Borel Matchings and Analogs of Hall's Theorem

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Allison Yiyun Wang

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ABSTRACT

In classical graph theory, Hall's theorem gives a necessary and sufficient condition for a bipartite graph to have a perfect matching. The analogous statement for Borel perfect matchings is false. If we instead consider Borel perfect matchings almost everywhere or Borel perfect matchings generically, results similar to Hall's theorem hold. We present Marks' proof that König's theorem, a special case of Hall's theorem, fails in the context of Borel perfect matchings. We then discuss positive results about the existence of Borel matchings that are close to perfect in the measure theory and Baire category settings.

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Chapter 1

INTRODUCTION

Given a standard Borel space X, a *(Borel) graph* on X is some $\boldsymbol{G} = (X, G)$ such that $G \subseteq X^2$ is a symmetric, irreflexive, (Borel) set. The elements of X are called the vertices of $\boldsymbol{G}$, and we think of the relation G as defining the edges of $\boldsymbol{G}$. If $x, y \in X$ satisfy $(x, y) \in G$, we say that x and y are adjacent and that these vertices are incident to the edge $\{x, y\}$. For $x \in X$, we denote the set of vertices adjacent to x by $N_G(x) := \{y \in X \mid (x, y) \in G\}$. For $A \subseteq X$, we define $N_G(A) := \cup_{x \in A} N_G(x)$. A set $A \subseteq X$ is an *independent set* if no two vertices in A are adjacent. The *degree* of a vertex $x \in X$ is the cardinality of $N_G(x)$. A graph $\boldsymbol{G}$ is *d-regular* if every vertex has degree d. We say that $\boldsymbol{G}$ has *bounded degree* if there is some $n \in \mathbb{N}$ such that every vertex has degree at most n. We call $\boldsymbol{G}$ *locally finite* if every vertex has finite degree, and we call $\boldsymbol{G}$ *locally countable* if the degree of every vertex is countable.

A *(Borel) matching* of $\boldsymbol{G}$ is a symmetric, irreflexive, (Borel) subset $M \subseteq G$ such that if $(x, y) \in M$ and $(x, z) \in M$ for some $x, y, z \in X$, then $y = z$. Then M is a subset of the edges of $\boldsymbol{G}$ such that no vertex is incident to more than one edge in M. Let X_M denote the set of vertices that are incident to edges in M. A (Borel) matching M is a *(Borel) perfect matching* if $X_M = X$. A matching M is *G-invariant* if $x \in X_M$ implies $N_G(x) \subseteq X_M$.

The *chromatic number* of a graph $\boldsymbol{G} = (X, G)$, written $\chi(\boldsymbol{G})$, is the least cardinality of a set Y for which there is a map $c : X \to Y$ such that $c(x) \neq c(y)$ if $(x, y) \in G$. A graph $\boldsymbol{G}$ is called *bipartite* if $\chi(\boldsymbol{G}) = 2$. The *Borel chromatic number* of $\boldsymbol{G} = (X, G)$, denoted $\chi_B(\boldsymbol{G})$, is the least cardinality of a standard Borel space Y for which there is a Borel map $c : X \to Y$ such that $c(x) \neq c(y)$ if $(x, y) \in G$.

In classical graph theory, Hall's theorem gives the following necessary and sufficient condition for a bipartite graph to have a perfect matching.

Theorem 1.0.1 (Hall's theorem)**.** *Let $\boldsymbol{G}$ be a finite bipartite graph with bipartition $\{A, B\}$. Then $\boldsymbol{G}$ has a perfect matching if and only if $|N_G(F)| \geq |F|$ for every $F \subseteq A$ and every $F \subseteq B$.*

In the case where $\boldsymbol{G}$ is d-regular, Hall's theorem specializes to König's theorem.

Theorem 1.0.2 (König's theorem)**.** *For all $d \geq 2$, every finite bipartite d-regular graph has a perfect matching.*

We are interested in similar results about Borel matchings. However, the direct analog of König's theorem in a Borel setting does not hold.

Theorem 1.0.3. *[Mar16, Theorem 1.5] For all $d \geq 2$, there is a d-regular acyclic Borel graph $\boldsymbol{G}$ on a standard Borel space X such that $\chi_B(\boldsymbol{G}) = 2$ and $\boldsymbol{G}$ has no Borel perfect matching.*

We present Marks's proof of Theorem 1.0.3 in Section 2.

Instead of Borel perfect matchings, we may consider Borel matchings that are almost perfect. In Section 3, we discuss such matchings in the context of measure theory. Suppose X is a standard Borel space with a Borel probability measure μ, and suppose $\boldsymbol{G} = (X, G)$ is a locally countable Borel graph. We call a Borel matching M of $\boldsymbol{G}$ a *Borel perfect matching μ-almost everywhere* (a.e.) if X_M is G-invariant and $\mu(X_M) = 1$. We say that $\boldsymbol{G}$ is *μ-measure preserving* if for every partial Borel bijection $f : Y \to Z$ between Borel subsets $Y, Z \subseteq X$ satisfying $\text{Graph}(f) \subseteq G$, we have $\mu(A) = \mu(f(A))$ for all Borel $A \subseteq Y$. In this setting, the following theorem by Lyons and Nazarov provides a sufficient condition similar to Hall's theorem.

Theorem 1.0.4. *[LN11, Remark 2.6] Let $\boldsymbol{G} = (X, G)$ be a Borel graph on (X, μ) that is locally finite, μ-measure preserving, and bipartite. Suppose there is some constant $c > 1$ such that for all Borel independent sets $A \subseteq X$, $\mu(N_G(A)) \geq c\mu(A)$. Then $\boldsymbol{G}$ has a Borel perfect matching μ-a.e.*

Finally, in Section 4, we consider Borel matchings that are almost perfect in a Baire category sense. Suppose X is a Polish space and $\boldsymbol{G} = (X, G)$ is a Borel graph. We say that a Borel matching M of G is a *Borel perfect matching generically* if X_M is G-invariant and comeager. Marks and Unger proved the following statement, which gives a sufficient condition analogous to that in Hall's theorem.

Theorem 1.0.5. *[MU16, Theorem 1.3] Let X be a Polish space, and let $\boldsymbol{G}$ be a locally finite bipartite Borel graph with a bipartition $\{B_0, B_1\}$. Suppose there is some $\varepsilon > 0$ such that for every finite set $F \subseteq B_0$ or $F \subseteq B_1$, $|N_G(F)| \geq (1 + \varepsilon)|F|$. Then $\boldsymbol{G}$ admits a Borel perfect matching generically.*

Chapter 2

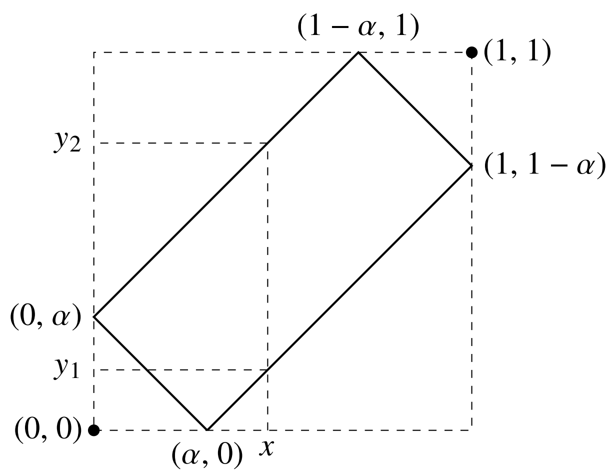
BOREL PERFECT MATCHINGS

One important result about matchings in classical graph theory is König's theorem.

Theorem 2.0.1 (König's theorem)**.** *For all $d \geq 2$, every finite bipartite d-regular graph has a perfect matching.*

2.1 Laczkovich's Example

In [Lac88], Laczkovich constructs the following example, which demonstrates that König's theorem does not hold in a Borel setting for $d = 2$. Fix some irrational $\alpha \in (0, 1)$. Let R denote the rectangle with vertices $(0, \alpha)$, $(\alpha, 0)$, $(1, 1 - \alpha)$, and $(1 - \alpha, 1)$, and let $R' := R \cup \{(0, 0), (1, 1)\}$, as shown in the diagram below.



Let X and Y be copies of $[0, 1]$, and let $\boldsymbol{G}$ be the bipartite graph on $X \sqcup Y$ where $x \in X$ and $y \in Y$ are adjacent in $\boldsymbol{G}$ if and only if $(x, y) \in R'$. For example, in the diagram above, the neighborhood of x is $\{y_1, y_2\}$. Observe that $\boldsymbol{G}$ is a 2-regular Borel graph with $\chi_B(\boldsymbol{G}) = 2$. Laczkovich proved that $\boldsymbol{G}$ is a counterexample to König's theorem in the Borel setting.

Theorem 2.1.1. *[Lac88] The graph $\boldsymbol{G}$ has no Borel perfect matching.*

Proof of Theorem 2.1.1. We follow Laczkovich's proof in [Lac88].

Let λ_1 denote the measure on R' such that $\lambda_1(p) = 0$ for all $p \in R'$ and such that for any disjoint line segments $A_1, A_2, \ldots, A_n \subseteq R'$,

$$\lambda_1(A_1 \sqcup A_2 \sqcup \ldots \sqcup A_n) = \sum_{i=1}^{n} \lambda(A_i),$$

where λ gives the Euclidean length of each line segment. Define the measure μ on R' by

$$\mu(H) := \frac{1}{2\sqrt{2}} \lambda_1(H)$$

for all measurable subsets $H \subseteq R'$. So $\mu(R') = 1$.

Note that a matching in $\boldsymbol{G}$ is equivalent to a subset $M \subseteq R'$ that is the graph of an injective involution from X to Y. We will prove that no such $M \subseteq R'$ is μ-measurable, which will imply that $\boldsymbol{G}$ does not have a Borel matching.

We will now define two functions $f : R' \to R'$ and $g : R' \to R'$. Let $f(x, y) = (x, y)$ if $x = 0$ or $x = 1$. If $x \in (0, 1)$, let $f(x, y) := (x, z)$, where z is the unique point in $[0, 1] \setminus \{y\}$ such that $(x, z) \in R'$. Similarly, let $g(x, y) = (x, y)$ if $y = 0$ or $y = 1$. If $y \in (0, 1)$, let $g(x, y) := (w, y)$, where w is the unique point in $[0, 1] \setminus \{x\}$ such that $(w, y) \in R'$. Note that $f = f^{-1}$ and $g = g^{-1}$ and that f and g are measure-preserving homeomorphisms from R' to itself.

We claim that $g \circ f$ is ergodic on R. Let $T = \mathbb{R}/\mathbb{Z}$ be the circle group with the Lebesgue measure. Let $h : R \to T$ be a measure-preserving homeomorphism satisfying

$$\begin{aligned} h(1, 1 - \alpha) &= 0; \\ h(1 - \alpha, 1) &= \frac{\alpha}{2}; \\ h(0, \alpha) &= \frac{1}{2}; \\ h(\alpha, 0) &= \frac{1 + \alpha}{2}. \end{aligned}$$

Define $k := h \circ g \circ f \circ h^{-1}$. Note that k is a measure-preserving homeomorphism from T to itself, so there is some constant $c \in T$ such that $k(t) = t + c$ for all $t \in T$ or $k(t) = -t + c$ for all $t \in T$. Since $k(0) = \alpha$ and $k(\frac{1}{2}) = \frac{1}{2} + \alpha$, $k(t) = t + \alpha$ for all $t \in T$. We chose α to be irrational, so k is ergodic on R. Therefore, $g \circ f$ is ergodic on R.

Suppose there is a μ-measurable $M \subseteq R'$ such that M is the graph of an injective involution from X to Y. Observe that $M \cap f(M)$, $R' \setminus (M \cup f(M))$, $M \cap g(M)$, and $R' \setminus (M \cup g(M))$ are finite sets, so $\mu(M) = \frac{1}{2}$. Using our earlier observation that $g = g^{-1}$, we note that $M \triangle (g \circ f)(M)$ is finite. Then $\mu(M) \in \{0, 1\}$ since $g \circ f$ is ergodic. This is a contradiction, so there is no such μ-measurable $M \subseteq R'$, and therefore, $\boldsymbol{G}$ does not have a Borel perfect matching. □

2.2 König's Theorem in Higher Degree

Laczkovich's example demonstrates that König's theorem fails in the Borel setting for degree $d = 2$. A theorem by Marks states that König's theorem fails for Borel matchings for all $d \geq 2$:

Theorem 2.2.1. *[Mar16, Theorem 1.5] For all $d \geq 2$, there is a d-regular acyclic Borel graph $\boldsymbol{G}$ on a standard Borel space X such that $\chi_B(\boldsymbol{G}) = 2$ and $\boldsymbol{G}$ has no Borel perfect matching.*

We follow the proof of Theorem 2.2.1 given in [Mar16]. We first introduce some definitions. Given a standard Borel space X and a countable group Γ, X^Γ is a standard Borel space under the product structure. We define the *left shift action* of Γ on X^Γ by

$$\alpha \cdot x(\beta) = x(\alpha^{-1}\beta)$$

for all $x \in X^\Gamma$ and $\alpha, \beta \in \Gamma$. We define $\mathrm{Free}(X^\Gamma)$ to be the set of all $x \in X^\Gamma$ for which $\gamma \cdot x \neq x$ for all $\gamma \in \Gamma \setminus \{e\}$. If X and Y are spaces equipped with actions of Γ and $f : X \to Y$ is a function satisfying $\gamma \cdot f(x) = f(\gamma \cdot x)$ for all $\gamma \in \Gamma$ and $x \in X$, then we say f is Γ-*equivariant*.

The proof of Theorem 2.2.1 relies on a result about equivalence relations. For an equivalence relation E on a standard Borel space X, a subset $A \subseteq X$ is a *complete section* if A intersects every E-class. If E and F are equivalence relations on X and there exist disjoint Borel sets $A, B \subseteq X$ such that A and B are complete sections for E and F, respectively, then E and F have *Borel disjoint complete sections*. We have the following theorem:

Theorem 2.2.2. *[Mar16, Theorem 3.7] Let Γ and Δ be countable groups. Define E_Γ to be the equivalence relation on $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that $x E_\Gamma y$ if and only if there exists some $\gamma \in \Gamma$ for which $\gamma \cdot x = y$. Define E_Δ similarly. Then E_Γ and E_Δ do not have Borel disjoint complete sections.*

To prove Theorem 2.2.2, we use several lemmas. We need the following definitions to state these lemmas. Given an equivalence relation E on a standard Borel space

X, E is a *countable Borel equivalence relation* if $E \subseteq X \times X$ is a Borel set and each E-class is countable. Suppose E is a countable Borel equivalence relation on a standard Borel space X, we define the following. A set $A \subseteq X$ is E-invariant if for every $x \in A$, the orbit of x is contained in A. We define $[X]^{<\infty}$ to be the standard Borel space of finite subsets of X, and we let $[E]^{<\infty}$ be the Borel subset of $[X]^{<\infty}$ consisting of the finite subsets of X whose elements are E-equivalent. The *intersection graph* on $[E]^{<\infty}$ is the graph with vertex set $[E]^{<\infty}$ where $A, B \in [E]^{<\infty}$ are adjacent exactly when $A \cap B \neq 0$ and $A \neq B$. Then we have the following lemma, which we state without proof:

Lemma 2.2.3. *[KM04, Lemma 7.3] Suppose E is a countable Borel equivalence relation on a standard Borel space X. Let $\boldsymbol{G}$ be the intersection graph on $[E]^{<\infty}$. Then $\boldsymbol{G}$ has a Borel $\mathbb{N}$-coloring.*

We also need the following definitions. Let $I \in \{1, 2, \ldots, \infty\}$, and let $\{E_i\}_{i<I}$ be a collection of equivalence relations on a standard Borel space X. If, for some $n \geq 2$, there is a sequence of distinct points $x_0, x_1, \ldots, x_n \in X$ and a sequence $i_0, i_1, \ldots, i_n \in \mathbb{N}$ such that $i_j \neq i_{j+1}$ for $0 \leq j \leq n-1$, $i_n \neq i_0$, and $x_0 E_{i_0} x_1 E_{i_1} x_2 \ldots x_n E_{i_n} x_0$, then we say that the E_i are *non-independent*. Otherwise, we say that the E_i are *independent*. Let $\bigvee_{i<I} E_i$ be the smallest equivalence relation that contains every E_i. The E_i are *everywhere non-independent* if for each $\bigvee_{i<I} E_i$-class $A \subseteq X$, the restrictions $E_i \restriction A$ are not independent. We state the following lemma without proof:

Lemma 2.2.4. *[Mar16, Lemma 2.3] Suppose $I \in \{1, 2, \ldots, \infty\}$, and suppose $\{E_i\}_{i<I}$ are countable Borel equivalence relations on a standard Borel space X that are everywhere non-independent. Then there is a Borel partition $\{A_i\}_{i<I}$ of X such that for all $i < I$, A_i^c intersects every E_i-class.*

We use Lemma 2.2.3 and Lemma 2.2.4 to prove the following lemma:

Lemma 2.2.5. *[Mar16, Lemma 2.1] Let Γ and Δ be countable groups. Suppose $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ is a Borel set. Then at least one of the following holds.*

1. *There is a continuous injective function $f : \mathrm{Free}(\mathbb{N}^{\Gamma}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that f is equivariant with respect to the left shift action of Γ and $\mathrm{ran}(f) \subseteq A$.*
2. *There is a continuous injective function $f : \mathrm{Free}(\mathbb{N}^{\Delta}) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that f is equivariant with respect to the left shift action of Δ and $\mathrm{ran}(f) \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$.*

We refer to words of the form $\gamma_0\delta_1\gamma_2\ldots$, where $\gamma_i \in \Gamma \setminus \{e\}$ and $\delta_i \in \Delta \setminus \{e\}$ for all i, as Γ-words. Similarly, we refer to words of the form $\delta_0\gamma_1\delta_2\ldots$, where $\gamma_i \in \Gamma \setminus \{e\}$ and $\delta_i \in \Delta \setminus \{e\}$, as Δ-words.

Proof of Lemma 2.2.5. First, we will define a game that produces an element $y \in \mathbb{N}^{\Gamma*\Delta}$, where player I defines $y(\alpha)$ when α is a Γ-word and player II defines $y(\alpha)$ when α is a Δ-word. Let $\gamma_0, \gamma_1, \ldots$, and $\delta_0, \delta_1, \ldots$ be enumerations of $\Gamma \setminus \{e\}$ and $\Delta \setminus \{e\}$, respectively. Let Y be the subset of $\mathbb{N}^{\Gamma*\Delta}$ consisting of the elements $x \in \mathbb{N}^{\Gamma*\Delta}$ such that for all x' in the orbit of x and $i \in \mathbb{N}$, $\gamma_i \cdot x' \neq x'$ and $\delta_i \cdot x' \neq x'$.

We now define a function $t : \Gamma * \Delta \setminus \{e\} \to \mathbb{N} \cup \{-1\}$ to determine which values of y are fixed on each turn of the game. Define $t(e) := -1$. For every $\alpha \in \Gamma * \Delta \setminus \{e\}$, we can write α uniquely in the form $\gamma_{i_0}\delta_{i_1}\gamma_{i_2}\ldots\pi_{i_m}$ or $\delta_{i_0}\gamma_{i_1}\delta_{i_2}\ldots\pi_{i_m}$, where $\pi_{i_m} \in \{\gamma_{i_m}, \delta_{i_m}\}$ based on the parity of m. Define $t(\alpha) := \max_{0\leq j\leq m}(i_j + j)$. Observe that if α is a Δ-word or $\alpha = e$, then $t(\gamma_i\alpha) = \max\{t(\alpha) + 1, i\}$. In particular, if $i \leq n$, then $t(\gamma_i\alpha) \leq n$ if and only if $t(\alpha) < n$. Similarly, if $i \leq n$ and α is a Γ-word or $\alpha = e$, then $t(\delta_i\alpha) \leq n$ if and only if $t(\alpha) < n$.

We will define the game G_k^B for any $k \in \mathbb{N}$ and any Borel set $B \subseteq Y$. The game G_k^B will produce some $y \in \mathbb{N}^{\Gamma*\Delta}$ satisfying $y(e) = k$. On each turn $n \in \mathbb{N}$, player I defines $y(\alpha)$ for all Γ-words α satisfying $t(\alpha) = n$, followed by player II defining $y(\alpha)$ for all Δ-words α satisfying $t(\alpha) = n$. We now specify who wins each run of the game. If $y \in B$, then player II wins, and if $y \in Y \setminus B$, then player I wins. Otherwise, $y \notin Y$, so one of the following must hold:

1. There is some $\alpha \in \Gamma * \Delta$ and $i \in \mathbb{N}$ such that $\gamma_i\alpha^{-1} \cdot y = y$. In this case, we say that (α, Γ) witnesses $y \notin Y$.

2. There is some $\alpha \in \Gamma * \Delta$ and $i \in \mathbb{N}$ such that $\delta_i\alpha^{-1} \cdot y = y$. In this case, we say that (α, Δ) witnesses $y \notin Y$.

We say that α witnesses $y \notin Y$ if either of the above statements holds. We specify that if (e, Γ) witnesses $y \notin Y$, then player II wins, and if (e, Δ) witnesses $y \notin Y$, then player I wins. If neither of these happens, then there must be a Γ-word or Δ-word witnessing $y \notin Y$. If there is some Δ-word α witnessing $y \notin Y$ such that no Γ-words β with $t(\beta) \leq t(\alpha)$ witness $y \notin Y$, then player I wins. Otherwise, player II wins.

We will use games of the form G_k^B to construct our desired function f for the given $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma*\Delta})$. Let E_Γ be the equivalence relation on Y such that $xE_\Gamma y$ if and only if $\gamma \cdot x = y$ for some $\gamma \in \Gamma$, and let E_Δ be defined similarly. Note that E_Γ and E_Δ are everywhere non-independent on $Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$. By applying Lemma 2.2.4 with $I = 2$, there exists some Borel $C \subseteq Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that C intersects every E_Δ-class on $Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ and C^c intersects every E_Γ-class on $Y \setminus \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$. Define $B_A := A \cup C$. Borel determinacy implies that for every $k \in \mathbb{N}$, player I or player II has a winning strategy for $G_k^{B_A}$. Then player I or player II must have a winning strategy for $G_k^{B_A}$ for infinitely many k. Suppose player II has a winning strategy for $G_k^{B_A}$ for infinitely many $k \in \mathbb{N}$, and let S be the set of all such k. We omit the case where player I has a winning strategy for $G_k^{B_A}$ for infinitely many $k \in \mathbb{N}$; a similar proof holds in that situation. For each $k \in S$, fix a winning strategy for player II in $G_k^{B_A}$.

There exists a continuous injective equivariant function $g : \mathrm{Free}(\mathbb{N}^\Gamma) \to \mathrm{Free}(S^\Gamma)$. So if we can construct a continuous injection $f : \mathrm{ran}(g) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that f is equivariant with respect to the left shift action of Γ and $\mathrm{ran}(f) \subseteq A$, then $f \circ g$ is a function satisfying the lemma.

We now define $f : \mathrm{ran}(G) \to \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$. Consider any $x' \in \mathrm{Free}(\mathbb{N}^\Gamma)$, and let $x := g(x')$. We will choose moves for player I in $G_{x'(\gamma^{-1})}^{B_A}$ such that the outcome of the game when player II plays according to their fixed winning strategy will be the value of $\gamma \cdot f(x)$. We will do so simultaneously for all $\gamma \in \Gamma$ such that f is an equivariant function and $f(x)(\gamma) = x'(\gamma)$ for all $\gamma \in \Gamma$.

We will define $(\gamma \cdot f(x))(\alpha)$ inductively on $t(\alpha)$. By our definition of $G_{x'(\gamma^{-1})}^{B_A}$, we have $(\gamma \cdot f(x))(e) = x'(\gamma^{-1})$ for all $\gamma \in \Gamma$. Suppose we have defined $(\gamma \cdot f(x))(\alpha)$ for all $\gamma \in \Gamma$ and all α for which $t(\alpha) < n$. We need to specify player I's move on turn n in each game; equivalently, we need to define $(\gamma \cdot f(x))(\beta)$ for all Γ-words β with $t(\beta) = n$. Suppose β is a Γ-word such that $t(\beta) = n$ and $\beta = \gamma_i \alpha$ for some $i \in \mathbb{N}$ and some α with $t(\alpha) < n$. By definition of the left shift action of Γ, we need $(\gamma \cdot f(x))(\gamma_i \alpha) = (\gamma_i^{-1} \gamma \cdot f(x))(\alpha)$ for all $\gamma \in \Gamma$. By assumption, we have already defined the value of $(\gamma_i^{-1} \gamma \cdot f(x))(\alpha)$, so this determines what player I should play for $(\gamma \cdot f(x))(\gamma_i \alpha)$. So we can determine player I's move on turn n in each game, which determines $(\gamma \cdot f(x))(\beta)$ for all $\gamma \in \Gamma$ and all Γ-words β such that $t(\beta) = n$. The values of $(\gamma \cdot f(x))(\beta)$ for Δ-words β is determined by player II's move on turn n in each game, which is specified by the winning strategies we fixed for player II. By induction, we can thus use the games $G_{x'(\gamma^{-1})}^{B_A}$ to define $\gamma \cdot f(x)$ for all $\gamma \in \Gamma$.

From our construction, f is injective, continuous, and equivariant with respect to the left shift action of Γ. We also know that each $f(x)$ is a winning outcome for player

II in $G_{x'}^{B_A}$. It suffices to show that $\mathrm{ran}(f) \subseteq A$. We will first show that $\mathrm{ran}(f) \subseteq Y$.

Consider any $x' \in \mathrm{Free}(\mathbb{N}^\Gamma)$, and let $x = g(x')$. Because $f(x)$ results from a winning strategy for player II, (e, Δ) does not witness $f(x) \notin Y$. Suppose (e, Γ) witnesses $f(x) \notin Y$. Then there exists $i \in \mathbb{N}$ such that for all $\gamma \in \Gamma$, $(\gamma_i \cdot f(x))(\gamma) = f(x)(\gamma)$. We can rewrite $(\gamma_i \cdot f(x))(\gamma)$ as $f(x)(\gamma_i^{-1}\gamma)$. By construction of f, we then have $x'(\gamma_i^{-1}\gamma) = x'(\gamma)$ for all $\gamma \in \Gamma$, or equivalently, $\gamma_i \cdot x' = x'$. But this contradicts $x \in \mathrm{Free}(\mathbb{N}^\Gamma)$. We conclude that (e, Γ) does not witnesses $f(x) \notin Y$. Therefore, e does not witness $f(x) \notin Y$.

We will show that for all $\alpha \in \Gamma * \Delta$ and all $x' \in \mathrm{Free}(\mathbb{N}^\Gamma)$, α does not witness $f(x) \notin Y$, where $x = g(x')$. We will use induction on $t(\alpha)$. Suppose this statement holds for all $\beta \in \Gamma * \Delta$ such that $t(\beta) < n$. First, we consider any Γ-word α satisfying $t(\alpha) = n$. We can find $\gamma \in \Gamma$ and $\beta \in \Gamma * \Delta$ such that $\alpha = \gamma\beta$ and $t(\beta) < n$. Then $\alpha^{-1} \cdot f(x) = \beta^{-1}\gamma^{-1} \cdot f(x)$, which we can rewrite as $\beta^{-1} \cdot f(\gamma^{-1} \cdot x)$ since f is Γ-equivariant. By our induction hypothesis, β does not witness $f(\gamma^{-1} \cdot x) \notin Y$, so α does not witness $f(x) \notin Y$. Now consider any Δ-word α satisfying $t(\alpha) = n$. By our induction hypothesis and our argument for Γ-words, there is no Γ-word β witnessing $f(x) \notin Y$ satisfying $t(\beta) \leq n$. Since $f(x)$ is consistent with player II's winning strategy, α cannot witness $f(x) \notin Y$.

Therefore, we have $\mathrm{ran}(f) \subseteq Y$. Because each $f(x)$ is the result of a winning strategy for player II in some game $G_k^{B_A}$, we must have $\mathrm{ran}(f) \subseteq B_A = A \cup C$. Note that $\mathrm{ran}(f)$ is Γ-invariant. By definition of C, C does not contain any non-empty Γ-invariant sets, so we must have $f(x) \subseteq A$. We conclude that f is a function of form (1), as desired. □

We now use these lemmas to prove Theorem 2.2.2.

Proof of Theorem 2.2.2. It suffices to show that it is impossible to find any Borel set $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ such that A is a complete section for E_Δ and $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$ is a complete section for E_Γ. Given any Borel set $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$, we can find a function f_A as in Lemma 2.2.5. If f_A satisfies statement (1) of Lemma 2.2.5, then $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$ cannot be a complete section for E_Γ. If f_A satisfies statement (2) of Lemma 2.2.5, then A cannot be a complete section for E_Δ. Therefore, we cannot have A be a complete section for E_Δ while $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$ is a complete section for E_Γ. □

Finally, we prove Theorem 2.2.1.

Proof of Theorem 2.2.1. Fix some $d \geq 2$, and let $\Gamma := \mathbb{Z}/d\mathbb{Z} =: \Delta$. Define E_Γ

and E_Δ as in Theorem 2.2.2, and let X be the standard Borel space consisting of equivalence classes of E_Γ and E_Δ. Let $\mathbf{G}$ be the intersection graph on X, and note that $\mathbf{G}$ is d-regular, acyclic, and Borel with $\chi_B(\mathbf{G}) = 2$.

We claim that $\mathbf{G}$ does not admit a Borel perfect matching. Suppose $M \subseteq X \times X$ is a Borel perfect matching for $\mathbf{G}$. Define $A \subseteq \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta})$ by

$$A := \{x \in \mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \mid \{x\} = R \cap S \text{ for some } (R, S) \in M\}.$$

Then A and $\mathrm{Free}(\mathbb{N}^{\Gamma * \Delta}) \setminus A$ are Borel disjoint complete sections for E_Γ and E_Δ, contradicting Theorem 2.2.2. Therefore, $\mathbf{G}$ does not admit a Borel perfect matching.

□

Chapter 3

BOREL MATCHINGS AND MEASURE THEORY

In graph theory, Hall's theorem states that bipartite graphs with a certain expansion property have perfect matchings. Recall that for any set of vertices F in a graph $\boldsymbol{G}$, we write $N_{\boldsymbol{G}}(F)$ to denote the set of vertices that are adjacent to F. Hall's theorem states the following:

Theorem 3.0.1 (Hall's theorem)**.** *Let $\boldsymbol{G}$ be a finite bipartite graph with bipartition $\{A, B\}$. Then $\boldsymbol{G}$ has a perfect matching if and only if $|N_G(F)| \geq |F|$ for every $F \subseteq A$ and every $F \subseteq B$.*

Theorem 2.2.1 states that König's theorem does not hold for Borel perfect matchings. Since König's theorem is a specific case of Hall's theorem, Hall's theorem likewise fails for Borel perfect matchings. Instead of Borel perfect matchings, we can consider Borel matchings that are perfect on "large" subsets of a graph. Lyons-Nazarov [LN11] and Marks-Unger [MU16] proved results analogous to Hall's theorem from the perspectives of measure theory and Baire category notions, respectively.

3.1 Bipartite Graphs

To present the theorem of Lyons and Nazarov, we first recall some definitions. Let $\boldsymbol{G} = (X, G)$ be a locally countable Borel graph on a standard Borel space X with some probability measure μ. Recall that if M is a Borel matching of $\boldsymbol{G}$ such that X_M is G-invariant and $\mu(X_M) = 1$, we call M a *Borel perfect matching μ-almost everywhere* (a.e.). Furthermore, recall that the graph $\boldsymbol{G}$ is *μ-measure preserving* if for every Borel automorphism $f : X \to X$ such that $\text{Graph}(f) \subseteq G$, $\mu(A) = \mu(f^{-1}(A))$ for all measurable $A \subseteq X$.

Lyons and Nazarov proved the following theorem, which uses a measure-theoretic concept of expansion in the setting of Borel perfect matchings μ-a.e.

Theorem 3.1.1. *[LN11, Remark 2.6] Let $\boldsymbol{G} = (X, G)$ be a Borel graph on a standard probability space (X, μ) such that $\boldsymbol{G}$ is locally finite, μ-measure preserving, and bipartite. Suppose there is some constant $c > 1$ such that for all Borel independent sets $A \subseteq X$, $\mu(N_G(A)) \geq c\mu(A)$. Then $\boldsymbol{G}$ has a Borel perfect matching μ-a.e.*

The proof of this statement relies on a result by Elek and Lippner about the lengths of augmenting paths in Borel matchings. A *path* of length k in $\boldsymbol{G} = (X, G)$ is a sequence of vertices $x_0, x_1, \ldots, x_k$ such that $(x_i, x_{i+1}) \in G$ for $0 \leq i < k$ and such that $x_i \neq x_j$ for $i \neq j$. Given a Borel matching M on a graph $\boldsymbol{G} = (x, G)$, a path $x_0, x_1, \ldots, x_{2n+1}$ in $\boldsymbol{G}$ is called an *augmenting path* if $(x_{2i}, x_{2i+1}) \notin M$ for $0 \leq i \leq n$, $(x_{2i+1}, x_{2i+2}) \in M$ for $0 \leq i < n$, and $x_0, x_{2n+1} \notin X_M$. Such a path is augmenting in the following sense: if we define M' to be the matching on $\boldsymbol{G}$ that reverses which edges in $x_0, x_1, \ldots, x_{2n+1}$ are contained in M and agrees with M on all other edges, then $X_{M'} = X_M \cup \{x_0, x_{2n+1}\}$. Elek and Lippner proved the following statement about the lengths of augmenting paths in Borel matchings.

Proposition 3.1.2. *[EL10, Proposition 1.1] Let X be a standard Borel space, and let $\boldsymbol{G} = (X, G)$ be a locally finite Borel graph on X. Fix any $T \geq 1$. For any Borel matching M of $\boldsymbol{G}$, there is a Borel matching M' of $\boldsymbol{G}$ such that $X_M \subseteq X_{M'}$ and such that every augmenting path in M' has length greater than $2T + 1$.*

Proof of Proposition 3.1.2. We follow the proof given in [Ant13].

Let Y be the set of paths of odd length at most $2T + 1$ in X. Let $\boldsymbol{H}$ be the graph with vertex set Y such that two paths $x_0, x_1, \ldots, x_{2n+1}$ and $y_0, y_1, \ldots, y_{2m+1}$ in Y are adjacent in $\boldsymbol{H}$ if and only if $x_i = y_j$ for some $0 \leq i \leq n$ and $0 \leq j \leq m$. Since $\boldsymbol{G}$ is locally finite, $\boldsymbol{H}$ is locally finite as well, an therefore, there is a Borel coloring $c : Y \to \mathbb{N}$ on $\boldsymbol{H}$.

Define $M_{-1} := M$. Let $c_0, c_1, c_2, \ldots$ be a sequence of elements in $\mathbb{N}$ such that each element of $\mathbb{N}$ appears infinitely many times in this sequence. To construct the desired matching M', we first construct a sequence of matchings $M_0, M_1, M_2, \ldots$ such that $X_{M_n} \subseteq X_{M_{n+1}}$ for all $n \in \mathbb{N}$. We do the following for each $i \in \mathbb{N}$ inductively. First, consider all paths in $c^{-1}(c_i)$. If $x_0, x_1, \ldots, x_{2i+1} \in c^{-1}(c_i)$ is an augmenting path in M_{i-1}, switch which edges of the path lie in the matching. In other words, remove each (x_{2j+1}, x_{2j+2}) from M_{i-1} for $0 \leq j < i$, and add (x_{2j}, x_{2j+1}) to M_{i-1} for $0 \leq j \leq i$. Since no two paths in $c^{-1}(c_i)$ share a vertex, the result of this procedure is still a matching; call this matching M_i. Note that $X_{M_{i-1}} \subseteq X_{M_i}$ for each $i \in \mathbb{N}$.

We claim that for every edge $(x, y) \in G$, there exists an N such that either $(x, y) \in X_n$ for all $n > N$ or $(x, y) \notin X_n$ for all $n > N$. Suppose not, and let $(x, y) \in G$ be a counterexample. Then (x, y) must be switched infinitely many times as part of an augmenting path. Define $B \subseteq X$ to be the collection of points $z \in X$ for which there is some $k \leq 2n$ and $x_1, x_2, \ldots, x_k \in X$ such that $x, x_1, x_2, \ldots, x_k, z$ is a path in $\boldsymbol{G}$. Note that every augmenting path that contains the vertex x must lie in B, so switching the edges in any augmenting path containing x increases the number of matched vertices in B by 1. Since (x, y) is switched infinitely many times, $|X_{M_n} \cap B|$ is unbounded as n increases, implying that B is an infinite set. However, B must be finite since $\boldsymbol{G}$ is locally finite. So such a counterexample cannot exist. We conclude that for every $(x, y) \in G$, there is some N such that $(x, y) \in X_n$ for all $n > N$ or $(x, y) \notin X_n$ for all $n > N$.

Let $M' \subseteq G$ consist of the edges (x, y) for which there exists some N such that $(x, y) \in M_n$ for all $n > N$. If (x, y) and (x, z) are edges in M', there is some n for which $(x, y) \in M_n$ and $(x, z) \in M_n$ by definition of M'. Since each M_n is a matching, $y = z$. Therefore, M' is a matching as well. Since each M_n is Borel, M' is a Borel matching. We claim that M' has the desired properties.

Consider any $x \in X_M$. Since $X_M \subseteq X_{M_n}$ for all $n \in \mathbb{N}$, we can find points $\{y_n\}_{n\in\mathbb{N}}$ such that $(x, y_n) \in M_n$ for each $n \in \mathbb{N}$. By local finiteness of $\boldsymbol{G}$, some y_m must occur infinitely many times. Our argument above implies that there is some N such that $(x, y_m) \in M_n$ for all $n > N$. Then $(x, y_m) \in M'$, so $x \in X_{M'}$. Therefore, $X_M \subseteq X_{M'}$.

It remains to show that M' does not contain any augmenting paths of length at most $2T + 1$. Suppose there is an augmenting path $x_0, x_1, \ldots, x_{2n+1}$ in M' with $n \leq T$. By definition of M', we can find some N such that for all $n > N$, we have $(x_{2i}, x_{2i+1}) \notin M_n$ for $0 \leq i \leq n$, and $(x_{2i+1}, x_{2i+2}) \in M_n$ for $0 \leq i < n$. Let k be the color assigned to this path by the coloring c. By definition of the sequence $c_0, c_1, c_2, \ldots$, we can find $N' > N$ such that $c_{N'} = k$. Then in round N', the edges of $x_0, x_1, \ldots, x_{2n+1}$ are switched. In other words, $(x_{2i}, x_{2i+1}) \in M_{N'}$ for $0 \leq i \leq n$, and $(x_{2i+1}, x_{2i+2}) \in M_{N'}$ for $0 \leq i < n$. This is a contradiction, so no such path exists. Therefore, M' does not contain any augmenting paths of length at most $2T + 1$.

Thus, M' is a Borel matching of $\boldsymbol{G}$ satisfying the desired conditions. □

We now use Proposition 3.1.2 to prove Theorem 3.1.1.

Proof of Theorem 3.1.1. We follow the proof given in [Ant13].

First, we recursively define a sequence $\{M_i\}_{i\in\mathbb{N}}$ of Borel matchings in $\boldsymbol{G}$. Let $M_0 := \emptyset$. Given the matching M_i, define M_{i+1} to be the Borel matching obtained via the construction in the proof of Proposition 3.1.2 above, using $T = i + 1$. Then $X_{M_i} \subseteq X_{M_{i+1}}$, and every augmenting path for M_{i+1} has length greater than $2(i+1)+1$. We bound the measure of $X \setminus X_{M_i}$ using the following lemma.

Lemma 3.1.3. *[LN11] Suppose M is a Borel matching of $\boldsymbol{G}$ such that every augmenting path for M has length greater than $2n+1$, and let $B := X \setminus X_M$. Then $\mu(B) \leq c^{-\frac{n}{2}}$.*

Proof of Lemma 3.1.3. We define a sequence of sets $\{B_i\}_{0\leq i\leq n}$ recursively. Let $B_0 := B$. Given B_{2k} for some $k \in \mathbb{N}$, define

$$B_{2k+1} := N_G(B_{2k}) = \{x \in X \mid \exists y \in B_{2k}\ (x, y) \in G\}.$$

Given B_{2k+1} for some $k \in \mathbb{N}$, define

$$B_{2k+2} := \{x \in X \mid \exists y \in B_{2k+1}\ (x, y) \in M\}.$$

First, we will show that B_{2k} is an independent set for $0 \leq 2k \leq n$. Suppose $x, y \in B_{2k}$ satisfy $(x, y) \in G$. By construction of B_{2k}, there is a path $x_0, x_1, \ldots, x_{2k} = x$ such that $x_j \in B_j$ for $0 \leq j \leq 2k$ and such that $(x_{2i+1}, x_{2i+2}) \in M$ for all $0 \leq i < k$. Similarly, there is a path $y_0, y_1, \ldots y_{2k} = y$ such that $y_j \in B_j$ for $0 \leq j \leq 2k$ and such that $(y_{2i+1}, y_{2i+2}) \in M$ for all $0 \leq i < k$. Then observe that $x_0, x_1, \ldots, x_{2k}, y_{2k}, y_{2k-1}, \ldots, y_1, y_0$ is an augmenting path for M of length $2k+1$, contradicting the definition of M. Therefore, we conclude that B_{2k} is an independent set.

Since each B_{2k} is a Borel independent set, our conditions on $\boldsymbol{G}$ imply that $\mu(B_{2k+1}) \geq c\mu(B_{2k})$ for each k. Now observe that $B_{2k+1} \subseteq X_M$: if some element of B_{2k+1} lies in $X \setminus X_M = B_0$, then $\boldsymbol{G}$ must contain an odd cycle, contradicting the assumption that $\boldsymbol{G}$ is bipartite. So we have $\mu(B_{2k+1}) = \mu(B_{2k+2})$ for each k since $\boldsymbol{G}$ is μ-measure preserving. Thus, we conclude that

$$\begin{aligned}\mu(B) &= \mu(B_0)\\ &\leq c^{-\frac{n}{2}}\mu(B_n)\\ &\leq c^{-\frac{n}{2}}.\end{aligned}$$

□

By Lemma 3.1.3, $\mu(X \setminus X_{M_i})$ converges to 0 as $i \to \infty$. Since $X \setminus X_{M_0} \supseteq X \setminus X_{M_1} \supseteq \ldots$, we have $\mu(\cup_{n\in\mathbb{N}} X_{M_n}) = 1$.

For each $x \in \cup_{n\in\mathbb{N}} X_{M_n}$, there exists N such that $x \in X_{M_n}$ for all $n > N$. For $n > N$, let $y_n \in X$ be defined such that $(x, y_n) \in M_n$. Since $\boldsymbol{G}$ is locally finite, there must be some y such that $y_n = y$ for infinitely many $n > N$. We claim that there is a set

$A' \subseteq \cup_{n\in\mathbb{N}} X_{M_n}$ of measure 1 such that for $x \in A'$, $y_n = y$ for cofinitely many such n. Let C be the set of all $x \in \cup_{n\in\mathbb{N}} X_{M_n}$ such that this does not hold. We wish to show that $\mu(C) = 0$.

For each $n \in \mathbb{N}$, define C_n to be the set of $x \in X$ such that there is some $y \in X$ for which $(x, y) \in M_n$ and $(x, y) \notin M_{n+1}$. Observe that $C \subseteq \limsup_{n\to\infty} C_n$. So

$$\begin{aligned}\mu(C) &\leq \mu(\limsup_{n\to\infty} C_n)\\ &\leq \mu(\lim_{k\to\infty} \bigcup_{n\geq k} C_n)\\ &\leq \lim_{k\to\infty} \sum_{n\geq k} \mu(C_n).\end{aligned}$$

Suppose $x \in C_n$. By the construction of M_{n+1} from M_n according to the proof of Proposition 3.1.2, x must be contained in exactly one augmenting path of length at most $2n + 1$ containing some element of $X \setminus X_{M_n}$. Then because $\boldsymbol{G}$ is μ-measure preserving, $\mu(C_n) \leq (2n + 1)\mu(X \setminus X_{M_n})$. Applying Lemma 3.1.3, we have $\mu(C_n) \leq (2n + 1)c^{-\frac{n}{2}}$. So $\lim_{k\to\infty} \sum_{n\geq k} \mu(C_n) = 0$, implying that $\mu(C) = 0$. Therefore, there is a set $A' \subseteq \cup_{n\in\mathbb{N}} X_{M_n}$ of measure 1 such that for $x \in A'$, $y_n = y$ for cofinitely many such n. Let $A \subseteq A'$ be a G-invariant set of measure 1.

We now define a matching M on A. For $x, y \in A$, we define $(x, y) \in M$ if and only if there are cofinitely many n such that $(x, y) \in M_n$. Since each M_n is a Borel matching, M is a Borel matching. Therefore, M is a Borel perfect matching μ-a.e. for $\boldsymbol{G}$. □

The requirement that $c > 1$ in Theorem 3.1.1 is necessary. Let $\boldsymbol{G}$ be the Cayley graph of the free part of the shift action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$. Then $\mu(N_G(A)) \geq \mu(A)$ for all Borel independent sets $A \subseteq 2^{\mathbb{Z}}$. If $\boldsymbol{G}$ has Borel perfect matching μ-a.e., then we obtain a Borel set that has measure $\frac{1}{2}$ and is invariant under applying the shift action twice, contradicting the fact that the action of $\mathbb{Z}$ on $2^{\mathbb{Z}}$ is mixing. Therefore, $\boldsymbol{G}$ does not have a Borel perfect matching μ-a.e., so the assumption $c > 1$ is necessary.

Furthermore, the expansion condition in Theorem 3.1.1 cannot be relaxed to one on finite sets, unlike in Hall's theorem.

Proposition 3.1.4. *[KM19, Proposition 15.1] Let (X, μ) be a standard probability space. For every $n \geq 1$, there is some Borel graph $\boldsymbol{G} = (X, G)$ that is μ-measure preserving and has bounded degree, satisfies $\chi_B(\boldsymbol{G}) = 2$, and has the property that*

$|N_G(F))| \geq n|F|$ for every finite independent set $F \subseteq X$, with no Borel perfect matching μ-a.e.

Conley and Miller proved a theorem about Borel perfect matchings μ-a.e. when the complexity of the edge relation is restricted. Let (X, μ) be a standard probability space, and let $\boldsymbol{G} = (X, G)$ be a locally countable Borel graph. If there is a G-invariant Borel set $A \subseteq X$ such that $\mu(A) = 1$ and such that, on A, the equivalence relation generated by G can be written as the increasing union of finite Borel equivalence relations, the graph $\boldsymbol{G}$ is *μ-hyperfinite*. Conley and Miller proved the following result about Borel matchings in acyclic μ-hyperfinite graphs:

Theorem 3.1.5. *[CM17, Theorem B] Let (X, μ) be a standard probability space, and let $\boldsymbol{G} = (X, G)$ be an acyclic, locally countable Borel graph. If $\boldsymbol{G}$ is μ-hyperfinite and every point in some G-invariant Borel set of measure 1 has degree at least 3, then $\boldsymbol{G}$ has a Borel perfect matching μ-a.e.*

3.2 A Shift Action

We conclude this section by presenting a theorem by Csóka and Lippner, which extends a result by Lyons and Nazarov in [LN11] by removing the assumption that the Cayley graph of the given group is bipartite.

Recall that for a standard Borel space X and a countable group Γ, the *shift action* of Γ on X^Γ is given by $\alpha \cdot x(\beta) = x(\alpha^{-1}\beta)$ for all $x \in X^\Gamma$ and $\alpha, \beta \in \Gamma$. Suppose S is a finite symmetric generating set of Γ, where S does not contain the identity. Then we define $\boldsymbol{G}(S, [0,1]^\Gamma)$ to be the graph on $[0,1]^\Gamma$ such that $x, y \in [0,1]^\Gamma$ are adjacent exactly when there is some $\gamma \in S$ such that $\gamma \cdot x = y$. Csóka and Lippner proved the following theorem about Borel matchings on $\boldsymbol{G}(S, [0,1]^\Gamma)$.

Theorem 3.2.1. *[CL17, Theorem 1.1] Let Γ be a non-amenable group with a finite symmetric generating set S not containing the identity. Let μ be the probability measure on $[0,1]^\Gamma$ defined as the product of the Lebesgue measure on each coordinate. Then $\boldsymbol{G}(S, [0,1]^\Gamma)$ has a Borel perfect matching μ-a.e.*

We first present several definitions.

Let $\boldsymbol{G} = (X, G)$ be a d-regular, infinite, connected graph. A *real cut* for $\boldsymbol{G}$ is a partition $X = A \sqcup A^c$ such that A is a finite set and $|A| \geq 2$. If $|A|$ is odd, we say that this partition is a *real cut into odd sets*. The *size* of the cut is $|\{(x, y) \in G \mid x \in A, y \in A^c\}|$, the number of edges between A and A^c, which is finite since $\boldsymbol{G}$ is

d-regular and A is finite. Csóka and Lippner proved the following lemma about the sizes of real cuts:

Lemma 3.2.2. *[CL17, Lemma 2.5] Let $\mathbf{G}$ be as above, and suppose that for every $x_1, x_2 \in X$, there is an automorphism of $\mathbf{G}$ sending $x_1 \mapsto x_2$. Then every real cut of $\mathbf{G}$ has size at least d, and there is a real cut with size exactly d if and only if every $x \in X$ lies in a unique d-clique.*

Now let X be a probability space with measure μ, and let $\mathbf{G} = (X, G)$ be a d-regular, μ-measure preserving, Borel graph on X. We define a measure on sets of edges as follows. For any symmetric, measurable set $H \subseteq G$, define

$$\mu^*(H) := \frac{1}{2} \int_X d_H(x) d\mu(x),$$

where $d_H(x)$ is the number of elements $y \in N_G(x)$ such that $(x, y) \in H$. For any measurable set of vertices $Y \subseteq X$, let

$$E(Y, Y^c) := \{(x, y) \in G \mid x \in Y, y \in Y^c\} \cup \{(y, x) \in G \mid x \in Y^c, y \in Y\}$$

be the collection of edges between Y and Y^c. For $c_0 > 0$, we say that $\mathbf{G}$ is a c_0-*expander* if for all measurable $Y \subseteq X$, we have

$$\mu^*(E(Y, Y^c)) \geq c_0 \mu(H) \mu(H^c).$$

If every real cut into odd sets for $\mathbf{G}$ has size at least $d+1$, and if there is some $c_0 > 0$ such that $\mathbf{G}$ is a c_0-expander, then we say that $\mathbf{G}$ is *admissible*. Csóka and Lippner proved the following theorem about augmenting paths in admissible graphs:

Theorem 3.2.3. *[CL17, Theorem 4.2] For any $c_0 > 0$ and any integer $d \geq 3$, there is a constant $c = c(c_0, d)$ such that the following holds: given any d-regular, μ-measure preserving, Borel graph $\mathbf{G}$ on a probability space (X, μ) such that $\mathbf{G}$ is admissible, and given any Borel matching M on $\mathbf{G}$ such that $\mu(X \setminus X_M) \geq \varepsilon$, there is an augmenting path for M of length at most $c(\log \frac{1}{\varepsilon})^3$.*

We can now prove Theorem 3.2.1, following [CL17].

Proof of Theorem 3.2.1. Let $d := |S|$, and note that $\mathbf{G} := \mathbf{G}(S, [0,1]^\Gamma)$ is d-regular. We also observe that for any $x_1, x_2 \in [0,1]^\Gamma$, there is an automorphism sending $x_1 \mapsto x_2$.

First, suppose there is a real cut of $\boldsymbol{G}$ with size d. Then by Lemma 3.2.2, every vertex of $\boldsymbol{G}$ lies in a unique d-clique. Then define M to be the collection of edges $(x, y) \in G$ such that x and y are in different d-cliques. Observe that M is a Borel perfect matching, as desired.

Now suppose every real cut of $\boldsymbol{G}$ has size at least $d+1$. Let $M_1 := \emptyset$, and define Borel matchings $M_1, M_2, M_3, \ldots$ inductively such that M_n has no augmenting paths of length at most $2n+1$ and such that M_{n+1} is obtained from M_n via the procedure described in the proof of Proposition 3.1.2. Let $U_n := X \setminus X_{M_n}$. Let $E_n = M_n \triangle M_{n+1}$ denote the set of edges in G that are switched between M_n and M_{n+1}. Note that every edge in E_n must lie along an augmenting path of length at most $2n+3$. Each augmenting path has both endpoints in U_n, and each point in U_n is contained in at most one augmenting path whose edges are switched, so $\mu^*(E_n) \leq (2n+3)\mu(U_n)$.

Suppose $\mu(U_n) = \varepsilon$. Since Γ is non-amenable, there is some $c_0 > 0$ such that $\boldsymbol{G}$ is a c_0-expander [LN11, Lemma 2.3]. We are assuming that every real cut of $\boldsymbol{G}$ has size at lest $d+1$, so $\boldsymbol{G}$ is admissible. Let $c = c(c_0, d)$ be a constant satisfying Theorem 3.2.3. Then we know that $\boldsymbol{G}$ has an augmenting path for M_n of length at most $c(\log \frac{1}{\varepsilon})^3$. By definition of M_n, we must have

$$c\left(\log\frac{1}{\varepsilon}\right)^3 > 2n+1,$$

which yields

$$\varepsilon < \exp\left(-\left(\frac{2n+1}{c}\right)^{\frac{1}{3}}\right).$$

So by the Borel-Cantelli lemma, the set E of edges that lie in E_k for infinitely many k satisfies $\mu^*(E) = 0$.

Define M to be the set of edges $(x, y) \in G$ that lie in M_n for cofinitely many n. Since each M_n is a Borel matching, M is a Borel matching. From above, we have that $\mu(X \setminus \cup_{n\in\mathbb{N}} X_{M_n}) = 0$ and $\mu^*(E) = 0$, so M is a Borel perfect matching μ-a.e. $\square$

Chapter 4

BOREL MATCHINGS AND BAIRE CATEGORY

An expansion condition on finite sets of vertices is sufficient to conclude that a Borel matching is almost perfect in a Baire category setting. Suppose $\boldsymbol{G} = (X, G)$ is a Borel graph on a Polish space X. We recall that a Borel matching M of G is called a *Borel perfect matching generically* if X_M is G-invariant and comeager. The following theorem by Marks and Unger adapts Hall's theorem to this setting.

Theorem 4.0.1. *[MU16, Theorem 1.3] Let X be a Polish space, and let $\boldsymbol{G} = (X, G)$ be a locally finite bipartite Borel graph with a bipartition $\{B_0, B_1\}$, which need not be Borel. Suppose there is some $\varepsilon > 0$ such that for every finite set $F \subseteq B_0$ or $F \subseteq B_1$, $|N_G(F)| \geq (1+\varepsilon)|F|$. Then $\boldsymbol{G}$ admits a Borel perfect matching generically.*

To prove this result, we need several definitions. Let $\boldsymbol{G} = (X, G)$ be a graph. For $x, y \in X$, let $d_G(x, y)$ be the length of the minimum-length path from x to y. Define $\boldsymbol{G_2} = (X, G_2)$ to be the graph on X such that $(x, y) \in G_2$ if and only if $d_G(x, y) = 2$. For any matching M of $\boldsymbol{G}$, we define the graph $\boldsymbol{G} - M$ to be the restriction $\boldsymbol{G} \upharpoonright (X \setminus X_M)$.

Now suppose $\boldsymbol{G}$ is bipartite with some bipartition $\{B_0, B_1\}$. The graph $\boldsymbol{G}$ satisfies *Hall's condition* if for every finite set $F \subseteq B_0$ or $F \subseteq B_1$, $|N_G(F)| \geq |F|$. If $\boldsymbol{G}$ satisfies Hall's condition, and if every G_2-connected finite set F with $|F| \geq n$ such that $F \subseteq B_0$ or $F \subseteq B_1$ satisfies $|N_G(F)| \geq (1 + \varepsilon)|F|$, we say that $\boldsymbol{G}$ satisfies $\text{Hall}_{\varepsilon,n}$.

Marks and Unger proved the following lemma.

Lemma 4.0.2. *[MU16, Lemma 3.1] Let X be a Polish space, and let $\boldsymbol{G} = (X, G)$ be a locally finite Borel graph on X. Given any funtion $f : \mathbb{N} \to \mathbb{N}$, there is some sequence $\{A_n\}_{n\in\mathbb{N}}$ of Borel sets in X such that $A := \cup_{n\in\mathbb{N}} A_n$ is comeager and G-invariant, and such that $d_G(x, y) > f(n)$ for all distinct $x, y \in A_n$.*

Proof of Lemma 4.0.2. We follow the proof in [MU16]. Let $\{U_i\}_{i\in\mathbb{N}}$ be a basis of open sets for X. For $i \in \mathbb{N}$ and $r > 0$, let $B_{i,r}$ be the set consisting of exactly those $x \in U_i$ such that for all $y \neq x$ with $d_G(x, y) \leq r$, $y \notin U_i$. Given any $r > 0$ and any

$x \in X$, the set $\{y \mid y \neq x, d_G(x,y) \leq r\}$ is finite since $\boldsymbol{G}$ is locally finite, so there is some i for which $x \in B_{i,r}$. Therefore, for any $r > 0$, $X = \cup_{i\in\mathbb{N}} B_{i,r}$.

Let $\{S_i\}_{i\in\mathbb{N}}$ be a set of Borel automorphisms that generate G, and denote the closure of this set under compositions and inverses by $\{T_i\}_{i\in\mathbb{N}}$. Let $\varphi : \mathbb{N} \to \mathbb{N}^2$ be a bijection, and write $\varphi(n) = (i_n, j_n)$. For any $n \in \mathbb{N}$, we have $X = \cup_{k\in\mathbb{N}} B_{k,f(n)}$, so we have $X = \cup_{k\in\mathbb{N}} T_{i_n}(B_{k,f(n)})$ since T_{i_n} is an automorphism. Then there is some $k \in \mathbb{N}$ such that $T_{i_n}(B_{k,f(n))})$ is non-meager in U_{j_n}. Define $A'_n := B_{k,f(n)}$, and observe that for distinct $x, y \in A'_n$, we have $d_G(x,y) > f(n)$. Let $A' := \cup_{n\in\mathbb{N}} A'_n$.

For each $i \in \mathbb{N}$, $T_i(A')$ is non-meager in every U_j, so $T_i(A')$ is a comeager set. Then $A := \cap_{i\in\mathbb{N}} T_i(A')$ is comeager. For each $i \in \mathbb{N}$ and $x \in X$, observe that $x \in A$ if and only if $T_i(x) \in A$ because $\{T_j\}_{j\in\mathbb{N}}$ is closed under compositions. Because $x, y \in X$ are G-connected if and only if $T_i(x) = y$ for some i, we conclude that A is G-invariant. Therefore, if we set $A_n := A \cap A'_n$ for each $n \in \mathbb{N}$, the sets $\{A_n\}_{n\in\mathbb{N}}$ satisfy the desired conditions. □

We now use Lemma 4.0.2 to prove Theorem 4.0.1, following the proof in [MU16].

Proof of Theorem 4.0.1. Fix an increasing function $f : \mathbb{N} \to \mathbb{N}$ such that $f(n) \geq 8$ for all $n \in \mathbb{N}$ and such that $\sum_{n\in\mathbb{N}} \frac{8}{f(n)} < \varepsilon$. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of Borel sets obtained by applying Lemma 4.0.2 with this function f, and let $A := \cup_{n\in\mathbb{N}} A_n$. It suffices to find a Borel perfect matching of $\boldsymbol{G} \restriction A$.

Let $\varepsilon_n := \varepsilon - \sum_{i\leq n} \frac{8}{f(i)}$ for each $n \in \mathbb{N}$. By our definition of f, $\varepsilon_n > 0$. Observe that it is enough to find a sequence of Borel matchings $\{M_n\}_{n\in\mathbb{N}}$ of $\boldsymbol{G}$ such that for all $n \in \mathbb{N}$, we have $\cup_{m\leq n} A_m \subseteq X_{M_n}$, $M_n \subseteq M_{n+1}$, and $\boldsymbol{G} - M_n$ satisfies $\mathrm{Hall}_{\varepsilon_n, f(n)}$. If we construct such a sequence, then $M := \cup_{n\in\mathbb{N}} M_n$ is a Borel perfect matching of $\boldsymbol{G} \restriction A$, implying that M is a Borel perfect generically of $\boldsymbol{G}$.

We define our sequence $\{M_n\}_{n\in\mathbb{N}}$ inductively. Let $M_{-1} := \emptyset$ and $\varepsilon_{-1} := \varepsilon$. By assumption, $\boldsymbol{G} - M_{-1} = \boldsymbol{G}$ satisfies $\mathrm{Hall}_{\varepsilon_{-1},1}$. Suppose we have defined M_{n-1} to satisfy the desired conditions. Let $X_{n-1} = X \setminus X_{M_{n-1}}$ be the set of vertices that are not matched by M_{n-1}. By assumption, $\boldsymbol{G} - M_{n-1}$ satisfies Hall's condition, so it has a perfect matching. So for any $x \in A_n \cap X_{n-1}$, there is some edge e that is incident to x in a perfect matching of $\boldsymbol{G} - M_{n-1}$. Note that $(\boldsymbol{G} - M_{n-1}) - \{e\}$ satisfies Hall's condition as well. We define a Borel set of such edges as follows. Let $\{T_i\}_{i\in\mathbb{N}}$ be Borel automorphisms generating G. For $x \in A_n \cap X_{n-1}$ and $i \in \mathbb{N}$ such that $x\ G\ T_i(x)$, let $e_{i,x}$ denote the edge between x and $T_i(x)$. Let e_x be the edge $e_{i,x}$ where $i \in \mathbb{N}$ is the minimal natural number for which $(\boldsymbol{G} - M_{n-1}) - \{e_{i,x}\}$

satisfies Hall's condition. We know e_x exists by our previous observations. Define $M'_n := \{e_x \mid x \in A_n \cap X_{n-1}\}$, and let $M_n := M_{n-1} \cup M'_n$.

Note that M_n is a matching since M_{n-1} is a matching and $d_G(x, y) > f(n) \geq 8$ for all distinct $x, y \in A_n \cap X_{n-1}$. Since M_{n-1} is Borel by our inductive hypothesis and M'_n was defined in a Borel manner, M_n is a Borel matching. By our construction, it is clear that $\cup_{m \leq n} A_m \subseteq X_{M_n}$ and $M_{n-1} \subseteq M_n$. We want to show that $\boldsymbol{G} - M_n$ satisfies $\mathrm{Hall}_{\varepsilon_n, f(n)}$. By the inductive hypothesis and our construction of M'_n, $\boldsymbol{G} - M_n$ satisfies Hall's condition. Let $X_n := X \setminus X_{M_n}$. It remains to show that for all finite $(G - M_n)_2$-connected sets F such that $|F| \geq f(n)$ and $F \subseteq X_n \cap B_0$ or $F \subseteq X_n \cap B_1$, we have $|N_{G-M_n}(F)| \geq (1 + \varepsilon_n)|F|$.

Let F be a finite $(G - M_n)_2$-connected subset of $X_n \cap B_0$ or $X_n \cap B_1$. Let $D := N_{G-M_{n-1}}(F) - N_{G-M_n}(F)$. We consider the cases $|D| \geq 2$ and $|D| \leq 1$ separately.

First, suppose $|D| \geq 2$. For each $x \in D$, there is some $y_x \in F$ such that xGy_x. Furthermore, there must exist some z such that $(x, z) \in M'_n$, and exactly one of x and z must be in A_n Let $\tilde{x} \in \{x, z\}$ be the point contained in A_n. Because M_n is a matching and F is contained in either B_0 or B_1, we have $\tilde{x} \neq \tilde{x'}$ for any distinct $x, x' \in D$. Then our definition of A_n implies that $d_G(y_x, y_{x'}) > f(n) - 4$. Because F is $(G - M_n)_2$-connected, there must be a path in $\boldsymbol{G} - M_n$ from y_x to $y_{x'}$. So there must be at least $\lfloor \frac{f(n)-4}{4} \rfloor$ elements $z \in F$ such that $d_G(y_x, z) \leq \frac{f(n)-4}{2}$. Since we know $d_G(y_x, y_{x'}) > f(n) - 4$ for distinct $x, x' \in D$, note that the sets $\{z \in F \mid d_G(y_x, z) \leq \frac{f(n)-4}{2}\}$ must be disjoint for all $x \in D$. Then $|F| \geq \lfloor \frac{f(n)-4}{4} \rfloor \cdot |D| \geq \frac{f(n)}{8} \cdot |D|$. So

$$\begin{aligned}
|N_{G-M_n}(F)| &= |N_{G-M_{n-1}}| - |D| \\
&\geq (1 + \varepsilon_{n-1})|F| - \frac{8}{f(n)}|F| \\
&\geq \left(\left(1 + \varepsilon - \sum_{i \leq n-1} \frac{8}{f(i)} \right) - \frac{8}{f(n)} \right) |F| \\
&= (1 + \varepsilon_n)|F|,
\end{aligned}$$

as desired.

We now consider the case $|D| \leq 1$. Since $|F| \geq f(n)$, we can rewrite $|D| =$

$|N_{G-M_{n-1}}(F)| - |N_{G-M_n}|(F) \leq 1$ as

$$\begin{aligned} |N_{G-M_n}(F)| &\geq |N_{G-M_{n-1}}(F)| - 1 \\ &\geq (1+\varepsilon_{n-1})|F| - \frac{1}{f(n)}|F| \\ &> (1+\varepsilon_{n-1})|F| - \frac{8}{f(n)}|F| \\ &= (1+\varepsilon_n)|F|. \end{aligned}$$

Thus, $\boldsymbol{G} - M_n$ satisfies $\mathrm{Hall}_{\varepsilon_n, f(n)}$. We have inductively constructed a sequence $\{M_n\}_{n\in\mathbb{N}}$ of Borel matchings satisfying the desired conditions. Therefore, by our earlier argument, $\boldsymbol{G}$ has a Borel perfect matching generically. □

Unlike in Hall's theorem, Theorem 4.0.1 does not hold when $\varepsilon = 0$.

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