1) and (2) hold, and that A, T are normal. Then $0 \in W(B)$ or $[A, B] = 0$.

PROOF. $T = ABA^{-1}B^{-1}$ implies $TA^{-1} = A^{-1}T = BAB^{-1}$. Put $N_1 = TA^{-1}$ and $N_2 = A^{-1}$. What follows shows $N_1 = N_2$. Hence $T = 1$ and $[A, B] = 0$.

Assume $0 \notin W(B)$. Since N_1 and N_2 are commuting normal transformations, they simultaneously have diagonal representations. Assume this representation is chosen so that $N_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ and $N_2 = \mathrm{diag}(\mu_1, \ldots, \mu_n)$. B has a matrix representation relative to the same orthonormal basis, $B = (b_{ij})_{i,j=1}^n$. $n = \dim \mathfrak{H}$.

That $N_1 B = BN_2$ implies $\lambda_i b_{ii} = \mu_i b_{ii}$, $i = 1, 2, \ldots, n$. However, by Proposition 1.3 iii, if P_i is the projection on the i^{th} basis vector, $b_{ii} = W(P_i B P_i) \subset W(B)$. $b_{ii} \neq 0$, $i = 1, 2, \ldots, n$, implies $\lambda_i = \mu_i$, $i = 1, 2, \ldots, n$, and $N_1 = N_2$. ■

Again Theorem 2.3 contains the Frobenius result, and the proof of Theorem 2.3 given above is still the most direct proof of Theorem 2.1. It is shown in [12] and in [17] that the literal extension of Theorem 2.3 to the infinite dimensional case is valid.

(2.4) THEOREM. For $T, A, B \in \mathfrak{B}(\mathfrak{H})$ suppose (1) and (2) hold and that T and A are normal. Then $0 \in W(B)$ or $[A, B] = 0$.

The proofs in [12] and [17] are virtually the same. The presentation of this proof is made in Section 3 where other related results are discussed.

A direct extension of Theorem 2.2 is made in [13]. The proof is suggested by DePrima and, independently, by the referee for [13].

(2.5) THEOREM. For $T, A, B \in \mathfrak{B}(\mathfrak{H})$, suppose (1) and (2) hold and that A is unitary. Then $0 \in W(B)^-$ or $[A, B] = 0$.

PROOF. $TB = ABA^{-1}$ and note that if $T^n B = A^n B A^{-n}$, then $T^{n+1}B = TA^n BA^{-n} = A^n TBA^{-n} = A^{n+1}BA^{-(n+1)}$ and $T^{n-1}B = T^{-1}A^n BA^{-n} =$

$A^{n-1}T^{-1}ABA^{-n} = A^{n-1}BA^{-(n-1)}$. Thus the relation

$$T^nB = A^nBA^{-n}, \quad n \in \mathbb{Z},$$

is valid. Because A is unitary, $W(T^nB) = W(A^nBA^{-n}) = W(B)$ and $w(T^nB) = w(B)$, $n \in \mathbb{Z}$. Hence by Theorem 1.2 iii, $\|T^nB\| \leq 2w(B)$, and $\|T^n\| \leq 2w(B)\|B^{-1}\|$, $n \in \mathbb{Z}$. This suggests application of a theorem of Sz.-Nagy [41]: $\|T^n\| \leq M$, $n \in \mathbb{Z}$ for some M, implies T is similar to a unitary operator. Hence $T = SUS^{-1}$, U unitary. But $\sigma(U) = \sigma(T) = \{1\}$ by Theorem 2.2 so $T = U = 1$.∎

3. THE MAIN THEOREM

The central result of this chapter is a theorem which contains every previous result. In some sense this theorem is the strongest possible result in the class of commutator theorems which have relations (1) and (2) in the hypothesis. A discussion of this aspect of the theorem is contained in Section 4.

Throughout the section the Fuglede-Putnam Theorem is used.

(2.6) THEOREM. [31]. <u>If</u> $N_1, N_2 \in \mathcal{B}(\mathfrak{H})$ <u>are normal and</u> $N_1B = BN_2$, $B \in \mathcal{B}(\mathfrak{H})$, <u>then</u> $N_1^*B = BN_2^*$.

A new proof of this theorem is given in Chapter 3, Section 5. Theorem 2.6 finds immediate application in the proof of the main result,

(2.7) THEOREM. <u>For</u> $T, A, B \in \mathcal{B}(\mathfrak{H})$, <u>suppose that</u> (1) <u>and</u> (2) <u>hold and that</u> A <u>is normal</u>. <u>Then</u> $0 \in W(B)^-$ <u>or</u> $[A, B] = 0$.

PROOF. As in the proof of Theorem 2.4 we examine the structure $TA^{-1} = A^{-1}T = BA^{-1}B^{-1}$. Put $N = A^{-1}$, $R = TA^{-1}$. $[R,N] = 0$, N is normal, and $RB = BN$.

For any Borel subset M of $\mathbb{C}$, let E_M denote the spectral projection of N for the set M. Since $[R,N] = 0$, $[R,N^*] = 0$ by Theorem 2.6, R commutes with any polynomial in N and N^*. Extending to weak limits R commutes with any Borel function of N and N^*. Hence $[R,E_M] = 0$, M a Borel subset of $\mathbb{C}$.

Noting this observe that

$$E_M RBE_M = E_M RE_M BE_M = E_M BE_M NE_M = E_M BNE_M \quad . \tag{3}$$

Putting $E_M RE_M = R_M$, $E_M BE_M = B_M$, and $E_M NE_M = N_M$ (3) implies

$$R_M B_M = B_M N_M \tag{4}$$

for any Borel set M.

Suppose $0 \notin W(B)^-$. Then if $E_M \neq 0$, Proposition 1.3 iii implies that $0 \notin W(B_M)^-$, where B_M is considered as an operator on $E_M \mathfrak{H}$. Thus B_M is invertible on $E_M \mathfrak{H}$.

Let $\epsilon > 0$ be given. It will be shown that $\|R - N\| < K\epsilon$ for some fixed K. Therefore $R = N$ and $T = 1$, as required.

Choose a family of Borel subsets of $\mathbb{C}$, $\{M_i\}_{i=1}^{*}$, such that

i) $\max(\mathrm{diam}(M_i)) \leq \epsilon, \qquad i = 1,2,\ldots,r,$

ii) $E_{M_i} \neq 0, \qquad i = 1,2,\ldots,r,$

iii) $\sum_{i=1}^{r} E_{M_i} = 1.$

The spectral theorem insures that such a selection is always possible.

Put $\delta = \text{dist}(0, W(B))$, then

$$\delta \|x\|^2 \leq |(B_{M_i}x, x)| \leq \|B_{M_i}x\| \, \|x\| , \quad x \in E_{M_i}\mathfrak{H} .$$

Thus $\|B_{M_i}^{-1}\| \leq \delta^{-1}$, where B_{M_i} is considered as an operator on $E_{M_i}\mathfrak{H}$.

Using (4) and taking $\lambda_i \in M_i$,

$$\begin{aligned} \|R_{M_i} - N_{M_i}\| &\leq \|R_{M_i}B_{M_i} - N_{M_i}B_{M_i}\|\delta^{-1} \\ &\leq \|B_{M_i}N_{M_i} - N_{M_i}B_{M_i}\|\delta^{-1} \\ &\leq (\|B_{M_i}(N_{M_i} - \lambda_i)\| + \|(N_{M_i} - \lambda_i)B_{M_i}\|)\delta^{-1} \\ &\leq 2\delta^{-1}\|B\| \cdot \epsilon . \end{aligned}$$

$\|N_{M_i} - \lambda_i\| \leq \epsilon$ because N_{M_i} is normal, $\sigma_{\mathcal{B}(E_{M_i}\mathfrak{H})}(N_{M_i}) \subset M_i$, and $r(N_{M_i} - \lambda_i) = \|N_{M_i} - \lambda_i\|$.

Finally, if $x \in \mathfrak{H}$, $x = \sum_{i=1}^{r} x_i$, $x_i \in E_{M_i}\mathfrak{H}$. Then

$$\begin{aligned} \|(R - N)x\|^2 &= \|(R - N)(\sum_{i=1}^{r} E_{M_i}x)\|^2 \\ &= \|\sum_{i=1}^{r} (R_{M_i} - N_{M_i})x_i\|^2 \\ &= \sum_{i=1}^{r} \|(R_{M_i} - N_{M_i})x_i\|^2 \\ &\leq (2\delta^{-1}\|B\|\epsilon)^2 \sum_{i=1}^{r} \|x_i\|^2 = (2\delta^{-1}\|B\|\epsilon)^2 \|x\|^2 . \blacksquare \end{aligned}$$

(2.8) COROLLARY. Let $R, N, B \in \mathcal{B}(\mathcal{H})$ and suppose

i) N is normal,

ii) $[N,R] = 0$,

iii) $RB = BN$.

Then $0 \in W(B)^-$ or $R = N$.

PROOF. Essentially it is the proof of this result which has been given above. To apply the theorem translate R and N by λ so that $R_\lambda = R - \lambda$ and $N_\lambda = N - \lambda$ are both invertible. Supposing that $0 \notin W(B)^-$, Theorem 2.7 implies $N_\lambda^{-1}R_\lambda = N_\lambda^{-1}BN_\lambda B^{-1} = 1$. $N_\lambda^{-1}R_\lambda = 1$ means $R = N$. ■

As mentioned in Section 2 the proof of Theorem 2.4 is similar to that of Theorem 2.7 and is suited for presentation here.

PROOF OF THEOREM 2.4. [12], [17]. Again the structure $A^{-1}T = TA^{-1} = BA^{-1}B^{-1}$ is examined. As before put $A^{-1}T = R$, $A^{-1} = N$. In this case, of course, both R and N are normal.

The family of spectral projections for R will be denoted F_M, M a Borel set. The family for N will be written E_M. By an argument similar to that used in the proof of Theorem 2.7 $E_{M_1}F_{M_2} = F_{M_2}E_{M_1}$, M_1, M_2 Borel.

Suppose $R \neq N$. Then for some Borel set M, $E_M \neq F_M$ and there exists a unit vector x such that either $E_Mx = x$ and $F_Mx = 0$, or $E_Mx = 0$ and $F_Mx = x$.

In either case because $RB = BN$ (and hence $F_MB = BE_M$),

$$(Bx, x) = (BE_Mx, x) = (F_MBx, x) = (Bx, F_Mx) = 0 \quad .$$

Thus $0 \in W(B)$. ■

In a sense this is a surprising result. As suggested at the beginning of the section one would expect the hypothesis $0 \in W(B)^-$ rather than $0 \in W(B)$ in the infinite dimensional setting.

An equivalent formulation of Theorem 2.4 is stated here as a corollary. This version of Theorem 2.4 is in the form taken by the main theorem of [17].

(2.9) COROLLARY. <u>For</u> $R, N, B \in \mathcal{B}(\mathcal{H})$ <u>suppose</u>

i) R <u>and</u> N <u>are normal</u>,

ii) $[N,R] = 0$,

iii) $RB = BN$.

<u>Then</u> $0 \in W(B)$ <u>or</u> $R = N$.

4. THE SHARPNESS OF THE MAIN RESULT

In view of the literal extension to the infinite dimensional case of Theorem 2.3 it is natural to ask if the condition in the hypothesis of Theorem 2.7, $0 \in W(B)^-$, can be weakened to $0 \in W(B)$. Unfortunately the answer to this question is not known. The impossibility of other weakenings is discussed in this section.

To see that the condition $0 \in W(B)^-$ can not be weakened to $0 \in \mathrm{co}\,\sigma(B)$ examine the example which follows.

For the example put

$$N = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -1 & 0 \\ 3 & 0 & 2 \\ 4 & 0 & 3 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These matrices represent transformations relative to an orthonormal basis for $\mathfrak{H} = \mathbb{C}^3$. Note that these transformations satisfy hypotheses i), ii), and iii) of either Corollary 2.8 or Corollary 2.9. Furthermore $\sigma(B) = \{1\}$ so that $0 \notin \mathrm{co}\,\sigma(B)$. However, that $N \neq R$ shows that the weakening being studied is invalid.

A relaxation of the condition i) of Corollary 2.8 from N, R normal to N, R diagonable also does not lead to a positive result. By a theorem of Williams [45], for any open set V such that $V \supset \mathrm{co}\,\sigma(B)$ there exists a similarity S_V for which $W(S_V B S_V^{-1}) \subset V$. Hence there exists an invertible S for the transformation B of the example above such that $0 \notin W(B')$ where $B' = SBS^{-1}$.

Putting $N' = SNS^{-1}$, $R' = SRS^{-1}$, the conditions of Corollary 2.9 are satisfied for R′, N′, B′ in place of R, N, B, except that the condition R, N normal is replaced by R′, N′ diagonable. Again because $R' \neq N'$ the weakening is not possible.

To see that three dimensions are required to find counterexamples note that the following holds:

(2.10) PROPOSITION. Let $\dim \mathfrak{H} = 2$. For $R, N, B \in \mathcal{B}(\mathfrak{H})$ suppose

i) N, R diagonable,

ii) $[R, N] = 0$,

iii) $RB = BN$.

Then $0 \in \mathrm{co}\,\sigma(B)$ or $N = R$.

PROOF. Suppose $N \neq R$ and that $0 \notin \sigma(B)$. It is shown here that $0 \in \mathrm{co}\,\sigma(B)$.

Since R and N are simultaneously diagonable by a similarity S, put $N' = SNS^{-1}$, $R' = SRS^{-1}$, and $B' = SBS^{-1}$. Because N′ and R′ are similar, $\sigma(N') = \sigma(R')$. Hence $N' = \mathrm{diag}(\alpha, \beta)$, $R' = \mathrm{diag}(\beta, \alpha)$, for some $\alpha, \beta \in \mathbb{C}$. However, since $R'B' = B'N'$, a computation shows that B′ has the form,

$$B' = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix} .$$

The trace of B′ being zero implies $0 \in \mathrm{co}\,\sigma(B') = \mathrm{co}\,\sigma(B)$. ■

5. REMARKS

It is noted in [12], that a result apparently stronger than Theorem 2.7 appears in the literature [16]. This result (Theorem III of [16]), having the same hypotheses as Theorem 2.7, has the conclusion $0 \in \mathrm{co}\,\sigma(B)$ or $T = 1$. That this is in error is seen from examining the example of the previous section. Actually what is shown in [16] is that under the hypotheses of Theorem 2.7 and under the additional assumption that A commute with $B^n A B^{-n}$, $n = 1, 2, \ldots$, then $0 \in \mathrm{co}\,\sigma(B)$ or $[A, B] = 0$.

Theorem 2.7 may be framed as a theorem for C*-algebras. If $\mathfrak{A}$ is a C*-algebra with unit, then the algebra numerical range will serve to replace the spatial range used in Theorem 2.7.

(2.11) THEOREM. Let $\mathfrak{A}$ be a C*-algebra with unit. For $A, B, T \in \mathfrak{A}$, suppose that (1) and (2) hold and that A is normal (i.e., $[A, A^*] = 0$). Then $0 \in V(\mathfrak{A}, B)$ or $[A, B] = 0$.

PROOF. $\mathfrak{A}$ is isometrically isomorphic to a self-adjoint subalgebra of $\mathcal{B}(\mathfrak{H})$, for some Hilbert space $\mathfrak{H}$. Let $\mathcal{B}$ be the weak closure of $\mathfrak{A}$ in $\mathcal{B}(\mathfrak{H})$.

Because $V(\mathfrak{A}, B) = V(\mathcal{B}, B)$ by the remarks of Chapter 1, Section 3 (page 9) and because the spectral theorem is valid in the W^*-algebra $\mathcal{B}$, all of the techniques used to prove Theorem 2.7 are valid in this setting. ∎

Observe that the C^*-algebra version of Theorem 2.4 is contained in Theorem 2.11. This happens because of the closedness of $V(\mathfrak{A}, B)$.

It should be mentioned that Theorem 2.4 (hence Theorem 2.7) has many interesting applications. A small collection of these are listed in [17]. One application which gives the general pattern of many of the others is due to Taussky [43] (see [43] and [47] for comments on the origin of the theorem).

(2.12) THEOREM. <u>For</u> $A \in \mathcal{B}(\mathfrak{H})$ <u>normal, suppose</u>

$$S^{-1}AS = A^*$$

<u>and that</u> $0 \notin W(S)$. <u>Then</u> $A = A^*$.

PROOF. $AS = SA^*$. Apply Corollary 2.9 with $R = A$, $N = A^*$, and $B = S$. ∎

It is of interest to observe that Theorem 2.7 may be viewed as a multiplicative analogue of an additive commutator theorem of Putnam [31], (2.13) THEOREM. Suppose $[A,[A,B]] = 0$ and that A is normal, then $[A,B] = 0$.

If the notation $\{A, B\} = ABA^{-1}B^{-1}$ is adopted, then Theorem 2.7 has the formulation

(2.14) THEOREM. Suppose $\{A, \{A, B\}\} = 1$ and that A is normal, then $0 \in W(B)^-$ or $\{A, B\} = 1$.

As is the case with Theorem 2.4 a proof of Theorem 2.7 in the case $\mathfrak{H}$ is finite dimensional is instructive.

PROOF OF THEOREM 2.7: DIM $\mathfrak{H}$ FINITE. Put $R = TA^{-1}$, $N = A^{-1}$. It suffices to show that $R = N$, supposing $0 \notin W(B)$.

By normality $N = \sum_{i=1}^{r} \lambda_i P_i$, P_i is the orthogonal projection on $\ker(N - \lambda_i)$. That $[R, N] = 0$ implies $RP_i = P_iR$, $i = 1, 2, \ldots, r$. Then since $RB = BN$

$$RP_iBP_i = P_iRBP_i = \lambda_i P_i BP_i, \qquad i = 1, 2, \ldots, r .$$

By Proposition 1.3 iii and the supposition $0 \notin W(B)$, P_iBP_i is invertible on $P_i\mathfrak{H}$.

Thus $RP_i = \lambda_i P_i$ or $P_i\mathfrak{H}$, and $R = \sum_{i=1}^{r} \lambda_i P_i = N$. ∎

CHAPTER 3
NUMERICAL RANGES AND DERIVATIONS

1. INTRODUCTION

Derivations, as do commutators, provide a structure with which commutativity properties of operators can be studied. This chapter is devoted to derivations, their numerical ranges, and an application of the derivation to a commutativity question. In particular, the numerical range of a derivation acting on an algebra of the form $\mathcal{B}(X)$, X a Banach space, is explicitly determined, and a numerical range proof of the Fuglede-Putnam Theorem, Theorem 2.6, is given.

Recall that, for the element T in a Banach algebra $\mathfrak{A}$, the derivation relative to T on $\mathfrak{A}$ is the linear mapping $\Delta_T : \mathfrak{A} \to \mathfrak{A}$ defined by

$$\Delta_T(A) = TA - AT , \qquad A \in \mathfrak{A} .$$

Related to the derivation is the intertwining operator on $\mathfrak{A}$ relative to elements S, T of $\mathfrak{A}$. This intertwining operator is the map $\Delta_{S,T} : \mathfrak{A} \to \mathfrak{A}$ defined by the relation

$$\Delta_{S,T}(A) = SA - AT , \qquad A \in \mathfrak{A} .$$

Of course, $\Delta_{T,T} = \Delta_T$.

R_T denotes the operation of multiplication on the right by T. L_T is multiplication on the left by T.

$R_T(A) = AT$ and $L_T(A) = TA$, $A \in \mathfrak{A}$. $\Delta_T = L_T - R_T$ and $\Delta_{S,T} = L_S - R_T$. Note that

$$V(\mathcal{B}(\mathfrak{A}), L_T) = \text{co}W(L_T)$$

$$= \text{co}\{f_A(TA) : f_A(A) = 1 = \|f_A\| = \|A\|, \quad A \in \mathfrak{A}\} , \qquad (1)$$

But $\{f_A(TA) : f_A(A) = 1 = \|f_A\| = \|A\|\} \subset V(\mathfrak{A}, T)$ since each functional $g(\cdot) = f_A(\cdot\, A)$ is a state of $\mathfrak{A}$. In the opposite direction if f is a state on $\mathfrak{A}$ then $f(\cdot) = f(\cdot\, 1)$ so that $V(\mathfrak{A}, T) \subset \{f_A(TA) : f_A(A) = 1 = \|f_A\| = \|A\|\}$. (1) and these observations imply

$$V(\mathcal{B}(\mathfrak{A}), L_T) = V(\mathfrak{A}, T) \quad . \qquad (2)$$

Similarly

$$V(\mathcal{B}(\mathfrak{A}), R_T) = V(\mathfrak{A}, T) \quad . \qquad (3)$$

It is evident that the following is valid:

(3.1) PROPOSITION. <u>Suppose</u> H, K <u>are hermitian elements of a complex unital Banach algebra</u> $\mathbb{C}$. <u>Then</u> $\Delta_{H,K}$ <u>is also hermitian in</u> $\mathcal{B}(\mathbb{C})$.

PROOF.

$$V(\mathcal{B}(\mathbb{C}), \Delta_{S,T}) = V(\mathcal{B}(\mathbb{C}), L_S - R_T)$$

$$\subset V(\mathcal{B}(\mathbb{C}), L_S) - V(\mathcal{B}(\mathbb{C}, R_T)$$

$$= V(\mathbb{C}, S) - V(\mathbb{C}, T) \subset \mathbb{R} \quad ,$$

by (2) and (3) and the hermiticity of S and T. ∎

Throughout the remainder of this chapter $\mathfrak{A}$ denotes the full algebra of bounded endomorphisms on a Banach space X, $\mathfrak{A} = \mathcal{B}(X)$. The techniques of this chapter yield best results when limited to this setting.

In the case $\mathfrak{A} = \mathcal{B}(\mathfrak{H})$ Stampfli [39] has computed the norm of a derivation in terms of the norm of the operator in $\mathfrak{H}$ defining the derivation.

(3.2) THEOREM. [39]. Let $T \in \mathcal{B}(\mathfrak{H})$. Then

$$\|\Delta_T\| = 2 \inf_{\lambda \in \mathbb{C}} \|\lambda - T\| . \tag{4}$$

It is clear that $\|\Delta_T\| \leq 2\|\lambda - T\|$, $\lambda \in \mathbb{C}$, since $\Delta_T = \Delta_{\lambda - T}$, $\lambda \in \mathbb{C}$. The location of a λ so that the inequality can be reversed is the content of the bulk of [39].

One of the results of this chapter is the formula for the numerical range of a derivation

$$V(\mathcal{B}(\mathfrak{A}), \Delta_T) = V(\mathfrak{A}, T) - V(\mathfrak{A}, T) , \tag{5}$$

where $\mathfrak{A}$ is of the form $\mathcal{B}(X)$. Because (4) does not hold for all algebras of the form $\mathfrak{A} = \mathcal{B}(X)$, it is surprising that (5) is always true.

Theorem 3.2 has an extension to the case of the intertwining operator. The same is true of the result described by (5). This extension to the intertwining case is the numerical range analog of Kleinecke's Theorem (see [28] for a discussion of this and related theorems) which completely characterizes the spectrum of intertwining operators:

(3.3) THEOREM. $\mathfrak{A}$ is of the form $\mathcal{B}(X)$, X a Banach space. For $S, T \in \mathfrak{A}$, $\sigma_{\mathcal{B}(\mathfrak{A})}(\Delta_{S,T}) = \sigma_{\mathfrak{A}}(S) - \sigma_{\mathfrak{A}}(T)$.

In addition to the result (5) and its extension to the intertwining operator case, normal elements of general Banach algebras are defined and properties of intertwining operators which are normal elements are discussed.

2. THE NUMERICAL RANGE OF AN INTERTWINING OPERATOR

As suggested above the exact analog of Kleinecke's Theorem on the spectrum of an intertwining operator is valid for the algebra numerical range.

(3.4) THEOREM. Let $\mathfrak{A}$ have the form $\mathfrak{A} = \mathfrak{B}(X)$. For $S, T \in \mathfrak{A}$

$$V(\mathfrak{B}(\mathfrak{A}), \Delta_{S,T}) = V(\mathfrak{A}, S) - V(\mathfrak{A}, T) \quad .$$

In more detail

$$W(\Delta_{S,T}) \supset W(S) - W(T) \quad . \tag{6}$$

PROOF. (6) is all that requires proof, for by Theorem 1.10 (6) implies

$$\begin{aligned} V(\mathfrak{B}(\mathfrak{A}), \Delta_{S,T}) &= co(W(\Delta_{S,T})^{-}) \\ &\supset co(W(S)^{-}) - co(W(T)^{-}) \\ &= V(\mathfrak{A}, S) - V(\mathfrak{A}, T) \quad . \end{aligned}$$

Here, the relation for compact sets $K, L \subset \mathbb{C}$

$$co(K + L) = coK + coL$$

is used. Containment in the other direction follows because $\Delta_{S,T} = L_S - R_T$. By (2) and (3)

$$\begin{aligned} V(\mathfrak{B}(\mathfrak{A}), \Delta_{S,T}) &= V(\mathfrak{B}(\mathfrak{A}), L_S - R_T) \\ &\subset V(\mathfrak{B}(\mathfrak{A}), L_S) - V(\mathfrak{B}(\mathfrak{A}), R_T) \\ &= V(\mathfrak{A}, S) - V(\mathfrak{A}, T) \quad . \end{aligned}$$

To prove (6) suppose $\lambda \in W(S)$, $\mu \in W(T)$. Then $\lambda = f_y(Sy)$, $\mu = f_x(Tx)$, $1 = \|x\| = \|y\| = \|f_x\| = \|f_y\| = f_x(x) = f_y(y)$, $x, y \in X$, $f_x, f_y \in X^*$. Put $Az = f_x(z)y$, $z \in X$. Then

$$\|Az\| = \|f_x(z)y\| = |f_x(z)| \leq \|z\|\,\|f_x\| = \|z\|$$

and

$$Ax = y$$

imply $\|A\| = 1$, $A \in \mathfrak{A}$.

Define $\varphi \in \mathfrak{A}^*$ by $\varphi(B) = f_y(Bx)$, $B \in \mathfrak{A}$. Here $\|\varphi\| \leq 1$ since $|\varphi(B)| = |f_y(Bx)| \leq \|Bx\| \leq \|B\|$, $B \in \mathfrak{A}$. Furthermore $\varphi(A) = f_y(Ax)$ $= f_y(y) = 1$ implies $\varphi(\Delta_{S,T}(A)) \in W(\Delta_{S,T})$. But

$$\begin{aligned} \varphi(\Delta_{S,T}(A)) &= f_y((SA - AT)x) \\ &= f_y(Sy) - f_y(f_x(Tx)y) \\ &= \lambda - \mu \ . \end{aligned}$$

Thus $\lambda - \mu \in W(\Delta_{S,T})$ and (6) is proved. ■

An alternate proof which has more numerical range orientation, but which applies only to the Hilbert space case follows.

(3.5) THEOREM. <u>For</u> $S, T \in \mathfrak{B}(\mathfrak{H})$,

$$V(\mathfrak{B}(\mathfrak{B}(\mathfrak{H})), \Delta_{S,T}) = V(\mathfrak{B}(\mathfrak{H}), S) - V(\mathfrak{B}(\mathfrak{H}), T) \ .$$

PROOF. Put $K = V(\mathfrak{B}(\mathfrak{B}(\mathfrak{H})), \Delta_{S,T})$ and $L = V(\mathfrak{B}(\mathfrak{H}), S) - V(\mathfrak{B}(\mathfrak{H}), T)$. Because both K and L are compact and convex it suffices to show that the projections of K and L on any line are the same.

Note that

$$e^{i\theta}\Delta_{S,T} = \Delta_{e^{i\theta}S,\, e^{i\theta}T} \quad .$$

Putting $T' = e^{i\theta}T$, $S' = e^{i\theta}S$, $\theta \in \mathbb{R}$,

$$\mathrm{Re}\, e^{i\theta}K = \mathrm{Re}\, V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), \Delta_{S',T'})$$

and

$$\mathrm{Re}\, e^{i\theta}L = \mathrm{Re}(V(\mathcal{B}(\mathfrak{h}), S') - V(\mathcal{B}(\mathfrak{h}), T')) \quad .$$

Hence, if $K' = V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), \Delta_{S',T'})$ and $L' = V(\mathcal{B}(\mathfrak{h}), S') - V(\mathcal{B}(\mathfrak{h}), T')$, then the theorem is established when $\mathrm{Re}K' = \mathrm{Re}L'$ is shown.

For $A \in \mathcal{B}(\mathfrak{h})$,

$$\begin{aligned} \Delta_{S',T'}(A) &= S'A - AT' \\ &= [(\mathrm{Re}S')A - A(\mathrm{Re}T')] + i[(\mathrm{Im}S')A - A(\mathrm{Im}T')] \\ &= \Delta_{\mathrm{Re}S',\,\mathrm{Re}T'}(A) + i\Delta_{\mathrm{Im}S',\,\mathrm{Im}T'}(A) \quad . \end{aligned}$$

Proposition 3.1, therefore, implies that $\Delta_{S',T'}$ has the form

$$\Delta_{S',T'} = H_1 + iH_2 \quad ,$$

H_1 and H_2 are hermitian in $\mathcal{B}(\mathcal{B}(\mathfrak{h}))$. Thus $\mathrm{Re}K' = V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), H_1)$. Applying Theorem 3.3 and Theorem 1.15,

$$\begin{aligned}
\operatorname{Re}K' &= V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), H_1) \\
&= \operatorname{co}\sigma_{\mathcal{B}(\mathcal{B}(\mathfrak{h}))}(H_1) \\
&= \operatorname{co}\{\sigma(\operatorname{Re}S) - \sigma(\operatorname{Re}T)\} \\
&= \operatorname{co}\sigma(\operatorname{Re}S) - \operatorname{co}\sigma(\operatorname{Re}T) \\
&= V(\mathcal{B}(\mathfrak{h}), \operatorname{Re}S) - V(\mathcal{B}(\mathfrak{h}), \operatorname{Re}T) \\
&= \operatorname{Re}\{V(\mathcal{B}(\mathfrak{h}), S) - V(\mathcal{B}(\mathfrak{h}), T\} \\
&= \operatorname{Re}L'. \blacksquare
\end{aligned}$$

As suggested in the introduction, Theorem 3.4 is somewhat surprising. This is because the numerical range is a norm property of an algebra element rather than an algebraic property. The formula in Theorem 3.4 determines the numerical range of an intertwining operator in terms of the numerical ranges of the operators defining the intertwining operator. Stampfli's formula for the norm of a derivation on $\mathcal{B}(\mathfrak{h})$ gives a determination of the derivation norm in terms of the inducing operator's norm,

$$\|\Delta_T\| = 2 \inf_{\lambda \in \mathbb{C}} \|\lambda - T\| .$$

What is surprising is that Stampfli's formula is not valid in arbitrary algebras of the form $\mathcal{A} = \mathcal{B}(X)$ (as shown by B. E. Johnson [23]), while the related assertion of Theorem 3.4, of course, is valid for any algebra $\mathcal{A} = \mathcal{B}(X)$.

The norm of a derivation can always be estimated from below by the numerical radius. Because of this Stampfli's formula remains valid for the special class of derivations in which

$$v(\Delta_T) = 2 \inf_{\lambda \in \mathbb{C}} \|\lambda - T\| \quad . \tag{7}$$

To see this note that the remarks following the statement of Theorem 3.1 assert that $\|\Delta_T\| \leq 2\|\lambda - T\|$, $\lambda \in \mathbb{C}$. Thus the inequality

$$v(\Delta_T) \leq \|\Delta_T\| \leq 2\|\lambda - T\|$$

and the assumption on the special class imply

$$\|\Delta_T\| = 2 \inf_{\lambda \in \mathbb{C}} \|\lambda - T\| \quad .$$

Because derivations induced by hermitians have property (7) Stampfli's formula holds in general for these derivations.

(3.6) COROLLARY. <u>If</u> $\mathfrak{A} = \mathfrak{B}(X)$ <u>and</u> H <u>is a hermitian element of</u> $\mathfrak{A}$, <u>then</u>

$$\|\Delta_H\| = 2 \inf_{\lambda \in \mathbb{C}} \|\lambda - H\| \quad .$$

More is true, however, as the determination of the norm of an intertwining operator induced by two hermitian elements is possible.

(3.7) COROLLARY. <u>If</u> $\mathfrak{A} = \mathfrak{B}(X)$ <u>and</u> H, K <u>are hermitian in</u> $\mathfrak{A}$, <u>then</u>

$$\|\Delta_{H,K}\| = \inf_{\lambda \in \mathbb{C}} \{\|\lambda - H\| + \|\lambda - K\|\} \quad . \tag{8}$$

PROOF. Let $V(\mathfrak{A}, H) = [r_1, r_2] \subset \mathbb{R}$, $V(\mathfrak{A}, K) = [s_1, s_2]$. $v(\Delta_{H,K})$ is either $s_2 - r_1$ or $r_2 - s_1$. To be specific suppose that $v(\Delta_{H,K}) = s_2 - r_1$. The argument in the case $v(\Delta_{H,K}) = r_2 - s_1$ follows the same pattern.

Note that because $s_2 - r_1 \geq r_2 - s_1$, $\frac{s_2 + s_1}{2} \geq \frac{r_2 + r_1}{2}$. Pick

$t \in \mathbb{R}$, $\frac{s_2+s_1}{2} \geq t \geq \frac{r_2+r_1}{2}$. Then by Theorem 1.15

$$\begin{aligned} v(\Delta_{H,K}) &= s_2 - r_1 = s_2 - t + t - r_1 \\ &= v(K-t) + v(t-H) \\ &= \|t-K\| + \|t-H\| \quad . \end{aligned}$$

Since, in general,

$$v(\Delta_{H,K}) \leq \|\Delta_{H,K}\| \leq \|\lambda - H\| + \|\lambda - K\|, \; \lambda \in \mathbb{C} \; ,$$

(8) holds. ■

3. CONSEQUENCES OF THE NUMERICAL RANGE CHARACTERIZATION

Several simple consequences of Theorem 3.4 are described in this section. The results derive from geometrical properties of the numerical range of the derivation characterized above.

The first result shows that the only elements of $\mathcal{B}(X)$ which commute with all of $\mathcal{B}(X)$ are the scalars.

(3.8) THEOREM. For $T \in \mathfrak{A} = \mathcal{B}(X)$ suppose that $[A, T] = 0$ for all $A \in \mathfrak{A}$. Then $T = \lambda$ for some $\lambda \in \mathbb{C}$.

PROOF. $\Delta_T = 0$. Therefore $V(\mathcal{B}(\mathfrak{A}), \Delta_T) = V(\mathfrak{A}, T) - V(\mathfrak{A}, T) = \{0\}$. Thus $V(\mathfrak{A}, T) = \{\lambda\}$ for some $\lambda \in \mathbb{C}$, or $T = \lambda$. ■

Recall that a convexoid operator is one for which the algebra numerical range coincides with the convex hull of the spectrum. That an intertwining operator inherits the property of being convexoid from the operators used to define it is the content of the next theorem.

(3.9) THEOREM. Let $\mathcal{O} = \mathcal{B}(X)$. If $S, T \in \mathcal{O}$ are convexoid, then $\Delta_{S,T}$ is convexoid in $\mathcal{B}(\mathcal{O})$.

PROOF. Take $\lambda - \mu \in V(\mathcal{B}(\mathcal{O}), \Delta_{S,T})$ extreme, $\lambda \in V(\mathcal{O}, S)$, $\mu \in V(\mathcal{O}, T)$. Should λ not be extreme in $V(\mathcal{O}, S)$, then $\lambda = \frac{\lambda_1 + \lambda_2}{2}$, $\lambda_1 \neq \lambda_2$, $\lambda_i \in V(\mathcal{O}, S)$, $i = 1, 2$. But then

$$\lambda - \mu = \frac{(\lambda_1 - \mu) + (\lambda_2 - \mu)}{2} ,$$

implies $\lambda - \mu$ is not extreme. This contradiction shows that λ is extreme in $V(\mathcal{O}, S)$ or that $\lambda \in \sigma_{\mathcal{O}}(S)$. Likewise $\mu \in \sigma_{\mathcal{O}}(T)$. Thus $\lambda - \mu \in \sigma_{\mathcal{B}(\mathcal{O})}(\Delta_{S,T})$ by Theorem 3.3. ∎

A related result is the observation that corners of the numerical range of an intertwining operator are spectral points.

(3.10) THEOREM. For $S, T \in \mathcal{B}(\mathcal{H})$, suppose λ is a corner in the boundary of $V(\mathcal{B}(\mathcal{B}(\mathcal{H})), \Delta_{S,T})$. Then $\lambda \in \sigma_{\mathcal{B}(\mathcal{B}(\mathcal{H}))}(\Delta_{S,T})$.

PROOF. It suffices to show that if $\lambda = \mu - \eta$, $\mu \in V(\mathcal{B}(\mathcal{H}), S)$ and $\eta \in V(\mathcal{B}(\mathcal{H}), T)$ that μ and η are corners of $V(\mathcal{B}(\mathcal{H}), S)$ and $V(\mathcal{B}(\mathcal{H}), T)$, respectively. Corners of Hilbert space numerical ranges are spectral points by Theorem 1.21.

If μ is not a corner of $V(\mathcal{B}(\mathcal{H}), S)$, then $\mu - \eta$ is not a corner of $V(\mathcal{B}(\mathcal{H}), S) - \eta \subset V(\mathcal{B}(\mathcal{H}), S) - V(\mathcal{B}(\mathcal{H}), T)$. Thus λ can not be a corner of $V(\mathcal{B}(\mathcal{B}(\mathcal{H})), \Delta_{S,T})$. The contradiction establishes μ as a corner. η is likewise a corner. ∎

Theorem 3.4 also implies a converse to Proposition 3.1.

(3.11) THEOREM. Let $\mathfrak{A} = \mathcal{B}(X)$. If $S, T \in \mathfrak{A}$ and $\Delta_{S,T}$ is hermitian, then there exists $\lambda \in \mathbb{C}$ such that both $\lambda - S$ and $\lambda - T$ are hermitian.

PROOF. Obvious. ∎

4. NORMAL ELEMENTS AND DERIVATIONS

The existence of hermitian elements in an arbitrary complex unital Banach algebra permits the existence of normal elements.

(3.12) DEFINITION. For the complex unital Banach algebra $\mathbb{L}$, $N \in \mathbb{L}$ is said to be normal if there exist hermitian elements $H, K \in \mathbb{L}$ such that $N = H + iK$ and $[H, K] = 0$.

This definition coincides with the definition of normality in $\mathcal{B}(\mathfrak{h})$ where $N \in \mathcal{B}(\mathfrak{h})$ is called normal if $[N, N^*] = 0$. In $\mathcal{B}(\mathfrak{h})$ $N = \mathrm{Re}N + i\mathrm{Im}N$ and it is easy to check that the normality of N is equivalent to the commutativity, $[\mathrm{Re}N, \mathrm{Im}N] = 0$.

Of interest here is the observation that if N_1 and N_2 are normal in a unital Banach algebra $\mathbb{L}$, then Δ_{N_1, N_2} is normal in $\mathcal{B}(\mathbb{L})$. To see this suppose $N_1 = H_1 + iK_1$, $N_2 = H_2 + iK_2$, where H_i, K_i are hermitian and $[H_i, K_i] = 0$, $i = 1, 2$. Then as described in the proof of Theorem 3.4

$$\Delta_{N_1, N_2} = \Delta_{H_1, H_2} + i\Delta_{K_1, K_2} \quad .$$

Also because of the commutativity, $[H_i, K_i] = 0$, $i = 1, 2$,

$$\Delta_{H_1,H_2}\Delta_{K_1,K_2}(A) = H_1[K_1A - AK_2] - [K_1A - AK_2]H_2$$
$$= K_1[H_1A - AH_2] - [H_1A - AH_2]K_2$$
$$= \Delta_{K_1,K_2}\Delta_{H_1,H_2}(A), \qquad A \in \mathbb{C} .$$

Thus because Δ_{H_1,H_2} and Δ_{K_1,K_2} are hermitian by Proposition 3.1 and because they commute, Δ_{N_1,N_2} is normal in $\mathcal{B}(\mathbb{C})$.

One important property of normal operators on Hilbert space which carries over to the general setting is that normal operators are convexoid. The proof of the following theorem is the promised proof of Proposition 1.3ii.

(3.13) THEOREM. [35] [10]. Let $\mathbb{C}$ be a complex unital Banach algebra. If $N \in \mathbb{C}$ is normal, then $co\,\sigma_{\mathbb{C}}(N) = V(\mathbb{C}, N)$.

PROOF. Let $N = H + iK$, H, K hermitian, $[H, K] = 0$. Notice that N normal implies $e^{i\theta}N = (\cos\theta H - \sin\theta K) + i(\sin\theta H + \cos\theta K)$ is normal. Since rotations of normals are normal the proof of theorem will be complete once

$$\sup\{\mathrm{Re}\lambda : \lambda \in \sigma_{\mathbb{C}}(N)\} = \sup\{\mathrm{Re}\lambda : \lambda \in V(\mathbb{C}, N)\}$$

is demonstrated.

Observe that for K hermitian, $t \in R$

$$\|A\| = \|Ae^{itK}e^{-itK}\| \leq \|Ae^{itK}\| \leq \|A\|, \qquad A \in \mathbb{C} .$$

Thus $\|A\| = \|Ae^{itK}\|$ and $\|A\| = \|e^{itK}A\|$, $t \in R$, $A \in \mathbb{C}$.

By Theorem 1.12, the hermiticity of K, and that $[H, K] = 0$,

$$\sup\{\mathrm{Re}\lambda : \lambda \in \sigma_{\mathbb{C}}(N)\} = \inf_{a>0} \frac{1}{a} \log\|\exp a(H+iK)\|$$

$$= \inf_{a>0} \frac{1}{a} \log\|\exp aH\| \quad .$$

By Theorems 1.12 and 1.15, the hermiticity of H, and that $[H,K] = 0$,

$$\inf_{a>0} \frac{1}{a} \log\|\exp aH\| = \sup_{a>0} \frac{1}{a} \log\|\exp aH\|$$

$$= \sup_{a>0} \frac{1}{a} \log\|\exp aN\|$$

$$= \sup\{\mathrm{Re}\lambda : \lambda \in V(\mathbb{C}, N)\} \quad . \blacksquare$$

A result on the kernel of a normal element in an arbitrary Banach algebra is the key to the derivation proof of the Fuglede-Putnam Theorem presented in the next section.

(3.14) DEFINITION. Let $\mathbb{C}$ be a complex unital Banach algebra. If $T \in \mathbb{C}$ is of the form $T = H + iK$, H, K hermitian, then the element $T^{\#} = H - iK$ is called the #-adjoint of T.

Note that this definition of the #-adjoint coincides with the usual adjoint for elements of $\mathcal{B}(\mathcal{H})$. A #-adjoint is defined for each element of $\mathcal{B}(\mathcal{H})$ because such elements always have a decomposition in terms of real and imaginary parts.

Trivially, the identities

$$\sigma_{\mathbb{C}}(N) = \overline{\sigma_{\mathbb{C}}(N^{\#})} \quad \text{and} \quad V(\mathbb{C}, N) = \overline{V(\mathbb{C}, N^{\#})}$$

hold for normal elements N in $\mathbb{C}$.

One additional property of normal operators on Hilbert space carries over to the general Banach algebra case.

(3.15) THEOREM. Let N be a normal element of the complex unital Banach algebra $\mathbb{C}$. Then $NA = 0$ if and only if $N^{\#}A = 0$, $A \in \mathbb{C}$.

PROOF. $NA = 0$ implies $\|A\| = \|(\exp zN)(A)\|$, $z \in C$. $N = H + iK$, H, K hermitian, $[H, K] = 0$. Therefore

$$\|A\| = \|(\exp irK)(\exp rH)A\| = \|(\exp rH)A\|, \quad r \in R .$$

Then

$$\begin{aligned}\|(\exp zH)A\| &= \|(\exp isH)(\exp rH)A\| \\ &= \|(\exp rH)A\| = \|A\|, \quad z = r + is .\end{aligned}$$

The function $g(z) = (\exp zH)A$ is entire and bounded, and, therefore, constant. That $HA = 0$ can be obtained by differentiating g and evaluating at zero.

$$NA = 0 \quad \text{and} \quad HA = 0 \quad \text{imply} \quad KA = 0 \quad \text{and} \quad N^{\#}A = 0 . \blacksquare$$

The same proof obviously holds in the case N is a normal element of the algebra $\mathcal{B}(X)$. $Nx = 0$, $x \in X$, implies $N^{\#}x = 0$ is the result in these circumstances.

(3.16) DEFINITION. An eigenvalue for a Banach algebra element T of the form $T = H + iK$, H, K hermitian, is called a normal eigenvalue if $(\lambda - T)A = 0$ implies $(\bar{\lambda} + T^{\#})A = 0$.

(3.17) COROLLARY. Eigenvalues of normal elements of Banach algebras are normal eigenvalues.

One additional fact about the nature of the kernel of a normal intertwining operator is appropriate.

(3.18) THEOREM. Let $N_1, N_2 \in \mathcal{B}(\mathcal{H})$ be normal. If $[N_1, N_2] = 0$ and $N_1 \neq N_2$ then $\Delta_{N_1,N_2}(B) = 0$ implies $0 \in W(B)$.

PROOF. This is a direct application of Corollary 2.9.■

5. THE FUGLEDE-PUTNAM THEOREM

The Fuglede-Putnam Theorem, Theorem 2.6, is an immediate consequence of Theorem 3.15 and the observation that an intertwining operator determined by normal elements is normal.

(3.19) THEOREM. Let $\mathbb{C}$ be a complex unital Banach algebra. For N_1, N_2 normal in $\mathbb{C}$, suppose $N_1A = AN_2$. Then $N_1^{\#}A = AN_2^{\#}$.

PROOF. Put $N_1 = H_1 + iK_1$, $N_2 = H_2 + iK_2$, H_i, K_i hermitian, $i = 1,2$. Then

$$\Delta_{N_1,N_2} = \Delta_{H_1,H_2} + i\Delta_{K_1,K_2} \qquad \text{so that}$$

$$\Delta_{N_1,N_2}^{\#} = \Delta_{H_1,H_2} - i\Delta_{K_1,K_2} = \Delta_{N_1^{\#},N_2^{\#}} .$$

By hypothesis $\Delta_{N_1,N_2}(A) = 0$. Therefore by the normality of Δ_{N_1,N_2} and Theorem 3.15, $\Delta_{N_1,N_2}^{\#}(A) = \Delta_{N_1^{\#},N_2^{\#}}(A) = 0$ or $N_1^{\#}A = AN_2^{\#}$. ■

While there are other simple proofs of this theorem (notably Rosenblum's [34]), this one seems to be a particularly natural proof. This is because only one object, the intertwining operator, receives attention rather than the two normal operators.

It is of interest to note that the Fuglede-Putnam Theorem presented here and the proof by Rosenblum [34] both rely on the Liouville Theorem on entire functions.

CHAPTER 4

NUMERICAL RANGES OF POWERS AND PRODUCTS

1. INTRODUCTION

As a means of discussing the behavior of numerical ranges of powers of operators a mapping theorem for numerical ranges of matrices with positive real part due to C. R. Johnson [24] is studied in detail in this chapter. The effort is to extend to the operator case and generalize Johnson's Theorem,

(4.1) THEOREM. Let dim $\mathfrak{H} < \infty$ and suppose $T \in \mathfrak{B}(\mathfrak{H})$. Then $T \geq 0$ if and only if $W(T^n) \subset \{z \in \mathbb{C} : \text{Re}z \geq 0\}$, $n = 1, 2, \ldots$.

The generalizations are partly motivated by the study of multiplicative commutators in Chapter 2.

A short discussion of the numerical range of the product of two operators is presented at the end of this chapter. The main result of the discussion is a containment result for the numerical range of a product, an extension of work of Stampfli [39] and Loewy [26].

Throughout the chapter primary attention will be devoted to the set of positive operators and the set of accretive operators. Because of this, the following definition is made here.

(4.2) DEFINITION. Putting

$$\Pi = \{z \in \mathbb{C} : \text{Re}z \geq 0\} \quad ,$$

define the set of accretive operators in a complex unital Banach algebra $\mathcal{O}$, $\mathbb{A}(\mathcal{O})$, *by*

$$\mathbb{A}(\mathcal{O}) = \{T \in \mathcal{O} : V(\mathcal{O}, T) \subset \Pi\}$$

and the set of positive operators in $\mathcal{O}$, $\mathbb{P}(\mathcal{O})$, *by*

$$P(\mathcal{O}) = \{T \in \mathcal{O} : T \in \mathbb{A}(\mathcal{O}) \quad \text{and} \quad V(\mathcal{O}, T) \subset \mathbb{R}\} \quad .$$

If $\mathcal{O} = \mathcal{B}(\mathfrak{H})$, *write* $A(\mathfrak{H})$ *for* $\mathbb{A}(\mathcal{O})$ *and* $\mathbb{P}(\mathfrak{H})$ *for* $\mathbb{P}(\mathcal{O})$.

2. KATO'S THEOREM

A theorem of Kato [25] is the key element in the extension of Theorem 4.1 to the infinite dimensional case. This result, which will be modified in this section to suit the purposes of the chapter more directly, is listed as

(4.3) THEOREM. *For* $T \in \mathbb{A}(\mathfrak{H})$, *suppose* f *is holomorphic in a neighborhood of* Π. *Then* $W(f(T)) \subset \mathrm{cof}(\Pi)^-$.

Here f(T) is defined by the Dunford calculus,

$$f(T) = \frac{1}{2\pi i} \int_{\Gamma} f(z)(z - T)^{-1}\, dz \quad . \tag{1}$$

Γ is a closed Jordan curve in the domain of holomorphy of f; Γ contains $\sigma(T)$ in the bounded component of its complement. The spectrum lays to the left as Γ is traversed.

It is shown by DePrima (see [13]) that Theorem 4.3 is a consequence of von Neumann's theory of spectral sets. This approach to the proof of Theorem 4.3 is basically different from that of Kato.

The interest in this section is in a modification of Kato's Theorem rather than in the proof of Theorem 4.3.

(4.4) LEMMA. Let $T \in \mathbb{A}(\mathfrak{h})$ and suppose $\sigma(T) \subset \mathring{\Pi}$. If f is holomorphic in $\mathring{\Pi}$, then $W(f(T)) \subset \mathrm{cof}(\mathring{\Pi})^-$.

PROOF. Note that f(T) is well defined by (1), since $\mathrm{dist}(\sigma(T), C\setminus\mathring{\Pi}) = \delta > 0$. For $\epsilon > 0$, supposing $2\epsilon < \delta$, define f_ϵ on $\mathring{\Pi} - \epsilon$ by $f_\epsilon(z) = f(z+\epsilon)$.

f_ϵ is holomorphic on a neighborhood of Π and $T \in \mathbb{A}(\mathfrak{h})$. Therefore by Theorem 4.3

$$W(f_\epsilon(T)) \subset \mathrm{cof}_\epsilon(\Pi)^- \subset \mathrm{cof}(\mathring{\Pi})^- \quad . \tag{2}$$

It is shown below that $f_\epsilon(T) \to f(T)$ as $\epsilon \to 0^+$. The continuity condition for the numerical range, Theorem 1.18 iii, and (2), therefore, imply the result.

Taking Γ to lie in $\mathring{\Pi}$, observe that

$$f_\epsilon(T) - f(T) = \frac{1}{2\pi i}\int_\Gamma (f(z+\epsilon) - f(z))(z-T)^{-1}\,dz \quad .$$

If $M = \max_{z\in\Gamma} \|(z-T)^{-1}\|$ and L is the length of Γ, then

$$\|f_\epsilon(T) - f(T)\| \leqslant \frac{1}{2\pi} ML \cdot \max_{z\in\Gamma} |f(z+\epsilon) - f(z)| \quad .$$

Since $f_\epsilon(z)$ converges uniformly to f(z) on Γ, $\|f_\epsilon(T) - f(T)\| \to 0$ as $\epsilon \to 0^+$, as required. ∎

Lenna 4.4 will be applied most often to functions of the form $f(z) = z^{\alpha}$. The facts needed about such functions of accretive operators are collected in the next proposition.

(4.5) DEFINITION. For $\alpha \in \mathbb{R}$, $0 \leq \alpha < 2$, put

$$S_{\alpha} = \{z \in \mathbb{C} : |\arg z| \leq \tfrac{\pi}{2} \alpha\} \cup \{0\} .$$

(4.6) PROPOSITION. Let $T \in \mathbb{A}(\mathfrak{H})$ with $\sigma(T) \subset \mathring{\Pi}$ and suppose $\alpha \in \mathbb{R}$, $0 < |\alpha| \leq 1$. Put $f(z) = z^{\alpha}$ with the normalization $f(1) = 1$. Then

i) $f(T) = T^{\alpha} \in \mathbb{A}(\mathfrak{H})$ and $W(T^{\alpha}) \subset S_{|\alpha|}$;

ii) if $\sigma(U) \subset \mathring{S}_{|\alpha|}$ and $U^{1/\alpha} = T$, then $T^{\alpha} = U$;

iii) for $\beta \in R$, T^{β} is invertible, T^{β} commutes with all operators commuting with T, and, if $\beta = n$, an integer, the definition of T^{n} given by (1) coincides with the usual algebraic power of T;

iv) the mapping $T \rightarrow T^{\alpha}$ is continuous in the uniform topology of $\mathfrak{B}(\mathfrak{H})$.

PROOF. i) is a direct application of Lemma 4.4 where $f(z) = z^{\alpha}$.

ii) is a consequence of the standard composition theorem for the Dunford calculus (see e.g., [15], Theorem VII.3.12).

The commutativity assertion of iii) follows from (1).

iv) is another consequence of the Dunford calculus; $z^{\beta} \rightarrow z^{\alpha}$ uniformly on $\sigma(T)$ as $\beta \rightarrow \alpha$.■

There is a strengthening of Proposition 4.6i which is useful in the sequel.

(4.7) PROPOSITION. For $T \in \mathbb{A}(\mathfrak{h})$, $\sigma(T) \subset \mathring{\Pi}$, suppose $0 < |\alpha| < 1$. Then $W(T^{\alpha})^{-} \subset S_{|\alpha|} \setminus \{0\}$. If $W(T)^{-} \subset \mathring{\Pi}$, then $W(T^{\alpha})^{-} \subset \mathring{S}_{|\alpha|}$.

PROOF. $W(T^{\alpha}) \subset S_{|\alpha|}$ from Proposition 4.6i. That $0 \notin W(T^{\alpha})^{-}$ follows from Theorem 1.21. To see this observe that $0 \in W(T^{\alpha})^{-}$ implies that 0 is a corner of $W(T^{\alpha})^{-}$, hence in $\sigma(T^{\alpha})$. However, $0 \in \sigma(T^{\alpha})$ implies $0 \in \sigma(T)$.

If $W(T)^{-} \subset \mathring{\Pi}$ then $\inf\{\operatorname{Re}\lambda : \lambda \in W(T)\} \geq 2\epsilon$, for some ϵ, $\epsilon > 0$. Putting $T_{\epsilon} = T - \epsilon$, then $W(T_{\epsilon})^{-} \subset \mathring{\Pi}$. Applying Lemma 4.4 to T_{ϵ} with $f_{\epsilon}(z) = (z + \epsilon)^{\alpha}$,

$$W(f_{\epsilon}(T_{\epsilon}))^{-} = W(T^{\alpha})^{-} \subset \operatorname{cof}_{\epsilon}(\mathring{\Pi})^{-} \subset \mathring{S}_{|\alpha|} \quad . \blacksquare$$

3. THE MAIN RESULTS

Preparation has now been made for the operator version of Theorem 4.1. The techniques used in the proof yield a more general result which is discussed later.

(4.8) THEOREM. For $T \in \mathcal{B}(\mathfrak{h})$, $T \in \mathbb{P}(\mathfrak{h})$ if and only if $T^{n} \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$.

PROOF. That $T^{n} \geq 0$, $n = 1, 2, \ldots$ for $T \geq 0$ is trivial, so the necessity is clear.

For the sufficiency note that for $\epsilon > 0$, $(T + \epsilon)^{n} \in \mathbb{A}(\mathfrak{h})$. In fact, $(T + \epsilon)^{n} = T^{n} + nT^{n-1}\epsilon + \cdots + nT\epsilon^{n-1} + \epsilon^{n}$. Hence

$$\begin{aligned} W((T + \epsilon)^{n}) &\subset W(T^{n}) + n\epsilon W(T^{n-1}) + \cdots + n\epsilon^{n-1}W(T) + \epsilon^{n} \\ &\subset \Pi + \epsilon^{n} \quad . \end{aligned}$$

Then,

$$\sigma((T+\epsilon)^n) = \sigma(T+\epsilon)^n \subset W((T+\epsilon)^n) \subset \Pi + \epsilon^n,\ n = 1,2,\ldots,\ \text{implies } \sigma(T+\epsilon) \subset R^+.$$

Applying Proposition 4.6i and ii to $(T+\epsilon)^n$ with $\alpha = 1/n$,

$$W(((T+\epsilon)^n)^{1/n}) = W(T+\epsilon) \subset S_{1/n} .$$

Since this holds for every n, $W(T+\epsilon) \subset S_0$. Because ϵ is arbitrary, $W(T) \subset S_0$ and $T \in \mathbb{P}(\mathfrak{h})$.■

As suggested these techniques yield a more general non-semigroup type of theorem.

(4.9) THEOREM. <u>Let</u> $\{T_n\}_{n=1}^{\infty}$ <u>be a sequence in</u> $\mathbb{A}(\mathfrak{h})$ <u>satisfying</u> $\sigma(T_n) \subset \mathring{\Pi}$, $n = 1, 2, \ldots,$ <u>and let</u> $\{r_n\}_{n=1}^{\infty}$ <u>be a sequence in</u> R^+ <u>converging to</u> 0. If $\lim_{n\to\infty} T_n^{r_n} = T$ <u>(in the uniform topology of</u> $\mathcal{B}(\mathfrak{h})$), <u>then</u> $T \in \mathbb{P}(\mathfrak{h})$.

PROOF. For some n_0, $n > n_0$ implies $r_n < 1$. By Proposition 4.6i $W(T_n^{r_n}) \subset S_{r_n}$, for $n > n_0$. Since $T_n^{r_n} \to T$, the continuity of W from Theorem 1.18 iii and the convergence $r_n \to 0$ as $n \to \infty$ imply $T \in \mathbb{P}(\mathfrak{h})$.■

Theorem 4.8 has two immediate corollaries. The first gives a perturbed form of Theorem 4.8, a prototype for the class of theorems which are investigated later in this chapter. The second is an application of the result to the theory of semigroups.

(4.10) COROLLARY. Let $H \in \mathbb{P}(\mathfrak{h}) \cap \mathcal{G}(\mathcal{B}(\mathfrak{h}))$. If $T^nH \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots,$ then $H^{-\frac{1}{2}}TH^{\frac{1}{2}} \in \mathbb{P}(\mathfrak{h})$.

PROOF. $(T^nHx, x) = (H^{-1}T^ny, y)$, $y = Hx$. Letting $(u, v)_H = (H^{-1}u, v)$ denote a new inner product, observe that T^n is accretive relative to the new inner product, $n = 1, 2, \ldots$. Thus by Theorem 4.8, $(H^{-1}Ty, y) \geq 0$, $y \in \mathfrak{h}$. But then $H^{-\frac{1}{2}}TH^{\frac{1}{2}} \in \mathbb{P}(\mathfrak{h})$.■

(4.11) COROLLARY. Any semigroup of accretive operators is necessarily a commutative semigroup of positive operators.

PROOF. For T accretive in the semigroup Σ, $T^n \in \Sigma$, $n = 1, 2, \ldots$. By Theorem 4.8, $T \in \mathbb{P}(\mathfrak{h})$. For $S, T \in \Sigma$, $ST \in \Sigma$. But $ST = (ST)^* = T^*S^*$ $= TS$.■

4. PERTURBATIONS OF THE HYPOTHESIS OF THE MAIN RESULT

The sequence $\{T^nB\}_{n=1}^{\infty}$ is studied in this section. Conditions are sought for which T is positive or T is self-adjoint.

To see that this structure arises naturally, recall from the discussion of multiplicative commutators that if

$$T = ABA^{-1}B^{-1} \quad \text{and} \quad [A, T] = 0 ,$$

then an induction argument shows that

$$T^nB = A^nBA^{-n}, \qquad n \in \mathbb{Z} . \tag{3}$$

In Theorem 2.5, with the additional assumptions A unitary and $0 \notin W(B)^-$, (3) is used to show $T = 1$. This section is directed toward generalizing this result.

The first theorem will help provide the promised proof of Theorem 2.2.

(4.12) THEOREM. If $T^nB \in \mathbb{A}(\mathfrak{H})$, $n = 0, 1, 2, \ldots$, then $0 \in W(B)^-$ or $\sigma(T) \subset R$.

PROOF. If $0 \notin W(B)^-$, then because B is accretive and because W(B) is convex, there exists α, $0 < \alpha < 1$, such that one of the sets $\exp(\pm i\frac{\pi}{2}\alpha)W(B)$ is contained in $S_{1-\alpha}$. By Theorem 1.19

$$\sigma(T)^n = \sigma(T^n) = \sigma(T^nBB^{-1}) \subset W(T^nB)^-/W(B)^- \quad . \tag{4}$$

Thus $\sigma(T)^n \subset K$, where K omits a nonvoid open sector with vertex at the origin and edge the negative real axis. Observe that for $r > 0$

$$(T+r)^nD = T^nD + nrT^{n-1}D + \cdots + nr^{n-1}TD + r^nD \quad .$$

Hence $(T+r)^nD \in \mathbb{A}(\mathfrak{H})$ for any $n \in \mathbb{N}$ and $r \geqslant 0$. The computation in (4) implies $\sigma((T+r)^n) \subset K$, for any $r \geqslant 0$, $n \in \mathbb{N}$.

Consequently, $\lambda \in \sigma(T)$ implies $(\lambda+r)^n \in K$ for any $n \in N$, $r \geqslant 0$. This is possible only for $\lambda \in \mathbb{R}$. ∎

To see that λ need not be non-negative, an example is presented. For the example, $\mathfrak{H} = \mathbb{C}^2$ and T and B have matrix representations in an orthonormal basis for $\mathbb{C}^2$:

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} \quad .$$

Observe that

$$T^nB = \begin{pmatrix} 1 & 0 \\ 0 & (-1)^n i \end{pmatrix}, \qquad n = 0, 1, 2, \ldots \quad .$$

Thus $\sigma(T) \subset S_0$ is not implied by $T^n B \in \mathbb{A}(\mathfrak{h})$, $n = 0, 1, 2, \ldots$ and $0 \notin W(B)^-$.

A stronger constraint on B gives a stronger conclusion.

(4.13) THEOREM. If $T^n B \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$ and $W(B)^- \subset \mathring{\Pi}$, then $\sigma(T) \subset S_0$.

PROOF. As before the hypothesis remain unchanged if T is replaced by $T + r$, $r > 0$. In this case, however, $W(B)^- \subset S_\beta$ for some β, $0 < \beta < 1$, so that by Theorem 1.19

$$\sigma((T+r)^n) = \sigma(T+r)^n \subset W((T+r)^n B)^- / W(B)^- \subset K .$$

$K \subset \mathbb{C}$ omits a nonvoid open sector of the plane with vertex at the origin symmetric about the negative real axis.

$\lambda \in \sigma(T)$ implies $(\lambda + r)^n \in K$ for any $r \geq 0$, $n \in N$. Thus $\lambda \in S_0$. ∎

To close an open issue the proof of Putnam's multiplicative commutator theorem, Theorem 2.2, is given as a corollary to these results.

(4.14) COROLLARY. For $T, A, B \in \mathcal{B}(\mathfrak{h})$, suppose $T = ABA^{-1}B^{-1}$, $[A, T] = 0$, and that A is unitary. Then $0 \in W(B)^-$ or $\sigma(T) = \{1\}$.

PROOF. An induction argument relying on the fact that $[A, T] = 0$ yields that

$$T^n B = A^n B A^{-n}, \qquad n \in \mathbb{Z} \tag{5}$$

(see the proof of Theorem 2.5).

Assume $0 \notin W(B)^-$ and note that because (5) is unchanged by multiplication by scalar factor of modulus 1, it can be and is assumed that $W(B)^- \subset \mathring{\Pi}$.

Because A is unitary,

$$W(T^n B) = W(A^n B A^{-n}) = W(B) \subset \mathring{\Pi}, \qquad n = 0, 1, 2, \ldots \quad . \tag{6}$$

By Theorem 4.13, $\sigma(T) \subset S_0$.

Furthermore by Theorem 1.19

$$\sigma(T^n) \subset W(T^n B)^- / W(B)^- = W(B)^- / W(B)^- \quad .$$

Since $W(B)^- / W(B)^-$ is a compact set omitting 0, $\lambda \in \sigma(T)$ implies $|\lambda| = 1$.

Thus $\sigma(T) = \{1\}$. ■

Recall that the hypothesis of Corollary 4.14 imply a stronger conclusion: $0 \in W(B)^-$ or $T = 1$ (see Theorem 2.5). The conclusion of Corollary 4.14 is obtained without using the full strength of the hypothesis, however. This is seen from (6) which uses (5) only on the non-negative integers.

Spectral properties of T are not the only properties of interest. The hermiticity of T or its positivity is the issue on which the remainder of the section is centered.

(4.15) CONJECTURE. <u>For</u> $T, B \in \mathcal{B}(\mathfrak{H})$ <u>suppose</u> $T^n B \in \mathbb{A}(\mathfrak{H})$, $n = 1, 2, \ldots$. <u>Then</u> $W(B)^- \not\subset \mathring{\Pi}$ <u>or</u> $T \in \mathbb{P}(\mathfrak{H})$.

Giving substance to the conjecture is the aim of the discussion which follows. To obtain a positive result a commutativity assumption is introduced.

(4.16) THEOREM. <u>For</u> $T, B \in \mathcal{B}(\mathfrak{h})$ <u>suppose</u> $T^n B \in \mathbb{A}(\mathfrak{h})$, $n = 0, 1, 2, \ldots$, <u>and</u> $[T, B] = 0$. <u>Then</u> $W(B)^- \not\subset \mathring{\Pi}$ <u>or</u> $T \in \mathbb{P}(\mathfrak{h})$.

PROOF. Suppose $W(B)^- \subset \mathring{\Pi}$. Put $T_\epsilon = T + \epsilon$, $\epsilon > 0$. The first task is to check that $T_\epsilon B^{1/n} = (T_\epsilon^n B)^{1/n}$. Proposition 4.6 ii implies that it need only be shown that

$$\sigma(T_\epsilon B^{1/n}) \subset \mathring{S}_{1/n} .$$

However, $\sigma(T_\epsilon) \subset S_0$ by Theorem 4.13 and $\sigma(B^{1/n}) \subset W(B^{1/n})^- \subset \mathring{S}_{1/n}$ by Proposition 4.7. By the commutativity

$$\sigma(T_\epsilon B^{1/n}) \subset \sigma(T_\epsilon)\sigma(B^{1/n}) \subset \mathring{S}_{1/n} .$$

Proposition 4.6 iv implies $B^{1/n} \to 1$. Hence $T_\epsilon B^{1/n} \to T_\epsilon$. Theorem 4.9 can be applied to the sequence of operators $\{T_\epsilon^n B\}_{n=1}^\infty$ and the sequence of positive real numbers $\{1/n\}_{n=1}^\infty$. Since it has been shown that $(T_\epsilon^n B)^{1/n} = T_\epsilon B^{1/n}$ and that $T_\epsilon B^{1/n} \to T_\epsilon$, the conclusion is that $T_\epsilon \in \mathbb{P}(\mathfrak{h})$.

Because ϵ is arbitrary $T \in \mathbb{P}(\mathfrak{h})$. ■

Theorem 4.16 can also be derived directly from Theorem 4.8. In this approach it is shown that if $W(U^n C)^- \subset \mathring{\Pi}$, $n = 0, 1, 2, \ldots$, then $W(U^n C^{\frac{1}{2}})^- \subset \mathring{\Pi}$, $n = 0, 1, 2, \ldots$ where $U \in \mathbb{A}(\mathfrak{h})$ and $[U, C] = 0$. Once this is demonstrated, repetition of the argument and a limit show that $W(U^n) \subset \mathring{\Pi}$, $n = 0, 1, 2, \ldots$ or that $U \geq 0$, by Theorem 4.8. Theorem 4.16 is obtained by letting $C = B^{1/2^m}$ for $m \in \mathbb{N}$ and $U = T + \epsilon$.

Demonstration of the required fact is made in the proof of

(4.17) PROPOSITION. For $U, C \in \mathcal{B}(\mathfrak{h})$ suppose $[U, C] = 0$. $W(U^nC)^- \subset \mathring{\Pi}$, $n = 0, 1, 2, \ldots$ implies $W(U^nC^{\frac{1}{2}})^- \subset \mathring{\Pi}$, $n = 0, 1, 2, \ldots$.

PROOF. Since $W(C)^- \subset \mathring{\Pi}$, Proposition 4.7 implies that $W(C^{\frac{1}{2}})^- \subset \mathring{S}_{\frac{1}{2}}$. By Theorem 1.19

$$\sigma(U^nC^{\frac{1}{2}}) \subset W(U^nC)^-/W(C^{\frac{1}{2}})^- \subset \mathring{S}_{3/2} \quad . \tag{7}$$

However, $UC = CU$ implies $UC^{\frac{1}{2}} = C^{\frac{1}{2}}U$, so $(U^nC^{\frac{1}{2}})^2 = U^{2n}C$. Because $\sigma(U^{2n}C) \subset \mathring{\Pi}$, the spectral mapping theorem shows that

$$\sigma(U^nC^{\frac{1}{2}}) \subset \mathring{S}_{\frac{1}{2}} \cup \{\mathbb{C}\backslash S_{3/2}\} \quad . \tag{8}$$

Combining (7) and (8), $\sigma(U^nC^{\frac{1}{2}}) \subset \mathring{S}_{\frac{1}{2}}$. Proposition 4.6 ii implies $(U^{2n}C)^{\frac{1}{2}} = (U^nC^{\frac{1}{2}})$. Hence $W(U^nC^{\frac{1}{2}}) \subset \mathring{S}_{\frac{1}{2}} \subset \mathring{\Pi}$. ∎

Without a commutativity assumption additional evidence to support the conjecture is available. Indeed, if T is assumed convexoid Theorem 4.13 can be applied directly to yield the conclusion of the conjecture in this special case. Another special case is one in which T is nilpotent. The proof of (4.18) is a simplification suggested by De Prima of the original.

(4.18) THEOREM. For $T, B \in \mathcal{B}(\mathfrak{h})$, suppose $TB \in \mathbb{A}(\mathfrak{h})$ and T is nilpotent. Then $0 \in W(B)$ or $T \in \mathbb{P}(\mathfrak{h})$.

PROOF. Suppose $T \neq 0$ and that $0 \notin W(B)$.

There exists a unit vector x such that

$$T^*x = y, \quad T^*y = 0, \quad y \neq 0, \quad (x, y) = 0 \quad . \tag{9}$$

Using (9) compute, putting $v = y/\|y\|$

$$(TB(\alpha x+\beta v),(\alpha x+\beta v)) = \|y\|[\,|\alpha|^2(Bx,v) + \bar{\alpha}\beta(Bv,v)] \in W(TB) \quad ,$$

for $|\alpha|^2 + |\beta|^2 = 1$.

Since $(Bv, v) \neq 0$, and because $|\alpha|^2(Bx, v)$ is dominated in modulus by $\bar{\alpha}\beta(Bv, v)$ for α close to zero, there exists $\alpha \neq 0$ such that $|\bar{\alpha}\beta(Bv, v)| > |\alpha|^2(Bx, v)$, $|\alpha|^2 + |\beta|^2 = 1$. Allowing β to vary in argument (for this fixed α) in the quantity

$$|\alpha|^2(Bx, v) + \bar{\alpha}\beta(Bv, v)$$

it is seen that W(TB) contains a neighborhood of 0. This is a contradiction.∎

Though limited in some respects Theorem 4.18 is of interest because it supports the conjecture with a weaker hypothesis. Here only that $TB \in \mathbb{A}(\mathfrak{h})$, rather than $T^nB \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$, is required; $W(B)^- \subset \mathring{\Pi}$ is replaced by $0 \notin W(B)$.

5. OTHER RELATED RESULTS

The sharpness of some of the preceding results are discussed in this section. Mention of some additional observations and extensions is made.

Two technical lemmas are of help. The first is a modification of Proposition 4.6 i.

(4.19) LEMMA. <u>For</u> $T \in \mathcal{B}(\mathfrak{h})$, <u>suppose</u> $T \in \mathbb{A}(\mathfrak{h}) \cap \mathcal{G}(\mathcal{B}(\mathfrak{h}))$, <u>then for</u> α, $0 < |\alpha| < 1$, $W(T^{\alpha}) \subset S_{|\alpha|}$.

PROOF. $T_\epsilon = T + \epsilon$. $T_\epsilon \in \mathbb{A}(\mathfrak{h})$, $\sigma(T_\epsilon) \subset \mathring{\Pi}$.

By Proposition 4.6i, $W((T+\epsilon)^{\alpha}) \subset S_{|\alpha|}$. Letting $\epsilon \to 0$ and noting that $(z+\epsilon)^{\alpha}$ approaches z^{α} uniformly on $\sigma(T)$, $(T+\epsilon)^{\alpha} \to T^{\alpha}$ and $W(T^{\alpha}) \subset S_{|\alpha|}$. ∎

(4.20) LEMMA. For $T \in \mathcal{B}(\mathfrak{h}) \cap \mathcal{G}(\mathcal{B}(\mathfrak{h}))$, suppose $W(T) \subset e^{i\theta}\Pi$, $-\pi/2 < \theta < \pi/2$, then for $0 < |\alpha| \leq 1$, $W(T^{\alpha}) \subset e^{i\alpha\theta}S_{|\alpha|}$.

PROOF. $W(T) \subset e^{i\theta}\Pi$ implies $e^{-i\theta}T \in \mathbb{A}(\mathfrak{h})$. Then Lemma 4.19 implies $W((e^{-i\theta}T)^{\alpha}) \subset S_{|\alpha|}$. However, $(e^{-i\theta}T)^{\alpha} = e^{-i\alpha\theta}T^{\alpha}$. Hence $W(T^{\alpha}) \subset e^{i\alpha\theta}S_{|\alpha|}$. ∎

An immediate consequence of this observation is a generalization of Theorem 4.9.

(4.21) THEOREM. Let $\{T_n\}_{n=1}^{\infty}$ be a sequence in $\mathcal{B}(\mathfrak{h}) \cap \mathcal{G}(\mathcal{B}(\mathfrak{h}))$, $\{r_n\}_{n=1}^{\infty}$ a sequence in R^+ converging to zero, and $\{\theta_n\}_{n=1}^{\infty}$ a sequence in R such that $\sup_n |\theta_n| = \delta < \pi/2$. If $W(T_n) \subset e^{i\theta_n}\Pi$, $n = 1, 2, \ldots$, and $\lim_{n\to\infty} T_n^{r_n} = T$, then $T \in \mathbb{P}(\mathfrak{h})$.

A related situation is examined in

(4.22) THEOREM. For $T \in \mathcal{B}(\mathfrak{h})$ suppose $W(T^n) \subset e^{i\theta}\Pi$, $n = 1, 2, \ldots$ for some θ, $\theta \in (-\pi, \pi]$. Then $T = T^*$. In particular, if $|\theta| < \pi/2$, $T \in \mathbb{P}(\mathfrak{h})$. If $|\theta| > \pi/2$, $T = 0$.

PROOF. For $|\theta| < \pi/2$, $\sigma(T) \subset S_0$ is readily verified. Taking $T_\epsilon = T+\epsilon$ note that all hypotheses remain intact for T_ϵ. By Theorem 4.21, with $T_n = (T+\epsilon)^n$, $r_n = 1/n$, and $\theta_n = \theta$, $T+\epsilon \in \mathbb{P}(\mathfrak{h})$. Thus $T \in \mathbb{P}(\mathfrak{h})$.

If $\theta = \pi/2$, then $\sigma(T) \subset \mathbb{R}$. Taking $U = T + \|T\| + 1$. $W(U^n) \subset e^{i\pi/2}\Pi$ so that $\exp(-i\pi/2)U^n \in \mathbb{A}(\mathfrak{h})$. Again applying Theorem 4.21, with $T_n = \exp(-i\pi/2)U^n$, $r_n = 1/n$ and $\theta = 0$, $U = U^*$. Thus $T = T^*$. The case $\theta = -\pi/2$ is similar.

$\sigma(T) = \{0\}$ in the case $|\theta| > \pi/2$. Hence $0 \in \partial W(T^n)$, $n = 1, 2, \ldots$. Theorem 1.22 can be applied to obtain $\|T\| \leq 8r(T) = 0$. Thus $T = 0$. ∎

To see that these results and the results of section 4 are sharp in some sense some examples are presented.

The condition $0 \notin W(B)^-$ in Theorem 4.12 can not be removed. To see this let

$$B = i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad T = i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Then

$$T^nB = \begin{cases} i^{n+1}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, & n \text{ odd} \\ i^{n+1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, & n \text{ even} \end{cases} .$$

Thus $T^nB \in \mathbb{A}(\mathfrak{h})$, $n = 0, 1, 2, \ldots$, $0 \in W(B)^-$ and $\sigma(T) \not\subset R$.

The hypothesis $W(B)^- \subset \mathring{\Pi}$ of Theorem 4.16 can not be replaced by the weaker $B \in \mathbb{A}(\mathfrak{h})$, $0 \notin W(B)$. For this examine the example which follows Theorem 4.12. There $T^nB \in \mathbb{A}(\mathfrak{h})$, $n = 0, 1, 2, \ldots$, $[T, B] = 0$, $0 \notin W(B)$, but $T \notin \mathbb{P}(\mathfrak{h})$.

A slight generalization of Theorem 4.16 holds.

(4.23) THEOREM. Let $T^n B \in \mathbb{A}(\mathfrak{h})$, $n = 0, 1, 2, \ldots$. If $[T, B] = 0$, then $0 \in W(B)^-$ or $T = T^*$.

PROOF. Suppose $0 \notin W(B)^-$. Theorem 4.12 implies $\sigma(T) \subset R$. Putting $U = T + \|T\| + 1$, note that the hypothesis of the theorem is satisfied by U and B in place of T and B. However, as in the proof of Theorem 4.16, $(U^n B)^{1/n} = UB^{1/n}$. By Theorem 4.21, with $T_n = U^n B$, $r_n = 1/n$, and $\theta_n = 0$, $U \in \mathbb{P}(\mathfrak{h})$ since $UB^{1/n} \to U$ as $n \to \infty$. $U \in \mathbb{P}(\mathfrak{h})$ implies $T = T^*$. ∎

6. THE MAIN RESULT IN AN ARBITRARY BANACH ALGEBRA

It is of interest to translate Theorem 4.8 into general Banach algebra terms and examine its validity. Unfortunately neither the necessity or the sufficiency of the theorem carry over to so general a setting. Special attention will be given to the C^*-algebra case where the translated Theorem 4.8 remains valid.

To explore the validity of Theorem 4.8 in a complex unital Banach algebra, $\mathfrak{A}$, results of Bollobas [6] are needed. Bollobas discusses the relationship between numerical ranges and entire functions of exponential type. Using this relationship, which is described below, extremal singly generated Banach algebras are constructed which have prescribed numerical range properties. These extremal algebras are useful in the search for counterexamples to the proposed extension of Theorem 4.8.

(4.24) DEFINITION. For a compact, convex set $K \subset \mathbb{C}$ determine the function k as follows:

$$k(K, \theta) = \sup\{\mathrm{Re}\lambda : \lambda \in e^{-i\theta}K\} \quad .$$

Call k the support functional of K.

(4.25) DEFINITION. Whenever K is compact, convex put

$$F(K) = \{f \text{ entire: } f(0) = 1 \quad \text{and} \quad |f(re^{i\theta})| \leq \exp rk(K, \theta), \ \theta, r \in \mathbb{R}\} \ .$$

Recall from Chapter 1

$$\sup\{\text{Re}\lambda : \lambda \in V(\mathfrak{A}, T)\} = \sup_{a>0} \frac{1}{a} \log\|\exp aT\|$$

where $\mathfrak{A}$ is a complex unital Banach algebra and $T \in \mathfrak{A}$. Therefore, if K is compact, convex, $V(\mathfrak{A}, T) \subset K$ if and only if $\|\exp zT\| \leq \exp rk(K, \theta)$, $z = re^{i\theta}$, $z \in \mathbb{C}$. Furthermore if $h \in \mathcal{S}(\mathfrak{A})$ the function $f(z) = h(\exp zT)$ is entire. In fact, by the previous remark

$$|f(z)| = |h(\exp zT)| \leq \|\exp zT\| \leq \exp rk(V(\mathfrak{A}, T), \theta) \ .$$

Hence $f \in F(V(\mathfrak{A}, T))$.

These remarks also lead to a containment result for the numerical range of a polynomial in T. Let p be a polynomial, $p(z) = \sum_{k=0}^{r} p_k z^k$. Observe that f of the last paragraph, $f \in F(V(\mathfrak{A}, T))$, can be written

$$f(z) = \sum_{k=0}^{\infty} \frac{h(T^k)z^k}{k!} \ , \qquad h \in \mathcal{S}(\mathfrak{A}) \ .$$

Thus

$$V(\mathfrak{A}, p(T)) = \{\sum_{k=0}^{r} p_k h(T^k) : h \in \mathcal{S}(\mathfrak{A})\}$$

$$\subset \{\sum_{k=0}^{r} p_k f_k : f(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{k!} \ , \ f \in F(V(\mathfrak{A}, T))\} \ . \qquad (10)$$

It is Bollobas' observation that this containment can be reversed in certain extremal algebras which he constructs. The facts about these constructions that are used below are collected in

(4.26) THEOREM. [6]. Let K be a compact convex subset of $\mathbb{C}$ and $v = \max\{|\lambda| : \lambda \in K\}$. There exists a complex unital Banach algebra $\mathfrak{A}$ and an element $T \in \mathfrak{A}$ such that if $g(z) = \sum_{k=0}^{\infty} g_k z^k$ is holomorphic in a neighborhood of the disc $\{z : |z| \leq v\}$, then

i) $V(\mathfrak{A}, T) = K$.

ii) $V(\mathfrak{A}, g(T)) = \{\sum_{k=0}^{\infty} f_k g_k : f(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{k!}$ and $f \in F(K)\}$.

iii) For any Banach algebra $\mathbb{B}$, $U \in \mathbb{B}$,

$$V(\mathbb{B}, U) \subset K \text{ implies } V(\mathbb{B}, g(U)) \subset V(\mathfrak{A}, g(T)) .$$

The containment (10) and its refinement, Theorem 4.26, are central to the discussion of Theorem 4.8 in a general Banach algebra setting. Some positive evidence toward establishing the necessity of 4.8 in the general setting is contained in

(4.27) THEOREM. If $T \in \mathfrak{A}$ and $V(\mathfrak{A}, T) \subset \mathbb{R}$, then $V(\mathfrak{A}, T^2) \subset \Pi$.

PROOF. T hermitian and Definition 4.25 imply $|f(it)| \leq 1$, $t \in \mathbb{R}$, for $f \in F(V(\mathfrak{A}, T))$. If $f(z) = \sum_{k=0}^{\infty} \frac{f_k z^k}{k!}$, $f_k = g_k + ih_k$, $g_k, h_k \in \mathbb{R}$, $k = 0, 1, \ldots$, then $f(it)$ can be calculated in terms of t, the g_k, and the h_k. By (10) the theorem is established once it is demonstrated that $\mathrm{Re} f_2 = g_2 \geq 0$.

$$1 \geq |f(it)|^2 = 1 + (-2h_1)t + (-g_2 + h_1^2 + g_1^2)t^2 + o(t^2) \quad . \tag{11}$$

From (11), $1 \geq 1 + (-2h_1)t + o(t)$. Since t approaches zero through both positive and negative values, $h_1 = 0$ is necessary to preserve the inequality. Similarly,

$$-g_2 + g_1^2 \leq 0 \quad \text{or} \quad 0 \leq g_1^2 \leq g_2 \quad . \blacksquare \tag{12}$$

(4.28) COROLLARY. Let $T \in \mathcal{Q}$ be hermitian, then $\mathrm{Re}V(\mathcal{Q}, T^2) \subset V(\mathcal{Q}, T)^2$.

PROOF. Theorem 1.15 implies $\|T\|^2 = \|T^2\|$. Thus since $\|T\| = \sup\{|\lambda| : \lambda \in V(\mathcal{Q}, T)\}$ and $V(\mathcal{Q}, T^2) \subset \{\lambda : |\lambda| \leq \|T\|^2\}$ it need only be shown that $\mu \in V(\mathcal{Q}, T^2)$ implies $\mathrm{Re}\mu \geq \inf\{\lambda^2 : \lambda \in V(\mathcal{Q}, T)\}$. This follows from (12). $\blacksquare$

These results represent the available positive evidence toward the establishment of Theorem 4.8 in this setting. Preparations are now made for displaying a counterexample.

A function f is exhibited such that $|f(z)| \leq \exp|\mathrm{Re}z|$. Let $f(z) = 5/9 + 4/9 \cosh z - 2/3 \sinh z$. For $z = it$, $f(it) = 5/9 + 4/9 \cos t - 2i/3 \sin t$. Hence

$$|f(it)|^2 = \frac{1}{81} (25 + 40 \cos t + 16 \cos^2 t + 36 \sin^2 t)$$
$$= \frac{1}{81} (61 + 20 \cos t (2 - \cos t)) \ .$$

Since $\cos t (2 - \cos t) \leq 1$, $|f(it)|^2 \leq 1$, $t \in \mathbb{R}$.

For $z = s$, $s \in \mathbb{R}$,

$$|f(s)| = \left|1 - \frac{2}{3} s + \frac{4}{9} \cdot \frac{s^2}{2!} - \frac{2}{3} \cdot \frac{s^3}{3!} + \cdots\right|$$
$$\leq 1 + \frac{2}{3} |s| + \frac{4}{9} \cdot \frac{|s|^2}{2!} + \frac{2}{3} \cdot \frac{|s|^3}{3!} + \cdots$$
$$\leq \exp|s| .$$

By a standard result in the theory of functions of exponential type, $|f(it)| \leq 1$ and $|f(s)| \leq \exp|s|$, $s, t \in R$, imply $|f(z)| \leq \exp|\mathrm{Re}\, z|$, (see e.g., Theorem 6.2.4 of [4]).

For $z = re^{i\theta}$, $|\mathrm{Re}\, z| = r|\cos\theta|$. Observe that for the set $K = [-1, 1]$, $k(K, \theta) = |\cos\theta|$. Thus the function f studied in the last paragraph belongs to F(K).

Theorem 4.26 states that there exists a Banach algebra $\mathcal{O}\!\mathcal{l}$ and an element $T \in \mathcal{O}\!\mathcal{l}$ such that $V(\mathcal{O}\!\mathcal{l}, T) = K$. Letting $U = T + 1$, f is used to compute an element of $V(\mathcal{O}\!\mathcal{l}, U^3)$.

$U^3 = (T+1)^3 = T^3 + 3T^2 + 3T + 1$. Hence, Theorem 4.26 ii implies $f_3 + 3f_2 + 3f_1 + 1 = -\frac{1}{3} \in V(\mathcal{O}\!\mathcal{l}, U^3)$. Because $U \in \mathbb{P}(\mathcal{O}\!\mathcal{l})$, it provides a counterexample to the extension of the necessity part of Theorem 4.8 to the Banach algebra case. This is summarized in

(4.29) THEOREM. <u>There exists a Banach algebra</u> $\mathcal{O}\!\mathcal{l}$ <u>and an element</u> $U \in P(\mathcal{O}\!\mathcal{l})$ <u>such that</u> $V(\mathcal{O}\!\mathcal{l}, U^3) \not\subset \Pi$.

Anderson [1] points out that a natural place to look for results on powers of hermitian elements is in the study of derivations. It is here that a counterexample to the sufficiency of an extended version of Theorem 4.8 is found.

In what remains P is an orthogonal projection in $\mathcal{B}(\mathfrak{h})$, $P \neq 0$, $P \neq 1$. Of interest is the element $\Delta_P \in \mathcal{B}(\mathcal{B}(\mathfrak{h}))$. By Proposition 3.1 Δ_P is hermitian in $\mathcal{B}(\mathcal{B}(\mathfrak{h}))$. Since P is an idempotent, the following is possible.

(4.30) LEMMA. For $A \in \mathcal{B}(\mathfrak{h})$, $(\Delta_P + 1)^n A = (2^n - 1)PA + (2 - 2^n)PAP - AP + A$, $n \in \mathbb{N}$.

PROOF. For n = 1 the relation is valid. Suppose further that it holds for some k. Then

$$\begin{aligned}(\Delta_P + 1)^{k+1} A &= (\Delta_P + 1)[(2^k - 1)PA + (2 - 2^k)PAP - AP + A] \\ &= (2^k - 1)PA + (2 - 2^k)PAP - PAP + PA \\ &\quad - (2^k - 1)PAP - (2 - 2^k)PAP + AP - AP \\ &\quad + (2^k - 1)PA + (2 - 2^k)PAP - AP + A \\ &= (2^{k+1} - 1)PA + (2 - 2^{k+1})PAP - AP + A \;. \blacksquare\end{aligned}$$

(4.31) THEOREM. $V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), (\Delta_P + 1)^n) \subset \Pi$, $n \in \mathbb{N}$ and $V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), \Delta_P^2) \not\subset \mathbb{R}$.

PROOF. To prove the first containment, application of Lemma 1.12 reduces the problem to showing that $\|\exp(-t(\Delta_P + 1)^n)\| \leq 1$, $t > 0$. From Lemma 4.30,

$$\begin{aligned}\exp(-t(\Delta_P + 1)^n)A &= \sum_{k=0}^{\infty} \frac{(-t)^k (\Delta_P + 1)^{nk}}{k!} A, \quad A \in \mathcal{B}(\mathfrak{h}) \\ &= PA \sum_{k=0}^{\infty} (-t)^k (2^{nk} - 1)/k! \\ &\quad + PAP \sum_{k=0}^{\infty} (-t)^k (2 - 2^{nk})/k! \\ &\quad + (A - AP) \sum_{k=0}^{\infty} (-t)^k/k! \; + \\ &= (e^{-2^n t} - e^{-t})PA + (2e^{-t} - e^{-2^n t})PAP \\ &\quad + e^{-t}(A - AP) \;.\end{aligned}$$

Letting $A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ be a representation of A relative to the decomposition of $\mathfrak{h} = P\mathfrak{h} \oplus (1-P)\mathfrak{h}$,

$$\exp(-t(\Delta_P + 1)^n)A = (e^{-2^n t} - e^{-t}) \begin{pmatrix} A_{11} & A_{12} \\ 0 & 0 \end{pmatrix}$$

$$+ (2e^{-t} - e^{-2^n t}) \begin{pmatrix} A_{11} & 0 \\ 0 & 0 \end{pmatrix}$$

$$- e^{-t} \begin{pmatrix} A_{11} & 0 \\ A_{21} & 0 \end{pmatrix} + e^{-t} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

$$= \begin{pmatrix} e^{-t} A_{11} & e^{-2^n t} A_{12} \\ 0 & e^{-t} A_{22} \end{pmatrix} .$$

Thus for $t > 0$, $\|\exp(-t(\Delta_P + 1)^n A\| \leq \|A\|$ (this is checked by a simple explicit calculation).

The second part is a calculation due to Anderson ([1], page 105) which is sketched here for completeness. In the notation of Chapter 3, $\Delta_P{}^2 = L_P + R_P - 2L_P R_P$. Since L_P and R_P are hermitian in $\mathfrak{B}(\mathfrak{B}(\mathfrak{h}))$, it suffices to show that $L_P R_P$ is not hermitian.

$$\exp(itL_P R_P)A = \exp(it)PAP + (A - PAP) .$$

Let $A = \begin{pmatrix} i/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$ on $\mathfrak{h} = P\mathfrak{h} \oplus (1-P)\mathfrak{h}$ and put $t = 3\pi/2$. Then

$\exp(itL_P R_P)A = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, an orthogonal projection (with norm 1).

However, $\|A\|^2 = \|A^*A\| = \frac{1}{2} + \frac{1}{2\sqrt{2}} < 1$. Thus $\|\exp(itL_PR_P)\| > 1$ for some t and L_PR_P is not hermitian. ∎

Theorem 4.31 shows that the sufficiency part of Theorem 4.8 can not be translated into the general Banach algebra setting. $(\Delta_P + 1)^2$ is not hermitian, but $V(\mathcal{B}(\mathcal{B}(\mathfrak{h})), (\Delta_P + 1)^{2n}) \subset \Pi$, n = 1, 2, ..., by Theorem 4.31.

In C^*-algebras the situation is close to that of $\mathcal{B}(\mathfrak{h})$. If $\mathfrak{A}$ is a C^*-algebra, then $\mathfrak{A}$ has a faithful *-representation as a closed self-adjoint subalgebra of $\mathcal{B}(\mathfrak{h})$ for some Hilbert space $\mathfrak{h}$ ([32], page 244). Moreover, for $T \in \mathfrak{A}$, $V(\mathfrak{A}, T) = W(t(T))^-$, where t(T) is the representation image of T in $\mathcal{B}(\mathfrak{h})$. Note that

$$t(P(\mathfrak{A})) = P(\mathfrak{h}) \cap t(\mathfrak{A}) .$$

Thus Theorem 4.8 (and the related results of sections 3, 4 and 5) can be trivially transplanted into the C^*-algebra setting.

If $\mathcal{K}(\mathfrak{h})$ is the closed two-sided ideal of compact operators in $\mathcal{B}(\mathfrak{h})$, $\mathfrak{h}$ infinite dimensional and separable, $\mathbb{C} = \mathcal{B}(\mathfrak{h})/\mathcal{K}(\mathfrak{h})$ is called the Calkin algebra [11]. It is well-known that $\mathbb{C}$ is a C^*-algebra [11]. Stampfli and Williams [18], [40] introduce the notion of an essential numerical range, $W_e(T)$, for $T \in \mathcal{B}(\mathfrak{h})$

$$W_e(T) = \bigcap_{K \in \mathcal{K}(\mathfrak{h})} W(T+K)^- .$$

It is demonstrated [18] that $W_e(T) = V(\mathbb{C}, \pi(T))$ where π is the canonical homomorphism of $\mathcal{B}(\mathfrak{h})$ into $\mathbb{C}$. The following result is a consequence of the fact that $\mathbb{C}$ is a C^*-algebra.

(4.32) THEOREM. <u>For</u> $T \in \mathcal{B}(\mathfrak{h})$, $W_e(T^n) \subset \Pi$, $n = 1, 2, \ldots$ <u>if and only if</u> $\mathrm{Im}\,T \in \mathcal{K}(\mathfrak{h})$ <u>and</u> $W_e(T) \geq 0$.

The validity of Theorem 4.8 in other settings (e.g., hermitian algebras, * normed algebras) deserves consideration. Study of these situations is hampered by the lack of numerical range mapping theorems.

7. THE FINITE DIMENSIONAL AND COMPACT CASES

It is of some interest to see that in the finite dimensional case the same results can be obtained by strictly finite dimensional techniques. How these techniques can yield a proof of Theorem 4.8 in the case the operator is compact is also shown.

The approach to a proof of Theorem 4.8 in the finite dimensional case is based on a reduction of the problem to a two dimensional case.

(4.33) LEMMA. <u>For a transformation</u> T <u>on</u> $\mathfrak{h} = \mathbb{C}^2$ <u>with representation</u>

$$T = \begin{pmatrix} 1 & a \\ 0 & d \end{pmatrix} ,$$

$d \geq 0$, $a > 0$, <u>there exists</u> $n \in N$, <u>such that</u> $W(T^n) \not\subset \Pi$.

PROOF. For $S \in \mathcal{B}(\mathfrak{h})$ put $\Lambda(S) = \inf\{\mathrm{Re}\lambda : \lambda \in W(S)\}$. By Theorem 1.9

$$\Lambda(S) = -\lim_{a \to 0+} \frac{\|1 + aS\| - 1}{a} = -\frac{1}{2} \lim_{a \to 0+} \frac{\|1 + aS\|^2 - 1}{a} . \qquad (13)$$

$\Lambda(S)$ is computed explicitly for transformations S with a representation

$$S = \begin{pmatrix} 1 & b \\ 0 & c \end{pmatrix} ,$$

$c \geq 0$, $b > 0$. Using that $\|1 + aS\|^2 = \|(1 + aS)(1 + aS^*)\|$ and that the norm of a hermitian element equals the spectral radius, the computation of the largest eigenvalue of $(1 + aS)(1 + aS^*)$ and the taking of the limit in (13) yields

$$\Lambda(S) = \tfrac{1}{2}(1 + c - [(d-1)^2 + b^2/2]^{\frac{1}{2}}) \quad .$$

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence defined successively, $a_1 = 1$, $a_2 = 2$, and $a_{n+1} = a_n^2$, $n \geq 2$.

$$T^{a_n} = \begin{pmatrix} 1 & a \prod_{k=1}^{n-1} (1+d^{a_k}) \\ 0 & d^{a_n} \end{pmatrix} \quad .$$

For the case, $0 \leq d < 1$

$$\Lambda(T^{a_n}) = \tfrac{1}{2}(1 + d^{a_n} - [(d^{a_n} - 1)^2 + \tfrac{1}{2}(a \prod_{k=1}^{n-1} (1 + d^{a_k}))^2]^{\frac{1}{2}})$$

$$\leq \tfrac{1}{2}(1 + d^{a_n} - [(d^{a_n} - 1)^2 + a^2/2]^{\frac{1}{2}}) \quad .$$

The right-hand side has limit $\frac{1}{2}(1 - (1 + a^2/2)^{\frac{1}{2}}) < 0$ as $n \to \infty$. For this case $\Lambda(T^{a_n}) < 0$ for n sufficiently large.

For $d \geq 1$, $\Lambda(T^{a_n}) \leq \frac{1}{2}(1 + d^{a_n} - [(d^{a_n} - 1)^2 + a2^{n-2}]^{\frac{1}{2}})$ and hence is negative for n sufficiently large. ■

(4.34) THEOREM. <u>Suppose</u> dim $\mathfrak{h} < \infty$. $T \in \mathbb{P}(\mathfrak{h})$ <u>if and only if</u> $T^n \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$.

PROOF. $T^n \in \mathbb{A}(\mathfrak{h})$ implies $\sigma(T) \subset S_0$. T is representable as an upper triangular matrix with real non-negative diagonal entries.

Suppose $T \neq T^*$ so that the triangular form has a non-zero entry above the diagonal. Let $S = T + 1$, $S^n \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$.

$S = (s_{ij})_{i,j=1}^{r}$, $\dim \mathfrak{h} = r$. Let (i, j) be the coordinates for an entry such that $s_{ij} \neq 0$, $j > i$, and such that for (k, l), $0 < l - k < j - i$, $s_{kl} = 0$. Such a coordinate pair exists by the supposition $T \neq T^*$.

P is the two dimensional projection on the i^{th} and j^{th} coordinate vectors. By the selection of (i, j)

$$(PSP)^n = PS^nP, \qquad n = 1, 2, \ldots \quad . \tag{14}$$

PSP has the representation

$$PSP = \begin{pmatrix} s_{ii} & s_{ij} \\ 0 & s_{jj} \end{pmatrix} \quad .$$

Both $s_{ii} > 0$ and $s_{jj} > 0$. s_{ij} can be assumed positive, as well (S can be altered by a diagonal unitary transformation so that this is the case).

Thus $P(\frac{1}{s_{ii}} S)P = \begin{pmatrix} 1 & a \\ 0 & d \end{pmatrix}$, $a = s_{ij}/s_{ii}$, $d = s_{jj}/s_{ii}$. From the observation (14), Proposition 1.3 iii and Lemma 4.33, there exists $m \in N$ such that $W(S^m) \not\subset \Pi$, hence $S^m \notin \mathbb{A}(\mathfrak{h})$, a contradiction.■

For T compact the proof is similar, but uses the theorem of Sinclair and Crabb, Theorem 1.22.

(4.35) THEOREM. <u>For</u> $T \in \mathfrak{B}(\mathfrak{h})$, T <u>compact</u>, $T \in \mathbb{P}(\mathfrak{h})$ <u>if and only if</u> $T^n \in \mathbb{A}(\mathfrak{h})$, $n = 1, 2, \ldots$.

PROOF. T is invariant on the space spanned by the eigenvectors of T. Call the closure of this space H. If T_1 is the operator induced by T on H, there is an orthonormal basis such that T_1 has an upper triangular representation (see [19], pages 16-17).

T, relative to the decomposition $\mathfrak{H} = H \oplus H^{\perp}$, has the form

$$T = \begin{pmatrix} T_1 & * \\ 0 & Q \end{pmatrix} ,$$

where Q is quasi-nilpotent and compact. Since $\{0\} = \sigma(Q)$ and $W(Q^n) \subset W(T^n) \subset \Pi$, $n = 1,2,\ldots$, $0 \in \partial W(Q^n)$, $n = 1,2,\ldots$. Theorem 1.22 implies $8r(Q) \geq \|Q\|$. Hence $Q = 0$.

Consequently $H = \mathfrak{H}$ and T has an upper triangular representation. The argument used to prove Theorem 4.34 yields that T is diagonal. Hence $T = T^*$. ∎

The techniques of this section rely heavily on an upper diagonal form. Accordingly there seems to be no way of extending these techniques to obtain an operator theoretic proof of Theorem 4.8.

8. THE NUMERICAL RANGE OF A PRODUCT

Independently, Loewy [26] and Stampfli [39] study the numerical range of the product of two positive operators. Loewy's results, obtained by matrix techniques, are shown to be consequences of the operator theoretic calculations of Stampfli. Some of the operator techniques are used to obtain more general results.

The first lemma is due to Stampfli [39].

(4.36) LEMMA. Suppose A, B, 1-A, 1-B $\in \mathbb{P}(\mathfrak{H})$, then $W(AB) \subset \Pi - \frac{1}{8}$.

PROOF. $Ax = \alpha x + \beta y$, $(x, y) = 0$, $\alpha = (Ax, x)$, $\|x\| = \|y\| = 1$, $\beta = (Ax, y)$. By the generalized Schwarz inequality for positive operators

$$|\beta|^2 \leqslant \alpha\lambda, \qquad \lambda = (Ay, y) \quad .$$

Because $1 - A \in \mathbb{P}(\mathfrak{h})$,

$$|((1-A)x, y)|^2 \leqslant (1-\alpha)(1-\lambda) \quad .$$

For $\alpha \neq 0, 1-\lambda \leqslant 1 - |\beta|^2/\alpha$. Hence

$$|\beta|^2 \leqslant \alpha(1-\alpha) \quad . \tag{15}$$

Note that (15) is valid for $\alpha = 0$.

The same process is applied to B. $Bx = \gamma x + \delta z$, $\gamma = (Bx, x)$, $\|z\| = 1$, $(x, z) = 0$, $\delta = (Bx, z)$. The result analogous to (15) is

$$|\delta|^2 \leqslant \gamma(1-\gamma) \quad . \tag{16}$$

Then

$$\begin{aligned}(ABx, x) &= (A(\gamma x + \delta z), x) \\ &= \alpha\gamma + \delta\bar{\beta}(z, y)\end{aligned}$$

and using (15) and (16)

$$\mathrm{Re}(ABx, x) \geqslant \alpha\gamma - [\alpha(1-\alpha)\gamma(1-\gamma)]^{\frac{1}{2}} \ . \tag{17}$$

Minimizing the right-hand side of (17), $\alpha, \gamma \in [0, 1]$, yields

$$\mathrm{Re}(ABx, x) \geqslant -\frac{1}{8} \ . \blacksquare$$

Stampfli [39] also exhibits an example to show that the value $-\frac{1}{8}$ is attained. For the example, put

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1/4 & \sqrt{3}/4 \\ \sqrt{3}/4 & 3/4 \end{pmatrix} . \tag{18}$$

The computation in the lemma is generalized to obtain the Loewy's estimate ([26], Theorem 3, page 59).

(4.37) THEOREM. <u>For</u> $A, B \in \mathbb{P}(\mathfrak{h})$ <u>suppose</u> $W(A) \subset [k,1]$ <u>and</u> $W(B) \subset [0,1]$, <u>then</u> $W(AB) \subset \Pi - \frac{1}{8}\frac{(1-k)^2}{1+k}$.

PROOF. Put $A' = \frac{A-k}{1-k}$. A' and B satisfy the hypothesis of the lemma.

From (17)

$$\mathrm{Re}(A'Bx, x) \geq \alpha\gamma - [\alpha(1-\alpha)\gamma(1-\gamma)]^{\frac{1}{2}} .$$

Hence

$$\mathrm{Re}(ABx, x) \geq (1-k)[\alpha\gamma - [\alpha(1-\alpha)\gamma(1-\gamma)]^{\frac{1}{2}}] + k\gamma . \tag{19}$$

Minimization of the right-hand side of (19), $\alpha, \gamma \in [0,1]$ gives $\alpha = \frac{1-\gamma^{\frac{1}{2}}}{2}$ and $\gamma^{\frac{1}{2}} = \frac{1}{2}\frac{1-k}{1+k}$. Hence

$$\mathrm{Re}(ABx, x) \geq -\frac{1}{8}\frac{(1-k)^2}{1+k} . \blacksquare$$

The hermiticity condition is removed and another class of operators is examined.

(4.38) THEOREM. <u>Let</u> $S = \{z : \mathrm{Re}\, z \in [0,1]$ <u>and</u> $\mathrm{Im}\, z \in [0,1]\}$. <u>For</u>

$A, B \in \mathcal{B}(\mathfrak{h})$ such that $W(A) \subset S$ and $W(B) \subset S$, then $W(AB) \subset \Pi + \alpha$, $\alpha = \frac{3}{4}(1 - \frac{\sqrt{3}}{2}) - \frac{1}{4}(1 + (1 + \frac{\sqrt{3}}{2})^{\frac{1}{2}})^2$.

PROOF. As in the proof of Lemma 4.36, put $Ax = \alpha x + \beta y$, $\|y\| = \|x\| = 1$, $(x, y) = 0$; $A^*x = \lambda x + \mu u$, $\lambda = \overline{\alpha}$, $\|u\| = 1$, $(u, x) = 0$; $Bx = \gamma x + \delta v$, $\gamma \in S$, $\|v\| = 1$, $(x, v) = 0$.

For convenience $\alpha = a + ib$, $\gamma = c + id$. As in the proof of the lemma

$$\begin{aligned} |(\mathrm{Re}Ax, y)|^2 &\leq a(1-a) \\ |(\mathrm{Re}A^*x, u)|^2 &\leq a(1-a) \\ |(\mathrm{Im}Ax, y)|^2 &\leq b(1-b) \\ |(\mathrm{Im}A^*x, u)|^2 &\leq b(1-b) \quad . \end{aligned} \tag{20}$$

Combining the second and fourth of the inequalities (20)

$$|(A^*x, u)| = |\mu| \leq (a(1-a))^{\frac{1}{2}} + (b(1-b))^{\frac{1}{2}} \quad .$$

The situation for B is the same

$$|(Bx, v)| = |\delta| \leq (c(1-c))^{\frac{1}{2}} + (d(1-d))^{\frac{1}{2}} \quad .$$

Then

$$(ABx, x) = \alpha\gamma + \delta\overline{\mu}(v, u) \quad .$$

$$\begin{aligned} \mathrm{Re}(ABx, x) \geq\ & ac - bd \\ & - [(a(1-a))^{\frac{1}{2}} + (b(1-b))^{\frac{1}{2}}][(c(1-c))^{\frac{1}{2}} + (d(1-d))^{\frac{1}{2}}] \quad . \end{aligned} \tag{21}$$

(21) is minimized with the help of the symmetry of a with c and b with

d. The minimum, calculated by standard procedures, is

$$\alpha = \frac{3}{4}\left(1 - \frac{\sqrt{3}}{2}\right) - \frac{1}{4}\left(1 + \left(\frac{\sqrt{3}}{2} + 1\right)^{\frac{1}{2}}\right)^2 . \blacksquare$$

A crude estimate of the minimum value of (21) is also possible.

$$ac - bd - [(a(1-a))^{\frac{1}{2}} + (c(1-c))^{\frac{1}{2}}][(b(1-b))^{\frac{1}{2}} + (d(1-d))^{\frac{1}{2}}]$$

$$= (ac - [a(1-a)c(1-c)]^{\frac{1}{2}}) + (-bd - [b(1-b)d(1-d)]^{\frac{1}{2}})$$

$$+(-[a(1-a)b(1-b)]^{\frac{1}{2}}) + (-[b(1-b)c(1-c)]^{\frac{1}{2}})$$

$$\geq -\frac{1}{8} - 1 - \frac{1}{4} - \frac{1}{4} = -\frac{13}{8} .$$

To within 0.01, $\alpha \cong -1.29$.

Using the operators described in (18), put $A' = i + A$, $B' = i + B$ and note that $W(A') \subset S$, $W(B') \subset S$, and $A'B' = AB - 1 + iA + iB$. Hence

$$\inf\{\mathrm{Re}\lambda : \lambda \in W(A'B')\} = -1 + \inf\{\mathrm{Re}\lambda : \lambda \in W(AB)\} = -9/8.$$

This is the closest computed value to the lower bound α .

Computations and estimates used to prove Theorem 4.38 can also be used to obtain a lower bound for

$$\inf\{\mathrm{Im}\lambda : \lambda \in W(AB)\}, \qquad W(A), W(B) \subset S .$$

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