A FOURIER INTEGRAL APPROACH TO AN AEOLOTROPIC MEDIUM

by

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SUMMARY

Chapter I:

The equations of equilibrium in terms of the displacement components for an axially symmetric aeolotropic medium are developed from the strain-energy function of the medium. Then follows a discussion of the literature of the subject, and an outline of the scope of the present thesis.

Chapter II:

The solution is carried through using Fourier Integral technique for the two dimensional plane strain case. Stresses and displacements are obtained for a concentrated line load.

Chapter III:

The results of Chapter II are applied to determine the surface settlements, vertical pressures, and shears for a symmetrically loaded strip called "the unit strip" of width two units. The following special load distributions are investigated: Concentrated, uniform, parabolic, inverted parabolic, hollow wall, and rigid wall. Extension is then made to a strip of any arbitrary width 2a, and settlements are obtained by means of influence factors, (Graph I). An examination is made of the influence of the type of load distribution, demonstrating St. Venant's principle of equipollent loads.

Chapter IV:

The equations of Chapter I are solved for an axially

symmetric loading by transforming to polar co-ordinates and using Fourier-Bessel Integral technique. The solution is carried through for the concentrated load case, and the results check those given by Mitchell [6].

Chapter V:

An investigation similar to that made in Chapter III is made for a loaded circular area of unit radius. The results are then extended to a circle of any arbitrary width a. Surface settlements are obtained quickly by means of influence factors (Graph II). In the latter part of the Chapter series expansions are obtained for the stresses and displacements at any point in the mass, and application is made to some of the more practical load distributions.

Chapter VI:

Corresponding results for an elastic isotropic medium, to those given in above Chapters, are obtained by the application of a limiting technique to above results. The ease with which the results are obtained is striking. A discussion is given of the infinite surface displacements that are usually obtained in two-dimensional problems

Chapter VII:

In this Chapter a review is made of the literature of the three constant medium. The physical significance of the assumptions and the measure of fulfillment of these assumptions by some types of wood, and by some crystals, is examined. Some errors are noted, and corrected. Finally all are shown to be just particular cases of the medium of Chapter II, without having the redeeming feature of simplicity over the more general theory.

Chapter VIII:

Results for Orthotropic plates are deduced from those given in Chapter II by a change of constants.

Chapter IX:

Typical problems in soil mechanics connected with a loaded column, and with a loaded wall, are worked out in detail. Graph III shows for a particular case the effect aeolotropy may have on the vertical stress distributions in a loaded soil. A brief outline is made of some other problems in an aeolotropic medium capable of solution by the methods of this thesis.

Appendix F:

Practical methods are given for the determination of the required constants. The value of skew samples is shown.

The results obtained in this thesis for an aeolotropic medium, apart from the concentrated case given by Mitchell[6], are new. A good test of the accuracy of the work is provided by the known isotropic elastic results obtained by a limiting procedure in Chapter VI. As far as the author is aware, some of the results of Chapter VI are new also. The direct application of Fourier Integral technique to the displacement equations of equilibrium is very rare in elastic problems. This thesis illustrates the power and simplicity of such an approach. Finally, as shown in Chapter IX the results are very readily adapted to practical use.

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INTRODUCTION

Engineers have long felt the need for a satisfactory mathematical theory for a loaded soil medium. Apart from the works of Wolf[1] and Weiskopf[2], little attempt has been made to depart fundamentally from the classical methods of isotropic elasticity. That most soils depart widely from this latter theory is recognized by all.

Many uniform soils may be regarded as having identical elastic properties in all horizontal planes, and hence may be said to have elastic symmetry about a vertical axis. The elastic isotropic case is reached in the limit, when the elastic properties on all planes are the same. This thesis presents a mathematical analysis of such aeolotropic systems. It necessitates the introduction of five independent elastic constants, which makes it more general than the corresponding elastic isotropic theory. Yet the end results are surprisingly simple, and are very easily adapted to practice.

However, this thesis does not allow for the possible variation of the elastic constants with depth. Such an analysis is practically mathematically impossible. Hence, this thesis is not the complete answer to all soil problems. However, since practical methods are given for measuring the constants involved, this thesis should have very useful application where the magnitude of the structure warrants the extra testing required to establish the necessary constants.

NOTATION

The notation employed is generally that used by Love[3] in his

"Mathematical Theory of Elasticity". The correspondence between this notation and that used by Timoshenko[4], "Theory of Elasticity", is as follows:

<u>Stresses</u>

$$\widehat{xx} = \sigma_x \qquad \widehat{yy} = \sigma_y \qquad \widehat{zz} = \sigma_z$$

$$\widehat{yz} = \tau_{yz} \qquad \widehat{zx} = \tau_{zx} \qquad \widehat{xy} = \tau_{xy}$$

<u>Strains</u>

$$e_{xx} = \epsilon_x \qquad e_{yy} = \epsilon_y \qquad e_{zz} = \epsilon_z$$

$$e_{yz} = \gamma_{yz} \qquad e_{zx} = \gamma_{zx} \qquad e_{xy} = \gamma_{xy}$$

Similar expressions hold in cylindrical coordinates where (r, θ, z) replace (x, y, z). An index of notation is given in Appendix H. In the coordinate systems used, z is always taken as vertical with the positive direction downwards into the medium.

CHAPTER I.

DEVELOPMENT OF AN AXIALLY SYMMETRIC AEOLOTROPIC MEDIUM WITH A DISCUSSION OF ITS LITERATURE

Consider a material that possesses a vertical axis of symmetry in the sense that all rays at right angles to this axis are equivalent. Taking the axis of symmetry as the axis of z, the strain-energy-function[3] becomes (Love 1944 Sec. 110)

$$2W = A(e_{xx}^2 + e_{yy}^2) + Ce_{zz}^2 + 2F(e_{yy} + e_{xx})e_{zz} + 2(A-2N)e_{xx}e_{yy} + L(e_{yz}^2 + e_{zx}^2) + Ne_{xy}^2 \tag{1.1}$$

from which

$$\begin{aligned}
\widehat{xx} &= Ae_{xx} + (A-2N)e_{yy} + Fe_{zz} \\
\widehat{yy} &= (A-2N)e_{xx} + Ae_{yy} + Fe_{zz} \\
\widehat{zz} &= Fe_{xx} + Fe_{yy} + Ce_{zz} \\
\widehat{yz} &= Le_{yz} \\
\widehat{zx} &= Le_{zx} \\
\widehat{xy} &= Ne_{xy}
\end{aligned} \tag{1.11}$$

By solving the above for strains in terms of stresses, the five constants A C F L and N can be expressed in terms of the better known moduli E_i, σ_i, μ_i (i = 1, 2, 3) where from symmetry

$$E_1 = E_2, \quad \mu_1 = \mu_2 \tag{1.12}$$

Writing Hooke's Law in terms of E_i, σ_i and μ_i we obtain on noting the equivalence of the x and the y directions

$$e_{xx} = \frac{\widehat{xx}}{E_1} - \frac{\sigma_1}{E_1}\widehat{yy} - \frac{\sigma_3}{E_1}\widehat{zz} \qquad e_{yz} = \frac{1}{\mu_3}\widehat{yz}$$

$$e_{yy} = \frac{\widehat{yy}}{E_1} - \frac{\sigma_1}{E_1}\widehat{xx} - \frac{\sigma_3}{E_3}\widehat{zz} \qquad e_{xz} = \frac{1}{\mu_3}\widehat{xz} \tag{1.13}$$

$$e_{zz} = -\frac{\sigma_1}{E_1}(\widehat{xx}+\widehat{yy}) + \frac{1}{E_3}\widehat{zz} \qquad e_{xy} = \frac{1}{\mu_1}\widehat{xy}$$

The physical interpretation of the constants E_i , σ_i and μ_i ($i = 1, 2, 3$) is evident from above equations.

and since

$$\frac{\partial^2 W}{\partial \widehat{xx}\,\partial \widehat{zz}} = \frac{\partial^2 W}{\partial \widehat{zz}\,\partial \widehat{xx}} \quad \text{i.e.} \quad \frac{\partial e_{xx}}{\partial \widehat{zz}} = \frac{\partial e_{zz}}{\partial \widehat{xx}}$$

$$\therefore \qquad \frac{\sigma_3}{E_3} = \frac{\sigma_2}{E_1} \tag{1.14}$$

On solving above equations for $\widehat{xx}$ and $\widehat{zz}$, and on comparing results with those in (1.11), we obtain

$$F = \frac{\sigma_3 E_1}{1-\sigma_1-2\sigma_2\sigma_3} \quad ; \quad C = \frac{(1-\sigma_1)E_3}{1-\sigma_1-2\sigma_2\sigma_3} \tag{1.15}$$

$$A = \frac{(1-\sigma_2\sigma_3)E_1}{(1+\sigma_1)(1-\sigma_1-2\sigma_2\sigma_3)} \quad ; \quad N = \frac{E_1}{2(1+\sigma_1)} = \mu_1$$

$$L = \mu_3$$

where μ_1 is modulus of rigidity of horizontal samples, and

μ_3 is modulus of rigidity of vertical samples.

In the case of an elastic isotropic body $E_1 = E_3 = E$ say, $\sigma_1 = \sigma_2 = \sigma_3 = \sigma$.

Therefore,

$$L = N = \frac{E}{2(1+\sigma)}$$

$$A = C = \frac{E(1-\sigma)}{(1+\sigma)(1-2\sigma)} \qquad F = \frac{\sigma E}{(1+\sigma)(1-2\sigma)} \tag{1.16}$$

Alternatively, we can express above in terms of λ and μ, where λ is Lame's elastic constant:

$$\begin{aligned} A &= C = \lambda + 2\mu \\ L &= N = \mu \\ F &= \lambda \\ G &= L + F = \lambda + \mu \end{aligned} \tag{1.17}$$

We shall use above form of the constants in Chapter 6. in deducing the elastic isotropic case by a limiting procedure.

The stress on any plane with normal $\overline{\nu}\,(l,m,n)$ are given by

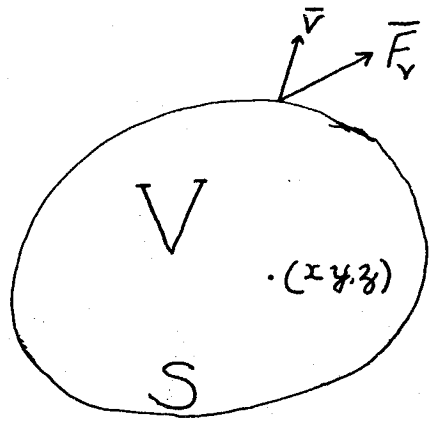
$$\overline{F}_\nu = (X_\nu\, ,\, Y_\nu\, ,\, Z_\nu)$$

where

$$\begin{aligned} X_\nu &= l\widehat{xx} + m\widehat{xy} + n\widehat{xy} \\ Y_\nu &= l\widehat{xy} + m\widehat{yy} + n\widehat{yz} \\ Z_\nu &= l\widehat{xz} + m\widehat{yz} + n\widehat{zz} \end{aligned} \tag{1.18}$$

General Derivation of the Equilibrium Equations

Let x y z represent a curvilinear orthogonal set of axes in an aeolotropic body. The normal stress on any plane $\overline{\nu}$ is $\overline{F}_\nu$ as given in (1.18).



Consider the motion of the material within a closed surface S enclosing a volume V surrounding a point (x, y, z) in the strained medium. Let $\bar{R} = (X, Y, Z)$ denote the body forces per unit volume and $\bar{S} = (u, v, w)$ be the displacement at the point (x, y, z) Considering motion in the x direction, we obtain on resolving

$$\int_V \rho \frac{\partial^2 u}{\partial t^2} dV = \int_V \rho X \, dV + \int_S X_\nu \, dS \tag{1.19}$$

But

$$\int_S X_\nu \, dS = \int_S (l\widehat{xx} + m\widehat{xy} + m\widehat{xz}) \, dS = \int_S (\widehat{xx}, \widehat{xy}, \widehat{xz}) . \bar{\nu} \, dS$$

$$= \int_V \text{div} \, (\widehat{xx}, \widehat{xy}, \widehat{xz}) \, dV$$

on applying the Divergence theorem.[5]

Hence, from (1.19)

$$\int_V \left[\rho \frac{\partial^2 u}{\partial t^2} - \rho X - \text{div} \, (\widehat{xx}, \widehat{xy}, \widehat{xz})\right] dV = 0$$

But this must hold for every volume V surrounding the point (x,y,z), therefore

$$\rho \frac{\partial^2 u}{\partial t^2} = \rho X + \text{div} \, (\widehat{xx}, \widehat{xy}, \widehat{xz})$$

Similarly

$$\rho \frac{\partial^2 v}{\partial t^2} = \rho Y + \text{div} \, (\widehat{xy}, \widehat{yy}, \widehat{yz}) \tag{1.20}$$

$$\rho \frac{\partial^2 w}{\partial t^2} = \rho Z + \text{div} \, (\widehat{xz}, \widehat{yz}, \widehat{zz}).$$

Note in the above expressions the proper curvilinear expressions for div. must be used. The above equations are the stress equations of motion; for the equilibrium case put

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 w}{\partial t^2} = 0$$

The solution of the system of equations (1.20) satisfying the appropriate boundary conditions must also satisfy the requirement that the displacements be single-valued. This is expressed by the compatibility equations connecting the second derivatives of the strain components. These are not nearly as simple for the aeolotropic case as for the isotropic case. Hence, a more direct approach to the problem will be made in this thesis by formulating the equilibrium equations in terms of the displacements. The author thinks this approach has been used all too sparingly in isotropic elasticity.

Mitchell's Solution to the Three Dimensional Case

Mitchell[6] 1900 (a) transforms equations (1.20) into equations in terms of the displacements, and then expresses the latter in terms of the three variables:

$$\text{(i)}\quad \epsilon = \operatorname{div} \overline{S} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\text{(ii)}\quad 2\varpi = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$$

and $$\text{(iii)}\quad \frac{\partial w}{\partial z}$$

The solution is then made to depend on the solution of the three simultaneous equations

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + k_i \frac{\partial^2}{\partial z^2}\right) V_i = 0 \qquad i = 1, 2, 3$$

where

$$V_1 = \epsilon + q_1 \frac{\partial w}{\partial z}$$

$$V_2 = \epsilon + q_2 \frac{\partial w}{\partial z}$$

$$V_3 = w$$

and k_i, q_1, q_2 depend on the elastic constants of the medium. The boundary conditions are likewise expressed in terms of the values of V_i on the boundary. Hence, the problem is brought within the scope of the potential theory, and the results are given in terms of functions representing the potentials of plane distributions at external points. The problem is carried through in detail for a point load, and the well known Bousinesque result is obtained by a limiting procedure. However, it is very difficult to obtain results in a practical form from his equations.

Scope of Present Thesis

The present work is concerned with

a) two dimensional systems

b) three dimensional systems with a vertical axis of symmetry, and loaded symmetrically about this axis.

Both these cases will be derived from a Fourier Integral solution of the displacement equations of equilibrium. As far as the author is aware, this direct approach has not hitherto been used on elastic problems. The elastic isotropic ase is obtained by a limiting process from the aeolotropic case.

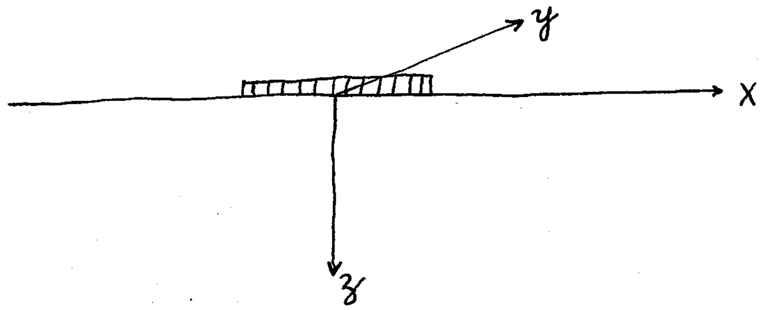
Green and Taylor[7] (1939) have developed two dimensional aeolotropic systems somewhat, using techniques analogous to the complex variable methods used in elastic isotropic cases[9]. This leads to a solution for an isolated force in an infinite plane, the two dimensional analogue of Kelvin's problem. However, no displacements are calculated. The technique for their calculation is similar to that used in Coker and Filon, though the labour is increased greatly.

Green[8] (1939) deals with generalized plane stress systems in an infinite aeolotropic strip and also in a semi-infinite plate bounded by a straight edge. The method of solution is similar to that used by Howland[10] (1929) for the corresponding problems in an isotropic material. It consists in obtaining a Fourier Integral representation for the stress function. Expressions are given for the stresses when a force acts on the boundary of a semi-infinite plate. However, no displacements are obtained and their calculation requires much additional labour.

This thesis obtains the displacements in a simple direct manner, and so should have many applications to soil mechanics where displacements are of paramount importance. No reference to part b) of the thesis could be traced. Biot[11] (1935) solves the analogous problem for an isotropic medium by a Fourier Integral expression for the stress function; again no displacements are calculated.

CHAPTER II.

TWO DIMENSIONAL PLANE STRAIN



For simplicity, consider a loading symmetrical about o y, and extend in either direction from O a considerable distance along the y axis, say from $y = \ell$ to $y = -\ell$. At regions, not close to $y = \pm\ell$ the state of strain is approximately plane: *i.e.*

$$e_{yy} = e_{xy} = e_{yz} = 0 \tag{2.1}$$

Note that $\widehat{yy}$ is not necessarily zero, as assumed by Weiskopf and discussed in Chapter 7. In soil mechanics, this corresponds to a long footing, or a loaded rectangle whose length (2ℓ) is much greater than its width. By a slight adjustment of constants, the problem can be treated as one of generalized plane stress and then furnishes a solution to the problem of a semi-infinite plate loaded along its boundary.

The strain-energy function now becomes

$$2W = Ae^2_{xx} + Ce^2_{zz} + 2Fe_{xx}e_{zz} + Le^2_{xz} \tag{2.11}$$

From which

$$\begin{aligned} \widehat{xx} &= Ae_{xx} + Fe_{zz} \\ \widehat{zz} &= Fe_{xx} + Ce_{zz} \\ \widehat{xz} &= Le_{xz} \end{aligned} \tag{2.12}$$

Substituting above values in equations (1.20) with $\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 w}{\partial t^2} = 0$

we obtain in terms of the strain components.

$$\rho X + A\frac{\partial e_{xx}}{\partial x} + F\frac{\partial e_{zz}}{\partial x} + L\frac{\partial e_{xz}}{\partial z} = 0 \qquad (2.13)$$

$$\rho Z + L\frac{\partial e_{xz}}{\partial x} + F\frac{\partial e_{xx}}{\partial z} + C\frac{\partial e_{zz}}{\partial z} = 0$$

Finally since[3]

$$e_{xx} = \frac{\partial u}{\partial x} \qquad e_{zz} = \frac{\partial w}{\partial z} \qquad e_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

we obtain the equilibrium equations in terms of the displacements with no body forces:

$$A\frac{\partial^2 u}{\partial x^2} + L\frac{\partial^2 u}{\partial z^2} + G\frac{\partial^2 w}{\partial x \partial z} = 0 \qquad (2.14)$$

$$C\frac{\partial^2 w}{\partial z^2} + L\frac{\partial^2 w}{\partial x^2} + G\frac{\partial^2 u}{\partial x \partial z} = 0$$

where

$$G = L + F \qquad (2.15)$$

As a useful practical case, consider loading symmetrically distributed with respect to both x and y axes. This requires that u be an odd function of x, and w an even function of x, hence type solutions are

$$u = U(z)\sin mx \qquad (2.16)$$

$$w = W(z)\cos mx$$

Let

$$D_1 \equiv \frac{\partial}{\partial x} \qquad D_3 \equiv \frac{\partial}{\partial z} \qquad D_{13}^2 \equiv \frac{\partial^2}{\partial x \partial z}$$

Then on eliminating w from (2.14), we obtain

$$\begin{vmatrix} A\,D_1^2 + LD_3^2 & G\,D_{13}^2 \\ G\,D_{13}^2 & LD_1^2 + CD_3^2 \end{vmatrix} u = 0$$

i.e. $$\left[A\,L\,D_1^4 + (L^2 + AC - G^2)D_{13}^4 + C\,L\,D_3^4\right] u = 0$$

Hence, on substituting $u = U \sin m x$ and $D_3 \equiv D = \frac{d}{dz}$.

$$\left[D^4 - \left(\frac{L^2 + AC - G^2}{CL}\right)m^2D^2 + \frac{A}{C}\,m\right] U = 0 \tag{2.17}$$

and the same equation holds also for W. The solutions to the above equations that tend to zero as $z \longrightarrow \infty$ are easily seen to be[10]

$$U = Re^{-s_1mz} + Se^{-s_2mz} \tag{2.18}$$

and $$W = R_1e^{-s_1mz} + S_1e^{-s_2mz}$$

where R, S, R_1 and S_1 are arbitrary constants connected by relations arising from the fact that (2.18) are solutions of the simultaneous equations (2.14). These constants may be functions of m, and s_1^2 and S_2^2 are the roots of the equation

$$S^4 - \frac{L^2 + AC - G^2}{CL}\,S^2 + \frac{A}{C} = 0 \tag{2.19}$$

These are both positive provided

$$L^2 + AC > G^2 = (L + F)^2$$

i.e. $$AC > 2LF + F^2$$

This inequality holds for all known materials. On using values (1.17) we easily obtain $\frac{L^2 + AC - G^2}{CL} = 2.$

And hence, $S_1^2 = S_2^2 = 1$ in the isotropic case. (2.20)

Relation Between Constants

Substituting the values (2.16) in the equations (2.14), and equating to zero the coefficients of $e^{-s_1 mz}$ and $e^{-s_2 mz}$ we obtain

$$R_1 = R\left(\frac{A-LS_1^2}{GS_1}\right) = h_1 R \tag{2.21}$$

$$S_1 = S\left(\frac{A-LS_2^2}{GS_2}\right) = h_2 S$$

where

$$h_1 = \frac{A-LS_1^2}{GS_1} \qquad\qquad h_2 = \frac{A-LS_2^2}{GS_2} \tag{2.22}$$

Boundary Conditions

On Z = 0, assume no shear stress, and a given normal stress:

$$0 = \widehat{xz}_{z=0} = \left.\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right|_{z=0} \tag{2.23}$$

$$f(x) = \widehat{zz} = Fe_{xx} + Ce_{zz} = \left.F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z}\right|_{z=0} \tag{2.24}$$

Consider the solution obtained by the superposition of simple solutions[13]:

$$u = \int_0^\infty \left[R_m\, e^{-s_1 mz} + S_m\, e^{-s_2 mz}\right] \sin mx\, dm \tag{2.25}$$

$$w = \int_0^\infty \left[h_1 R_m\, e^{-s_1 mz} + h_2 S_m\, e^{-s_2 mz}\right] \cos mx\, dm \tag{2.26}$$

where R_m and S_m are functions of m to be determined from the boundary conditions (2.23) and (2.24).

From (2.12) the shear stress is given by

$$\widehat{xz} = -L\int_0^\infty m\left[R_m\,(s_1+h_1)\,e^{-s_1 mz} + S_m(s_2+h_2)\,e^{-s_2 mz}\right]\sin mx\,dm \qquad (2.27)$$

applying (2.23) we obtain

$$0 = \int_0^\infty m\left[R_m\,(s_1+h_1) + S_m\,(s_2+h_2)\right]\sin mx\,dm$$

Clearly a sufficient condition for this is that

$$m\left[R_m\,(s_1+h_1) + S_m\,(s_2+h_2)\right] = 0 \qquad (2.28)$$

<u>Stresses and Displacements in Terms of R_m</u>

Again from (2.12) and (2.28)

$$\widehat{zz} = F\frac{\partial u}{\partial x} + C\frac{\partial w}{\partial z} = \int_0^\infty m\left[R_m\,(F-Ch_1s_1)\,e^{-s_1 mz} + S_m\,(F-Ch_2s_2)\,e^{-s_2 mz}\right]\cos mx\,dm$$

$$\therefore\quad \widehat{zz}(s_2+h_2) = \int_0^\infty m\,R_m\left[s_3e^{-s_1 mz} - s_4e^{-s_2 mz}\right]\cos mx\,dm \qquad (2.29)$$

also

$$\widehat{xz} = L\,(s_1+h_1)\int_0^\infty m\,R_m\left[-e^{-s_1 mz} + e^{-s_2 mz}\right]\sin mx\,dm \qquad (2.30)$$

and $$\widehat{xx} = A\frac{\partial u}{\partial x} + F\frac{\partial w}{\partial z} = \int_0^\infty m\left[R_m(A-Fh_1s_1)\,e^{-s_1 mz} + S_m(A-Fh_2s_2)\,e^{-s_2 mz}\right]\cos mx\,dm$$

Hence,

$$\widehat{xx}\,(s_2+h_2) = \int_0^\infty m\,R_m\left[s_5e^{-s_1 mz} - s_6e^{-s_2 mz}\right]\cos mx\,dm \qquad (2.31)$$

Similarly

$$u\,(s_2+h_2) = \int_0^\infty R_m\left[(s_2+h_2)\,e^{-s_1 mz} - (s_1+h_1)\,e^{-s_2 mz}\right]\sin mx\,dm \qquad (2.32)$$

$$w\,(s_2+h_2) = \int_0^\infty R_m\left[s_7e^{-s_1 mz} - S_8e^{-s_2 mz}\right]\cos mx\,dm.$$

Where the new constants introduced above are given by

$$s_3 = (F - Ch_1 s_1)(s_2+h_2), \quad s_4 = (F- Ch_2 s_2)(s_1+h_1) \tag{2.34}$$

$$s_5 = (A - Fh_1 s_1)(s_2+h_2), \quad s_6 = (A - Fh_2 s_2)(s_1+h_1)$$

$$s_7 = h_1(s_2+h_2) \quad , \quad s_8 = h_2 (s_1+h_1)$$

Evaluation of R_m

Applying (2.24) to (2.29) we obtain

$$(s_2+h_2)f(x) = \int_0^\infty m\, R_m\, (s_3 - s_4) \cos m x\, dm \tag{2.35}$$

This is an integral equation for R_m, and can be solved by the Fourier Integral theorem(13), or its equivalent, the method of the Fourier transform(14). Hence,

$$m R_m = \frac{2}{\pi} \frac{s_2+h_2}{s_3-s_4} \int_0^\infty f(x) \cos mx\, dm = s_9 \int_0^\infty f(x) \cos m x\, dm \equiv s_9 U_m \tag{2.36}$$

where

$$U_m = \int_0^\infty f(x) \cos mx\, dm. \tag{2.37}$$

and

$$s_9 = \frac{2}{\pi} \frac{s_2+h_2}{s_3-s_4}$$

A sufficient condition(12) for validity of above transform is that $m R_m$ be of the class $L^2 (0, \infty)$

i.e., $\int_0^\infty | m R_m |^2 dm < \infty$ in the Lebesque sense. Using Parseval's formula,(14) this relation becomes

$$\int_0^\infty | m R_m |^2 dm = s_9^2 \int_0^\infty | f(x) |^2 dx < \infty \tag{2.38}$$

In the examples considered in this thesis, $f(x) = 0$, $|x| > a$ i.e., the loading is over a finite strip only. Hence, condition (2.38) is always satisfied, except in the case of a concentrated load.

It is much more difficult to show that the conditions on the Fourier Integral[14] are satisfied. It can be done in a manner similar to that adopted for the Fourier-Bessel Integral in Chapter 4.

Convergence of Integrals (2.29) - (2.33)

Integrals (2.29) - (2.33) are convergent for $z > 0$ provided $m R_m$ is bounded[(16)]. On $z = 0$ the integrals are convergent provided $\int_0^\infty m R_m \, dm$ is convergent. These requirements are satisfied for all cases discussed, except the concentrated load. Equations (2.32) and (2.33) are convergent at infinity, but they need investigation at $m = 0$. Integral (2.32) is convergent under above conditions since $R_m \sin m x$ is of order $m R_m$ as $m \to 0$, and this is bounded in the cases considered.

However, in (2.33) $R_m \cos m x \to 0 \left(\frac{1}{m}\right)$ and hence is divergent unless $\int_0^\infty f(x)\,dx = 0$ i.e., unless the applied loading has zero resultant on plane $z = 0$. Mathematically this latter condition means that $w \to \underset{\delta \to 0}{\mathrm{Lt}} \int_\delta^\infty R_m \left[s_7 e^{-s_1 mz} - s_8 e^{-s_2 mz}\right] \cos mx \, dm$

This integral enables us to calculate relative displacements at points close to and in the loaded area when R_m is given by formula (2.36) where $f(x)$ is taken as due to the loading on the finite part of the plane only. The equilibrating load at infinity does not affect the stresses in the finite part of the plane, and it makes w finite by superposing an infinite displacement in the opposite direction to that produced by the load in the finite part of the plane. This question is dealt with mathematically in Chapter 6. for the case of an isotropic body, and clearly the same reasoning applies to the aeolotropic case.

Surface Settlement (w_s)

From (2.33) and (2.37) we obtain

$$w_s = \frac{2}{\pi(s_3-s_4)} \operatorname*{Lt}_{z\to 0} \int_0^\infty \frac{U_m}{m} \left[s_7 e^{-s_1 mz} - s_8 e^{-s_2 mz} \right] \cos mx \, dm$$

$$\therefore \quad w_s \frac{\pi}{2} s_{13} = - \operatorname*{Lt}_{\upsilon\to 0} \int_0^\infty \frac{U_m}{m} e^{-m\upsilon} \cos mx \, dm \qquad (2.39)$$

where $s_{13} = - \dfrac{s_3 - s_4}{s_7 - s_8}$

Normal Vertical Pressure

$$p = - \widehat{zz} \Big|_{x=0}$$

Hence, from (22.9)

$$\widehat{zz} = \frac{2}{\pi(s_3-s_4)} \int_0^\infty U_m \left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz} \right] dm$$

$$= 2\, s_{10} \int_0^\infty U_m \left[\frac{1}{s_1} e^{-s_1 mz} - \frac{1}{s_2} e^{-s_2 mz} \right] dm \qquad (2.40)$$

Shear Stress on Plane x = 0

From (2.30) we note that $\widehat{xz} = 0$ when $x = 0$, and, therefore, $\widehat{zz}$ and $\widehat{xx}$ are principal stresses at every point on this plane. Hence, the maximum shear stress in the material (τ_M) is given by

$$\tau_M = \tfrac{1}{2} (\widehat{zz}-\widehat{xx}) = \frac{1}{\pi(s_3-s_4)} \int_0^\infty U_m \left[(s_3-s_5) e^{-s_1 mz} - (s_4-s_6) e^{-s_2 mz} \right] dm \qquad (2.41)$$

Concentrated Load at the Origin

Consider a load P uniformly distributed over width 2ϵ where $\epsilon \to 0$.

Then on z=0 $f(x) = -\frac{P}{2\epsilon}$, $|x| \leq \epsilon$

$= 0$, $|x| > \epsilon$

Hence from (2.36) formally we obtain

$$m\,R_m = -\frac{s_9 P}{2}\ \underset{\epsilon \to 0}{\mathrm{Lt}}\ \frac{1}{\epsilon}\int_0^{\epsilon} \cos mx\, dm = -s_9 P/2 \qquad (2.42)$$

It is rather difficult to justify the limiting process by which $m\,R_m$ was derived. Following Carslaw and Jaeger[17], the simplest procedure is to show that the stresses and strains obtained do actually satisfy the boundary and equilibrium conditions for a concentrated line load. This is easily shown.

Evaluation of (2.29) - (2.33)

The integrals involved is evaluating expressions (2.29) - (2.33) in the above case, can be calculated by replacing z by s_1z or s_2z in the appropriate integrals in Appendix A. Let

$$\overset{sz}{I}_1 \equiv \int_0^{\infty} e^{-smz} \cos mx\, dm \qquad (2.43)$$

$$\overset{sz}{I}_2 \equiv \int_0^{\infty} e^{-smz} \sin mx\, dm$$

$$r_i^2 = x^2 + (s_i z)^2 \quad ; \quad \tan \theta_i = \frac{s_i z}{x} \qquad i = 1, 2$$

Hence from (2.29)

$$\widehat{zz}\,(s_2+h_2) = -\frac{s_9 P}{2}\left[s_3 \overset{s_1 z}{I}_1 - s_4 \overset{s_2 z}{I}_1 \right]$$

$$= -\frac{s_9 P}{2}\, z \left[\frac{s_3 s_1}{r_1^2} - \frac{s_4 s_2}{r_2^2} \right]$$

$$\therefore \widehat{zz} = - s_{10} \frac{Pz}{} \left[\frac{1}{r_1^2} - \frac{1}{r_2^2} \right] \tag{2.44}$$

since from Appendix C-2 $s_1 s_3 = s_2 s_4$

$$\text{Similarly } \widehat{xz} = s_{10}\, Px \left[\frac{1}{r_1^2} - \frac{1}{r_2^2} \right] \tag{2.45}$$

where as proved in Appendix C-4

$$- L\,(s_1+h_1)\,\frac{s_9}{2} = \frac{s_9 s_4 s_2}{2(s_2+h_2)} \equiv S_{10} \tag{2.46}$$

Similarly

$$\widehat{xx}\,(s_2+h_2) = - \frac{s_9 Pz}{2} \left[\frac{s_1 s_5}{r_1^2} - \frac{s_2 s_6}{r_2^2} \right]$$

$$\therefore \quad \widehat{xx} = - \frac{Pz}{\pi} \left[\frac{s_1 s_2 (s_2 s_2 - s_1 s_6) z^2 + (s_1 s_5 - s_2 s_6)\, x^2}{(s_3 - s_4) \quad r_1^2 r_2^2} \right]$$

$$\widehat{xx} = s_{11} \frac{Px^2 z}{\pi r_1^2 r_2^2} \quad \text{on using C-5 and C-6} \tag{2.47}$$

<u>Displacements</u>

$$u(s_2+h_2) = - \frac{s_9 P}{2} \int_0^\infty \left[(s_2+h_2) e^{-s_1 mz} - (s_1+h_1) e^{-s_2 mz} \right] \frac{\sin mx}{m} \, dm$$

$$= - \frac{s_9 P}{2} \left[(s_2+h_2) \, {}^{s_1 z}I_2^{-1} - (s_1+h_1) \, {}^{s_2 z}I_2^{-1} \right]$$

$$= - \frac{s_1 P}{2} \left[\frac{\pi}{2} (s_2+h_2-s_1-h_1) - (s_2+h_2)\,\theta_1 + (s_1+h_1)\,\theta_2 \right]$$

$$\therefore u = \frac{s_9 P}{2} \left[\frac{\pi}{2} (s_{12}-1) - s_{12}\,\theta_2 + \theta_1 \right] \tag{2.48}$$

where $\qquad s_{12} = \dfrac{s_1 + h_1}{s_2 + h_2}$

and

$$w(s_2+h_2) \longrightarrow - s_9\frac{P}{2} \left[s_7\, \delta \mathcal{I}_{1}^{s_1 z\;-1} - S_8\, \delta \mathcal{I}_{1}^{s_2 z\;-1} \right]$$

$$\longrightarrow - s_9\frac{P}{2} \left[\text{constant} - s_7 \log r_1 + s_8 \log r_2 \right]$$

$$\therefore \quad w \rightarrow \frac{s_9 P}{2} \left[\text{constant} + h_1 \log r_1 - h_2 s_{12} \log r_2 \right] \tag{2.49}$$

On the surface $z = 0$ above result becomes

$$w \rightarrow \frac{P}{\pi s_{13}} \left[\text{constant} - \log x \right] \tag{2.50}$$

<u>Stress on Plane Z = Constant</u>

The stress is given by its components $\widehat{zz}$ and $\widehat{xz}$ (2.43 and (2.44). Clearly the resultant of these is always directed away from the origin and is of a magnitude

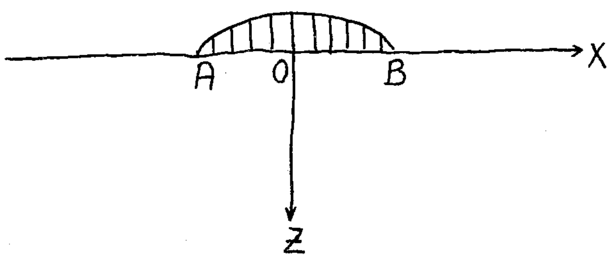
$$F_z = s_{10}\, Pr \left[\frac{1}{r_2^2} - \frac{1}{r_2^2} \right]$$

The well known result $F_z = - \frac{2P}{\pi} z^2/r^3$ for the elastic isotropic case follows easily from above on using the limiting procedure developed in Chapter 6.

CHAPTER III.

LOADED INFINITE STRIP

Unit Strip



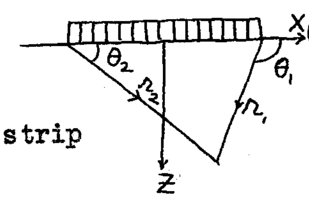
$$\widehat{zz} = -f(x), \quad |x| \leq 1$$
$$= 0 \quad |x| \geq 1$$

Consider loading on an infinite strip of width 2 units hereafter called the unit strip. We shall calculate the surface displacements, and vertical pressures for the following special cases of practical importance.

(3.1)

Concentrated Load [A]

Uniform Load Distribution [B]

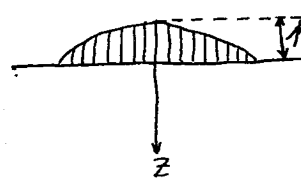


(3.11)

$$\widehat{zz}\Big|_{z=0} = -p_0 \quad |x| \leq 1$$
$$= 0 \quad |x| > 1$$

$\therefore \quad P = 2\,p_0$ per unit length of strip

Parabolic Load Distribution [C]

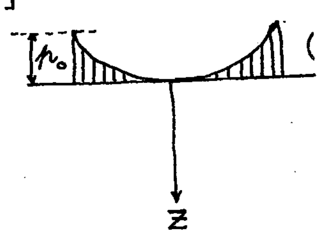


(3.12)

$$\widehat{zz}\Big|_{z=0} = -p_0\,(1-x^2) \quad |x| \leq 1$$
$$= 0 \quad |x| > 1$$

$$\therefore \quad P = 2\,p_0 \int_0^1 (1-x^2)\,dx = 4\,\frac{p_0}{3}$$

Inverted Parabolic Load Distribution [D]



(3.13)

$$\widehat{zz}\Big|_{z=0} = -p_0\,x^2 \quad |x| \leq 1$$
$$= 0 \quad |x| > 1$$

$$\therefore \quad P = 2\,p_0 \int_0^1 x^2\,dx = 2\,\frac{p_0}{3}$$

Hollow Wall [E] (3.14)

$$\widehat{zz}\Big|_{z=0} = -\frac{P}{2\epsilon}\ , \qquad 1-\epsilon \leq |x| \leq 1\ , \qquad \epsilon > 0$$

$$= 0 \qquad \text{all other values of } x$$

Rigid Wall [F]

$$w\Big|_{z=0} = \text{Const.} = w_0\ , \ |x| \leq 1 \qquad (3.15)$$

$$\widehat{zz}\Big|_{z=0} = 0 \quad , \qquad |x| > 1$$

Case A

This case is worked in detail (2.44) and (2.49). The results are

$$w_s\, \pi\, s_{13} = P\left[\text{const.} - \log\ x\right] \qquad (3.16)$$

and $\widehat{zz}\Big|_{x=0} = -\ s_{10}\,\frac{P}{z}\left[\frac{1}{s_1^2} - \frac{1}{s_2^2}\right] \equiv -\ s_{10}\,\frac{P}{z}\left[\frac{1}{s^2}\right]_{s_2}^{s_1}$

Case B

$$U_m = \int_0^\infty f(x)\cos mx\, dm = -\int_0^1 p_0 \cos mx\, dx$$

$$= -\frac{P}{2}\ \frac{\sin m}{m}$$

Then from (2.39) on substituting

$$w_s\, \pi\, s_{13} = P_2 \int_\delta^\infty \frac{\sin m\ \cos mx\ dm}{m^2}$$

$$= \frac{P}{2}\left[{}^{1+x}_{\delta}\mathrm{I}^{-2}_{2} + {}^{1+x}_{\delta}\mathrm{I}^{-2}_{2} \right]_{z=0} \qquad \text{Using A·12}$$

$$= \frac{P}{2}\left[(x-1)\log|x-1| - (x+1)\log|x+1| + \text{const.}\right] \qquad (3.17)$$

Note that this expression is finite everywhere on the surface, provided the constant is taken as finite, since $\underset{t\to 0}{\mathrm{Lt}}\ t\log t = 0$

Also from (2.40) we obtain

$$\widehat{zz}\Big|_{x=0} = -\frac{P}{\pi(s_3-s_4)}\int_0^\infty \frac{\sin m}{m}\left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] dm$$

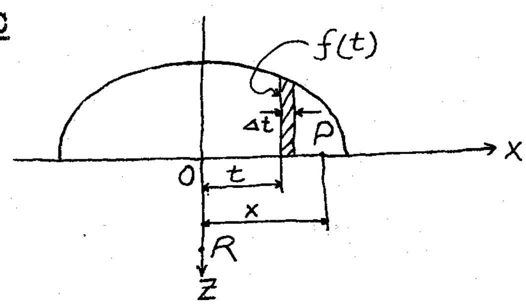
$$= -\frac{P}{\pi(s_3-s_4)}\left[s_3 \, I_{2}^{s_1 z,\,-1} - S_4 \, I_{2}^{s_2 z,\,-1}\right] \quad \text{on using A·8} \tag{3.18}$$

$$= -\frac{P}{\pi(s_3-s_4)}\left[s_3 \tan^{-1}\frac{1}{s_1 z} - s_4 \tan^{-1}\frac{1}{s_2 z}\right]$$

$$= S_{10} P\left[\frac{1}{s}\tan^{-1}\frac{1}{sz}\right]_{s_2}^{s_1}$$

The results for the remaining cases may be established in a similar manner. The integrals necessary for their calculation are given in Appendix A. However, it is somewhat simpler to obtain them by integration of the concentrated load case as follows:

Case C



The deflection at any point P due to the elemental loading $f(t)\Delta t$ at the point t is, on using (2.50), given by

$$\Delta w_p = \frac{f(t)\Delta t}{\pi s_{13}}\left[C - \log t\right]$$

Hence on summing up for parabolic loading

$$w_p = \frac{p_o}{\pi s_{13}} \underset{\epsilon \to 0}{\mathrm{Lt}} \left[\int_{-1}^{x-\epsilon} \left\{ C - \log(x-t) \right\} (1-t^2)dt + \int_{x+\epsilon}^{1} \left\{ C - \log(t-x) \right\} (1-t^2)dt \right]$$

Both of these improper integrals are convergent. They can be evaluated by integration by parts, provided we evaluate as a principal value. Hence we obtain

$$w\pi s_{13} = \frac{P}{4}\left[\text{const.} - 2x^2 + (x+1)^2(x-2)\log|x+1| - (x-1)^2(x+2)\log|x-1| \right] \tag{3.19}$$

Normal Vertical Stress

The normal vertical stress at any point R on the z axis due to the elemental loading $f(t)\Delta t$ at t, on using (2.44), is given by

$$\Delta \widehat{zz}_R = - s_{10}\, f(t)\Delta t\, z \left[\frac{1}{t^2+s_1^2 z} - \frac{1}{t^2+s_2^2 z} \right]$$

$\therefore$ on integrating for parabolic loading

$$\widehat{zz}\Big|_{x=0} = - 2\, s_{10} p_o z \int_0^1 \left[\frac{1-t^2}{t^2+s_1^2 z} - \frac{1-t^2}{t^2+s_2^2 z} \right] dt$$

$$= - 2\, s_{1o} p_o \left[\frac{1+s_1^2 z}{s_1} \tan^{-1}\frac{1}{s_1 z} - \frac{1+s_2^2 z}{s_2} \tan^{-1}\frac{1}{s_2 z} \right]$$

$$= - \frac{3\, P\, s_{10}}{2} \left[\frac{1+s^2 z^2}{s} \tan^{-1}\frac{1}{sz} \right]_{s_2}^{s_1} \tag{3.20}$$

Case D

This can be obtained by combining [B] and [C] i.e, 3[B] - 2[C]

Hence

$$w \pi s_{13} = \frac{P}{2}\left[\text{Const.} + 2x^2 - (x^3+1)\log|x+1| + (x^3-1)\log|x-1|\right] \tag{3.21}$$

and

$$\widehat{zz}\Big|_{x=0} = +\,3\, s_{10}\, P\, z^2 \left[s \tan^{-1}\frac{1}{sz}\right]_{S_2}^{S_1} \tag{3.22}$$

<u>Case E</u>

Using results for a concentrated load at the origin, we easily obtain

$$w_s \pi s_{13} = \text{Const.} - \frac{P}{2}\log|x^2 - 1| \tag{3.23}$$

and

$$\widehat{zz}\Big|_{x=0} = -\, s_{10}\, P\, z\left[\frac{1}{1+s^2 z^2}\right]_{S_2}^{S_1}$$

<u>Case F</u>

Consider a rigid wall of width 2 with boundary conditions as given in (3.15). Hence from (2.29) and (2.33), provided the integrals can be shown to exist in a physical sense, we require:

$$0 = \int_0^\infty m R_m \cos(mx)\, dm \quad , \quad |X| > 1 \tag{3.25}$$

and $$w_0 = \frac{s_7 - s_8}{s_2 + h_2} \operatorname*{Lt}_{\nu\to 0} \int_0^\infty e^{-m\nu} R_m \cos(mx)\, dm \; , \; |X| \le 1 \tag{3.26}$$

From Watson[18] "B.F." we obtain (13·42)

$$\int_0^\infty J_0(m)\cos mx\, dm = \frac{1}{\sqrt{1-x^2}} \quad \text{or zero,} \quad |X| \lessgtr 1 \tag{3.27}$$

Hence by comparison with (3.25)

$R_m = C\,\dfrac{J_0(m)}{m}$, provided this also satisfies

(3.26) We note in passing that $m\, R_m$ satisfies conditions (3.23).

The pressure distribution under the wall is from (2.29)

$$\widehat{zz}\Big|_{z=0} = \frac{s_3 - s_4}{s_2 + h_2} \quad \frac{C}{\sqrt{1 - x^2}}$$

If the pressure at the center line is p_o then

$$p_o = - C \; \frac{s_3 - s_4}{s_2 + h_2} \tag{3.28}$$

and hence the contact pressure is

$$\widehat{zz}\Big|_{z=0} = - \frac{p_o}{\sqrt{1 - x^2}} = - \frac{2P}{\pi\sqrt{1-x^2}} \quad , \; |x| \leq 1 \tag{3.29}$$

where

$$P = 2\, p_o \int_0^1 \frac{dx}{\sqrt{1-x^2}} = \pi/p_o = \text{total load per unit length.}$$

Surface Settlement

From (2.33)

$$w_3\, \pi\, s_{13} = P \operatorname*{Lt}_{\nu \to 0} \int_0^\infty e^{-m\nu} \frac{J_o(m)}{m} \cos mx \, dm$$

$$\longrightarrow P \int_\delta^\infty J_o(m) \, \frac{\cos mx}{m} \, dm \quad \text{where} \quad \delta > 0$$

This latter integral is divergent due to singularity at $m = 0$. However as in previous examples we may calculate relative displacements by bounding m away from zero as shown.

Since $K_\delta = \int_\delta^\infty J_o(m) \, \frac{\cos mx}{m} \, dm = - \int_0^\infty J_o(m) \left\{ \int^x \sin mx \, dx \right\} dm$

$$= - \int^x \left[\int_\delta^\infty J_o(m) \sin mx \, dm \right] dx \qquad \delta > 0$$

And from Watson's[18] "B.F's" 13·42 we have

$$\int_0^\infty J_0(m) \sin mx \, dm = 0 \quad \text{or} \quad \frac{1}{\sqrt{x^2-1}} \qquad \text{according as } |x| \gtrless 1$$

Above operations are permissible since K_δ is uniformly convergent $\delta > 0$, and $\int_0^\infty J_0(m) \sin mx \, dm \rightarrow \int_\delta^\infty J_0(m) \sin mx \, dm$ as $\delta \rightarrow 0$. Therefore

$$K_\delta \longrightarrow C \quad \text{or} - \int^x \frac{dx}{x^2-1} \qquad \text{according as} \quad |x| \lessgtr 1$$

$$\therefore K_\delta = C$$
$$= C_1 - \log_e(x + \sqrt{x^2-1})$$

since K_δ is continuous at $x = 1$ $\therefore$ $C_1 = C$

Hence we obtain

$$w_s \pi s_{13} = P\, C \tag{3.30}$$
$$= P\left[C + -\log_e(x + \sqrt{x^2-1})\right]$$

where C is arbitrary, since above procedure gives only relative deflections

The vertical pressure at points along z axis is given by (2.29) on substituting value of $m\, R_m$:

$$\widehat{zz}\Big|_{x=0} = - \frac{P}{\pi(s_3-s_4)} \int_0^\infty J_0(m) \left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] dm$$

$$= - \frac{P}{\pi(s_3-s_4)} \left[\frac{s_3}{\sqrt{1+s_1^2 z^2}} - \frac{s_4}{\sqrt{1+s_2^2 z^2}}\right] \text{ from B·3 with n=0}$$

$$= - s_{10} P \left[\frac{1}{s\sqrt{1+s^2 z^2}}\right]_{s_2}^{s_1} \tag{3.31}$$

Summary

All solutions obtained in above cases are arbitrary to the extent of an additive constant. A value must be assigned to this constant to compare results obtained. Results obtained later in the case of the loaded circular disc indicate that the load distribution is unimportant at points distant more than two diameters from the center of the loaded area. This result was to be expected from St. Venant's principle. The settlement at a point distant two diameters from the centre is found to be about $12\frac{1}{2}\%$ of that produced at the load centre by a uniform load. Clearly the superposition principle indicates that above results hold even more strongly in the two dimensional case. Hence it will be safe to assume that at $x=4$, the settlement is only $12\frac{1}{2}\%$ of that produced by a uniform load along its load axis. The arbitrary constants will now be evaluated on the above basis. This gives when $x=4$

$$\frac{w_s\, s_{13}}{P} = 0\cdot 108 \qquad (3.32)$$

Approximate Surface Settlements $|x| \leq 4$

Concentrated [A]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{\pi}\left[1.726 - \log_e |x|\right] \qquad (3.33)$$

Uniform [B]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{2\pi}\left[5\cdot 430 - (x+1)\log_e |x+1| + (x-1)\log_e |x-1|\right] \qquad (3.34)$$

Parabolic [C]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{4\pi}\Big[12{\cdot}212 - 2x^2 + (x+1)^2(x-2)\log_e|x+1| - (x-1)^2(x+2)\log_e|x-1|\Big] \qquad (3.35)$$

Inverted Parabola [D]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{2\pi}\Big[4{\cdot}078 + 2x^2 - (x^3+1)\log_e|x+1| + (x^3-1)\log_e|x-1|\Big] \qquad (3.36)$$

Hollow Wall [E]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{2\pi}\Big[3{\cdot}387 - \log_e|x^2-1|\Big] \qquad (3.37)$$

Rigid Wall [F]

$$\frac{w_s\, s_{13}}{P} = \frac{1}{\pi}\Big[2{\cdot}403 - \log_e\left(x + \sqrt{x^2-1}\right) \qquad , \quad |x| \geq 1$$

$$= \frac{2{\cdot}403}{\pi} \qquad , \quad |x| < 1$$

Note: All logarithms above are to the natural base e.

TABLE I.

Settlement Influence Factors $\frac{W_s S_{13}}{P} = N(x)$

Loading \ X	0	.25	.50	.75	1	2	4
Concentrated		0.990	0.771	0.642	0.550	0.330	0.108
Uniform	0.865	0.854	0.822	0.764	0.645	0.340	0.108
Parabolic	0.972	0.946	0.860	0.731	0.593	0.336	0.108
Inverted Parabola	0.650	0.668	0.753	0.827	0.746	0.348	0.108
Hollow Wall	0.539	0.550	0.585	0.671		0.364	0.108
Rigid Wall	0.764	0.764	0.764	0.764	0.764	0.348	0.108

Effect of Arbitrary Width of Strip

Stresses

Consider a strip of width 2a, carrying a load P per unit width with a distribution $\widehat{zz} = - f(x)$

From (2.29) and (2.36)

$$\widehat{zz}\Big|_{x,z} = \frac{}{\pi(s_3-s_4)}\int_0^\infty\left\{\int_0^a f(t)\cos mt\, dt\right\}\left\{s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz} \cos mx\right\} dm. \qquad (3.39)$$

If we introduce dimensionless coordinates given by $x = x'a$ $y = z'a$ $t = t'a$ $m = m'/a$, the surface distribution becomes $\widehat{z'z'} = - a\, f(x'a) = - f'(x')$ say

Then from (3.39)

$$\widehat{zz}\Big/_{x,z} = \frac{1}{a}\cdot\frac{2}{\pi(s_3-s_4)}\int_0^\infty\left\{\int_0^a f(t')\cos m't\, dt'\right\}\left\{s_3 e^{-s_1 m'z'} - s_4 e^{-s_2 m'z'}\right\} dm' = \frac{1}{a}\,\widehat{z'z'}\Big|_{x',z'} \qquad (3\cdot 40)$$

where $\widehat{z'z'}$ is a stress component at the dimensionless point (x', z'). Or if in addition we take $P' = 1$ in dimensionless system, and denote corresponding stresses by $\widehat{z'z'}$, $\widehat{x'x'}$ etc., then

$$\widehat{zz}\Big|_{x,z} = \frac{P}{a}\, \widehat{z'z'}\Big|_{x',z'} \qquad (3.41)$$

Similar results follow for the other stress components. From this we see that under a given form of load distribution, <u>corresponding stress components are directly proportional to the total load per unit length of wall and inversely proportional to its width.</u> Or since P/a is proportional to p, the intensity of loading on the surface, it follows that <u>corresponding stress components are proportional to the intensity of loading on the surface.</u> (3.42)

<u>Displacements</u>

From (2.33) and (2.37) we obtain for the settlement

$$w\Big|_{x,z} = \frac{2}{\pi(s_3-s_4)}\int_0^\infty \frac{1}{m}\left[\int_0^a f(t)\cos mt\, dt\right]\left[s_7 e^{-s_1 mz} - s_8 e^{-s_2 mz}\right] dm$$

On introducing dimensionless coordinates as above, we obtain

$$w\Big|_{x,z} = \frac{2}{\pi(s_3-s_4)}\int_0^\infty m'\left[\int_0^1 f'(t')\cos m't'\, dt'\right]\left[s_7 e^{-s_1 m'z'} - s_8 e^{-s_2 m'z'}\right] dm' = Pw\Big|_{x',z'} \qquad (3.43)$$

A similar result holds for u.

From this we see that under a given form of load distribution, <u>corresponding displacements are directly proportional to the</u> (3.44) <u>total load or they are jointly proportional to the intensity of loading on the surface, and the width of the strip.</u> These results are in accord with the elastic theory[19].

Practical Calculation of Settlements

Result (2.43) furnishes a very rapid method for calculating settlements in terms of the settlement influence factor $N(x')$ of Table I appropriate to the distribution. On rewriting (3.43) becomes

$$W_s\Big|_x = \frac{P}{s_{13}} N(X') \qquad (3.45)$$

Normal Vertical Stress $\widehat{zz}$ under loaded Strip 2a

Too many variables are involved in the expression for $\widehat{zz}\Big|_{o,z}$ to permit tabulation. Hence it appears desirable to write out complete expressions for $\widehat{zz}\Big|_{o,z}$. This is easily done with the aid of (3.41) and the equations (3.18), (3.20), (3.22), (3.24) and (3.31) Hence we obtain

Concentrated Load [A]

$$\widehat{zz}\Big|_{o,z} = - s_{10} \frac{P}{z} \left[\frac{1}{s^2}\right]_{S_2}^{S_1} \qquad (3.46)$$

Uniform Loading [B]

$$\widehat{zz}\Big|_{o,z} = - s_{10} \frac{P}{a} \left[\frac{1}{s} \tan^{-1} \frac{a}{sz}\right]_{S_2}^{S_1} \qquad (3.47)$$

Parabolic [C]

$$\widehat{zz}\Big|_{o,z} = - 3 \frac{s_{10} P}{2a^3} \left[\frac{a^2+s^2z^2}{s} \tan^{-1} \frac{a}{sz}\right]_{S_2}^{S_1} \qquad (3.48)$$

Inverted Parabola [D]

$$\widehat{zz}\Big|_{o,z} = + 3 s_{10} \frac{Pz^2}{a^3} \left[s \tan^{-1} \frac{a}{sz}\right]_{S_2}^{S_1} \qquad (3.49)$$

Hollow Wall[E]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, Pz \left[\frac{1}{a^2+s^2z^2} \right]_{S_2}^{S_1} \tag{3.50}$$

Rigid Wall[F]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, P \left[\frac{1}{s\,\sqrt{a^2+s^2z^2}} \right]_{S_2}^{S_1} \tag{3.51}$$

Stresses for Large Z

As a good check on the above results, we shall find the stresses for large values of z by expanding in powers of $\frac{1}{z}$. As expected the results approach that for the concentrated load (3.46), providing an example of St. Venants principle(3) of equipollent load systems. The results obtained from the Binomial theorem and the expansion $\tan^{-1}w = w - \frac{1}{3}w^3 + \frac{1}{5}w^5 - \ldots$ are:

Concentrated Load[A]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, \frac{P}{z} \left[\frac{1}{s^2} \right]_{S_2}^{S_1} \tag{3.52}$$

Uniform Loading[B]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, \frac{P}{z} \left[\frac{1}{s^2} \left(1 - \frac{1}{3}\frac{a^2}{s^2z^2} + \frac{1}{5}\frac{a^4}{s^4z^4} - \ldots\right) \right]_{S_2}^{S_1} \tag{3.53}$$

Parabolic[C]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, \frac{P}{z} \left[\frac{1}{s^2} \left(1 - \frac{1}{5}\frac{a^2}{s^2z^2} + \frac{3}{35}\frac{a^4}{s^4z^4} - \ldots\right) \right]_{S_2}^{S_1} \tag{3.54}$$

Inverted Parabola[D]

$$\widehat{zz}\Big|_{o,z} = - s_{10}\, \frac{P}{z} \left[\frac{1}{s^2} \left(1 - \frac{3}{5}\frac{a^2}{s^2z^2} + \frac{3}{7}\frac{a^4}{s^4z^4} - \ldots\right) \right]_{S_2}^{S_1} \tag{3.55}$$

Hollow Wall [E]

$$\widehat{zz}\Big|_{0,z} = - s_{10}\,\frac{P}{z}\left[\frac{1}{s^2}\left(1 - \frac{a^2}{s^2z^2} + \frac{a^4}{s^4z^4} - \cdots\right)\right]_{S_2}^{S_1} \tag{3.56}$$

Rigid Wall

$$\widehat{zz}\Big|_{0,z} = - s_{10}\,\frac{P}{z}\left[\frac{1}{s^2}\left(1 - \frac{1}{2}\,\frac{a^2}{s^2z^2} + \frac{3}{16}\,\frac{a^4}{s^4z^4} - \cdots\right)\right]_{S_2}^{S_1} \tag{3.57}$$

Maximum Shear Stress at Points on $x = 0$

From (2.41)

$$\tau_M = \frac{1}{\pi(s_3 - s_4)}\left[(s_3 - s_5)\,{}^{z}\mathcal{I}_{S_1} - (s_4 - s_6)\,{}^{z}\mathcal{I}_{S_2}\right]$$

On using results C·2 and C·5 this becomes

$$_M = \pi\left\{s_{14}\left[S\,{}^{z}\mathcal{I}_{S}\right]_{S_2}^{S_1} + s_{10}\left[\frac{1}{S}\,{}^{z}\mathcal{I}_{S}\right]_{S_2}^{S_1}\right\} \tag{3.58}$$

where

$$s_{14} = -\frac{1}{\pi}\,\frac{s_5}{s_1(s_3 - s_4)}$$

$$\tag{3.59}$$

and

$$s_{10} = \frac{1}{\pi}\,\frac{s_4 s_2}{s_3 - s_4}$$

The value of τ_M can easily be obtained when ${}^{z}\mathcal{I}_{S}$ is known. Noting from (2.40) that

$$zz = 2\,s_{10}\left[\frac{1}{S}\,{}^{z}\mathcal{I}_{S}\right]_{S_2}^{S_1} \tag{3.60}$$

we can readily write down the appropriate values of ${}^{z}\mathcal{I}_{S}$ by comparing (3.60) with results (3.46) - (3.51). This yields for strip of width 2a

Concentrated [A]

$$ {}^{z}\!\!\int_{s} = -\frac{P}{2z}\left(\frac{1}{s}\right) \tag{3.61} $$

Uniform [B]

$$ {}^{z}\!\!\int_{s} = -\frac{P}{2a}\tan^{-1}\frac{a}{sz} \tag{3.62} $$

Parabolic [C]

$$ {}^{z}\!\!\int_{s} = -\frac{3P}{4a^3}\left(a^2+s^2z^2\right)\tan^{-1}\frac{a}{sz} $$

Inverted Parabola [D]

$$ {}^{z}\!\!\int_{s} = \frac{3Pz^2}{2a^3}\,s^2\tan^{-1}a/sz $$

Hollow Wall [E]

$$ {}^{z}\!\!\int_{s} = -\frac{Pz}{2}\,\frac{s}{a^2+s^2z^2} $$

Rigid Wall [F]

$$ {}^{z}\!\!\int_{s} = -\frac{P}{2\;\;a^2+s^2z^2} $$

The values of τ_M can readily be obtained from (3.58) using above values for ${}^{z}\!\!\int_{s}$.

CHAPTER IV.

THREE DIMENSIONAL SYMMETRIC CASE

When the loading is symmetrical with respect to a vertical axis, the problem is most easily treated by the use of cylindrical co-ordinates r, θ, z. If the displacements along these coordinate axes are u, v, w, then from symmetry

$$v = 0 \quad \text{and} \quad \frac{\partial u}{\partial \theta} = \frac{\partial w}{\partial \theta} = 0 \tag{4.00}$$

The strains are[4]

$$e_{rr} = \frac{\partial u}{\partial r} \qquad e_{\theta\theta} = \frac{u}{r}$$

$$e_{zz} = \frac{\partial w}{\partial z} \qquad e_{r\theta} = e_{\theta z} = 0 \tag{4.01}$$

$$e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

and the dilatation $\epsilon = e_{rr} + e_{\theta\theta} + e_{zz} = \dfrac{\partial u}{\partial r} + \dfrac{u}{r} + \dfrac{\partial w}{\partial z}$.

With no body forces, the stress equations of equilibrium become from (1.20)

$$\frac{\partial \widehat{rr}}{\partial r} + \frac{\partial \widehat{rz}}{\partial z} + \frac{\widehat{rr} - \widehat{\theta\theta}}{r} = 0 \tag{4.02}$$

$$\frac{\partial \widehat{rz}}{\partial r} + \frac{\partial \widehat{zz}}{\partial z} + \frac{\widehat{rz}}{r} = 0$$

where from (1.11)

$$\widehat{rr} = A\, e_{rr} + (A-2N)\, e_{\theta\theta} + F e_{zz}$$

$$\widehat{zz} = F\, e_{rr} + F\, e_{\theta\theta} + C e_{zz} \tag{4.03}$$

$$\widehat{\theta\theta} = (A-2N)\, e_{rr} + A\, e_{\theta\theta} + F e_{zz}$$

$$\widehat{rz} = L\, e_{rz}$$

Hence on substituting in (4.02), the strain equations of equilibrium are:

$$A \frac{\partial \epsilon}{\partial r} + (F-A) \frac{\partial e_{zz}}{\partial r} + L \frac{\partial e_{rz}}{\partial z} = 0 \qquad (4.04)$$

$$F \frac{\partial \epsilon}{\partial r} + (C-F) \frac{\partial e_{zz}}{\partial z} + L \frac{\partial e_{rz}}{\partial r} + \frac{L}{r} e_{rz} = 0$$

and on using (4.01) these become

$$A\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{u}{r^2} + \frac{L}{A}\frac{\partial^2 u}{\partial z^2}\right) + (G)\frac{\partial^2 w}{\partial r \partial z} = 0 \qquad (4.05)$$

$$(G)\left(\frac{\partial^2 u}{\partial r \partial z} + \frac{1}{r}\frac{\partial u}{\partial z}\right) + L\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{C}{L}\frac{\partial^2 w}{\partial z^2}\right) = 0$$

Try for solutions of the type

$$u = e^{-\lambda z} U(r)$$

$$w = e^{-\lambda z} W(r) \qquad (4.06)$$

Substituting in (4.05) we obtain

$$A\left(U'' + \frac{U'}{r} - \frac{U}{r^2} + \frac{L}{A}\lambda^2 U\right) - (G)\lambda W' = 0 \qquad (4.07)$$

$$(G)\ (U' + U/r) + L(W'' + W'/r + \frac{C}{L}\lambda^2 W) = 0$$

where the dashes denote differentiation with respect to r *i.e.*

$$U'' = \frac{d^2\, U(r)}{dr^2}$$

The pattern of the Bessel equation is evident in the combinations of U and W in (4.07). Substitute in (4.07)

$$U = R_m J_1(mr) \qquad (4.09)$$

$$W = R'_m J_o(mr)$$

where R_m and R'_m are arbitrary functions of m. After some simplification we obtain

$$R_m A(-m^2 + \frac{L}{A}\lambda^2)\, J_1(mz) + R'_m(F+L)\, m\lambda J_1(mz) = 0 \qquad (4.10)$$

$$-R_m \lambda\, m\, (F+L)\, J_o(mz) + R'_m L(-m^2 + \frac{C}{L}\lambda^2)\, J_o(mz) = 0$$

Hence for $z > 0$ on eliminating R_m and R'_m from above equations after removal of their respective factors $J_1(mz)$, $J_o(mz)$ we obtain

$$\begin{vmatrix} -Am^2 + L\lambda^2 & \lambda m(F+L) \\ -\lambda m\, (F+L) & -LM^2 + C\lambda^2 \end{vmatrix} = 0$$

$$\therefore \lambda^4 - \frac{L^2 + AC - G^2}{CL}\lambda^2 m^2 + \frac{A}{C} m^4 = 0 \qquad (4.11)$$

This is the same as equation (2.19) in the two dimensional case, and so its roots are

$$\left.\begin{aligned} \lambda_1^2 &= m^2 s_1^2 \\ \lambda_2^2 &= m^2 s_2^2 \end{aligned}\right\} \quad \text{or} \quad \begin{aligned} \lambda_1 &= ms_1\ , & \lambda_3 &= -ms_1 \\ \lambda_2 &= ms_2\ , & \lambda_4 &= -ms_2 \end{aligned} \quad s_1,\ s_2 > 0 \qquad (4.12)$$

and clearly the relationship between the constants is

$$R'_i = \frac{A - Ls_i^2}{Gs_i} R_i = h_i R_i \quad , \quad i = 1, 2, 3, 4 \qquad (4\cdot 13)$$

where $R'_i = R'_m$ and $R_i = R_m$ and $h_i = \dfrac{A - Ls^2_i}{Gs_i}$

Hence type solutions are

$$u = \sum_{i=1}^{4} R_i\, e^{-\lambda_i z}\, J_1(mr)$$

$$w = \sum_{i=1}^{4} h_i R_i\, e^{-\lambda_i z} J_o(mr) \qquad (4\cdot 14)$$

and by superposition of such solutions, we obtain

$$u = \int_0^\infty \sum_{i=1}^{4} R_i \, e^{-\lambda_i z} J_1(mr)dm$$
$$w = \int_0^\infty \sum_{i=1}^{4} h_i R_i \, e^{-\lambda_i z} J_o(mr)dm \qquad (4.15)$$

provided R_i is such as to make the integrals uniformly convergent and continuous in the whole range, and $z > 0$.

Boundary Conditions on Semi-Infinite Body Bounded by Z = 0

A. Stresses and strains tend to zero as z tends to infinity. This obviously requires

$$R_3 = R_4 = 0 \quad \text{since} \quad \lambda_3 < 0, \; \lambda_4 < 0 \qquad (4.16)$$

B. Assume $z = 0$ is free from tangential stress, and so has only a normal traction. This requires

$$0 = \widehat{rz}\Big|_{z=0} = L\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}\right)_{z=0}$$

$$\widehat{zz} = f(r) = Fe_{rr} + Ce_{zz} + Fe_{\theta\theta}\Big|_{z=0} \qquad (4.17)$$

$$= F\frac{\partial u}{\partial r} + C\frac{\partial w}{\partial z} + F\frac{u}{r}\Big|_{z=0}$$

Now $\frac{\widehat{rz}}{L} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$

$$= -\int_0^\infty m\left[R_1(s_1+h_1)e^{-s_1 mz} + R_2(s_2+h_2)e^{-s_2 mz}\right] J_1(mr)dm \qquad (4.18)$$

Applying boundary condition (4.17)

$$\widehat{rz}\Big|_{z=0} = 0 = \int_0^\infty m\left[R_1(s_1+h_1) + R_2(s_2+h_2)\right] J_1(mr)dm$$

A sufficient condition for this is that

$$R_1(s_1+h_1) + R_2\,(s_2+h_2) \equiv 0 \qquad (4.19)$$

We now write out the stresses and displacements in terms of $R_1 \equiv R_m$.

Using equations (4.15), (4.16), (4.19) and (4.01) we obtain:

Stresses and Displacements

$\widehat{zz} = F \frac{\partial u}{\partial r} + C \frac{\partial w}{\partial z} + F \frac{u}{r}$. Using the result $J\ (t) + \frac{1}{t} J\ (t) = J_0(t)$

$$\therefore \quad \widehat{zz}(s_2+h_2) = \int_0^\infty m\, R_m \left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz} \right] J_0(mr)\,dm \qquad (4.20)$$

$$\widehat{rz} = L(s_1+h_1) \int_0^\infty m\, R_m \left[-e^{-s_1 mz} + e^{-s_2 mz} \right] J_\phi(mr)\,dm \qquad (4.21)$$

and

$$rr = A\left(\frac{\partial u}{\partial r}\right) + F\left(\frac{\partial w}{\partial z}\right) + (A-2N)\frac{u}{r} \quad \text{from } (4.03)$$

$$\therefore \quad \widehat{rr}(s_2+h_2) = \int_0^\infty m\, R_m \left[s_5 e^{-s_1 mz} - s_6 e^{-s_2 mz} \right] \qquad (4.22)$$

$$J_0(mr)\,dm - 2N \frac{u}{r} (s_2+h_2)$$

also

$$u(s_2+h_2) = \int_0^\infty R_m \left[(s_2+h_2) e^{-s_1 mz} - (s_1+h_1)\, e^{-s_2 mz} \right] \qquad (4.23)$$

$$J_1(mr)\,dm$$

$$w(s_2+h_2) = \int_0^\infty R_m \left[s_7 e^{-s_1 mz} - s_8 e^{-s_2 mz} \right] J_0(mr)\ dm. \qquad (4.24)$$

The similarity between above formulae and the corresponding two dimensional ones (2.29)-(2.33) is striking. The constants are as defined in Appendix G.

Determination of R_m

Boundary condition (4.17) requires that

$$(s\ +h\ )f\ (r) = \int_0^\infty m\, R_m\ (s_3-s_4)\ J_0(mr)\ dm$$

This integral equation canbe solved for R_m by the Fourier-Bessel Integral Theorem[18]. This gives

$$R_m = \frac{s_2+h_2}{s_3-s_4} \int_0^\infty t\ f(t)\ J_0(tm)\ dt.$$

provided R_m satisfies the following sufficient conditions:

(i) $\int_0^\infty \sqrt{m}\, R_m \, dm$ exists and is absolutely convergent (4.25)

(ii) R_m has bounded variation for all m. (4.26)

Any sectionally continuous function of m, for which left and right hand derivatives exist at $m > 0$ satisfies this requirement.

Then

$$R_m = \pi/2\ s_9 \int_0^\infty t\ f(t)\ J_o(tm)\ dt \equiv \pi/2\ s_9\ U_m \tag{4.27}$$

where

$$U_m = \int_0^\infty t\ f(t)\ J_o(mt)\ dt \tag{4.270}$$

Consider a load uniformly distributed over a unit circle :

$$f(t) = -p_o \quad , \quad |r| \leq 1$$
$$= 0 \quad , \quad |r| > 1$$

Then from (4.27)

$$R_m = -\pi/2\ s_9\ p_o \int_0^1 t\ J_o(tm)\ dt = -\pi/2\ s_9 p_o \frac{J_1(m)}{m} \tag{4.2701}$$

This function is continuous and has a continuous differential coefficient, and therefore satisfies (4.26). However, it does not satisfy (4.25), since as $m \to \infty$ $|J_1(m)| = 0\ \frac{1}{\sqrt{m}}$

and $\therefore \int_{m_1}^\infty \left|\sqrt{m}\, R_m\right| dm = 0 \int_{m_1}^\infty \frac{dm}{m}$ is divergent.

However, the above analysis may be modified as follows:

Let $\widehat{zz} = \phi(r, z)$ and then physical conditions demand $\underset{z\to 0}{Lt}\ \phi(r, z$

$\longrightarrow$ f(r)uniformly

From 4.20 $(s_2+h_2)\phi(r,z) = \int_0^\infty m\, R_m \left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] J_0(mr)\, dm, \; z > 0$

This gives on inverting(15)

$$m\, R_m \frac{\left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right]}{s_2 + h_2} = \int_0^\infty t\,\phi(t,z) J_0(tm)\, dt. \qquad (4.2702)$$

Taking the limit of both sides as $z \to 0$

$$m\, R_m \frac{(s_3 - s_4)}{s_2 + h_2} = \underset{z\to 0}{\mathrm{Lt}} \int_0^\infty t\,\phi(t,z)\, J_0(t,m)\, dt.$$

The limiting can be taken inside the integral sign since $\phi(r,z) \to f(r)$ uniformly as $z \to 0$, and for the cases considered $f(r) = 0$, $|r| > 1$

Hence as before, on proceeding to the limit under the integral sign,

$$R_m = \pi/2\; s_9\, U_m\,.$$

The validity conditions now depend on (4.2702), and are on the function

$$R(m,z) \equiv R_m\,(s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz})$$

This expression is clearly continuous and has continuous differential coefficients in the uniform load case. Also (4.25) is now satisfied for $z > 0$ provided $\sqrt{m}\, R_m$ is bounded.

All functions considered in this thesis satisfy above conditions.

<u>Surface Deflection</u> (w_s)

From (4.24) and (4.27), on noting $\dfrac{\pi\, s_9}{2(s_2+h_2)} = \dfrac{1}{s_3 - s_4}$

we obtain

$$w_s = \frac{1}{s_3 - s_4} \underset{z\to 0}{\mathrm{Lt}} \int_0^\infty U_m \left[s_7 e^{-s_1 mz} - s_8 e^{-s_2 mz}\right] J_0(mr)\, dm \qquad (4.28)$$

$$\therefore\; w_s\, s_{13} = -\underset{v\to 0}{\mathrm{Lt}} \int_0^\infty U_m\, e^{-mv}\, J_0(mr)\, dm$$

Normal Axial Stress $\left(p = -\, \widehat{zz}\Big|_{r=0}\right)$

Similarly from (4.20)

$$\widehat{zz}\Big|_{r=0} = +\,\frac{1}{s_3-s_4}\int_0^{\infty} m\, U_m\left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] dm \qquad (4.29)$$

Maximum Axial Shear Stress (τ_A)

From (4.21) we note that $\widehat{rz} = 0$ and $u = 0$ when $r = 0$. Hence at any point on $r = 0$, $\widehat{zz}$ and $\widehat{rr}$ are principal stresses, and therefore the maximum shear stress is at $\pi/4$ to the vertical and is given by

$$\tau_A = \tfrac{1}{2}\,(\widehat{zz} + \widehat{rr})\Big|_{r=0}$$

Hence from (4.20) and (4.22)

$$\tau_A = \frac{1}{2(s_3-s_4)}\int_0^{\infty} m\, U_m\left[(s_3+s_5)\, e^{-s_1 mz} - (s_4+s_6)\, e^{-s_2 mz}\right] dm \qquad (4.30)$$

Concentrated Load at Origin

Consider a concentrated load P as being uniformly distributed over a small area of radius ϵ, hence

$$f(r) = -\,\frac{P}{\pi\epsilon^2}\ , \quad |r| \le \epsilon$$

$$= 0\ , \quad |r| > \epsilon \qquad \epsilon \to 0$$

Therefore from (4.27)

$$R_m = -\,\frac{s_9\, P}{2}\ \underset{\epsilon\to 0}{\mathrm{Lt}}\ \frac{1}{\epsilon^2}\int_0^{\epsilon} J_0(tm)\,t\, dt$$

$$\text{since}\quad J_0(tm) = 1 + O(tm)_{\,t\to 0}$$

$$\therefore\ R_m = -\,\frac{s_9 P}{2}\ \underset{\epsilon\to 0}{\mathrm{Lt}}\ \frac{1}{\epsilon^2}\left[\frac{\epsilon^2}{2} + O(\epsilon^3)\right]$$

$$= -\,\frac{s_9 P}{4} \qquad (4.31)$$

Evaluations

The required integral are obtained by substituting for z the values $s_1 z$ or $s_2 z$ in the appropriate results in Appendix B.

Definitions

$$R_i^2 = r^2 + (s_i z)^2 \qquad i = 1,2$$

$$\tan\theta_i = \frac{s_i z}{x} \qquad \tan\theta = z/x$$

Hence we obtain from (4.20) and B·7

$$\widehat{zz}(s_2+h_2) = -\, s_9 \frac{Pz}{4}\left[\frac{s_3 s_1}{R_1^3} - \frac{s_4 s_2}{R_2^3}\right]$$

As shown in Appendix C, $s_3 s_1 = s_4 s_2$ and $\dfrac{s_9 s_4 s_2}{s_2+h_2} = \dfrac{1}{\pi}\dfrac{s_4 s_2}{s_3-s_4} \equiv s_{10}$

therefore,

$$\widehat{zz} = -\, s_{10} \frac{Pz}{2}\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] \qquad (4.311)$$

Also

$$\widehat{rz} = -\, L(s_1+h_1)\, s_9 \frac{Pr}{4}\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right]$$

$$= s_{10}\frac{Pr}{2}\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] \qquad \text{from Appendix B·7} \quad (4.312)$$

Similarly

$$u = \frac{s_9 P}{4r}\left[(s_{12}-1) + s_{12}\sin\theta_2 - \sin\theta_1\right] \qquad (4.313)$$

$$w = -\frac{P}{2\pi(s_3-s_4)}\left[\frac{s_7}{R_1} - \frac{s_8}{R_2}\right] \qquad (4.314)$$

$$rr = -\frac{Pz}{2\pi(s_3-s_4)}\left[\frac{s_1 s_5}{R_1^3} - \frac{s_2 s_6}{R_2^3}\right] - 2\frac{Nu}{r} \qquad (4.315)$$

Here, as in the two dimensional case, the resultant stress on any plane z = constant is from (4.311) and (4.312) always directed away from the origin and is of magnitude

$$F_z = \sqrt{\widehat{zz}^2 + \widehat{rz}^2} = s_{10}\,\frac{PR}{2}\left[\frac{1}{R_1^3} - \frac{1}{R_2^3}\right] \tag{4.316}$$

This corresponds to the elastic isotropic case where the resultant stress on any plane z = constant is always directed towards the origin and is of magnitude[3]

$$F_z = \frac{3P}{2\pi}\,\frac{z^2}{R^4}$$

This latter result can easily be obtained from (4.316) by a limiting process as illustrated in Chapter 6.

It is very difficult to justify mathematically the process by which R_m was obtained in the case of a concentrated load, due to the limiting process involved in the definition of a concentrated load. The method used by Carslaw and Jaeger[17] for similar problems dealing with impulsive forces can be resorted to here, i.e., show that the solutions obtained do actually satisfy all conditions of the problem. The boundary conditions involved now become

$$\begin{aligned} &1)\ \text{on } z = 0 \quad \widehat{rz} = 0, \quad \widehat{zz} = 0 \quad r \neq 0 \\ &2)\ z > 0 \quad \int_0^\infty \widehat{zz}\; 2\pi r\, dr = -P \end{aligned} \tag{4.317}$$

It is a simple exercise to show that solutions obtained above, do satisfy (4.317), and the differential equations (4.05). Above results for w and F_z check those given by Mitchell[6] 1900(b), obtained as described in Chapter I.

CHAPTER V.

A LOADED CIRCULAR AREA

Surface Settlement of Unit Circle

Assume

$$\widehat{zz} = f(r), \quad r \leq 1$$
$$= 0, \quad r > 1$$

Then from (4.270)

$$U_m = \int_0^1 t\, f(t)\, J_0\, (mt)\, dt \tag{5.01}$$

and hence from (4.28)

$$w_s\, s_{13} = - \int_0^\infty \int_0^1 t\, f(t)\, J_0\, (mt)\, J_0\, (mr)\, dt\, dm \tag{5.02}$$

If f(t) satisfies conditions necessary for inverting the order of integration, then we obtain

$$w_s\, s_{13} = - \int_0^1 t\, f(t) \left[\int_0^\infty J_0\, (mt)\, J_0\, (mr)\, dm \right] dt \tag{5.03}$$

$$\text{Let } I_1 = \int_0^\infty J_0\, (mt)\, J_0\, (mr)\, dm \tag{5.14}$$

Using result 2 (13.4) Watson[18]

$$\begin{aligned} I_1 &= \frac{1}{t}\, {}_2F_1\, (\tfrac{1}{2}, \tfrac{1}{2}; 1, r^2/t^2), \quad t > r \\ &= \frac{1}{r}\, {}_2F_1\, (\tfrac{1}{2}, \tfrac{1}{2}; 1, t^2/r^2), \quad t < r \end{aligned} \tag{5.05}$$

The above hypergeometric functions can be expressed as complete elliptic functions of the first kind[20]

Hence

$$\begin{aligned} I_1 &= \frac{2}{\pi t}\, K\left(\frac{1}{k^2}\right), \quad |k| > 1 \quad \textit{i.e.}\ t > r \\ &= \frac{2}{\pi r}\, K(k^2), \quad |k| < 1 \quad \textit{i.e.}\ t < r \end{aligned} \tag{5.06}$$

where $k^2 = t^2/r^2$ and $K(k^2) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$

Hence substituting for I_r in (5.03) we obtain

$$w\,\frac{\pi}{2}\,s_{13} = \underset{\substack{\epsilon_1\to 0\\ \epsilon_2\to 0}}{\mathrm{Lt}}\left[\int_0^{r-\epsilon_1} t/r\; f(t)K(k^2)\,dt + \int_{r+\epsilon_2}^{1} f(t)K\left(\frac{1}{k^2}\right)dt\right] \qquad r < 1$$

Provided f(t) is finite and continuous in neighborhood of t=r, then we can easily show that both above limits exist separately. Hence,

$$w\,\frac{\pi}{2}\,s_{13} = \int_0^r \frac{t}{r}\,f(t)K(k^2)\,dt + \int_r^1 f(t)K\left(\frac{1}{k^2}\right)dt \quad , \quad r \leqslant 1$$
$$= \int_0^1 \frac{t}{r}\,f(t)K(k^2)\,dt \quad , \quad r > 1 \qquad (5.07)$$

More suitable integrals for evaluation are obtained by using the substitutions $x = k^2 = t^2/r^2$ and $x = \frac{1}{k^2} = r^2/t^2$ respectively is above integrals. These give

$$w\,\frac{\pi}{2}\,s_{13} = \frac{r}{2}\int_0^1 f(r\sqrt{x})K(x)\,dx + \frac{r}{2}\int_{r^2}^1 x^{-3/2} f\left(\frac{r}{\sqrt{x}}\right)K(x)\,dx \quad , \quad 0 \leqslant r \leqslant 1$$
$$= \frac{r}{2}\int_0^{1/r^2} f(r\,x)K(x)\,dx \quad , \quad r > 1 \qquad (5.08)$$

where $K(x) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-x\sin^2\theta}}$

Evaluations can be made by use of recurrence formula[21]

$$(2n+1)^2 I_n - 4n^2 I_{n-1} = 2x^n\left[E - (2n+1)(1-x)K\right] \qquad (5.09)$$

where $I_n = \int K(x)x^n\,dx$, and $x = k^2$

E is the complete elliptic function of the second kind. Above formula holds for all values of n, positive negative, integral and fractional except $n = -\frac{1}{2}$, provided the integrals involved are convergent.

Reduction of Integrals in (5-08) by Recurrence

The reduction of integrals in (5.08) by means of recurrence formula (5.09), depends on the limits involved:

(i) Limits 0 to 1

Recurrence relation now becomes

$$(n+\tfrac{1}{2})^2 I_n - n^2 I_{n-1} = \tfrac{1}{2} \qquad (5.10)$$

After s applications of above ($s \geqslant 2$) we obtain

$$2 I_n = \frac{1}{(n+\frac{1}{2})^2}\left[1 + \left(\frac{n}{n-\frac{1}{2}}\right)^2 + \left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2}\right)^2 + \ldots\ldots \right.$$

$$\left. + \left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2} \cdots\cdots \frac{n-s+2}{n-s+3/2}\right)^2 + I_{n-s}\, 2\left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2} \cdots \frac{n-s+2}{n-s+3/2} \quad n-s+1\right)^2\right] \qquad (5.11)$$

(ii) Limits r^2 to 1 ($r^2 < 1$)

Recurrence relation now becomes

$$(n+\tfrac{1}{2})^2 \, {}_{r^2}^{\;1}I_n - n^2 \, {}_{r^2}^{\;1}I_{n-1} = \tfrac{1}{2} - r^{2n}\left[\frac{E(r^2)}{2} - (1-r^2)(n+\tfrac{1}{2})\, K(r^2)\right]$$

$$\equiv s(r,n) \qquad (5.12)$$

Hence after s applications we obtain

$$= \frac{1}{(n+\frac{1}{2})^2}\left[s(r,n) + \left(\frac{n}{n-\frac{1}{2}}\right)^2 s(r,n-1) + \left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2}\right)^2 S(r,n-2) \right.$$

$$+ \ldots\ldots + \left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2} \cdots \frac{n-s+2}{n-s+3/2}\right)^2 s(r,n-s+1)$$

$$\left. + I_{n-s}\left(\frac{n}{n-\frac{1}{2}} \cdot \frac{n-1}{n-3/2} \cdots \frac{n-s+2}{n-r+3/2}\right)^2 (n-s+1)^2\right] \qquad (5.13)$$

(iii) Limits 0 to $1/r^2$ ($r^2 > 1$)

Then (5.09) becomes

$$(n+\tfrac{1}{2})^2 \, {}_0^{1/r^2}I_n - n^2 \, {}_0^{1/r^2}I_{n-1} = \frac{1}{r^{2n}}\left[\frac{E(1/r^2)}{2} - (1-1/r^2)(n+\tfrac{1}{2})K(1/r^2)\right]$$

$$\equiv s_1\left(\frac{1}{r},n\right) = \tfrac{1}{2} - s\left(\frac{1}{r}, n\right) \qquad (5.14)$$

Hence after s applications we obtain

$$
{}_0\mathcal{I}_n^{1/2} = \frac{1}{(n+\frac{1}{2})^2}\left[S_1\left(\frac{1}{r},n\right) + \left(\frac{n}{n-\frac{1}{2}}\right)^2 S_1\left(\frac{1}{r},n-1\right) + \left(\frac{n}{n-\frac{1}{2}}\cdot\frac{n-1}{n-3/2}\right)^2 S_1\left(\frac{1}{r},n-2\right) + \right. \tag{5.15}
$$

$$
\left. + \left(\frac{n}{n-\frac{1}{2}}\cdot\frac{n-1}{n-\frac{3}{2}}\cdot\frac{n-2}{n-\frac{5}{2}}\cdots\frac{n-s+2}{n-s+\frac{3}{2}}\right)^2 S_1\left(\frac{1}{r},n-s+1\right) + \left(\frac{n}{n-\frac{1}{2}}\cdot\frac{n-1}{n-\frac{3}{2}}\cdots\frac{n-s+2}{n-s+\frac{3}{2}}\right)^2 (n-s+1)^2\,\mathcal{I}_{n-s}\right]
$$

Above reduction formulae are useful when f(r), the normal loading on the surface is a finite polynomial. The numerical work becomes increasingly tedious as the degree of the polynomial increases. For other forms of f(r) we can approximate by a finite polynomial, or alternatively use graphical or numerical integration to evaluate (5.07). In practice it is not possible to take more than a relatively small number of pressure measurements under the loaded area, and a polynomial can easily be fitted to these measurements.

Evaluation of Surface Settlement (5.08)

Continued application of above developed reduction formulae, makes the value of (5.08), when f(t) is a polynomial, depend on one of the following integrals:

$$
\mathcal{I}_0 = \int_0^1 K(x)\,dx = 2x\,B(x)\Big|_0^1 = 2 \tag{5.16}
$$

$$
\mathcal{I}_{-1/2} = \int_0^1 \frac{K(x)}{\sqrt{x}}\,dx = 2\int_0^1 K(k^2)\,dk = 4G = 3.6639
$$

where G is Catalan's constant

$$
\mathcal{I}_{1/2} = 1.4160 \quad \text{from Appendix } D
$$

$$ {}_{r^2}^{1}I_{-3/2} = \int_{r^2}^{1} x^{-3/2} K(x)dx = -2 \frac{E(x)}{\sqrt{x}} \Big|_{r^2}^{1} = 2\left[\frac{E(r^2)}{r} - 1\right] \quad (5.17) $$

$$ {}_{r^2}^{1}I_{-1} = \int_{r^2}^{1} \frac{K(x)}{x}\, dx \text{ — tabulated in Appendix D} $$

$$ {}_{r^2}^{1}I_{1/2} = \int_{r^2}^{1} \sqrt{x}\, K(x)dx \text{ — tabulated in Appendix D} $$

$$ {}_{0}^{1/r^2}I_{0} = \int_{0}^{1/r^2} K(x)dx = 2x\, B(x) \Big|_{0}^{1/r^2} = 2/r^2\, B(1/r^2) \quad (5.18) $$

$$ {}_{0}^{1/r^2}I_{1/2} = \int_{0}^{1/r^2} \sqrt{x}\, K(x)dx \text{ — tabulated in Appendix D} $$

Some of the above integrals are given in JAHNKE-EMDE(21). The remainder are calculated and tabulated in Appendix D.

Special Cases

Concentrated Load [A]

Let $a \to 0$, and so $w \to \infty$ under the load. However at some distance from load when $a^2/r^2 \to 0$ IN 5.20

$$ w_s S_{13} = \frac{2}{\pi^2} \cdot \frac{1}{r} \cdot \frac{\pi}{4} = \frac{P}{2\pi} \cdot \frac{1}{r} \quad (5.21) $$

This checks with result found previously (4.314)

Uniform Load Distribution [B]

Let $f(r) = -p_o$ where p_o is the loading intensity. Then from

(5.08) $\qquad P = \pi p_o$ = Total Load.

$\underline{r \leq 1}$

$$ w_s\, \pi/2\, s_{13} = p_o \frac{r}{2} \left[\int_0^1 K(x)dx + \int_{r^2}^{1} x^{-3/2} K(x)dx \right] $$

$$ \therefore \quad w_s\, s_{13} = p_o \frac{r}{\pi} \left[I_0 + {}_{r^2}^{1}I_{-3/2} \right] = \frac{2P}{\pi^2 r} B\left(\frac{1}{r^2}\right) $$

Or if the load P is distributed uniformly over the circle $r = a$, then on applying scale factor to above results we obtain

$\underline{r \leq a}$ $\qquad$ (5.20)

$$ w_s\, s_{13} = \frac{2P}{\pi^2 a} E\,(r^2/a^2) $$

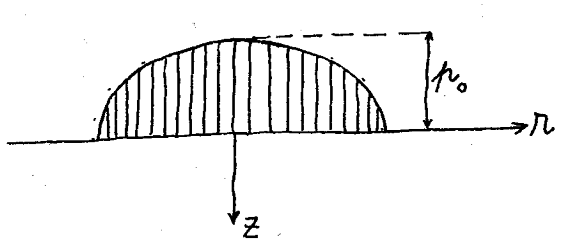
<u>$r > a$</u>

$$w_s\, s_{13} = \frac{2P}{\pi^2 r}\, B\left(\frac{a^2}{r^2}\right)$$

$$\text{as } r \to \infty \qquad B(a^2/r^2) \to B(o) = \frac{\pi}{4}$$

$$r \to a \qquad B(a^2/r^2) \to B(1) = 1$$

<u>Parabolic Load Distribution C</u>



$$f(r) = -p_o(1-r^2)$$

$$\therefore \text{ Total Load } P = \int_0^1 f(r)\, 2\pi r\, dr = \frac{\pi p_o}{2}$$

From (5.08) we obtain

$$w_s\, s_{13} = \frac{2P}{\pi^2}\, r\left[\int_0^1 (1-r^2x)K(x)\,dx + \int_{r^2}^1 (x^{-3/2} - r^2x^{-5/2})K(x)\,dx\right] \qquad r \leqslant 1$$

$$= \frac{2P}{\pi^2}\, r\left[\int_0^{1/r^2} (1-r^2x)K(x)\,dx\right], \quad r > 1$$

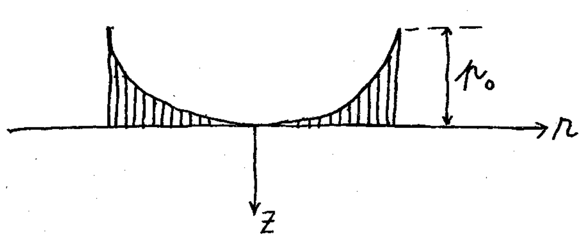
Using evaluations given in Appendix D, and in (5.16) above becomes

$$\frac{w_s\, s_{13}}{P} = \frac{8}{9\pi^2}\left[E(r2)(4-2r2) - (1-r2)K(r2)\right] \qquad r < 1$$

$$= \frac{4r}{\pi^2}\left[\left(\frac{1}{r^2} - \frac{4}{9}\right) B\left(\frac{1}{r^2}\right) - \frac{1}{9} E\left(\frac{1}{r^2}\right) + \left(1-\frac{1}{r^2}\right) K\left(\frac{1}{r^2}\right)\right] \qquad (5.22)$$

$$r > 1$$

<u>Inverted Parabolic Load D</u>



$$f(r) = -rp_o$$

$$\therefore P = p_o \int_0^1 2\pi r^3\, dr = \frac{\pi p_o}{2}$$

From (5.08) we obtain

$$w_s \; s_{13} = \frac{2Pr}{\pi^2}\left[r^2 \int_0^1 x \, K(x)\,dx + r^2 \int_{r^2}^1 x^{-5/2} \, K(x)\,dx \right] \qquad r < 1$$

$$= \frac{2Pr^3}{\pi^2} \int_0^{1/r^2} x \, K(x)\,dx \qquad r > 1$$

Results can be written down from those of [B] and [C] or directly from the evaluation of the above integrals. Hence

$$\frac{w_s \; s_{13}}{P} = \frac{4}{9\pi^2}\left[E(r^2) \quad 1+4r^2 \quad + 2(1-r^2)K(r^2) \right] \qquad r > 1$$

$$= \frac{4r}{\pi^2}\left[\frac{4}{9} B\left(\frac{1}{r^2}\right) + \frac{1}{9} E\left(\frac{1}{r^2}\right) - \frac{(1-1/r^2)K(1/r^2)}{3} \right] \qquad r < 1 \qquad (5.23)$$

<u>Hollow Column</u> [E]

Consider

$$f(r) = -\frac{P}{2\pi\epsilon}, \qquad 1-\epsilon \le r \le 1 \qquad (5.231)$$

$$= 0 \qquad \text{for all other values of } r.$$

From (5.01)

$$R_m = -\frac{\pi s_9}{2} \; \underset{\epsilon \to 0}{\mathrm{Lt}} \int_{1-\epsilon}^{1} \frac{P}{2\pi\epsilon} \; t \, J_0(mt)\,dt$$

$$= -\frac{Ps_9}{4} \; J_0(m)$$

Therefore from (4.24) surface settlement is given by

$$\frac{w_s(s_2+h_2)}{s_7-s_8} = -\int_0^\infty R_m \, J_0(mr)\,dm$$

$$\therefore \quad w_s \; s_{13} = \frac{P}{2\pi} \int_0^\infty J_0(m) \; J_0(mr) \; dm$$

and therefore from (5.06)

$$I_1 = \frac{2}{\pi r} K\left(\frac{1}{r^2}\right) \quad , \quad r^2 > 1$$

$$= \frac{2}{\pi} K(r^2) \quad , \quad r^2 < 1$$

$$\therefore \; w_s \, s_{13} = \frac{P}{\pi^2} K(r^2) \quad , \quad r^2 < 1$$

$$= \frac{P}{\pi^2 r} K\left(\frac{1}{r^2}\right) \quad , \quad r^2 > 1$$

Rigid Disc [F]

Consider a rigid disc of radius 1, with boundary conditions

$$\widehat{zz} = 0 \quad , \quad |r| > 1$$

$$w = w_o \quad , \quad |r| \leq 1 \quad \text{where } w_o \text{ is constant.}$$

Hence from (4.20) and (4.24), provided the integrals exist, we require

$$0 = \int_0^\infty m \, R_m \, J_o(mr) \, dm \quad , \quad r > 1$$

and $$w_o = \frac{s_7 - s_8}{s_2 + h_2} \int_0^\infty R_m \, J_o(mr) \, dm \quad , \quad r < 1$$

From Watson(18) (13.42) we obtain

$$\int_0^\infty J_o(mr) \sin m \, dm = \frac{1}{\sqrt{1-r^2}} \quad \text{or} \quad 0, \quad r < 1 \text{ or } > 1$$

$$\int_0^\infty J_o(mr) \frac{\sin m}{m} \, dm = \pi/2 \quad \text{or} \quad \sin^{-1} 1/r \, , \quad r < 1 \text{ or } > 1$$

Hence by comparison

$$R_m = C \frac{\sin m}{m}$$

and $$\therefore \; w_o = \frac{s_7 - s_8}{s_2 + h_2} \, \pi/2 \; C \quad , \quad r \leq 1$$

The pressure distribution is given by

$$\widehat{zz} = \frac{s_3 - s_4}{s_2 + h_2} \, \frac{C}{\sqrt{1 - r^2}}$$

If the pressure at the centre is p_o

$$\text{then} \quad p_o = -C\,\frac{s_3 - s_4}{s_2 + h_2}$$

and hence $\widehat{zz} = -\dfrac{p_o}{\sqrt{1-r^2}}$

$$\therefore \quad \text{Total Load} \quad P = p_o \int_0^1 \frac{2\pi\, r\, dr}{\sqrt{1-r^2}} = 2\pi\, p_o$$

Hence we obtain for the surface settlement

$$w_s\, s_{13} = p_o\, \pi/2 = \frac{P}{4} \quad , \quad r \leq 1 \tag{5.25}$$

$$= p_o \sin^{-1} 1/r = \frac{P}{2\pi} \sin^{-1} 1/r \quad , \quad r > 1$$

The contact pressure is given by

$$\widehat{zz} = -\frac{p_o}{\sqrt{1-r^2}} = -\frac{P}{2\pi\sqrt{1-r^2}} \quad , \quad r \leq 1 \tag{5.26}$$

<u>Normal Stress Along Axis</u>

From (4.20)

$$\widehat{zz}\Big|_{r=0} = -\frac{p_o}{s_3 - s_4}\int_0^\infty \sin m \left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] dm$$

$$= -\frac{P}{2\pi(s_3 - s_4)}\left[\frac{s_3}{1+s_1^2 z^2} - \frac{s_4}{1+s_2^2 z^2}\right] \quad \text{from A1} \tag{5.27}$$

$$= -\frac{P s_{10}}{2}\left[\frac{1}{s(1+s^2 z^2)}\right]_{s_2}^{s_1}$$

TABLE II

Settlement Influence Factors $\dfrac{w_s\, s_{13}}{P} = N(r)$

Loading \ r	0	0.25	0.50	0.75	1.00	2	4
Concentrated	∞	0.636	0.318	0.213	0.159	.0795	.0398
Uniform	0.318	0.312	0.296	0.266	0.202	.0826	.0400
Parabolic	0.424	0.403	0.348	0.265	0.180	.0804	.0399
Inverted Parabola	0.212	0.220	0.244	0.267	0.224	.0849	.0404
Hollow Wall	0.159	0.162	0.171	0.193	∞	.0852	.0405
Rigid Wall	0.250	0.250	0.250	0.250	0.250	.0833	.0402

Stresses at Points on Axis

In practice the only stresses likely to require investigation are those along the axis of symmetry r = 0. Hence from (5.14) we obtain

$$\widehat{zz}\left(s_3 - s_4\right) = \int_0^\infty m \left[\int_0^1 t\ f(t)\ J_0(mt)\ dt\right]\left[s_3 e^{-s_1 mz} - s_4 e^{-s_2 mz}\right] dm \tag{5.28}$$

Let
$$^{z}I_s = \int_0^\infty m\ e^{-smz}\left[\int_0^1 t\ f(t)\ J_0(mt)\ dt\right]\ dm \tag{5.29}$$

Since the infinite integral is absolutely and uniformly convergent for $z > 0$, and the finite integral is everywhere continuous and finite we can invert the order of integration. Hence

$$^{z}I_s = \int_0^1 t\ f(t)\left[\int_0^\infty m\ e^{-smz}\ J_0(mt)\ dm\right]\ dt.$$

From Appendix B :

$$\int_0^\infty m e^{-smz}\ J_0(mt)\ dm = \frac{P_1(\cos\theta)}{R^2} \qquad z > 0$$

where $R^2 = s^2z^2+t^2$, $\tan\theta = sz/t$ $P_1(\cos\theta) = \cos\theta$

$= sz/R$

Hence

$$^{z}I_s = \int_0^1 \frac{t\ f(t)\ P_1(\cos\theta)}{s^2z^2+t^2}\ dt \tag{5.30}$$

$$= sz\int_0^1 \frac{t\ f(t)}{\left[s^2z^2+t^2\right]^{3/2}}\ dt$$

<u>Evaluation</u>

A If f(t) is an even polynomial in t as it is in the cases considered in this thesis, then (5.30) can be evaluated by the substitution

$$T = s^2z^2 + t^2$$

and therefore $f(t) = f(\sqrt{T-s^2z^2})$ is a polynomial in $(T-s^2z^2)$

Hence (5.30) becomes

$$^{z}I_s = \frac{sz}{2}\int_{s^2z^2}^{1+s^2z^2} \frac{f(T-s^2z^2)}{T^{3/2}}\ dt \quad \text{and can easily be} \tag{5.31}$$

evaluated.

B If f(t) is any polynomial in t, other than an even one, then the evaluation is more tedious.

We note $^{z}I_s = -\frac{1}{s}\frac{\partial}{\partial z}\left[\int_0^{\alpha}\frac{t\ f(t)}{\left[s^2z^2+t^2\right]^{1/2}}dt\right.$

Substituting $t = sz\ \sinh\theta$ $\sinh\alpha = \frac{1}{sz}$ we obtain

$$^{z}I_s = -\frac{1}{s}\frac{\partial}{\partial z}\left[\int_0^{\alpha} sz\ \sinh\theta\, f\ (\sinh\theta)\ d\theta.\right. \tag{5.32}$$

The differentiation can be performed first[13] giving

$$^{z}I_s = \int_0^{\alpha}\sinh\theta\, f(\sinh\theta)\, d\theta - z\ \sinh\alpha\ \ f(\sinh\alpha)\ \frac{\partial\alpha}{\partial z}$$

since $\frac{\partial\alpha}{\partial z} = -\frac{1}{z\sqrt{s^2z^2+1}}$ and $\sinh\alpha = \frac{1}{sz}$ we obtain

$$^{z}I_s = -\int_0^{\alpha} \sinh\theta\, f(\sinh\theta)\, d\theta + \frac{1}{sz\sqrt{s^2z^2+1}} \tag{5.33}$$

Assuming f(t) is continuous, as it will be in practice, then the above integral involves only integrals of the type $\int_0^{\alpha} \sinh^n\theta\, d\theta$, where n is an integer. Two methods can be used for integrating this integral

(i) If $J_n \equiv \int_0^{\alpha} \sinh^n\theta\, d\theta$

then easily by using integration by parts, we can establish the recurrence formula:

$$J_n = \frac{\sinh^{n-1}\alpha\,\cosh\alpha}{n} - \frac{n-1}{n} J_{n-2} \tag{5.34}$$

The final result is thus obtained in powers of $\sinh\alpha$ and $\cosh\alpha$, and these can very easily be expressed in terms of z.

(ii) If the value of the integral is desired for some given numerical value of z, an alternative to the above procedure is obtained by using

$$\sinh^{2n}\theta = \frac{1}{2^{2n-1}}\left[\cosh 2n\theta - {}^{2n}C_1\cosh(2n-2)\theta + {}^{2n}C_2\cosh(2n-4)\theta + \dots (-)^n\, {}^{2n}C_n\right] \tag{5.35}$$

$$\sinh^{2n+1}\theta = \frac{1}{2^{2n}}\left[\sinh(2n+1)\theta - {}^{2n+1}C_1\sinh(2n-1)\theta \dots (-)^n\, {}^{2n+1}C_n\sinh\theta\right]$$

where ${}^{2n}C_r \quad \frac{n!}{r!\,2n-r!}$. The integrals now involved are simply

$$\int_0^{\alpha} \sinh r\theta \, d\theta = \frac{\cosh r\alpha - 1}{r}$$

$$\int_0^{\alpha} \cosh r\theta \, d\theta = \frac{\sinh r\alpha}{r}$$

Hence their numerical values can easily be obtained

Finally from (5.28)

$$\left.\widehat{zz}\right|_{r=0} = \frac{s_3 \, {}^{z}\mathcal{I}_{s_1} - s_4 \, {}^{z}\mathcal{I}_{s_2}}{s_3 - s_4}$$

$$= \pi s_{10} \left[\frac{1}{s_1} {}^{z}\mathcal{I}_{s_1} - \frac{1}{s_2} {}^{z}\mathcal{I}_{s_2} \right]$$

since $s_4 s_2 = s_3 s_1$ from C.

$$= \pi s_{10} \left[\frac{1}{s} {}^{z}\mathcal{I}_{s} \right]_{s_2}^{s_1} \qquad (5.36)$$

WHERE

$$s_{10} = \frac{1}{\pi} \frac{s_4 s_2}{s_3 - s_4}$$

where $s_{10} = \dfrac{1}{\pi} \dfrac{s_4 s_2}{s_3 - s_4}$

and $\left[f(s)\right]_{s_2}^{s_1} = f(s_1) - f(s_2)$ as in the integral calculus (5.361)

<u>Maximum Axial Shear Stress</u> (ie. For points on r=0)

From (4.30)

$$\tau_A = \frac{1}{2(s_3 - s_4)} \left[(s_3 - s_5) \, {}^{z}\mathcal{I}_{s_1} - (s_4 - s_6) \, {}^{z}\mathcal{I}_{s_2} \right]$$

Using results in Appendix C, C.2, and C.5 :

$$\therefore \tau_A = \frac{\pi}{2} \left\{ s_{14} \left[s \, {}^{z}\mathcal{I}_{s} \right]_{s_2}^{s_1} + s_{10} \left[\frac{1}{s} {}^{z}\mathcal{I}_{s} \right]_{s_2}^{s_1} \right\} \qquad (5.362)$$

where

$$s_{14} = -\frac{1}{\pi} \frac{s_5}{s_1(s_3 - s_4)}$$

<u>Special Cases</u>

We now proceed to evaluate ${}^{z}\mathcal{I}_{s}$ in the cases for which surface displacements have been obtained in (5.20) to (5.26).

Concentrated Load [A]

From (5.30) and (5.36), or directly from (4.311) we obtain

$$\widehat{zz}\Big|_{r=0} = - s_{10}\frac{P}{2z^2}\left(\frac{1}{s^3} - \frac{1}{s^3}\right) = - \frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\right]_{s_2}^{s_1} \tag{5.37}$$

Uniform Load [B]

$$f(r) = - p_o \ , \ r \leqslant 1 \qquad P = \pi p_o$$
$$= 0 \ , \ r > 1$$

Then from (5.30)

$$-\frac{1}{s}\,{}^{z}I_{s} = zp_o \int_0^1 \frac{t\,dt}{\left[s^2z^2+t^2\right]^{3/2}} = - \frac{sz\,p_o}{\sqrt{s^2z^2+t^2}}\Bigg|_0^1$$

$$= zp_o \left[\frac{1}{sz} - \frac{1}{\sqrt{s^2z^2+1}}\right]_0^1$$

∴ from (5.36)

$$\widehat{zz}\Big|_{r=0} = - P\, s_{10} \left[\frac{1}{s} - \frac{z}{\sqrt{s^2z^2+1}}\right]_{s_2}^{s_1} \tag{5.371}$$

Parabolic Load [C]

$$f(r) = - p_o(1-r^2) \ , \quad r \leqslant 1$$
$$= 0 \ , \quad r > 1 \qquad\qquad P = \frac{\pi}{2} p_o$$

From (5.30)

$$-\frac{1}{s}\,{}^{z}I_{s} = zp_o \int_0^1 \frac{t(1-t^2)\,dt}{\left[s^2z^2+t^2\right]^{3/2}} \qquad \text{Let } T = s^2z^2+t^2$$

$$= z\,\frac{p_o}{2}\int_{s^2z^2}^{1+s^2z^2} \frac{1+s^2z^2-T}{T^{3/2}}\,dT$$

$$= zp_o \left[2\,sz + \frac{1}{sz} - 2\sqrt{s^2z^2+1}\right]_{s_2}^{s_1}$$

Hence from (5.36)

$$\widehat{zz}\Big|_{r=0} = -\ 2P\ s_{10}\ z\ \Big[2sz + \frac{1}{sz} - 2\sqrt{s^2z^2+1}\Big]_{S_2}^{S_1} \qquad (5.38)$$

Inverted Parabolic Load [D]

$$f(r) = -r^2\ p_0 \quad , \quad r \leqslant 1 \qquad\qquad P = \frac{\pi}{2}\ p_0$$

$$= 0 \qquad\qquad r > 1$$

Then directly from (5.30) and (5.36), or alternatively by combining $2[B] - [C]$ we easily obtain

$$\widehat{zz}\Big|_{r=0} = -\ 2\ Ps_{10}\ z\ \Big[2\sqrt{1+s^2z^2} - \frac{1}{\sqrt{1+s^2z^2}} - 2sz\Big]_{S_2}^{S_1} \qquad (5.39)$$

Hollow Column [E]

$$f(r) = -\ \frac{P}{2\pi\epsilon} \qquad\qquad 1 - \epsilon \leqslant r \leqslant 1$$

$$= 0 \quad \text{for all other values of r.}$$

Then from (5.30)

$$-\ \frac{1}{s}\ {}^{z}I_s = \frac{Pz}{2\pi}\ \underset{\epsilon\to 0}{Lt}\ \frac{1}{\epsilon}\int_{1-\epsilon}^{1} \frac{t\ dt}{[s^2z^2+t^2]^{3/2}}$$

$$= \frac{Pz}{2\pi}\Big[\frac{1}{[s^2z^2+1]^{3/2}}\Big]$$

and from (5.36)

$$\widehat{zz}\Big|_{r=0} = -\ s_{10}\ \frac{Pz}{2}\ \Big[\frac{1}{[s^2z^2+1]^{3/2}}\Big]_{S_2}^{S_1} \qquad (5.40)$$

As a check on the accuracy of above results, we can find the value in each case of $\widehat{zz}\Big|_{r=0}$ when z is large by expanding in powers of $\frac{1}{z}$. This gives for the various distributions treated above

[A] $$\widehat{zz}\Big|_{r=0} = -\ \frac{s\ P}{2z^2}\ \Big[\frac{1}{s^3}\Big]_{S_2}^{S_1}$$

$$[B] \quad \widehat{zz}\Big|_{r=0} = -\frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\left(1 - \frac{3}{4s^2z^2} + \frac{5}{8s^4z^4} + \cdots\cdots\right)\right]_{s_2}^{s_1}$$

$$[C] \quad \widehat{zz}\Big|_{r=0} - \frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\left(1 - \frac{1}{2s^2z^2} + \frac{5}{16s^4z^4} - \cdots\cdots\right)\right]_{s_2}^{s_1} \qquad (5.41)$$

$$[D] \quad \widehat{zz}\Big|_{r=0} = -\frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\left(1 - \frac{1}{s^2z^2} + \frac{15}{16s^4z^4} - \cdots\cdots\right)\right]_{s_2}^{s_1}$$

$$[E] \quad \widehat{zz}\Big|_{r=0} = -\frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\left(1 - \frac{3}{2s^2z^2} + \frac{15}{8s^4z^4} - \cdots\cdots\right)\right]_{s_2}^{s_1}$$

$$[F] \quad \widehat{zz}\Big|_{r=0} = -\frac{s_{10}P}{2z^2}\left[\frac{1}{s^3}\left(1 - \frac{1}{s^2z^2} + \frac{1}{s^4z^4} - -\right)\right]_{s_2}^{s_1}$$

Clearly we see that all distributions approach the value for the concentrated load as z becomes large. This is in accordance with St Venant's principle, and provides a good check on the derivations (5.37) to (5.40)

Maximum Axial Shear Stress

This can readily be obtained in the above cases by substituting in the formula (5.362) the appropriate values of $\left[s\,{}^{z}I_s\right]_{s_2}^{s_1}$ and $\left[\frac{1}{s}\,{}^{z}I_s\right]_{s_2}^{s_1}$

The values of ${}^{z}I_s$ are from previous work.

[A] Concentrated $\quad {}^{z}I_s = \frac{P}{2\pi z^2}\left[\frac{1}{s^2}\right]$

[B] Uniform $\quad {}^{z}I_s = -\,zp_o\left[\frac{1}{z} - \frac{s}{s^2z^2+1}\right]$

[C] Parabolic $\quad {}^{z}I_s = -\,zp_o\left[2sz^2 + \frac{1}{z} - 2s\sqrt{s^2z^2+1}\right]$

[D] Inverted Parabola

$$ {}^{z}I_s = -\,sz\,p_o\left[2\sqrt{1+s^2z^2} - \frac{1}{\sqrt{1+s^2z^2}} - 2\,sz\right] \qquad (5.41)$$

[E] Hollow Column

$$ {}^{z}\mathcal{I}_s = -\frac{3\,Pz}{2\pi}\;\frac{1}{(s^2z^2+1)^{3/2}} $$

[F] Rigid Column

$$ {}^{z}\mathcal{I}_s = -\frac{P}{2\pi}\;\frac{1}{1+s^2z^2} $$

Loaded Circular Area of Any Radius

Consider loading distributed according to law $\widehat{zz}\big|_{z=0}=f(r)$, the total load being P.

From (4.20) and (4.27),

$$ \widehat{zz}\Big|_{r,z} = \frac{1}{s^3-s^4}\int_0^\infty m\left\{\int_0^a tf(t)J_0(mt)\,dt\right\}\left\{s_3e^{-s_1mz}-s_4e^{-s_2mz}\right\}J_0(mr)\,dm $$

As in the two dimensional case, if we introduce dimensionless coordinates $z = z'a$, $r = r'a$ $t = t'a$ $m = m'/a$, the surface distribution becomes $\widehat{z'z'} = -\,a^2f(r'a) = -\,f'(r')$ say

Then from above we obtain

$$ \widehat{zz}\Big|_{r,z} = \frac{1}{a^2}\,\frac{1}{s_3-s_4}\int_0^\infty m\left\{\int_0^1 t'f'(t')J_0(m't')\,dt'\right\}\left\{s_3e^{-s_1m'z'}-s_4e^{-s_2m'z'}\right\} J_0(m'r')\,dm' $$

$$ = \frac{1}{a^2}\;\widehat{z'z'}\Big|_{r',z'} $$

where $\widehat{z'z'}$ is stress component at the dimensionless point (r',z'). Or take $P' = 1$ in the dimensionless system, and denote the corresponding influence stresses by $\widehat{z'z'}$, $\widehat{r'r'}$, etc., then

$$ \widehat{zz}\Big|_{r,z} = \frac{P}{a^2}\;\widehat{z'z'}\Big|_{r',z'} \tag{5.42} $$

Hence, corresponding stress components are directly proportional to the total load, and inversely proportional to the square of (5.43)

<u>the radius of the loaded</u> area. Or since P/a^2 is proportional to the surface stress, <u>we conclude as in the two dimensional case corresponding stress components are proportional to the intensity of loading on the surface.</u> (5.44)

A similar consideration of the surface displacements (4.24) and (4.27) leads to the result

(5.45)

$$w\Big|_{r,z} = \frac{P}{a}\,\hat{w}'\Big|_{r',z'}$$

Hence, we have

$$w_s\Big|_{r} = \frac{P}{a\,s_{13}}\,N(r') \qquad (5.46)$$

where $N(r')$ is the appropriate influence factor for the distribution, and is tabulated in Table II. Since P/a is proportional to a p_0, we conclude from (5.46) <u>that corresponding displacements</u> (i.e. at point $r' = r/a$) <u>are directly proportional to the radius of the loaded area, and the intensity of the applied surface loading.</u>

<u>Normal Stress Along Axis</u>

Expressions for the normal stress along the axis due to a load P distributed in a given manner over a circle of radius a are easily obtained on applying result (5.42) to equations (5.37) (5.40)

This gives:

[B] <u>Uniform Load</u>

$$\widehat{zz}\Big|_{0,z} = \frac{-P\,s_{10}}{a^2}\left[\frac{1}{s} - \frac{z'}{\sqrt{s^2 z'^2+1}}\right]_{s_2}^{s_1} = -\frac{P\,s_{10}}{a^2}\left[\frac{1}{s} - \frac{z}{\sqrt{s^2 z^2+a^2}}\right]_{s_2}^{s_1} \qquad (5.47)$$

[C] Parabolic Load

$$\widehat{zz}\Big|_{o,\,z} = -\,2\,\frac{Ps_{10}\,z}{a^4}\left[2sz + \frac{a^2}{sz} - 2\sqrt{s^2z^2+a^2}\,\right]_{s_2}^{s_1} \qquad (5.48)$$

[D] Inverted Parabola

$$\widehat{zz}\Big|_{o,\,z} = -\,2\,\frac{Ps_{10}\,z}{a^4}\left[2\sqrt{s^2z^2+a^2} - \frac{a^2}{\sqrt{s^2z^2+a^2}} - 2sz\right]_{s_2}^{s_1} \qquad (5.481)$$

[E] Hollow Column

$$\widehat{zz}\Big|_{o,\,z} = -\,s_{10}\,\frac{Pz}{2}\left[\frac{1}{\left[s^2z^2+a^2\right]^{3/2}}\right]_{s_2}^{s_1} \qquad (5.49)$$

[F] Rigid Disc

$$\widehat{zz}\Big|_{o,\,z} = -\,\frac{Ps_{10}}{2}\left[\frac{1}{s(s^2z^2+a^2)}\right]_{s_2}^{s_1} \qquad (5.50)$$

Maximum Axial Shear Stress ($\mathcal{Z}_A$)

Similarly transforming the factors ${}^{z}\mathcal{I}_{s}$, used in equation (5.362) for the calculation of $\mathcal{Z}_A$, by means of result (5.42), we obtain

A Concentrated $\quad {}^{z}\mathcal{I}_{s} = -\,\dfrac{P}{2\pi z^2}\left[\dfrac{1}{s^2}\right]$

B Uniform $\quad {}^{z}\mathcal{I}_{s} = -\,\dfrac{P}{\pi a^2}\left[1 - \dfrac{sz}{\sqrt{s^2z^2+a^2}}\right]$

C Parabolic $\quad {}^{z}\mathcal{I}_{s} = -\dfrac{2\,Pz}{\pi a^4}\left[2s^2z + \dfrac{a^2}{z} - 2s\sqrt{s^2z^2+a^2}\right]$

D Inverted Parabola $\qquad (5.51)$

$${}^{z}\mathcal{I}_{s} = -\,\frac{2Pzs}{\pi a^4}\left[2\sqrt{s^2z^2+a^2} - \frac{a^2}{\sqrt{s^2z^2+a^2}} - 2sz\right]$$

E. Hollow Column

$$= - \frac{Pz}{2\pi} \cdot \frac{s}{\left[s^2z^2+a^2\right]^{3/2}}$$

F Rigid Disc

$$= - \frac{P}{2\pi} \; \frac{1}{(s^2z^2+a^2)}$$

Stresses and Displacements at Any Point in Mass

The previous analysis, using Eliptic integrals, is limited to a determination of the surface displacements. No closed form can be obtained for calculation of the stresses and displacements at an arbitrary point of the mass. However, the desired quantities can be obtained as infinite series in the following manner:

Substituting value of U_m from (5.01) into (4.20), we obtain

$$\widehat{zz}\ (s_2+h_2) = \frac{\pi\delta g}{2}\int_0^\infty m\left[\int_0^1 t\ f(t)\ J_o(mt)\ dt\right]\left[s_3e^{-s_1mz} - s_4e^{-s_2mz}\right] J_o\ (mr)dm \qquad (5.52)$$

with similar expressions for the other stresses and displacements. Clearly the evaluation depends on the evaluation of

$$^{z}I_s \equiv \int_0^\infty m\left[\int_0^1 t\ f(t)\ J_o\ (mt)dt\right]\left[e^{-smz}\ J_o\ (mr)\right]dm \qquad (5.53)$$

and this depends on the form of f(t). Physical conditions demand that in any contact process f(t) should be finite and continuous and so have a Taylor expansion around the origin

$$f(t) = \sum_{n=0}^{\infty} a_n\ t^n \qquad (5.54)$$

Another form that suggests itself when f(t) is an even function is

$$f(t) = \sum_{n=0}^{\infty} b_n(1-t^2)^n \qquad (5.55)$$

Assuming that we can invert the order of integration in (5.53) we obtain

$$^{z}I_s = \int_0^1 t\ f(t)\left[\int_0^\infty e^{-smz}m\ J_o\ (mt)\ J_o(mr)dm\right] dt$$

On substituting the expansion(20) $J_o(mt) = \sum_{r=0}^{\infty} \frac{(-)^r\left(\frac{mt}{2}\right)^{2r}}{(r!)^2}$

in the above integral, we note that the infinite series under the integral sign is uniformly convergent for all finite m and t. Hence term by term integration can be used over the infinite range 0 to for m, provided the resulting series is absolutely convergent[22].

On using results in Appendix B, we obtain

$$ {}^{z}I_{S_i} = \sum_{r=0}^{\infty} \frac{(-)^r\,(2r+1)!}{2^{2r}\,(\lfloor r)^2} \; \frac{P_{2r+1}(\cos\theta_i)}{R_i^{2r+2}} \int_0^1 t^{2r+1}\, f(t)\,dt $$

$$ = \frac{2}{\sqrt{\pi}} \sum_{r=0}^{\infty} (-)^r \frac{\Gamma(r+3/2)}{\lfloor r} \; \frac{P_{2r+1}(\cos\theta_i)}{R_i^{2r+2}} \int_0^1 t^{2r+1} f(t)\,dt. \tag{5.56} $$

where

$$ R_i^2 = r^2 + (s_i z)^2 \qquad \cos\theta_i = \frac{s_i z}{R_i} $$

Clearly the rth term in this series is of order $\frac{1}{R_i^2}$* and so is uniformly convergent when $|R_i| > 1$ and so derivation of (5.56) is valid in this range. When the form of f(t) is known the evaluation can be carried out quite readily. The series is in effect an asymptotic series.

---

*Since $\left| P_{2r+1}(\cos\theta_i) \right| \leq 1$

and $\left| \int_0^1 t^{2r+1} f(t)\,dt \right| < \int_0^1 |f(t)|\,dt$ = Total Load P.

Case $|R_i|<1$.

Here the only course is to first evaluate

$$U_m = \int_0^1 t\, f(t)\, J_o\,(mt)\, dt \text{ in terms of Bessel functions.}$$

If in (5.54) f(t) is given as a finite series (as it will be in practice) then the result is obtained by application of the reduction formula.

$$T_n = \frac{J_1(m)}{m} + \frac{(n-1)}{m^2}\,\underline{J_o(m)} - \frac{(n-1)^2}{m^2}\, T_{n-2} \qquad (5.57)$$

when

$$T_n = \int_0^1 t^n J_o\,(mt)\,dt.$$ This can readily be established by integrating by parts.

A much neater result is obtained on using form (5.55) for f(t).

Then

$$U_m = \sum b_n \int_0^1 t\,(1-t^2)^n \;\; J_o\,(mt)\,dt$$

On putting $t = \sin\theta$

$$U_m = \sum b_n \int_0^1 J_o(m \sin\theta)\,\sin\theta\;\cos^{2n+1}\theta\;\,d\theta$$

$$= \sum b_n\; 2^n\,\Gamma(n+1)\,\frac{J_{n+1}(m)}{m^{n+1}} \qquad \text{(Watson}^{(18)}\text{, 12.11)} \qquad (5.58)$$

where $n > -1$ for convergence. Having evaluated R_m in terms of Bessel Functions (5.15) can now be evaluated for $|R_i|<1$ by expanding $J_o(mr)$ in series, and integrating term by term.

Using (5.58) for U_m we obtain

$$^{z}I_s = \sum_{n=0} b_n\; 2^n\,\Gamma(n+1) \int_0^\infty e^{-smz}\,\frac{J_{n+1}(m)}{m^n} \sum_{v=0} \frac{(-)^v}{(\lfloor v)^2}\left(\frac{mr}{2}\right)^{2v} dm$$

$$= \sum_{n=0}\sum_{v=0} (-)^v\; b_n\; 2^{n-2v}\,\lfloor n\;\; \frac{r^{2v}}{(\lfloor v)^2} \int_0^\infty e^{-smz}\, J_{n+1}(m)\,m^{-n+2v}\,dm$$

From Appendix B·3

$$\int_0^\infty e^{-smz} J_{n+1}(m)\, m^{-n+2v}\, dm = \frac{\Gamma(2v+2)}{(1+s^2z^2)^{\frac{-n+2v+1}{2}}}\, P_{-n+2v}^{-(n+1)}(\cos\theta_i')$$

$$\therefore\ {}^{z}I_s = \sum_{n=0}\sum_{v=0} (-)^v b_n\, 2^{n-2v}\, \frac{\lfloor n\ \lfloor 2v+2}{(\lfloor v)^2}\, \frac{P_{-n+2v}^{-(n+1)}(\cos\theta_i')\, r^{2v}}{(1+s^2z^2)^{\frac{-n+2v+1}{2}}}$$

(5.59)

If (5.55) is a finite series then (5.59) is convergent

$$\frac{r^2}{1+s^2z^2} < 1 \qquad ,/, \qquad |R_i|^2 < 1 + 2s^2z^2$$

The other stress and displacement components may be similarly obtained.

<u>Evaluation of Vertical Pressure at Any Point</u>

<u>B</u> <u>Uniform Load</u>

$$f(r) = -\,p_o \qquad\qquad \therefore\ -\int_0^1 t^{2r+1} f(t)\,dt = \frac{p_o}{2r+2}$$

Hence for $|R| > 1$ from (5.56) we obtain

$$ {}^{z}I_s = \sum_{r=0}^{\infty} p_o\, \frac{(-)^r\ \lfloor 2r+1}{2^{2r}\ (\lfloor r)^2\ (2r+2)}\ \frac{P_{2r+1}(\cos\theta_i')}{R^{2r+2}} \tag{5.60}$$

From (5.59) for $|R| < 1 + 2s^2z^2$

$$ {}^{z}I_s = p_o \sum_{v=0}^{\infty} (-)^v\, \frac{\lfloor 2v+2}{2^{2v}\,(\lfloor v)^2}\ \frac{P_{2v}^{-1}(\cos\theta_i)}{(1+s^2z^2)^{v+\frac{1}{2}}}\ r^{2v} \tag{5.61}$$

Therefore from (5.52)

$$\widehat{zz} = \frac{s_3\, {}^{z}I_{s_1} - s_4\, {}^{z}I_{s_2}}{s_3 - s_4} = -\,\pi S_{10} \left[\frac{1}{s}\, {}^{z}I_s\right]_{s_2}^{s_1} \tag{5.62}$$

where

$$\left[\frac{1}{s}\, {}^{z}I_s\right]_{s_2}^{s_1} \equiv \frac{1}{s_1}\, {}^{z}I_{s_1} - \frac{1}{s_2}\, {}^{z}I_{s_2} \tag{5.63}$$

C Parabolic Load

$$f(t) = -p_0(1-t^2) \qquad \therefore \quad \text{only } b_1 \text{ exists}$$

$$-\int_0^1 t^{2r+1} f(t)\,dt = -p_0 \int_0^1 t(1-t^2)\,dt \qquad (\xi = t^2)$$

$$= \frac{p_0}{2} \int_0^1 \xi^r (1-\xi)\,d\xi = \frac{p_0}{2}\, B(r+1,2)$$

$$= p_0 \frac{\Gamma(r+1)}{\Gamma(r+3)} = \frac{p_0}{2(r+2)(r+1)}$$

Hence $|R| > 1$ from (5.56)

$$ {}^{z}I_s = p_0 \sum_{r=0} \frac{(-)^r \lfloor 2r+1}{2^{2r+1} \lfloor r+2 \lfloor r} \frac{P_{2r+1}(\cos\theta_i)}{R^{2r+2}} \tag{5.64}$$

and $|R| < 1$ from (5.59), n = 1 is only term in n summation

$\therefore$

$$ {}^{z}I_s = p_0 \sum_{v=0} 2^{1-2v} \frac{\lfloor 2v+2}{(\lfloor v)^2} \frac{P^{-2}_{-2+2v}(\cos\theta)}{(1+s^2z^2)^v} r^{2v} \tag{5.56}$$

and $\widehat{zz}$ can be obtained from (5.62) with above values for ${}^{z}I_s$

D Inverted Parabolic Load

$$f(t) = -p_0 t^2 \qquad |t| < 1$$

$$= 0 \qquad |t| > 1 \qquad \therefore \quad P = \frac{\pi p_0}{2}$$

$$\therefore \quad -\int_0^1 t^{2r+1} f(t)\,dt = p_0/2r+4$$

Hence from equation (5.56) for $|R| > 1$

$$ {}^{z}I_s = p_0 \sum_{r=0}^{\infty} \frac{(-)^r \lfloor 2r+1}{2^{2r} (\lfloor r)^2 (2r+4)} \frac{P_{2r+1}(\cos\theta)}{R^{2r+2}} \tag{5.66}$$

However it is the region $|R| < 1$ in which we are essentially interested, because for outside of this region Saint Venant's principle can be employed, permitting the loading to be taken as concentrated.

For $|R| < 1$ $f(t)$ being an even function can be expressed in form (5.55) since

$$f(t) = p_o\, r^2 = p_o \left[1 - (1-r^2)\right]$$

Hence the solution can be written down from (5.59), noting that n takes only two values o and 1. In the present case, it is easier to use superposition of $[A]$ and $[B]$

Hence

$$\widehat{zz} = 2\,\widehat{zz}_{\text{distrib.}} - \widehat{zz}_{\text{parabolic.}} \tag{5.67}$$

E Hollow Column

With $f(r)$ defined as in (5.231)

$$-\int_0^1 t^{2r+1} f(t)\,dt = \underset{\epsilon\to 0}{Lt} \int_{1-\epsilon}^{1} t^{2r+1} \frac{P}{2\pi\epsilon}\,dt = P/2\pi$$

Hence from equation (5.56) for $|R| > 1$ we obtain

$$^{z}\mathcal{I}_s = \frac{P}{2\pi} \sum_{r=0}^{\infty} \frac{(-)^r \,\lfloor 2r+1}{2^{2r}(\lfloor r)^2} \frac{P_{2r+1}(\cos\theta)}{R^{2r+2}} \tag{5.68}$$

And for $|R| < 1$ from (5.01)

$$U_m = \frac{P}{2\pi} J_o(m)$$

Therefore

$$^{z}\mathcal{I}_s = \frac{P}{2\pi} \int_0^{\infty} e^{-smz}\, m\, J_o(m)\, J_o(mr)\, dm \tag{5.69}$$

This can be expressed in closed form by Watson[18] (13·22)

since

$$\int_0^\infty e^{-smz} J_o(m) J_o(mr)\,dm = \frac{1}{\pi\sqrt{r}}\, Q_{-\frac{1}{2}}\left(\frac{s^2z^2+r^2+1}{2r}\right)$$

Hence differentiating partially w.r.t. (sz)

$$ {}^{z}I_{S} = -\frac{sz}{\pi r^{3/2}}\, Q'_{-\frac{1}{2}}\left(\frac{R^2+1}{2r}\right)$$

$$= \frac{sz}{\pi r^3}\left[\frac{E\left(\frac{R^2+1}{2r}\right)}{\sqrt{2\left\{\frac{R^2-1}{2r}-1\right\}}}\right]^{S} \qquad (\text{Magnus}^{(20)})$$

∴ As in (5.62)

$$\widehat{zz} = -\frac{s_{/0}\, z}{r^3}\left[\frac{E\left(\frac{R^2+1}{2r}\right)}{\sqrt{2\left\{\frac{R^2+1}{2r}-1\right\}}}\right]_{S_2}^{S_1} \qquad (5.71)$$

where $R^2 = r^2 + s^2z^2$. Note this expression does not hold when $r = 0$.

CHAPTER VI.

ELASTIC ISOTROPIC CASE

In the elastic isotropic case $E_1 = E_3 \equiv E$, $\sigma_1 = \sigma_2 = \sigma_3 \equiv \sigma$, and the constants L, N, A, C, F are given by equations (1.34) and (1.35). The roots of the characteristic equation are $s_1 = s_2 = 1$. However for these values all the results become indeterminate and so must be evaluated by a limiting procedure. We could take $s_1 = 1 + \delta_1$, $s_2 = 1 + \delta_2$ and take limits as $\delta_1 \to 0$, and $\delta_2 \to 0$. However, noting that all results are determinate for $s_1 = 1$, $s_2 \neq 1$, we can approach the elastic isotropic case more easily by taking $s_1 = 1$ and $s_2 = 1 + \delta$ where $\delta \to 0$.

Limits Required

For the constants defined in Appendix G, when $s_1 = 1$ and $s_2 = 1 + \delta$, to the first order in δ we obtain

$$s_2 = s_1 + \delta \,;\quad h_1 = \frac{A - L}{G} = \frac{\lambda + \mu}{\lambda + \mu} = 1 \tag{6.01}$$

$$h_2 = \frac{A - Ls_2^2}{Gs_2} \longrightarrow 1 + \delta \frac{\partial}{\partial s_2}\left[\frac{A - Ls_2^2}{Gs_2}\right]_{s_2 = 1}$$

$$= 1 - \delta\left(\frac{A+L}{G}\right) = 1 - \delta\left(\frac{\lambda + 3\mu}{\lambda + \mu}\right)$$

$$s_1 + h_1 = 2$$

$$s_2 + h_2 \longrightarrow 2 - \delta \frac{2\mu}{\lambda + \mu} \quad \therefore \quad \Delta(s_2 + h_2) = -\delta \frac{2\mu}{\lambda + \mu} \tag{6.02}$$

$$s_3 = (F - C)(s_2 + h_2) \longrightarrow -2\mu\left[2 - \delta\frac{2\mu}{\lambda + \mu}\right]$$

$$h_2 s_2 \longrightarrow 1 - \delta \frac{2\mu}{\lambda + \mu}$$

$$s_4 = 2\left[F - C\, h_2 s_2\right] \longrightarrow 2\left[-2\mu + \delta\, \frac{2(\lambda + 2\mu)\mu}{\lambda + \mu}\right]$$

$$\therefore \quad s_4 - s_3 \rightarrow \delta(4\mu) \equiv \Delta s_3 \tag{6.03}$$

$$s_5 = (A - F)(s_2+h_2) \rightarrow (\lambda+\mu)\left[\,2 - \delta\frac{2\mu}{\lambda+\mu}\right]$$

$$s_6 = 2\left[A - F\; h_2 s_2\right] \rightarrow 2\left[\lambda + \mu + \delta\frac{2\lambda\mu}{\lambda+\mu}\right]$$

$$\therefore \quad s_6 - s_5 \rightarrow \delta\frac{2\mu(3\lambda+\mu)}{\lambda+\mu} \equiv \Delta s_5 \tag{6.04}$$

$$s_7 = s_2+h_2 \rightarrow 2 - \delta\frac{2\mu}{\lambda+\mu}$$

$$s_8 = 2\,h_2 \rightarrow 2 - \delta\, 2\frac{(\lambda+3\mu)}{\lambda+\mu}$$

$$\therefore \quad s_8 - s_7 \rightarrow -\delta\frac{2(\lambda+2\mu)}{\lambda+\mu} \equiv \Delta s_7 \tag{6.05}$$

$$s_9 \rightarrow \frac{2}{\pi}\left[\frac{2-\delta\frac{2\mu}{\lambda+\mu}}{-4\mu\delta}\right] \rightarrow -\frac{1}{\pi\mu\delta} \tag{6.06}$$

$$s_{10} = \frac{s_4 s_2}{\pi(s_3-s_4)} \rightarrow \frac{1}{\pi\delta} \tag{6.07}$$

$$s_{11} = -2 \text{ (Limit only required)} \tag{6.08}$$

$$s_{12} = \frac{2}{s_2+h_2} \rightarrow 1 + \delta\frac{\mu}{\lambda+\mu} \equiv 1 + \Delta s_{12} \tag{6.09}$$

$$s_{13} = -\frac{s_3-s_4}{s_7-s_8} = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \tag{6.090}$$

$$\underset{s_2\to 1}{Lt}\,\Delta\left(\frac{1}{R_2}\right) = \Delta s_2 \frac{\partial}{\partial s_2}\left(\frac{1}{R_2}\right)_{s_2=1} = -\Delta s_2 \frac{z^2}{R^3} \rightarrow -\delta\frac{z^2}{R^3} \tag{6.091}$$

To preserve symmetry in calculating limits required, it is desirable to set $\delta \equiv \Delta s_1$, and so $s_2 = s_1 + \Delta s_1$ where $s_1 = 1$

<u>Two Dimensional Elastic Isotropic Case</u>

For the loading treated in Chapter II, we have from (2.36)

$$m\,R_m = s_9\,T_m$$

where

$$T_m = \int_0^\infty f(x) \cos mx \, dx. \tag{6.10}$$

Hence from (2.29) with $s_1 = 1$ $s_2 = 1 + \delta$, and noting that

$\dfrac{s_9}{s_2+h_2} = \dfrac{2}{\pi(s_3-s_4)}$, we obtain

$$\widehat{zz} = \underset{\delta\to 0}{Lt} \frac{2}{\pi} \int_0^\infty T_m \left[\frac{s_3e^{-mz} - s_4e^{-s_2mz}}{s_3 - s_4} \right] \cos mx \, dm \tag{6.11}$$

This last integral is uniformly convergent and a continuous function of m for T_m continuous and $z > 0$, and so the operations of proceeding to the limit and of integrating may be intercharged provided the resulting integral is convergent. This is always the case for $z > 0$, since the resulting integrals are all majorized by the factor e^{-mz}

And $$\underset{\delta\to 0}{Lt} \frac{s_3e^{-mz} - s_4e^{-s_2mz}}{s_3 - s_4} = \underset{\substack{\delta\to 0\\ \Delta s_3\to 0}}{Lt} \frac{s_3e^{-mz} - (s_3+\Delta s_3)e^{-(1+\Delta s_1)mz}}{-\Delta s_3}$$

$$= \underset{\Delta s_1\to 0}{Lt} \frac{\Delta s_3e^{-mz} - \Delta s_1 \, mz \, s_3e^{-mz}}{\Delta s_3}$$

$$= e^{-mz}\,(1+mz) \tag{6.12}$$

since $s_3 \dfrac{\Delta s_1}{\Delta s_3} = \dfrac{\delta(-4\mu)}{\delta(4\mu)} = -1$ from (6.03) .

Hence from (6.11)

$$\widehat{zz} = \frac{2}{\pi} \int_0^\infty (1 + mz)\, e^{-mz}\, T_m \cos mx \, dm$$

Similarly it can be shown that

$$\widehat{xz} = \frac{2}{\pi} \int_0^\infty mz\, e^{-mz}\, T_m \sin mx \, dm$$

$$\widehat{xx} = \frac{2}{\pi} \int_0^\infty (1 - mz)\, e^{-mz}\, T_m \cos mx \, dm \tag{6.13}$$

Also from (2.32)

$$u = \underset{\delta \to 0}{Lt}\ \frac{2}{\pi} \int_0^\infty \frac{T_m}{m} \left[\frac{(s_2+h_2)e^{-mz} - (s_1+h_1)e^{-s_2 mz}}{s_3 - s_4} \right] \sin mx \, dm$$

$$= \frac{2}{\pi} \int_0^\infty \frac{T_m}{m} \underset{\delta \to 0}{Lt} \left[\frac{\Delta(s_2+h_2)e^{-mz} + 2mz \, \Delta s_2 \, e^{-mz}}{-\Delta s_3} \right] \sin mx \, dm \qquad (6.14)$$

$$= \frac{1}{\pi} \int_0^\infty \left[\frac{1}{\lambda+\mu} \cdot \frac{1}{m} - \frac{z}{\mu} \right] e^{-mz} \, T_m \sin mx \, dm$$

Similarly

$$w = -\frac{1}{\pi} \int_0^\infty \left[\frac{\lambda+2\mu}{\mu(\lambda+\mu)} \frac{1}{m} + \frac{z}{\mu} \right] e^{-mz} \, T_m \cos mx \, dm$$

T_m is finite for an applied load that is finite per unit length of wall and if applied load has a non-zero resultant, then $T_m \neq 0$ when $m = 0$. Hence all integrals in (6.13) and (6.14) are uniformly convergent $z > 0$ excepting the integral for w. The remarks on the w integral in the aeolotropic case, apply here with equal force, and so the integral may be used to obtain relative deflections at short distances from the origin. This same difficulty is encountered in the treatment of the above problem by the method of singularities (see Love's[(3)] "Theory of Elasticity", Page 211). Later in this chapter the infinity is removed by placing an equilibrating load at a great distance from the origin to secure zero resultant on the plane $z = 0$.

Concentrated Line Load

The stresses and displacements due to a concentrated line load may be obtained directly from equations (6.13) and (6.14). However, for the purpose of the present thesis, it serves as a check on results (2.44) - (2.49), to deduce the corresponding elastic isotropic results by a limiting procedure.

Stresses

From (2.44) on using result (6.07) ne obtain

$$\widehat{zz} = \underset{\delta\to 0}{Lt} - Pz \frac{L}{\pi\Delta s_1}\left[\frac{1}{r_1^2} - \frac{1}{r_1^2} - \Delta s_1 \frac{\partial}{\partial s_1}\left(\frac{1}{r_2^2}\right)\right]$$

$$\therefore \quad \widehat{zz} = \frac{2P}{\pi}\frac{z^3}{r^4} \tag{6.15}$$

Similarly

$$\widehat{xz} = -\frac{2P}{\pi}\frac{x\,z^2}{r^4} \quad ; \quad \widehat{xx} = -\frac{2P}{\pi}\frac{x^2 z}{r^4}$$

Displacements

In (2.49) place constant = 0, as we only need relative values. On using results (6.06) and (6.09) we easily obtain

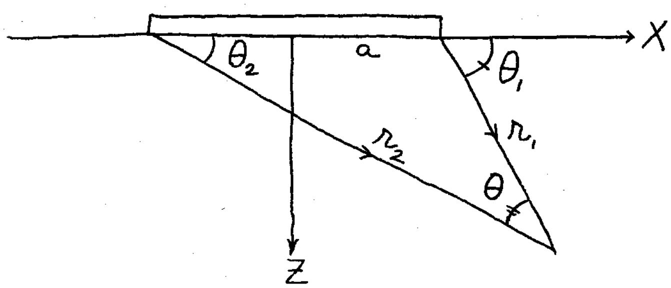
$$\begin{aligned} w &\rightarrow \frac{P}{2\pi\mu}\underset{\delta\to 0}{Lt}\frac{1}{\Delta s_1}\left[\Delta\left(h_2\, s_{12} \log r_2\right)\right. \\ &\rightarrow \frac{P}{2\pi\mu}\underset{\delta\to 0}{Lt}\frac{1}{\Delta s_1}\left[\log r\left\{\Delta h_2 + \Delta s_{12}\right\} + \Delta\left\{\log r_2\right\}\right] \\ &\rightarrow -\frac{P(\lambda+2\mu)}{2\pi\mu(\lambda+\mu)} + \frac{P}{2\pi\mu}\frac{z^2}{r^2} \end{aligned} \tag{6.16}$$

Similarly

$$\begin{aligned} u &= \frac{P}{2\pi\mu}\underset{\delta\to 0}{Lt}\frac{1}{\Delta s_1}\left[\frac{\pi}{2}\Delta s_{12} - \Delta(s_{12}\theta_2)\right] \\ &= \frac{P}{2\pi\mu}\underset{\delta\to 0}{Lt}\frac{1}{\Delta s_1}\left[\left(\frac{\pi}{2} - \theta\right)\Delta s_{12} - \Delta\theta_2\right] \\ &= \frac{P}{2\pi(\lambda+\mu)}(\theta - \pi/2) + \frac{P}{2\pi\mu}\frac{xz}{r^2} \end{aligned} \tag{6.17}$$

All above results check with those given by Love,[3] Page 211 "Theory of Elasticity" except (6.16). This latter agrees with Loves's expression if we take the constant in the problem = - 1. Hence the above example provides a very fine check on the accuracy of the resultsn(2.44) - (2.49)

<u>Evaluation of Uniform Load Case Directly From Integrals</u>



Consider a load w_o per unit area distributed over $|x| \leq a$

$$\therefore \quad f(x) = -w_o \quad , \qquad |x| \leq a$$
$$= 0 \quad , \qquad |x| > a$$

And from (6.10)

$$T_m = \int_0^\infty f(x)\cos mx\,dx = -w_o\int_0^a \cos mx\,dx$$
$$= w_o \frac{\sin ma}{m}$$

Substituting in equations (6.13) and (6.14) we obtain

$$\widehat{xx} = \frac{2w_o}{\pi}\int_0^\infty (-1+mz)\,e^{-mz}\,\frac{\sin ma}{m}\cos mx\,dm$$
$$\widehat{xz} = -\frac{2w_o z}{\pi}\int_0^\infty e^{-mz}\sin ma\,\sin mx\,dm \qquad (6.18)$$
$$\widehat{zz} = -\frac{2w_o}{\pi}\int_0^\infty (1+mz)\,e^{-mz}\,\frac{\sin ma}{m}\cos mx\,dm$$
$$u = \frac{w_o}{\pi}\int_0^\infty \left[\frac{z}{\mu} - \frac{1}{\lambda+\mu}\,\frac{1}{m}\right] e^{-mz}\,\frac{\sin ma}{m}\sin mx\,dm$$

and (6.19)

$$w = \frac{w_0}{\pi}\int_0^\infty \left[\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\cdot\frac{1}{m} + \frac{z}{\mu}\right] e^{-mz}\,\frac{\sin ma}{m}\,\cos mx\,dm$$

As discussed previously all the above integrals are convergent except the integral for w. In this latter case relative deflections can be obtained by bounding m away from zero. The evaluations are easily performed by means of the integrals obtained in Appendix A.

Evaluations

$$\widehat{xx} = \frac{w_0}{\pi}\left[z I_3 - I_3^{-1}\right] = \frac{w}{\pi}\left[\frac{z(x+a)}{r^2} - \frac{z(x-a)}{r_1^2} - \theta_1 + \theta_2\right]$$

$$\widehat{xz} = -\frac{w_0 z}{\pi} I_4 = -\frac{w z^2}{\pi}\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] \qquad (6.20)$$

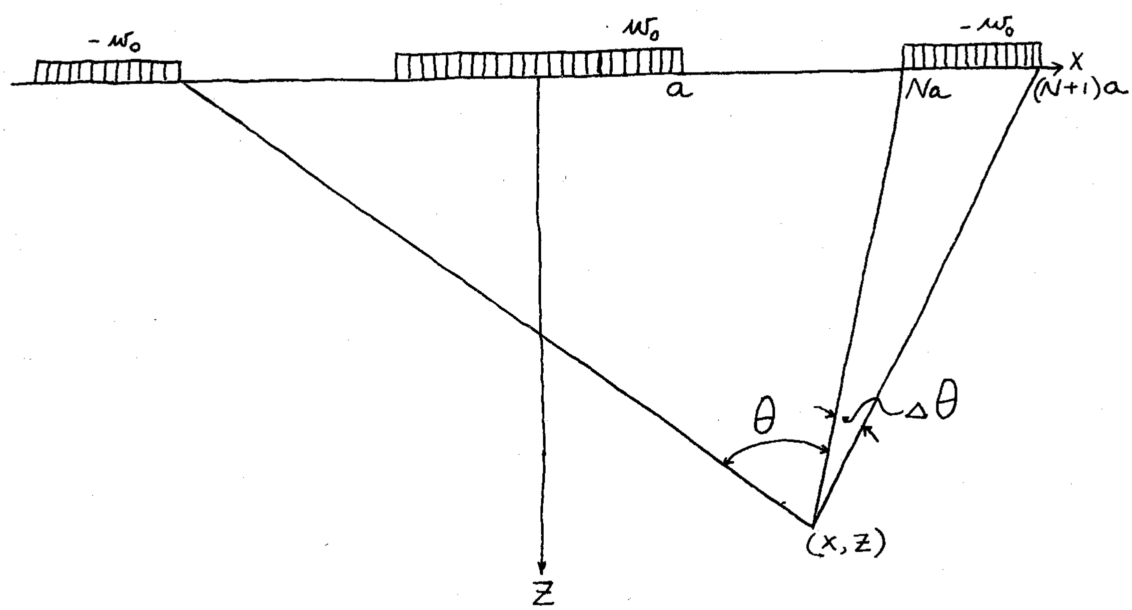
$$\widehat{zz} = \frac{w_0}{\pi}\left[z I_3 + I_3^{-1}\right] = \frac{w}{\pi}\left[\frac{z(x+a)}{r_2^2} - \frac{z(x-a)}{r_1^2} + \theta_1 - \theta_2\right]$$

$$u = \frac{w_0}{2\pi}\left[\frac{z}{\mu} I_4^{-1} - \frac{1}{\lambda+\mu} I_4^{-2}\right] = \frac{w}{2\pi}\left[\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\, z \log\left(\frac{r_2}{r_1}\right) - \frac{x}{\lambda+\mu}(\theta_1 - \theta_2)\right] \qquad (6.21)$$

$$w = \frac{w_0}{2\pi}\left[\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\,\delta I_3^{-2} + \frac{z}{\mu} I_3^{-1}\right] = \frac{w}{2\pi}\left[-\frac{z}{\mu}(\theta_1 - \theta_2) + \frac{\lambda+2\mu}{\mu(\lambda+\mu)}\left(-x\log\frac{r_2}{r_1} + a\log r_1 r_2\right)\right] + \text{const.}$$

Equilibrating Load at Great Distance from Origin

Integral (6.14) for w is finite at m = 0 if $\int_0^\infty f(x)\,dx = 0$ i.e. if the applied loading on z = 0 has zero resultant. One way in which this result can be produced is by placing uniform strip loads of intensity -w as shows in following sketch.



Accordingly

$$f(x) = -w_o \quad , \qquad |x| \leq a$$
$$= w_o \quad , \qquad Na \geqslant |x| \geqslant (N+1)a$$
$$= 0 \quad , \qquad \text{for all other values of } x$$

and from (6.10)

$$T_m = \int_0^\infty f(x) \cos mx \, dx = - w_o \left[\frac{\sin ma}{m} + \frac{\sin Nam}{m} - \frac{\sin(N+1)ma}{m} \right] \qquad (6.22)$$

Hence the corresponding stresses and displacements can be written down from results (6.21)

Influence of Equilibrating Loading on Stresses and Displacements Near Origin

Stresses

Consider a region in the neighbourhood of the loaded area, and take

Na large compared with either x or z. Then from (6.20) we obtain as the contribution to $\widehat{xx}$ say $\widehat{xx}_1$ made by the terms $\left[\frac{\sin Nam}{m} - \frac{\sin(N+1)am}{m}\right]$

is (6.22):

$$\widehat{xx}_1 = \frac{w}{\pi}\left\{\left[\frac{z(x+Na')}{z^2+(x+Na')^2} - \frac{z(x+\overline{N+1}a')}{z^2+(x+\overline{N+1}a')^2}\right]_{a'=-a}^{a'=a} + \theta\Big|_{Na}^{\overline{N+1}\,a}\right\}$$

From above diagram, we easily see that

$$\theta\Big|_{Na}^{\overline{N+1}\,a} = 2\Delta\theta$$

is of the order $\frac{1}{N}$ when Na is large compared with either x or z

Hence

$\widehat{xx}_1$ is at least of order $\frac{1}{N}$ i.e. $O(\frac{1}{N})$

Similarly we can show

$\widehat{zz}_1$, $\widehat{xz}_1$ are also $O(\frac{1}{N})$

Hence the contributions made to the stresses in the region near the origin by the equilibrating forces on z = 0 may be neglected.

Displacements

u: From (6.21)

$$u_1 = \frac{w}{2\pi}\left\{\left[\frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\,\frac{z}{2}\log\frac{z^2+(x+Na')^2}{z^2+(x+\overline{N+1}a')^2}\right]_{a'=-a}^{a'=a} + \frac{x}{\lambda+\mu}\,\theta\Big|_{Na}^{\overline{N+1}\,a}\right\}$$

We have already shown

$$\theta\Big|_{Na}^{\overline{N+1}\,a} = 2\Delta\theta = O(\tfrac{1}{N})$$

and $\log\frac{z^2+(x+Na)^2}{z^2+(x+\overline{N+1}a)^2} = \log\left[1 - \frac{a(2x+2\overline{N+1}\,a)}{z^2+(x+\overline{N+1}\,a)^2}\right]$

$$\longrightarrow -a\,\frac{(2x+2\overline{N+1}\,a)}{z^2+(x+\overline{N+1}\,a)^2} \longrightarrow O(\tfrac{1}{N})$$

when Na is large with respect to either x or z. Therefore $u_1 = O(\frac{1}{N})$ and so contribution to u may be neglected.

<u>w</u>. The integral is now uniformly convergent, and can be evaluated from Appendix A and result (6.21) as follows:

$$w = \underset{\delta\to 0}{Lt}\left\{\frac{\lambda+2\mu}{\mu(\lambda+\mu)}\left[{}_{\delta}^{a}I_{3}^{-2} + {}_{\delta}^{Na}I_{3}^{-2} - {}_{\delta}^{\overline{N+1}a}I_{3}^{-2}\right] + \frac{z}{\mu}\left[{}^{a}I_{3}^{-1} + {}^{Na}I_{3}^{-1} - {}^{\overline{N+1}a}I_{3}^{-1}\right]\right\}$$

The above limit now exists since the integral for w is uniformly convergent. The equilibrating loading superposes an infinite displacement to neutralize the infinite displacement due to the loading on $|x| \leqslant a$. Hence a finite displaced is produced at the origin. Moreover as in previous examples the finite contribution of the equilibrating loading to the displacement in the neighbourhood of the origin is $O(\frac{1}{N})$ and so can be neglected. Thus an analytical explanation is obtained for our procedure both in the aeolotropic and in the isotropic case of bounding the variable of integration m away from zero i.e. $m > \delta > 0$.

<u>Results From Loaded Strip</u>

As noted in the aeolotropic case the displacements depend only on the settlement constant s_{13}. Hence the surface settlements in both the aeolotropic and the isotropic cases are similar in form, and have the magnitude ratio

$$\text{aeolotropic} : \text{isotropic} = \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} : s'_{13} \tag{6.23}$$

This may account for the observed fact that the actual deflections are less than those calculated by the isotropic theory. This is the case if

$$s_{13} > \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}$$

However the axial pressures, and the axial shear stress patterns are dissimilar in the two cases. This dissimilarity could also affect the surface displacements considerably in the case of a multi-layered soil of dissimilar materials. The above remarks apply to an aeolotropic axially symmetric soil medium under all conditions of loading. It may be useful to calculate the normal vertical stresses for a loaded isotropic strip by a limiting procedure.

Loaded Isotropic Strip - Normal Vertical Stresses Along Axis

From equations (3.52) - (3.58) on applying the limiting procedure developed in this chapter we obtain

[A] Concentrated Load

$$\widehat{zz}\Big|_{o,z} = + \underset{\delta \to 0}{Lt} \frac{P}{\pi \Delta s_1 z} \cdot \frac{\partial}{\partial s}\left(\frac{1}{s^2}\right)\Big|_{s=1} \Delta s_1 = - \frac{2P}{\pi z} \tag{6.24}$$

[B] Uniform Load

$$\widehat{zz}\Big|_{o,z} = + \frac{P}{\pi a} \underset{\delta \to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[\frac{1}{s} \tan^{-1} \frac{a}{sz}\right]\Bigg|_{s=1} \Delta s_1 \tag{6.25}$$

$$= - \frac{P}{\pi a}\left[+ \tan^{-1} \frac{a}{z} + \frac{a\,z}{a^2 + z^2}\right]$$

[C] Parabolic Load

$$\widehat{zz}\Big|_{o,z} = \frac{3P}{2\pi a^3} \underset{\delta \to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[\frac{a^2+s^2z^2}{s} \tan^{-1} \frac{a}{sz}\right]_{s=1} \Delta s_1 \tag{6.26}$$

$$= - \frac{3P}{2\pi a^3}\left[(a^2-z^2) \tan^{-1} \frac{a}{z} + az\right]$$

[D] Inverted Parabola

$$\widehat{zz}\Big|_{o,z} = -\frac{3\,Pz^2}{\pi\, a^3}\ \underset{\delta\to 0}{Lt}\ \frac{1}{\Delta s_1}\frac{\partial}{\partial s}\left[s\tan^{-1}\frac{a}{sz}\right]_{s=1}\Delta s_1 \tag{6.27}$$

$$= -\frac{3\,Pz^2}{\pi a^3}\left[\tan^{-1}\frac{a}{z} - \frac{az}{a^2+z^2}\right]$$

[E] HOLLOW WALL

$$\widehat{zz}\Big|_{o,z} = \frac{P\ z}{\pi}\ \underset{\delta\to 0}{Lt}\ \frac{1}{\Delta s_1}\frac{\partial}{\partial s}\left[\frac{1}{a^2+s^2z^2}\right]_{s=1}\Delta s_1 \tag{6.28}$$

$$= -\frac{2\,Pz^3}{\pi(a^2+z^2)^2}$$

[F] Rigid Wall

$$\widehat{zz}\Big|_{o,z} = \frac{P}{\pi}\ \underset{\delta\to 0}{Lt}\ \frac{1}{\Delta s_1}\frac{\partial}{\partial s}\left[\frac{1}{s\sqrt{a^2+s^2z^2}}\right]_{s=1}\Delta s_1 \tag{6.29}$$

$$= -\frac{P}{\pi}\left[\frac{1}{\sqrt{a^2+z^2}} + \frac{z^2}{(a^2+z^2)^{3/2}}\right]$$

[B] Three Dimensional - Axially Symmetric

For the loading treated in Chapter IV , and

$$R_m = \pi/2\ s_9\ U_m = \frac{s_2+h_2}{s_3-s_4}\ U_m \tag{6.30}$$

where
$$U_m = \int_0^\infty f(t)\ J_o(tm)\ t\ dt.$$

we have from (4.22)

$$\widehat{zz} = \underset{\delta\to 0}{Lt}\int_0^\infty m\ U_m\left[\frac{s_3e^{-mz} - s_4e^{-s_2mz}}{s_3 - s_4}\right] J_o(mr)\ dm$$

$$= \int_0^\infty m(1+mz)\ e^{-mz}\ U_m\ J_o(mr)\ dm \tag{6.31}$$

on using result (6.12)

Similarly it can be shown that

$$\widehat{rz} = 2L \int_0^\infty m^2 z\, e^{-mz}\, U_m\, J_1\,(mr)\, dm \tag{6.32}$$

and

$$u = 2 \underset{\delta\to 0}{Lt} \int_0^\infty U_m \left[\frac{e^{-mz} - s\quad e^{-s_2 mz}}{s_3 - s_4}\right] J_1\,(mr)\, dm$$

$$= 2 \int_0^\infty U_m\, J_1\,(mr) \underset{\delta\to 0}{Lt} \left[\frac{mz\,\Delta s_1 - \Delta s_{12}}{-\Delta s_3}\right] e^{-mz}\, dm$$

$$= \frac{1}{2} \int_0^\infty \left[\frac{1}{\lambda+\mu} - \frac{mz}{\mu}\right] e^{-mz}\, U_m\, J_1\,(mr)\, dm \tag{6.33}$$

Similarly

$$w = \int_0^\infty U_m \underset{\delta\to 0}{Lt} \left[\frac{s_7 e^{-mz} - s_8 e^{-s_2 mz}}{s_3 - s_4}\right] J_o\,(mr)\, dm$$

$$= \int_0^\infty U_m \underset{\delta\to 0}{Lt} \left[\frac{s_8 mz\,\Delta s_1\; e^{-mz} - \Delta s_7 e^{-mz}}{-\Delta s_3}\right] J_o(mr)\, dm$$

$$= -\frac{1}{2}\int_0^\infty U_m \left[\frac{\lambda + 2\mu}{\mu(\lambda+\mu)} + \frac{mz}{\mu}\right] e^{-mz}\; J_o(mr)\, dm \tag{6.34}$$

and finally from (4.22)

$$\widehat{rr} = \frac{1}{2}\int_0^\infty m\, U_m \underset{\delta\to 0}{Lt} \left[\frac{s_5 e^{-mz} - s_6 e^{-s_2 mz}}{s_3 - s_4}\right] J_o(mr)\, dm - 2\mu \frac{u}{r}$$

Since

$$\underset{\delta\to 0}{Lt} \left[\frac{s_5 e^{-mz} - s_6 e^{-s_2 mz}}{s_3 - s_4}\right] = \underset{\delta\to 0}{Lt} \frac{(s_6 mz\Delta s_1 - \Delta s_5)}{-\Delta s_3}\, e^{-mz}$$

$$= \left[-\frac{\lambda+\mu}{2\mu}\; mz + \frac{3\lambda+2\mu}{2(\lambda+\mu)}\right] e^{-mz}$$

$$\widehat{rr} = \frac{1}{2}\int_0^\infty m\, U_m \left[\frac{3\lambda+\mu}{\lambda+\mu} - \frac{\lambda+\mu}{\mu}\, mz\right] e^{-mz}\, J_o(mr)\, dm - 2\mu\frac{u}{r} \tag{6.35}$$

Above results check with these given in a little known paper by Lamb(23)

Surface Deflection (w_s)

Clearly from (6.34)

$$w_s = -\frac{\lambda+2\mu}{2\mu(\lambda+\mu)} \underset{z\to 0}{Lt} \int_0^\infty U_m e^{-mz} J_0(mr)\ dm \tag{6.36}$$

and the Axial Pressure p is obtained from (6.31)

$$p = -\ \widehat{zz}\Big|_{r=0} = \int_0^\infty m\ U_m\ (1+mz)e^{-mz}\ dm \tag{6.37}$$

Axial Shear Stress

From (4.273) $\quad s_4 \to s_3 + \Delta s_3 \qquad s_6 \to s_5 + \Delta s_5$

$$s_2 \to 1 + \Delta s_1$$

$$\tau_A = \frac{1}{2} \underset{\delta\to 0}{Lt} \int_0^\infty m\ U_m \left[\frac{\Delta\{(s_3+s_5)e^{-s_2 mz}\}}{\Delta s_3} \right] dm\ .$$

and $$\underset{\delta\to 0}{Lt}\ \frac{\Delta\{(s_3+s_5)e^{-s_2 mz}\}}{\Delta s_3} = \underset{\delta\to 0}{Lt}\ e^{-mz} \left\{ \frac{\Delta(s_3+s_5) - mz(s_3+s_5)\Delta s_1}{4\mu\ \Delta s_1} \right\}$$

$$= -\frac{e^{-mz}}{2}\left[+\frac{\lambda-\mu}{\lambda+\mu} + \frac{\lambda-\mu}{\mu}\ mz \right]$$

Hence

$$\tau_A = \frac{\lambda-\mu}{4} \int_0^\infty m\ U_m \left[(\lambda-\mu) + (\lambda+\mu)mz \right] e^{-mz}\ dm. \tag{6.38}$$

Concentrated Load

The Bousinesque solution for the case of a load P acting perpendicular to the plane boundary of a semi-infinite body can easily be obtained directly from equations (6.30) - (6.35).

However, as a check on corresponding results for an aeolotropic body, we shall now obtain the isotropic elastic case by a limiting procedure from equations (4.311) - (4.315)

From (4.311)

$$\widehat{zz} = \frac{Pz}{2\pi} \underset{\delta \to 0}{Lt} \frac{1}{\Delta s_1} \left[\frac{1}{R^3} - \frac{1}{R^3} - \Delta s_1 \frac{\partial}{\partial s_2} \left(\frac{1}{R_2^3} \right)_{s_2=1} \right] \qquad (6.39)$$

$$= - \frac{3 P z^3}{2\pi R^5}$$

This is the well known Bousinesque result.

Similarly

$$\widehat{rz} = \frac{3 P r z^2}{2\pi R^5} \qquad (6.40)$$

Also from (4.314)

$$w \to \frac{P}{4\pi\mu} \underset{\delta \to 0}{Lt} - \frac{1}{\Delta s_1} \Delta \left(\frac{h_2 s_{12}}{R_2} \right)$$

$$\therefore \quad w \to - \frac{P}{4\pi\mu} \underset{\delta \to 0}{Lt} \frac{1}{\Delta s_1} \left[\frac{1}{R} \Delta s_{12} + \frac{1}{R} \Delta h_2 + \Delta \left(\frac{1}{R_2} \right) \right]$$

$$\to - \frac{P}{4\pi\mu} \left[\frac{1}{R} \left(\frac{\mu}{\lambda+\mu} - \frac{\lambda+3\mu}{\lambda+\mu} \right) - \frac{z^2}{R^3} \right]$$

$$\therefore \quad w = \frac{P}{4\pi\mu} \frac{z^2}{R^3} + \frac{P(\lambda+2\mu)}{4\pi\mu(\lambda+\mu)} \cdot \frac{1}{R} \qquad (6.40)$$

on using results (6.091), (6.01, (6.09).

In a like manner we can show that

$$u = \frac{P \sin\theta}{4\pi\mu R} \left[\cos\theta - \frac{\mu}{\lambda+\mu} \frac{1}{1+\cos\theta} \right] \qquad (6.41)$$

where $\cos\theta = z/R$ $\sin\theta = r/R$

All above results check those given by Love[3] Page 191.

<u>Loaded Isotropic Circular Area - Normal Vertical Stress Along Axis</u>

From equations (5.47)-(5.51) on applying a limiting procedure we obtain:

A Concentrated Load

$$\widehat{zz}\Big|_{o,z} = \frac{P}{2\pi z^2} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left(\frac{1}{s^3}\right)_{s=1} \Delta s_1, \qquad (6.42)$$

$$= -\frac{3P}{2\pi z^2}$$

B Uniform Load

$$\widehat{zz}\Big|_{o,z} = \frac{P}{\pi a^2} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[\frac{1}{s} - \frac{z}{\sqrt{s^2z^2+a^2}}\right]_{s=1} \Delta s_1 \qquad (6.43)$$

$$= -\frac{P}{\pi a^2}\left[1 - \frac{z^3}{(a^2+z^2)^{3/2}}\right]$$

C Parabolic Load

$$\widehat{zz}\Big|_{o,z} = \frac{2Pz}{\pi a^4} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[2sz + \frac{a^2}{sz} - 2\sqrt{s^2z^2+a^2}\right]_{s=1} \Delta s_1$$

$$= -\frac{2Pz}{\pi a^4}\left[-2z + \frac{a^2}{z^2} + \frac{2z^2}{\sqrt{a^2+z^2}}\right] \qquad (6.44)$$

D Inverted Parabola

$$\widehat{zz}\Big|_{o,z} = \frac{2Pz}{\pi a^4} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[2\sqrt{s^2z^2+a^2} - \frac{a^2}{\sqrt{s^2z^2+a^2}} - 2sz\right]_{s=1} \Delta s_1$$

$$= -\frac{2Pz^2}{\pi a^4}\left[2 \cdot - \frac{a^2 z}{(a^2+z^2)^{3/2}} - \frac{2z}{\sqrt{a^2+z^2}}\right] \qquad (6.45)$$

E Hollow Column

$$\widehat{zz}\Big|_{o,z} = -\frac{P\ z}{2\pi} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[(s^2z^2+a^2)^{-3/2}\right]_{s=1} \Delta s_1$$

$$= -\frac{3\ P\ z^3}{2\pi(a^2+z^2)^{5/2}} \qquad (6.46)$$

F Rigid Disc

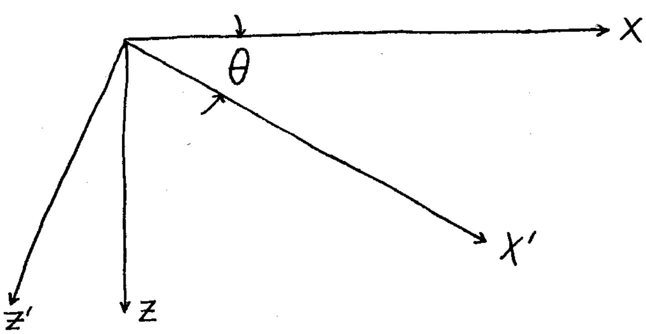
$$\widehat{zz}\Big|_{0,z} = \frac{P}{2\pi} \underset{\delta\to 0}{Lt} \frac{1}{\Delta s_1} \frac{\partial}{\partial s}\left[\frac{1}{s(s^2z^2+a^2)}\right]_{s=1} \Delta s_1 \tag{6.47}$$

$$= -\frac{P}{2\pi}\left[\frac{1}{a^2+z^2} + \frac{2z^2}{(a^2+z^2)^2}\right]$$

CHAPTER VII.

ANALYSIS OF THREE CONSTANT MEDIUMS

Beginning with Wolf[1] (1936) several papers have appeared attempting to simplify for use in soils, in woods, and in crystals the two dimensional axially-symmetric aeolotropic theory. To evaluate the physical meanings of the assumptions on which these simplifications are based, we need expressions for E and μ associated with any arbitrary directions in the medium.



Consider a two dimensional medium, with the orthogonal pair of axes ox', oz' making an angle θ with ox and oz respectively.

Expression for E_θ

Consider pure tension along the arbitrary direction OX'. Referred to the axes ox', oz' the stress condition is then specified by

$$\widehat{x'x'} = T \quad \text{(a constant)} \tag{7.1}$$

$$\widehat{z'z'} = 0 \qquad \widehat{x'z'} = 0$$

and the strain along ox' is specified by $e_{x'x'}$. The above stress system may be referred to the axes ox, oy by the well-known equations[3] for transformation of stress.

These give

$$\widehat{xx} = \ell^2 T \qquad \widehat{zz} = n^2 T \qquad \widehat{xz} = \ell n T \tag{7.11}$$

where ox' has the direction cosines $(\ell, n) = (\cos\theta, \sin\theta)$. Also the equations for transformation of strain give

$$e_{x'x'} = \ell^2 e_{xx} + n^2 e_{zz} + \ell n \, e_{xz} \tag{7.12}$$

The relations between strain and stress for two dimensional plane stress, or generalized plane stress are given by (1.13) with $\widehat{yy} = 0$. These may be presented in a notation more suitable to the present needs in a notation introduced by Voigt.

$$\begin{aligned} e_{xx} &= s_{11}\widehat{xx} + s_{13}\widehat{zz} \\ e_{zz} &= s_{13}\widehat{xx} + s_{33}\widehat{zz} \\ e_{xz} &= s_{66}\widehat{xz} \end{aligned} \tag{7.13}$$

where

$$s_{11} = \frac{1}{E_1} \; ; \quad s_{13} = \frac{\sigma_3}{E_3} = \frac{\sigma_2}{E_1} \; ; \quad s_{33} = \frac{1}{E_3} \tag{7.14}$$

$$s_{66} = \frac{1}{\mu_3}$$

Substituting values (7.13) in (7.12), and using (7.11) we obtain

$$\frac{1}{E_\theta} = \frac{e_{x'x'}}{T} = s_{11}\ell^4 + (2s_{13} + s_{66})\ell^2 n^2 + s_{33} n^4. \qquad (7.$$

Expression for μ_θ

Consider pure shear along the two perpendicular directions ox', oz'. Referred to these axes the stress condition is then specified by

$$\widehat{x'z'} = S \quad \text{(a constant)} \tag{7.16}$$

$$\widehat{x'x'} = 0 \qquad \widehat{z'z'} = 0$$

Transforming to axes ox, oy we obtain

$$\widehat{xx} = 2ll_1 \, S \qquad \widehat{zz} = 2nn_1 \, S \qquad \widehat{xz} = (ln_1 + l_1 n)S \tag{7.17}$$

where oz, has the direction cosines $(l_1, n_1) = (-\sin\theta, \cos\theta)$

The equation for the transformation of shear strain(3) gives

$$e_{x'y'} = 2\,ll_1 e_{xx} + 2\,mm_1 e_{zz} + (ln_1 + l_1 n)e_{xz} \tag{7.18}$$

substituting in above from (7.16) and using (7.17) we obtain

$$\frac{1}{\mu_\theta} = \frac{e_{x'y'}}{S} = 4\,s_{11}(ll_1)^2 + 4\,s_{33}(nn_1)^2 \tag{7.19}$$

$$+ 8\,s_{13}(ll_1 mm_1) + s_{66}(ln_1 + l_1 n)^2$$

On substituting for direction cosines in terms of θ, above becomes

$$\frac{1}{\mu_\theta} = (s_{11} + s_{33} - 2s_{13} - s_{66})\sin^2 2\theta + s_{66} \tag{7.20}$$

This shows that attains a maximum or a minimum at $\theta = \frac{\pi}{4}$ according as $s_{11} + s_{33} <$ or $> 2\,s_{13} + s_{66}$. When the material is isotropic then $E_1 = E_2$ and μ_θ is a constant. This requires $s_{11} + s_{33} = 2\,s_{13} + s_{66}$. On using (7.14) this gives the familiar relation

$$\mu = \frac{E}{2(1+\sigma)}$$

We now proceed to a brief review of the literature on three constant mediums.

(i) Wolf's Paper[1] (1935)

Wolf in his paper assumes that

$$\mu = \frac{E_1 E_3}{E_1 + E_3(1+2\sigma)} \tag{7.21}$$

He does not discuss the physical implication of this assumption, having adopted it entirely for mathematical expediency. He presents a plane strain treatment suitable for an aeolotropic soil medium. By a rather laborious process that follows closely the stress function pattern of two dimensional isotropic elasticity he obtains solutions for the concentrated load, and the uniformly distributed load. He then considers the case of a uniform load distributed over a circular area, but in addition to the assumption (7.21), he sets $\sigma_1 = \sigma_3 = 0$. No settlements are obtained, nor is any technique given for their derivation.

We can easily show that assumption (7.21) is equivalent to

$$s_{11} + s_{33} = 2\, s_{13} + s_{66} \tag{7.21}$$

From (7.20) we see that this implies that the material is everywhere isotropic with regard to shear, however as seen from (7.15) it is not isotropic with regard to direct stress.

(ii) Sen's Paper[24] (1939)

Sen in his paper gives an erroneous derivation of (7.21) by considering the deformation of a rectangle with sides parallel to axes. This paper is an extension to the aeolotropic two-dimensional case of his previous work in isotropic elasticity. The assumption (7.21) is mathematically necessary in his treatment, in order that the solution may depend on that of the harmonic equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z_1^2} \right) \theta = 0$$

where $z_1 = \sqrt{\dfrac{E_1}{E_2}}\, z$ and $\theta = \widehat{xx} + \widehat{zz}$.

His method, based on analytic functions, is mathematically very elegant, and leads to the direct determination of stresses. The displacements are then obtained, but the method is rather laborious. Application is made to an infinitely large plate with a horizontal straight boundary. The moduli are different in the horizontal and vertical directions. What the material is, he does not specify.

(iii) Okubo's[25] Paper (1939)

Obubo bases his treatment on the system proposed by Wolf for the solution of stress problems in crystal plates when $E_1 \to E_3$. To evaluate his system it is best to revert to the general equation for the two dimensional aeolotropic stress - function χ as given by Huber[26] (1938)

$$\left(\frac{\partial^2}{\partial x^2} + \alpha_1 \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2}{\partial x^2} + \alpha_2 \frac{\partial^2}{\partial z^2}\right)\chi = 0 \qquad (7.22)$$

where $\alpha_1 \alpha_2 = \dfrac{s_{11}}{s_{33}}$ and $\alpha_1 + \alpha_2 = \dfrac{s_{66} + 2\, s_{13}}{s_{33}}$

Okubo's fundamental equation is

$$\left(\frac{\partial^2}{\partial x^2} + \frac{1}{k}\frac{\partial^2}{\partial z^2}\right)^2 \chi = 0 \qquad (7.23)$$

where $k^2 = \dfrac{s_{33}}{s_{11}}$

This latter equation is derived from (7.22) by the use of assumption (7.21) together with $k \to 1$. For his examples the latter requirement obtains, but he makes no investigation of how closely requirement (7.21) is satisfied.

Let $W_F \equiv W_{F_{1,3}} = \dfrac{s_{11} + s_{33}}{2s_{13} + s_{66}}$, then (7.21) demands (7.24)

that $W_F = 1$. It is instructive to compare values of k and W_F for some crystals that have $k \to 1$. Using data given by Voigt[27] (Page 761), we find:

TABLE 3

CRYSTALS

Crystal	$10^6\,s_{11}$ $10^6\,s_{22}$ $10^6\,s_{33}$ mm^2/kg	$-10^6\,s_{13}$ $-10^6\,s_{12}$ $-10^6\,s_{23}$ mm^2/Kg	$10^6\,s_{44}$ $10^6\,s_{55}$ $10^6\,s_{66}$ mm^2/Kg	W_F	k
Barytes	161.3 185.7 104.2	18.8 88.0 24.6	823 342 353	1.10	1.07
Aragonit	68.4 129 120	-4.2 29.8 23.3	238 382 230	1.35	1.04
Topaz	43.4 34.6 37.7	8.4 13.5 6.5	91 74 75	1.17	1.045

From above it can be concluded that Wolfe's assumption is not very well satisfied for crystals, and this is a serious defect in Okubo's otherwise excellent paper. Note that the approximation is much worse than would be suspected from the deviation of k from unity. This is due to the fact that many crystals are definitely not isotropic with regard to shear.

It is instructive to calculate W_F factors for planks or boards, as a measure of the applicability of Wolf's and Sen's results to wooden plates. The y axes is taken perpendicular to the radial rings of the wood, and the x and z axes are respectively tangential and radial to the rings. Accordingly planks may be cut in the planes y x and y z. We shall in the following table calculate $W_{F_{21}}$ and $W_{F_{23}}$ corresponding to above planks for different types of wood. The values of the stress coefficients s_{ij} are taken from Horig[28].

TABLE 4.

WOOD

Type	s_{11} s_{22} s_{33} mm^2/kg	$-s_{13}$ $-s_{21}$ $-s_{32}$ mm^2/kg	s_{44} s_{55} s_{66} mm^2/kg	$\frac{W_{F_{21}}}{W_{F_{23}}}$
Oak	10.15 1.72 4.57	3.00 0.87 0.55	7.60 25.0 12.8	1.07 — 0.97
Ash	12.2 0.62 6.45	4.90 0.46 0.48	7.31 36.3 11.0	1.19 — 1.11
Birch	15.9 0.600 8.88	6.40 0.26 0.29	8.36 52.7 10.8	1.61 — 1.22
Oregon Pine	10.9 0.599 7.55	4.5 0.23 0.22	8.35 123 10.8	1.11 — 1.03
Spruce	15.5 0.587 12.1	5.17 0.33 0.22	15.7 279 11.5	1.49 — 0.83

The results above indicate that assumption (7.21) may be used for oak and Oregon pine but certainly not for the other types. However as shown in next chapter, the results obtained for two dimensional plane strain may very easily be extended to orthotropic two dimensional generalized plane stress problems. Hence this treatment should supersede the works of Wolf and of Sen discussed above.

Weiskopf's[2] Paper

In this paper a soil system is developed based on three independent elastic constants E, σ and μ. This is just a particular case of the system in Chapter II by putting

$$E_1 = E_2 \tag{7.25}$$

Weiskopf justifies this assumption from the observed fact that in a sandy medium, due to the slipping of the granules on each other the resistance to shear is much less than in a solid. This means that μ is less than its value in an elastic solid. $\frac{E}{2(1+\sigma)}$. The physical implications of above assumption can be seen from equations (7.15) and (7.19) on putting $s_{11} = s_{33}$ and noting that $\mu \neq \frac{E}{2(1+\sigma)}$ i.e. $s_{66} \neq 2(s_{11} - s_{13})$

Hence both E_θ and μ_θ are seen to be non-isotropic, attaining extreme values when $\theta = \frac{\pi}{4}$. Physically this is rather an unlikely soil medium, since we should expect an extreme value of E only at $\theta = \frac{\pi}{2}$, if the value of E varies with the angle θ.

His equation for the two-dimensional stress function is somewhat in error, due to an error in equations (4) of his paper. Plane strain is the only tenable assumption for a two dimensional soil medium, as such things as soil plates are nebulous. Yet in equations (4) he tacitly puts $\widehat{yy} = \sigma_y = 0$. In other words he assumes that plane strain and plane stress can exist simultaneously in a body. If we correct equations (4) we obtain

$$e_{xx} = \frac{1}{E}\left[\widehat{xx} - \sigma(\widehat{yy} + \widehat{zz})\right]$$
$$0 = e_{yy} = \frac{1}{E}\left[\widehat{yy} - \sigma(\widehat{xx} + \widehat{zz})\right]$$
$$e_{zz} = \frac{1}{E}\left[\widehat{zz} - \sigma(xx + yy)\right]$$

From the second of which

$$\widehat{yy} = (\widehat{xx} + \widehat{zz})\,\sigma$$

Accordingly on substituting above values into compatibility equation

$$\frac{\partial^2 e_{xx}}{\partial z^2} + \frac{\partial^2 e_{zz}}{\partial x^2} = \frac{\partial^2 e_{xz}}{\partial x \partial z}$$

with

$$\widehat{xx} = \frac{\partial^2 \varphi}{\partial z^2} \; ; \quad \widehat{zz} = \frac{\partial^2 \varphi}{\partial x^2} \, , \quad \widehat{xz} = - \frac{\partial^2 \varphi}{\partial x \partial z}$$

we obtain as the correct form for his equation (6) governing φ

$$\frac{\partial^4 \varphi}{\partial x^4} + \frac{1}{1-\sigma^2}\left(\frac{E}{\mu} - 2\sigma - 2\sigma^2\right)\frac{\partial^4 \varphi}{\partial x^2 \partial z^2} + \frac{\partial^4 \varphi}{\partial z^4} = 0 \qquad (7.25)$$

This alters somewhat the value of the soil constant C introduced by him.

In three dimensions his assumptions are even more drastic, and less likely, with $E_1 = E_2 = E_3$, $\sigma_1 = \sigma_2 = \sigma_3$, $\mu_1 = \mu_2$ $\mu_3 = \frac{E}{2(1+\sigma)}$. The assumption on μ_3 is actually a necessity in the axially symmetric theory as presented in Chapter I.

Westegardes' Paper[29]

Westegarde's assumes that the soil medium, an elastic isotropic, medium is reinforced horizontally by inextensible membranes uniformly distributed, yet volumetrically infinitesimal. He then assumes that the membranes prevent all horizontal movement of the soil. Physical conditions show that above will yield conservative estimates for the normal stress, but it is likely to underestimate the surface settlement. It represents the extreme case of the stratified condition that exists frequently in sedimentary soils. But then such soils are readily treated by the methods of this thesis. Taylor[30] presents a practical adaptation of Westergarde's work. He also gives an excellent discussion of the applicability of elastic theory to soils, in which he points out that the most important requirement is the proportionality between stress and strain.

Conclusions

All the above systems arose from mathematical expediency, and not from any deep physical reasons. They can all be derived as particular cases of the general axially symmetric - four constant - system developed in this thesis by mere numerical substitution. Therefore, it seems pointless to use them, when they have not even the merit of numerical simplification over the more general theory.

CHAPTER VIII.

ORTHOTROPIC PLATES

An orthotropic medium possesses at each point three planes of symmetry at right angles to each other. Many types of crystals possess this type of symmetry (Love[3] #110). The directions of the planes of symmetry need not be invariant, e.g. a circular tree trunk if we assume all rings have equal strength. This latter type is called curvilinear aeolotropy. Two dimensional plane strain problems are identical in the above medium, and in the medium discussed in Chapters II and III. Assuming plane strain the strain-energy function becomes from Love[3] (#110)

$$2\,W = A\,e^2_{xx} + C\,e^2_{zz} + 2\,G e_{xx} e_{zz} + M\,e^2_{xz} \tag{8.1}$$

This is identical with (1.1) where F replaces G and N replaces M. A, C, G and M can easily be found in terms of s_{11} s_{22} s_{33} s_{13} s_{23} and s_{12}, the constants introduced by Voigt. These in turn can easily be determined practically.

Orthotropic plates form an important two dimensional application of the above medium. Approximations have been discussed in the previous chapter. Now a complete solution will be deduced for the case of generalized plane stress, the system usually used in plate problems.

Generalized Plane Stress

The assumptions made in this type of stress are, for a plate bounded by $y = \pm h$:

$$\begin{aligned} &\widehat{yy} = 0 && \text{throughout the plane} \\ &\left.\begin{aligned}\widehat{xy} &= 0 \\ \widehat{yz} &= 0\end{aligned}\right] && \text{only on edges } y = \pm h. \end{aligned} \tag{8.11}$$

Average displacements, strains, and stresses are defined as follows:

$$\bar{u} = \frac{1}{2h}\int_{-h}^{h} u\,\delta y \qquad \bar{w} = \frac{1}{2h}\int_{-h}^{h} w\,\delta y$$

$$\overline{e_{xx}} = \frac{1}{2h}\int_{-h}^{h} e_{xx}\,\delta y \qquad \overline{e_{zz}} = \frac{1}{2h}\int_{-h}^{h} e_{zz}\,\delta y$$

$$\overline{e_{xz}} = \frac{1}{2h}\int_{-h}^{h} e_{xz}\,\delta y \tag{8.12}$$

and

$$\breve{xx} = \frac{1}{2h}\int_{-h}^{h} \widehat{xx}\,\delta y \qquad \breve{zz} = \frac{1}{2h}\int_{-h}^{h} \widehat{zz}\,\delta y$$

$$\breve{xz} = \frac{1}{2h}\int_{-h}^{h} \widehat{xz}\,\delta y$$

Now

$$\frac{1}{2h}\int_{-h}^{h} \frac{\partial \widehat{xy}}{\partial y}\,\delta y = \frac{1}{2h}\Big[\widehat{xy}\Big]_{-h}^{+h} = 0 \quad \text{from (8.11)} \tag{8.13}$$

similarly

$$\frac{1}{2h}\int_{-h}^{h} \frac{\partial \widehat{yz}}{\partial y}\,\delta y = 0$$

The stress equilibrium equations (1.20) when integrated with respect to y between $y = \pm h$, on using (8.13) and definitions (8.12) become:

$$\frac{\partial \breve{xx}}{\partial x} + \frac{\partial \breve{xz}}{\partial z} = 0$$

$$\frac{\partial \breve{xz}}{\partial x} + \frac{\partial \breve{zz}}{\partial z} = 0 \tag{8.14}$$

Note that all the average quantities are independent of y from their definitions. The stress-strain relations for an orthotropic medium are obtained from its strain-energy function.

$$2W = A\,e^2_{xx} + B\,e^2_{yy} + C\,e^2_{zz} + 2F\,e_{yy}e_{zz} + 2G\,e_{zz}e_{xx} + 2H\,e_{xx}e_{yy} + L\,e^2_{yz} + M\,e^2_{zx} + N\,e^2_{xy}\ . \tag{8.15}$$

These are

$$\begin{aligned}\widehat{xx} &= A\ e_{xx} + H\ e_{yy} + G\ e_{zz}\\ \widehat{yy} &= H\ e_{xx} + B\ e_{yy} + F\ e_{zz}\\ \widehat{zz} &= G\ e_{xx} + F\ e_{yy} + C\ e_{zz}\\ \widehat{yz} &= L\ e_{yz} \qquad \widehat{zx} = M\ e_{zx} \qquad \widehat{xy} = N\ e_{xy}\end{aligned} \tag{8.16}$$

Since $\widehat{yy} = 0$ from (8.11)

therefore from (8.16)

$$e_{yy} = -\frac{H}{B}\, e_{xx} - \frac{F}{B}\, e_{zz} \tag{8.17}$$

Hence on substituting for e_{yy} in expressions for $\widehat{xx}$ and $\widehat{zz}$, we obtain

$$\begin{aligned}\widehat{xx} &= P\ e_{xx} + Q\ e_{zz}\\ \widehat{zz} &= Q\ e_{xx} + R\ e_{zz}\end{aligned} \tag{8.18}$$

where

$$P = A - \frac{H^2}{B} \qquad Q = G - \frac{FH}{B}$$

$$R = C - \frac{F^2}{B} \tag{8.19}$$

P, Q and R may be called Plate Stress Constants. These are similar to the two plate stress constants introduced by Coker and Filon[9], to deal with similar problems in isotropic elasticity.

Integrating equations (8.18) with respect to y between $y = \pm h$, we obtain using (8.12)

$$\begin{aligned}\breve{xx} &= P\ e_{xx} + Q\ e_{zz}\\ \breve{zz} &= Q\ e_{xx} + R\ e_{zz}\end{aligned} \qquad \breve{zx} = M\ e_{zx} \tag{8.20}$$

Accordingly generalized plane stress solution of any plate problem requires the solution of equations (8.14) and (8.20) subject to the appropriate boundary conditions. But these equations are exactly similar to those for the plane strain problems discussed in Chapter II, where A B C and L are replaced by P Q R and M respectively. Hence solutions to generalized plane stress problems can be deduced immediately from the corresponding solutions for plane strain obtained in Chapter II. This solves the problems discussed by Sen and Wolf in a relatively simple manner. Besides no unwarranted assumptions are necessary. Of course the above problems are all connected with a semi-infinite plate, bounded by one straight edge under a specified loading. However, the techniques developed are capable of extension to an infinite elastic strip, and possibly to circular plates. The author plans to return to these problems at an early date. It should have an important bearing on aircraft structural analysis for wooden, or plywood members.

CHAPTER IX

APPLICATIONS TO SOIL MECHANICS

This thesis presents in Chapters III and V relatively simple methods for calculating the surface settlements and the vertical pressures for a loaded wall, or for a loaded circular foundation. The laboratory tests necessary to establish the required constants are discussed in Appendix E.

Examples will now be worked to illustrate the procedure:

Experimental Data

Suppose that tests on a soil have furnished the following values for the required constants:

$$E_1 = 18{,}000 \text{ p.s.i.} \qquad E_3 = 22{,}000 \text{ p.s.i.} \tag{9.1}$$
$$\mu = 4{,}500 \text{ p.s.i.} \qquad \sigma_3 = 0.38$$
$$\sigma_1 = 0.35$$

Consider the following problems:

Problem 1.

Find the relative settlement of a long wall of width 6 ft., carrying a load P = 15 tons per foot length of wall. Also what is the maximum vertical pressure 10' below the ground level.

Problem 2.

A circular column 5' in diameter supports a load of 100 tons. Find the relative settlement of the column, and the vertical pressure 5' below ground level. What is maximum shear in the material 5' below ground level?

Using equations (1.14) and (1.15), we obtain

$$\sigma_2 = .38 \frac{18}{22} = .31$$

$$\sigma_2 \sigma_3 = (.31)(.38) = .12$$

$$1 - \sigma_1 - 2\sigma_2\sigma_3 = 1 - .35 - .24 = .41$$

$$A = \frac{(.88)(18,000)}{(1.35)(.41)} = 28,600 \text{ p.s.i.}$$

$$C = \frac{(.65)(22,000)}{.41} = 34,900 \text{ p.s.i.}$$

$$F = \frac{(.38)(18,000)}{.41} = 16,700 \text{ p.s.i.} \tag{9.11}$$

$$N = \frac{18,000}{2.70} = 6,700 \text{ p.s.i}$$

$$L = 4,500 \text{ p.s.i} \qquad G = L + F = 21,200 \text{ p.s.i}$$

Therefore

$$\frac{L^2+AC-G^2}{CL} = \frac{4.5^2+(28.6)(34.9)-21.2^2}{(34.9)(4.5)} = 3.62$$

and $$\frac{A}{C} = \frac{28.6}{34.9} = 0.822$$

Hence on using G'1 the characteristic equation becomes

$$y^2 - 3.62y + 0.822 = 0 \tag{9.12}$$

solving this on the slide rule we obtain

$$s_1^2 = y_1 = 3.38 , \qquad s_2^2 = y_2 = 0.243$$

$$\therefore \tag{9.13}$$

$$s_1 = 1.84 \qquad s_2 = 0.493$$

From G·2

$$h_1 = \frac{28.6 - (4.5)(3.38)}{(21.2)(1.84)} = 0.345$$

$$h_2 = \frac{28.6 - (4.5)(0.243)}{(21.2)(0.493)} = 2.63$$

Hence

$$h_1 s_1 = 0.635 \qquad h_2 s_2 = 1.30$$

substituting in expressions G·3 we easily obtain

$$\begin{aligned} s_3 &= -17.0 \times 10^3 & s_4 &= -63.0 \times 10^3 \\ s_5 &= 56.6 \times 10^3 & s_6 &= 15.1 \times 10^3 \\ s_7 &= 1.09 & s_8 &= 5.75 \end{aligned} \qquad (9.14)$$

Checks on the accuracy of our computations are provided by results C·2 and C·5 in Appendix C. These are easily seen to be satisfied:

$$\begin{aligned} s_1 s_3 &= 17.1 \times 10^3 & s_2 s_4 &= 17.1 \times 10^3 \\ s_2 s_5 &= 27.9 \times 10^3 & s_1 s_6 &= 27.8 \times 10^3 \end{aligned}$$

Therefore

$$s_1 s_3 = s_2 s_4 \qquad \text{and} \qquad s_2 s_5 = s_1 s_6 \quad \text{in}$$

accordance with results C·2 and C·3

Using values in (9.14) we obtain

$$s_3 - s_4 = 46.0 \times 10^3$$

$$s_7 - s_8 = -4.66$$

and hence substituting in G·4, G·5 and G·6:

$$\begin{aligned} s_{10} &= -\frac{1}{\pi}\,\frac{63.0 \times 0.493}{46.0} = -\frac{0.675}{\pi} \\ s_{13} &= \frac{46.0 \times 10^3}{4.66} = 9{,}860 \\ s_{14} &= -\frac{1}{\pi}\,\frac{56.6}{(1.84)(46.0)} = -\frac{0.670}{\pi} \end{aligned} \qquad (9.15)$$

Approximate Corresponding Elastic Isotropic Medium

To compare results from the aeolotropic theory with the isotropic theory, we might take for E and σ the respective mean values of these quantities in the soil medium. This gives

$$\sigma = 0.35 \qquad E = \frac{2\text{x}18,000 + 22,000}{3} = 19.3\text{x}10^3 \text{ p.s.i.} \tag{9.16}$$

$$A = 30.7 \times 10^3$$

Hence

$$\mu = \frac{E}{2(1+\sigma)} = \frac{19,300}{2.7} = 7,150 \text{ p.s.i.}$$

and from G·5

$$s'_{13} = \frac{2(7.15)(23.55)}{30.7} = 11,000 \tag{9.17}$$

Having calculated the necessary constants for the soil medium we can now proceed to the proposed problems.

Problem I

a) Settlement:

Using the results of Chapter V, we note from the calculated values of s_{13} and s'_{13}, that the actual settlement in the aeolotropic theory is 90% of that given by the isotropic theory. This also applies to the column of Problem 2. It is independent of the type of pressure distribution that exists under the wall. Assume a parabolic pressure distribution, then from Graph I or Table I with $x' = x/a$ we have for

$$x = 0 \;,\; x' = 0 \qquad N(x') = 0.972$$

$$x = 3 \;,\; x' = 1 \qquad N(x') = 0.593$$

On using (3.45) i.e. $w_s = \dfrac{BN(x')}{s_{13}}$ we have

$$x = 0 \qquad (w_s)_{x=0} = \frac{15\text{x}2000\text{x}.972}{9,860} = 2.96''$$

$$x = 3 \qquad (w_s)_{x=3} = \frac{15\text{x}2000}{9,860} \text{ x } .593 = 1.78''$$

Hence the relative settlement is 1.18". Other distributions can be investigated in a similar manner.

b) Pressure:

Again assuming parabolic pressure distribution under the wall, the required pressure may be obtained from 3.48 i.e.

$$\widehat{zz}\Big|_{o,z} = - \frac{3\, s_{10}\, P}{2a^3} \left[\frac{a^2+s^2z^2}{s} \tan^{-1} \frac{a}{sz} \right]_{S_2}^{S_1}$$

Let

$$f(s_1) \equiv \frac{a^2 + s^2z^2}{s} \tan^{-1} \frac{a}{sz} \quad ; \; a = 3, \quad z=10$$

Then

$$f(s_1\,) = f(1.84) = \frac{9+100(1.84)^2}{1.84} \tan^{-1}\left(\frac{3}{18.4}\right) = 30.6$$

$$f(s_2) = f(0.493) = 36.8$$

Hence from (3.48)

$$\widehat{zz}\Big|_{o,10} = + \frac{3\text{x}.675\text{x}30\text{x}10^3}{2\pi\ 3^3} \left(30.6-36.8 \right) \tag{9.18}$$

$$= - 2,210 \text{ p.s.f.}$$

Note that on the surface $\widehat{zz}$ at r=0 is one and a half times the average value of $\widehat{zz}$ on the surface.

Hence

$$\widehat{zz}\Big|_{o,o} = \frac{3}{2} \text{ x } \frac{30,000}{6} = - 7,500 \text{ p.s.f.}$$

Problem II

Assume a parabolic distribution of pressure under the column.

a) Settlement

Using Graph II or Table II, with $r' = r/a$. we obtain:

$r = 0 \qquad r' = 0 \qquad N(r') = 0.424$

$r = 5/2 \qquad r' = 1 \qquad N(r') = 0.180$

On using (5.46) i.e. $w_s = \frac{P}{a s_{13}} N(r')$ we have

$$r = 0 \qquad w_s \Big|_{r=0} = \frac{2x10^5}{5/2x9,860} \text{ x } 0.424 = 3.44''$$

$$r = 5/2 \qquad w_s \Big|_{r=5/2} = 3.44x\frac{180}{424} = 1.46''$$

Hence the relative settlement is 1.98"

b) Pressure:

This may be obtained from (5.48) i.e.

$$\widehat{zz}\Big|_{0,z} = -\ 2P\ \frac{s_{10}\ z}{a^4}\ \left[2sz + \frac{a^2}{sz} - 2\sqrt{a^2+s^2z^2}\right]_{S_2}^{S_1}$$

Let

$$f(s) \equiv 2sz + \frac{a^2}{sz} - 2\sqrt{a^2+s^2z^2}$$

with $z = 5, \quad a = 5/2$

$\therefore \qquad f(s_1) = f(1.84) = 0.027$

$\qquad f(s_2) = f(0.493) = 0.468$

Hence from (5.48)

$$\widehat{zz}\Big|_{0,5} = -\ \frac{4x10^5x5x.675}{(5/2)^4x\pi} \text{ x } 0.441 = -\ 4,880 \text{ ps.f.} \qquad (9.19)$$

Note that on the surface $\widehat{zz}$ is twice the average value of $\widehat{zz}$ on the surface. Hence

$$\widehat{zz}\Big|_{o,o} = -\frac{4 \times 10}{\pi(5/2)^2} = 20,400 \quad \text{p.s.f.}$$

c) <u>Shear Stress</u> Using (5.51) [C]

$$^{z}I_{s} = -2\frac{P\ z\ s}{\pi\ a^4}\ f(s) = -\frac{2\text{x}10^5\text{x}10}{\pi(5/2)^4}\ s\ f(s)$$

$$= -16.35 \times 10^2 \quad s\ f\ (s)$$

$$\therefore \quad ^{z}I_{s_1} = -1500$$

$$^{z}I_{s_2} = -3700$$

Hence from (5.362)

$$\tau_A = \frac{\pi}{2}\left\{ S_{14}\left[s\ ^{z}I_s \right]_{s_2}^{s_1} + S_{10}\left[\frac{1}{s}\ ^{z}I_s \right]_{s_2}^{s_1} \right\}$$

$$= \frac{1}{2}\left\{ .670\left[2760 - 1825 \right] + .675\left[815 - 7525 \right] \right\}$$

$$= \frac{1}{2}\left[625 - 4500 \right] = -1,937 \quad \text{p.s.f.}$$

Hence

$$\left| \tau_A \right| = 1,937 \quad \text{p.s.f.} \tag{9.20}$$

Corresponding results for vertical pressures along the axis of a loaded circular area when the distribution is parabolic may be obtained using (6.44) viz,

$$\widehat{zz}\Big|_{o,z} = -\frac{2\ P\ z}{\pi\ a^4}\left[-2z + \frac{a^2}{z} + \frac{2z^2}{\sqrt{a^2+z^2}} \right]$$

We note that this and the results for the other distributions (6.42)-(6.47) are independent of the elastic constants of the medium. In Problem II when z = 5

$$\widehat{zz}\Big|_{o,5} = -\frac{2x10^6}{\pi 95/2)^4}\left[-10 + 1.25 + \frac{50}{\sqrt{31.25}}\right] \qquad (9.21)$$

$$= -3,200 \text{ p.s.f.}$$

We shall now calculate the vertical pressure distribution for the column in Problem II. The aeolotropic results are obtained from (5.48), while the isotropic results are from (6.44). The results are:

TABLE V.

Column diameter 5'

Depth in Ft.	Pressure, 10^3 p.s.i. (Aeolotropic)	Pressure, 10^3 p.s.i. (Isotropic)
0	20.40	20.40
1	17.25	16.35
2	12.48	10.67
3	8.92	6.83
4	6.46	4.53
5	4.88	3.20
6	3.65	2.40
7	2.84	1.68
8	2.21	1.47
9	1.85	1.30
10	1.46	0.82

A study of above results shows that the isotropic theory seriously underestimates the pressures under the column. This is especially important, where weak layers occur in the soil medium as these may be subject to pressures for greater than those predicted by the isotropic theory. Of course, a rigorous investigation of this case would require the analysis of a layered system. Burmister[31] has given such an analysis for two and three layered isotropic systems. The Author plans at an early date to give a corresponding analysis for aeolotropic systems. The only significant deviation from isotropy is the hypothetical figures 9.1 is in shear. Weiskopf [2] indicates that such deviations do occur. The results are represented graphically in Graph III.

Conclusion

This thesis is merely an introduction to the subject of aeolotropic axially symmetric systems. It is doubtful if such progress could have been made without the use of the Fourier Integral, a tool that appears eminently suitable for the further exploration of the subject. Mindlin's[32] problem of a force within a semi-infinite mass, and Kelvin's[3] problem of a force in an infinite mass, can be solved quite easily for aeolotropic systems by the methods of this thesis. Similarly vibrations[33], and layered systems[31] can be investigated, and the corresponding isotropic elastic cases can be obtained by a limiting procedure as in Chapter VI. Of course, it is highly desirable that considerable experimental research be done on the results given in this thesis. The Author is confident that equipment now being developed in the Soil Mechanics Laboratory of the California Institute of Technology, will prove adequate and convenient for obtaining the necessary soil constants. The Author hopes to pursue this fascinating subject further at University College Cork, Ireland, where he has been appointed to a position in Applied Mathematics and Soil Mechanics.

APPENDIX A

Integrals required in Two Dimensional Case

Let

$$\mathcal{I}_1 \equiv \int_0^\infty e^{-mz} \cos mx \, dm$$

$$\mathcal{I}_2 \equiv \int_0^\infty e^{-mz} \sin mx \, dm$$

therefore

$$\mathcal{I}_1 + i\mathcal{I}_2 = \int_0^\infty e^{m(ix-z)} dm = -\frac{1}{ix-z}$$

$$\therefore \mathcal{I}_1 = z/r^2$$

and

$$\mathcal{I}_2 = x/r^2 \qquad \text{A·1}$$

Integrals as Derivatives of $\mathcal{I}_1$ or $\mathcal{I}_2$

Let $\mathcal{I}_1^k \equiv \int_0^\infty m^k e^{-mz} \cos mx \, dm$ where k is any integer $k > 0$.

$$= \int_0^\infty \frac{d^k}{dz^k} (e^{-mz}) \cos mx \, dm$$

$$= (-)^k \frac{d^k}{dz^k} \int_0^\infty e^{-mz} \cos mx \, dm$$

$$= (-)^k \frac{d^k}{dz^k} \mathcal{I}_1$$

$$\therefore \mathcal{I}_1^k = (-)^k \frac{d^k \mathcal{I}_1}{dz^k} \qquad \text{(A·2)}$$

Interchange of the orders of differentiation and integration is permissible, since $\mathcal{I}_1^{k-1}$ is uniformly convergent in for $z > 0$, and $\mathcal{I}_1^k$ is convergent $z > 0$.

Similarly

$$\mathcal{I}_2^k \equiv \int_0^\infty m^k e^{-mz} \sin mx \, dm$$

$$= (-)^k \frac{d^k}{dz^k} \mathcal{I}_2 \qquad \text{(A·3)}$$

We require integrals of the type

$$ {}_{\delta}^{z}I_{1}^{-k} = \int_{\delta}^{\infty} m^{-k} e^{-mz} \cos mx \, dm \qquad \delta > 0 $$

$$ {}_{\delta}^{z}I_{2}^{-k} = \int_{\delta}^{\infty} m^{-k} e^{-mz} \sin mx \, dm \tag{A.4} $$

These can be reduced by repeated integration by parts to one of the integrals

$$ {}_{\delta}^{z}I_{1}^{-1} \equiv \int_{\delta}^{\infty} \frac{e^{-mz}}{m} \cos mx \, dm $$

$$ {}_{\delta}^{z}I_{2}^{-1} = \frac{e^{-mz}}{m} \sin mx \, dm \tag{A.5} $$

All the integrals involved are uniformly and absolutely convergent for $z > 0$; also all the integrands are continuous in z, x, and m. Hence by Abel's theorem

$$ {}_{\delta}^{0}I_{1}^{-1} = \underset{z \to 0}{Lt} \int_{\delta}^{\infty} \frac{e^{-mz}}{m} \cos mx \, dm $$

with similar results for the integrals in (A.4)

<u>Evaluation of (A.5) Integrals</u>

Omitting z superscript, since there is no danger of ambiguity, we have

$$ {}_{\delta}I_{1,2}^{-1} \equiv {}_{\delta}I_{1}^{-1} + i \, {}_{\delta}I_{2}^{-1} = \int_{0}^{\infty} \frac{e^{-m(z-ix)}}{m} \, dm $$

This integral is uniformly convergent $z > 0$, $\delta > 0$, is continuous in z, x and m, and hence we can differentiate under the integral sign.

$$ \therefore \quad \frac{\partial}{\partial z}\left[{}_{\delta}I_{1,2}^{-1} \right] = -\int_{\delta}^{\infty} e^{-m(z-ix)} dm \to -\int_{0}^{\infty} e^{-m(z-ix)} dm \text{ as } \delta \to 0 $$

$$ = -\frac{1}{z-ix} \tag{A.6} $$

$$ \therefore \quad {}_{\delta}I_{1,2}^{-1} \to -\log(z-ix) + g_{1}(x) \quad \text{for } \delta \text{ small} $$

Similarly

$$\frac{\partial}{\partial x}\left[{}_\delta I_{1,2}^{-1}\right] = i\int_\delta^\infty e^{-m(z-ix)}\,dm \rightarrow \frac{i}{z-ix}$$

(A.7)

$$\therefore\quad {}_\delta I_{1,2}^{-1} \rightarrow -\log(z-ix) + f_1(y)$$

where $g_1(x)$ and $f_1(y)$ are arbitrary functions of integration. On comparing the two results we obtain

$$g_1(x) = f_1(y) = \text{Constant} = K(\delta)$$

As $z \rightarrow \infty$ ${}_\delta I_{1,2}^{-1} \rightarrow 0$, hence K is an infinite constant of the type $\log R + O\left(\frac{1}{\delta}\right)$. However, we can drop an arbitrary constant from the expressions for u and w without affecting the stresses, or relative displacements. This is merely equivalent to superposing a rigid body displacement. Hence

$${}_\delta I_{1,2}^{-1} \rightarrow -\log(z-ix) = -\log r + i(\pi/2-\theta)$$

Taking real and imaginary parts we obtain

(A.8)

$${}_\delta I_1^{-1} = -\log r$$

$${}_\delta I_2^{-1} = -\tan^{-1} x/z$$

Integrals (A.4) Required for Special Cases.

The integrals required are

$${}_\delta I_{1,2}^{-2}, \quad {}_\delta I_{1,2}^{-3} \quad \text{and} \quad {}_\delta I_{1,2}^{-4}$$

Applying integration by parts, we obtain for $z > 0$

$${}_\delta I_{1,2}^{-2} \equiv \int_\delta^\infty \frac{e^{-m(z-ix)}}{m^2}\,dm = -\left.\frac{e^{-m(z-ix)}}{m}\right|_\delta^\infty - (z-ix)\int_\delta^\infty \frac{e^{-m(z-ix)}}{m}\,dm$$

(A.9)

$$\longrightarrow \quad ix + (z-ix)\log(z-ix) + O\left(\frac{1}{\delta}\right)$$

Also

$$ {}_{\delta}\overline{I}_{1,2}^{-3} \equiv \int_{\delta}^{\infty} \frac{e^{-m(z-ix)}}{m^3}\, dm = -\frac{e^{-m(z-ix)}}{2m^2}\Bigg|_{\delta}^{\infty} - \frac{(z-ix)}{2}\int_{\delta}^{\infty} \frac{e^{-m(z-ix)}}{m^2}\, dm \tag{A.10} $$

$$ \longrightarrow \quad -\frac{ix}{2}(z-ix) + \frac{(z-ix)^2}{2}\log(z-ix) + 0\left(\frac{1}{\delta^2}\right) $$

and

$$ {}_{\delta}\overline{I}_{1,2}^{-4} \equiv \int_{\delta}^{\infty} \frac{e^{-m(z-ix)}}{m^4}\, dm = -\frac{e^{-m(z-ix)}}{3m^3}\Bigg|_{\delta}^{\infty} - \frac{(z-ix)}{3}\int_{\delta}^{\infty} \frac{e^{-m(z-ix)}}{m^3}\, dm $$

$$ \longrightarrow \quad \frac{ix}{6}(z-ix)^2 - \frac{(z-ix)^3}{6}\log(z-ix) + 0\left(\frac{1}{\delta^3}\right) \tag{A.11} $$

Taking real and imaginary parts of above, we obtain the following required integrals.

From (A.9)

$$ {}_{\delta}\overline{I}_{1}^{-2} \longrightarrow z\log r + x\tan^{-1} x/z \qquad r^2 = x^2+z^2 $$

$$ {}_{\delta}\overline{I}_{2}^{-2} \longrightarrow x - x\log r + z\tan^{-1}(x/z) \qquad \tan\theta = z/x \tag{A.12} $$

From (A.10)

$$ {}_{\delta}\overline{I}_{1}^{-3} \longrightarrow -\frac{x^2}{2} + \frac{(z^2-x^2)}{2}\log r - xz\tan^{-1} x/z $$

$$ {}_{\delta}\overline{I}_{2}^{-3} \longrightarrow -\frac{xz}{2} - xz\log r - \frac{(z^2-x^2)}{2}\tan^{-1} x/z \tag{A.13} $$

and from (A.11)

$$ {}_{\delta}\overline{I}_{1}^{-4} \longrightarrow +\frac{x^2 z}{3} - \frac{(z^3-3zx^2)}{6}\log r - \frac{(3z^2x-x^3)}{6}\tan^{-1} x/z \tag{A.14} $$

$$ {}_{\delta}\overline{I}_{2}^{-4} \longrightarrow \frac{x}{6}(z^2-x^2) + \frac{(3z^2x-x^3)}{6}\log r - \frac{(z^3-3zx^2)}{6}\tan^{-1} x/z $$

APPENDIX B

Evaluation of Integrals Involving Bessel Functions

Let $$K_1^n = \int_0^\infty e^{-mz} J_0(mr)\, m^n\, dm\,, \quad z>0 \qquad \text{(B.1)}$$

On substituting $z = R\cos\theta$, $r = R\sin\theta$, $t = mR$ $\quad \cos\theta > 0$

$$J_0(\eta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\eta\cos\phi}\, d\phi$$

we obtain

$$K_1^m = \frac{1}{R^{n+1}}\,\frac{1}{2\pi}\int_0^\infty t^n\, e^{-t\cos\theta}\left[\int_{-\pi}^{\pi} e^{-it\sin\theta\cos\varphi}\, d\phi\right] dt$$

The order of integration may be interchanged since both integrals are uniformly convergent and continuous for $\cos\theta > 0$. Hence

$$K_1^n = \frac{1}{2\pi R^{n+1}}\int_{-\pi}^{\pi}\left[\int_0^\infty t^n\, e^{-t(\cos\theta + \sin\theta\cos\phi)}\, dt\right] d\phi \qquad \text{(B.2)}$$

Also from the definition of the Gama Function by a slight substitution it follows that

$$\int_0^\infty t^n e^{-\alpha t}\, dt = \frac{\Gamma(n+1)}{\alpha^{n+1}} \qquad \alpha > 0\,, \quad n > -1$$

Also since

$$P_n(\cos\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{d\phi}{\left[\cos\theta + \sin\theta\cos\phi\right]^{n+1}}$$

Hence from (B.2) on using above results,

$$K_1^n = \frac{\Gamma(n+1)}{2\pi R^{n+1}}\int_{-\pi}^{\pi} \frac{d\phi}{\left[\cos\theta + \sin\theta\cos\phi\right]^{n+1}} \qquad \text{(B.3)}$$

$$= \frac{\Gamma(N+1)}{R^{n+1}}\, P_n(\cos\theta)$$

Also

$$\frac{\partial}{\partial r}\left[K_1^n\right] = \int_0^\infty e^{-mz} J_0'(mr)m^{n+1}\,dm$$

$$= -\int_0^\infty e^{-mz} J_1(mr)m^{n+1}\,dm \quad \text{since } J_0'(\eta) = -J(\eta)$$

$$\equiv - K_2^{n+1} \quad \text{by definition} \tag{B.4}$$

Hence it follows from (B.3) that

$$K_2^{n+1} = -\frac{\partial}{\partial r}\left[K_1^n\right] = \Gamma(n+2)\frac{r}{R^{n+3}}P_n(\cos\theta) + \Gamma(n+1)\frac{rz}{R^{n+4}}P_n'(\cos\theta) \tag{B.5}$$

Substituting for $P_n'(\cos\theta)$ its value from the recurrence formulae for Legendre functions

$$P_n'(\cos\theta) = \frac{-n\cos\theta\, P_n(\cos\theta) + n\,P_{n-1}(\cos\theta)}{\sin^2\theta}$$

we obtain

$$K_2^{n+1} = \frac{n!}{R^{n+2}}\left[\frac{-nz^2+(n+1)r^2}{rR}P_n(\cos\theta) + \frac{nz}{r}P_{n-1}(\cos\theta)\right] \tag{B.6}$$

Jahnke Emde "Tables of Functions" Page 124 gives tables both of $P_n(\cos\theta)$ and $P_n'(\cos\theta)$, hence it is slightly less numerical work to use the first form (B.5) for K_2^{n+1}

Special Cases

$$K_1' = \frac{1}{R^2}P_1(\cos\theta) = \frac{\cos\theta}{R^2} = \frac{z}{R^3}$$

$$K_2' = \frac{r}{R^3} \tag{B.7}$$

APPENDIX C.

Relations Between the Constants in Solution

From (2.22)

$$h_1\ h_2 = \frac{(A-Ls_1^2)(A-Ls_2^2)}{G^2 s_1 s_2}$$

$$= \left[\frac{A^2 - AL(s_1^2+s_2^2) + L\ s_1^2 s_2^2}{G^2 s_1 s_2}\right]$$

Applying the theory of equations to (2.19) we obtain

$$s_1^2 + s_2^2 = \frac{L^2+AC-G^2}{CL} \qquad s_1^2\ s_2^2 = A/C$$

and hence on substitution

$$h_1\ h_2 = \sqrt{A/C} \quad = \quad s_1 s_2 \tag{C.1}$$

In the isotropic elastic case the coefficients of $\widehat{xz}$ and $\widehat{zz}$ are equal numerically, and a limiting procedure suggests it should hold for the aeolotropic case. Also this requires proving that

$$s_4 s_2 = s_3 s_1 \tag{C.2}$$

$$\frac{s_4 s_2}{s_2+h_2} = -\ L(s_1+h_1) \tag{C.3}$$

Now on substituting for s_4 and s_3 in C.2 we obtain

$$s_4 s_2 - s_3 s_1 = s_2(F-Ch_2 s_2)(s_1+h_1) - s_1(F-Ch_1 s_1)(s_2+h_2)$$

$$\therefore \frac{s_4 s_2 - s_3 s_1}{s_1 s_2} = F\Big(\frac{h_1}{s_1} - \frac{h_2}{s_2}\ - C(h_2 s_2 - h_1 s_1) - C(s_2^2 - s_1^2)$$

$$= (s_2^2 - s_1^2)\left[\frac{FC}{G} + \frac{CL}{G} - C\right] \quad = 0$$

on substituting for h_1, h_2 and s_1^2, s_2^2. Hence (C.2) follows.

To prove (C.3) consider substituting for s_4

$$s_2(F-Ch_2s_2) + L(s_2+h_2)$$

Substituting for h_2 and simplifying we obtain

$$CL\, s_2^4 + s_2^2 \left[L^2 + AC - G^2\right] + AL$$

This is zero since s_2 is a root of equation (2.19)

Hence result (C.3) follows:

<u>Evaluation of s_{10}</u>: <u>The Stress Constant</u>

By the definition of s_9 in Appendix G.

$$-2\, s_{10} = L(s_1+h_1)\, s_9 = \frac{2}{\pi} \frac{L(s_1+h_1)(s_2+h_2)}{s_3-s_4}$$

$$= -\frac{2}{\pi}\frac{s_4 s_2}{s_3-s_4} \qquad \text{on using (C.3)}$$

$\therefore$

$$s = \frac{1}{\pi}\frac{s_4 s_2}{s_3-s_4} \tag{C.4}$$

As in (C.2) we can show that

$$s_2 s_5 = s_1 s_6 \tag{C.5}$$

Also a reduction required in the concentrated load case for $\widehat{xx}$ is

$$\frac{s_1 s_5 - s_2 s_6}{s_3-s_4} = \frac{s_6\left[\dfrac{s_1^2-s_2^2}{s_2}\right]}{s_4\left[\dfrac{s_2}{s_1}-1\right]} = -\frac{s_6}{s}(s_1+s_2)$$

$$\equiv -s_{11} \qquad \text{by definition} \tag{C.6}$$

APPENDIX D

Integrals Required for Loaded Circular Area

Consider the integrals

$$U_{-1}(x) = \int \frac{K(x)}{x}\,dx \qquad x \neq 0 \text{ for convergence} \tag{D.1}$$

$$U_{\frac{1}{2}}(x) = \int x^{\frac{1}{2}} K(x)\,dx \tag{D.11}$$

Now $K(x) = \pi/2 \; {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1, x) = \tfrac{1}{2} \sum_{n=0}^{\infty} \left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^n}{n}$

On substituting the series for K(x) in (D.1), and on integrating term by term, over a range in which the series under the integral is uniformly convergent, we obtain

$$U_{-1}(x) = \pi/2 \log x + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^n}{n} \tag{D.12}$$

since $\displaystyle \mathop{Lt}_{n\to\infty} \frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1-n)} \longrightarrow n^{-1}$

the above series is of order $\frac{x}{n^2}$, and so is convergent $|x| \leqslant 1$. However as $x \to 1-$ the convergence is rather slow for computation. A more rapidly convergent series can be obtained as follows:

since $E(x) = \pi/2 \; {}_2F_1(-\tfrac{1}{2}, \tfrac{1}{2}, 1, x) = -\tfrac{1}{4} \sum_{n=0}^{\infty} \left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^n}{n-\frac{1}{2}}$, $|x| < 1$

therefore

$$U_{-1}(x) + 2E(x) = \pi/2 \log x + \pi - \frac{1}{4} \sum_{n=1}^{\infty} \left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^n}{n=\frac{1}{2}} \tag{D.13}$$

The above infinite series has convergence of order $\frac{x}{n^3}$, and so is much better for computation than (D.12)

Define $V_1(x) \equiv \sum_{n=1}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^n}{n(n-\frac{1}{2})}$ (D.14)

Similarly from (D.11)

$$U_{\frac{1}{2}}(x) = \frac{1}{2} \sum_{n=0}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^{n+3/2}}{n+3/2} \quad , |x|<1$$

and therefore

$$U_{\frac{1}{2}}(x) + 2x^{3/2} E(x) = -\sum_{n=0}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^{n+3/2}}{(n+3/2)(n-\frac{1}{2})} \qquad \text{(D.15)}$$

Define

$$V_2(x) \equiv -\sum_{n=0}^{\infty}\left[\frac{\Gamma(\frac{1}{2}+n)}{\Gamma(1+n)}\right]^2 \frac{x^{n+3/2}}{(n+3/2)(n-\frac{1}{2})} \qquad \text{(D.16)}$$

$V_1(x)$ and $V_2(x)$ were obtained with the aid of a calculating machine, for $0 \leq x \leq 1$. These results are believed accurate to four places of decimals, as six places were used in all calculations. The results are given in Table III.

for intervals of 0.2 in x and $\sqrt{x}$. Other values may be obtained by graphical or arithmetical interpolation.

TABLE III

$\sqrt{x}$	x	$V_1(x)$	$V_2(x)$
0	0	0	0
0.2		0.0631	0.0333
0.4		0.2553	0.2615
	0.2	0.3204	0.3631
0.6		0.5868	0.8532
	0.4	0.6499	0.9927
	0.6	1.0075	1.7536
0.8		1.0806	1.9158
	0.8	1.3842	2.5824
1	1	1.8009	3.4162

Hence the integrals required in (5.16) and (5.17) can be obtained as follows from (D.13)

$$
\begin{aligned}
{}_{r^2}^{\,1}I_{-1} &\equiv \int_{r^2}^{1} \frac{K(x)\,dx}{x} = \left[-\,2\,E(x) + \pi - \frac{1}{4}\,V_1(x) \right]_{r^2}^{1} \\
&= 2\left[E(r^2) - 1\right] - \frac{1}{4}\left[V_1(1) - V_1(r^2)\right] - \pi \log r \\
&= -2.4502 + 2\,E(r^2) + \frac{1}{4}\,V_1(r^2) - \pi \log r
\end{aligned}
\qquad (D.17)
$$

$$
\begin{aligned}
{}_{0}^{1/r^2}I_{1/2} &\equiv \int_{0}^{1/r^2} x^{1/2}\,K(x)\,dx = \left[-\,2x^{3/2}E(x) + V_2(x) \right]_{0}^{1/r^2} \\
&= V_2\!\left(\frac{1}{r^2}\right) - \frac{2}{r^3}\,E\left(\frac{1}{r^2}\right) \qquad r^2 > 1
\end{aligned}
\qquad (D.18)
$$

Also $${}^{1}_{r^2}I_{\frac{1}{2}} = {}^{1}_{0}I_{\frac{1}{2}} - {}^{r^2}_{0}I_{\frac{1}{2}} \tag{D.19}$$

On putting $n = \frac{1}{2}$ in (5.10), and on substituting for $I_{-\frac{1}{2}}$ from (5.16), we obtain

$$I_{\frac{1}{2}} = \frac{1}{4} I_{-\frac{1}{2}} + \frac{1}{2} = 1.4160 \tag{D.20}$$

Table IV gives values of ${}^{1}_{r^2}I_{-1}$ and ${}^{1/r^2}_{0}I_{\frac{1}{2}}$ for $0 \leqslant r^2 \leqslant 1$, in intervals of 0.2. Intermediate values can be obtained by interpolation.

TABLE IV

$r, 1/r$	${}^{1}_{r^2}I_{-1}$	${}^{1/r^2}_{0}I_{\frac{1}{2}}$	${}^{1}_{r^2}I_{\frac{1}{2}}$
0	∞	0	1.4162
0.2	2.8715	0.0084	1.4078
0.4	1.8754	0.0687	1.3475
0.6	1.2295	0.2406	1.1756
0.8	0.6770	0.6089	0.8073
1	0	1.4162	0

Integrals Required for the Parabolic Case

$$I_1 \equiv \int_0^1 x\,K(x)\,dx = \frac{4}{9} I_0 + \frac{2}{9} = \frac{10}{9} \quad (5.10,\ n=1) \tag{D.21}$$

$$\begin{aligned} {}^{1/r^2}_{0}I_1 &\equiv \int_0^{1/r^2} x\,K(x)\,dx = \frac{4}{9}\,{}^{1/r^2}_{0}I_0 + \frac{4}{9} S_1(1/r,1) \\ &= \frac{4}{9}\,{}^{1/r^2}_{0}I_0 + \frac{2}{9r^2}\left[E(1/r^2) - 3(1-1/r^2)\,K(1/r^2)\right] \end{aligned} \tag{D.22}$$

On using (5.14) with $n = 1$

$$ {}_{r^2}^{\;1}I_{-5/2} = \int_{r^2}^{1} x^{-5/2}\, K(x)\, dx = \frac{4}{9}\left[{}_{r^2}^{\;1}I_{-3/2} - S(r,-3/2) \right]^{*} $$

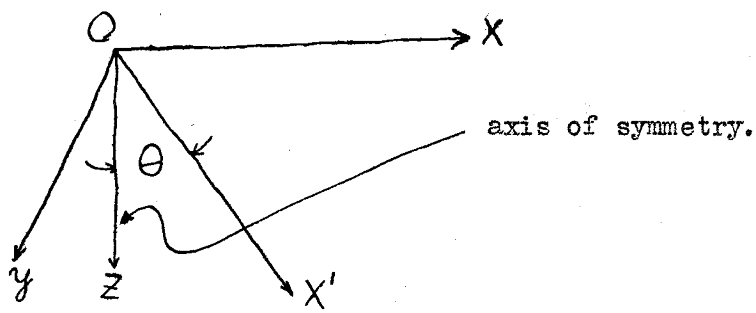
$$ = \frac{4}{9}\left[E(r^2)\left(\frac{2}{r} + \frac{1}{2r^3}\right) - \frac{5}{2} + \frac{r'^2}{r^3} K\,(r^2) \right], \; r'^2 = 1 - \frac{1}{r^2}, \quad (D.23) $$

* On using (5.12) with $n = -3/2$

APPENDIX E

Practical Determination of the Constants

From results (1.15) we see that the determination of the constants A, C, F, L and N for an aeolotropic axially symmetric medium depends on the values of E_1 E_3 σ_2 μ_3 and μ_1 or σ_1. We need an expression for Young's Modulus E_θ at a direction θ to the axis of symmetry, say along ox' in the following figure:



Consider when state of stress in body is due to a stress $\widehat{x'x'}$ in the direction ox'. Take oy' and oz' to form an orthogonal set of axes with ox'.

Referred to original x y z axes the direction-cosines of ox' are $(\sin\theta, o, \cos\theta)$, since due to the axial symmetry of the medium ox' can be taken in plane of x O z without any loss of generality. Using the well known relations for the transformations of stress and strain,[3] we have

$$\begin{aligned} \widehat{xx} &= \sin^2\theta\ \widehat{x'x'} & \widehat{yy} &= 0 \\ \widehat{zz} &= \cos^2\theta\ \widehat{x'x'} & \widehat{yz} &= 0 \qquad \text{(E.1)} \\ \widehat{xz} &= \cos\theta\sin\theta\ \widehat{x'x'} & \widehat{xy} &= 0 \end{aligned}$$

Also

$$e_{x'x'} = \sin^2\theta\, e_{xx} + \cos^2\theta\, e_{zz} + \cos\theta\sin\theta\, e_{xz} \qquad (E.2)$$

where

e_{xx}, e_{zz}, e_{xz} are given by 1·13.

On substituting these values in (E.2), and then on substituting values (E.1) for $\widehat{xx}$ etc., we easily obtain

$$\frac{1}{E_\theta} = \frac{e_{x'x'}}{\widehat{x'x'}} = \frac{1}{E_1}\sin^4\theta + \left(\frac{1}{\mu_3} - \frac{2\sigma_2}{E_1}\right)\cos^2\theta\sin^2\theta + \frac{1}{E_3}\cos^4\theta \qquad (E.3)$$

The constants may now be obtained as follows:

(i) A triaxial test on a sample taken parallel to axis of symmetry yields values for E_3 and σ_2

(ii) An unconfined compression test on a sample taken perpendicular to the axis of symmetry yields value of E_1. A triaxial test on such a sample is of little value as Poisson's ratio is different at all points on the perimeter of the sample.

(iii) Result (E.3) enables us to find μ_3 by taking an unconfined compression test (or a compression test at constant lateral pressure) on an oblique sample.

(iv) It remains to determine μ_1, or σ_1. This can be done by a torsion test on a sample taken parallel to the axis of symmetry as in (i). This sample alone shears in a plane, the xy plane, where all directions have the same Poisson's Ratio (σ_1).

Since the above procedure is rather difficult to carry out successfully an alternative is got by considering the cubical dilatation ϵ.

Now

$$\epsilon = \frac{dV}{V} \quad \text{i.e. increase in Volume per unit volume}$$

$$= e_{xx} + e_{yy} + e_{zz}$$

Hence if we take a soil sample subject to a uniform pressure p, we have

$$-\frac{dV}{V} = p \quad \frac{2(1-\sigma_1-\sigma_2)}{E_1} + \frac{1}{E_3}(1-2\sigma_3) \qquad \text{(E.4)}$$

The quantity $-\frac{dV}{V}$ can be measured, and then σ_1 can be obtained from above equation, since all the other quantities are known.

APPENDIX G

Constants Introduced

s_1^2 , s_2^2 are the roots of the quadratic equation (2.19):

$$y^2 - \frac{L^2 + AC - G^2}{CL}y + A/C = 0 \qquad (G.1)$$

$$h_1 = \frac{A - Ls_1^2}{G\, s_1} \quad ; \qquad h_2 = \frac{A - Ls_2^2}{Gs_2} \quad ; \quad G = L+F \qquad (G.2)$$

$$s_3 = (F - Ch_1 s_1)(s_2 + h_2)$$

$$s_4 = (F - Ch_2 s_2)(s_1 + h_1)$$

$$s_5 = (A - Fh_1 s_1)(s_2 + h_2) \qquad (G.3)$$

$$s_6 = (A - Fh_2 s_2)(s_1 + h_1)$$

$$s_7 = h_1(s_2 + h_2)$$

$$s_8 = h_2(s_1 + h_1)$$

$$s_9 = \frac{2}{\pi}\,\frac{s_2 + h_2}{s_3 - s_4}$$

$$s_{11} = \frac{s_6}{s_3}(s_1 + s_2)$$

$$s_{12} = \frac{s_1 + h_1}{s_2 + h_2}$$

Stress Constant

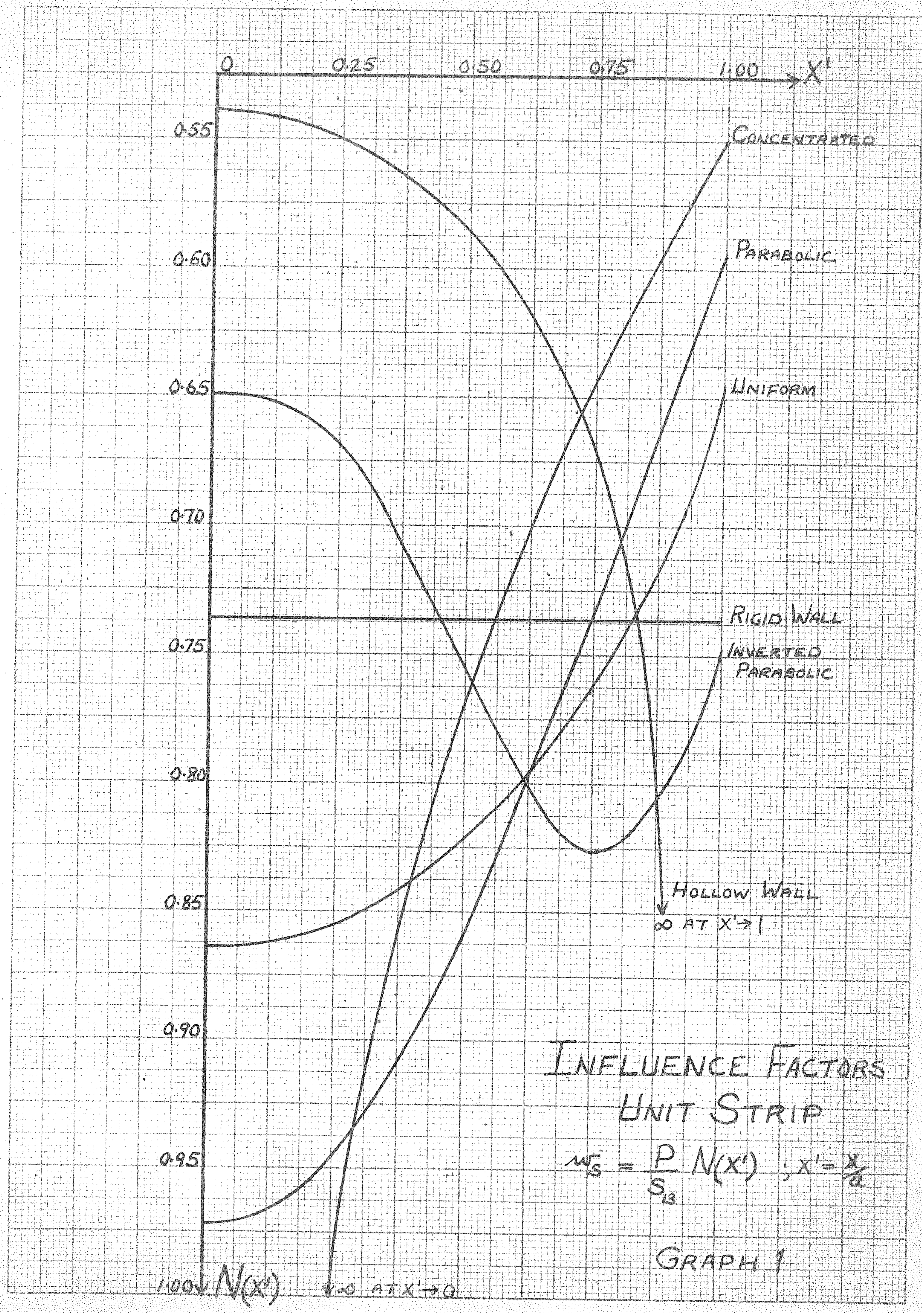
$$s_{10} = \frac{1}{\pi}\,\frac{s_4\ s_2}{s_3 - s_4} \qquad \text{from (C.4)} \qquad (G.4)$$

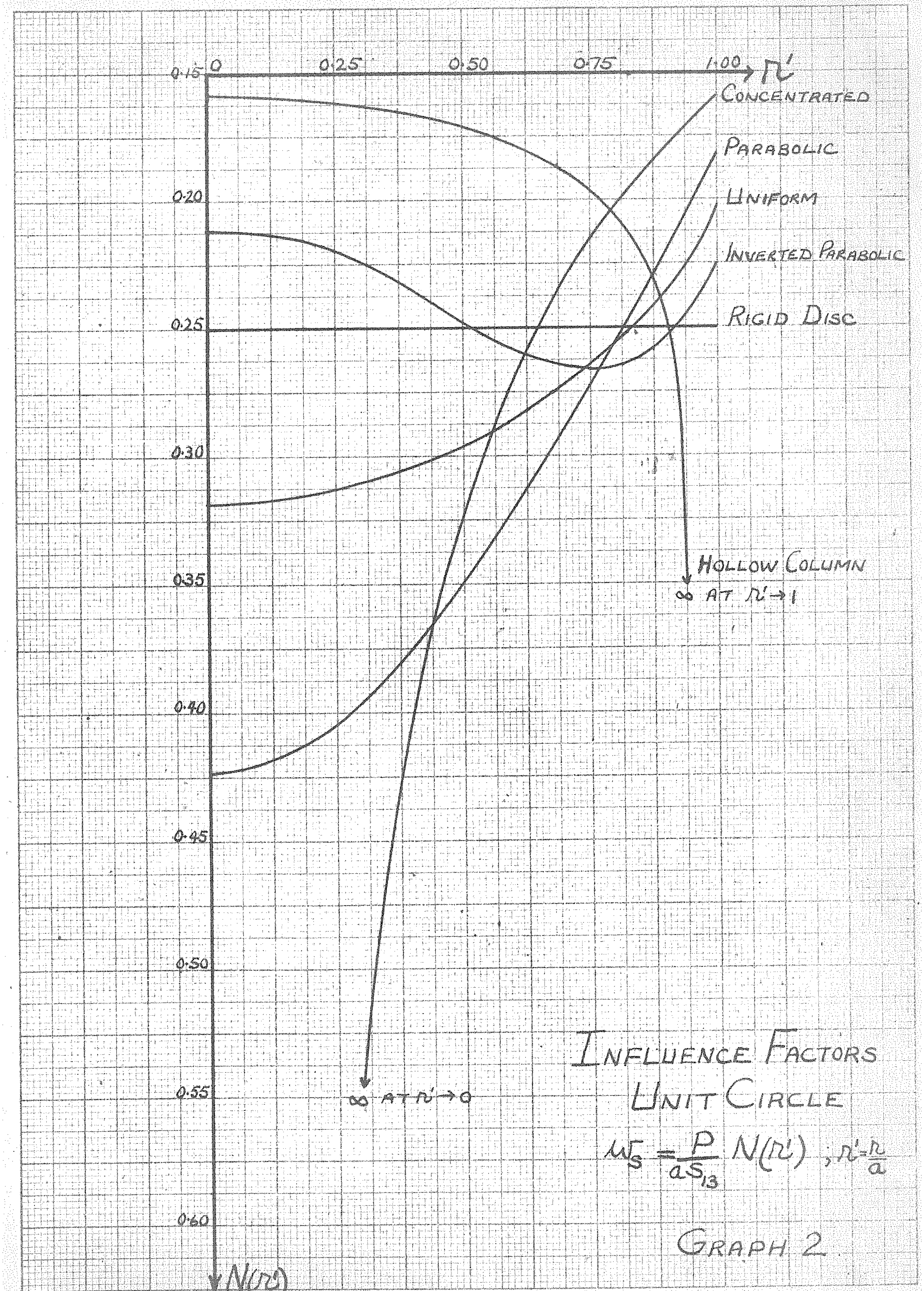
Settlement Constant

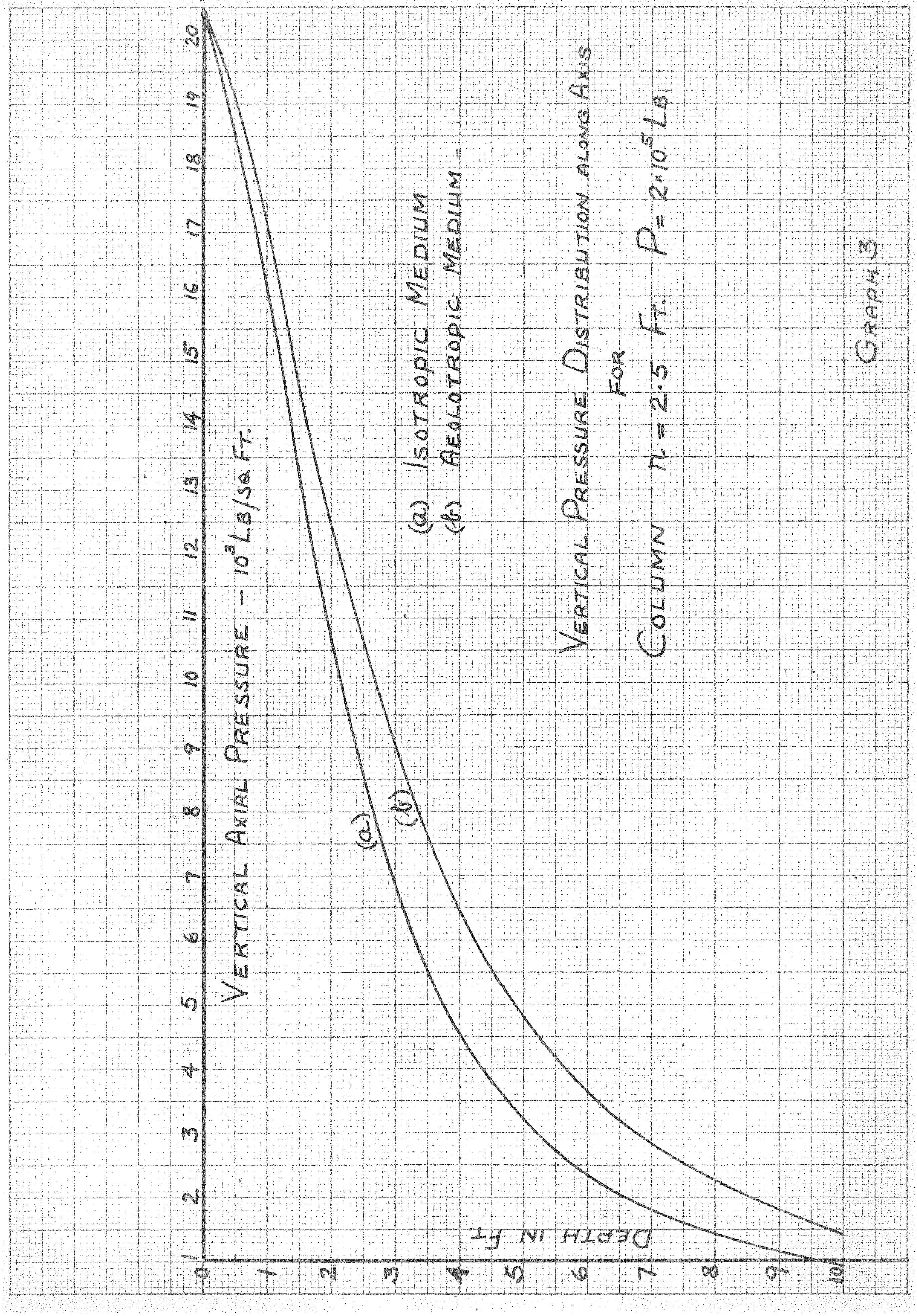
$$s_{13} = -\frac{s_3 - s_4}{s_7 - s_8} \qquad (G.5)$$

$$= \frac{2\mu(\lambda + \mu)}{\lambda + 2\mu} = \frac{2N(A-N)}{A} \equiv s_{13}' \text{ in isotropic case}$$

$$s_{14} = -\frac{1}{\pi}\,\frac{s_5}{s_1(s_3 - s_4)} \qquad (G.6)$$


0
0.25
0.50
0.75
1.00
X'
0.55
0.60
0.65
0.70
0.75
0.80
0.85
0.90
0.95
1.00
N(X')
CONCENTRATED
PARABOLIC
UNIFORM
RIGID WALL
INVERTED PARABOLIC
HOLLOW WALL
∞ AT X'→1
∞ AT X'→0
INFLUENCE FACTORS
UNIT STRIP
$w_S = \frac{P}{S_{13}} N(X')$; $X' = X/a$
GRAPH 1


0.15
0
0.25
0.50
0.75
1.00
r'
CONCENTRATED
PARABOLIC
UNIFORM
INVERTED PARABOLIC
RIGID DISC
0.20
0.25
0.30
0.35
0.40
0.45
0.50
0.55
0.60
HOLLOW COLUMN
∞ AT $r' \to 1$
∞ AT $r' \to 0$
$N(r')$
INFLUENCE FACTORS
UNIT CIRCLE
$w_s = \frac{P}{aS_{13}} N(r')$; $r' = \frac{r}{a}$
GRAPH 2


VERTICAL AXIAL PRESSURE — 10³ LB/SQ FT.
(a) ISOTROPIC MEDIUM
(b) AEOLOTROPIC MEDIUM
VERTICAL PRESSURE DISTRIBUTION ALONG AXIS
FOR
COLUMN n = 2·5 FT. P = 2×10⁵ LB.
GRAPH 3
DEPTH IN FT.
(a)
(b)
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
0 1 2 3 4 5 6 7 8 9 10

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