TWO PIGEON HOLE PRINCIPLES AND UNIONS OF CONVEXLY DISJOINT SETS

Thesis by

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In Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

California Institute of Technology

Pasadena, California

1973

(Submitted October 25, 1973)

Dedication

Chapter I: To Drs. Hare and Kenelly, University of Clemson, for not showing me Drs. Motzkin and Grünbaum's original paper and conjecture on the Helly number for special unions of convex sets (and thereby having me formulate for myself the problem and the best probable method to solve the conjecture).

Chapter II: To Drs. Motzkin and Grünbaum, for making this conjecture in the first place (which eventually led to this dissertation).

Gratitude

I wish to express my deepest gratitude for all aid, friendship and humor that Dr. Bohnenblust has extended to me during my years as a Graduate student at Caltech.

ABSTRACT

In Euclidean space R^n, a set is convex if the set contains every straight line segment whose endpoints are in the given set. Suppose that a set C consisted of convex sets in R^n, and that for any choice of $n+1$ sets in C the $n+1$ sets had a point in common. Then Helly's theorem states that any finite number of sets in C have a common point. $(n+1)$ is known as the Helly number of convex sets in R^n. One may ask if unions of k convex sets have a similar Helly's number. This paper puts convexity in the abstract, and imposes conditions on a set $\mathcal{A}$ (consisting of sets that are unions of k convex sets) such that $\mathcal{A}$ can be shown to have a Helly's number. This paper also considers an abstraction of the notion of "polygonally connected sets" from an abstract convexitist's point of view.

In showing that certain sets of unions of convex sets have an a Helly's number, a special case of a generalized pigeon hole principle is used. This paper also proves two generalized pigeon hole principles, and in many cases gives the best possible results. Both generalized pigeon hole principles make the following assumptions on a matrix A:

(1) there are n rows

(2) each row has at most ℓ zero's

(3) every submatrix of A, that does not have any zero entries, has at most k distinct (not identical) rows

(4) that numbers h and/or t are given.

One generalized pigeon hole principle states there exists a function $\chi a(h, k, \ell)$ such that if $n \geq \chi a(h, k, \ell)$, then there must exist some $h+1$

columns such that along every row of the matrix those $h+1$ columns have the same entry with possibly ℓ exceptions. The other generalized pigeon hole principle states that there exists a function $\chi e(h, k, \ell, t)$ such that if $n \geq \chi e(h, k, \ell, t)$, then there must exist some $sh+t$ $(s > 0)$ columns that can be partitioned into s sets of columns such that it is possible to make suitable changes to the zero entries in each of these $sh+t$ columns in order to make those s sets of columns into s sets of equal columns. It is also shown that for certain values of h, k, ℓ, and t that $\chi a(h, k, \ell) = hk+1$ and $\chi e(h, k, \ell, t) = kh+t+(k-1)\ell$. It is also shown that there exists examples such that $\chi e(h, k, \ell, t) > hk+t+(k-1)\ell$.

Introduction

In any Euclidean space, a set is convex if the set contains all line segments whose endpoints are in the set. In an n-dimensional Euclidean space, the intersection of all the sets in a finite collection of convex sets is not empty if and only if every intersection of n+1 (or fewer) convex sets in the collection of convex sets is not empty (Helly's theorem). If (n+1) had been replaced with any smaller number, the previous sentence would not have been a true statement. (n+1) is known as the Helly's number for convex sets in an n-dimensional Euclidean space. The question arises asking if collections of sets that are unions of two (or three, or four) convex sets have a Helly's number in an n-dimensional Euclidean space. Unless those collections of unions of k ($1 < k < \infty$) convex sets have some additional conditions imposed on them, the answer in general is no. Drs. Motzkin and Grünbaum, were the first to impose conditions on collections of unions of two convex sets that implied the collection had a Helly's number. Drs. Grünbaum and Motzkin proved their results in an abstract setting, and showed that their results were the best possible given only their given conditions. They imposed similar conditions on collections of unions of three (or four, or five, etc.) and conjectured that they also had a Helly number, and furthermore they conjectured what the best possible results would be. Their conjecture was obvious for one-dimensional Euclidean spaces. Dr. Larman showed that their conjectures were true for unions of three convex sets in Euclidean space. This paper, among other things, proves in the abstract both of Grünbaum and Motzkin's conjectures, except for (ironically) the special case of the best possible results when the Euclidean analog of our abstract space is one-dimensional. This paper also relaxes the conditions imposed by Drs. Grünbaum and Motzkin, and proves results that are more general than those originally conjectured.

This paper is divided into two chapters. The first chapter proves an extension of the pigeon hole principle. A special case of the pigeon hole principle (Section 4, Chapter I) will be used to prove an extension of Helly's theorem (see previous paragraph). The second chapter will extend the notation of polygonally connected sets. The second chapter will also prove the indicated extension of Helly's theorem, and give some indications of the differences between the conditions I imposed and the conditions imposed in Drs. Motzkin and Grünbaum's conjectures. In this paper, the extended Helly's theorem looks like a contrived application of an extended pigeon hole principle developed in Chapter I, but the extended Helly's theorem was conjectured first. I will admit, however, that I decided on a method of proof before trying to prove the extended Helly's theorem. [I was working under the assumption that a reasonable line of reasoning would either lead to a proof or to a counter example.] My method of proof tacitly led to proving or assuming that a special case (Section 4, Chapter I) of the extended pigeon hole principle was true. Eventually I proved the extended Helly's theorem and was urged to expand and if possible prove the extended pigeon hole principles that I had used to prove the extended Helly's theorem. I succeeded, and obtained better results than I had at first expected to be true.

ERRATA

Through both Chapter I and II, my typists read my κ (the Greek letter Kappa) as a script K. Hence $K(A|X)$ is an integral valued function and not a set of sets.

In Theorem 23 $P_i = \{Hi, (Z \sim Hi), \phi\}$.

The leading paragraph of explanation in Section 3 of Chapter I should have been deleted. It was a leftover from a constructive grind-out proof using Algorithm #1 and Algorithm #2 instead of a simplified inductive proof.

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CHAPTER I: A Generalization of the Pigeon Hole Principle

Introduction:

The classical pigeon hole principle can be stated in many forms. Two forms of the pigeon hole principle are of particular importance to us in the sequel, they are:

Form 1: Assume that n letters have been delivered to at most k addresses. If $n \geq hk + 1$ then one addressee has at least $(h + 1)$ letters.

Form 2: Assume that n letters have been placed in at most k mailboxes, and subsequently up to h letters have been removed from each of those mailboxes. If $n \geq hk + t$ $(t > 0)$, then at least t letters are remaining in the mailboxes.

A generalization of this principle will be proven in this chapter, and a special case of this generalized pigeon hole principle will be used in Chapter II to help prove a conjecture made by Dr. Grünbaum and Dr. Motzkin. This generalization of the pigeon hole principle will be stated in the second section of Chapter I. This generalization will have two forms that will correspond to Form I, and Form 2, respectively. The rest of this introduction will reformulate the classical pigeon hole principle in several ways. Eventually, the reformulations of the pigeon hole principle will lead to the generalized forms of the pigeon hole principle that will be presented in this paper.

If the columns of a matrix are identified with the letters to be delivered, and if the entries in each row of a column correspond to part of the address that column's letter was sent to (i.e., every mailbox would have its own special r-digit zip code, and listing the sequence of digits uniquely identifies the mailbox the letter was sent to), then Form 1 and Form 2 can be rewritten in the following two forms:

Both forms assume a matrix with n columns (letters), which contains at most k unequal columns (different addresses).

Form 3: If $n \geq hk + 1$, then at least $(h + 1)$ columns of the matrix are equal to each other.

Form 4: If $n \geq hk + t$ $(t > 0)$, then for some positive integer s, there exists $(sh + t)$ columns of the matrix for which the submatrix consisting of those $(sh + t)$ columns contains at most s unequal columns.

The next reformulation is based on the fact that a matrix has at most k unequal columns iff every one of its submatrices has at most k unequal columns. The common assumption of Form 3 and Form 4 can thus be changed to read:

Both forms assume a given matrix of n columns, for which every submatrix of the matrix contains at most k unequal columns.

The final reformulation has no change in content, and again affects only the common assumptions.

Common assumptions: A matrix is assumed

(1) to be zero free (i.e., no entry equals zero)

(2) to have n columns

(3) to have no zero-free submatrix with more than k unequal columns.

Conclusions: The same as Form 3 and Form 4.

The mention of zero-entries points to the generalization at the pigeon hole principle. The given matrix will be allowed to have exceptional entries, which for convenience may as well be zero. Part (3) of the common assumptions will then have more (at least different) significance.

The Generalized Pigeon Hole Principle

In addition to the three positive integer valued parameters n, k, and t, used to describe the classical pigeon hole principle, a further (non-negative) integer valued parameter ℓ is introduced. The classical pigeon hole principle will correspond to the value $\ell = 0$.

The generalized principle will consist of two forms, which are based on the same assumptions.

Common assumptions: A matrix is assumed with the three conditions,

(1) Each row has at most ℓ zero entries

(2) The matrix has n columns

(3) Each zero free submatrix contains at most k different columns.

To facilitate an explanation of the conclusions, two definitions are introduced.

Let $\vec{\nu}_1, \vec{\nu}_2, \ldots, \vec{\nu}_p$ be p column vectors, with the components $\nu_i(j)$, $j = 1, \ldots, q$.

Definition 1: The vectors $\vec{\nu}_1, \ldots, \vec{\nu}_p$ are 'essentially' equal if for each component j, the values $\nu_i(j)$ are equal to a common value, or zero; i.e., for all j, and all i_1, i_2,

$$\nu_{i_1}(j)\, \nu_{i_2}(j)\, (\nu_{i_1}(j) - \nu_{i_2}(j)) = 0.$$

Definition 2: The vectors $\vec{\nu}_1, \ldots, \vec{\nu}_p$ are (ℓ)-essentially equal if there exists a vector $\vec{\nu}$ whose components are all non-zero, and such that for each component j,

$$\vec{\nu}_i(j) = \nu(j) \text{ except for at most } \ell \text{ values of } i.$$

<u>Form A:</u> Under the common assumptions, if $n \geq kh + 1$, and if h is sufficiently large (compared to some function of k and ℓ), then the matrix must contain (h + 1) columns which are (ℓ)-essentially equal (Def. 2).

<u>Form B:</u> Under the common assumptions, if $n \geq kh + t + (k - 1)\ell$ and if h is sufficiently large (compared to some function of k and ℓ) then for some positive integer s, the matrix contains some (sh + t) columns that can be partitioned into s-sets of 'essentially' equal columns.

A set of columns of which any two columns are'essentially equal' can be considered either as (1) a submatrix (with no rows deleted from the original matrix) of which no zero-free sub-submatrix contains two unequal columns or (2) a submatrix (with no rows deleted from the original matrix) at which it is possible to change all the zero-entries (individually) of the submatrix so that the columns all become equal to each other.

A set of (ℓ)-essentially equal columns can be considered as a submatrix (with no rows deleted from the original matrix) such that by changing at most ℓ-entries in every row (the entries need not be zero) of the submatrix, the columns all become equal to each other.

In both Form A and Form B, h must be greater than some function of k and ℓ. In Chapter I, such functions will be constructed. For Form B, a counter example will show that form B is not true for all h. However, for both form A and form B, there exists finite functions $g_A(h, k, \ell)$, and $g_B(h, k, \ell, t)$ such that if $n \geq g_A(h, k, \ell)$ (or if $n \geq g_B(h, k, \ell, t)$) the conclusions of form A (as form B) remain true for any choice of h (independent of k and ℓ).

Section 0. Notation

Elements of a set will be represented by small italicized Roman letters. The letters h through t (inclusive) will always stand for integers. Functions, whose range is the set of integers, will be denoted by Greek letters.

Sets (that do not contain sets) will be represented by capital Roman letters. The letters I and J will be reserved for sets of integers. Functions, whose range is a set of sets will be represented by combinations of Roman letters, the first of which will be capitalized.

Sets of sets will be represented by italicized capital Roman letters. Functions whose range is a set of sets of sets will be represented by a combination of script (or italicized) Roman letters, the first letter of which will be capitalized.

Sets of sets of sets will be represented by double capital Roman letters. Functions whose range is a set of sets of sets of sets will be represented by a combination of double Roman letters, the first two of which will be capitalized.

Normal set notation will be used. $A \subset B$ (or $\mathit{A} \subset \mathit{B}$, or $AA \subset BB$) means that A(or *A*, or AA) is a subset of B(or *B*, or BB). The intersection of two sets A & B (or *A* & *B*, or AA & BB) will be written as $A \cap B$ (or $\mathit{A} \cap \mathit{B}$, or $AA \cap BB$). The union of two sets A & B, etc., will be written as $A \cup B$. Finally the set containing all the points of a set A (or *A*, or AA) that are not contained in the set B(or *B*, or BB) will be written as $A \sim B$ (or $\mathit{A} \sim \mathit{B}$, or $AA \sim BB$); i.e., $A \sim B$ is the set such that $(A \sim B) \subset A \subset B \cup (A \sim B)$ and $B \cap (A \sim B) = \phi$, (and ϕ represents the null set).

Section 1

Basic Partition Theory

Definition 1: The cardinal number of a set A will be denoted by $|A|$. The cardinal number of $A \cap X$ will be denoted by $|A|_X = |A \cap X|$.

Definition 2: A collection P of sets is said to be a partion if:

(1) $\phi \in P$

(2) $\forall$ A, B $\in P$, either $A \cap B = \phi$ or $A = B$.

Definition 3: The support of any nonempty set A of sets is defined to be

$$\mathrm{Supp}(\mathcal{A}) = \bigcup_{A \in \mathcal{A}} A$$

The support of the null set is the null set (i.e., $\mathrm{Supp}(\phi) = \phi$). Also the following notation will be used: $\mathrm{Supp}(\mathcal{A}|X) = \mathrm{Supp}(\mathcal{A}) \cap X$.

Definition 4: (a) A partion P is said to partition X if $X \subset \mathrm{Supp}(P)$.
(b) A partion P is said to be an incomplete partition of X if $\mathrm{Supp}(P) \subset X$.

Definition 5: The residue, $\mathrm{Res}(P|X)$, of a set P of sets with respect to a set X is the set $X \sim \mathrm{Supp}(P)$, i.e., $\mathrm{Res}(P|X) = X \sim \mathrm{Supp}(P) = X \sim \mathrm{Supp}(P|X)$.

Definition 6: If P is a partion, and if $x \in \mathrm{Supp}(P)$, then $\mathrm{Mat}(x; P)$ is the set such that $x \in \mathrm{Mat}(x;P) \in P$. If P is a partion, and if $y \notin \mathrm{Supp}(P)$, then $\mathrm{Mat}(y;P) = \phi$.

Note: By definition 2, Mat(z; P) is both uniquely defined and a member of the set P.

Several incomplete partions of a set X will have to be considered simultaneously. It is convenient to consider a collection AA of incomplete partitions of a set X as a matrix. The columns correspond to the elements of X. Each row corresponds to an incomplete partition in AA. The entry, in the column corresponding to an element $x \in X$, and in the row corresponding to a partion $P \in AA$, will be Mat(x; P). Note that two elements of a row are equal iff the corresponding elements of X both belong either to the same set P of the partion P of AA, or to the set Res(P |X). In the introduction of Chapter I, zero was used in the matrix instead of the null set since introducing the notion of a matrix whose entries were sets would have unnecessarily complicated the introduction.

If P is a partion,then P induces an incomplete partition of X for any set X. This incomplete partition is obtained by intersecting each set of P with the set X. In particular, if $X \subset \text{Supp}(P)$, then the induced incomplete partition of X partitions X. In any case, the induced incomplete partition will be written as $P \wedge [X]$.

Definition 7: Let AA be a collection of partions. We define the intersection of all the partions of AA as

$$\bigwedge_{P \in AA} P = \{R \mid R = \bigcap_{P \in AA} \text{Mat}(x; P),\ x \in \bigcup_{P \in AA} \text{Supp}(P)\} \cup \{\phi\}$$

Note to the reader: If you are following the matrix concept

$$R \in \bigwedge_{P \in AA} P$$

if, for the matrix induced by AA and for

$$X = \bigcup_{P \in AA} \mathrm{Supp}(P)$$

R is the intersection of all the sets in a column of the induced matrix.

Theorem 1: The intersection of a collection AA of partions is a partion.

Proof of theorem 1:

(1) $\phi \in \bigwedge_{P \in AA} P$, by definition

(2) Suppose $R_1 \in \bigwedge_{P \in AA} P$, and $R_2 \in \bigwedge_{P \in AA} P$, and $R_1 \cap R_2 \neq \phi$.

Let $x \in R_1 \cap R_2$, then $\forall\ P \in AA$, $R_1 \subset \mathrm{Mat}(x; P)$ and $R_2 \subset \mathrm{Mat}(x; P)$. (by the uniqueness of the Mat(x; P) function) and furthermore

$$R_1 = \bigcap_{P \in AA} \mathrm{Mat}(x; P) = R_2 \qquad \text{Q.E.D.}$$

Theorem 2: For any collection AA of partions, and for any set X:

(1) $\mathrm{Supp}\,(\bigwedge_{P \in AA} P) = \bigcap_{P \in AA} \mathrm{Supp}(P)$; $\mathrm{Supp}(\bigwedge_{P \in AA} P \,|X) = \bigcap_{P \in AA} \mathrm{Supp}(P\,|X)$.

(2) $$\operatorname{Res}\left(\bigwedge_{P \in AA} P \,\middle|\, X\right) = \bigcup_{P \in AA} \operatorname{Res}(P \,|\, X)$$

Proof of theorem 2: Suppose

$$x \in \operatorname{Supp}\left(\bigwedge_{P \in AA} P\right), \quad \text{then} \quad \exists\, R \in \bigwedge_{P \in AA} P$$

such that $x \in R$. Thus

$$x \in R = \bigcap_{P \in AA} \operatorname{Mat}(x; P) \subset \bigcap_{P \in AA} \operatorname{Supp}(P) \Rightarrow x \in \operatorname{Supp}(P) \;\forall\, P \in AA.$$

Suppose

$x \in \bigcap_{P \in AA} \operatorname{Supp}(P)$, then $x \in \bigcap_{P \in AA} \operatorname{Mat}(x; P) \in \bigwedge_{P \in AA} P$. $\therefore \operatorname{Supp}(\bigcap_{P \in AA} P) = \bigcap_{P \in AA} \operatorname{Supp}(P)$.

Both

$$\operatorname{Supp}\left(\bigwedge_{P \in AA} P \,\middle|\, X\right) = \bigcap_{P \in AA} \operatorname{Supp}(P \,|\, X), \text{ and}$$

$$\operatorname{Res}\left(\bigwedge_{P \in AA} P \,\middle|\, X\right) = \bigcup_{P \in AA} \operatorname{Res}(P \,|\, X)$$

follow from DeMorgan's Laws.

Definition 8: If P is a partion, then $a \simeq b (\text{mod } P)$ if there exists a $P \in P$ such that both a and b are elements of P. Note, if $y \notin \operatorname{Supp}(P)$, then $y \not\simeq y \;(\text{mod } P)$.

Theorem 3: $\simeq$ (mod P) is an equivalence relation over the support of P for any partion P.

Proof of theorem 3: If a, b $\in$ Supp(P), then a $\simeq$ b (mod P) iff Mat(a; P) = Mat(b; P). Q.E.D.

Theorem 4: For any collection AA of partions,

$$a \simeq b \ (\mathrm{mod} \bigwedge_{P \in AA} P) \Leftrightarrow \forall \ P \in AA, \ a \simeq b \ (\mathrm{Mod}\ P).$$

Proof is obvious.

Note to reader: In the matrix notation, two columns (neither of which has the null set as an entry) are equivalent iff for every row of the matrix the two columns have the same entry (this notion of equivalence should be of no surprise to anyone).

Definition 9: For any collection of sets P, and for any set Y, we define $\langle P \rangle_Y$ as $\langle P \rangle_Y = |\mathrm{Res}(P\,|\,Y)|$.

Theorem 5: For any pair of partions P and Q, and for any set Y

$$\langle P \wedge Q \rangle_Y + |Y \sim (\mathrm{Supp}(P) \cup \mathrm{Supp}(Q))| = \langle P \rangle_Y + \langle Q \rangle_Y.$$

Proof of theorem 5:

(*) $\mathrm{Res}(P \wedge Q\,|\,Y) = \mathrm{Res}(P\,|\,Y) \cup \mathrm{Res}(Q\,|\,Y)$ by theorem 2

$$|\mathrm{Res}(P \wedge Q\,|Y)| = |\mathrm{Res}(P\,|Y) \cup \mathrm{Res}(Q\,|Y)| =$$

$$= |\mathrm{Res}(P\,|Y)| + |\mathrm{Res}(Q\,|Y)| - |\mathrm{Res}(P\,|Y)| \cap |\mathrm{Res}(Q\,|Y)|,\ \text{i.e.,}$$

$$\langle P \wedge Q \rangle_Y + |(Y \sim \mathrm{Supp}(P)) \cap (Y \sim \mathrm{Supp}(Q))| = \langle P \rangle_Y + \langle Q \rangle_Y,\ \text{i.e.,}$$

$$\langle P \wedge Q \rangle_Y + |Y \sim (\mathrm{Supp}(P) \cup \mathrm{Supp}(Q))| = \langle P \rangle_Y + \langle Q \rangle_Y.$$

Corollary to theorem 5: For any pair of partions P and Q, and for any set Y,

$$\langle P \wedge Q \rangle_Y \leq \langle P \rangle_Y + \langle Q \rangle_Y.$$

Corollary to theorem 2: For any pair of partions P and Q, and any set Y

$$\langle P \wedge Q \rangle_Y \geq \max\ (\langle P \rangle_Y, \langle Q \rangle_Y)$$ (see *previous page).

Definition 10: If Y is any set, then $[Y] = \{\phi\} \cup \{Y\}$. Note that $[Y]$ is a partion, and Supp $([Y]) = Y$. Also note that $[X] \wedge [Y] = [X \cap Y]$. Further note that if P is an incomplete partion of Y, then $P = P \wedge [Y]$ - i.e., $[Y]$ is a unitary incomplete partition of Y.

Definition 11: For any partion P, and for any set Y.

$$||P||_Y = |(P \wedge [Y]) \sim \{\phi\}|.$$

Note, that when working with a collection of incomplete partions of a set Z, occasionally $||P||_Z$ will be written as $||P||$. In any case $||P||$ will always be $||P|| = ||P||_{\mathrm{Supp}(P)}$: Definition 11 will also be applied to any set F of sets if $R \cup \{\phi\}$ is a partion.

Theorem 6: If $|Y| < \infty$, and P is any partion, then

$$||P||_Y = 0 \Leftrightarrow \langle P \rangle_Y = |Y|.$$

Proof of theorem 6: If $||P||_Y = 0$, then $P \wedge [Y] = \{\phi\}$, or $\mathrm{Supp}(P) \cap Y = \phi$. $\therefore$ $\mathrm{Res}(P|Y) = Y \Rightarrow \langle P \rangle_Y = |Y|$.
If $\langle P \rangle_Y = |Y|$, then $|\mathrm{Res}(P|Y)| = |Y| = |\mathrm{Res}(P|Y)| + |Y \cap \mathrm{Supp}(P)|$
$\therefore$ $|Y \cap \mathrm{Supp}(P)| = 0 \Rightarrow Y \cap \mathrm{Supp}(P) = \phi \Rightarrow P \wedge [Y] = \{\phi\} \Rightarrow ||P||_Y = 0.$

Theorem 7: For any pair of partions P and Q, and for any set Y, the following must be true:

(1) $$(P \wedge Q) \wedge [Y] = \bigcup_{P \in P} (Q \wedge [P \cap Y]) = \bigcup_{Q \in Q} (P \wedge [Q \cap Y]).$$

(2) $$||P \wedge Q||_Y = \sum_{P \in P} ||Q||_{P \cap Y} = \sum_{Q \in Q} ||P||_{Q \cap Y}$$

(In (1) I'm treating partions as if they were sets, and they are sets.)

Proof of theorem 7: Suppose $R \in (P \wedge Q) \wedge [Y]$, then there exists both a $P \in P$, and a $Q \in Q$ such that $R = P \cap Q \cap Y \in P \wedge [Q \cap Y] \Rightarrow$
$$(P \wedge Q) \wedge [Y] \subset \bigcup_{Q \in Q} (P \wedge [Q \cap Y]).$$

Suppose $R \in \bigcup_{Q \in Q} (P \wedge [Q \cap Y])$, then there exist both a $P \in P$, and a $Q \in Q$ such that $R = P \cap Q \cap Y \in (P \wedge Q) \wedge [Y]$ - $\therefore$ $P \wedge Q \wedge [Y] = \bigcup_{Q \in Q} (P \wedge [Q \cap Y])$. Similarly $P \wedge Q \wedge [Y] = \bigcup_{P \in P} (Q \wedge [P \cap Y])$ and (1) follows.

Since the members of any partion are pairwise disjoint, it follows that

$$||P\wedge Q||_Y = \sum_{Q \in Q} ||P||_{Q\cap Y} = \sum_{P \in P} ||Q||_{P\cap Y}.$$

Definition 12: For any pair of partions P and Q, and for any set Y, $P \overset{Y}{=} Q$ if $P\wedge[Y] = Q\wedge[Y]$.

Obviously $\overset{Y}{=}$ is an equivalence relation for partions.

Theorem 8: If P and Q both partition the set Y, then either

(1) $||P\wedge Q||_Y > \max(||P||_Y, ||Q||_Y)$ or

(2) $P \overset{Y}{=} P\wedge Q$ or

(3) $Q \overset{Y}{=} P\wedge Q$ must be true.

Proof of theorem 8: By the corollary to theorem 2 (after the corollary to theorem 5), $||P\cap Q||_Y \geq \max(||P||_Y, ||Q||_Y)$.

$\therefore$ Either (1) is true or w.l.o.g. $||P\wedge Q||_Y = ||P||_Y$.

But

$$||P\wedge Q||_Y = \sum_{P \in P} ||Q||_{P\cap Y} = \sum_{\substack{P \in P \\ P\cap Y \neq \phi}} ||Q||_{P\cap Y} \geq \sum_{\substack{P \in P \\ P\cap Y \neq \phi}} 1 = ||P||_Y$$

(the next to last step is true since $\mathrm{Supp}(Q) \cap P \cap Y = P\cap Y \neq \phi$, implies that $||Q||_{P\cap Y} \geq 1$ by theorem 6).

$$\therefore\ ||Q||_{P\cap Y} = 1,\ \forall\ P \in P \text{ such that } P\cap Y \neq \phi.$$

$\Rightarrow [P\cap Y]\wedge Q = [P\wedge Y]$ (since $\phi \subset \mathrm{Res}(Q\,|P\cap Y) \subset \mathrm{Res}(P\,|Y) = \phi$).

$$\Rightarrow P\wedge Q\wedge Y = Q\wedge(P\wedge[Y]) = \bigcup_{\substack{P \in P \\ P\cap Y \neq \phi}} (Q\wedge[P\cap Y]) = \bigcup_{\substack{P \in P \\ P\cap Y \neq \phi}} ([P\cap Y]) = P\wedge[Y].$$

$$\therefore P \overset{Y}{\equiv} P \wedge Q. \quad \text{Q.E.D.}$$

Theorem 9: Given

(1) $\mathrm{Supp}(P|Y) = \mathrm{Supp}(Q|Y)$ for partions P and Q.

(2) $\forall$ a, b $\in \mathrm{Supp}(P|Y)$, $a \simeq b \pmod{P} \leftrightarrow a \simeq b \pmod{Q}$.

Conclusion: $P \overset{Y}{\equiv} Q$.

Proof of theorem 9: Suppose that $\mathrm{P} \in P \wedge [Y]$, then all the elements of P are equivalent (mod P) to each other (theorem 3). By hypothesis all the elements of P are equivalent to each other (mod Q). Therefore there must exist a $\mathrm{Q} \in Q \wedge [Y]$ such that $\mathrm{P} \subset \mathrm{Q}$. Similarly there exists a $\mathrm{P}' \in P \wedge [Y]$ such that $\mathrm{P} \subset \mathrm{Q} \subset \mathrm{P}'$. If $\mathrm{P} \neq \phi$, then $\mathrm{P} \cap \mathrm{P}' = \mathrm{P} \neq \phi \Rightarrow \mathrm{P} = \mathrm{P}' \Rightarrow \mathrm{P} = \mathrm{Q}$. $\therefore \mathrm{P} \in Q \wedge [Y]$. Thus $P \wedge [Y] \subset Q \wedge [Y]$ and similarly $Q \wedge [Y] \subset P \wedge [Y] \therefore Q \wedge [Y] = P \subset [Y]$.

Note that if $Y \supset \mathrm{Supp}(P) \cup \mathrm{Supp}(Q)$ in theorem 9, the conclusion is $P = Q$.

Theorem 10: If for all $Y \subset X$, $||P||_Y = ||Q||_Y$ for partions P and Q, then $P \overset{X}{\equiv} Q$.

Proof of theorem 10:

$0 = ||P||_{\mathrm{Res}(P|X)} = ||Q||_{\mathrm{Res}(P|X)} \therefore \mathrm{Res}(P|X) \subset \mathrm{Res}(Q|X)$

and similarly, $\mathrm{Res}(Q|X) \subset \mathrm{Res}(P|X) \Rightarrow \mathrm{Supp}(P|X) = \mathrm{Supp}(Q|X)$. Suppose $\mathrm{P} \in P \wedge [X]$, then $||P||_{\mathrm{P}} = ||Q||_{\mathrm{P}} = 1$, i.e., there exists a $\mathrm{Q} \in Q \wedge [X]$ such that $\mathrm{P} \subset \mathrm{Q}$. Similarly, there must exist a $\mathrm{P}' \in P \wedge [X]$ such that $\mathrm{P} \subset \mathrm{Q} \subset \mathrm{P}'$. If $\mathrm{P} \neq \phi$, then $\mathrm{P} = \mathrm{P} \cap \mathrm{P}' = \mathrm{Q} \Rightarrow$

$P\wedge[X] \subset Q \wedge [X]$. Similarly $Q\wedge[X] \subset P\wedge[X]$ $\therefore$ $P \overset{X}{=} Q$.

Q. E. D.

Theorem 11: Given:

(1) a collection AA of partions,

(2) a partion Q_0

(3) a set Y

(4) $Y \subset \operatorname{Supp}(\bigwedge_{P \in AA} P)$ (5) $1 < ||\bigwedge_{P \in AA} P||_Y = k < \infty$

(6) $\bigwedge_{P \in AA} P\wedge Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P$ (7) $k > ||Q_0||_Y = \ell \geq 1$.

Conclusion: $\exists$ BB $\subset$ AA such that $|BB| \leq k - \ell$ and that

$$\bigwedge_{P \in BB} P\wedge Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P$$

Proof of theorem 11:

$$\bigwedge_{P \in AA} P\wedge Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P, \therefore Y = Y \cap (\bigcap_{P \in AA} \operatorname{Supp}(P)) \subset \operatorname{Supp}(Q_0|Y)$$

so Q_0 partitions Y, and by (4) all $P \in$ AA partitions Y. Either there exists a $P_1 \in$ AA such that $||Q_0 \wedge P_1||_Y > ||Q_0||_Y$ or by theorem 8, $\forall\, P \in$ AA,

$$P\wedge Q_0 \overset{Y}{=} Q_0 \Rightarrow Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P\wedge Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P$$

$$\Rightarrow ||Q_0||_Y = ||\bigcap_{P \in AA} P||_Y \text{ (a contradiction).}$$

$\therefore$ Let $Q_1 = P_1 \wedge Q_0$. [Remember that $||Q_0 \wedge P_1||_Y > ||Q_0||_Y$). It is also obvious that

$$\bigwedge_{P \in AA} P \wedge Q_1 = \bigwedge_{P \in AA} P \wedge Q_0 \overset{Y}{=} \bigwedge_{P \in AA} P.$$

Having found P_1, Q_1 P_2, Q_2, P_3, Q_3, ..., P_{j+1}, Q_{j+1}($Q_{i+1} = Q_i \wedge P_{i+1}$, $||Q_{i+1}||_Y > ||Q_i||_Y$ for $0 \leq i \leq j$. Also $||Q\,||_Y \geq \ell + i$, and

$$\bigwedge_{P \in AA} P \wedge Q_i \overset{Y}{=} \bigwedge_{P \in AA} P \quad \text{for}$$

$0 \leq i \leq j + 1$). Either $||Q_{j+1}|| \geq k$, or we may replace Q_0 with Q_{j+1} in the above argument and find a P_{j+2} and a Q_{j+2}. Since $\ell + j < ||Q_{j+1}||_Y$, it can be assumed that the above process stops with some $j < k - \ell$. Then $||Q_{j+1}||_Y \geq k$. But by theorem 8, $||Q_{j+1}||_Y \leq ||\bigwedge P \wedge Q_{j+1}||_Y = k$, so $||Q_{j+1}||_Y = k$. $\therefore$ for all $P \in AA$, $||Q_{j+1} \wedge P||_Y = ||Q_{j+1}||_Y$, then for all $P \in AA$ must $Q_{j+1} \wedge P \overset{Y}{=} Q_{j+1}$.

$$\therefore \bigwedge_{P \in AA} P \wedge Q_{j+1} \overset{Y}{=} Q_{j+1} = Q_0 \wedge \left(\bigwedge_{i=1}^{j+1} P_i\right).$$

Let $BB = \{Pi \mid 1 \leq i \leq j + 1 \leq k - \ell\}$, then $|BB| = j + 1 \leq k - \ell$, and

$$\bigwedge_{P \in BB} P \wedge Q_0 = Q_{j+1} \overset{Y}{=} \bigwedge_{P \in BB} P.$$

Q. E. D.

Corollary to theorem 11: Given

(1) A collection AA of partions

(2) A set $Y \subset \mathrm{Supp}(\bigwedge_{P \in AA} P)$.

(3) $||\bigwedge_{P \in AA} P||_Y = k > 1$

Conclusion: There exists a set BB $\subset$ AA such that

(4) $|BB| < k$

(5) $\bigwedge_{P \in BB} P \overset{Y}{\equiv} \bigwedge_{P \in AA} Pi$

Proof of corollary to theorem 11: Let $Q_0 = [Y]$. $||Q_0||_Y = 1$.
By theorem 11, $\exists$ BB $\subset$ AA, $|BB| \leq k - 1 < k$ such that

$$\bigwedge_{P \in BB} P \wedge [Y] \overset{Y}{\equiv} \bigwedge_{P \in AA} P \Rightarrow \bigwedge_{P \in BB} P \overset{Y}{\equiv} \bigwedge_{P \in AA} P \qquad \text{Q.E.D.}$$

In the remaining sections of Chapter I, theorem 11 and the corollary to theorem 5 will probably be the most useful facts that were derived in Section 1.

Section 2

The Generalized Pigeon Hole Principles. Formulation and Useful Counter Examples

In this section, unless specified otherwise, AA will be a collection of partions; X, Y, Z will be sets such that $X \subset Z$ and $Y \subset Z$.

Definition 13: $\eta(AA \mid Z) = \max_{P \in AA} \langle P \rangle_Z = \max_{P \in AA} |\mathrm{Res}(P \mid Z)|$

Note to Reader: To those of you that who the matrix notation useful, $\eta(AA \mid Z)$ counts the maximum number of times the null set (or zero) appears in any row of the matrix induced by the set Z and the collection AA of partions. Hence,

$$\eta(AA \mid Z) = \max_{P \in AA} |\{z \mid z \in Z,\ \mathrm{Mat}(z; P) = \phi\}|.$$

Theorem 12: $\eta(AA \mid X) \leq \eta(AA \mid Z)$. If BB is any collection of partions, then

$$\eta(AA \mid Z) \leq \eta(AA \cup BB \mid Z).$$

Theorem 12 is quite obvious, and the proof is omitted.

Definition 14: $K(AA \mid Z) = \max_{BB \subset AA} \left| \left| \bigwedge_{P \in BB} P \right| \right|_Z.$

Note to the reader: In the matrix induced by the set Z and the collection AA of partions, $K(AA \mid Z)$ is the maximal number of non-equivalent columns of any submatrix not containing the null set as an entry, (i.e., zero-free).

Theorem 13: $K(AA|X) \leq K(AA|Z)$. If CC is any collection of partions, then $K(AA|Z) \leq K(AA \vee CC|Z)$.

Theorem 13 is also obvious, and the proof will be omitted.

Definition 15: $\mathcal{F}a(AA|Z) = \{F | F \subset Z,$ and

$$\forall \mathcal{P} \in AA, \exists P \in \mathcal{P} \ni |F \sim P| \leq \eta(AA|Z)\}.$$

Definition 16: $\mathcal{F}e(AA|Z) = \{F | F \subset Z, \text{ and } ||\mathcal{P}||_F \leq 1 \ \forall \mathcal{P} \in AA\}.$

Theorem 14: $\mathcal{F}a(AA|X) \subset \mathcal{F}a(AA|Z)$.

$\mathcal{F}e(AA|X) \subset \mathcal{F}e(AA|Z)$.

The proof of theorem 14 is obvious and is omitted.

Theorem 15: $F \in \mathcal{F}e(AA|Z)$ iff $\forall \mathcal{P} \in AA, \exists P \in \mathcal{P} \ni F \subset (P \cup Res(\mathcal{P}|Z)) \cap Z$.

Proof of theorem 15: If for a set F, and if for all $\mathcal{P} \in AA$ there exists a $P \in \mathcal{P}$ such that $F \subset (P \cup Res(\mathcal{P}|Z)) \cap Z$, then obviously ($\forall \mathcal{P} \in AA$)

(1) $F \subset Z$ and (2) $||\mathcal{P}||_F \leq ||\mathcal{P}||_{P \cup Res(\mathcal{P}|Z)} \leq ||\mathcal{P}||_P + ||\mathcal{P}||_{Res(\mathcal{P}|Z)}$

$$= 1 + 0 = 1$$

so by definition, $F \in \mathcal{F}e(AA)$.

If $F \in \mathcal{F}e(AA|Z)$, then $||\mathcal{P}||_F \leq 1 \ \forall \mathcal{P} \in AA$. That means there can exist at most one $P \in \mathcal{P}$ such that $F \cap P \neq \phi$. (If no such P exists, then

$F \subset \text{Res}(P|Z) = (\phi \cup \text{Res}(P|Z)) \cap Z$. One may as well assume that such a P exists). For that $P \in P$, $F \cap (\text{Supp}(P) \sim P) = \phi$; but $F \subset Z$ so then $F \subset P \cup (Z \sim \text{Supp}(P))$, i.e., $F \subset (P \cup \text{Res}(P|Z) \cup P) \cap Z$.
$\therefore \forall P \in AA$, $\exists\, P \in P$ such that $F \subset (\text{Res}(P|Z) \cup P) \cap Z$. Q.E.D.

Definition 17: If $P \in AA$, and if F_1 & F_2 are both members of $Fe(AA|Z)$, then $F_1 \simeq F_2$ (Mod P) if there exists both a $f_1 \in F_1$ and a $f_2 \in F_2$ such that $f_1 \simeq f_2$ (mod P).

Note: $F_1 \simeq F_2$ (Mod P) is an extention of the notion of $a \simeq b$ (mod P) since (1) $a \simeq b$ (mod P) iff $\{a\} \simeq \{b\}$ (Mod P) and (2) $F \in Fe(AA|Z)$ iff $\text{Supp}(F) \subset \text{Mat}(f;P)$ for all $P \in AA$ and for all $f \in \text{Supp}(P|F)$. For the set $M = \{M | M \in Fe(AA|Z),\ M \not\subset \text{Res}(P|Z)\}$, $\simeq$ (Mod P) is an equivalence relation. If $BB \subset AA$, and if

$$B = \bigwedge_{P \in BB} P \quad ,$$

then definition 17 may also be used to define $\simeq$ (Mod B).

Theorem 16: If both F & G are members of $Fe(AA|Z)$, then either

(1) $\qquad (F \cup G) \in Fe(AA|Z)$, or

(2) $\qquad \exists\, P \in AA \ni ||P||_{F \cup G} = 2 \Rightarrow F \not\simeq G$ (Mod P).

Proof of theorem 16:

F, G, $\in \mathit{Fe}(AA|Z)$, so both $||P||_F \leq 1$ and $||P||_G \leq 1 \ \forall P \in AA$.

$$\therefore \forall P \in AA, \ ||P||_{F \cup G} \leq ||P||_F + ||P||_G \leq 2.$$

$\therefore$ Either (1) $||P||_{F \cup G} \leq 1 \ \forall \ P \in AA \Rightarrow (F \cap G) \in Fe(AA|Z)$ or

(2) $\exists \ P \in AA$ such that $||P||_{F \cup G} = 2$. Q. E. D.

Theorem 17: $\mathit{Fe}(AA|Z) \subset \mathit{Fa}(AA|Z)$.

Proof of theorem 17: Suppose $F \in \mathit{Fe}(AA|Z)$ and that $P \in AA$. Then by theorem 15, there exists a $P \in P$ such that $F \subset (P \cup Res(P|Z)) \cap Z$.

$$\therefore F \sim P \subset [(P \cup Res(P|Z)) \cap Z] \sim P = (Res(P|Z) \cap Z) \sim P$$

$$F \sim P \subset Res(P|Z) \sim P = Res(P|Z).$$

$$\therefore |F \sim P| \leq |Res(P|Z)| = \langle P \rangle_Z \leq \eta(AA|Z) \Rightarrow F \in \mathit{Fa}(AA|Z)$$

$$\Rightarrow \mathit{Fe}(AA|Z) \subset \mathit{Fa}(AA|Z). \text{ Q. E. D.}$$

Theorem 18: Given:

(1) F & H are both members of $Fe(AA|Z)$.

(2) $|H| = h + m \ (1 \leq m \leq \eta(AA|Z))$

(3) $F \cap H = \phi$

(4) $|F| > \eta(AA|Z) - m$.

Conclusion:

Either (5) $\exists \ M \in \mathit{Fa}(AA|Z), \ |M| > h + \eta(AA|Z)$

or (6) $\exists \ P \in AA, \ F \not\equiv H \ (\text{Mod } P)$ and $\text{Supp}(P|H) \leq h$.

Proof of theorem 18: Suppose (6) is not true, then let $X \subset F$ such that $|X| = \eta(AA|Z) + 1 - m$, and let $M = H \cup X$. Note that $|M| = h + \eta(AA|Z) + 1 > h + \eta(AA|Z)$.

Case I: $\mathcal{P} \in AA$, $H \simeq F$ (Mod $\mathcal{P}$). Then there exists a $P \in \mathcal{P}$ such that $|M \sim P| = |M \sim \text{Supp}(\mathcal{P})| \leqslant |Z \sim \text{Supp}(\mathcal{P})| \leqslant \eta(AA|Z)$.

Case II: $\mathcal{P} \in AA$, $F \not\simeq H$ (Mod $\mathcal{P}$), but $|\text{Supp}(\mathcal{P}|H)| > h$. Then there exists a $P \in \mathcal{P}$ such that $|P \cap H| \geqslant h+1$, hence $|M \sim P| = |M| - |P \cap H| - |X \cap H| \leqslant (h + \eta(AA|Z) + 1) - (h+1) - 0 \leqslant \eta(AA|Z)$.

Therefore, for all $\mathcal{P} \in AA$ there exists a $P \in \mathcal{P}$ such that $|M \sim P| \leqslant \eta(AA|Z)$, so $M \in \mathcal{F}a(AA|Z)$.

Note to reader: Theorems 16 and 18 are simple but powerful. In Section 3, applying these two theorems in Algorithm No. 2 is essential in proving this papers generalized pigeon-hole principles.

Theorem 19: Suppose AA and BB are both collections of partions, then

(1) $$\mathcal{F}a(AA|Z) \cap \mathcal{F}a(BB|Z) \subset \mathcal{F}a(AA \cup BB|Z)$$

(2) $$\mathcal{F}e(AA|Z) \cap \mathcal{F}e(BB|Z) = \mathcal{F}e(AA \cup BB|Z).$$

Proof of theorem 19: Obviously $\eta(AA \cup BB | Z) = \max(\eta(AA|Z), \eta(BB|Z))$. Suppose that $F \in Fa(AA|Z) \cap Fa(BB|Z)$. Then (a) $F \subset Z$, and (b) $\forall \mathcal{P} \in AA$, $\exists P \in \mathcal{P} \ni |F \sim P| \leq \eta(AA|Z) \leq \eta(AA \cup BB|Z)$, and (c) $\forall \mathcal{P} \in BB$, $\exists P \in \mathcal{P} \ni |F \sim P| \leq \eta(BB|Z) \leq \eta(AA \cup BB|Z)$, so we have (d, 1) $F \subset Z$, and (d, 2) $\forall \mathcal{P} \in AA \cup BB$, $\exists P \in \mathcal{P} \ni |F \sim P| \leq \eta(AA \cup BB|Z)$ which by definition implies that $F \in \mathcal{F}a(AA \cup BB|Z)$. Suppose that $F \leq Fe(AA|Z) \cap Fe(BB|Z)$ then if $\mathcal{P} \in AA$, $||\mathcal{P}||_F \leq 1$, or if $\mathcal{P} \leq BB$, $||\mathcal{P}||_F \leq 1$. Therefore, if $\mathcal{P} \in AA \cup BB$, then $||\mathcal{P}||_F \leq 1$. Since $F \in \mathcal{F}e(AA|Z)$, then $F \subset Z$, so we now have by definition that $F \in \mathcal{F}e(AA \cup BB|Z)$. Suppose that $F \in \mathcal{F}e(AA \cup BB|Z)$, then if $\mathcal{P} \in AA \cup BB$, then $||\mathcal{P}||_F \leq 1$. But $F \subset Z$, and $AA \subset AA \cup BB$, and $BB \subset AA \cup BB$, so $F \in Fe(AA|Z)$ and $F \in Fe(BB|Z)$ and finally $F \in Fe(AA|Z) \cap Fe(BB|Z)$. Q. E. D.

Definition 18:

(a) $$\mu a(AA|Z) = \underset{F \in Fa(AA|Z)}{\text{maximum}} |F|$$

(b) $$\mu e(AA|Z) = \underset{F \in Fe(AA|Z)}{\text{maximum}} |F|.$$

Corollary to theorem 17: $\mu a(AA|Z) \leq \mu e(AA|Z)$.

Definition 19:

(a) $GGem(AA|Z) = \{\mathcal{G} \,|\, \mathcal{G} \subset \mathcal{F}e(AA|Z),\ (\mathcal{G} \cup \{\phi\}) \text{ is a partion}\}$.

(b) $GGet(O;AA|Z) = \{\mathcal{G} \,|\, \mathcal{G} \in GGem(AA|Z)$, and if both G_1 and G_2 are in $\mathcal{G}$ with $G_1 \neq \phi \neq G_2$, then $(G_1 \cup G_2) \in \mathcal{F}e(AA|Z) \Leftrightarrow G_1 = G_2\}$.

(c) $\mathrm{GGet}(r; AA|Z) = \{\mathcal{G} \mid \mathcal{G} \in \mathrm{GGet}(O; AA|Z),\ |G| \geqslant r\ \forall G \in \mathcal{G}\}$.

Note: It is obvious that if $r < s$, then $\mathrm{GGet}(s; AA|Z) \subset \mathrm{GGet}(r; AA|Z)$. Also, $\phi \in \mathrm{GGet}(r; AA|Z)$ for any cardinal r.

Definition 20: $\tau(h;AA|Z) = \max \sum_{G \in \mathcal{G}} (|G| \sim h)$, over all $\mathcal{G} \in \mathrm{GGem}(AA|Z)$.

Note: It should be obvious that if $\mathcal{G} \in \mathrm{GGem}(AA|Z)$ and if $\sum_{G \in \mathcal{G}} (|G| - h) = \tau(h; AA|Z)$ then $\mathcal{G} \in \mathrm{GGet}(h; AA|Z)$.

Furthermore, there always exists a $\mathcal{G} \in \mathrm{GGet}(h+1; AA|Z)$ such that

$$\sum_{G \in \mathcal{G}} (|G| - h) = \tau(h; AA|Z).$$

Theorem 20: $\mathrm{GGem}(AA|X) \subset \mathrm{GGem}(AA|Z)$.

$\mathrm{GGet}(r; AA|X) \subset \mathrm{GGet}(r; AA|Z)$ for any cardinal r.

Theorem 21: $\tau(h; AA|X) \leqslant \tau(h; AA|Z)$.

Definition 21: $\sigma(r;AA|Z) = \underset{\mathcal{G} \in \mathrm{GGet}(r;AA|Z)}{\mathrm{maximum}} \|\mathcal{G}\|$.

Theorem 22: If $\mathcal{G} \in \mathrm{GGet}(r;AA|Z)$, then (if $r > 0$)

$$|\mathrm{Supp}(\mathcal{G})| \leqslant \sigma(r;AA|Z) \cdot h + \tau(h;AA|Z).$$

Proof of theorem 22:

$$\tau(h;AA|Z) \geq \sum_{G \in \mathcal{G}} (|G| - h) = \sum_{G \in \mathcal{G}} |G| - \sum_{G \in \mathcal{G}} h \geq |\mathrm{Supp}(\mathcal{G})| - h \cdot \sigma(r;AA|Z).$$

$$|\mathrm{Supp}(\mathcal{G})| \leq \sigma(r;AA|Z) \cdot h + \tau(h;AA|Z).$$

The next two definitions define this paper's generalized pigeon hole principle, for finite cases. Infinite cases will not be discussed in this paper (and to do such would require a few minor changes in my previous definitions), but infinite cases are quite doable (with use of the axiom of choice).

Definition 22: $\chi a(h, k, \ell)$ is the minimal integer such that if (1) $|Z| \geq \chi a(h, k, \ell)$, and if (2) $K(AA|Z) \leq k$, and if (3) $\eta(AA|z) \leq \ell$, then $\mu a(AA|Z)$ must be greater than $h(\mu a(AA|Z) > h)$.

Definition 23: $\chi e(h,k, \ell, t)$ is the minimal integer such that if (1) $|Z| \geq \chi e(h, k, \ell, t)$, and if (2) $K(AA|Z) \leq k$, and if (3) $\eta(AA|Z) \leq \ell$, then $\tau(h;AA|Z)$ must be at least t $(\tau(h;AA|Z) \geq t)$.

Note: It will always be assumed that $h \geq 0$, $k > 0$, $\ell \geq 0$, and $t > 0$. Both $\chi a(h,k, \ell)$ and $\chi e(h, k, \ell, t)$ are non-decreasing in each of their variables.

Theorem 23: (A counter example) $\chi a(h,k, \ell) \geq hk + 1$.

Proof of theorem 23: Take any k pairwise disjoint sets $H_1, H_2, \ldots, H_k$ such that $|H_1| = |H_2| = \cdot = |H_k| = h$. Let

$$Z = \bigcup_{i=1}^{k} H_i.$$

Let AA = $\{\{Hi, (Z \sim Hi), \phi\} \quad 1 \leq i \leq k\}$. Obviously $K(AA|Z) = k$, and $\eta(AA|Z) = 0 \leq \ell$. Suppose $F \in \mathcal{F}a(AA|Z)$, then there exists an i such that $F \cap Hi \neq \phi$ (or $F = \phi$). But $||P_i||_F \leq 1$, so $F \subset Hi$ (since $\eta(AA|Z) = 0 \Rightarrow \text{Res}(Pi|Z) = \phi$). $\therefore |F| \leq |Hi| \leq h$. $\therefore \mu a(AA|Z) \leq h$. Therefore, by definition of $\chi a(h, k, \ell)$, $hk = |Z| < \chi a(h, k, \ell)$, which proves theorem 23.

Theorem 24: $\chi a(h, k, 0) = hk + 1$. Suppose we have a set Z, $|Z| \geq kh + 1$, and that we have a collection AA of partions with the two properties of $K(AA|Z) \leq k$ and $\eta(AA|Z) = 0$. We must now show that $\mu a(AA|Z) > h$. Let

$$R = \bigwedge_{P \in AA} P .$$

Note that $||R||_Z \leq K(AA|Z) \leq k$ and furthermore that $\forall P \in AA$, $R \wedge P = R$. $\eta(AA|Z) = 0$, so $\forall P \in AA$, $\text{Res}(P|Z) = \phi$. $\therefore$ (By theorem 2), $\text{Res}(R|Z) = \phi \Rightarrow R$ partitions Z and

$$Z = \bigcup_{R \in R\wedge[Z].} R$$

By theorem 8, $||R||_Z = K(AA|Z) \leq k$. Let $R \in R\wedge[Z]$, then $\forall P \in AA$, $||P||_R \leq ||P \wedge R||_R = 1$, so $R \in Fe(AA|Z)$. We now have the $|Z| \geq kh + 1$ elements of Z partitioned into $K(AA|Z) \leq k$ non-void disjoint sets of $R\wedge[Z]$, so by the classical pigeon hole principle (and this is the tie-in) there exists an $R^* \in R\wedge[Z]$ such that $|R^*| > h$ and furthermore (since $R^* \in Fe(AA|Z) \subset Fa(AA|Z)$) $\mu a(AA|Z) \geq |R^*| > h$. Q.E.D.

Theorem 25: (Another counter example). If $h \geq \ell$, $\chi e(h,k,\ell,t) \geq kh + t + (k-1)\ell$.

Proof of theorem 25: Take pairwise disjoint sets H_0, $H_1, \ldots, H_{k-1}$, $L_1, \ldots, L_{k-1}$, such that (1) $|H_0| = h + t - 1$, (2) $|Hi| = h$ for $1 \leq i < k$, and (3) $Li = \ell$ for $1 \leq i < k$. Let $L_0 = \phi$, let

$$Z = \bigcup_{i=0}^{k-1} (Hi \cup Li),$$

let $P_i = \{Li, (Z \sim Li), \phi\}$ for $1 \leq i < k$, let $Qi = \{Z \sim (Hi \cup Li)), Hi, \phi\}$ for $1 \leq i < k$, and let $AA = \{Pi | i \leq i < k\} \cup \{Qi | 1 \leq i < k\}$. Obviously, $Pi \wedge Qi = Qi$, $\eta(AA|Z) = \ell$, and $K(AA|Z) \leq K$. ($K(AA|Z) \leq k$ is not so obvious, but rests on the fact that for all $P \in AA$ and for any $i > 0$, either $H_0 \simeq Hi$ (Mod P) or $Li \subset Res(P|Z)$. This is also true for any partion that is the intersection of the partions in an arbitrary subset of AA. So for any arbitrary intersection A of partions in AA, and for any $i > 0$, $||A||_{H_0 \cup Li \cup Hi} \leq 2$ which implies that $||A|| \leq 1 + (k-1) = k$. Suppose $F \in \mathcal{F}e(AA|Z)$.

Case I: $F \cap Li \neq \phi$ for some i. By theorem 15, (and the fact that $Pi \in AA$) $F \subset Li \cup Res(Pi|Z) = Li \cup \phi \Rightarrow |F| \leq |Li| = \ell \leq h$.

Case II: $F \cap Hi \neq \phi$ $(i > 0)$. By theorem 15 we have

(1) $Fi \subset (Z \sim Li) \cup Res(Pi|Z) = Z \sim Li$ and

(2) $Fi \subset Hi \cup Res(Qi|z) = H \cup Li$.

$\therefore$ $F \subset (Z \sim Li) \cap (H \cup Li) = Hi \Rightarrow |F| \leq |Hi| = h$.

Case III: $F \subset H_0$.

By the note after definition 20, $\tau(h; AA|Z) \leq |H_0| - h = t - 1 < t$

$\therefore \chi e(h, k, \ell, t) > |Z| = hk + (t - 1) + (k - 1)\ell$ for all $h \geq \ell$.

Important note to the reader: This paper will eventually show in Section 3 that for h sufficiently large (k and ℓ fixed) that $\chi a(h, k, \ell) = hk + 1$ (Theorem 35) and that $\chi e(h, k, \ell, t) = hk + t + (k-1)\ell$ (Theorem 34). To show that $\chi e(h, k, \ell, t)$ is not always $hk + t + (k-1)\ell$ we have the following theorem.

Theorem 26: (Another counter example) for $k \geqslant 2$,

$$\chi e(\ell, k, \ell, 1) \geqslant 2k\ell + 1 \;(> (2k-1)\ell + 1).$$

Proof of theorem 26: Let $L_1, L_2, \ldots, L_{2k}$ be pairwise disjoint sets such that $|Li| = \ell$ for $1 \leqslant i \leqslant 2k$. Let $L_1 = L_{2k+1}$ and and let $L_2 = L_{2k+2}$. Set

$$Z = \bigcup_{i=1}^{2k} Li.$$

Let $Pi = \{Li, \phi, Z \sim (Li \cup L_{i+1})\}$ for $1 \leqslant i \leqslant 2k + 1$. Let $AA = \{Pi \mid 1 \leq i \leq 2k\}$. By construction, $\eta(AA|Z) = \ell$. Also by construction, if $BB \subset AA$, then

$$\Big|\bigcup_{P \in BB} P\Big| \leqslant |BB| + 1 \text{ and } \Big|\bigcap_{P \in BB} P\Big| \leqslant 2k - |BB|.$$

Therefore

$$BB \subset AA, \quad \sum_{P \in BB} ||P|| \leqslant [\tfrac{1}{2}(|BB| + 1 + 2k - |BB|)] = [\tfrac{1}{2}(2k+1)] = k,$$

$\therefore K(AA|Z) \leqslant k$. If $F \in \mathcal{F}e(AA|Z)$, and if $F \neq \phi$, then there exists an Li ($1 \leqslant i \leqslant 2k$) such that $F \cap Li \neq \phi$. By theorem 15, (1) $F \subset Li \cup Res(\mathcal{P}i|Z) = Li \cup L_{i+1}$ and furthermore (2) $F \subset (Z \sim (L_{i+1} \cup L_{i+2})) \cup Res(\mathcal{P}_{i+1}|Z) = Z \sim L_{i+1}$. $\therefore F \subset (Li \cup L_{i+1}) \cap (Z \sim L_{i+1}) = Li \Rightarrow \forall F \in \mathcal{F}(AA|Z)$, $|F| \leq \ell \Rightarrow \tau(\ell, AA|Z) = 0$, $\chi e(\ell, k, \ell, 1) > |Z| = 2k\ell$. Q. E. D.

Note: In the previous theorem, the trouble with K = 1 is that it is assumed that $Li \subset Z \sim (L_{i+1} \cup L_{i+2})$, but for k = 1, $Z \sim (Li \cup L_{i+1}) = \phi \Rightarrow \phi \neq Li \subset \phi$, but that is not possible.

Theorem 27: $\chi e(h, k, 0, t) = hk + t$. Suppose Z is a set and AA is a collection of partions such that (1) $|Z| \geqslant hk + t$, (2) $K(AA|Z) \leqslant k$, and (3) $\eta(AA|Z) = 0$. We must show that $\tau(h;AA|Z) \geqslant t$. As in theorem 24, let

$$\mathcal{R} = \bigwedge_{P \in AA} P .$$

As in theorem 24, it can be shown that $Supp(\mathcal{R}|Z) = Z$, $||\mathcal{R}||_Z = K(AA|Z) \leqslant k$, and that $(\mathcal{R} \wedge [Z] \sim \{\phi\} \in GGem(AA|Z)$. $\therefore \tau(h;AA|Z) \geqslant \Sigma(|R| - h) = Supp(\mathcal{R}|Z) - h \cdot K(AA|Z) \geqslant (kh + t) - kh \geqslant t$. Q. E. D. $R \in (\mathcal{R} \wedge [Z]) \sim \{\phi\}$.

Note to Reader: Again for $\eta(AA|Z) = 0$, the fact that $Supp(\mathcal{R}|Z) = Z$, and the fact that $(\mathcal{R} \wedge [Z]) \sim \{\phi\} \in GGem(AA|Z)$ implies that the points of Z were split up between at most k disjoint sets of $Fe(AA|Z)$, so the pigeon hole principle had to apply and the result followed immediately.

Theorem 28: $\chi a(h, 1, \ell) = h+1$, $\chi e(h, 1, \ell, t) = h + t$.

Proof of theorem 28: Suppose we have a set Z, and any collection of partions such that $K(AA|Z) = 1$. Then $\forall P \in AA$, $||P||_Z \leqslant K(AA|Z) = 1$. $\therefore Z \in Fe(AA|Z) \Rightarrow \mu a(AA|Z) = |Z|$ and $\tau(h, AA|Z) = \max(O, |Z| - h)$. $\therefore \mu a(AA|Z) > h$ iff $|Z| > h$ ($\Rightarrow \chi a(h, 1, \ell) = h + 1$). $\therefore \tau(h;AA|Z) \geqslant t$, iff $|Z| \geqslant h + t$ ($\Rightarrow \chi b(h, 1, \ell, t) = h + t$). Q.E.D.

Section 3

Proof of the Generalized Pigeon Hole Principles

The proofs of the generalized pigeon hole principle will be basically an induction on ℓ in nature. Eventually, an initial set X_0 and an initial collection AA_0 of partions will be given, from which will be constructed an X_1, AA_1; X_2, AA_2, X_3, AA_3; etc., such that $X_{i+1} \subset X_i$, $AA_{i+1} \subset AAi$, $\eta(AA_{i+1} | X_{i+1}) < \eta(AAi | Xi)$, and $\tau(h;AA_{i+1} | X_{i+1}) \leq \tau(h;AAi | Xi)$. (h will be specified to be at least so large in this process).

Theorem 29: Let AA and BB be collections of partions and let X be a set. Let

$$B = \bigwedge_{P \in BB} P$$

Assume that:

(1) $BB \subset AA$

(2) $||B||_X = K(AA | X)$

(3) $0 < \eta(AA | Z) < \infty$

(4) $\eta(BB | X) = 0$ (i.e. $\forall\ P \in BB,\ \langle P \rangle_X = 0$).

Then there exists an AA*, X* such that:

(5) $BB \subset AA^* \subset AA$

(6) $X^* \subset X$

(7) $\eta(AA^* | X^*) < \eta(AA | X)$

(8) $Fe(AA^*|X^*) = Fe(AA|X^*) \subset Fe(AA|X)$.

(9) $|X \sim X^*| = ||\{B | B \in B, |B|_X \leq \eta(AA|X)\}||_X$

Proof of theorem 29: For every $B \in B$ such that $0 < |B|_X \leq \eta(AA|X)$, delete an element of $B \cap X$ from X. The remaining set is X*. Properties (6) and (9) are obviously true. Let $AA^* = \{P | P \in AA, |Res(P|X^*)| < \eta(AA|X)\}$. Properties (5) and (7) are obvious, and (8) is the only property which remains to be proven. Since $\forall\ P \in BB$ $Res(P|X) = 0$, theorem 15 implies that $F \in Fe(BB|X)$ iff $\exists$ B such that $F \subset B \in B\wedge[X]$.

Lemma: If $B \in B\wedge[X]$, and if $P \in AA \sim AA^*$, then $\langle P \rangle_B < |B|$.

Case I: $|B| > \eta(AA|X)$, then $\langle P \rangle_B \leq \eta(AA|B) \leq \eta(AA|X) < |B|$.

Case II: $|B| < \eta(AA|X)$. But $P \in (AA \sim AA^*)$, so $\langle P \rangle_{X^*} = \eta(AA|X)$. But $\langle P \rangle_{X \sim X^*} = \langle P \rangle_X - \langle P \rangle_{X^*} \leq \eta(AA|X) - \eta(AA|X) = 0$, $\therefore \langle P \rangle_{X \sim X^*} = 0$ and $(X \sim X^*) \subset Supp(P)$. By construction, $B \cap (X \sim X^*) \neq \phi$, so $\phi \neq B \cap Supp(P) \Rightarrow Res(P|B) \neq B$. $\therefore \langle P \rangle_B \neq |B|$. End of lemma.

Lemma: If $B \in B\wedge[X]$, and if $P \in (AA \sim AA^*)$, then $||P||_B \leq 1$.

By theorem 6, (if $B \neq \phi$) and by the previous lemma, $\forall\ P \in (AA \sim AA^*)$, $||P||_B \geq 1$. $\forall\ P \in (AA \sim AA^*)$, we have (Theorem 7) that,

$$||B\wedge P||_X = \sum_{B \in B\wedge[X]} ||P||_B \geq \sum_{\substack{B \in B\wedge[X] \\ B \neq \phi}} 1 = ||B||_X = K(AA|X)$$

$\therefore \forall\, P \in AA \sim AA^*$, $||\mathcal{B} \wedge P||_X = K(AA|X)$. Since equality holds, the applications of theorem 7 yields that $||P||_B = 1$ for all $B \in \mathcal{B} \wedge [X]$ $(B \neq \phi)$. End of Lemma.

Proof of property (8). The lemma just proved implies that $Fe(BB|X) \subset Fe(AA \sim AA^*|X)$. By theorem 19 we have $Fe(AA|X^*) = Fe(AA \sim AA^*|X) \cap Fe(AA^*|X^*)$. Hence

$$Fe(AA|X^*) \supset Fe(BB|X^*) \cap Fe(AA^*|X^*) \supset Fe(AA|X^*).$$

Equality must hold, and the proof is complete.

Theorem 30: For all h, k, ℓ, t finite, $\chi e(h, k, \ell, t)$ is finite. Furthermore, $\chi e(h, k, \ell+1, t) \leq (k-1)(\ell+1) + \chi e(h, k, \ell, t)$ for $h > \ell \geq 0$.

Proof of theorem 30: (By induction on ℓ).

$\chi e(h, k, 0, t) = hk + t$ by theorem 27. Suppose $\chi e(h, k, \ell, t)$ is finite. Suppose AA is a collection of partions, and that Z is a set such that $K(AA|Z) \leq k$, $\eta(AA|Z) \leq \ell + 1$, and $\tau(h; AA|Z) < t$. Let $BB^* \subset AA$ by any set such that

$$\Big|\Big| \bigwedge_{P \in BB} P \Big|\Big|_Z = K(AA|Z).$$

By the corollary to theorem 11, there exists a $BB \subset BB^*$ such that (1) $|BB| < K(AA|Z)$ and (2)

$$\Big|\Big| \bigwedge_{P \in BB} P \Big|\Big|_Z = K(AA|Z)$$

Let $$X = Z \cap \operatorname{Supp}(\bigwedge_{P \in BB} P)$$

then

$$|Z \sim X| = |\operatorname{Res}(\bigwedge_{P \in BB} P|Z)| = |\bigcup_{P \in BB} \operatorname{Res}(P|Z)| \leq |BB| \cdot \eta(BB|Z) \leq (k-1)(\ell+1)$$

Note that (3) $BB \subset AA$ is a set of partions such that $\eta(BB|X) = 0$, and

(4) $$||\bigwedge_{P \in BB} P||_X = K(AA|X)$$

By theorem 29, there exists a $X_1 \subset X$, and an $AA_1 \subset AA$ such that

(5) $$|X \sim X_1| \leq ||\{B | B \in \bigwedge_{P \in BB} P, \ |B|_X \leq \eta(AA|X)\}||_X$$

and that (6) $Fe(AA_1|X_1) \subset Fe(AA|X)$, and (7) $\eta(AA|X_1) < \eta(AA|X) = \ell + 1$. Since $\tau(h;AA_1|X_1) \leq \tau(h;AA|X) \leq \tau(h;AA|Z) < t$, we must have $|X_1| < \chi_1(h,k,\ell,t)$. Combining all this information, we have:

$$|Z| = |Z \sim X| + |X \sim X_1| + |X_1|$$

(7) $$|Z| < (k-1)(\ell+1) + ||\{B | B \in \bigwedge_{P \in BB} P, \ |B|_X \leq \eta(AA|Z)\}||_X + \chi e(h,k,\ell,t)$$

$$|Z| < (k-1)(\ell+1) + ||\bigwedge_{P \in BB} P||_X + \chi e(h,k,\ell,t)$$

$$|Z| < (k-1)(\ell+1) + K(AA|Z) + \chi e(h,k,\ell,t)$$

(8) $$|Z| < (k-1)(\ell+1) + k + \chi e(j,k,\ell,t).$$

We now know that if for some Z, $|Z| \geq (k-1)(\ell+1) + k + \chi e(h,k,\ell,t)$, and that if for some collection of partions, $\eta(AA|Z) = \ell + 1$

and $K(AA|Z) \leq k$, then $\tau(h;AA|Z) \geq t$. That implies that $\forall\, \ell \geq 0$ $\chi e(h,k,\ell+1,t) \leq (k-1)(\ell+1) + k + \chi e(h,k,\ell,t)$ which implies that for all $\ell \geq 0$, $\chi e(h,k,\ell,t)$ is finite.

Special case: Before, we assumed that $|X \sim X_1| \leq K(AA|Z) \leq k$. But if $|X| \geq k(\ell+1)+1$, then by the ordinary pigeon hole principle,

$$\exists\, B \in \bigwedge_{P \in BB} P$$

such that $|B|_X > \ell + 1$ (since there are at most

$$k\ B\text{'s} \in \bigwedge_{P \in BB} P$$

such that $|B|_X = 0$, and that

$$\sum_{B \in \bigwedge_{P \in BB} P} |B|_X = |X|.)$$

That would imply that $|X \sim X'| \leq K(AA|X) - 1 \leq k - 1$ and that one could reduce the estimate by one.

But for Z with $|Z| \geq hk + t + (k-1)(\ell+1)$ - (the minimal possible value of $\chi e(h,k,\ell+1,t)$ for $h > \ell$ by theorem 25) - $|X|$ must be at least $kh + t$. But $kh + t \geq k(\ell + 1) + 1$ whenever $h > \ell$. $\therefore$ If $h > \ell \geq 0$, then $\chi e(h,k,\ell+1,t) \leq (k-1)(\ell+2) + \chi e(h,k,\ell,t)$. (The same estimate as before except that it's been reduced by one). Q.E.D.

Corollary: $\chi e(h,k,\ell,t) \leq hk + t + \frac{1}{2}(k-1)\ell(\ell+3)$ for $h \geq \ell$. [In fact, the above can also be shown to be true whenever $kh + t + \frac{1}{2}(k-1)(\ell-1)(\ell+2) \geq k\ell + 1$.]

Corollary: $\chi a(h, k, \ell)$ is finite for all finite h, k, ℓ. (Since $\chi a(h, k, \ell) \leq \chi e(h, k, \ell, 1)$).

The rest of this section is devoted to showing that for h sufficiently large (compared to some function of k and ℓ), $\chi a(h+\ell, k, \ell) = k(h+\ell) + 1$ and $\chi e(h, k, \ell, t) = kh + t + (k-1)$. First we will assume a useful condition on a set X_1 and a collection AA_1 of partions. Next, we obtain a set $X_2 \subset X_1$ and a collection $AA_2 \subset AA_2$ of partions. X_2 and AA_2 will have the same useful condition, and $\eta(AA_2 | X_2) < \eta(AA_1 | X_1)$. Next we prove (by induction) size estimates of $|X_1|$. This size estimate of $|X_1|$ will be eventually used to estimate both $\chi e(h, k, \ell, t)$ and $\chi a(h, k, \ell)$. Next, a process giving from a set Z and a collection AA of partions to a set $X \subset Z$ and the collection AA at partions, such that X and AA have the useful condition that we had previously only assumed. The process will give the indicated results for $\chi e(h, k, \ell, t)$; and with a little more investigation of the process, the indicated result for $\chi a(h, k, \ell)$ will be obtained.

Definition 24: A triplet *AA, BB, X* consisting of two collections AA and BB of partions, and a set X is called <u>acceptable of type r</u> if:

(1) $BB \subset AA$, $BB \neq \phi$

(2) $\eta(BB | X) = 0$

(3) $||B||_X = K(AA | X) \geq 1$, for $B = \wedge P$, $P \in BB$.

(4) $|G| = ||B||_{Supp(G)}$ $\forall$ $G \in GGet(r; AA | X)$.

Algorithm No. 1: If $*AA, BB, X*$ is an acceptable triplet of type r, and $\eta(AA|X) > 0$, do the following (If $\eta(AA|X) = 0$, do nothing).

Step 1: Choose $Y \subset X$ such that if $B \in \bigwedge_{P \in BB} P$, then

$$|B \sim Y| = \begin{cases} 1 \text{ if } 0 < |B|_X \leq \eta(AA|X) \\ 0 \text{ otherwise} \end{cases}$$

Step 2: Let $AA' = \{P \mid P \in AA, |Res(P|Y)| < \eta(AA|X)\}$.

(Note that Step 1 and Step 2 are both done in theorem 29. Since $\eta(BB|X) = 0$, we must have $BB \subset AA'$).

Step 3: Choose BB′ such that (1) $BB \subset BB' \subset AA'$,

$$(2) \qquad \Big|\Big| \bigwedge_{P \in BB'} P \Big|\Big|_Y = K(AA'|Y)$$

and (3) $|BB' \sim BB|$ is minimal over all choices of BB′ satisfying (1) and (2).

Step 4: Let $X' = Supp(\bigwedge_{P \in BB'} P \wedge [Y])$

End of Algorithm.

Algorithm No. 1 starts with AA, BB, and X and ends with AA′, BB′, and X′. We shall abbreviate Algorithm No. 1 as:

$$*AA, BB, X* \xrightarrow{\#1, r} *AA', BB', X'*$$

Theorem 31: If *AA, BB, X* is acceptable of type r, and if $\eta(AA\,|X) > 0$, then *AA′, BB′, X′* from *AA, BB, X* $\xrightarrow{\#1,\,r}$ *AA′, BB′, X′* is an acceptable triplet of type r.

Proof of theorem 31:

(1) $BB' \subset AA'$, $BB' \supset BB \neq \phi$ by construction.

(2) $\eta(BB'\,|X') = 0$ since $X' \subset \mathrm{Supp}(\bigwedge_{P \in BB'} P)$

(3) $$\Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big|_{X'} = \Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big|_{Y} = K(AA'\,|Y) \geq K(AA'\,|X') \geq \Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big|_{X'}$$

(4) Suppose $G \in \mathrm{GGet}(r;AA'\,|X') \subset \mathrm{GGet}(r;AA\,|X)$

(since by theorem 29, $Fe(AA'\,|X') \subset Fe(AA\,|X)$, which implies that $\mathrm{GGet}(r;AA\,|X') \subset \mathrm{GGet}(r;AA\,|X)$.

$$\therefore\ |G| = \Big|\Big|\bigwedge_{P\in BB} P\Big|\Big|_{\mathrm{Supp}(G)}$$

Now

$$\Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big|_{\mathrm{Supp}(G)} \leq \sum_{G\in G} \Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big| \leq \sum_{G\in G} 1 = |G|$$

But $\Big|\Big|\bigwedge_{P\in BB'} P\Big|\Big|_{\mathrm{Supp}(G)} \geq \Big|\Big|\bigwedge_{P\in BB} P\Big|\Big|_{\mathrm{Supp}(G)} = |G|$ (see Theorem 8).

Note: In going from *AA, BB, X* to a *AA′, BB′, X′* via Algorithm 1, *AA′, BB′, X′, is not necessarily unique. Since we have

GGet(r, AA′ |X′) ⊂ GGet(r, AA′ |X′), we must also have $\sigma(r;AA' \mid X') \leq \sigma(r;AA \mid X)$.

Theorem 32: If *AA, BB, X* is acceptable of type $r > K(AA \mid X) \cdot \eta(AA \mid X)$, then there exists an s, $0 \leq s \leq \sigma(r;AA \mid X)$, such that

$$|X| \leq sh + \tau(AA \mid X) + (K(AA \mid X) - s) \cdot \left[\frac{\eta(AA \mid X)\ (\eta(AA \mid X) + 1)}{2} + r - 1\right]$$

Proof of theorem 32: (By induction on $\ell = \eta(AA \mid X)$.) If $\ell = 0$, then

$$\bigwedge_{P \in BB} P \in GGem(AA \mid X)$$

and,

$$X = \bigcup_{\substack{B \in \wedge P \\ P \in BB}} (B \cap X) = \Big(\bigcup_{\substack{B \in \wedge B \\ P \in BB \\ |B|_X < r}} (B \cap X)\Big) \cup \Big(\bigcup_{\substack{B \in \wedge P \\ P \in BB \\ |B|_X \geq r}} (B \cap X)\Big)$$

Now $\Big| \big|\{B \mid B \in \wedge P,\ |B|_X \geq r\}\big| \Big|_X = s \leq \sigma(r;AA \mid X)$,
$\qquad P \in BB$

and

$$\Big| \big|\{B \mid B \in \wedge P,\ |B|_X < r\big| \Big| = K(AA \mid X) - s$$
$\qquad\qquad P \in BB$

$$\therefore \quad \Big| \bigcup_{\substack{B \in \wedge P \\ P \in BB \\ |B|_X < r}} (B \cap X) \Big| \leq (K(AA \mid X) - s)\ (r - 1);\ \text{and since}$$

$$\{B \mid B \in \bigwedge_{P \in BB} P,\ |BB|_X \geq r\} \in \mathrm{GGem}(AA|X), \text{ then}$$

$$\left|\bigcup (B \cap X)\right| \leq \tau(h;AA|X) + sh. \quad \text{Therefore, for } \ell = 0$$

$$B \in \bigwedge_{P \in BB} P, \quad |B|_X \geq r$$

$$|X| \leq sh + \tau(h;AA|X) + (K(AA|X) - s)(r - 1).$$

Now suppose that theorem 32 is true as long as $\eta(AA|X) \leq \ell$. Also suppose that *AA, BB, X* is acceptable of type $r > (\ell) \cdot K(AA|Z)$ with $\eta(AA|X) = \ell + 1$. Apply Algorithm # 1 to obtain another acceptable *AA', BB', X'* with $\eta(AA'|X) \leq \ell$. Let $s_0 = \sigma(r;AA|X)$, and let $s_1 = \sigma(r;AA'|X') \leq s_0$. Let's take a closer look at *AA, BB, X* $\xrightarrow{\#1,\, r}$ *AA', BB', X'*. In step 1, it is obvious that $(s_0 + |X \sim Y|) \leq \left|\bigwedge_{P \in BB} P\right|_X = K(AA|X)$.

Also, in step one, it is clear that $\mathrm{GGet}(r;AA|Y) = \mathrm{GGet}(r;AA|X)$ since $r > K(AA|X)\, \eta(AA|X) \geq \eta(AA|X)$.

Let $s_0' = \underset{G \in \mathrm{GGet}(r;AA|X)}{\text{maximum}} \big||G|\big|_{X'} \leq s_0$. Now by step 3 of algorithm No. 1, and by theorem 11, $|BB' \sim BB| \leq K(AA|Y) - s_0' \leq K(AA|X) - s_0'$. Combining that information with steps 2, 3, and 4, we have

$$|Y \sim X'| = \langle \bigwedge_{P \in BB'} P \rangle_Y \leq |BB' \sim BB| \cdot \eta(AA'|Y) \leq (K(AA|X) - s_0') \cdot \ell$$

but $\langle \bigwedge P \rangle \geq (s_0 - s_0')r \Rightarrow r \cdot (s_0 - s_0') \leq (K(AA|X) - s_0')\ell < r$

$$\therefore \quad (s_0 - s_0') = 0 \Rightarrow |Y \sim X'| \leq (K(AA|X) - s_0)\ell.$$

By our induction hypothesis, there exists an s, $0 \leqslant s \leqslant s_1$, such that

$$|X'| \leqslant sh + \tau(h;AA'|X') + \tfrac{1}{2}(K(AA'|X') - s)\cdot[\eta(AA|X)\cdot(\eta(AA|X) + 1) + r-1]$$

$$|X'| \leqslant sh + \tau(h;AA|X) + \tfrac{1}{2}(K(AA|X) - s)\cdot[\ell\cdot(\ell + 1) + r - 1].$$

$$|Y \sim X'| \leqslant \ell(K(AA|X) - s_0) \leqslant (K(AA|X) - s_1)\ell \leqslant (K(AA|X)-s)\cdot\ell$$

$$|X \sim Y| \leqslant K(AA|X) - s_0 \leqslant K(AA|X) - s.$$

But

$$|X| = |X \sim Y| + |Y \sim X'| + |X'|, \text{ so}$$

$$|X| \leqslant sh + \tau(h;AA'|X') + (K(AA|X) - s)\left[\frac{(\ell + 1)(\ell + 2)}{2} + r - 1\right]. \text{ Q.E.D.}$$

Algorithm No. 2: Given a set Z, a collection AA of partions, and h and r such that $(h + 1) \geqslant r \geqslant K(AA|Z)\cdot\eta(AA|Z)$, and a $G_0 \in GGet(r;AA|Z)$ such that $|G_0| = \sigma(r;AA|Z)$. Do the following.

Step 1: Order the members of G_0 ($G_0 = \{G_i | 1 \leqslant i \leqslant \sigma(r;AA|Z\}$)

Step 2: Find a P_2 (see theorem 16) such that $G_1 \not\simeq G_2 \pmod{P_2}$. If $|G_2| > h$ and $\mu a(AA|Z) \leqslant h + \ell$, be sure that $Supp(P_2|G_2) \leqslant h$ (see theorem 18).

Step 3: Having found $P_2, P_3, \ldots, P_n$, choose any $i \leqslant n$ such that for $2 \leqslant j \leqslant n$, $G_i \simeq G_{n+1} \pmod{P_j}$. (If no such i exists, let i = n). The value of i is uniquely determined. Choose a P_{n+1} such that $G_i \not\simeq G_{n+1} \pmod{P_{n+1}}$. If $|G_{n+1}| > h$, and if $\mu a(AA|Z) \leq h + \ell$, be sure that $\mathrm{Supp}(P_{n+1} | G_{n+1}) \leq h$.

Step 4: Choose a $BB \subset AA$ such that (1) $P_i \in BB, 2 \leq i \leq \sigma(r; AA|Z)$ (2) $\| \bigwedge_{P \in BB} P \|_X = K(AA|\mathrm{Supp}(\bigwedge_{i=2}^{s_0} P_i))$ where $s_0 = \sigma(r; AA|Z)$, and where (3) $|BB|$ is minimum over all choice of BB that satisfy (1) and (2).

Step 5: Let $X = \mathrm{Supp}(\bigwedge_{P \in BB} P | Z)$.

Theorem 33: For the AA, BB, and X produced in Algorithm #2, *AA, BB, X* is acceptable of type r if $K(AA|Z) > 1$.

Proof of Theorem 33: Obviously $BB \subset AA$, and $BB \neq \phi$ if $K(AA|Z) > 1$. By construction

$$X \subset \mathrm{Supp}(\bigwedge_{P \in BB} P) \Rightarrow \eta(BB|X) = 0 \quad .$$

$K(AA|X) \geq 1$, as long as $X \neq \phi$. X can be the empty set iff

$$X = \mathrm{Res}\,(\bigwedge_{i=2}^{s_0} P_i | Z) \quad ,$$

but $X = \phi$ implies either that

$$s_0 r \leq \langle \bigwedge_{i=2}^{s_0} P_i | Z \rangle \leq (s_0 - 1)\ell < s_0 r \quad ,$$

and if $\sigma(r; AA|Z) > 0$ (that is impossible), or that $0 = |X| \geq K(AA|Z) > 0$ (which impossible) if $\sigma(r; AA|Z) = 0$. Therefore, $K(AA|X) \geq 1$. By construction

$$| \bigwedge_{P \in BB} P |_X = K(AA|X) \quad .$$

All that remains to be shown is that if $G \in \text{GGet}\ (r; AA|X)$, that

$$|G| = \|\bigwedge_{P \in BB} P\|_{\text{Supp}(G)} .$$

Since

$$\text{Supp}(G) \subset \text{Supp}(\bigwedge_{P \in BB} P), \text{then } |G| = \|\bigwedge_{P \in BB} P\|_{\text{Supp}(G)}$$

unless there exists a $\{F_1, F_2\} \in \text{GGet}(r; AA|X)$ such that $F_1 \simeq F_2$ (Mod $\bigwedge_{P \in BB} P$). By construction, there can be at most one $G \in G_0$ such that $G \simeq F_1 \simeq F_2$ (Mod P_i) for $2 \leqslant i \leqslant s_0$. If such a G exists, let $G_1 = \{F_1, F_2\} \cup G_0 \sim \{G\}$. Otherwise, let $G_1 = \{F_1, F_2\} \cup G_0$. In either case, $G_1 \in \text{GGet}(r; AA|Z)$, but $|G_1| > |G_1| = \sigma(r; AA|Z)$ which is a contradiction to the definition of $\sigma(r; AA|Z)$. Hence, for all $G \in \text{GGet}(r; AA|X)$, $|G| = |\bigwedge_{P \in BB} P|_{\text{Supp}}(G)$.

Theorems 34 and 35: (34) $\chi e(h, k, \ell, t) = kh + t + (k-1)\ell$ for $h \geqslant k\ell + \frac{1}{2}\ell(\ell+3)$

(35) $\chi a(h+\ell, k, \ell) = k(h+\ell) + 1$ for $h \geqslant k\ell + \frac{1}{2}\ell(\ell+3)$.

Proof of Theorems 34 and 35: Suppose we are given any set Z, and a collection AA of portions such that $\eta(AA|Z) \leqslant \ell$ and $K(AA|Z) \leqslant k$. Choose any $G_0 \in \text{GGet}(k\ell+1; AA|Z)$ such that (1) $|G_0| = \sigma(k\ell+1; AA|Z) = s_0$; (2) that for all $x \in \text{Res}(G_0|Z)$ there does not exist a $G \in G_0$ such that $G \cup \{x\} \in Fe(AA|Z)$, and (3) that $|\{H | H \in G_0, |H| > h\}|$ is maximal over all choices of G_0' that satisfy just (1) and (2). Apply Algorithm #2 to obtain an *AA, BB, X* that is acceptable of type $(k\ell+1)$. Let's estimate $|Z \sim X|$. Let

$$s_0' = \max_{G \in \text{GGet}(k\ell+1; AA|Z)} |G|_X ,$$

and let $s_1 = \sigma(k\ell+1;AA|X)$. By Theorems 11 and 33, $|Z \sim X| \leq (s_0-1)\ell + (k-s_0')\ell$. But $|Z \sim X| \geq (s_0-s_0')(k\ell+1)$, hence $(s_0-s_0')(k\ell+1) \leq (k-1)\ell + (s_0-s_0')\ell$. Therefore, $(s_0-s_0')[(k-1)\ell+1] \leq (k-1)\ell$, which implies that $s_0 = s_0'$. Furthermore $|Z \sim X| \leq (k-1)\ell$. By Theorem 32, we have that there exists an s, $0 \leq s \leq s_1$ such that $|X| \leq sh + \tau(h;AA|X) + (k-s)[\frac{1}{2}\ell(\ell+3) + k\ell]$. But $|Z| = |Z \sim X| + |X|$, hence $|Z| \leq (k-1)\ell + sh + (k-s)[\frac{1}{2}\ell(\ell+3) + k\ell] + \tau(h;AA|X)$, or in other words, $\tau(h;AA|X) \geq |Z| - (k-1)\ell - kh + (k-s)[h - \frac{1}{2}\ell(\ell+3) - k\ell]$. If $h \geq \frac{1}{2}\ell(\ell+3) + k\ell$, then $\tau(h;AA|X) \geq |Z| - (k-1)\ell - kh$. Furthermore, if $|Z| \geq kh + t + (k-1)\ell$, then we have that $\tau(h;AA|Z) \geq \tau(h;AA|X) \geq t$, which proves Theorem 34. If $|Z| \geq k(h+\ell)+1$, then $\tau(h;AA|Z) \geq \ell + 1$. Suppose that $\mu a(AA|Z) \leq h + \ell$, then for $\tau(h;AA|Z) \geq \ell+1$ to be true, $\sigma(h+1;AA|Z) \geq 2$. (Otherwise $\mu a(AA|Z) \geq \mu e(AA|X) \geq h + \tau(h;AA|X) \geq h+\ell+1$.) Let $\boldsymbol{H}_0 \in GGet(h+1;AA|Z)$, $|\boldsymbol{H}_0| = 2$. It is possible that for one $H^* \in \boldsymbol{H}_0$ that

$$H^* \simeq G_1(\in \boldsymbol{G}_0)\,(\mathrm{Mod} \bigwedge_{\boldsymbol{P}\in BB} \boldsymbol{P}) \quad ,$$

but for any other $H(\neq H^*) \in \boldsymbol{H}_0$,

$$H \not\simeq G_1\ (\mathrm{Mod} \bigwedge_{\boldsymbol{P}\in BB} \boldsymbol{P}) \quad .$$

But there must exist a $G \in \boldsymbol{G}_0$ such that

$$H \simeq G\ (\mathrm{Mod} \bigwedge_{\boldsymbol{P}\in BB} \boldsymbol{P}) \quad .$$

(Otherwise $\boldsymbol{G}_0 \cup \{H\} \in GGet(k\ell+1;AA|Z)$, which would imply the contradictory fact that $|\boldsymbol{G}_0 \cup \{H\}| > |\boldsymbol{G}_0| = s_0$). Furthermore, for that

$G \in \mathcal{G}_0$ such that

$$H \simeq G(\mathrm{Mod} \bigwedge_{P \in BB} P), \quad |G| > h .$$

(Otherwise, $(\mathcal{G}_0 \cup \{H\}) \sim \{G\}$ would indicate that $\mathcal{G}_0$ did not satisfy (3).) What is $|H \cap G|$? If $|H \cap G| > \ell$, then $H \cup G \in \mathcal{F}e(AA|Z)$. Since $|H \sim G| \geq 1$ (by step 3 of Algorithm #2, we would have a contradiction that BB satisfied (2)). Hence $|H \cap G| \leq \ell$. Consider the set $H \sim G$. Now $|H \sim G| = |H| - |H \cap G| \geq (h+1) - \ell \geq k\ell + 1$. But for all $G' \in \mathcal{G}_0$, $G' \cup (H \sim G) \notin \mathcal{F}e(AA|Z)$, and $G' \cap (H \sim t) = \phi$. Hence $\mathcal{G}_0 \cup \{H \sim G\} \in GGet(k\ell+1;AA|Z)$, but this contradicts the fact that BB satisfies (1). Hence for $|Z| \geq k(h+\ell)+1$, $\mu a(AA|Z) > h + \ell$, and Theorem 35 is proved.

Section 4

Special Case: Evaluating $\chi e(h, k, 1, t)$ and $\chi a(h, k, 1)$

Theorems 36 and 37: (36) $\chi e(h, k, 1, t) = kh + t + k - 1$ for $h \geq 2$

(37) $\chi a(h, k, 1) = kh + 1$ for $h \geq 3$

Proof of Theorems 36 and 37: Given: AA, Z, $\eta(AA|Z) = 1$, $k(AA|Z) \leq k$, $|Z| < \infty$. Choose any $BB \subset AA$ such that for $\mathcal{B} = \bigwedge_{P \in BB} P$

(1) $||\mathcal{B}||_Z = K(AA|X)$ and

(2) $||\{B|B \in \mathcal{B}, |B|_Z \leq 1\}||_Z = m$ is minimal over all choices of BB that satisfy (1).

Theorem 11 insures us that BB can also be chosen to satisfy

(3) $|BB| < K(AA|Z) \leq k$.

Lemma: If $\mu a(AA|Z) \leq h + 1$, then BB can be chosen to satisfy

(4) $\tau(h;AA|\mathrm{Supp}(\mathcal{B}|Z)) \leq 1$.

Proof of Lemma: Take any BB that satisfies (1), (2), and (3). Suppose that $H \subset \mathrm{Supp}(\mathcal{B})$, that $H \in \mathcal{F}e(AA|Z)$, and that $|H| = h+1$. By Theorem 18, $\forall x \in (Z \sim H)$, there exists an $P_{x,H} \in AA$ such that $H \not\leq \{x\}$ (Mod $P_{x,H}$), and that $\mathrm{Supp}(P_{x,H}|H) = h$. (Note that for this special case of $\eta(AA|Z) = 1$, that $H \sim \mathrm{Supp}(P_{x,H}) = \mathrm{Res}(P_{x,H}|Z)$.) In particular, we can conclude that if $x \in (\mathrm{Supp}(\mathcal{B}) \sim H)$, that $\{x\} \not\leq H$ (Mod $\mathcal{B}$). (Otherwise $||P_{x,H} \wedge \mathcal{B}||_Z > ||\mathcal{B}||_Z = K(AA|Z)$, a contradiction to the definition of $K(AA|Z)$.) Hence $H \in \mathcal{B} \wedge [Z]$. Let $\mathcal{H}_0 = \{H|H \subset \mathrm{Supp}(\mathcal{B}), H \in \mathcal{F}e(AA|Z), |H| = h+1\}$. It is obvious that

$H_0 \in \text{GGet}(h+1;AA|Z)$. It is also obvious that $|H_0| = \sigma(h+1;AA|\text{Supp}(B))$. Applying Algorithm #2 to Supp(B), AA, H_0, r = h + 1, h = h (ignoring the restriction on r), one obtains a set BB_0^* such that for

$$B_0^* = \bigwedge_{P \in BB_0^*} P \quad ,$$

$||B_0^*||_{\text{Supp}(H_0)} = |H_0|$, and that $\forall\, H_i, H_j \in H_0$ (i < j) $H_i \not\preceq H_j$ (Mod B_0^*) (the ordering of the members of H_0 was done by Algorithm #2), and that $|H_1| = h+1$; $|H_i| = h$ for i > 1. (Again the fact that $\eta(AA|Z) = 1$ uniquely determines $|\text{Res}(B_0^*|H_i)|$ and $|\text{Res}(B_0^*|(Z \sim \text{Supp}(H_0)))|$.) Hence, $||B \wedge B_0^*|| = ||B|| = K(AA|Z)$ and furthermore, $\{B|B \in (B \wedge B_0^* \wedge [Z]),\ |B| \leq 1\} = \{B|B \in B \wedge [Z],\ |B| \leq 1\}$. Theorem 11, insures us that there exists a BB_1, with $BB_0^* \subset BB_1 \subset BB_0^* \cup BB$ such that BB_1 satisfies (1), (2), and (3). Let

$$B_1 = \bigwedge_{P \in BB_1} P \quad .$$

Again if $H \in H_0$, and if $x \in Z \sim H$ there exist a $P_{x,H} \in AA$ such that $\{x\} \not\preceq H$ (Mod $P_{x,H}$), and such that $\text{Supp}(P_{x,H}|H) = h$. From which we can conclude that for any such $P_{x,h}$, if $h \geq 2$, then $||P_{x,H} \wedge B_1|| \geq K(AA|Z) + \langle B_1 \rangle_{\{x\}} (2 - \langle B_1 \rangle_{H \cup \{x\}})$. In particular, if $x \in (\text{Supp}(B_1) \sim H)$, then $x \not\preceq H$ (Mod B_1). In other words, $\forall\ H \in H_0$, $H \cap \text{Supp}(B_1) \in B_1 \wedge [Z]$. Let $H_1 = \{H|H \subset \text{Supp}(B_1),\ H \in Fe(AA|Z),\ |H| = h+1\} \sim \{H_1\}$ ($H_1 \in H_0$, and was so named H_1 by the application of Algorithm #2). For all $H \in H_1$, $H \cap \text{Supp}(H_0) = \phi$ by construction. And as before, $||B_1||_{\text{Supp}[H_0 \cup H_1)]} = |H_0 \cup H_1|$, $||H_0 \cup H_1|| = \sigma(h+1;AA|\text{Supp}(H_0 \cup H_1))$, $H_0 \cup H_1 \in \text{GGett}(h+1;AA|Z)$. Again, we may apply a watered-down version of Algorithm #2 (ignoring the restriction on r), one obtains a BB_1^* such that $BB_0^* \subset BB_1^*$ (by appropriate ordering of $H_0 \cup H_1$ and choosing the P_i appropriately to correspond to the previous time Algorithm #2 was applied). This in turn generates a B_1^*, and finally a BB_2, $BB_1^* \subset BB_2 \subset AA$ that satisfy (1), (2), and (3). In a like manner, one can proceed to find a H_3, BB_4, etc. Since

$$\bigcup_{i=1}^{\infty} H_i \cup \{\phi\}$$

is an incomplete portion of Z, then

$$\sum_{i=1}^{\infty} |\mathrm{Supp}(H_i)| \leq |Z| < \infty \quad .$$

Therefore, there exists an H_n, $n < \infty$ such that $H_n = \phi$. Then for BB_n

$$(\text{and } B_n = \bigwedge_{P \in BB_n} P)$$

BB_n satisfies (1), (2), and (3), and also

(4) $\tau(h; AA\,|\,\mathrm{Supp}(B_n\,|\,Z)) = |H_1| - h = 1$. (Here I tacitly assumed that $H_0 \neq \phi$; otherwise (4) $\tau(h; AA\,|\,\mathrm{Supp}(B\,|\,Z)) = 0$).

End of Lemma.

Having found a BB, and B that satisfy (1), (2), (3), and perhaps even (4) (if $\mu a(AA\,|\,Z) \leq h+1$) let

$$X = \bigcup_{\substack{B \in B^\wedge[Z] \\ |B| > 1}} B \quad ,$$

and let $AA' = \{P \mid P \in AA, \langle P \rangle_X = 0\}$. By Theorem 29, $\mathcal{F}b(AA' \mid X) = \mathcal{F}b(AA \mid X) \subset \mathcal{F}b(AA \mid Z)$, and $||B||_X = K(AA \mid Z) - m$. Since $\eta(AA' \mid X) = 0$, then

$$\Big|\Big| \bigwedge_{P \in AA'} P \Big|\Big| = K(AA' \mid X) \quad .$$

By Theorem 11, there exist a $BB' \subset AA'$ such that

(5) $$BB \subset BB',$$

(6) $$\bigwedge_{P \in BB'} P \overset{X}{\cong} \bigwedge_{P \in AA'} P \quad , \text{ and}$$

(7) $$|BB' \sim BB| \leqslant K(AA' \mid X) + m - K(AA \mid Z) \quad .$$

By construction,

$$B' = \bigwedge_{P \in AA'} P \wedge [X] \in \mathrm{GGet}(0;AA' \mid X) \subset \mathrm{GGet}(0;AA \mid Z) \quad .$$

Let $B^* = \bigwedge_{P \in BB'} P$.

Lemma: $\Big|\Big| \bigwedge_{P \in BB'} P \Big|\Big|_Z = K(AA \mid Z)$.

Proof of Lemma: $||B^*||_Z = ||B \wedge B^*||_Z$

$$= \sum_{B \in \mathcal{B}} ||B^*||_{B \cap Z} \geqslant [\,||B^*||_X + (m - \langle B^* \rangle_{\mathrm{Supp}(B)})$$

$$\therefore K(AA \mid Z) \geqslant ||B^*|| \geqslant ||B^*||_X + m - |BB' \sim BB|$$

$$\geqslant K(AA' \mid X) + m - [K(AA \mid Z)\,] = K(AA \mid Z) \quad .$$

End of Lemma.

By construction, $\|\{B \mid B \in \mathcal{B}^*, \ |B|_Z \leqslant 1\}\|_Z \geqslant m$. If $h \geqslant 1$, (and since $\mathcal{B}' \in \mathrm{GGet}(0;AA \mid Z)$), then

$$\mathrm{Supp}(\mathcal{B}^* \mid Z) \leqslant m + h \cdot (K(AA \mid Z) - m) + \tau(h;AA' \mid X)$$

$$\mathrm{Supp}(\mathcal{B}^* \mid Z) \leqslant m + h \cdot (K(AA \mid Z) - m) + \tau(h;AA \mid X) \quad .$$

But
$$\begin{aligned} |Z| &\leqslant |\mathrm{Res}(\mathcal{B}^* \mid Z)| + |\mathrm{Supp}(\mathcal{B}^* \mid Z)| \\ &\leqslant |BB'| + [m + (K(AA \mid Z) - m) \cdot h + \tau(h;AA \mid X)] \\ &\leqslant [\,|BB| + |BB' \sim BB|\,] + m + (K(AA \mid Z) - m)h + \tau(h;AA \mid X) \\ &\leqslant [K(AA \mid Z) - 1] + [K(AA \mid X) + m - K(AA \mid Z)] + m \\ &\qquad\qquad + (K(AA \mid Z) - m)h + \tau(h;AA \mid X) \\ &\leqslant [K(AA \mid X) - 1] + h \cdot K(AA \mid Z) - m(h-2) + \tau(h;AA \mid X) \end{aligned}$$

$$\therefore \ |Z| \leqslant K(AA \mid Z) - 1 + h \cdot K(AA \mid Z) + \tau(h;AA \mid X) \quad \text{if } h \geqslant 2.$$

Hence, $\tau(h;AA \mid Z) \geqslant \tau(h;AA \mid X) \geqslant |Z| - K(AA \mid Z) \cdot h - [K(AA \mid Z) - 1]$ for $h \geqslant 2$. Therefore if $|Z| \geqslant kh + t + k - 1)$, and if $K(AA \mid Z) \leqslant K$, then $\tau(h;AA \mid Z) \geqslant t$. (Theorem 36).

In particular if $|Z| \geqslant k(h+1) + 1 = kh + 2 + (k-1)$, then $\tau(h;AA \mid X) \geqslant 2$. $\therefore$ (4) cannot hold, and hence $\mu a(AA \mid Z) > h+1$ for $h \geqslant 2$ (Theorem 37).

Section 5

Conjectures and Other Remarks

Chapter I, definitely showed that for $h \geqslant (k-1)\ell + \frac{1}{2}\ell(\ell+3)$ that $\chi a(h+\ell, k, \ell) = k(h+\ell)+1$ and that $\chi e(h, k, \ell, t) = hk + t + (k-1)\ell$. Not presented at this time are results that show

(1) if $h \geqslant \frac{1}{2}\ell(\ell+3) + 2\ell$ then $\chi e(h, k, \ell, t) = kh + t + (k-1)\ell$, and

(2) $h \geqslant \frac{1}{2}\ell(\ell+3) + (k-1)^{\frac{1}{3}}\ell^{\frac{2}{3}} + o(k^{\frac{1}{3}}\ell^{\frac{2}{3}})$, then $\chi a(h, k, \ell) = hk+1$, and

(3) if $h \leqslant \ell$, then $\chi e(h, k, \ell, 1) > k(h+\ell)$.

I would like to conjecture on even better results:

Conjecture #1: $\chi a(h, k, 1) = hk + 1$ for all h.

Conjecture #2: $\chi e(h, k, \ell, t) = kh + t + (k-1)\ell$ for all $h \geqslant \frac{1}{2}\ell(\ell+3)$ (and perhaps for all $h \geqslant 2\ell$.)

Conjecture #3: $\chi e(h, k, \ell, t) \leqslant kh + t + k\ell$ for all h.

Conjecture #4: For some values of h, k, and ℓ, $\chi e(h, k, \ell, t)$ will not be a polynomial in t (i.e., there is some nontrivial t dependence).

Conjecture #5: There does not exist a function $\alpha(\ell, t)$ such that for $k > 1$ that $\chi e(h, k, \ell, t) = kh + t + (k-1)\ell$ iff $h \geqslant \alpha(\ell, t)$. [I expect something freaky in either the range $2 \leqslant k \leqslant 2\ell$ or the range $2 \leqslant k \leqslant \frac{1}{2}\ell(\ell+3)$. I also expect this strange behavior to differentiate between small values of k and medium values of k.]

Conjecture #6: There exists a function $\alpha(k, \ell, t)$ such that

(1) $\chi e(h, k, \ell, t) = kh + t + (k-1)\ell$ iff $h \geqslant \alpha(k, \ell, t)$

(2) $\alpha(k, \ell, t)$ is a nonincreasing function of t.

CHAPTER II: "An Abstraction of Polygonally Connected Sets" and "On the Unions of Convexly Disjoint Convex Sets"

This chapter is mainly concerned with Grünbaum and Motzkin's conjecture on the Helly's number of certain special collections of unions of convex sets. In Grünbaum and Motzkin's original paper, it is by no means clear what the unions of convex sets should look like, or even what the convex sets themselves could be. To alleviate that difficulty, I place no conditions on the convex sets themselves other than that the system of convex sets must be closed under finite intersection, and I develop in detail my definition of convexly disjoint sets. In an Euclidean space, it turns out that two nonvoid convex sets C_1 and C_2 are convexly disjoint iff their union is not polygonally connected. For a finite-dimensional Euclidean space, it turns out that two convex sets C_1 and C_2 are convexly disjoint iff $C_1 \cap \overline{C}_2 = \overline{C}_1 \cap C_2 = \phi$, where $\overline{C}_1$ and $\overline{C}_2$ are the topological closures of C_1 and C_2, respectively. In Euclidean spaces, unions of three, four, five, etc., nonvoid convex sets can be similarly defined. In developing my definition of convexly disjoint sets, I proceed to develop the abstract notion of polygonally connected sets. I do not know whether anyone else has developed the abstract notion of polygonally connected sets, and I do not know if any of the theorems concerning polygonally connected sets (in the abstract sense) is original in this paper.

The notations of Chapter I will also be used in Chapter II. All the theorems and definitions (occasionally slightly reworded) of Chapter I will be assumed in Chapter II. Also, the notation for the power set $\mathcal{Q}^X = \{Y \mid Y \subset X\}$ will be used.

Definition 25: $\mathcal{C} \subset \mathcal{Q}^X$ is said to be an abstract convexity of X if the members of $\mathcal{C}$ are closed under finite intersection (i.e., if A and B $\in \mathcal{C}$, then $A \cap B \in \mathcal{C}$), and if both ϕ and X $\in \mathcal{C}$. Elements of $\mathcal{C}$ will be called convex sets or abstract convex sets. Unless otherwise indicated, any abstract convexity will be an abstract convexity of the set X.

Definition 26: $\mathcal{A} \subset \mathcal{Q}^X$ is said to be a collection of convexly disjoint sets (with respect to some abstract convexity $\mathcal{C}$) if $\mathcal{A}$ has the following properties:

(1) $\mathcal{A}$ is an incomplete partition of X.

(2) $\forall C \subset \text{Supp}(\mathcal{A})$, where $C \in \mathcal{C}$, there exists an $A \in \mathcal{A}$ such that $C \in \mathcal{A}$.

Note: (2) can also be written as $\forall C \subset \text{Supp}(\mathcal{A})$ (with $C \in \mathcal{C}$), and $\forall\ c \in C$, $C \subset \text{Mat}(c;\mathcal{A})$.

Theorem 38: If $\mathcal{A}$ and $\mathcal{B}$ are both collections of convexly disjoint sets, then $\mathcal{R} = (\mathcal{A} \wedge \mathcal{B})$ is also a collection of convexly disjoint sets.

Proof of Theorem 38: By Theorem 1, $\mathcal{D}$ is an incomplete partion of X. If $C \subset \text{Supp}(\mathcal{R})$, and if $C \in \mathcal{C}$, then $C \subset \text{Supp}(\mathcal{A}) \cap \text{Supp}(\mathcal{B})$. Hence $C \subset \text{Supp}(\mathcal{A})$ and $C \subset \text{Supp}(\mathcal{B})$. But both $\mathcal{A}$ and $\mathcal{B}$ are collections of convexly disjoint sets, so there exists an $A \in \mathcal{A}$ and a $B \in \mathcal{B}$ such that $C \subset A$ and $C \subset B$. Therefore, $C \subset (A \cap B) \in \mathcal{R}$, which is all that was needed to be shown.

Theorem 39: For all $Y \subset X$, and for any abstract convexity of X, $[Y]$ is a collection of convexly disjoint sets.

Proof of Theorem 39 is obvious.

Corollary to Theorems 38 and 39: If $\mathcal{A}$ is a collection of convexly disjoint sets, and if $Y \subset X$, then $\mathcal{B} = \mathcal{A} \wedge [Y] = \{B \mid B = A \cap Y,\ A \in \mathcal{A}\}$ is also a collection of convexly disjoint sets.

Theorem 40: If $\mathcal{A}$ is a collection of convexly disjoint sets, and if $\mathcal{B}$ is an incomplete partition of X such that for all $B \in \mathcal{B}$ there exists an $\mathcal{A}' \subset \mathcal{A}$ such that $B = \text{Supp}(\mathcal{A}')$, then $\mathcal{B}$ is a collection of convexly disjoint sets.

Proof of Theorem 40: Suppose that $C \in \mathcal{C}$, and that $C \subset \text{Supp}(\mathcal{B})$, then $C \subset \text{Supp}(\mathcal{B}) \subset \text{Supp}(\mathcal{A})$. Therefore for any $c \in C$, $C \subset \text{Mat}(c; \mathcal{A})$. But $\text{Mat}(x; \mathcal{A}) \subset \text{Mat}(x; \mathcal{B})$ for all $x \in \text{Supp}(\mathcal{B})$, therefore for any $c \in C$, $C \subset \text{Mat}(c; \mathcal{A}) \subset \text{Mat}(c; \mathcal{B})$. Q.E.D.

Theorem 41: If $\mathcal{A}$ is a collection of convexly disjoint sets such that $\text{Supp}(\mathcal{A}) \in \mathcal{C}$ (where $\mathcal{C}$ is the abstract convexity), then $\mathcal{A} = [\text{Supp}(\mathcal{A})]$.

Proof of Theorem 41: $\text{Supp}(\mathcal{A}) \in \mathcal{C}$, and $\mathcal{A}$ is a collection of convexly disjoint sets; so, $\forall\ a \in \text{Supp}(\mathcal{A})$, $\text{Supp}(\mathcal{A}) \subset \text{Mat}(a; \mathcal{A}) \subset \text{Supp}(\mathcal{A})$. Hence $\text{Supp}(\mathcal{A}) \in \mathcal{A}$. But $\mathcal{A}$ is a partion, and it is obvious that $\mathcal{A} = [\text{Supp}(\mathcal{A})]$, since $\text{Supp}(\mathcal{A}) \in \mathcal{A}$.

Definition 27: A set D is connected in a topology $\mathcal{T}$, if in the relative topology $\mathcal{T}_D = \{T \mid T = S \cap D,\ S \in \mathcal{T}\}$ the set D cannot be represented by the union of two disjoint non-void sets of $\mathcal{T}_D$.

Definition 28: F is a collection of relatively open sets in a topology T if for all F ϵ F, there exist a T ϵ T such that F = T $\cap$ Supp(F).

Theorem 42: If all the members of C (an abstract convexity) are connected in a topology T, and if A is an incomplete partition of X, and if A is a collection of relatively open sets in T, then A is a collection of convexly disjoint sets.

Proof of Theorem 42: Suppose C ϵ C, and C $\subset$ Supp(A), then $[C] \wedge A$ is a collection of relatively open sets. But C is connected, so there exists an A ϵ A such that C $\subset$ A $\Rightarrow$ theorem.

Definition 29: E is a unit set if [E] is the only collection A of convexly disjoint sets such that E = Supp(A).

Note: The notion of unit sets is my abstraction of 'polygonally connected sets'. All convex sets are unit sets. The tie-in of unit sets with polygonally connected sets will become obvious.

Theorem 43: If all the members of C are connected in some topology T, and if Y is a unit set, then Y is connected in T.

Proof of Theorem 43: Suppose Y = Supp(Z) where Z is an incomplete portion of Y and where Z is a collection of relatively open sets. By Theorem 42, Z is a collection of convexly disjoint sets. By Definition 29, $Z = [Y] = \{\phi\} \cup \{Y\}$. $\therefore$ Y is connected in T.

Theorem 44: If all the members of C are connected in some topology T, and if for $A \subset \mathcal{Q}^X$ ($|A| < \infty$) A is not a collection of convexly disjoint sets then either there exists an $A \in \mathcal{A}$ such that $\bar{A} \cap (\mathrm{Supp}(\mathcal{A} \sim [A]) \neq \phi$ (where $\bar{A}$ is the closure of the set A in the topology $\mathcal{T}$) or $\phi \notin \mathcal{A}$.

Proof of Theorem 44: Suppose for all $A \in \mathcal{A}$, $\bar{A} \cap \mathrm{Supp}(\mathcal{A} \sim \{A\}) = \phi$. For all $A \in \mathcal{A}$, $\mathrm{Ext}(A) = (X \sim \bar{A})$ is open. For all other $A' \in \mathcal{A}$ $(A' \neq A)$, $A' \subset \mathrm{Supp}(\mathcal{A} \sim \{A\}) \subset \mathrm{Ext}(A)$. Hence

$$A' \subset \bigcap_{\substack{A \in \mathcal{A} \\ A \neq A'}} \mathrm{Ext}(A) \quad .$$

But since $|\mathcal{A}|$ is finite

$$\bigcap_{\substack{A \in \mathcal{A} \\ A \neq A'}} \mathrm{Ext}(A) \in \mathcal{T} \quad .$$

Therefore $\mathcal{A}$ is a collection of relatively open sets. But $\bar{A} \cap \mathrm{Supp}(\mathcal{A} \sim \{A\}) = \phi$ implies that for all $A' \in \mathcal{A}$ $(A' \neq A)$ that $A' \cap A \subset \mathrm{Supp}(\mathcal{A} \sim \{A\}) \cap \bar{A} = \phi$. Hence, $\mathcal{A} \cup \{\phi\}$ is an incomplete partition of X. By Theorem 42, $\mathcal{A} \cup \{\phi\}$ is a collection of convexly disjoint sets. We must then conclude that $\phi \notin \mathcal{A}$. Q.E.D.

Restating Theorem 44: If all the members of a convexity $\mathcal{C}$ are connected in a topology $\mathcal{T}$, and if $\mathcal{A}$ is a collection of sets such that $\forall\ A \in \mathcal{A}$, $\bar{A} \cap \mathrm{Supp}(\mathcal{A} \sim \{A\}) = \phi$, then $(\mathcal{A} \cup \{\phi\})$ is a collection of convexly disjoint sets.

Note: For convex sets in $\mathcal{R}^m$(Euclidean space) the condition

$$\bar{C}_i \cap (\bigcup_{j \neq i} C_j) = \phi$$

for $1 \leqslant i \leqslant n$ is both a <u>necessary</u> and sufficient condition that

$\{C_1, C_2, \ldots, C_n, \phi\}$ be a collection of convexly disjoint <u>convex</u> sets. In general, it is not necessary that

$$\bar{C}_i \cap \Big(\bigcup_{j \neq i} C_j \Big) = \phi \qquad (1 \leq i \leq n)$$

for $\{C_1, C_2, \ldots, C_n, \phi\}$ to be a collection of convexly disjoint sets whenever the convexity $\mathcal{C}$ has a topology $\mathcal{T}$ such that all the members of $\mathcal{C}$ are connected. (As an example, in R^2 let the set of convex sets be rectangles whose edges are either parallel to the x-axis or to the y-axis. An open rectangle sharing a corner with a closed rectangle would be a pair of convexly disjoint connected convex sets that doesn't satisfy $\bar{C}_1 \cap C_2 = \phi = C_1 \cap \bar{C}_2$.)

Definition 30: For any abstract convexity $\mathcal{C}$ of X, and for any $Y \subset X$, define

$$Gp_0(x;Y) = \begin{cases} \{x\} & \text{if} \quad x \in Y \\ \phi & \text{if} \quad x \notin Y \end{cases}$$

$$Gp_1(x;Y) = \begin{cases} \bigcup\limits_{x \in C \in \mathcal{C},\, C \subset Y} C & \text{if} \quad x \in Y \\ \phi & \text{if} \quad x \notin Y \end{cases}$$

for $n > 1$,

$$Gp_{n+1}(x;Y) = \begin{cases} \bigcup\limits_{x^* \in Gp_n(x;Y)} Gp_1(x^*;Y) & \text{if} \quad Gp_n(x;Y) \neq \phi \\ \phi & \text{if} \quad Gp_n(x;Y) = \phi \end{cases}$$

$$Gpa(x;Y) = \bigcup_{i=0}^{\infty} Gp_i(x;Y) .$$

Note that: $Gp_n(x; Y) \subset Gp_{n+1}(x; Y)$.

Theorem 46: If $y \in Gp_n(x; Y)$, then

(1) $x \in Gp_n(y; Y)$ and

(2) $Gp_k(y; Y) \subset Gp_{n+k}(x; Y)$ $\forall k \geq 0$.

The proof of Theorem 46 is obvious.

Theorem 47: $Gpa(x; Y)$ is a unit set.

Proof of Theorem 47:

Case I: $x \notin Y$, then $Gpa(x; Y) = \phi \in C$. $\therefore$ $Gpa(x; Y)$ is a unit set.

Case II: $x \in Y$. Suppose $Gpa(x; Y) = Supp(\mathcal{A})$ where $\mathcal{A}$ is a collection of convexly disjoint sets. Let $A = Mat(x; \mathcal{A})$. Suppose that $A \neq Gpa(x; Y)$, then there exists an n ($< \infty$) such that $Gp_n(x; Y) \subset A$, but $Gp_{n+1}(x; Y) \not\subset A$. (Otherwise $Gpa(x; Y) = A$). Let $z \in Gp_{n+1}(x; Y) \sim A \neq \phi$. But by definition, there exists an $x^* \in Gp_n(x; Y)$ and there exists a $C \in C$ such that (1) $z \in C$, $x^* \in C$, and (2) $C \subset Y$. Hence $x^* \not\approx z$ (Mod $\mathcal{A}$) $\Rightarrow$ $2 = ||\mathcal{A}||_{\{x^*, z\}} \leq ||\mathcal{A}||_C = 1$. (A contradiction.) Hence $A = Gpa(x; Y)$, and $\mathcal{A} = [A]$. Hence $Gpa(x; Y)$ is a unit set.

Theorem 48: If $\mathcal{A}$ is a collection of convexly disjoint sets, and if $a \in \mathbf{A} \in \mathcal{A}$, then $Gpa(a; Supp(A)) \subset A$.

Proof of Theorem 48: $\mathcal{A} \wedge [Gpa(a; Supp(\mathcal{A})]$ is a collection of convexly disjoint sets. But $Gpa(a; Supp(\mathcal{A})) \subset Supp(\mathcal{A} \wedge [Gpa(a; Supp(\mathcal{A})])$ $\subset Gpa(a;Supp(\mathcal{A}))$. Since $Gpa(a;\mathcal{A})$ is a unit set, (and since $\mathcal{A} \wedge [Gpa(a; Supp(\mathcal{A})]$ is a collection of convexly disjoint sets), $\mathcal{A} \wedge [\mathrm{Gpa}(a; A)] = [\mathrm{Gpa}(a; A)]$. Hence $\mathrm{Gpa}(a; \mathrm{Supp}(\mathcal{A})) = A \cap \mathrm{Gpa}(a; \mathrm{Supp}(\mathcal{A}))$, which yields that $\mathrm{Gpa}(a; \mathrm{Supp}(\mathcal{A})) \subset A$. Q.E.D.

Theorem 49: If $z \in \mathrm{Gpa}(x; Y)$, then $\mathrm{Gpa}(x; Y) = \mathrm{Gpa}(z; Y)$.

Proof of Theorem 49: If $z \in \mathrm{Gpa}(x; Y)$, then there exists an $n < \infty$ such that $z \in \mathrm{Gp}_n(x; Y)$. By Theorem 46, $x \in \mathrm{Gp}_n(z; Y)$, and furthermore for all $k > 0$, $\mathrm{Gp}_{n+k}(x; Y) \subset \mathrm{Gp}_k(z; Y)$ and $\mathrm{Gp}_{n+k}(z; Y) \subset \mathrm{Gp}_k(x; Y)$. Hence $\mathrm{Gpa}(x; Y) \subset \mathrm{Gpa}(z; Y)$ and $\mathrm{Gpa}(z; Y) \subset \mathrm{Gpa}(x; Y)$. Therefore $\mathrm{Gpa}(x; Y) = \mathrm{Gpa}(z; Y)$. Q.E.D.

Theorem 50: For all $Y \subset X$, there exists a unique collection $\mathcal{A}$ of convexly disjoint unit sets such that $Y = \mathrm{Supp}(\mathcal{A})$.

Proof of Theorem 50: By Theorem 48, it suffices to show that $\mathcal{A} = \{\phi\} \cup \{\mathrm{Gpa}(y; Y) \mid y \in Y\}$. Suppose that (1) $C \in \mathcal{C}$, that (2) $C \subset Y$, and that (3) $C \cap \mathrm{Gpa}(x; Y) \neq \phi$. Then there exists a $z \in C \cap \mathrm{Gap}(x; Y)$. By Theorem 50, $\mathrm{Gpa}(z; Y) = \mathrm{Gpa}(x; Y)$. By definition $C \subset \mathrm{Gp}_1(z; Y) \subset \mathrm{Gpa}(z; Y)$. Also if $\mathrm{Gpa}(a; Y) \cap \mathrm{Gpa}(b; Y) \neq \phi$, again Theorem 50 assures us that $\mathrm{Gpa}(a; Y) = \mathrm{Gpa}(b; Y)$. Hence $\mathcal{A}$ is a collection of convexly disjoint unit sets.

Definition 31: For a given convexity $\mathcal{C}$, and a set $Y \subset X$, let $\mathcal{P}t(Y) = \{\phi\} \cup \{\mathrm{Gpa}(y; Y) \mid y \in Y\}$.

Corollary to Theorem 50: Suppose A is the union of some collection of convexly disjoint sets, then the convexly disjoint convex sets of which A is the union are uniquely determined.

Proof: Convex sets are unit sets.

Theorem 51: Suppose both A and B are unions of two (not necessarily different) collections of convexly disjoint sets, then
$\mathcal{P}t(A) \wedge \mathcal{P}t(B) = \mathcal{P}t(A \cap B)$.

Proof of Theorem 51: By Theorem 50, both $\mathcal{P}t(A)$ and $\mathcal{P}t(B)$ are collections of convexly disjoint convex sets. Since $\mathcal{C}$ is closed under finite intersection, $\mathcal{P}t(A) \wedge \mathcal{P}t(B)$ is a collection of convexly disjoint convex sets. Obviously, $A \cap B = \mathrm{Supp}(\mathcal{P}t(A) \wedge \mathcal{P}t(B)) = \mathrm{Supp}(\mathcal{P}t(A \cap B))$. The uniqueness property in Theorem 50 insures us that
$\mathcal{P}t(A) \wedge \mathcal{P}t(B) = \mathcal{P}t(A \cap B)$.

Theorem 52: For any sets A and $B \in \mathcal{Q}^X$, and any point $x \in X$,
(1) $\mathrm{Gpa}(x; A \cap B) \subset \mathrm{Gpa}(x; A) \cap \mathrm{Gpa}(x; B)$
(2) $\mathrm{Gpa}(x; A) \cup \mathrm{Gpa}(x; B) \subset \mathrm{Gpa}(x; A \cup B)$.

Proof of Theorem 52: (1) follows from Theorems 50 and 38.
From (1) it follows that if $Y \subset Z$, then

(1.5) $\mathrm{Gpa}(x; Y) \subset \mathrm{Gpa}(x; Z)$, hence

$\mathrm{Gpa}(x; A) \subset \mathrm{Gpa}(x; A \cup B)$

$\mathrm{Gpa}(x; B) \subset \mathrm{Gpa}(x; A \cup B)$

$\therefore$ (2) $\mathrm{Gpa}(x; A) \cup \mathrm{Gpa}(x; B) \subset \mathrm{Gpa}(x; A \cup B)$.

Note: Theorems 43, 44, 50, and 52 are the main reasons why I consider my unit sets to be an abstraction of the notion of polygonally connected sets.

Definition 32: A collection A of indexed sets that are unions of convexly disjoint sets is said to have property $[k, m]$ $(m \geq k)$ if for all $D_{\alpha_1}, D_{\alpha_2}, \ldots, D_{\alpha_m}$ (the α_i are pairwise different) $\in A, \left|\left| \bigwedge_{i=1}^{m} Pt(D_{\alpha_i}) \right|\right| \leq k$.

Definition 33: A collection of sets S has Helly's # h if there does not exist a finite subcollection $R \subset S$ such that

(1) $\bigcap_{R \in R} R = \phi$ and

(2) for all $R' \subset R(|R'| = h)$, $\bigcap_{R \in R'} R \neq \phi$.

Note: The minimum Helly's number for the set of convex sets in R^n (Euclidean space) is $h = n+1$.

Theorem 53: (The generalized Helly's theorem). Suppose that A was a collection of indexed sets which were unions of convexly disjoint convex sets (of some abstract convexity C), and that A had property $[k, m]$ $(m \geq k)$. Also suppose that C had Helly's number $h \geq 2$. Then the set A has Helly's number $q = \max(hk, m+k)$ for $h \geq 3$ and the set has Helly's number $q = \max(3k-1, k+m)$ for $h = 2$.

Note: For $h = 2$, theorem 53 is not the best result possible.

Proof of theorem 53: We may as well assume that we are given C with Helly's # h, and a collection A of sets with property $[k, m]$. And for the appropriate q (determined by h, k, and m), we may further assume that for all $\alpha_1, \alpha_2, \ldots, \alpha_q$ (the α_i are pairwise disjoint) that

$$\bigcap_{i=1}^{q} A_{\alpha_i} \neq \phi,$$

(the $A_{\alpha_i} \in \mathcal{A}$ of course). It is sufficient to show that there does not exist an $\alpha_1, \alpha_2, \ldots, \alpha_n$ ($n < \infty$) such that the α_i are pairwise distinct and such that

$$\bigcap_{i=1}^{n} A_{\alpha_i} = \phi \quad .$$

For $n \leqslant q$, we have already assumed the above statement to be true. We shall assume the statement in question is true so long as $n \leqslant p$ and proceed to show the statement is true for $n = p+1$ (i.e., an induction proof). Choose any $A_1, A_2, \ldots, A_{p+1} \in \mathcal{A}$. We may assume that the A_i are pairwise distinct (otherwise

$$\bigcap_{i=1}^{p+1} A_i = \bigcap_{i=1}^{p} A_{\alpha_i} \neq \phi$$

by the induction assumption). For each i ($1 \leqslant i \leqslant p+1$) choose an $x_i \in \bigcap_{j \neq i} A_j$ ($\neq \phi$ by the induction hypothesis). Let $Z = \{x_i \mid 1 \leqslant i \leqslant p+1\}$. Let $AA = \{\mathcal{P}t(A_i) \mid 1 \leqslant i \leqslant p+1\}$. Note that for $1 \leqslant i \leqslant p+1$, $\langle \mathcal{P}t(A_i) \rangle_Z \leqslant 1$. If $\langle \mathcal{P}t(A_i) \rangle_Z = 0$, for some i, then

$$x_i \in \bigcap_{i=1}^{p+1} A_i$$

(and that would prove the theorem). So, we may as well assume that $\{x_i\} = \mathrm{Res}(\mathcal{P}t(A_i) \mid Z)$ for $1 \leqslant i \leqslant p+1$.

Lemma: $\mathcal{K}(AA \mid Z) \leqslant k$.

Proof of Lemma: Suppose that there existed a $BB \subset AA$ such that for $\mathcal{B} = \bigwedge_{\mathcal{P} \in AA} \mathcal{P}$, $\|\mathcal{B}\|_Z > k$. Then there exists a set I of integers such that $|I| = k+1 = \|\mathcal{B}\|_{\{x_i | i \in I\}}$. By theorem 11, we may as well assume that $\|BB\| \leqslant k \leqslant m$. Since $|Z| \geqslant p+1 \geqslant q+1 \geqslant k+m+1$, then there exists a set of integers J such that $I \cap J = \phi$, such that $|J| = m$, and such that $BB \subset \{\mathcal{P}t(A_j) | j \in J\}$. Let $X = \{x_i | i \in I\}$. Both $\mathcal{B}$ and $\bigwedge_{j \in J} \mathcal{P}$ partition X. Now $|\mathcal{B}|_X = k+1$, so by the corollary to heorem 3,

$$\|\bigwedge_{j \in J} \mathcal{P}_j\|_X = \|\bigwedge_{j \in J} \mathcal{P}_j \wedge \mathcal{B}\|_X \geqslant k+1 \ .$$

But this violates the assumption that $\mathcal{A}$ had property $[k, m]$. Hence $\mathcal{K}(AA | Z) \leqslant k$. End of Lemma.

We have now reached the conclusion that either theorem 53 is true, or that $\mathcal{K}(AA | Z) \leqslant k$, $\eta(AA | Z) = 1$. Since $|Z| \geqslant \chi a(h, k, 1)$ (note that $\chi a(h, k, 1) - 1$ is the first term in the maximum function that determines the Helly #), there exists a $H \subset Z$, $|H| = h+1$ such that for $1 \leqslant i \leqslant p+1$ there exists an $C_i \in \mathcal{P}t(A_i)$ such that $|C_i|_H \geqslant h$. By construction, for any I, ($|I| \leqslant h$, $I \subset \{i | 1 \leqslant i \leqslant p+1\}$), $\bigcap_{i \in I} C_i \neq \phi$. By the assumption that $\mathcal{C}$ had Helly's #h,

$$\bigcap_{i=1}^{p+1} C_i \neq \phi \ .$$

Hence,

$$\phi \neq \bigcap_{i=1}^{p+1} C_i \subset \bigcap_{i=1}^{p+1} A_i \ .$$

Q.E.D.

Note to Readers: When I was deciding on a probable method of proof of the generalized Helly's Theorem, I was very much impressed with Radon's proof of Helly's Theorem. The emphasis in Chapter I and II on sets labeled H was due to the anticipated use of a set H, with $|H| = h+1$, etc., in the final part of the proof of the generalized Helly's Theorem. In fact, if I had been proving the generalized Helly's Theorem just for unions of convex sets in Euclidean space, I could have used Radon's Theorem for convex sets instead of Helly's Theorem for convex sets.

The next few remarks and definitions are to compare Grünbaum's and Motzkin's original definitions and conditions with my corresponding definitions and conditions. (See Bibliography)

Definition 34: A convexity C is γ-non-additive (for a finite of infinite cardinal $\gamma \geqslant 2$) if for every subfamily $C' \subset C$, with $1 < ||C'|| < \gamma + 1$, such that C' is a partition, we have that C' is a collection of convexly disjoint sets. The family C is non additive if it is γ-non-additive for every cardinal $\gamma \geqslant 2$.

Off hand, it looks like γ-non-additive is no better than a condition that guarantees that unions of disjoint convex sets are actually unions of convexly disjoint sets. On the other hand, a collection A of unions of convexly disjoint convex sets with property [k, m] can always be imbedded into a convexity that is non-additive. (But the non-additive convexity would really be a convexity on $\mathcal{X}^X$ instead of a convexity on X.)

Definition 35: A family $\mathcal{C}$ has the "Helly property of order h" with limit γ (h, γ cardinals with $2 \leq h \leq \gamma$) If for each subfamily $\mathcal{C}' \subset \mathcal{C}$ with $|\mathcal{C}'| < \gamma + 1$ the condition

$$"\bigcap_{C \in \mathcal{C}^*} C \neq \phi \text{ for all } \mathcal{C}^* \subset \mathcal{C}', \text{ with } |\mathcal{C}^*| < h+1"$$

implies that $\bigcap_{C \in \mathcal{C}'} C \neq \phi$. The family $\mathcal{C}$ has unlimited Helly property of order h if it has the Helly property of order h with limit γ for every $\gamma > h$.

In theorem 53, I could have added the condition that the Convexity $\mathcal{C}$ had Helly's property with limit $\gamma \geq \aleph_0$, and then concluded that $\mathcal{A}$ with property [k, m] also had Helly's property with limit γ. To prove that additional bit of information, it would be a simple exercise in using the axiom of choice. Since I have not made any use of families with Helly's property of limit γ, I decided just mentioning that property in passing would suffice in this paper.

Concluding remarks: In theorem 53, for h = 2 it is possible that $q = \max(2k, k+m) = k+m$ without $\chi a(2, k, 1) = 2k+1$. In the proof of Theorem 53, I only used the fact that: if I could find three points such that each set A of $\mathcal{A}$ contained at least two points in a convex set $C \subset A$, then the intersection of any finite subcollection of $\mathcal{A}$ is not empty. I never considered the fact that if I could find $2n+1$ point such that each set $A \in \mathcal{A}$ contained at least $n+1$ points in a convex set $C \subset A$, then the intersection of any finite subcollection of $\mathcal{A}$ is not empty. Many other such possibilities exist (a countable set of them in fact). It is possible to construct a set X and a convexity C of X such that the minimal Helly's number is determined solely by those enumerated possibilities. (Note to convexitists: for that constructed set X and constructed convexity C, it is possible to include in the construction that the abstract convexity C admits no finite Radon's number nor a finite Carathéodory's number.)

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