THE DEFLECTION CURVE FOR A CENTRIFUGALLY LOADED TAPE TRAVERSING AN ANNULAR CAP

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ABSTRACT

This paper is an analysis of the deflection curve formed by a centrifugally loaded tape which traverses an annular gap. The derived differential equation which describes the system is non-linear but possesses an exact closed form solution in terms of elliptic integrals. A simplified solution of the approximate linearized differential equation is also obtained.

The theoretically obtained results were checked by obtaining an actual trace of a curve on a model which represented the problem as stated. The theoretical and experimental results agreed within the limits of accuracy imposed by the apparatus used. Several different cases are discussed which indicate the effect of the significant parameters.

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I. INTRODUCTION

The problem considered arose in conjunction with work done at the International Business Machines research laboratory at San Jose, California during the summer of 1957. As formulated here however, the problem is of rather limited practical significance, as far as is known, but it is interesting when treated as a purely academic problem. This is especially true since it is a system described by a non-linear differential equation which possesses an exact closed form solution. Any one familiar with non-linear problems knows that a problem of this type seldom occurs.

The problem consists of finding an expression for a curve formed in the following manner. If there is uniform density length medium which traverses an annular gap between an outer rim and an inner hub, the medium will assume a particular curve when the entire system is rotated. A thin tape gives a good approximation of a uniform density length medium, therefore this medium will hereafter be referred to as a tape. Since the system is rotating, the tape will be acted upon by centrifugal force and if the annular gap is in a horizontal plane the effect of gravity is eliminated. The problem is strictly two-dimensional. The tape must, of course, be fastened to the hub and rim since it is under tension and for this problem it must be fastened at some distance from the points of contact, i.e. there must be no discontinuity in the slope of the tape at the point of contact. This may be stated another way as follows: the curve of the tape is to be tangent to the concentric circles formed by the hub and rim at the point of contact with each other. If there are several layers of tape on the hub and rim the tape will then be essentially fastened to each by frictional force in a way that will satisfy the problem as stated. That, then, is the problem to be analyzed in this paper.

II. THEORETICAL SOLUTION OF THE CURVE

The method used to derive the differential equations which describe the system was to take an element of the tape and sum the forces in the radial direction and the forces in the direction perpendicular to the radius which act on the element. The only external force acting on the tape is that due to the centripetal acceleration and is directed along the radius.

Referring to figure 1, R_1 is the radius of the hub, and R_2 is the radius of the rim, i.e. R_1 and R_2 are the inner and outer radii of the gap respectively. The angle θ is the angle measured from the point of contact with the hub to any point on the tape of radius R in the gap. θ_{max} is the angle measured to the point of contact with the rim. Therefore

$$\theta = 0 \qquad \text{when} \qquad R = R_1$$

$$\theta = \theta_{max} \qquad \text{when} \qquad R = R_2$$

The angle α is the angle between the tangent to the curve of the tape at any point and a line perpendicular to the radius at that point.

Referring to figure 2, taking an element of the tape and summing the forces acting on it in a direction perpendicular to the radius gives:

$$T \cos \alpha - (T + \Delta T) \cos (\alpha + \Delta T - \Delta\theta) = \frac{\rho R\omega^2 (R\Delta\theta)}{\cos \alpha} \cdot \sin \frac{\Delta\theta}{2} \qquad (1)$$

where

T = tension force

ρ = mass per unit length

ω = angular velocity

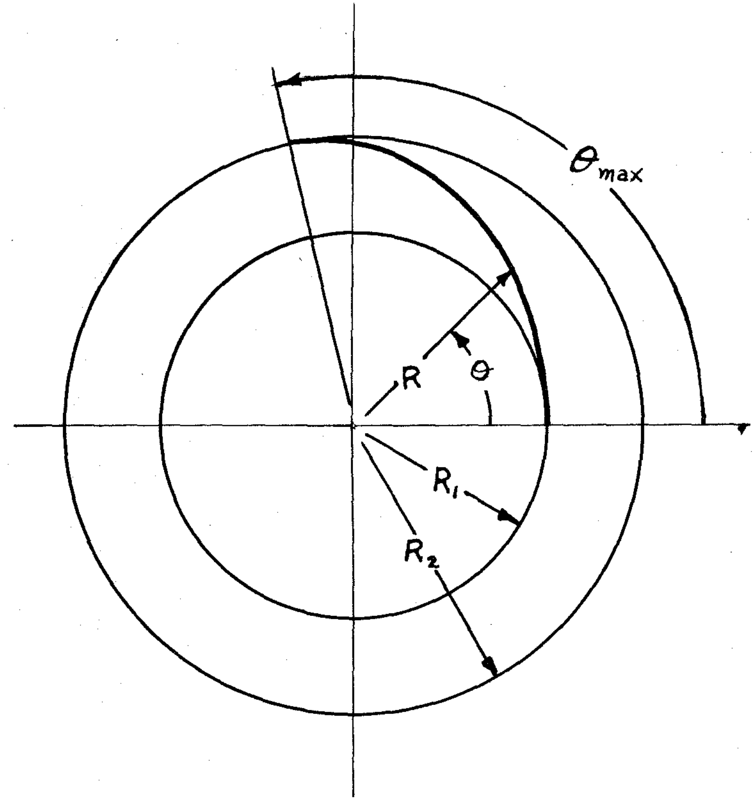

FIGURE 1. Axial view of the curve in the gap

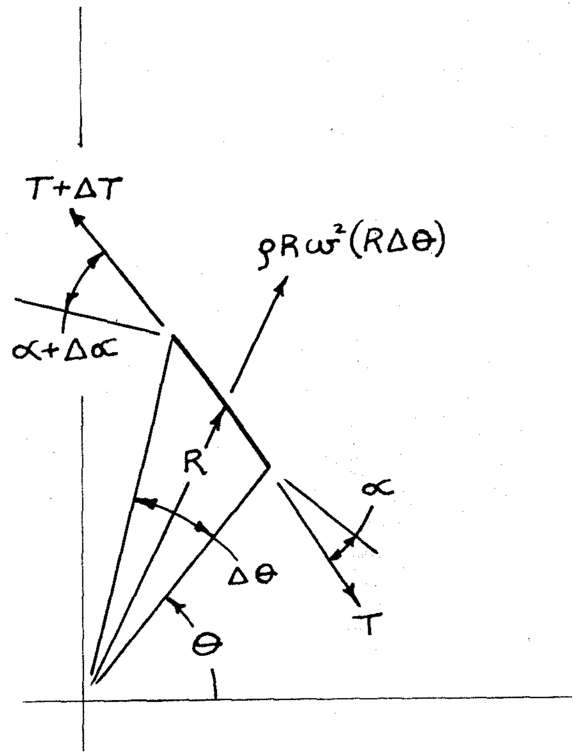

FIGURE 2. A typical element of the curve

Expanding this gives:

$$T\cos\alpha - T\cos\alpha\cos\Delta\alpha\cos\Delta\theta + T\sin\alpha\sin\Delta\alpha\cos\Delta\theta$$
$$- T\sin\alpha\cos\Delta\alpha\sin\Delta\theta - T\cos\alpha\sin\Delta\alpha\sin\Delta\theta$$
$$- \Delta T\cos\alpha\cos\Delta\alpha\cos\Delta\theta + \Delta T\sin\alpha\sin\Delta\alpha\sin\Delta\theta$$
$$- \Delta T\sin\alpha\cos\Delta\alpha\sin\Delta\theta - \Delta T\cos\alpha\sin\Delta\alpha\sin\Delta\theta$$
$$= \frac{\rho R^2\omega^2}{\cos\alpha}\sin\frac{\Delta\theta}{2}$$

Dividing by $\Delta\theta$ and taking the limit as $\Delta\theta \to 0$:

$$T\sin\alpha\frac{d\alpha}{d\theta} - T\sin\alpha - \frac{dT}{d\theta}\cos\alpha = 0$$

or

$$\frac{dT}{d\theta}\cos\alpha - T\sin\alpha\left(\frac{d\alpha}{d\theta} - 1\right) = 0 \qquad (2)$$

Referring again to figure 2, summing the forces in the radial direction gives:

$$T\sin\alpha - (T + \Delta T)\sin(\alpha + \Delta\alpha - \Delta\theta) = \frac{\rho R\omega^2(R\Delta\theta)}{\cos\alpha}\cdot\cos\frac{\Delta\theta}{2} \qquad (3)$$

Expanding this gives:

$$T\sin\alpha - T\sin\alpha\cos\Delta\alpha\cos\Delta\theta - T\cos\alpha\sin\Delta\alpha\cos\Delta\theta$$
$$+ T\cos\alpha\cos\Delta\alpha\sin\Delta\theta - T\sin\alpha\sin\Delta\alpha\sin\Delta\theta$$
$$- \Delta T\sin\alpha\cos\Delta\alpha\cos\Delta\theta - \Delta T\cos\alpha\sin\Delta\alpha\cos\Delta\theta$$
$$+ \Delta T\cos\alpha\cos\Delta\alpha\sin\Delta\theta - \Delta T\sin\alpha\sin\Delta\alpha\sin\Delta\theta$$
$$= \frac{\rho R^2\omega^2\Delta\theta}{\cos\alpha}\cos\frac{\Delta\theta}{2}$$

Dividing by $\Delta\theta$ and taking the limit as $\Delta\theta \to 0$:

$$-T\cos\alpha\frac{d\alpha}{d\theta} + T\cos\alpha - \frac{dT}{d\theta}\sin\alpha = \frac{\rho R^2\omega^2}{\cos\alpha} \qquad (4)$$

Equations 2 and 4 are the differential equations which describe the system. Referring to figure 1,

$$\tan \alpha = \frac{dR}{Rd\theta} \tag{5}$$

from which we see that the curve may be found by solving the integral:

$$\theta = \int_{R_1}^{R} \frac{dR}{R \tan \alpha} \tag{6}$$

In order to do this we must be able to express the angle α as a function of the radius R. This can be found from equations 3 and 4 together with the boundary conditions. Dividing equation 2 by $\cos \alpha$ and substituting equation 5 gives:

$$\frac{dT}{d\theta} - T \tan \alpha \frac{d\alpha}{d\theta} + \frac{T}{R}\frac{dR}{d\theta} = 0 \tag{7}$$

multiplying by $\frac{d\theta}{T}$ gives:

$$\frac{dT}{T} - \tan \alpha \, d\alpha + \frac{dR}{R} = 0 \tag{8}$$

Equation 8 is integrable in T, α, and R with the result that:

$$\ln T + \ln \cos \alpha + \ln R = \text{const.} \tag{9}$$

Taking the anti-log of both sides gives:

$$TR \cos \alpha = C_1 \tag{10}$$

Multiplying equation 2 by $\cos \alpha$ and equation 4 by $\sin \alpha$ and subtracting gives:

$$\frac{dT}{d\theta} = - \frac{\rho R^2 \omega^2 \sin \alpha}{\cos \alpha} \tag{11}$$

Again using equation 5 and multiplying by $d\theta$ gives:

$$dT = -\rho R \omega^2 dR \tag{12}$$

Integrating equation 12 gives:

$$T = -\frac{\rho R^2\omega^2}{2} + C_2 \qquad (13)$$

To evaluate the constants of integration in equations 10 and 13, the boundary conditions must be established.

Since the curve of the tape must be tangent to the concentric circles of radii R_1 and R_2, this provides the boundary conditions for the problem which are:

$$\alpha = 0 \qquad \text{at} \qquad R = R_1$$

$$\alpha = 0 \qquad \text{at} \qquad R = R_2$$

Substituting into equation 10 gives:

$$T_{(R_1)}R_1 = T_{(R_2)}R_2 \qquad (14)$$

but from equation 13:

$$T_{(R_1)} = \frac{\rho R_1^{\,2}\omega^2}{2} + C_2 \qquad (15)$$

and

$$T_{(R_2)} = -\frac{\rho R_2^{\,2}\omega^2}{2} + C_2 \qquad (16)$$

Substituting equations 15 and 16 into equation 14 gives:

$$-\frac{\rho\omega^2}{2}R_1^{\,3} + C_2R_1 = \frac{\rho\omega^2}{2}R_2^{\,3} + C_2R_2$$

and solving for C_2:

$$C_2 = \frac{\rho\omega^2}{2}\,\frac{(R_2^{\,3} - R_1^{\,3})}{(R_2 - R_1)} \qquad (17)$$

Substituting equation 17 into equation 15 gives:

$$T = \frac{\rho\omega^2}{2}\left[\frac{R_2^{\,3} - R_1^{\,3}}{R_2 - R_1} - R^2\right] \tag{18}$$

It is convenient to normalize the problem by letting $r = R/R_2$. Also let the ratio $R_1/R_2 = \eta$. Equation 18 then becomes:

$$T = \frac{\rho\omega^2}{2}\,(1 + \eta + \eta^2 - r^2) \tag{19}$$

where:

$$\eta \leq r \leq 1$$

Evaluating equation 10 at either end point gives:

$$C_1 = \frac{\rho\omega^2}{2}\,\eta(\eta + 1) \tag{20}$$

and equation 10 gives:

$$\cos\alpha = \frac{\eta(\eta + 1)}{r(1 + \eta + \eta^2 - r^2)} \tag{21}$$

Equation 20 gives the desired relationship, i.e. it expresses the angle α as a function of r. It is interesting to note here that since α is independent of ρ and ω, the curve given by the integration of equation 6 will be independent of the density of the tape and also independent of the speed of rotation.

For the normalized case, equations 5 and 6 become:

$$\tan\alpha = \frac{dr}{r\,d\theta} \tag{22}$$

and

$$\theta = \int_{\eta}^{r} \frac{dr}{r\tan\alpha} \tag{23}$$

Tan α as a function of r may be found using equation 21 and the trigonometric relationship:

$$\tan \alpha = \left[\frac{1}{\cos^2 \alpha} - 1 \right]^{1/2}$$

which gives:

$$\tan \alpha = \left[r^2 \left(\frac{1 + \eta + \eta^2 - r^2}{\eta(\eta + 1)} \right)^2 - 1 \right]^{1/2}$$

$$= \left[r^2 \left(\frac{1}{\eta(\eta + 1)} + 1 - \frac{1}{\eta(\eta + 1)} r^2 \right)^2 - 1 \right]^{1/2}$$

$$= \left[r^2 (1 + \beta - \beta r^2)^2 - 1 \right]^{1/2} \qquad (24)$$

where:

$$\frac{1}{\beta} = \eta(\eta + 1)$$

Substituting equation 24 into equation 23 gives:

$$\theta = \int_{\eta}^{r} \frac{dr}{r[r^2(1 + \beta - \beta r^2)^2 - 1]^{1/2}} \qquad (25)$$

Since the quantity within the radical contains terms of r^2, r^4 and r^6, to reduce this to a cubic polynomial, let $x = r^2$. Then equation 25 becomes:

$$\theta = \int_{\eta^2}^{r^2} \frac{dx}{2x[x(1 + \beta - \beta x)^2 - 1]^{1/2}} \qquad (26)$$

Expanding the quantity in the radical gives:

$$\theta = \frac{1}{2\beta} \int_{\eta^2}^{r^2} \frac{dx}{x[x^3 - \frac{2(1+\beta)}{\beta} x^2 + \frac{(1+\beta)^2}{\beta^2} x - \frac{1}{\beta^2}]^{1/2}} \qquad (27)$$

The polynomial in the brackets will have three zeros given by the roots of the equation

$$x^3 - \frac{2(1 + \beta)}{\beta} x^2 + \frac{(1 + \beta)^2}{\beta^2} x - \frac{1}{\beta^2} = 0 \qquad (28)$$

Referring to the boundary conditions we note that

$$\frac{d\theta}{dR} = \infty \quad \text{at} \quad R = R_1 \text{ and } R = R_2$$

or with the change of variable

$$\frac{d\theta}{dr} = \infty \quad \text{at} \quad r = \eta \text{ and } r = 1$$

and

$$\frac{d\theta}{dx} = \infty \quad \text{at} \quad x = \eta^2 \text{ and } x = 1.$$

Therefore the denominator of the integral in equation 27 must be zero for $x = \eta^2$ and $x = 1$. These values must be two of the roots of equation 28 which checks by substitution. The third root must therefore be real and is easily checked by using the fact that the constant term is equal to the product of the roots, i.e.

$$\frac{1}{\beta^2} = \eta^2(\eta + 1)^2 = e_1 e_2 e_3 = \eta^2 e_1$$

where e_1, e_2, and e_3 are the roots of equation 28. This gives:

$$e_1 = (\eta + 1)^2$$

Therefore equation 27 can now be written in the form:

$$\theta = \frac{1}{\beta} \int_{\eta^2}^{r^2} \frac{dx}{x[p(x)]^{1/2}} \qquad (29)$$

where

$$p(x) = [4(x - e_1)(x - e_2)(x - e_3)]$$

and

$$e_1 = (\eta + 1)^2$$

$$e_2 = 1$$

$$e_3 = \eta^2$$

The integral is now in the form of integral 8a9, I 243 of reference 1.

The solution of equation 29 is:

$$\theta = \frac{(\eta + 1)}{\eta(2\eta + 1)^{1/2}} \; \Pi\,(\phi,\lambda,k) \tag{30}$$

where

$$\lambda = \frac{1 - \eta^2}{\eta^2}$$

$$\sin^2 \phi = \frac{\eta^2 - r^2}{\eta^2 - 1}$$

$$k^2 = \frac{1 - \eta^2}{2n + 1}$$

and

$$\Pi(\phi,\lambda,k) = \int_0^{\phi} \frac{d\xi}{(1 + \lambda \sin^2\xi)(1 - k^2\sin^2\xi)^{1/2}}$$

In equation 30 the symbol $\Pi(\phi,\lambda,k)$ means the incomplete elliptic integral of the third kind where ϕ is the argument, k is the modulus, and λ is the parameter. The general solution of an elliptic integral of the third kind has been found and is given in the literature in terms of a series. For the range of parameters encountered in this problem, the solution of equation 30 is given by integral 2d, I 241 of reference 1. Unfortunately, tabulated values for elliptic integrals of the third kind are not available. Therefore, if a plot of the curve given by the exact solution of equation 30 is desired, the series solution must be evaluated to the desired degree of accuracy.

However, the maximum angle θ_{max}, may be found more easily. Referring to equation 29

$$\theta_{max} = \frac{1}{\beta}\int_{\eta^2}^{1} \frac{dx}{x[p(x)]^{1/2}} \tag{31}$$

This integral is evaluated in reference 2, integral number 4a, II222 and after evaluating the constants the result is:

$$\theta_{max} = \frac{\eta + 1}{\eta(2\eta + 1)^{1/2}} \; \Pi\left(\frac{\pi}{2}, \lambda, k\right) \tag{32}$$

where λ and k are as defined previously for equation 30. The integral $\Pi(\frac{\pi}{2}, \lambda, k)$ is a complete elliptic integral of the third kind. This may be evaluated in terms of elliptic integrals of the first and second kind as given by equation 13a, II221 of reference 2. The result is:

$$\theta_{max} = \frac{\eta}{(\eta+1)(2\eta+1)^{1/2}} K(k) + [E(k) - K(k)]F(\psi, k')$$
$$+ K(k)E(\psi, k) \tag{33}$$

where

$$\sin\psi = \frac{(2\eta + 1)^{1/2}}{\eta + 1}$$

$$k^2 = \frac{1 - \eta^2}{2\eta + 1}$$

$$k'^2 = 1 - k^2$$

$K(k)$ = complete elliptic integral of the first kind

$E(k)$ = complete elliptic integral of the second kind

$F(\psi,k)$ = incomplete elliptic integral of the first kind

$E(\psi,k)$ = incomplete elliptic integral of the second kind.

It is advantageous to evaluate equation 32 in terms of elliptic integrals of the first and second kind since they have been tabulated. A particularly good table is found in reference 3. Reference 4 may also be used but is not as complete.

Equation 32 states that θ_{max} is a function of η only. Equation 33 was used to compute the curve of θ_{max} vs. η in figure 3. A sample calculation is given in the appendix. It is interesting to note that θ_{max} increases as the gap width becomes smaller although one might intuitively expect the opposite to be true.

Since the evaluation of equation 33 for $\eta = 0$ involves indeterminate forms, the following form can be used:

$$\theta_{max} = \frac{\eta + 1}{\eta(2\eta + 1)^{1/2}} \quad \frac{1}{1 + \lambda} \ \ln \frac{4}{k'} + \frac{\sqrt{\lambda}}{1 + \lambda} \ \text{arc tan} \sqrt{\lambda}$$

$$+ 0(k')^2 \qquad (34)$$

since for $\eta = 0$, $k' = 0$. Equation 34 is evaluated for $\eta = 0$ in the appendix.

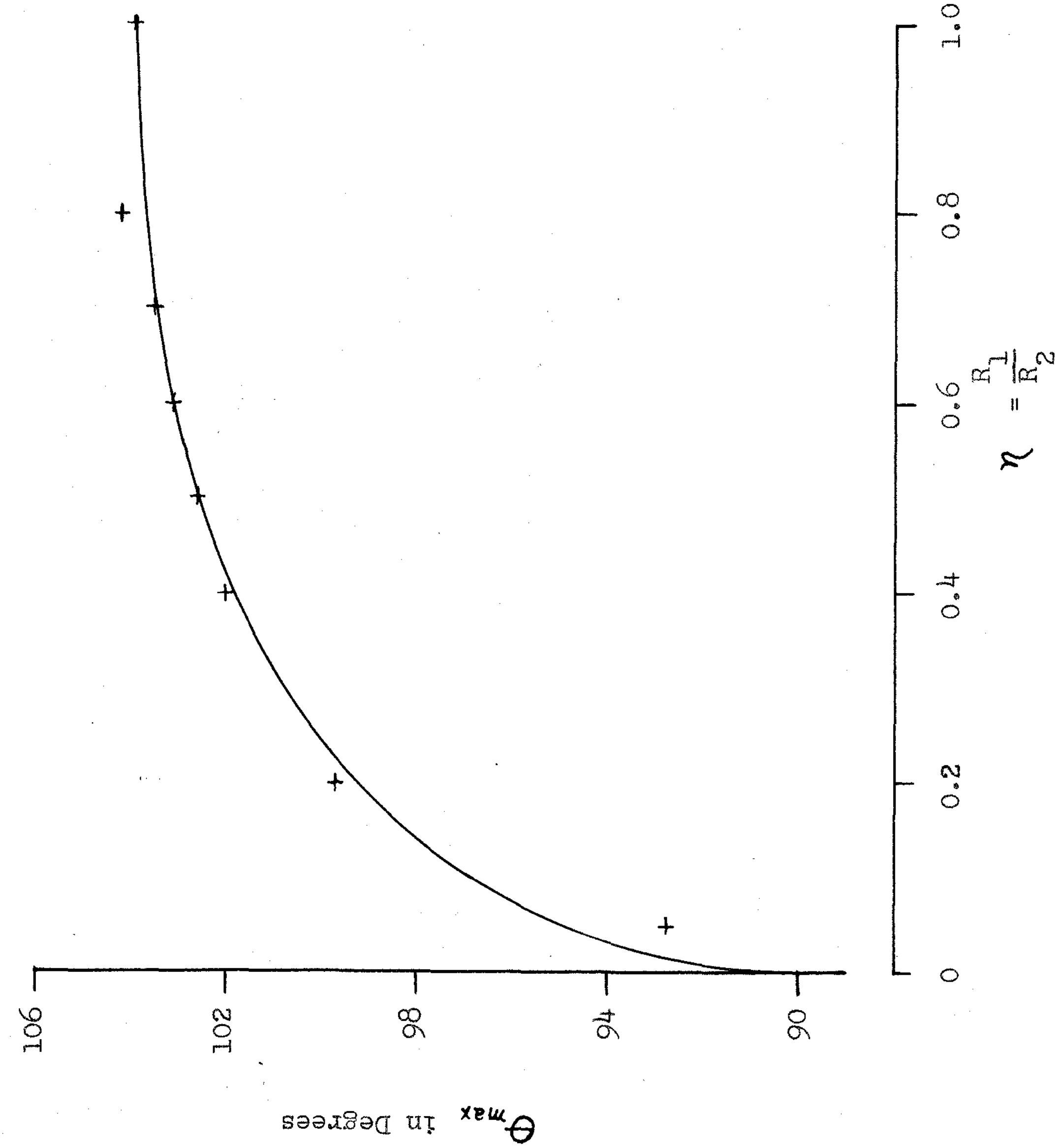

FIGURE 3. θ_{max} vs. η

III. THEORETICAL SOLUTION FOR THE LENGTH OF TAPE IN THE GAP

It is also possible to solve for the length of tape in the gap. Referring to figure 2:

$$ds = \frac{dR}{\sin\alpha} = R_2 \frac{dr}{\sin\alpha} \tag{35}$$

Therefore the arc length is given by:

$$S = R_2 \int_\eta^r \frac{dr}{\sin\alpha} = R_2 \int_\eta^r \frac{dr}{(1 - \cos^2\alpha)^{1/2}} \tag{36}$$

Substituting equation 21 into the above gives:

$$S = R_2 \int_\eta^r \frac{r(1 + \eta + \eta^2 - r^2)\,dr}{[r^2(1 + \eta + \eta^2 - r^2)^2 - 1]^{1/2}} \tag{37}$$

Substituting $\frac{1}{\beta} = \eta(\eta + 1)$ and making the change of variable $x = r^2$, equation 36 becomes:

$$S = R_2 \int_{\eta^2}^{r^2} \frac{(1 + \beta - \beta x)\,dx}{[4x(1 + \beta - \beta x)^2 - 1]^{1/2}} \tag{38}$$

The denominator of the integral can be factored as before with the result that:

$$S = \frac{R_2(1 + \beta)}{\beta} \int_{\eta^2}^{r^2} \frac{dx}{[p(x)]^{1/2}} - R_2 \int \frac{x\,dx}{[p(x)]^{1/2}} \tag{39}$$

where $p(x)$ is as defined in equation 29. The above integrals are evaluated in reference 2 in terms of incomplete elliptic integrals of the first and second kind. The result is:

$$S = R_2(2\eta + 1)^{1/2}\, E(\phi,k) - \frac{R_2\eta}{(2\eta + 1)^{1/2}}\, F(\phi,k) \tag{40}$$

where

$$k^2 = \frac{1 - \eta^2}{2\eta + 1}$$

$$\sin^2 \phi = \frac{\eta^2 - r^2}{\eta^2 - 1}$$

Equation 40 can be checked against the result by equation 33 for $\eta = 1$ since for $R_2 = 1$, S should equal θ_{max}. Evaluating equation 40 gives:

$$S = \sqrt{3}\,\frac{\pi}{2} - \frac{\pi}{2\sqrt{3}} = \frac{\pi}{\sqrt{3}}$$

This agrees with the value of θ_{max} shown in figure 3 for $\eta = 1$. Also the equation 39 can be checked for the limiting value of $\eta = 0$ since then the tape would lie along the radius and the length should be equal to the radius R_2. Evaluating equation 40 for $\eta = 0$ gives

$$S = R_2 \cdot 1 - 0 = R_2$$

This agrees with the previous statement.

The results which have been obtained can also be applied to the symmetric case of the same problem, i.e. where there is no outer rim and the tape is returned to the hub so that a loop is formed as shown in figure 11. It is seen then that the same differential equations which were developed for the original problem will also apply to this case. The boundary conditions are also the same where R_1 is again the radius of the hub and R_2 is now the maximum radius of the loop. Therefore the original problem is exactly the same as one half of this symmetric case.

IV. SOLUTION OF THE LINEARIZED DIFFERENTIAL EQUATION

It is interesting to note that a linearized differential equation for the system can be obtained by making certain approximations. Consider the case where the gap is small, i.e. η is nearly unity. Multiplying equation 2 by sin α and equation 4 by cos α and adding gives:

$$\frac{d\alpha}{d\theta} - 1 + \frac{\rho R^2 \omega^2}{T} = 0 \tag{41}$$

but

$$\alpha = \tan^{-1} \frac{dR}{R d\theta}$$

differentiating

$$\frac{d\alpha}{d\theta} = \frac{\frac{1}{R}\frac{d^2R}{d\theta^2} - \frac{1}{R^2}\left(\frac{dR}{d\theta}\right)^2}{1 + \frac{1}{R^2}\left(\frac{dR}{d\theta}\right)^2} \tag{42}$$

From equation 24 for η nearly unity tan α is small. Therefore $\frac{dr}{d\theta}$ is small and $\frac{dr}{d\theta}^2$ can be neglected. With this assumption equation 42 becomes:

$$\frac{d\alpha}{d\theta} \approx \frac{1}{R} \frac{d^2R}{d\theta^2} \tag{43}$$

Substituting into equation 41 gives:

$$\frac{d^2R}{d\theta^2} - R + \frac{\rho R^3 \omega^2}{T} = 0 \tag{44}$$

Let

$$R = R_0(1 + q) \tag{45}$$

where R_0 is the mean radius of the gap. Then $R_1 = R_0(1 - \varepsilon)$ and $R_2 = R_0(1 + \varepsilon)$. Since the gap is small, terms involving q^2, q^3, ε^2, and ε^3 can be neglected.

Equation 18 gives:

$$T = \frac{\rho\omega^2}{2}\left[\frac{R_0{}^3(1 + 3\varepsilon) - R_0{}^3(1 - 3\varepsilon)}{R_0(1 + \varepsilon) - R_0(1 - \varepsilon)} - R_0{}^2(1 + q)\right]$$

$$= \rho\omega^2 R_0{}^2(1 - q) \qquad (46)$$

Substituting equations 45 and 46 into equation 44 and neglecting all higher order terms in q gives:

$$\frac{d^2q}{d\theta^2} + 3q = 0 \qquad (47)$$

The solution of this is:

$$q = A \cos \sqrt{3}\,\theta + B \sin \sqrt{3}\,\theta \qquad (48)$$

Since $\frac{dR}{d\theta} = 0$ at $R = R_1$, then $\frac{dq}{d\theta} = 0$ at $\theta = 0$ and therefore $B = 0$. Also $q = -\varepsilon$ at $\theta = 0$ which when substituted into equation 48 gives:

$$q = -\varepsilon \cos \sqrt{3}\,\theta \qquad (49)$$

Therefore

$$R = R_0(1 - \varepsilon \cos \sqrt{3}\,\theta) \qquad (50)$$

The value of θ_{max} is given by:

$$\theta_{max} = \theta_{q=\varepsilon} - \theta_{q=-\varepsilon} = \frac{\pi}{\sqrt{3}}$$

This agrees with the exact solution when $\eta = 1$.

V. AN APPROXIMATE NON-LINEAR SOLUTION

Another method for obtaining an approximate solution is as follows. Equation 29 may be written:

$$\theta = \frac{1}{2\beta}\int_{\eta^2}^{r^2} \frac{1}{x} \cdot \frac{1}{[(1+\eta)^2 - x]^{1/2}} \cdot \frac{1}{(1-x)^{1/2}} \cdot \frac{1}{(x-\eta^2)^{1/2}}\, dx \quad (51)$$

For values of η near unity, the variation in the term $1 \Big/ (1+\eta)^2 - x$ is small in the interval $\eta^2 \le x \le 1$. Therefore by evaluating this term for $\eta = 1$ and considering it a constant, it can be taken out from the integral. Substituting into equation 51 gives:

$$\theta = \frac{1}{2\beta\sqrt{3}}\int_{\eta^2}^{r^2} \frac{dx}{x[(1-x)(x-\eta^2)]^{1/2}}$$

Rewriting gives:

$$\theta = \frac{\eta(\eta+1)}{2\sqrt{3}}\int_{\eta^2}^{r^2} \frac{dx}{x[-x^2 + (1+\eta^2)x + \eta^2]^{1/2}} \quad (52)$$

This integral may be found evaluated in reference 5. The result is:

$$\theta = \frac{\eta+1}{2\sqrt{3}}\left[\sin^{-1}\left(\frac{r^2(\eta^2+1) - 2\eta^2}{r^2(\eta^2-1)}\right) - \frac{\pi}{2}\right] \quad (53)$$

The value of θ_{max} given by equation 53 is found by evaluating for $r = 1$, and is:

$$\theta_{max} = \frac{\eta+1}{2\sqrt{3}}\pi$$

If this is evaluated for $\eta = 1$, the result is:

$$\theta_{max} = \frac{\pi}{\sqrt{3}}$$

This agrees with the exact solution of equation 30.

VI. NUMERICAL INTEGRATION PROCEDURE

It is desirable to compare the experimental results with theoretical results which were derived from the differential equations 2 and 4. Therefore it is necessary to obtain the curve of θ as a function of the radius. Two approximate methods have been derived which would probably give reasonably good accuracy for values of η close to unity but would not be good for small values of η, i.e. a wide gap. Therefore the complete curve was found by numerical integration using equation 25, i.e.

$$\theta = \int_{\eta}^{r} \frac{dr}{r[r^2(1+\beta-\beta r^2)^2 - 1]^{1/2}} \qquad (25)$$

However this equation cannot be used at the end points since for $r = \eta$ and $r = 1$, the integrand of equation 25 is infinite.

To evaluate the function at the end points it was expanded in a Taylor's series using the first three terms. In the neighborhood of $r = \eta$ the expansion is:

$$r = \eta + \left(\frac{dr}{d\theta}\right)_{\eta} \Delta\theta + \frac{1}{2}\left(\frac{d^2r}{d\theta^2}\right)_{\eta} \Delta\theta^2 + \ldots \qquad (54)$$

But since $\alpha = 0$ at $r = R_1$, $\left(\frac{dr}{d\theta}\right)_{\eta} = 0$ and rewriting equation (54) gives:

$$\Delta r = r - \eta = \frac{1}{2}\left(\frac{d^2r}{d\theta^2}\right)_{\eta} \Delta\theta^2 \qquad (55)$$

solving for $\Delta\theta$ gives:

$$\Delta\theta = \left[\frac{2\Delta r}{\left(\frac{d^2r}{d\theta^2}\right)_{\eta}}\right]^{1/2} \qquad (56)$$

The same result is also applicable in the neighborhood of $r = R_2$, i.e.

$$\Delta\theta = \left[\frac{2\Delta r}{\left(\frac{d^2 r}{d\theta^2}\right)_1} \right]^{1/2} \tag{57}$$

To evaluate equations 56 and 57 it is necessary to evaluate $\frac{d^2 r}{d\theta^2}$ at the end points. Equation 22 may be written:

$$\frac{dr}{d\theta} = r \tan \alpha \tag{58}$$

Differentiating gives:

$$\frac{d^2 r}{d\theta^2} = \tan \alpha \frac{dr}{d\theta} + r \sec^2 \alpha \frac{d\alpha}{d\theta}$$

$$= \tan \alpha \frac{dr}{d\theta} + r \sec^2 \alpha \frac{d\alpha}{dr} \cdot \frac{dr}{d\theta}$$

$$= \tan \alpha \frac{dr}{d\theta} + r^2 \sec^2 \alpha \tan \alpha \frac{d\alpha}{dr} \tag{59}$$

but

$$\cos \alpha = f(r) = \frac{\eta(\eta + 1)}{r(1 + \eta + \eta^2 - r^2)}$$

Differentiating:

$$\frac{d\alpha}{dr} = \frac{f'(r)}{\sin \alpha}$$

Substituting into equation 59 gives:

$$\frac{d^2 r}{d\theta^2} = \tan \alpha \frac{dr}{d\theta} - r^2 \sec^3 f'(r) \tag{60}$$

but

$$f'(r) = \frac{-\eta(\eta + 1)(1 + \eta + \eta^2 - 3r^2)}{r^2(1 + \eta + \eta^2 - r^2)} \tag{61}$$

Therefore:

$$\frac{d^2r}{d\theta^2} = \tan\alpha \frac{dr}{d\theta} + \frac{\sec^3\alpha\ \eta(\eta + 1)(1 + \eta + \eta^2 - 3r^2)}{(1 + \eta + \eta^2 - r^2)^2} \tag{62}$$

Then evaluating equation 62 at $r = \eta$ and $r = 1$ gives

$$\left(\frac{d^2r}{d\theta^2}\right)_\eta = \frac{\eta(1 + \eta - 2\eta^2)}{(1 + \eta)} \tag{63}$$

$$\left(\frac{d^2r}{d\theta^2}\right)_1 = \frac{\eta^2 + 2\eta - 2}{\eta(\eta + 1)} \tag{64}$$

The numerical integration was performed using Simpson's rule with equation 25 over the central portion of the gap and using equation 56 at the end points. This was done for three different values of η, namely

$\eta = 0.2$

$\eta = 0.4$

$\eta = 0.8$

A sample calculation is shown in the appendix. The results are plotted in figures 4, 5, and 6.

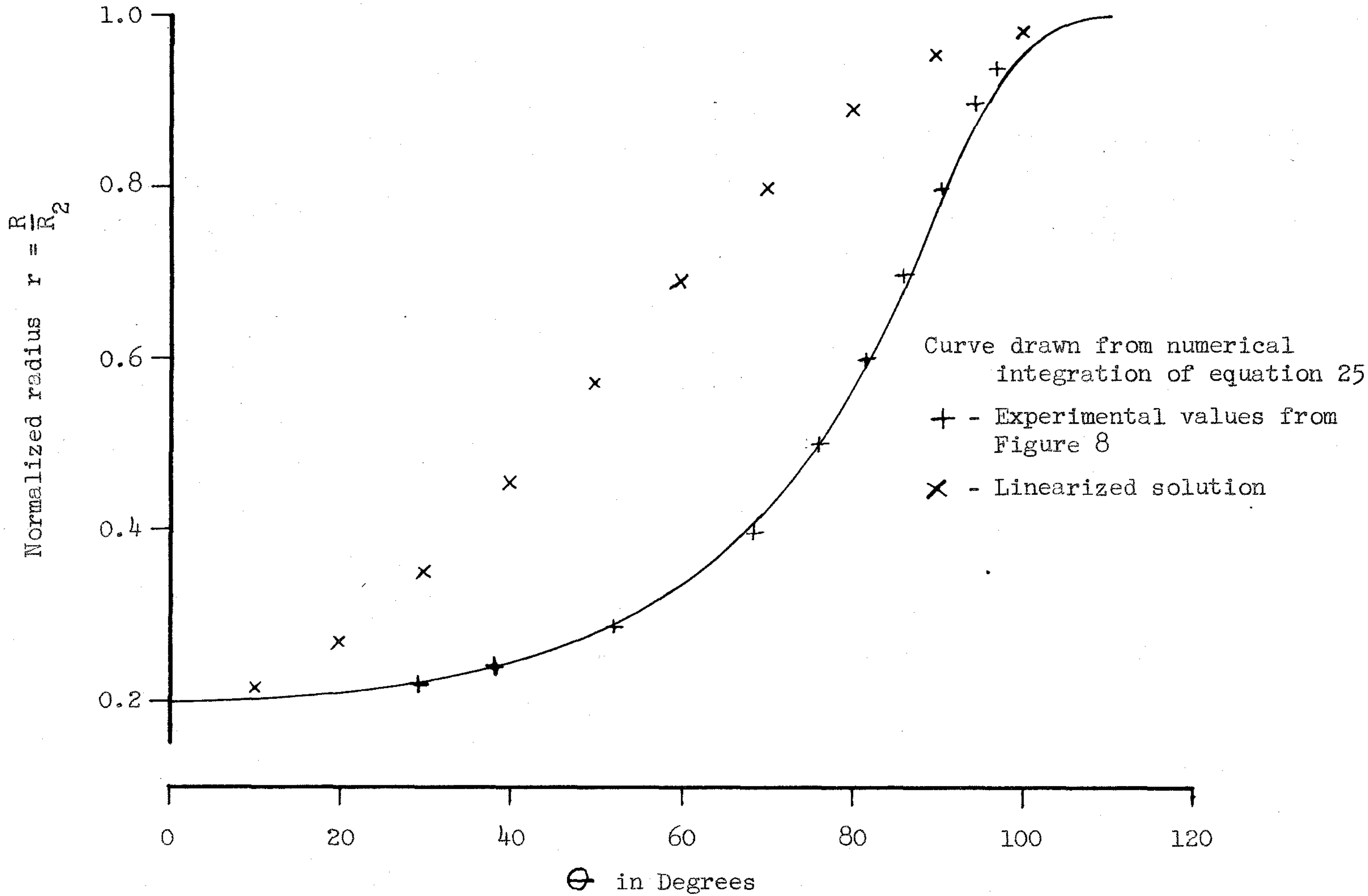

FIGURE 4. Theoretical and experimental curves of θ vs. r for η =0.2

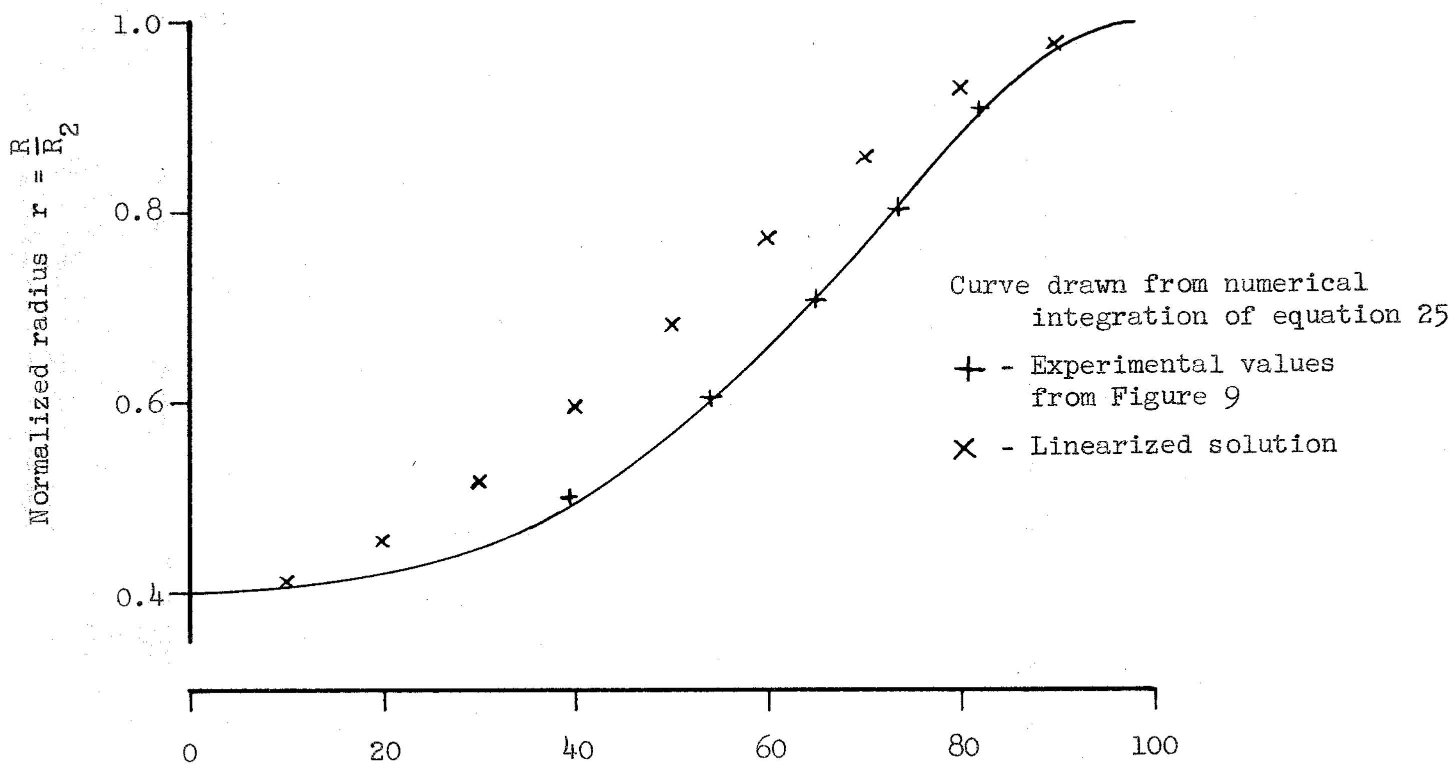

FIGURE 5. Theoretical and experimental curves of θ vs. r for $\eta = 0.4$

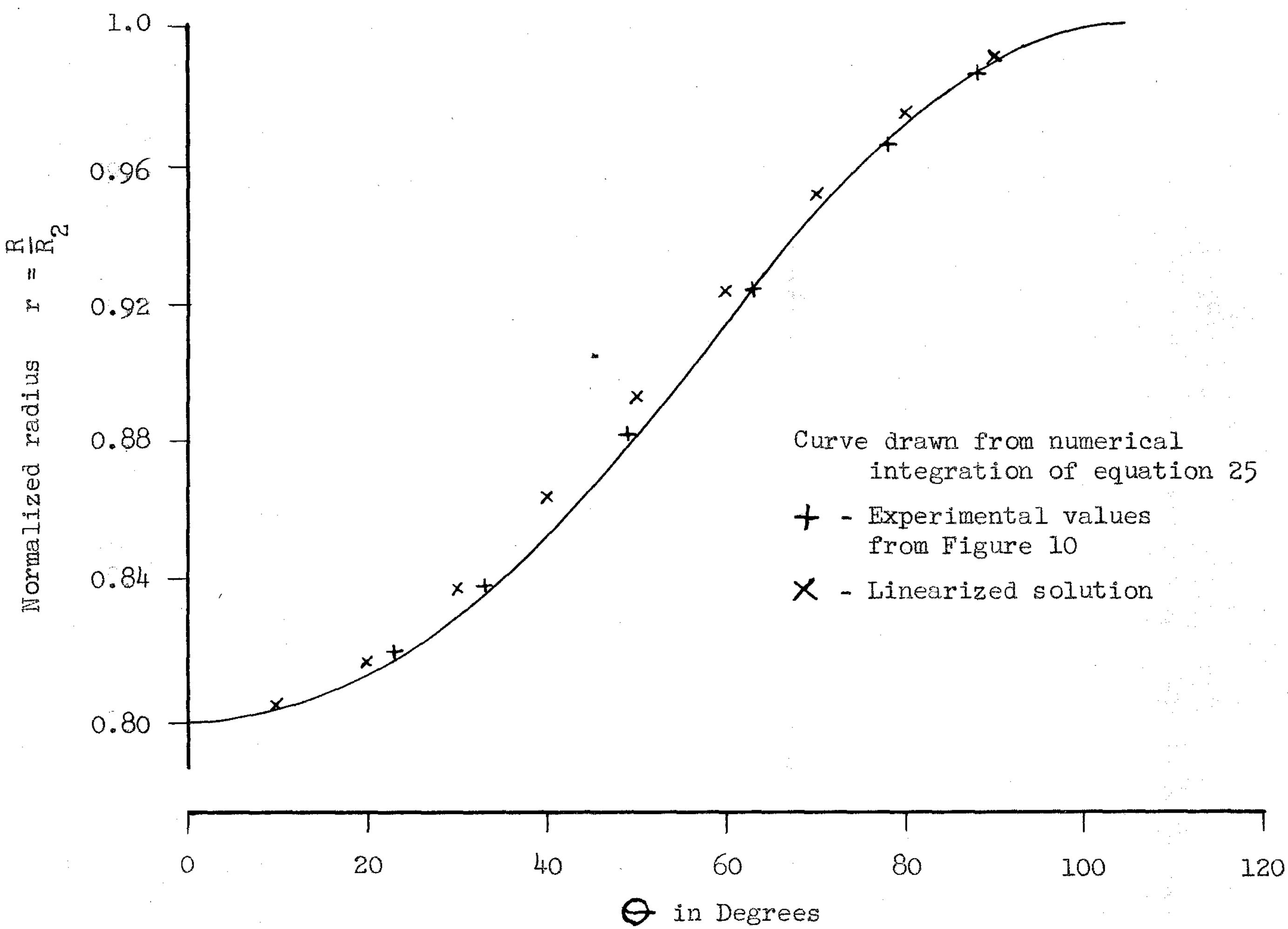

FIGURE 6. Theoretical and experimental curves of θ vs. r for η =0.8

VII. EXPERIMENTAL PROCEDURE

A relatively simple apparatus was used to verify the theoretical results. The equipment used is shown in the photograph of figure 7. The curve was formed on a ten inch diameter wooden disc. Around the circumference of the disc was a thin steel band which protruded about 1/8 inch above the surface of the disc. This formed the rim. The hub was formed by a circular plywood plug which could be fastened to the center of the 10" disc. Several plugs of different diameters were made. This provided the means for varying the gap width and hence the value of η, i.e. R_2 was fixed but R_1 was variable.

To provide values of $\eta = 0.2$, 0.4, and 0.8, hubs of 1", 2", and 4" radius were used. Actually the hub diameters were less than the nominal dimension by an amount equal to the diameter of the chain so that the radius to center line of the chain would be exactly equal to the nominal radius. Unfortunately, the rim radius was made slightly less than what it was intended to be so that the radius to the center line of the cahin at the rim, R_2, is slightly less than the nominal dimension of 5 inches. However, the actual dimension was used in calculating the curves.

The disc was mounted in a horizontal plane directly on the shaft of variable speed, thyratron controlled, D. C. motor. With the disc in a horizontal plane the effect of gravity is nullified. The chain was then fastened to the hub and the rim in such a way that the natural curve was formed by the chain when the disc was rotated. This was checked as follows. Upon rotating the disc the chain assumed an equilibrium curve. The points at which the chain was fastened to the

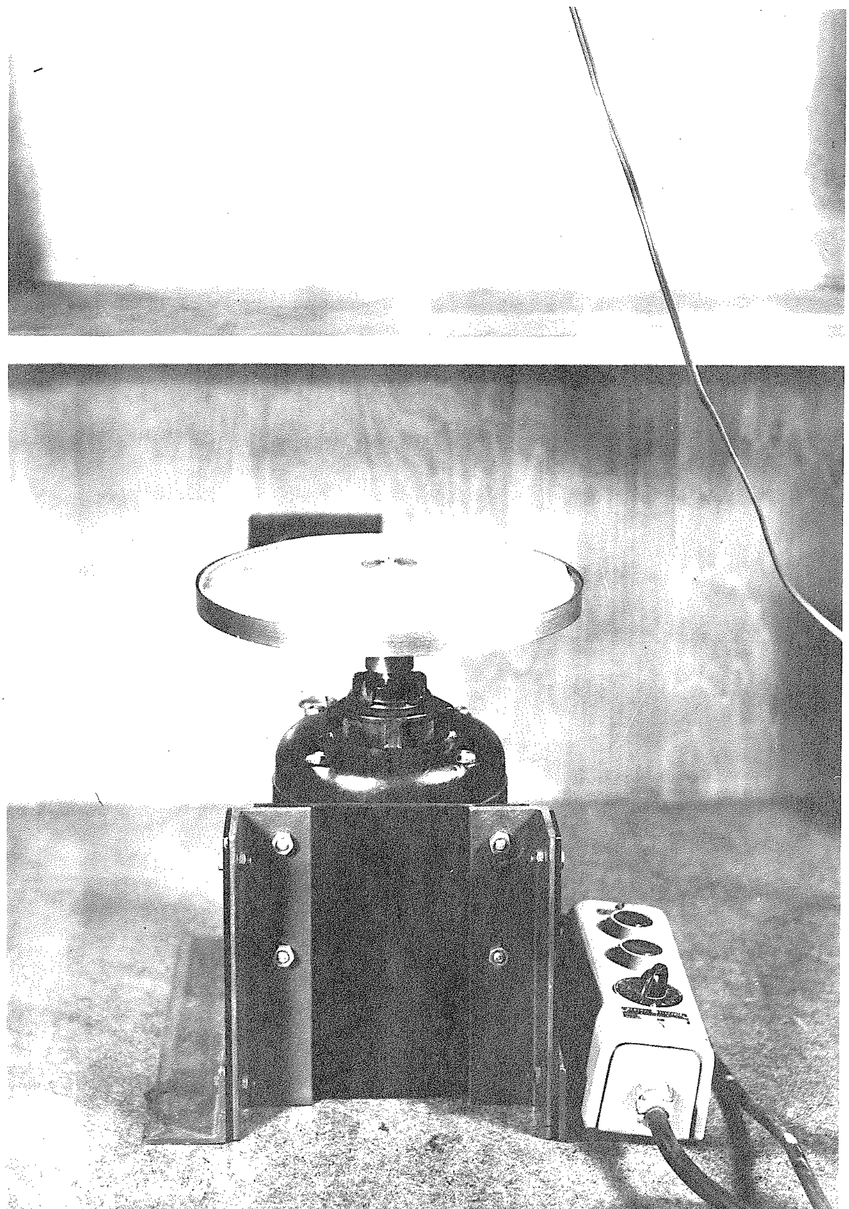

FIGURE 1.   Experimental Apparatus

hub and rim had to be at some distance from the points of tangency in order to assure that there was no discontinuity in the slope of the chain at these points.

The curve was formed by an ordinary ball and socket chain such as is used on light fixtures. This obviously meets the requirement of having zero stiffness. The only difficulty experienced with this chain was due to the fact that it is not truly a continuous medium since the links are rigid members of finite length. Since the links of the chain used were small, this effect was noticeable only on the portion of the curve where the curvature was large.

To record the trace of the curve a technique was used which gave the image of the chain curve directly on an ultra violet sensitive paper. This was done in the following manner.

Directly on the disc was placed a piece of ultra violet sensitive paper. The type used was Driazo No. 1200SS which is an extra rigid speed paper. Over this was placed a transparent circular-radial graph paper. The hub held both of these flat against the surface of the disc. The chain was then placed on the disc on top of the graph paper and fastened to the hub and rim in the appropriate manner. After the chain had assumed its natural curve when the disc was rotated, the disc was exposed to an ultra violet light source for a period of approximately one minute while the disc was rotating which was sufficient to expose the ozalid paper. This left the image of the chain with a reference grid provided by the graph paper on the sensitized paper. This paper was then developed in an "Ozalid" machine. This technique had a number of features which made it advantageous to use. They were:

1. Since there was no relative motion between the chain, the the graph paper, and the sensitized paper with the disc rotating, the exposure did not have to be instantaneous. Therefore no high speed flash equipment was required and an ordinary lamp socket type ultra violet light bulb was adequate as the ultra violet source.
2. The work could be done in ordinary ambient light since the sensitized paper is insensitive to tungsten light.
3. Since the grid lines were on the paper before the paper was developed, there is no relative distortion between the trace of the curve and the grid lines due to shrinkage which may occur during the developing process.
4. The resulting trace was on a large scale and there were no parallax problems.

VIII. RESULTS

The actual curve traces obtained by the described experimental method are shown in figures 8, 9, 10, and 11.

To compare the theoretical and experimental curves they were plotted together in figures 4, 5, and 6 for the different cases considered. The theory states that the angle as a function of the normalized radius should depend only on the parameter η. Therefore, the normalized curves should be identical for the cases where:

$$R_1 = 1'' \text{ and } R_2 = 2.5'' \text{ - symmetric case}$$

and

$$R_1 = 2'' \text{ and } R_2 = 5.0'' \text{ - non-symmetric case}$$

since $\eta = 0.4$ for both cases. A comparison of the curves shows this to be true. Also plotted in figures 4, 5, and 6 are the cosine curves approximations from equation 50 which are given by the solution of the linearized differential equation.

The plotting of the curve from the experimentally obtained picture to the rectangular coordinate plots of figures 4, 5, and 6 presented the problem of orienting the entire curve. It is very difficult to establish the points of tangency on the experimental curves. Also, the greatest error in the curves obtained by numerical integration occurred at the end points since the series expansion was used to give the first and last intervals. Therefore, a point on the curve at approximately the mean radius of the gap was used as a reference point between the theoretical and experimental curves. This gave a good "best fit" match between the curves. A smooth curve was drawn through the points given by the numerical integration procedure and the points

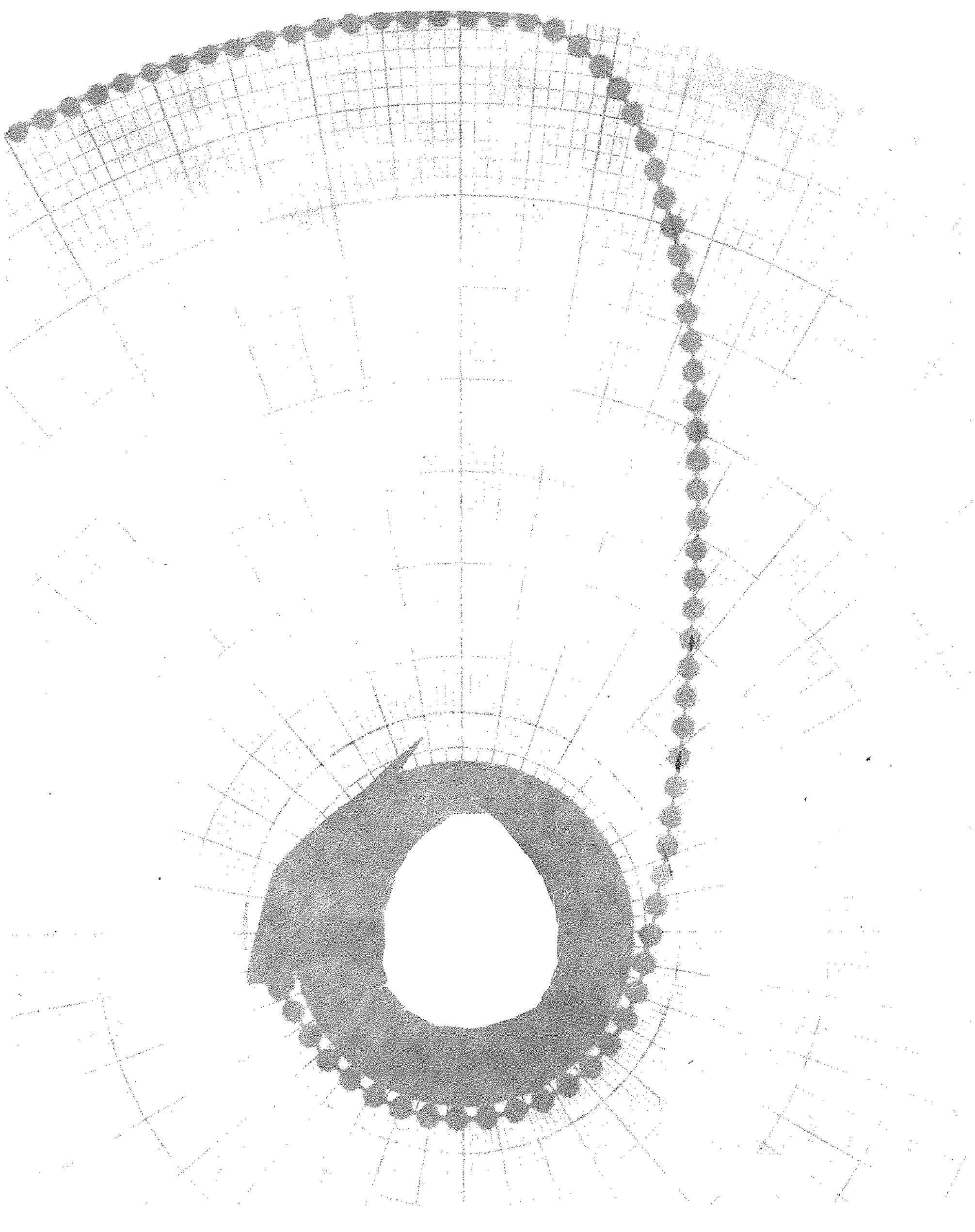

FIGURE 8. Experimental curve for η =0.2

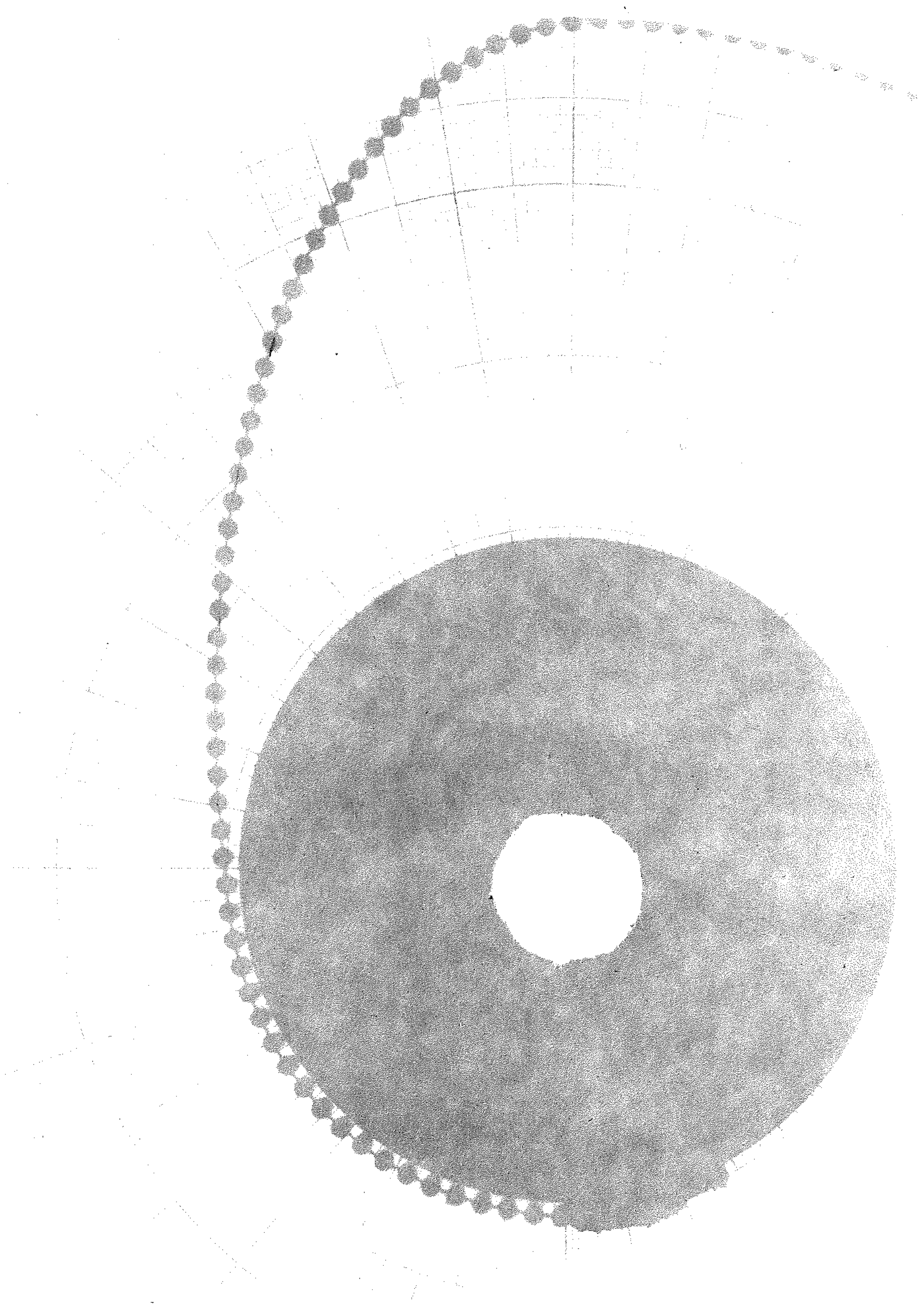

FIGURE 9. Experimental curve for $\eta = 0.4$

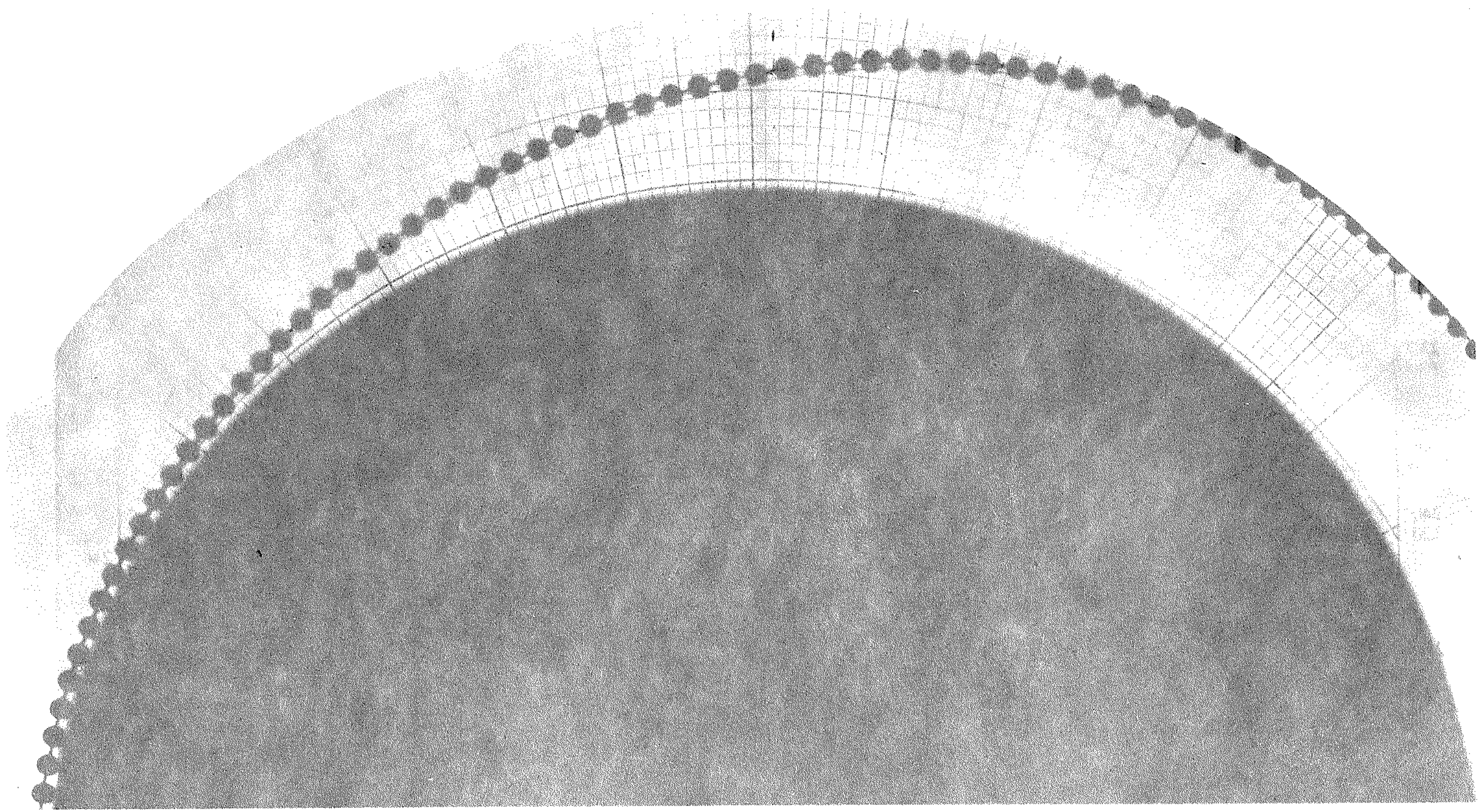

FIGURE 10. Experimental curve for $\eta = 0.8$

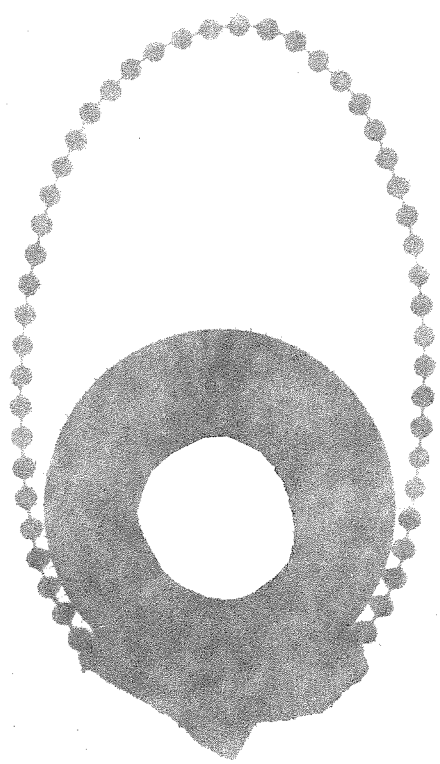

FIGURE 11. Experimental curve for symmetric case where $\eta = 0.4$

from the experimental curve and the linearized solution were superimposed on this.

To check whether the curve was independent of the speed of rotation, the disc was rotated at several different speeds while observing the curve with the aid of a strobe light.

IX. DISCUSSION OF RESULTS AND CONCLUSIONS

The first result predicted by the theory was that the curve should be independent of the speed of rotation. This was observed to be true by checking with a strobe light. There was no variation in the curve over a wide range of rotational speed. Since the density term, ρ, in the equation cancelled out exactly as did the rotational speed term, ω, it was assumed that the curve would also be independent of the density.

A result shown by the analytical solution which would not likely be predicted intuitively is that θ_{max} increases when the gap width decreases as shown in figure 3. This curve was difficult to check by the experimental curves because of the difficulty of establishing the points of tangency with the hub and rim. However, it is clear from a comparison of the experimental curves for $\eta = 0.2$ and $\eta = 0.8$ that θ_{max} is greater for the latter.

The most conclusive verification of the theory is the comparison of the experimental curves and the curves obtained by numerical integration of equation 25, shown in figures 4, 5, and 6. The curves agree within the limits of the accuracy imposed by the experimental technique and the numerical integration procedure. The discrepancy indicated near the outer portion of the curve for $\eta = 0.2$ is probably due to the effect of the finite lengths of the chain links since the curvature is largest there. However, the results indicate that equation 25 does accurately describe the system. Since equation 30 is the exact solution of equation 25, it then must be the solution for the curve in the gap.

Figures 4, 5, and 6 show that the solution of the approximate linearized differential equation is quite good for large values of η but inaccurate for small values of η. However, this was to be expected from the assumptions made in the linearization procedure. The effects of the non-linearities in the system are most important for small values of η.

Although the problem discussed in this paper has been treated in an academic manner, it has practical significance also since it arose from a problem encountered at the IBM research laboratory and may have further applications elsewhere. However its main significance lies in the fact that it is a non-linear problem which possesses an exact analytical solution. There are indeed many commonly occurring phenomena which can only be described accurately by a non-linear equation but relatively few of these have an exact solution.

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APPENDIX

Sample Calculations

A. Evaluation of θ_{max} given by equation 32.

$$\theta_{max} = \frac{\eta}{(\eta + 1)(2\eta + 1)^{1/2}} K(k) + \{[E(k) - K(k)]F(\psi,k') + K(k)E(\psi,k')\}$$

where

$$k^2 = \frac{1 - \eta^2}{2\eta + 1} \qquad k'^2 = 1 - k^2 \qquad \psi = \sin^{-1} \frac{(2\eta + 1)^{1/2}}{\eta + 1}$$

The following table was evaluated for different values of η where $0 < \eta < 1$. The calculations for $\eta = 0.3$ are shown here

1	η	0.3
2	k^2	0.5688
3	$\log k^2$	9.75496-10
4	$\log k$	9.87748-10
5	$\sin^{-1} k$	48°57'
6	$\sin^{-1}k' = 90 - \sin^{-1}k$	41°3'
7	$\log \sin \psi$	9.98812-10
8	ψ	76°40'
9	$F(k)$	1.92
10	$E(k)$	1.31
11	$F(\psi,k')$	1.49
12	$E(\psi,k')$	1.21
13	$\eta K(k)/(\eta+1) \times (2\eta+1)^{1/2}$	0.3502
14	$E(k) - K(k)$	-0.61
15	$14 \cdot F(\psi,k')$	-0.91
16	$K(k)E(\psi,k')$	2.32
17	θ(rad.) = 13 + 15 + 16	1.76
18	θ (degrees)	100.8

B. Evaluation of θ_{max} given by equation 33 for $\eta = 0$.

$$\theta_{max} = \frac{\eta + 1}{\eta(2\eta + 1)^{1/2}} \left[\frac{1}{1 + \lambda} \ln \frac{4}{k'} + \frac{\lambda}{1 + \lambda} \tan^{-1}\sqrt{\lambda} + 0(k')^2 \right]$$

where

$$\lambda = \frac{1 - \eta^2}{\eta^2}$$

$$k^2 = \frac{1 - \eta^2}{2\eta + 1}$$

$$k'^2 = 1 - k^2$$

For $\eta = 0$,

$$\lambda = \infty$$

$$k' = 0$$

Taking the first term,

$$\lim_{\eta \to 0} \frac{(\eta + 1)\ln(4/k')}{\eta(2\eta + 1)^{1/2}(1 + \lambda)} = \lim_{\eta \to 0} \frac{(\eta + 1)\ \eta\ \ln 4/\sqrt{\frac{\eta(\eta + 2)}{2\eta + 1}}}{(2\eta + 1)^{1/2}} = 0$$

This leaves only the second term which gives:

$$\theta_{max} = \lim_{\eta \to 0} \frac{(\eta + 1)\sqrt{\lambda}}{\eta\sqrt{(2\eta + 1)}(1 + \lambda)} \tan^{-1}\sqrt{\lambda}$$

$$= \lim_{\eta \to 0} \frac{(\eta + 1)(1 - \eta^2)^{1/2}\ \eta^2}{\eta(2\eta + 1)^{1/2}\eta} \tan^{-1}\sqrt{\lambda}$$

$$= \tan^{-1}(\infty) = \frac{\pi}{2}$$

C. To integrate numerically the curve for $\eta = 0.8$, equation 62 was first evaluated.

$$\left(\frac{d^2r}{d\theta^2}\right)_\eta = \frac{\eta(1+\eta-2\eta^2)}{(1+\eta)} = \frac{0.8(1.8-1.28)}{1.8} = \frac{0.4(0.52)}{0.9} = 0.2311$$

$$\left(\frac{d^2r}{d\theta^2}\right)_1 = \frac{\eta^2+\eta-2}{\eta(\eta+1)} = \frac{0.64+0.8-2}{0.64+0.8} = -\frac{0.56}{4.44} = -0.3889$$

Then from equation 55 with $\Delta r = 0.02$

$$(\Delta\theta)_{r=\eta} = \left[\frac{2\Delta r}{\left(\frac{d^2r}{d\theta^2}\right)_{r=\eta}}\right]^{1/2} = \left[\frac{0.04}{0.2311}\right]^{1/2} = [0.1731]^{1/2} = 0.4160$$

Equation 24 was integrated over the curve from $r = 0.82$ to $r = 0.98$ using Simpson's rule.

Simpson's rule states that:

$$\int_a^{a+2n\Delta r} f(r)dr = \frac{\Delta r}{3}[f(a) + 4f(a+\Delta r) + 2f(a+2\Delta r)$$
$$+ 4f(a+3\Delta r) + \ldots 4f[a+(2n-1)\Delta r] + f(a+2n\Delta r)]$$

For this problem

$$f(r) = \frac{1}{r[(1+\beta-\beta r^2)^2-1]^{1/2}}$$

The following table was computed using $\Delta r = 0.02$.

NUMERICAL INTEGRATION OF EQUATION 24 FOR $\eta = 0.8$

r	r^2	βr^2	$\gamma = 1+\beta-\beta r^2$	$r^2\gamma^2-1$	$\sqrt{r^2\gamma^2-1}$	$r\sqrt{r^2\gamma^2-1}$	$\frac{1}{r\sqrt{r^2\gamma^2-1}}$	$\theta \cdot \frac{3}{\Delta r}$	θ (degrees)
0.80	0.64	0.4444	1.2500	0.0000	0.0000	0.0000	∞	0	0
0.82	0.6724	0.4669	1.2275	0.0132	0.1149	0.0942	10.62	62.4	23.84
0.84	0.7056	0.4900	1.2044	0.0235	0.1533	0.1288	7.764		
0.86	0.7396	0.5136	1.1808	0.0312	0.1766	0.1519	6.583	110.66	42.27
0.88	0.7744	0.5377	1.1567	0.0385	0.1892	0.1665	6.042		
0.90	0.8100	0.5625	1.1319	0.0378	0.1944	0.1750	5.714	147.12	56.20
0.92	0.8464	0.5877	1.1067	0.0367	0.1916	0.1763	5.672		
0.94	0.8836	0.6136	1.0808	0.0321	0.1792	0.1684	5.938	181.46	69.32
0.96	0.9216	0.6400	1.0544	0.0246	0.1568	0.1505	6.645		
0.98	0.9604	0.6669	1.0275	0.0140	0.1183	0.1159	8.628	222.61	85.04
1.00	1.0000	0.6944	1.0000	0.0000	0.0000	0.0000	∞	270.78	103.4

D. Evaluation of the curve given by the approximate solution of equation 50 for $\eta = 0.4$.

$$R = R_0(1 - \varepsilon \cos \quad 3\,\theta) \qquad (50)$$

For $\eta = 0.2$:

R_1 = 1.0 inches

R_2 = 4.965 inches

R_0 = 2.982 inches

ε = 1.982 inches

θ degrees	cos 3 θ	ε cos 3 θ	R inches
10	0.955	1.892	1.090
20	0.823	1.631	1.351
30	0.617	1.223	1.759
40	0.354	0.702	2.280
50	0.0593	0.118	2.864
60	-0.240	-0.476	3.458
70	-0.519	-1.030	4.012
80	-0.749	-1.485	4.467
90	-0.911	-1.806	4.788
100	-0.991	-1.964	4.946