California Institute of Technology

Physics Senior Thesis

Energy Non-Conservation in Quantum Mechanics

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May 29th, 2020

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Abstract

Conservation of energy is an integral component of modern physics, but questions remain in quantum mechanics. In traditional quantum mechanics, superpositions of energy eigenstates collapse into a single energy eigenstate upon measurement, and any change in energy after measurement is thought to be lost/gained in the measurement process. However, we argue that energy non-conservation in quantum mechanics cannot be entirely accounted for by leakage to the apparatus/environment. We first present a comprehensive Hamiltonian where energy is not conserved in any considered interpretation *according to an observer*. Next, we present a protocol for developing experiments to observe this non-conservation, then we use our protocol to develop an example thought experiment. In both the comprehensive Hamiltonian and thought experiment examples, energy is not conserved *to an observer* in all considered interpretations, but it is conserved when considering the universe's global wave function in Everettian quantum mechanics. Finally, we discuss implications for the status of conservation of energy and other conservation laws. The work presented in this thesis will be adapted into an upcoming paper, Carroll and Lodman (2020).

1 Introduction

The law of conservation of energy remains one of the most important physical laws, and it is used in problems that range from simple mechanics exercises in high school physics classes to complicated loop corrections in quantum field theory. Unfortunately, as with many aspects of physics, the move from classical to quantum systems brings new challenges. In particular, the need of an "interpretation" of quantum mechanics to connect mathematics with observations remains a controversial topic. However, as interpretations result in (generally; see Section (3.2.3)) the same predictions, there remain no methods to directly test (most) interpretations.

The conflict between interpretations connects directly with questions regarding energy conservation in quantum mechanics. The indeterminacy of measurement results in quantum mechanics can lead to apparent problems with the law as stated. In traditional quantum mechanics, when measured, superpositions of energy eigenstates collapse into a single eigenstate, whose eigenvalue is then observed [1]. This means that, for wave functions in superpositions of multiple energy eigenstates with *different* energy eigenvalues, only one outcome can be measured, leading to an apparent change in energy. What happened to the difference between superposition and eigenstate energies? Can we account for that change with leakage to the environment/apparatus?

In this work, we argue that, no, environmental leakage cannot account for all of the apparent violations of conservation of energy in quantum mechanics. We present an expressly time-independent, comprehensive Hamiltonian for which conservation of energy would be violated, *according to an observer,* in all interpretations we cover, and results in an actual violation in all interpretations except for Everettian QM. We then proceed to develop a protocol for designing experiments to test whether energy is conserved by minimizing energy leakage between system/environment, and present an example thought experiment. Said thought experiment results in the same non-conservation scenario as the comprehensive Hamiltonian previously presented. Finally, we tackle theoretical questions which arise from our work and illuminate our future directions.

In brief, the law of conservation of energy in quantum mechanics can appear to be violated by certain time-independent, comprehensive Hamiltonians regardless of the interpretation of quantum mechanics an observer subscribes to, and that apparent violation is an actual violation in all interpretations except for Everettian QM. This violation should be observable using an experiment developed from our protocol.

1.1 Our Definition of Energy

Perhaps unsurprisingly, the definition of energy in a quantum system has been a topic of much debate among physicists and philosophers of physics. Let us begin by explaining the origins of our definition. Beginning with Schrödinger's equation,

$$\frac{\partial |\Psi\rangle}{\partial t} = -\frac{i}{\hbar}\hat{H}\,|\Psi\rangle\,, \tag{1}$$

where $|\Psi\rangle$ and $\hat{H}$ are the universe's wave function and Hamiltonian respectively, we can write $|\Psi\rangle$ as

$$|\Psi\rangle = \sum_i c_i\,|E_i\rangle\,, \tag{2}$$

where $|E_i\rangle$ are the universe's energy eigenstates and c_i are the associated complex coefficients. Since energy should not be entering or exiting the universe, $\hat{H}$ should be time-independent, thus

$$|\Psi(t)\rangle = \sum_i c_i e^{\frac{-iE_i t}{\hbar}}\,|E_i\rangle\,. \tag{3}$$

The universe existing in an energy eigenstate corresponds to

$$|\Psi(t)\rangle = c_i e^{\frac{-iE_i t}{\hbar}}\,|E_i\rangle\,, \tag{4}$$

for some i. If the universe is indeed in an energy eigenstate, all that evolves in time is the phase, which is not physical. Therefore, no actual time evolution occurs. Meanwhile, if the universe is not in an energy eigenstate, there exists time evolution, as the different energy eigenstates interact with each other. Because we observe time evolution occurring, we infer that the universe is not in an energy eigenstate (see Section (2.2) for discussion of the emergence of time) [2].

With the previous discussion in mind, a pertinent question is, "what features would we desire in our definition of energy?" We would like it to always be defined, regardless of whether the system is in an eigenstate or not, and equal to the energy eigenvalue if it is in an energy eigenstate. We would also like it to be dependent only on the wave function and conserved under unitary evolution of a time-independent Hamiltonian. Therefore, we define the energy of a system with wave function $|\Psi\rangle$ under the Hamiltonian $\hat{H}$ as

$$E = \langle\Psi|\hat{H}|\Psi\rangle \,. \tag{5}$$

In mixed states, we use $E = \mathrm{Tr}(\hat{\rho}\hat{H})$, where $\hat{\rho}$ is the system's density matrix. This definition of energy satisfies the criteria presented above, while no commonly used alternatives fulfill all four.

1.2 Alternative Energy Definitions

Eq.(5) is not the only possible definition of energy. The Eigenvector-Eigenvalue (EE) rule is an alternative prominent in traditional (Standard Collapse Postulate/Copenhagen interpretation) quantum mechanics. It states that a quantity can only be said to have a value x of an operator $\hat{O}$ if x is the eigenvalue of an eigenstate of $\hat{O}$ and the quantum system is currently in that eigenstate. Using this definition of energy, systems in mixed quantum states would not have well defined energy [2]. However, since time evolution occurs in our universe, we assume the universe is not in an energy eigenstate, thus energy is never well defined under this definition.

If one assumes that Schrödinger's equation is not correct in its current form, as is the case in Dynamical Collapse models, the definition of energy, especially if one attempts to construct a conserved quantity, can quickly become much more complicated. However, proponents of such interpretations generally accept that these theories do not conserve energy, and use one of the already mentioned definitions (see Section (3.2.3) for a more thorough discussion of Dynamical Collapse models) [3, 2].

2 Conservation of Energy in Classical Physics

2.1 Non-Relativistic Physics

Conservation of energy is relatively straight forward to define in non-relativistic classical physics. However, as we shall quickly observe, the same cannot be said for quantum mechanics, as wave function collapse and measurement complicate proceedings. Classically, we define energy as

$$E = H = \sum_{i=1}^{N} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L, \tag{6}$$

where q_i are the generalized coordinates of the system, L is the Lagrangian of the system and a function of $q_i, \dot{q}_i$, and t (thus energy is the same as the system's Hamiltonian, and we will refer to them interchangeably from now on) [4].

Conservation of energy was connected to symmetries in 1918 by Emmy Noether in the theorem that bares her name [5]. Noether's theorem states that every symmetry in a Lagrangian is associated with a conserved quantity [6]. For time-invariance, the conserved quantity is energy. In the language of Lagrangian dynamics, a Lagrangian is considered time-invariant if

$$\frac{\partial L}{\partial t} = 0, \tag{7}$$

thus the system has no explicit time dependence [4]. Therefore,

$$\begin{aligned}\frac{dE}{dt} &= \frac{d}{dt}(\sum_{i=1}^{N}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L)\\ &= \sum_{i=1}^{N}(\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\ddot{q}_i) - \frac{dL}{dt}\\ &= \sum_{i=1}^{N}((\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i) - (\frac{\partial L}{\partial \dot{q}_i}\dot{q}_i + \frac{\partial L}{\partial \ddot{q}_i}\ddot{q}_i)) - \frac{\partial L}{\partial t}\\ &= -\frac{\partial L}{\partial t},\end{aligned} \tag{8}$$

by the Euler-Lagrange Equations of Motion (proof taken from [4]). However, since we assumed that the Lagrangian had no explicit time dependence,

$$\frac{dE}{dt} = -\frac{\partial L}{\partial t} = 0, \tag{9}$$

as desired. Conservation of energy in non-relativistic classical physics can be easily derived and connected to time-invariance using Noether's theorem [5, 6, 4].

2.2 Conservation of Energy in General Relativity

The "energy" conservation law most commonly invoked in introductory general relativity is the conservation of energy-momentum of matter. It is written as

$$T^{\alpha\beta}{}_{,\beta} = 0 \tag{10}$$

in special relativity, and as

$$T^{\alpha\beta}{}_{;\beta} = 0 \tag{11}$$

in general relativity, using the normal conversion of "comma-to-semicolon" [7]. This definition, though almost uniformly accepted as the correct statement of the conservation law, is not without its issues [8]. Hoefer argues that Eq.(11) is actually not a statement of conservation of energy, because Eq.(11) cannot be written as an integral relating the flow of energy into/out of a volume via Gauss's law. In fact, Einstein himself stated that this was not a statement of local energy-momentum conservation in his original paper, and instead represents "interchange of energy and momentum between gravitational and ordinary stress-energy" [9, 8]. Attempts to define the conservation law as

$$(T^{\alpha\beta} + t^{\alpha\beta})_{,\beta} = 0, \tag{12}$$

where $t^{\alpha\beta}$ is the gravitational stress-energy psuedotensor, fail, since $t^{\alpha\beta}$ is a pseudotensor, thus the result of applying Gauss's law to create an integral conservation law is not well-defined for physical situations [8].

However, conservation of energy-momentum is not the entire story. In the Arnowitt-Deser-Misner (ADM) formalism, general relativity is rewritten in a Hamiltonian formalism. This formalism begins with the Einstein-Hilbert Lagrangian,

$$\mathcal{L} = R\sqrt{-g} \tag{13}$$

where g is the four-dimensional metric tensor and R is the associated Ricci scalar. Then, considering the 3+1 decomposition of the Einstein field, the generalized coordinates are the spatial metric tensor components g_{ij}. The conjugate momenta can then be expressed as

$$\pi^{ij} = \sqrt{-g}(\Gamma_p{}^0{}_q - g_{pq}\Gamma_r{}^0{}_s g^{rs})g^{ip}g^{jq}, \tag{14}$$

where Γ are the four-dimensional Christoffel symbols. The Hamiltonian can then be determined from the Lagrangian in the usual method. However, that Hamiltonian is in the form of a boundary term, leading to the constraint

$$H_{ADM} = -\sqrt{g}({}^3R + \frac{1}{g}(\frac{1}{2}\pi^2 - \pi^{ij}\pi_{ij})) = 0, \tag{15}$$

where H_{ADM} is the ADM formalism Hamiltonian of the universe and ${}^3R$ is the curvature scalar formed from g_{ij} [10]. This equation contains no time evolution, so why do we appear to observe time evolution? The answer is that time is emergent, as seen in the Wheeler-Dewitt equation, a step towards quantum gravity. The Wheeler-Dewitt equation says that

$$\hat{H}_{WD} |\Psi\rangle = 0, \tag{16}$$

a similar Hamiltonian constraint as in the ADM formalism, though the ADM formalism is purely classical, while Wheeler-Dewitt includes quantum mechanics. We can write $\hat{H}_{WD}$ from Eq.(16) in the form

$$\hat{H}_{WD} = \hat{H}_{eff} + i\hbar\frac{\partial}{\partial\lambda}, \tag{17}$$

where $\hat{H}_{eff}$ is the effective ("observed") Hamiltonian and λ is the emergent time parameter. We can then rewrite Eq.(16) as

$$\frac{\partial}{\partial\lambda} |\Psi\rangle = -\frac{i}{\hbar}\hat{H}_{eff} |\Psi\rangle, \tag{18}$$

which is identical to Schrödinger's equation if we make the substitution $\lambda \to t$ [11, 12, 13]. Therefore, when we refer to conservation of energy, we are implicitly referring to $\hat{H}_{eff}$, not $\hat{H}_{WD}$. We make no further statements on the emergence of time in this work.

Although the law of conservation of energy is treated as ironclad in classical physics, questions still arise relativistically. Since the definition of conservation of energy is subtle classically, the existence of further subtleties in quantum mechanics should not be surprising.

3 Conservation of Energy in Quantum Mechanics

3.1 The Basics of Energy Conservation

In quantum mechanics, evolution controlled by Schrödinger's equation is known as unitary evolution, which is expressed, as before, as

$$\frac{\partial |\Psi\rangle}{\partial t} = \frac{-i}{\hbar}\hat{H} |\Psi\rangle \tag{19}$$

[1]. An operator $\hat{X}$ represents a conserved quantity under a Hamiltonian $\hat{H}$ if

$$[\hat{X}, \hat{H}] = 0, \tag{20}$$

thus $\hat{X}$ and $\hat{H}$ share a common set of eigenstates [2, 3]. Therefore, energy is conserved if:

$$[\hat{H}(t_1), \hat{H}(t_2)] = 0 \tag{21}$$

for any t_1 and t_2. For time-independent Hamiltonians, $\hat{H}(t) = \hat{H}$, thus energy is always conserved under unitary evolution of a time-independent Hamiltonian [2, 1]. Barring theories that modify Schrödinger's equation in some way (see Section (3.2.3)), this much is generally agreed upon by practitioners [2, 3, 1].

3.2 Measurement and Interpretations of Quantum Mechanics

However, the evolution of systems is not always unitary in traditional quantum mechanics. During "measurement," a process whose definition is incredibly controversial, a single eigenvalue of an operator is observed by a measurement apparatus. Probabilities are assigned to each eigenstate by the Born rule, which states that the probability of a certain state being observed is equal to its amplitude squared (normalized to unity) [14]. To discuss measurement further, we must narrow our focus to particular "interpretations" of quantum mechanics.

Interpretations of quantum mechanics connect observations with the mathematics behind them [15]. The status of wave function is much less agreed upon by practitioners than classical objects tend to be. This, coupled with the lack of methods to experimentally test (most, see Section (3.2.3)) interpretations, leads their study to be dismissed by some researchers. However, their ontological differences lead to dramatic consequences for conservation laws.

3.2.1 Standard Collapse Postulate QM/Copenhagen Interpretation

Standard Collapse Postulate QM, better known as the Copenhagen interpretation, is the interpretation of quantum mechanics typically taught to students. This interpretation does not elaborate on how measurement chooses an eigenvalue, only that it does so and the result is observed [3]. However, this immediately leads to an obvious problem.

For example, take a system that is represented by a wave function

$$|\Psi\rangle = \alpha\,|E_1\rangle + \beta\,|E_2\rangle\,, \tag{22}$$

where the kets correspond to energy eigenstates and $|\alpha|^2 + |\beta|^2 = 1$. Before measurement, the system has, by our definition of energy (Eq.(5)), $E_0 = |\alpha|^2 E_1 + |\beta|^2 E_2$, while after measurement, the system has either E_1 or E_2. Assuming $E_1 \neq E_2$, where did the missing energy go/gained energy come from? Before we continue, it should be noted that using the EE rule, which is often done in this interpretation, energy would not have been well defined before measurement, thus even though the system is now in an energy eigenstate, one cannot say that energy was truly conserved.

If this question was raised in an introductory physics course, the answer would likely be, "it was lost in the measurement process." Of course, the measurement apparatus was not treated as part of our quantum system, but as part of the classical world surrounding the system [15]. The Von Neumann measurement apparatus was formulated to treat the measuring apparatus as a quantum object as well as the system, by sending the total Hilbert space $\tilde{H}_{tot}$ to

$$\tilde{H}_{tot} = \tilde{H}_{sys} \rightarrow \tilde{H}_{sys} \otimes \tilde{H}_a, \tag{23}$$

where $\tilde{H}_{sys}$ and $\tilde{H}_a$ are the system's and apparatus's Hilbert spaces respectively, and

$$\begin{aligned} \tilde{H}_{sys} &\in \mathrm{span}\{|x_{n_s}\rangle\},\ n \in \{1, ..., d_s\} \\ \tilde{H}_a &\in \mathrm{span}\{|q_{m_a}\rangle\},\ m \in \{1, ..., d_a\} \end{aligned} \tag{24}$$

[2, 15].

Although it may not be obvious, the Von Neumann measurement apparatus "measures" a quantity by entangling the system and apparatus. To demonstrate this remarkable property, take a system/apparatus pair initially in an unentangled state

$$|\Psi(0)\rangle = |\Psi_{sys}\rangle \otimes |q_0\rangle\,, \tag{25}$$

where $|\Psi_{sys}\rangle$ is the system's wave function and $|q_i\rangle$ are possible apparatus states. Eq.(25) acts under the Hamiltonian

$$\hat{H} = \hat{H}_{sys} + \hat{H}_a + \hat{H}_{int}, \tag{26}$$

where $\hat{H}_{sys}, \hat{H}_a$, and $\hat{H}_{int}$ are the system's self Hamiltonian, the apparatus's self Hamiltonian, and the interaction Hamiltonian between the system and apparatus respectively. Let us decide to measure a quantity $\hat{X}$ with our apparatus, and for ease, let $\hat{X}$ commute with $\hat{H}_{sys}$ and $\hat{H}_a$, so we may just consider $\hat{H}_{int}$. $\hat{X}$ may be represented as

$$\hat{X} = \sum^{n} x_n\,|x_n\rangle\,\langle x_n|\,. \tag{27}$$

With the benefit of foresight, we choose

$$\begin{aligned} H_{int} &= f(t)\hat{X}_s \otimes \hat{P}_A \\ \int_o^t & f(x)dx = 1. \end{aligned} \tag{28}$$

Thus, $f(t)$ refers to the interaction remaining on from $t = 0$ to time t. From this,

$$\begin{aligned} |\Psi(t)\rangle &= U(t)\,|\Psi(0)\rangle \\ U(t) &= e^{-i\int \hat{\mathcal{H}}_{int}dt}. \end{aligned} \tag{29}$$

It can then be easily shown that

$$|\Psi(t)\rangle \approx \sum^{n} x_n |x_n\rangle |q_n\rangle \,, \tag{30}$$

since

$$e^{-ix\hat{P}_A} |q_i\rangle = |q_{i+x}\rangle \,. \tag{31}$$

As the eigenstates of the system and apparatus are now paired, the apparatus measures the state of the system via entanglement, as the state of the system can be determined from knowing the state of the apparatus. In summary, the Von Neumann measurement apparatus allows a quantum apparatus to measure a quantum system by becoming entangled with it [15].

While the Von Neumann measurement apparatus is a step in the right direction, unanswered questions remain. It does not specify how the observer measures the result on the apparatus, nor how the specific result is chosen [15]. It is at this point that the standard formalism tends to end its analysis of measurement.

3.2.2 Everettian Quantum Mechanics

Everettian quantum mechanics, also known under the name "Many Worlds" interpretation, is incredibly similar to Standard Collapse Postulate QM, except there is no collapse postulate. All that exists in Everettian QM is the wave function, which evolves unitarily through Schrödinger's equation [16]. This interpretation treats everything as quantum, or

$$\tilde{H} = \tilde{H}_{sys} \otimes \tilde{H}_a \otimes \tilde{H}_e, \tag{32}$$

where $\tilde{H}_e$ is the environment's Hilbert space and the others are defined as they were in the Von Neumann measurement apparatus [15, 16]. The process of "measurement" in Everettian QM is broken down into three steps: measurement, decoherence, and observation, *but evolution remains completely unitary.* Measurement occurs when the system becomes entangled with the measurement apparatus. After the system and apparatus become entangled, the environment states quickly decohere, as different states have approximately no overlap ($\langle e_i|e_j\rangle \approx \delta_{ij}$) with each other. Finally, an observer becomes entangled with the system-apparatus-environment when they observe it [15]. Therefore, just as how in the Von Neumann measurement formalism, the apparatus measures the quantum state of a system through entanglement, the environment measures the state of the system/apparatus in Everettian QM [16]. The combination of measurement, decoherence, and observation is known as "branching." Due to the lack of overlap between environment states, the branches do not interact with each other. The branches are the "worlds" which give the "Many Worlds" interpretation its name.

As the observer decoheres, one observer evolves into n observers (one for each decohered state). Observers only see the outcome associated with their branch. However, the entire wave function continues to exist, but remains inaccessible to observers located on a particular branch! To summarize,

$$\begin{aligned}
|\Psi(t_0)\rangle &= \sum_{i=1}^{n} \phi_i |\phi_i\rangle |a_0\rangle |e_0\rangle |O_0\rangle \\
|\Psi(t_1)\rangle &\approx \sum_{i=1}^{n} \phi_i |\phi_i\rangle |a_i\rangle |e_0\rangle |O_0\rangle \;\text{ Measurement} \\
|\Psi(t_2)\rangle &\approx \sum_{i=1}^{n} \phi_i |\phi_i\rangle |a_i\rangle |e_i\rangle |O_0\rangle \;\text{ Decoherence} \\
|\Psi(t_3)\rangle &\approx \sum_{i=1}^{n} \phi_i |\phi_i\rangle |a_i\rangle |e_i\rangle |O_i\rangle \;\text{ Observation,}
\end{aligned} \tag{33}$$

where the states labeled by $|\phi_i\rangle, |O_i\rangle, |a_i\rangle$, and $|e_i\rangle$ refer to the system, observer, apparatus, and environment states respectively, and $\langle e_i|e_j\rangle \approx \delta_{ij}$ [16].

Questions of why probabilities exist at all in this interpretation, and how they should be interpreted, remain relevant [16, 17]. The existence of probabilities occurs because of the "Self-Locating Uncertainty," or that even if the wave function of the entire universe was known to an observer, that observer would not know what branch they were on, as measurement would cause their current branch to split [16]. If probabilities exist, Gleason's theorem identifies them as those assigned by the Born rule [17, 14]. However, what exactly do probabilities correspond to? In truth, while this is an open question in the field, we use the interpretation that the probabilities given by the Born rule correspond to the probability that an observer would find themselves on a particular branch if they performed the appropriate measurement [18].

Let us return to the example begun with Eq.(22) in the previous section. An observer on one branch would observe that the energy of the system is E_1, while on the other branch, the observer would find the energy of the system to be E_2. However, since an observer has the probability $|\alpha|^2$ to observe they are on the E_1 branch and probability $|\beta|^2$ to observe that they are on the E_2 branch, the expectation value of the energy is still

$$E_f = |\alpha|^2 E_1 + |\beta|^2 E_2 = E_0. \tag{34}$$

Therefore, while observers would not see it as conserved, energy is conserved in the universe as a whole. Note that this example does not take the apparatus/environment into account, but merely illustrates the idea that, while energy can appear to be non-conserved in Everettian QM *from an observer's prospective*, in reality, it is conserved in the universe's wave function.

In brief, in Everettian QM, only the wave function exists, and systems are measured through decoherence after being entangled with their environment. All outcomes of a measurement exist, however observers only see the outcome associated with their branch of the wave function, as observers branch with the rest of the quantum universe. Therefore, apparent violations may exist even if actual violations do not [16, 17].

3.2.3 Dynamical Collapse Models

Dynamical Collapse models are interpretations of quantum mechanics that modified Schrödinger's equation with non-linear, stochastic terms that make collapse a physical process [2, 3]. However, modifying Schrödinger's equation means that evolution is expressly non-unitary, thus there is no guarantee of energy conservation even outside of collapse [19].

The two most popular models in this category are Spontaneous Localization (SL), aka Ghirardi-Rimini-Weber (GRW) theory, and its modified version Continuous Spontaneous Localization (CSL). In GRW theory, an extra type of evolution is added in addition to unitary evolution. Randomly, the wave function spontaneously localizes itself at a particular point in space. More precisely, for a one particle system, let a particular $\vec{q}$ be chosen with probability

$$P(\vec{q}) = \int_{-\infty}^{\infty} |\psi(\vec{x} = \vec{q})|^2 dx. \tag{35}$$

Then this new type of evolution is defined by

$$|\psi(\vec{x})\rangle \to g(\vec{q}, \vec{x}) \, |\psi(\vec{x})\rangle \,, \tag{36}$$

where

$$g(\vec{q}, \vec{x}) = K e^{\frac{-(\vec{q}-\vec{x})^2}{2d^2}} \,, \tag{37}$$

where K is the normalization constant and $d \sim 10^{-5}$ is known as the localization accuracy [19, 20]. For a single particle, this evolution must occur with a rate of $\frac{10^{-16}}{s}$ to match observations, but that rate is proportional to particle number density, so macroscopic objects localize approximately instantaneously. There is an associated increase in energy due to narrowing of the wave function, thus energy is understood to not be conserved in this interpretation [2, 3].

In CSL, collapse is a continuous process instead of a random event, but evolution is still expressly non-unitary [2]. A classical field interacts with particle number density (or mass density; can be expressed multiple ways), causing the system to collapse towards eigenstates of mass/energy density, again leading to an increase in energy [3, 2]. Therefore, energy is expressly non-conserved. This lack of conservation has been used to constrain CSL parameters [3]. For example, Collett et al. used Ge atoms to constrain localization rates in CSL. Overtime, the constituent electrons of the Ge atoms should gain energy, thus eventually become excited. The theorized excitation rate was then compared to the rate that seemingly random X-ray pulses are generated by Ge atoms. This produced an experimental bound on localization rate of electrons in CSL [21].

Dynamical Collapse models are interpretations of quantum mechanics that explicitly lack energy conservation under traditional definitions, but have remained of interest to researchers due to their potential testability, which is rare among interpretations [19, 21].

3.2.4 Hidden Variable Theories/Bohmian Mechanics

Hidden Variable theories, such as Bohmian Mechanics, grew out of the idea that there exist so-called "hidden variables," or variables that cannot be observed, but make the seemingly indeterministic quantum mechanics deterministic [22]. However, Bell's theorem proved that the results of quantum mechanics cannot be replicated if said hidden variables are local [23]. This led to the growth of non-local hidden variables, of which de-Broglie-Bohm theory, aka Bohmian Mechanics, is the most influential.

There exists two equations of merit,

$$\frac{\partial |\Psi\rangle}{\partial t} = \frac{-i}{\hbar}\hat{H}|\Psi\rangle \tag{38}$$

$$\frac{dX_i}{dt} = \frac{\hbar}{m_i}\mathrm{Im}(\frac{\vec{\nabla}_i \cdot \Psi}{\Psi})(\{X_1, ..., X_n\}(t)), \tag{39}$$

where the first equation is the usual form of Schrödinger's equation, the latter is called the "Guidance Equation," and the X_i are particle positions [19, 2]. These equations are, as promised, deterministic. Schrödinger's equation guides the particles, but is not affected by them, leading to the alternative name of "Pilot-Wave" theory. Bohmian Mechanics recovers the results of Standard Collapse Postulate/Everettian QM as the configuration of particles before measurement is not known, thus if one begins with a configuration that follows Born statistics, it will remain in one after measurement [19].

Measurement in Bohmian Mechanics is similar to measurement in the Von Neumann measurement apparatus. First, divide the universe into a system and apparatus, or

$$\{\vec{X_i}\} = \{\vec{s_a}, \vec{y_n}\}, \tag{40}$$

where $\{\vec{s_a}\}$ and $\{\vec{y_n}\}$ refer to the possible system/apparatus hidden variable values respectively. If we begin in an unentangled state

$$\Psi(t=0) = [\sum_{\alpha} \psi_\alpha(\{s_a\})]\phi_0(\{\vec{y_n}\}), \tag{41}$$

where ϕ is a pointer, unitary evolution brings us to

$$\Psi(t=0) = \sum_{\alpha} \psi_\alpha(\{S_a\})]\phi_\alpha(\{\vec{y_n}\}), \tag{42}$$

thus the pointer has become completely entangled with the system. If we measure $\{\vec{y_n}\} = \{\vec{Y_n}\}$, thus $\{\vec{Y_n}\}$ is the actual apparatus hidden variable value, the conditional wave function $\Psi(\{s_a\})$ is

$$\psi(\{s_a\}) = \Psi(\{s_a\}, \{\vec{Y_n}\}). \tag{43}$$

For a well chosen pointer, $\{\vec{Y_n}\}$ will fix ϕ_α. Therefore, if $\{\vec{Y_n}\}$ is associated with ϕ_{α_i}, Eq.(43) can be rewritten as

$$\psi(\{s_a\}) = \psi_{\alpha_i}(\{s_a\})\phi_{\alpha_i}(\{\vec{Y_n}\}). \tag{44}$$

The conditional wave function is then used to determine the evolution of the particle positions ($\{X_i\}$) using the Guidance Equation [19].

One of the unique properties of Hidden Variable theories are that many quantities are contextual, or dependent on the particular experiment, and do not exist independently. In Bohmian Mechanics, all variables except particle positions, including energy, are taken as contextual. What does that mean for energy conservation? Energy can by defined by Eq.(5) or the EE rule, however, proponents of this interpretation take issue with these definitions, as they only involve contextual quantities. Even so, there does not exist a quantity dependent on non-contextual quantities (positions) that is both generally accepted as energy and always conserved for time-independent Hamiltonians. This is an area of active research among proponents [2].

In summary, Hidden Variable theories, such as Bohmian Mechanics, posit that there exist some unknowable variables that, if known, would make quantum mechanics deterministic, but the concept of energy is not well defined [19, 22, 2].

4 Statement of the Problem and Previous Work

From Eq.(21), energy is conserved under unitary evolution of a time-independent Hamiltonian. However, during non-unitary evolution, such as during wave function collapse, no such statement can be made (see Section(3.2.1); existence of non-unitary evolution depends on interpretation). Energy can appear to not be conserved (see Eq.(22) discussion), but it is possible that all of the energy involved is simply not accounted for (e.g. lost to apparatus/environment) during measurement. In other words, is the apparent non-conservation of energy due to our sloppiness or does conservation of energy, as currently formulated/interpreted, simply not always hold in particular interpretations of quantum mechanics?

To answer these questions and more, we first considered previous works related to these questions. Then, we looked to improve on the simple example in Eq.(22) to account for the environment, and found that it is possible to write a manifestly time-independent, comprehensive Hamiltonian that would not appear, to an observer, to conserve energy in all previously discussed interpretations. After that, we developed a protocol to develop experiments to observe this (apparent) non-conservation. Finally, using our protocol, we designed a realistic thought experiment that could be used to achieve an arbitrarily high level of (apparent) non-conservation. However, in both the comprehensive Hamiltonian and thought experiment examples, the violation is only apparent in Everettian QM, but real in the other discussed interpretations.

4.1 Previous Work

Pearle argues that only a set of measure zero (points enclosed by arbitrary small interval) of quantum states satisfy conservation of momentum (all states that satisfy momentum conservation satisfy conservation of energy as well), and only a measure zero set of those states are "macroscopically" distinct [3, 24]. For the system

$$|\Psi\rangle = |\Phi\rangle\,|\phi\rangle\,, \tag{45}$$

where $|\Phi\rangle$ is the C.O.M wave function and $|\phi\rangle$ is the internal wave function, if it begins in the mixed state

$$|\Psi(0)\rangle = \alpha_1\,|\Psi\rangle_1 + \alpha_2\,|\Psi\rangle_2\,,\ |\alpha_1|^2 + |\alpha_2|^2 = 1, \tag{46}$$

states must satisfy

$$\langle\Psi_1|p\rangle\,\langle p|\Psi_2\rangle\, = 0 \tag{47}$$

to conserve momentum (and energy). This condition extends to Hilbert spaces including the apparatus and environment [3]. Therefore, Pearle argues that conservation of energy-momentum is realistically unobtainable, and because of that, Dynamical Collapse models should not be dismissed just because they do not conserve energy.

While Pearle focuses on a more theoretical discussion, Aharonov et al. postulate a thought experiment that has the potential to show energy non-conservation using super oscillators. Super oscillator are functions that can oscillate faster than their fastest Fourier component over a small region. An example of such a function is

$$f(x) = \sum_{n=0}^{N} c(n,N,\alpha)e^{ix(\frac{2n}{N}-1)} \tag{48}$$

$$c(n,N,\alpha) = \frac{1}{2^N}\binom{N}{n}(1+\alpha)^n(1-\alpha)^{N-n}. \tag{49}$$

In the region $|x| \lessapprox \sqrt{N}$, Taylor expanding this function gives

$$\begin{aligned} f(x) &\approx (\frac{1+\alpha}{2}(1+i\frac{x}{N})\frac{1-\alpha}{2}(1-i\frac{x}{N}))^N \\ &= (1+\frac{i\alpha x}{N})^N \\ &\approx e^{i\alpha x}. \end{aligned} \tag{50}$$

Since α can be any real number, the super oscillator can oscillate faster than its fastest Fourier component.

The experiment setup involves a box of length $2\pi Na$ with a single photon inside. The photon has wave function $|\psi(x)\rangle = \frac{i}{N}(f(\frac{x}{a}) - f^*(\frac{x}{a}))$ and $\alpha > 1$. This corresponds to the photon in energy eigenstates of $E_n = |\frac{2n}{N} - 1| \leq 1$ (in natural units). However, near the center of the box, the photon appears to be a plane wave with $E_\alpha = \alpha > 1$. Therefore, $E_\alpha > E_n$ for any E_n.

There is a mechanism in the box to insert and remove a mirror from the center of the box. The mirror is in the box for $t \lessapprox \sqrt{N}$, and the photon has the probability $\int_{-t}^{t} |\psi(x)|^2 dx$ to hit the mirror and fly out of the box. If it does so, what is the energy of the photon that is measured? The only information that had time to reach the mirror was that the photon was a plane wave of $E = \alpha$, which is greater than the energy of any one Fourier component, and can, in fact, be arbitrarily high. Does this violate conservation of energy? Unfortunately, this work was supposed to be the first in a two part series, with the author's conclusion in the second part, which was apparently never published [25].

5 Comprehensive Hamiltonian Example

Is it possible to write a time-independent, comprehensive (accounts for both system/environment) Hamiltonian that would result in an (apparent) violation of conservation of energy? We provide an example that clearly shows it is possible, and energy is, *according to an observer,* not conserved under any mentioned interpretation, but is conserved in the universe's global wave function in Everettian QM. In the remaining interpretations, the apparent violation is real.

5.1 Explicit Comprehensive Example

While Pearle just argues that it is possible for time-independent Hamiltonians to not conserve energy, we will provide an actual example of a comprehensive Hamiltonian which does not conserve energy, *from the perspective of an observer,* in any aforementioned interpretation [3]. Begin with the Hilbert space $\tilde{H}$,

$$\tilde{H} = \tilde{H}_s \otimes \tilde{H}_e \tag{51}$$

$$\tilde{H}_s = \text{span}\{|1\rangle_s, |2\rangle_s\} \tag{52}$$

$$\tilde{H}_e = \text{span}\{|0\rangle_e, |1\rangle_e, |2\rangle_e\}, \tag{53}$$

where $\tilde{H}_s$ and $\tilde{H}_e$ refer to the system/environment Hilbert spaces respectively. Let the initial condition of the wave function be

$$|\psi(0)\rangle = (\alpha |1\rangle_s + \beta |2\rangle_s) |0\rangle_e, \tag{54}$$

so that the system begins completely unentangled with the environment and $|\alpha|^2 + |\beta|^2 = 1$. Our Hamiltonian takes the form

$$\hat{H} = \hat{H}_{self} + \hat{H}_{int}, \tag{55}$$

where $\hat{H}_{self}$ and $\hat{H}_{int}$ refer to the self and interaction Hamiltonians respectively. We assign energy E_1 to $|1\rangle_s$ and energy E_2 to $|2\rangle_s$, where $E_1 \neq E_2$, while the environment states do not differ in energy. This corresponds to

$$\hat{H}_{self} = (E_1 |1\rangle_s \langle 1| + E_2 |2\rangle_s \langle 2|) \otimes \mathbb{1}_e. \tag{56}$$

To entangle the system and environment states, we chose

$$\hat{H}_{int} = -i\lambda[|1\rangle_s \langle 1| \otimes (|0\rangle_e \langle 1| - |1\rangle_e \langle 0|) + |2\rangle_s \langle 2| \otimes (|0\rangle_e \langle 2| - |2\rangle_e \langle 0|)]. \tag{57}$$

We exponentiate the Hamiltonian to find the unitary time evolution operator

$$\hat{U}(t) = \exp(i\hat{H}t) = e^{-iE_1 t} |1\rangle_s \langle 1| \otimes \begin{pmatrix} \cos\lambda t & -\sin\lambda t & 0 \\ \sin\lambda t & \cos\lambda t & 0 \\ 0 & 0 & 1 \end{pmatrix}_e + e^{-iE_2 t} |2\rangle_s \langle 2| \otimes \begin{pmatrix} \cos\lambda t & 0 & -\sin\lambda t \\ 0 & 1 & 0 \\ \sin\lambda t & 0 & \cos\lambda t \end{pmatrix}_e . \tag{58}$$

Combining Eq.(54) and Eq.(58), we find that

$$|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle = \alpha e^{-iE_1 t} |1\rangle_s (\cos\lambda t |0\rangle_e + \sin\lambda t |1\rangle_e) + \beta e^{-iE_2 t} |2\rangle_s (\cos\lambda t |0\rangle_e + \sin\lambda t |2\rangle_e). \tag{59}$$

At time $t^* = \frac{\pi}{2\lambda}$,

$$|\psi(t^*)\rangle = \alpha e^{-iE_1 t^*} |1\rangle_s |1\rangle_e + \beta e^{-iE_2 t^*} |2\rangle_s |2\rangle_e , \tag{60}$$

thus the system is fully entangled. The energy at t^*, using Eg.(5), is

$$E = \langle\psi(t^*)|\hat{H}|\psi(t^*)\rangle = \langle\psi(t^*)|\hat{H}_{self}|\psi(t^*)\rangle + \langle\psi(t^*)|\hat{H}_{int}|\psi(t^*)\rangle , \tag{61}$$

where

$$\begin{aligned} \langle\psi(t^*)|\hat{H}_{self}|\psi(t^*)\rangle &= |\alpha|^2 E_1 + |\beta|^2 E_2 \\ \langle\psi(t^*)|\hat{H}_{int}|\psi(t^*)\rangle &= 0, \end{aligned} \tag{62}$$

thus,

$$E = \alpha|^2 E_1 + |\beta|^2 E_2. \tag{63}$$

Therefore, if an observer finds the system to be in the state $|1\rangle_s$, the energy of the system is E_1, and similarly for $|2\rangle_s$ and E_2. Since $E_1 \neq E_2$,

$$E_f = E_1 \text{ or } E_2 \neq |\alpha|^2 E_1 + |\beta|^2 E_2 = E_0 \tag{64}$$

$$E_0 \neq E_f , \tag{65}$$

and energy is not conserved according to an observer.

5.2 Discussion

This example shows, *from an observer's prospective*, a clear change in the system's energy that is not accompanied by a corresponding change in the environment's energy. However, whether that apparent change corresponds to an actual change of the universe's total energy depends on one's interpretation of quantum mechanics. For example, in Standard Collapse Postulate QM, energy is not conserved, as only one outcome exists. Meanwhile, energy is always non-conserved in Dynamical Collapse models, thus it is not conserved in this example either, and energy is not well-defined in Bohmian Mechanics (see Section (3.2.4)) [19, 2]. This leaves Everettian QM, where, from the perspective of an observer, energy is not conserved, but this non-conservation is only apparent. Measurement does not "collapse" Eq.(60) into one of its eigenstates; the entire wave function still exists after measurement. The energies of the branches are weighted by the respective probability of an observer finding themselves on that branch, thus the energy after measurement is still equal to Eq.(63) [18]. Although an individual observer will measure the same result regardless of interpretation, and will see an apparent non-conservation of energy, only in Everettian QM is energy actually conserved in the universe.

An astute reader might point out that the system immediately begins to recohere after t^*. We acknowledge that this is indeed the case, and is due to the relatively few possible environment eigenstates. By adding more environment states, the system will remain decohered for an arbitrarily long time.

If we can construct a Hamiltonian that does not conserve energy (at least from an observer's perspective), it should be possible to construct an experiment to observe that lack of conservation. Designing such an experiment, in the form of a thought experiment, is the subject of the remaining part of this work.

6 Testing Energy Non-Conservation in QM

If conservation of energy does not always hold, it should have experimental implications. Therefore, it should be possible to design an experiment which would measure the change in energy. As this is a theoretical work, this experiment will take the form of a thought experiment. However, we hope that this work will be of use to experimenters who might seek to actually design such an experiment. For convenience, we will use natural units in this section unless explicitly noted.

6.1 Experiment Protocol

In order to view energy non-conservation experimentally, we must measure the energy of a system while minimizing the potential for energy to enter/leave the system during the measurement process. The level of non-conservation must also be too massive to be accounted for by experiment errors. To develop experiments to observe energy non-conservation, whether apparent or actual, we developed the following protocol:

1. Begin with the system in a quantum superposition of energy eigenstates with different energy eigenvalues.
2. Entangle system with a probe, minimizing the potential for energy transference.
3. Measure the state of the probe system, again minimizing the potential for energy transference.
4. End with the system in an approximate energy eigenstate with a substantial difference in energy from prior to the measurement.

We will now outline an experiment that follows the above protocol and can produce an arbitrarily high level of non-conservation.

6.2 Example Thought Experiment

6.2.1 Brief Overview

Imagine that we initially have a proton (particle 1) sitting at rest at the origin of our system (say held in place in a Penning Trap) [26]. While Penning Traps consist of both electric and magnetic fields, it is only the constant magnetic field $B\vec{(r)} = B\hat{z}\theta(R-r)$, which extends to a radius R from the origin and is symmetric about it, that plays a roll beyond preventing the proton from moving.

At the beginning of the experiment, a neutron (particle 2) is fired at the system, moving at a velocity v. It begins at $(vt_i, b, 0)$, where t_i corresponds to the beginning of the experiment, and ends at $(vt_f, b, 0)$, where t_f corresponds to the end of the experiment. We make the approximation that the neutron travels at less than relativistic speeds (<0.01c), and will travel at a constant velocity in the $\hat{x}$ direction, while remaining at the same position in the $\hat{y}/\hat{z}$ directions. We can make this assumption as the change in the kinetic energy should be much less than the change in the particles' potential energies, and we focus on the produced entanglement. We cover the actual classical path in Appendix (A).

Particle 2 will fly past particle 1, becoming entangled with it, but with minimal opportunity to exchange energy, as the proton is held in place by the Penning Trap [26]. We will choose parameters/initial conditions to maximally entangle particles 1 and 2. After passing particle 1, particle 2 will continue to move in a straight line until it leaves the magnetic field. Once that occurs, the spin of particle 2 can be measured. Because we chose parameters to maximally entangle the particles, the spin of particle 2 will tell us the spin of particle 1. This will allow us to determine particle 1's energy, as the energy of particle 1, with the distance between particles 1 and 2 $\approx \pm\infty$, is

$$\begin{aligned} E_{1_\uparrow} &= \frac{\omega_1 B}{2} \\ E_{1_\downarrow} &= -\frac{\omega_1 B}{2}. \end{aligned} \tag{66}$$

Since the energy of particle 1 is proportional to the magnetic field, the difference in energy before/after measurement can be made arbitrarily large, preventing the non-conservation from being accounted for by energy leakage between the system and environment. This example experiment minimizes the chance for energy to enter/leave the system, allowing experimenters to test if the seemingly "gained/lost" energy is gained from/lost into the apparatus/environment or if conservation of energy may not always hold. Fig.(1) presents a graphical representation of the experiment as described in the previous paragraphs.

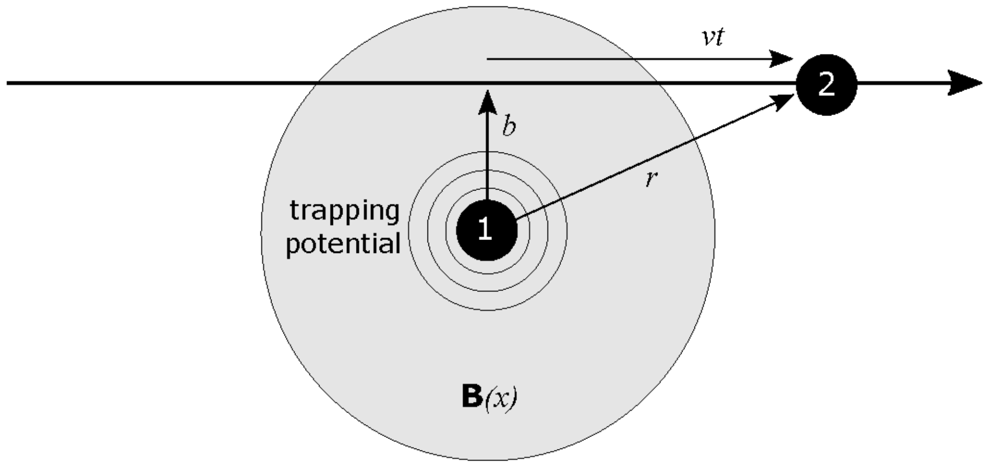

Figure 1: The thought experiment layout as described in Section (6.2.1).

6.2.2 Time Evolution of System/Apparatus

Because of the considerations mentioned in the previous section, we only need to consider the spin-dependent Hamiltonian. Therefore, the system's Hamiltonian $\hat{H}$ is defined as

$$\hat{H} = \omega_1 S_{z_1} + \omega_2 S_{z_2} \Theta(R - r) + g\frac{((\vec{S_1} \cdot \vec{S_2}) - 3(\vec{S_1} \cdot \hat{r})(\vec{S_2} \cdot \hat{r}))}{4\pi r^3}, \tag{67}$$

where $S_{i_n} = \frac{1}{2}\sigma_i$ (the operator acts on particle n), σ_i are the Pauli matrices, r is the distance between particle 2 and particle 1, R is the radius of the magnetic field, γ_n is the nth particle's gyromagnetic ratio, $g = \mu_0\gamma_1\gamma_2\hbar^2$, μ_0 is the magnetic constant, and $\omega_n = \gamma_n B$ (particle n's Larmor frequency) [27, 28]. In SI units, for particle 1, a proton, $\gamma_1 = \gamma_p = 2.6752 \times 10^8 (s \cdot T)^{-1}$, and for particle 2, a neutron, $\gamma_2 = \gamma_n = 1.8324 \times 10^8 (s \cdot T)^{-1}$ [29, 30].

We write the spin wave function as

$$|\psi(t)\rangle = \alpha(t) |\uparrow\uparrow\rangle + \beta(t) |\uparrow\downarrow\rangle + \gamma(t) |\downarrow\downarrow\rangle + \delta(t) |\downarrow\uparrow\rangle \,, \tag{68}$$

where the first $\uparrow$ in $|\uparrow\uparrow\rangle$ refers to particle 1 in the spin up state, and the second $\uparrow$ to the spin of particle 2 in the spin up state. Similarly, a $\downarrow$ in the first/second position refers to particle 1/2 in the spin down state etc.

The differential equations controlling the time evolution of the coefficients can be determined using

$$|\Psi(t)\rangle = \sum_k c_k(t) |\Psi_k\rangle \tag{69}$$

$$\frac{\partial c_k(t)}{\partial t} = \frac{-i}{\hbar} \sum_n \langle k|H|n\rangle \, c_n(t), \tag{70}$$

where $|\Psi_k\rangle$ are a complete set of orthogonal states that satisfy the time-independent Schrödinger's equation.

Inserting Eq.(67) into Eq.(70), we find

$$
\begin{aligned}
\frac{\partial\alpha(t)}{\partial t} &= -i(\alpha(t)(\frac{g}{16\pi r^3}+\frac{\omega_1}{2}+\frac{\omega_2\Theta(R-r)}{2})-\gamma(t)(3\frac{g}{4\pi r^5}((vt)^2-b^2-2ibvt)) \\
\frac{\partial\beta(t)}{\partial t} &= -i(\beta(t)(-\frac{g}{16\pi r^3}+\frac{\omega_1}{2}-\frac{\omega_2\Theta(R-r)}{2})-\delta(t)\frac{g}{16\pi r^3}) \\
\frac{\partial\gamma(t)}{\partial t} &= -i(\gamma(t)(\frac{g}{16\pi r^3}-\frac{\omega_1 B}{2}-\frac{\omega_2\Theta(R-r)}{2})-\alpha(t)(3\frac{g}{4\pi r^5}((vt)^2-b^2+2ibvt)) \\
\frac{\partial\delta(t)}{\partial t} &= -i(\delta(t)(-\frac{g}{16\pi r^3}-\frac{\omega_1}{2}+\frac{\omega_2\Theta(R-r)}{2})-\beta(t)\frac{g}{16\pi r^3}),
\end{aligned}
\tag{71}
$$

where $r=\sqrt{(vt)^2+b^2}$. Note that $x(t)=vt$, $y(t)=b$, and $z(t)=0$ (for a more precise calculation of the classical path, see Appendix (A)). The energy of the system before measurement for finite r is

$$
\begin{aligned}
E(t) = {} & |\alpha(t)|^2\left(\frac{1}{2}B(\omega_2\theta(R-r)+\omega_1)+\frac{g}{(16\pi)r^3}\right)+|\beta(t)|^2\left(\frac{1}{2}B(\omega_1-\omega_2\theta(R-r))-\frac{g}{(16\pi)r^3}\right) \\
& +|\gamma(t)|^2\left(\frac{g}{(16\pi)r^3}-\frac{1}{2}B(\omega_2\theta(R-r)+\omega_1)\right)+|\delta(t)|^2\left(-\frac{1}{2}B(\omega_1-\omega_2\theta(R-r))-\frac{g}{(16\pi)r^3}\right) \\
& -\frac{\alpha(t)\gamma(t)^\dagger\left(3g\left(x^2+2ixy-y^2\right)\right)}{(16\pi)r^5}-\frac{\gamma(t)\alpha(t)^\dagger\left(3g\left(x^2-2ixy-y^2\right)\right)}{(16\pi)r^5} \\
& -\beta(t)\delta(t)^\dagger\left(\frac{g}{(16\pi)r^3}\right)-\delta(t)\beta(t)^\dagger\left(\frac{g}{(16\pi r^3)}\right)
\end{aligned}
. \tag{72}
$$

6.2.3 Initial Conditions

Let the system start completely unentangled at $t=t_i$. Thus, the eigenstates of our system can be expressed as

$$|\psi(t_i)\rangle = |\eta_i\rangle_1\otimes|\rho_i\rangle_2\,, \tag{73}$$

where $|\eta_i\rangle_1$ is the wave function of particle 1 and $|\rho_i\rangle_2$ is the wave function of particle 2. Begin with particle 2 in an arbitrary superposition of the spin up/spin down, or

$$|\rho_i\rangle = a_1|\uparrow\rangle_2+a_2|\downarrow\rangle\,, \tag{74}$$

where $|a_1|^2+|a_2|^2=1$. Meanwhile, while we can take particle 1 as being equally split between the spin up/spin down states, we do not actually know the relative phase ϕ due to Larmor precession. Therefore,

$$|\eta_i\rangle_1 = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1+e^{-i\phi}|\downarrow\rangle_1). \tag{75}$$

Putting everything together, our initial condition is

$$
\begin{aligned}
|\psi(t_i)\rangle &= \frac{1}{\sqrt{2}}(|\uparrow\rangle_1+e^{-i\phi}|\downarrow_1\rangle)(a_1|\uparrow\rangle_2+a_2|\downarrow\rangle) \\
&= \frac{a_1}{\sqrt{2}}|\uparrow\rangle_1|\uparrow\rangle_2+\frac{a_2}{\sqrt{2}}|\uparrow\rangle_1|\downarrow\rangle_2+\frac{a_2e^{-i\phi}}{\sqrt{2}}|\downarrow\rangle_1|\downarrow\rangle_2+\frac{a_1e^{-i\phi}}{\sqrt{2}}|\downarrow\rangle_1|\uparrow\rangle_2 \\
&= \frac{a_1}{\sqrt{2}}|\uparrow\uparrow\rangle+\frac{a_2}{\sqrt{2}}|\uparrow\downarrow\rangle+\frac{a_2e^{-i\phi}}{\sqrt{2}}|\downarrow\downarrow\rangle+\frac{a_1e^{-i\phi}}{\sqrt{2}}|\downarrow\uparrow\rangle\,,
\end{aligned}
\tag{76}
$$

where $|\uparrow\uparrow\rangle=|\uparrow\rangle_1|\uparrow\rangle_2$ is used for convenience.

6.2.4 Maximizing Entanglement

If particle 1 and particle 2 are maximally entangled, measuring the spin of particle 2 will tell us the spin, and thus the energy, of particle 1 (see Eq.(66)). We can determine when the system is maximally entangled using the system's density matrix $\hat{\rho}$, defined as

$$\hat{\rho} = |\psi(t)\rangle\langle\psi(t)|\,, \tag{77}$$

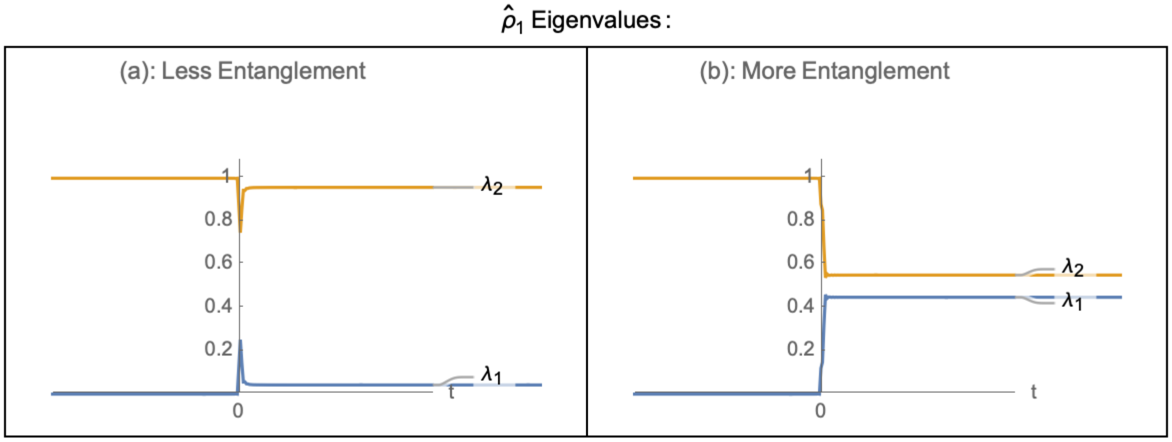

Figure 2: The values of λ_i for two different parameter sets. Even slight changes in parameter values can dramatically change the level of entanglement that results from the particle interaction.

where $|\psi(t)\rangle$ is defined in Eq.(68). We want to trace over the states of particle 2 to find the reduced density matrix $\hat{\rho}_1$, or

$$\hat{\rho}_1 = \mathrm{Tr}_2\hat{\rho} = \sum_i \langle \eta_{2_i}|\hat{\rho}|\eta_{2_i}\rangle \,, \tag{78}$$

where $|\eta_{2_i}\rangle$ are the eigenstates of particle 2. With some simple calculations, we find that

$$\begin{aligned}\hat{\rho}_1 = &|\alpha(t)|^2 \left|\uparrow\right\rangle_1 \left\langle\uparrow\right| + \delta(t)\alpha^\dagger(t) \left|\downarrow\right\rangle_1 \left\langle\uparrow\right| + \delta(t)^\dagger\alpha(t) \left|\uparrow\right\rangle_1 \left\langle\downarrow\right| + |\delta(t)|^2 \left|\downarrow\right\rangle_1 \left\langle\downarrow\right| \\ &+ |\beta(t)|^2 \left|\downarrow\right\rangle_1 \left\langle\downarrow\right| + \gamma(t)\beta^\dagger(t) \left|\downarrow\right\rangle_1 \left\langle\uparrow\right| + \gamma^\dagger(t)\beta(t) \left|\uparrow\right\rangle_1 \left\langle\downarrow\right| + |\gamma(t)|^2 \left|\downarrow\right\rangle_1 \left\langle\downarrow\right| ,\end{aligned} \tag{79}$$

where $|\uparrow\rangle$ / $|\downarrow\rangle$ refer to the eigenstates of particle 1. This can be written in matrix form as

$$\hat{\rho}_1 = \begin{pmatrix} |\alpha(t)|^2 + |\beta(t)|^2 & \alpha(t)\delta(t)^\dagger + \beta(t)\gamma^\dagger(t) \\ \alpha(t)^\dagger\delta(t)^+\beta(t)^\dagger\gamma(t) & |\gamma(t)|^2 + |\delta(t)|^2 \end{pmatrix}. \tag{80}$$

[31]. Particles 1 and 2 will be maximally entangled when the eigenvalues of Eq.(80) are

$$\lambda_1 = \frac{1}{2} = \lambda_2. \tag{81}$$

Two examples of the evolution of the eigenvalues of Eq.(80) can be found in Fig.(2). From this figure, it is clear that changing parameter values can result in dramatic changes in the level of entanglement between particles 1 and 2.

6.2.5 Evolution of System Energy

When particle 2 is at $r \approx \pm\infty$, the energy of particle 1 reduces to that of a spin-$\frac{1}{2}$ particle in a constant magnetic field of magnitude B (as defined in Eq.(66)). Therefore, the energy of particle 1 at $t = t_i$ is, neglecting interaction energy due to sufficient distance between particles,

$$E_1(t = t_i) = 0. \tag{82}$$

Meanwhile, particle 2 begins and ends outside the magnetic field, thus only has kinetic energy, which is assumed to be constant throughout the experiment. Therefore,

$$E_2(t = t_i) = E_2(t = t_f). \tag{83}$$

We want to measure the spin along the axis defined by $\hat{n}$, whose components we will now determine. As the particles have become maximally entangled, at $t = t_f$, we can write

$$|\psi(t_f)\rangle = \frac{1}{\sqrt{2}}(\left|\uparrow\right\rangle_1 \left|\zeta_\uparrow\right\rangle_2 + \left|\downarrow\right\rangle_2 \left|\zeta_\downarrow\right\rangle_2). \tag{84}$$

$|\zeta_\uparrow\rangle_2 = c_1 |\uparrow\rangle_2 + c_2 |\downarrow_2\rangle\rangle$ is an eigenstate of spin along $\hat{n}$, where

$$\begin{aligned} n_x &= 0 \\ n_y &= -2i\frac{c_1 c_2}{c_1^2 - c_2^2} \\ n_z &= \frac{c_1^2 + c_2^2}{c_1^2 - c_2^2}. \end{aligned} \tag{85}$$

If particle 2 is spin up along $\hat{n}$, then the energy of particle 1 is $E_1 = E_{1_\uparrow} = \frac{\omega_1 B}{2}$, and if it is spin down along that axis, $E_1 = -\frac{\omega_1 B}{2}$. Therefore, the total energy of the universe has changed by

$$\Delta E = \pm\frac{\omega_1 B}{2}, \tag{86}$$

where the sign of ΔE depends on the measured spin of particle 2.

6.2.6 Numerical Results

The differential equations found in Eq.(71) lack analytical solutions, requiring the use of numerical methods. This makes results heavily dependent on the numerical value of parameters. We used Mathematica's NDSolve function to solve for the time-dependent coefficients.

In natural units, $\hbar = c = 1, \mu_0 = 4\pi$, $\gamma_1 = \frac{0.2543}{GeV}, \gamma_2 = \frac{0.1742}{GeV}$, and $g = \frac{0.5565}{GeV^2}$ [32]. NDSolve worked best with all coefficients of order unity, thus we would need a magnetic field of order $\frac{1}{GeV}$. This corresponds to a field of $1.444 \times 10^{-12}T$. This field strength would correspond to a change in the universe's energy of $1.3 \times 10^{-19}eV$. This is an absolutely minuscule amount of energy, but we are not attempting to measure that energy directly. However, this does not mean that we can only work with magnetic fields of order $10^{-12}T$, merely that an alternative system to Mathematica's NDSolve function would be required to determine what parameters result in maximum entanglement. In reality, we would not want to work with such minuscule magnetic fields, as we want the (apparent) change in the universe's energy to be large enough that it could not be accounted for by experiment error. We plan to use an alternate program to perform a wider parameter search in the future.

6.3 Challenges of Maximizing Entanglement

We want to maximize entanglement using only parameters that could be controlled in a laboratory setting. ϕ denotes the phase between the spin up and spin down eigenstates of particle 1 at $t = t_i$, but this phase does not affect the overall energy of the system (see Eq.(75)). It also precesses in a constant magnetic field at frequency ω_n, where ω_n is defined above for a positively charged particle (add a minus sign for a negatively charged particle). We asked the question, "would experimenters need to know ϕ in order to maximize entanglement?"

The answer seems to be, unfortunately, yes. We define the quantity $\Delta\lambda$ as

$$\Delta\lambda = |\lambda_1 - \lambda_2|, \tag{87}$$

where the λ_i are the eigenvalues of the reduced density matrix $\hat{\rho}_1$ (Eq.(80)). The particles will be maximally entangled when $\Delta\lambda = 0$. Fig.(3) shows the dependence of $\Delta\lambda$ on ϕ for two different sets of parameters. There is a clear sinusoidal dependence shown, thus the magnitude of $\Delta\lambda$ (and level of entanglement) are clearly affected by the value of ϕ. However, how much $\Delta\lambda$ is affected, or the amplitude of the change in $\Delta\lambda$, varies between the presented cases. Therefore, experimenters will likely have to know ϕ in order to maximize entanglement, unless there happens to be a set of parameters where the entanglement is approximately maximized for all values of ϕ.

6.4 Discussion

There is still much work to be done in developing our thought experiment. Performing a more extensive parameter search to find a combination that maximizes entanglement is a priority. Due to Mathematica's

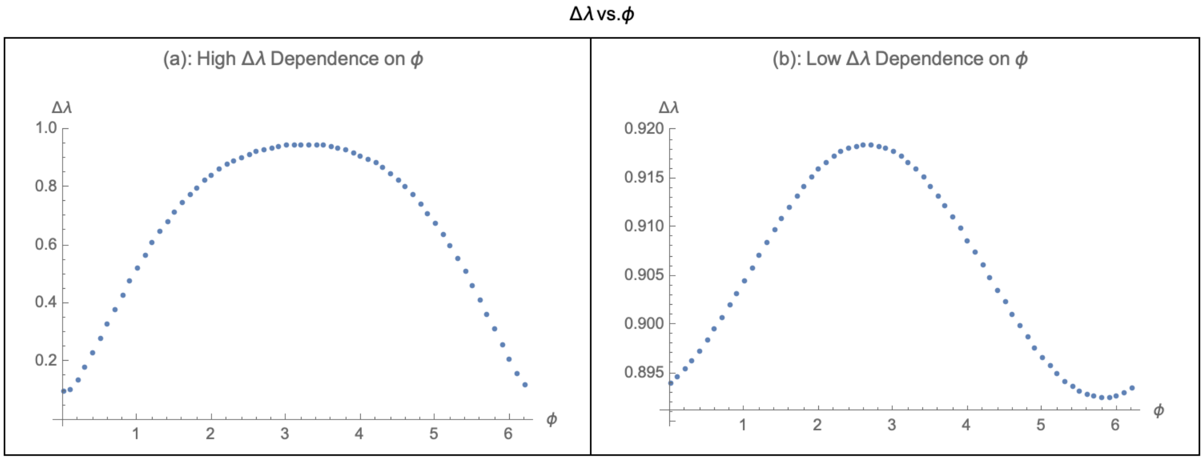

Figure 3: $\Delta\lambda$ (Eq.(87) vs. relative phase ϕ (Eq.(75)) for different initial parameters. There is a clear dependence between $\Delta\lambda$ and ϕ, but how much the dependence affects the entanglement varies based on initial parameters.

NDSolve limitations, we will likely perform the search using Python's SciPy and NumPy packages. We also want to include particle 2's real classical path in our calculations (see Appendix (A)). However, a more pressing issue is that of the entanglement's dependence on ϕ (Eq.(75)). If an experimenter cannot control ϕ, it may not be possible to maximize entanglement even if all other experiment parameters can be controlled. A wider parameter search would help determine how much of a problem ϕ presents, as there may be a set of parameters where the system is approximately maximally entangled for all values of ϕ. There may also be a way to determine/control ϕ that we have not yet identified. Finally, we are analyzing what can be said about energy conservation in situations where the particles do not become completely entangled, but that analysis is not finished at this time. We plan to continue developing our experiment to observe energy non-conservation by engaging in further collaboration with experimenters.

When/if a set of parameters is found that maximizes entanglement, technological limitations will be of upmost concern for potential experimenters. For example, as of writing, the largest magnetic field produced in a lab was on the order of 10^3 T [33]. Any experiment that would require a larger magnetic field would be not feasible at this time. Similar limits exist for electric fields, particle velocity, and other parameters. It is very possible that there are factors, such as technological constraints, outside of our control that make this thought experiment more or less feasible to convert into an actual experiment. Our purpose in developing this experiment was to illuminate how it might be possible for someone with the appropriate experience to develop a feasible experiment to observe energy non-conservation in quantum mechanics.

Experimenters interested in this topic might wonder what experimental limits on conservation of energy currently exist. Unfortunately, this is a harder question to answer than it might appear. Most experimenters assume conservation of energy holds when both designing their experiment and analyzing results, thus those results are not easily convertible to experimental bounds on conservation of energy. Conversations with experimenters indicate that obtaining values out of past/current experiments may indeed be possible, but doing so would require enough effort for a paper of its own.

One issue astute readers might point to is that we made a few assumptions and simplifications in our thought experiment design. For instance, Penning Traps do not hold charged particles exactly still (they are quantum objects after all), and the neutron's classical path is not exactly a straight line (see Appendix (A)) [26]. However, the beauty of our experiment's design is that the change in the universe's energy is directly proportional to the magnetic field present (see Eq.(86)). The change in energy can, theoretically, be arbitrarily increased to the point that these potential sources of error could not be the cause of the level of apparent non-conservation observed. Therefore, while we could treat the system as a quantum mechanical wave packet scattering problem, we do not need to for the result we desire, and doing so should not greatly impact our results.

On the completely theoretical side of things, the change in the universe's energy exists *according to an observer* (using Eq.(5) as their definition for the energy) regardless of which interpretation of quantum mechanics one supports. However, in Everettian QM, energy is conserved when one considers the wave function of the universe instead of just the branches' wave functions individually. In the other interpretations, energy is either actually not conserved (Standard Collapse Postulate QM), always not conserved (Dynamical Collapse models), or not well defined (Bohmian Mechanics). Energy is conserved under unitary evolution, and only in Everettian QM is evolution completely unitary *and* well defined. Therefore, if conservation of energy is accepted *a priori*, only Everettian QM, or an unmentioned interpretation with a similar branching structure, fulfills the requirement.

7 Discussion and Future Work

From the work presented in this thesis, we believe that there is clearly more to the topic of conservation of energy in quantum mechanics than is normally considered. In both our example comprehensive Hamiltonian and realistic thought experiment, energy is not conserved, *from an observer's perspective*, in any studied interpretation. It is not conserved in Standard Collapse Postulate QM, as the system must exist in a single energy eigenstate after the energy is measured. Because the system was in a superposition of energy eigenstates of different energies before measurement, the energy before measurement could not equal the energy after measurement. Bohmian Mechanics has no definition of energy dependent on non-contextual quantities [2]. Conservation of energy was already known to fail in Dynamical Collapse models, and was actually used as a reason to dismiss those interpretations by many detractors [3]. Only in Everettian QM is energy actually conserved in the universe's global wave function, though individual observers would not see it as such. Our work in this project should not be taken as an attack against conservation of energy. Instead, we show that apparent non-conservation of energy is not a problem, at least in Everettian QM.

However, even if Everettian QM is the correct interpretation of quantum mechanics, we (and other observers) cannot view the universe's global wave function. Therefore, we should observe this non-conservation. Conservation of energy has endured in the era of quantum mechanics because energy appears to be (or was assumed to be) conserved in experiments. Why then, if conservation of energy can be violated, has it appeared solid for so long? We believe it is because, while time evolution means that the universe is not in an energy eigenstate (see Section (2.2)), the universe must be in a superposition of energy eigenstates of similar energies. This is because energy eigenstates of vastly different energies decohere quickly, since they interact with the environment differently. To illustrate this idea, consider a particle of dust floating in space while being bombarded by photons. This dust particle exists in a superposition of two macroscopically distinct position eigenstates, or

$$\phi_{dust} \propto e^{-(\vec{x}-\vec{x_1})^2} + e^{-(\vec{x}-\vec{x_2})^2}. \tag{88}$$

If a photon is in a collision course with the dust particle only if it is located at x_1, then either the dust particle is found to be there, and the photon scatters off, or it is not, and the photon travels past x_1 unmolested. In both cases, the dust particle's position is "measured" by the photons. In this case, the rate of decoherence is proportional to the dust particle's cross-sectional area [34]. The same principle is true of the universe's energy eigenstates. Large differences in energy would lead to physical differences that would cause rapid decoherence. Therefore, it is quite difficult, as the lengths at which we went to construct our thought experiment clearly demonstrate, to produce potentially observable energy non-conservation.

If conservation of energy can be violated in QM, what about other conservation laws, such as charge? Can they be violated as well? While this is a direction we have not yet pursued heavily, if it turns out that charge etc. are conserved, we have an idea as to why. We assume that the universe cannot be in an energy eigenstate, as time evolution occurs [2, 11]. However, in the case of charge, Gauss's law tells tells us that the net charge in the universe must be zero, since

$$\Phi = \oint \vec{E} \cdot d\vec{A} = \frac{Q}{\epsilon_0}, \tag{89}$$

where Φ is the flux passing through a surface and Q is the total charge enclosed by the system. If the universe is closed, then

$$\Phi = 0 = Q, \tag{90}$$

and the net charge in the universe is uniformly zero and conserved [35]. Thus, even if there are (*apparent*) violations of conservation of energy, we can make no such claim about other conservation laws. Each conservation law would likely require an individual project.

8 Conclusion

There exist issues related to conservation of energy in quantum mechanics that are often overlooked. When a system in a superposition of energy eigenstates of different energy eigenvalues is measured, one of the eigenvalues is observed with probability given by Born's rule [14, 1]. The traditional explanation for why energy conservation still holds in that scenario is that the energy has been lost to the apparatus/environment, which are usually (but not always) treated classically, during measurement. However, we developed an expressly time-independent, comprehensive (environment is also a quantum system) Hamiltonian, Eq.(57), where energy is not conserved *from the perspective of an observer* in any considered interpretation of quantum mechanics. Even so, energy is conserved if one considers the universe's global wave function in Everettian QM, but an observer would still not see it as conserved.

The statement that the law of conservation of energy is violable is, to many, only as good as the experiment that could show deviation from the law. To that end, we developed a protocol to create experiments that could show energy non-conservation, then used that protocol to design an example thought experiment. Said experiment, though it may not be the most feasible, shows that it is possible to achieve an arbitrarily high level of non-conservation (change in the universe's energy) from an observer's perspective. This should allow the level of non-conservation to be observable in experiments (see Eq.(86)). As with the comprehensive Hamiltonian, all explored interpretations of quantum mechanics lead to apparent violations of the conservation of energy, but in Everettian QM, energy is still conserved in the wave function of the universe. Therefore, mandating conservation of energy be upheld mandates Everettian QM (or another interpretation with the same branching structure) to be the correct interpretation of quantum mechanics. Finally, we proposed that, since conservation of energy has appeared non-violable for so long, the universe must be in a superposition of energy eigenstates with similar energy eigenvalues. Our work resulted in many implications and questions, both theoretical and on experiment feasibility. For example, are other conservation laws similarly violated in different interpretations of quantum mechanics? We hope to explore this question and more in the future.

In brief, though it is a cornerstone of all areas of physics, the law of conservation of energy can be apparently violated by time-independent Hamiltonians regardless of the interpretation one believes in, while only in Everettian QM is conservation of energy actually upheld in the universe, *but not to an observer*.

9 Acknowledgments

I would like to start off by thanking my amazing thesis advisor, Sean Carroll, for putting up with me as an advisee for four years. I would not be where I am today without your help, and you are an all around amazing advisor. To Mark Wise, thank you for all of your advice and help with graduate school admissions over the past two years. I know I owe you.

On a similar note, I would like to thank Clifford Cheung, Kathryn Zurek, Michele Papucci, Andrea Mitridate, Ashmeet Singh, and the many other members of the Particle Theory Group lunch for welcoming me to the group and to regular lunches. One of the things I have missed most this term is getting to interact regularly with all of you.

To Tom Prince, Charles Steinhardt, Nick Hutzler, Yanbei Chen, Saul Teukolsky, and Vijay Varma, thanks for your time/funding for my undergraduate research projects. To Jann Lacoss, thanks for helping me get here in the first place. To Gil Refael and Aidan Chatwin-Davies, thanks for being a consistent part of my Caltech experience.

I would like to thank my wonderful house, Dabney, for being my home for the past four years. Dabney may not have been my first choice during rotation, but it ended up being the right fit.

Back home, I would like to thank my first physics professor, Hector Garcia-Villa, for teaching me the fundamentals of basic physics and scientific writing skills. To Noah Endermann, my high school chemistry teacher, thanks for both providing an unparalleled education in chemistry and being there for me. You were a bright spot in my otherwise bad high school experience.

Finally, I would like to thank my parents for all their love and support for the past 21 years and for allowing me to graduate from Caltech debt free. To my cat, Mercury, thanks for being by my side throughout the past three years.

Thanks to everyone, mentioned and not, who helped me on my way.

A The Classical Path Revisited

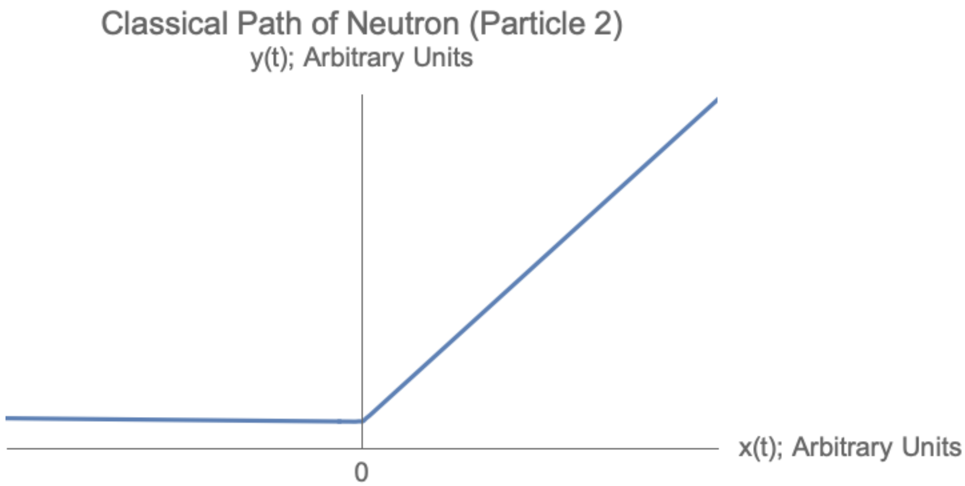

Figure 4: The classical path of particle 2. Distances are given in arbitrary units.

Previously, we assumed that particle 2 traveled in a straight line with $x(t) = vt, y(t) = b$, and $z(t) = 0$. However, in reality, particle 2 should scatter off of the potential at an angle with respective to the x-axis (see Fig.(4)). We assumed the straight path as the change in kinetic energy should be much less than the change in potential energy, and we were focused on the entanglement produced. The real classical path is as follows.

The spin dependent part Hamiltonian can be thought of as a potential $U(x, y, z, t)$. From kinematics,

$$\vec{F} = -\vec{\nabla}U(x, y, z, t), \tag{91}$$

where $\vec{F}$ is the force vector. In our case,

$$\begin{aligned} U(x, y, z = 0, t) = {} & |\alpha(t)|^2(\frac{g}{16\pi r^3} + \frac{\omega_1 B}{2} + \frac{\omega_2 B\Theta(R - r)}{2}) - \alpha^\dagger(t)\gamma(t)(3\frac{g}{4\pi r^5}(x(t)^2 - y(t)^2 - 2ix(t)y(t))) \\ & + |\beta(t)|^2(-\frac{g}{16\pi r^3} + \frac{\omega_1 B}{2} - \frac{\omega_2 B\Theta(R - r)}{2}) - \beta^\dagger(t)\delta(t)\frac{g}{16\pi r^3} \\ & + |\gamma(t)|^2(\frac{g}{16\pi r^3} - \frac{\omega_1 B}{2} - \frac{\omega_2 B\Theta(R - r)}{2}) - \gamma^\dagger(t)\alpha(t)(3\frac{g}{4\pi r^5}(x(t)^2 - y(t)^2 + 2ix(t)y(t))) \\ & + |\delta(t)|^2(-\frac{g}{16\pi r^3} - \frac{\omega_1 B}{2} + \frac{\omega_2 B\Theta(R - r)}{2}) - \delta^\dagger(t)\beta(t)\frac{g}{16\pi r^3}. \end{aligned} \tag{92}$$

Note that because $z = 0$ initially, there is no force in the $\hat{z}$ direction due to the shape of the potential. Thus, particle 2 remains at $z = 0$. The classical path of particle 2 can be written as

$$\begin{aligned} m_2\ddot{x} = F_x = -\frac{\partial U(x, y, z = 0, t)}{\partial x} \\ m_2\ddot{y} = F_y = -\frac{\partial U(x, y, z = 0, t}{\partial y}, \end{aligned} \tag{93}$$

where m_2 is the mass of particle 2. The total system of differential equations becomes

$$\begin{aligned}
\frac{\partial\alpha(t)}{\partial t} &= -i(\alpha(t)(\frac{g}{16\pi r^3} + \frac{\omega_1}{2} + \frac{\omega_2\Theta(R-r)}{2}) - \gamma(t)(3\frac{g}{4\pi r^5}((x(t))^2 - y(t)^2 - 2ix(t)y(t))) \\
\frac{\partial\beta(t)}{\partial t} &= -i(\beta(t)(-\frac{g}{16\pi r^3} + \frac{\omega_1}{2} - \frac{\omega_2\Theta(R-r)}{2}) - \delta(t)\frac{g}{16\pi r^3}) \\
\frac{\partial\gamma(t)}{\partial t} &= -i(\gamma(t)(\frac{g}{16\pi r^3} - \frac{\omega_1 B}{2} - \frac{\omega_2\Theta(R-r)}{2}) - \alpha(t)(3\frac{g}{4\pi r^5}((x(t))^2 - y(t)^2 + 2ix(t)y(t)) \\
\frac{\partial\delta(t)}{\partial t} &= -i(\delta(t)(-\frac{g}{16\pi r^3} - \frac{\omega_1}{2} + \frac{\omega_2\Theta(R-r)}{2}) - \beta(t)\frac{g}{16\pi r^3}) \\
m_2\ddot{x} &= F_x = -\frac{\partial U(x,y,z=0,t)}{\partial x} \\
m_2\ddot{y} &= F_y = -\frac{\partial U(x,y,z=0,t)}{\partial y}.
\end{aligned} \tag{94}$$

An example of the classical path of particle 2 can be seen in Fig.(4). Because the change in potential energy should be much less than the change in potential energy, we can simplify the problem as we did in Section (6.2.1). Even so, we want eventually want to incorporate the real classical path in our thought experiment calculations.

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