Topological Phases of Matter: Classification, Stacking Law, and Relation to Topological Quantum Field Theory

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Minyoung You

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Minyoung You
ORCID: 0000-0002-8251-953X

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ABSTRACT

We study aspects of gapped phases of matter, focusing on their classification, including the group law under stacking, and their relation to topological quantum field theories (TQFT). In one spatial dimension, it is well-known that Matrix Product States (MPS) efficiently approximate ground states of gapped systems; by showing that these states arise naturally in $1+1$-dimensional lattice TQFT, which in turn are closely related to continuum TQFT, we provide a concrete connection between ground states of lattice systems and TQFT in $1+1$ dimensions. We generalize this to systems with symmetries and fermions, and obtain a classification and group law for the stacking of $1+1$-dimensional symmetry-protected topological phases. Further, we study the effect of turning on/off interactions for the classification: the phase classification of a given symmetry class of Hamiltonians can be different depending on whether we allow interactions or not, and in low dimensions we provide some concrete formulas relating the phases under the non-interacting classification and those under the interacting classification. Lastly, we study the phases of the $2+1$-dimensional topological superconductor, and show that for all 16 phases braiding statistics of vortices, which determine the underlying TQFT, can be obtained by stacking layers of the basic $p+ip$ superconductor.

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Chapter 1

PHASES OF MATTER: THE LANDAU PARADIGM AND BEYOND

1.1 Phases characterized by symmetry

In the Landau paradigm, phases are understood via symmetry breaking: two systems belong to different phases if their ground states transform differently under the symmetry of the free energy or the Hamiltonian (Landau, 1937).

For example, take water: liquid water has continuous translation symmetry, but this is broken in the solid phase down to a discrete translation symmetry of ice. Liquid and vapor water, on the other hand, have the same continuous translation symmetry, which means that, in spite of the existence of a first-order phase transition between them, they actually belong to the same phase. Indeed, it is possible to avoid the phase transition and deform the system smoothly from liquid to vapor by going beyond the critical point.

The critical point itself also gives an illustration of the Landau picture: it is a point of second-order phase transition, where $\mathbb{Z}_2$ symmetry is broken. In this case, however, it is not the transition between the liquid and the vapor phases (as they do not belong to different phases). Instead, the transition is between the line of first-order phase transition (boiling) and the region of the phase diagram where there is no transition (supercritical fluid). Above the critical point, there is a $\mathbb{Z}_2$ symmetry which exchanges the liquid and vapor (since the two are indistinct now), but below the critical point, it is broken.

Such critical points are "universal," in that different systems undergo the same type of transitions. For example, the critical point of the Ising model, where the spins transition from being ordered to disordered, belongs to the same universality class as that of the boiling-supercritical transition in water: they have the same critical exponents – for example, the specific heat behaves in the same way as we approach the critical temperature –, and are both characterized by spontaneous breaking of $\mathbb{Z}_2$-symmetry. There is an underlying conformal field theory (CFT) which captures the universal properties of such transitions (Ginsparg, 1988).

Quantum symmetry-breaking phases

At zero temperature, we may have quantum phases where phase transitions are driven by quantum rather than thermal fluctuations. Then we work with quantum Hamiltonians and quantum symmetries (operators which commute with the Hamiltonian), but the basic idea is the same: different phases are characterized by how they break the overall symmetry group. More precisely, consider a Hamiltonian with a symmetry generated by Q, so that $[H, Q] = 0$. The ground states $|\Omega\rangle$ of H may not be invariant under symmetry: $Q|\Omega\rangle \neq e^{i\alpha}|\Omega\rangle$. Then we say that the symmetry is spontaneously broken. Phases with a given symmetry group G are classified by the unbroken subgroup $G' \subset G$.

As an example, consider the transverse-field Ising model in 1 dimension, which is described by the Hamiltonian

$$H = -J \sum_i \sigma_i^z \sigma_{i+1}^z - h \sum_i \sigma_i^x \tag{1.1}$$

where σ^j are Pauli operators. Since the Hamiltonian has a $\mathbb{Z}_2$ symmetry corresponding to flipping all the spins in the z-direction, the system can be in two phases: the $\mathbb{Z}_2$ is either broken or unbroken. Indeed, when $h > J$, there is a unique ground state; in the $J = 0$ limit, this is simply $|+\rangle \otimes \cdots \otimes |+\rangle$, where $|+\rangle$ is the eigenstate of σ^x with eigenvalue +1. On the other hand, when $J > h$, we have two ground states – when $h = 0$, these are the states where every spin is up or every spin is down in the z-basis. This is the symmetry-broken phase, and indeed the two states are exchanged under the $\mathbb{Z}_2$ spin-flip symmetry. When $J = h$, we have a phase transition, and its critical exponents are described by the 1 + 1d minimal model with central charge $\frac{1}{2}$, also called the Ising CFT (Ginsparg, 1988).

1.2 Topological phases: quantum phases beyond symmetry-breaking

We say that two gapped quantum systems belong to different phases if we can smoothly deform one Hamiltonian into the other without closing the gap (in the thermodynamic limit).

If the ground state of a system is a product state, we call it the trivial system, and any system which can be smoothly deformed to the trivial system without closing the gap will belong to the trivial phase. More precisely, we consider local Hamiltonians, which are constructed as sums of local terms: $H = \sum_i H_i$, where i labels the sites and H_i acts on a limited number of sites. The total Hilbert space is a tensor product of the Hilbert space of each site. For example, if we consider N spin-$\frac{1}{2}$s, the total Hilbert space is $\mathcal{H} = (\mathbb{C}^2)^{\otimes N}$. A system is topologically trivial if its ground state is a product state in terms of this local basis: $|\Omega\rangle = \bigotimes_i |\psi_i\rangle$.

It turns out that with this definition, there are non-trivial quantum phases which do not break symmetry and hence do not fit into the Landau paradigm. We call these phases topological. A well-known example is the quantum Hall effect (Klitzing, Dorda, and Pepper, 1980; Laughlin, 1981): at low temperatures and strong magnetic fields, the Hall conductance is quantized. When the magnetic field is increased sufficiently, the Hall conductance will jump, and the system will be in a different phase – even though no symmetry has been broken. Hence the Hall conductance can be considered a topological invariant, and is in fact related to the Chern number of the bundle of filled states over the Brillioun zone. Moreover, the quantum Hall system is described at low energies by Chern-Simons theory, which is a nontrivial topological quantum field theory sensitive to the topology of the manifold on which it is defined (Tong, 2016). For these reasons, we call such phases topological.

There are different ways in which a gapped phase can be topological: it can have intrinsic topological order, or symmetry-protected topological (SPT) order (Chen, Gu, and Wen, 2011b).

Phases with intrinsic topological orders (which we also simply call topological orders) can exist without any symmetry. They are characterized by ground state degeneracy which depends on the topology of the spatial manifold on which they are defined, fractional excitations (particle-like in $2+1$ dimensions), and long-range entanglement (LRE), which means that their ground states cannot be transformed to a product state via local unitary transformations. Fractional quantum Hall systems, for example, exhibit topological order in this sense.

SPT phases, on the other hand, have a unique ground state (on a closed manifold), and this ground state can be transformed to a product state via local unitary transformations if we ignore the symmetry protecting the phase. For this reason, they are also called short-range-entangled (SRE). The Hamiltonian of an SPT phase can be smoothly deformed to that of the trivial phase if we are allowed to add symmetry-breaking terms. This is why they are called "symmetry-protected": symmetry is what protects them from deformation to the trivial phase. Unlike topological orders, these phases have a unique ground state on any compact spatial manifold. Well-known examples include the Haldane phase and topological insulators.

It is possible for a given system to exhibit both SPT order and topological order, in which case we say that the system has symmetry-enriched topological (SET) order.

Stacking and the monoid structure

The most basic classification of gapped phases consists of simply finding the set $\mathcal{S}$ of distinct equivalence classes of systems. Beyond this, however, we can stack two system to get a third system (physically, putting two systems on top of each other and allowing local interactions), which gives a commutative operation $\mathcal{S} \times \mathcal{S} \to \mathcal{S}$ on the set of equivalence classes of systems or phases. Moreover, since stacking any phase with the trivial phase results in the same phase, the trivial phase acts as the unit for this operation. This turns the set $\mathcal{S}$ into a commutative monoid.

Certain phases have an inverse under the stacking operation, and these phases will form a group under stacking. Kitaev gives an alternative definition of SRE phases as invertible phases. Clearly, the invertible phases form a group. Non-invertible phases are then call LRE phases (Kitaev, 2011). When the distinction is relevant, we will use this definition of SRE phases unless otherwise noted. For example, the Kitaev chain (Section 2.1) is an LRE phase according to the definition involving local unitary transformations, but it is an SRE according to the invertibility definition. The E_8 phase in $2+1$ dimensions, which has a nontrivial CFT on its edge but no bulk topological excitations, is also an SRE as it is invertible, but it is not protected by any symmetry. Such phases are sometimes called invertible topological orders (Kitaev, 2011; Lan, Kong, and Wen, 2016b).

1.3 Description of contents

In the rest of the thesis, we study various aspects of such topological phases. Chapter 2 and Chapter 3 provide the relevant background on SPT phases and topological quantum field theory. In Chapter 4, based on (Kapustin, Turzillo, and You, 2017; Kapustin, Turzillo, and You, 2018; Turzillo and You, 2019), we make a concrete connection between the Matrix Product State formalism, useful for classifying SPT phases in 1 dimension, and topological quantum field theory in $1+1$ dimensions. We generalize this to the fermionic case and cases with anti-unitary as well as unitary symmetries, and obtain a classification and stacking law for $1+1$d fermionic SPTs. In Chapter 5, (based on Chen et al., 2019) we study the relation between the classification of SPT phases with and without interactions. In 0 and 1 dimension, we provide a concrete map from the set of free phases (those without interactions) to the set of phases with interactions. In Chapter 6, based on (You, 2020), we study vortices in topological superconductors in $2 + 1$ dimensions, which have a $\mathbb{Z}_{16}$-classification, and show that the braiding statistics of vortices for all 16 phases can be obtained by repeatedly stacking the basic $p + ip$ superconductor.

Chapter 2

SYMMETRY-PROTECTED TOPOLOGICAL PHASES

SPT phases have a unique ground state (on a spatial manifold without boundary), and can be deformed to the trivial phase if we ignore symmetry. Given a specific symmetry group, the type of system (bosonic/fermionic, interacting/free), and dimension, we can ask the question: what is the set of distinct SPT phases? Moreover, since SPT phases form a group under stacking, we ask: what is the group structure on this set?

Here we focus on reviewing two well-known classes of examples which will be relevant later: phases which can be constructed from free fermions with time-reversal, particle-hole, and chiral symmetries, and bosonic phases in 1 dimension (with arbitrary unitary symmetry), which have a matrix product state representation.

2.1 Phases of free fermions

Certain SPT phases can be constructed simply from fermions without any interactions.

A general non-interacting fermionic Hamiltonian in zero dimension involving N fermions can be written as:

$$H = \psi^{\dagger i} M_{ij} \psi^j + \psi^{\dagger i} N_{ij} \psi^{\dagger j} + \psi^i O_{ij} \psi^j \tag{2.1}$$

where ψ^i and $\psi^{\dagger i}$, $i = 1, ..., N$, are creation and annihilation operators. This can be written more compactly as

$$H = \Upsilon^{\dagger I} \mathcal{H}_{IJ} \Upsilon^J \tag{2.2}$$

in terms of the Nambu spinors $\Upsilon = \begin{pmatrix} \psi^1 \\ \vdots \\ \psi^N \\ \psi^{\dagger 1} \\ \vdots \\ \psi^{\dagger N} \end{pmatrix}$.

Υ satisfies the constraint

$$(\Upsilon^\dagger)^T = \tau_1 \Upsilon \tag{2.3}$$

where τ_1 is the matrix $\begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$ exchanging ψ^i and $\psi^{\dagger i}$.

In higher dimensions, we have

$$H = \sum_k \Upsilon^{\dagger I}(k)\mathcal{H}_{IJ}(k)\Upsilon^J(k) \tag{2.4}$$

where momentum k runs over the Brillouin zone.

Equivalently, we could also write the Hamiltonian in terms of Majorana fermions Γ^I, $I = 1, ..., 2N$, as

$$H = \Gamma^I A_{IJ} \Gamma^J \tag{2.5}$$

with A a real anti-symmetric matrix.

The ten-fold way

For systems of free fermions, there are ten special symmetry classes involving time-reversal, particle-hole, and chiral symmetries.

Time-reversal symmetry is anti-unitary

$$T : \psi^i \mapsto (U_T)_{ij}\psi^j \tag{2.6}$$

while particle-hole symmetry exchanges the creation and annihilation operators:

$$C : \psi^i \mapsto (U_C^*)_{ij}\psi^{\dagger j} \tag{2.7}$$

where U_T and U_C are unitary matrices.

The invariance of the "second-quantized" Hamiltonian H under T or C,

$$\begin{aligned} THT^{-1} &= H \\ CHC^{-1} &= H \end{aligned} \tag{2.8}$$

leads to the following conditions on the "single-particle Hamiltonian" $\mathcal{H}$,

$$U_T^\dagger \mathcal{H}^* U_T = \mathcal{H}$$
$$U_C^\dagger \mathcal{H}^T U_C = -\mathcal{H}. \tag{2.9}$$

Moreover, there is a "chiral" or "sublattice" symmetry S, which acts like the combination of T and C:

$$S : \psi^i \mapsto (U_S)_{ij} \psi^{\dagger j} \tag{2.10}$$

where $U_S = U_T U_C$. Invariance under S leads to the condition

$$U_S^\dagger \mathcal{H} U_S = -\mathcal{H}. \tag{2.11}$$

The action of T or C on the fermions can square to $+1$ or -1 (or, of course, the symmetry could simply be absent). Thus there are three possibilities for the action of T or C and nine possible cases when combined. In each case, the action of S is determined by $S = T \cdot C$. In addition, there is the possibility that we have S without T or C. This results in ten different symmetry classes.

These symmetry classes are classes of the single-particle Hamiltonian $\mathcal{H}$, which can be thought of as a matrix coupling the different creation and annihilation operators – this scheme ignores the underlying physical system. For example, consider a superconducting system with no extra symmetry, written in terms of a BdG Hamiltonian $H = \Psi^{\dagger I} \mathcal{H}_{IJ} \Psi^J$. The constraint that $(\Upsilon^\dagger)^T = \tau_1 \Upsilon$ is not a symmetry, but nevertheless it leads to the condition $\tau_1 \mathcal{H}^T \tau_1 = -\mathcal{H}$. This means that $\mathcal{H}$ satisfies the condition Eq. (2.9) with $U_C = \tau_1$, and hence belongs to Class D.

The phase classification for free fermionic systems with these symmetries, in all dimensions, are given in Table 2.1 (Kitaev, Lebedev, and Feigel'man, 2009; Ryu et al., 2010; Chiu et al., 2016). The classification only depends on the dimension mod 8. For each symmetry class in a particular dimension, the phases are labeled by either an integer or a $\mathbb{Z}_2$ number; sometimes, there are no nontrivial phases at all.

Example: Kitaev chain

The Kitaev chain (Fidkowski and Kitaev, 2011) is an example of a 1D fermionic SPT. Consider a system of fermions on a 1D lattice. Each site has Hilbert space $\mathbb{C}^2$,

Class	T	C	S	0	1	2	3	4	5	6	7
A	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AIII	0	0	+1	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AI	+1	0	0	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
BDI	+1	+1	1	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$
D	0	+1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$	0
DIII	−1	+1	1	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$2\mathbb{Z}$
AII	−1	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
CII	−1	−1	+1	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
C	0	−1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
CI	+1	−1	+1	0	0	0	$2\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$

Table 2.1: Classification of phases with time-reversal, particle-hole, and chiral symmetries. The number refers to the spatial dimension (due to Bott periodicity, the table repeats itself from dimension 8 on).

where we have fermion creation and annihilation operators $a_j^\dagger$ and a_j acting on the jth site. We can form Majorana operators, two for each site, as follows:

$$\begin{aligned} c_{2j-1} &= -i(a_j - a_j^\dagger) \\ c_{2j} &= a_j + a_j^\dagger. \end{aligned} \tag{2.12}$$

The trivial system is given by:

$$H = -\frac{i}{2}\sum_{j=1}^{N} c_{2j-1}c_{2j} = \sum_j \left(a_j^\dagger a_j - \frac{1}{2}\right). \tag{2.13}$$

Evidently, it is simply a series of uncoupled harmonic oscillators, and there is a unique ground state, $|0\rangle^{\otimes N}$, in which each site is unoccupied by the fermion. In the Majorana form, we see that this Hamiltonian only couples Majorana fermions within the same physical site.

Now consider:

$$H = -\frac{i}{2}\sum_{j=1}^{N-1} c_{2j}c_{2j+1} = \frac{1}{2}\sum_j \left(-a_j^\dagger a_{j+1} - a_j a_{j+1}^\dagger + a_j^\dagger a_{j+1}^\dagger + a_{j+1}a_j\right). \tag{2.14}$$

This Hamiltonian now couples Majorana fermions in different physical sites (c_{2j} belongs to physical site j, but c_{2j+1} belongs to physical site $j+1$). This Hamiltonian leads to “edge modes” c_1 and c_N, which commute with the Hamiltonian. We cannot gap them out by a local term: a term such as ic_1c_N would be non-local, since c_1 and c_N live on different ends of the chain. This gives us a signature of a topological phase.

It is more difficult to write down the exact form of the ground state for this system, but we know that there should be two. We can form a complex fermion out of the edge modes by

$$\begin{aligned} d &= c_1 + ic_N \\ d^\dagger &= c_1 - ic_N, \end{aligned} \tag{2.15}$$

and there are two states which are occupied or unoccupied with respect to this fermion. Both have the same energy, since the Hamiltonian commutes with $d^\dagger$ and d. This ground state degeneracy is robust to local perturbations, since we would need to add a non-local term to the Hamiltonian in order to lift the degeneracy. This gives us another signature of a topological phase.

What happens if we stack the system with itself, i.e. put another layer on top of one, and allow coupling by local terms? For convenience, we will consider chains with a single edge. Then, we have

$$H = H_1 + H_2 = \frac{-i}{2}\left(c_2^1c_3^1 + c_4^1c_5^1 + \dots\right) + \frac{-i}{2}\left(c_2^2c_3^2 + c_4^2c_5^2 + \dots\right) \tag{2.16}$$

where the superscript labels the layer, and the subscript labels the site within the layer. We have two edge modes, c_1^1 and c_1^2, but now it is possible to gap them out by a term such as $ic_1^1c_1^2$, which is local. Hence, the stacked system is no longer topologically nontrivial: its gapless edge modes are not protected from being gapped out. Thus, we have a $\mathbb{Z}_2$ classification.

Time-reversal invariant Kitaev chain

Note that the original system has time-reversal symmetry, which acts by:

$$\begin{aligned} a_j &\mapsto a_j \\ a_j^\dagger &\mapsto a_j^\dagger \\ i &\mapsto -i. \end{aligned} \tag{2.17}$$

On the Majorana operators, we see that

$$\begin{aligned} c_{2j-1} &\mapsto -c_{2j-1} \\ c_{2j} &\mapsto c_{2j}. \end{aligned} \tag{2.18}$$

Now consider two systems stacked with each other. As before, we have the boundary modes c_1^1 and c_1^2. Time-reversal acts separately on each layer, so the Hermitian term $ic_1^1c_1^2$ maps to $-ic_1^1c_1^2$ under time-reversal. Hence, we cannot gap out the boundary modes if we require that time-reversal invariance is preserved.

If we stack more copies of the Kitaev chain, we get a number of boundary modes corresponding to the number of copies, and we cannot gap out any of them with a time-reversal-invariant quadratic term. Hence the free classification is $\mathbb{Z}$ (the classes of systems labeled by negative integers can be understood as differences between phases).

We could consider interactions, i.e. higher-order terms. It turns out if we have eight copies of the system, we can construct an interaction which gaps out the edge modes (Fidkowski and Kitaev, 2010). Hence the interacting classification is $\mathbb{Z}_8$.

Topological Superconductors in $2 + 1$ dimensions

We start with fermions in $2 + 1$ dimensions with p-wave pairing (Read and Green, 2000; Bernevig, 2013):

$$H = \sum_p \Psi_p^\dagger \mathcal{H}_{BdG} \Psi_p = \frac{1}{2} \sum_p \begin{pmatrix} c_p^\dagger & c_{-p} \end{pmatrix} \begin{pmatrix} \frac{p^2}{2m} - \mu & 2i\Delta(p_x + ip_y) \\ -2i\Delta^*(p_x - ip_y) & -\frac{p^2}{2m} + \mu \end{pmatrix} \begin{pmatrix} c_p \\ c_{-p}^\dagger \end{pmatrix}. \tag{2.19}$$

This can be written as

$$\mathcal{H} = \frac{1}{2}\mathbf{d}(p) \cdot \sigma \tag{2.20}$$

with $\mathbf{d} = (-2|\Delta|p_y, -2|\Delta|p_x, \frac{p^2}{2m} - \mu)$, where σ^i are Pauli matrices.

We can define a topological invariant called the Chern number,

$$\nu = \frac{i}{16\pi} \int \mathrm{Tr}[QdQdQ] \tag{2.21}$$

where $Q(p)$ is the matrix with the same eigenvectors as $\mathcal{H}_{BdG}$, but with eigenvalues normalized to ± 1.

In the case at hand, this takes the form

$$\nu = \frac{1}{8\pi} \int d^2 p \epsilon^{ij} \hat{d} \cdot \left(\partial_{p_i} \hat{d} \times \partial_{p_j} \hat{d} \right) \tag{2.22}$$

Where $\mu > 0$ gives the trivial phase with Chern number $\nu = 0$, while $\mu < 0$ leads to the topological phase with $\nu = 1$. If we have n copies of the nontrivial system, we get $\nu = n$: hence $\nu \in \mathbb{Z}$ tells us the number of stacked layers of the p-wave superconductor. This is consistent with the classification of Class D systems in 2 dimensions.

2.2 Interacting bosonic phases in $1 + 1$ dimensions and matrix product states

Now let us consider interacting bosonic systems in $1 + 1$ dimensions, such as spin chains. In one spatial dimension, ground states of gapped Hamiltonians are efficiently approximated by an ansatz called a matrix product state (MPS) (Hastings, 2007), and we can use this formalism to classify these systems. [1] Here we will focus on the parent Hamiltonian approach, which constructs Hamiltonians associated to the MPS ground states and considers the equivalnce classes of those Hamiltonians in order to arrive at a classification (Schuch, Perez-Garcia, and Cirac, 2011); there is also an approach focusing on the ground states themselves and local unitary transformations between these states (Chen, Gu, and Wen, 2010).

First, let us define what an MPS is. Consider a closed chain of N sites, each with a copy of a *physical* Hilbert space $A \simeq \mathbb{C}^d$, so that the total Hilbert space is $A^{\otimes N}$. To each physical site, we attach two copies V^L, V^R of a *virtual* space $\mathbb{C}^D$. We identify $V^L = V$ and $V^R = V^*$ and choose a Hilbert space structure on V. Between each adjacent pair $(s, s + 1)$ of sites, place the maximally entangled state

$$|\omega\rangle_{s,s+1} = \sum_{i=1}^{D} |i\rangle \otimes |i\rangle \in V_s^R \otimes V_{s+1}^L. \tag{2.23}$$

An MPS tensor[2] is a linear map $\mathcal{P} : V^L \otimes V^R \to A$. The MPS associated to $\mathcal{P}$ is

[1] We only consider translationally-invariant MPS.

[2] More generally, the tensors $\mathcal{P}_s$ may depend on the site index s. But any translationally-invariant state has an MPS representation with a site-independent tensor (Perez-Garcia et al., 2006).

the state

$$|\psi_{\mathcal{P}}\rangle = (\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) \\ (|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes |\omega\rangle_{N1}) \tag{2.24}$$

in $A^{\otimes N}$. Since $|\psi_{\mathcal{P}}\rangle$ lies in the image of $\mathcal{P}^{\otimes N}$, we do not lose generality by truncating A to $\operatorname{im} P$. We will assume we have done so in the following. Equivalently, we assume that the adjoint MPS tensor $T = \mathcal{P}^\dagger$ is injective[3]. To be precise, T is related to $\mathcal{P}$ via

$$\mathcal{P} = \sum_{i,\alpha,\beta} T(e_i)_{\alpha\beta} |i\rangle\langle\alpha,\beta| \tag{2.25}$$

where $|\alpha,\beta\rangle \in V^L \otimes V^R$ and $|i\rangle \in A$.

The MPS wavefunction can be expressed as a trace of a product of matrices, hence its name. In the basis $\{e_i\}_{i=1,\ldots,d}$ of A, the conjugate state takes the form

$$\langle\psi_T| = \sum_{i_1\cdots i_N=1}^{d} \operatorname{Tr}\big[T(e_{i_1})\cdots T(e_{i_N})\big]\langle i_1\cdots i_N|. \tag{2.26}$$

There may be many different ways to represent a given state in $A^{\otimes N}$ in an MPS form. Even the dimension of the virtual space V is not uniquely defined. In general, it is not immediate to read off the properties of the state ψ_T from the tensor T.

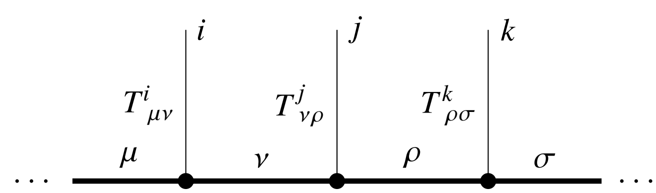


Figure 2.1: An MPS represented as a tensor network

For the tensor T, one can construct a LCP Hamiltonian H_T, called the *parent Hamiltonian*[4] of $|\psi_T\rangle$, which has $|\psi_T\rangle$ as a ground state. It is given as a sum of 2-site terms $h_{s,s+1}$ that project onto the orthogonal complement of $\ker h = (\mathcal{P} \otimes \mathcal{P})(V \otimes |\omega\rangle \otimes V^*)$. Explicitly,

[3]To avoid confusion, we stress that injectivity of T is unrelated to the notion of an injective MPS in the sense of (Schuch, Perez-Garcia, and Cirac, 2011). In particular, while we will always assume that T is injective, we will not assume that the ground state of the parent Hamiltonian is unique.

[4]There is a more general notion of a parent Hamiltonian where h is any operator with this kernel; however, we will always take h to be the projector.

$$H_T = \sum_s h_{s,s+1} \qquad \text{where}$$
$$h_{s,s+1} = \mathbb{1} - (\mathcal{P}_s \otimes \mathcal{P}_{s+1})\delta(\mathcal{P}_s^+ \oplus \mathcal{P}_{s+1}^+) \tag{2.27}$$

where δ is the projector onto $(V_s \otimes |\omega\rangle \otimes V_{s+1}^*)$ and $\mathcal{P}_s^+ := (T_s\mathcal{P}_s)^{-1}T_s$ is a left inverse of $\mathcal{P}_s$. The local projectors $h_{s,s+1}$ commute, so H_T is gapped. $|\psi_T\rangle$ is annihilated by $h_{s,s+1}$, $\forall s$ and therefore also by H_T.

Given the parent Hamiltonian, we can classify its phase by the usual principles: two Hamiltonians, and hence the corresponding MPS ground states, are in the same phase if we can deform them into each other without closing the gap. In this way, an MPS, even though it is just a state, defines a specific phase in the usual sense given by Hamiltonians.

When the Hamiltonian has a symmetry G which acts on-site on the physical Hilbert spaces as

$$\begin{aligned} g : &A^{\otimes N} \to A^{\otimes N} \\ &|\psi\rangle \mapsto R(g)^{\otimes N}|\psi\rangle \end{aligned} \tag{2.28}$$

with R a representation of G on A, we have a projective representation Q on the bond space; this acts as $Q(g) \otimes Q(g)^\dagger$ on $V \otimes V^*$, such that

$$R(g)\mathcal{P} = \mathcal{P}\left(Q(g) \otimes Q(g)^\dagger\right). \tag{2.29}$$

The projective nature of Q is encoded in the function (cochain) ω on $G \times G$ in the following way: $Q(g)Q(h) = e^{2\pi i\omega(g,h)}Q(gh)$. From the associativity condition

$$Q(g)\left(Q(h)Q(k)\right) = \left(Q(g)Q(h)\right)Q(k) \tag{2.30}$$

we get the cocycle condition

$$\omega(g,h) + \omega(gh,k) - \omega(h,k) - \omega(g,hk) = 0 \mod 1. \tag{2.31}$$

We can redefine $Q(g)$ by a g-dependent phase factor so that $Q'(g) = e^{2\pi i f(g)}Q(g)$. Then, we see that

$$\omega'(g,h) = \omega(g,h) + f(g) + f(h) - f(gh) \tag{2.32}$$

and we identify $\omega' \sim \omega$. The set of such functions ωs modulo the equivalence relation given above (and this set moreover forms a group under the addition of ωs) is called the second group cohomology group of G with $U(1)$ coefficients, $H^2(G, U(1)$.

If we have two systems with symmetry action in the same cohomology class $[\omega]$, we can continuously deform one into the other, while maintaining the symmetry. On the other hand, it can be shown that any smooth deformation of the parent Hamiltonian and hence the ground state and the MPS tensors will never change the class $[\omega]$; hence, systems with different classes of projective actions will belong to different phases. Thus, $1+1$ dimensional (bosonic) systems with symmetry G are classified by $H^2(G, U(1))$ (Schuch, Perez-Garcia, and Cirac, 2011).

Example: Haldane phase

Consider a chain of spin-1s with the following Hamiltonian:

$$H = \sum_i \left(\vec{S}_i \cdot \vec{S}_{i+1} + \frac{1}{3}(\vec{S}_i \cdot \vec{S}_{i+1})^2 \right). \tag{2.33}$$

The interaction term results in a higher energy cost for neighboring spins forming a spin-2 state, compared to spin-1 or 0.

The physical on-site Hilbert space is $\mathbb{C}^3$ spanned by the three spin states $|-1\rangle$, $|0\rangle$, and $|1\rangle$. We can take the bond space to be $V = \mathbb{C}^2$. Then the ground state is an MPS with MPS tensor

$$\mathcal{P} = \Pi_{S=1} \circ (\mathbb{1} \otimes i\sigma^y). \tag{2.34}$$

Equivalently, we can write it as

$$|\Omega\rangle = \sum_{i_1,\ldots,i_N} \mathrm{Tr}\left[T^{i_1} \cdots T^{i_N}\right] |i_1 \cdots i_N\rangle \tag{2.35}$$

with the matrices

$$
\begin{aligned}
T^{(-1)} &= -\frac{1}{\sqrt{2}}(\sigma^x - i\sigma^y) \\
T^{(0)} &= -\sigma^z \\
T^{(1)} &= \frac{1}{\sqrt{2}}(\sigma^x + i\sigma^y).
\end{aligned} \tag{2.36}
$$

The Hamiltonian is invariant under an $SO(3)$ spin rotation symmetry, represented on the physical sites by standard spin-1 matrices. On the bond spaces $\mathbb{C}^2$, the symmetry acts by the spin-$\frac{1}{2}$ representation, which is projective. This action satisfies the equivariance condition

$$
T(R(g)a) = Q(g)T(a)Q(g)^{-1}. \tag{2.37}
$$

Since we have a projective representation of the symmetry group, we cannot deform the system to the trivial phase without breaking the symmetry. In fact, we only need a subgroup $\mathbb{Z}_2 \times \mathbb{Z}_2 \in SO(3)$ to get a nontrivial 2-cocycle and hence a nontrivial phase (Schuch, Perez-Garcia, and Cirac, 2011; Zeng et al., 2015).

Chapter 3

TOPOLOGICAL QUANTUM FIELD THEORY AND TOPOLOGICAL ORDER

3.1 Topological quantum field theory

Topological quantum field theory (TQFT) is a quantum field theory whose partition function depends only on the topology, and not the geometry, of spacetime (Atiyah, 1989; Kapustin, 2010). For example, the Chern-Simons action

$$S_{CS} = \frac{k}{4\pi}\int_X \left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) \tag{3.1}$$

has no explicit dependence on the metric of the spacetime-manifold X. There are also the so-called Witten-type or cohomological TQFTs, whose action contains explicit dependence on the metric, but whose correlation functions do not end up depending on the metric.

Under renormalization group flow, quantum field theories with a mass gap will flow to a TQFT in the far infrared (while a massless theory will flow to a CFT). Hence it is believed that TQFTs (with the relevant structures, such as G-bundles for G-symmetric phases) ultimately classify gapped topological phases.

Unifying these into a formal language, Atiyah proposed a definition of TQFT as a functor (Atiyah, 1989). A d-dimensional TQFT with no extra structure is given by a functor from the category of d-dimensional cobordisms, Bord_d, to the category of vector spaces over $\mathbb{C}$, Vect. (The latter could be a more general category such as the category of modules over some commutative ring, but we will focus on the physically relevant case of complex vector spaces.)

The category Bord_d consists of the following data: objects are closed oriented $d-1$-dimensional manifolds. The morphism between two objects M and N is given by a d-dimensional bordism, i.e. a d-dimensional manifold X whose boundary ∂X consists of the disjoint union of M and $\bar{N}$, the orientation-reversal of N.

A TQFT Z attaches a vector space $Z(M)$ to each $d-1$-dimensional closed manifold M, which can be thought of as the space of states living on a spatial slice M. To a d-dimensional bordism X from M to N, the TQFT attaches a linear map $Z(X)$ from

M to N. To the empty manifold, the TQFT attaches the ground field, which in our case is $\mathbb{C}$. A closed d-dimensional manifold P can be thought of as a bordism from the empty manifold to the empty manifold; hence, $Z(P)$ is a linear map between $\mathbb{C}$ and $\mathbb{C}$, which is just a complex number. This number is the partition function of the theory on P.

The TQFT functor has to be monoidal: to a disjoint union M and N, it attaches the tensor product of state spaces, $Z(M) \otimes Z(N)$.

We can consider different types of TQFTs by introducing extra structure on both sides of the fuctor. For example, we can consider a functor which attaches $\mathbb{Z}_2$-graded vector spaces (or super-vector spaces) to closed $d-1$-dimensional manifolds with spin structure; this is the appropriate type of TQFT to consider when we are dealing with fermionic theories.

3.2 Topological quantum field theory in 1+1 dimensions

As a simple example, let us consider the $1+1$-dimensional case. In $1+1$ dimensions, there is only one type of closed oriented connected 1-dimensional manifold, the circle, and this makes things particularly simple: the data of a $1+1$d TQFT is equivalent to that of a commutative Frobenius algebra.

A $1+1$ TQFT associates a space of states $\mathcal{A}$ to an oriented circle, and a vector space $\mathcal{A}^{\otimes n}$ to n disjoint oriented circles. To a bordism Σ from n circles to l circles, the TQFT associates a linear map from $\mathcal{A}^{\otimes n}$ to $\mathcal{A}^{\otimes l}$. This map is invariant under diffeomorphisms. Gluing bordisms, taking care that orientations agree, corresponds to composing linear maps.

If Σ is a pair-of-pants bordism from two circles to one circle, the TQFT gives us a corresponding map $m : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ which defines an associative, commutative product on $\mathcal{A}$. The cap bordism defines a symmetric trace function $\mathrm{Tr} : \mathcal{A} \to \mathbb{C}$ such that the scalar product $\eta(a, b) = \mathrm{Tr}(ab)$ is symmetric and non-degenerate. These data make $\mathcal{A}$ into a commutative Frobenius algebra. It is known that a two-dimensional TQFT is completely determined by the commutative Frobenius algebra structure on $\mathcal{A}$.(Atiyah, 1989; Moore and Segal, 2006; Abrams, 1996) The state-operator correspondence identifies $\mathcal{A}$ with the algebra of local operators. This Frobenius algebra encodes the 2- and 3-point functions on the sphere, from which all other correlators, including the partition function, can be reconstructed.

In 2d, there is an essentially trivial family of unitary oriented TQFTs parameterized by a positive real number λ. The partition function of such a TQFT on a closed oriented 2d manifold Σ is $\lambda^{\chi}(\Sigma)$, while the Hilbert space attached to a circle is one-dimensional. Such 2d TQFTS are called invertible, since the partition function is a nonzero number for any Σ. Since, by the Gauss-Bonnet theorem, $\chi(\Sigma)$ can be expressed as an integral of scalar curvature, tensoring a 2d TQFT by an invertible 2d TQFT is equivalent to redefining the TQFT action by a local counterterm which depends only on the background curvature. One usually disregards such counterterms – we will follow this practice and regard TQFTs related by tensoring with an invertible TQFT as equivalent.

$1+1$**d spin-TQFT**

As mentioned above, we can take additional structures into account to obtain a generalization of TQFT. One example, which is relevant for fermionic systems, is the spin-TQFT, where we replace the category of cobordisms with the category of cobordisms with spin strcutures, and the category of vector spaces with the category of $\mathbb{Z}_2$-graded vector spaces (Moore and Segal, 2006). Now, instead of just the circle, there are two different kinds of simple 1-manifolds: the circle with periodic or Ramond (R) spin structure, and the circle with anti-periodic or Neveu-Schwarz (NS) spin structure. To these, the spin-TQFT associates vector spaces A_r and A_{ns}, respectively. The automorphisms of the spin structures induce an involution on the vector spaces $A_{r,ns}$, allowing us to define a $\mathbb{Z}_2$-grading on the vector spaces.

3.3 Topological quantum field theory in 2+1 dimensions

In $2+1$ dimensions, a TQFT attaches a space of states $Z(\Sigma)$ to every closed oriented surface Σ, and a linear map between such spaces to cobordisms between surfaces. An important and well-studied example is the Chern-Simons theory. It starts from the action

$$S_{CS} = \frac{k}{4\pi}\int_X \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) \tag{3.2}$$

which is the integral of the Chern-Simons form over a 3-manifold X (here A is a G-gauge field for some gauge group G). We can study correlation functions of Wilson loops $W(\gamma_i) = \exp\left\{\oint_{\gamma_i} A\right\}$, and these give invariants of knots and links (Witten, 1989): if n loops γ_i, $i = 1, ..., n$ form a link, the correlation function

$$\langle W(\gamma_1)\cdots W(\gamma_n)\rangle \tag{3.3}$$

depends only on the topological class of the link (i.e. is unchanged when the link, or the ambient manifold, is smoothly deformed). These links can be thought of as trajectories of quasiparticle excitations (or quasiparticle-antiquasiparticle pairs). It was also shown in (Witten, 1989) that by canonically quantizing on $\Sigma \times \mathbb{R}^1$, the Chern-Simons theory (with gauge group G and level k) attaches to Σ a Hilbert space which is the space of conformal blocks of a level k G-Wess-Zumino-Witten model, a chiral rational conformal field theory (RCFT). These surfaces can be further pierced by quasiparticle trajectories, which correspond to insertions of fields on the 2d RCFT level.

Modular tensor categories and topological order

There is a different but related way of characterizing a 2+1d TQFT, which focuses on quasiparticle excitations and their properties under braiding and fusion. The types of quasiparticles and their braiding and fusion data, known as the R- and F-symbols, determine a mathematical structure known as a modular tensor category, which in turn determines a 2 + 1d TQFT (Kitaev, 2006; Bais and Slingerland, 2009). The boundary RCFT is also constrained by this data, which in that context is known as the Moore-Seiberg data (Moore and Seiberg, 1989).

In 2 + 1d, pointlike excitations can have anyonic statistics, in addition to the usual bosonic and fermionic statistics; hence, we will call them anyons in general. An anyon type can equivalently be thought of as a superselection sector: since it is impossible to create or destroy anyons by acting with local operators on the vacuum, states with different types of anyons belong to different superselection sectors. We always have the vacuum superselection sector, usually denoted by 1.

Two anyons can fuse into different types of anyons. We encode this through the fusion rules

$$a \times b = \sum_c N_{ab}^c c. \tag{3.4}$$

This states that anyons a and b can fuse to a particle type c whenever $N_{ab}^c \neq 0$. If $N_{ab}^c > 1$, we get multiple copies of the anyon c. All anyons fuse trivially with the vacuum:

$$1 \times a = a \tag{3.5}$$

and fusion is commutative and associative:

$$a \times b = b \times a$$
$$a \times (b \times c) = (a \times b) \times c. \tag{3.6}$$

For each anyon a, there exists a unique anti-particle $\bar{a}$, with which it can fuse to the vacuum:

$$a \times \bar{a} = 1 + ... \tag{3.7}$$

where ... represents some other anyons which may potentially be present. Another way to put this is that $N^1_{a\bar{a}} = 1$.

When we fuse a and b, we can get states $|\psi^{ab}\rangle$, which form a Hilbert space V_{ab}. We can decompose this space into different superselection sectors labeled by c: $V_{ab} = \oplus_c V^c_{ab}$. The spaces V^c_{ab} are called fusion spaces, with dimension $\dim V^c_{ab} = N^c_{ab}$ which corresponds to the number of ways of fusing a and b into c.

When we fuse three anyons, there are multiple ways of getting to the final result. The F-symbol encodes the relation (basis transformation) between them.

More relevant for us will be the R-symbol, which encodes the braiding data. If we exchange two anyons a and b, we get a map $R : V^{ab}_c \rightarrow V^{ba}_c$, called the braiding coefficient (or the braiding matrix, if $\dim V^{ab}_c > 1$). The braiding coefficient of a and b depends on the fusion channel c, and is denoted by R^{ab}_c.

When $a \neq b$, R^{ab}_c does not have a gauge-invariant meaning. Only the full braiding of one type of anyon around the other, $R^{ba}_c R^{ab}_c$, is a topological invariant. We denote this by M:

$$M^{ab}_c = R^{ba}_c R^{ab}_c. \tag{3.8}$$

F and R satisfy consistency conditions know as the pentagon equation and the hexagon equation.

Relation to abelian Chern-Simons theory

In the case of an abelian Chern-Simons theory, described by an $N \times N$ K-matrix, we can obtain the braiding statistics of anyons in a simple manner (Belov and Moore, 2005; Stirling, 2008; Kapustin and Saulina, 2011).

The anyons are characterized by a vector $q^I \in \mathbb{Z}^N$ describing how it is charged under each A_I; the phase acquired under a full braiding of two anyons described by q^I, q'^J is given by

$$\exp\left\{2\pi i q^I (K^{-1})_{IJ} q'^J\right\}. \tag{3.9}$$

Under exchange of two anyons of the same type q^I, we get a phase

$$\exp\left\{\pi i q^I K^{-1}_{IJ} q^J\right\}. \tag{3.10}$$

Two charge vectors related by $q \mapsto q + Kl$ for $l \in \mathbb{Z}^n$ are equivalent; indeed, the modified q leads to the same braiding statistics.

We can think of the charge vector q as living on a lattice $\Lambda = \mathbb{Z}^N$; then the distinct charges (anyon types) are elements of the discriminant group

$$\mathcal{D} = \Lambda / K\Lambda \tag{3.11}$$

and their braiding statistics is encoded in a quadratic form Q, such that for $x, y \in \mathcal{D}$,

$$Q(x+y) - Q(x) + Q(y) = K^{-1}(\tilde{x}, \tilde{y}) = K^{-1}_{IJ} \tilde{x}^I \tilde{y}^J \tag{3.12}$$

where $\tilde{x}$ and $\tilde{y}$ are lifts of x, y from $\mathcal{D}$ to Λ, and

$$Q(nx) = n^2 Q(x) \tag{3.13}$$

for any integer n.

Note that $Q(x) = \frac{1}{2} K^{-1}(\tilde{x}, \tilde{x})$: we can recover Q from the braiding statistics.

$\mathcal{D}$ and Q, together with the chiral central charge c mod 24 ($\mathcal{D}$ and Q by themselves only determine the central charge mod 8), fully determine an abelian Chern-Simons theory (Belov and Moore, 2005).

Example: Toric code

Kitaev's toric code model proposed in (Kitaev, 2003) has four anyons $1, e, m, \psi$ with fusion rules

$$\begin{aligned} e \times e = m \times m = \psi \times \psi &= 1 \\ e \times m &= \psi \\ e \times \psi &= m \\ m \times \psi &= e. \end{aligned} \tag{3.14}$$

The braiding coefficients are

$$\begin{aligned} R_1^{ee} = R_1^{mm} &= 1 \\ R_1^{\psi\psi} &= -1 \\ M_m^{e\psi} = M_e^{m\psi} &= -1 \\ M_\psi^{em} &= -1. \end{aligned} \tag{3.15}$$

This system can be represented as an abelian Chern-Simons theory with

$$K = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}. \tag{3.16}$$

In $\Lambda/K\Lambda$, we have $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. By labeling

$$\begin{aligned} \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= 1 \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} &= e \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} &= m \\ \begin{pmatrix} 1 \\ 1 \end{pmatrix} &= \psi \end{aligned} \tag{3.17}$$

we immediately see that they satisfy the fusion rules (3.14), and using $K^{-1} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$ and the equations (3.9) and (3.10), we recover the braiding statistics for the toric code (3.15).

Example: Ising TQFT and p-wave superconductors

This theory consists of three anyons $1, \sigma, \psi$ with the fusion rules

$$\begin{aligned} \sigma \times \sigma &= 1 + \psi \\ \sigma \times \psi &= \sigma \\ \psi \times \psi &= 1. \end{aligned} \tag{3.18}$$

The braiding coefficients are given by:

$$\begin{aligned} R_1^{\sigma\sigma} &= \theta e^{i\alpha\frac{\pi}{4}} \\ R_\psi^{\sigma\sigma} &= \theta e^{-i\alpha\frac{\pi}{4}}. \end{aligned} \tag{3.19}$$

A p-wave superconductor in the presence of a vortex (modelled by the winding behavior of the phase of the order parameter, $\Delta = \Delta_0(r)e^{i\varphi}$ where $\varphi(\theta) = \theta$) has a zero-energy Majorana solution to the BdG equations. The Majorana zero mode $\gamma = \int r dr d\theta \left(u(r,\theta)c(r,\theta) + v(r,\theta)c^\dagger(r,\theta)\right)$ is exponentially localized to the vortex:

$$\gamma = \int r dr d\theta i g(r) \left[-e^{i\theta/2}c(r,\theta) + e^{-i\theta/2}c^\dagger(r,\theta)\right] \tag{3.20}$$

where $g(r)$ is exponentially localized at $r = 0$.

If we have two vortices, we obtain two Majorana zero modes γ_1 and γ_2, each localized to the respective vortex core. Exchanging these vortices results in (Ivanov, 2001)

$$\begin{aligned} \gamma_1 &\mapsto \gamma_2 \\ \gamma_2 &\mapsto -\gamma_1. \end{aligned} \tag{3.21}$$

With two Majorana modes, we can form a complex fermion

$$\begin{aligned} a &= \frac{1}{2}(\gamma_1 + i\gamma_2) \\ a^\dagger &= \frac{1}{2}(\gamma_1 - i\gamma_2) \end{aligned} \tag{3.22}$$

and the occupied and unoccupied states with respect to this fermion, $|0\rangle$ and $|1\rangle$, span the space of ground states $\mathbb{C}^2$. These states correspond to the superselection sectors 1 (vacuum) and ψ (fermion) in the Ising TQFT.

The operator on $\mathbb{C}^2$ which accomplishes Eq. (3.21) by conjugation is

$$R = \theta e^{-\frac{\pi}{4}\gamma_1\gamma_2} = \theta e^{-i\frac{\pi}{4}(i\gamma_2\gamma_1)}. \tag{3.23}$$

From this we can recover Eq. (3.19).

Chapter 4

TOPOLOGICAL QUANTUM FIELD THEORY AND MATRIX PRODUCT STATES

4.1 Introduction and overview

It is a widely held belief that the universal long-distance behavior of a quantum phase of matter at zero temperature can be encoded into an effective field theory.[1] In the case of gapped phases of matter, the extreme infrared should be described by a topological quantum field theory, discussed in 3.1. On the other hand, we have seen in 2.2 that the ground state of any gapped 1+1d Hamiltonian with a short-range interaction can be approximated by a Matrix Product State (MPS). This representation is very efficient, especially in the translationally-invariant case, and is well-suited to the Renormalization Group analysis. In particular, it leads to a classification of Short-Range Entangled Phases of 1+1d matter in terms of group cohomology (Chen, Gu, and Wen, 2011a; Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011). It is natural to ask about the connection between these two approaches to gapped phases of matter. This chapter will attempt to answer this question.

First, we consider the bosonic case without symmetry. We show that a standard-form MPS is naturally associated with a module M over a finite-dimensional semisimple algebra A. The universality class of the MPS depends only on the center $Z(A)$. On the other hand, every unitary 2d TQFT has a state-sum construction which uses a semisimple algebra as an input. Further, given a module M over this algebra, one naturally gets a particular state in the TQFT space of states. We show that this state is precisely the MPS associated to the pair (A, M). Since the TQFT depends only on $Z(A)$, we reproduce the fact that the universality class of the MPS depends only on $Z(A)$.

In the case of an MPS with a symmetry G, a similar story holds. A G-equivariant MPS is encoded in a G-equivariant module M over a G-equivariant semisimple algebra A. Such an algebra can be used to give a state-sum construction of a G-equivariant TQFT, while every G-equivariant module M gives rise to a particular

[1] It is hard to make this rigorous since neither the notion of a phase of matter nor that of an effective field theory has been formalized.

state. This state is an equivariant MPS state. Again, different A can give rise to the same TQFT. This leads to an equivalence relation on G-equivariant algebras which is a special case of Morita equivalence. An indecomposable phase with symmetry G is therefore associated with a Morita-equivalence class of indecomposable G-equivariant algebras. The classification of such algebras is well known (Ostrik, 2003) and leads to an also well-known (Chen, Gu, and Wen, 2011a; Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011) classification of bosonic 1+1d gapped phases of matter with symmetry G. In the special case of Short-Range Entangled gapped phases, we recover the group cohomology classification of SPT phases.

We then move on to the relation between fermionic MPS and spin-TQFTs. We review the state-sum construction of spin-TQFTs in two space-time dimensions from $\mathbb{Z}_2$-graded algebras following (Novak and Runkel, 2014; Gaiotto and Kapustin, 2016). We also show that stacking fermionic systems together corresponds to taking the supertensor product of the corresponding algebras. This gives a very clean and simple derivation of the spin-statistics relation in the topological case. We evaluate the annulus diagram and show that it gives rise to a generalized MPS both in the Neveu-Schwarz and the Ramond sector, and work out the commuting projector Hamiltonian starting from the TQFT data describing an invertible spin-TQFT. We show that for a nontrivial spin-TQFT, the resulting Hamiltonian describes the Majorana chain (Fidkowski and Kitaev, 2011). In section 4.8, we discuss G-equivariant spin-TQFT and G-equivariant fermionic MPS. We show that fermionic SRE phases with a symmetry G times the fermion parity are in 1-1 correspondence with invertible G-equivariant spin-TQFTs, and that the TQFT data give rise to fermionic G-equivariant MPS. We also discuss the case when the symmetry is a nontrivial extension $\mathcal{G}$ of G by fermion parity, which is related to $\mathcal{G}$-Spin TQFTs. We then derive interpretations of the invariants on the closed chain that extend the results of (Kapustin and Thorngren, 2017), which were discovered in the context of spin-TQFT. Next, time-reversing symmetries and their relation to spatial parity are discussed. The generalizations of the three invariants to phases with such symmetries are derived and interpreted. In Section 4.9, a general stacking law (4.174) is derived for fermionic SRE phases with a symmetry $\mathcal{G}$ that is a central extension by fermion parity of a bosonic symmetry group that may contain anti-unitary symmetries. We contrast this result with the bosonic group structure and emphasize the origin of the difference. In Section 4.10, we demonstrate our result with several examples, recovering the $\mathbb{Z}/8$ classification of fermionic SRE phases in the symmetry class BDI ($T^2 = 1$) and the $\mathbb{Z}/2$ classification in the class DIII

$(T^2 = P)$.

4.2 Matrix Product States at RG Fixed Points

Matrix product states

In this section, we review matrix product states (MPS) and extract the algebraic data that characterizes them at fixed points of the Renormalization Group (RG). We find that a fixed point MPS is described by a module over a finite-dimensional semisimple algebra. We discuss the notion of a gapped phase and argue that they are classified by finite-dimensional semisimple commutative algebras. Given a fixed point MPS and the corresponding semisimple algebra A, the commutative algebra characterizing the gapped phase is the center $Z(A)$ of A, denoted $\mathcal{A}$.

We are interested in Hamiltonians with an energy gap that persists in the thermodynamic limit of an infinite chain. A large class of examples of gapped systems come from local commuting projector (LCP) Hamiltonians; that is, $H = \sum h_{s,s+1}$, where the $h_{s,s+1}$ are projectors that act on sites s, $s+1$ and commute with each other. Since the local projectors commute, an eigenstate of H is an eigenstate of each projector. It follows that the gap of H is at least 1. Thus LCP Hamiltonians are gapped in the thermodynamic limit. As noted in 2.2, these are efficiently approximated by MPSs in one spatial dimension, and an MPS is given by a tensor $T : A \to V \otimes V^*$, or, equivalently, $\mathcal{P} : V \otimes V^* \to A$, where V is the virtual space and A is the physical Hilbert space on each lattice site. From this, one can construct the parent Hamiltonian H_T, which has the associated MPS as its ground state. Here we will discuss the properties of the parent Hamiltonian in more detail.

While H_T is constructed so that it has the MPS with conjugate wavefunction (2.26) as its ground state, H_T can have other ground states in general. Consider a state of the form

$$\left|\psi_T^X\right\rangle = (\mathcal{P}_1 \otimes \mathcal{P}_2 \otimes \cdots \otimes \mathcal{P}_N) \left(|\omega\rangle_{12} \otimes |\omega\rangle_{23} \otimes \cdots \otimes \left|\omega^X\right\rangle_{N1}\right) \tag{4.1}$$

for some virtual state

$$\left|\omega^X\right\rangle = \sum_{i=1}^{D} X_{ij} \, |i\rangle \otimes |j\rangle \in V^* \otimes V \tag{4.2}$$

where X is a matrix that commutes with $T(a)$ for all $a \in A$. Note that $\left|\omega^{\mathbb{1}}\right\rangle = |\omega\rangle$ and so $\left|\psi_T^{\mathbb{1}}\right\rangle = |\psi_T\rangle$. The states (4.1) are clearly annihilated by $h_{s,s+1}$ for $s \neq N$. To see that they are annihilated by h_{N1}, note that tensor $T(e_i)XT(e_j)$ is expressible as a linear combination of tensors $T(e_i)T(e_j)$ if and only if X commutes with every $T(e_i)$. The conjugate states have wavefunctions

$$\langle\psi_T^X| = \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_N|. \tag{4.3}$$

We will refer to these states as *generalized MPS*.

It turns out that all ground states of H_T can be written as generalized MPS. One can always take T to be an isometry with respect to some inner product on A and the standard inner product

$$\langle M|N\rangle = \mathrm{Tr}\big[M^\dagger N\big] \quad M, N \in \mathrm{End}(V) \tag{4.4}$$

on $\mathrm{End}(V)$. For an orthogonal basis $\{e_i\}$ of A, $\mathrm{Tr}\big[T(e_i)^\dagger T(e_j)\big] = \delta_{ij}$. Consider the case $N = 1$. An arbitrary state

$$\langle\psi| = \sum_i a_i \langle i| \tag{4.5}$$

can be written in generalized MPS form (4.3) if one takes

$$X = \sum_j a_j T(e_j)^\dagger. \tag{4.6}$$

Thus generalized MPS with commuting X are the only ground states. Neither the number of generalized MPS nor the number of ground states depends on N; thus, the argument extends to all N.

Suppose the data (A_1, V_1, T_1) and (A_2, V_2, T_2) define two MPS systems with parent Hamiltonians H_1 and H_2. Consider the composite system $(A_1\otimes A_2, V_1\otimes V_2, T_1\otimes T_2)$. It has $\mathcal{P} = \mathcal{P}_1 \otimes \mathcal{P}_2$ and $\delta = \delta_1 \otimes \delta_2$. Then

$$\begin{aligned} h_{A\otimes B} &= \mathbb{1}_{A_1\otimes A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1\otimes A_2} \\ &= \mathbb{1}_{A_1}\otimes\mathbb{1}_{A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1}\otimes\mathcal{P}^2\delta\mathcal{P}^{+2}_{A_2} \\ &= (\mathbb{1}_{A_1} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_1})\otimes\mathbb{1}_{A_2} + \mathbb{1}_{A_1}\otimes(\mathbb{1}_{A_2} - \mathcal{P}^2\delta\mathcal{P}^{+2}_{A_2}) \\ &= h_{A_1}\otimes\mathbb{1}_{A_2} + \mathbb{1}_{A_1}\otimes h_{A_2} \end{aligned} \tag{4.7}$$

where the penultimate line follows from the fact that $\mathcal{P}^2\delta\mathcal{P}^{+2}$ is a projector. Therefore, the composite parent Hamiltonian is

$$H_{A\otimes B} = H_{A_1}\otimes\mathbb{1}_{A_2} + \mathbb{1}_{A_1}\otimes H_{A_2}. \tag{4.8}$$

RG-fixed MPS and gapped phases

Under real-space renormalization group (RG) flow (Verstraete et al., 2005), adjacent pairs of sites are combined into blocks with physical space $A \otimes A$. The MPS form of the state is preserved, with the new MPS tensor being

$$T'(a \otimes b) = T(a)T(b), \tag{4.9}$$

where on the r.h.s. the multiplication is matrix multiplication. We also define $\mathcal{P}' = T'^{\dagger}$. Though an RG step squares the dimension of the codomain of the MPS tensor, the rank is bounded above by D^2, and so the truncated physical space $\mathrm{im}(\mathcal{P}')$ never grows beyond dimension D^2.

An *RG fixed MPS tensor* is an MPS tensor such that $\mathcal{P}$ and $\mathcal{P}'$ have isomorphic images and are identical (up to this isomorphism) as maps. That is, there exists an injective map $\mu : A \to A \otimes A$ such that

$$\mu \circ \mathcal{P} = \mathcal{P}'. \tag{4.10}$$

If we denote $m = \mu^{\dagger}$, this is equivalent to

$$T(m(a \otimes b)) = T(a)T(b). \tag{4.11}$$

Since T was assumed to be injective, this equation completely determines m. Similarly, the fact that matrix multiplication is associative implies that $m : A \otimes A \to A$ is an associative multiplication on A. The map $T : A \to \mathrm{End}(V)$ then gives V the structure of a module over A. Since T is injective, this module is faithful (all nonzero elements of A act nontrivially). The statement that X commutes with T in the ground state of the parent Hamiltonian is the statement that X is a module endomorphism of V.

As previously stated, a state in $A^{\otimes N}$ may have multiple distinct MPS descriptions. One can always choose T to have a certain *standard form* (Schuch, Perez-Garcia, and Cirac, 2011) – regardless of whether it is RG fixed. When this is done, the matrices $T(a)$ are simultaneously block-diagonalized, for all $a \in A$. Moreover, if we denote by $T^{(\alpha)}$ the α^{th} block, say of size $L_\alpha \times L_\alpha$, then the matrices $T^{(\alpha)}(e_i)$ span the space of $L_\alpha \times L_\alpha$ matrices. That is, $T^{(\alpha)}$ defines a surjective map from A to the space of $L_\alpha \times L_\alpha$ matrices.

For an RG-fixed MPS tensor in its standard form, one can easily see that A is a direct sum of matrix algebras. Indeed, each block A^α defines a surjective homomorphism

T^α from A to the algebra of $L_\alpha \times L_\alpha$ matrices, and if an element of A is annihilated by all these homomorphisms, then it must vanish. Thus we get a decomposition

$$A = \oplus_\alpha A^\alpha, \tag{4.12}$$

where each $A^\alpha = (\ker T^\alpha)^\perp$ is isomorphic to a matrix algebra. We stress that some of these homomorphisms might be linearly dependent, so the number of summands may be smaller than the number of blocks in the standard form of T. An algebra of such a form is semisimple, that is, any module is a direct sum of irreducible modules. More specifically, any module over a matrix algebra of $L \times L$ matrices is a direct sum of several copies of the obvious L-dimensional module. This basic module is irreducible. If, for a particular A^α, T contains more than one copy of the irreducible module, the corresponding blocks in the standard form of T are not independent.

The ground-state degeneracy is simply related to the properties of the algebra A. Namely, the number of ground states is equal to the number of independent blocks in a standard-form MPS, or equivalently the number of summands in the decomposition (4.12). Since the center of a matrix algebra consists of scalar matrices and thus is isomorphic to $\mathbb{C}$, one can also say that the number of ground states is equal to the dimension of $\mathcal{A} = Z(A)$.

Two gapped systems are said to be in the same phase if their Hamiltonians can be connected by a Local Unitary (LU) evolution, i.e. if they are related by conjugation with a finite-time evolution operator for a local time-dependent Hamiltonian (Chen, Gu, and Wen, 2010). Clearly, the ground-state degeneracy is the same for all systems in a particular phase. In fact, for 1+1d gapped bosonic systems, it completely determines the phase (Schuch, Perez-Garcia, and Cirac, 2011; Chen, Gu, and Wen, 2011a).

It is convenient to introduce an addition operation $\oplus$ on systems and phases. Given two 1+1d systems with local Hilbert spaces A_1 and A_2, we can form a new 1+1d system with the local Hilbert space $A_1 \oplus A_2$. The Hamiltonian is taken to be the sum of the Hamiltonians of the two systems plus projectors which enforce the condition that neighboring "spins" are either both in the A_1 subspace or in the A_2 subspace. The ground state degeneracy is additive under this operation. A phase is called decomposable if it is a sum of two phases, otherwise it is called indecomposable. Clearly, it is sufficient to classify indecomposable phases.

It is easy to see that if A decomposes as a sum of subalgebras, the corresponding phase is decomposable. Further, an indecomposable semisimple algebra A is isomorphic to a matrix algebra. The corresponding ground state is unique. Moreover, while the parent Hamiltonians for different matrix algebras are different, they all correspond to the same phase,(Chen, Gu, and Wen, 2011a) i.e. are related by a Local Unitary evolution. Hence the phase is determined by the number of components in the decomposition (4.12), or in other words, by $Z(A)$.

4.3 Topological quantum field theory

We have seen above that an RG-fixed MPS state is associated with a finite-dimensional semisimple algebra A, and that the universality class of the corresponding phase depends only on the center of A. On the other hand, it is known since the work of Fukuma, Hosono, and Kawai (1994) that for any finite-dimensional semisimple algebra A with an invariant scalar product, one can construct a unitary 2D TQFT, and that the isomorphism class of the resulting TQFT depends only on the center of A. In this section, we show that this is not a mere coincidence, and that the ground states of this TQFT can be naturally written in an MPS form, with an RG-fixed MPS tensor.

State-sum construction of 2d TQFTs

We have seen in 3.2 that a (closed) 2D TQFT associates a space of states $\mathcal{A}$ to an oriented circle, and a vector space $\mathcal{A}^{\otimes n}$ to n disjoint oriented circles.

Every unitary oriented 2d TQFT[2] has an alternative construction called the state-sum construction (Fukuma, Hosono, and Kawai, 1994), which is combinatorial and manifestly local. The input for this construction is a finite-dimensional semisimple algebra A, which is not necessarily commutative. To compute the linear maps associated to a particular bordism Σ, one needs to choose a triangulation of Σ. Nevertheless, the result is independent of the choice of the triangulation. The connection between the not-necessarily commutative algebra A and the commutative algebra $\mathcal{A}$ is that $\mathcal{A}$ is $Z(A)$, the center of A. From the perspective of open-closed TQFTs, A is the algebra of states on the interval for a particular boundary condition. The scalar product on $\mathcal{A}$ is also fixed by the structure of A.

Let us describe the state-sum construction for the partition function Z_Σ of a closed oriented 2D manifold Σ, following (Fukuma, Hosono, and Kawai, 1994). Fix a

[2] More precisely, every equivalence class of unitary oriented 2d TQFTs, in the sense explained in the previous paragraph.

basis e_i, $i \in S$, of A. We define the following tensors:

$$\eta_{ij} = \eta(e_i, e_j) = \mathrm{Tr}_A\, P_i P_j, \qquad C_{ijk} = \mathrm{Tr}_A\, P_i P_j P_k \tag{4.13}$$

Here $P_i : A \to A$ is the operator of multiplication by e_i. The tensor η_{ij} is symmetric and non-degenerate (if the algebra A is semi-simple); the tensor C_{ijk} is cyclically symmetric. We also denote by η^{ij} the inverse to the tensor η_{ij}. Note also that C_{ijk} is related to the structure constants $C^i{}_{jk}$ in this basis by

$$C^i{}_{jk} = \sum_l \eta^{il} C_{ljk}. \tag{4.14}$$

Let $T(\Sigma)$ be a triangulation of Σ. A coloring of a 2-simplex F of $T(\Sigma)$ is a choice of a basis vector e_i for each 1-simplex $E \in \partial F$. A coloring of $T(\Sigma)$ is a coloring of all 2-simplices of $T(\Sigma)$. Note that each 1-simplex of $T(\Sigma)$ has two basis vectors attached to it, one from each 2-simplex that it bounds. The weight of a coloring is the product of C_{ijk} over 2-simplices and η^{ij} over 1-simplices, where the cyclic ordering of indices for each 2-simplex is determined by the orientation of Σ. The partition function is the sum of these weights over all colorings.

Topological invariance of Z_Σ can be shown as follows. It is known that any two triangulations of a smooth manifold are related by a finite sequence of local moves (Pachner, 1991). In two dimensions, there are two moves – the 2-2 move and the 3-1 move, depicted in Figure 4.1 – which swap two or three faces of a tetrahedron with their complement. Invariance of the state-sum under the 2-2 "fusion" move reads

$$C_{ij}{}^p C_{pk}{}^l = C_{jk}{}^p C_{ip}{}^l. \tag{4.15}$$

Similarly the 3-1 move reads

$$C_i{}^{mn} C_{nl}{}^k C^l{}_{mj} = C_{ij}{}^k. \tag{4.16}$$

These axioms are satisfied by any finite-dimensional semisimple algebra A (Fukuma, Hosono, and Kawai, 1994); therefore, the partition sum is a topological invariant[3].

Open-closed 2d TQFT

So far we have discussed what is known as closed 2D TQFTs. That is, the boundary circles were interpreted as spacelike hypersurfaces, and thus each spatial slice had an empty boundary. The notion of a TQFT can be extended to incorporate spatial

[3] In two dimensions, there is no difference between topological and smooth manifolds.

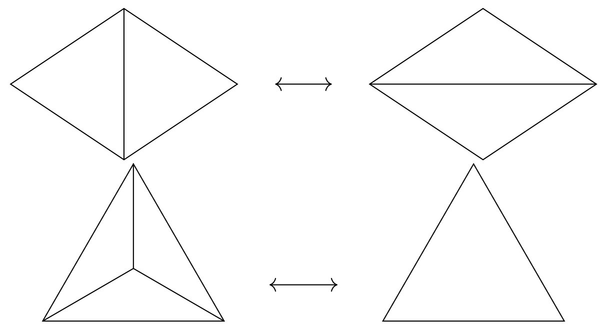


Figure 4.1: The 2-2 and the 3-1 Pachner moves.

boundaries; such theories are called open-closed TQFTs. In such a theory, a spatial slice is a compact oriented manifold, possibly with an nonempty boundary. That is, it is a finite collection of oriented intervals and circles. A bordism between such spatial slices is a smooth oriented *surface with corners*: paracompact Hausdorff spaces for which each point has a neighborhood homeomorphic to an open subset of a half-plane. Surfaces with corners are homeomorphic, but typically not diffeomorphic, to smooth surfaces with a boundary.

The corner points subdivide the boundary of the bordism into two parts: the initial and final spatial slices, and the rest. We will refer to the initial and final spatial slices as the cut boundary, while the rest will be referred to as the brane boundary. The cut boundary can be thought of as spacelike, while the brane boundary is timelike. Bordisms are composed along their cut boundary (hence the name), while on the brane boundary, one needs to impose boundary conditions (known as D-branes in the string theory context, hence the name). More precisely, if C is the set of boundary conditions, one needs to label each connected component of the brane boundary with an element of C.

An open-closed 2d TQFT associates a vector space $V_{MM'}$ to every oriented interval with the endpoints labeled by $M, M' \in C$, and a vector space $\mathcal{A}$ to every oriented circle. To a collection of thus labeled compact oriented 1D manifolds, it attaches the tensor product of spaces $V_{MM'}$ and $\mathcal{A}$. To every bordism with corners labeled in the way explained above, it attaches a linear map from a vector space of the "incoming" cut boundary to the vector space of the "outgoing" cut boundary. Gluing bordisms along their cut boundaries corresponds to composing the linear maps.

Just like in the case of a closed 2d TQFT, one can describe algebraically the data which are needed to construct a 2d open-closed TQFT. These axioms were discovered by (Lazaroiu, 2001), and we also refer to (Moore and Segal, 2006) for

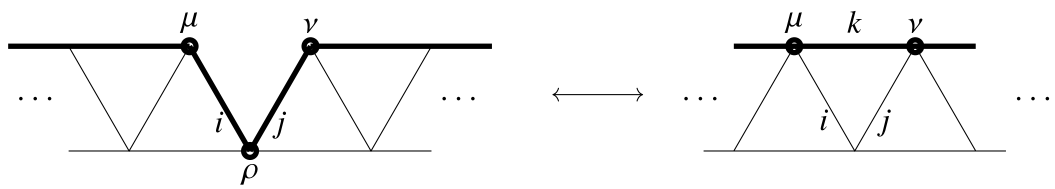


Figure 4.2: An elementary shelling representing $T^{\mu}_{\rho i} T^{\rho}_{\nu j} = C^k_{ij} T^{\mu}_{\nu k}$ (4.17). The thick line is a physical boundary.

details. Suffice it to say that each space V_{MM} is a (possibly noncommutative) Frobenius algebra, and each space $V_{MM'}$ is a left module over V_{MM} and a right module over $V_{M'M'}$. That is, to every element $x \in V_{MM}$, one associates a linear operator $T^M(x) : V_{MM'} \to V_{MM'}$ so that composition of elements of V_{MM} corresponds to the composition of linear operators: $T^M(x)T^M(x') = T^M(xx')$ (and similarly for $V_{M'M'}$). Also, for every $M \in C$, there is a map $\iota^M : \mathcal{A} \to V_{MM}$ which is a homomorphism of Frobenius algebras. The dual map $\iota_M : V_{MM} \to \mathcal{A}$ is known as the generalized boundary-bulk map. In particular, if we act with ι_M on the identity element of the algebra V_{MM}, we get a distinguished element $\psi_M \in \mathcal{A}$ called the boundary state corresponding to the boundary condition M. Geometrically, ψ_M is the element of $\mathcal{A}$ which the open-closed TQFT associates to an annulus whose interior circle is a brane boundary labeled by M, while the exterior circle is an outgoing cut boundary.

One may wonder if it is possible to reconstruct the open-closed TQFT from the closed TQFT. The answer turns out to be yes if $\mathcal{A}$ is a semisimple, i.e. if every module over $\mathcal{A}$ is a sum of irreducible modules (Moore and Segal, 2006).[4] Then C is the set of finite-dimensional modules over $\mathcal{A}$, and $V_{MM'}$ is the space of linear maps from the module M to the module M' commuting with the action of $\mathcal{A}$ (i.e. $V_{MM'}$ is the space of module homomorphisms). Conversely, one can reconstruct the algebra $\mathcal{A}$ from any "sufficiently large" brane $M \in C$: if we assume that the module M is faithful (i.e. all nonzero elements of $\mathcal{A}$ act nontrivially), then $\mathcal{A} = Z(V_{MM})$.

The state-sum construction generalizes to the open-closed case (Lauda and Pfeiffer, 2007). Let us describe it for a semisimple $\mathcal{A}$, assuming that the bordism Σ only has a brane boundary. Each connected component of $\partial\Sigma$ is then labeled by a brane $M \in C$. We pick a sufficiently large brane M_0 such that $\mathcal{A} = Z(V_{M_0 M_0})$. Let $A = V_{M_0 M_0}$. We also choose a basis f^M_μ, $\mu \in S_M$ in each module M. Denote the

[4]This might seem like a rather uninteresting case, since by the Wedderburn theorem, every commutative semisimple algebra is isomorphic to a sum of several copies of $\mathbb{C}$. But as explained below, unitarity forces $\mathcal{A}$ to be semisimple. Also, in the case of TQFTs with symmetries and fermionic TQFTs, the classification of semisimple algebras is more interesting.

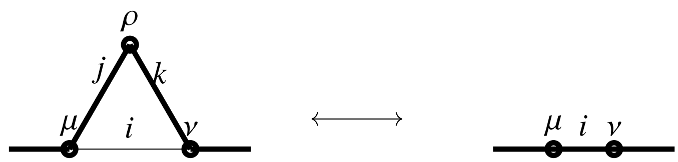


Figure 4.3: An elementary shelling representing $T^{\mu}_{\rho j} T^{\rho}_{\nu k} C^{jk}_{i} = T^{\mu}_{\nu i}$ (4.18).

matrix elements of the action of A on M by $T^{\mu}_{M\nu i}$. We choose a triangulation of Σ, which also gives us a triangulation of each connected component of the boundary. 2-simplices of Σ are labeled as before. We label the boundary 0-simplices on any M-labeled boundary component by the basis vectors f^{M}_{μ}. Thus each boundary 1-simplex is labeled by a basis vector of A and a pair of basis vectors of a module. We assign a weight to each 2-simplex and each interior 1-simplex as before. We also assign a weight to each boundary 1-simplex as follows. Suppose the boundary 1-simplex is labeled by $e_i \in A$ and $f^{M}_{\mu}, f^{M}_{\nu} \in M$. Then the weight of the boundary 1-simplex is $T^{\mu}_{M\nu i}$. The total weight is the product of weights of all 2-simplices and all 1-simplices (both interior and exterior).

Due to the introduction of brane boundaries, there are two more moves, called the 2-2 and 3-1 *elementary shellings* and depicted in Figures 4.2 and 4.3, that must be considered when demonstrating topological invariance (Lauda and Pfeiffer, 2007). They yield conditions

$$T^{\mu}_{M\rho i} T^{\rho}_{M\nu j} = C^{k}_{ij} T^{\mu}_{M\nu k} \tag{4.17}$$

and

$$T^{\mu}_{M\rho j} T^{\rho}_{M\nu k} C^{jk}_{i} = T^{\mu}_{M\nu i}, \tag{4.18}$$

respectively. The first one is the definition of a module, and the second one follows from the semisimplicity of A. Therefore the state-sum is a well-defined open-closed TQFT. Moreover, such structures are precisely those required to define a topologically invariant state-sum.

Unitary TQFTs and semisimplicity

The state-sum construction defines a perfectly good topological invariant for any finite-dimensional semisimple algebra A; however, if it is to model an actual physical system, its space of states must carry a Hilbert space structure, and linear maps corresponding to bordisms must be compatible in some sense with this structure. To be precise, for any oriented bordism Σ whose source is a disjoint union of n circles and whose target is a disjoint union of l circles, let $-\Sigma$ denote its orientation-reversal.

$-\Sigma$ has l circles in its source and n circles in its target. A 2d TQFT attaches to Σ a linear map $\mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}$, and to $-\Sigma$ a linear map $\mathcal{A}^{\otimes l} \to \mathcal{A}^{\otimes n}$. A unitary structure on a 2d TQFT is a Hilbert space structure on $\mathcal{A}$ such that the maps corresponding to Σ and $-\Sigma$ are adjoint to each other. For an open-closed 2D TQFT, we require that the state-space assigned to each boundary-colored interval has a non-degenerate Hermitian metric, and that cobordisms with nonempty brane boundary also satisfy the Hermiticity condition. In particular, the product m and coproduct μ are adjoints. It then follows from the Pachner moves that μ is an isometry. Likewise, the module structure T is an isometry.

Let $\langle a, b\rangle$ denote the Hilbert space inner product of $a, b \in \mathcal{A}$. Since $\mathcal{A}$ also has a bilinear scalar product η, we can define an antilinear map

$$* : \mathcal{A} \to \mathcal{A}, \quad a \mapsto a^*, \tag{4.19}$$

such that $\langle a, b\rangle = \eta(a^*, b)$. It can be shown that this map is an involution (i.e. $a^{**} = a$) and an anti-automorphism (i.e. $(ab)^* = b^* a^*$) (Turaev, 2010). This can also be expressed by saying that $\mathcal{A}$ is a $*$-algebra. Conversely, one can show that any commutative Frobenius $*$-algebra such that the sesquilinear product $\eta(a^*, b)$ is positive-definite gives rise to a unitary 2d TQF (Turaev, 2010).

A corollary of this result is that for a unitary 2d TQFT, the algebra $\mathcal{A}$ is semisimple. To see this, note first that any nonzero self-adjoint element a, $a = a^*$, cannot be nilpotent. Indeed, if n is the smallest n such that $a^n = 0$, then $a^{2m} = 0$, where $m = \lfloor (n+1)/2 \rfloor$. Then $\langle a^m | a^m \rangle = \left\langle 1 | a^{2m} | 1 \right\rangle = 0$, and therefore $a^m = 0$. Since $n \le m$, repeat with $n' = m$ until $n = 1$, i.e. $a = 0$. Now we can use a result (Kapustin, 2013) which says that a $*$-algebra with no nilpotent self-adjoint elements (apart from zero) is semisimple.

By the Artin-Wedderburn theorem, a finite-dimensional semisimple algebra over complex numbers is isomorphic to a sum of matrix algebras. Since $\mathcal{A}$ is also commutative, this means that it is isomorphic to a sum of several copies of $\mathbb{C}$.

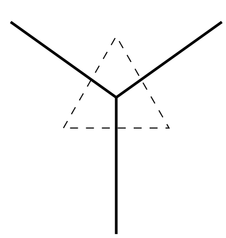


Figure 4.4: The Poincare dual of a triangle.

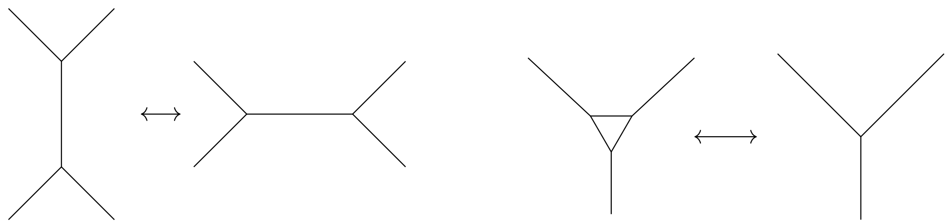


Figure 4.5: The dual 2-2 and 3-1 Pachner moves.

Frobenius and $*$-algebra structures exist and are unique up to isomorphism. This means that the only invariant of the 2d TQFT is the dimension of $\mathcal{A}$, i.e. the ground-state degeneracy of the corresponding phase.

As discussed above, for a semisimple algebra $\mathcal{A}$, boundary conditions correspond to finite-dimensional modules over $\mathcal{A}$. It is easy to see that for the open-closed TQFT to be unitary, the algebra V_{MM} must also have a Hilbert space structure such that

$$T(a)^{\dagger} = T(a^{*}). \tag{4.20}$$

Such a structure always exists and is unique. Thus a boundary condition for a unitary 2d TQFT can be simply identified with a module over $\mathcal{A}$. One can use any faithful module over $\mathcal{A}$ as an input for the state-sum construction.

State-sum construction of the space of states

We have discussed above the state-sum construction of the partition function $Z(\Sigma)$ for an oriented 2d manifold Σ without boundary (or more generally, with only brane boundary). More generally, one also needs to describe in similar terms the state space $\mathcal{A}$ and a linear map $\mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}$ for every bordism Σ whose source is a disjoint union of n circles and target is a disjoint union of l circles. That is, one needs to describe $Z(\Sigma)$ for the case when Σ has nonempty cut boundary.

Consider a bordism Σ with a nonempty cut boundary. For simplicity, let us assume that there is no brane boundary; the general case is a trivial generalization, but requires a more cumbersome notation. We choose a triangulation $\mathcal{T}$ of Σ. It induces a triangulation of each boundary circle. We label the edges of 2-simplices with basis elements of A, as before. The only difference is that boundary 1-simplices have only one label rather than two. If we assign the weights to every 2-simplex and every internal 1-simplex as before and sum over the labelings of internal 1-simplices, we get a number $Z_{\mathcal{T}}(\Sigma)$ which depends on the labelings of the boundary 1-simplices. Suppose some boundary circle is divided into N intervals. Then a labeling by $e_{i_1}, \ldots, e_{i_N}$ corresponds to a vector

$$e_{i_1} \otimes \ldots \otimes e_{i_N} \in A^{\otimes N}. \tag{4.21}$$

We can think of the number $Z_{\mathcal{T}}(\Sigma)$ computed by the state-sum as a matrix element of a linear map

$$A^{\otimes N_1} \otimes \ldots \otimes A^{\otimes N_n} \longrightarrow A^{\otimes M_1} \otimes \ldots \otimes A^{\otimes M_l}, \tag{4.22}$$

where $N_1, \ldots, N_n$ denote the number of 1-simplices in the source circles, and $M_1, \ldots, M_l$ denote the number of 1-simplices in the target circles of Σ. It can be shown (Fukuma, Hosono, and Kawai, 1994) that the map $Z_{\mathcal{T}}(\Sigma)$ does not depend on the triangulation of Σ, provided we fix the triangulation of the boundary circles.

$Z_{\mathcal{T}}(\Sigma)$ is not yet the desired $Z(\Sigma)$ because it depends on the way the boundary circles are triangulated. To get rid of this dependence, we need to restrict this map to a certain subspace in each source factor $A^{\otimes N_i}$ and project to a certain subspace in each target factor $A^{\otimes M_j}$. Both tasks are accomplished by means of projectors $C_N : A^{\otimes N} \to A^{\otimes N}$. The projector C_N is simply $Z_{\mathcal{T}_N}(C)$, where C is a cylinder and $\mathcal{T}_N$ is any triangulation of C such that both boundary circles are subdivided into N intervals. The image of each C_N is a certain subspace of $A^{\otimes N}$ isomorphic to $Z(A)$ (Fukuma, Hosono, and Kawai, 1994). Restricting $Z_{\mathcal{T}}(\Sigma)$ to these subspaces and then projecting to the image of each C_{M_j} gives us the desired map

$$Z(\Sigma) : \mathcal{A}^{\otimes n} \to \mathcal{A}^{\otimes l}, \tag{4.23}$$

where $\mathcal{A} = Z(A)$.

MPS from TQFT

Let us consider the special case when Σ is an annulus such that one of the circles is a cut boundary, while the other one is a brane boundary corresponding to an A-module M. Let $T(a) \in \mathrm{Hom}(M, M)$ represent an action of $a \in A$ in this module. For definiteness, we choose the cut boundary to be the source of Σ, while the target is empty. Thus $Z(\Sigma)$ is a linear map $\mathcal{A} \to \mathbb{C}$. It is the dual of the boundary state corresponding to the module M.

Let us now pick a triangulation of the annulus such that the cut boundary is divided into N intervals. Then $Z_{\mathcal{T}}(\Sigma)$ is a linear map $A^{\otimes N} \to \mathbb{C}$ which depends only on $\mathcal{T}$ and N. We claim that this map is the dual of the MPS state with the dual MPS tensor given by $T : A \to \mathrm{Hom}(M, M)$.

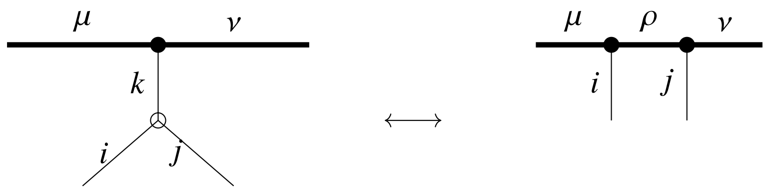


Figure 4.6: The dual shelling of (4.17). A filled dot represents T, while an empty dot represents C.

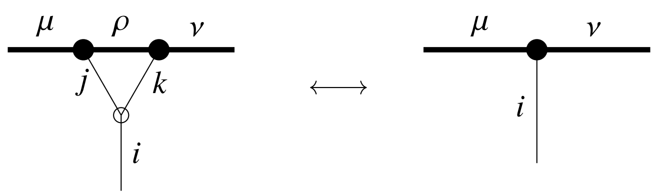
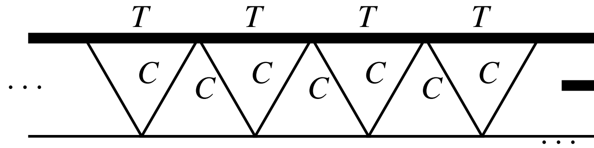


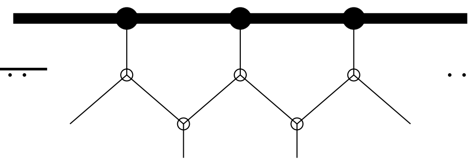
Figure 4.7: The dual shelling of (4.18), representing $T^j_{\mu\rho}T^k_{\rho\nu}C_{ijk} = T^i_{\mu\nu}$
.

To see this, it is convenient to reformulate the state-sum on the Poincare dual complex. This complex is built from the triangulation $\mathcal{T}(\Sigma)$ by replacing k-cells with $(2-k)$-cells, as in Figure 4.4. The dual of a triangulation is not a simplicial complex but a more general cell complex; since we will only be interested in the edges and vertices of this dual complex, we will refer to it as a *skeleton* for Σ. The Pachner moves are the same for skeleton as for triangulations, see Figure 4.5. Recall that for a unitary TQFT, one can choose $\eta_{ij} = \delta_{ij}$, so that indices may be freely raised and lowered; nonetheless, keeping track of index positions now will pay off later when we generalize to equivariant theories. Choose a direction for each edge; the state-sum does not depend on this choice. Choose these directions so that all edges on incoming boundaries are incoming and all edges on outgoing boundaries are outgoing. To define a state-sum on a skeleton, label its non-boundary edges with elements e_i and assign structure coefficients C to each non-boundary vertex according to orientation and using lower indices for incoming arrows and upper for outgoing. With these conventions, the Pachner moves algebrize to (4.15) and (4.16) as before. To incorporate brane boundaries, color brane boundary edges by elements v_μ and attach the module tensor T to each boundary vertex. The boundary moves recover (4.17) and (4.18). The dual state-sum is naturally a tensor network: it defines a circuit between the incoming and outgoing legs. Note that the "virtual" module indices are all contracted, so these legs are physical.

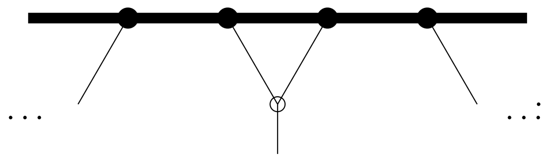
Consider the triangulation, shown in Figure 4.8a, of the annulus with boundary condition T on one of its boundary components. Its state-sum defines a state in the



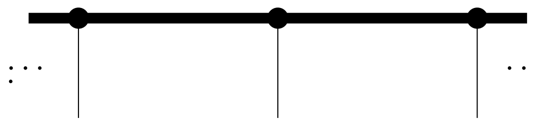
(a) Annulus with upper boundary colored by module T



(b) The Poincare dual of **(a)**



(c) The move (4.17) applied to **(b)**



(d) The move (4.18) applied to **(c)**

Figure 4.8: The equivalence of the annulus to the tensor network representation of an MPS.

physical space $\mathcal{A}^N$. We claim that this state is the fixed point MPS $|\psi_T\rangle$. The proof of this fact is straightforward: by Pachner invariance, the annulus and MPS tensor networks are equivalent, see Figure 4.8.

More generally, one can insert a local observable on the brane boundary of the annulus. Such a local observable is parameterized by $X \in \mathrm{Hom}(M, M)$ which commutes with $T(a)$ for all $a \in A$. The corresponding dual state is $\mathrm{Tr}\left[X^\dagger TT \cdots T\right]$, i.e. it is a generalized MPS state, with A being the physical space.

Since the linear operators $T(a)$ satisfy $T(a)T(b) = T(ab)$, all these MPS states are RG-fixed MPS states. The RG-step is described by the algebra structure on A, $m : A \otimes A \to A$. Moreover, the MPS is automatically in a standard form. The module $T : A \to \mathrm{End}(V)$ is semisimple, so it has a decomposition into simple modules $T^{(\alpha)} : A \to \mathrm{End}(V^{(\alpha)})$. The collection of spaces $\mathrm{End}(V^{(\alpha)})$ form a block-diagonal subspace of $\mathrm{End}(V)$. Since $V^{(\alpha)}$ is simple, $T^{(\alpha)}$ surjects onto the block $\mathrm{End}(V^{(\alpha)})$. Moreover, as we have seen, unitarity of the TQFT enforces that T is an isometry.

The parent Hamiltonian of the MPS on an N-site closed chain has a TQFT interpretation as well: it is the linear map $C_N = Z_{\mathcal{T}_N}(C) : A^{\otimes N} \to A^{\otimes N}$ assigned to a triangulated cylinder C whose boundary consists of two circles triangulated into N intervals. As previously stated, C_N projects onto a subspace $\mathcal{A} = Z(A) \subset A^{\otimes N}$, precisely the space of ground states of the parent Hamiltonian. In the continuum TQFT, topological invariance requires that the cylinder is the identity; this is consistent with our already having projected to $\mathcal{A}$ in defining the continuum state spaces.

We have seen that a unitary TQFT is completely determined by its space of states $\mathcal{A}$ on a circle and that each finite-dimensional commutative algebra $\mathcal{A}$ defines a unitary TQFT. Therefore, the classification of unitary TQFTs is quite simple: there is one for every positive integer n, in agreement with the MPS-based classification of gapped phases (Chen, Gu, and Wen, 2011a; Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011).

4.4 Equivariant TQFT and equivariant MPS

In this section, we generalize the relation between 2D TQFT and MPS states to systems with a global symmetry G. We show that both G-equivariant TQFTs and G-equivariant RG-fixed MPS states are described by semisimple G-equivariant algebras. In particular, we show that invertible G-equivariant TQFTs correspond to short-range entangled phases with symmetry G, and that both are classified by $H^2(G, U(1))$.

G-equivariant matrix product states

Let G be a finite symmetry group acting on the physical space A via a unitary representation R, $g \mapsto R(g) \in \mathrm{End}(A)$. A G-invariant MPS tensor is a map $\mathcal{P} : U \otimes U^* \to A$ equivariant in the following sense:

$$R(g)\mathcal{P}(X) = \mathcal{P}\left(Q(g)XQ(g^{-1})\right), \tag{4.24}$$

where the linear maps $Q(g) \in \mathrm{End}(U)$ form a projective representation of G. Let $T = \mathcal{P}^\dagger$. In terms of T, the equivariance condition looks as follows:

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}, \tag{4.25}$$

for any $a \in A$ and any $g \in G$. The dual MPS state corresponding to T is

$$\langle\psi_T| = \sum_{i_1,\ldots,i_N} \mathrm{Tr}_U[T(e_{i_1}) \ldots T(e_{i_N})]\langle i_1 \ldots i_N|. \tag{4.26}$$

It is easy to see that the state ψ_T is G-invariant, thanks to the equivariance condition on P. More generally, let $X \in \mathrm{End}(U)$. Note that $\mathrm{End}(U)$ is a genuine (not projective) representation of G. Then the generalized MPS state $\mathrm{Tr}[XTT \ldots T]$ transforms in the same way as X.

G-equivariant TQFT

Roughly speaking, a definition of a G-equivariant TQFT is obtained from the definition of an ordinary TQFT by replacing oriented manifolds with oriented manifolds with principal G-bundles. This reflects the intuition that a model with a global non-anomalous symmetry G can be coupled to a background G gauge field. (For a finite group G, there is no difference between a G gauge field and a principal G-bundle.)

Some care is required regarding marked points and trivializations. Namely, each source and each target circle must be equipped with a marked point and a trivialization of the G-bundle at this point. This means that the holonomy of the gauge field around the circle is a well-defined element $g \in G$, rather than a conjugacy class. A G-equivariant TQFT associates a vector space $\mathcal{A}_g$ to a circle with holonomy g. A generic G-equivariant bordism has more than one marked point, and the holonomies between marked points along chosen paths are well-defined elements of G as well. Of course, these holonomies depend only on the homotopy classes of paths. For example, a G-equivariant cylinder bordism has two marked points (one for each boundary circle) and depends on two arbitrary elements of G. On the other hand, a G-equivariant torus, regarded as bordism with an empty source and empty target, has no marked points and depends on two commuting elements of G defined up to an overall conjugation.

One can describe a G-equivariant TQFT purely algebraically in terms of a G-crossed Frobenius algebra (Turaev, 2010; Moore and Segal, 2006). This notion generalizes the commutative Frobenius algebra $\mathcal{A}$ and encodes the linear maps $Z(\Sigma, \mathcal{P})$ in a fairly complicated way.

We will use instead a state-sum construction of 2D equivariant TQFTs which is manifestly local. Its starting point is a finite-dimensional semisimple G-equivariant algebra A. This is an algebra with an action of G that preserves the multiplication $m : A \otimes A \to A$. That is, G acts on A via a linear representation $R(g)$, $g \in G$, such that

$$m(R(g)a \otimes R(g)b) = R(g)m(a \otimes b). \tag{4.27}$$

This condition implies that the group action also preserves the scalar product η defined in (4.13):

$$\eta(R(g)a, R(g)b) = \eta(a, b). \tag{4.28}$$

The condition (4.28) says that $R(g)$ is orthogonal with respect to η. As a consequence, if $R(g)$ commutes with the anti-linear map (4.19), it is unitary with respect to the Hilbert space inner product.

A large class of examples of G-equivariant algebras is obtained by taking $A = \mathrm{End}(U)$, where U is a vector space, and G acts on U via a projective representation $Q(g)$. It is clear that this gives rise to a genuine action of G on $\mathrm{End}(U)$ which preserves the usual matrix multiplication on $\mathrm{End}(U)$. Moreover, the standard Frobenius structure

$$\eta(a, b) = \mathrm{Tr}(ab) \tag{4.29}$$

is clearly G-invariant.

A G-equivariant module over a G-equivariant algebra A is a vector space V with compatible actions of both A and G. That is, for every $a \in A$ we have a linear map $T(a) : V \to V$ such that $T(a)T(a') = T(aa')$, and for every $g \in G$ we have an invertible linear map $Q(g) : V \to V$ such that $Q(g)Q(g') = Q(gg')$. The compatibility condition that they satisfy reads

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}. \tag{4.30}$$

If we take $A = \mathrm{End}(U)$, where U is a projective representation of G with a 2-cocycle $\omega \in H^2(G, U(1))$, then U is not a G-equivariant module over A unless ω vanishes. However, if W is a projective representation of G with a 2-cocycle $-\omega$, then $U \otimes W$ is a G-equivariant module.[5]

Equivariant TQFTs admit a lattice description as well. It is simplest to describe a Poincare dual formulation in the sense of Section 3.5; spaces in this formulation also have direct interpretations as tensor networks. A trivialized background gauge field is represented on a skeleton as a decoration of each oriented edge with an element $g \in G$. Flipping the orientation of the edge replaces g with g^{-1}. We require that the field is flat: that the product of the group elements around the boundary of each face is the identity element.[6] In a basis e_i, $i \in S$ of A, the weight of a coloring of the skeleton is the product of the structure constants C^{ijk} over vertices (with the cyclic order given by the orientation) and a factor $\eta(R(g)e_i, e_j) = R(g)^k{}_i \eta_{jk}$ for each edge directed from i to j labeled by g. The partition sum is the sum of these weights over all colorings; we emphasize that the group labels represent a background gauge field and are not summed. To incorporate brane boundaries, choose a G-equivariant module V over A. Fix a basis f_μ of V. For each brane boundary vertex, label its

[5] In fact, the category of projective representations of G with a 2-cocycle $-\omega$ is equivalent to the category of G-equivariant modules over $\mathrm{End}(U)$, and the equivalence sends a projective representation W to $U \otimes W$.

[6] In the triangulation picture, we require the product of all group elements corresponding to edges entering a particular vertex to be the identity element.

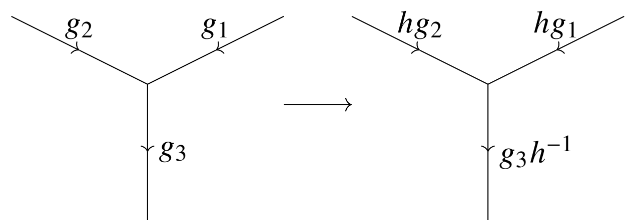


Figure 4.9: A gauge transformation at the vertex by h

adjacent boundary edges each with a basis element, so that each boundary edge has a total of two labels. The weight of a skeleton with a brane boundary is a product of C's and R's as well as a module tensor T for each brane boundary vertex and a matrix element $Q(g)^{\mu}_{\nu}$ for each brane boundary edge.

As before, topological invariance of the state-sum amounts to checking the conditions (4.15), (4.16), (4.17), and (4.18). These are satisfied by any finite-dimensional semisimple A. In order for the equivariant state-sum to constitute a well-defined equivariant TQFT, it must also be independent of the choice of trivialization of the background gauge field; in order words, it must be gauge invariant. A gauge transformation by $h \in G$ on a vertex acts by changing the decorations of the three edges whose boundary contains the vertex: incoming edges with g become hg, outgoing gh^{-1}, as in Figure 4.9. Invariance under a gauge transformation on a vertex in the interior is ensured by axioms (**??**) and (4.28) of a G-equivariant algebra. For vertices in the brane boundary, the analogous result follows from the G-equivariant module condition (4.30).[7] Finally, invariance under simultaneously reversing an edge direction and inverting its group label is enforced by the axiom (4.28).

G-equivariant semisimple algebras

The classic Wedderburn theorem implies that every finite-dimensional semisimple algebra is a sum of matrix algebras. Let us discuss a generalization of this result to the G-equivariant case following (Ostrik, 2003) and (Etingof, 2015).

First, we can write every G-equivariant semisimple algebra as a sum of indecomposable ones, so it is sufficient to classify indecomposable G-equivariant semisimple algebras. A large class of examples is given by algebras of the form $\mathrm{End}(U)$, where U is a projective representation of G. Another set of examples is obtained as follows: let $H \subset G$ be a subgroup. Consider the space of complex-valued functions on G invariant with respect to left translations by H, i.e. $f(h^{-1}g) = f(g)$ for all $g \in G$

[7] Here it is crucial that linear transformations $Q(g)$ form an ordinary (i.e. not projective) representation of G.

and all $h \in H$. The group G acts on this space by right translations:

$$(R(g)f)(g') = f(g'g). \tag{4.31}$$

Pointwise multiplication makes this space of functions into an associative algebra, and it is clear that the G-action commutes with the multiplication. This G-equivariant algebra is indecomposable for any H.

The most general indecomposable G-equivariant semisimple algebra is a combination of these two constructions called the induced representation $\mathrm{Ind}_H^G \mathrm{End}(U)$ (Ostrik, 2003; Etingof, 2015). One picks a subgroup $H \subset G$ and a projective representation (U, Q) of H. Here U is a vector space and Q is a map $H \to \mathrm{End}(U)$ defining a projective action with a 2-cocycle $\omega \in H^2(H, U(1))$. Then one considers the space of functions on G with values in $\mathrm{End}(U)$ which have the following transformation property under the left H action:

$$f(h^{-1}g) = Q(h)f(g)Q(h)^{-1}. \tag{4.32}$$

It is easy to check that the right G translations act on this space of functions. Pointwise multiplication makes this space into a G-equivariant algebra, and one can show that it is indecomposable. To summarize, indecomposable G-equivariant semisimple algebras are labeled by triples (H, U, Q), where $H \subset G$ is a subgroup, and (U, Q) is a projective representation of H. All these algebras are actually Frobenius algebras: the trace function $A \to \mathbb{C}$ is given by

$$\sum_{g \in G} \mathrm{Tr}_U f(g). \tag{4.33}$$

A G-equivariant module over such an algebra A is obtained as follows. Start with an H-equivariant module (M, Q) over $\mathrm{End}(U)$. Here M is a module over $\mathrm{End}(U)$ and $Q : H \to \mathrm{End}(M)$ is a compatible action of H on M. As explained above, M must have the form $U \otimes W$, where W carries a projective action $S(h)$ of H with a 2-cocycle $-\omega$. Then consider functions on G with values in M which transform as follows under the left H-translations:

$$m(h^{-1}g) = (Q(h) \otimes S(h))m(g), \quad m : G \to U \otimes W. \tag{4.34}$$

The group G acts on this space by right translations, and it is easy to see that the pointwise action of $A = (H, U, Q)$ makes it into a G-equivariant module over A. One can show that any G-equivariant module over such an A is a direct sum of modules of this sort.

G-equivariant MPS from G-equivariant TQFT

It is sufficient to consider indecomposable TQFTs and G-equivariant algebras. Let us begin with the case $H = G$. Then the algebra $A = (G, U, Q)$ is isomorphic to the algebra $\mathrm{End}(U)$, and a G-equivariant module over it is simply a vector space M with a G-equivariant action of $\mathrm{End}(U)$. In other words, $M = U \otimes W$, where U carries a projective representation of G with the 2-cocycle ω, and W carries a projective representation of G with a 2-cocycle $-\omega$.

Consider an annulus whose outer boundary is labeled by a brane M and whose inner boundary is a cut boundary. Let us triangulate both boundary circles into N intervals. Let $g_{i,i+1}$ be the element of G labeling the interval from the $(i+1)^{\mathrm{th}}$ to the i^{th} points on the boundary. We also assume that the holonomy of the gauge field between the points labeled by 1 on the two boundary circles is trivial. We get the following dual state:

$$\langle\psi_T| = \sum \mathrm{Tr}_{U\otimes W}[T(e_{i_1})Q(g_{1,2})\cdots \cdots T(e_{i_N})Q(g_{N,1})]\langle i_1\cdots i_N|. \tag{4.35}$$

Note that although $T(e_i)$ is an operator on $U \otimes W$, it has the form $T(e_i) \otimes \mathbf{1}_W$. Therefore, if $g_{i,i+1} = 1$ for all i, the trace over W gives an overall factor $\dim W$, and up to this factor we get the equivariant MPS (4.26). Inserting an observable $X \in \mathrm{End}(U)$ on the brane boundary, we get a generalized equivariant MPS. The case when $X \in \mathrm{End}(U \otimes W)$ does not give anything new, since the trace over V factors out.

The generalized equivariant MPS (cf. eq (4.3))

$$\langle\psi_T^X| = \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n| \tag{4.36}$$

may be charged under the action of $h \in G$:

$$\begin{aligned} R(h)^{\otimes N}\left\langle\psi_T^X\right| &= \sum \mathrm{Tr}\big[X^\dagger T(e_{i_1})\cdots T(e_{i_n})\big]\langle (h^{-1}\cdot i_1)\cdots(h^{-1}\cdot i_n)| \\ &= \sum \mathrm{Tr}\big[X^\dagger T(h\cdot e_{i_1})\cdots T(h\cdot e_{i_n})\big]\langle i_1\cdots i_n| \\ &= \sum \mathrm{Tr}\big[Q(h^{-1})X^\dagger Q(h)T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n|. \end{aligned} \tag{4.37}$$

Let us now consider the case when H is a proper subgroup of G and $A = \mathrm{Ind}_H^G\mathrm{End}(U)$, for some projective representation U of H. If we choose right H-coset representatives g_a, $a \in H\backslash G$, and a basis e_i in $\mathrm{End}(U)$, then a basis in A is given by e_i^a. Similarly, if f_μ is a basis in an H-equivariant module $U \otimes W$, then a basis in the corresponding G-equivariant module M is f_μ^a.

The action of A on M is diagonal as far as the a index is concerned. Therefore the dual state corresponding to a triangulated annulus with $g_{i,i+1} = 1$ for all i vanishes unless all a indices are the same. Then

$$\langle\psi_T| = \dim(W) \sum_{a,i_1,\ldots,i_N} \mathrm{Tr}_U[T(e_{i_1})\cdots \\ \cdots T(e_{i_N})]\langle i_1 a\, i_2 a\, \cdots i_N a|. \tag{4.38}$$

This state has equal components along all $|H\backslash G|$ directions. We can get a state concentrated at a particular value of a by inserting a suitable observable $X \in \mathrm{End}(M)$ on the brane boundary. Such an observable must commute with the action of A, so it must have the form $X^{\mu a}_{\nu\ b} = f(a)\delta^\mu_\nu \delta^a_b$. Choosing the function $f(a)$ to be supported at a particular value of a gives a generalized MPS state supported at this value of a.

The symmetry group G acts transitively on $H\backslash G$. This suggests that we are dealing with a phase where the symmetry G is spontaneously broken down to H, so that we get $|H\backslash G|$ sectors labeled by the index a. To confirm this, consider the partition function of this TQFT on a closed oriented 2-manifold Σ with a trivial G-bundle. After we choose a skeleton of Σ, we can represent this G-bundle by labeling every 1-simplex with the identity element of G. In addition, every 1-simplex is labeled by a pair of basis vectors of A. Since both the multiplication in the algebra A and the scalar product are pointwise in $H\backslash G$, the partition function receives contributions only from those labelings where all a labels are the same. Furthermore, turning on a gauge field which takes values in H does not destroy this property. We conclude that the theory has superselection sectors labeled by elements of $H\backslash G$, and each sector has unbroken symmetry H.

Twisted-sector states

Now let us not assume that $g_{i,i+1} = 1$, but instead allow the gauge field around the circle to have a nontrivial holonomy. Let us take $H = G$ first, i.e. the case of unbroken symmetry. Consider the MPS (4.35). Applying a gauge transformations (by $g_{1,2}g_{2,3}\cdots g_{k-1,k}$ at vertex k) to the boundary vertices, it can be written as

$$\langle\psi_{T,g}| = \sum \mathrm{Tr}_{U\otimes W}[Q(g)T(e_{i_1})\cdots T(e_{i_N})] \\ \otimes_{k=1}^N R(g_{1,2}\cdots g_{k-1,k})^{i_k}{}_{j_k}\langle j_k| \tag{4.39}$$

where $g = g_{1,2}g_{2,3}\cdots g_{N,1}$ is the holonomy of the gauge field. This is LU equivalent to the state

$$\langle\psi_{T,g}| = \sum \mathrm{Tr}_{U\otimes W}[\mathcal{Q}(g)T(e_{i_1})\cdots T(e_{i_N})]\langle i_1\cdots i_N| \tag{4.40}$$

so we have effectively set $g_{i,i+1} = 1$ for all $i \neq N$ and $g_{N,1} = g$. Note that $\mathcal{Q} = Q \otimes S$, so the trace factors into a product of a trace over U and a trace over W. The latter gives us an overall factor, and we have

$$\begin{aligned}\langle\psi_{T,g}| = \mathrm{Tr}_W[S(g)]\sum \mathrm{Tr}_U[Q(g)T(e_{i_1})\cdots \\ \cdots T(e_{i_N})]\langle i_1\cdots i_N|.\end{aligned} \tag{4.41}$$

This state transforms under $h \in G$ into

$$\begin{aligned}R(h)^{\otimes N}\left\langle\psi_{T,g}\right| &= (\mathrm{Tr}_W[S(g)])\sum \mathrm{Tr}\big[Q(g)T(e_{i_1})\cdots T(e_{i_n})\big]\langle(h^{-1}\cdot i_1)\cdots(h^{-1}\cdot i_n)| \\ &= (\mathrm{Tr}_W[S(g)])\sum \mathrm{Tr}\big[Q(h)^{-1}Q(g)Q(h)T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n| \\ &= (\mathrm{Tr}_W[S(g)])\omega(g,h)\omega(h^{-1},gh)\sum \mathrm{Tr}\big[Q(h^{-1}gh)T(e_{i_1})\cdots T(e_{i_n})\big]\langle i_1\cdots i_n|.\end{aligned} \tag{4.42}$$

Note that the g-twisted sector becomes the hgh^{-1}-twisted sector.

Now suppose H is a proper subgroup of G. Since T acts pointwise in the a label, while G acts on $a \in H\backslash G$ by right translations, the annulus state vanishes unless the holonomy around the circle is in H. This confirms once again that H is the unbroken subgroup. Indeed, when the holonomy does not belong to the unbroken subgroup, there must be a domain wall somewhere on the circle. Its energy is nonzero in the thermodynamic limit, so the TQFT space of states must be zero-dimensional for holonomies not in H.

If $\mathcal{A}_g$ denotes the space of states in the g-twisted sector, the space $\mathcal{A} = \oplus_g\mathcal{A}_g$ has an automorphism $\alpha_h := R(h)^{\otimes N}$ for each $h \in G$ such that $\alpha_h(\mathcal{A}_g) \subset \mathcal{A}_{hgh^{-1}}$. $\mathcal{A}$ is the G-graded vector space underlying the G-crossed Frobenius algebra that defines the associated G-equivariant TQFT (Turaev, 2010; Moore and Segal, 2006).

Morita equivalence

We have seen that to any semisimple G-equivariant algebra, one can associate a G-equivariant 2d TQFT. But different algebras may give rise to the same TQFT.

In particular, we would like to argue that the TQFT corresponding to an indecomposable algebra $A = (H, U, Q)$, where (U, Q) is a projective representation of H, depends only on the subgroup H and the 2-cocycle ω, but not on the specific choice of (U, Q).

To show this, note first of all that the partition function vanishes if the holonomy does not lie in H (this again follows from the fact that multiplication in the algebra A is pointwise with respect to the a index). Thus it is sufficient to consider oriented 2-manifolds with H-bundles. Further, if U and U' are projective representations of H with the same 2-cocycle, then $U' = U \otimes W$, where W is an ordinary representation of H. Thus we only need to show that the partition functions corresponding to algebras (H, U, Q) and $(H, U \otimes W, Q \otimes S)$ are the same, where $S : H \to \mathrm{End}(W)$ is a representation of H. But it is clear from the state sum construction that the two partition functions differ by a factor which is the partition function of two dimensional H-equivariant TQFT corresponding to the algebra (H, W, S).

We reduced the problem to showing that the H-equivariant TQFT constructed from the algebra (H, W, S) is trivial when (W, S) is an ordinary (not projective) representation of H. This is straightforward: the equation $S(h_1) \ldots S(h_n) = S(h_1 \ldots h_n)$ and the flatness condition for the H gauge field imply that the partition function is independent of the H-bundle, and for the trivial H-bundle the partition function is the same as for the trivial TQFT with $A = \mathbb{C}$.

From the mathematical viewpoint, G-equivariant algebras with the same H and ω are *Morita-equivalent*[8] (Ostrik, 2003). Thus we have shown that Morita-equivalent algebras lead to identical G-equivariant TQFTs.[9]

Stacking phases

Consider two gapped systems built from algebras A_1 and A_2. Recall from Section 4.2 that the stacked system (4.8) is built from the tensor product algebra $A_1 \otimes A_2$. Although we have not discussed parent Hamiltonians of G-equivariant MPS, an analogous stacking operation can be defined for G-symmetric gapped phases by way of the connection to TQFT. Now suppose A_1 and A_2 are G-equivariant algebras. It is clear from the G-equivariant state sum construction that the partition functions for

[8] More accurately, algebras with the same H and ω, *up to conjugation in* G, are Morita-equivalent. In physical contexts, however, it is typical to keep track of the embedding of the unbroken symmetry H in the full symmetry group G. Therefore, the classification of physical gapped G-symmetric phases is slightly more refined than that of Morita classes.

[9] Strictly speaking, we only showed this for closed 2d TQFTs, but the argument easily extends to the open-closed case.

(H,ω)	type of phase	name
$(\langle a,b\rangle,1)$	trivial	1
$(\langle a,b\rangle,\omega_1)$	symmetry-protected	ω
$(\langle a\rangle,1)$	broken symmetry	A
$(\langle b\rangle,1)$	broken symmetry	B
$(\langle ab\rangle,1)$	broken symmetry	C
$(1,1)$	broken symmetry	0

Figure 4.10: Indecomposable phase classification for the $G = \mathbb{Z}_2 \times \mathbb{Z}_2$.

the algebra $A_1 \otimes A_2$ are products of those for A_1 and A_2 and that the Hilbert spaces are tensor products. Thus the MPS ground states, which determine a phase and which are realized in TQFT, stack like the tensor product of G-equivariant algebras.

It is a tedious but straightforward exercise to check that the result of stacking the phase labeled by subgroup-cocycle pair (H,ω) with the phase (K,ρ) is the phase

$$(H\cap K, \omega|_{H\cap K} + \rho|_{H\cap K})^{\oplus[G:HK]} \tag{4.43}$$

where $\omega|_{H\cap K}$ denotes the restriction of ω to the intersection subgroup $H\cap K$ and $[G:HK]$ denotes the index of the subgroup HK in G, assuming H and K are normal in G.

Let us consider a simple example: take $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle a,b\rangle$, where a and b are commuting elements of order 2. For the subgroup $H = G$, there are two cohomology classes $\omega \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1))$. Let ω_1 denote the nontrivial class. For each of the other subgroups $H = \langle a\rangle$, $\langle b\rangle$, $\langle ab\rangle$, 1, there is a unique cocycle. Thus the classification of $\mathbb{Z}_2 \times \mathbb{Z}_2$-equivariant phases is like Figure 4.10.

According to (4.43), the stacking rules are

$$1\otimes 1 = 1,\quad 1\otimes\omega=\omega,\quad 1\otimes A = A,\quad 1\otimes B = B,\quad 1\otimes C = C,\quad 1\otimes 0 = 0$$
$$\omega\otimes\omega = 1,\quad \omega\otimes A = A,\quad \omega\otimes B = B,\quad \omega\otimes C = C,\quad \omega\otimes 0 = 0$$
$$A\otimes A = A^{\oplus 2},\quad B\otimes B = B^{\oplus 2},\quad C\otimes C = C^{\oplus 2},\quad A\otimes B = 0,\quad B\otimes C = 0,\quad C\otimes A = 0$$
$$A\otimes 0 = 0^{\oplus 2},\quad B\otimes 0 = 0^{\oplus 2},\quad C\otimes 0 = 0^{\oplus 2},\quad 0\otimes 0 = 0^{\oplus 4}.$$

Symmetry-protected topological phases

Finally, let us discuss the case of Short-Range Entangled (SRE) phases with symmetry G. According to one definition (Kitaev, 2015), an SRE phase is one that is invertible under the aforementioned stacking operation. Such phases have a one-dimensional space of ground states for every G-bundle on a circle. Since the space of states of a decomposable TQFT on a circle with a trivial bundle has a dimension greater than one, a TQFT corresponding to an SRE phase must be indecomposable. We showed that when H is a subgroup of G, the space of states is zero-dimensional whenever the holonomy does not lie in H. Hence an equivariant TQFT built from an indecomposable G-equivariant algebra (H, U, Q) cannot correspond to an SRE unless $H = G$.

These SRE phases are all Symmetry Protected Topological (SPT) phases - phases that are trivial if we ignore symmetry. A G-equivariant algebra of the form $\mathrm{End}(U)$, where U is a projective representation of G, is simply a matrix algebra if we ignore the G action. Hence the corresponding non-equivariant TQFT is trivial; the corresponding Hamiltonian is connected to the trivial one by a Local Unitary transformation. Hence SPT phases with symmetry G are labeled by 2-cocycles $\omega \in H^2(G, U(1))$. This is a well-known result (Chen, Gu, and Wen, 2011a; Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011).

4.5 Spin-TQFTs

$\mathbb{Z}_2$-graded semi-simple algebras

Now, let us consider a fermionic version of the relation between MPS and TQFT. The algebraic input for the fermionic state-sum construction is a $\mathbb{Z}_2$-graded semisimple Frobenius algebra A (Novak and Runkel, 2014; Gaiotto and Kapustin, 2016).[10] A Frobenius algebra is a finite-dimensional algebra over $\mathbb{C}$ with a non-degenerate symmetric scalar product $\eta : A \otimes A \to \mathbb{C}$ satisfying $\eta(a, bc) = \eta(ab, c)$ for all $a, b, c \in A$. A $\mathbb{Z}_2$-grading on A is a decomposition $A = A_+ \oplus A_-$ such that

$$\begin{aligned} A_+ \cdot A_+ \subset A_+, \quad A_- \cdot A_- \subset A_+, \\ A_- \cdot A_+ \subset A_-, \quad A_+ \cdot A_- \subset A_-. \end{aligned} \tag{4.44}$$

Equivalently, a $\mathbb{Z}_2$-grading is an operator $\mathcal{F} : A \to A$ such that $\mathcal{F}^2 = 1$ and $\mathcal{F}(a) \cdot \mathcal{F}(b) = \mathcal{F}(a \cdot b)$. The operator $\mathcal{F}$ is called fermion parity and is traditionally denoted $(-1)^F$. We also assume that the scalar product η is $\mathcal{F}$-invariant:

$$\eta(\mathcal{F}(a), \mathcal{F}(b)) = \eta(a, b). \tag{4.45}$$

[10]While it is possible to relax the semi-simplicity condition (Novak and Runkel, 2014), here we are interested in unitary TQFTs, and for such TQFTs, one may assume that A is semi-simple (Kapustin, Turzillo, and You, 2017).

Note that $\mathcal{F}$ defines an action of $\mathbb{Z}_2$ on A which makes A into a $\mathbb{Z}_2$-equivariant algebra. This observation is the root cause of the bosonization phenomenon: there is a 1-1 map between 1+1d phases of bosons with $\mathbb{Z}_2$ symmetry and 1+1d phases of fermions. For now, we use this fact to describe the classification of $\mathbb{Z}_2$-graded simple algebras. Namely, since the only proper subgroup of $\mathbb{Z}_2$ is the trivial one, and $H^2(\mathbb{Z}_2, U(1)) = 0$, a simple $\mathbb{Z}_2$-graded algebra is isomorphic either to $\mathrm{End}(V)$ for some $\mathbb{Z}_2$-graded vector space $V = V_+ \oplus V_-$, or to $C\ell(1) \otimes \mathrm{End}(V)$ for some purely even vector space $V = V_+$ (Kapustin, Turzillo, and You, 2017). Here $C\ell(1)$ denotes the Clifford algebra with one generator, i.e. an algebra with an odd generator Γ satisfying $\Gamma^2 = 1$.

As explained in (Kapustin, Turzillo, and You, 2017), the bosonic phase depends only on the Morita-equivalence class of A. The choice of V does not affect the Morita-equivalence class of the algebra, so there are only two Morita equivalence classes of $\mathbb{Z}_2$-graded algebras: the trivial one, corresponding to the algebra $\mathbb{C}$, and the nontrivial one, corresponding to the algebra $C\ell(1)$. In the bosonic case, the former one corresponds to the trivial gapped phase with a $\mathbb{Z}_2$ symmetry, while the latter one corresponds to the phase with a spontaneously broken $\mathbb{Z}_2$.

The fermionic interpretation is different. As briefly mentioned in (Gaiotto and Kapustin, 2016) and discussed in more detail below, the algebra $C\ell(1)$ describes a gapped fermionic phase which is equivalent to the nontrivial Majorana chain. This is in accord with the intuition that fermion parity cannot be spontaneously broken.

Spin structures

A spin structure on an oriented manifold enables one to define a spin bundle. For a 1d manifold X, a spin bundle is a real line bundle L plus an isomorphism $L \otimes L \to TX$. Thus a spin bundle is a square root of the tangent bundle. Since TX is trivial, such L are classified by elements of $H^1(X, \mathbb{Z}_2)$. Since $H^1(S^1, \mathbb{Z}_2) = \mathbb{Z}_2$, there are two possible spin structures on a circle, called the R (Ramond) an NS (Neveu-Schwarz) spin structures in the string theory literature. The R structure corresponds to a trivial L, while NS structure corresponds to the "Möbius band" L. In other words, if we give L a metric and compute the holonomy of the unique connection compatible with it along S^1, we get 1 for the R case, and -1 for the NS case.

For an oriented 2d manifold Σ, we can regard $T\Sigma$ as a complex line bundle, and then a spin bundle on Σ is a complex line bundle S equipped with an isomorphism $S \otimes S \to T\Sigma$. One can show that such an S always exists. If S and S' are two spin bundles, they differ by a line bundle which squares to a trivial line bundle on Σ. The latter are classified by elements of $H^1(\Sigma, \mathbb{Z}_2)$. Thus there are as many spin structures as there are elements of $H^1(\Sigma, \mathbb{Z}_2)$. But in general there is no natural way to identify elements of $H^1(\Sigma, \mathbb{Z}_2)$ with spin structures.[11]

It is easy to see that a spin structure s on an oriented 2d manifold Σ induces a spin structure on any oriented 1d manifold γ embedded into Σ. Define $\sigma_s(\gamma) = +1$ if the induced structure is of the NS type and $\sigma_s(\gamma) = -1$ if the induced structure is of the R type. That is, $\sigma_s(\gamma)$ is the negative of the holonomy of the connection corresponding to the induced spin structure. It is easy to show that $\sigma_s(\gamma)$ depends only on the homology class of γ and thus defines a function $\sigma_s : H_1(\Sigma, \mathbb{Z}_2) \to \mathbb{Z}_2$. With more work, one can show that this function satisfies

$$\sigma_s([\gamma] + [\gamma']) = \sigma_s([\gamma])\sigma_s([\gamma'])(-1)^{\langle[\gamma],[\gamma']\rangle}. \tag{4.46}$$

That is, it is a quadratic $\mathbb{Z}_2$-valued function on $H_1(\Sigma, \mathbb{Z}_2)$ whose corresponding bilinear form is the intersection pairing on $H_1(\Sigma, \mathbb{Z}_2)$. In fact, it is a theorem of Atiyah (Atiyah, 1971) that for a closed Σ, the spin structure is determined by such a quadratic function, and that any such quadratic function determines a spin structure. Note that the ratio of two such quadratic functions is a linear function on $H_1(\Sigma, \mathbb{Z}_2)$, or equivalently an element of $H^1(\Sigma, \mathbb{Z}_2)$. Thus we recover the result that two spin structures differ by an element of $H^1(\Sigma, \mathbb{Z}_2)$.

We record for future use another property of the function σ_s:

$$\sigma_{s+a}([\gamma]) = (-1)^{\int_\gamma a}\sigma_s([\gamma]), \tag{4.47}$$

where a is an arbitrary element of $H^1(\Sigma, \mathbb{Z}_2)$. Thus $\sigma_s([\gamma])$ is an affine-linear function of s and a quadratic function of $[\gamma]$.

We will also need a version of this result for the case when Σ has a nonempty boundary. As in the case of equivariant TQFT, it is convenient to choose, along with a spin structure s, a point on every connected component of $\partial\Sigma$ and a normalized basis vector for the real spin bundle L at this point. This simplifies the gluing of spin manifolds. We will denote by $\partial_0\Sigma$ the set of all marked points, and will call a spin structure on Σ together with a trivialization of L at $\partial_0\Sigma$ a spin structure on the pair $(\Sigma, \partial_0\Sigma)$. The group $H^1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ acts freely and transitively on the set of spin structures on $(\Sigma, \partial_0\Sigma)$. Despite this, there is no canonical way to identify spin

[11]The case of a torus is an exception, since then $T\Sigma$ is trivial. This is why one can talk about periodic and anti-periodic spin structures on a torus.

structures with elements of $H^1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$. To get an algebraic description of spin structures, one can proceed as follows (Segal, 1988). First, note that $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ can be identified with $H_1(\Sigma_*, \mathbb{Z}_2)$, where Σ_* is a closed oriented 2d manifold obtained by gluing a sphere with holes onto Σ. This identification depends on the choice of a cyclic order of the set of boundary circles of Σ. Thus the intersection form on $H_1(\Sigma_*, \mathbb{Z}_2)$ induces a non-degenerate symmetric bilinear form on $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$. There is also an identification of the set of spin structures on $(\Sigma, \partial_0\Sigma)$ and the set of spin structures on Σ^* (Segal, 1988). Thus the set of spin structures on $(\Sigma, \partial_0\Sigma)$ can be identified with the set of $\mathbb{Z}_2$-valued quadratic functions on $H_1(\Sigma, \partial_0\Sigma; \mathbb{Z}_2)$ refining the intersection form. This identification still depends on a choice of a cyclic order on the set of boundary circles of Σ. One can determine which spin structure is induced on any particular connected component of $\partial\Sigma$ by evaluating this quadratic function on the closed curve wrapping that component.

State-sum construction of the spin-dependent partition function

To define the partition function of a spin-TQFT on a closed oriented 2-manifold Σ with a spin structure, we choose a skeleton of Σ, i.e. a trivalent graph Γ on Σ whose complement is homeomorphic to a disjoint union of disks. Equivalently, one may think of Γ as the Poincaré dual of a triangulation $\mathcal{T}$ of Σ.[12] For every vertex $v \in \Gamma$, let $\Gamma(v)$ denote the edges containing v. Orientation of Σ gives rise to a cyclic order on $\Gamma(v)$ for all v. This is sufficient to produce the partition function of a bosonic TQFT based on the algebra A, but in order to construct the fermionic partition function, we need to choose an actual order on $\Gamma(v)$. We can do it by picking one special edge $e_0(v) \in \Gamma(v)$ for every v. We also choose an orientation for each edge of Γ. (In Ref. (Gaiotto and Kapustin, 2016) both an orientation of edges and a choice of $e_0(v)$ arose from a branching structure on $\mathcal{T}$, but here we follow Ref. (Novak and Runkel, 2014) and choose them independently.) These choices are called a *marking* of Γ.

We also need to describe a choice of spin structure on Σ. This is a cellular 1-cochain s valued in $\mathbb{Z}_2$ (i.e. an assignment of elements of $\mathbb{Z}_2$ to edges of Γ) with its coboundary being a certain 2-cocycle w_2 whose cohomology class is the second Stiefel-Whitney class $[w_2](\Sigma)$. Following (Novak and Runkel, 2014), we write the constraint $\delta s = w_2$ as

$$(\delta s)(f) = 1 + K + D \bmod 2 \tag{4.48}$$

[12] One can formulate the construction either in terms of triangulations or in terms of skeletons, but the latter approach gives a bit more flexibility when we allow Σ to have a nonempty boundary.

where f is a particular cell in $\Sigma\backslash\Gamma$, K is the number of clockwise oriented edges in ∂f, and D is the number of vertices v for which the counterclockwise-oriented curve homologous to ∂f in Γ enters v through $e_0(v)$. Two solutions s, s' of this constraint are regarded equivalent, $s \sim s'$, if $s - s' = \delta t$ for some 0-cochain t. Two solutions s, s' define isomorphic spin structures on Σ if and only if $s \sim s'$ (Novak and Runkel, 2014; Gaiotto and Kapustin, 2016). Thus we recover the fact that the number of distinct spin structures on Σ is equal to $|H^1(\Sigma, \mathbb{Z}_2)|$.

One can give an explicit description of the holonomy function $\sigma_s(\gamma)$ corresponding to the 1-cochain s in terms of the marking of Γ along a closed oriented curve γ; see eq. (3.45) of Ref. (Novak and Runkel, 2014). This formula can be written as

$$\sigma_s(\gamma) = -(-1)^{s(\gamma)+K+D+L}, \tag{4.49}$$

where K is the number of edges anti-aligned with γ, D is the number of special edges through which γ enters a vertex, and L is the number of special edges to the left of γ. For example, when γ is a counterclockwise-oriented curve bounding a single cell in $\Sigma\backslash\Gamma$, L vanishes, and so, by (4.48), we have $\sigma_s(\gamma) = +1$. One can show that this function depends only on the homology class of γ and is a quadratic refinement of the intersection form.

Choose a basis e_i in A whose elements are eigenvectors of $\mathcal{F}$. Let $\eta_{ij} = \eta(e_i, e_j)$. Since η is non-degenerate, it has an inverse η^{ij}. Let $C^i{}_{jk}$ denote the structure constants of A. Define $C_{ijk} = \eta_{il} C^l{}_{jk}$. It can be shown that the tensor C_{ijk} is cyclically symmetric (Kapustin, Turzillo, and You, 2017). Denote by $(-1)^{\beta_i}$ the eigenvalue of $\mathcal{F}$ corresponding to e_i.

Now we can explain the recipe for computing the partition function for a surface Σ with a marked skeleton Γ and a spin structure s. Each edge of Γ is colored with a pair of basis vectors $e_i \in A$, and we have a factor of C_{ijk} for each vertex and η^{ij} for each edge. Since A is $\mathbb{Z}_2$-graded, η^{ij} vanishes unless $\beta_i = \beta_j$, and C_{ijk} vanishes unless $\beta_i + \beta_j + \beta_k = 0$. Hence the function $\beta : e_i \mapsto \beta_i$ on the set of edges of Γ defines a mod-2 1-cycle on Σ. The contribution of a particular coloring of Γ is the product of all C_{ijk} and η^{ij}, the spin-dependent sign factor

$$(-1)^{s(\beta)} = (-1)^{\sum_e s(e)\beta(e)}, \tag{4.50}$$

and the Koszul sign $\sigma_0(\beta)$. The partition function is obtained by summing over all colorings. Note that

$$Z_{\text{ferm}}(A, \eta) = \sum_{\beta} Z_{\text{bose}}(A, \beta)\sigma_s(\beta), \tag{4.51}$$

where $Z_{\text{bose}}(A,\beta)$ is the sum over all colorings with a fixed 1-cycle β. Using the isomorphism $H_1(\Sigma,\mathbb{Z}_2) \simeq H^1(\Sigma,\mathbb{Z}_2)$, one can interpret β as a $\mathbb{Z}_2$ gauge field on a dual triangulation and $Z_{\text{bose}}(A,\beta)$ as the partition function of a bosonic system with a global $\mathbb{Z}_2$ symmetry coupled to β. Equation ((4.51)) is a manifestation of the bosonization phenomenon.

It remains to explain how the Koszul sign $\sigma_0(\beta)$ is evaluated. Consider a vertex whose edges are labeled by i, j, k starting from the special edge and going counter-clockwise. Assign to it an element $C_v = C_{ijk} e_i \otimes e_j \otimes e_k$ in $A \otimes A \otimes A$. Tensoring over vertices, we get an element C_Γ of $A^{\otimes 3N}$, where N is the number of vertices of Γ. Now consider an oriented edge of Γ labeled by i, j. It corresponds to an ordered pair of factors in C_Γ. Permute the factors of C_Γ until these two are next to each other and in order, keeping track of the fermionic signs

$$e_i \otimes e_j \mapsto (-1)^{\beta_i \beta_j} e_j \otimes e_i \tag{4.52}$$

one incurs in the process, and then contract using the scalar product η. Continuing in this fashion, we are left with the product of all C_{ijk} and η^{ij} times a sign. This sign is the Koszul sign $\sigma_0(\beta)$. It is clear that it depends on the coloring of Γ only through the 1-cycle β. Note that the elements C_v as well as the pairs of factors for each edge are all even, so one does not need to order the set of vertices or the set of edges. One can also define $\sigma_0(\beta)$ as a Grassmann integral, as was originally done in (Gu and Wen, 2014). The product of the Koszul sign $\sigma_0(\beta)$ and the spin-dependent factor $(-1)^{s(\beta)}$ is nothing but the quadratic function $\sigma_s(\beta)$ (Gaiotto and Kapustin, 2016).

One can show (Novak and Runkel, 2014; Gaiotto and Kapustin, 2016) that the partition function thus defined depends only on the spin surface (Σ, s) and not the skeleton Γ, its marking, or the particular 1-cochain representing s. Finally, it is clear that if A is purely even, both the Koszul sign and the spin-dependent sign factor are trivial, and the partition function reduces to the bosonic partition function associated with A.

Stacking and the supertensor product

It is interesting to determine the behavior of the partition function under stacking systems together. Given a pair of fermionic systems encoded in a pair of $\mathbb{Z}_2$-graded Frobenius algebras A_1, A_2, stacking these systems together gives us a system with a partition function $Z_{\text{ferm}}(A_1,\eta)Z_{\text{ferm}}(A_2,\eta)$. It turns out that

$$Z_{\text{ferm}}(A_1,\eta)Z_{\text{ferm}}(A_2,\eta) = Z_{\text{ferm}}(A_1 \mathbin{\widehat{\otimes}} A_2,\eta), \tag{4.53}$$

where $\widehat{\otimes}$ is the supertensor product of $\mathbb{Z}_2$-graded algebras. Let us recall what this means. The usual tensor product of algebras $A_1 \otimes A_2$ obeys the multiplication rule

$$(a_1 \otimes a_2) \cdot (a_1' \otimes a_2') = (a_1 \cdot a_1') \otimes (a_2 \cdot a_2'). \tag{4.54}$$

If the algebras A_1, A_2 are $\mathbb{Z}_2$-graded, $A_1 \otimes A_2$ is also $\mathbb{Z}_2$-graded in an obvious way. On the other hand, for the supertensor product, the multiplication is defined as follows:

$$(a_1 \widehat{\otimes} a_2) \cdot (a_1' \widehat{\otimes} a_2') = (-1)^{|a_2|\cdot|a_1'|}(a_1 \cdot a_1') \widehat{\otimes} (a_2 \cdot a_2'), \tag{4.55}$$

where $(-1)^{|a|}$ is the fermionic parity of a.

To derive (4.53), we first note that

$$Z_{\text{bose}}(A_1, \beta_1) Z_{\text{bose}}(A_2, \beta_2) = Z_{\text{bose}}(A_1 \otimes A_2, \beta_1, \beta_2), \tag{4.56}$$

where we used the fact that the stacking of two bosonic systems with symmetry $\mathbb{Z}_2$ has a symmetry $\mathbb{Z}_2 \times \mathbb{Z}_2$ and thus can be coupled to a pair of $\mathbb{Z}_2$ gauge fields β_1, β_2. Next, it is easy to see that

$$Z(A_1 \widehat{\otimes} A_2, \beta_1, \beta_2) = (-1)^{\langle[\beta_1],[\beta_2]\rangle} Z(A_1 \otimes A_2, \beta_1, \beta_2). \tag{4.57}$$

These two identities together with (4.46) imply (4.53).

As an illustration, consider $A = C\ell(1)$. Since apart from 1, this algebra has a single odd basis element γ, β completely determines the coloring of Γ. With the proper normalization of Z_{bose}, one gets

$$Z_{\text{ferm}}(s) = 2^{-b_1(\Sigma)/2} \sum_{[\beta]} \sigma_s([\beta]). \tag{4.58}$$

The r.h.s. is called the Arf invariant of the spin structure s and is denoted $\mathrm{Arf}(s)$. One can show that it takes values ± 1. If we stack two such systems together, we will get the partition function which is 1 for all spin structures and all Σ, i.e. a trivial spin-TQFT.

It is easy to see that $C\ell(1) \widehat{\otimes} C\ell(1)$ is the Clifford algebra with two generators, $C\ell(2)$. This algebra is non-trivial, but it is Morita-equivalent to the trivial algebra $\mathbb{C}$. One can show that, just as in the bosonic case (Kapustin, Turzillo, and You, 2017), spin-TQFT constructed from A depends only on the Morita equivalence class of A. This explains why the spin-TQFT corresponding to $C\ell(2)$ is trivial.

We see that $A = C\ell(1)$ corresponds to a nontrivial SRE phase in the fermionic case (it is its own inverse). On the other hand, $C\ell(1) \otimes C\ell(1)$ is a commutative algebra isomorphic to a sum of two copies of $C\ell(1)$. Therefore the bosonic phase corresponding to $C\ell(1)$ is not invertible. This example illustrates that bosonization does not preserve the stacking operation.

Including boundaries

When Σ has a non-empty boundary, Γ is allowed to have univalent vertices which all lie on the boundary $\partial\Sigma$. Let M be the number of boundary vertices. For every vertex v, we color each element of $\Gamma(v)$ with a basis vector of A, so that a vertex on the boundary has only a single label. As before, the weight of each coloring is a product of three factors: the product of C_{ijk} over all trivalent vertices and η^{ij} over all edges, the Koszul sign, and the spin-dependent sign. When summing over colorings, the labels of the boundary vertices remain fixed. The result of the summation can be interpreted as a value of a map

$$Z_\Gamma(\Sigma) : A^{\otimes M} \to \mathbb{C}, \tag{4.59}$$

on a particular basis vector in $A^{\otimes M}$.

It is implicit here that the map depends on the spin structure on every connected component of $\partial\Sigma$. It can be read off from the function $\sigma_s(\gamma)$ evaluated on the boundary components. The spin structure is Neveu-Schwarz if $\sigma_s = 1$ and Ramond if $\sigma_s = -1$.

We can also consider open-closed spin-TQFT, i.e. spin-TQFT in the presence of topological boundary conditions (branes). Such boundary conditions are encoded in $\mathbb{Z}_2$-graded modules over A. A $\mathbb{Z}_2$-graded module over a $\mathbb{Z}_2$-graded algebra A is a $\mathbb{Z}_2$-graded vector space $U = U_+ \oplus U_-$ with the structure of an A-module $T : A \to \mathrm{End}(U)$ such that $T(A_+)U_\pm \subseteq U_\pm$ and $T(A_-)U_\pm \subseteq U_\mp$. Equivalently, U is an A-module equipped with an involution P such that $T(\mathcal{F}(a)) = PT(a)P^{-1}$.

For each boundary component of Σ, choose a $\mathbb{Z}_2$-graded A-module U and a homogeneous basis f^U_μ of U. Label each boundary edge with a basis vector of U. The weight of the coloring is a product of the C's and η's and a sign $\sigma_s(\beta)$, as well as a module tensor $T^\mu{}_{\nu i}$ for each boundary vertex. The sign is computed as before as a product of the spin-structure-dependent sign and the Koszul sign.

4.6 Fermionic MPS

Fermionic matrix product states and the annulus diagram

In this section, we will extract MPS wavefunctions from the spin-TQFT by considering the special case when Σ is an annulus. Take one of the boundary circles to

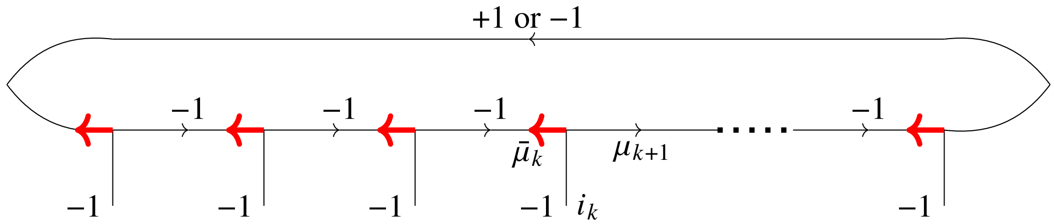


Figure 4.11: Black arrows are edge orientations, and red arrows are special edges. All of the spin signs are −1 except possibly the one on the N-to-1 edge, which is +1 in the NS sector and −1 in the R sector.

be a source cut boundary and the other to be a brane boundary corresponding to a $\mathbb{Z}_2$-graded A-module U with action $T(a) \in \mathrm{End}(U)$. Choose a triangulation of Σ. It was shown in (Kapustin, Turzillo, and You, 2017) that one can deform the skeleton to look like Figure 4.11.

Give the skeleton a marking and spin signs that models the spin structure on Σ. It is convenient to make the choices shown in Figure 4.11. The sign on the N-to-1 edge is +1 if the spin structure induced on the boundary circles is NS and −1 if it is R. To get the sign (4.50), we insert a factor of P for each +1.

Following the procedure detailed in Section 4.5 to evaluate the diagram in Figure 4.11, one finds

$$\begin{aligned} Z(\Sigma_{T,NS}) = \sum_{I=\{i_k,\mu_k,\bar{\mu}_k\}} & \sigma_0(\beta_I) \times T^{\bar{\mu}_N i_1 \mu_1} T^{\bar{\mu}_1 i_2 \mu_2} \\ & \times \cdots T^{\bar{\mu}_{N-1} i_N \mu_N} \delta_{\mu_1 \bar{\mu}_1} \delta_{\mu_2 \bar{\mu}_2} \\ & \times \cdots P_{\mu_N \bar{\mu}_N} \langle i_1 i_2 \cdots i_N | \end{aligned} \tag{4.60}$$

in the NS sector and

$$\begin{aligned} Z(\Sigma_{T,R}) = \sum_{I=\{i_k,\mu_k,\bar{\mu}_k\}} & \sigma_0(\beta_I) \times T^{\bar{\mu}_N i_1 \mu_1} T^{\bar{\mu}_1 i_2 \mu_2} \\ & \times \cdots T^{\bar{\mu}_{N-1} i_N \mu_N} \delta_{\mu_1 \bar{\mu}_1} \delta_{\mu_2 \bar{\mu}_2} \\ & \cdots \delta_{\mu_N \bar{\mu}_N} \langle i_1 i_2 \cdots i_N | \end{aligned} \tag{4.61}$$

in the R sector, where the Koszul sign is given as a Grassmann integral

$$\sigma_0(\beta_I) = \int d\theta_1^{|\mu_1|} d\bar{\theta}_1^{|\bar{\mu}_1|} d\theta_2^{|\mu_2|} d\bar{\theta}_2^{|\bar{\mu}_2|} \cdots d\theta_N^{|\mu_N|} d\bar{\theta}_N^{|\bar{\mu}_N|}$$
$$\times d\theta_{i_1}^{|i_1|} d\theta_{i_2}^{|i_2|} \cdots d\theta_{i_N}^{|i_N|} \bar{\theta}_N^{|\bar{\mu}_N|} \theta_{i_1}^{|i_1|} \theta_1^{|\mu_1|} \bar{\theta}_1^{|\bar{\mu}_1|} \theta_{i_2}^{|i_2|} \theta_2^{|\mu_2|}$$
$$\times \cdots \bar{\theta}_{N-1}^{|\bar{\mu}_{N-1}|} \theta_{i_N}^{|i_N|} \theta_N^{|\mu_N|}. \tag{4.62}$$

Evaluating the integral amounts to reordering the variables in the integrand to match the ordering in the measure while recording the sign

$$\theta_1^{s_1} \theta_2^{s_2} = (-1)^{s_1 s_2} \theta_2^{s_2} \theta_1^{s_1}. \tag{4.63}$$

Moving $\bar{\theta}_N^{|\bar{\mu}_N|}$ across the integrand gives a sign $(-1)^{|\bar{\mu}_N|}$. Then moving each $\theta_{i_k}^{|i_k|}$ to the right gives a sign $+1$. Therefore the total sign is

$$\sigma_0(\beta_I) = (-1)^{|\bar{\mu}_N|}. \tag{4.64}$$

Noting that $\delta_{\mu_n \bar{\mu}_N} (-1)^{|\bar{\mu}_N|} = P_{\mu_n \bar{\mu}_N}$, we find that the MPS wavefunctions take the forms

$$\begin{aligned} \left\langle \psi_{T,NS} \right| &= Z(\Sigma_{T,NS}) \\ &= \sum_{i_1, i_2, \cdots, i_N} \mathrm{Tr}\left[T(e_{i_1}) T(e_{i_2}) \cdots T(e_{i_N}) \right] \langle i_1 i_2 \cdots i_N | \end{aligned} \tag{4.65}$$

and

$$\begin{aligned} \left\langle \psi_{T,R} \right| &= Z(\Sigma_{T,R}) \\ &= \sum_{i_1, i_2, \cdots, i_N} \mathrm{Tr}\left[P T(e_{i_1}) T(e_{i_2}) \cdots T(e_{i_N}) \right] \langle i_1 i_2 \cdots i_N | \,. \end{aligned} \tag{4.66}$$

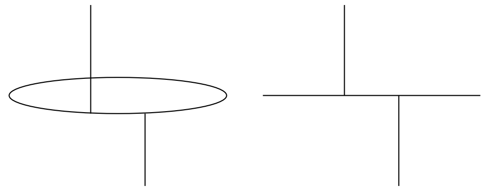
More general states, called *generalized MPS*, on the closed chain are obtained from the spin-TQFT by inserting a local observable on the brane boundary of the annulus. Such observables are parametrized by linear maps $X : U \to U$ and can be either even or odd; that is, $PX = XP$ or $PX = -XP$, respectively.

The NS sector MPS resulting from the insertion of X has conjugate wavefunction

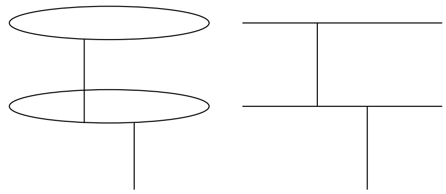
$$\left\langle \psi_{T,NS}^X \right| = \sum_{i_1 \cdots i_N} \mathrm{tr}\left[X^\dagger T(e_{i_1}) \cdots T(e_{i_N}) \right] \langle i_1 \cdots i_N | \,. \tag{4.67}$$

In the R sector,

$$\left\langle \psi_{T,R}^X \right| = \sum_{i_1 \cdots i_N} \mathrm{tr}\left[P X^\dagger T(e_{i_1}) \cdots T(e_{i_N}) \right] \langle i_1 \cdots i_N | \,. \tag{4.68}$$



(a) Skeleton of an annulus with cut boundaries



(b) An annulus with one brane and one cut boundary

Figure 4.12

Note that the generalized MPS corresponding to the trivial observable $X = \mathbb{1}$ are the states $\langle \psi_T |$ (4.65)(4.66).

The state $\left| \psi^X_{T,NS/R} \right\rangle$ has the same fermionic parity as the observable X since

$$
\begin{aligned}
&\mathcal{F}^{\otimes N} \left\langle \psi^X_{T,NS(R)} \right| \\
&= \sum \operatorname{Tr}\left[(P) X^\dagger T(\mathcal{F} \cdot e_{i_1}) \cdots T(\mathcal{F} \cdot e_{i_n})\right] \langle i_1 \cdots i_n | \\
&= \sum \operatorname{Tr}\left[(P) P X^\dagger P T(e_{i_1}) \cdots T(e_{i_n})\right] \langle i_1 \cdots i_n | \\
&= (-1)^{|X|} \left\langle \psi^X_{T,NS(R)} \right| .
\end{aligned} \tag{4.69}
$$

Parent Hamiltonians

Hamiltonians appear in TQFT as cylinders. There is one for each of the NS and R sectors. To be precise, the Hamiltonian is the linear map

$$H_{NS(R)} = \mathbb{1} - Z(C_{NS(R)}), \tag{4.70}$$

where $C_{NS(R)}$ denotes the cylinder with NS (R) spin structure. The composition of two cylinder cobordisms is again a cylinder, so $Z(C)$ is a projector, and therefore so is H. Ground states are those with eigenvalue 1 under $Z(C)$. It is convenient to specialize to the case of a single site, $N = 1$. Since these Hamiltonians arise from a topologically-invariant theory, properties of the $N = 1$ system must hold more generally. Consider the skeleton of the cylinders depicted in Figure 4.12.

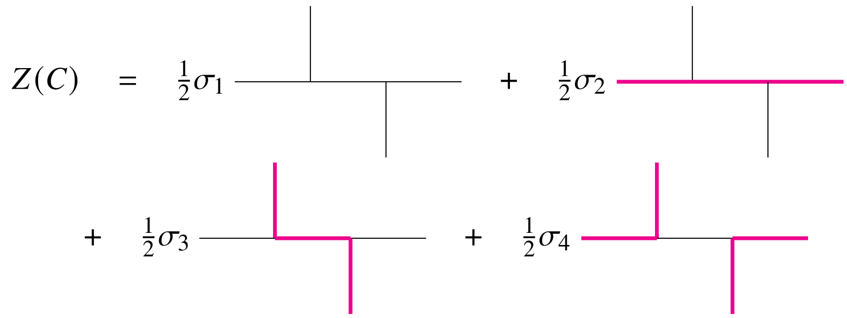


Figure 4.13: The cylinder partition sum $Z(C)$ factors as a signed sum of four colored diagrams: $\sigma(\beta_1)C_1+\sigma(\beta_2)C_2+\sigma(\beta_3)C_3+\sigma(\beta_4)C_4 = C_1+\eta C_2+C_3-\eta C_4$. Magenta lines indicate odd edges.

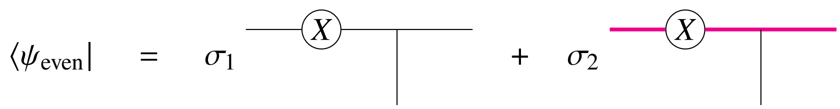


Figure 4.14: $\langle\psi_{\text{even}}| = \sigma(\beta_1)\,\langle\psi_1| + \sigma(\beta_2)\,\langle\psi_2| = \langle\psi_1| + \eta\,\langle\psi_2|$.

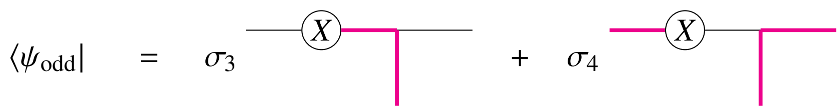


Figure 4.15: $\langle\psi_{\text{odd}}| = \sigma(\beta_3)\,\langle\psi_3| + \sigma(\beta_4)\,\langle\psi_4| = \langle\psi_3| + \eta\,\langle\psi_4|$.

By exploiting (4.51), we will not need the full machinery of lattice spin structures to understand the Hamiltonians and their ground states. The path integrals for the cylinders can be expressed as a sum over the four relative 1-cycles $\beta_1, \ldots, \beta_4$ depicted in Figure 4.13. The first colored diagram corresponds to the trivial cycle β_1 and has no odd labels, so its sign is trivial, $\sigma_s(\beta_1) = 1$. The second one corresponds to the equator of the cylinder and comes with the sign $\sigma(\beta_2) := \eta$, which is +1 in the NS sector and −1 in the R sector. The relative cycles β_3 and β_4 sum to β_2 and have intersection number 1, where the intersection pairing is defined by gluing another annulus onto the annulus, to get a torus $C^* = T^2$, as explained in Section 4.5. Therefore (4.46) says that there is a relative sign

$$\begin{aligned}\sigma_s(\beta_3)\sigma_s(\beta_4) &= \sigma_s(\beta_3+\beta_4)(-1)^{\langle\beta_3,\beta_4\rangle} \\ &= \sigma_s(\beta_2)(-1) = -\eta. \end{aligned} \tag{4.71}$$

One can choose a spin structure on the closed space $C^* = T^2$ such that $\sigma_s(\beta_3) = 1$; this amounts to fixing trivializations of the spin structures induced on each component of ∂C at the univalent vertices.

Similarly, an even MPS can be expressed as the sum in Figure 4.14, where $\sigma_1 = 1$

and $\sigma_2 = \eta$, and an odd MPS as the sum in Figure 4.15 with $\sigma_1 = 1$ and $\sigma_2 = \eta$.

Now we are ready to argue that the parent Hamiltonian has a generalized MPS $\langle \psi_T^X |$ as a ground state if X *supercommutes* with $T(a)$; that is, if an even observable satisfies

$$XT(a) = T(a)X \quad \forall a \in A, \tag{4.72}$$

and an odd observable satisfies

$$XT(a) = (-1)^{|a|} T(a)X \quad \forall a \in A. \tag{4.73}$$

Linear maps satisfying these conditions are called even and odd $\mathbb{Z}_2$-graded module endomorphisms.

The maps C_3 and C_4 correspond to diagrams with odd legs, and so annihilate even states $\langle \psi_{\text{even}} |$. Therefore

$$Z(C) \langle \psi_{\text{even}} | = \tfrac{1}{2}(C_1 + \eta C_2)(\langle \psi_1 | + \eta \langle \psi_2 |). \tag{4.74}$$

By the sequence of diagram moves depicted in Figures 4.16, B.1, B.2, and B.3, one can show that

$$\begin{aligned} C_1 \langle \psi_1 | = \langle \psi_1 | , \quad C_2 \langle \psi_1 | = \eta_X \langle \psi_2 | , \\ C_1 \langle \psi_2 | = \langle \psi_2 | , \quad C_2 \langle \psi_2 | = \eta_X \langle \psi_1 | , \end{aligned} \tag{4.75}$$

where η_X denotes the sign due to commuting X with odd $T(a)$. According to the rule (4.72), $\eta_X = 1$, so

$$Z(C) \langle \psi_{\text{even}} | = \tfrac{1}{2}(1 + \eta_X) \langle \psi_{\text{even}} | = \langle \psi_{\text{even}} | . \tag{4.76}$$

Similarly, the cylinder acts on odd states as

$$Z(C) \langle \psi_{\text{odd}} | = \tfrac{1}{2}(C_3 - \eta C_4)(\langle \psi_3 | + \eta \langle \psi_4 |). \tag{4.77}$$

Commuting X with the vertex gives $\langle \psi_4 | = \eta_X \langle \psi_3 |$, which means $\langle \psi_{\text{odd}} | = (1 + \eta \eta_X) \langle \psi_3 |$. According to the rule (4.73), $\eta_X = -1$, so the only odd ground state in the NS sector is $\langle \psi | = 0$. This agrees with (Moore and Segal, 2006).

In the Ramond sector, one can have nonzero odd ground states. The sequence of moves of Figures B.4 and B.5 shows

$$C_3 \langle \psi_3 | = \langle \psi_3 | , \quad C_4 \langle \psi_3 | = \langle \psi_3 | , \tag{4.78}$$

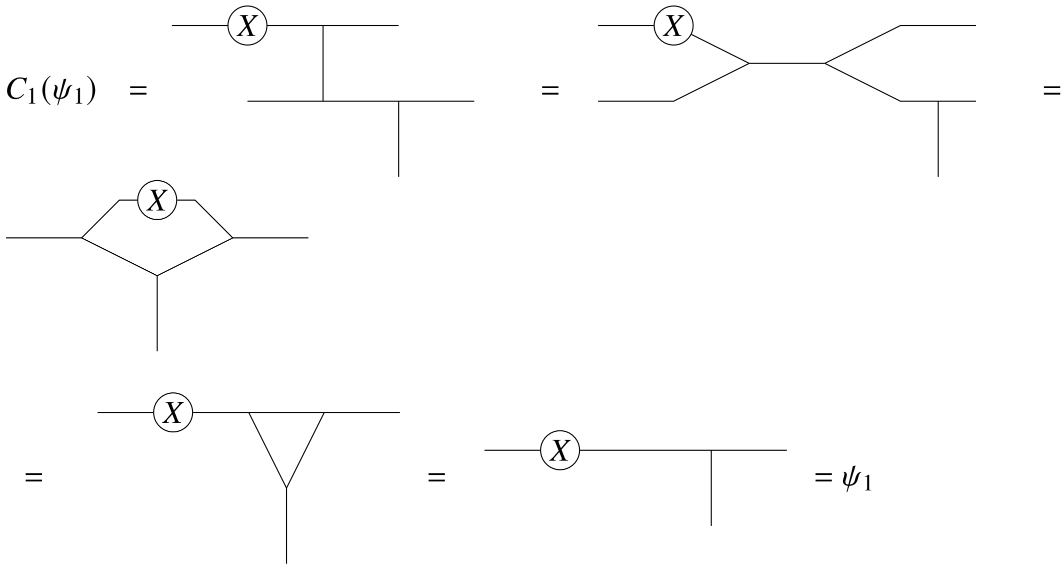


Figure 4.16: Diagrammatic proof of $C_1\langle\psi_1| = \langle\psi_1|$. The topmost line represents the physical boundary, with module indices living on it. The others are depicted in Appendix B.1.

so

$$Z(C)\,\langle\psi_{\text{odd}}| = \tfrac{1}{2}(1-\eta)\,\langle\psi_{\text{odd}}| = \langle\psi_{\text{odd}}| \text{ (in the R sector).} \tag{4.79}$$

Therefore $\langle\psi_T^X|$ is indeed a ground state of $H_{NS(R)}$ provided X is a $\mathbb{Z}_2$-graded module endomorphism.

Next we argue that every ground state of H of the form (4.67) or (4.68) for arbitrary X can be written as a generalized MPS where X supercommutes with T. A result of (Moore and Segal, 2006) (c.f. eq 3.18) implies that

$$Z(C_{NS})\,|ij\rangle = (-1)^{|i||j|+|i|} Z(C_{NS})\,|ji\rangle \tag{4.80}$$

and

$$Z(C_R)\,|ij\rangle = (-1)^{|i||j|} Z(C_R)\,|ji\rangle\,. \tag{4.81}$$

In Appendix B.2, we rederive this result in the Novak-Runkel formalism. Then, since $|X| = |i| + |j|$,

$$\begin{aligned} &Z(C_{NS})\,\mathrm{Tr}\big[XT(e_i)T(e_j)\big]\,|ij\rangle \\ &\quad= (-1)^{|i||X|} Z(C_{NS})\,\mathrm{Tr}\big[T(e_i)XT(e_j)\big]\,|ji\rangle \end{aligned} \tag{4.82}$$

and

$$\begin{aligned} &Z(C_R)\,\mathrm{Tr}\big[PXT(e_i)T(e_j)\big]\,|ij\rangle \\ &\quad= (-1)^{|i||X|} Z(C_R)\,\mathrm{Tr}\big[PT(e_i)XT(e_j)\big]\,|ji\rangle\,. \end{aligned} \tag{4.83}$$

For ground states, i.e. eigenstates of $Z(C)$ with eigenvalue 1, this means that X supercommutes with T.

It turns out that all ground states of H can be written as generalized MPS. As discussed in (Kapustin, Turzillo, and You, 2017), in a unitary theory, T is an isometry with respect to some inner product on A and the standard inner product

$$\langle M|N\rangle = \mathrm{Tr}\left[M^\dagger N\right] \quad M, N \in \mathrm{End}(V) \tag{4.84}$$

on $\mathrm{End}(V)$. For an orthogonal basis $\{e_i\}$ of A, $\mathrm{Tr}\left[T(e_i)^\dagger T(e_j)\right] = \delta_{ij}$. Consider the case $N = 1$. An arbitrary state

$$\langle\psi| = \sum_i a_i \langle i| \tag{4.85}$$

can be written in generalized MPS form (4.67)(4.68) if one takes

$$X_{NS} = \sum_j a_j T(e_j)^\dagger \quad \text{or} \quad X_R = \sum_j a_j PT(e_j)^\dagger. \tag{4.86}$$

Thus generalized MPS with supercommuting X are the only ground states. Neither the number of generalized MPS nor the number of ground states depends on N; thus, the argument extends to all N.

A consequence of supercommutativity and (4.69) is that there are no odd ground states in the NS sector. Suppose that X is an odd observable. For $a \in A_-$, the matrix $X^\dagger$ anticommutes with $T(a)$, so the coefficient $\mathrm{Tr}\left[X^\dagger T(a)\right]$ vanishes. For $a \in A_+$, the matrix $X^\dagger T(a)$ maps $U_\pm$ to $U_\mp$, so again it vanishes in the trace. Therefore the state (4.67) is zero for odd X, which is to say that the NS sector does not support odd states. The argument fails for the state (4.68); generically, the R sector supports both even and odd states. The lack of odd states in the NS sector can also be seen directly from (4.80), which implies $|C\,|ij\rangle\,| = |i| + |j| = 0$.

Stacking fermionic MPS

Bosonization establishes a 1-1 correspondence between 1d bosonic systems with $\mathbb{Z}_2$ symmetry and 1d fermionic systems. In the gapped case, the corresponding topological phases are described by the same algebraic data, namely by a $\mathbb{Z}_2$-graded algebra A. But bosonization does not preserve a crucial physical structure: stacking systems together. From the mathematical viewpoint, either bosonic or fermionic topological phases of matter form a commutative monoid (a set with a commutative associative binary operation and a neutral element, but not necessarily with an inverse for every element), but bosonization does not preserve the monoid structure (i.e. it does not preserve the product). A well-known example is given by the fermionic SRE phases: the non-trivial fermionic SRE phase (the Majorana chain) is mapped to the bosonic phase with a spontaneously broken $\mathbb{Z}_2$. The former one is invertible, while the latter one is not. Both phases correspond to the algebra $C\ell(1)$.

In the bosonic case, it was shown in (Kapustin, Turzillo, and You, 2017) that, given two algebras A_1 and A_2 with bosonic Hamiltonians H_1 and H_2, the tensor product system $A_1 \otimes A_2$ has a Hamiltonian $H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$. That is, stacking bosonic systems together corresponds to the tensor product of algebras.

On the other hand, in Section 4.5 we have shown that for fermionic systems, stacking corresponds to the supertensor product (4.55). We can now see that the supertensor product rule is consistent with the way fermionic generalized MPS are defined (while the usual tensor product is not).

Suppose H_1 is the Hamiltonian for the MPS system built from a $\mathbb{Z}_2$-graded algebra A_1 that acts on a $\mathbb{Z}_2$-graded module U_1 by T_1. Its ground states are parametrized by $\mathbb{Z}_2$-graded module endomorphisms X_1 of U_1. Consider stacking H_1 with a second system H_2 defined by $T_2 : A_2 \to \mathrm{End}(U_2)$ with ground states parametrized by X_2. The stacked system is the MPS system with physical space $A_1 \otimes A_2$ and Hamiltonian $H = H_1 \otimes \mathbb{1}_2 + \mathbb{1}_1 \otimes H_2$. It has bond space $U_1 \otimes U_2$ and MPS tensor $T = T_1 \otimes T_2$.

The ground states are generalized MPS, and so correspond to $\mathbb{Z}_2$-graded endomorphisms of the module $U_1 \otimes U_2$. Since the MPS tensor is $T = T_1 \otimes T_2$, the state $\langle \psi_T^X |$ is trivial unless X is of the form $X_1 \otimes X_2$. We also know that X supercommutes with T:

$$\begin{aligned}(X_1 \otimes X_2)&(T_1 \otimes T_2)\\ &= (-1)^{(|X_1|+|X_2|)(|T_1|+|T_2|)}(T_1 \otimes T_2)(X_1 \otimes X_2).\end{aligned} \tag{4.87}$$

There are two ways one might define the composition of tensor products of operators[13]:

$$(X_1 \otimes X_2)(T_1 \otimes T_2) = X_1 T_1 \otimes X_2 T_2 \tag{4.88}$$

and

$$(X_1 \widehat{\otimes} X_2)(T_1 \widehat{\otimes} T_2) = (-1)^{|X_2||T_1|} X_1 T_1 \widehat{\otimes} X_2 Y_2. \tag{4.89}$$

[13]These correspond to the two symmetric monoidal structures on the category of $\mathbb{Z}_2$-graded vector spaces.

Since X_1 supercommutes with T_1 and X_2 with T_2, only the second notion (4.89) of composition is consistent with (4.87). The composition rule is an algebra structure on $\mathrm{End}(U_1) \otimes \mathrm{End}(U_2)$ and pulls back by T to an algebra structure on $A_1 \otimes A_2$ given by the rule (4.55).

An important assumption in this argument is that isomorphic TQFTs correspond to equivalent gapped phases. Assuming this is true, we can easily see that the group of fermionic SRE phases is isomorphic to $\mathbb{Z}_2$. Indeed, one can easily see that a phase which is invertible must correspond to an indecomposable algebra (i.e. the algebra which cannot be decomposed as a sum of algebras). Since all our algebras are semisimple, this means that invertible phases must correspond to simple algebras. It is well-known that there are exactly two Morita-equivalence classes of $\mathbb{Z}_2$-graded algebras: the trivial one and the class of $C\ell(1)$. The square of the nontrivial class is the trivial class. Hence the group of invertible fermionic phases is isomorphic to $\mathbb{Z}_2$. In the next section, we will show explicitly that $C\ell(1)$ corresponds to the nontrivial Majorana chain.

4.7 Hamiltonians for fermionic SRE phases

The trivial SRE phase

An example of a system in the trivial phase is the trivial Majorana chain (Fidkowski and Kitaev, 2011). On a circle, this system has only bosonic states: one in the NS sector and one in the R sector. We will now demonstrate that this is the same phase as the MPS system built out of the Clifford algebra $C\ell(2) = \mathrm{End}(\mathbb{C}^{1|1})$.

The algebra $A = C\ell(2)$ is expressed in terms of its odd generators as $\mathbb{C}[x, y]/(x^2 - 1, y^2 - 1, xy + yx)$. Let A act on $U = \mathbb{C}^{1|1}$ by

$$T : x \mapsto [\sigma_x]_\pm \quad , \quad y \mapsto [\sigma_y]_\pm \tag{4.90}$$

where $[\cdot]_\pm$ denotes a matrix in the homogeneous basis of U. This action is graded and faithful. The fermion parity operator P acts by σ_z.

The even ground states of this system are parametrized by matrices that commute with σ_x, σ_y, and σ_z. Thus X is proportional to the identity $\mathbb{1}$. The corresponding NS sector state has the wavefunction $\mathrm{Tr}\big[T(e_{i_1}) \cdots T(e_{i_N})\big]$. There is also an even state in the R sector given by $\mathrm{Tr}\big[PT(e_{i_1}) \cdots T(e_{i_N})\big]$.

The odd ground states are parametrized by matrices that commute with $T(a)$ – in particular, $T(xy) = \sigma_z$ – and anticommute with $P = \sigma_z$. This is impossible, so there are no odd states in either sector.

In summary, the ground states of the $A = C\ell(2)$ MPS system are a bosonic one in the NS sector and a bosonic one in the R sector, just like the ground states of the trivial Majorana chain.

One can show that the MPS parent Hamiltonian (c.f. (Kapustin, Turzillo, and You, 2017; Schuch, Perez-Garcia, and Cirac, 2011)) is a nearest-neighbor Hamiltonian with the two-body interaction $H_T = -\sum_{\alpha=1}^{4} |v_\alpha\rangle \langle v_\alpha|$ where

$$\begin{aligned} v_1 &= 1 \otimes 1 - x \otimes x - y \otimes y - xy \otimes xy \\ v_2 &= 1 \otimes x + x \otimes 1 + y \otimes xy - xy \otimes y \\ v_3 &= 1 \otimes y + y \otimes 1 + xy \otimes x - x \otimes xy \\ v_4 &= 1 \otimes xy + xy \otimes 1 + x \otimes y - y \otimes x. \end{aligned} \tag{4.91}$$

It is not obvious that H_T is equivalent to the Hamiltonian of the trivial Majorana chain

$$H = \sum_j (a_j^\dagger a_j - 1) \tag{4.92}$$

but it should be possible to construct an LU transformation between the two Hamiltonians (after some blocking), as the systems have the same spaces of ground states and so lie in the same phase.

The nontrivial SRE phase

An example of a fermionic system in a nontrivial SRE phase is the Majorana chain with a two-body Hamiltonian (Fidkowski and Kitaev, 2011)

$$H_j = \frac{1}{2}\left(-a_j^\dagger a_{j+1} - a_{j+1}^\dagger a_j + a_j^\dagger a_{j+1}^\dagger + a_{j+1} a_j\right). \tag{4.93}$$

This system has one bosonic and one fermionic ground state on the interval arising from one Majorana zero mode at each end. In the continuum limit, this system becomes a free Majorana fermion with a negative mass. In the NS sector, there is a unique ground state which is bosonic, while in the R sector there is a unique ground state which is fermionic (this is most easily seen from the continuum field theory).

In order to get this phase from a spin TQFT, we let $A = C\ell(1)$. To see the full space of ground states, we need a faithful graded module over A. Let $U = U_+ \oplus U_-$, where each $U_\pm$ is spanned by a single vector $u_\pm$. Let A act on U by

$$T : \Gamma \mapsto [\sigma_x]_\pm = u_+ \otimes u_-^* + u_- \otimes u_+^*. \tag{4.94}$$

In other words, U is A regarded as a module over itself.

The even ground states of this system are parametrized by matrices that commute with $P = [\sigma_z]_\pm$ and $T(\Gamma) = [\sigma_x]_\pm$. Such matrices are proportional to $\mathbb{1}$. The corresponding NS sector state has wavefunction $\mathrm{Tr}\big[T(e_{i_1})\cdots T(e_{i_N})\big]$. There is no even state in the R sector as the trace $\mathrm{Tr}\big[PT(e_1)\cdots T(e_{i_N})\big]$ vanishes.

The odd ground states are parametrized by matrices that anticommute with P and $T(\Gamma)$. Such matrices X are all proportional to $[\sigma_y]_\pm$. By the general argument of Section 4.6, we know that the NS sector has no odd states. The wavefunction $\mathrm{Tr}\big[PX^\dagger T(e_{i_1})\cdots T(e_{i_N})\big]$ defines an odd state in the R sector.

In summary, the ground states of the $A = C\ell(1)$ MPS system are a bosonic one in the NS sector and a fermionic one in the R sector, just like the ground states of the nontrivial Majorana chain.

We can also observe the equivalence of the two systems from the standpoint of Hamiltonians. We build the MPS parent Hamiltonian for the $A = C\ell(1)$ system by following (Kapustin, Turzillo, and You, 2017; Schuch, Perez-Garcia, and Cirac, 2011). The adjoint $\mathcal{P} = T^\dagger$ is given by

$$\mathcal{P} : 2u_\pm \otimes u_\pm^* \mapsto 1\otimes 1 + \Gamma\otimes\Gamma \ , \quad 2u_\pm \otimes u_\mp^* \mapsto 1\otimes\Gamma + \Gamma\otimes 1. \tag{4.95}$$

With respect to the inner products on A and U for which 1 and Γ and u_+ and u_- are unit vectors, the graded module structure T is an isometry, so the left inverse $\mathcal{P}^+$ is simply T. Putting these pieces together, we find

$$H_T = |11\rangle\langle\Gamma\Gamma| - |1\Gamma\rangle\langle\Gamma 1| - |\Gamma 1\rangle\langle 1\Gamma| + |\Gamma\Gamma\rangle\langle 11| \tag{4.96}$$

where $|ab\rangle\langle cd|$ denotes the element $a\otimes b\otimes c^*\otimes d^* \in \mathrm{End}(A\otimes A)$. In terms of the annihilation operators $a_j = \sqrt{2}\,|1\rangle\langle\Gamma|_j$ and their adjoints, the hopping (top row) and pairing (bottom) terms look like

$$\begin{aligned} a_j^\dagger \otimes a_{j+1} &= 2\,|\Gamma 1\rangle\langle 1\Gamma| \qquad & a_{j+1}^\dagger \otimes a_j &= 2\,|1\Gamma\rangle\langle\Gamma 1| \\ a_j^\dagger \otimes a_{j+1}^\dagger &= 2\,|\Gamma\Gamma\rangle\langle 11| \qquad & a_{j+1}\otimes a_j &= 2\,|11\rangle\langle\Gamma\Gamma| \end{aligned} \tag{4.97}$$

so the Hamiltonians (4.93) and (4.96) agree. The variables a_j satisfy fermionic anti-commutation relations. For example,

$$\begin{aligned} \{a_j, a_{j+1}\} &= (a\otimes\mathbb{1})(\mathbb{1}\otimes a) + (\mathbb{1}\otimes a)(a\otimes\mathbb{1}) \\ &= a\otimes a + (-1)^{|a||a|} a\otimes a = 0 \end{aligned} \tag{4.98}$$

if we are careful to use the fermionic tensor product (4.55). The other relations can be checked similarly.

4.8 Equivariant spin-TQFT and equivariant fermionic MPS

$(\mathcal{G}, p)$-equivariant algebras and modules

Let $(\mathcal{G}, p)$ be a finite supergroup, i.e. a finite group $\mathcal{G}$ with a distinguished involution $p \in \mathcal{G}$ called *fermion parity*. We assume that the involution p is central in $\mathcal{G}$, which means that there are no supersymmetries. Every supergroup $(\mathcal{G}, p)$ arises as a central extension of a group $G_b \simeq \mathcal{G}/\mathbb{Z}_2$ of bosonic symmetries by $\mathbb{Z}_2 = \{1, p\}$; that is, there is an exact sequence

$$1 \to \mathbb{Z}_2 \xrightarrow{i} \mathcal{G} \xrightarrow{b} G_b \to 1. \tag{4.99}$$

A trivialization of $(\mathcal{G}, p)$ is a function $t : \mathcal{G} \to \mathbb{Z}_2$ such that $t \circ i$ is the identity on $\mathbb{Z}_2$. Given a trivialization, one can encode the multiplication rule for $\mathcal{G}$ in terms of the product on G_b and a $\mathbb{Z}_2$-valued group 2-cocycle ρ of G_b. Consider the following product on the set $G_b \times \mathbb{Z}_2$ (denoted $G_b \times_\rho \mathbb{Z}_2$). For $\bar{g}, \bar{h} \in G_b$, $f, f' \in \mathbb{Z}_2$,

$$(\bar{g}, f) \cdot (\bar{h}, f') = (\bar{g}\bar{h}, \rho(\bar{g}, \bar{h}) + f + f'). \tag{4.100}$$

Denote $\bar{g} := b(g)$. The map $b \times_\rho t : g \mapsto (\bar{g}, t(g))$ defines a group isomorphism $\mathcal{G} \xrightarrow{\sim} G_b \times_\rho \mathbb{Z}_2$; that is,

$$\begin{aligned} g \cdot h = (\bar{g}, t(g)) \cdot (\bar{h}, t(h)) &= (\bar{g}\bar{h}, \rho(\bar{g}, \bar{h}) + t(g) + t(h)) \\ &= (\overline{gh}, t(gh)) = gh, \end{aligned} \tag{4.101}$$

if and only if

$$\rho(\bar{g}, \bar{h}) = t(gh) + t(g) + t(h). \tag{4.102}$$

Suppose t' is another trivialization. Since $t = t'$ on the image of i and the sequence (4.99) is exact, the map $t - t'$ defines a 1-cochain of G_b. Thus, upon replacing t with t', ρ is modified by the coboundary $\delta(t - t')$, so only the cohomology class $[\rho]$ of c is an invariant of the extension. If $[\rho]$ is trivial, $\mathcal{G}$ is isomorphic to the direct product group $G_b \times \mathbb{Z}_2$ and we say the extension *splits*; in general, this is not the case. Some discussions of fermionic phases in the physics literature assume that $(\mathcal{G}, p)$ is split, but we will consider both cases simultaneously. Note that (Fidkowski and Kitaev, 2011) considered both cases as well.

An action R of $(\mathcal{G}, p)$ on a vector space V endows it with a distinguished $\mathbb{Z}_2$-grading

$$V_\pm = \{v \in V : R(p)v = \pm v\}. \tag{4.103}$$

Centrality of p ensures that $R(g)$ is even with respect to this grading, for all $g \in \mathcal{G}$. A $(\mathcal{G}, p)$-equivariant Frobenius algebra is a Frobenius algebra (A, m, η) with an action of $(\mathcal{G}, p)$ that satisfies

$$m(R(g)a \otimes R(g)b) = R(g)m(a \otimes b) \tag{4.104}$$

and

$$\eta(R(g)a, R(g)b) = \eta(a, b) \tag{4.105}$$

for all $a, b \in A, g \in G$. As was true for the special case $\mathcal{G} = \mathbb{Z}_2$, there are two notions of tensor product of these algebras: the usual one that forgets the distinguished $\mathbb{Z}_2$-grading and a supertensor product (4.55) that remembers it. In both cases, the symmetry acts on the product as

$$R(g)(a_1 \otimes a_2) = R_1(g)a_1 \otimes R_2(g)a_2 \tag{4.106}$$

which is a special case of the rule

$$(\phi_1 \otimes \phi_2)(a_1 \otimes a_2) = (-1)^{|\phi_2||a_1|}\phi_1(a_1) \otimes \phi_2(a_2) \tag{4.107}$$

for $\phi_1 \otimes \phi_2 \in \mathrm{End}(A_1) \otimes \mathrm{End}(A_2)$, where we have taken $R(g) = R_1(g) \otimes R_2(g)$.

We have argued in (Kapustin, Turzillo, and You, 2017) that bosonic phases with symmetry G are classified by G-equivariant symmetric Frobenius algebras and that stacking of phases corresponds to the usual tensor product of their algebras. Here we will argue the fermionic analog: $(\mathcal{G}, p)$-equivariant symmetric Frobenius algebras classify fermionic phases with symmetry $(\mathcal{G}, p)$, for which stacking is governed by the supertensor product. In this language, bosonization means taking a $(\mathcal{G}, p)$-equivariant algebra to a $\mathcal{G}$-equivariant algebra by forgetting the distinguished involution p. Generically, if $\mathcal{G}$ has more than one central involution, this map is many-to-one.

An equivariant module over a $(\mathcal{G}, p)$-equivariant algebra A is vector space V with compatible actions of A and $(\mathcal{G}, p)$; that is, for every $a \in A$, we have a linear map $T(a) \in \mathrm{End}(V)$ such that $T(a)T(b) = T(ab)$, and for every $g \in G$, a linear map $Q(g)$ such that $Q(g)Q(h) = Q(gh)$. The compatibility condition reads

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1}. \tag{4.108}$$

Note that T automatically respects the $\mathbb{Z}_2$-grading.

For a review of the classification of equivariant algebras and modules, we refer the reader to the prequel (Kapustin, Turzillo, and You, 2017), which compiles some algebraic facts from Ostrik, 2003 and Etingof, 2015. There are two classes of algebras that will be especially useful in the present context, as they describe fermionic SRE phases. One class of algebras is those of the form $\mathrm{End}(U)$ for a projective representation U of $\mathcal{G}$. Each pair (Q, U) has an associated class $[\omega] \in H^2(\mathcal{G}, U(1))$ that measures the failure of Q to be a homomorphism:

$$Q(g)Q(h) = \exp(2\pi i\omega(g,h))Q(gh). \tag{4.109}$$

Each $[\omega]$ defines a Morita class of algebras and therefore a phase. Equivariant modules over $\mathrm{End}(U)$ are all of the form $U \otimes W$, where W is a projective representation with class $-[\omega]$. When $\mathcal{G}$ can be written as $G_b \times \{1, p\}$ for some group G_b of bosonic symmetries, another class of equivariant algebras is those of the form $\mathrm{End}(U_b) \otimes C\ell(1)$ for a projective representation (U_b, Q_b) of G_b. The group G_b acts by conjugation on $\mathrm{End}(U_b)$. It also acts on the generator of $C\ell(1)$ by

$$\bar{g} : \Gamma \mapsto (-1)^{\beta(\bar{g})}\Gamma, \tag{4.110}$$

where $\beta : G_b \longrightarrow \mathbb{Z}_2$ is a homomorphism. Up to Morita-equivalence, algebras of this type depend only on the 1-cocycle β and the 2-cocycle α on G_b corresponding to the projective representation Q_b. While the bosonic phases built from these algebras have a broken $\mathbb{Z}_2$, their fermionic duals are nonetheless SRE phases.

Equivariant fermionic MPS

Let $(\mathcal{G}, p)$ be a supergroup acting on the physical space A by a unitary representation R. A $(\mathcal{G}, p)$-invariant MPS tensor is a map $T : A \mapsto \mathrm{End}(U)$ such that $T(a)T(b) = T(ab)$ and

$$T(R(g)a) = Q(g)T(a)Q(g)^{-1} \tag{4.111}$$

where the linear maps $Q(g) \in \mathrm{End}(U)$ form a projective representation of $(\mathcal{G}, p)$ on U. For $X \in \mathrm{End}(U)$ satisfying the supercommutation rule (4.72) or (4.73), the conjugate generalized MPS is

$$\left\langle \psi_T^X \right| = \mathrm{Tr}_U[XT(e_{i_1}) \cdots T(e_{i_N})] \left\langle i_1 \cdots i_N \right| \tag{4.112}$$

in the NS sector and

$$\left\langle \psi_T^X \right| = \mathrm{Tr}_U[PXT(e_{i_1}) \cdots T(e_{i_N})] \left\langle i_1 \cdots i_N \right| \tag{4.113}$$

in the R sector, where P denotes $Q(p)$. More generally, we can insert $Q(g)$ instead of P:

$$\left\langle \psi_T^X \right| = \mathrm{Tr}_U[Q(g)XT(e_{i_1}) \cdots T(e_{i_N})] \left\langle i_1 \cdots i_N \right|. \tag{4.114}$$

These are twisted sector states. When $\mathcal{G} = G_b \times \{1, p\}$, states with twist $Q(\bar{g}, 1)$ correspond to NS spin structure on a circle and a G_b gauge field of holonomy $\bar{g}$, while states with twist $Q(\bar{g}, p)$ correspond to the R spin structure on a circle and a G_b gauge field of holonomy $\bar{g}$. When $\mathcal{G}$ is non-split, one does not have spin structures and gauge fields, but a $\mathcal{G}$-Spin structure, as discussed in Section 4.8.

Note that $\mathrm{End}(U)$ carries a genuine (not projective) action of $(\mathcal{G}, p)$. By arguing as in (4.69), one can show that $\langle \psi_T^X |$ transforms under $(\mathcal{G}, p)$ in the same way as X.

Fermionic SRE phases and their group structure

In this section, we restrict our attention to fermionic SRE phases, i.e. topological fermionic phases that are invertible under the stacking operation. These phases form a group under stacking. According to (Fidkowski and Kitaev, 2011), if the symmetry group $\mathcal{G}$ splits as $G_b \times \mathbb{Z}_2$, each fermionic SRE phase corresponds to an element of the set

$$(\alpha, \beta, \gamma) \in H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}_2) \times \mathbb{Z}_2. \tag{4.115}$$

If $G_b = \{1\}$, the two elements $(0, 0, 0)$ and $(0, 0, 1)$ correspond to the trivial and nontrivial Majorana chains, respectively. More generally, elements of the form $(\alpha, \beta, 0)$ correspond to fermionic SRE phases that remain invertible after bosonization, while the bosonic duals of the fermionic SREs $(\alpha, \beta, 1)$ are not SREs (they have a spontaneously broken $\mathbb{Z}_2$ but unbroken G_b).

If $\mathcal{G}$ does not split, we claim that fermionic SRE phases are classified by pairs (α, β), where $\beta \in H^1(G_b, \mathbb{Z}_2)$, and α is a 2-cochain on G_b with values in $U(1)$ satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$, i.e. for $\bar{g}, \bar{h}, \bar{k}, \in G_b$,

$$\alpha(\bar{g}, \bar{h}) + \alpha(\bar{g}\bar{h}, \bar{k}) = \alpha(\bar{h}, \bar{k}) + \alpha(\bar{g}, \bar{h}\bar{k}) + \frac{1}{2}\rho(\bar{g}, \bar{h})\beta(\bar{k}). \tag{4.116}$$

Here ρ is the 2-cocycle on G_b which encodes the multiplication in $\mathcal{G}$. Certain pairs (α, β) correspond to equivalent SRE phases. Namely, adding to α an exact 2-cochain gives an equivalent SRE. Also, if we add to the 2-cocycle ρ a coboundary of a 1-cochain μ, α is shifted by $\frac{1}{2}\mu \cup \beta$.

This classification can be understood from the standpoint of bosonization. Recall that $\mathcal{G}$-invariant bosonic SREs are classified by group cohomology classes $[\omega] \in H^2(\mathcal{G}, U(1))$ and arise from algebras of the form $A = \mathrm{End}(U)$ where U is a projective representation of class $[\omega]$. Unlike the linear maps $R(g)$ of a genuine representation, the $Q(g)$ can be either even or odd with respect to $P := Q(p)$. Using (4.109) and the centrality of p, it can be shown that $Q(g)$ and $Q(gp)$ have the same parity $\omega(p,g) - \omega(g,p)$; thus, one can define $\beta(\bar{g}) := |Q(g)|$, i.e.

$$PQ(g)P^{-1} = e^{i\pi\beta(b(g))}Q(g). \tag{4.117}$$

The function β is clearly a homomorphism, and so defines a $\mathbb{Z}_2$-valued group 1-cocycle of G_b. Given a trivialization t, one can re-express ω in terms of β and a $U(1)$-valued group 2-cochain α of G_b satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$ as follows: [14]

$$\omega(g,h) = \alpha(\bar{g},\bar{h}) + \frac{1}{2}t(g)\beta(\bar{h}). \tag{4.118}$$

Using (4.102), one can verify that (4.116) is equivalent to the cocycle condition for ω. We prove in Appendix B.3 that (4.118) defines an isomorphism between $H^2(\mathcal{G}, U(1))$ and the set of pairs (α, β), up to coboundaries.

When $\mathcal{G}$ does not split, it is impossible to break $\mathbb{Z}_2$ without breaking $G_b = \mathcal{G}/\mathbb{Z}_2$, so all fermionic SRE phases arise as fermionized bosonic SRE phases. Then the analysis above agrees with the result of (Fidkowski and Kitaev, 2011) that, in the non-split case, fermionic SREs are classified by elements of $H^2(\mathcal{G}, U(1))$ (modulo identifications).

But when $\mathcal{G}$ splits, it is possible to break $\mathcal{G}$ and still get an invertible fermionic phase. One can break $\mathcal{G}$ down to any subgroup H such that the quotient $\mathcal{G}/H$ is a $\mathbb{Z}_2$ generated by p. Any such subgroup takes the form $H_\beta = \{g \in \mathcal{G} : t(g) = \beta(\bar{g})\}$ for some homomorphism $\beta : G_b \to \mathbb{Z}_2$, and all homomorphisms give such a subgroup. This gives rise to a second class of fermionic SPTs - those whose bosonic duals are not invertible.

The algebras corresponding to these phases are of the form $A = \mathrm{End}(U_\beta) \otimes C\ell(1)$ for some projective representation (U_β, Q_β) of H_β. Let $h \in H_\beta$, $M \in \mathrm{End}(U_\beta)$, $m \in \mathbb{Z}_2$. The subgroup and quotient act on A as

$$R(h) : M \otimes \Gamma^m \mapsto Q_\beta(h)^{-1} M Q_\beta(h) \otimes \Gamma^m, \tag{4.119}$$

$$R(p) : M \otimes \Gamma^m \mapsto (-1)^m M \otimes \Gamma^m. \tag{4.120}$$

[14] When the extension splits, both α and β are cocycles, and their equivalence to ω can be seen from the Künneth theorem for homology and the fact that $H^2(\mathcal{G}, U(1))$ is the Pontryagin dual of $H_2(\mathcal{G}, \mathbb{Z})$.

This action is a special case of the more general rule discussed in Section 4.3 of (Kapustin, Turzillo, and You, 2017). In terms of $\mathcal{G}$,

$$\begin{aligned} R(g) &= R(\bar{g}, \beta(\bar{g})) \cdot R(p)^{t(g)+\beta(\bar{g})} : \\ M &\otimes \Gamma^m \\ &\mapsto (-1)^{m(t(g)+\beta(\bar{g}))} Q_\beta(\bar{g}, \beta(\bar{g}))^{-1} M Q_\beta(\bar{g}, \beta(\bar{g})) \otimes \Gamma^m \end{aligned} \tag{4.121}$$

as claimed in (4.110) (after setting $t(\bar{g}, 1) = 0$). Note that β, which encodes the action of the symmetry on fermions, can be offset by changing the trivialization t, i.e. the splitting isomorphism $\mathcal{G} \xrightarrow{\sim} G_b \times \mathbb{Z}_2$. As a projective representation, Q_β is characterized by a class $[\alpha] \in H^2(H, U(1)) \simeq H^2(G_b, U(1))$.

We have shown that $(\mathcal{G}, p)$-equivariant fermionic SRE phases can be characterized by pairs (α, β) and - if $\mathcal{G}$ is split - an additional $\mathbb{Z}_2$ label γ that represents a $C\ell(1)$ factor in the algebra. This parameterization is useful for discussing stacking of fermionic phases, which is different from the standard group structure on $H^2(\mathcal{G}, U(1))$ (the latter describes bosonic stacking). First, since $C\ell(1) \mathbin{\widehat{\otimes}} C\ell(1) \simeq C\ell(2)$ is Morita-equivalent to $\mathbb{C}$, the γ parameters must simply add up under stacking. Second, if we consider two phases with parameters $(\alpha_1, \beta_1, 0)$ and $(\alpha_2, \beta_2, 0)$ corresponding to two $\mathcal{G}$-equivariant algebras (Q_1, U_1) and (Q_2, U_2), the supertensor product is a $\mathcal{G}$-equivariant algebra (Q, U), where $U = U_1 \mathbin{\widehat{\otimes}} U_2$ and $Q = Q_1 \mathbin{\widehat{\otimes}} Q_2$. We can easily compute:

$$\begin{aligned} Q(g)Q(h) &= (Q_1(g) \mathbin{\widehat{\otimes}} Q_2(g))(Q_1(h) \mathbin{\widehat{\otimes}} Q_2(h)) \\ &= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} Q_1(g)Q_1(h) \mathbin{\widehat{\otimes}} Q_2(g)Q_2(h) \\ &= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} e^{2\pi i \alpha_1(\bar{g},\bar{h})} (-1)^{t(g)\beta_1(\bar{h})} \\ &\quad \times e^{2\pi i \alpha_2(\bar{g},\bar{h})} (-1)^{t(g)\beta_2(\bar{h})} Q_1(gh) \mathbin{\widehat{\otimes}} Q_2(gh) \\ &= \exp\left(2\pi i (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_2 \cup \beta_1)\right)(\bar{g}, \bar{h}) \\ &\quad \times (-1)^{t(g)(\beta_1+\beta_2)(\bar{h})} Q(gh). \end{aligned} \tag{4.122}$$

Thus the group structure in this case is

$$\begin{aligned} &(\alpha_1, \beta_1, 0) + (\alpha_2, \beta_2, 0) \\ &\quad = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0). \end{aligned} \tag{4.123}$$

Note that $\beta_1 \cup \beta_2$ differs from $\beta_2 \cup \beta_1$ by an exact term, and thus the difference between them is inessential. Based on these two special cases, it is easy to guess that the group structure induced by stacking is

$$\begin{aligned}(\alpha_1, \beta_1, \gamma_1) + (\alpha_2, \beta_2, \gamma_2) \\ = (\alpha_1 + \alpha_2 + \frac{1}{2}\beta_1 \cup \beta_2, \beta_1 + \beta_2, \gamma_1 + \gamma_2).\end{aligned} \tag{4.124}$$

This will be verified later when we incorporate anti-unitary symmetries 4.9.

The set of triples (α, β, γ) with this group law is isomorphic to the spin-cobordism group $\Omega^2_{Spin}(BG_b)$ (Gaiotto and Kapustin, 2016). This agrees with the proposal of (Kapustin et al., 2015) about the classification of fermionic SRE phases. In the non-split case, the group structure is given by the same formulas, except that γ is set to zero, and α is not closed, but satisfies the equation $\delta\alpha = \frac{1}{2}\rho \cup \beta$.

If $\mathcal{G}$ splits, the isomorphism $\mathcal{G} \simeq G_b \times \mathbb{Z}_2$ may be taken as part of the physical data. This means that one fixes the action of G_b on fermions as well as on bosons. Alternatively, if one regards this isomorphism as unphysical, one only fixes the action of G_b on bosons, while the action on fermions is fixed only up to certain signs. So far we have been taking the former viewpoint. If we take the latter viewpoint, we also need to understand how the parameters (α, β, γ) change when we change the action of G_b on fermions. Given a particular action of $\bar{g} \in G_b$, any other action which acts in the same way on bosons differs from it by $p^{\mu(\bar{g})}$, where p is fermion parity and $\mu : G_b \to \mathbb{Z}_2$ is a homomorphism. If we define $\tilde{Q}(\bar{g}) = Q(\bar{g})P^{\mu(\bar{g})}$, we have

$$\tilde{Q}(\bar{g})\tilde{Q}(\bar{h}) = \exp\big(2\pi i\alpha(\bar{g}, \bar{h})\big)(-1)^{\mu(\bar{g})\beta(\bar{h})}\tilde{Q}(\bar{g}\bar{h}), \tag{4.125}$$

and

$$P\tilde{Q}(\bar{g})P^{-1} = (-1)^{\beta(\bar{g})}\tilde{Q}(\bar{g}). \tag{4.126}$$

This implies that for $\gamma = 0$, the parameter β is unchanged, while $\alpha \mapsto \alpha + \frac{1}{2}\mu \cup \beta$. For $\gamma = 1$, the situation is different, since fermion parity acts trivially on U, and thus α is not modified. But it acts nontrivially on the generator of $C\ell(1)$, so that the new G_b transformation multiplies it by $(-1)^{\beta(\bar{g})+\mu(\bar{g})}$. Thus $\beta \mapsto \beta + \mu$. Thus if we do not fix the action of G_b on fermions, all fermionic SRE phases with $\gamma = 1$ and a fixed $[\alpha]$ are equivalent. This agrees with (Fidkowski and Kitaev, 2011).

Two examples with $G_b = \mathbb{Z}_2$

Let us consider the case $G_b = \mathbb{Z}_2 = \{1, b\}$. There are two extensions of G_b by fermionic parity $\mathbb{Z}_2^{\mathcal{F}} = \{1, p\}$: one is $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathbb{Z}[b]/(b^2) \times \mathbb{Z}[p]/(p^2)$; the other is $\mathbb{Z}_4 = \mathbb{Z}[b, p]/(b^2 - p)$.

First take $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$. Consider algebras of the form $A = \text{End}(U)$, where U is a projective representation of $\mathcal{G}$. Each is characterized by a class $[\omega] \in H^2(\mathbb{Z}_2 \times \mathbb{Z}_2, U(1)) = \mathbb{Z}_2$. The two options for $[\omega]$ have cocycle representatives

$$\omega_0(g, h) = 0 \quad \text{and} \quad \omega_1(g, h) = \frac{1}{2} g_2 h_1 \tag{4.127}$$

where $g = (g_1, g_2)$, $h = (h_1, h_2)$. On the bosonic side of the duality, we think of ω_0 as describing the trivial phase and ω_1 as describing a nontrivial SRE. Alternatively, one can replace each ω by a pair (α, β). There is only the trivial $[\alpha] \in H^2(\mathbb{Z}_2, U(1))$. There are two β's: $\beta_0(b) = 0$ and $\beta_1(b) = 1$. These correspond to ω_0 and ω_1, respectively, as

$$\omega_i(g, h) = \frac{1}{2} t(g) \beta_i(b(h)) \tag{4.128}$$

where $t(g) = g_2$ and $b(h) = h_1$. On the fermionic side, β_0 describes a trivial phase and β_1 a nontrivial SRE.

Now consider breaking the symmetry down to any of the three $\mathbb{Z}_2$ subgroups of $\mathcal{G}$; this means considering algebras $A = \text{Ind}_H^{\mathcal{G}}(\text{End}(U))$ for projective representations U of the unbroken $H = \mathbb{Z}_2$. Since $H^2(\mathbb{Z}_2, U(1))$ is trivial, the only possibility (up to Morita equivalence) is $A = C\ell(1)$, graded by $\mathcal{G}/H$. On the bosonic side, each choice of H is a different non-invertible phase. As fermionic phases, the G_b-graded $C\ell(1)$ is a symmetry-broken phase, while the $\mathbb{Z}_2^{\mathcal{F}}$-graded $C\ell(1)$ is a nontrivial Majorana-chain phase $(0, \beta_0, 1)$. Breaking down to the diagonal $\mathbb{Z}_2$ gives a p-graded $C\ell(1)$ on which the bosonic symmetry acts non-trivially, i.e. $(0, \beta_1, 1)$.

Now take $\mathcal{G} = \mathbb{Z}_4$. The extension class is represented by the 2-cocycle $\rho(b, b) = 1$. There is only the trivial class $[\omega] \in H^2(\mathbb{Z}_4, U(1)) = \{1\}$. Meanwhile, there are two β's: β_0 and β_1 as before. They satisfy $\rho \cup \beta_0 = 0$ and $\rho \cup \beta_1(b, b, b) = 1$. The trivial α is the unique solution to $\delta\alpha = \rho \cup \beta_0$, and one can show that there are no solutions to $\delta\alpha = \rho \cup \beta_1$. In summary, there is only one pair (α, β) – it is the trivial one.

Consider breaking the only subgroup $\mathbb{Z}_2^{\mathcal{F}}$. The corresponding algebra is the G_b-graded $C\ell(1)$, which, as before, describes a symmetry-broken phase in both the bosonic and fermionic pictures.

State-sum for the equivariant fermionic theory

In Section 4.6, we observed that fermionic MPS arise from the state-sum for a spin-TQFT evaluated on an annulus diagram. A similar story can be told about

bosonic	(H,ω)	(α,β,γ)	fermionic
trivial	$(\mathcal{G},\omega_0)$	$(0,\beta_0,0)$	trivial
BSRE	$(\mathcal{G},\omega_1)$	$(0,\beta_1,0)$	FSRE
SB	$(G_b,1)$	$(0,\beta_0,1)$	FSRE
SB	$(\langle bp\rangle,1)$	$(0,\beta_1,1)$	FSRE
SB	$(\mathbb{Z}_2^{\mathcal{F}},1)$	n/a	SB
SB	(1,1)	n/a	SB

(a) Phases with $\mathcal{G} = \mathbb{Z}_2 \times \mathbb{Z}_2$

bosonic	(H,ω)	(α,β)	fermionic
trivial	$(\mathcal{G},\omega_0)$	$(0,\beta_0)$	trivial
SB	$(\mathbb{Z}_2^{\mathcal{F}},1)$	n/a	SB
SB	(1,1)	n/a	SB

(b) Phases with $\mathcal{G} = \mathbb{Z}_4$

Figure 4.17: Phase classification for the $G_b = \mathbb{Z}_2$ symmetry groups.

equivariant fermionic MPS. Now we will define a state-sum for equivariant spin-TQFTs and recover the MPS (4.114) as states on an annulus.

We will focus on the case where the total symmetry group $\mathcal{G}$ splits as a product of G_b and $\mathbb{Z}_2$ and then indicate the modifications needed in the non-split case. A G_b-equivariant spin-TQFT is defined in the same way as an ordinary spin TQFT, except that spin manifolds are replaced with spin manifolds equipped with principal G_b-bundles. Since G_b is finite, a G_b-principal bundle is completely characterized by its holonomies on non-contractible cycles. We will denote by $\mathcal{A}$ the collection of all holonomies. When working on manifolds with boundaries, it is convenient to fix a marked point and a trivialization of the bundle at this point on each boundary, so that the holonomy around each of these circles is a well-defined element of G_b rather than a conjugacy class.

The algebraic input for the state-sum construction is $G_b \times \mathbb{Z}_2$-equivariant semisimple Frobenius algebra A. The geometric data are a closed oriented two-dimensional manifold Σ equipped with a G_b-bundle and a spin structure. To define the state-sum, we also choose a marked skeleton Γ. Then a trivialized G_b-bundle can be represented as a decoration of each oriented edge with an element $g \in G_b$. Reversing an edge orientation replaces g with g^{-1}. We impose a flatness condition: the product of group labels around the boundary of each 2-cell is the identity. Equivalently, we can use the dual triangulation Γ^*: each dual edge is labeled by a group element, and the flatness condition says that the cyclically-ordered product of group elements on dual edges meeting at each dual vertex is the identity. One can think of the dual edges as domain walls and the dual edge labels as the G_b transformations due to moving across them.

The state-sum is defined as follows. Given a skeleton with a principal bundle, color the edges with pairs of elements e_i of some homogeneous basis of A. The weight of a coloring is the product of structure constants C_{ijk} over vertices (with indices cyclically ordered by orientation) and terms $R(g)^i{}_k \eta^{kj}$ over edges times the spin-dependent Koszul sign σ_s. The partition sum is the sum of the weights over colorings; the holonomies $\mathcal{A}$, which represent a background gauge field, are not summed over.

To incorporate brane boundaries, choose a $G_b \times \mathbb{Z}_2$-equivariant A-module U for each boundary component. Color the boundary edges by pairs of elements f^U_μ of a homogeneous basis of U – one for each vertex sharing the edge. The weight of a coloring is the usual weight times a factor of $T^{i\mu}{}_\nu$ for each boundary vertex and $Q(g)^\mu{}_\nu$ for each boundary edge.

As in the non-equivariant case, the partition sum is a spin-topological invariant. It also does not depend on the choice of trivialization of the principal bundle; in other words, it is gauge invariant. Invariance is ensured by the equivariance conditions (4.104), (4.105), and (4.108). In fact, one can evaluate the partition function in a closed form when the boundary is empty. Let $A = \mathrm{End}(U) \otimes C\ell(1)$ for some projective representation of G_b with a 2-cocycle α, and the action of G_b on $C\ell(1)$ determined by a homomorpism $\beta : G_b \to \mathbb{Z}_2$. It is easy to see that the partition function factorizes into a product of the partition function corresponding to $\mathrm{End}(U)$ and the partition function corresponding to $C\ell(1)$. The former factor is the partition function of a bosonic SRE phases, i.e. $\exp\left(2\pi i \int_\Sigma \alpha\right)$ (Kapustin, Turzillo, and You, 2017). The latter one is essentially the Arf invariant, modified by additional signs from the edges e for which $\beta(e) = 1$:

$$2^{-b_1(\Sigma)/2} \sum_{[a]\in H_1(\Sigma,\mathbb{Z}_2)} \sigma_s(a)(-1)^{\sum_{e\in a}\beta(\mathcal{A}(e))}. \tag{4.129}$$

Using the property (4.47), the definition of the Arf invariant, and the identity $\mathrm{Arf}(s + a) = \mathrm{Arf}(s)\sigma_s(a)$ (Atiyah, 1971), we can write this as

$$\mathrm{Arf}(s + \beta(\mathcal{A})) = \mathrm{Arf}(s)\sigma_s(\beta(\mathcal{A})). \tag{4.130}$$

Thus partition function of the fermionic SRE with the parameters $(\alpha, \beta, 1)$ is

$$\exp\left(2\pi i \int_\Sigma \alpha\right)\sigma_s(\beta(\mathcal{A}))\mathrm{Arf}(s). \tag{4.131}$$

Tensoring with another copy of $C\ell(1)$ multiplies this by another factor $\mathrm{Arf}(s)$, so that the partition function of the fermionic SRE with the parameters $(\alpha, \beta, 0)$ is

$$\exp\left(2\pi i \int_\Sigma \alpha\right)\sigma_s(\beta(\mathcal{A})). \tag{4.132}$$

We can also recover the equivariant MPS wavefunctions from the state sum. First suppose $A = \mathrm{End}(U)$, i.e. the parameter $\gamma = 0$. An equivariant module over A is of the form $M = U \otimes W$, where (U, Q) and (W, S) have projective actions of $\mathcal{G}$ characterized by opposite cocycles. Consider the annulus where one boundary is a brane boundary labeled by M and the other is a cut boundary. We work with a skeleton on the annulus such that each boundary is divided into N intervals, and let $g_{i,i+1}$ denote the group label between vertices i and $i+1$. A computation similar to that of Section 4.6 gives the state

$$\begin{aligned}\langle\psi_T| = \sum \mathrm{Tr}_{U\otimes W}[T(e_{i_1})\mathcal{Q}(g_{12})\cdots T(e_{i_N})\mathcal{Q}(g_{N1})] \\ \times \langle i_1 \cdots i_N|\end{aligned} \tag{4.133}$$

which, after performing gauge transformations and LU transformations, can be put in the form

$$\langle\psi_T| = \sum \mathrm{Tr}_{U\otimes W}[\mathcal{Q}(g)T(e_{i_1})\cdots T(e_{i_N})]\,\langle i_1 \cdots i_N| \tag{4.134}$$

where $g = g_{12}\cdots g_{N1}$. Since $\mathcal{Q} = Q \otimes S$ and $T(e_i)$ has the form $T(e_i) \otimes \mathbb{1}_W$, the trace factorizes:

$$\begin{aligned}\langle\psi_T| = \mathrm{Tr}_W[S(g)] \sum \mathrm{Tr}_U[Q(g)T(e_{i_1})\cdots T(e_{i_N})] \\ \times \langle i_1 \cdots i_N|\,.\end{aligned} \tag{4.135}$$

Up to normalization, this is the MPS (4.114).

The case $A = \mathrm{End}(U_\beta) \otimes C\ell(1)$ is similar. An indecomposable module over A is of the form $U \otimes W \otimes V$, where U and W carry projective H_β actions of opposite cocycles and $V = \mathbb{C}^{1|1}$ is the $C\ell(1)$-module considered in Section 4.7. The action of $\mathcal{G}$ is determined by $\mathcal{Q}(h) = Q_\beta(h) \otimes S(h) \otimes \mathbb{1}$ and $\mathcal{Q}(p)(M \otimes u_\pm) = \pm M \otimes u_\pm$. The argument proceeds as before, with the trace over W factoring out. We are left with an expression of the form (4.114) where the trace is over $U \otimes V$, the most general indecomposable MPS tensor over A.

Let us now discuss the non-split case. If $\mathcal{G}$ is a nontrivial extension of G_b by fermion parity, it is no longer true that a $\mathcal{G}$-equivariant algebra defines a G_b-equivariant spin-TQFT. Rather, it defines a $\mathcal{G}$-Spin TQFT (Kapustin et al., 2015). A $\mathcal{G}$-Spin structure on a manifold X is a G_b gauge field $\mathcal{A}$ on X together with a trivialization of the $\mathbb{Z}_2$-valued 2-cocycle $w_2 - \rho(\mathcal{A})$, where $\rho(\mathcal{A})$ is the pull-back of ρ from BG_b to X and w_2 is a 2-cocycle representing the 2^{nd} Stiefel-Whitney class of X. Now, if X is a Riemann surface Σ, $[w_2]$ is always zero, so $[\rho(\mathcal{A})]$ must be trivial too. Instead of choosing a trivialization of $w_2 - \rho(\mathcal{A})$, we can choose a trivialization s of w_2 and a trivialization τ of $\rho(\mathcal{A})$. That is, we choose $\mathbb{Z}_2$-valued 1-cochains s and τ such that $\delta s = w_2$ and $\delta\tau = \rho(\mathcal{A})$. These data are redundant: we can shift both s and τ by $\psi \in H^1(\Sigma, \mathbb{Z}_2)$.

We can now proceed as in the split case. Instead of a triple (α, β, γ), we have a pair (α, β) where $\beta \in H^1(G_b, \mathbb{Z}_2)$ and α is a 2-cochain on G_b with values in $U(1)$ satisfying $\delta\alpha = \frac{1}{2}\rho \cup \beta$. These data parameterize a 2-cocycle on $\mathcal{G}$. As shown above, the pairs (α, β) and $(\alpha + \frac{1}{2}\mu \cup \beta, \beta)$ correspond to the same 2-cocycle on $\mathcal{G}$, for any $\mu \in H^1(G_b, \mathbb{Z}_2)$. The partition function is evaluated exactly in the same way as in the split case, except that α is no longer closed, and an extra correction factor is needed to ensure the invariance of the partition function under a change of triangulation or a G_b gauge transformation. This correction factor is

$$(-1)^{\int_\Sigma \tau \cup \beta(\mathcal{A})} \tag{4.136}$$

where τ is a trivialization of $\rho(\mathcal{A})$ which is part of the definition of the $\mathcal{G}$-Spin structure on Σ. Thus the partition function is

$$\exp\left(2\pi i \int_\Sigma \alpha(\mathcal{A})\right)(-1)^{\int_\Sigma \tau \cup \beta(\mathcal{A})} \sigma_s(\beta(\mathcal{A})). \tag{4.137}$$

Using Eq. (4.47), one can easily see that the partition function is invariant under shifting both τ and s by any $\psi \in H^1(\Sigma, \mathbb{Z}_2)$. One can also see that the partition function is invariant under shifting α by $\frac{1}{2}\mu \cup \beta$ for any $\mu \in H^1(G_b, \mathbb{Z}_2)$ if we simultaneously shift $\tau \mapsto \tau + \mu(\mathcal{A})$.

Returning to the split case, we can examine the effect of treating the isomorphism $\mathcal{G} \simeq G_b \times \mathbb{Z}_2$ as unphysical. Every two such isomorphisms differ by a homomorphism $\mu : G_b \to \mathbb{Z}_2$. The effect this has on the data (α, β, γ) has been described in section

4.8:

$$\alpha \mapsto \alpha + (1-\gamma)\frac{1}{2}\mu \cup \beta, \qquad \beta \mapsto \beta + \gamma\mu, \qquad \gamma \mapsto \gamma. \tag{4.138}$$

Using the properties of σ_s and the Arf invariant, it is easy to check that the partition function is unaffected by these substitutions if we simultaneously shift the spin structure:

$$s \mapsto s + \mu(\mathcal{A}). \tag{4.139}$$

This can be interpreted as a special case of an equivalence relation between different spin structures which define the same $\mathcal{G}$-Spin structure.

Invariants of fermionic SRE phases

We have seen in 4.8 that a fermionic equivariant algebra A is characterized by the invariants α, β, and γ. These can also be extracted from an SRE fermionic MPS system without reference to the algebra A. Inspired by (Kapustin and Thorngren, 2017), we give a physical interpretation of these invariants as observable quantities.

We begin by studying how the MPS in the g-twisted sector transforms under the action of a unitary symmetry $h \in \mathcal{G}_0$. Let ω be the cocycle that characterizes the projective action Q on the module. Then

$$\begin{aligned} R(h)\cdot \operatorname{Tr}\left[Q(g)XT^i\right] \langle i| &\\ = \operatorname{Tr}\left[Q(g)XQ(h)^{-1}T^iQ(h)\right] \langle i| &\\ = e^{2\pi i(\omega(h,g)+\omega(hg,h^{-1})-\omega(h,h^{-1}))} &\\ \operatorname{Tr}\left[Q(hgh^{-1})[Q(h)XQ(h)^{-1}]T^i\right] \langle i|\,. \end{aligned} \tag{4.140}$$

We have used the fact that

$$\omega(h,h^{-1}) = \omega(h^{-1},h), \tag{4.141}$$

which follows from the cocycle condition.[15] We see that under the action of a unitary symmetry h,

1. The g-twisted sector maps to the hgh^{-1}-twisted sector.

2. The operator X is conjugated by $Q(h)$.

3. States also pick up a phase of

$$e^{2\pi i(\omega(h,g)+\omega(hg,h^{-1})-\omega(h,h^{-1}))}. \tag{4.142}$$

[15] We always work in a gauge $Q(1) = \mathbb{1}$.

We are now ready to interpret the three invariants.

Gamma.

Suppose $h = p$ and $g \in \{1, p\}$. Then the phase (4.142) vanishes, but there is still a sign coming from the conjugation of X by P. It is always +1 if the algebra is of the form $\mathrm{End}(U)$ (i.e. if $\gamma = 0$). If the algebra is of the form $\mathrm{End}(U_b) \otimes \mathbb{C}\ell(1)$ (i.e. if $\gamma = 1$), this sign is +1 in the NS sector and −1 in the R sector. Therefore we can conclude that the invariant $(-1)^\gamma$ is detected as the fermion parity (p-charge) of the R sector state.

Beta.

Continuing to take $h = p$, in the g-twisted sector, the phase (4.142) becomes

$$^1\!/\!_2\beta(g) := \omega(p, g) - \omega(g, p). \tag{4.143}$$

This term satisfies $\beta(pg) = \beta(g)$ and takes values in $\{0, ^1\!/\!_2\}$; in fact, it defines a $\mathbb{Z}/2$-valued cocycle of G_b. See the appendix for a proof. When $\gamma = 0$, the sign $(-1)^{\beta(g_b)}$ is the fermion parity of the g-twisted sector for g with $b(g) = g_b$. If $\mathcal{G}$ splits, one can equivalently say that $(-1)^{\beta(g)}$ is the parity of the $b(g)$-twisted NS and R sectors. If $\mathcal{G}$ splits, it is possible that $\gamma = 1$. In this case, one must choose a splitting to make sense of β. Then $(-1)^{\beta(g)}$ is still the parity of the $b(g)$-twisted NS sector, but the parity of the $b(g)$-twisted R sector receives a contribution of −1 from conjugation of X by P, in addition to the $\beta(g)$ term.

Note that $\beta(g)$ also describes the g-charge of the p-twisted (Ramond) sector for systems with $\gamma = 0$. This is no coincidence: the phase (4.142) agrees with Equation 4.11 of Ref. (Kapustin and Turzillo, 2017), where it was derived from bosonic (i.e. $X = \mathbb{1}$) TQFT. If g and h commute, one can sew together the ends of the cylinder to create a torus with holonomies g and h around its cycles. This torus evaluates to the phase

$$\omega(h, g) + \omega(hg, h^{-1}) - \omega(h, h^{-1}) = \omega(h, g) - \omega(g, h). \tag{4.144}$$

This surface can also be evaluated as a torus with holonomies h and g^{-1}, respectively, yielding

$$\begin{aligned} \omega(g^{-1}, h) + &\omega(g^{-1}h, g) - \omega(g^{-1}, g) \\ &= \omega(h, g) + \omega(g^{-1}, hg) - \omega(g^{-1}, g) \\ &= \omega(h, g) - \omega(g, h). \end{aligned} \tag{4.145}$$

These are equal, as is required by consistency of the TQFT. In terms of states, the h-charge of the g-twisted sector is the same as the g^{-1}-charge of the h-twisted sector, as long as g and h commute. There is no analogous statement for systems with $\gamma = 1$. Recall that $\beta(g)$ measures whether or not g acts as σ_z on the second factor of $\mathrm{End}(U) \otimes \mathbb{C}\ell(1)$. Then $Q(g)$ anticommutes with $X = \mathbb{1} \otimes \sigma_y$, and so the state picks up an extra charge of $\beta(g)$ which cancels with the sign (4.142) for a total g-charge of $+1$ in the R sector.

Alpha.

Consider the MPS state on a circle with two adjacent domain walls, parametrized by bosonic symmetries $g_b, h_b \in G_b$, as in Figure 4.18. Upon fusing them, the state picks up a phase:

$$\begin{aligned} &\mathrm{Tr}\left[Q(s(g_b))Q(s(h_b))T^i\right] \langle i| \\ &\qquad = e^{2\pi i \omega(s(g_b), s(h_b))}\, \mathrm{Tr}\left[Q(s(g_b)s(h_b))T^i\right] \langle i| \,. \end{aligned} \tag{4.146}$$

These phases define a G_b-cochain

$$\alpha(g_b, h_b) = \omega(s(g_b), s(h_b)). \tag{4.147}$$

If $\mathcal{G}$ splits, then the fact that ω is a cocycle implies that α is as well. If the extension $\mathcal{G}$ is instead defined by a nontrivial ρ, then α has coboundary $1/2\beta \cup \rho$. See the appendix for details. Redefining each $X = \mathbb{1}$ by a sector-dependent phase shifts α by a $\mathcal{G}$-coboundary with arguments in G_b, as expected.

Note that when β and γ are trivial, there are no fermionic states and the system is insensitive to spin structure. In this sense, α captures purely bosonic features of the system.

In summary:

- $(-1)^\gamma$ is the fermion parity of the untwisted R sector.
- If $\gamma = 0$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g-twisted sector for either of the two g's with $b(g) = g_b$. Alternatively, $(-1)^{\beta(g_b)}$ is the g-charge of the untwisted R sector. If $\gamma = 1$, $(-1)^{\beta(g_b)}$ is the fermion parity of the g_b-twisted NS sector, as determined by the choice of splitting.
- $e^{2\pi i \alpha(g_b, h_b)}$ is the phase due to fusing g_b and h_b domain walls.

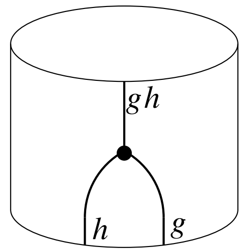


Figure 4.18: Fusion of domain walls.

$$\begin{array}{ccccc} & & \mathbb{Z}_2^F & & \\ & & \downarrow & & \\ & & \mathcal{G} & & \\ & & \downarrow b & & \\ G_0 & \longrightarrow & G_b & \xrightarrow{x} & \mathbb{Z}_2^T \end{array}$$

Figure 4.19: Symmetry data.

Anti-unitary and orientation-reversing symmetries

More generally, a fermionic system may be invariant under anti-unitary symmetries as well as unitary ones. In this case, the full symmetry group $\mathcal{G}$ is a central extension by $\mathbb{Z}_2^F$ of a bosonic symmetry group G_b, which is itself an extension of $\mathbb{Z}_2^T$ by a finite group G_0, as in Figure 4.19. The symmetry class $(\mathcal{G}, p, x)$ is determined by a central $p \in \mathcal{G}$ and a map $x : G_b \to \mathbb{Z}/2$ that encodes whether a bosonic symmetry is unitary or anti-unitary. Note that the composition $x \circ b$, which we also call 'x,' satisfies $x(p) = 0$. Let $\mathcal{G}_0$ denote its kernel.

A fixed point MPS system of symmetry class $(\mathcal{G}, p, x)$ consists of a finite-dimensional semisimple associative algebra A and a faithful module $T : A \to \mathrm{End}(V)$, satisfying the equivariance conditions (4.25) and (4.27) as before; only now the group action may be anti-unitary. In particular, the projective action on V is given by a unitary operator $Q(g)$ for each $g \in \mathcal{G}_0$ and an anti-unitary operator $Q(g)$ for each $g \notin \mathcal{G}_0$ that satisfy

$$Q(g)Q(h) = e^{2\pi i \omega(g,h)} Q(gh) \tag{4.148}$$

for phases $\omega(g, h)$. By comparing $[Q(g)Q(h)]Q(k)$ and $Q(g)[Q(h)Q(k)]$, we find the x-twisted cocycle condition:

$$\omega(g, h) + \omega(gh, k) = (-1)^{x(g)} \omega(h, k) + \omega(g, hk). \tag{4.149}$$

Redefining each $Q(g)$ by a g-dependent phase corresponds to shifting ω by an x-twisted coboundary. Therefore the action of $\mathcal{G}$ on the module V is characterized by a twisted cohomology class $[\omega] \in H^2(\mathcal{G}, U(1)_T)$. The group action R on A is defined via (4.25). It will be convenient to define linear maps $M(g)$ by

$$M(g) = \begin{cases} Q(g) & g \in \mathcal{G}_0 \\ Q(g)K & g \notin \mathcal{G}_0 \end{cases} \tag{4.150}$$

where K denotes complex conjugation.

Unitary symmetries that reverse the orientation of one-dimensional space can also be described in this language. Let x measure whether a symmetry reverses orientation. The natural generalization of (4.25) is

$$\begin{aligned} T(R(g)a) &= M(g)T(a)M(g)^{-1} \qquad \text{for } g \in \mathcal{G}_0 \\ T(R(g)a) &= M(g)T(a)^T M(g)^{-1} \qquad \text{for } g \notin \mathcal{G}_0. \end{aligned} \tag{4.151}$$

Let us introduce the following shorthand. For a matrix $O \in \mathrm{End}(V)$, define

$$\begin{aligned} O^{T0} &= O, \qquad & O^{T1} &= O^T, \\ \{O\}^0 &= O, \qquad & \{O\}^1 &= [O^{-1}]^T. \end{aligned} \tag{4.152}$$

Since R is a group homomorphism,

$$\begin{aligned} M(g)\{M(h)\}^{x(g)}&T(a)^{Tx(gh)}M(h)^{Tx(g)}M(g)^{-1} \\ &= T(R(g)R(h)a) \\ &= T(R(gh)a) \\ &= M(gh)T(a)^{Tx(gh)}M(gh)^{-1}. \end{aligned} \tag{4.153}$$

This implies that there exists a number $\omega(g,h) \in \mathbb{R}/\mathbb{Z}$ such that

$$M(g)\{M(h)\}^{x(g)} = e^{2\pi i \omega(g,h)} M(gh). \tag{4.154}$$

By comparing the two equal expressions $M(g)\{M(h)\}^{x(g)}\{M(k)\}^{x(gh)}$ and $M(g)\{M(h)M(k)^{x(h)}\}^{x(g)}$, one recovers the x-twisted cocycle condition (4.149) for ω.

From the perspective of two-dimensional spacetime, it is not surprising that time-reversal[16] and space-reversal should be treated similarly. To make the connection

[16]By a well-known result of Wigner, an anti-unitary symmetry reverses the direction of time.

more explicit, note that the physical Hilbert space carries the action of an anti-linear involution $*$, which we regard as CPT (see Ref. (Kapustin, Turzillo, and You, 2017)). Using equivariance of the multiplication and (anti-)unitarity of $R(g)$ with respect to the inner product on the Hilbert space, it may be shown that $*$ commutes with $R(g)$ for all $g \in \mathcal{G}$. With respect to the product on A, this map is an anti-automorphism. If $R(g)$ denotes the action of a time-reversing symmetry, $R(g)*$ is a unitary symmetry that reverses the orientation of space. Then

$$T(R(g) * a) = M(g)\overline{T(*a)}M(g)^{-1} = M(g)T(a)^T M(g)^{-1}. \tag{4.155}$$

Moreover, since $*$ commutes with $R(g)$, the equivariance condition (4.25) implies that $M(g)$ is unitary (up to a phase), so (4.148) and (4.154) are equivalent (up to a coboundary). For the remainder of the paper, we suppress $*$ and simply write R to denote a time-reversing or space-reversing symmetry.

Invariants of fermionic SRE phases with anti-unitary symmetries

As in the case of unitary symmetries, fermionic SRE systems at fixed points correspond to even algebras of the form $\mathrm{End}(U)$ and odd algebras of the form $\mathrm{End}(U_b) \otimes \mathbb{C}\ell(1)$. However, when the symmetries may act anti-unitarily, the cohomology class characterizing the Morita class (and hence the SRE phase) is twisted.

We now discuss the meaning of the invariants α, β, and γ in the anti-unitary context, following the previous analysis. The form of the MPS conjugate wavefunction is (4.114) as before. Consider the action of an anti-unitary symmetry $h \notin \mathcal{G}_0$ on an MPS in the g-twisted ($g \in \mathcal{G}_0$) sector:

$$\begin{aligned}
R(h)\cdot\, & \mathrm{Tr}\big[Q(g)XT^i\big]\,\langle i| \\
&= \mathrm{Tr}\big[M(g)XM(h^{-1})(T^i)^T M(h^{-1})^{-1}\big]\,\langle i| \\
&= \mathrm{Tr}\big[M(h^{-1})^T X^T M(g)^T M(h^{-1})^{-1T} T^i\big]\,\langle i| \\
&= e^{2\pi i\omega(h,h^{-1})}\mathrm{Tr}\,\big[M(h^{-1})^T M(hg)^{-1}M(hg) \\
&\qquad X^T[M(h)M(g)^{-1T}]^{-1}T^i\big]\,\langle i| \\
&= e^{2\pi i(\omega(h,h^{-1})-\omega(h,g))}\mathrm{Tr}\,\big[[M(hg)M(h^{-1})^{-1T}]^{-1} \\
&\qquad [M(hg)X^T M(hg)^{-1}]T^i\big]\,\langle i| \\
&= e^{2\pi i(\omega(h,g^{-1})+\omega(hg^{-1},h^{-1})+\omega(g,g^{-1})-\omega(h,h^{-1}))} \\
&\qquad \mathrm{Tr}\big[Q(hg^{-1}h^{-1})[M(hg)X^T M(hg)^{-1}]T^i\big]\,\langle i|
\end{aligned} \tag{4.156}$$

where in the last line we use the fact that

$$\begin{aligned}\omega(hg^{-1}h^{-1}, hgh^{-1}) \\ = -\omega(h,g) - \omega(hg,h^{-1}) - \omega(hg^{-1},h^{-1}) \\ - \omega(h,g^{-1}) - \omega(g^{-1},g) - 2\omega(h^{-1},h),\end{aligned} \tag{4.157}$$

which can be verified by repeated application of the twisted cocycle condition. We see that under the action of an anti-unitary symmetry h,

1. The g-twisted sector maps to the $hg^{-1}h^{-1}$-twisted sector.
2. The operator X is transposed, then conjugated by $M(hg)$.[17]
3. States also pick up a phase of

$$e^{2\pi i(\omega(h,g^{-1})+\omega(hg^{-1},h^{-1})+\omega(g,g^{-1})-\omega(h,h^{-1}))}. \tag{4.158}$$

The phase matches Equation 4.12 of Ref. (Kapustin and Turzillo, 2017). In particular, when g acts on the R sector, it is

$$1/2\beta(g) := \omega(g,p) - \omega(p,g) + \omega(p,p), \qquad g \notin \mathcal{G}_0. \tag{4.159}$$

This phase satisfies $\beta(pg) = \beta(g)$, takes values in $\mathbb{Z}/2$, and, together with (4.143), is a G_b-cocycle. Refer to the appendix for a proof. When $\gamma = 0$, this is the g-charge of the R sector. However, when $\gamma = 1$, the charge receives an additional contribution from the transformation of X. Similarly to the unitary case detailed above, the total charge is the β-independent quantity $(-1)^{x(g)}$, so this interpretation of β fails.

The invariant β also has an interpretation in terms of edge states, like (4.117).[18] A time-reversing symmetry $g \notin G_0$ maps V to its dual space V^*, on which p acts as P^{-1}, so the parity of $Q(g)$ is read off of

$$P^{-1}Q(g)P^{-1} = e^{i\pi\beta(\bar{g})}Q(g), \qquad g \notin \mathcal{G}_0. \tag{4.160}$$

A similar interpretation holds if g reverses the orientation of space. Let $V^* \otimes V$ represent the tensor product of left and right edge state spaces. On this space, g acts as

$$\begin{aligned}\psi_L \otimes \psi_R \mapsto \\ Q(g)^{-1}(\psi_L \otimes \psi_R)^T Q(g) = Q(g)^{-1}\psi_R \otimes Q(g)\psi_L.\end{aligned} \tag{4.161}$$

[17] If X is Hermitian, this is the same as X being conjugated by the anti-linear operator $Q(hg)$.

[18] If g is anti-linear, the expression (4.117) is not invariant under the change of gauge $\omega \mapsto \omega + \delta\Lambda$.

β appears as the result of acting by $P \otimes P^{-1}$, g, and then by $P^{-1} \otimes P$:

$$\psi \otimes 1 \mapsto 1 \otimes PQ(g)P\psi = e^{i\pi\beta(\bar{g})}(1 \otimes \psi). \tag{4.162}$$

The meaning of α (4.147) is more difficult to describe in Hamiltonian language.[19] The lack of twisted sectors for anti-unitary symmetries means that $\alpha(g_b, h_b)$ has an interpretation as the phase due to fusing domain walls only when g_b and h_b are unitary. The rest of α appears in other places. It is convenient to first describe the invariant ω. For two unitary symmetries $g, h \in \mathcal{G}_0$, the phase $\omega(g, h)$ is due to fusing domain walls. It was shown in Ref. (Kapustin and Turzillo, 2017) that two extra families of phases – which we now describe – together with ω restricted to $\mathcal{G}_0$, determine the full ω on $\mathcal{G}$. The first family is the phases (4.158) due to acting on the g-twisted sector by an anti-unitary symmetry h. The second family consists of the relative phases due to comparing, for each anti-unitary symmetry $g \notin \mathcal{G}_0$, the crosscap state (see Refs. (Shiozaki and Ryu, 2017; Kapustin and Turzillo, 2017)) $\mathrm{Tr}\left[Q(g)Q(g)T^i\right] \langle i|$ to the MPS state in the g^2-twisted sector. These phases have the simple form $\omega(g, g)$. Note that these data are not gauge invariant, and the equivalence classes of them under shifting ω by a twisted coboundary do not take a simple form. Now that we have described the full ω, the full α can be recovered by restricting to G_b. As we demonstrate in the appendix, the result is a G_b cochain whose x-twisted coboundary is $\beta \cup \rho$.

Finally, γ is the fermion parity of the untwisted Ramond sector, as in the unitary case.

4.9 The fermionic stacking law

Gapped fermionic phases form a commutative monoid under the operation of stacking. The result of stacking fixed point systems corresponding to algebras A_1 and A_2 is the system corresponding to the supertensor product $A_1 \widehat{\otimes} A_2$, defined by the multiplication law $(a_1 \widehat{\otimes} a_2)(b_1 \widehat{\otimes} b_2) = (-1)^{|a_2||b_1|} a_1 b_1 \widehat{\otimes} a_2 b_2$ (Bultinck et al., 2017; Kapustin, Turzillo, and You, 2018). SRE phases are precisely those that are invertible under stacking, so they form a group. The goal of this section is to derive this group structure on the set of SRE phases in terms of the invariants α, β, and γ. We will derive (4.124) and its generalization which takes into account the possibility that $Q(g)$ is anti-linear. We summarize the results at the end of the section.

[19] In the Lagrangian picture, we expect α to be related to trivalent junctions of possibly orientation-reversing domain walls.

The following discussion relies on a result proven in the appendix: that one can choose a gauge such that the twisted cocycle ω is related to α and β by, for all $g, h \in \mathcal{G}$, where $\bar{g}$ is short for $b(g)$,

$$\omega(g,h) = \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2\beta(\bar{g})t(h). \tag{4.163}$$

There are three cases to consider: the stacking of 1) two even algebras, 2) an even and an odd algebra, 3) two odd algebras. When $\mathcal{G}$ does not split, there are no odd algebras so we need only consider the first case.

Even-even stacking

Consider the even algebras $\mathrm{End}(U_1)$ and $\mathrm{End}(U_2)$. Their tensor product is $\mathrm{End}(U_1 \mathbin{\widehat{\otimes}} U_2)$, where $U_1 \mathbin{\widehat{\otimes}} U_2$ carries a projective representation $Q = Q_1 \mathbin{\widehat{\otimes}} Q_2$. Then

$$\begin{aligned} Q(g)Q(h) &= \left(Q_1(g) \mathbin{\widehat{\otimes}} Q_2(g)\right)\left(Q_1(h) \mathbin{\widehat{\otimes}} Q_2(h)\right) \\ &= (-1)^{\beta_2(\bar{g})\beta_1(\bar{h})} Q_1(g)Q_1(h) \mathbin{\widehat{\otimes}} Q_2(g)Q_2(h) \\ &= (-1)^{(\beta_2\cup\beta_1)(\bar{g},\bar{h})} e^{2\pi i(\alpha_1(\bar{g},\bar{h}) + {}^1\!/\!_2\beta_1(\bar{g})t(h))} \\ &\qquad e^{2\pi i(\alpha_2(\bar{g},\bar{h}) + {}^1\!/\!_2\beta_2(\bar{g})t(h))} Q_1(gh) \mathbin{\widehat{\otimes}} Q_2(gh) \\ &= e^{2\pi i(\alpha_1+\alpha_2+{}^1\!/\!_2\beta_2\cup\beta_1)(\bar{g},\bar{h}) + {}^1\!/\!_2(\beta_1+\beta_2)(\bar{g})t(h)} Q(gh). \end{aligned} \tag{4.164}$$

Thus the invariants of the stacked phase are $\alpha = \alpha_1 + \alpha_2 + {}^1\!/\!_2(\beta_1 \cup \beta_2)$[20] and $\beta = \beta_1 + \beta_2$. Since the stacked algebra is again even, $\gamma = 0$. The presence of anti-unitary symmetries does not affect even-even stacking.

Even-odd stacking

Now consider the even algebra $A_1 = \mathrm{End}(U_1)$, where U_1 carries a projective representation Q_1 of $\mathcal{G}$, and the odd algebra $A_2 = \mathrm{End}(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective representation Q_2 of G_b. Their tensor product $\mathrm{End}(U_1) \mathbin{\widehat{\otimes}} (\mathrm{End}(U_2) \otimes \mathbb{C}\ell(1))$ is isomorphic as an algebra to the odd algebra $\mathrm{End}(U_1 \otimes U_2) \otimes \mathbb{C}\ell(1)$ by the map

$$JW : M_1 \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma^m) \mapsto M_1 P^m \otimes M_2 \otimes \Gamma^{m+|M_1|}, \tag{4.165}$$

which has inverse

$$JW^{-1} : M_1 \otimes M_2 \otimes \Gamma^m \mapsto M_1 P^{m+|M_1|} \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma^{m+|M_1|}), \tag{4.166}$$

where the parity of M_1 is defined by Q_1: $P_1 M_1 P_1 = (-1)^{|M_1|} M_1$. This isomorphism respects the $\mathbb{Z}/2$-grading defined by the standard action of fermion parity on even and odd algebras.

[20] We have used the fact that $\beta_2 \cup \beta_1$ is cohomologous to $\beta_1 \cup \beta_2$ in $\mathbb{Z}/2$.

It remains to determine the G_b action on the odd algebra. For $g \in \mathcal{G}$ with $t(g) = 0$,

$$\begin{aligned} &JW \circ g \circ JW^{-1} \cdot (M_1 \otimes M_2 \otimes \Gamma^m) \\ &\quad = JW \circ g \cdot \left(M_1 P^{m+|M_1|} \widehat{\otimes} (M_2 \otimes \Gamma^{m+|M_1|}) \right) \\ &\quad = JW \cdot \Big(Q_1(g) M_1 P^{m+|M_1|} Q_1(g)^{-1} \widehat{\otimes} \\ &\quad (Q_2(\bar{g}) M_2 Q_2(\bar{g})^{-1} \otimes (-1)^{(m+|M_1|)\beta_2(\bar{g})} \Gamma^{m+|M_1|}) \Big) \\ &\quad = (-1)^{(m+|M_1|)(\beta_1(\bar{g})+\beta_2(\bar{g}))} Q_1(g) M_1 Q_1(g)^{-1} \\ &\qquad \otimes Q_2(\bar{g}) M_2 Q_2(\bar{g})^{-1} \otimes \Gamma^m. \end{aligned} \tag{4.167}$$

In order to read off the invariants from this group action, we must rewrite it in the standard form by defining $\tilde{Q}_1(g) = Q_1(g) P^{\beta_1(g)+\beta_2(g)}$[21] and $Q(g) = \tilde{Q}_1(g) \otimes Q_2(\bar{g})$. Then, continuing from (4.167),

$$\begin{aligned} &g \cdot (M_1 \otimes M_2 \otimes \Gamma^m) \\ &\quad = (-1)^{m(\beta_1(\bar{g})+\beta_2(\bar{g}))} (\tilde{Q}_1(g) \otimes Q_2(\bar{g})) M_1 \\ &\qquad \otimes M_2 (\tilde{Q}_1(g)^{-1} \otimes Q_2(\bar{g})^{-1}) \otimes \Gamma^m, \end{aligned} \tag{4.168}$$

from which we read off the stacked invariant $\beta = \beta_1 + \beta_2$. In addition,

$$\begin{aligned} Q(g)Q(h) &= \left(\tilde{Q}_1(g) \otimes Q_2(\bar{g})\right) \left(\tilde{Q}_1(h) \otimes Q_2(\bar{h})\right) \\ &= Q_1(g) P^{\beta_1(\bar{g})+\beta_2(\bar{g})} Q_1(h) P^{\beta_1(\bar{h})+\beta_2(\bar{h})} \otimes Q_2(\bar{g}) Q_2(\bar{h}) \\ &= (-1)^{\beta_1(\bar{h})(\beta_1(\bar{g})+\beta_2(\bar{g}))} \\ &\qquad Q_1(g) Q_1(h) P^{\beta_1(\bar{gh})+\beta_2(\bar{gh})} \otimes Q_2(\bar{g}) Q_2(\bar{h}) \\ &= e^{2\pi i (\alpha_1(g,h)+\alpha_2(g,h)+1/2(\beta_2 \cup \beta_1)(g,h)+1/2(\beta_1 \cup \beta_1)(g,h))} Q(gh), \end{aligned} \tag{4.169}$$

from which we see $\alpha = \alpha_1 + \alpha_2 + 1/2 \beta_1 \cup \beta_2 + 1/2 \beta_1 \cup \beta_1$. There is no asymmetry: the $1/2 \beta_1 \cup \beta_1$ term always comes from the β of the even algebra.[22] Finally, $\gamma = 1$ since the stacked algebra is odd.

Odd-odd stacking

Consider the odd algebras $A_1 = \mathrm{End}(U_1) \otimes \mathbb{C}\ell(1)$, where U_1 carries a projective representation Q_1 of G_b, and $A_2 = \mathrm{End}(U_2) \otimes \mathbb{C}\ell(1)$, where U_2 carries a projective

[21] Adding a phase factor to $\tilde{Q}$ would have shifted the resulting 2-cocycle α by an irrelevant coboundary. For example, if we had chosen a factor $i^{\beta_1(g)}$ as in Ref. (Kapustin, Turzillo, and You, 2018), we would have gotten $\beta_1 \cup x$ instead of $\beta_1 \cup \beta_1$ in the final answer.

[22] Note that while $\beta_1 \cup \beta_1$ is an ordinary coboundary and hence could be ignored for phases without time-reversal, it is not a twisted coboundary and so cannot be ignored when time-reversing symmetries are present. By adding a twisted coboundary, we can put it in the form $\beta_1 \cup x$, which makes the dependence on time-reversal symmetry manifest.

representation Q_2 of G_b. Their tensor product is given by $A_1 \mathbin{\widehat{\otimes}} A_2 \simeq \mathrm{End}(U_1 \otimes U_2 \otimes \mathbb{C}^{1|1})$, since $\mathbb{C}\ell(1) \mathbin{\widehat{\otimes}} \mathbb{C}\ell(1) \simeq \mathbb{C}\ell(2) \simeq \mathrm{End}(\mathbb{C}^{1|1})$, via an isomorphism

$$(M_1 \otimes \Gamma_1^m) \mathbin{\widehat{\otimes}} (M_2 \otimes \Gamma_2^n) \mapsto M_1 \otimes M_2 \otimes \sigma_1^m \sigma_2^n, \tag{4.170}$$

where σ_1 and σ_2 are any two distinct Pauli matrices. With respect to the action of fermion parity on the $\mathrm{End}(\mathbb{C}^{1|1})$ factor as conjugation by $\sigma_3 = -i\sigma_1\sigma_2$, this map is an isomorphism of $\mathbb{Z}/2$-graded algebras.

One choice[23] of G_b-action Q on $U_1 \otimes U_2 \otimes \mathbb{C}^2$, with respect to which (4.170) is equivariant, is

$$g : u_1 \otimes u_2 \otimes v \mapsto Q_1(\bar{g})u_1 \otimes Q_2(\bar{g})u_2 \otimes \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} v, \tag{4.171}$$

for $g \in \mathcal{G}$ with $t(g) = 0$, where K denotes complex conjugation in a basis in which σ_1 and σ_2 are real. Then

$$\begin{aligned} &Q(\bar{g})Q(\bar{h}) \\ &\quad = \left(Q_1(\bar{g}) \otimes Q_2(\bar{g}) \otimes \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})}\right) \\ &\qquad\qquad \left(Q_1(\bar{h}) \otimes Q_2(\bar{h}) \otimes \sigma_1^{\beta_2(\bar{h})} \sigma_2^{\beta_1(\bar{h})} K^{x(\bar{h})}\right) \\ &\quad = e^{2\pi i \alpha_1(\bar{g},\bar{h})} Q_1(\bar{gh}) \otimes e^{2\pi i \alpha_2(\bar{g},\bar{h})} Q_2(\bar{gh}) \otimes \\ &\qquad\qquad (-1)^{\beta_1(\bar{g})\beta_2(\bar{h})} \sigma_1^{\beta_1(\bar{gh})} \sigma_2^{\beta_2(\bar{gh})} K^{x(\bar{gh})} \\ &\quad = e^{2\pi i (\alpha_1 + \alpha_2 + 1/2 \beta_1 \cup \beta_2)(\bar{g},\bar{h})} Q(\bar{gh}), \end{aligned} \tag{4.172}$$

from which we see that $\alpha = \alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2$. Since $U_1 \otimes U_2$ is purely even, the parity of Q comes from

$$\begin{aligned} &P\sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} P \\ &\quad = (-i\sigma_1\sigma_2)\sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} (-i\sigma_1\sigma_2) \\ &\quad = (-1)^{\beta_2(\bar{g})+\beta_1(\bar{g})} \sigma_1^{\beta_2(\bar{g})} \sigma_2^{\beta_1(\bar{g})} K^{x(\bar{g})} (-1)^{x(\bar{g})}. \end{aligned} \tag{4.173}$$

We read off $\beta = \beta_1 + \beta_2 + x$. Finally, the stacked algebra is even, so $\gamma = 0$.

In summary

The stacking law for the invariants (α, β, γ) is given by

$$\begin{aligned} (\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 0) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2, \beta_1 + \beta_2, 0) \\ (\alpha_1, \beta_1, 0) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2 + 1/2\beta_1 \cup \beta_1, \beta_1 + \beta_2, 1) \\ (\alpha_1, \beta_1, 1) \cdot (\alpha_2, \beta_2, 1) &= (\alpha_1 + \alpha_2 + 1/2\beta_1 \cup \beta_2, \beta_1 + \beta_2 + x, 0). \end{aligned} \tag{4.174}$$

[23] Again, had we chosen a different G_b-action on $\mathbb{C}^{1|1}$ compatible with the action $g_b : \Gamma_i \mapsto (-1)^{\beta_i(g_b)}\Gamma_i$ on the $\mathbb{C}\ell(1)$ factors, the 2-cocycle α would be shifted by a twisted coboundary.

This group law inherits the properties commutativity and associativity from the tensor product of algebras. When $\mathcal{G}$ does not split, γ is not present, and the stacking law is simply

$$(\alpha_1, \beta_1) \cdot (\alpha_2, \beta_2) = (\alpha_1 + \alpha_2 + {}^1\!/\!_2\beta_1 \cup \beta_2, \beta_1 + \beta_2). \tag{4.175}$$

We emphasize that while data $[\alpha, \beta]$ are equivalent to $[\omega] \in H^2(\mathcal{G}, U(1)_T)$, the group structure on $H^2(\mathcal{G}, U(1)_T)$ differs from (4.174). On the other hand, the stacking of bosonic SRE phases, which are also characterized by classes $[\omega]$, is described by the usual group structure on $H^2(\mathcal{G}, U(1)_T)$.

4.10 Examples

Class BDI fermions: $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$

Let us consider SRE phases with symmetry $\mathcal{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2^T$. The two classes $\alpha \in H^2(\mathbb{Z}_2^T, U(1)_T) = \mathbb{Z}/2$, two classes $\beta \in H^1(\mathbb{Z}_2^T, \mathbb{Z}/2) = \mathbb{Z}/2$, and two classes $\gamma \in \mathbb{Z}/2$ make for a total of eight phases. A straightforward application of the general stacking law (4.174) reveals that these phases stack like the cyclic group $\mathbb{Z}/8$. In this section, we will reproduce this group law by exploiting the relationship between $\mathcal{G}$-equivariant algebras, real super-division algebras, and Clifford algebras, which have Bott periodicity $\mathbb{Z}/8$.

We begin by describing simple $\mathcal{G}$-equivariant algebras. The matrix algebra $M_2\mathbb{C}$ represents the sole Morita class of simple complex algebras. This algebra has a unitary structure $*$ given by conjugate transposition. Its action fixes a basis $\{\mathbb{1}, X, Y, Z = -iXY\}$. On $C\ell_2\mathbb{C} \simeq M_2\mathbb{C}$, $*$ acts by Clifford transposition and complex conjugation of coefficients with respect to a pair of generators that square to $+1$.

There are two distinct real structures on $M_2\mathbb{C}$ given by complex conjugation T on the second component of $M_2\mathbb{C} \simeq M_2\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ and $M_2\mathbb{C} \simeq \mathbb{H} \otimes_\mathbb{R} \mathbb{C}$. The unitary structure $*$ of $M_2\mathbb{C}$ acts by transposition on $M_2\mathbb{R}$, complex conjugation on $\mathbb{C}$, and inversion of the generators $\hat{\imath}$ and $\hat{\jmath}$ of $\mathbb{H}$; that is, its fixed bases are

$$\begin{gathered} \{\mathbb{1} \otimes 1, X \otimes 1, iY \otimes i, Z \otimes 1\} \in M_2\mathbb{R} \otimes_\mathbb{R} \mathbb{C}, \\ \{\mathbb{1} \otimes 1, \hat{\imath} \otimes i, \hat{\jmath} \otimes i, \hat{k} \otimes i\} \in \mathbb{H} \otimes_\mathbb{R} \mathbb{C}. \end{gathered} \tag{4.176}$$

These bases have the same T-eigenvalues as they do $*T$-eigenvalues, where $*T$ acts as transposition on $M_2\mathbb{R}$ and inversion of generators on $\mathbb{H}$. Under the algebra isomorphisms $M_2\mathbb{R} \simeq C\ell_{1,1}\mathbb{R} \simeq C\ell_{2,0}\mathbb{R}$ and $\mathbb{H} \simeq C\ell_{0,2}\mathbb{R}$, $*T$ acts by inverting the generators and products of generators that square to -1.

Let us derive the invariants α of these real structures. Pulled back from $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $M_2\mathbb{C}$, T acts like complex conjugation and $*T$ like transposition; that is $M(t) = \mathbb{1}$. Then

$$M(t)M(t)^{-1T} = \mathbb{1}, \tag{4.177}$$

which means $\alpha(t,t) = \omega(t,t) = 0$. Pulled back from $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, T acts like complex conjugation and conjugation by Y, while $*T$ acts like transposition and conjugation by Y; that is $M(t) = Y$. Then $\alpha(t,t) = 1/2$ since

$$M(t)M(t)^{-1T} = e^{i\pi}\mathbb{1}. \tag{4.178}$$

By the Skolem-Noether theorem, a superalgebra structure on $M_2\mathbb{C}$ is given by conjugation by an element that squares to one. If this element is $\mathbb{1}$, the $\mathbb{Z}/2$-grading is purely even; otherwise, it has two even dimensions and two odd ones. All structures of the latter type are isomorphic in the absence of a real structure.

In the presence of the real structure $M_2\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, there are three distinct gradings. First, there is the purely even grading, given by $P = \mathbb{1}$. This structure has $1/2\beta(t) = \omega(t,p) = 0$. Second, there is conjugation by Z (or X), which gives $M_2\mathbb{R}$ the superalgebra structure of $C\ell_{1,1}\mathbb{R}$. Again, $\beta(t) = 0$ since $P = Z$ means

$$PM(t)P^T = Z\mathbb{1}Z^T = \mathbb{1}. \tag{4.179}$$

The matching of the invariants alludes to the fact that the real superalgebra structures $M_2\mathbb{R}$ and $C\ell_{1,1}\mathbb{R}$ are graded Morita equivalent. Third, there is conjugation by Y; that is, $P = Y$. Then $\beta(t) = 1$ since

$$PM(t)P^T = Y\mathbb{1}Y^T = e^{i\pi}\mathbb{1}. \tag{4.180}$$

The corresponding real Clifford algebra is $C\ell_{2,0}\mathbb{R}$ and represents a distinct Morita class.

On the real structure $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C}$, there are two distinct gradings. First, there is the purely even grading $P = \mathbb{1}$, which has $\beta(t) = 0$. The second grading is given by conjugation by Z (or X or Y) on $M_2\mathbb{C}$ and gives $\mathbb{H}$ the superalgebra structure of $C\ell_{0,2}\mathbb{R}$. Then $\beta(t) = 1$ since

$$PM(t)P^T = ZYZ^T = e^{i\pi}Y. \tag{4.181}$$

Now consider algebras of the form $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$. The second component $C\ell_1\mathbb{C}$ has a unitary structure $*$ given by complex conjugation of coefficients of the generator Γ that squares to $+1$. There are two distinct real structures on $C\ell_1\mathbb{C}$ given by complex conjugation T on the second component of $C\ell_{1,0}\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}$ and $C\ell_{0,1}\mathbb{R}\otimes_{\mathbb{R}}\mathbb{C}$. The unitary structure $*$ of $C\ell_1\mathbb{C}$ acts by complex conjugation on $\mathbb{C}$ and inversion of generators that square to -1; that is, the fixed bases are $\{1 \otimes 1, \gamma \otimes 1\}$ for $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ and $\{1 \otimes 1, \gamma \otimes i\}$ for $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$. The map $*T$ is trivial on $C\ell_{1,0}\mathbb{R}$ and inversion of the generator on $C\ell_{0,1}\mathbb{R}$. Therefore, pulled back from $C\ell_{1,0}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$ to $C\ell_1\mathbb{C}$, T is complex conjugation and $*T$ is trivial. From $C\ell_{0,1}\mathbb{R} \otimes_{\mathbb{R}} \mathbb{C}$, $*T$ is inversion of Γ.

As discussed in Ref. (Kapustin, Turzillo, and You, 2018), we need only to consider a single $\mathbb{Z}/2$-grading on $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ – the one where $M_2\mathbb{C}$ is purely even and the generator of $C\ell_1\mathbb{C}$ is odd. The algebra $M_2\mathbb{C} \otimes C\ell_1\mathbb{C}$ has four real structures: a choice of $M_2\mathbb{R}$ or $\mathbb{H}$ for the first component and $C\ell_{1,0}\mathbb{R}$ or $C\ell_{0,1}\mathbb{R}$ for the second. As was true for even algebras, the first choice determines whether $M(t)$ is $\mathbb{1}$ or Y; that is, whether $\alpha(t,t)$ is 0 or $1/2$. The second choice determines whether $*T$ inverts the odd generator; this is $\beta(t)$.

Due to the Morita equivalence $M_2\mathbb{R} \sim \mathbb{R}$, several of the eight Morita classes are represented by algebras of lower dimension; for example, $C\ell_{1,0}\mathbb{R}$ instead of $M_2\mathbb{R} \otimes_{\mathbb{R}} C\ell_{1,0}\mathbb{R}$. Up to this substitution, the eight real-structured superalgebras we found are complexifications of the eight *central real super-division algebras* – real superalgebras with center $\mathbb{R}$ that are invertible under supertensor product up to graded Morita equivalence (Wall, 1964; Trimble, 2005). They constitute a set of representatives of the eight graded Morita classes of real superalgebras. These algebras appear in the second column of Figure 4.20, next to their invariants in the third column.

Another set of Morita class representatives is the Clifford algebras $C\ell_{n,0}\mathbb{R}$. In terms of these algebras, stacking is simple, as

$$C\ell_{n,0}\mathbb{R} \mathbin{\widehat{\otimes}} C\ell_{m,0}\mathbb{R} \simeq C\ell_{n+m,0}\mathbb{R} \tag{4.182}$$

and

$$C\ell_{n,0}\mathbb{R} \sim C\ell_{m,0}\mathbb{R} \qquad \text{for } n \equiv m \bmod 8. \tag{4.183}$$

Each central super-division algebra can be matched with the Clifford algebra $\mathbb{C}\ell_{n,0}\mathbb{R}$, $n < 8$ in its Morita class (Trimble, 2005), as in the first column of Figure 4.20. This determines a $\mathbb{Z}/8$ stacking law on central super-division algebras and their invariants that agrees with the more general law (4.174).

Physically speaking, the $\mathbb{Z}/8$ classification is generated by the time-reversal-invariant Majorana chain (Fidkowski and Kitaev, 2011; Fidkowski and Kitaev, 2010). While the symmetry protects pairs of dangling Majorana zero modes from being gapped out, turning on interactions can gap out these modes in groups of eight. Fidkowski and Kitaev formulate their stacking law in terms of three invariants that are equivalent to α, β, and γ. Their results match ours.

For contrast, we list the invariants of the corresponding bosonic phases in the rightmost column of Figure 4.20. There, H denotes the subgroup of unbroken symmetries and ω denotes 2-cocycle characterizing the SPT order. These invariants can be obtained from the fermionic invariants (Kapustin, Turzillo, and You, 2018). We observe that invertibility is not preserved by bosonization; in particular, only the fermionic SREs with $\gamma = 0$ become bosonic SREs. The four bosonic SRE phases have a $\mathbb{Z}/2 \times \mathbb{Z}/2$ stacking law. We also include the two *non-central* super-division algebras $\mathbb{C}$ and $\mathbb{C}\ell_1$ at the bottom of the table. These correspond to symmetry-breaking (SB) phases.

$C\ell_{n,0}$	A_{div}	α, β, γ	fermionic	bosonic	(H, ω)
0	$\mathbb{R}$	$0, 0, 0$	trivial	trivial	$(\mathcal{G}, 0)$
1	$C\ell_{1,0}$	$0, 0, 1$	SRE	SB	$(\mathbb{Z}_2^T, 0)$
2	$C\ell_{2,0}$	$0, 1, 0$	SRE	SPT	$(\mathcal{G}, \omega_1)$
3	$\mathbb{H} \otimes C\ell_{0,1}$	$1, 1, 1$	SRE	mixed	$(\mathbb{Z}_2^{\text{diag}}, \alpha)$
4	$\mathbb{H}$	$1, 0, 0$	SRE	SPT	$(\mathcal{G}, \omega_2)$
5	$\mathbb{H} \otimes C\ell_{1,0}$	$1, 0, 1$	SRE	mixed	$(\mathbb{Z}_2^T, \alpha)$
6	$C\ell_{0,2}$	$1, 1, 0$	SRE	SPT	$(\mathcal{G}, \omega_1 + \omega_2)$
7	$C\ell_{0,1}$	$0, 1, 1$	SRE	SB	$(\mathbb{Z}_2^{\text{diag}}, 0)$
-	$\mathbb{C}$	-	SB	SB	$(\mathbb{Z}_2^F, 0)$
-	$\mathbb{C}\ell_1$	-	SB	SB	$(1, 0)$

Figure 4.20: The ten-fold way of $\mathbb{Z}_2^F \times \mathbb{Z}_2^T$-symmetric fermionic phases.

Class DIII fermions: $\mathcal{G} = \mathbb{Z}_4^{FT}$

In the following, $\mathcal{G} = \mathbb{Z}_4^{FT}$ denotes the non-trivial extension of $G_b = \mathbb{Z}_2^T$ by fermion parity. Let us consider fermionic SRE phases with this symmetry. There are two distinct classes $\beta \in H^1(\mathbb{Z}_2^T, \mathbb{Z}/2)$, determined by $\beta(t) = 0$ and $\beta(t) = 1$. The trivial β has a single α, the trivial one, that satisfies $\delta_T \alpha = {}^1\!/\!_2 \beta \cup \rho$, up to the proper equivalence.[24] The nontrivial β also has a single compatible α, up to equivalence: $\alpha(t,t) = 1/4$.

The trivial phase is represented by the algebra $\mathbb{C}$ with trivial actions of p and t, as always. For the nontrivial phase, consider $A = \mathrm{End}(U)$, where P and T act on U as

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{and} \qquad M(t) = \begin{pmatrix} 0 & 1 \\ -i & 0 \end{pmatrix}. \tag{4.184}$$

Then the invariants can be recovered:

$$M(t)M(t)^{-1T} = e^{2\pi i/4}P \qquad \Rightarrow \qquad \alpha(t,t) = 1/4 \tag{4.185}$$

$$PM(t)P^T = e^{i\pi}M(t) \qquad \Rightarrow \qquad \beta(t) = 1. \tag{4.186}$$

According to the rule (4.174), stacking two copies of this phase results in the trivial phase:

$$(1/4, 1) \cdot (1/4, 1) = (1/4 + 1/4 + 1/2 \cdot 1 \cdot 1, 1 + 1) = (0, 0). \tag{4.187}$$

We find that fermionic SRE phases with symmetry $\mathbb{Z}_4^{FT}$ have a $\mathbb{Z}/2$ classification, in agreement with the condensed matter literature (Ryu et al., 2010; Kitaev, Lebedev, and Feigel'man, 2009). The nontrivial phase appears as a Majorana chain with two dangling modes protected by the symmetry.

Unitary $\mathbb{Z}/2$ symmetry

As a last set of examples, let us consider systems with a unitary bosonic symmetry group $G_0 = \mathbb{Z}/2$, in addition to time-reversal and fermion parity. There are many ways to organize these symmetries into a full symmetry class $(\mathcal{G}, p, x)$. Here, we consider the five abelian possibilities, which are listed with their fermionic and bosonic phase classifications in Figure 4.21. The first three have $G_b = \mathbb{Z}_2 \times \mathbb{Z}_2^T$, the last two $G_b = \mathbb{Z}_4^T$. In the two cases where the central extension of G_b by $\mathbb{Z}_2^F$ splits, we use a superscript γ to denote the subgroup of the fermionic classification that contains the odd phases.

[24]The cocycle $\alpha(t,t) = 1/2$ is nontrivial in $H^2(G_b, U(1)_T)$, but is trivialized by adding a 2-coboundary on $\mathcal{G}$ satisfying the proper conditions. See the Appendix for details.

symmetry class	fermionic	bosonic
$\mathbb{Z}_2 \times \mathbb{Z}_2^T \times \mathbb{Z}_2^F$	$\mathbb{Z}_4 \times \mathbb{Z}_8^\gamma$	$(\mathbb{Z}_2)^4$
$\mathbb{Z}_2 \times \mathbb{Z}_4^{FT}$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^T \times \mathbb{Z}_4^F$	$\mathbb{Z}_4$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_2^F \times \mathbb{Z}_4^T$	$\mathbb{Z}_2 \times \mathbb{Z}_4^\gamma$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathbb{Z}_8^{FT}$	$\mathbb{Z}_2$	$\mathbb{Z}_2$

Figure 4.21: Fermionic phases with unitary and anti-unitary symmetries.

Chapter 5

FREE VERSUS INTERACTING PHASES OF FERMIONS

5.1 Introduction

It is well-known by now that short-range-entangled (SRE) phases of free fermions on a lattice can be classified using K-theory (Kitaev, Lebedev, and Feigel'man, 2009), or equivalently using the topology of symmetric spaces (Schnyder et al., 2009; Ryu et al., 2010). Originally, the classification was done in the framework of the ten-fold way, where the only allowed symmetries are charge conservation, time-reversal, particle-hole symmetry, or a combination thereof, as explained in section 2.1. But the K-theory framework can also be extended to systems with more general symmetries, both on-site and crystallographic (Teo, Fu, and Kane, 2008; Mong, Essin, and Moore, 2010; Fu, 2011; Kruthoff et al., 2017; Freed and Moore, 2013; Ando and Fu, 2015). The answer is encoded in an abelian group, with the group operation corresponding to the stacking of phases.

The first goal of this chapter is to derive the classification of free fermionic SRE phases with a unitary on-site symmetry $\hat{G}$ in arbitrary dimensions. We show that in any dimension, representation-theoretic considerations reduce the problem to classifying systems of class D, A, and C. Since the solution of the latter problem is well-known, the key step in the derivation is the reduction from a general symmetry $\hat{G}$ to ten-fold symmetry classes. Such a reduction is not new and has been described in detail in Ref. (Heinzner, Huckleberry, and Zirnbauer, 2005). But since (Heinzner, Huckleberry, and Zirnbauer, 2005) works with complex fermions, and for our purposes it is more convenient to use Majorana fermions, we give a new proof of the reduction. The classification is described succinctly in Table 5.1. We have included in Table 5.2 the results of applying this general classification formula to some common symmetry groups.

When we consider systems with symmetries other than the ten-fold way symmetries, it is no longer useful to adopt the ten-fold way nomenclature. For example, a fermionic system with a $U(1) \times G$ symmetry, where the generator of $U(1)$ is the fermion number, can equally well be regarded as a symmetry-enriched class A system and as a symmetry-enriched class D system. On the other hand, the distinction between unitary and anti-unitary symmetries remains important. If we denote by

$\hat{G}$ the total symmetry group (including the fermion parity $\mathbb{Z}_2^F$), this information is encoded in a homomorphism

$$\rho : \hat{G} \to \mathbb{Z}_2^T. \tag{5.1}$$

We also need to specify an element $P \in \hat{G}$ which generates the subgroup $\mathbb{Z}_2^F$. This element satisfies $P^2 = 1$ and is central.[1] Since P is unitary, we must have $\rho(P) = 1$ (here we identify $\mathbb{Z}_2$ with the set $\{1, -1\}$). The symmetry of a fermionic system is encoded in a triplet $(\hat{G}, P, \rho)$. For example, class D systems correspond to a triplet $(\mathbb{Z}_2, -1, \rho_0)$, where ρ_0 is the trivial homomorphism (sends the whole $\hat{G}$ to the identity), while class A systems correspond to a triplet $(U(1), -1, \rho_0)$. In this chapter, we study only systems with unitary symmetries, i.e. we always set $\rho = \rho_0$. We allow $\hat{G}$ to be an arbitrary compact Lie group, with the exception of section 3.3, where $\hat{G}$ is assumed to be finite.

The reader might notice that many of our results on the classification of free systems can be naturally expressed in terms of equivariant K-theory. The connection between free systems with an arbitrary (not necessarily on-site or unitary) symmetry and equivariant K-theory has been studied in detail in (Freed and Moore, 2013). However, in this paper we prefer to use more elementary methods, such as representation theory of compact groups. This has the advantage of making clear the physical meaning of K-theory invariants, which is crucial for the purpose of comparison with interacting systems.

The second goal of this chapter is to study the relationship between the free classification and the classification of short-range entangled interacting fermionic phases. When interactions of arbitrary strength are allowed, the classification of SRE phases of fermions is much more complicated, but in low dimensions[2], the answer is known for an arbitrary finite on-site symmetry $\hat{G}$ (Chen, Gu, and Wen, 2011b; Fidkowski and Kitaev, 2011; Gu and Wen, 2014; Bhardwaj, Gaiotto, and Kapustin, 2017; Wang, Lin, and Gu, 2017; Kapustin, Turzillo, and You, 2018; Wang and Gu, 2018; Kapustin and Thorngren, 2017). It is also given by an abelian group, where the group operation is stacking.

Every free fermionic system can be regarded as an interacting one (where the quartic and higher order interaction terms are set to zero), and this gives a homomorphism from the abelian group of free SRE phases to the abelian group of interacting ones

[1] Centrality is equivalent to the assumption that all symmetries are bosonic, i.e. do not exchange bosons with fermions.

[2] An answer in an arbitrary number of dimensions was conjectured in (Kapustin et al., 2015).

(with the same symmetry). In general, this homomorphism is neither injective nor surjective. The homomorphism may have a non-trivial kernel because some non-trivial free SRE phases can be destabilized by interactions. It may fail to be surjective because some interacting SRE phases are *intrinsically interacting*, *i.e.* cannot be realized by free fermions. The most familiar example of the former phenomenon occurs in 1d systems of class BDI (Fidkowski and Kitaev, 2011): while free SRE phases in this symmetry class are classified by $\mathbb{Z}$, the interacting ones are classified by $\mathbb{Z}_8$. An example of the latter phenomenon apparently occurs in dimension 6, where the cobordism classification of systems in class D predicts $\mathbb{Z} \times \mathbb{Z}$, while the free phases in the same symmetry class are classified by $\mathbb{Z}$. We study both phenomena more systematically in low dimensions. In particular, we will see that already in zero and one dimensions, there exist fermionic SRE phases protected by a unitary symmetry which cannot be realized by free fermions.

To address such questions, it is very useful to have an efficient way to compute the interacting invariants of any given band Hamiltonian with any on-site symmetry $\hat{G}$. One of the results of our paper is the computation of these invariants for arbitrary 0d and 1d band Hamiltonians. We also propose a partial answer in the 2d case. In the 1d case, we identify one of the invariants as a charge-pumping invariant.

The content of the chapter is as follows. In Section 5.2, we derive the classification of free SRE phases with a unitary symmetry $\hat{G}$ in an arbitrary number of dimensions. In particular, we show that for $d = 3$, all such phases are trivial. In Section 5.3 we describe the map from free to interacting SRE phases for $d = 0, 1,$ and 2. Appendices C.1 and C.2 contain some of the mathematical background. In Appendix C.3, we show that one of the invariants for free 1d SPT systems can be interpreted as a charge-pumping invariant.

5.2 Free fermionic systems with a unitary symmetry

Reduction to the ten-fold way

In this section, we show that the classification of free fermionic SRE systems with a unitary symmetry $\hat{G}$ in dimension d reduces to the classification of systems of class D, A, and C in the same dimension. The group $\hat{G}$ is assumed to be a compact Lie group. This includes finite groups as a special case. For simplicity, we show this for the case of 0d systems, from which the general case can be deduced. For systems of dimension $d > 0$, the Majorana fermions have an additional index (the coordinate label). Accordingly, all matrices except $r(\hat{g})$ (defined below) become infinite.

However, since the symmetry is on-site, all representation-theoretic manipulations remain valid, and the conclusions are unchanged.

Consider a general quadratic 0d Hamiltonian

$$H = \frac{i}{2} A_{IJ} \Gamma^I \Gamma^J, \tag{5.2}$$

where A_{IJ}, $I, J = 1, \ldots, 2N$ is a real skew-symmetric matrix and Γ^I are Majorana fermions satisfying

$$\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}. \tag{5.3}$$

This Hamiltonian is known as the Majorana representation of the Bogoliubov-de Gennes Hamiltonian and may be straightforwardly obtained from its more familiar complex-fermion representation as, for example, in Ref. (Chiu et al., 2016). Suppose the Hamiltonian is invariant under a linear action of a group $\hat{G}$:

$$\hat{g} : \Gamma^I \mapsto \hat{R}(\hat{g})^I_J \Gamma^J. \tag{5.4}$$

This defines a homomorphism $\hat{R} : \hat{G} \to O(2N)$.

Let us decompose $\hat{R}$ into real irreducible representations of $\hat{G}$. Suppose the irreducible representation r_α enters with multiplicity n_α. The sum of all these copies of r_α will be called a block. It is clear that the Hamiltonian can only couple the fermions in the same block, so the matrix A is block-diagonal.

Let us focus on a particular block corresponding to an irreducible real representation r. There are three kinds of real irreducibles which are distinguished by the set of matrices which commute with all $r(\hat{g})$, $\hat{g} \in \hat{G}$ (Bröcker and Dieck, 1985). This set is known as the commutant of r. It is easy to see that it is closed under multiplication, and thus the commutant is an algebra. By Schur's lemma, if r is irreducible, the commutant must be a real division algebra, so we have irreducibles of type $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$, corresponding to the algebras of real numbers, complex numbers, and quaternions.[3] The corresponding block A_r can be thought of as an operator on the space $r \otimes \mathbb{R}^n$, where n is the multiplicity of r. $\hat{G}$-invariance of the Hamiltonian implies that this operator commutes with the $\hat{G}$-action. The resulting constraint on A_r depends on the type of the representation r.

If r is of $\mathbb{R}$-type, only scalar matrices commute with all $r(\hat{g})$. (Hence $r \otimes_{\mathbb{R}} \mathbb{C}$ is a complex irreducible representation of $\hat{G}$. This is an equivalent characterization of

[3]The reader may be more familiar with Schur's lemma for complex representations, where there is only one possible commutant: the unique complex division algebra $\mathbb{C}$ corresponding to matrices proportional the identity.

$\mathbb{R}$-type irreducibles.) Hence A_r must have the form

$$A_r = 1 \otimes \mathcal{A}, \tag{5.5}$$

where $\mathcal{A}$ is a real skew-symmetric matrix of size $n \times n$. There are no further constraints on $\mathcal{A}$, so such a block can be thought of as describing $\dim r$ copies of a system of class D, i.e. a free fermion system whose only symmetry is the fermion parity. In particular,

$$H = \frac{i}{2} A_{IJ} \Gamma^I \Gamma^J = \frac{i}{2} \sum_{\mu}^{\dim r} \mathcal{A}_{ij} \Gamma^i_\mu \Gamma^j_\mu \tag{5.6}$$

where we have relabeled fermions $\Gamma^I \mapsto \Gamma^i_\mu$ by indices $i = 1, \ldots, n$ and $\mu = 1, \ldots, \dim r$.

If r is of $\mathbb{C}$-type, then the algebra of matrices commuting with all $r(\hat{g})$ is spanned by 1 and an element J satisfying $J^2 = -1$. Since J^T must be proportional to J, this means that $J^T = -J$. The most general $\hat{G}$-invariant A_r must have the form

$$A_r = 1 \otimes \mathcal{A} + J \otimes C, \tag{5.7}$$

where $\mathcal{A}$ is skew-symmetric and C is symmetric. We can equivalently parametrize such a Hamiltonian by a complex Hermitian matrix

$$h = C + i\mathcal{A} \; . \tag{5.8}$$

Upon complexification, we can decompose r into eigenspaces of J with eigenvalues $\pm i$. These eigenspaces are complex irreducible representations of $\hat{G}$, and it is clear that they are conjugate to each other. We will denote them q and $\bar{q}$. (An equivalent definition of a $\mathbb{C}$-type representation is that $r \otimes_{\mathbb{R}} \mathbb{C}$ is a sum of two complex irreducible representations q and $\bar{q}$ which are complex-conjugate and inequivalent). The $n \cdot \dim r$ Majorana fermions can be equivalently described by $\frac{1}{2} n \cdot \dim r$ complex fermions Ψ^a_k, $a = 1, \ldots, n$, $k = 1, \ldots, \frac{1}{2} \dim r$ satisfying the commutation relations

$$\{\Psi^a_k, \bar{\Psi}^b_l\} = \delta^a_b \delta^l_k . \tag{5.9}$$

In terms of these fermions, the Hamiltonian takes the form

$$H = \sum_{k,a,b} \bar{\Psi}^b_k h^b_a \Psi^a_k . \tag{5.10}$$

Thus a $\mathbb{C}$-type block can be thought of as describing $\dim q = \frac{1}{2} \dim r$ copies of a system of class A.

If r is of $\mathbb{H}$-type, then the algebra of matrices commuting with all $r(\hat{g})$ is spanned by 1 and three elements I, J, K which are skew-symmetric and obey the relations

$$I^2 = J^2 = K^2 = -1, \quad IJ = K. \tag{5.11}$$

Accordingly, A_r must have the form

$$A_r = 1 \otimes \mathcal{A} + I \otimes \mathcal{B} + J \otimes C + K \otimes \mathcal{D}, \tag{5.12}$$

where $\mathcal{A}$ is skew-symmetric and $\mathcal{B}, C, \mathcal{D}$ are symmetric. Equivalently, we can introduce a Hermitian $2n \times 2n$ matrix

$$Z = \begin{pmatrix} C + i\mathcal{A} & \mathcal{B} + i\mathcal{D} \\ \mathcal{B} - i\mathcal{D} & -(C + i\mathcal{A})^T \end{pmatrix}. \tag{5.13}$$

This is the most general Hermitian matrix satisfying the particle-hole (PH) symmetry condition

$$C^\dagger Z^T C = -Z, \tag{5.14}$$

where $C = i\sigma_2 \otimes 1$. Since $C^* C = -1$, such a PH-symmetric system belong to class C.

To make this relationship with class C systems explicit, we again decompose $r \otimes_{\mathbb{R}} \mathbb{C}$ into a pair of complex-conjugate representations q and $\bar{q}$. These two representations are equivalent, with the intertwiner being given by the tensor I. We also can think of I as a non-degenerate skew-symmetric pairing $q \otimes q \to \mathbb{C}$. This implies that $\dim q$ is divisible by 2 (and hence $\dim r$ is divisible by four). As in the $\mathbb{C}$-type case, we can describe the system by $n \cdot \dim q$ complex fermions. However, the presence of an $\hat{G}$-invariant tensor I means that the most general $\hat{G}$-invariant Hamiltonian is

$$H = \bar{\Psi}(1 \otimes h)\Psi + \frac{1}{2}\left(\Psi^T (I \otimes Y)\Psi + h.c.\right), \tag{5.15}$$

where h is a Hermitian matrix, and Y is a complex symmetric matrix. This is a BdG Hamiltonian, which can be re-written in terms of Dirac-Nambu fermions

$$\Phi = \begin{pmatrix} \Psi \\ (I \otimes 1)\bar{\Psi}^T \end{pmatrix}. \tag{5.16}$$

The Dirac-Nambu spinors are defined so that the upper and lower components transform in the same way under $\hat{G}$. They take values in $q \otimes \mathbb{C}^2 \otimes \mathbb{C}^n$, where $\mathbb{C}^2$ is the Dirac-Nambu space. The particle-hole (PH) symmetry acts by

$$C : \Phi \mapsto (I \otimes \sigma_1 \otimes 1)\, \bar{\Phi}^T \tag{5.17}$$

and satisfies $C^2 = -1$. In terms of Dirac-Nambu spinors, the Hamiltonian takes the form

$$H = \bar{\Phi}(1 \otimes Z)\Phi, \tag{5.18}$$

where

$$Z = \frac{1}{2}\begin{pmatrix} h & -Y^\dagger \\ -Y & -h^T \end{pmatrix}. \tag{5.19}$$

Such matrices describe the most general class C system. Thus an $\mathbb{H}$-type block can be thought of as describing $\dim q = \frac{1}{2}\dim r$ copies of a system of class C.

Classification of free SRE phases with a unitary symmetry

We always make the physically reasonable assumption that the generator of $\mathbb{Z}_2^F$ acts on all fermions by negation, i.e.

$$\hat{R}(P) = -1. \tag{5.20}$$

The same must be true for all irreducible representations r_α which appear with nonzero multiplicity. We will call such irreducible representations *allowed*. The set of all irreducible real representations of a compact group $\hat{G}$ will be denoted $\mathrm{Irr}(\hat{G})$, while the set of all allowed irreducible real representations will be denoted $\mathrm{Irr}'(\hat{G})$. The set of allowed irreducible representations of type K ($K = \mathbb{R}, \mathbb{C}, \mathbb{H}$) will be denoted $\mathrm{Irr}'(\hat{G}, K)$. If $\hat{G} = \mathbb{Z}_2^F \times G$, we can identify $\mathrm{Irr}'(\hat{G}, K)$ with the set $\mathrm{Irr}(G, K)$.

Let us recall the classification of class D, A, and C systems from the periodic table. Here we are listing only the "strong" invariants which do not depend on translational invariance.

	0	1	2	3	4	5	6	7
Class D ($\mathbb{R}$-type)	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$				$\mathbb{Z}$	
Class A ($\mathbb{C}$-type)	$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$		$\mathbb{Z}$	
Class C ($\mathbb{H}$-type)			$\mathbb{Z}$		$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	

These results together with those of the previous subsection allow us to deduce the classification of free fermionic SREs with an arbitrary unitary symmetry $\hat{G}$. In the physically interesting dimensions $d \leq 3$, the classification is given in Table 5.1. This does not contradict the fact that there are interesting interacting fermionic 3d SREs.

$d = 0$	$\oplus_{r\in\mathrm{Irr}'(\hat{G},\mathbb{R})}\mathbb{Z}_2 \times \oplus_{r\in\mathrm{Irr}'(\hat{G},\mathbb{C})}\mathbb{Z}$
$d = 1$	$\oplus_{r\in\mathrm{Irr}'(\hat{G},\mathbb{R})}\mathbb{Z}_2$
$d = 2$	$\oplus_{r\in\mathrm{Irr}'(\hat{G})}\mathbb{Z}$
$d = 3$	trivial

Table 5.1: The classification of free phases protected by on-site unitary symmetry $\hat{G}$ in physical dimensions.

In what follows, an invariant attached to a particular irreducible representation r_α will be denoted ϱ_α. Depending on the spatial dimension and the type of r_α, ϱ_α will take values either in $\mathbb{Z}_2$ or $\mathbb{Z}$. An invariant of free SRE phases will thus be a "vector" with components ϱ_α. If $\hat{G}$ is finite, then the number of allowed irreducible representations is finite, and the "vector" has a finite length. If $\hat{G}$ is a compact Lie group, the number of allowed irreducible representations may be infinite, and then the space of "vectors" has infinite dimension (although all but a finite number of ϱ_α are zero for a particular SRE phase). These vectors can be interpreted as elements of the (twisted) equivariant K-theory, whose relevance to the classification of gapped band Hamiltonians is explained in (Freed and Moore, 2013).

The above results can be simplified a bit when $\hat{G}$ is a product of G and $\mathbb{Z}_2^F$. In this case the sums over allowed representations of $\hat{G}$ can be replaced with the sums over all representations of G.

The $\mathbb{Z}$ and $\mathbb{Z}_2$ invariants that appear in K-theory are relative invariants; that is, they detect something non-trivial about the junction between two phases. If one chooses a phase to regard as trivial (typically the phase containing the product state ground state in dimension $d > 0$), the invariant for the junction of a phase $[H]$ with the trivial phase may be regarded as an absolute invariant of $[H]$.

Examples

Let us consider a few examples of free classifications for common symmetry groups.

- Superconductors with spin parity symmetry. $\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_2$. The action of $\mathbb{Z}_2^F$ on fermions is fixed, so we only need to choose the action of the second $\mathbb{Z}_2$. Overall, there are two allowed irreducible representations, both of them of $\mathbb{R}$-type. Thus free phases with this symmetry are classified by $\mathbb{Z}_2 \times \mathbb{Z}_2$ in 0d and 1d, and by $\mathbb{Z} \times \mathbb{Z}$ in 2d.

$\hat{G}$	$\mathbb{Z}_2^F$	$U(1)^F$	$SU(2)^F$	$\mathbb{Z}_2^F \times \mathbb{Z}_{2n}$	$\mathbb{Z}_{4n}^F$	$\mathbb{Z}_{4n+2}^F \simeq \mathbb{Z}_2^F \times \mathbb{Z}_{2n+1}$
$d = 0$	$\mathbb{Z}_2$	$\mathbb{Z}^{\mathbb{N}}$	0	$\mathbb{Z}_2^2 \times \mathbb{Z}^{n-1}$	$\mathbb{Z}^n$	$\mathbb{Z}_2 \times \mathbb{Z}^n$
$d = 1$	$\mathbb{Z}_2$	0	0	$\mathbb{Z}_2^2$	0	$\mathbb{Z}_2$
$d = 2$	$\mathbb{Z}$	$\mathbb{Z}^{\mathbb{N}}$	$\mathbb{Z}^{\mathbb{N}}$	$\mathbb{Z}^{n+1}$	$\mathbb{Z}^n$	$\mathbb{Z}^{n+1}$
$d = 3$	0	0	0	0	0	0

Table 5.2: Free classification results for some common symmetry groups.

- Charge-4e superconductors. $\hat{G} = \mathbb{Z}_4$, where the $\mathbb{Z}_2$ subgroup is fermion parity. $\mathbb{Z}_4$ has three irreducible real representations, of dimensions 1, 1, and 2, but only the 2-dimensional representation is allowed. It is of $\mathbb{C}$-type, hence free 0d and 2d phases with this symmetry are classified by $\mathbb{Z}$, while those in 1d have a trivial classification.

- $\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_4$. Allowed irreducible representations of $\hat{G}$ are equivalent to the 1, 1, and 2 dimensional irreducible representations of $G = \mathbb{Z}_4$. Therefore the 0d classification is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}$, the 1d classification is $\mathbb{Z}_2 \times \mathbb{Z}_2$, and the 2d classification is $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$.

- Class A insulators. $\hat{G} = U(1)$, with the obvious $\mathbb{Z}_2^F$ subgroup. There is one real representation for every non-negative integer, but only odd integers are allowed. All of these representations are of $\mathbb{C}$-type, so free 0d phases with this symmetry are classified by $\mathbb{Z}^{\mathbb{N}}$, that is, by a product of countably many copies of $\mathbb{Z}$. Note that although the symmetry is the same as for class A insulators, the classification is different. This is because it is usually assumed that complex fermions have charge 1 with respect to $U(1)$, while we allow arbitrary odd charges. In 1d, there are no free phases with this symmetry, while in 2d there is again a $\mathbb{Z}^{\mathbb{N}}$ classification.

- $\hat{G} = SU(2)$ with $\mathbb{Z}_2^F$ being the center. In this case, only representations of half-integer spin are allowed. All these representations are of $\mathbb{H}$-type, hence all free Hamiltonians with this symmetry are in the same (trivial) phase in both 0d and 1d. In 2d, the classification is $\mathbb{Z}^{\mathbb{N}}$.

5.3 Interacting invariants of band Hamiltonians

Zero dimension

The only invariant of a general gapped fermionic 0d Hamiltonian with a unique ground state and symmetry $\hat{G}$ is the charge[4] of the ground state

$$\omega \in H^1(\hat{G}, U(1)). \tag{5.21}$$

As usual, this charge suffers from ambiguities, so it is better to consider the relative charge of two ground states. Let us compute this relative charge for the free Hamiltonian corresponding to a representation $\hat{R}$. We decompose it into irreducibles, compute the charge in each sector separately, and then add up the results.

Let us start with $\mathbb{C}$-type representations. The corresponding Hamiltonian is described by a non-degenerate Hermitian matrix h of size $n_r \times n_r$. Suppose we are given two such matrices h and h', with the number of negative eigenvalues m_r and m'_r. We can consider a path deforming h' to h. Every time an eigenvalue of h' changes from a positive one to a negative one, the ground state is multiplied by an operator

$$\prod_a \bar{\Psi}^a_i v^i, \tag{5.22}$$

where v^i is the corresponding eigenvector of h. Since $\bar{\Psi}^a_i$ transforms under $\hat{g} \in \hat{G}$ as

$$\bar{\Psi}^a_i \mapsto \bar{q}(\hat{g})^a_b \bar{\Psi}^b_i, \tag{5.23}$$

the above operator has charge $\det \bar{q}(\hat{g})$. Thus a $\mathbb{C}$-type irreducible representation r_α contributes a relative charge

$$(\det \bar{q}_\alpha(\hat{g}))^{\varrho_\alpha}, \tag{5.24}$$

where $\varrho_\alpha = m_\alpha - m'_\alpha \in \mathbb{Z}$ is the relative topological invariant of a pair of gapped class A Hamiltonians.

For an $\mathbb{R}$-type representation r, the Hamiltonian is described by a non-degenerate skew-symmetric real matrix $A_{r,ij}$ of size $n_r \times n_r$. Any two such matrices A_r and A'_r are related by

$$A_r = O^T A'_r O, \quad O \in O(n_r). \tag{5.25}$$

To compute the relative charge of the ground states, we recall that the orthogonal group is generated by hyperplane reflections. Without loss of generality, we can

[4] Recall that $H^1(\hat{G}, U(1))$ is the group of one-dimensional unitary complex representations of $\hat{G}$.

assume that the hyperplane is orthogonal to the 1st coordinate axis. Let us compute the change in the ground state charge due to a reflection of the 1st coordinate axis. This corresponds to the following map on fermions:

$$\Gamma^1_a \mapsto -\Gamma^1_a, \quad a = 1, \dots, \dim r, \tag{5.26}$$

while the rest of the fermions remain invariant. We need to treat separately the cases when $\dim r$ is even and when it is odd.

If $\dim r$ is even, the map on fermions is in $SO(n_r \cdot \dim r)$, even though it arises from an element of $O(n_r)$ with determinant -1. On the Hilbert space, this map is represented by a bosonic operator proportional to

$$\prod_{a=1}^{\dim r} \Gamma^1_a. \tag{5.27}$$

This operator carries charge $\det r(\hat{g})$ under $\hat{g} \in \hat{G}$, hence the relative charge of the ground state corresponding to a hyperplane reflection is $\det r(\hat{g})$.

If $\dim r$ is odd, the map on fermions is an orthogonal transformation with determinant -1, and thus must be represented on the Hilbert space by a fermionic operator. This fermionic operator is proportional to

$$\prod_{j=2}^{n_r} \prod_{a=1}^{\dim r} \Gamma^j_a. \tag{5.28}$$

It carries charge $(\det r(\hat{g}))^{n_r - 1} = \det r(\hat{g})$ under $\hat{g} \in \hat{G}$. Hence the relative charge of the ground state is again $\det r(\hat{g})$.

We conclude that when $O \in O(n_r)$ is a hyperplane reflection, the relative charge of the ground state under $\hat{g} \in \hat{G}$ is $\det r(\hat{g})$. Since $\det r(\hat{g}) = \pm 1$ and every element of $SO(n_r)$ is a product of an even number of hyperplane reflections, this implies that the relative charge is trivial when $O \in SO(n_r)$. Since every element of $O(n_r)$ is a product of a hyperplane reflection and an element of $SO(n_r)$, the relative charge of the ground state for an O which is not in $SO(n_r)$ is $\det r(\hat{g})$.

To summarize, the relative charge contribution from an $\mathbb{R}$-type representation r_α is

$$(\det r_\alpha(\hat{g}))^{\varrho_\alpha}, \tag{5.29}$$

where $\varrho_\alpha \in \mathbb{Z}_2$ is the relative invariant of a pair of gapped class D Hamiltonians.

Finally, $\mathbb{H}$-type representations do not contribute to the relative charge since all 0d class C systems are deformable into each other.

In summary, the map from free to interacting phases in 0d is

$$\{\varrho_\alpha\} \mapsto \omega(\hat{g}) = \prod_{\alpha \in Irr'(\hat{G},\mathbb{R})} (\det r_\alpha(\hat{g}))^{\varrho_\alpha} \prod_{\alpha \in Irr'(\hat{G},\mathbb{C})} (\det \bar{q}_\alpha(\hat{g}))^{\varrho_\alpha}. \tag{5.30}$$

In what follows, we often find it more convenient to identify $U(1)$ with $\mathbb{R}/\mathbb{Z}$, i.e. write the abelian group operation on 1-cocycles additively rather than multiplicatively. This amounts to taking the logarithm of both sides of (5.30) and dividing by $2\pi i$. Then ω becomes as sum of two terms, $\omega = \omega_1 + \omega_2$. The first term

$$\omega_1(\hat{g}) = \sum_{\alpha \in \mathrm{Irr}'(\hat{G},\mathbb{R})} \frac{1}{2\pi i} \varrho_\alpha \log\det r_\alpha(\hat{g}) \tag{5.31}$$

can be interpreted as the weighted sum of the 1st Stiefel-Whitney classes of the representations r_α (see Appendix C.2 for an explanation of this terminology). More precisely, the 1st Stiefel-Whitney class $w_1(r_\alpha)$ is an element of $H^1(\hat{G}, \mathbb{Z}_2)$, while ω_1 involves the corresponding class in $H^1(\hat{G}, \mathbb{R}/\mathbb{Z})$ which we denote $w_1^{U(1)}(r_\alpha)$:

$$\omega_1 = \sum_{\alpha \in \mathrm{Irr}'(\hat{G},\mathbb{R})} \varrho_\alpha w_1^{U(1)}(r_\alpha) \in H^1(\hat{G}, \mathbb{R}/\mathbb{Z}). \tag{5.32}$$

The 2nd term which arises from $\mathbb{C}$-type representations can be interpreted in terms of the 1st Chern class of the complex representations q_α:

$$\omega_2 = \sum_{\alpha \in \mathrm{Irr}'(\hat{G},\mathbb{C})} \beta^{-1}(c_1(\varrho_\alpha q_\alpha)) \in H^1(\hat{G}, \mathbb{R}/\mathbb{Z}). \tag{5.33}$$

Here β^{-1} is the inverse of the Bockstein isomorphism $\beta : H^1(\hat{G}, \mathbb{R}/\mathbb{Z}) \to H^2(\hat{G}, \mathbb{Z})$. In the 0d case, it seems superfluous to express determinants in terms of Stiefel-Whitney and Chern classes, but in higher dimensions, characteristic classes of representations become indispensable. They are briefly reviewed in Appendix C.2.

It is clear that the map from $\{\varrho_\alpha\}$ to ω is many-to-one for almost all $\hat{G}$. In fact, for Lie group symmetries, such as $U(1)$ or $SU(2)$, a single interacting phase corresponds to an infinite number of free phases.

More surprisingly, the map may fail to be surjective. A class $\omega \in H^1(\hat{G}, U(1))$ defines a one-dimensional complex representation q of $\hat{G}$. If this representation is allowed (i.e. if $\omega(P) = -1$), we can take a complex fermion $\bar{\Psi}$ and its Hermitian conjugate Ψ and let them transform in the representations q and $\bar{q}$, respectively. Now the two $\hat{G}$-invariant Hamiltonians

$$H_{\pm} = \pm\left(\bar{\Psi}\Psi - \frac{1}{2}\right) \tag{5.34}$$

have relative ground-state charge ω. But if the representation q is not allowed, $\omega(P) = 1$, then the situation is more complicated. For certain $\hat{G}$, there are no allowed one-dimensional representations at all, but one could try to use higher-dimensional allowed representation to get the relative ground-state charge ω.

Let us exhibit an example of a group $\hat{G}$ where certain relative charges ω cannot be obtained from free systems. This shows that the map from free to interacting 0d phases is not surjective in general. Consider extending the group $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ by $\mathbb{Z}_2$. If the extension class in $H^2(G, \mathbb{Z}_2)$ maps to a non-trivial element in $H^2(G, U(1))$, the group $\hat{G}$ may be presented in terms of generators A, B, P, where P is central and

$$P^2 = A^4 = B^4 = 1 \qquad \text{and} \qquad AB = PBA. \tag{5.35}$$

The group of one-dimensional representations of $\hat{G}$ is then the same as the group of one-dimensional representations of G, i.e. $\mathbb{Z}_4 \times \mathbb{Z}_4$, defined by $q(A), q(B) \in \{\pm 1, \pm i\}$. All sixteen of these are disallowed, as $q(P) = +1$. Up to equivalence, only four irreducible representations remain. They are two-dimensional and of the form $q(P) = -\mathbb{1}_2$ (allowed), $q(A) = i^a \sigma_z$, and $q(B) = i^b \sigma_x$, for $a, b \in \{0, 1\}$. Each is related to a complexification of a real irreducible representation r by $r_{\mathbb{C}} = q \oplus \bar{q}$ and has $\det q(\hat{g}) \in \{\pm 1\}$. This means that twelve out of sixteen cocycles (those with $\omega(\hat{g}) = \pm i$ for some $\hat{g}$) do not arise from free systems.

One dimension

Let us begin by recalling invariants of interacting fermionic SRE phases in 1d and their interpretation in terms of boundary zero modes. Any fermionic 1d SRE phase has an invariant $\gamma \in \mathbb{Z}_2$ (Fidkowski and Kitaev, 2011). (From now on, we will write $\mathbb{Z}_2$ additively, i.e. identify it with the set $\{0, 1\}$, unless stated otherwise.) It tells us whether the number of fermionic zero modes on the boundary is even or odd. Algebraically, if $\gamma = 0$, the algebra of boundary zero modes A_b is a matrix algebra, while for $\gamma = 1$, it is a sum of two matrix algebras. In both cases, A_b is simple provided we regard it as a $\mathbb{Z}_2^F$-graded algebra. In the case $\gamma = 0$, the graded center of A_b is isomorphic to $\mathbb{C}$, while for $\gamma = 1$, it is isomorphic to $C\ell(1)$. The odd generator of $C\ell(1)$ is denoted $\hat{Z}$.

If the system also has a unitary symmetry $\hat{G}$, then there are further invariants whose form depends on the value of γ (Fidkowski and Kitaev, 2011). If $\gamma = 0$, the additional invariant is $\hat{\alpha} \in H^2(\hat{G}, U(1))$. If $\gamma = 1$, the additional invariants are a homomorphism $\mu : \hat{G} \to \mathbb{Z}_2$ such that $\mu(P) = 1$ (the generator of $\mathbb{Z}_2$) and $\alpha \in H^2(G, U(1))$. A homomorphism μ allows one to define an isomorphism $\hat{G} \simeq G \times \mathbb{Z}_2^F$ as follows:

$$\hat{g} \mapsto (g, \mu(\hat{g})). \tag{5.36}$$

So if $\hat{G}$ is not isomorphic to the product $G \times \mathbb{Z}_2^F$, the case $\gamma = 1$ is impossible.

Note that there is a homomorphism $H^2(\hat{G}, U(1)) \to H^1(G, \mathbb{Z}_2)$ whose kernel is non-canonically isomorphic to $H^2(G, U(1))$. To see this, let us define the group law on $\hat{G}$ using a $\mathbb{Z}_2$-valued 2-cocycle ρ on G:

$$(g, \epsilon) \circ (g', \epsilon') = (gg', \epsilon + \epsilon' + \rho(g, g')), \quad g, g' \in G, \quad \epsilon, \epsilon' \in \{0, 1\}. \tag{5.37}$$

Then $\hat{\alpha}$ can be parameterized by a pair of cochains $(\alpha, \beta) \in C^2(G, U(1)) \times C^1(G, \mathbb{Z}_2)$ satisfying $\delta\beta = 0$ and $\delta\alpha = \frac{1}{2}\rho \cup \beta$, modulo $\alpha \mapsto \alpha + \delta\lambda$, $\lambda \in C^1(G, U(1))$. The map from $H^2(\hat{G}, U(1))$ to $H^1(G, \mathbb{Z}_2)$ sends the pair (α, β) to β.

The boundary interpretation of the additional invariants also depends on whether $\gamma = 0$ or $\gamma = 1$. For $\gamma = 0$, the algebra A_b is a matrix algebra, and therefore $\hat{G}$ acts on it by conjugation:

$$\hat{g} : a \mapsto V(\hat{g}) a V(\hat{g})^{-1}, \quad a \in A_b. \tag{5.38}$$

One can even choose the invertible elements $V(\hat{g}) \in A_b$ to be unitary (A_b is actually a C^*-algebra, so the notion of a unitary element makes sense). The elements $V(\hat{g})$ are well-defined up to a $U(1)$ factor and satisfy

$$V(\hat{g}) V(\hat{g}') = \hat{\alpha}(\hat{g}, \hat{g}') V(\hat{g}\hat{g}'), \tag{5.39}$$

where $\hat{\alpha}$ is a 2-cocycle on $\hat{G}$.

On the other hand, if $\gamma = 1$, then the same considerations apply to the even part of the graded algebra A_b, and one gets an invariant $\alpha \in H^2(G, U(1))$ in the same way. In addition, one can ask how the group $\hat{G}$ acts on the odd central element $\hat{Z} \in A_b$. One must have

$$\hat{g} : \hat{Z} \mapsto (-1)^{\mu(\hat{g})} \hat{Z}, \tag{5.40}$$

where $\mu : \hat{G} \to \mathbb{Z}_2$ is a homomorphism satisfying $\mu(P) = 1$.

As explained above, free SRE 1d systems with symmetry $\hat{G}$ are classified by a sequence of invariants $\varrho_\alpha \in \mathbb{Z}_2$, one for each real irreducible representation of $\hat{G}$ of $\mathbb{R}$-type. The physical meaning of ϱ_α is simple. The group $\hat{G}$ acts on the boundary zero modes (assumed to form a Clifford algebra) via a real representation[5]

$$\mathcal{R} = \oplus \nu_\alpha r_\alpha. \tag{5.41}$$

The integer ν_α reduced modulo 2 is the free topological invariant ϱ_α discussed in Section 5.2.

Let us now describe the map from free to interacting invariants. For a free system, the algebra of boundary zero modes is $A_b = C\ell(M)$, so one has $\gamma = M \bmod 2$. Equivalently, using the decomposition (5.41), we get

$$\gamma = \sum_\alpha \varrho_\alpha \dim r_\alpha \quad \bmod 2. \tag{5.42}$$

Now let us determine the remaining invariants. Consider the case $\gamma = 0$ first. Then $O(M)$ is a non-trivial extension of $SO(M)$ by $\mathbb{Z}_2$. We can interpret $A_b = C\ell(M)$ as the algebra of operators on a Fock space of dimension $2^{M/2}$, and the group $\hat{G}$ acts projectively on this space. The cohomology class of the corresponding cocycle is $\hat{\alpha}$. Clearly, it is completely determined by the representation $\mathcal{R} : \hat{G} \to O(M)$.

From the group-theoretic viewpoint, a projective action of $\hat{G}$ on the Fock space is the same as a homomorphism $\hat{G} \to Pin_c(M)$, where $Pin_c(M)$ is a certain non-trivial extension of $O(M)$ by $U(1)$. $Pin_c(M)$ and related groups are reviewed in Appendix C.1. Thus $\hat{\alpha}$ is the obstruction to lifting $\mathcal{R}$ to a homomorphism $\hat{G} \to Pin_c(M)$. As discussed in Appendix C.2, this obstruction is the image of the 2nd Stiefel-Whitney class of $\mathcal{R}$ under the homomorphism $H^2(\hat{G}, \mathbb{Z}_2) \to H^2(\hat{G}, U(1))$. We denote it $w_2^{U(1)}(\mathcal{R})$. The Whitney formula for Stiefel-Whitney classes says

$$w_2(\mathcal{R}) = w_2\left(\oplus \nu_\alpha r_\alpha\right) = \sum_\alpha \varrho_\alpha w_2(r_\alpha) + \sum_{\alpha<\beta} \varrho_\alpha \varrho_\beta w_1(r_\alpha) \cup w_1(r_\beta). \tag{5.43}$$

Therefore

$$\hat{\alpha} = w_2^{U(1)}(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_2^{U(1)}(r_\alpha) + \sum_{\alpha<\beta} \frac{1}{2} \varrho_\alpha \varrho_\beta w_1(r_\alpha) \cup w_1(r_\beta). \tag{5.44}$$

[5] One should not confuse the "boundary" representation $\mathcal{R}$ with the on-site representation $\hat{R}$. The former can be odd-dimensional, while the latter is always even-dimensional. Also, $\hat{R}$ takes values in $SO(2N)$, while $\mathcal{R}$ in general takes values in the orthogonal group.

Note that $Pin_c(M)$ is a $\mathbb{Z}_2$-graded group, i.e. it is equipped with a homomorphism to $\mathbb{Z}_2$. The value of this homomorphism tells us if $V(\hat{g})$ is an even or odd element in $C\ell(M)$. It is easy to see that this homomorphism is precisely $\beta(g)$. On the other hand, as explained in Appendix C.2, the said homomorphism is simply $\det\mathcal{R}(\hat{g})$. Thus

$$\beta = w_1(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_1(r_\alpha). \tag{5.45}$$

In Appendix C.3, we give an alternate characterization of β as a charge-pumping invariant.

Now consider the case $\gamma = 1$, where $A_b \simeq C\ell(M)$ with odd M. In agreement with (Fidkowski and Kitaev, 2011), the map $\hat{g} \mapsto \det\mathcal{R}(\hat{g})$ defines a splitting of $\hat{G}$, i.e. an isomorphism $G \times \mathbb{Z}_2^F \simeq \hat{G}$. This means

$$\mu = w_1(\mathcal{R}) = \sum_\alpha \varrho_\alpha w_1(r_\alpha). \tag{5.46}$$

We can define a new representation $\tilde{\mathcal{R}} : G \to SO(M)$ by

$$\tilde{\mathcal{R}}(g) = \mathcal{R}(\hat{g})\det\mathcal{R}(\hat{g}). \tag{5.47}$$

Here $\hat{g} \in \hat{G}$ is any lift of $g \in G$. Thus we get a homomorphism

$$G \times \mathbb{Z}_2^F \to O(M) \simeq SO(M) \times \mathbb{Z}_2^F, \quad (g, \epsilon) \mapsto (\tilde{\mathcal{R}}(g), \epsilon). \tag{5.48}$$

By definition, α is the obstruction for lifting $\tilde{\mathcal{R}}$ to a homomorphism $G \to Spin_c(M)$. Thus

$$\alpha = w_2^{U(1)}(\tilde{\mathcal{R}}). \tag{5.49}$$

Using a formula for Stiefel-Whitney classes of a tensor product (see Appendix C.2), one can show that $w_2(\tilde{\mathcal{R}}) = w_2(\mathcal{R})$, and thus one can also write

$$\alpha = w_2^{U(1)}(\mathcal{R}). \tag{5.50}$$

We note that the map from free to interacting 1d SRE phases is compatible with the stacking law derived in (Kapustin, Turzillo, and You, 2018; Gaiotto and Kapustin, 2016). For example, if we consider for simplicity the case $\gamma = 0$, then the stacking law takes the form

$$\hat{\alpha} \circ \hat{\alpha}' = \hat{\alpha} + \hat{\alpha}' + \tfrac{1}{2}\beta \cup \beta'. \tag{5.51}$$

On the other hand, stacking two SRE systems characterized by representations $\mathcal{R}$ and $\mathcal{R}'$ gives an SRE system corresponding to the representation $\mathcal{R} \oplus \mathcal{R}'$. If we set

$\alpha = w_2^{U(1)}(\mathcal{R}) = \frac{1}{2}w_2(\mathcal{R})$ and $\beta = w_1(\mathcal{R})$, then the stacking law (5.51) follows from the Whitney formulas

$$w_1(\mathcal{R} \oplus \mathcal{R}') = w_1(\mathcal{R}) + w_1(\mathcal{R}'), \tag{5.52}$$

$$w_2(\mathcal{R} \oplus \mathcal{R}') = w_2(\mathcal{R}) + w_2(\mathcal{R}') + w_1(\mathcal{R}) \cup w_1(\mathcal{R}'). \tag{5.53}$$

It is clear that the map from free to interacting phases is not injective. Let us discuss surjectivity. We have seen that for free systems, the invariants $\hat{\alpha}$ and α are always of order 2. Hence to get an example of a fermionic SRE phase which cannot be realized by free fermions, it is sufficient to pick a $\hat{G}$ and a non-trivial 2-cocycle which is not of order 2. For example, if we take $\hat{G} = \mathbb{Z}_2^F \times \mathbb{Z}_3 \times \mathbb{Z}_3$, and take α to be any non-trivial element of $H^2(\mathbb{Z}_3 \times \mathbb{Z}_3, U(1)) = \mathbb{Z}_3$, then such a phase cannot be realized by free fermions.

One might hope that perhaps every $\hat{\alpha}$ or α of order 2 can be realized by free fermions, but this is not the case either. The reason for this is that for any orthogonal representation $\mathcal{R}$ of $\hat{G}$, the 2-cocycle $w_2(\mathcal{R})$ satisfies some relations (Strickland, n.d.). This is explained in Appendix C.2. These relations need not hold for a general 2-cocycle on $\hat{G}$. Unfortunately, the simplest example of $\hat{G}$ for which this happens is rather non-trivial (Gunarwardena, Kahn, and Thomas, 1989).

While not every fermionic 1d SRE phase can be realized by free fermions, every fermionic 1d SRE phase with $\hat{G} \simeq G \times \mathbb{Z}_2^F$ can be realized by stacking bosonic 1d SRE phases with free fermions. First, we can change γ of an SRE phase at will by stacking with the Kitaev chain. If we make $\gamma = 0$ by such stacking, then we can change β at will by stacking with two copies of the Kitaev chain on which the group G acts by

$$(\gamma_1, \gamma_2) \mapsto ((-1)^{\beta(g)}\gamma_1, \gamma_2). \tag{5.54}$$

Finally, since α is an arbitrary element of $H^2(G, U(1))$ in this case, one can change it at will by stacking with bosonic SRE phases with symmetry G.

Two dimensions

To every fermionic 2d SRE phase, one can attach an integer invariant κ. It measures the chiral central charge for the boundary CFT.

If the SRE has a unitary symmetry $\hat{G}$, there are further invariants. For simplicity, let us assume that we are given an isomorphism $\hat{G} \simeq G \times \mathbb{Z}_2^F$. We will also assume that G is finite, rather than merely compact. Then the invariants are a 1-cocycle

$\gamma \in H^1(G, \mathbb{Z}_2)$, a 2-cocycle $\beta \in H^2(G, \mathbb{Z}_2)$, and a 3-cochain $\alpha \in C^3(G, U(1))$ satisfying

$$\delta\alpha = \frac{1}{2}\beta \cup \beta. \tag{5.55}$$

There are certain non-trivial identifications on these data, see (Gu and Wen, 2014; Gaiotto and Kapustin, 2016). The abelian group structure corresponding to stacking the systems is also quite non-trivial. We just note for future use that if we ignore α, the group law is

$$(\beta, \gamma) + (\beta', \gamma') = (\beta + \beta' + \gamma \cup \gamma', \gamma + \gamma'). \tag{5.56}$$

The physical meaning of these invariants is somewhat complicated, with the exception of $\gamma(g)$: it measures the number of Majorana zero modes on a g-vortex, reduced modulo 2.

On the other hand, a free 2d SRE is characterized by a sequence of invariants $\varrho_\alpha \in \mathbb{Z}$, one for each real irreducible representation of G.

It is easy to determine the chiral central charge κ for such a free SRE. A basic system of class D has $\kappa = 1/2$. For example, a $p + ip$ superconductor has a single chiral Majorana fermion on the boundary which has chiral central charge $1/2$.[6] A basic system of class A has $\kappa = 1$. For example, the basic Chern insulator has a single chiral complex fermion on the boundary which has chiral central charge 1. Two basic class C systems[7] have chiral central charge 2. For example, two copies of the basic Chern insulator can be regarded as the basic class C system tensored with a two-dimensional representation of $SU(2)$, and thus has $\kappa = 2$. Consequently, the chiral central charge is given by

$$\kappa = \frac{1}{2} \sum_{r_\alpha \in \mathrm{Irr}(G)} \varrho_\alpha \dim r_\alpha. \tag{5.57}$$

The other interacting invariants are harder to deduce. We will propose natural candidates for γ and β based on experience with lower-dimensional cases.

Given an orthogonal representation $r : G \to O(n)$, we can define a 1-cocycle

$$\det r(g) \in H^1(G, \mathbb{Z}_2). \tag{5.58}$$

[6] In the literature on fermionic SRE phases, it is common to re-write systems of class D, which only have a $\mathbb{Z}_2^F$ symmetry, as systems with both a $U(1)$ symmetry and a particle-hole symmetry (Bernevig, 2013; Chiu et al., 2016). This entails doubling the number of degrees of freedom, and therefore doubling κ.

[7] Since $\dim q$ is even for $\mathbb{H}$-type representations, only an even number of class C systems can occur.

It is sometimes called the 1st Stiefel-Whitney class of r, for reasons explained in Appendix C.2. We will denote it $w_1(r)$. For irreducible representations of type $\mathbb{C}$ and $\mathbb{H}$, it is trivial.[8]

Similarly, we can define the 2nd Stiefel-Whitney class of G as an obstruction to lifting $r : G \to O(n)$ to $\tilde{r}_+ : G \to Pin_+(n)$. One can lift each $r(g)$ to an element $\tilde{r}_+(g) \in Pin_+$, but the composition law will only hold up to a 2-cocycle $\lambda(g, g')$ with values in ± 1. Thus we get a well-defined element $w_2(r) \in H^2(G, \mathbb{Z}_2)$. One might also consider an obstruction to lifting r to a homomorphism $\tilde{r}_- : G \to Pin_-(n)$, but it is expressed in terms of $w_2(r)$ and $w_1(r)$ (namely, the Pin_- obstruction is $w_2 + w_1^2$).

A natural guess for the contribution of an irreducible r_α to γ is $\varrho_\alpha w_1(r_\alpha)$. Assuming this, the formula for the invariant γ is

$$\gamma = \sum_{r_\alpha \in \mathrm{Irr}(G,\mathbb{R})} \varrho_\alpha w_1(r_\alpha) = w_1(\mathcal{R}), \tag{5.59}$$

where we defined a "virtual representation"[9]

$$\mathcal{R} = \oplus_\alpha \varrho_\alpha r_\alpha. \tag{5.60}$$

Note that only $\mathbb{R}$-type representations contribute to γ, since only those representations can have nonzero $w_1(r)$. On the other hand, $\mathcal{R}$ includes all representations.

There are two natural guesses for the contribution of a single irreducible r to β: $w_2(r)$ or $\tilde{w}_2(r) = w_2(r) + w_1(r)^2$. To derive β for a general virtual representation $\mathcal{R}$, we note that the Whitney formula for Stiefel-Whitney classes says

$$w_2(\mathcal{R} + \mathcal{R}') = w_2(\mathcal{R}) + w_2(\mathcal{R}') + w_1(\mathcal{R}) \cup w_1(\mathcal{R}'). \tag{5.61}$$

The same formula applies to $\tilde{w}_2(r)$. This formula looks just like the stacking law for β and γ, if we identify γ with w_1 and β with w_2 (or $\tilde{w}_2$). Hence for a general $\mathcal{R}$, we have either $\beta(\mathcal{R}) = w_2(\mathcal{R})$ or $\beta(\mathcal{R}) = w_2(\mathcal{R}) + w_1(\mathcal{R})^2$.

A non-trivial check on both of these candidates is that they are compatible with the group supercohomology equation. This equation implies that $\beta \cup \beta \in H^4(G, \mathbb{Z}_2)$ maps to a trivial class in $H^4(G, U(1))$. This is automatically satisfied for both $\beta = w_2(\mathcal{R})$ and $\beta = w_2(\mathcal{R}) + w_1(\mathcal{R})^2$, as shown in Appendix C.2.

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[8] For $\mathbb{C}$-type representations, we have $\det r(g) = \det q(g) \det\{\bar{q}\}(g) = 1$, while for $\mathbb{H}$-type representations, $\det q(g) = 1$ since $q(g)$ takes values in the unitary symplectic group.

[9] The word "virtual" reflects the fact that the numbers ϱ_α can be both positive and negative. Thus $\mathcal{R}$ is best thought of as an element of the K-theory of the representation ring of G.

Is there any way to decide between the two candidates for β? Not without understanding better the physical meaning of β. Indeed, formally, a change of variables $\beta \mapsto \beta + \gamma \cup \gamma$ is an automorphism of the group of fermionic SRE phases in 2d. This automorphism maps one candidate for β to the other one. Thus formally they are equally good. One can pick one over another only if one assigns β a particular physical meaning. The same is even more true about $\alpha \in C^3(G, U(1))$, since it depends on various choices in a complicated way.

Let us make a few remarks about surjectivity of the map from free to interacting SRE phases in the 2d case. It is clear that every value of the parameter $\gamma \in H^1(G, \mathbb{Z}_2)$ can be realized by free fermionic systems. One can just take two copies of the basic system of class A with opposite values of the chiral central charge κ (for example, a $p + ip$ superconductor stacked with a $p - ip$ superconductor) and let G act only on the first copy via a 1-dimensional real representation of G given by the 1-cocycle γ. This construction was used in Ref. (Gu and Levin, 2014) for the case $G = \mathbb{Z}_2$.

One can also ask if every β that solves the supercohomology equation can be realized by free fermions. The answer appears to be no (Strickland, n.d.), for a sufficiently complicated G. The reason is again some highly non-trivial relations satisfied by Stiefel-Whitney classes. Thus not all supercohomology phases in 2d can be realized by free fermions. At the moment, we do not know how to find a concrete example of a finite group G for which this happens. It would be interesting to study this question further and in particular determine both α and β for a general 2d band Hamiltonian with symmetry G.

Chapter 6

BRAIDING STATISTICS OF VORTICES IN TOPOLOGICAL SUPERCONDUCTORS

6.1 Introduction

As discussed in Section 2.1, a p-wave superconductor in $2+1$ dimensions supports Majorona zero modes on vortices. These Majorana modes σ, together with the quasiparticle ψ, satisfy the fusion and braiding rules of the Ising TQFT (Section 3.3). We can stack layers of the p-wave superconductor, which has Chern number $\nu = 1$, and ν tells us the number of layers that have been stacked, yielding us an integer classification of topological superconductors. On the other hand, by considering the underlying TQFT, or the braiding statistics of vortices, we obtain a $\mathbb{Z}_{16}$-classification (Kitaev, 2006) (Bernevig and Neupert, 2015). In (Kitaev, 2006), the underlying TQFTs for the 16 phases was computed, and the bulk-boundary correspondence was invoked to match the Chern number (number of layers of the basic $p + ip$ superconductor) to the TQFTs. In this chapter, we construct effective Hamiltonians describing the statistical interaction of vortices for each of the 16 phases, and show that we obtain the correct vortex braiding statistics for all 16 phases by starting with the basic $\nu = 1$ system (or any chosen phase) and stacking layers of the system.

6.2 Review of the 16-fold way and anyon condensation

The interacting classification: the 16-fold way

In the absence of interactions, Class D superconductors in $2+1$ dimensions have the Chern number invariant ν, which tells us the net number of layers of the $p + ip$ superconductor (Section 2.1). In the presence of interactions, this integer classification breaks down to a $\mathbb{Z}_{16}$-classification, which is based on the underlying TQFT. In particular, they can be distinguished by the the braiding statistics of vortices (Kitaev, 2006; Bernevig and Neupert, 2015). We shall denote these phases by $\mathcal{P}_\nu$, $\nu = 1, \ldots, 16$. Here we summarize the results which will be relevant. R_c^{ab} will denote the braiding coefficient of a and b in fusion channel c, and $M_c^{ab} = (R_c^{ab})^2$ will denote the phase due to the double exchange of a and b in fusion channel c: when a and b are of different types, only the double exchange yields a topologically invariant phase factor.

When $\nu \in \mathbb{Z}_{16}$ is odd, we have the Ising topological order, consisting of three anyons $1, \sigma, \psi$ with the fusion rules Eq. (3.18)

$$\begin{aligned} \sigma \times \sigma &= 1 + \psi \\ \sigma \times \psi &= \sigma \\ \psi \times \psi &= 1. \end{aligned} \tag{6.1}$$

The braiding coefficients are given by Eq. (3.19):

$$\begin{aligned} R_1^{\sigma\sigma} &= \theta e^{i\alpha\frac{\pi}{4}} \\ R_\psi^{\sigma\sigma} &= \theta e^{-i\alpha\frac{\pi}{4}} \end{aligned} \tag{6.2}$$

where $\theta := \theta(\nu) = e^{\pi i\nu/8}$ and $\alpha = (-1)^{(\nu+1)/2}$.

When ν is even, we have an abelian theory, but the exact fusion rules depend on whether $\nu = 0$ or 2 mod 4. If $\nu = 0$ mod 4, we have the toric code fusion rules, or the $\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules: four anyons $1, e, m, \psi$ with fusion rules

$$\begin{aligned} e \times e = m \times m = \psi \times \psi &= 1 \\ e \times m &= \psi \\ e \times \psi &= m \\ m \times \psi &= e. \end{aligned} \tag{6.3}$$

The braiding coefficients for vortices (e and m) are

$$\begin{aligned} R_1^{ee} = R_1^{mm} &= e^{\pi i\nu/8} \\ M_\psi^{em} &= -e^{\pi i\nu/4}. \end{aligned} \tag{6.4}$$

When $\nu = 2$ mod 4, we have the $\mathbb{Z}_4$ fusion rules: four anyons $1, a, \psi, \bar{a}$ with

$$\begin{aligned} a \times a = \bar{a} \times \bar{a} &= \psi \\ a \times \bar{a} &= 1 \\ a \times \psi &= \bar{a} \\ \bar{a} \times \psi &= a \end{aligned} \tag{6.5}$$

and the braiding coefficients are

$$\begin{aligned} R_{\psi}^{aa} = R_{\psi}^{\bar{a}\bar{a}} &= e^{\pi i \nu/8} \\ M_1^{a\bar{a}} &= e^{-\pi i \nu/4}. \end{aligned} \tag{6.6}$$

Fermionic stacking and anyon condensation

The topological superconductor may be described by the Ising topological order, but we need to be careful about stacking. Naively stacking two Ising TQFTs, $\mathcal{P}_1 \boxtimes \mathcal{P}_1$ leads to a theory with 9 anyons. We need to stack them as fermionic phases. Roughly speaking, for the Ising category without extra structure, the fermion is treated as just another anyon, a non-local (topological) excitation, while as a fermionic theory, we should consider the fermion to a local excitation (possible to create or annihilate by a local operator) (Lan, Kong, and Wen, 2016b). From this perspective, the Ising TQFT or the toric code theory, which contain a fermion, are modular extensions of the trivial fermionic theory consisting only of 1 and ψ (Lan, Kong, and Wen, 2016b). The correct stacking law is defined in (Lan, Kong, and Wen, 2016a), and in this case reduces to using the naive stacking law and condensing the condensable bosons (Neupert et al., 2016). We will denote this fermionic stacking by $\boxtimes_f$.

An anyon which has trivial braiding with itself can be condensed and is called a condensable boson (for a non-abelian anyon, we require that it has trivial self-braiding in at least one of the fusion channels, though we will only have to deal with condensing abelian anyons) (Burnell, 2018). After condensation, several things happen: (1) anyons which have nontrivial braiding with the condensed boson become confined; (2) anyons related by fusion with the condensed boson are identified; (3) other anyons can split into different anyons.

Let us discuss some cases which will be relevant. If we (bosonically) stack two theories with Ising fusion rules, we obtain a theory with nine anyons: $(1,1)$, $(1,\sigma)$, $(1,\psi)$, $(\sigma,1)$, (σ,σ), (σ,ψ), $(\psi,1)$, (ψ,σ), (ψ,ψ). We can condense (ψ,ψ), which is a boson. Then,

$$\begin{aligned} (1,\sigma) &\sim (1,\sigma)\times(\psi,\psi) = (\psi,\sigma) \\ (\sigma,1) &\sim (\sigma,1)\times(\psi,\psi) = (\sigma,\psi) \\ (\psi,1) &\sim (\psi,1)\times(\psi,\psi) = (1,\psi) \end{aligned} \tag{6.7}$$

and $(1,\sigma)$ and $(\sigma,1)$ are confined. We are left only with $(1,1)$, (σ,σ), and $(1,\psi)$. $(1,1)$ clearly takes the role of the vacuum, which we denote by $\tilde{1}$, and $(1,\psi)$ is a fermion, which we denote by $\tilde{\psi}$.

Note that

$$(\sigma,\sigma)\times(\sigma,\sigma) = (1,1)+(1,\psi)+(\psi,1)+(\psi,\psi) \sim \tilde{1}+\tilde{1}+\tilde{\psi}+\tilde{\psi}. \tag{6.8}$$

As (σ,σ) fuses with itself to two copies of the vacuum in the condensed phase, (σ,σ) cannot be a single type of anyon – it actually splits into two anyons.

One possibility is that it splits as $(\sigma,\sigma)\mapsto e+m$ with

$$\begin{aligned} e\times e = m\times m &= \tilde{1} \\ e\times m &= \tilde{\psi}. \end{aligned} \tag{6.9}$$

It is easily verified that $(e+m)\times(e+m) = \tilde{1}+\tilde{1}+\tilde{\psi}+\tilde{\psi}$. Then, we end up with four anyons $\tilde{1},\tilde{\psi},e,m$ with $\mathbb{Z}_2\times\mathbb{Z}_2$ fusion rules.

Another possibility is $(\sigma,\sigma)\mapsto a+\bar{a}$ with the fusion rules

$$\begin{aligned} a\times\bar{a} &= \tilde{1} \\ a\times a = \bar{a}\times\bar{a} &= \tilde{\psi}. \end{aligned} \tag{6.10}$$

This also satisfies the condition that $(a+\bar{a})\times(a+\bar{a}) = \tilde{1}+\tilde{1}+\tilde{\psi}+\tilde{\psi}$. Then we obtain a theory with four anyons $\tilde{1},a,\bar{a},\tilde{\psi}$ with $\mathbb{Z}_4$ fusion rules.

Which kind of theory we end up with depends on the exact braiding coefficients of the Ising theory we are stacking, and can be determined algebraically: see (Neupert et al., 2016). We shall give another way of determining which kind of fusion rules we obtain, in Section 6.4.

6.3 Effective Hamiltonian for vortices of an odd ν phase

Braiding coefficients and superselection sectors

Consider two vortices, with the corresponding Majorana zero modes γ_1 and γ_2, respectively. These combine into a single set of creation and annihilation operators,

$$a = \frac{1}{2}(\gamma_1 + i\gamma_2)$$

$$a^\dagger = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

and act on a Hilbert space $\mathbb{C}^2$ spanned by $|0\rangle$ and $|1\rangle$, which are respectively unoccupied and occupied with respect to $a, a^\dagger$.

As discussed in Sec. 2.1, braiding two vortices results in

$$\begin{aligned}\gamma_1 &\mapsto \gamma_2 \\ \gamma_2 &\mapsto -\gamma_1.\end{aligned} \tag{6.11}$$

This can also be derived by the following reasoning: the states $|0\rangle$ and $|1\rangle$ formed from two Majorana modes differ by a fermion. If we do a 2π rotation of the whole configuration, it is equivalent to two braids between v_1 and v_2, and hence should give us a R^2. On the other hand, a fermion acquires a sign under 2π rotation, so $|0\rangle \mapsto |0\rangle$ and $|1\rangle \mapsto -|1\rangle$, i.e. $R^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = i\gamma_2\gamma_1$, i.e. it just acts by fermionic parity and hence reverses the sign of each γ. This is achieved by a single R taking $\gamma_1 \mapsto \gamma_1$, $\gamma_2 \mapsto -\gamma_2$ or vice versa, since γ have to be real.

The operators γ_1 and γ_2 generate $Cl(2) \simeq Mat(2, \mathbb{C})$ which acts on the $\mathbb{C}^2$ spanned by the states $|0\rangle$ and $|1\rangle$. From the fermionic point of view, which considers fermions to be fundamental particles, the two basis states belong to the same superselection sector. From a bosonic point of view, however, they belong to different superselection sectors: the bosonic operators 1 and $\gamma_1\gamma_2$ are both diagonal in this basis, so there is no way to move from one state to another if we employ only bosonic operators.

Recall that the Ising topological order describes a bosonized picture of the topological superconductor: the anyon ψ corresponds to a nontrivial superselection sector since there is no local bosonic operator which can create it out of the vacuum 1. Let us denote the states in the two superselection sectors 1 and ψ, which are the two possibilities we can land on when fusing two σ particles (which carry Majorana modes), as $|\sigma\sigma; 1\rangle$ and $|\sigma\sigma; \psi\rangle$. These should correspond to the states $|0\rangle$ and $|1\rangle$, which are even and odd, respectively, under the fermonic parity $i\gamma_2\gamma_1$. Since we do not *a priori* know which one is odd and which is even, we write

$$i\gamma_2\gamma_1|\sigma\sigma;1\rangle = -\alpha|\sigma\sigma;1\rangle$$
$$i\gamma_2\gamma_1|\sigma\sigma;\psi\rangle = +\alpha|\sigma\sigma;\psi\rangle \tag{6.12}$$

for some $\alpha = \pm 1$.

The operator on $\mathbb{C}^2$ which accomplishes Eq. 6.11 by conjugation is

$$R = \theta e^{-\frac{\pi}{4}\gamma_1\gamma_2} = \theta e^{-i\frac{\pi}{4}(i\gamma_2\gamma_1)} \tag{6.13}$$

where θ is a phase factor. By noting how $i\gamma_2\gamma_1$ acts on the states $|\sigma\sigma;1\rangle$ and $|\sigma\sigma;\psi\rangle$, we see that $R_1^{\sigma\sigma} = \theta e^{i\alpha\frac{\pi}{4}}$ and $R_\psi^{\sigma\sigma} = \theta e^{-i\alpha\frac{\pi}{4}}$. By invoking the bulk-boundary correspondence, Kitaev determines this α to be $(-1)^{(\nu+1)/2}$ and θ to be $e^{\pi i\nu/8}$, so we get Eq. (6.2). θ can be interpreted as the topological spin of the σ anyon (Kitaev, 2006).

If $\nu = 1 \bmod 4$, $\alpha = -1$. Since $i\gamma_2\gamma_1$ is fermionic parity, this means that $(-)^F|\sigma\sigma;1\rangle = +|\sigma\sigma;1\rangle$, i.e. the fusion channel 1 corresponds to the "unoccupied" state $|0\rangle$; similarly, ψ corresponds to the "occupied" state $|1\rangle$. On the other hand, if $\nu = 3 \bmod 4$, $\alpha = +1$, and we have $i\gamma_2\gamma_1|\sigma\sigma;1\rangle = -|\sigma\sigma;1\rangle$, etc. The fusion channel 1 corresponds to $|1\rangle$ and ψ to $|0\rangle$.

Note that R as an operator acting on $\mathbb{C}^2$ is fixed to be of the form $\theta e^{-\frac{\pi}{4}}\gamma_1\gamma_2$; This difference between $\mathcal{P}_{4n+1}$ and $\mathcal{P}_{4n+3}$ in how one interprets the fusion channels in terms of fermionic states will be important for stacking.

The effective Hamiltonian

Consider an odd ν system. Two σ vortices interacting with each other can be effectively described by

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + i\gamma_2\gamma_1 V(|\vec{r}_1 - \vec{r}_2|). \tag{6.14}$$

acting on the Hilbert space $L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2) \otimes \mathbb{C}^2$.

$$\vec{A}_1 = \frac{F(\nu)}{|\vec{r}_1 - \vec{r}_2|^2}\left(-(y_1 - y_2), x_1 - x_2\right) \tag{6.15}$$

is the gauge field felt by the vortex 1 due to vortex 2. $F(\nu)$ is a ν-dependent factor valued in $\mathrm{End}(\mathbb{C}^2)$; it takes the form

$$F(\nu) = -\frac{i}{4}\gamma_2\gamma_1 + \frac{\nu}{8}. \tag{6.16}$$

It is clear that $F(-\nu) = -F(\nu)$. The expression for $\vec{A}_2$ is similar.

V is some Hermitian potential ($i\gamma_2\gamma_1$ is itself Hermitian) which splits the energies of the two states in $\mathbb{C}^2$. (Cheng et al., 2009) calculate the splitting energy to be $V(R) \approx -2\frac{\Delta_0}{\pi^{\frac{3}{2}}}\frac{\cos p_F R + \frac{\pi}{4}}{\sqrt{p_F R}}e^{-R/\xi}$ for large separation $R \gg \xi$ where ξ is the superconducting coherence length and Δ_0 is the mean-field value of the superconducting order parameter Δ. Unless otherwise noted, we will assume that the vortices are far enough apart that we can ignore the potential energy, and focus on the universal properties of their braiding.

In terms of complex coordinates $z = x + iy$, $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and

$$\vec{A}_1 \cdot d\vec{r}_1 = F(\nu)\left(\frac{1}{2i}\frac{dz_1}{z_1 - z_2} - \frac{1}{2i}\frac{d\bar{z}_1}{\bar{z}_1 - \bar{z}_2}\right).$$

If vortex 1 encircles vortex 2, which corresponds to a double-braiding, the wave-function changes by $e^{i\oint \vec{A}_1\cdot d\vec{r}_1} = \exp\{2\pi i F(\nu)\} = \exp\left\{\frac{\pi}{2}\gamma_2\gamma_1\right\}e^{\frac{i\pi\nu}{4}}$ which produces the correct double-braiding coefficients, i.e. the square of Eq. (6.13).

When ν is even, we have multiple types of vortices and hence the effective Hamiltonian describing the interaction of vortices depends on the specific types of vortices we consider. The Hamiltonians for even ν will be written down when we discuss stacking. Moreover, in Section 6.5 we will show that given the effective Hamiltonian for any one odd ν phase, we can obtain those for the others by stacking.

6.4 Stacking: even from odd-odd

Stacking two $\nu = 1$ systems

Take $\nu = 1$ system with two vortices, of Eq. 6.14:

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + i\gamma_2\gamma_1 V(|\vec{r}_1 - \vec{r}_2|). \tag{6.17}$$

We stack it with the same system; we write the second layer as

$$\bar{H} = \frac{1}{2m}\left((\vec{\bar{p}}_1 - \vec{A}_1)^2 + (\vec{\bar{p}}_2 - \vec{A}_2)^2\right) + i\bar{\gamma}_2\bar{\gamma}_1 V(|\vec{\bar{r}}_1 - \vec{\bar{r}}_2|) \tag{6.18}$$

where the bars simply denote that we have different coordinate and momentum variables, as well as different Majorana operators, from the first layer, even though the two are formally the same. The gauge fields $\vec{A}_i$ on the second layer are written in terms of the barred Majorana operators $\bar{\gamma}_i$ and the barred coordinates $\bar{r}_i$.

Stacking these two systems, we obtain

$$H' = H \otimes \mathbb{1} + \mathbb{1} \otimes \bar{H} \tag{6.19}$$

acting on $(L^2(\mathbb{R}^2))^{\otimes 4} \otimes \mathbb{C}^4$. H' depends on four coordinates $z_1, z_2, \bar{z}_1, \bar{z}_2$, which are the positions of the first and second vortex on the two layers. Recall that we need to condense the (ψ, ψ) anyon in the stacked phase in order to get to the resultant fermionic phase. This condensation does three things: confinement, identification, and splitting.

Confinement occurs for the $(\sigma, 1) \sim (\sigma, \psi)$ and $(1, \sigma) \sim (\psi, \sigma)$ anyons. This can be achieved by introducing a potential such as $V \sim e^{|z_i - \bar{z}_i|}$, which forces the position of the vortices on each layer to be the same – there is no way to move z_i independently of $\bar{z}_i$, so $(\sigma, 1)$ and $(1, \sigma)$ are confined. After confinement, we obtain:

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + i(\gamma_2\gamma_1 + \bar{\gamma}_2\bar{\gamma}_1)V(|\vec{r}_1 - \vec{r}_2|), \tag{6.20}$$

where $\bar{\gamma}_1$ and $\bar{\gamma}_2$ are the Majorana modes of the second $\nu = 1$ layer, and we see that

$$\vec{A'}_1 \cdot d\vec{r}_1 = \left(-\frac{1}{4}(i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1) + \frac{1}{4}\right)\frac{1}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right).$$

This leads to the braiding matrix $R = (e^{\pi i/8})^2 \exp\{-\frac{\pi}{4}\gamma_1\gamma_2\} \exp\{-\frac{\pi}{4}\bar{\gamma}_1\bar{\gamma}_2\}$. Since $\alpha = -1$ for both layers, $R|00\rangle = R_1^{\sigma\sigma} R_1^{\sigma\sigma}|00\rangle = (e^{i\pi/8})^2 e^{-2\pi i/4}|00\rangle = e^{-\pi i/4}$, etc. The full braiding matrix in the $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis is:

$$R = \begin{pmatrix} e^{-i\pi/4} & 0 & 0 & 0 \\ 0 & e^{i\pi/4} & 0 & 0 \\ 0 & 0 & e^{i\pi/4} & 0 \\ 0 & 0 & 0 & e^{i3\pi/4} \end{pmatrix}. \tag{6.21}$$

After confinement, we are left with the (σ, σ) anyon, and since it fuses with (ψ, ψ) to itself, there is no further identification of anyons needed. The remaining question is the splitting of (σ, σ) into $a + \bar{a}$.

Just as two σ anyons on a single layer behave as either 1 or ψ when zoomed out and considered together, the two σ anyons on two different layers behave as either a or $\bar{a}$ when considered together. Since they differ by a fermion ($\bar{a} = a \times \psi$ and $a = \bar{a} \times \psi$), we look for eigenstates of the fermionic parity operator localized to a single vortex – that is, we change the basis from the eigenbasis of $i\gamma_2\gamma_1$ and $i\bar{\gamma}_2\bar{\gamma}_1$ to the eigenbasis of $i\bar{\gamma}_1\gamma_1$ and $i\bar{\gamma}_2\gamma_2$. With respect to localized fermionic parity, we will denote the even state by $|a\rangle$ and the odd state by $|\bar{a}\rangle$. The expression for the new basis states in terms of the old basis is given by:

$$
\begin{aligned}
|aa\rangle &= \frac{|01\rangle - i|10\rangle}{\sqrt{2}} \\
|a\bar{a}\rangle &= \frac{|00\rangle - i|11\rangle}{\sqrt{2}} \\
|\bar{a}a\rangle &= \frac{|00\rangle + i|11\rangle}{\sqrt{2}} \\
|\bar{a}\bar{a}\rangle &= \frac{|01\rangle + i|10\rangle}{\sqrt{2}}.
\end{aligned} \tag{6.22}
$$

Under braiding, $|aa\rangle$ and $|\bar{a}\bar{a}\rangle$ transform with a phase of $e^{\pi i/4}$ while $|a\bar{a}\rangle \mapsto e^{-\pi i/4}|\bar{a}a\rangle$. These are the correct braiding coefficients for $\mathcal{P}_2$, Eq. (6.6). Note that $|aa\rangle$ should belong to the superselection sector ψ since $a \times a = \psi$. Since $|aa\rangle$ is a linear combination of $|01\rangle$ and $|10\rangle$, each of which belongs to the superselection sector ψ, this is consistent with the fusion rules Eq. (6.5). The same holds for the other three states.

Let us discuss what the effective Hamiltonian looks like. The stacked Hamiltonian acts on $\mathbb{C}^4$. If we take the $|aa\rangle$ sector,

$$
\left(-\frac{1}{4}(i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1) + \frac{1}{4}\right)|aa\rangle = \frac{1}{4}\left(|01\rangle - |01\rangle - i(-|10\rangle + |10\rangle)\right) + \frac{1}{4}|aa\rangle = \frac{1}{4}|aa\rangle. \tag{6.23}
$$

So in this sector, the gauge field takes the form $\vec{A'}_1 \cdot d\vec{r}_1 = \frac{1}{4}\frac{1}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right)$.

The Hamiltonian is

$$
H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) \tag{6.24}
$$

with $\vec{A}$ taking the above form. The potential term vanishes as it should (no ground state degeneracy in this sector) since $i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1$ vanishes on $|aa\rangle$. We could still have a simple $V(|\vec{r}_1 - \vec{r}_2|)$ that depends only on the distance.

The $|\bar{a}\bar{a}\rangle$ sector works similarly. On the other hand, since $|a\bar{a}\rangle$ and $|\bar{a}a\rangle$ transform into each other after braiding, we need to consider these states together. The sector spanned by $|a\bar{a}\rangle$ and $|\bar{a}a\rangle$ is $\mathbb{C}^2 = \text{Span}\{|00\rangle, |11\rangle\}$, and

$$F(\nu) = \left(-\frac{1}{4}(i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1) + \frac{1}{4}\right) = \frac{1}{4}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}. \tag{6.25}$$

Restricting this to the subspace $\mathbb{C}^2$, we see that the gauge field takes the form

$$\vec{A'}_1 \cdot d\vec{r}_1 = \frac{1}{4}\begin{pmatrix} -1 & 0 \\ 0 & 3 \end{pmatrix} \frac{1}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right).$$

in the $|00\rangle$ and $|11\rangle$ basis.

Now we convert this to the $|a\bar{a}\rangle$ and $|\bar{a}a\rangle$ basis. Since

$$\left(-\frac{1}{4}(i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1) + \frac{1}{4}\right)|a\bar{a}\rangle = \frac{1}{4}\left(-|a\bar{a}\rangle + 2|\bar{a}a\rangle\right)$$

and

$$\left(-\frac{1}{4}(i\gamma_2\gamma_1 + i\bar{\gamma}_2\bar{\gamma}_1) + \frac{1}{4}\right)|\bar{a}a\rangle = \frac{1}{4}\left(2|a\bar{a}\rangle - |\bar{a}a\rangle\right),$$

in the $|a\bar{a}\rangle, |\bar{a}a\rangle$ basis the gauge field takes the form

$$\vec{A'}_1 \cdot d\vec{r}_1 = \frac{1}{4}\begin{pmatrix} -1 & 2 \\ 2 & -1 \end{pmatrix} \frac{1}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right).$$

The Hamiltonian describing the interaction between an a vortex and a $\bar{a}$ vortex will take the form

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + \sqrt{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} V(|\vec{r}_1 - \vec{r}_2|) \tag{6.26}$$

in the $|a\bar{a}\rangle, |\bar{a}a\rangle$ basis, where $\vec{A}$ takes the above form. The form of the potential comes from noting how $i(\gamma_2\gamma_1 + \bar{\gamma}_2\bar{\gamma}_1)$ acts on $|a\bar{a}\rangle$ and $|\bar{a}a\rangle$.

Stacking $\nu = 1$ with $\nu = -1$

We start with H for two vortices in the phase $\mathcal{P}_1$:

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + i\gamma_2\gamma_1 V(|\vec{r}_1 - \vec{r}_2|) \tag{6.27}$$

and add a second layer in the phase $\mathcal{P}_{-1}$:

$$\tilde{H} = \frac{1}{2m}\left((\vec{\tilde{p}}_1 - \vec{\tilde{A}}_1)^2 + (\vec{\tilde{p}}_2 - \vec{\tilde{A}}_2)^2\right) - i\tilde{\gamma}_2\tilde{\gamma}_1 V(|\vec{\tilde{r}}_1 - \vec{\tilde{r}}_2|). \tag{6.28}$$

$\tilde{\gamma}_1$ and $\tilde{\gamma}_2$ are Majorana modes of the second layer.

(Note that the gauge field is odd under time-reversal, and $F(-1) = -F(1)$, so $\mathcal{P}_{-1}$ is the time-reversal of $\mathcal{P}_1$.)

After stacking, the total Hamiltonian is again $H' = H \otimes \mathbb{1} + \mathbb{1} \otimes \tilde{H}$, acting on $\left(L^2(\mathbb{R}^2)\right)^{\otimes 4} \otimes \mathbb{C}^4$. The condensation process proceeds in the same way as in the $\mathcal{P}_1 \boxtimes_f \mathcal{P}_1$ case, and after confinement we obtain:

$$H' = \frac{1}{2m}\left((\vec{p'}_1 - \vec{A'}_1)^2 + (\vec{p'}_2 - \vec{A'}_2)^2\right) + i(\gamma_2\gamma_1 - \tilde{\gamma}_2\tilde{\gamma}_1)V(|\vec{r}_1 - \vec{r}_2|). \tag{6.29}$$

where $\vec{A'}_1 \cdot d\vec{r}_1 = \frac{1}{4}\frac{(\gamma_1\gamma_2 + \tilde{\gamma}_1\tilde{\gamma}_2)}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right)$. Note that the overall phase factors $\theta(\nu)$ cancel each other out, since $\theta(1)\theta(-1) = 1$.

The braiding matrix will then be

$$R = \exp\left\{-\frac{\pi}{4}\gamma_1\gamma_2\right\}\exp\left\{-\frac{\pi}{4}\tilde{\gamma}_1\tilde{\gamma}_2\right\}. \tag{6.30}$$

We follow the same steps as in the $\mathcal{P}_1 \boxtimes_f \mathcal{P}_1$ case. Now, $\alpha = -1$ for the first layer and $\alpha = +1$ for the second layer, so the state $|00\rangle = |0\rangle \otimes |0\rangle$ corresponds to the state in the fusion channel 1 on the first layer and ψ on the second layer. Thus we have $R|00\rangle = (R^{\nu=1})_1^{\sigma\sigma}(R^{\nu=-1})_\psi^{\sigma\sigma}|00\rangle = \theta(1)e^{-\pi i/4}\theta(-1)e^{-\pi i/4}|00\rangle = e^{-\pi i/2}|00\rangle$. Repeating this for the other states, we compute the braiding matrix in this basis to be

$$R = \begin{pmatrix} -i & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i \end{pmatrix}. \tag{6.31}$$

Now we consider the system in a different basis: instead of thinking of $\mathbb{C}^4$ as $\mathbb{C}^2_{\text{layer1}} \otimes \mathbb{C}^2_{\text{layer2}}$, we think of it as $\mathbb{C}^2_{\text{vortex1}} \otimes \mathbb{C}^2_{\text{vortex2}}$, where each vortex carries two Majorana modes γ_i, $\tilde{\gamma}_i$. Each vortex carries a space $\mathbb{C}^2$ whose states are eigenstates of the vortex-localized fermionic parity operator $i\tilde{\gamma}_1\gamma_1$ or $i\tilde{\gamma}_2\gamma_2$. On each $\mathbb{C}^2_{\text{vortex}i}$ we have an even state $|e\rangle$ and an odd state $|m\rangle$; the total fermionic Hilbert space $\mathbb{C}^4$ is spanned by the basis $|ee\rangle \equiv |e\rangle_1 \otimes |e\rangle_2$, $|em\rangle$, $|me\rangle$, and $|mm\rangle$. We can write these states in terms of the old basis states as:

$$|ee\rangle = \frac{|01\rangle - i|10\rangle}{\sqrt{2}}$$

$$|em\rangle = \frac{|00\rangle - i|11\rangle}{\sqrt{2}}$$

$$|me\rangle = \frac{|00\rangle + i|11\rangle}{\sqrt{2}}$$

$$|mm\rangle = \frac{|01\rangle + i|10\rangle}{\sqrt{2}}.$$

Since we know how the states $|00\rangle$, $|01\rangle$, etc. transform under braiding, we can compute the behavior of the new basis states under braiding. We see that $R_1^{ee} = R_1^{mm} = 1$; and also that $R|em\rangle = -i|me\rangle$ and $R|me\rangle = -i|em\rangle$, from which we see that $R_1^{em}R_1^{me} = M_1^{em} = -1$. These are indeed the correct braiding coefficients for the toric code, Eq. 6.4.

The effective Hamiltonians involving different types of vortices can be obtained from this braiding matrix in the same manner as the $\mathcal{P}_1 \boxtimes_f \mathcal{P}_1$ case.

Action of time-reversal

In the $\mathcal{P}_1 \boxtimes_f \mathcal{P}_{-1}$ system, time-reversal (TR) acts as (Bernevig, 2013)

$$\begin{aligned} \gamma_i &\mapsto -\tilde{\gamma}_i \\ \tilde{\gamma}_i &\mapsto \gamma_i. \end{aligned} \tag{6.32}$$

This flips the sign of the fermionic parity operator on each vortex:

$$i\tilde{\gamma}_i\gamma_i \mapsto -i\tilde{\gamma}_i\gamma_i. \tag{6.33}$$

Hence, $|e\rangle$ and $|m\rangle$ map to each other under time-reversal.

Stacking a $p+ip$ (which belongs to $\mathcal{P}_1$) and a $p-ip$ superconductor (which belongs to $\mathcal{P}_{-1}$), we obtain a superconductor in Class DIII, a system that is protected by time-reversal symmetry from deformation to the trivial system. If we break time-reversal symmetry, we can deform it to the s-wave superconductor, which has the toric code as its underlying topological order (Hansson, Oganesyan, and Sondhi, 2004). In the s-wave superconductor, the vortex m and the sector which has a vortex and a fermion $e = m \times \psi$ are unrelated by time-reversal symmetry, whereas we have seen that in the nontrivial Class DIII TR-invariant superconductor, the TR operation exchanges e and m. Thus, on the level of the TQFT, this nontrivial TR action distinguishes it from the trivial phase – without it, it would be identical to the toric code *simpliciter*, which is the unit for the group of phases $\mathbb{Z}_{16}$.

With the two systems stacked, we could have terms like $i\gamma_1\tilde{\gamma}_1 V_1$ which is now local (unlike $i\gamma_2\gamma_1 V$). This would break the degeneracy between the e and m particles, since $i\gamma_1\tilde{\gamma}_1|e\rangle = +|e\rangle$, $i\gamma_1\tilde{\gamma}_1|m\rangle = -|m\rangle$. However, under time-reversal, $i\gamma\tilde{\gamma} \mapsto -i\tilde{\gamma}(-)\gamma = -i\gamma\tilde{\gamma}$, so such terms are not TR-invariant.

In $\nu = 2$ phase, there is no TR symmetry, so nothing prevents us from adding such terms, which would lift the degeneracy between a and $\bar{a}$. As discussed in (Bernevig and Neupert, 2015), there are no stable Majorana bound states in even ν phases, unless we protect them by a symmetry.

$\mathbb{Z}_2 \times \mathbb{Z}_2$ fusion rules or $\mathbb{Z}_4$ fusion rules?

Stacking gives us a way to determine whether a given even-ν phase $\mathcal{P}_{2n}$ is described by $\mathbb{Z}_2 \times \mathbb{Z}_2$ anyons $\{1, e, m, \psi\}$ or by $\mathbb{Z}_4$ anyons $\{1, a, \psi, \bar{a}\}$. We can distinguish between the two by noting that two vortices of the same type fuse to 1 in the former case, but they fuse to ψ in the latter case; 1 will of course have trivial braiding, while ψ will acquire a phase -1 under exchange. In the former case, we need $R_1^{ee} = 1^{1/4}$, and in the latter case, $R_\psi^{aa} = (-1)^{1/4}$. If we denote a basic vortex of the phase in question by v, we can consider the braiding of v with itself, which gives us a phase R^{vv}. If $(R_{vv})^4 = 1$, the phase is of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ type, while if $(R_{vv})^4 = -1$, the phase is of $\mathbb{Z}_4$ type.

Consider the above examples. The $\nu = 0$ phase had $|ee\rangle \mapsto |ee\rangle$ under braiding. $1^4 = 1$, so it is indeed of $\mathbb{Z}_2 \times \mathbb{Z}_2$ type. The $\nu = 2$ phase on the other hand had $|aa\rangle \mapsto e^{i\pi/4}|aa\rangle$ under braiding, and since $(e^{\pi i/4})^4 = -1$, it is of $\mathbb{Z}_4$ type.

6.5 Stacking: odd from even-odd

$\mathcal{P}_3 = \mathcal{P}_2 \boxtimes_f \mathcal{P}_1$

Let us first consider this stacking on the level of anyon condensation: we stack the $\nu = 2$ phase consisting of $1, a, \bar{a}, \psi$ and the $\nu = 1$ phase consisting of $1, \sigma, \psi$, and condense the (ψ, ψ) anyon. Most of the combinations are confined, and we are left with

$$\begin{aligned} 1' &= (1, 1) \\ \sigma' &= (a, \sigma) \sim (\bar{a}, \sigma) \\ \psi' &= (1, \psi) \sim (\psi, 1) \end{aligned} \tag{6.34}$$

with the usual fusion and braiding rules for the Ising TQFT, Eqs. (3.18) and (3.19).

As we saw in section 6.4, there are four different $\nu = 2$ Hamiltonians for two vortices, corresponding to the sectors $|aa\rangle$, $|a\bar{a}\rangle$, $|\bar{a}a\rangle$, and $|\bar{a}\bar{a}\rangle$; each Hamiltonian acts on a Hilbert space $L^2(\mathbb{R}^2) \otimes L^2(\mathbb{R}^2)$. Let us consider the $|aa\rangle$ sector, which has the Hamiltonian Eq. 6.24 with

$$\vec{A'}_1 \cdot d\vec{r}_1 = \frac{1}{4}\frac{1}{2i}\left(\frac{dz_1}{z_1 - z_2} - \frac{dz_1^*}{z_1^* - z_2^*}\right). \tag{6.35}$$

Let us stack this with a $\nu = 1$ system, which is just the Hamiltonian in Eq. 6.14,

$$H = \frac{1}{2m}\left((\vec{p}_1 - \vec{A}_1)^2 + (\vec{p}_2 - \vec{A}_2)^2\right) + i\gamma_2\gamma_1 V(|\vec{r}_1 - \vec{r}_2|) \tag{6.36}$$

with

$$\vec{A}_1 \cdot d\vec{r}_1 = \left(-\frac{i}{4}\gamma_2\gamma_1 + \frac{1}{8}\right)\left(\frac{1}{2i}\frac{dz_1}{z_1 - z_2} - \frac{1}{2i}\frac{d\bar{z}_1}{\bar{z}_1 - \bar{z}_2}\right). \tag{6.37}$$

The total Hilbert space becomes $\left(L^2(\mathbb{R}^2)\right)^{\otimes 4} \otimes \mathbb{C}^2$, but after condensation, forcing the vortex (a or $\bar{a}$ on the first layer and σ on the second) position to be the same on the two layers, we are left with $\left(L^2(\mathbb{R}^2)\right)^{\otimes 2} \otimes \mathbb{C}^2$.

Suppose the original $\nu = 2$ Hamiltonian was in the $|aa\rangle$ sector. After stacking with a $\nu = 1$ Hamiltonian with two vortices, we obtain a Hamiltonian for two (a, σ) particles, which has an internal space $\mathbb{C}^2$ from

$$(a, \sigma) \times (a, \sigma) = (\psi, 1) + (\psi, \psi) \sim \psi + 1. \tag{6.38}$$

The result actually should be the same if we had started with the $|\bar{a}\bar{a}\rangle$ sector or the sector containing $|\bar{a}a\rangle$ and $|a\bar{a}\rangle$, since $(a,\sigma) \sim (\bar{a},\sigma)$. Regardless of which Hamiltonian we chose for the $\nu = 2$ phase, after fermionic stacking, we end up with a single type of vortex, described by a Hamiltonian of the type Eq. 6.14.

Let us confirm that we get the correct braiding coefficients. First, consider the case where we have started with the $|aa\rangle$ sector. The braiding matrix for the $\nu = 1$ phase is $diag(e^{-i\pi/8}, e^{3\pi i/8})$ for the braiding of two σ vortices. However, for the $\nu = 3$ phase, we need to switch the two components: the $\nu = 3$ vortex $\sigma' = (a,\sigma)$ has fusion $\sigma' \times \sigma' = (\psi,1) + (\psi,\psi) = \psi' + 1'$, hence if we are in the 1 sector of the $\nu = 1$ phase that is being stacked, we are in the ψ' sector of the $\nu = 3$ phase, and vice versa. Thus, $R^{\sigma'\sigma'}_{1'} = e^{3\pi i/8}$ and $R^{\sigma'\sigma'}_{\psi'} = e^{-\pi i/8}$, up to the additional phase coming from the as. After multiplying by a phase $e^{i\pi/4}$ from the exchange of two as, we get

$$\begin{aligned} R^{\sigma'\sigma'}_{1'} &= e^{5\pi i/8} \\ R^{\sigma'\sigma'}_{\psi'} &= e^{\pi i/8}. \end{aligned} \tag{6.39}$$

These are indeed the braiding coefficients for the $\nu = 3$ phase, Eq. 6.2. Since $R^{\bar{a}\bar{a}}_{\psi} = R^{aa}_{\psi}$, the same argument would hold, had we started out in the $|\bar{a}\bar{a}\rangle$ sector.

Now, let us consider the vortices coming from the $|a\bar{a}\rangle$ and $|\bar{a}a\rangle$ sector. First, we note that

$$(a,\sigma) \times (\bar{a},\sigma) = (1,1) + (1,\psi) = 1' + \psi'. \tag{6.40}$$

So the 1 sector of the $\nu = 1$ phase gives rise to the $1'$ sector of the $\nu = 3$ phase, and the ψ sector of the $\nu = 1$ phase gives rise to the ψ' sector of the $\nu = 3$ phase. Therefore, the braiding coefficients of $\nu = 3$ inherit those of $\nu = 1$ directly: $e^{-i\pi/8}$ and $e^{3\pi i/8}$. Multiplying these by the phase $e^{-\pi i/4}$ from the braiding of a and $\bar{a}$, we obtain $(R^{\sigma'\sigma'}_{1'}, R^{\sigma'\sigma'}_{\psi'}) = (e^{-3\pi i/8}, e^{\pi i/8})$.

We consider the double-braiding, where all coefficients are gauge-invariant (Kitaev, 2006). After a double-braid we get the phases

$$\begin{aligned} R^{\sigma'\sigma'}_{1'} &= e^{-6\pi i/8} = e^{10\pi i/8} \\ R^{\sigma'\sigma'}_{\psi'} &= e^{2\pi i/8} \end{aligned} \tag{6.41}$$

which is the same as the double-braiding coefficients obtained from stacking with the $|aa\rangle$ sector, the square of Eq. 6.39.

General braiding coefficients from stacking

Recall that for any odd ν, we have

$$R_1^{\sigma\sigma} = \theta(\nu)e^{\alpha\pi i/4}$$
$$R_\psi^{\sigma\sigma} = \theta(\nu)e^{-\alpha\pi i/4} \tag{6.42}$$

where $\theta(\nu) = e^{\frac{\nu\pi i}{8}}$ and $\alpha = -1$ for $\nu = 1 \bmod 4$ and $+1$ for $\nu = 3 \bmod 4$.

The value of α can be understood from the stacking perspective in the following way. A $\nu = 1 \bmod 4$ phase is obtained by stacking $\mathcal{P}_1$ with a $\mathcal{P}_{4n}$; the latter phase has e and m type vortices. After stacking, we get the vortex $\sigma' = (\sigma, e) \sim (\sigma, m)$, with the fusion rule

$$\sigma' \times \sigma' = (1, 1) + (\psi, 1) = 1' + \psi', \tag{6.43}$$

so the sectors $1'$ and ψ' of $\mathcal{P}_{4n+1}$ correspond to the sectors 1 and ψ of $\mathcal{P}_1$. Hence we get $\alpha = -1$ (since $\mathcal{P}_1$ has $\alpha = -1$).

On the other hand, $\mathcal{P}_{4n+3} = \mathcal{P}_1 \boxtimes_f \mathcal{P}_{4n+2}$, and since $\mathcal{P}_{4n+2}$ has a and $\bar{a}$ type vortices, $\mathcal{P}_{4n+3}$ has the vortex $\sigma' = (\sigma, a) \sim (\sigma, \bar{a})$ with the fusion rule

$$\sigma' \times \sigma' = (1, \psi) + (\psi, \psi) = \psi' + 1', \tag{6.44}$$

so the sectors $1'$ and ψ' of $\mathcal{P}_{4n+3}$ correspond to the sectors ψ and 1 of $\mathcal{P}_1$, respectively. This means that the braiding coefficients $R_1^{\sigma\sigma}$ and $R_\psi^{\sigma\sigma}$ need to change places, compared to those for $\mathcal{P}_1$ (and $\theta(\nu)$ is unaffected since it is common to both). Thus we see that $\alpha = +1$ for $\mathcal{P}_{4n+3}$.

We can also think of an odd phase $\mathcal{P}_\nu$ as the stacking of $\mathcal{P}_2$ with some other odd phase $\mathcal{P}_{\nu-2}$. By the above logic, stacking with $\mathcal{P}_2$ changes the sign of α; on the other hand, the braiding coefficients for the vortex σ' of $\mathcal{P}_\nu$ also acquires a phase $e^{i\pi/4}$ from the braiding of the a vortices of $\mathcal{P}_2$. Hence the overall phase behaves as

$$\theta(\nu) = e^{2\pi i/8}\theta(\nu - 2). \tag{6.45}$$

Thus we see that, whenever ν advances by 2, going from an odd phase to an odd phase, the value of α gets reversed and $\theta(\nu)$ increases by $e^{2\pi i/8}$. This means that once we are given the braiding coefficients for one odd phase, we can obtain those of all the other odd phases immediately.

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Appendix A

APPENDICES FOR CHAPTER 4

A.1 Relations between bosonic and fermionic invariants

Lemma A.1.1. *For a twisted cocycle $\omega \in Z^2(\mathcal{G}, U(1)_T)$, the 1-cochain defined by*

$$\begin{aligned} {}^1\!/\!_2\beta(g) &:= \omega(g,p) - \omega(p,g) + x(g)\omega(p,p) \\ &= \begin{cases} \omega(g,p) - \omega(p,g) & g \in G_0 \\ \omega(g,p) - \omega(p,g) + \omega(p,p) & g \notin G_0 \end{cases} \end{aligned} \tag{A.1}$$

is gauge-invariant, satisfies $\beta(gp) = \beta(g)$, takes values in $\{0, {}^1\!/\!_2\}$, and defines a G_b-cocycle.

Proof. First, ${}^1\!/\!_2\beta(g)$ picks up a factor of

$$\begin{aligned} (L(g)+(-1)^{x(g)}L(p) - L(gp)) - (L(p) + L(g) - L(gp)) \\ + x(g)(L(p) + L(p) - L(1)) \\ = -2x(g)L(p) + 2x(g)L(p) - x(g)L(1) = 0 \end{aligned} \tag{A.2}$$

under a transformation $\omega \mapsto \omega + \delta_T L$ for some 1-cochain L of $\mathcal{G}$ satisfying $L(1) = 0$.[1]

Second,

$$\begin{aligned} {}^1\!/\!_2\beta(gp) &= \omega(gp,p) - \omega(p,gp) + x(gp)\omega(p,p) \\ &= \omega(g,p) - \omega(p,g) + x(g)\omega(p,p) - \delta_T\omega(p,g,p) \\ &= {}^1\!/\!_2\beta(g). \end{aligned} \tag{A.3}$$

Third,

$$\begin{aligned} &\omega(g,p) - \omega(p,g) \\ &\quad = (-1)^{x(g)}\omega(p,p) - \omega(g,p) - \omega(p,gp) \\ &\qquad\quad - (\delta_T\omega)(g,p,p) + (\delta_T\omega)(p,g,p) \\ &\quad = (-1)^{x(g)}\omega(p,p) - \omega(g,p) - \omega(p,p) \\ &\qquad\quad + \omega(p,g) + (\delta_T\omega)(p,p,g) \\ &\quad = -\omega(g,p) + \omega(p,g) - (1 - (-1)^{x(g)})\omega(p,p) \end{aligned} \tag{A.4}$$

[1] This condition on L ensures that $Q(1) = \mathbb{1}$ is preserved.

means that $\sfrac{1}{2}\beta$ takes values in the $\mathbb{Z}/2$ subgroup of $U(1)$.

Therefore $\sfrac{1}{2}\beta$ defines a $\beta \in C^1(G_b, \mathbb{Z}/2)$. Let $g_b, h_b \in G_b$ and choose any lifts g, h to $\mathcal{G}$.

Fourth,

$$
\begin{aligned}
&(\delta\beta)(g_b, h_b) \\
&\quad = \tfrac{1}{2}(\beta(g) + \beta(h) - \beta(ghp^{\rho(\bar{g},\bar{h})})) \\
&\quad = \tfrac{1}{2}(\beta(g) + \beta(h) - \beta(gh)) \\
&\quad = \omega(g,p) - \omega(p,g) + x(g)\omega(p,p) \\
&\qquad + \omega(h,p) - \omega(p,h) + x(h)\omega(p,p) \\
&\qquad - \omega(gh,p) + \omega(p,gh) - x(gh)\omega(p,p) \\
&\quad = \omega(g,p) - \omega(p,g) + \omega(h,p) - \omega(p,h) \\
&\qquad + \omega(g,h) - \omega(g,hp) - (-1)^{x(g)}\omega(h,p) \\
&\qquad - \omega(g,h) + \omega(p,g) + \omega(pg,h) + 2x(g)x(h)\omega(p,p) \\
&\quad = \omega(g,p) + 2x(g)\omega(h,p) - \omega(p,h) \\
&\qquad + (-1)^{x(g)}\omega(p,h) - \omega(g,p) + 2x(g)x(h)\omega(p,p) \\
&\quad = 2x(g)(\omega(h,p) - \omega(p,h) + x(h)\omega(p,p)) \\
&\quad = 2x(g)\cdot \tfrac{1}{2}\beta(h) \\
&\quad = 0.
\end{aligned} \tag{A.5}
$$

□

Lemma A.1.2. *Each cohomology class $H^2(\mathcal{G}, U(1)_T)$ contains an element ω that satisfies, for all $g, h \in \mathcal{G}$,*

$$\omega(pg, h) = \omega(g, h) \tag{A.6}$$

$$\omega(g, ph) = \omega(g, h) + \omega(g, p). \tag{A.7}$$

Proof. For an arbitrary 2-cocycle $W \in Z^2(\mathcal{G}, U(1)_T)$, define

$$\omega = W - \delta_T L \tag{A.8}$$

where $L \in C^1(\mathcal{G}, U(1)_T)$ satisfies

$$
\begin{aligned}
L(1) &= 0 \\
L(p) &= \tfrac{1}{2}W(p,p) \ \text{ or } \ \tfrac{1}{2}W(p,p) + \tfrac{1}{2} \\
L(p\bar{g}) &= L(\bar{g}) - W(p,\bar{g}) + L(p).
\end{aligned} \tag{A.9}
$$

Here, we abuse notation by letting $\bar{g}$ denote a $g \in \mathcal{G}$ with $t(g) = 0$. This implies $L(p) = {}^1\!/\!_2 W(p,p)$. We have fixed $L(p\bar{g})$ in terms of $L(\bar{g})$ and $L(p)$, but left $L(\bar{g})$ undetermined, while $L(p)$ is fixed up to a ${}^1\!/\!_2$.

We see that

$$\begin{aligned} \omega(p,p) &= W(p,p) - (-1)^{x(p)} L(p) - L(p) + L(1) \\ &= W(p,p) - 2 \cdot {}^1\!/\!_2 W(p,p) = 0 \end{aligned} \tag{A.10}$$

and

$$\begin{aligned} \omega(p,\bar{g}) &= W(p,\bar{g}) - \Big((-1)^{x(p)} L(\bar{g}) + L(p) - L(p\bar{g})\Big) \\ &= W(p,\bar{g}) - W(p,\bar{g}) + {}^1\!/\!_2 W(p,p) - {}^1\!/\!_2 W(p,p) = 0. \end{aligned} \tag{A.11}$$

Next we show that any ω satisfying (A.10) and (A.11) must also satisfy the gauge conditions (A.6) and (A.7). First,

$$\begin{aligned} \omega(p,p\bar{g}) = -\delta_T \omega(p,p,\bar{g}) &- (-1)^x(p)\omega(p,\bar{g}) \\ &+ \omega(p,p) + \omega(1,\bar{g}) = 0. \end{aligned} \tag{A.12}$$

Similarly, computing $0 = \delta_T \omega(p,\bar{g},\bar{h})$ shows that $\omega(\bar{g}p,\bar{h}) = \omega(\bar{g},\bar{h})$ and computing $0 = \delta_T \omega(p,\bar{g},\bar{h}p)$ shows that $\omega(\bar{g}p,\bar{h}p) = \omega(\bar{g},\bar{h}p)$. Putting these together, we see that (A.6) is satisfied.

Now we compute $0 = \delta_T \omega(\bar{g},p,\bar{h})$ which shows that $\omega(\bar{g},p\bar{h}) = \omega(\bar{g},\bar{h}) + \omega(\bar{g},p)$ and $0 = \delta_T \omega(\bar{g},p,p\bar{h})$ which shows that $\omega(\bar{g},\bar{h}) = \omega(\bar{g},p\bar{h}) + \omega(\bar{g},p)$. Putting these together, we see that (A.7) is satisfied.

□

Lemma A.1.3. *Given a trivialization t, the map*

$$\omega(g,h) = \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2 \beta(\bar{g}) t(h) \tag{A.13}$$

defines a bijection from pairs $(\alpha,\beta) \in C^2(G_b, U(1)_T) \times C^1(G_b, \mathbb{Z}/2)$ *that satisfy* $\delta_T \alpha = {}^1\!/\!_2 \beta \cup \rho$ *and* $\delta\beta = 0$ *(where* ${}^1\!/\!_2 \beta$ *is regarded as a* $U(1)_T$*-valued cocycle) to twisted cocycles* $\omega \in Z^2(\mathcal{G}, U(1)_T)$ *that satisfy* (A.6) *and* (A.7) *for all* $g,h \in \mathcal{G}$*. In particular, for all* $g_b, h_b \in G_b$*, this map has an inverse*

$$\begin{aligned} \alpha(g_b,h_b) &= \omega(s(g_b), s(h_b)) \\ {}^1\!/\!_2 \beta(g_b) &= \omega(s(g_b), p). \end{aligned} \tag{A.14}$$

Proof. First we show that ω is a twisted cocycle:

$$\begin{aligned}
&(\delta_T\omega)(g,h,k)\\
&= (-1)^{x(g)}\omega(h,k) + \omega(g,hk) - \omega(g,h) - \omega(gh,k)\\
&= (-1)^{x(g)}\alpha(\bar{h},\bar{k}) + \alpha(\bar{g},\bar{hk}) - \alpha(\bar{g},\bar{h}) - \alpha(\bar{gh},\bar{k})\\
&\qquad + {}^1\!/\!_2(-1)^{x(g)}\beta(\bar{h})t(k) + {}^1\!/\!_2\beta(\bar{g})t(hk)\\
&\qquad - {}^1\!/\!_2\beta(\bar{g})t(h) - {}^1\!/\!_2\beta(\bar{gh})t(k)\\
&= (\delta_T\alpha)(\bar{g},\bar{h},\bar{k}) + {}^1\!/\!_2(\delta_T\beta)(\bar{g},\bar{h})t(k) - {}^1\!/\!_2\beta(\bar{g})(\delta t)(h,k)\\
&= 0.
\end{aligned} \tag{A.15}$$

Next we verify that ω satisfies the gauge conditions:

$$\begin{aligned}
\omega(pg,h) &= \alpha(\bar{pg},\bar{h}) + {}^1\!/\!_2\beta(\bar{pg})t(h)\\
&= \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2\beta(\bar{g})t(h)\\
&= \omega(g,h)
\end{aligned} \tag{A.16}$$

$$\begin{aligned}
\omega(g,ph) &= \alpha(\bar{g},\bar{ph}) + {}^1\!/\!_2\beta(\bar{g})t(ph)\\
&= \alpha(\bar{g},\bar{h}) + {}^1\!/\!_2\beta(\bar{g})(t(h)+1)\\
&= \omega(g,h) + \omega(g,p).
\end{aligned} \tag{A.17}$$

Then we check the conditions for α and β. For these two calculations, let $\bar{g}$ denote g_b and g denote $s(g_b)$. Note that $s(\bar{gh}) = p^{\rho(\bar{g},\bar{h})}gh$. Then

$$\begin{aligned}
&(\delta_T\alpha)(\bar{g},\bar{h},\bar{k})\\
&= (-1)^{x(g)}\alpha(\bar{h},\bar{k}) + \alpha(\bar{g},\bar{hk}) - \alpha(\bar{g},\bar{h}) - \alpha(\bar{gh},\bar{k})\\
&= (-1)^{x(g)}\omega(h,k) + \omega(g,p^{\rho(\bar{h},\bar{k})}hk)\\
&\qquad - \omega(g,h) - \omega(p^{\rho(\bar{g},\bar{h})}gh,k)\\
&= (\delta_T\omega)(g,h,k) + \{\text{terms of the form } \omega(p^{-},-)\}\\
&\qquad + \omega(g,p^{\rho(\bar{h},\bar{k})}) + (\delta_T\omega)(p^{\rho(\bar{g},\bar{h})},gh,k)\\
&\qquad - (\delta_T\omega)(g,p^{\rho(\bar{h},\bar{k})},hk) + (\delta_T\omega)(p^{\rho(\bar{h},\bar{k})},g,hk)\\
&= {}^1\!/\!_2\beta(\bar{g})\rho(\bar{h},\bar{k}).
\end{aligned} \tag{A.18}$$

The object ${}^1\!/\!_2\beta$ defined in (A.14) is the gauge-fixed form of (A.1). Then, by A.1.1, it defines a $\beta \in Z^1(G_b, \mathbb{Z}/2)$.

It remains to show that these maps are indeed inverses. Since ${}^1\!/\!_2\beta$ is the image of a $\mathbb{Z}/2$-valued cocycle β, ω can be written with a minus sign like $\omega = \alpha - {}^1\!/\!_2\beta \cup t$.

Note also that $s(\bar{g}) = p^{t(g)}g$. Then

$$\begin{aligned}\omega(g,h) &= \alpha(\bar{g},\bar{h}) - {}^1\!/\!_2\beta(\bar{g})t(h) \\ &= \omega(p^{t(g)}g, p^{t(h)}h) - \omega(p^{t(g)}g, p)t(h) \\ &= \omega(g,h),\end{aligned} \tag{A.19}$$

$$\begin{aligned}\alpha(g_b,h_b) &= \omega(s(g_b), s(h_b)) \\ &= \alpha(g_b,h_b) + {}^1\!/\!_2\beta(g_b)t(s(g_b)) \\ &= \alpha(g_b,h_b),\end{aligned} \tag{A.20}$$

$$\begin{aligned}{}^1\!/\!_2\beta(g_b) &= \omega(s(g_b), p) \\ &= \alpha(g_b,1) + {}^1\!/\!_2\beta(g_b)t(p) \\ &= {}^1\!/\!_2\beta(g_b).\end{aligned} \tag{A.21}$$

□

Theorem A.1.4. *$H^2(\mathcal{G}, U(1)_T)$ equals, as a set, the set of pairs (α,β) (see A.1.3), modulo the following equivalence relation: $(\alpha',\beta) \sim (\alpha,\beta)$ if $\alpha' = \alpha + \delta_T\lambda$ with λ a cochain in $C^1(\mathcal{G}, U(1)_T)$ satisfying $\lambda(s(g_b)p) = \lambda(s(g_b)) + \lambda(p)$.* [2]

Proof. The preceding lemmas show that the set of twisted cocycles ω satisfying the gauge conditions (A.6) and (A.7) is equivalent to the set of pairs (α,β). After transforming ω into this gauge, there remains freedom to choose $L(g)$ for each $g \in \mathcal{G}$ such that $t(g) = 0$, and to shift $L(p)$ by $^1\!/\!_2$. We have already seen that β is invariant under an arbitrary gauge transformation. However, there is some residual gauge freedom for α.

Let $\omega' = \omega + \delta_T\lambda$ be another 2-cocycle satisfying the gauge conditions. It takes the form $W - \delta_T L'$, with L' possibly differing from L in its values on $s(g_b)$ and p. We see from $\delta_T\lambda = \omega' - \omega = \delta_T(L' - L)$ that $\lambda = L' - L + \kappa$ where κ is a twisted 1-cocycle. Then, by (A.9), $\lambda(s(g_b)p) = \lambda(s(g_b)) + \lambda(p)$. The quantities $L(p)$, $L'(p)$, $\kappa(p)$, and therefore $\lambda(p)$, can each be chosen to be 0 or $^1\!/\!_2$. Finally, by (A.14), this freedom in gauge-fixed ω translates into the desired freedom in α.

□

[2] Had we chosen a different representative $\rho' = \rho + \delta\mu$ of $[\rho]$ to describe the extension of G_b by $\mathbb{Z}_2^F$, we would have considered a different set of cochains α (modulo coboundaries), shifted by $^1\!/\!_2\beta \cup \mu$, but their counting would be the same.

Appendix B

APPENDICES FOR CHAPTER 5

B.1 Diagrams for the ground states

These diagrams are used in the argument of Section 4.6.

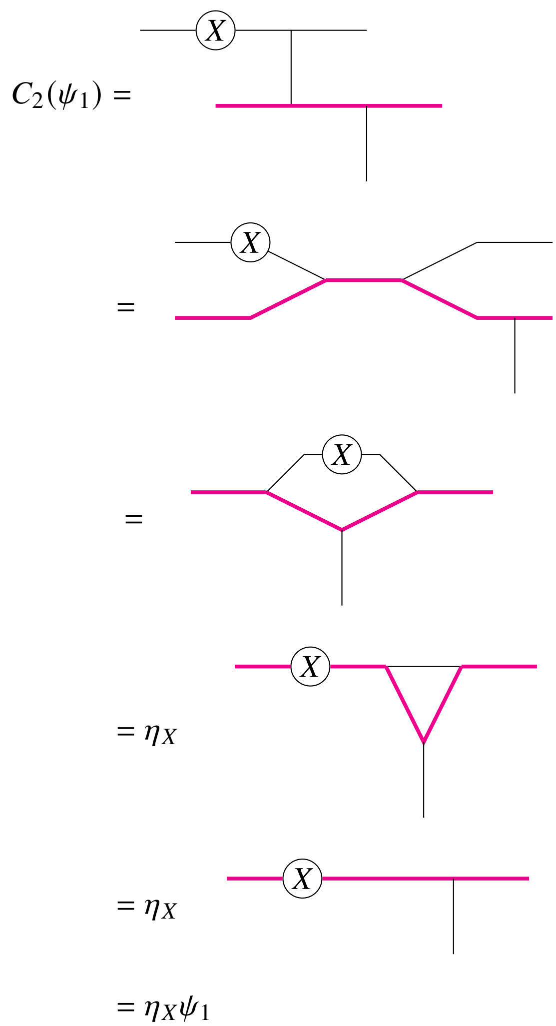


Figure B.1: Diagrammatic proof of $C_2\langle\psi_1| = \eta_X\langle\psi_2|$.

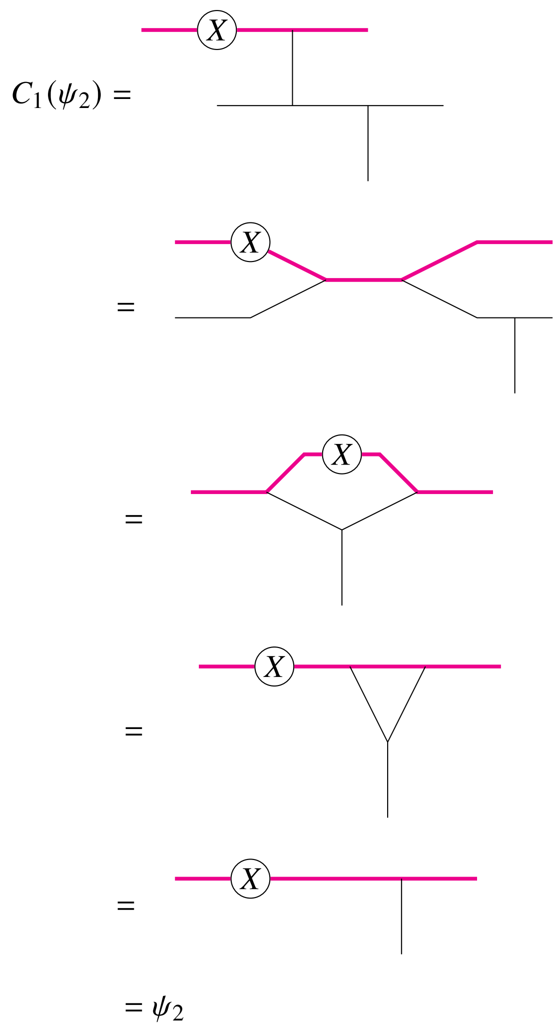


Figure B.2: Diagrammatic proof of $C_1\langle\psi_2| = \langle\psi_2|$.

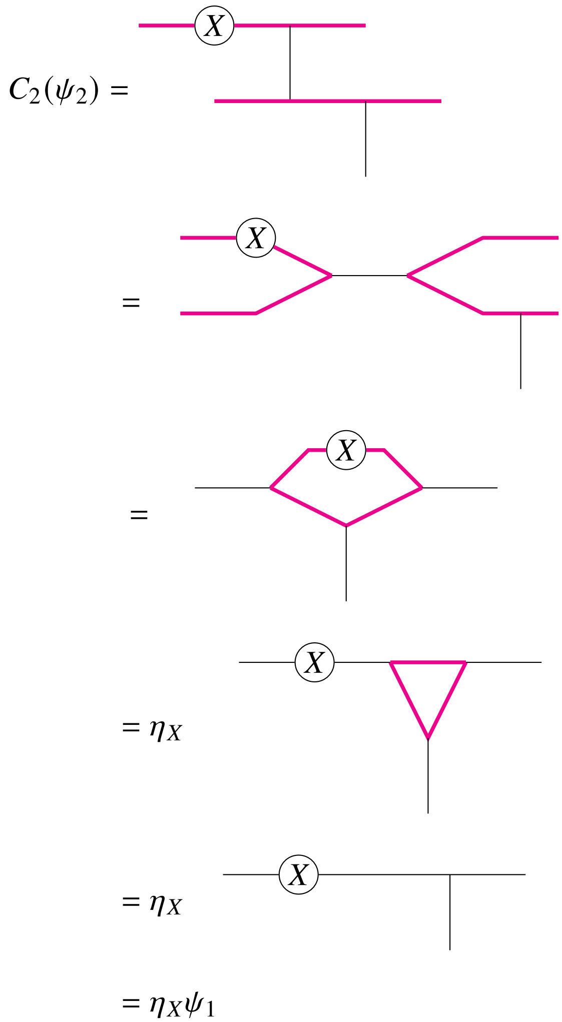
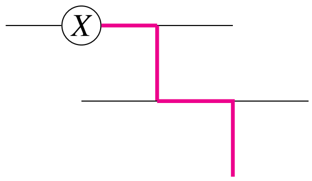
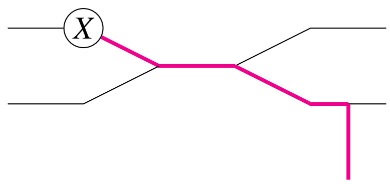


Figure B.3: Diagrammatic proof of $C_2\langle\psi_2| = \eta_X\langle\psi_1|$.

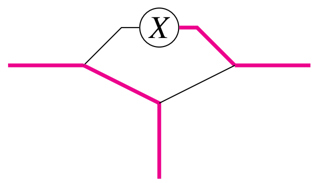
$C_3(\psi_3) =$



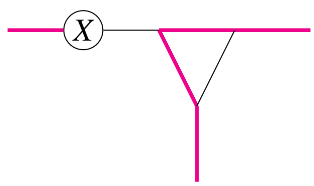
$=$



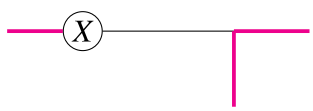
$=$



$= \eta_X$



$= \eta_X$



$= \eta_X \psi_4 = \psi_3$

Figure B.4: Diagrammatic proof of $C_3 \langle \psi_3 | = \langle \psi_3 |$.

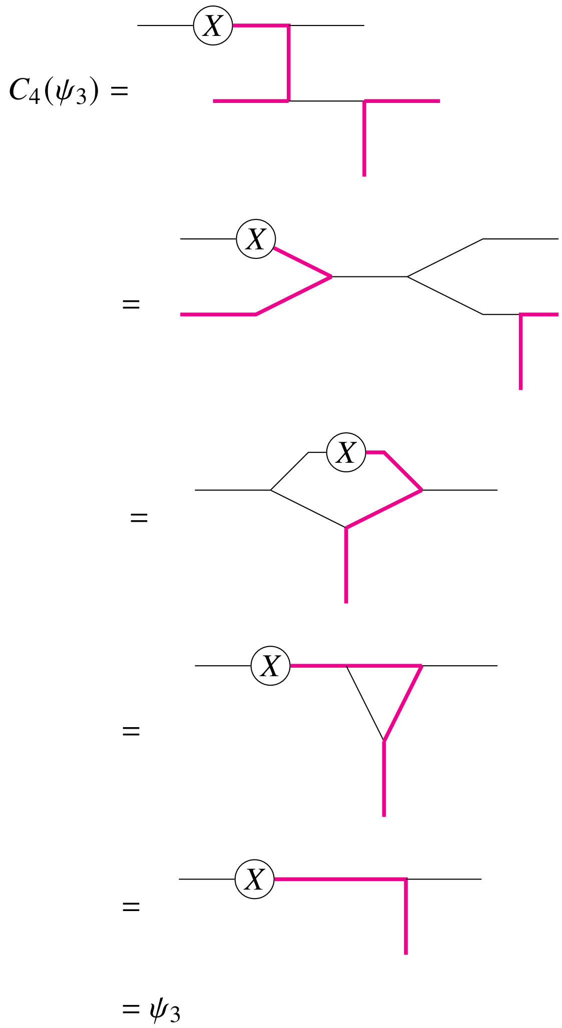
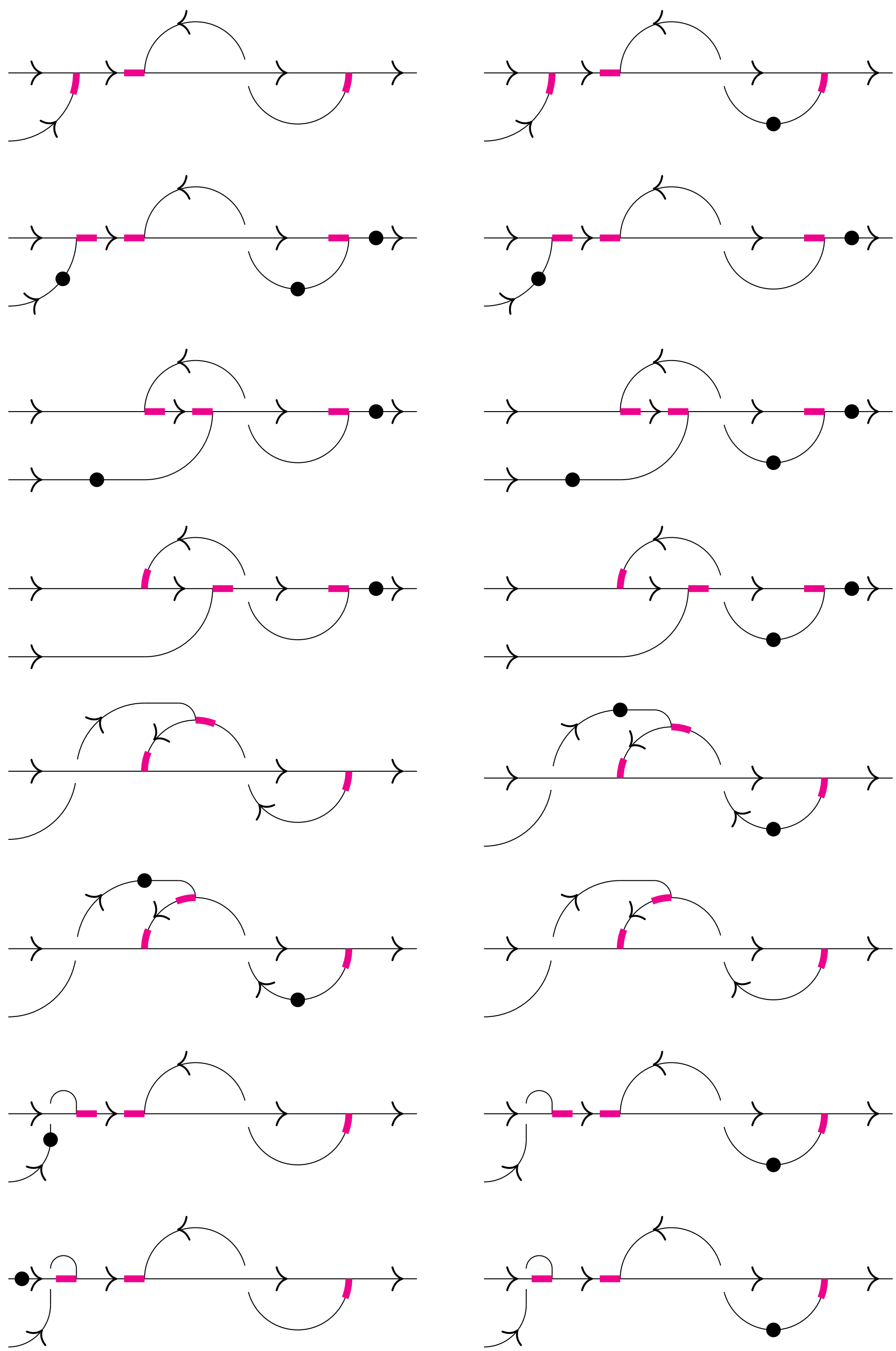


Figure B.5: Diagrammatic proof of $C_4\langle\psi_3| = \langle\psi_3|$.

B.2 Necessity of supercommutativity

This appendix is a derivation of the results (4.80) and (4.81) from the lattice spin formalism introduced in Section 4.5. Consider acting on the state $|ij\rangle$ with the cylinder map $Z(C)$; this is represented in the top diagram of each column of Figure B.6. To manipulate these diagrams into the diagrams at the bottom of each column, one applies a series of "moves" that are like Pachner moves, but are compatible with the lattice spin structure (see (Novak and Runkel, 2014) for details). Finally, one unbraids the legs at the cost of a sign $(-1)^{|i||j|}$.



(a) NS sector: $C_{ij} = (-1)^{|i||j|+|i|} C_{ji}$ (b) R sector: $C_{ij} = (-1)^{|i||j|} C_{ji}$

Figure B.6: A proof of equations (4.80) and (4.81). Arrows denote edge directions, magenta line segments denote special edges, and black dots denote spin signs +1, i.e. insertions of $\mathcal{F}$.

B.3 Description of ω in terms of pairs (α, β)

Start with some $[\omega] \in H^2(\mathcal{G}, U(1))$. We denote by $\bar{g}$ either an element of G_b or the corresponding element in $\mathcal{G}$ whose $t(g) = 0$, i.e. $(\bar{g}, 0)$. A general element of $\mathcal{G}$ then takes the form of either $\bar{g}$ or $\bar{g}p$.

Given an arbitrary ω, we can shift it by a coboundary δB where $B \in C^1(\mathbb{Z}_2, U(1))$ such that $B(0) = 0$ and $B(p) = \frac{1}{2}\omega(p,p)$ so that our new ω satisfies $\omega(p,p) = 0$. Then we can add a coboundary δA with $A \in C^1(\mathcal{G}, \mathbb{Z}_2)$ satisfying $A(\bar{g}p) = A(\bar{g}) - \omega(\bar{g}, p)$ to ω to make $\omega(\bar{g}, p) = 0$ for all $\bar{g} \in G_b$.

Evaluating the 3-cochain $\delta\omega$ on $(\bar{g}, p, p)$, $(\bar{g}, \bar{h}, p)$, and $(\bar{g}p, \bar{h}, p)$, and using the fact that $\delta\omega = 0$, we see that changing the second argument of ω by p does not affect its value, i.e. $\omega(g, h) = \omega(g, hp), \forall g, h \in \mathcal{G}$.

Then, evaluating $\delta\omega$ on $(\bar{g}, p, \bar{h})$ gives $\omega(\bar{g}p, \bar{h}) = \omega(\bar{g}, \bar{h}) + \omega(p, \bar{h})$. Defining $\alpha(\bar{g}, \bar{h}) := \omega(\bar{g}, \bar{h})$ and $\beta(\bar{g}) := \omega(p, \bar{g})$, $\omega = \alpha + t \cup \beta$, and we can check that $\delta\beta = 0$, and hence $\delta\alpha = -\delta t \cup \beta = \rho \cup \beta$. With our gauge choice, one can show that this definition of β is consistent with $\beta(\bar{g}) = |Q(g)|$. The residual gauge freedom, shifts ω by a coboundary $\delta\lambda$ for λ which is a pull-back from G_b. This leaves β invariant, but shifts α by a G_b-coboundary. Hence $\alpha \sim \alpha + \delta\lambda$, and we see that equivalence classes of ω correspond to equivalence classes of pairs (α, β) satisfying $\delta\alpha = \rho \cup \beta$ and $\delta\beta = 0$ with $(\alpha, \beta) \sim (\alpha + \delta\lambda, \beta)$.

When $\mathcal{G}$ splits, ρ is trivial and we have $\delta\alpha = 0$, so the set of equivalence classes of α is $H^2(G_b, U(1))$. The set of equivalence classes of β is of course $H^1(G_b, \mathbb{Z}_2)$. This confirms $H^2(\mathcal{G}, U(1)) \simeq H^2(G_b, U(1)) \times H^1(G_b, \mathbb{Z}_2)$, which we already knew from more abstract arguments.

Appendix C

APPENDICES FOR CHAPTER 5

C.1 Pin groups

Here we review the definition and some properties of Pin groups following Ref. (Atiyah, Bott, and Shapiro, 1963). Just as $Spin(M)$ is a non-trivial extension of $SO(M)$ by $\mathbb{Z}_2$, $Pin_+(M)$ and $Pin_-(M)$ are extensions of $O(M)$ by $\mathbb{Z}_2$. Since $O(M)$ has two connected components, so do $Pin_\pm(M)$. The connected component of the identity for both $Pin_+(M)$ and $Pin_-(M)$ is $Spin(M)$.

The groups $Pin_\pm(M)$ can be defined using the Clifford algebra $C\ell(M)$. To define $Pin_+(M)$, one considers the Clifford algebra for the positive metric:

$$\{\Gamma^I, \Gamma^J\} = 2\delta^{IJ}, \quad I, J = 1, \dots, M. \tag{C.1}$$

This is a $\mathbb{Z}_2$-graded algebra. For any $a \in Cl(M)$, we let $\epsilon(a) = a$ if a is even and $\epsilon(a) = -a$ if a is odd. Invertible elements in the Clifford algebra form a group. $Pin_+(M)$ is a subgroup generated by elements of the form $\not{v} = \Gamma^I v^I$, where v_I is a unit vector in $\mathbb{R}^M$. To define the homomorphism $Pin_+(M) \to O(M)$, consider the "twisted conjugation map"

$$\Gamma^J \mapsto \epsilon(a)\Gamma^J a^{-1}, \quad a \in Cl(M). \tag{C.2}$$

If $a = \not{v}$, then this map becomes

$$\Gamma^J \mapsto -\not{v}\Gamma^J \not{v}^{-1} = \Gamma^J - 2v^J \not{v}. \tag{C.3}$$

This is a hyperplane reflection on the space spanned by Γ^J. Since the whole group $O(M)$ is generated by hyperplane reflections, twisted conjugation by elements of $Pin_+(M)$ gives a surjective homomorphism from $Pin_+(M)$ to $O(M)$. The kernel of this map is the $\mathbb{Z}_2$ generated by -1. The subgroup $Spin(M) \subset Pin_+(M)$ consists of products of an even number of hyperplane reflections. Note that every hyperplane reflection $\not{v}$ squares to the identity in $Pin_+(M)$.

The group $Pin_-(M)$ is defined similarly, except that one starts with the "negative" Clifford algebra

$$\{\Gamma^I, \Gamma^J\} = -2\delta^{IJ}, \quad I, J = 1, \dots, M. \tag{C.4}$$

In this case, hyperplane reflections $\not{v}$ square to -1, which generates the kernel of the homomorphism $Spin(M) \to SO(M)$. In other words, for $Pin_-(M)$, hyperplane reflections square to fermion parity.

Finally, the group $Pin_c(M)$ is defined as $(Pin_+(M) \times U(1))/\mathbb{Z}_2^{diag}$, and its subgroup $Spin_c(M) \subset Pin_c(M)$ is defined as $(Spin(M) \times U(1))/\mathbb{Z}_2$. $Pin_c(M)$ is an extension of $O(M)$ by $U(1)$, while $Spin_c(M)$ is an extension of $SO(M)$ by $U(1)$. It is easy to show that the group $(Pin_-(M) \times U(1))/\mathbb{Z}_2$ is isomorphic to $Pin_c(M)$. The significance of $Pin_c(M)$ is the following: if we regard the complexification of the Clifford algebra as the algebra of observables of a fermionic system, then $Pin_c(M)$ can be identified with the subgroup of those unitaries which act linearly on the generators of the Clifford algebra. Thus lifting a real linear action of a group G on the Clifford generators Γ^I to a unitary action on the Fock space is equivalent to lifting the corresponding homomorphism $G \to O(M)$ to a homomorphism $G \to Pin_c(M)$. Similarly, if we are given a homomorphism $G \to SO(M)$, lifting it to a unitary action on the Fock space is the same as lifting it to a homomorphism $G \to Spin_c(M)$.

C.2 Characteristic classes of representations of finite groups

The theory of characteristic classes of vector bundles (a classic reference is (Milnor and Stasheff, 1974)) is familiar to physicists. A version of this construction also gives rise to characteristic classes of representations of a finite group which take values in cohomology of the said group (Atiyah, 1961). Real representations give rise to Stiefel-Whitney and Pontryagin classes, while complex representations give rise to Chern classes.

To define these classes, it is best to think of a real representation of G of dimension n as a homomorphism $R : G \to O(n)$, which then induces a continuous map of classifying spaces $\tilde{R} : BG \to BO(n)$. The map $\tilde{R}$ is defined up to homotopy only, but this suffices to define cohomology classes on BG by pull-back from $BO(n)$. Any cohomology class ω on $BO(n)$ thus defines a cohomology class $\tilde{R}^*\omega$ on BG. Cohomology classes on $BO(n)$ are precisely characteristic classes of real vector bundles, and their pull-backs via $\tilde{R}$ are called characteristic classes of the representation R. Similarly, given a complex representation $R : G \to U(n)$, we get a continuous map $\tilde{R} : BG \to BU(n)$, and can define Chern classes of R by pull-back.

In low dimensions, these classes have a concrete representation-theoretic interpretation. For example, the 1st Stiefel-Whitney class $w_1(R) \in H^1(G, \mathbb{Z}_2)$ of a real representation R is the obstruction for $R : G \to O(n)$ to descend to homomorphism $R' : G \to SO(n)$. Obviously, $w_1(r)(g)$ is given by $\det R(g)$.

Similarly, the 1st Chern class $c_1(R) \in H^2(G, \mathbb{Z})$ of a complex representation R can be interpreted as an obstruction for R to descend to $R' : G \to SU(n)$. The obstruction $\det R(g)$ is a 1-cocycle on G with values in $U(1)$. The corresponding class in $H^2(G, \mathbb{Z})$ is obtained by applying the Bockstein homomorphism (which for finite groups is an isomorphism). Explicitly:

$$c_1(R)(g,h) = \frac{1}{2\pi i}\left(\log\det R(gh) - \log\det R(g) - \log\det R(h)\right). \tag{C.5}$$

The 2nd Stiefel-Whitney class $w_2(R) \in H^2(G, \mathbb{Z}_2)$ is an obstruction to lifting R to a homomorphism $R' : G \to Pin_+(n)$. One can always define R' as a projective representation, and the corresponding 2-cocycle represents $w_2(R)$. The image of $w_2(R)$ in $H^2(G, U(1))$ under the embedding $\mathbb{Z}_2 \to U(1)$ is an obstruction to lifting R to a homomorphism $R' : G \to Pin_c(n)$. In the main text, it is denoted $w_2^{U(1)}(R)$. By the isomorphism $H^2(G, U(1)) \simeq H^3(G, \mathbb{Z})$ (valid for finite groups), this class can be interpreted as an element of $H^3(G, \mathbb{Z})$. Then it is known as the 3rd integral Stiefel-Whitney class W_3.

By functoriality, known relations between cohomology classes of $BO(n)$ and $BU(n)$ lead to relations between characteristic classes of representations. Let us describe those of them which we have used in the main text. First of all, the Whitney formula expresses Stiefel-Whitney (or Chern) classes of $R + R'$ in terms of Stiefel-Whitney (or Chern) classes of R and R':

$$w_k(R + R') = \sum_{p=0}^{k} w_p(R) \cup w_{k-p}(R'). \tag{C.6}$$

There are also more complicated formulas expressing characteristic classes of $R \otimes R'$ in terms of those of R and R' (Milnor and Stasheff, 1974). We will only need a particular case: let R be a real representation of odd dimension M, and L be a one-dimensonal real representation, then

$$w_2(R \otimes L) = w_2(R). \tag{C.7}$$

In Section 5.3, we propose that given a gapped 2d band Hamiltonian, the invariant $\beta \in H^2(G, \mathbb{Z}_2)$ of 2d fermionic SRE phases with symmetry $G \times \mathbb{Z}_2^F$ is given either by $w_2(R)$ or $w_2(R) + w_1(R)^2$, where R is a certain representation of G. The supercohomology equation implies that $\beta \cup \beta \in H^4(G, \mathbb{Z}_2)$ maps to a trivial class in $H^4(G, U(1))$. To show that this is automatically the case for our two candidates, we note that for finite groups $H^4(G, U(1)) \simeq H^5(G, \mathbb{Z})$. The class in $H^5(G, \mathbb{Z})$ corresponding to $\beta \cup \beta$ can be obtained by applying the Bockstein homomorphism $H^4(G, \mathbb{Z}_2) \to H^5(G, \mathbb{Z})$. A mod-2 class is annihilated by the Bockstein homomorphism if and only if it is a mod-2 reduction of an integral class. Now recall the well-known relation between Stiefel-Whitney classes and Pontryagin classes (Milnor and Stasheff, 1974):

$$w_2^2 = p_1 \bmod 2. \tag{C.8}$$

Hence w_2^2 is indeed annihilated by the Bockstein homomorphism. The same is true if we replace w_2 with $w_2 + w_1^2$. Indeed, since

$$(w_2 + w_1^2)^2 = w_2^2 + w_1^4, \tag{C.9}$$

it is sufficient to show that w_1^4 maps to a trivial class in $H^4(G, U(1))$. Now we recall that w_1^2 is cohomologous to $\delta\omega/2$, where ω is an integral lift of w_1. Therefore w_1^2 is cohomologous to $\frac{1}{2}\delta\omega \cup \frac{1}{2}\delta\omega$, which is a coboundary of $\frac{1}{4}\omega \cup \delta\omega$.

In Section 5.3, we show that for a band Hamiltonian, the invariant $\hat{\alpha} \in H^2(\hat{G}, U(1))$ of 1d fermionic SRE phases with symmetry $\hat{G}$ is equal to the image of $w_2(R)$ under the map $\iota : H^2(\hat{G}, \mathbb{Z}_2) \to H^2(\hat{G}, U(1))$, for a particular representation R. Obviously, any element in the image of ι has order 2, so in general not every element in $H^2(\hat{G}, U(1))$ can be realized by a band Hamiltonian. But we claimed that for some $\hat{G}$, even certain elements of order 2 in $H^2(\hat{G}, U(1))$ cannot be realized by band Hamiltonians. This happens because not every element in $H^2(\hat{G}, \mathbb{Z}_2)$ arises as $w_2(R)$ for some representation R. The reason is again the relation (C.8). It implies that for any representation R of $\hat{G}$, the Bockstein homomorphism annihilates $w_2(R)^2$. On the other hand, a generic element of $H^2(\hat{G}, \mathbb{Z}_2)$ need not have this property. An example of a finite group $\hat{G}$ for which some elements of $H^2(\hat{G}, \mathbb{Z}_2)$ do not arise as $w_2(R)$ for any R is given in (Gunarwardena, Kahn, and Thomas, 1989).

C.3 Beta as a charge pumping invariant

As discussed in Section 5.3, fermionic SRE phases in 1d with symmetry $\hat{G}$ have an invariant $\beta \in H^1(G, \mathbb{Z}_2)$. More precisely, this invariant is defined if the invariant γ (the number of boundary fermionic zero modes modulo 2) vanishes. The definition of β given in (Fidkowski and Kitaev, 2011) relies on the properties of boundary zero modes. Namely, $\beta(g) = 1$ (resp. $\beta(g) = 0$) if $g \in G$ acts on the boundary Hilbert space by a fermionic (resp. bosonic) operator. Here we explain an alternative formulation of $\beta \in H^1(G, \mathbb{Z}_2)$ as a charge pumping invariant. Any symmetry $\hat{g} \in \hat{G}$ gives rise to a loop in the space of 1d band Hamiltonians. The net fermion parity pumped through any point is a $\mathbb{Z}_2$-valued invariant of the loop. This is a special case of the Thouless pump (Teo and Kane, 2010; Moore and Balents, 2007).

Given $\hat{g} \in \hat{G}$ which is a symmetry of a band Hamiltonian $H(k)$, we can define a loop in the space of band Hamiltonians as follows. Since $SO(2N)$ is a connected group, we can choose a path $\eta : [0,1] \to SO(2N)$ such that $\eta(0) = 1$ and $\eta(1) = \hat{R}(\hat{g})$. Next we define $H(k,t) = \eta(t)H(k)\eta(t)^{-1}$. Since $\hat{R}(\hat{g})$ commutes with $H(k)$, $H(k,1) = H(k,0)$. Thus $H(k,t)$ is a loop in the space of 1d band Hamiltonians. A general argument (Teo and Kane, 2010; Moore and Balents, 2007) shows that the net fermion parity $(-1)^{B(\hat{g})}$ pumped through one cycle of this loop does not depend on the choice of path η. This immediately implies that $B(\hat{g}\hat{g}') = B(\hat{g}) + B(\hat{g}')$. Thus $B(\hat{g})$ defines an element of $H^1(\hat{G}, \mathbb{Z}_2)$.

To evaluate $B(\hat{g})$, we apply the general formula from (Teo and Kane, 2010) for Hamiltonians in class D. One simplification is that locally in k, t, the Berry connection can be taken as $\eta^{-1}\partial_t\eta$, and thus its curvature vanishes. Then

$$B(\hat{g}) = \frac{1}{2\pi} \int \mathrm{Tr}\left[(P_+(0) - P_+(\pi))\eta(t)^{-1}\partial_t\eta(t)\right] dt \qquad \text{(C.10)}$$

where $P_+(k)$ is the projector to positive-energy states at momentum k.

Next we decompose $\hat{R}$ into real irreducible representations r_α. Obviously, each representation contributes independently to $B(\hat{g})$. Representations of $\mathbb{C}$-type and $\mathbb{H}$-type do not contribute at all, since the corresponding Hamiltonians can be deformed to trivial ones. A Hamiltonian $A_{r,ij}$ corresponding to an $\mathbb{R}$-type representation r_α is of class D and can be deformed either to a trivial one or to a trivial one stacked with a single Kitaev chain. In the former case, both the boundary invariant $(-1)^{\beta(\hat{g})}$ and the charge-pumping invariant $B(\hat{g})$ are trivial (equal to 1). In the latter case, we get a single Majorana zero mode for each of the $d_r = \dim r$ basis vectors of r, so the boundary invariant $(-1)^{\beta(\hat{g})}$ is equal to $\det r(g)$. We just need to verify that $B(\hat{g})$ is also equal to $\det r(g)$ for d_r copies of the Kitaev chain. The on-site representation of $\hat{G}$ is given by $\hat{R} = r \oplus r$ in this case.

For d_r copies of the Kitaev chain, the projector to positive-energy states is

$$P_+(k) = \tfrac{1}{2}\left(\mathbb{1}_2 - \sigma_y \sin k + \sigma_z \cos k\right) \otimes \mathbb{1}_{d_r}, \qquad \text{(C.11)}$$

which commutes with $\hat{R}(\hat{g}) = \mathbb{1}_2 \otimes r(\hat{g})$ and satisfies $P_+(0)\hat{R}(\hat{g}) = r(\hat{g}) \oplus 0$ and $P_+(\pi)\hat{R}(\hat{g}) = 0 \oplus r(\hat{g})$. Let $\eta(t)$ be a path in $SO(2d_r)$ from 1 to $\hat{R}(\hat{g})$. We may choose it to belong to the $U(d_r)$ subgroup of matrices that commute with $P_+(0)$ and $P_+(\pi)$. Then $\eta(t) = q(t) \oplus \bar{q}(t)$ for a path $q(t)$ through $U(d_r)$ from $\mathbb{1}$ to $r(\hat{g})$.

Substituting all this into (C.10), we get

$$\begin{aligned} B(\hat{g}) &= \frac{1}{2\pi}\int \mathrm{Tr}\Big((P_+(0) - P_+(\pi))\eta(t)^{-1}\partial_t\eta(t)\Big)\,dt \\ &= \frac{1}{2\pi}\int \mathrm{Tr}\Big(q(t)^{-1}\partial_t q(t) - \bar{q}(t)^{-1}\partial_t\bar{q}(t)\Big)\,dt. \end{aligned} \tag{C.12}$$

Note that this vanishes whenever $q(t) = \bar{q}(t)$ at all t. We now show how to recover $(-1)^{B(\hat{g})} = \det r(\hat{g})$.

If $r(\hat{g})$ has determinant +1, it lives in $SO(d_r)$, which is path-connected. Hence the path $q(t)$ from $\mathbb{1}$ to $r(\hat{g})$ may be taken to lie in $SO(d_r) \subset U(d_r)$. Therefore $q(t) = \bar{q}(t)$ is real, and so $B(\hat{g}) = 0$.

If $r(\hat{g})$ has determinant -1, we construct $q(t)$ as follows. First connect $\mathbb{1}$ to $\mathrm{diag}(-1, +1, +1, \ldots, +1)$ by $\mathrm{diag}(\exp(it), +1, +1, \ldots, +1)$. Now that the determinant is -1, we may get to $r(\hat{g})$ through a real path in the identity-disconnected component of $O(d_r)$. This second segment of the path contributes nothing to $B(\hat{g})$. It remains to compute the contribution of the first segment, where $q(t) = \exp(it) \oplus \mathbb{1}$:

$$B(\hat{g}) = \frac{1}{2\pi}\int \left(e^{-it}\partial_t e^{it} - e^{it}\partial_t e^{-it}\right)\,dt = 1. \tag{C.13}$$

This completes the proof that $B(\hat{g}) = \beta(\hat{g})$. In particular, $B(P) = 0$, i.e. B is really a homomorphism from $G = \hat{G}/\mathbb{Z}_2^F$ to $\mathbb{Z}_2$.

The interpretation of $\beta(g)$ in terms of a fermion-parity pump has the following intuitive reason. Assume that one can make a "Wick rotation" of the pump. Then the twist by $\hat{g}$ along the "time" direction gets reinterpreted as a twist along the spatial direction. The invariant $B(\hat{g})$ can be re-interpreted as the fermionic parity of the ground state of the system with an $\hat{g}$-twist, or equivalently as the fermionic parity of the $\hat{g}$ domain wall. On the other hand, it is known (Kapustin, Turzillo, and You, 2018) that this is yet another interpretation of the invariant β.

To conclude this section, we show how to compute $B(g) = \beta(g)$ from the holonomy of the Berry connection between $k = 0$ and $k = \pi$. This makes the topological nature of $B(g)$ explicit. Recall first how the holonomy is defined. If there are

$2N$ Majorana fermions per site, a free 1d Hamiltonian can be described by a non-degenerate $2N \times 2N$ matrix $X(k)$, where k is the momentum (Chiu et al., 2016). At $k = 0$ and $k = \pi$, this matrix is real and skew-symmetric. We can bring $X(0)$ to the standard form X_0 using an orthogonal transformation $O(0) \in O(2N)$. Similarly, we use $X(\pi)$ to define $O(\pi) \in O(2N)$. The holonomy of the Berry connection is $O = O(\pi)O(0)^{-1}$. The invariant $(-1)^{\gamma}$ is equal to the sign of $\det O$ (Budich and Ardonne, 2013). If γ vanishes, then $\det O(0)$ and $\det O(\pi)$ have the same sign, and by a choice of basis we may assume that both $O(\pi)$ and $O(0)$ lie in $SO(2N)$.

To define a topological invariant associated to an element $\hat{g} \in \hat{G}$, we choose a path $\eta(t) : [0,1] \to SO(2N)$ from the identity to $\hat{R}(\hat{g})$. Consider now the following map from $[0,1]$ to $SO(2N)$:

$$\Pi(t) = \begin{cases} \eta(2t), & 0 \leq t \leq 1/2, \\ O\eta(2-2t)O^{-1}, & 1/2 \leq t \leq 1. \end{cases} \tag{C.14}$$

Since $O \equiv O(\pi)O(0)^{-1}$ is the holonomy of the Berry connection from 0 to π, it commutes with all symmetries of the Hamiltonian, including $\hat{R}(\hat{g})$ for all $\hat{g} \in \hat{G}$. This implies that $\Pi(t)$ is a loop in $SO(2N)$. We claim that $B(\hat{g})$ is the class of this loop in $\pi_1(SO(2N)) = \mathbb{Z}_2$.

This definition is independent of the path from 1 to $\hat{R}(\hat{g})$. Any two paths differ (in the sense of homotopy theory) by a loop in $SO(2N)$. Thus changing the path will result in composing $\Pi(t)$ with a loop and its conjugation by O. Since these two loops are homotopic, the homotopy class $[\Pi]$ is unchanged.

To prove that the homotopy class of the loop Π coincides with $B(\hat{g})$, one can follow the same strategy as before: use homotopy-invariance to reduce to the case of a single Kitaev chain, and then compute the invariant by choosing a particularly convenient path.