THE INTERACTION OF FLOW DISCONTINUITIES WITH SMALL DISTURBANCES IN A COMPRESSIBLE FLUID

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SUMMARY

The interaction of a random small disturbance field in a compressible fluid with shock waves and flame fronts is analyzed. The disturbance field, which may consist of fluctuations of pressure, entropy, and vorticity, is found to be modified in passing through the shock or flame.

In the case of the shock wave, it is found that all of the three types of disturbances are generated in comparable strength in the downstream flow by the presence of any of the three in the upstream flow. Moderate fluctuations of either vorticity (turbulence) or entropy will produce intense noise fields in the downstream flow. If the shock is normal, the frequency of this noise field is much lower for very weak shocks than for strong shocks, given the same upstream velocity and disturbance wave length. If the weak shock is oblique to the flow, the frequency of the noise is increased.

For the flame front, also, it is found that all three types of disturbances are generated in the downstream flow by the presence of one of them in the upstream flow. In this case, however, the normal propagation Mach number of the flame enters as a small parameter. It is found that the intensity of the downstream turbulence generated by sound waves impinging on the upstream face of the flame is proportional to the reciprocal of this Mach number times the intensity of the upstream pressure fluctuation. Hence, rather strong turbulence may be generated downstream of a flame by comparatively weak sound upstream. The

pressure amplitudes of the sound fields generated by entropy and vorticity fluctuations in the upstream flow are proportional, respectively, to the Mach number squared and cubed. For ordinary hydrocarbon flames, ten percent turbulent velocity fluctuations, or one percent entropy (temperature) fluctuations will cause audible sound to be emitted. The frequency is in the range of 20 to perhaps 100 cycles per second for an input disturbance wave length of one inch.

Although the analysis is carried out for an isolated, infinite discontinuity, it is felt that the results are applicable, at least qualitatively, to the complicated configurations of shock waves found in under or overexpanded nozzles, and to the flame configurations found in actual combustion processes.

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I. INTRODUCTION

The problem to be considered here is an outgrowth of recent interest in noise generation by fluid mechanical processes. It is a matter of common experience that such processes do produce noise; high speed jets and turbulent Bunsen burner flames are two examples. However, the production of noise by such mechanisms had not received much study until the advent of the modern jet-propelled airplane. The propulsive jets of these craft produce so much noise as to be quite objectionable if operated near densely populated areas.

The manner in which this noise is produced is not completely understood, however it is likely that most of the noise from present-day jet engines with subsonic exhaust jets is produced by turbulence in the jet. Lighthill(5) has studied the generation of noise by turbulence and applied the theory to the production of noise by gaseous jets. One of the results of this theory, which is verified by experiments, is that the intensity of the noise produced varies approximately as the eighth power of the jet velocity. Thus, as the jet velocity of engines is increased by improvements in design or by adding afterburners, the noise from this source is likely to become very severe.

If the velocity of the jet is supersonic another, perhaps equally important mechanism for the production of noise exists; it is noise generation by the interaction of turbulence with shock waves. This mechanism was first recognized by Ribner(7) and Moore(8). Ribner computed the interaction of a sinusoidal velocity profile of small amplitude with a shock wave, finding that small disturbances of pressure and entropy, as well as a modified velocity profile, were produced downstream of the shock. He realized that since a field of turbulence might be represented by a superposition of such velocity profiles, the resultant superposed pressure fields would then constitute a noise field, which would be very intense by acoustical standards. Moore carried out a similar analysis for the case of plane sound waves interacting with shock waves, finding that both velocity and entropy disturbances, as well as pressure disturbances, arose downstream of the shock.

Small disturbances in a compressible fluid may be separated, as shown by Kovasznay(6) and Moyal(4), into three types, which consist respectively of fluctuations of pressure, entropy, and vorticity. These three types of disturbances obey separate differential equations, which are linear if the disturbances are small so that quadratic terms are negligible, and which are not coupled if viscous and heat conductive effects are also small. Hence, to this approximation, the three types of small disturbances do not interact with each other.

If the amplitudes of the disturbances are not small, the interaction terms become appreciable, and each type of disturbance may be produced by each of the others. Lighthill's theory considers the generation of noise by relatively strong turbulence (vorticity). On the other hand, even if the disturbances are weak, they will interact if carried through a region of high gradients in the mean flow quantities, since the interaction terms again become large in this region. There are three common flow phenomena in which very large gradients are encountered. They are shock waves, flame fronts, and shear layers. The first two are distinguished from the last by the fact that the gas flows through, rather than along the surface of them. The shear layer will not be considered because it would interact substantially only with pressure waves, not with the entropy and vorticity fields.

The problem to be considered here then, is the interaction of discontinuities, in particular shock waves and flames, with small disturbances of pressure, entropy, and vorticity. Since these small disturbances are ordinarily random, of the nature of turbulence, only the statistical properties of the disturbances can be specified, and hence only the influence of the discontinuity on these properties can be determined. Specifically, the mean squares of the pressure, entropy, and vorticity fluctuations are the quantities which will be regarded as known inputs to the interaction process. It will be assumed that the input disturbances are isotropic. The quantities to be determined are the mean squares of the pressure, entropy, and vorticity fluctuations which result from the interaction of the input disturbances with the discontinuity. For simplicity, it will be assumed that the discontinuity is an infinite plane, separating two semi-infinite regions of uniform flow perturbed by the small disturbance fields. The extent to which this idealized model approximates the actual physical problem of finite regions of nearly uniform flow, separated by discontinuities and perhaps bounded by solid surfaces, may be understood from the nature of the small flow disturbances themselves. The vorticity and entropy disturbances may be thought of as small amplitude velocity and temperature variations. As Kovasznay has shown, they are carried along by the main flow, in such a way that an observer moving with the mean flow velocity would see no changes in these quantities with time. Thus the nature of the vorticity and entropy fluctuations at a point in the fluid is dependent only on the character of the disturbances along the stream-tube passing thru that point, and the solution for these quantities from the infinite discontinuity should be exactly applicable to the case of the finite discontinuity. On the other hand, the pressure disturbances are essentially sound waves, which propagate in the fluid. Therefore a given point in the fluid may be affected by pressure disturbances originating at all other points of the fluid, if the flow is subsonic. In the case of an infinite discontinuity a given point in the flow is affected by pressure disturbances having all possible directions of propagation away from one side of the discontinuity, a given point of origin on the discontinuity corresponding to each direction of propagation. Now if the disturbance is restricted to a finite region of the discontinuity, only those pressure disturbances whose lines of constant phase are such as to pass thru both the finite region and the point in the flow will influence the pressure field at that point. On the other hand, if the finite discontinuity is contained in a channel, or bounded by any sound-reflecting surfaces, the reflection processes may well make a given point in the fluid accessible to pressure disturbances having nearly all possible directions of propagation, in which case the solution of the problem of an infinite discontinuity will be a reasonable approximation to that of a real, finite discontinuity. It is therefore felt that the analysis should provide at least a qualitative understanding of the influence of discontinuities on small disturbances in actual flow processes. Some possible applications will be mentioned here.

There are a number of cases in which the interaction of shocks with small disturbances may be of importance. One of these, as mentioned above, is the supersonic, over- or under-expanded nozzle. In either case there will be shocks in the jet. If the flow is turbulent as it leaves the nozzle, this turbulence will be carried thru the shocks. Similarly, if there are temperature non-uniformities in the flow, due to combustion irregularities in the engine carrying the nozzle, these temperature fluctuations will be carried through the shocks. The interaction of the shocks with the turbulence or temperature fluctuations will give rise to a sound field in the flow downstream of the shocks. If the intensity of the turbulence and temperature fluctuation can be estimated, the noise level within the jet can then be estimated from this analysis. To determine the amount of noise radiated from the jet is another problem; however since the jet is bounded by constant pressure boundaries, and the temperature is in general lower outside than inside, it seems that the noise level just outside the jet should be of the same order of magnitude as that within the jet. Furthermore the frequency of sound waves would be conserved in the transmission process thru the jet boundaries, so the estimates of sound frequency given by this analysis should apply directly. Another interesting case is the interaction of the leading edge shock of a supersonic airplane with atmospheric turbulence, temperature fluctuations, or sound. In particular, the influence of the noise field so produced on the airplane skin or on the boundary layer may be of interest.

It is harder to find clear cut examples of plane flame fronts which may interact with disturbances. However, if a turbulent flame may be thought of as a plane flame front the position of which fluctuates with time*, this analysis should be applicable, and should for example, be useful in predicting the noise level generated by turbulent flames.

Since the inception of this analysis, solutions of the interaction of the vorticity mode with shocks and flames have been published by Ribner[9] and Tucker[10], respectively. The present analysis includes the results of these papers as special cases, and in addition considers some aspects of these problems not contained in the above papers. The most important of these are a determination of the frequency of the sound generated by shock-turbulence interaction, and a treatment of the sound generated by flame-turbulence interaction.

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* For a recent investigation of this question, see Reference 16 .

II. FORMULATION OF PROBLEM

The problem has been idealized, as explained in the introduction, to the case of two semi-infinite regions of uniform flow, separated by an infinite plane discontinuity. On the upstream side of the discontinuity there is supposed to exist a random small disturbance field which interacts with the discontinuity and is regarded as the input to the interaction process. It is desired to determine the disturbance field which results from the interaction process.

Because the differential equations governing the small flow disturbances are linear, if the input disturbance were specified completely, the problem could be solved exactly by the well-known methods of linear analysis. Actually, since only the statistical properties of the input disturbance are specified, a complete solution is not needed; however, because of the ease with which complete solutions are obtained, the usual procedure in linear problems such as this is to define completely a model which has the same statistical properties as the input disturbance, solve the interaction process completely for this model, then extract the desired statistical properties from the complete solution. For some examples of this procedure, see Ref. (11).

In the present problem, the model must be capable of representing the statistical properties of a distribution of random vorticity, pressure, and entropy fluctuations. It will be assumed that all statistical properties of this distribution are stationary in time. Then since the relevant statistical properties of the model are to be independent of time, the time dependence of the model may be chosen arbitrarily. A convenient choice is to represent the pressure field by a superposition of plane sinusoidal sound waves, and the vorticity and entropy fields by plane sinusoidal velocity and entropy profiles. The disturbance field may then be regarded as composed of elementary disturbances, each of which satisfies the equations of the fluid, so that the interaction process may be computed separately for the individual disturbances, and the desired results obtained by superposition.

The analysis is divided into four sections. In the first, the detailed interaction process is solved for the three types of input disturbances, insofar as it can be without specifying the precise nature of the discontinuity. A linear transformation between the upstream normal velocity, pressure, and entropy and the corresponding downstream quantities is defined, the coefficients of which may be derived from the conservation laws for a given type of discontinuity. From this transformation and the assumption that the tangential component of velocity is conserved in passing thru the discontinuity, a set of boundary conditions is derived relating the amplitudes of the resultant sinusoidal disturbances of vorticity, pressure, and entropy to the amplitudes of the input disturbances with the same spatial and temporal periodicity at the unperturbed position of the discontinuity. This set of relations is found to be one short of sufficient, the other condition being a characteristic of the discontinuity which is not contained in the conservation laws. Finally a set of transfer functions is defined which is sufficient to describe the influence of the discontinuity on the disturbance field.

In the second section, the statistical properties of the resultant perturbation field are calculated in terms of the statistical properties of the input disturbances and the transfer functions defined in the previous section. The superposition of plane sinusoidal disturbances naturally leads to use of the Fourier transform technique. The statistical results are thus formulated as a series of integrals involving the transfer functions and the input statistical properties.

The final two sections are devoted to calculation of the transfer functions and statistical properties for the two specific types of discontinuities: shock waves and flame fronts. For the shock wave, the Rankine - Hugoniot equations, which are derivable from the conservation laws, and the fact that it is impossible for disturbances originating at the shock to influence the region upstream of the shock, provide the information necessary for completion of the problem. In the case of the flame front, the analogous relations are the Hugoniot equation, and the relationship between the normal burning velocity and the upstream temperature and pressure.

In both cases, the transfer functions are obtained in analytical form such that most of the integrations required for evaluation of the statistical properties can be done analytically, whereas the corresponding integrations in Refs. (9) and (10) were done numerically.

PART I

INTERACTION OF ELEMENTARY DISTURBANCES WITH THE DISCONTINUITY

A. Governing Equations of the Fluid

Assuming that the effects of viscosity and heat conduction are small during the time intervals involved in the interaction of the turbulence with the discontinuity*, the equations governing the fluid are:

Momentum,

$$\rho^* \frac{D\underline{u}^*}{Dt} + \nabla p^* = 0 \tag{1.1}$$

Continuity,

$$\frac{\partial \rho^*}{\partial t} + \nabla \cdot (\rho^* \underline{u}^*) = 0 \tag{1.2}$$

Energy,

$$\frac{Ds^*}{Dt} = 0 \tag{1.3}$$

State,

$$p^* = \rho^* R T^* \tag{1.4}$$

If the disturbance velocities are small compared to the main stream velocity, and the pressure, density and entropy fluctuations and their derivatives are small compared to their respective undisturbed values, the equations may be linearized by taking as new variables,

$$\begin{aligned} u_1 &= u_1^* - U \\ u_2 &= u_2^* \\ u_3 &= u_3^* \\ p &= p^* - P \\ \delta\rho &= \rho^* - \rho \\ s &= s^* - s_0 \end{aligned}$$

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* See Appendix C for a discussion of the neglected terms.

where U , P , ρ and S_0 are the average values of the various quantities. The resulting system of equations is

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}\right)\underline{u} + \frac{1}{\rho}\nabla p = 0 \tag{1.5}$$

$$\frac{1}{a^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}\right)^2 p - \nabla^2 p = 0 \tag{1.6}$$

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}\right) s = 0 \tag{1.7}$$

where Eq. (1.4) has been used to eliminate the density as a variable. Now taking the curl of Eq. (1.5), and letting the vorticity associated with the disturbance be $\underline{\Omega} = \nabla\times\underline{u}$, the equation governing the vorticity is found to be,

$$\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x_1}\right)\underline{\Omega} = 0 \tag{1.8}$$

Equations (1.6), (1.7) and (1.8) show that pressure, vorticity and entropy disturbances are, to first order, independent, since they obey separate differential equations. Equations (1.7) and (1.8) indicate that the entropy and vorticity are convected by the main flow in such a way that an observer moving with the main flow would see vorticity and entropy fields independent of time. Equation (1.6) is a wave equation, indicating that pressure disturbances propagate in the fluid.

B. Fourier Analysis of the Disturbance Field

As shown by Eqs. (1.6), (1.7) and (1.8), a general disturbance field in a fluid may be thought of as composed of pressure, vorticity and entropy fluctuations. It is possible to represent this field by means of Fourier integrals as follows:

$$\frac{u}{U} = \int e^{i[\underline{K}\cdot\underline{X} + \omega t]} \, d\underline{Z}_u(\underline{K},\omega)$$

$$\frac{p}{P} = \int e^{i[\underline{K}\cdot\underline{X} + \omega t]} \, dZ_p(\underline{K},\omega)$$

$$\frac{s}{c_p} = \int e^{i[\underline{K}\cdot\underline{X} + \omega t]} \, dZ_s(\underline{K},\omega)$$

where the X are chosen so that the mean velocity is zero. Now in terms of the stationary coordinate system, these become

$$\frac{u}{U} = \int e^{i[K_1(X_1 - Ut) + K_2X_2 + K_3X_3 + \omega t]} \, d\underline{Z}_u(\underline{K},\omega) \tag{1.9}$$

$$\frac{p}{P} = \int e^{i[K_1(X_1 - Ut) + K_2X_2 + K_3X_3 + \omega t]} \, dZ_p(\underline{K},\omega) \tag{1.10}$$

$$\frac{s}{c_p} = \int e^{i[K_1(X_1 - Ut) + K_2X_2 + K_3X_3 + \omega t]} \, dZ_s(\underline{K},\omega) \tag{1.11}$$

Equations (1.6), (1.7) and (1.8) impose certain easily determined conditions on ω if it is assumed that the integrals of Eqs. (1.9), (1.10) and (1.11) represent superpositions of elementary flows, so that the integrands themselves must satisfy the differential equations. Thus, for the pressure field, from Eqs. (1.10) and (1.6),

$$\omega^2 = a^2 (K_1^2 + K_2^2 + K_3^2)$$

or

$$\omega = \pm a K \tag{1.12}$$

and for the vorticity and entropy fields,

$$\omega = 0 \quad .$$

Part of the velocity field is associated with the pressure field, and part with the vorticity field. For the part associated with the pressure, the vorticity is zero, or

$$\nabla \times \frac{\underline{u}_p}{U} = 0$$

Using Eq. (1.9), this may be shown to be equivalent to

$$\underline{K} \times d\underline{Z}_u (\underline{K}, \omega) = 0$$

which means that $d\underline{Z}_u$ and hence $\underline{u}_p$ is parallel to $\underline{K}$. From Eqs. (1.9), (1.10) and (1.5), the velocity field associated with the pressure field is,

$$\frac{\underline{u}_p}{U} = \frac{1}{\gamma M} \int e^{i[K_1(x_1 - Ut) + K_2 x_2 + K_3 x_3 - aKt]} \frac{\underline{K}}{K} dZ_p (\underline{K}, \omega) \tag{1.13}$$

where $\omega = -aK$ has been chosen from Eq. (1.12).

The divergence of the part of the velocity field associated with the vorticity is zero, or

$$\nabla \cdot \underline{u}_v = 0 \, ,$$

which is equivalent to

$$\underline{K} \cdot \underline{u}_v = 0 \, ,$$

meaning that $\underline{u}_v$ is perpendicular to $\underline{K}$.

The general problem is represented diagrammatically in (Fig. 1), where the discontinuity in the flow occurs at the plane surface S , which is parallel to the X_3 axis, and makes the angle θ with the x_1 axis. It consists of determining, from the known disturbance field upstream of S , the disturbance field downstream of S .

Upstream of S ,

$$\frac{\underline{u}_v}{U} = \int e^{i[K_1(X_1-Ut)+K_2X_2+K_3X_3]} \, d\underline{Z}_v(\underline{K}) \tag{1.14}$$

$$\frac{s}{c_p} = \int e^{i[K_1(X_1-Ut)+K_2X_2+K_3X_3]} \, dZ_s(\underline{K}) \tag{1.15}$$

$$\frac{p}{P} = \int e^{i[K_1(X_1-Ut)+K_2X_2+K_3X_3-aKt]} \, dZ_p(\underline{K}) \tag{1.16}$$

$$\frac{\underline{u}_p}{U} = \frac{1}{\gamma M}\int e^{i[K_1(X_1-Ut)+K_2X_2+K_3X_3-aKt]} \, \frac{\underline{K}}{K} \, dZ_p(\underline{K}) \tag{1.17}$$

A similar set of equations applies for the downstream flow, the X_1' axis being parallel to U' , the X_3' axis coincident with the X_3 axis, and the X_2' axis orthogonal to X_1' and X_3 .

$$\frac{\underline{u}_v'}{U'} = \int e^{i[K_1(X_1'-U't)+K_2X_2'+K_3X_3]} \, d\underline{Z}_v(\underline{K}) \tag{1.18}$$

$$\frac{s'}{c_p} = \int e^{i[K_1(x_1'-U't) + K_2 x_2' + K_3 x_3]} dZ_s'(\underline{K}) \tag{1.19}$$

$$\frac{p'}{P'} = \int e^{i[K_1(x_1'-U't) + K_2 x_2' + K_3 x_3 - a'Kt]} dZ_p'(\underline{K}) \tag{1.20}$$

$$\frac{u_p'}{U'} = \frac{1}{\gamma M'} \int e^{i[K_1(x_1'-U't) + K_2 x_2' + K_3 x_3 - a'Kt]} \frac{\underline{K}}{K} dZ_p'(\underline{K}) \tag{1.21}$$

C. Transformation to the Case of a Normal Discontinuity

In order to simplify the boundary conditions at the discontinuity, it is desirable to transform to a coordinate system in which the discontinuity appears normal to the stream velocity. This requires a simple rotation and a translation along the discontinuity, in the plane of x_1 and x_2. Therefore, let

$$\begin{aligned} x_1 &= \xi \sin\theta + \eta \cos\theta + Ut\cos^2\theta \\ x_2 &= -\xi \cos\theta + \eta \sin\theta + Ut\cos\theta\sin\theta \end{aligned} \tag{1.22}$$

where x_1, x_2 and θ may refer to either the upstream or downstream flow, because of the fact that the main stream velocity tangential to the discontinuity is equal on the two sides.

Also, let

$$
\begin{aligned}
K_1 &= k_1 \sin\theta + k_2 \cos\theta \\
K_2 &= -k_1 \cos\theta + k_2 \sin\theta \\
K_3 &= k_3 \\
U &= \frac{U_n}{\sin\theta}
\end{aligned}
\tag{1.23}
$$

Using these relations, Eqs. (1.14) through (1.17) may be written as

$$\frac{u_v}{U_n} = \int e^{i[k_1(\xi - U_n t) + k_2\eta + k_3 x_3]} \frac{d\underline{Z}_v(\underline{k})}{\sin\theta} \tag{1.24}$$

$$\frac{s}{c_p} = \int e^{i[k_1(\xi - U_n t) + k_2\eta + k_3 x_3]} \, dZ_s(\underline{k}) \tag{1.25}$$

$$\frac{p}{P} = \int e^{i[k_1(\xi - U_n t) + k_2\eta + k_3 x_3 - akt]} \, dZ_p(\underline{k}) \tag{1.26}$$

$$\frac{u_p}{U_n} = \frac{1}{\gamma M_n} \int e^{i[k_1(\xi - U_n t) + k_2\eta + k_3 x_3 - akt]} \frac{\underline{k}}{k} \, dZ_p(\underline{k}) \tag{1.27}$$

and for the downstream flow,

$$\frac{u_v'}{U_n'} = \int e^{i[k_1(\xi - U_n' t) + k_2\eta + k_3 x_3]} \frac{d\underline{Z}_v'(\underline{k})}{\sin\theta'} \tag{1.28}$$

$$\frac{s'}{c_p} = \int e^{i[k_1(\xi - U_n' t) + k_2\eta + k_3 x_3]} \, dZ_s'(\underline{k}) \tag{1.29}$$

$$\frac{p'}{P'} = \int e^{i[k_1(\xi'-U_k't)+k_2\eta+k_3x_3-a'kt]} dZ_p'(\underline{k}) \tag{1.30}$$

$$\frac{u_p'}{U_k'} = \frac{1}{\gamma M_k'} \int e^{i[k_1(\xi'-U_k't)+k_2\eta+k_3x_3-a'kt]} \frac{\underline{k}}{k} dZ_p'(\underline{k}) \tag{1.31}$$

In general, the relation between $dZ(\underline{k})$ and $dZ(\underline{\kappa})$ is given by the rotation (1.23). If, however, the upstream disturbance field is isotropic, then $dZ(\underline{k})$ is equal to $dZ(\underline{\kappa})$.

The entropy and pressure fields are completely specified by $dZ_s(\underline{k})$ and $dZ_p(\underline{k})$ respectively. However, the specification of the vorticity field requires the three components of $d\underline{Z_v}(\underline{k})$. The incompressible continuity equation provides one more relation between these three quantities, in the form,

$$k_1 dZ_{1v}(\underline{k}) + k_2 dZ_{2v}(\underline{k}) + k_3 dZ_{3v}(\underline{k}) = 0$$

or

$$k_1 u_{1v} + k_2 u_{2v} + k_3 u_{3v} = 0 \tag{1.32}$$

so that two components of $d\underline{Z_v}(\underline{k})$ are sufficient to define the vorticity field.

D. Boundary Conditions at the Discontinuity

The matching of the downstream disturbance field, as represented by Eqs. (1.28) through (1.31), to the upstream field, Eqs. (1.24) through (1.27), at the discontinuity will be considered in two parts. First, for given upstream wave numbers, k_1 , k_2 , k_3 , the corresponding downstream wave numbers will be determined to give the same periodicity in η , x_3 and t for ξ' equal to zero. Then the $dZ(k)$ will be determined in such a way as to satisfy the boundary conditions for the velocity, pressure and entropy.

1. Matching of Wave Numbers

For the purpose of matching the known and unknown wave numbers, the general problem may be divided into two cases:

1. The known upstream disturbance consists only of convected disturbances (vorticity or entropy). The unknown wave numbers are then associated with the convected downstream disturbances, downstream pressure disturbances and upstream pressure disturbances, the last being present because the pressure disturbances propagate in the fluid. Comparison of Eqs. (1.24) through (1.27) with Eqs. (1.28) through (1.31) indicates that in all cases,

$$k_2' = k_2$$
$$k_3' = k_3$$

Accordingly, take

$$k_4 = \sqrt{k_2^2 + k_3^2} \tag{1.33}$$

where k_4 is the same for all disturbances which match in periodicity at the discontinuity. Matching of the time dependence yields for the convected downstream disturbances,

$$k_1' = m\, k_1 \tag{1.34}$$

where $$m = \frac{U_n}{U_n'}$$

and for the downstream pressure disturbance,

$$k_{1\varphi}' U_n' + k_\varphi' a' = k_1 \tag{1.35}$$

and

$$\frac{k_{1\varphi}'}{k_1} = -\frac{m M_n'^2}{1-M_n'^2}\left[1 - \frac{1}{M_n'}\sqrt{1 - \frac{1-M_n'^2}{m^2 M_n'^2}\left(\frac{k_4}{k_1}\right)^2}\;\right] \; ; \quad \begin{matrix} k_1 > 0 \\ M_n' < 1 \end{matrix} \tag{1.36}$$

For the special case of $M_n' = 1$,

$$\frac{k_{1\varphi}'}{k_1} = \frac{m^2 - \left(\frac{k_4}{k_1}\right)^2}{2m} \tag{1.37}$$

Similarly, for the upstream pressure disturbance,

$$\frac{k_{1\varphi R}}{k_1} = -\frac{M_n^2}{1-M_n^2}\left[1 + \frac{1}{M_n}\sqrt{1 - \frac{1-M_n^2}{M_n^2}\left(\frac{k_4}{k_1}\right)^2}\;\right] \; ; \quad \begin{matrix} k_1 > 0 \\ M_n < 1 \end{matrix} \tag{1.38}$$

The values of $k_{1\varphi}'$ and $k_{1\varphi R}$, as given by Eqs. (1.36) and (1.38), may be real or complex. If they are real, the pressure waves are periodic in ξ' . If they are complex, the pressure waves are exponentially attenuated periodic functions of ξ' . The downstream pressure waves are not attenuated if

$$\frac{1-M_n'^2}{m^2 M_n'^2}\left(\frac{k_4}{k_1}\right)^2 \leq 1 \tag{1.39}$$

It is apparent that if $M_n' = 1$, the waves are not attenuated (see Eq. 1.37), and that if $M_n' = 0$, the waves are all attenuated.

The upstream pressure waves are not attenuated if

$$\frac{1-M_n^2}{M_n^2}\left(\frac{k_4}{k_1}\right)^2 \leq 1 \tag{1.40}$$

2. The known upstream disturbance consists only of pressure waves. In this case there can be no upstream convected disturbance. For the downstream convected disturbances,

$$k_1' = m\left(k_{1p} + \frac{1}{M_n}\sqrt{k_{1p}^2 + k_4^2}\right) \tag{1.41}$$

For the downstream pressure wave,

$$\frac{k_{1p}'}{k_1'} = -\frac{M_n'^2}{1-M_n'^2}\left[1 - \frac{1}{M_n'}\sqrt{1 - \frac{1-M_n'^2}{M_n'^2}\left(\frac{k_4}{k_1'}\right)^2}\;\right] \tag{1.42}$$

For the special case $M_n' = 1$,

$$\frac{k_{1p}'}{k_1'} = \frac{1-\left(\frac{k_4}{k_1'}\right)^2}{2} \tag{1.43}$$

For the reflected upstream pressure wave,

$$\frac{k_{1pR}}{k_1'} = \frac{-2M_n^2 - M_n(1+M_n^2)\;\frac{k_{1p}}{k_p}}{m(1-M_n^2)\left[1 + M_n\frac{k_{1p}}{k_p}\right]} \tag{1.44}$$

2. Boundary Conditions on the Velocity

Since the elementary upstream disturbance is plane, the interaction with the discontinuity is essentially two-dimensional. Let r be a coordinate in the η , x_3 , plane and perpendicular to the line of intersection of the planes of constant phase of the disturbance with the plane of the discontinuity, as in (Fig. 3). Also, let α be the angular coordinate as shown in (Fig. 3). Let u_r be the resultant of u_2 and u_3 in the direction of k_4, and u_α be the resultant of u_2 and u_3 perpendicular to k_4.

$$u_\alpha = -u_2 \frac{k_3}{k_4} + u_3 \frac{k_2}{k_4} \tag{1.45}$$

For the pressure waves, $u_\alpha = 0$. For the vorticity disturbance,

$$u_\alpha = -u_{2v} \frac{k_3}{k_4} + u_{3v} \frac{k_2}{k_4}$$

Now since in crossing the discontinuity the disturbance does not change along lines perpendicular to r,

$$u_\alpha' = u_\alpha \tag{1.46}$$

Then, from Eqs. (1.32), (1.45) and (1.46)

$$u_{2v} = \frac{-u_{1v} \frac{k_{1v}}{k_3} - u_\alpha \frac{k_4}{k_2}}{\frac{k_2}{k_3} + \frac{k_3}{k_2}}$$

$$u_{3v} = \frac{-u_{1v} \frac{k_{1v}}{k_2} + u_\alpha \frac{k_4}{k_3}}{\frac{k_2}{k_3} + \frac{k_3}{k_2}} .$$

Since $$u_r = u_2 \frac{k_2}{k_4} + u_3 \frac{k_3}{k_4} ,$$

$$\frac{u_r}{U_n} = -\frac{u_{1v}}{U_n}\frac{k_{1v}}{k_4} + \frac{u_{2p}}{U_n}\frac{k_{2p}}{k_4} + \frac{u_{3p}}{U_n}\frac{k_{3p}}{k_4} \tag{1.47}$$

The analysis may now be completed in the plane of ξ' and r, as follows:

Assume that the discontinuity is disturbed, so that its angle to the r axis is σ , and that the disturbance propagates in the r direction at the same velocity, V , as the upstream disturbance. For convected inputs,

$$\frac{V}{U_n} = \frac{k_1}{k_4} \tag{1.48}$$

And for pressure input,

$$\frac{V}{U_n} = \frac{k_1}{k_4} + \frac{\sqrt{k_1^2 + k_4^2}}{k_4 M_n} \tag{1.49}$$

Denoting by u_n and u_t the instantaneous local perturbations to the velocities normal and tangential to the discontinuity,

$$\begin{aligned} \frac{u_n}{U_n} &= \frac{u_1}{U_n} - \sigma \frac{V}{U_n} \\ \frac{u_t}{U_n} &= \frac{u_r}{U_n} - \sigma \end{aligned} \tag{1.50}$$

where u_t is in the ξ , r plane. Similarly, for the downstream side,

$$\begin{aligned} \frac{u_1'}{U_n'} &= \frac{u_n'}{U_n'} + \sigma \frac{V}{U_n'} \\ \frac{u_r'}{U_n'} &= \frac{u_t'}{U_n'} + \sigma \end{aligned} \tag{1.51}$$

From the conservation of momentum in the tangential direction,

$$\frac{u_t'}{U_n'} = m \frac{u_t}{U_n} \tag{1.52}$$

Relationships may be found between $\frac{u_n'}{U_n'}$, $\frac{p'}{P'}$, $\frac{S'}{c_p}$, and the upstream values $\frac{u_n}{U_n}$, $\frac{p}{P}$, $\frac{s}{c_p}$, for any given type of discontinuity. For the present these relations will be indicated by:

$$\frac{u_n'}{U_n'} = b_1 \frac{u_n}{U_n} + b_2 \frac{s}{c_p} + b_3 \frac{p}{P} \tag{1.53}$$

$$\frac{p'}{P'} = b_4 \frac{u_n}{U_n} + b_5 \frac{s}{c_p} + b_6 \frac{p}{P} \tag{1.54}$$

$$\frac{s'}{c_p} = b_7 \frac{u_n}{U_n} + b_8 \frac{s}{c_p} + b_9 \frac{p}{P} \tag{1.55}$$

The boundary conditions which are applicable at the discontinuity then follow from Eqs. (1.50) through (1.55). They are:

$$\frac{u_1'}{U_k'} = b_1 \frac{u_1}{U_n} + b_2 \frac{s}{c_p} + b_3 \frac{p}{P} + (m - b_1)\frac{V}{U_n}\sigma \tag{1.56}$$

$$\frac{u_r'}{U_k'} = m \frac{u_r}{U_n} + (1-m)\sigma \tag{1.57}$$

$$\frac{p'}{P'} = b_4 \frac{u_1}{U_n} + b_5 \frac{s}{c_p} + b_6 \frac{p}{P} - b_4 \frac{V}{U_n}\sigma \tag{1.58}$$

$$\frac{s'}{c_p} = b_7 \frac{u_1}{U_n} + b_8 \frac{s}{c_p} + b_9 \frac{p}{P} - b_7 \frac{V}{U_n}\sigma \tag{1.59}$$

From Eqs. (1.24) through (1.31) and (1.47),

$$\frac{u_1}{U_n} = \frac{dZ_{1v}}{\sin\theta} - C_{1I}\frac{k_{1pI}}{k_4}dZ_{pI} - C_{1R}\frac{k_{1pR}}{k_4}dZ_{pR}$$
$$\frac{s}{c_p} = dZ_s$$
$$\frac{p}{P} = dZ_{pI} + dZ_{pR} \tag{1.60}$$
$$\frac{u_r}{U_n} = -\frac{k_{1v}}{k_4}\frac{dZ_{1v}}{\sin\theta} - C_{1I}\,dZ_{pI} - C_{1R}\,dZ_{pR}$$

$$\frac{u_1'}{U_n'} = \frac{dZ_{1v}'}{\sin\theta'} - C_2\frac{k_{1p}'}{k_4}dZ_p'$$
$$\frac{s'}{c_p} = dZ_s'$$
$$\frac{p'}{P'} = dZ_p' \tag{1.61}$$
$$\frac{u_r'}{U_n'} = -\frac{k_{1v}'}{k_4}\frac{dZ_{1v}'}{\sin\theta'} - C_2\,dZ_p'$$

where $C_1 = -\frac{1}{\gamma M_n}\,\frac{k_4}{k_p}$, $C_2 = -\frac{1}{\gamma M_n'}\,\frac{k_4}{k_p'}$ and the subscripts I and R mean incident and reflected.

Equations (1. 44) and (1. 45) are understood to be multiplied by the common exponential factor obtained from Eqs. (1. 24) through (1. 31) by setting $\xi' = 0$ and the appropriate matching of the wave numbers. This factor cancels out of the boundary conditions. It will be assumed that, with the same factor omitted,

$$\sigma = dZ_\sigma \tag{1.63}$$

Substitution of Eqs. (1.60), (1.61) and (1.63) in Eqs. (1.56) through (1.59) leads to four equations for the five unknown quantities $\frac{dZ'_{1v}}{\sin\theta'}$, dZ'_s, dZ'_p, dZ_{pR}, and dZ_σ. The other relation which is necessary to make the equations determinate depends on the particular discontinuity. For example, for shock waves, since pressure disturbances do not propagate upstream of the shock, $dZ_{pR} = 0$. For a flame front, the kinetics of the combustion process gives a connection between $\frac{u_n}{U_n}$, $\frac{s}{c_p}$ and $\frac{p}{P}$.

E. Definition of Transfer Function

The result of solution of the set of equations obtained above for the unknown dZ_i may be expressed as follows:

$$dZ_i(\underline{k}) = T_i^j \, dZ_j(\underline{k}) \tag{1.64}$$

where dZ_i denotes $\frac{dZ'_{1v}}{\sin\theta'}$, dZ'_s, dZ'_p, dZ_{pR}, dZ'_{rv}; dZ_j denotes $\frac{dZ_{1v}}{\sin\theta}$, dZ_s, dZ_{pI}, and the repeated index is to be summed over. The T_i^j may be interpreted as transfer functions. For example, T_p^v is the transfer function giving the amplitude of the downstream pressure field which results from vorticity convected through the discontinuity.

F. Effect of Obliqueness of the Discontinuity

If the input perturbation field is isotropic, as will be assumed in the present work, an oblique discontinuity has exactly the same influence on the perturbation field as a normal discontinuity with the same normal velocity, if it is observed from a coordinate system traveling along the discontinuity as in (Fig. 2). Furthermore, only the spacial orientation and time dependence of the downstream field will be different when it is viewed from the ordinary stationary coordinate system. All mean squares of fluctuating quantities will be unchanged. Accordingly, in the following, all results will be referred to the ξ , η , x_3 coordinate system, except where a time dependent characteristic, such as the frequency of the downstream sound waves, is involved.

PART II

EVALUATION OF STATISTICAL PROPERTIES

A. Statistical Specification of the Disturbance Field

A complete knowledge of the input perturbation field would permit a complete analysis into modes, as in Eqs. (1.14) through (1.17), and a complete calculation of the downstream perturbation field. Practically, however, since the perturbation field is random, it can only be described statistically, and in fact, knowledge of the statistical nature of the field is limited to the two-point correlation tensor, although in principle, more complete information could be given by means of the three-point and higher correlations.

The correlation tensor is defined by:

$$R_{ij}(\underline{\delta}, \underline{X}) = \overline{q_i(\underline{X})\, q_j(\underline{X}+\underline{\delta})} \tag{2.1}$$

where q_i and q_j denote any two quantities associated with the perturbation field, and $\underline{\delta}$ is the vector separation between the points at which they are measured. The bar indicates an ensemble average, that is, an average over a large number of possible configurations such as represented by Eqs. (1.14) through (1.17).

The spectral tensor is defined as the Fourier transform of the correlation tensor,

$$\Phi_{ij}(\underline{K}, \underline{X}) = \frac{1}{8\pi^3} \int R_{ij}(\underline{\delta}, \underline{X})\, e^{-i\underline{K}\cdot\underline{\delta}}\, d\underline{\delta} \tag{2.2}$$

and therefore,

$$R_{ij}(\underline{\delta}, \underline{X}) = \int \Phi_{ij}(\underline{K}, \underline{X})\, e^{i\underline{K}\cdot\underline{\delta}}\, d\underline{K} \tag{2.3}$$

As is indicated in Eqs. (2.2) and (2.3), the spectral and correlation tensors are in general functions of $\underline{X}$. If the perturbation field is homogeneous, they are independent of translations of the coordinates, and if isotropic, of rotations of the coordinates.

It will be assumed that the known upstream field is homogeneous and isotropic, so that,

$$R_{ij}(\underline{\delta}) = \int \Phi_{ij}(\underline{K}) e^{i\underline{K}\cdot\underline{\delta}} d\underline{K} \tag{2.4}$$

and

$$\Phi_{ij}(\underline{K}) = \int R_{ij}(\underline{\delta}) e^{i\underline{K}\cdot\underline{\delta}} d\underline{\delta} \tag{2.5}$$

The elements of the downstream spectral tensor which do not contain the pressure fluctuations also will be homogeneous; however, because part of it is attenuated, the downstream pressure field will lead to elements of the spectral tensor which are dependent on the distance from the discontinuity. Furthermore, in the cases where there is both an incident and a reflected upstream pressure field, the resultant pressure field will in general be inhomogeneous, because of the correlation between the incident and reflected waves.

The discussion of the downstream correlation tensor may then be given in two parts, the first dealing with the homogeneous fields, which include the vorticity, entropy and sound wave disturbances, and the second, with the non-homogeneous fields associated with the attenuated pressure waves, and upstream sound waves.

For the sake of simplicity, only the mean square values of the various perturbation quantities, hence only the diagonal elements of the spectral tensor, will be considered.

B. Homogeneous Fields

Substituting any of Eqs. (1.14) through (1.21) formally in Eq. (2.1),

$$R_{ii}(\underline{\xi}) = \int e^{i[-\underline{\kappa}\cdot\underline{x} + \underline{\kappa}'(\underline{x}+\underline{\xi})]} \overline{dZ_i^*(\underline{\kappa}')\, dZ_i(\underline{\kappa})}$$

where the integration is over $\underline{\kappa}$ and $\underline{\kappa}'$. In the case of homogeneous fields, however, $\overline{dZ_i^*(\underline{\kappa}')\, dZ_i(\underline{\kappa})} = 0$ unless $\underline{\kappa}' = \underline{\kappa}$ [11], so the integral reduces to

$$R_{ii}(\underline{\xi}) = \int e^{i\underline{\kappa}\cdot\underline{\xi}} \overline{dZ_i^*(\underline{\kappa})\, dZ_i(\underline{\kappa})} \tag{2.6}$$

The connection between the representation of the perturbation field given by Eqs. (1.14) through (1.21) and the statistical representation in terms of the spectral tensor is [11],

$$\Phi_{ii}(\underline{\kappa})d\underline{\kappa} = \overline{dZ_i^*(\underline{\kappa})\, dZ_i(\underline{\kappa})} \tag{2.7}$$

which seems reasonable from comparison of Eq. (2.3) with Eq. (2.6).

Equation (1.64) gives the relationships between the known and unknown Fourier coefficients, dZ_j and dZ_i, and Eq. (2.7) then allows the downstream spectral tensor to be given in terms of the upstream spectral tensor. Thus,

$$\Phi'_{ii}(\underline{K}')d\underline{K}' = \overline{dZ_i'^{*}(\underline{K}')\,dZ_i'(\underline{K}')}$$
$$= \overline{\left[T_i^{j*}dZ_j^{*}(\underline{K}_j)\right]\left[T_i^{k}dZ_k(\underline{K}_k)\right]}$$

whence

$$\Phi'_{ii}(\underline{K}')d\underline{K}' = T_i^{j*}T_i^{k}\,\overline{dZ_j^{*}(\underline{K}_j)\,dZ_k(\underline{K}_k)} \tag{2.8}$$

Again, it will be assumed for simplicity that the cross-correlations of the upstream pressure, vorticity and entropy modes are negligible. Equation (2.8) then becomes,

$$\Phi'_{ii}(\underline{K}')d\underline{K}' = T_i^{j*}T_i^{j}\;\Phi_{jj}(\underline{K}_j)\,d\underline{K}_j \tag{2.9}$$

This equation gives the diagonal elements of the downstream spectral tensor in terms of the known diagonal elements of the upstream spectral tensor.

1. Mean Square Values

The unknown mean square values are obtained from Eq. (2.3) by setting $\underline{\delta}$ equal to zero. Thus,

$$\overline{q_i^2} = \int T_i^{j*}T_i^{j}\;\Phi_{jj}(\underline{K}_j)\,d\underline{K}_j \tag{2.10}$$

Here it has been assumed that the transfer functions, which were found from the elementary disturbance analysis as functions of $\underline{k}$, may be converted by Eq. (1.23) to functions of $\underline{K}$. In the particular case where the known upstream field is isotropic, however, it is simpler to proceed as follows:

$$\Phi_{jj}(\underline{K}_i)\,d\underline{K}_i = \Phi_{jj}(\underline{k}_j)\,d\underline{k}_i \qquad \text{(by isotropy)}$$

and

$$\overline{q_i^2} = \int T_i^{j*} T_i^{j} \Phi_{jj}(\underline{k}_i)\,d\underline{k}_i \tag{2.11}$$

C. Non-homogeneous Fields

1. Attenuated Pressure Waves

In the case of the attenuated pressure waves, the wave number normal to the discontinuity, k_{1p}' or k_{1pR}, is complex, and the expression analogous to Eq. (2.6) contains a real exponential factor, so that if $\frac{k_{1p}}{k_1} = \alpha + i\beta$, the relation corresponding to Eq. (2.11) is,

$$\overline{\left(\frac{p}{P}\right)^2_{att.}} = \int e^{-2\beta k_1 \xi} T_p^{j*} T_p^{j} \Phi_{jj}(\underline{k}_j)\,d\underline{k}_i \tag{2.12}$$

2. Reflected Upstream Sound

The equation analogous to Eq. (2.9) for the upstream pressure field which results from incident sound waves is somewhat more complicated than Eq. (2.9), due to the fact that the pressure is the sum of the incident and reflected pressure waves, the latter being spatially correlated with the former. Thus, from Eq. (1.26)

$$dZ_p = \left[1 + T_{pR}^{p} e^{i\left[(k_{1R}-k_{1I})(\xi - U_u t) - a(k_R - k_I)t\right]}\right] dZ_{pI}$$

if

$$\frac{p}{\rho} = \int e^{i[k_{1I}(\xi - U_n t) + k_2\eta + k_3 x_3 - a k_I t]} dZ_p$$

Now define $T_{pR}^{p''}$ by

$$dZ_p = (1 + T_{pR}^{p''}) dZ_{pI}$$

Then

$$\overline{dZ_p^* \, dZ_p} = (1 + T_{pR}^{p''*} + T_{pR}^{p''} + T_{pR}^{p''*} T_{pR}^{p''}) \overline{dZ_{pI}^* \, dZ_{pI}}$$

and the relation analogous to Eq. (2.9) is

$$\Phi_{pp}(\underline{K}) d\underline{K}\Big|_{total} = (1 + T_{pR}^{p''*} + T_{pR}^{p''} + T_{pR}^{p''*} T_{pR}^{p''}) \Phi_{pp}(\underline{K}) d\underline{K}\Big|_{incident} \quad (2.13)$$

where

$$T_{pR}^{p''} = T_{pR}^{p} \, e^{i[(k_{1R} - k_{1I})(\xi - U_n t) - a(k_R - k_I)t]}$$

D. Downstream Pressure Field

As has been indicated previously, the downstream pressure field consists partly of sound waves and partly of attenuated pressure waves. The mean square of each of these types will be determined, and in addition the form of the spectral function will be determined for the sound waves. Also, the one-dimensional frequency spectrum of the sound will be determined.

1. Mean Square Pressure Fluctuation

It is necessary to evaluate Eqs. (2.11) and (2.12) for the pressure. It will be found later that the transfer functions are most conveniently expressed in spherical polar coordinates. Therefore, let

$$
\begin{aligned}
k_1 &= k \cos\varphi \\
k_2 &= k \sin\varphi \sin\psi \\
k_3 &= k \sin\varphi \cos\psi \\
k_4 &= k \sin\varphi \\
J\left(\frac{k}{k,\varphi,\psi}\right) &= k^2 \sin\varphi
\end{aligned}
\tag{2.14}
$$

Using these relations,

$$
\begin{aligned}
\overline{\left(\frac{p'}{\rho'}\right)^2} &= 2\pi \int\int_0^\infty T_p^{j*} T_p^j \Phi_{jj}(\underline{k})\, k^2 \sin\varphi\, dk\, d\varphi \\
&+ 2\pi \int\int_0^\infty e^{-2\beta k \cos\varphi\, \xi} T_p^{j*} T_p^j \Phi_{jj}(\underline{k})\, k^2 \sin\varphi\, dk\, d\varphi
\end{aligned}
\tag{2.15}
$$

where the range of integration in φ is that which gives sound waves for the first integral, and that which gives attenuated pressure waves for the second integral.

Since the upstream field has been assumed isotropic, the form of the spectral functions in φ may be given. For the vorticity [11],

$$
\Phi_{vv} = \frac{E(k_v)}{4\pi U_k^2 k_v^2} \sin^2\varphi \tag{2.16}
$$

and, since ς and p are scalars,

$$\Phi_{ss} = G_s(k_s) \tag{2.17}$$

$$\Phi_{pp} = G_p(k_p) \tag{2.18}$$

Inserting these in Eq. (2.15), the integrations over k and φ may be separated in the first integral, because the transfer functions are independent of k .

$$\begin{aligned}\overline{\left(\frac{p'}{P'}\right)^2_{sound}} &= \frac{1}{2U_k^2}\int_0^\infty E(k_v)\,dk_v \int T_p^{v*}T_p^v \sin^3\varphi\, d\varphi \\ &+2\pi\int_0^\infty G_s(k_s)k_s^2\,dk_s\int T_p^{s*}T_p^s \sin\varphi\, d\varphi \\ &+2\pi\int_0^\infty G_p(k_p)k_p^2\,dk_p\int T_p^{p*}T_p^p \sin\varphi\, d\varphi\end{aligned} \tag{2.19}$$

The first integral in each term gives just the upstream mean square value of that input, since

$$\begin{aligned}\overline{\left(\frac{u_{1v}}{U_k}\right)^2} &= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^\infty \Phi_{vv}\,k_v^2 \sin\varphi\, dk_v\, d\varphi\, d\psi \\ &= \frac{2}{3U_k^2}\int_0^\infty E(k_v)\,dk_v\end{aligned} \tag{2.20}$$

$$\begin{aligned}\overline{\left(\frac{S}{C_p}\right)^2} &= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^\infty G_s(k_s)\,k_s^2 \sin\varphi\, dk_s\, d\varphi\, d\psi \\ &= 4\pi\int_0^\infty G_s(k_s)\,k_s^2\,dk_s\end{aligned} \tag{2.21}$$

$$\overline{\left(\frac{p}{P}\right)^2} = 4\pi \int_0^\infty G_p(k_p)\, k_p^2\, dk_p \tag{2.22}$$

Inserting these relations in Eq. (2.19),

$$\overline{\left(\frac{p'}{P'}\right)^2_{sound}} = \frac{3}{4}\overline{\left(\frac{u_{1v}}{U_n}\right)^2} \int T_p^{v*} T_p^{v} \sin^3\varphi\, d\varphi$$

$$+\frac{1}{2}\overline{\left(\frac{S}{C_p}\right)^2} \int T_p^{s*} T_p^{s} \sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2} \int T_p^{p*} T_p^{p} \sin\varphi\, d\varphi \tag{2.23}$$

Evaluation of the attenuated part of the mean-square pressure fluctuation as a function of ζ requires knowledge of $E(k_v)$, $G_s(k_s)$ and $G_p(k_p)$. It is not possible to even estimate G_s or G_p. These would be determined by the conditions of a given problem. However, at least an estimate can be made of the probable form of $E(k_v)$. It is known that for decaying isotropic turbulence $E(k_v)$ varies as $k_v^{-5/3}$ for k_v large, and also that for isotropic turbulence it varies as k_v^4 for k_v small. An interpolation formula suggested in Ref. (15) is

$$E(\zeta) = \frac{A}{k_0} \frac{\zeta^4}{(1+\zeta^2)^{17/6}} \tag{2.24}$$

where $\zeta = \frac{k_v}{k_0}$ and A is a normalization factor. This form will be adopted for $E(\zeta)$, as an example, although in a specific case $E(\zeta)$ might be quite different from this. The important trends in the influence of this form of $E(\zeta)$ on the properties of the resultant disturbance field will be quite clear, so the choice is not critical. The constant A may be referred to the input through Eq. (2.20),

$$\overline{\left(\frac{u_{1v}}{U_n}\right)^2} = \frac{2A}{3U_n^2}\int_0^\infty \frac{\xi^4}{(1+\xi^2)^{17/6}}\,d\xi = \frac{2A}{3U_n^2}\,(1.0347) \tag{2.25}$$

Now let

$$I(\varphi) = \int_0^\infty e^{-g(\varphi)\xi}\,\frac{\xi^4}{(1+\xi^2)^{17/6}}\,d\xi$$

where

$$g(\varphi) = 2k_0\,\xi'\beta\cos\varphi$$

Then

$$\overline{\left(\frac{p'}{P'}\right)^2_{att.}} = .725\,\overline{\left(\frac{u_{1v}}{U_n}\right)^2}\int I(\varphi)\,T_p^{v*}T_p^{v}\sin^3\varphi\,d\varphi \tag{2.26}$$

$I(\varphi)$ is given in (Fig. 5).

2. Spectral Function of Downstream Sound Waves

Given the form (2.24) for $E(k_v)$, it is possible to determine $\Phi'_{pp}(\underline{k}'_p)$, the spectral function for the downstream sound waves, by transforming the appropriate integral of Eq. (2.11) to an integral over $\underline{k}'_p$. From Eq. (1.35),

$$\begin{aligned} k_{1v} &= \frac{1}{m}\left(k'_{1p} + \frac{1}{M'_n}\sqrt{k'^2_{1p} + k_+^2}\right) \\ k_{2v} &= k'_{2p} \\ k_{3v} &= k'_{3p} \end{aligned} \tag{2.27}$$

$$J\left(\frac{\underline{k}_v}{\underline{k}'_p}\right) = \frac{1}{m}\left(1 + \frac{k'_{1p}}{M'_n\,k'_p}\right) \tag{2.28}$$

then

$$\overline{\left(\frac{p'}{P'}\right)^2_{sound}} = \int T_p^{v*} T_p^{v} \Phi_{vv}(k_p') J\left(\frac{k_v}{k_p'}\right) dk_p'$$

$$= \int \Phi'_{pp}(k_p') dk_p'$$

and

$$\Phi'_{pp}(k_p') = T_p^{v*} T_p^{v} \Phi_{vv}(k_p') J\left(\frac{k_v}{k_p'}\right)$$

This function is most conveniently expressed in polar coordinates, by Eq. (2.14). Using this transformation, and integrating out the ψ', which is trivial since it occurs only in the differential,

$$\Phi'_{pp}(\varphi', k_p') = 4\pi T_p^{v*} T_p^{v} \Phi_{vv}(\varphi') J\left(\frac{k_v}{k_p'}\right) J\left(\frac{k_p'}{k_p', \varphi', \psi'}\right) \tag{2.29}$$

The extra factor of 2 arises from the fact that for φ varying from 0 to φ_c, $0 < \varphi_c < \pi/2$, φ' ranges from 0 to φ_c', $\pi/2 < \varphi_c' < \pi$.

$$J\left(\frac{k_v}{k_p'}\right) = \frac{1}{m}\left(1 + \frac{\cos\varphi'}{M_u}\right)$$

$$J\left(\frac{k_p'}{k_p', \varphi', \psi'}\right) = k_p'^2 \sin\varphi'$$

and from Eq. (2.16),

$$\Phi_{vv} = \frac{E(k_v)}{4\pi U_k^2 k^2}\left[\frac{k_4^2}{k_1^2 + k_4^2}\right]$$

$$k^2 = k_\varphi'^2\left[\frac{1}{m^2}\left(\cos\varphi' + \frac{1}{M_k'}\right)^2 + \sin^2\varphi'\right]$$

$$k_4^2 = k_\varphi'^2 \sin^2\varphi'$$

Hence,

$$\Phi_{vv} = \frac{E(k_v)}{4\pi U_k^2 k_\varphi'^2}\left\{\frac{\sin^2\varphi'}{\left[\frac{1}{m^2}\left(\cos\varphi' + \frac{1}{M_k'}\right)^2 + \sin^2\varphi'\right]^2}\right\}$$

Collecting these relations and substituting in Eq. (2. 29),

$$\frac{\Phi_{pp}'\left(\varphi', \frac{k_\varphi'}{k_0}\right)}{\left(\frac{\overline{u_{1v}}}{U_k}\right)^2} = 1.45\,\frac{T_p^{v*}T_p^v}{m}\left(1 + \frac{\cos\varphi'}{M_k'}\right)\frac{\sin^3\varphi'}{(1+\zeta^2)^{17/6}}\left(\frac{k_\varphi'}{k_0}\right)^4 \qquad (2.30)$$

where

$$\zeta^2 = \left(\frac{k_\varphi'}{k_0}\right)^2\left[\frac{1}{m^2}\left(\cos\varphi' + \frac{1}{M_k'}\right)^2 + \sin^2\varphi'\right]$$

Here φ' is the inclination of the downstream sound wave and $\frac{k_\varphi'}{k_0}$ is the ratio of its wave number to the reference wave number of the upstream turbulence. The range of φ' is from 0 to φ_c', where

$$\varphi_c' = \tan^{-1}\left(-\frac{\sqrt{1-M_k'^2}}{M_k'}\right) \qquad (2.31)$$

3. One-Dimensional Frequency Spectrum of Downstream Sound Waves

Equation (2. 30) gives the distribution of intensity of the downstream sound waves over the inclination, φ', and the wave number, k_φ'. However, the quantity which is of interest acoustically is the distribution of energy over the frequency which would be noted by a stationary observer downstream.

Considering a single shear wave input, its resultant downstream sound wave will have a frequency which is equal to the frequency of the upstream wave as it is convected past a stationary point. Thus, if the discontinuity is normal ; $\theta = \frac{\pi}{2}$, (Fig. 2),

$$\nu = \frac{U_n k_v \cos\varphi}{2\pi} \tag{2.32}$$

An average upstream frequency may be defined by

$$\nu_0 = \frac{U_n k_0 \int_0^{\pi/2} \cos\varphi \sin\varphi \, d\varphi}{2\pi \int_0^{\pi/2} \sin\varphi \, d\varphi} = \frac{U_n k_0}{4\pi}$$

and the dimensionless, downstream frequency ν' by

$$\nu' = \frac{\nu}{\nu_0} = 2\,\frac{k_v}{k_0} \cos\varphi \tag{2.33}$$

The downstream sound intensity is, from Eqs. (2.15), (2.16), (2.24), and (2.25),

$$\left(\frac{\overline{p'}}{p'}\right)^2 = .725 \left(\frac{\overline{U_{1v}}}{U_n}\right)^2 \int \int_0^\infty \frac{\zeta^4}{(1+\zeta^2)^{17/6}} T_p^{v\,*} T_p^v \sin^3\varphi \, d\zeta \, d\varphi$$

where $\qquad \zeta = \frac{k_v}{k_0}$

From Eq. (2.33),

$$d\zeta = \frac{1}{2\cos\varphi}\, d\nu'$$

Therefore

$$\overline{\left(\frac{\varphi'}{p'}\right)^2} = \int_0^\infty \Phi_\nu(\nu')\,d\nu'$$

where

$$\Phi_\nu(\nu') = .363\left(\frac{\overline{u_{1v}}}{U_n}\right)^2 \int \frac{\zeta^4}{(1+\zeta^2)^{17/6}} T_\varphi^{v*} T_\varphi^{v} \frac{\sin^3\varphi}{\cos\varphi}\,d\varphi \qquad (2.34)$$

and

$$\zeta = \frac{\nu'}{2\cos\varphi}$$

Equation (2.34) may be simplified for ζ both small and large. The range of $\cos\varphi$ is $\cos\varphi_c \leq \cos\varphi \leq 1$, where φ_c varies with M_u', being less than $\pi/2$ for $M_u' < 1$.

For ζ small, that is for $\frac{\nu'}{2\cos\varphi_c} << 1$,

$$\Phi_\nu(\nu') = .0226\left(\frac{\overline{u_{1v}}}{U_n}\right)^2 \nu'^4 \int T_\varphi^{v*} T_\varphi^{v} \frac{\sin^3\varphi}{\cos^5\varphi}\,d\varphi \qquad (2.35)$$

For ζ large, that is for $\frac{\nu'}{2} >> 1$,

$$\Phi_\nu(\nu') = 1.15\left(\frac{\overline{u_{1v}}}{U_n}\right)^2 \nu'^{-5/3} \int T_\varphi^{v*} T_\varphi^{v} \cos^{2/3}\varphi\,\sin^3\varphi\,d\varphi \qquad (2.36)$$

If the discontinuity is not normal ($\theta < \pi/2$, Fig. 2), Eq. (2.32) for ν must be replaced by

$$\nu = \frac{U_n\,k v}{2\pi\cos\varphi}\cos\left(\theta - \frac{\pi}{2} + \varphi\right)$$

and ν_0 may be chosen as

$$\nu_0 = \frac{U_K k_0}{4\pi \cos\theta}$$

Therefore,

$$\nu' = 2 \frac{k v}{k_0} \cos\left(\theta - \frac{\pi}{2} + \varphi\right) \tag{2.37}$$

and Eq. (2.34) becomes

$$\Phi_\nu(\nu') = .363 \left(\frac{\overline{u_1 v}}{U_K}\right)^2 \int \frac{\xi^4}{(1+\xi^2)^{17/6}} T_p^{v*} T_p^v \frac{\sin^3\varphi}{\cos(\theta - \frac{\pi}{2} + \varphi)} \, d\varphi \tag{2.38}$$

where
$$\xi = \frac{\nu'}{2\cos(\theta - \frac{\pi}{2} + \varphi)}$$

Similarly, Eq. (2.35) becomes

$$\Phi_\nu(\nu') = .0226 \left(\frac{\overline{u_1 v}}{U_K}\right)^2 \nu'^4 \int T_p^{v*} T_p^v \frac{\sin^3\varphi}{\cos^5(\theta - \frac{\pi}{2} + \varphi)} \, d\varphi \tag{2.39}$$

and Eq. (2.36) becomes

$$\Phi_\nu(\nu') = 1.15 \left(\frac{\overline{u_1 v}}{U_K}\right)^2 \nu'^{-5/3} \int T_p^{v*} T_p^v \cos^{2/3}\left(\theta - \frac{\pi}{2} + \varphi\right) \sin^3\varphi \, d\varphi \tag{2.40}$$

E. Downstream Convected Fields

The downstream convected fields, that is, the vorticity and entropy fields, may be computed in the same manner as the downstream pressure field. The vorticity field has two significant mean square values, the velocities perpendicular to and parallel to the shock. The entropy field is described by one mean square, since it is a scalar field.

Each of these quantities may be computed by

$$\overline{q_i^2} = \frac{3}{4}\left(\overline{\frac{u_{1v}}{U_n}}\right)^2 \int T_i^{v*} T_i^{v} \sin^3\varphi \, d\varphi$$
$$+\frac{1}{2}\left(\overline{\frac{S}{c_p}}\right)^2 \int T_i^{s*} T_i^{s} \sin\varphi \, d\varphi + \frac{1}{2}\left(\overline{\frac{p}{P}}\right)^2_I \int T_i^{p*} T_i^{p} \sin\varphi \, d\varphi \tag{2.41}$$

where, for the entropy field, $\overline{q_i^2} = \left(\overline{\frac{S'}{c_p}}\right)^2$ and for the vorticity field,

$$\overline{q_i^2} = \left(\overline{\frac{u_{1v}'}{U_k}}\right)^2 \quad or \quad \left(\overline{\frac{u_{rv}'}{U_k'}}\right)^2 .$$

However, it will be recalled from the analysis of the interaction of a single vorticity wave with the discontinuity that one component of the velocity, u_α , carries through without change. Thus, the actual mean square velocity fluctuation associated with the vorticity and parallel to the discontinuity is a combination of this unchanged component and u_{rv}' . Let $u_{\ell v}'$ be the lateral component of velocity associated with the vorticity, that is, the vector sum of u_α and u_{rv}' . From Eq. (1.64),

$$dZ_{rv}' = T_{rv}^{v} \frac{dZ_{1v}}{\sin\theta}$$

and from Eq. (1.46)

$$dZ_{\alpha v}' = m \, dZ_{\alpha v}$$

Also,

$$dZ_{\ell v}^{\prime *}\, dZ_{\ell v}^{\prime} = T_{rv}^{v*} T_{rv}^{v} \frac{dZ_{1v}^{*}\, dZ_{1v}}{\sin^2\theta} + m^2\, dZ_{\alpha v}^{*}\, dZ_{\alpha v}$$

but

$$dZ_{rv}^{*}\, dZ_{rv} = \left(\frac{k_{1v}}{k_4}\right)^2 \frac{dZ_{1v}^{*}\, dZ_{1v}}{\sin^2\theta}$$

and from Eq. (1.60),

$$dZ_{\alpha v}^{*}\, dZ_{\alpha v} = dZ_{\ell}^{*}\, dZ_{\ell} - dZ_{rv}^{*}\, dZ_{rv}$$

whence

$$dZ_{\ell v}^{\prime *}\, dZ_{\ell v}^{\prime} = \left(T_{rv}^{v*} T_{rv}^{v} - \frac{m^2}{\tan^2\varphi}\right) \frac{dZ_{1v}^{*}\, dZ_{1v}}{\sin^2\theta} + m^2\, dZ_{\ell}^{*}\, dZ_{\ell}$$

Substituting in the first integral of Eq. (2.41) and carrying out the integral of $\dfrac{m^2}{\tan^2\varphi}$,

$$\overline{\left(\frac{u_{\ell v}^{\prime}}{U_k^{\prime}}\right)^2} = \overline{\left(\frac{u_{1v}}{U_k}\right)^2}\left[-\frac{m^2}{2} + \frac{3}{4}\int T_{rv}^{v*} T_{rv}^{v} \sin^3\varphi\, d\varphi\right] + m^2\, dZ_{\ell}^{*}\, dZ_{\ell}$$

Assuming isotropy,

$$dZ_{\ell}^{*}\, dZ_{\ell} = 2\,\overline{\left(\frac{u_{1v}}{U_k}\right)^2}$$

and the first term of Eq. (2.41), for $\overline{g_i^2} = \overline{\left(\dfrac{u_{\ell v}^{\prime}}{U_k^{\prime}}\right)^2}$ is

$$\overline{\left(\frac{u_{\ell v}^{\prime}}{U_k^{\prime}}\right)^2} = \overline{\left(\frac{u_{1v}}{U_k}\right)^2}\left[\frac{3}{2} m^2 + \frac{3}{4}\int T_{rv}^{v*} T_{rv}^{v} \sin^3\varphi\, d\varphi\right] \qquad (2.42)$$

The other terms are unchanged, since $u_{\alpha} = 0$ for them.

F. Reflected Upstream Pressure Field

The reflected upstream pressure field from a pressure input is in general inhomogeneous. It consists only of sound waves [see Eq. (1.44)] ; therefore, k_{1pR} is real. The reflected sound wave intensity will be taken as the total sound intensity minus the incident sound intensity. Thus, from Eq. (2.13),

$$\overline{\left(\frac{p}{P}\right)^2_R} = \int \left[T_{pR}^{p*} T_{pR}^{p} + T_{pR}^{p''*} + T_{pR}^{p''} \right] \Phi_{pp}(\underline{k}_p)\, d\underline{k}_p$$

where

$$T_{pR}^{p''} = T_{pR}^{p}\, e^{i(k_{1R} - k_{1I})\xi'}$$

Now T_{pR}^{p} is independent of k, so that the integrations over φ and k can be separated in the first term, giving

$$\overline{\left(\frac{p}{P}\right)^2_R} = \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2_I} \int T_{pR}^{p*} T_{pR}^{p} \sin\varphi\, d\varphi + \int \left[T_{pR}^{p''*} + T_{pR}^{p''} \right] \Phi_{pp}(\underline{k}_p)\, d\underline{k}_p \qquad (2.43)$$

Since T_{pR}^{p} may be complex, the last term is best written:

$$2\int Re(T_{pR}^{p}) \cos(k_{1R} - k_{1I})\xi'\; \Phi_{pp}(\underline{k}_p)\, d\underline{k}_p \qquad (2.44)$$

PART III

SHOCK WAVES

Shock waves, like other flow discontinuities, may be characterized by certain relationships between the velocities normal to the shock front, the pressure, and the entropy on the two sides of the shock. For the shock wave, these relations are contained in the famous Rankine-Hugoniot equations.

Completion of the preceding general analysis for this special case then entails the computation of the coefficients of Eqs. (1.53) through (1.55) from the Rankine-Hugoniot equations, solution of the system of Eqs. (1.56) through (1.59) for the transfer functions of the shock wave, and finally evaluation of the integrals giving the desired statistical properties of the downstream disturbance field.

A. Completion of Analysis of the Interaction of the Discontinuity with the Elementary Disturbances

1. Characteristics of the Shock Wave

The velocity ratio across a shock wave is given by:

$$m = \frac{U_n}{U_n'} = \frac{\frac{\gamma+1}{2} M_n^2}{1 + \frac{\gamma-1}{2} M_n^2} \qquad (3.1)$$

For small perturbations of U_n , U_n' and M_n , by using the thermodynamic relation,

$$\frac{\delta T}{T} = \frac{\delta s}{c_p} + \frac{\gamma-1}{\gamma} \frac{\delta p}{p} , \qquad (3.2)$$

Eq. (3.1) may be written in the form,

$$\frac{u_n'}{U_n'} = -\left(1 - 2\frac{\gamma-1}{\gamma+1} m\right)\frac{u_n}{U_n} + \left(1 - \frac{\gamma-1}{\gamma+1} m\right)\left[\frac{s}{c_p} + \frac{\gamma-1}{\gamma}\frac{p}{P}\right]. \qquad (3.3)$$

The following two equations may be shown, from the equations of momentum, continuity and state, to be valid for any discontinuity:

$$\frac{p'}{P'} = \left[1-(\gamma-1)(m-1)M_n'^2\right]\frac{p}{P} + \gamma M_n'^2\left[(2m-1)\frac{u_n}{U_n} - \frac{u_n'}{U_n'}\right] - \gamma M_n'^2(m-1)\frac{s}{c_p} \tag{3.4}$$

$$\frac{s'}{c_p} = \frac{1}{\gamma}\left(\frac{p'}{P'} - \frac{p}{P}\right) + \left(\frac{u_n'}{U_n'} - \frac{u_n}{U_n}\right) + \frac{s}{c_p} \tag{3.5}$$

Using Eq. (3.3) to eliminate the $\frac{u_n'}{U_n'}$ from Eqs. (3.4) and (3.5),

$$\frac{p'}{P'} = \left[\frac{4\gamma m}{\gamma+1}M_n'^2\right]\frac{u_n}{U_n} - \left[\frac{2\gamma m}{\gamma+1}M_n'^2\right]\frac{s}{c_p} + \left[1-2\frac{\gamma-1}{\gamma+1}mM_n'^2\right]\frac{p}{P} \tag{3.6}$$

$$\frac{s'}{c_p} = \left[\frac{4m}{\gamma+1}M_n'^2 - 2\left(1-\frac{\gamma-1}{\gamma+1}m\right)\right]\frac{u_n}{U_n} - \left[\frac{2m}{\gamma+1}M_n'^2 - 2 + \frac{\gamma-1}{\gamma+1}m\right]\frac{s}{c_p} + \frac{\gamma-1}{\gamma}\left[1-\frac{\gamma-1}{\gamma+1}m - \frac{2m}{\gamma+1}M_n'^2\right]\frac{p}{P} \tag{3.7}$$

Comparing Eqs. (1.53) through (1.55) with Eqs. (3.3), (3.6) and (3.7) gives the coefficients b_i for the shock wave:

$$
\begin{aligned}
b_{1s} &= -2b_{2s} + 1 \\
b_{2s} &= 1 - \frac{\gamma-1}{\gamma+1} m \\
b_{3s} &= \frac{\gamma-1}{\gamma} b_{2s} \\
b_{4s} &= \frac{4\gamma m}{\gamma+1} M_n'^2 \\
b_{5s} &= -\frac{1}{2} b_{4s} \\
b_{6s} &= 1 - \frac{\gamma-1}{2\gamma} b_{4s} \\
b_{7s} &= \frac{1}{\gamma} b_{4s} - 2 b_{2s} \\
b_{8s} &= -\frac{1}{2\gamma} b_{4s} + b_{2s} + 1 \\
b_{9s} &= -\frac{\gamma-1}{2\gamma^2} b_{4s} + \frac{\gamma-1}{\gamma} b_{2s}
\end{aligned}
\tag{3.8}
$$

2. Calculation of the Transfer Functions

Equations (1.56) through (1.59), together with Eqs. (1.60) and (1.61), represent four equations for the five unknown quantities, $\frac{dZ_{1v}'}{\sin\theta'}$, dZ_s', dZ_φ', dZ_{pR} and dZ_σ. In the case of the shock wave, the system is made determinate by the impossibility of upstream propagation of pressure waves. Thus dZ_{pR} is equal to zero. Equation (1.57) may be used to eliminate dZ_σ as an unknown from Eqs.(1.56), (1.58) and (1.59), the result being three simultaneous equations for the unknowns of the downstream perturbation field,

$\frac{dZ'_{1v}}{\sin\theta'}$, dZ'_p and dZ'_s . These are:

$$\left[1-C_3\frac{k'_{1v}}{k_4}\right]\frac{dZ'_{1v}}{\sin\theta'}-C_2\left[\frac{k'_{1p}}{k_4}+C_3\right]dZ'_p =$$

$$\left[b_{1s}-mC_3\frac{k_{1v}}{k_4}\right]\frac{dZ_{1v}}{\sin\theta}+b_{2s}\,dZ_s+\left[b_{3s}-C_1\left(b_{1s}\frac{k_{1p}}{k_4}+mC_3\right)\right]dZ_p \tag{3.9}$$

$$\left[1+C_2\frac{b_{4s}}{m-1}\frac{V}{U_n}\right]dZ'_p+\left[\frac{b_{4s}}{m-1}\frac{V}{U_n}\frac{k'_{1v}}{k_4}\right]\frac{dZ'_{1v}}{\sin\theta'} =$$

$$b_{4s}\left[1+\frac{m}{m-1}\frac{V}{U_n}\frac{k_{1v}}{k_4}\right]\frac{dZ_{1v}}{\sin\theta}+b_{5s}\,dZ_s+\left[b_{6s}-C_1b_{4s}\left(\frac{k_{1p}}{k_4}-\frac{m}{m-1}\frac{V}{U_n}\right)\right]dZ_p \tag{3.10}$$

$$\left[\frac{b_{7s}}{m-1}\frac{V}{U_n}\frac{k'_{1v}}{k_4}\right]\frac{dZ'_{1v}}{\sin\theta'}+\left[\frac{b_{7s}}{m-1}C_2\frac{V}{U_n}\right]dZ'_p+dZ'_s =$$

$$b_{7s}\left[1+\frac{m}{m-1}\frac{V}{U_n}\frac{k_{1v}}{k_4}\right]\frac{dZ_{1v}}{\sin\theta}+b_{8s}\,dZ_s+\left[b_{9s}-C_1b_{7s}\left(\frac{k_{1p}}{k_4}-\frac{m}{m-1}\frac{V}{U_n}\right)\right]dZ_p \tag{3.11}$$

where

$$C_3 = \frac{m-b_{1s}}{m-1}\,\frac{V}{U_n}$$

The coefficients in these equations, and therefore the transfer functions, are functions only of ratios of wave numbers, and are most simply expressed in terms of polar coordinates as defined by Eqs. (2. 14). The evaluation of the coefficients in Eqs. (3. 9) through (3. 11) depends on whether the input is convected or propagating.

For convected inputs:

$$
\begin{aligned}
\frac{k_1}{k_4} &= \frac{1}{\tan\varphi} \\
\frac{V}{U_n} &= \frac{1}{\tan\varphi} \\
\frac{k_{1v}'}{k_4} &= \frac{m}{\tan\varphi} \\
\frac{k_{1p}'}{k_1} &= -\frac{1}{ma_1}\left(1-\frac{X}{M_n'}\right) \quad ; \quad X=\sqrt{1-a_1\tan^2\varphi} \\
C_2 &= -\frac{ma_1}{\gamma}\left[\frac{\tan\varphi}{1-M_n'X}\right]
\end{aligned}
\tag{3.12}
$$

For propagating (pressure wave) input:

$$
\begin{aligned}
\frac{k_1}{k_4} &= \frac{1}{\tan\varphi} \\
\frac{V}{U_n} &= \frac{1}{y} \\
\frac{k_{1v}'}{k_4} &= \frac{m}{y} \\
\frac{k_{1p}'}{k_{1v}'} &= -\frac{1}{m^2a_1}\left[1-\frac{Z}{M_n'}\right] \quad ; \quad Z=\sqrt{1-a_1y^2} \\
C_{1I} &= -\frac{\sin\varphi}{\gamma M_n} \\
C_2 &= -\frac{ma_1}{\gamma}\left[\frac{y}{1-M_n'Z}\right] \\
y &= \frac{\sin\varphi}{\cos\varphi+\frac{1}{M}}
\end{aligned}
\tag{3.13}
$$

where

$$a_1 = \frac{1 - M_n'^2}{m^2 M_n'^2}$$

$$a_2 = \frac{4}{\gamma+1}\left(\frac{m}{m-1}\right) \qquad (3.14)$$

$$a_3 = \left(\frac{m}{m-1}\right)\left(1 + \frac{3-\gamma}{\gamma+1} m\right)$$

With this notation, for convected inputs, take

$$\Delta_{v\,\mathrm{or}\,s} = 1 - \frac{a_3}{\tan^2\varphi} - m a_1 a_2 M_n'^2 \frac{1 - \frac{1}{a_1}\left[\frac{1 - \frac{X}{M_n'}}{\tan^2\varphi}\right]}{1 - M_n' X} \qquad (3.15)$$

and for a pressure input,

$$\Delta_p = 1 - \frac{a_3}{y^2} - m a_1 a_2 M_n'^2 \frac{1 - \frac{1}{a_1}\left[\frac{1 - \frac{Z}{M_n'}}{y^2}\right]}{1 - M_n' Z} \qquad (3.16)$$

In terms of these quantities, the transfer functions for the shock wave, which will be denoted by S_i^j are, for the downstream pressure field,

$$\Delta_v S_p^v = b_{4s}\left[1 - \frac{m}{\tan^2\varphi}\right] \qquad (3.17)$$

$$\Delta_s S_p^s = -\frac{b_{4s}}{2}\left[1 - \frac{m}{\tan^2\varphi}\right] \qquad (3.18)$$

$$\Delta_p S_p^p = -\frac{m}{m-1} b_{4s}\left[\left(\frac{\sin\varphi}{\gamma M_n}\right)\frac{1}{y} + \left(b_{3s} + \frac{b_{1s}}{\gamma M_n}\cos\varphi\right)\frac{1}{y^2}\right]$$
$$+\left[b_{6s} + \frac{b_{4s}}{\gamma M_n}\cos\varphi\right]\left(1 - \frac{a_3}{y^2}\right) \qquad (3.19)$$

For the vorticity,

$$\Delta_v S_{1v}^{v} = b_{1s} - \frac{a_3}{\tan^2\varphi} - a_2 M_n'^2 \left[\frac{m^2 a_1 - \left(m-1+\frac{m}{\tan^2\varphi}\right)\left(1-\frac{x}{M_n'}\right)}{1-M_n' x}\right] \tag{3.20}$$

$$\Delta_s S_{1v}^{s} = b_{2s} + \frac{2m}{\gamma+1} M_n'^2 \left[\frac{m a_1 - \left(1-\frac{x}{M_n'}\right)}{1-M_n' x}\right] \tag{3.21}$$

$$\begin{aligned}\Delta_\varphi S_{1v}^{\varphi} &= b_{3s} + \frac{\sin\varphi}{\gamma M_n}\left[\frac{b_{1s}}{\tan\varphi} + \frac{a_3}{y}\right] \\ &- \frac{m a_1}{\gamma}\left[\frac{\frac{a_3}{m} - \frac{\gamma-1}{2\gamma} b_{4s} + \frac{m}{m-1}\frac{b_{4s}}{\gamma M_n}\cos\varphi - \left[b_{6s} - \frac{b_{4s}}{\gamma M_n}\left(\frac{m}{m-1}\frac{1}{y} - \frac{1}{\tan\varphi}\right)\sin\varphi\right]\frac{1-\frac{z}{M_n'}}{m a_1}}{1-M_n' z}\right]\end{aligned} \tag{3.22}$$

For the lateral component of the vorticity,

$$\Delta_v S_{rv}^{v} = -\frac{m}{\tan\varphi}\Delta_v S_{1v}^{v} \tag{3.23}$$

$$\Delta_s S_{rv}^{s} = -\frac{m}{\tan\varphi}\Delta_s S_{1v}^{s} \tag{3.24}$$

$$\Delta_\varphi S_{rv}^{\varphi} = -\frac{m}{y}\Delta_\varphi S_{1v}^{\varphi} \tag{3.25}$$

and for the entropy,

$$\Delta_v S_{s}^{v} = b_{7s}\left[1-\frac{m}{\tan^2\varphi}\right] \tag{3.26}$$

$$\Delta_s S_{s}^{s} = b_{8s} - m\left(b_{8s} + \frac{2b_{2s}}{m-1}\right)\frac{1}{\tan^2\varphi} - m a_1 a_2 M_n'^2 \frac{1-\frac{1-\frac{x}{M_n'}}{a_1\tan^2\varphi}}{1-M_n' x} \tag{3.27}$$

$$S_s^p = b_{9s} + \frac{b_{7s}}{m-1}\left[\frac{\sin\varphi}{\gamma M_n}\left(\frac{m-1}{\tan\varphi} - \frac{m}{y}\right) - \frac{m}{y^2} S_{1v}^p + \frac{m a_1}{\gamma}\left(\frac{S_p^p}{1 - M_n' Z}\right)\right] \qquad (3.28)$$

These transfer functions may be real or complex. They are real if the downstream pressure wave is a sound wave, and complex if it is attenuated. The physical significance of the complex transfer functions is that the phase of the associated quantity is shifted relative to that of the input.

3. Limiting Cases

For the convected inputs, a simple limiting case is that for $\varphi = \frac{\pi}{2}$. The lines of constant phase are then coincident with streamlines, and the problem is that of a steady sinusoidal velocity or entropy profile imposed on a shock. For this case,

$$\Delta_{v \text{ or } s} = 1 \qquad \frac{k_{1p}'}{k_4} = \frac{i}{\sqrt{1 - M_n'^2}}$$

and

$$\begin{aligned}
S_p^v &= b_{4s} \\
S_p^s &= -\tfrac{1}{2} b_{4s} \\
S_{1v}^v &= 2m-1 \\
S_{rv}^v &= 0 \\
S_{1v}^s &= 1-m \\
S_s^v &= b_{7s} \\
S_s^s &= b_{8s}
\end{aligned} \qquad (3.29)$$

For the pressure input, an interesting special case is that for which $\cos\varphi = -\frac{1}{M_n}$. The upstream pressure waves are then Mach waves. For this case, $\Delta_p = 1$; $\quad \frac{k_{1p}'}{k_4} = \frac{i}{\sqrt{1-M_n'^2}}$
and

$$
\begin{aligned}
S_p^p &= b_{6s} - \frac{b_{4s}}{\gamma M_n^2} \\
S_{1v}^p &= b_{3s} - \frac{b_{1s}}{\gamma M_n^2} + \frac{1}{\gamma M_n'^2}\left[b_{6s} - \frac{b_{4s}}{\gamma M_n^2}\right] \\
S_{rv}^p &= 0 \\
S_s^p &= b_{9s} - \frac{b_{7s}}{\gamma M_n^2}
\end{aligned}
\qquad (3.30)
$$

The downstream pressure field is, of course, attenuated, so that only S_{1v}^p and S_s^p are of real interest. These are shown in (Fig. 6). For convenience in interpreting this figure and those to follow, the variation of M_n with m is shown in (Fig. 7).

B. Statistical Properties of the Downstream Disturbance Field

The diagonal elements of the transformation matrix will be evaluated; that is, the mean square sound pressure, normal and lateral vorticity, and entropy will be evaluated in terms of the mean squares of these quantities upstream. In addition, the mean square pressure in the attenuated pressure waves will be evaluated for vorticity input, and the spectral function and one-dimensional frequency spectrum will be evaluated for the sound waves resulting from vorticity input.

1. Mean Square Sound, Vorticity and Entropy

From Eqs. (2.23) and (2.41),

$$\overline{\left(\frac{p'}{P'}\right)^2}_{sound} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_n}\right)^2}\int_0^{\varphi_{c1}} S_\varphi^{v^2}\sin^3\varphi\, d\varphi + \overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\varphi_{c1}} S_\varphi^{s^2}\sin\varphi\, d\varphi$$

$$+\frac{1}{2}\overline{\left(\frac{p}{P}\right)^2}\int_{0\,,\,\varphi_{c2}}^{\varphi_{c1}\,,\,\pi} S_\varphi^{p^2}\sin\varphi\, d\varphi \tag{3.31}$$

$$\overline{\left(\frac{u'_{1v}}{U'_n}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_n}\right)^2}\int_0^{\pi/2} S_{1v}^{v*}S_{1v}^{v}\sin^3\varphi\, d\varphi$$

$$+\overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\pi/2} S_{1v}^{s*}S_{1v}^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2}\int_0^{\pi} S_{1v}^{p*}S_{1v}^{p}\sin\varphi\, d\varphi \tag{3.32}$$

$$\overline{\left(\frac{u'_{rv}}{U'_n}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_n}\right)^2}\int_0^{\pi/2} S_{rv}^{v*}S_{rv}^{v}\sin^3\varphi\, d\varphi$$

$$+\overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\pi/2} S_{rv}^{s*}S_{rv}^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2}\int_0^{\pi} S_{rv}^{p*}S_{rv}^{p}\sin\varphi\, d\varphi \tag{3.33}$$

$$\overline{\left(\frac{s'}{c_p}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_n}\right)^2}\int_0^{\pi/2} S_s^{v*}S_s^{v}\sin^3\varphi\, d\varphi$$

$$+\overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\pi/2} S_s^{s*}S_s^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2}\int_0^{\pi} S_s^{p*}S_s^{p}\sin\varphi\, d\varphi \tag{3.34}$$

where for convected inputs (vorticity and entropy), φ_{c1} is given by

$$\tan^2\varphi_{c1} = \frac{1}{a_1} \tag{3.35}$$

and for pressure input, φ_{c1} and φ_{c2} are the two roots of

$$y_c^2 = \frac{1}{a_1} \tag{3.36}$$

These critical angles are presented in (Figs. 8 and 9).

The above integrals have been evaluated analytically for the cases of convected inputs. The details of this calculation are given in Appendix B. The greater complexity of the transfer functions for pressure input made analytical evaluation of these integrals impractical; however, they were evaluated numerically.

For convenience, define average values of the transfer functions by:

$$\begin{aligned}
\overline{\left(\frac{p'}{P'}\right)^2_{sound}} &= \overline{S_p^{v\,2}}'\,\overline{\left(\frac{u_{1v}}{U_h}\right)^2} + \overline{S_p^{s\,2}}'\,\overline{\left(\frac{s}{c_p}\right)^2} + \overline{S_p^{p\,2}}'\,\overline{\left(\frac{p}{P}\right)^2} \\
\overline{\left(\frac{u'_{1v}}{U'_h}\right)^2} &= \overline{S_{1v}^{v\,2}}\,\overline{\left(\frac{u_{1v}}{U_h}\right)^2} + \overline{S_{1v}^{s\,2}}\,\overline{\left(\frac{s}{c_p}\right)^2} + \overline{S_{1v}^{p\,2}}\,\overline{\left(\frac{p}{P}\right)^2} \\
\frac{1}{2}\overline{\left(\frac{u'_{\ell v}}{U'_h}\right)^2} &= \overline{S_{\ell v}^{v\,2}}\,\overline{\left(\frac{u_{1v}}{U_h}\right)^2} + \overline{S_{\ell v}^{s\,2}}\,\overline{\left(\frac{s}{c_p}\right)^2} + \overline{S_{\ell v}^{p\,2}}\,\overline{\left(\frac{p}{P}\right)^2} \\
\overline{\left(\frac{s'}{c_p}\right)^2} &= \overline{S_s^{v\,2}}\,\overline{\left(\frac{u_{1v}}{U_h}\right)^2} + \overline{S_s^{s\,2}}\,\overline{\left(\frac{s}{c_p}\right)^2} + \overline{S_s^{p\,2}}\,\overline{\left(\frac{p}{P}\right)^2}
\end{aligned} \tag{3.37}$$

These average values of the transfer functions are given in (Figs. 10 through 13).

(Figure 10) shows the $\overline{S_p}$, that is, the root-mean-square, downstream, dimensionless pressure fluctuation divided by the appropriate root-mean-square input. $\overline{S_p^v}$ and $\overline{S_p^s}$ exhibit rather peculiar behavior near $m = 1$, with an extremely rapid decrease to zero at $m = 1$. The limiting behavior of $\overline{S_p^v}$ for m approaching 1 is given by

$$\overline{S_{\varphi}^{v}}' = \left(\frac{8}{5}\right)^{\frac{1}{2}} \frac{\gamma}{\gamma+1} \left(\frac{\gamma+1}{2}\right)^{\frac{1}{4}} (m-1)^{\frac{1}{4}} + O(m-1)^{5/4} \tag{3.38}$$

which is in agreement with Ref. (4). The derivation of this expression is given in Appendix B.

Equation (3.38) represents the limiting behavior of the derived formula for $\overline{S_{\varphi}^{v}}'$ as m approaches 1; however, two of the assumptions which were made in the linearization of the problem are violated as m approaches 1, so it does not necessarily represent the actual behavior of the sound intensity.

First, the assumption that all perturbation quantities and their gradients must be small is violated. For m near 1, all of the contribution to the sound intensity is from the sound waves which result from shear waves having φ very near φ_{c1}. This may be seen from the limiting form of $\overline{S_{\varphi}^{v^2}}'$. From Eq. (48), Appendix B,

$$\overline{S_{\varphi}^{v^2}}' = \frac{3}{2} b_{45}^2 \sqrt{a_1} \int_0^1 \frac{x(1-x)\,dx}{(x+1)^3 \sqrt{1-x^2}} \tag{3.39}$$

where

$$x = \sqrt{1 - \frac{\gamma+1}{2}(m-1)\tan^2\varphi}$$

The contribution to the integral is distributed over the range $0 \leq x \leq 1$, with the integrand zero at 0 and 1. However, for m approaching 1, x is very nearly 1 except for $\tan^2\varphi$ very large, so that all the contribution to the integral comes from a small range of φ near $\varphi_{c1} = \tan^{-1}\sqrt{\frac{2}{\gamma+1}\left(\frac{1}{m-1}\right)}$. The quantity $\frac{p'}{p'}\frac{k_{\varphi}'}{k_4}$ may be taken as a measure of the change in the pressure gradient in the downstream sound wave as φ varies. For φ equal to φ_{c_1}, and m approaching 1, this quantity is

$$\frac{p'}{p'}\frac{k_p'}{k_+} = -4\sqrt{2}\left(\frac{\gamma}{\gamma+1}\right)\frac{1}{\sqrt{\gamma+1}}\frac{1}{\sqrt{m-1}} \tag{3.40}$$

so that the gradients become very large as m approaches 1.

Secondly, the assumption that boundary conditions may be applied at the unperturbed position of the shock is violated. As m approaches 1, the angle φ_c' which corresponds to φ_c approaches $\frac{\pi}{2}$, so that the principal contribution to the sound is from waves whose fronts are nearly parallel to the unperturbed shock. The limiting relation for φ_c' is

$$\varphi_c' = \pi - \sqrt{\frac{\gamma+1}{2}(m-1)} \tag{3.41}$$

Thus, for $m = 1.01$, $\varphi_c' = 173.7^\circ$, and the wave fronts are just 6.3° from parallel to the unperturbed shock. Now since the shock is actually wavy, if $\pi - \varphi_c'$ becomes too small, the sound waves may actually intersect the shock at a point removed from their origin and be absorbed.

Because of these difficulties, the behavior of $\overline{S_p^v}'$ cannot be definitely determined from the present analysis for m very close to 1, that is, for very weak shocks. However, for m greater than about 1.01, the assumptions are valid, and for $m = 1$, the sound level is zero, since then the shock is a Mach wave and has no first order interaction with the turbulence. Therefore, it can be said that the sound generated by interaction of turbulence with a shock should increase initially very rapidly as the normal Mach number of the shock is increased from 1.

The behavior of $\overline{S_p^s}'$ is quite similar to that of $\overline{S_p^v}'$, the limiting expression for m approaching 1 being

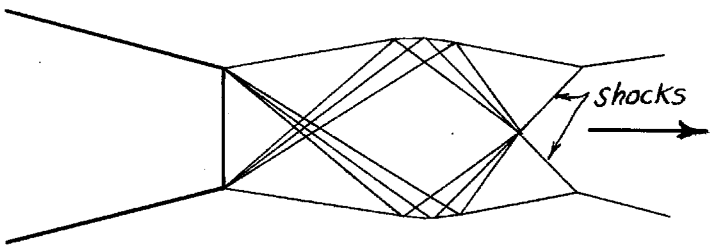
$$\overline{S_p^s}' = \left(\frac{4}{15}\right)^{\frac{1}{2}} \frac{\gamma}{\gamma+1} \left(\frac{\gamma+1}{2}\right)^{\frac{1}{4}} (m-1)^{\frac{1}{4}} + O(m-1)^{5/4} \tag{3.42}$$

The fact that $\overline{S_p^s}'$ is of magnitude comparable to that of $\overline{S_p^v}'$ indicates that temperature spottiness in the upstream flow is as important as the turbulence level in determining the noise production by shocks. In fact, in the exhaust of an afterburning turbojet, for example, it is not unlikely that the temperature spottiness would be more important than the turbulence.

(Figure 11) gives the averages of the transfer functions for the component normal to the shock of the velocity associated with the vorticity mode, and (Fig. 12) gives the component parallel to the shock. Comparison of (Figs. 11 and 12) shows that these two components are of the same order of magnitude for all three inputs, so that the downstream turbulence is essentially isentropic.

(Figure 13) indicates that a considerable amount of entropy spottiness is generated by either sound or shear waves interacting with the shock, although for weak shocks, the entropy generation by vorticity goes to zero quite rapidly. This is consistent with the well-known result that for weak shocks the entropy increase varies as $(M_u - 1)^3$. (13)

As an example of the usefulness of the average transfer functions of (Figs. 10 through 13), consider the following problem. A convergent nozzle operating at a pressure ratio in excess of that required for choking will develop a flow pattern with an alternating set of expansions and re-compressions, the number depending on the pressure ratio.



Assume that just ahead of the shock, the root-mean-square fluctuations of velocity are one percent of the free stream, and the entropy spottiness is one percent. The latter might arise from combustion inhomogeneities, especially if an afterburner is mounted on the engine. From (Fig. 10), the mean square downstream pressure fluctuation is given by

$$\overline{\left(\frac{p}{P}\right)^2} = 10^{-4}\left(\overline{S_p^{v}}^{\,\prime 2} + \overline{S_p^{s}}^{\,\prime 2}\right) = 8.2 \times 10^{-6}$$

This pressure intensity corresponds to a noise level of 149 decibels within the jet. If an appreciable fraction of this intensity is radiated out of the jet by interaction with the boundaries, this will be an important source of noise from choked nozzles.

2. Mean Square Pressure Fluctuation in Attenuated Waves Due to Vorticity

Evaluation of Eq. (2.26) is more difficult than that of Eq. (2.23) because of the exponential and algebraic factors, each of which involve both variables of integration. In the range of attenuated waves, S_p^v

is complex, which means that the downstream pressure waves are shifted in phase relative to the shear wave input. After forming the complex conjugate, multiplying and simplifying,

$$S_p^{v*} S_p^{v} = \frac{b_{45}^2 (\tan^2\varphi - m) \left[\left(m + \frac{1}{m a_1} \right)^2 - \frac{1}{(a_1 m M_u')^2} + \frac{\tan^2\varphi}{1 - M_u'^2} \right]}{\left[a_3 m - \frac{1}{a_1} + \left(a_2 - m - \frac{1}{m a_1} \right) \tan^2\varphi \right]^2 + (a_1 \tan^2\varphi - 1) \left[\frac{m + \tan^2\varphi}{a_1 m M_u'} \right]^2} \tag{3.43}$$

From Eq. (1.36),

$$\beta = \frac{\sqrt{a_1 \tan^2\varphi - 1}}{m a_1 M_u'}$$

and

$$g(\varphi) = 2 k_0 \xi' \left[\frac{\cos 2\varphi}{a_1 m M_u'} \sqrt{a_1 \tan^2\varphi - 1} \right] \tag{3.44}$$

Equation (2.26) becomes

$$\overline{\left(\frac{p'}{P'} \right)^2_{att.}} = 1.45 \left(\overline{\frac{u_{1v}}{U_u}} \right)^2 \int_{\varphi_{c1}}^{\frac{\pi}{2}} I(\varphi)\, S_p^{v*} S_p^{v} \sin^3\varphi \, d\varphi \tag{3.45}$$

where

$$\tan^2\varphi_{c1} = \frac{1}{a_1}$$

$I(\varphi)$ is obtained from (Fig. 5) and Eq. (3.44) as a function of φ, and Eq. (3.45) may then be integrated numerically using Eq. (3.43). The result is shown in (Fig. 14), for $m = 2$. The intensity of pressure fluctuations in the sound waves has been added for comparison. At the shock, the intensity in the attenuated waves is about 7.5 times as high as that in the sound waves; however, at a distance $\xi' = .139\,\lambda_0$, where λ_0 is 1.55 times the wave length associated with the peak of $E(\xi)$ Eq. (2.24), it has decayed to the same magnitude as the sound waves. If, for example, the peak energy were at a wave length of one inch, then the distance at which the intensities are equal would be .21 inches, and at a distance of one inch the attenuated waves would be comparatively weak. These trends would not be significantly different for other Mach numbers. The expression for $\overline{S_{\varphi}^{v}}''$, the transfer function for the amplitude of the attenuated waves at the discontinuity, for m approaching 1, is

$$\overline{S_{\varphi}^{v}}'' = 4\sqrt{3}\left(\frac{\gamma}{\gamma+1}\right)\left(\frac{\gamma+1}{2}\right)^{\frac{1}{4}}(m-1)^{\frac{1}{4}} + O(m-1)^{5/4} \tag{3.46}$$

which is $\sqrt{30} \approx 5.5$ times that of the sound waves, as given by Eq. (3.38).

3. Spectral Function for Downstream Sound Waves

Equation (2.30) has been evaluated for a single stream Mach number ($m = 2$, $M_n = 1.58$) to show the distribution of energy over the wave number k'_{φ} and angle φ' of the sound waves. The results are shown in (Figs. 15 and 16). (Figure 15) gives the variation with wave number for $\varphi' = 120^{\circ}$, and (Fig. 16), the variation with φ' for $\frac{k'_{\varphi}}{k_0} = 1.6$. The variations with wave number for other angles are very nearly the same as in (Fig. 15), only the amplitude varying as in (Fig. 16). The variations with wave number for angles other than 120° are very nearly as in (Fig. 16), with the amplitude varying as in (Fig. 15).

The variation with $\frac{k_\varphi'}{k_0}$ is, of course, a direct result of the assumed energy distribution, Eq. (2.24) ; however, the distribution over φ' is characteristic of the interaction process. The distribution of energy over φ varies as $\sin^3\varphi$ [Eq. (2.20)]. The distribution over φ' is roughly as shown in (Fig. 16) for all Mach numbers, except that as m approaches 1, the entire contribution comes from the peak near φ_c', which is 132.4° in (Fig. 16). For proof of this, see Eq. (3.39). This means that for m approaching 1, all the sound is concentrated in waves propagating upstream and nearly parallel to the shock.

4. One-Dimensional Frequency Spectrum of Downstream Sound Waves

Equation (2.34) will first be evaluated for a range of m for normal shocks. Then, since the downstream sound spectrum depends on the obliqueness of the shock [θ in (Fig. 2)] as well as its strength, the behavior with θ will be found for weak oblique shocks. For normal shocks, Eq. (2.34) becomes

$$\Phi_\nu(\nu') = .725\left(\frac{\overline{u_{1v}}}{U_k}\right)^2 \int_0^{\varphi_c} \frac{\zeta^4}{(1+\zeta^2)^{17/6}} S_\varphi^{v\,2} \frac{\sin^3\varphi}{\cos\varphi}\, d\varphi \tag{3.47}$$

where

$$\zeta = \frac{\nu'}{2\cos\varphi}$$

and Eqs. (2.35) and (2.36) become, for ζ small,

$$\Phi_\nu(\nu') = .0453\left(\frac{\overline{u_{1v}}}{U_k}\right)^2 \nu'^4 \int_0^{\varphi_c} S_\varphi^{v\,2} \frac{\sin^3\varphi}{\cos^5\varphi}\, d\varphi \tag{3.48}$$

for ξ large,

$$\Phi_\nu(\nu') = 2.30\left(\frac{\overline{u_1 v}}{U_k}\right)^2 \nu'^{-\frac{5}{3}} \int_0^{\varphi_c} S_p^{v\,2} \cos^{2/3}\varphi \sin^3\varphi \, d\varphi \tag{3.49}$$

For ν' small, Φ_ν varies as ν'^4, as shown by Eq. (3.48), and for ν' large, it varies as $\nu'^{-\frac{5}{3}}$, as shown by Eq. (3.49). These limiting behaviors are, of course, characteristic of the assumed form of $E(\xi)$ [Eq. (2.24)]. The computed curves of (Fig. 17) show that for intermediate ν' the behavior is still roughly that of $E(\xi)$, with the exception that as m decreases, the peak shifts toward zero. If the peak of $\frac{\Phi_\nu}{\left(\frac{p'}{P'}\right)^2}$ occurs at ν' = 1.55, the distribution of energy over frequency is about the same as for the upstream flow, where by frequency in the upstream flow is meant the apparent frequency of a velocity profile as it is convected past a stationary observer. If the peak is to the left of 1.55, the distribution of energy has shifted to lower frequencies, and if it is to the right, the distribution of energy has shifted to higher frequencies. (Figure 18) shows that for m above 2, the shift is to slightly higher frequency, but for m near 1, the effective down-stream frequency becomes comparatively small. The fact that, for m approaching 1, all the contribution to the sound intensity comes from a small range of angles near φ_c permits an approximate analytical evaluation of Φ_ν for this case.

$$\frac{\Phi_\nu(\nu')}{\left(\frac{p'}{P'}\right)^2} = \frac{.483}{\cos\varphi_c} \frac{\xi_c^4}{(1+\xi_c^2)^{17/6}}$$

where

$$\varsigma_c = \frac{\nu'}{2\cos\varphi_c}$$

The peak of this distribution is at $\varsigma_c = 1.55$, hence at

$\nu'_{max.} = 3.10 \cos\varphi_c$, so that as m approaches 1 and φ_c approaches $\frac{\pi}{2}$ [(Fig. 8)], the peak of the distribution of energy over frequency approaches zero frequency.

The character of the downstream sound field resulting from convection of either turbulence or entropy spottiness through a normal shock may now be described roughly as follows. For vanishingly weak shocks, there will be no sound. As the strength of the shock is increased slightly, the sound intensity will increase very rapidly, and the frequency will increase very rapidly also, until a value of $m = 1.1$ ($M_n = 1.06$) is reached. The intensity then will behave as in (Fig. 10) and the frequency will continue to increase gradually as shown in (Fig. 18).

For oblique shocks, Eq. (2.38) may be evaluated for m near 1, since the major contribution to the integral still comes from φ near φ_c. Thus,

$$\frac{\Phi_\nu(\nu')}{\left(\frac{\psi'}{p'}\right)^2} = \frac{.483}{\cos(\theta-\frac{\pi}{2}+\varphi_c)} \frac{\varsigma_c^4}{(1+\varsigma_c^2)^{17/6}}$$

where

$$\varsigma_c = \frac{\nu'}{2\cos(\theta-\frac{\pi}{2}+\varphi_c)}$$

and

$$\varphi_c = \frac{\pi}{2} - \sqrt{\frac{\gamma+1}{2}(m-1)}$$

The maximum of $\frac{\Phi_\nu}{\left(\frac{\psi'}{p'}\right)^2}$ will be at $\varsigma_c = 1.55$, as before, so that

$$\nu'_{max} = 3.10\left[\cos\theta\cos\left(\frac{\pi}{2}-\varphi_c\right)+\sin\theta\sin\left(\frac{\pi}{2}-\varphi_c\right)\right]$$

Therefore, for a given m near 1, since $\frac{\pi}{2}-\varphi_c$ is small, ν'_{max} increases as θ decreases from $\frac{\pi}{2}$, and in fact for θ appreciably less than $\frac{\pi}{2}$ this is the dominating effect, so that for θ small, as it would be for weak shocks at high stream Mach numbers, much higher frequency sound would be expected than from a normal shock of the same strength. The above formulas should be reasonably accurate for m up to 1.5, or M_n up to 1.3.

C. Summary of Results for Shock Wave

A small disturbance field of vorticity, entropy or pressure is in general amplified in passing through a shock wave. Furthermore a given type of disturbance will produce disturbances of all three kinds, of comparable magnitudes, in the downstream flow. Both the amplification of the original disturbance and the production of the other types in general increase continuously as the shock strength increases. The intensity of sound generated by vorticity and entropy disturbances is an exception. In this case the intensity increases very rapidly as the strength of the shock is increased to a velocity ratio of about 1.1. It then decreases somewhat and finally increases continuously. The frequency of the sound generated by vorticity or entropy fluctuations depends on the strength of the shock, not just on its propagation velocity and the scale of the fluctuation. If the shock is normal, as its strength (velocity ratio) is increased for given disturbance wave length and velocity of propagation, the frequency increases from 0 for a velocity ratio of 1 to a value equal to that of the convected upstream disturbance for a velocity ratio of 1.75 (Mach number of 1.43). Beyond this point it remains nearly constant. If the shock is oblique and not very strong, the frequency of the sound varies as the cosine of the angle between the shock and the flow direction.

In addition to the statistical results for random disturbance fields, which have been summarized above, the analysis also provides a complete set of transfer functions for the interaction of shock waves with sound waves and stationary small amplitude velocity and temperature profiles in the upstream flow.

PART IV

FLAME FRONTS

If the flame front is considered as a very thin region through which the flow is altered by heat addition, it represents a discontinuity across which the normal velocity, the pressure and the entropy may change, as for the shock wave. The principal differences are two: first, the flame front propagates at a much lower velocity than the shock, and second, because of this, pressure disturbances arising at the flame may propagate upstream.

A. Completion of Analysis of Interaction of the Flame with the Elementary Disturbances

1. Characteristics of the Flame Front

A flame front, considered as a discontinuity, may be characterized by two quantities: the heat release Q, and the normal burning velocity U_n. Both of these quantities are dependent on the reaction taking place at the flame front. In the present analysis, Q will be considered dependent only on the composition of the combustible mixture. U_n will be considered a function of the temperature and pressure, the exact functional form to be determined empirically.

From the usual conservation relations, the velocity ratio across the flame front may be written,

$$m = \frac{U_n}{U_n'} = \frac{1 + \gamma M_n^2 + \sqrt{(1+\gamma M_n^2)^2 - 2(\gamma+1) M_n^2 \left(1 + \tau + \frac{\gamma-1}{2} M_n^2\right)}}{2\left(1 + \tau + \frac{\gamma-1}{2} M_n^2\right)} \tag{4.1}$$

where

$$\tau = \frac{Q}{c_p T}$$

Now, since M_n is small for ordinary flame fronts (of the order of 10^{-3}), it is appropriate to expand Eq. (4.1) in M_n^2. The result is

$$m = \frac{1}{1+\tau}\left[1 + f(\tau) M_n^2 + \cdots\right]$$

where

$$f(\tau) = \gamma - \frac{\gamma-1}{2}(1+\tau) - \frac{3(\gamma-1)}{4}\left(\frac{1}{1+\tau}\right)$$

$f(\tau)$ is shown in (Fig. 19). Since $f(\tau)$ is not large, the above relation may be approximated by

$$m = \frac{1}{1+\tau} \tag{4.2}$$

For small perturbations of velocity, pressure and entropy upstream, this may be written, using Eq. (3.2), as

$$\frac{u_n'}{U_n'} = \frac{u_n}{U_n} - \left(\frac{\tau}{1+\tau}\right)\left(\frac{s}{c_p} + \frac{\gamma-1}{\gamma}\frac{p}{P}\right) \tag{4.3}$$

From Eqs. (3.4) and (3.5), using the fact that if M_n is small, M_n' is also small,

$$\frac{p'}{P'} = \frac{p}{P} \tag{4.4}$$

$$\frac{s'}{c_p} = \left(\frac{1}{1+\tau}\right)\frac{s}{c_p} - \frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)\frac{p}{P} \tag{4.5}$$

The constants of Eqs. (1.53) through (1.55) are then as follows for the flame front:

$$
\begin{aligned}
b_{1f} &= 1 \\
b_{2f} &= -\frac{\tau}{1+\tau} \\
b_{3f} &= -\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right) \\
b_{4f} &= 0 \\
b_{5f} &= 0 \\
b_{6f} &= 1 \\
b_{7f} &= 0 \\
b_{8f} &= \frac{1}{1+\tau} \\
b_{9f} &= b_{3f}
\end{aligned}
\tag{4.6}
$$

2. Calculation of the Transfer Functions

Equations (1.56) through (1.59) represent four equations for the five quantities $\frac{dZ_{1v}'}{\sin\theta'}$, dZ_s' , dZ_p' , dZ_{pR} and dZ_σ . In the case of the flame, the functional dependence of the normal burning velocity on the temperature and pressure, that is, on the reaction kinetics, is sufficient to make the system determinate. This relation will be indicated by:

$$\frac{u_n}{U_n} = f_1(s,p)\frac{s}{c_p} + f_2(s,p)\frac{p}{P} \tag{4.7}$$

Using Eq. (1.50), this becomes

$$\frac{u_1}{U_n} - \sigma\frac{V}{U_n} = f_1(s,p)\frac{s}{c_p} + f_2(s,p)\frac{p}{P} \tag{4.8}$$

Equation (4.8) together with Eqs. (1.56) through (1.59), represent a complete set of equations for the parameters of the downstream perturbation field, and the upstream reflected pressure field. Using

Eqs. (1.60) and (1.61), and eliminating dZ_σ by Eq. (1.57),

$$\left(1-\frac{V}{U_n}\frac{k_{1v}'}{k_4}\right)\frac{dZ_{1v}'}{\sin\theta'}-C_2\left(\frac{k_{1\varphi}'}{k_4}+\frac{V}{U_n}\right)dZ_\varphi'+\left[\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)+C_{1R}\left(\frac{k_{1\varphi R}}{k_4}+m\frac{V}{U_n}\right)\right]dZ_{\varphi R}=$$
$$\left(1-m\frac{V}{U_n}\frac{k_{1v}}{k_4}\right)\frac{dZ_{1v}}{\sin\theta}-\left(\frac{\tau}{1+\tau}\right)dZ_s-\left[\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)+C_{1I}\left(\frac{k_{1\varphi I}}{k_4}+m\frac{V}{U_n}\right)\right]dZ_{\varphi I} \tag{4.9}$$

$$\left(\frac{1}{m-1}\frac{V}{U_n}\frac{k_{1v}'}{k_4}\right)\frac{dZ_{1v}'}{\sin\theta'}+C_1\left(\frac{1}{m-1}\frac{V}{U_n}\right)dZ_\varphi'+\left[f_2+C_{1R}\left(\frac{k_{1\varphi R}}{k_4}-\frac{m}{m-1}\frac{V}{U_n}\right)\right]dZ_{\varphi R}=$$
$$\left(1+\frac{m}{m-1}\frac{V}{U_n}\frac{k_{1v}}{k_4}\right)\frac{dZ_{1v}}{\sin\theta}-f_1\,dZ_s-\left[f_2+C_{1I}\left(\frac{k_{1\varphi I}}{k_4}-\frac{m}{m-1}\frac{V}{U_n}\right)\right]dZ_{\varphi I} \tag{4.10}$$

$$dZ_\varphi'-dZ_{\varphi R}=dZ_{\varphi I} \tag{4.11}$$

$$dZ_s'+\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)dZ_{\varphi R}=-\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)dZ_{\varphi I}+\frac{1}{1+\tau}dZ_s \tag{4.12}$$

As for the shock wave, the transfer functions are most simply expressed in polar coordinates. In addition to Eqs. (3.12) and (3.13), the following are needed, for convected inputs,

$$\frac{k_{1\varphi R}}{k_1}=-\frac{1}{a_4}\left(1+\frac{v}{M_n}\right)\quad ;\quad v=\sqrt{1-a_4\tan^2\varphi}$$
$$C_{1R}=-\frac{a_4}{\gamma}\frac{\tan\varphi}{1+M_n v} \tag{4.13}$$

for pressure input,

$$\frac{k_{1\varphi R}}{k_1'}=-\frac{1}{m\,a_4}\left(1+\frac{\omega}{M_n}\right)\quad ;\quad \omega=\sqrt{1-a_4 y^2}$$
$$C_{1R}=-\frac{a_4}{\gamma}\frac{y}{1+M_n\omega} \tag{4.14}$$

and

$$a_4=\frac{1-M_n^2}{M_n^2}$$

With this notation, for convected inputs take

$$\Delta_{v\,or\,s} = f_2\left(1-\frac{m}{\tan^2\varphi}\right)+\frac{\gamma-1}{\gamma}\left(\frac{1}{1+\tau}\right)\frac{1}{\tan^2\varphi} - \frac{a_4}{\gamma\tau}\left[\frac{1-\frac{1}{a_4}\left(\tau+\frac{m}{\tan^2\varphi}\right)\left(1+\frac{v}{M_u}\right)}{1+M_u v}\right]$$
$$+\frac{a_1}{\gamma\tau}\left[\frac{1-\frac{1}{a_1}\,\frac{1-\frac{X}{M_u'}}{\tan^2\varphi}}{1-M_u' X}\right] \qquad (4.15)$$

and for pressure input,

$$\Delta_{p} = f_2\left(1-\frac{m}{y^2}\right)+\frac{\gamma-1}{\gamma}\left(\frac{1}{1+\tau}\right)\frac{1}{y^2} - \frac{a_4}{\gamma\tau}\left[\frac{1-\frac{1}{a_4}\left(\tau+\frac{m}{y^2}\right)\left(1+\frac{\omega}{M_u}\right)}{1+M_u \omega}\right]$$
$$+\frac{a_1}{\gamma\tau}\left[\frac{1-\frac{1}{a_1}\,\frac{1-\frac{Z}{M_u'}}{y^2}}{1-M_u' Z}\right] \qquad (4.16)$$

The denominators of the transfer functions are somewhat more complicated than the analogous relations for the shock wave [Eqs. (3.15) and (3.16)] because of the extra term which results from the influence of the pressure wave which propagates upstream. However, they can be simplified in specific cases by making use of the fact that M_u is very small. It is convenient to consider the convected and pressure inputs separately.

For convected inputs, since a_1 and a_4 become large for M_u small, it is evident from Eqs. (3.12) and (4.13) that $\frac{k_{1p}'}{k_1}$ and $\frac{k_{1pR}}{k_1}$ are complex for practically the entire range of φ, which means, of course, that the upstream and downstream pressure waves are attenuated for practically all φ . It is therefore a good approximation to neglect the range of φ yielding sound waves, in the evaluation of the downstream vorticity and entropy fields, so that the transfer functions for the entropy and vorticity will be given in the simplified form which is obtained in the limit of M_n approaching zero. A similar argument applies to the transfer functions for the attenuated pressure field, so that they too will be given only in the limiting form for M_n approaching zero. However, the range of φ in which sound waves are generated is appreciable though small. Therefore, transfer functions which are valid in this small range of φ will be given for the sound fields resulting from convected inputs.

For pressure inputs, the transfer functions may be simplified by straightforward expansion in terms of M_n . The range of φ yielding sound waves downstream is not small in this case because the propagation velocity of the incident sound effectively increases the velocity of propagation along the flame, V .

The simplified results which are applicable for M_n approaching zero, excluding the sound generated by convected inputs, are then as follows:

for convected inputs,

$$x = \frac{i}{m\, M_u'} \tan\varphi$$

$$\frac{k_{1\varphi}'}{k_1} = -m\, M_u'^{\,2} + i \tan\varphi$$

$$C_2 = -\frac{1}{\gamma M_u'^{\,2}} \left[\frac{\tan\varphi}{m - i\tan\varphi} \right] \tag{4.17}$$

$$v = \frac{i}{M_u} \tan\varphi$$

$$\frac{k_{1\varphi R}}{k_1} = -M_u^2 - i\tan\varphi$$

$$C_{1R} = -\frac{1}{\gamma M_u^2} \left[\frac{\tan\varphi}{1 + i\tan\varphi} \right]$$

and for pressure input,

$$y = M_u \sin\varphi$$

$$z = \sqrt{1 - (1+\tau)\sin^2\varphi}$$

$$w = \cos\varphi \tag{4.18}$$

$$\frac{k_{1\varphi R}}{k_1'} = -\frac{M_u}{m}\left[\cos\varphi + M_u(2 - \cos^2\varphi) \right]$$

$$C_{1R} = C_{1I} = -\frac{\sin\varphi}{\gamma M_u}$$

$$a_1 = \frac{1}{m^2 M_u'^{\,2}}$$

$$a_4 = \frac{1}{M_u^2}$$

With these simplifications,

$$\Delta_{vors} = \frac{\left(1+\frac{\tan^2\varphi}{m}\right) + i\left[\left(\frac{2}{m\tau}+1\right)\tan\varphi + \frac{1+m}{\tau}\frac{1}{\tan\varphi}\right]}{\gamma M_u^2\left[\left(1+\frac{\tan^2\varphi}{m}\right) - i\left(\tau\tan\varphi\right)\right]} \tag{4.19}$$

$$\Delta_\varphi = \frac{m}{\gamma M_u^3\,\tau}\left[\frac{\cos 2\varphi + \sqrt{m}\,z}{\sin 2\varphi}\right] \quad ; \quad \varphi < \frac{\pi}{2} \tag{4.20}$$

Equation (4.20) is relevant only for $\varphi < \frac{\pi}{2}$, since for $\varphi > \frac{\pi}{2} + \epsilon$ the upstream sound waves would not interact with the discontinuity, but would propagate upstream from it.

The transfer functions are:

$$\Delta_v F_\varphi^v = 1 - \frac{m}{\tan^2\varphi} \tag{4.21}$$

$$\Delta_s F_\varphi^s = -f_1 + m(f_1-1)\frac{1}{\tan^2\varphi} \tag{4.22}$$

$$\Delta_\varphi F_\varphi^\varphi = \frac{2m}{\gamma M_u^3\,\tau}\;\frac{\cos\varphi}{\sin^2\varphi} \tag{4.23}$$

$$\Delta_v F_{1v}^v = \frac{1}{\gamma M_u^2}\;\frac{m+\tan^2\varphi - i\left[\frac{\tau-2}{\tau}\tan\varphi - \frac{m+1}{\tau}\frac{1}{\tan\varphi}\right]}{\left[1+\frac{\tan^2\varphi}{m}\right] - i\left[\tau\tan\varphi\right]} \tag{4.24}$$

$$\Delta_s F_{1v}^s = \frac{1}{\gamma M_u^2}\;\frac{\left[\frac{\tau(f_1-1)}{1+\tau} + \tau(f_1+1)\tan^2\varphi\right] + i\left[\left(4f_1 - 2 - \frac{\tau}{1+\tau}\right)\tan\varphi\right]}{\left[1+\frac{\tan^2\varphi}{m}\right] - i\left[\tau\tan\varphi\right]} \tag{4.25}$$

$$\Delta_\varphi F_{1v}^{\varphi} = \frac{2(m-1)\cos\varphi}{\gamma^2 M_u^3 \tau} \tag{4.26}$$

$$\Delta_v F_{rv}^{v} = -\frac{m}{\tan\varphi}\Delta_v F_{1v}^{v} \tag{4.27}$$

$$\Delta_s F_{rv}^{s} = -\frac{m}{\tan\varphi}\Delta_s F_{1v}^{s} \tag{4.28}$$

$$\Delta_\varphi F_{rv}^{\varphi} = -\frac{m}{M_u \sin\varphi}\Delta_\varphi F_{1v}^{\varphi} \tag{4.29}$$

$$F_{\varphi R}^{v} = F_{\varphi}^{v} \tag{4.30}$$

$$F_{\varphi R}^{s} = F_{\varphi}^{s} \tag{4.31}$$

$$F_{\varphi R}^{\varphi} = F_{\varphi}^{\varphi} - 1 \tag{4.32}$$

$$F_s^{v} = -\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right)F_{\varphi}^{v} \tag{4.33}$$

$$F_s^{s} = m \tag{4.34}$$

$$F_s^{\varphi} = -\frac{\gamma-1}{\gamma}\left(\frac{\tau}{1+\tau}\right) \tag{4.35}$$

In all the above simplified transfer functions, M_n occurs only as a multiplicative factor, if at all, so that the influence of burning velocity may be seen quite easily. The orders of magnitude of the various transfer functions are shown in (Fig. 20) as powers of M_u. The transfer function for lateral vorticity from a pressure input is worthy of note, since it varies as $\frac{1}{M_u}$. This is a result of the fact that the Mach number of propagation of the sound wave along the flame front is of the order of one, rather than of the order of M_u, as for convected inputs. Thus, the lines of constant phase of the downstream velocity profile lie almost parallel to the flame, and the velocity in the shear flow is nearly parallel to the flame. Since the normal component of this velocity is of the same order as the input, the tangential component becomes large.

3. Transfer Functions for Sound Generated by Convected Inputs

The range of φ for which sound waves arise is

$$0 \leq \tan^2\varphi \leq \frac{1}{a_+} = M_u^2$$

Using the fact that $\tan^2\varphi$ is small then, the following approximate form may be obtained for Δ_{vors}, from Eq. (4.15)

$$\Delta_{vors} = \frac{1}{\gamma\tau M_u}\left[\frac{mv + \sqrt{m'}\,x}{\tan^2\varphi}\right]$$

whence

$$F_p^v = -\gamma m M_u \tau \frac{1}{mv + \sqrt{m'}\,x} \tag{4.36}$$

and

$$F_\varphi^s = \gamma m M_u \tau \frac{f_1 - 1}{mv + \sqrt{m'}\,x} \tag{4.37}$$

It is evident that these transfer functions are not particularly small*, so

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* The transfer function for the attenuated waves is of order M_u^2.

that even though the range of φ giving sound is small, the sound intensity may be appreciable.

B. Statistical Properties of the Resultant Homogeneous Disturbance Fields

As for the shock wave, the quantities to be determined are the mean squares of the pressure, entropy and velocity fluctuations, in terms of the corresponding input quantities. Because the flame propagates subsonically, pressure fields result both upstream and downstream. The downstream sound intensity will be denoted by $\overline{\left(\frac{p'}{p'}\right)^2}$. To distinguish the incident upstream sound intensity, which is the pressure input, from the upstream sound intensity resulting from the interaction, they will be denoted by $\overline{\left(\frac{p}{p}\right)^2_I}$ and $\overline{\left(\frac{p}{p}\right)^2_R}$, respectively. The actual upstream sound intensity is, of course, the sum of these.

The downstream vorticity and entropy intensities are readily obtained from Eqs. (2.41) and the appropriate transfer functions of Eqs. (4.21) to (4.35). Since the transfer functions were derived for the limiting case of M_n approaching zero, the range of integration may be extended from 0 to $\pi/2$.

Upstream and downstream sound waves arise only from the sound wave input for the case of M_n approaching zero. This sound intensity is obtained from Eqs. (2.23) and (2.43). Only the homogeneous part of the reflected sound intensity will be included here. The inhomogeneous term will be considered separately later.

There is also a small amount of sound produced by convected inputs, since M_u is actually not zero, though it is small. This sound intensity is obtained by integration of Eqs. (4.36) and (4.37) over the appropriate small ranges of φ. The mean square sound, entropy and velocity are then given by

$$\overline{\left(\frac{p'}{P'}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_u}\right)^2}\int_0^{\varphi_c} F_p^{v\,2}\sin^3\varphi\, d\varphi + \overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\varphi_c} F_\varphi^{s\,2}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)_I^2}\int_0^{\varphi_c} F_\varphi^{p\,2}\sin\varphi\, d\varphi \tag{4.38}$$

$$\overline{\left(\frac{p}{P}\right)_R^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_u}\right)^2}\int_0^{\varphi_c} F_{pR}^{v*}F_{pR}^{v}\sin^3\varphi\, d\varphi + \overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\varphi_c} F_{pR}^{s*}F_{pR}^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)_I^2}\int_0^{\pi/2} F_{pR}^{p*}F_{pR}^{p}\sin\varphi\, d\varphi \tag{4.39}$$

$$\overline{\left(\frac{u_{1v}'}{U_u'}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_u}\right)^2}\int_0^{\pi/2} F_{rv}^{v*}F_{rv}^{v}\sin^3\varphi\, d\varphi + \overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\pi/2} F_{rv}^{s*}F_{rv}^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)_I^2}\int_0^{\pi/2} F_{rv}^{p*}F_{rv}^{p}\sin\varphi\, d\varphi \tag{4.40}$$

$$\overline{\left(\frac{u_{rv}'}{U_u'}\right)^2} = \frac{3}{2}\overline{\left(\frac{u_{1v}}{U_u}\right)^2}\int_0^{\pi/2} F_{rv}^{v*}F_{rv}^{v}\sin^3\varphi\, d\varphi + \overline{\left(\frac{s}{c_p}\right)^2}\int_0^{\pi/2} F_{rv}^{s*}F_{rv}^{s}\sin\varphi\, d\varphi + \frac{1}{2}\overline{\left(\frac{p}{P}\right)^2}\int_0^{\pi/2} F_{rv}^{p*}F_{rv}^{p}\sin\varphi\, d\varphi \tag{4.41}$$

$$\overline{\left(\frac{s'}{c_p}\right)^2} = \overline{\left(\frac{s}{c_p}\right)^2} \int_0^{\pi/2} F_s^{s*} F_s^s \sin\varphi \, d\varphi + \frac{1}{2} \overline{\left(\frac{p}{P}\right)_I^2} \int_0^{\pi/2} F_s^{p*} F_s^p \sin\varphi \, d\varphi \tag{4.42}$$

As before, let

$$\overline{\left(\frac{p'}{P'}\right)^2} = \overline{F_p^{v2}}\, \overline{\left(\frac{u_{1v}}{U_u}\right)^2} + \overline{F_p^{s2}}\, \overline{\left(\frac{s}{c_p}\right)^2} + \overline{F_p^{p2}}\, \overline{\left(\frac{p}{P}\right)_I^2}$$

$$\overline{\left(\frac{p}{P}\right)_R^2} = \overline{F_{pR}^{v2}}\, \overline{\left(\frac{u_{1v}}{U_u}\right)^2} + \overline{F_{pR}^{s2}}\, \overline{\left(\frac{s}{c_p}\right)^2} + \overline{F_{pR}^{p2}}\, \overline{\left(\frac{p}{P}\right)_I^2}$$

$$\overline{\left(\frac{u_{1v}'}{U_u'}\right)^2} = \overline{F_{1v}^{v2}}\, \overline{\left(\frac{u_{1v}}{U_u}\right)^2} + \overline{F_{1v}^{s2}}\, \overline{\left(\frac{s}{c_p}\right)^2} + \overline{F_{1v}^{p2}}\, \overline{\left(\frac{p}{P}\right)_I^2}$$

$$\frac{1}{2}\overline{\left(\frac{u_{\ell v}'}{U_u'}\right)^2} = \overline{F_{\ell v}^{v2}}\, \overline{\left(\frac{u_{1v}}{U_u}\right)^2} + \overline{F_{\ell v}^{s2}}\, \overline{\left(\frac{s}{c_p}\right)^2} + \overline{F_{\ell v}^{p2}}\, \overline{\left(\frac{p}{P}\right)_I^2}$$

$$\overline{\left(\frac{s'}{c_p}\right)^2} = \overline{F_s^{s2}}\, \overline{\left(\frac{s}{c_p}\right)^2} + \overline{F_s^{p2}}\, \overline{\left(\frac{p}{P}\right)_I^2}$$

For the computation of the values of $\overline{F_p^{s2}}$ and $\overline{F_{1v}^{s2}}$, it was necessary to estimate the value of f_1. From Ref. (12), the value of f_1 is as follows:

for Iso-Octane and air,	1.40
for Propane and air,	1.16
for Ethylene and air,	1.18

A representative value of 5/4 was chosen therefore for the calculations.

(Figures 21 and 22) show the transfer functions for the downstream vorticity field. They indicate that the turbulent intensity, measured as a fraction of the stream velocity, is strongly reduced by the flame. The actual magnitudes of the mean square velocities are almost unchanged as shown in (Fig. 23). The latter figure is in agreement with the results of Ref. (10).

The turbulence generated by upstream entropy fluctuations is strongly anisotropic, the normal component of velocity being almost 20 times the lateral component. These transfer functions are of such a size that the downstream turbulence is of about the same fractional intensity as the entropy fluctuation which induces it. The dimensionless entropy fluctuation used here is, in the absence of sound, equal to the temperature fluctuation divided by the absolute temperature [Eq. (3.2)], so a variation of one percent in the upstream temperature would produce a turbulence level of one percent downstream.

(Figure 22) shows that the generation of turbulence by the interaction of sound waves with a flame may also be quite important, since $\overline{F_{lv}^{p}}$ is proportional to $\frac{1}{M_n}$. Thus, for $\tau = 6$, $U_n = 100$ cm/sec = 3.3 ft/sec, $a = 1120$ ft/sec, and atmospheric upstream pressure, if the upstream sound level is 100 decibels, the downstream root-mean-square lateral velocity fluctuation is .23 percent of the downstream normal velocity. If the sound level is 120 decibels, it is 2.3 percent, and if the sound level is 140 decibels, the turbulence level is 23 percent. Such sound intensities may be expected in the combustion chamber or afterburner of a turbojet, for example.

Sound waves impinging on a flame are largely reflected as shown by (Figs. 25 and 26). This has, of course, been known for some time Ref. (14), although in previous analyses, the flame has been considered a temperature discontinuity, without through flow.

(Figures 27 and 28) show that there is some sound generated by convected disturbances passing through a flame. It is of low intensity for small M_k, being proportional to M_k^3 for vorticity input and M_k^2 for entropy input. However, since sound is observable at extremely low intensities, it is not negligible. For example, if U_k = 3.3 ft/sec (100 cm/sec), a = 1120 ft/sec, and atmospheric pressure, a turbulence level of ten percent would yield sound intensities of 16.4 and 21.2 decibels downstream and upstream, respectively, for $\tau = 6$.

An entropy fluctuation of one percent would, for the same case, give 40 decibels downstream and 46.5 decibels upstream. These are not high noise levels, but they are definitely in the audible range.

The frequency of the noise would be equal to that of the input disturbance as it is convected by a stationary observer. Thus, for the above example, if the wave length of the input disturbance were one inch, the frequency would be 40 cycles per second for a normal flame. Because the sound comes from input waves having φ very near zero, this frequency is characteristic of the normal burning velocity, not of the flow velocity, so that an oblique flame would give sound of the same frequency as that from a normal flame.

1. Mean Square Pressure Fluctuation in Attenuated Waves Due to Vorticity

The evaluation of Eq. (2.26) may be carried out as it was for the shock wave. Thus

$$F_\varphi^{v*}F_\varphi^{v} = \gamma M_k^4 \frac{\left(1-\frac{m}{\tan^2\varphi}\right)^2\left[\left(1+\frac{\tan^2\varphi}{m}\right)^2+\tau^2\tan^2\varphi\right]}{\left[1+\frac{\tan^2\varphi}{m}\right]^2+\left[\left(\frac{2}{m\tau}+1\right)\tan\varphi+\frac{1+m}{\tau}\frac{1}{\tan\varphi}\right]^2} \tag{4.43}$$

From Eq. (1.36)

$$\beta = \tan\varphi \quad \text{and} \quad g(\varphi) = 2k_0\xi\sin\varphi$$

$$\left(\frac{\overline{p'}}{P'}\right)^2_{att.} = 1.45\left(\frac{\overline{u_{1v}}}{U_k}\right)^2\int_0^{\pi/2} I(\varphi)\,F_\varphi^{v*}F_\varphi^{v}\sin 3\varphi\,d\varphi \tag{4.44}$$

where $I(\varphi)$ is again given by (Fig. 5). This equation has been evaluated numerically for a typical case of $\tau = 6$. The result is shown in (Fig. 29). This result is not very sensitive to τ, for τ in the range $5 < \tau < 10$. The principal dependence is on M_k, as indicated in (Fig. 27). For the case mentioned above, with U_k = 3.3 ft/sec, a = 1120 ft/sec, one percent initial turbulence would yield a noise level of 50 decibels directly behind the flame. At a distance of about half the wave length corresponding to the peak of $E(\xi)$, the sound would be 20 decibels weaker, and it would continue to attenuate approximately in this manner. Increasing the turbulence level to 10 percent would raise the noise level 20 decibels at each point.

The contribution to the integral of Eq. (4.44) is fairly evenly distributed over the entire range of φ, so that the distribution of energy over frequency for this sound should be approximately the same as the distribution of energy over frequency for the upstream shear waves.

Thus, the peak intensity would occur at the frequency corresponding to the peak of $E(\nu)$, which is, for a normal flame, [Eq. (2.32)],

$$\nu = \frac{U_u k}{4\pi} \quad ;$$

for $U_u = 3.3$ ft/sec, if $\frac{2\pi}{k}$ = one inch,

$$\nu = 20 \text{ cycles/sec.}$$

The shorter wave length eddies would, however, produce higher frequencies. Furthermore, since the contribution to Eq. (4.44) is uniformly distributed over φ, the frequency of the sound produced is determined by the flow velocity, not the velocity normal to the flame, so that the frequency would be higher for oblique flames than for normal flames, if the normal burning velocity were the same in both cases.

C. Inhomogeneous Reflected Upstream Sound

The inhomogeneous contribution to the reflected sound intensity is given by Eq. (2.44). In the present case of M_u approaching zero,

$$\frac{k_{1\varphi R}}{k_1} = -\left(1 + 2\frac{M_u}{\cos\varphi}\right)$$

and to first order, the inhomogeneous factor becomes

$$\cos\left(2 k_{\varphi I}\, \xi' \cos\varphi\right)$$

so that

$$\overline{\left(\frac{p}{P}\right)^2_R} = 4\pi \int Rl(F^{+}_{\varphi k}) G(k_\varphi) k_\varphi^2 \cos(2 k_{\varphi I} \xi \cos\varphi) \sin\varphi \, dk_\varphi \, d\varphi \qquad (4.45)$$

where

$$F_{pR}^{p} = \frac{\cos\varphi - \sqrt{m}\sqrt{1-(1+\tau)\sin^2\varphi}}{\cos\varphi + \sqrt{m}\sqrt{1-(1+\tau)\sin^2\varphi}}$$

Assume that $G_p(k_{pI})$ is a delta function at k_0, then

$$\overline{\left(\frac{p}{P}\right)^2_R} = \overline{\left(\frac{p}{P}\right)^2_I} \int_0^{\pi/2} Rl\,(F_{pR}^{p}) \cos(2k_0\xi\cos\varphi)\sin\varphi\,d\varphi \tag{4.46}$$

Now let $\cos\varphi = x$, then

$$F_{pR}^{p} = \frac{x - \sqrt{x^2-(1-m)}}{x+\sqrt{x^2-(1-m)}} \tag{4.47}$$

$m = \frac{1}{1+\tau}$ is about 0.1, and x varies from 1 to 0, so that F_{pR}^{p} is complex over most of the range of φ (roughly from 17° to 90°). In this range,

$$Rl\,(F_{pR}^{p}) = \frac{2}{1-m}x^2 - 1$$

$$\overline{\left(\frac{p}{P}\right)^2_R} = \overline{\left(\frac{p}{P}\right)^2_I} \int_0^{\sqrt{1-m}} \left(\frac{2}{1-m}x^2-1\right)\cos ax\,dx \quad ; \quad a = 2k_0\xi$$

and

$$\overline{\left(\frac{p}{P}\right)^2_R} = \overline{\left(\frac{p}{P}\right)^2_I}\,\frac{1}{a}\left(1+\frac{4}{a^2(1-m)}\right)\cos\left[a\sqrt{1-m} - \tan^{-1}\left(\frac{a\sqrt{1-m}}{4} - \frac{1}{a\sqrt{1-m}}\right)\right] \tag{4.48}$$

For a approaching zero, this becomes

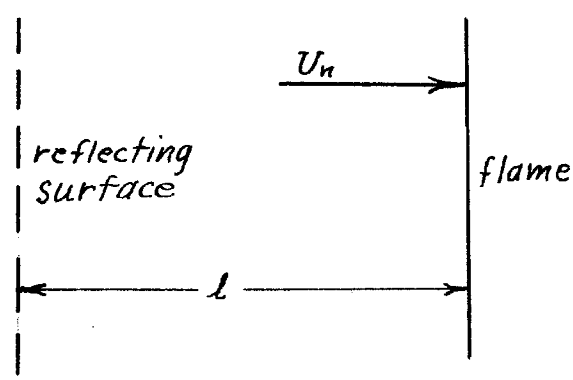
$$\overline{\left(\frac{p}{P}\right)^2_R} = -\frac{1}{3}\sqrt{1-m}\,\overline{\left(\frac{p}{P}\right)^2_I} \tag{4.49}$$

and for a very large, it approaches zero, with oscillations in between. For a given k_0, then, a fluctuation in the sound intensity should be noted as the observer moves upstream from the flame. This is shown in (Fig. 30), for $m = 0$. For any real distribution of intensity of the incident waves over wave number, this pattern would be smeared out, but there might still be a noticeable periodicity.

D. Reflection and Re-impingement of Sound Waves on the Upstream Face of the Flame

Since the sound intensity reflected back upstream is a large fraction of the incident sound intensity when sound waves impinge on a flame front [see (Fig. 26)], the question arises as to the possibility of a resonant condition occurring, with a large increase in pressure intensity, if there is a reflecting surface upstream of the flame.

A simple model which contains the essential features of the problem is shown below.



The reflecting surface will be assumed to reflect the wave like a solid wall, except that the amplitude of the reflected wave is a fraction of the amplitude which would result from reflection by a solid wall.

Assume, now, that a sound wave is generated at the flame by its interaction with sound, vorticity or entropy waves. This wave is described by:

$$\left(\frac{p}{P}\right)^{(1)} = e^{i[k_1^{(1)}(\xi' - U_u t) + k_4 r - a k^{(1)} t]} dZ_p^{(1)} \tag{4.50}$$

The notation is a little different from that used previously. The superscript indicates the order of the wave. Thus 1 denotes the original sound wave, 2 the sound wave reflected from the upstream surface, and so on. This reflected wave is given by

$$\left(\frac{p}{P}\right)^{(2)} = e^{i[k_1^{(2)}(\xi' - U_u t) + k_4 r - a k^{(2)} t + \psi^{(2)}]} dZ_p^{(2)} \tag{4.51}$$

where, by matching the time dependence and the ξ' dependence at $\xi' = -\ell$,

$$\frac{k_1^{(2)}}{k_1^{(1)}} = -(1 + 2M_u \sec \varphi^{(1)})$$

and

$$\psi^{(2)} = -2\ell k_1^{(1)}(1 + M_u \sec \varphi^{(1)})$$

If the upstream surface reflects like a solid wall, the amplitude of the reflected wave is given by:

$$dZ_p^{(2)} = \alpha \, dZ_p^{(1)}$$

where α will be taken less than one. The wave reflected off the upstream surface now interacts with the flame, which gives off an upstream sound wave,

$$\left(\frac{p}{P}\right)^{③} = e^{i\left[k_1^{③}(\xi - U_u t) + k_4 r - a k^{③} t + \psi^{③}\right]} dZ_\varphi^{③} \tag{4.52}$$

where

$$\frac{k_1^{③}}{k_1^{②}} = -\left(1 + 2 M_u \sec\varphi^{②}\right)$$

$$\psi^{③} = \psi^{②}$$

and

$$dZ_\varphi^{③} = \frac{\cos\varphi^{②} - \sqrt{m}\sqrt{1-(1+r)\sin^2\varphi^{②}}}{\cos\varphi^{②} + \sqrt{m}\sqrt{1-(1+r)\sin^2\varphi^{②}}}\, dZ_\varphi^{②} \tag{4.53}$$

The process of reflection continues, the pressure in the region finally being given by:

$$\frac{p}{P} = e^{i\left[k_1^{①}(\xi - U_u t) + k_4 r - a k^{①} t\right]} dZ_\varphi^{①} \left\{1 + \alpha e^{i\left[(k_1^{②} - k_1^{①})(\xi - U_u t) - a(k^{②} - k^{①})t - 2 l k_1^{①}(1 + M_u \sec\varphi^{①})\right]}\right\}$$

$$\times \sum_{n=0}^{\infty} \left\{\alpha F_{\varphi k}^{p} e^{-i\left[2 l k_1^{①}(1 + M_u \sec\varphi^{①})\right]}\right\}^n$$

The series is readily summed, so that

$$\frac{p}{P} = \frac{1 + \alpha e^{i\left[(k_1^{②} - k_1^{①})(\xi - U_u t) - a(k^{②} - k^{①})t - 2 l k_1^{①}(1 + M_u \sec\varphi^{①})\right]}}{1 - \alpha e^{i\left[2 l k_1^{①}(1 + M_u \sec\varphi^{①})\right]}}$$

$$\times e^{i\left[k_1^{①}(\xi - U_u t) + k_4 r - a k^{①} t\right]} dZ_\varphi^{①} \tag{4.54}$$

Define a new transfer function by:

$$\frac{p}{P} = F' e^{i[k_1^{(0)}(\xi - U_u t) + k_q r - a k^{(0)} t]} \, dZ_p^{(0)} \tag{4.55}$$

The pressure intensity in the region between the flame and the reflecting surface is given by:

$$\overline{\left(\frac{p}{P}\right)^2} = 2\pi \int F'^* F' \, \Phi_{pp}(k_p^{(0)}) k_p^{(0)2} \sin \varphi^{(0)} \, dk_p^{(0)} \, d\varphi^{(0)}$$

For the sake of simplicity, assume that $\Phi_{pp}(k_p^{(0)})$ is a delta function in $k_p^{(0)}$, at k_0, and uniform in $\varphi^{(0)}$ for $\frac{\pi}{2} < \varphi^{(0)} < \pi$, then

$$\overline{\left(\frac{p}{P}\right)^2} = \overline{\left(\frac{p}{P}\right)^2_g} \int_{\pi/2}^{\pi} F'^* F' \sin \varphi^{(0)} \, d\varphi^{(0)} \tag{4.56}$$

where g denotes the pressure intensity generated by the original disturbance interacting with the flame. If there is to be a large amplification of the pressure, most of the contribution to the integral of Eq. (4.56) will come from a value of $\varphi^{(0)}$ where the denominator of F'^*F' is small. To zero order in M_u, F'^*F' is

$$F'^*F' = \frac{1 + \frac{2\alpha}{1+\alpha^2 F_{pR}^{p*} F_{pR}^{p}} \cos[2k_0(\ell+\xi)\cos\varphi^{(0)}]}{1 - \frac{2\alpha}{1+\alpha^2 F_{pR}^{p*} F_{pR}^{p}} \left[Rl(F_{pR}^{p}) \cos(2\ell k_0 \cos\varphi^{(0)}) + Im(F_{pR}^{p}) \sin(2\ell k_0 \cos\varphi^{(0)}) \right]}$$

F_{pR}^{p} may be real or complex. In the range of where it is real, F_{pR}^{p} has a maximum of 1 at $\sin^2\varphi^{(0)} = m$. In the range of $\varphi^{(0)}$ where it is complex $F_{pR}^{p*} F_{pR}^{p} = 1$, and the real part of F_{pR}^{p} has a maximum of 1 at $\sin^2\varphi = m$, where the imaginary part of $F_{pR}^{p} = 0$. Therefore, F'^*F' may be large at $\sin^2\varphi^{(0)} = m$ if α is near 1 and the cosine in the denominator is near 1 for $\sin^2\varphi^{(0)}$ near m. F_{pR}^{p}

varies rather slowly with $\varphi^{(1)}$ near $\sin^2\varphi^{(1)} = m$, so that near this value of $\varphi^{(1)}$,

$$F'^{*}F' \approx \frac{1+\beta\cos\left[2k_0(\ell+\xi')\sqrt{1-m^2}\right]}{1-\beta\cos\left[2k_0\ell\sqrt{1-m^2}\right]}$$

where

$$\beta = \frac{2\alpha}{1+\alpha^2}$$

which will be large if $k_0\ell = \frac{n\pi}{\sqrt{1-m}}$ and β approaches one. It thus appears that if the reflecting upstream surface reflects the sound waves without too much attenuation, and if the wave length of the sound is $\frac{2\ell\sqrt{1-m^2}}{n}$, n an integer, there is a possibility of a high pressure intensity developing. The actual magnitude of the pressure intensity depends strongly on how well the upstream surface reflects the sound waves.

E. Summary of Results for Flames

A small disturbance field of pressure, vorticity or entropy fluctuations will generate disturbance fields of all three types in interacting with a flame front. The amplitudes of the generated disturbances are of comparable magnitude to that of the original disturbance, and depend only on the heat release of the flame, with the following exceptions. Sound waves incident on the upstream face of the flame generate turbulence in the downstream flow. The amplitude of the turbulent velocity parallel to the flame is proportional to the reciprocal of the propagation Mach number of the flame. Turbulence (vorticity mode) or temperature fluctuations (entropy mode) when convected thru a flame, produce sound or noise, the root-mean square pressure fluctuation of which is proportional to the propagation Mach number of the flame cubed for vorticity input and squared for entropy input. Either type of input, of reasonable amplitude, will produce audible, though not particularly loud sound. The amplitudes of all disturbances resulting from convection of entropy disturbances through the flame also depend on the relation between the entropy of the fluid upstream of the flame and its velocity of propagation.

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APPENDIX A

NOTATION

A	Equation (2.24)
C	Equation (1.61)
E	Equation (2.16)
F	Transfer functions for flame [subscript indicates output, superscript input, Eq. (1.64)]. See also equations following Eq. (4.42)
G	Equation (2.17)
I	Integral [Eq. (2.25)]
J	Jacobian
M	Mach number
P	Unperturbed or average pressure
Q	Heat release
R	Gas constant or correlation tensor [Eq. (2.1)]
S	Transfer functions for shock [subscript indicates output, superscript input, Eq. (1.64)]. See also Eq. (3.37)
T	Transfer function - general Eq. (1.64)
U	Unperturbed or average stream velocity
V	Velocity of propagation of disturbance along discontinuity (Fig. 4)
dZ	Fourier coefficient Eqs. (1.9 - 1.11)
a	Speed of sound. Also functions of Mach number in transfer functions
b	Coefficients in relations specifying nature of discontinuity Eqs. (1.53 - 1.55)
f	Coefficients specifying dependence of burning velocity of flame on pressure and temperature Eq. (4.7)

g	Function of φ [Eq. (2.25)]
k	Wave number referred to transformed axes [Eq. (1.23)]
m	Velocity ratio,
p	Disturbance pressure
q	Any of the small disturbance quantities
r	Coördinate parallel to discontinuity (Fig. 4)
s	Entropy fluctuation
t	Time
u	Disturbance velocities
v	Special variable [Eq. (4.13)]
w	Special variable [Eq. (4.14)]
x	Coordinate or special variable [Eq. (3.12)]
y	Special variable [Eq. (3.13)]
z	Special variable [Eq. (3.13)]
Δ	Denominator of transfer function
Φ	Spectral tensor
$\underline{\Omega}$	Vorticity vector
α	Equation preceding Eq. (4.52)
β	Imaginary part of $\frac{k_{1p}}{k_1}$ Eq. (2.12)
γ	Ratio of specific heats, c_p/c_v.
$\underline{\delta}$	Separation vector [Eq. (2.1)]
ζ	$\frac{k_v}{k_0}$ [Eq. (2.24)]

η	Axis parallel to discontinuity
θ	Angle between flow direction and discontinuity (Fig. 1)
κ	Wave number referred to original axis
λ	Wave length
ν	Frequency Eq. (2.33)
ξ	Coördinate in direction of normal velocity (Fig. 2)
ρ	Density
σ	Perturbation angle of discontinuity (Fig. 4)
τ	Dimensionless heat release Eq. (4.1)
φ	Angular coordinate Eq. (2.14)
ω	Circular frequency

<u>Subscripts</u>

I	Incident
R	Reflected
c	Critical
n	Normal to discontinuity
p	Pressure
r	In r direction
s	Entropy
t	Tangential to discontinuity
u	Velocity
v	Velocity associated with vorticity
1, 2, 3	In x_1, x_2, x_3, or ξ, η, x_3 directions
4	In r direction when used in k_4

APPENDIX B

INTEGRATIONS

SHOCK WAVE

A. Mean Square Outputs for Convected Inputs

For evaluation, the integrals of Eqs. (3.31) to (3.34) may be divided into two parts; first the range in which the transfer functions are real, and second, the range in which they are complex. For the convected inputs, the transfer functions are real for $0<\varphi<\varphi_{c_1}$, and complex for $\varphi_{c_1}<\varphi<\pi/2$.

1. Evaluation for $0<\varphi<\varphi_{c_1}$

Using the substitutions,

$$\left.\begin{aligned} x &= \sqrt{1-a_1\tan^2\varphi} \\ \tan^2\varphi &= \frac{1-x^2}{a_1} \\ \sin\varphi &= \sqrt{\frac{1-x^2}{b^2-x^2}} \\ b^2 &= a_1+1 \\ d\varphi &= -\sqrt{a_1}\left[\frac{x\,dx}{(b^2-x^2)\sqrt{1-x^2}}\right] \end{aligned}\right\} \qquad \text{B 1}$$

Eq. (3.15) may be written

$$\frac{1}{\Delta_{vors}} = -\frac{(x+a_6)(1-x^2)}{x^3+a_9x^2+a_{10}x+a_{11}}$$

and the transfer functions become

$$S_\varphi^v = b_{45}\,\frac{x^3+a_6x^2+a_7x+a_8}{x^3+a_9x^2+a_{10}x+a_{11}} = b_{45}\,\frac{N_1}{D_1} \qquad \text{B 2}$$

$$S_\varphi^s = -\frac{1}{2}\,S_\varphi^v \qquad \text{B 3}$$

$$S_{1v}^{v} = a_{12}\frac{x^3+a_{13}x^2+a_{14}x+a_{15}}{x^3+a_9x^2+a_{10}x+a_{11}} = a_{12}\frac{N_2}{D_1} \qquad \text{B 4}$$

$$S_{1v}^{s} = a_{16}\frac{(1-x^2)(x+a_{17})}{x^3+a_9x^2+a_{10}x+a_{11}} = a_{16}\frac{N_3}{D_1}(1-x^2) \qquad \text{B 5}$$

$$S_{rv}^{v} = -m\sqrt{\frac{a_1}{1-x^2}}\, S_{1v}^{v} \qquad \text{B 6}$$

$$S_{rv}^{s} = -m\sqrt{\frac{a_1}{1-x^2}}\, S_{1v}^{s} \qquad \text{B 7}$$

$$S_{s}^{v} = \frac{b_{7s}}{b_{4s}}\, S_{p}^{v} \qquad \text{B 8}$$

$$S_{s}^{s} = b_{8s}\frac{x^3+a_{20}x^2+a_{21}x+a_{22}}{x^3+a_9x^2+a_{10}x+a_{11}} = b_{8s}\frac{N_4}{D_1} \qquad \text{B 9}$$

where

$$a_6 = -\frac{1}{M_u'}$$

$$a_7 = m a_1 - 1$$

$$a_8 = a_6 a_7$$

$$a_9 = 2M_u' - \frac{1}{M_u'}$$

$$a_{10} = -\left[1 + a_1(a_2 m - a_3)\right]$$

$$a_{11} = \frac{1 - a_1 a_3}{M_u'}$$

$$a_{12} = b_{1s} + a_2(m-1) \qquad \text{B 10}$$

$$a_{13} = \frac{1}{a_{12}}\left[\frac{a_2 - b_{1s}}{M_u'} - m M_u' a_2\right]$$

$$a_{14} = \frac{1}{a_{12}}\left[a_1(a_3 - a_2 m) - a_2(m-1) - b_{1s}\right]$$

$$a_{15} = \frac{1}{a_{12}}\left[\frac{b_{1s} - a_1 a_3}{M_u'} + a_2 M_u'(m-1)(1 - a_1 m)\right]$$

$$a_{16} = -b_{2s} + \frac{2m}{\gamma+1}$$

$$a_{17} = \frac{M_u'}{a_{16}}\left[-\frac{2m}{\gamma+1}(1 - m a_1) + \frac{b_{2s}}{M_u'^2}\right]$$

$$a_{20} = M_u' \frac{m a_1 a_2}{b_{8s}} - \frac{1}{M_u'}$$

$$a_{21} = \frac{m a_1}{b_{8s}}\left(\frac{2 b_{2s}}{m-1} - a_2\right) + m a_1 - 1$$

$$a_{22} = \frac{1}{M_u'}\left[1 - m a_1\left(1 + \frac{2 b_{2s}}{b_{8s}(m-1)}\right)\right]$$

The parts of Eqs. (3.31) to (3.34) due to convected inputs, and in the range $0<\varphi<\varphi_{c1}$ are then,

$$\overline{S_p^{v^2}}' = \frac{3}{2} b_{45}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_1}{D_1}\right)^2 \frac{x(1-x^2)}{(b^2-x^2)^2\sqrt{b^2-x^2}}\, dx \quad ; \quad m=1,\ n=2 \qquad \text{B 11}$$

$$\overline{S_p^{s^2}}' = \frac{1}{4} b_{45}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_1}{D_1}\right)^2 \frac{x}{(b^2-x^2)\sqrt{b^2-x^2}}\, dx \quad ; \quad m=0,\ n=1 \qquad \text{B 12}$$

$$\overline{S_{1v}^{v^2}}' = \frac{3}{2} a_{12}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_2}{D_1}\right)^2 \frac{x(1-x^2)}{(b^2-x^2)^2\sqrt{b^2-x^2}}\, dx \quad ; \quad m=1,\ n=2 \qquad \text{B 13}$$

$$\overline{S_{1v}^{s^2}}' = a_{16}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_3}{D_1}\right)^2 \frac{x(1-x^2)}{(b^2-x^2)\sqrt{b^2-x^2}}\, dx \quad ; \quad m=1,\ n=1 \qquad \text{B 14}$$

$$\overline{S_{rv}^{v^2}}' = \frac{3}{2} m^2 a_1^{3/2} a_{12}^2 \int_0^1 \left(\frac{N_2}{D_1}\right)^2 \frac{x}{(b^2-x^2)^2\sqrt{b^2-x^2}}\, dx \quad ; \quad m=0,\ n=2 \qquad \text{B 15}$$

$$\overline{S_{rv}^{s^2}}' = m^2 a_1^{3/2} a_{16}^2 \int_0^1 \left(\frac{N_3}{D_1}\right)^2 \frac{x}{(b^2-x^2)\sqrt{b^2-x^2}}\, dx \quad ; \quad m=0,\ n=1 \qquad \text{B 16}$$

$$\overline{S_s^{v^2}}' = \left(\frac{b_{75}}{b_{45}}\right)^2 \overline{S_p^{v^2}}'$$

$$\overline{S_s^{s^2}}' = b_{65}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_4}{D_1}\right)^2 \frac{x}{(b^2-x^2)\sqrt{b^2-x^2}}\, dx \quad ; \quad m=0,\ n=1 \qquad \text{B 17}$$

The factor multiplying the radical in each of the integrands may be written,

$$f(x) = \left(\frac{N}{D_1}\right)^2 \frac{x(1-x^2)^m(-1)^h}{(x^2-b^2)^n} = (-1)^h \sum_{i=1}^{5} \left[\frac{A_i}{x-r_i} + \frac{B_i}{(x-r_i)^2} \right]$$

where $m = 0, 1$ as indicated in Eqs. (B 15) to (B 21)
$n = 1, 2$

and r_1, r_2, r_3 are the roots of $x^3 + a_9 x^2 + a_{10} x + a_{11} = 0$

$r_4 = b$

$r_5 = -b$

Each of the integrals may thus be decomposed into a set of integrals of the forms,

$$I_i = \int_0^1 \frac{dx}{(x-r_i)\sqrt{b^2-x^2}} = \frac{1}{\sqrt{b^2-r_i^2}} \log \frac{(-r_i)\left[-r_i + b^2 + \sqrt{(b^2-1)(b^2-r_i^2)}\right]}{(1-r_i)\left[b^2 + \sqrt{b^2(b^2-r_i^2)}\right]} \quad ; r_i^2 < b^2$$

B 18

$$= \frac{1}{\sqrt{r_i^2-b^2}} \left\{ \sin^{-1}\left[\frac{b^2-r_i}{b|1-r_i|}\right] - \sin^{-1}\frac{b}{|r_i|} \right\} \quad ; r_i^2 > b^2$$

$$= -\frac{1}{b}\left[\sqrt{\frac{b+1}{b-1}} - 1\right] \quad ; r_i = b > 0$$

$$= -\frac{1}{b}\left[\sqrt{\frac{b-1}{b+1}} - 1\right] \quad ; r_i = -b < 0$$

$$I_i' = \int_0^1 \frac{dx}{(x-r_i)^2\sqrt{b^2-x^2}} = -\frac{1}{(b^2-r_i^2)}\left[\frac{\sqrt{b^2-1}}{1-r_i} + \frac{b}{r_i} - r_i I_i\right] \quad ; \; r_i^2 < b^2$$

$$= \frac{1}{3b}\left[\frac{\sqrt{b^2-1}}{(1-b)^2} - \frac{1}{b} - I_i\right] \quad ; \; r_i = b \qquad \text{B 19}$$

$$= -\frac{1}{3b}\left[\frac{\sqrt{b^2-1}}{(1+b)^2} - \frac{1}{b} - I_i\right] \quad ; \; r_i = -b$$

The A_i and B_i are computed as follows:

$$B_i = \left[f(x)(x-r_i)^2\right]_{x=r_i}$$

$$A_i = \frac{d}{dx}\left[f(x)(x-r_i)^2\right]_{x=r_i} \qquad \begin{cases} n=2, & i=1,2,3,4,5 \\ n=1, & i=1,2,3 \end{cases}$$

$$A_i = \left[f(x)(x-r_i)\right]_{x=r_i} \qquad n=1, \; i=4,5$$

$$B_i = \left[\frac{N^2 x(1-x^2)^m (x-r_i)^2}{\prod_{j=1}^{3}(x-r_j)^2 \prod_{j=4}^{5}(x-r_j)^n}\right]_{x=r_i} \qquad \text{B 20}$$

$$A_i = B_i\left[\frac{2}{N}\frac{dN}{dx} + \frac{1}{x} - \frac{2mx}{1-x^2} - \sum_{\substack{j=1\\ \neq i}}^{3}\frac{2}{(x-r_j)} - \sum_{\substack{j=4\\ j\neq i}}^{5}\frac{n}{(x-r_j)}\right]_{x=r_i} ; \begin{cases} n=2, & i=1,2,3,4,5 \\ n=1, & i=1,2,3 \end{cases} \qquad \text{B 21}$$

$$A_i = \left[\frac{N^2 x(1-x^2)^m (x-r_i)}{\prod_{j=1}^{3}(x-r_j)^2 \prod_{j=4}^{5}(x-r_j)}\right]_{x=r_i} \quad ; \; n=1, \; i=4,5 \qquad \text{B 22}$$

where N , n and m are different for each integral, but the r_i are the same for all of them. The final results are then,

$$\overline{S_{\varphi}^{v2}}' = \frac{3}{2} b_{45}^2 \sqrt{a_1} \left[\sum_{i=1}^{5} (A_i I_i + B_i I_i') \right]$$

$$\overline{S_{\varphi}^{s2}}' = \frac{1}{4} b_{45}^2 \sqrt{a_1} \left[\sum_{i=1}^{5} (A_i I_i) + \sum_{i=1}^{3} (B_i I_i') \right]$$

$$\overline{S_{1v}^{v2}}' = \frac{3}{2} a_{12}^2 \sqrt{a_1} \left[\sum_{i=1}^{5} (A_i I_i + B_i I_i') \right]$$

$$\overline{S_{1v}^{s2}}' = a_{16}^2 \sqrt{a_1} \left[\sum_{i=1}^{5} (A_i I_i) + \sum_{i=1}^{3} (B_i I_i') \right]$$

$$\overline{S_{rv}^{v2}}' = \frac{3}{2} m^2 a_1^{3/2} a_{12}^2 \left[\sum_{i=1}^{5} (A_i I_i + B_i I_i') \right] \qquad \text{B 23}$$

$$\overline{S_{rv}^{s2}}' = m^2 a_1^{3/2} a_{16}^2 \left[\sum_{i=1}^{5} (A_i I_i) + \sum_{i=1}^{3} (B_i I_i') \right]$$

$$\overline{S_{s}^{v2}}' = \left(\frac{b_{75}}{b_{45}} \right)^2 \overline{S_{\varphi}^{v2}}'$$

$$\overline{S_{s}^{s2}}' = b_{85}^2 \sqrt{a_1} \left[\sum_{i=1}^{5} (A_i I_i) + \sum_{i=1}^{3} (B_i I_i') \right]$$

2. Evaluation for $\varphi_c < \varphi < \pi/2$

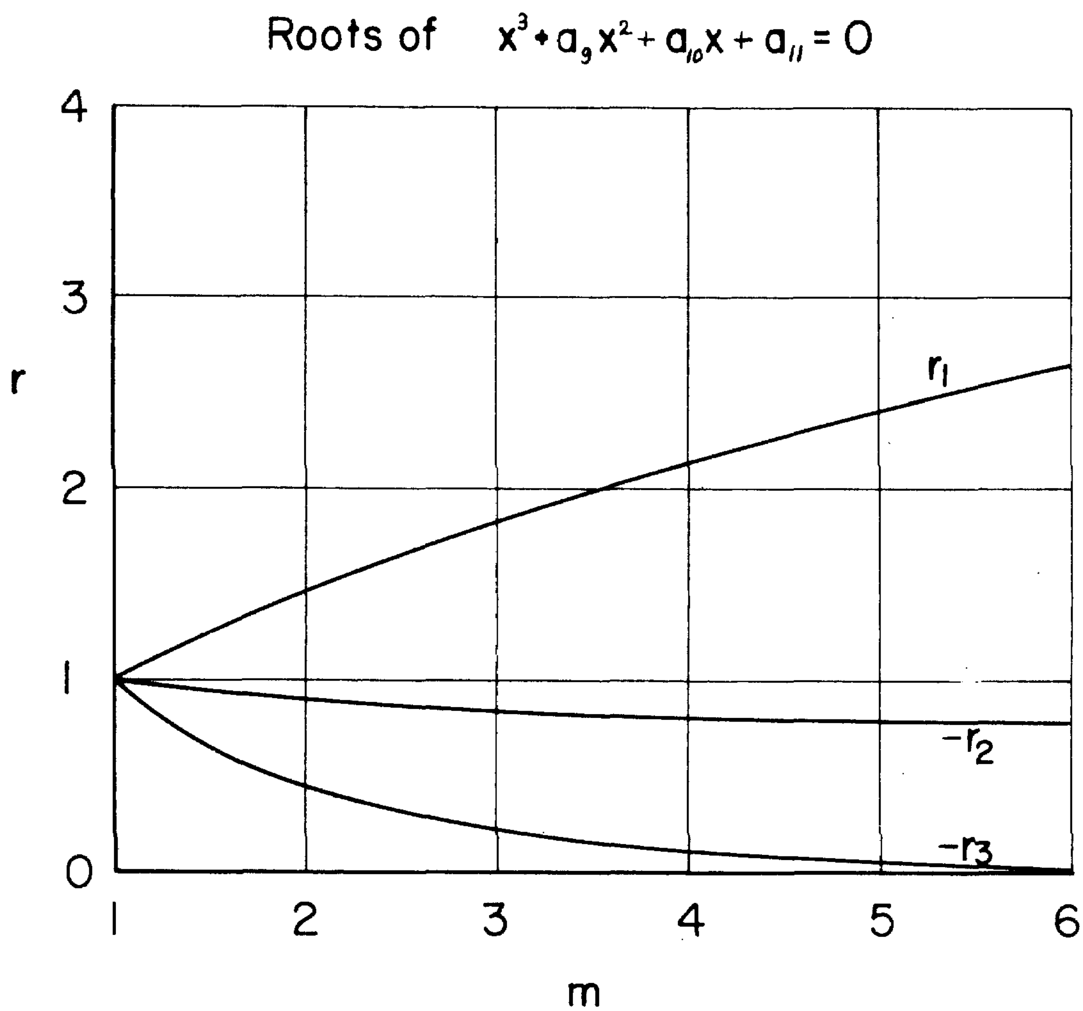
In this range of φ , x is imaginary. Let

$$z = a_1 \tan^2 \varphi - 1$$

$$\tan^2 \varphi = \frac{1+z}{a_1} \qquad \text{B 24}$$

$$\sin \varphi = \sqrt{\frac{1+z}{b^2+z}}$$

$$d\varphi = \frac{1}{2} \sqrt{a_1} \left[\frac{dz}{(b^2+z)\sqrt{1+z}} \right]$$


Roots of $x^3 + a_9 x^2 + a_{10} x + a_{11} = 0$
r
4
3
2
1
0
1
2
3
4
5
6
m
r_1
$-r_2$
$-r_3$

and the transfer functions become

$$S_{\varphi}^{v*} S_{\varphi}^{v} = b_{45}^{2} \frac{Z^3 + a_{23} Z^2 + a_{24} Z + a_{25}}{Z^3 + a_{26} Z^2 + a_{27} Z + a_{28}} = b_{45}^{2} \frac{N_5}{D_2} \qquad \text{B 25}$$

$$S_{\varphi}^{s*} S_{\varphi}^{s} = \frac{1}{4} S_{\varphi}^{v*} S_{\varphi}^{v} \qquad \text{B 26}$$

$$S_{1v}^{v*} S_{1v}^{v} = a_{12}^{2} \frac{Z^3 + a_{29} Z^2 + a_{30} Z + a_{31}}{Z^3 + a_{26} Z^2 + a_{27} Z + a_{28}} = a_{12}^{2} \frac{N_6}{D_2} \qquad \text{B 27}$$

$$S_{1v}^{s*} S_{1v}^{s} = a_{16}^{2} \frac{(1+Z)^2 (Z + a_{32})}{Z^3 + a_{26} Z^2 + a_{27} Z + a_{28}} = a_{16}^{2} \frac{N_7}{D_2} (1+Z)^2 \qquad \text{B 28}$$

$$S_{rv}^{v*} S_{rv}^{v} = \frac{m^2 a_1}{1+Z} S_{1v}^{v*} S_{1v}^{v} \qquad \text{B 29}$$

$$S_{rv}^{s*} S_{rv}^{s} = \frac{m^2 a_1}{1+Z} S_{1v}^{s*} S_{1v}^{s} \qquad \text{B 30}$$

$$S_{s}^{v*} S_{s}^{v} = \left(\frac{b_{75}}{b_{45}} \right)^2 S_{\varphi}^{v*} S_{\varphi}^{v} \qquad \text{B 31}$$

$$S_{s}^{s*} S_{s}^{s} = b_{85}^{2} \frac{Z^3 + a_{35} Z^2 + a_{36} Z + a_{37}}{Z^3 + a_{26} Z^2 + a_{27} Z + a_{28}} = b_{85}^{2} \frac{N_8}{D_2} \qquad \text{B 32}$$

where

$$\begin{aligned}
a_{23} &= a_6^2 - 2a_7 \\
a_{24} &= a_7^2 - 2a_6 a_8 \\
a_{25} &= a_8^2 \\
a_{26} &= a_9^2 - 2a_{10} \\
a_{27} &= a_{10}^2 - 2a_9 a_{11} \\
a_{28} &= a_{11}^2 \\
a_{29} &= a_{13}^2 - 2a_{14} \\
a_{30} &= a_{14}^2 - 2a_{13} a_{15} \\
a_{31} &= a_{15}^2 \\
a_{32} &= a_{17}^2 \\
a_{35} &= a_{20}^2 - 2a_{21} \\
a_{36} &= a_{21}^2 - 2a_{20} a_{22} \\
a_{37} &= a_{22}^2
\end{aligned} \qquad \text{B 33}$$

The parts of Eqs. (128) to (131) due to convected inputs, and in the range $\varphi_{c_1} < \varphi < \pi/2$ are then,

$$\overline{S_\varphi^{v^2}}'' = \frac{3}{4} b_{45}^2 \sqrt{a_1} \int_0^\infty \frac{N_5}{D_2} \frac{1+z}{(b^2+z^2)^2 \sqrt{b^2+z}}\, dz \qquad ; \; m=1,\; n=2 \qquad \text{B 34}$$

$$\overline{S_\varphi^{s^2}}'' = \frac{1}{8} b_{45}^2 \sqrt{a_1} \int_0^\infty \frac{N_5}{D_2} \frac{1}{(b^2+z)\sqrt{b^2+z}}\, dz \qquad ; \; m=0,\; n=1 \qquad \text{B 35}$$

$$\overline{S_{1v}^{v\,2}}'' = \frac{3}{4} a_{12}^2 \sqrt{a_1} \int_0^\infty \frac{N_6}{D_2} \frac{1+z}{(b^2+z)^2 \sqrt{b^2+z}} dz \qquad ; \; m=1, n=2 \qquad \text{B 36}$$

$$\overline{S_{1v}^{s\,2}}'' = \frac{1}{2} a_{16}^2 \sqrt{a_1} \int_0^\infty \frac{N_7}{D_2} \frac{(1+z)^2}{(b^2+z)\sqrt{b^2+z}} dz \qquad ; \; m=2, n=1 \qquad \text{B 37}$$

$$\overline{S_{rv}^{v\,2}}'' = \frac{3}{4} m^2 a_1^{3/2} a_{12}^2 \int_0^\infty \frac{N_6}{D_2} \frac{1}{(b^2+z)^2 \sqrt{b^2+z}} dz \qquad ; \; m=0, n=2 \qquad \text{B 38}$$

$$\overline{S_{rv}^{s\,2}}'' = \frac{m^2}{2} a_1^{3/2} a_{16}^2 \int_0^\infty \frac{N_7}{D_2} \frac{1+z}{(b^2+z)\sqrt{b^2+z}} dz \qquad ; \; m=1, n=1 \qquad \text{B 39}$$

$$\overline{S_s^{v\,2}}'' = \left(\frac{b_{7s}}{b_{4s}}\right)^2 \overline{S_p^{v\,2}}'' \qquad \text{B 40}$$

$$\overline{S_s^{s\,2}}'' = \frac{1}{2} b_{8s}^2 \sqrt{a_1} \int_0^\infty \frac{N_8}{D_2} \frac{1}{(b^2+z)\sqrt{b^2+z}} dz \qquad ; \; m=0, n=1 \qquad \text{B 41}$$

The factor multiplying the radical may be written,

$$f(z) = \frac{N}{D_2} \frac{(1+z)^m}{(b^2+z)^n} = \sum_{i=1}^{4} \left(\frac{A_i}{z-r_i} \right) + \frac{B_4}{(z+b^2)^2}$$

where $m = 0, 1, 2$

$n = 1, 2$

r_1, r_2, r_3 are the roots of $z^3 + a_{26} z^2 + a_{27} z + a_{28} = 0$

$r_4 = -b^2$

Each of the integrals may thus be decomposed into a set of integrals of the form

$$I_i = \int_0^\infty \frac{dz}{(z-r_i)\sqrt{z+b^2}} = \frac{1}{\sqrt{r_i+b^2}} \log \frac{b+\sqrt{r_i+b^2}}{b-\sqrt{r_i+b^2}} \quad ; \quad r_i+b^2>0$$

$$= \frac{2}{\sqrt{-(r_i+b^2)}} \tan^{-1} \frac{1}{b}\sqrt{-(r_i+b^2)} \quad ; \quad r_i+b^2<0 \qquad \text{B 42}$$

$$= \frac{2}{b} \quad ; \quad r_i = -b^2$$

$$I_4' = \int_0^\infty \frac{dz}{(z+b^2)^2\sqrt{z+b^2}} = \frac{2}{3b^3} \qquad \text{B 43}$$

The A_i and B_i are computed as follows

$$A_i = \left[\frac{N\,(1+z)^m\,(z-r_i)}{\left[\prod_{j=1}^{3}(z-r_j)\right](z+b^2)^n} \right]_{z=r_i} \qquad \left\{ \begin{array}{l} n=1,\ i=1,2,3,4 \\ n=2,\ i=1,2,3 \end{array} \right. \qquad \text{B 44}$$

$$B_4 = \left[\frac{N\,(1+z)^m}{\prod_{j=1}^{3}(z-r_j)} \right]_{z=-b^2} \qquad n=2 \qquad \text{B 45}$$

$$A_4 = B_4 \left[\frac{1}{N}\frac{dN}{dx} + \frac{m}{1+z} - \sum_{j=1}^{3} \frac{1}{z-r_j} \right]_{z=-b^2} \quad ; \quad n=2,\ i=4 \qquad \text{B 46}$$

The final results are then,

$$\overline{S_p^{v\,2}}'' = \frac{3}{4} b_{45}^2 \sqrt{a_1}\left[\sum_{i=1}^{4}(A_i I_i) + B_4 I_4'\right]$$

$$\overline{S_p^{s\,2}}'' = \frac{1}{8} b_{45}^2 \sqrt{a_1}\left[\sum_{i=1}^{4}(A_i I_i)\right]$$

$$\overline{S_{1v}^{v\,2}}'' = \frac{3}{4} a_{12}^2 \sqrt{a_1}\left[\sum_{i=1}^{4}(A_i I_i) + B_4 I_4'\right]$$

$$\overline{S_{1v}^{s\,2}}'' = \frac{1}{2} a_{16}^2 \sqrt{a_1}\left[\sum_{i=1}^{4}(A_i I_i)\right]$$

$$\overline{S_{rv}^{v\,2}}'' = \frac{3}{4} m^2 a_1^{3/2} a_{12}^2 \left[\sum_{i=1}^{4}(A_i I_i) + B_4 I_4'\right] \qquad \text{B 47}$$

$$\overline{S_{rv}^{s\,2}}'' = \frac{m^2}{2} a_1^{3/2} a_{16}^2 \left[\sum_{i=1}^{4}(A_i I_i)\right]$$

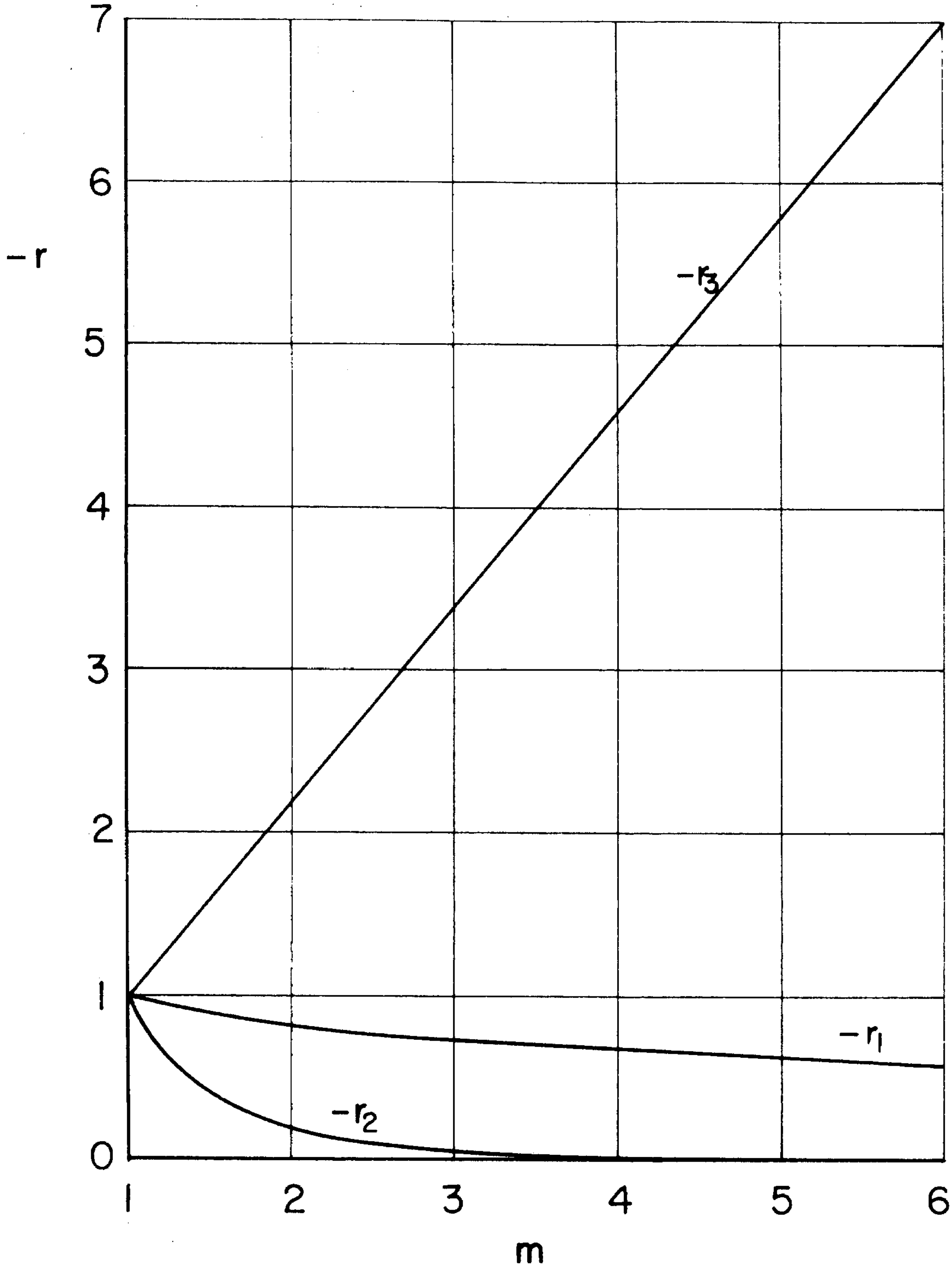
$$\overline{S_s^{v\,2}}'' = \left(\frac{b_{75}}{b_{45}}\right)^2 \overline{S_p^{v\,2}}''$$

$$\overline{S_s^{s\,2}}'' = \frac{1}{2} b_{85}^2 \sqrt{a_1}\left[\sum_{i=1}^{4}(A_i I_i)\right]$$

The total mean square value is, for example

$$\overline{S_{1v}^{v\,2}} = \overline{S_{1v}^{v\,2}}' + \overline{S_{1v}^{v\,2}}''$$

Roots of $z^3 + a_{26}z^2 + a_{27}z + a_{28} = 0$



B. Limiting Form of $\overline{S_p^v}'$ and $\overline{S_p^s}'$ for $m \rightarrow 1$

From Eq. (B 11)

$$\overline{S_p^{v\,2}}' = \frac{3}{2} b_{45}^2 \sqrt{a_1} \int_0^1 \left(\frac{N_1}{D_1}\right)^2 \frac{x(1-x^2)}{(b^2-x^2)^2\sqrt{b^2-x^2}}\, dx$$

where
$$\frac{N_1}{D_1} = \frac{x^3 + a_6 x^2 + a_7 x + a_8}{x^3 + a_9 x^2 + a_{10} x + a_{11}}$$

It is desired to expand the above expression in powers of $(m-1)$. The coefficients in $\frac{N_1}{D_1}$ may be expressed as follows: [see Eq. (B 10)].

$$a_6 = -\left[1 + \frac{\gamma+1}{4}(m-1)\right]$$

$$a_7 = -\left[1 - \frac{\gamma+1}{2}(m-1)\right]$$

$$a_8 = 1 - \frac{\gamma+1}{4}(m-1)$$

$$a_9 = 1 - \frac{3}{4}(\gamma+1)(m-1)$$

$$a_{10} = -\left[1 + 2(m-1)\right]$$

$$a_{11} = -\left[1 + \frac{\gamma+1}{4}(m-1)\right]$$

also,

$$a_1 = \frac{\gamma+1}{2}(m-1)$$

$$b^2 = 1 + \frac{\gamma+1}{2}(m-1)$$

Therefore $\left(\frac{N_1}{D_1}\right)^2$ may be expanded as a power series in $(m-1)$, as follows,

$$\left(\frac{N_1}{D_1}\right)^2 = \left(\frac{x-1}{x+1}\right)^2 + f(x)\,(m-1) + \cdots$$

so that the integral becomes

$$\int_0^1 \left(\frac{x-1}{x+1}\right)^2 \frac{x(1-x^2)\,dx}{(1-x^2)^2\sqrt{1-x^2}} + O(m-1) = \frac{1}{15} + O(m-1) \qquad \text{B 48}$$

The coefficient of the integral becomes

$$\frac{3}{2} b_{45}^2 \sqrt{a_1} = 24\left(\frac{\gamma}{\gamma+1}\right)^2 \sqrt{\frac{\gamma+1}{2}}\,(m-1)^{\frac{1}{2}} + O(m-1)$$

so that

$$\overline{S_p^{v^2}}' = \frac{8}{5}\left(\frac{\gamma}{\gamma+1}\right)^2 \sqrt{\frac{\gamma+1}{2}}\,(m-1)^{\frac{1}{2}} + O(m-1)^{3/2}$$

and

$$\overline{S_p^{v}}' = \left(\frac{8}{5}\right)^{\frac{1}{2}} \frac{\gamma}{\gamma+1}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{4}}(m-1)^{\frac{1}{4}} + O(m-1)^{5/4} \qquad \text{B 49}$$

The limiting form of $\overline{S_p^{s}}'$ differs from that of $\overline{S_p^{v}}'$ only in a factor of $\sqrt{\frac{1}{6}}$, thus

$$\overline{S_p^{s}}' = \left(\frac{4}{15}\right)^{\frac{1}{2}} \frac{\gamma}{\gamma+1}\left(\frac{\gamma+1}{2}\right)^{\frac{1}{4}}(m-1)^{\frac{1}{4}} + O(m-1)^{5/4} \qquad \text{B 50}$$

FLAME FRONT

A. Mean Square Outputs

The integrals of Eqs. (4.32) to (4.36) may be divided into two categories, those arising from convected inputs, and those from pressure input.

1. Convected Inputs

let $z = \tan^2 \varphi$

then $\sin\varphi = \sqrt{\frac{z}{1+z}}$ B 51

$$d\varphi = \frac{dz}{2(1+z)\sqrt{z}}$$

and the transfer functions may be written as follows:

$$F_{1v}^{v*} F_{1v}^{v} = m^2 \left[\frac{z^3 + a_{55} z^2 + a_{56} z + a_{57}}{z^3 + a_{50} z^2 + a_{51} z + a_{52}} \right] = m^2 \frac{N_9}{D_3} \qquad \text{B 52}$$

$$F_{1v}^{s*} F_{1v}^{s} = a_{58} \left[\frac{(z^2 + a_{59} z + a_{60}) z}{z^3 + a_{50} z^2 + a_{51} z + a_{52}} \right] = a_{58} \frac{N_{10}}{D_3} z \qquad \text{B 53}$$

$$F_{rv}^{v*} F_{rv}^{v} = \frac{m^2}{z} F_{1v}^{v*} F_{1v}^{v} \qquad \text{B 54}$$

$$F_{rv}^{s*} F_{rv}^{s} = \frac{m^2}{z} F_{1v}^{s*} F_{1v}^{s} \qquad \text{B 55}$$

$$F_{s}^{s*} F_{s}^{s} = m^2 \qquad \text{B 56}$$

where

$$
\begin{aligned}
a_{50} &= 2m + \left(\tfrac{2}{\tau} + m\right)^2 \\
a_{51} &= m^2 + 2\left(\tfrac{1+m}{\tau}\right)\left(\tfrac{2m}{\tau} + m^2\right) \\
a_{52} &= \left[\frac{m(m+1)}{\tau}\right]^2 \\
a_{55} &= 2m + \left(\tfrac{\tau-2}{\tau}\right)^2 \\
a_{56} &= m^2 - \tfrac{2m}{\tau}(2+\tau)\left(\tfrac{\tau-2}{\tau}\right) \\
a_{57} &= \left[\tfrac{m}{\tau}(2+\tau)\right]^2 \\
a_{58} &= \left[m\tau(f_1+1)\right]^2 \\
a_{59} &= \left[\frac{4f_1 - 2 - m\tau}{\tau(f_1+1)}\right]^2 \\
a_{60} &= m^2\left(\frac{f_1-1}{f_1+1}\right)^2
\end{aligned}
\qquad \text{B 57}
$$

The integrals to be evaluated are then

$$\overline{F_{1v}^{v\,2}} = \frac{3}{4}m^2 \int_0^\infty \frac{N_9}{D_3} \frac{z\,dz}{(1+z)^2\sqrt{1+z}} \quad ; \quad m=1,\ n=2 \qquad \text{B 58}$$

$$\overline{F_{1v}^{s\,2}} = \frac{a_{58}}{2} \int_0^\infty \frac{N_{10}}{D_3} \frac{z\,dz}{(1+z)\sqrt{1+z}} \quad ; \quad m=1,\ n=1 \qquad \text{B 59}$$

$$\overline{F_{rv}^{v\,2}} = \frac{3}{4}m^4 \int_0^\infty \frac{N_9}{D_3} \frac{dz}{(1+z)^2\sqrt{1+z}} \quad ; \quad m=0,\ n=2 \qquad \text{B 60}$$

$$\overline{F_{rv}^{s\,2}} = \frac{a_{58}\,m^2}{2} \int_0^\infty \frac{N_{10}}{D_3} \frac{dz}{(1+z)\sqrt{1+z}} \quad ; \quad m=0,\ n=1 \qquad \text{B 61}$$

$$\overline{F_5^{s2}} = m^2 \int_0^{\pi/2} \sin\varphi \, d\varphi = m^2 \qquad \text{B 62}$$

The factor multiplying the radical in each integral may be written:

$$f(z) = \frac{N}{D_3} \frac{z^m}{(1+z)^n} = \sum_{i=1}^{4} \left(\frac{A_i}{z-r_i} \right) + \frac{B_4}{(1+z)^2}$$

where $m = 0, 1$
$n = 1, 2$

$r_i = r_1, r_2, r_3$ are roots of $z^3 + a_{50} z^2 + a_{51} z + a_{52} = 0$

$r_4 = -1$

The integrals are then of the same form as for the shock;

$$I_i = \frac{1}{\sqrt{r_i+1}} \log \frac{1+\sqrt{r_i+1}}{1-\sqrt{r_i+1}} \quad ; \quad r_i + 1 > 0 \qquad \text{B 63}$$

$$= \frac{2}{\sqrt{-(r_i+1)}} \tan^{-1} \sqrt{-(r_i+1)} \quad ; \quad r_i + 1 < 0 \qquad \text{B 64}$$

$$= 2 \quad ; \quad r_i = -1 \qquad \text{B 65}$$

$$I_4' = \frac{2}{3} \qquad \text{B 66}$$

and the A_i and B_i are computed as follows:

$$A_i = \left[\frac{N z^m (z-r_i)}{\left[\prod_{j=1}^{3} (z-r_j) \right] (z+1)^n} \right]_{z=r_i} \quad \begin{cases} n=1, & i=1,2,3,4 \\ n=2, & i=1,2,3 \end{cases} \qquad \text{B 67}$$

$$B_4 = \left[\frac{N \, z^m}{\prod_{j=1}^{3}(z-r_j)} \right]_{z=-1} \qquad n=2 \qquad \text{B 68}$$

$$A_4 = B_4 \left[\frac{1}{N}\frac{dN}{dx} + \frac{m}{z} - \sum_{j=1}^{3} \frac{1}{(z-r_j)} \right]_{z=-1} \qquad n=2\ ,\ i=4 \qquad \text{B 69}$$

The final results are

$$\overline{F_{1v}^{v\,2}} = \tfrac{3}{4} m^2 \left[\sum_{i=1}^{3} (A_i I_i) + 2A_4 + \tfrac{2}{3} B_4 \right]$$

$$\overline{F_{1v}^{s\,2}} = \frac{a_{58}}{2} \left[\sum_{i=1}^{3} (A_i I_i) + 2A_4 \right] \qquad \text{B 70}$$

$$\overline{F_{rv}^{v\,2}} = \tfrac{3}{4} m^4 \left[\sum_{i=1}^{3} (A_i I_i) + 2A_4 + \tfrac{2}{3} B_4 \right]$$

$$\overline{F_{rv}^{s\,2}} = \frac{a_{58} m^2}{2} \left[\sum_{i=1}^{3} (A_i I_i) + 2A_4 \right]$$

$$\overline{F_s^{s\,2}} = m^2$$

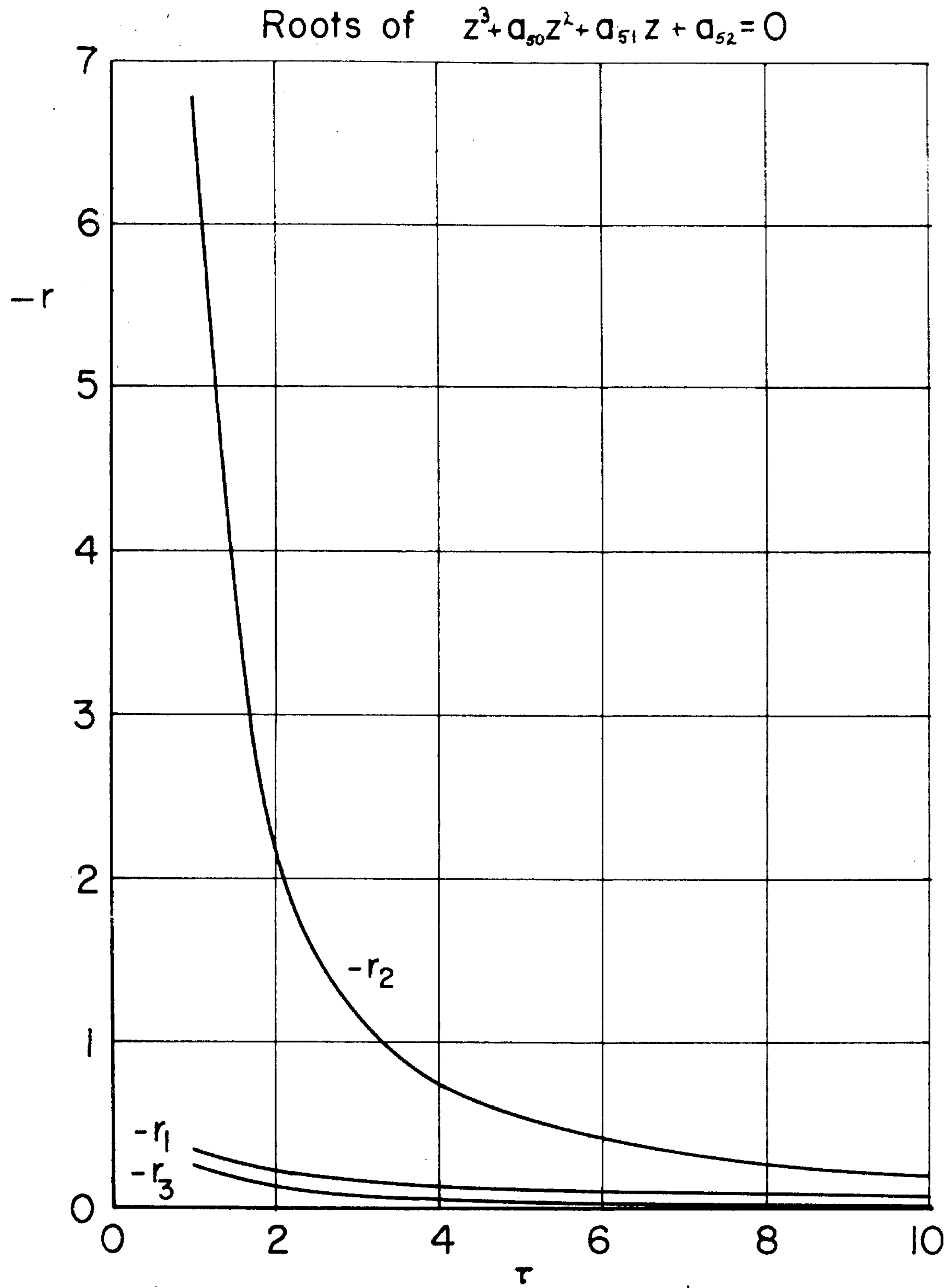
2. Pressure Input

Again two separate integrations are necessary, for the range where the transfer functions are real and complex, that is $0 < \varphi < \varphi_c$ and $\varphi_c < \varphi < \pi/2$ respectively. Evaluation for $0 < \varphi < \varphi_c$:

$$x = \sqrt{1-(1+\tau)\sin^2\varphi}$$

$$\sin\varphi = \sqrt{m}\,\sqrt{1-x^2} \qquad \text{B 71}$$

$$d\varphi = \frac{-x\,dx}{\sqrt{1-x^2}\,\sqrt{\tau+x^2}}$$


Roots of $z^3 + a_{50}z^2 + a_{51}z + a_{52} = 0$
7
6
5
4
3
2
1
0
$-r$
$-r_2$
$-r_1$
$-r_3$
0
2
4
6
8
10
τ

then from Eq. (4.20),

$$\Delta_p = \frac{m^{\frac{1}{2}}}{\gamma M_u^3 \tau}\left[\frac{x+\sqrt{\tau+x^2}}{1-x^2}\right]$$

and the transfer functions become:

$$F_p^p = 2\,\frac{\sqrt{\tau+x^2}}{x+\sqrt{\tau+x^2}} \qquad \text{B 72}$$

$$F_{1v}^p = \frac{2(m-1)}{\gamma}\;\frac{(1-x^2)\sqrt{\tau+x^2}}{x+\sqrt{\tau+x^2}} \qquad \text{B 73}$$

$$F_{rv}^p = -\frac{2m^{\frac{1}{2}}(m-1)}{\gamma M_u}\;\frac{\sqrt{1-x^2}\,\sqrt{\tau+x^2}}{x+\sqrt{\tau+x^2}} \qquad \text{B 74}$$

$$F_s^p = -\frac{\gamma-1}{\gamma}\,m\tau\,F_p^p \qquad \text{B 75}$$

$$F_{p_R}^p = F_p^p - 1 \qquad \text{B 76}$$

The integrals to be evaluated are:

$$\overline{F_p^{p\,2}}' = 2\sqrt{m}\int_0^1 \frac{\sqrt{\tau+x^2}}{\left[x+\sqrt{\tau+x^2}\right]^2}\,x\,dx \qquad \text{B 77}$$

$$\overline{F_{p_R}^{p\,2}}' = \frac{\sqrt{m}}{2}\int_0^1 \left[\frac{2\sqrt{\tau+x^2}}{x+\sqrt{\tau+x^2}} - 1\right]^2 \frac{x\,dx}{\sqrt{\tau+x^2}} \qquad \text{B 78}$$

$$\overline{F_{1v}^{\varphi^2}}' = \frac{2(m-1)^2}{\gamma^2}\sqrt{m}\int_0^1 \frac{(1-x^2)^2\sqrt{\tau+x^2}}{\left[x+\sqrt{\tau+x^2}\right]^2}\, x\,dx \qquad \text{B 79}$$

$$\overline{F_{rv}^{\varphi^2}}' = \frac{2(m-1)^2}{\gamma^2 M_u^2}\, m^{3/2}\int_0^1 \frac{(1-x^2)\sqrt{\tau+x^2}}{\left[x+\sqrt{\tau+x^2}\right]^2}\, x\,dx \qquad \text{B 80}$$

$$\overline{F_s^{\varphi^2}}' = \left(\frac{\gamma-1}{\gamma}m\tau\right)^2 \overline{F_p^{\varphi^2}}' \qquad \text{B 81}$$

Upon evaluation,

$$\overline{F_p^{\varphi^2}}' = \frac{2\sqrt{m}}{\tau^2}\left[\frac{1}{m^{3/2}}\left(\frac{2}{5m}-\frac{\tau}{3}\right)-\tau\left(\frac{2}{3}+\frac{\tau^{3/2}}{15}\right)-\frac{2}{5}\right]$$

$$\overline{F_{pR}^{\varphi^2}}' = \frac{\sqrt{m}}{2\tau^2}\left[\frac{8}{5}(\tau+1)^{5/2}-\frac{8\tau}{3}(\tau+1)^{3/2}+\tau^2(\tau+1)^{\frac{1}{2}}+\frac{\tau^{5/2}}{15}-\frac{4\tau}{3}-\frac{8}{5}\right] \qquad \text{B 82}$$

$$\overline{F_{1v}^{\varphi^2}}' = \frac{2\sqrt{m}}{\gamma^2\tau^2}(m-1)^2\Big\{-\frac{4+\tau/3}{7}\left[(\tau+1)^{7/2}-\tau^{7/2}\right]+\frac{1}{5}\left[(1-\tau)+\tau\left(4+\frac{\tau}{3}\right)\right]\left[(\tau+1)^{5/2}-\tau^{5/2}\right]$$

$$+\frac{\tau}{3}\left[1-2(1-\tau)-\tau\left(4+\frac{\tau}{3}\right)\right]\left[(\tau+1)^{3/2}-\tau^{3/2}\right]+\frac{2}{9}(\tau+1)^{3/2}-\frac{2}{3}\tau+\frac{2}{5}(2\tau-1)+\frac{2}{7}(2-\tau)-\frac{2}{9}\Big\}$$

$$M_u^2\,\overline{F_{rv}^{\varphi^2}}' = \frac{2m^{3/2}(m-1)^2}{\gamma^2\tau^2}\Big\{-\frac{2}{7}(\tau+1)^{7/2}+\left(\frac{3\tau+2}{5}-\frac{\tau}{3}\right)(\tau+1)^{5/2}+\frac{2}{7}\tau^{7/2}-\left(\frac{3\tau+2}{5}-\frac{1+\tau}{3}\right)\tau^{5/2}$$

$$+\frac{2}{7}-\frac{2}{5}(1-\tau)-\frac{2}{3}\tau\Big\}$$

Evaluation for $\varphi_c < \varphi < \pi/2$

let $z = (1+\tau)\sin^2\varphi - 1$

$$\overline{F_{pR}^{\varphi^2}}'' = \frac{\sqrt{m}}{2}\int_0^{\tau}\frac{dz}{\sqrt{\tau-z}} \qquad \text{B 83}$$

$$\overline{F_{1v}^{p\,2}}'' = \frac{(m-1)^2 m^{\frac{1}{2}}}{\gamma^2 \tau}\int_0^\tau (z^2+2z+1)\sqrt{\tau - z}\,dz \qquad \text{B 84}$$

$$\overline{F_{rv}^{p\,2}}'' = \frac{2m^{3/2}(m-1)^2}{\gamma^2 M_n^2 \tau}\int_0^\tau (1+z)\sqrt{\tau - z}\,dz \qquad \text{B 85}$$

$$\overline{F_s^{p\,2}}'' = \frac{2m^{\frac{1}{2}}}{\tau}\left(\frac{\gamma-1}{\gamma} m\tau\right)^2 \int_0^\tau \sqrt{\tau - z}\,dz \qquad \text{B 86}$$

after evaluation,

$$\overline{F_{pk}^{p\,2}}'' = m^{\frac{1}{2}} \tau^{\frac{1}{2}}$$

$$\overline{F_{1v}^{p\,2}}'' = \frac{(m-1)^2 m^{\frac{1}{2}}}{\gamma^2}\left[\frac{2}{3}\tau^{\frac{1}{2}} + \frac{8}{15}\tau^{3/2} + \frac{16}{105}\tau^{5/2}\right]$$

$$\overline{F_{rv}^{p\,2}}'' = \frac{2(m-1)^2 m^{3/2}}{\gamma^2 M_n^2}\left[\frac{2}{3}\tau^{\frac{1}{2}} + \frac{4}{15}\tau^{3/2}\right] \qquad \text{B 87}$$

$$\overline{F_s^{p\,2}}'' = \frac{4}{3} m^{\frac{1}{2}}\left(\frac{\gamma-1}{\gamma} m\tau\right)^2 \tau^{\frac{1}{2}}$$

B. Approximate Evaluation of $\overline{F_\varphi^v}'$ and $\overline{F_\varphi^s}'$ for $M_n \to 0$

The range of φ for which sound waves arise in the downstream flow is that for which

$$\tan^2\varphi \le \frac{1}{a_1} \approx m^2 M_n'^2 = m M_n^2$$

Now for M_n small, Δ_{vors} may be written approximately as

$$\Delta_{vors} = \frac{1}{\gamma\tau M_n}\left[\frac{m v + \sqrt{m}\, x}{\tan^2\varphi}\right] \qquad \text{B 88}$$

where
$$v = \sqrt{1 - a_4 \tan^2\varphi} \quad ; \quad a_4 = \frac{1}{M_u^2}$$
$$x = \sqrt{1 - a_1 \tan^2\varphi} \quad ; \quad a_1 = \frac{1}{m M_u^2}$$

1. Downstream Sound Intensity

Since $a_4 = m a_1$, where m is about 1/5 to 1/10, it is reasonable to take $v \approx 1$, so that for $0 \leq x \leq 1$ for this case Δ_v becomes

$$\Delta_v = \frac{\sqrt{m}}{\gamma \tau M_u}\left[\frac{\sqrt{m} + x}{\tan^2\varphi}\right]$$

Then

$$F_p^v = \frac{\gamma \tau M_u}{\sqrt{m}}\left[\frac{\tan^2\varphi - m}{\sqrt{m} + x}\right] \qquad \text{B 89}$$

and from Eq. (2. 23),

$$\overline{F_p^{v\,2}}' = \frac{3}{2}\gamma^2 m^3 \tau^2 M_u^6 \int_0^1 \left[\frac{1}{\sqrt{m} + x}\right]^2 x(1-x^2)\,dx$$

After evaluation of the integral,

$$\overline{F_p^{v\,2}}' = \frac{3}{2}\gamma^2 (1-m)^2 m \left[2\sqrt{m} - \frac{1-m}{1+\sqrt{m}} + (1-3m)\log\frac{1+\sqrt{m}}{\sqrt{m}}\right] M_u^6 \qquad \text{B 90}$$

2. Upstream Sound Intensity

The range of φ for which sound waves arise in the upstream flow is that for which

$$\tan^2\varphi \leq \frac{1}{a_4} \approx M_u^2$$

Since $m < 1$, this range is greater than that for downstream sound. Therefore the upstream sound intensity will consist of a contribution from the range of φ which gives downstream sound, plus a contribution from the range where the downstream waves are attenuated, but the upstream waves are not. The total sound intensity is the sum of these two contributions, the first of which is exactly equal to the downstream sound intensity. [see Eq. (4.30)]. In the second region, χ is imaginary [Eq. (B 88)], so that $\Delta^*_{vors}\,\Delta_{vors}$ may be written

$$\Delta^*_{vors}\,\Delta_{vors} = \frac{1-m}{\gamma^2\tau^2 M_u^2}\;\frac{1-(1+m)v^2}{\tan^2\varphi}$$

whence

$$F_p^{v*} F_p^{v} = \frac{\gamma^2\tau^2 M_u^2}{1-m}\;\frac{m^2}{1-(1+m)v^2} \qquad \text{B 91}$$

Also,

$$\sin\varphi = \sqrt{\frac{1-v^2}{1+a_4-v^2}}$$

$$d\varphi = -\sqrt{a_4}\;\frac{v\,dv}{(1+a_4-v^2)\sqrt{1-v^2}}$$

and from Eq. (B 86)

$$\overline{F_{pR}^{v\,2}}' - \overline{F_p^{v\,2}}' = \frac{3}{2}\gamma^2\tau^2\frac{m^2}{1-m^2}M_u^6\int_0^{\sqrt{1-m}}\frac{v(1-v^2)}{\frac{1}{1+m}-v^2}\,dv$$

whence

$$\overline{F_{pR}^{v\,2}}' - \overline{F_p^{v\,2}}' = \frac{3}{4}\gamma^2\frac{m(1-m)}{(1+m)^2}\log\left(\frac{1}{m^2}\right)M_u^6 \qquad \text{B 92}$$

The approximate evaluation of $\overline{F_p^{s\,2}}'$ and $\overline{F_{pR}^{s\,2}}'$ is quite similar, the results being

$$\overline{F_p^{s\,2}}' = \gamma^2(1-m)^2(f_1-1)^2 M_u^4\left(\log\frac{1+\sqrt{m}}{\sqrt{m}} - \frac{1}{1+\sqrt{m}}\right) \qquad \text{B 93}$$

and

$$\overline{F_{pp}^{s\,2}}' - \overline{F_{p}^{s\,2}}' = \frac{\gamma^2}{2}\,\frac{1-m}{1+m}\,(f_1-1)^2 M_n^4 \log\left(\frac{1}{m^2}\right) \qquad \text{B 94}$$

The change in the power of M_n from 6 in Eqs. (B 90) and (B 92) to 4 in Eqs. (B 93 and (B 94) is due to the different angular dependence of the upstream spectral function for vorticity and entropy inputs [see Eq. (2.33)] . Since φ is small $\sin\varphi \simeq \varphi \approx M_n$, so that the integral for $\overline{F_{p}^{v\,2}}'$ contains an additional factor of order M_n^2 .

APPENDIX C

SECOND ORDER THEORY

A. Differential Equations of the Fluid

The equations describing the motion of the fluid are

$$\frac{d\underline{u}}{dt} = -\frac{1}{\rho}\nabla p + \frac{1}{3}\nu\nabla(\nabla\cdot\underline{u}) + \nu\nabla^2\underline{u} \qquad \text{C 1}$$

$$\frac{dT}{dt} = \frac{1}{\rho c_p}\frac{dp}{dt} + \frac{\Phi}{\rho c_p} + \frac{K}{c_p\rho}\nabla^2 T \qquad \text{C 2}$$

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0 \qquad \text{C 3}$$

$$\frac{p}{p_0} = \frac{\rho T}{\rho_0 T_0} \qquad \text{C 4}$$

where it has been assumed that c_p , ν and K are constant. Now let $\pi = \log\frac{p}{p_0}$, $\theta = \log\frac{T}{T_0}$, $\zeta = \log\frac{\rho}{\rho_0}$. From the usual relation for the entropy of a perfect gas

$$\sigma = \frac{S - S_0}{c_p} = \theta - \frac{\gamma-1}{\gamma}\pi \qquad \text{C 5}$$

Using Eqs. (C 4) and (C 5) and assuming that the Prandtl number is 3/4, Eqs. (C 1), (C 2) and (C 3) may be rearranged as follows:

$$\frac{\partial\underline{u}}{\partial t} + \frac{a^2}{\gamma}\nabla\pi = \nu\left[\nabla^2\underline{u} + \frac{1}{3}\nabla(\nabla\cdot\underline{u})\right] + \underline{u}\times\nabla\times\underline{u} - \frac{1}{2}\nabla u^2 \qquad \text{C 6}$$

$$\frac{\partial\sigma}{\partial t} - \frac{4}{3}\nu\nabla^2\sigma - \frac{4}{3}\left(\frac{\gamma-1}{\gamma}\right)\nu\nabla^2\pi =$$

$$\frac{\gamma-1}{a^2}\frac{\Phi}{\rho} + \frac{4}{3}\nu\left[\left(\frac{\partial(\nabla\cdot\underline{u})}{\partial x}\right)^2 + \left(\frac{\partial(\nabla\cdot\underline{u})}{\partial y}\right)^2 + \left(\frac{\partial(\nabla\cdot\underline{u})}{\partial z}\right)^2\right] - \underline{u}\cdot\nabla\sigma \qquad \text{C 7}$$

$$\nabla \cdot \underline{u} = -\frac{1}{\gamma}\frac{\partial \pi}{\partial t} + \frac{\partial \sigma}{\partial t} - \underline{u}\cdot\nabla\tau \qquad \text{C 8}$$

Now take the curl of Eq. (C 6) letting $\nabla \times \underline{u} = \underline{\omega}$

$$\frac{\partial \underline{\omega}}{\partial t} - \nu \nabla^2 \underline{\omega} = -(\nabla\cdot\underline{u})\underline{\omega} - RT_0 e^{\theta}(\nabla\theta)\times(\nabla\pi) \qquad \text{C 9}$$

Take the divergence of Eq. (C 6), substitute Eq. (C 8) for $\nabla\cdot\underline{u}$, and eliminate $\frac{\partial^2\sigma}{\partial t^2}$ by Eq. (C 7), then

$$\frac{1}{a^2}\frac{\partial^2\pi}{\partial t^2} - \nabla^2\pi - \frac{4}{3}\frac{\gamma\nu}{a^2}\frac{\partial \nabla^2\pi}{\partial t} = \frac{4}{3}\frac{\gamma\nu}{a^2}\nabla^2(\underline{u}\cdot\nabla\tau) - \frac{1}{2}\frac{\gamma}{a^2}\nabla^2 u^2 + \frac{\gamma}{a^2}\left(\frac{\partial\tau}{\partial t}\right)^2$$

$$-\frac{\gamma}{a^2}\frac{\partial}{\partial t}\left\{\frac{1}{\gamma}\underline{u}\cdot\nabla\pi - \frac{\gamma-1}{a^2}\frac{\Phi}{\rho} + \frac{4}{3}\nu\left[\left(\frac{\partial(\nabla\cdot\underline{u})}{\partial x}\right)^2 + \left(\frac{\partial(\nabla\cdot\underline{u})}{\partial y}\right)^2 + \left(\frac{\partial(\nabla\cdot\underline{u})}{\partial z}\right)^2\right]\right\} + e^{\theta}(\nabla\theta)\cdot(\nabla\pi) \qquad \text{C 10}$$

Equations (C 7), (C 9) and (C 10) give a complete description of the behavior of the fluid, in terms of the dependent variables $\underline{\omega}$, π and σ. The advantage of this particular form is that if $\underline{u}$, π and σ are small, and the viscosity ν is small, the equations immediately reduce to three linear equations for the three variables $\underline{\omega}$, π and σ. The significance of the small non-linear terms may best be seen by making the equations completely dimensionless, then separating out the first and second order terms according to the classical perturbation scheme.

A characteristic length may be introduced by imagining the disturbances as sinusoidal; then the wave length is the characteristic length desired. The speed of sound is the natural choice for the other dimensional quantity. In terms of these, define

$$t^* = kat \qquad ; \quad k = \frac{2\pi}{\lambda}$$
$$\underline{x}^* = k\underline{x}$$
$$\underline{u}^* = \frac{\underline{u}}{a}$$
$$\nabla^* = \frac{\nabla}{k}$$
$$\underline{\omega}^* = \frac{\underline{\omega}}{ka}$$

and let

$$\underline{\omega}^* = \mathcal{E}\,\underline{\omega}_1 + \mathcal{E}^2\underline{\omega}_2 + \cdots$$
$$\underline{u}^* = \mathcal{E}\,\underline{u}_1 + \mathcal{E}^2\,\underline{u}_2 + \cdots$$
$$\pi = \mathcal{E}\,\pi_1 + \mathcal{E}^2\,\pi_2 + \cdots$$
$$\sigma = \mathcal{E}\,\sigma_1 + \mathcal{E}^2\,\sigma_2 + \cdots$$
$$\theta = \mathcal{E}\,\theta_1 + \mathcal{E}^2\,\theta_2 + \cdots$$

where $\underline{\omega}_1$, $\underline{u}_1$, $\underline{\omega}_2$, $\underline{u}_2$ are assumed to be of order one, so that $\mathcal{E}$ is a measure of the size of $\underline{\omega}^*$, $\underline{u}^*$ etc. Also, take $\alpha = \frac{k\nu}{a}$ to be a small quantity, of order $\mathcal{E}$ or less, then the first and second order terms of Eqs. (C 7), (C 9) and (C 10) may be separated, as follows:

$$\frac{\partial \underline{\omega}_1}{\partial t^*} = 0 \qquad \text{C 11}$$

$$\frac{\partial \sigma_1}{\partial t^*} = 0 \qquad \text{C 12}$$

$$\frac{\partial^2 \pi_1}{\partial t^{*2}} - \nabla^{*2}\pi_1 = 0 \qquad \text{C 13}$$

$$\frac{\partial \underline{\omega}_2}{\partial t^*} = \frac{\alpha}{\mathcal{E}}\left(\nabla^{*2}\underline{\omega}_1\right) - (\nabla\cdot\underline{u}_1^*)\,\underline{\omega}_1^* - (\nabla^*\theta_1)\times(\nabla^*\pi_1) \qquad \text{C 14}$$

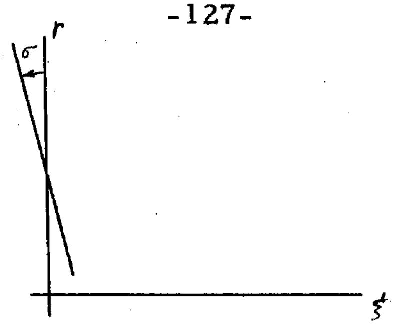
$$\frac{\partial \sigma_2}{\partial t^*} = \frac{\alpha}{\epsilon}\left(\frac{4}{3}\nabla^{*2}\sigma_1 + \frac{4}{3}\frac{\gamma-1}{\gamma}\nabla^{*2}\pi_1\right) + \underline{u}_1 \cdot \nabla^*\sigma_1 \qquad \text{C 15}$$

$$\frac{\partial^2 \pi_2}{\partial t^{*2}} - \nabla^{*2}\pi_2 = \frac{\alpha}{\epsilon}\left(\frac{4}{3}\gamma\frac{\partial \nabla^{*2}\pi_1}{\partial t^*}\right) - \frac{\gamma}{2}\nabla^{*2}u_1^2 + \gamma\left(\frac{\partial \tau_1}{\partial t^*}\right)^2$$
$$- \frac{\partial}{\partial t^*}\left(\underline{u}_1 \cdot \nabla^*\pi_1\right) + (\nabla^*\theta_1)\cdot(\nabla^*\pi_1) \qquad \text{C 16}$$

Equations (C 11), (C 12) and (C 13) are of course equivalent to Eqs. (1.6), (1.7) and (1.8), page 11. Equations (C 14), (C 15) and (C 16) for the second order quantities are the same, except that first order quantities act as sources on the right side. The validity of the assumptions made in deriving Eqs. (1.6), (1.7) and (1.8) may now be assessed. First, it is evident that $\alpha = \frac{k\nu}{a}$ must be small, so that k may not be too large. Since the ratio $\frac{\nu}{a}$ is roughly proportional to the temperature, the assumptions may not be valid at extremely high temperatures. Secondly, since the terms on the right of Eqs. (C 14) to (C 16) involve the derivatives of the various quantities, these derivatives must be small, as well as the quantities themselves. As is pointed out in the discussion of the downstream sound field from a shock, this condition is not always satisfied in the preceding analysis.

B. Boundary Conditions at the Discontinuity

The discontinuity is, as before, assumed to be disturbed, its angle to the r axis being σ, and its velocity of propagation in the r direction, V. The boundary conditions are most readily formulated



in terms of the velocities, pressure and entropy. Thus, let

$$\frac{u_1}{U_h} = \epsilon u_{1,1} + \epsilon^2 u_{1,2} + \cdots$$

$$\frac{u_r}{U_h} = \epsilon u_{r,1} + \epsilon^2 u_{r,2} + \cdots \qquad \text{C 17}$$

$$\frac{p}{P} = \epsilon p_1 + \epsilon^2 p_2 + \cdots$$

$$\frac{s}{C_p} = \epsilon s_1 + \epsilon^2 s_2 + \cdots$$

Where $u_{1,1}$ is the first order velocity associated with π_1, and ω_1, in the direction of the ξ' axis, $u_{r,1}$ in the direction of the r axis, $u_{1,2}$ is the second order velocity, etc. Now denote the actual position of the discontinuity by

$$\delta = \epsilon \delta_1 + \epsilon^2 \delta_2 + \cdots \qquad \text{C 18}$$

The connection between δ and σ being,

$$\frac{d\delta}{dr} = \tan\sigma \qquad \text{C 19}$$

The values of Eq. (C 17) at the actual position of the discontinuity, to order ϵ^2 are then

$$\frac{u_1}{U_n}(\delta) = \varepsilon\, u_{1,1}(0) + \varepsilon^2\left[u_{1,2}(0) + \frac{\partial u_{1,1}(0)}{\partial \xi}\,\delta_1\right]$$

$$\frac{u_r}{U_n}(\delta) = \varepsilon\, u_{r,1}(0) + \varepsilon^2\left[u_{r,2}(0) + \frac{\partial u_{r,1}(0)}{\partial \xi}\,\delta_1\right] \qquad \text{C 20}$$

$$\frac{p}{P}(\delta) = \varepsilon\, p_1(0) + \varepsilon^2\left[p_2(0) + \frac{\partial p_1(0)}{\partial \xi}\,\delta_1\right]$$

$$\frac{s}{c_p}(\delta) = \varepsilon\, s_1(0) + \varepsilon^2\left[s_2(0) + \frac{\partial s_1(0)}{\partial \xi}\,\delta_1\right]$$

The instantaneous local velocities normal and tangential to the discontinuity are then

$$\frac{u_n}{U_n} = \varepsilon\left[u_{1,1}(0) - \frac{V}{U_n}\sigma_1\right] + \varepsilon^2\left[u_{1,2}(0) + \frac{\partial u_{1,1}(0)}{\partial \xi}\delta_1 - \frac{\sigma_1^2}{2} - \frac{V}{U_n}\sigma_2\right] \qquad \text{C 21}$$

$$\frac{u_t}{U_n} = \varepsilon\left[u_{r,1}(0) - \sigma_1\right] + \varepsilon^2\left[u_{r,2}(0) + \frac{\partial u_{r,1}(0)}{\partial \xi}\delta_1 - \sigma_2 - u_{1,1}(0)\sigma_1\right]$$

The characteristics of a given discontinuity may be expanded in the form

$$\frac{u_n'}{U_n'} = \varepsilon\left[b_1 \frac{u_n}{U_n}(\delta) + b_2 \frac{s}{c_p}(\delta) + b_3 \frac{p}{P}(\delta)\right]$$

$$+\varepsilon^2\left[b_{11}\left(\frac{u_n}{U_n}(\delta)\right)^2 + b_{22}\left(\frac{s}{c_p}(\delta)\right)^2 + b_{33}\left(\frac{p}{P}(\delta)\right)^2 + b_{12}\frac{u_n}{U_n}(\delta)\frac{s}{c_p}(\delta)\right. \qquad \text{C 22}$$

$$\left. + b_{13}\frac{u_n}{U_n}(\delta)\frac{p}{P}(\delta) + b_{23}\frac{s}{c_p}(\delta)\frac{p}{P}(\delta)\right]$$

and similar relations for $\frac{p'}{P'}$ and $\frac{s'}{c_p}$.

Now the problem consists of inverting the relations analogous to Eq. (C 21) for the downstream side of the discontinuity, using Eqs. (C 22) and (C 21) to obtain expressions to first and second order for $\frac{u_1'}{U_n'}(\delta)$ in terms of $\frac{u_1}{U_n}(\delta)$, $\frac{p}{P}(\delta)$ and $\frac{s}{c_p}(\delta)$, to which the solutions for the downstream flow may be matched.

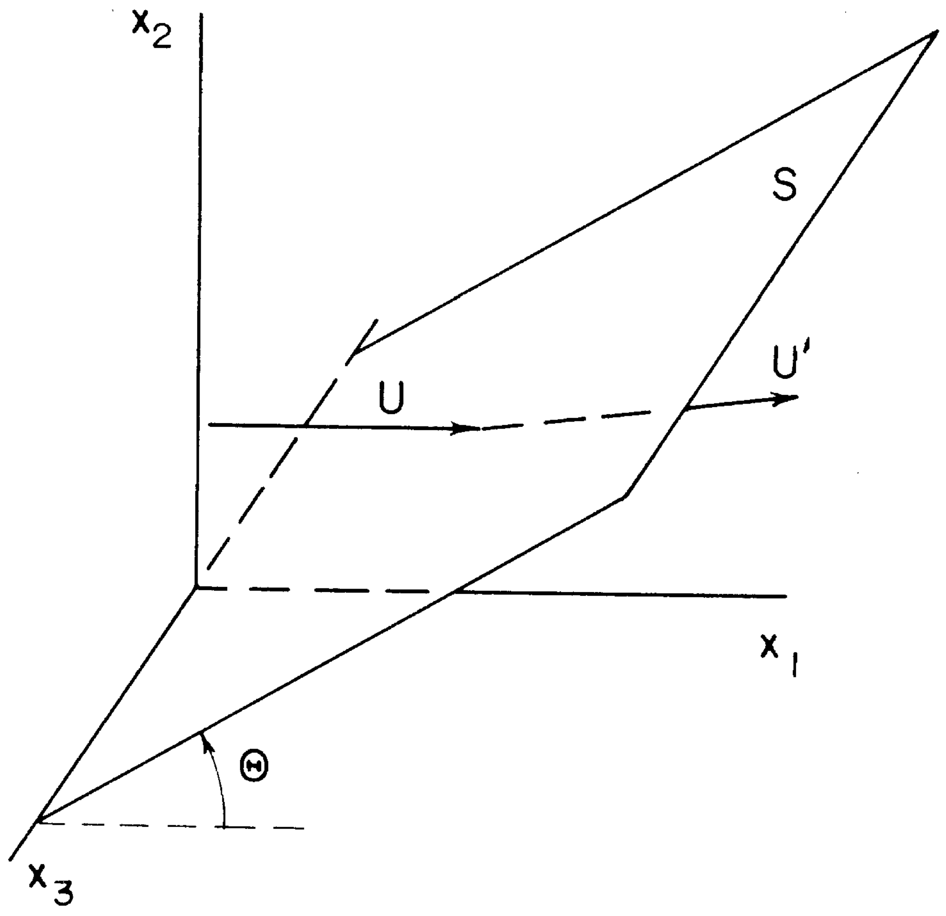


Figure 1: Orientation of discontinuity, S, with respect to mean upstream velocity, U, mean downstream velocity, U', and coordinates X1, X2, X3.

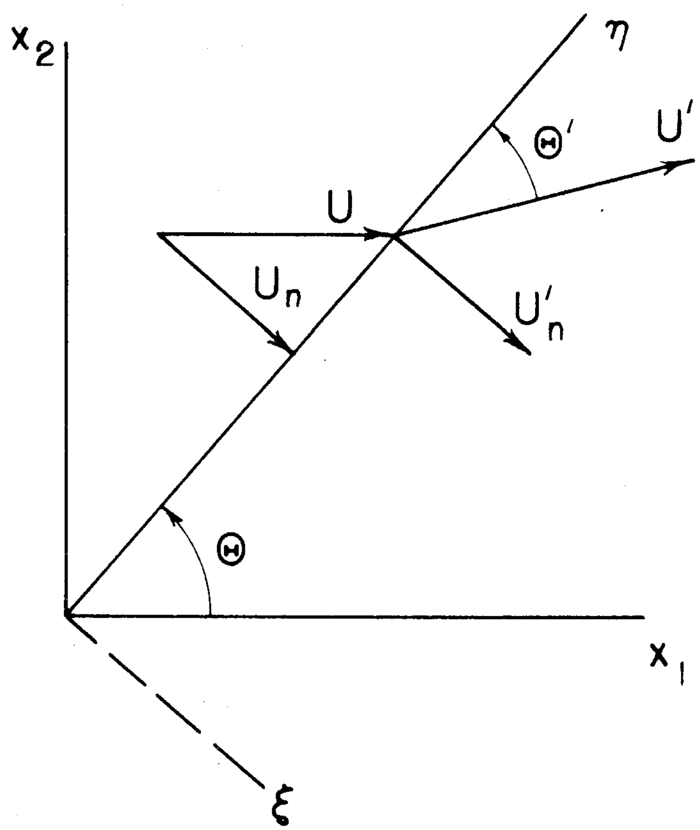


Figure 2: Transformation from case of oblique discontinuity in X1, X2, X3 coordinate system to case of normal discontinuity in ξ, η, x_3 coordinate system

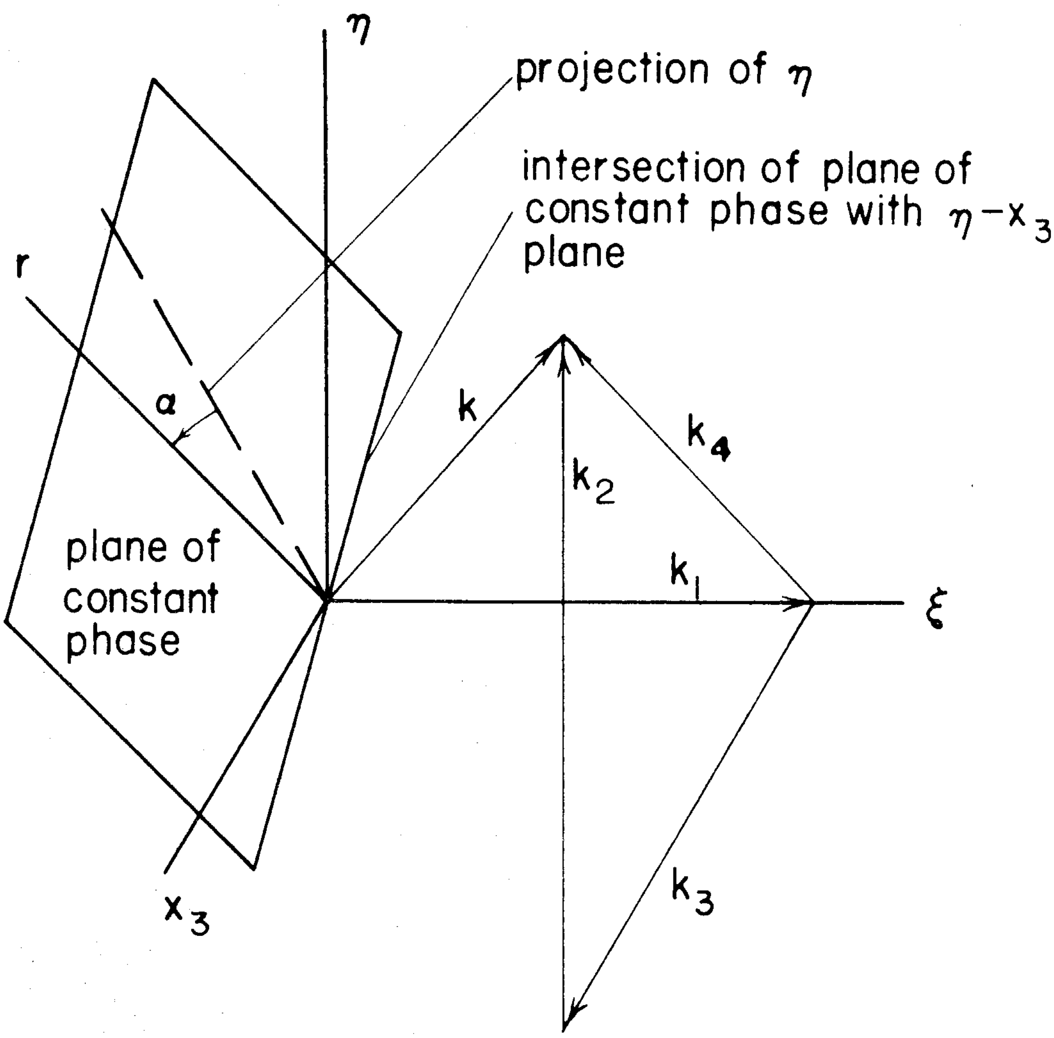


Figure 3: Reduction to two-dimensional probel inξ, r plane.

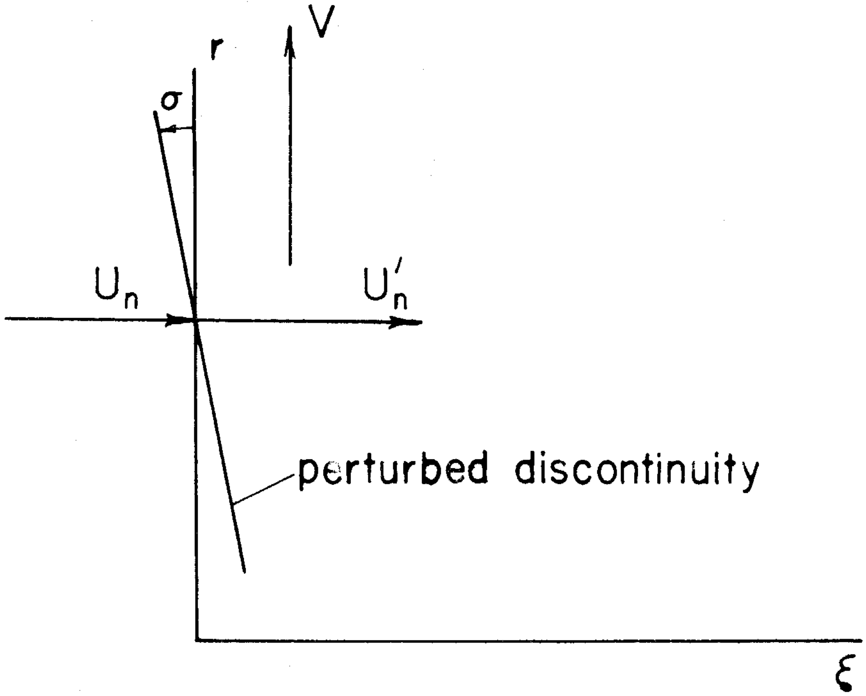


Figure 4 - Perturbed discontinuity in ξ, r plane.

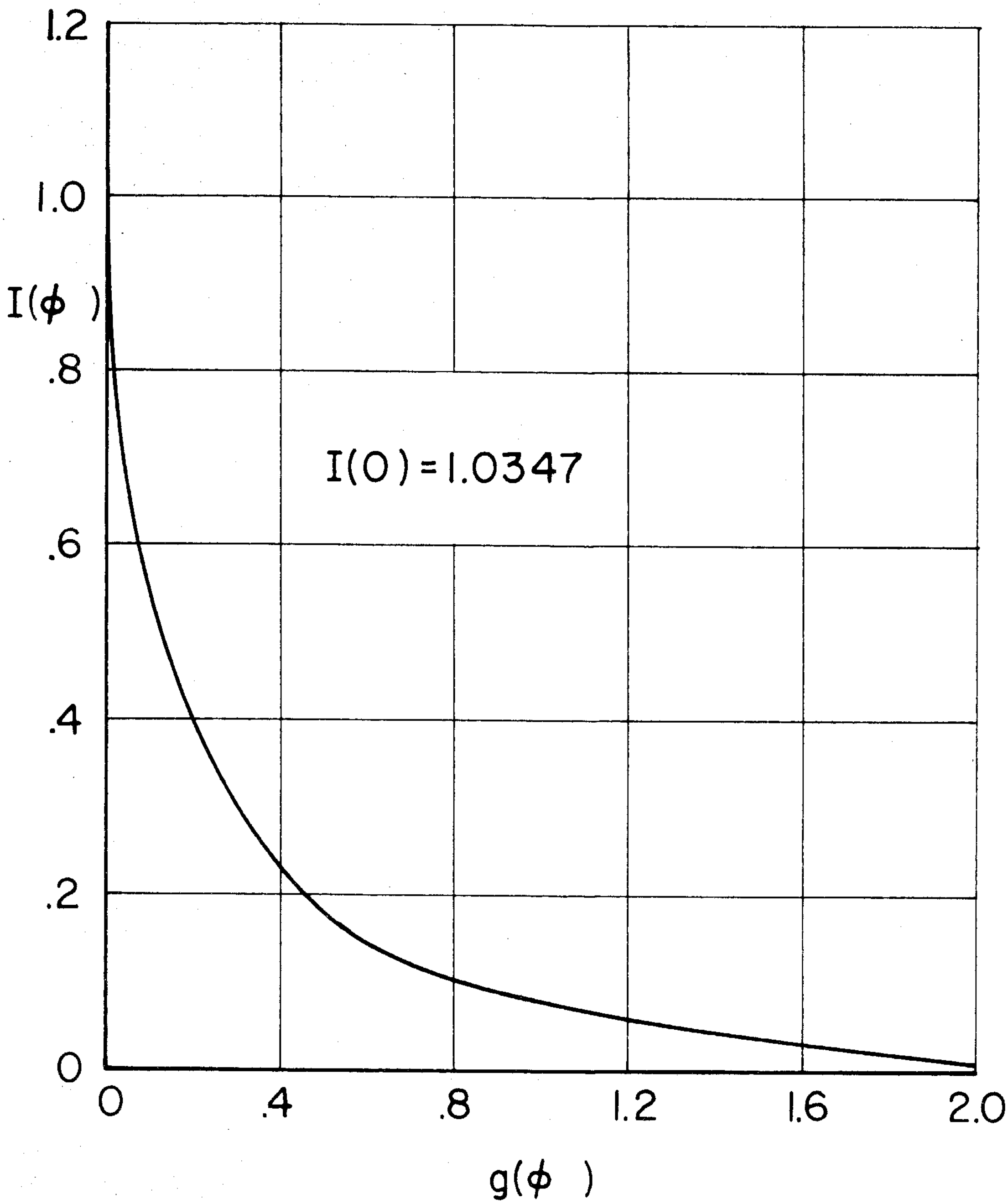


Figure 3 Magnitude of the integral involved in evaluation of attenuated pressure waves.

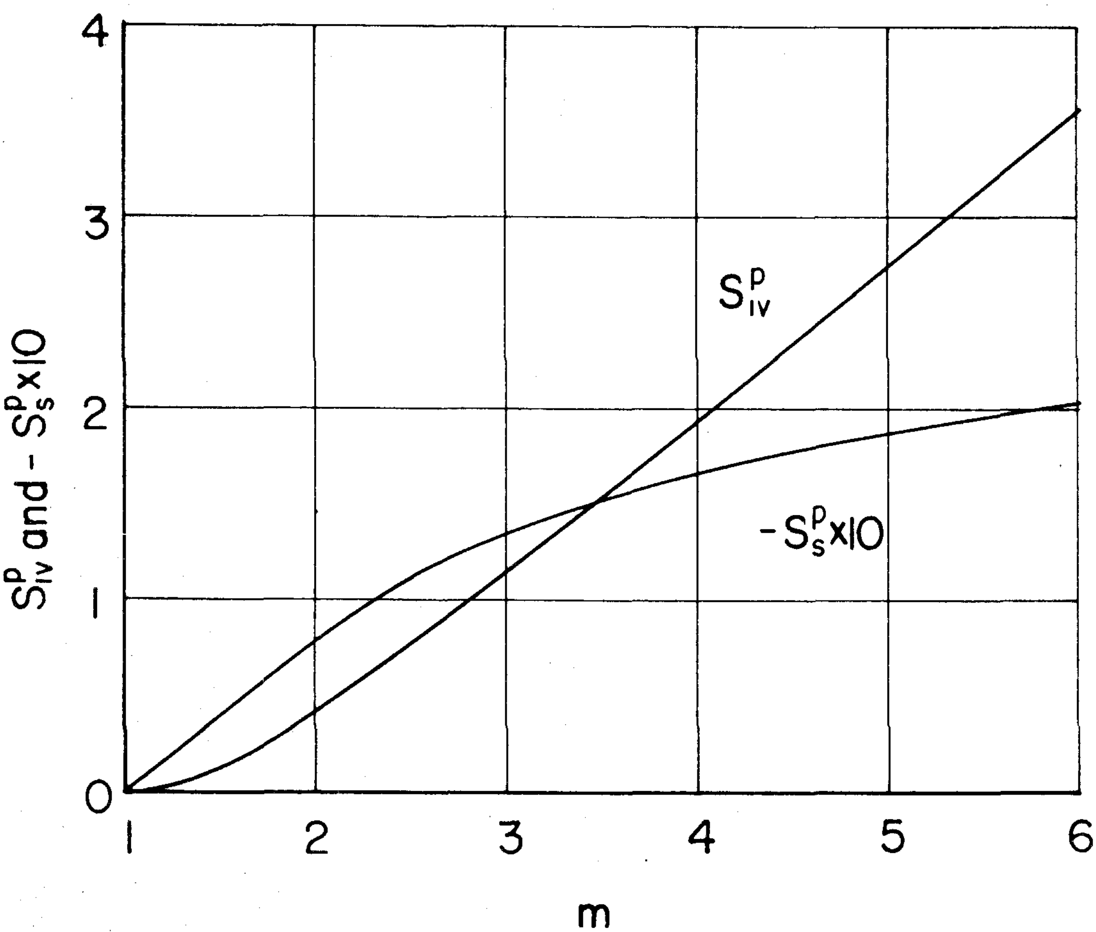


Figure [illegible]. Transfer functions for shock wave interacting with Mach waves.

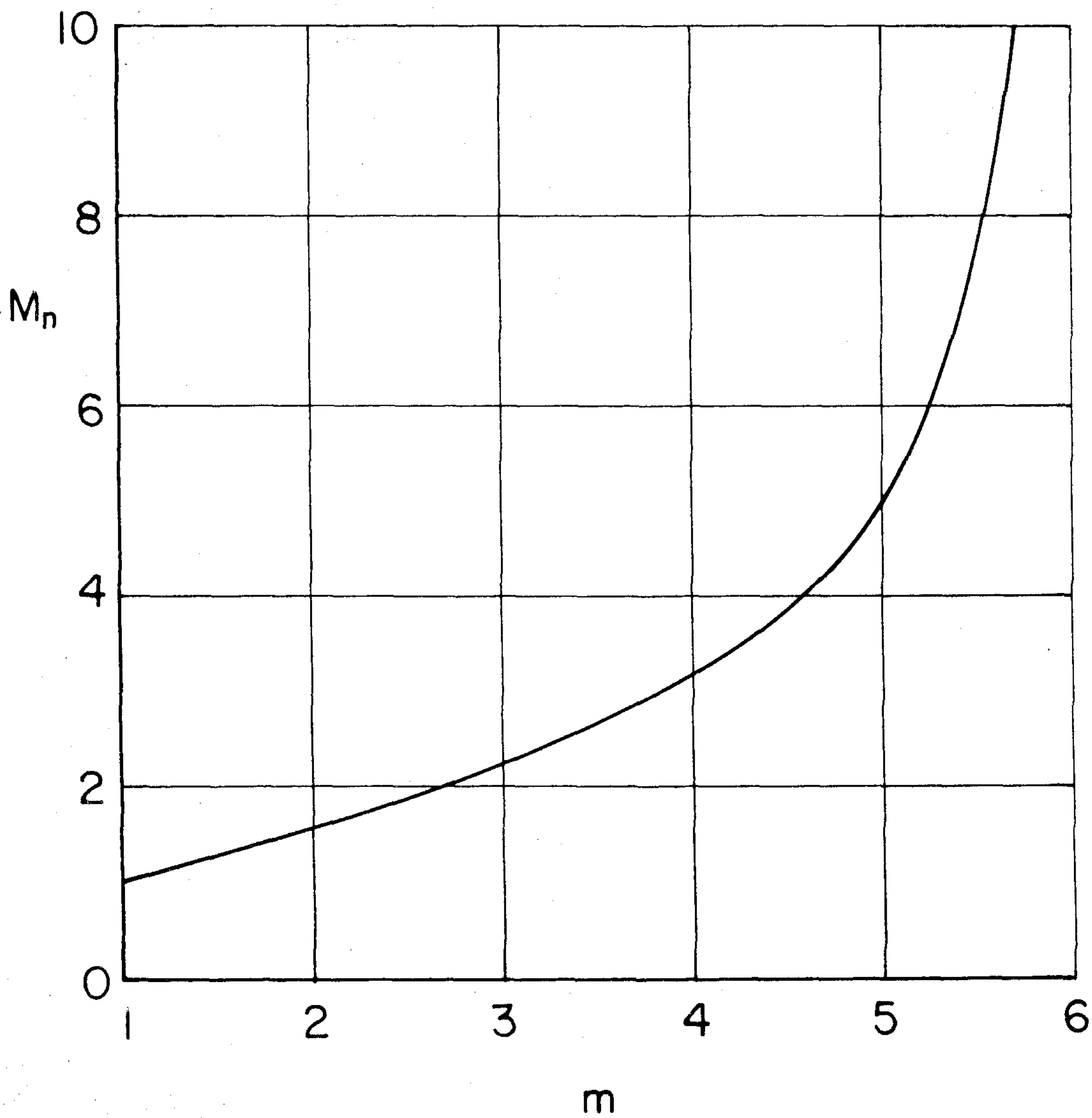


Figure 7 Normal upstream Mach number of shock as a function of velocity ratio across shock.

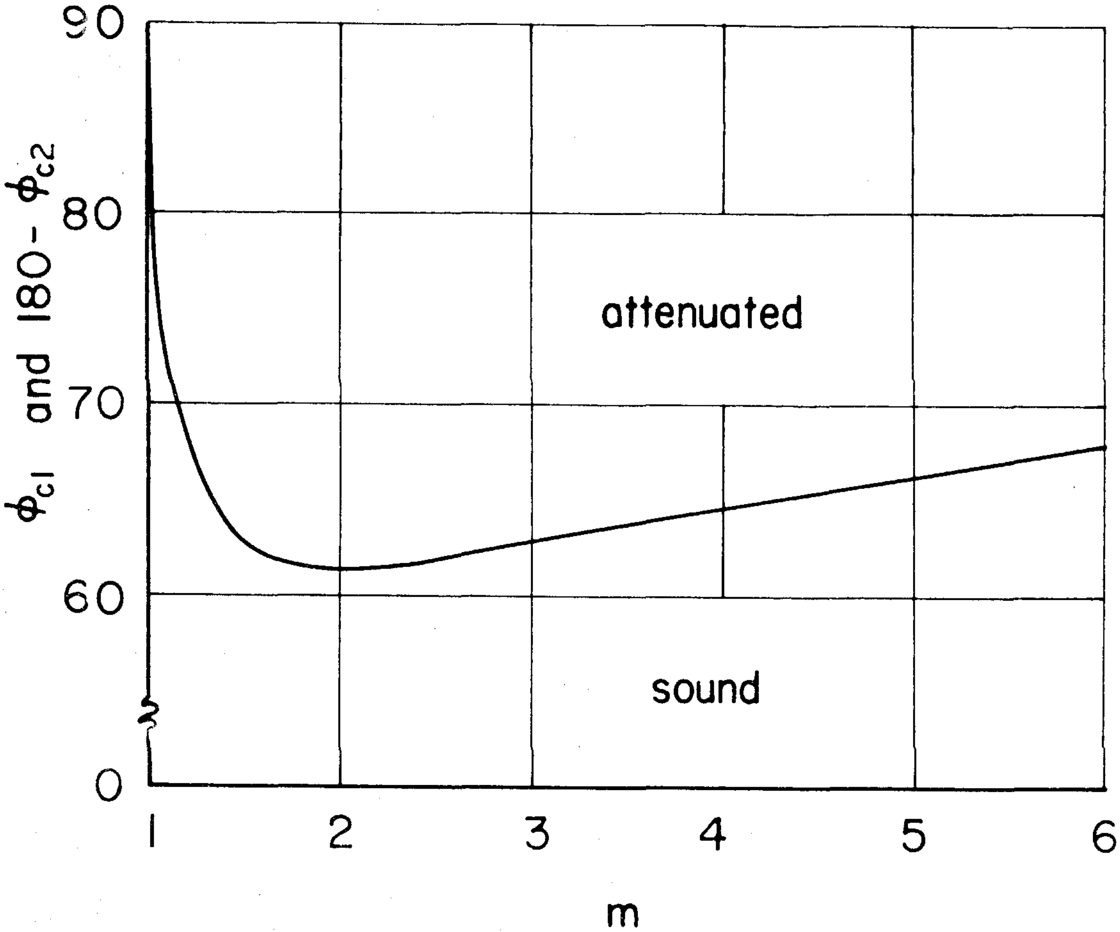


Figure [illegible] Critical angle for convected disturbances interacting with shock.

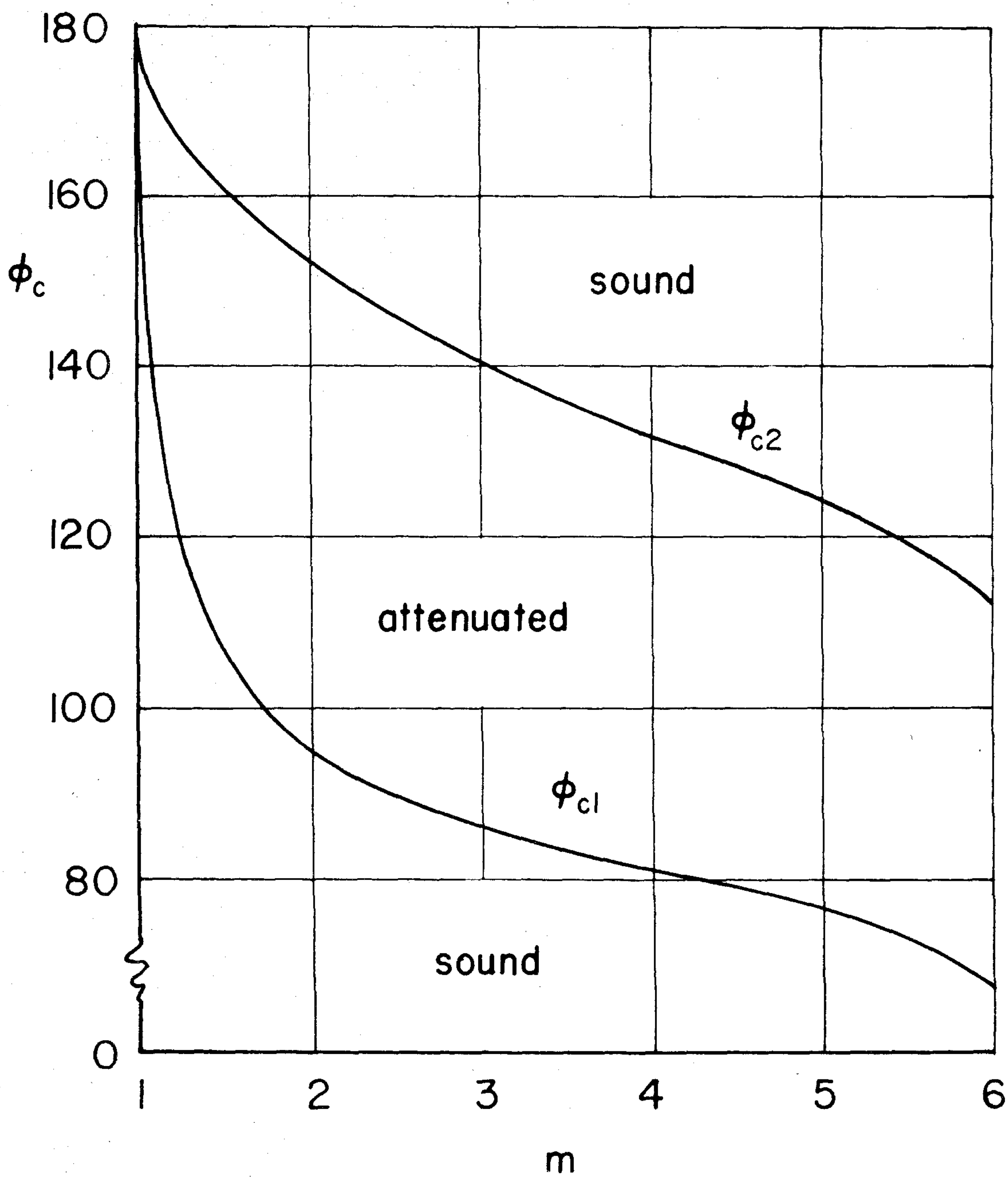


Figure 9 Critical angles for pressure waves interacting with shock.

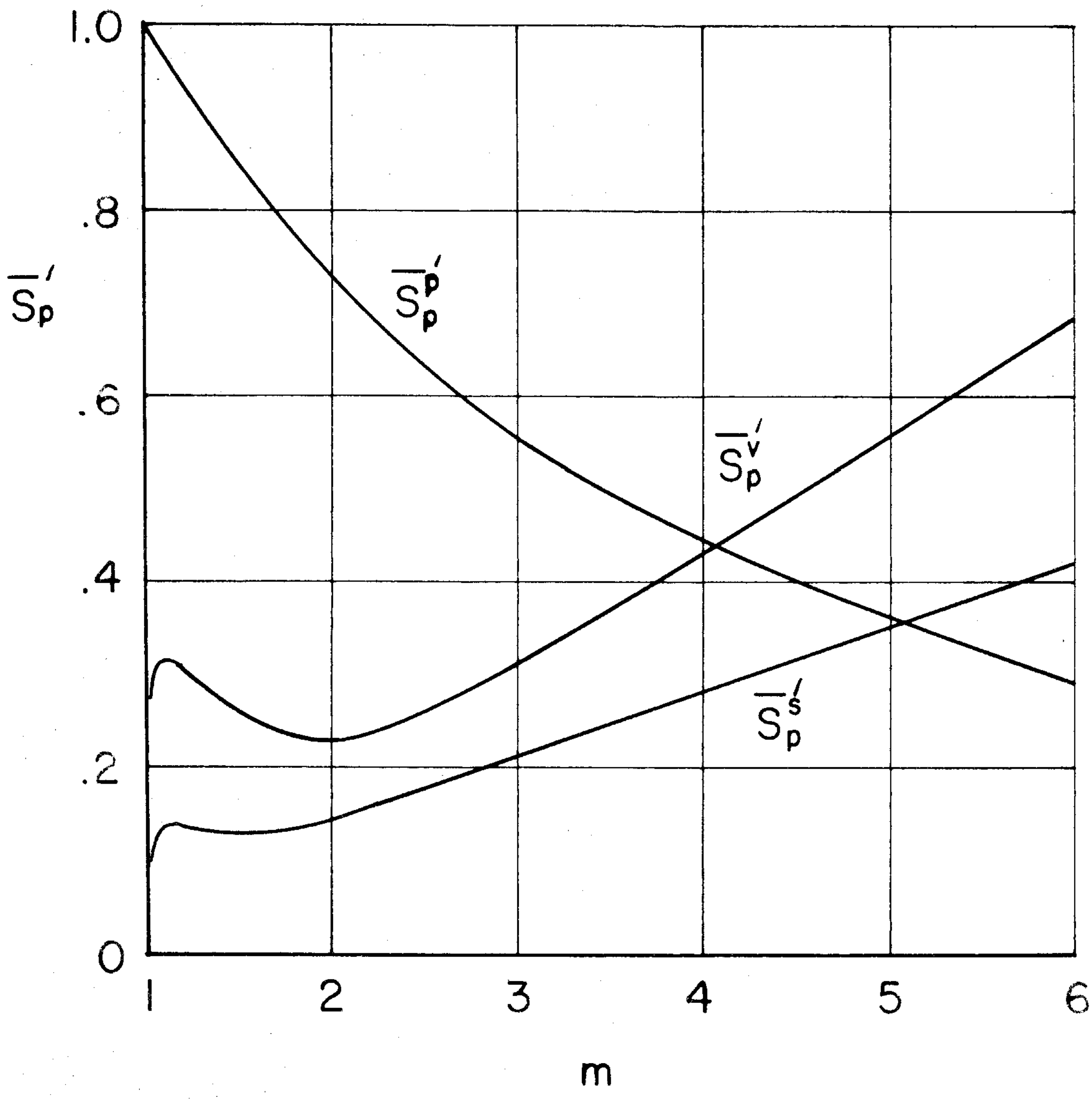


Figure 10a Average transfer functions for downstream sound pressure resulting from interaction of shock wave with pressure, vorticity, and entropy disturbances.

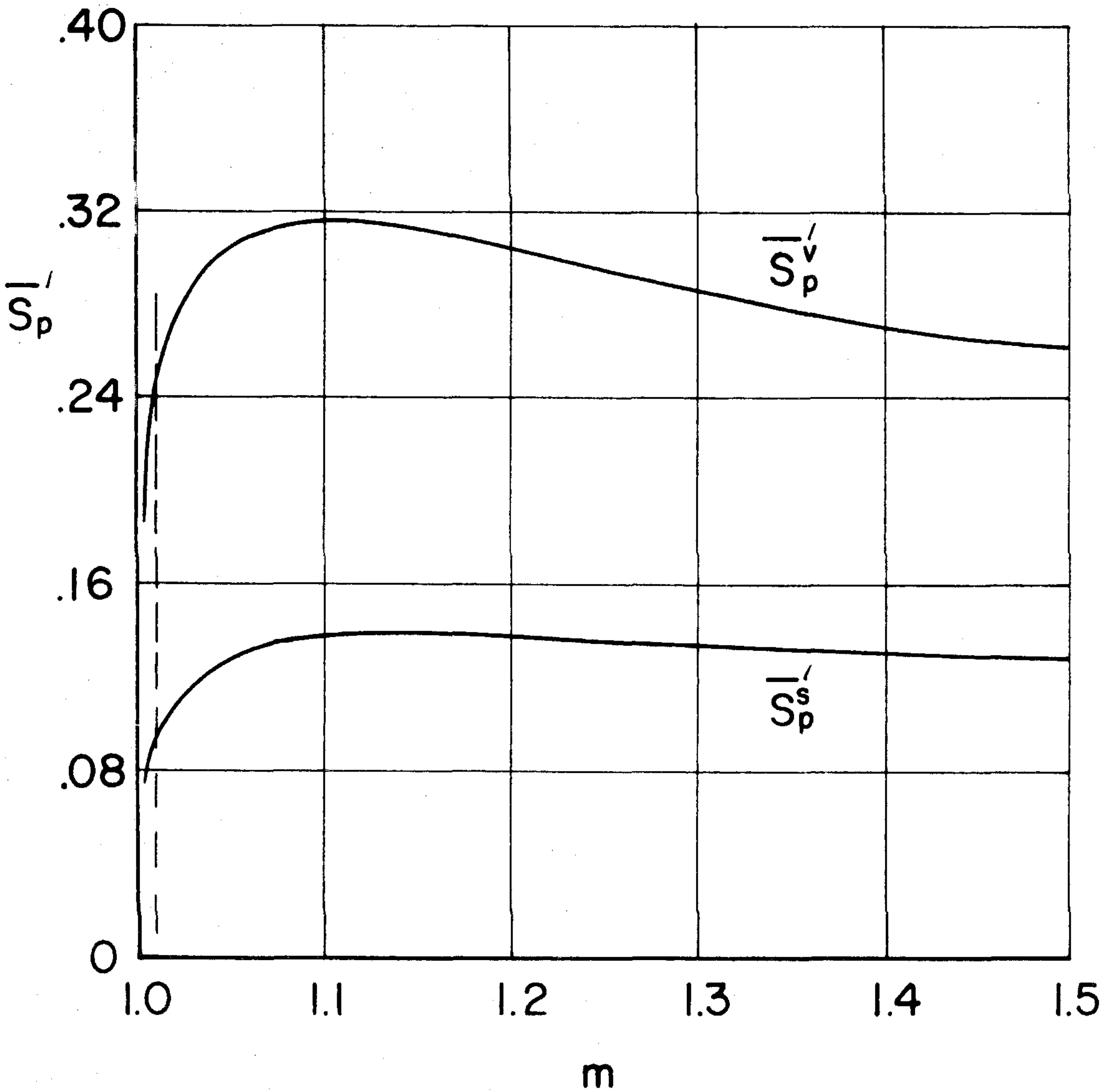


Figure 10b Average transfer functions for downstream sound pressure resulting from interaction of shock wave with vorticity and entropy disturbances, for m near 1.

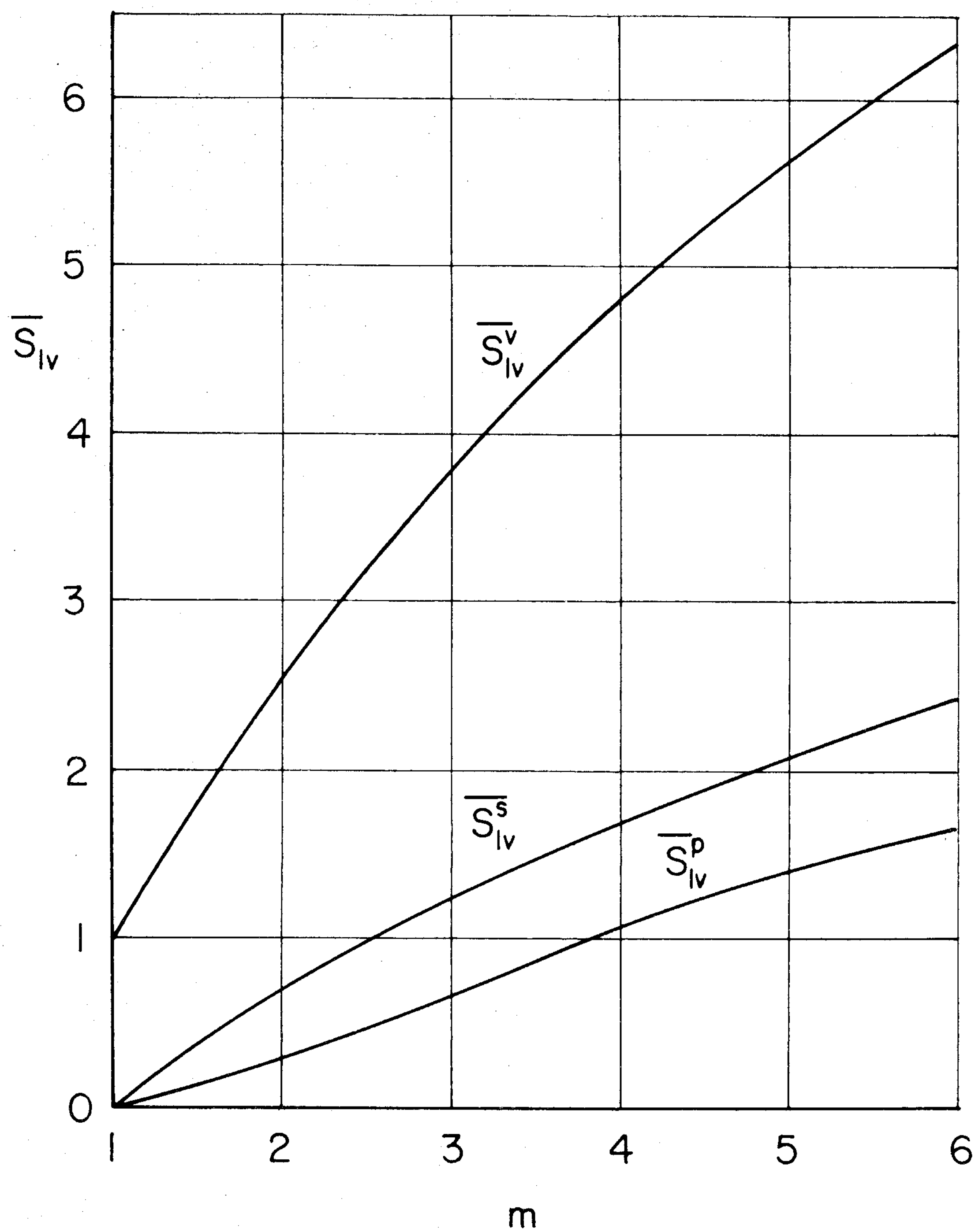


Figure 11 Average transfer functions for downstream vorticity (shear velocity normal to shock) resulting from interaction of shock with pressure, vorticity, and entropy disturbances.

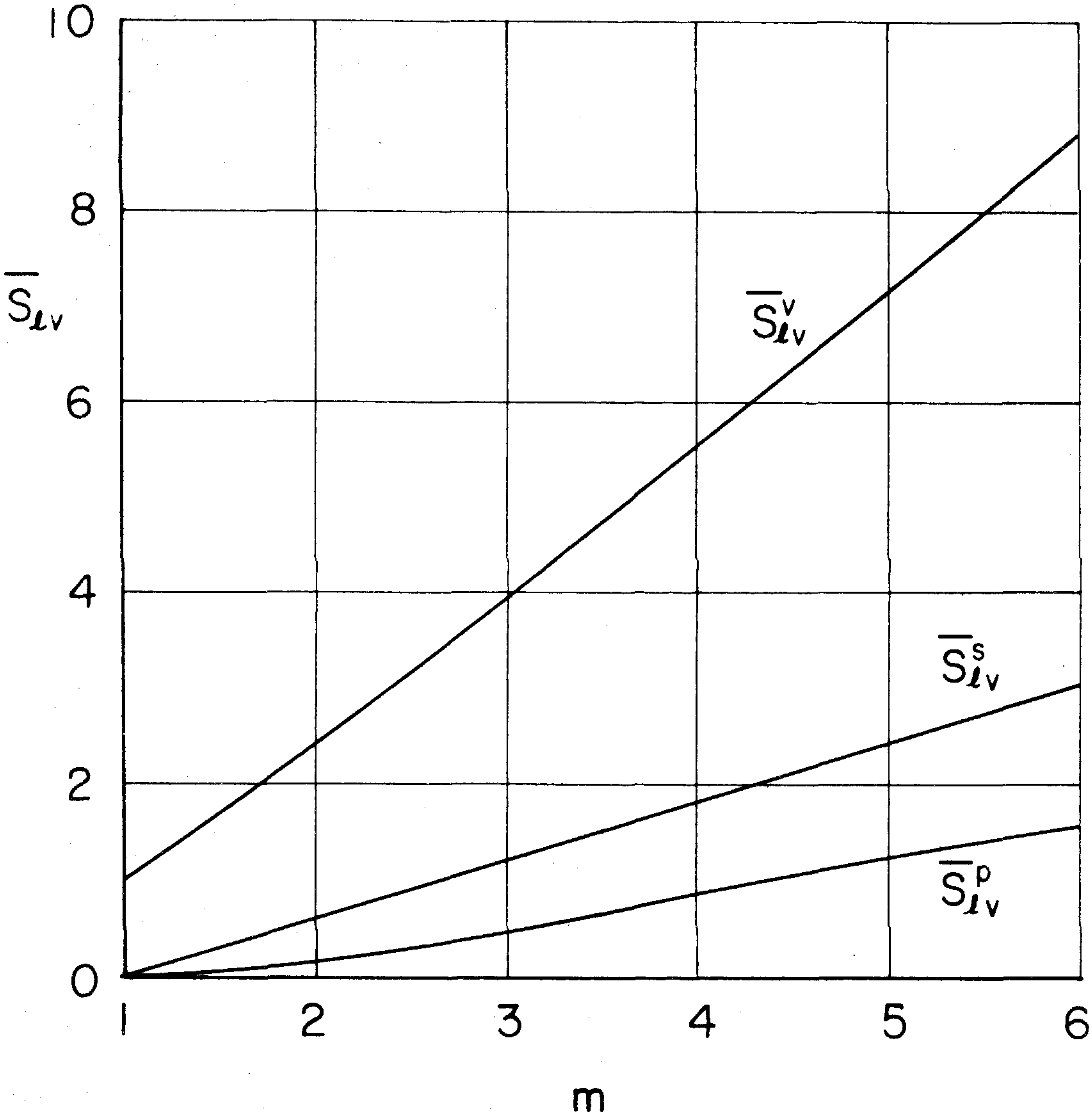


Figure 12. Average transfer functions for downstream lateral vorticity (shear velocity parallel to shock) resulting from interaction of shock wave with pressure, vorticity, and entropy disturbances.

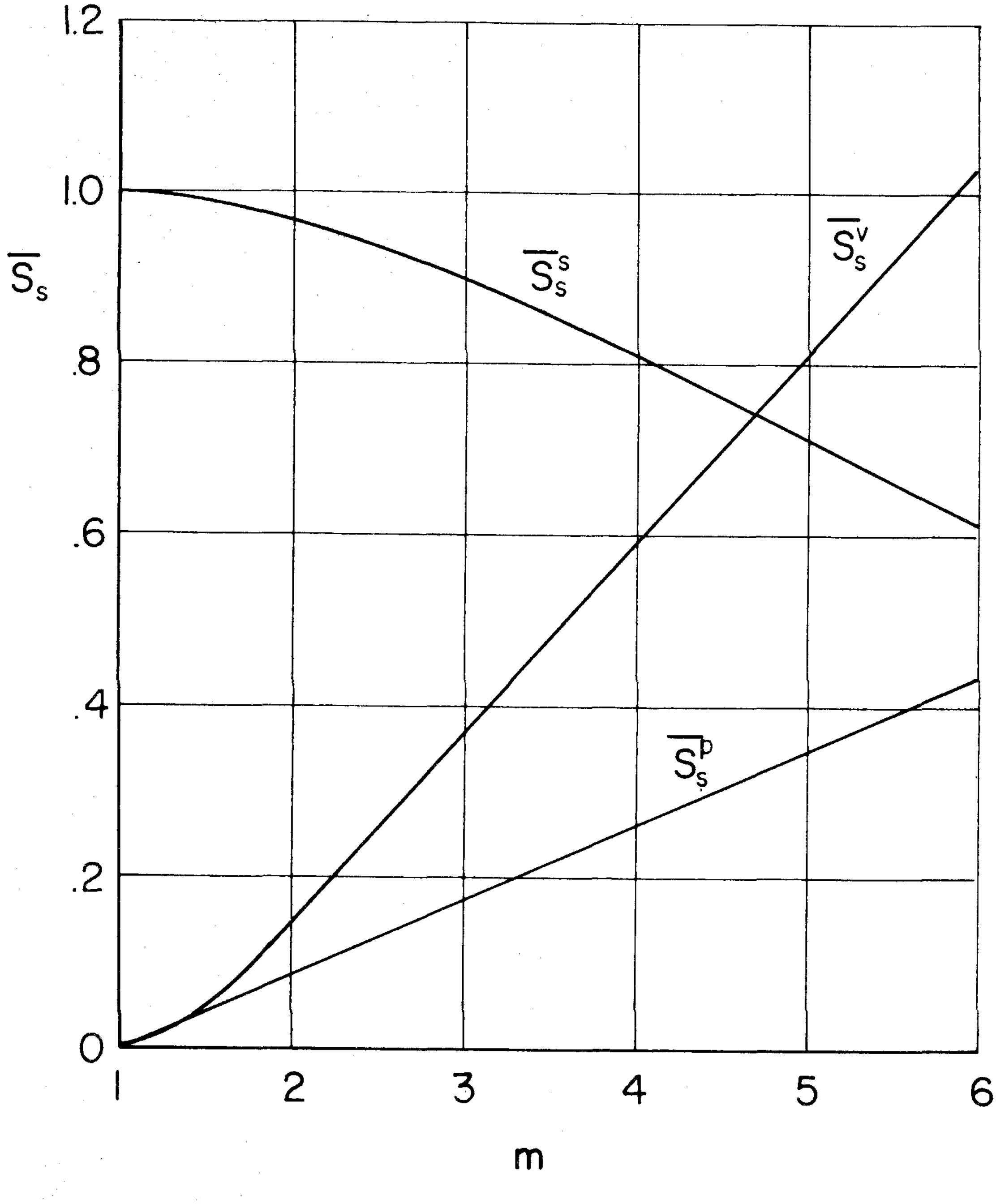


Figure 13 Average transfer functions for downstream entropy resulting from interaction of shock wave with pressure, vorticity, and entropy disturbances.

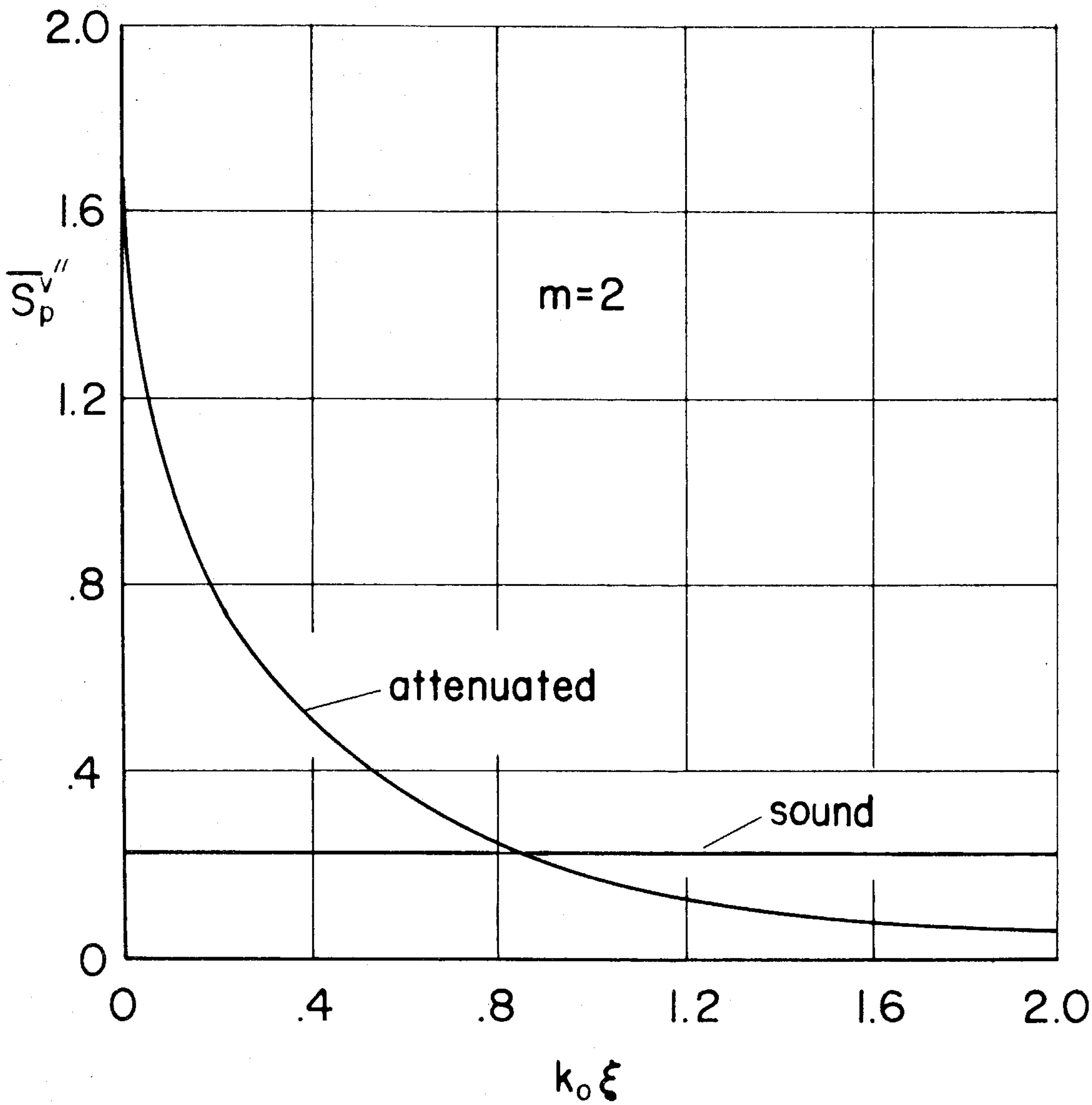


Figure 14 Average transfer function for attenuated downstream pressure field resulting from interaction of shock wave with vorticity disturbance (for $m=2$).

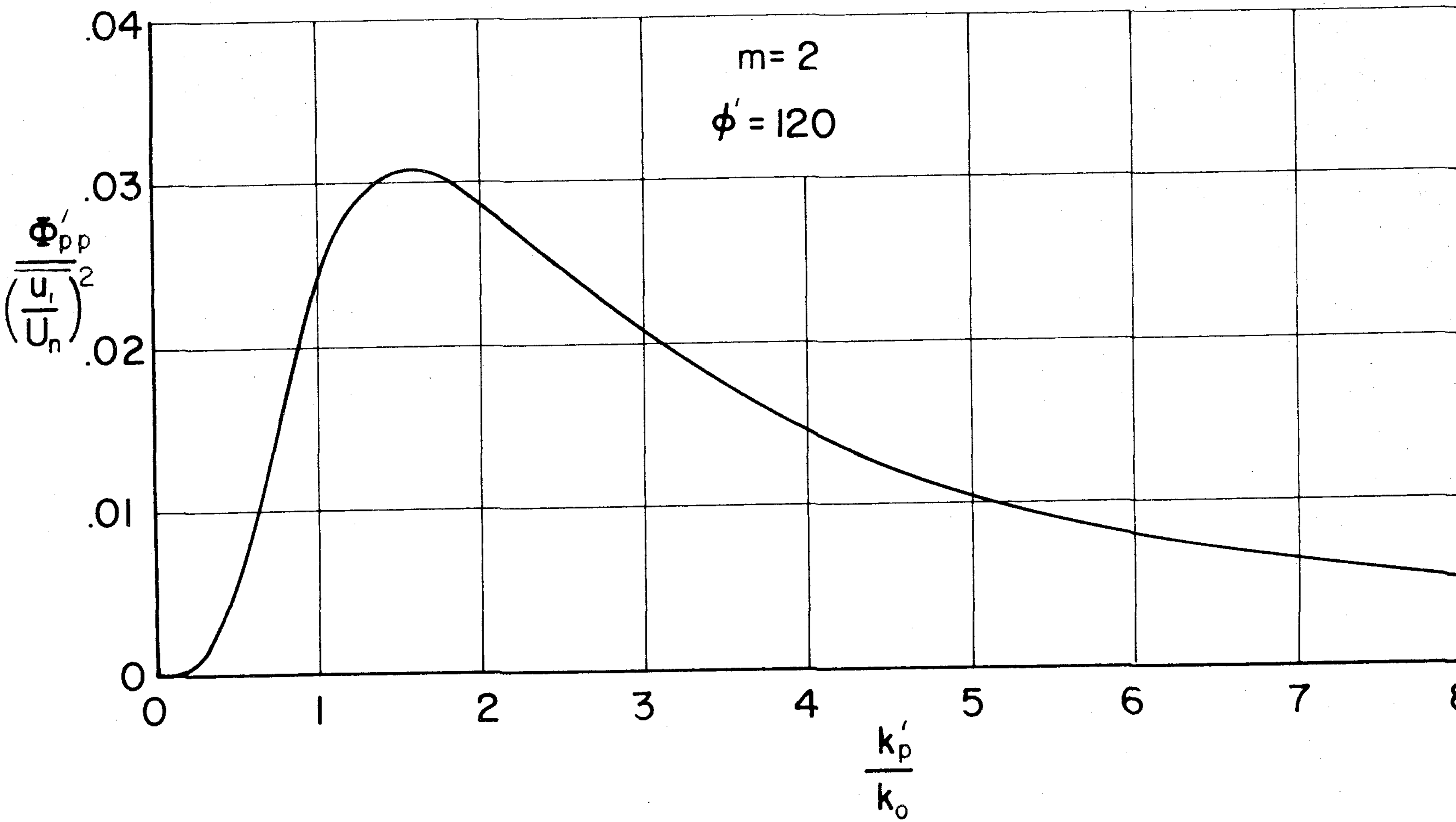


Figure 15 Distribution of downstream sound intensity in wave number (for $m=2$).

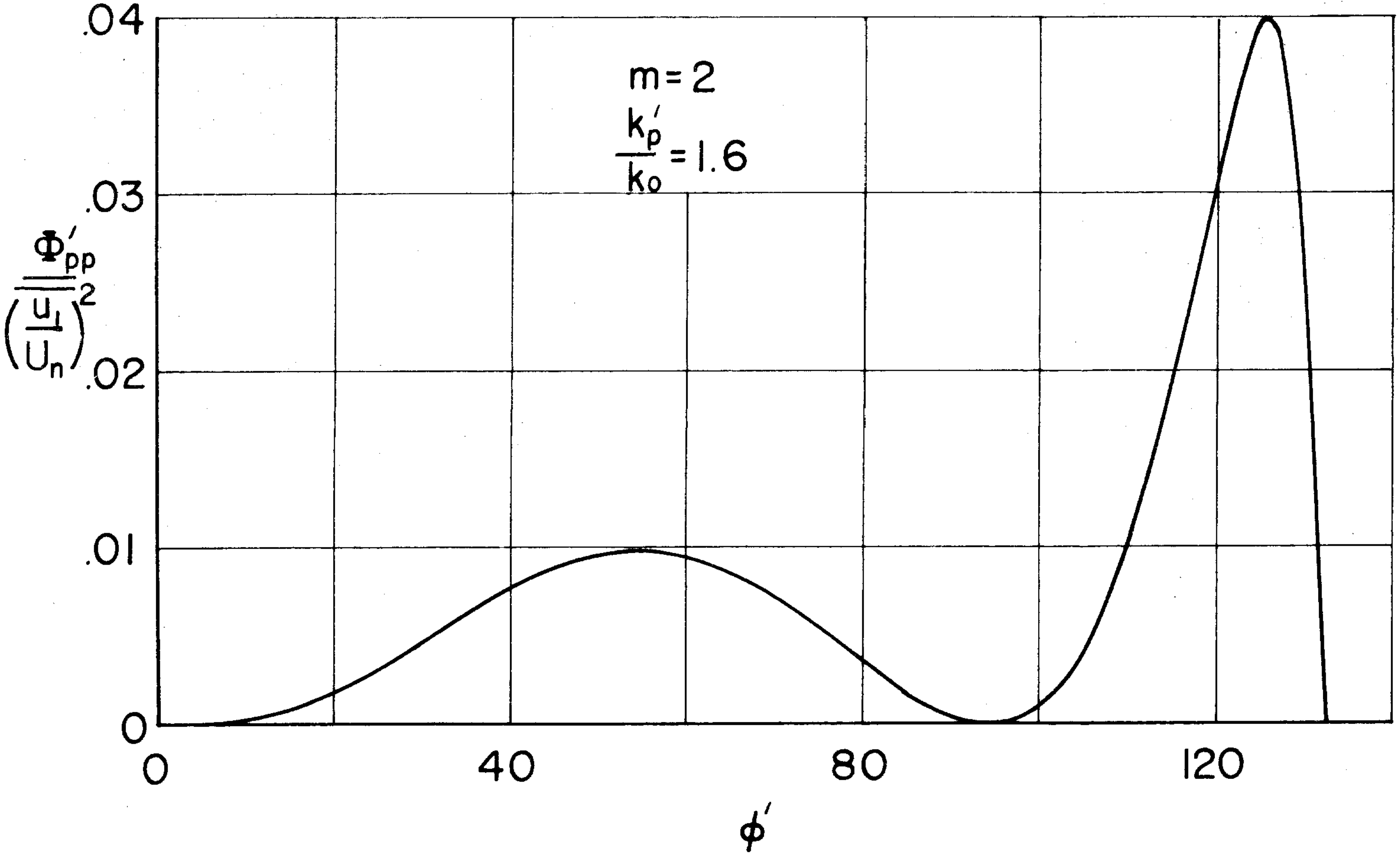


Figure 16 Distribution of downstream sound intensity in wave angle (for $m = 2$).

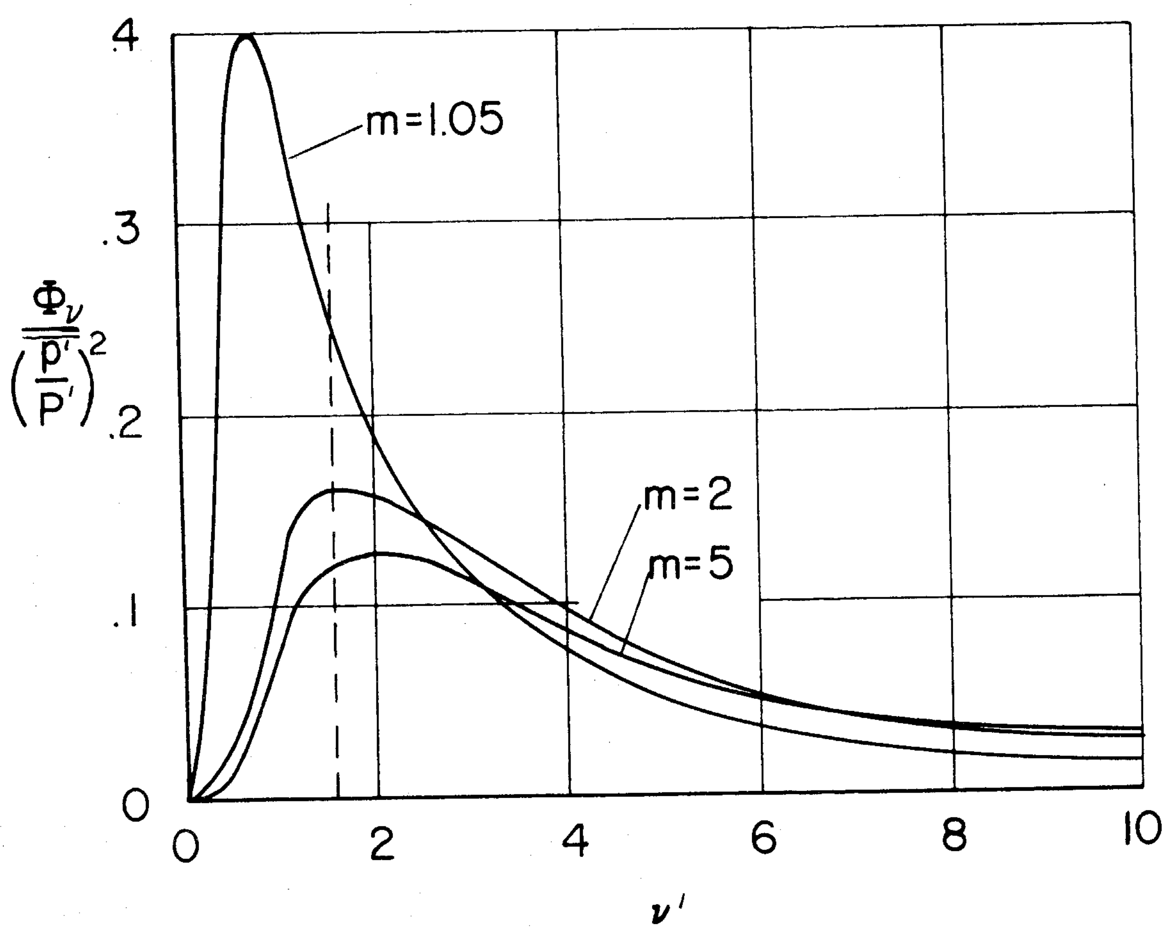


Figure 17 Frequency spectrum of downstream sound waves.

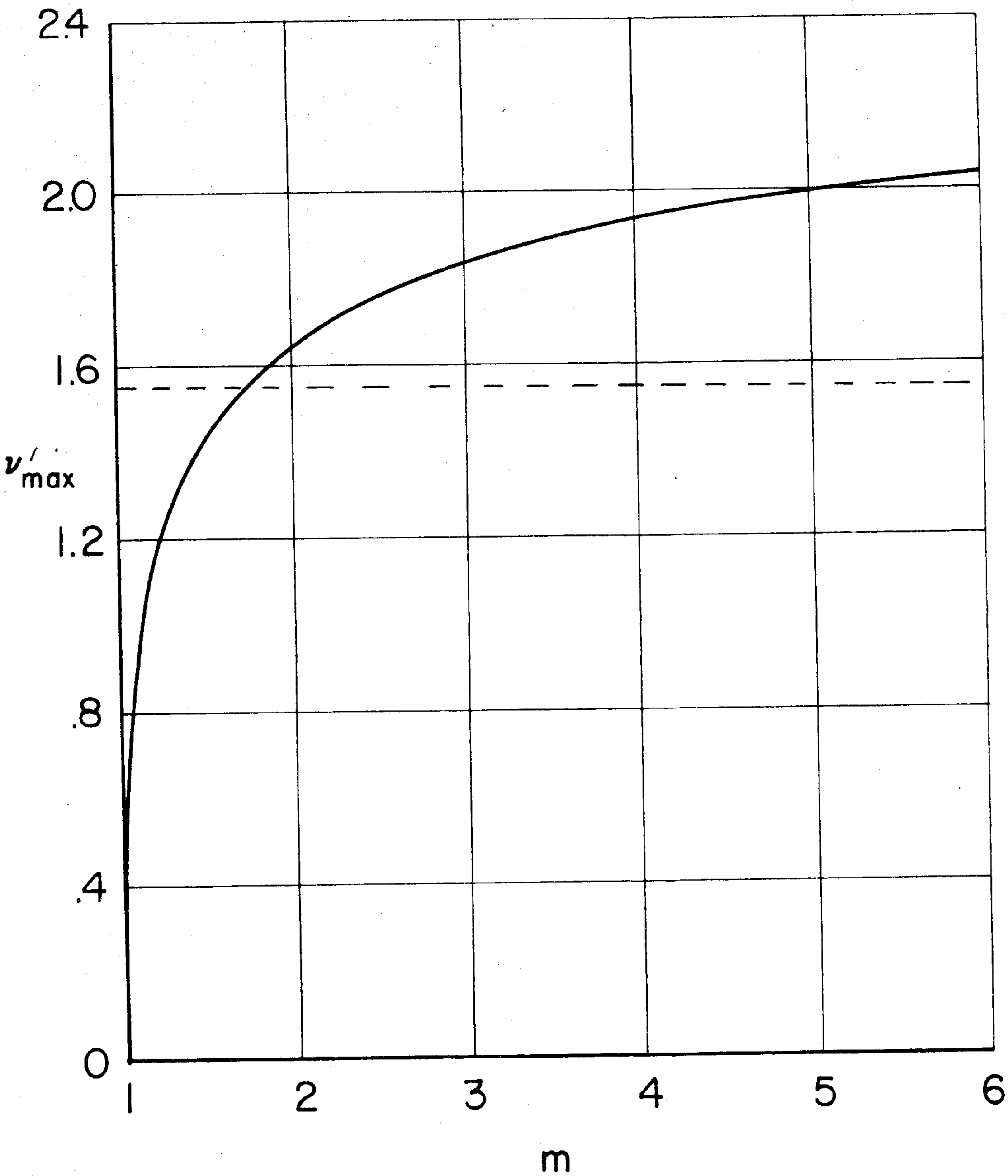


Figure 18 Change in peak of frequency spectrum of downstream sound waves (effective downstream sound frequency) with strength of shock.

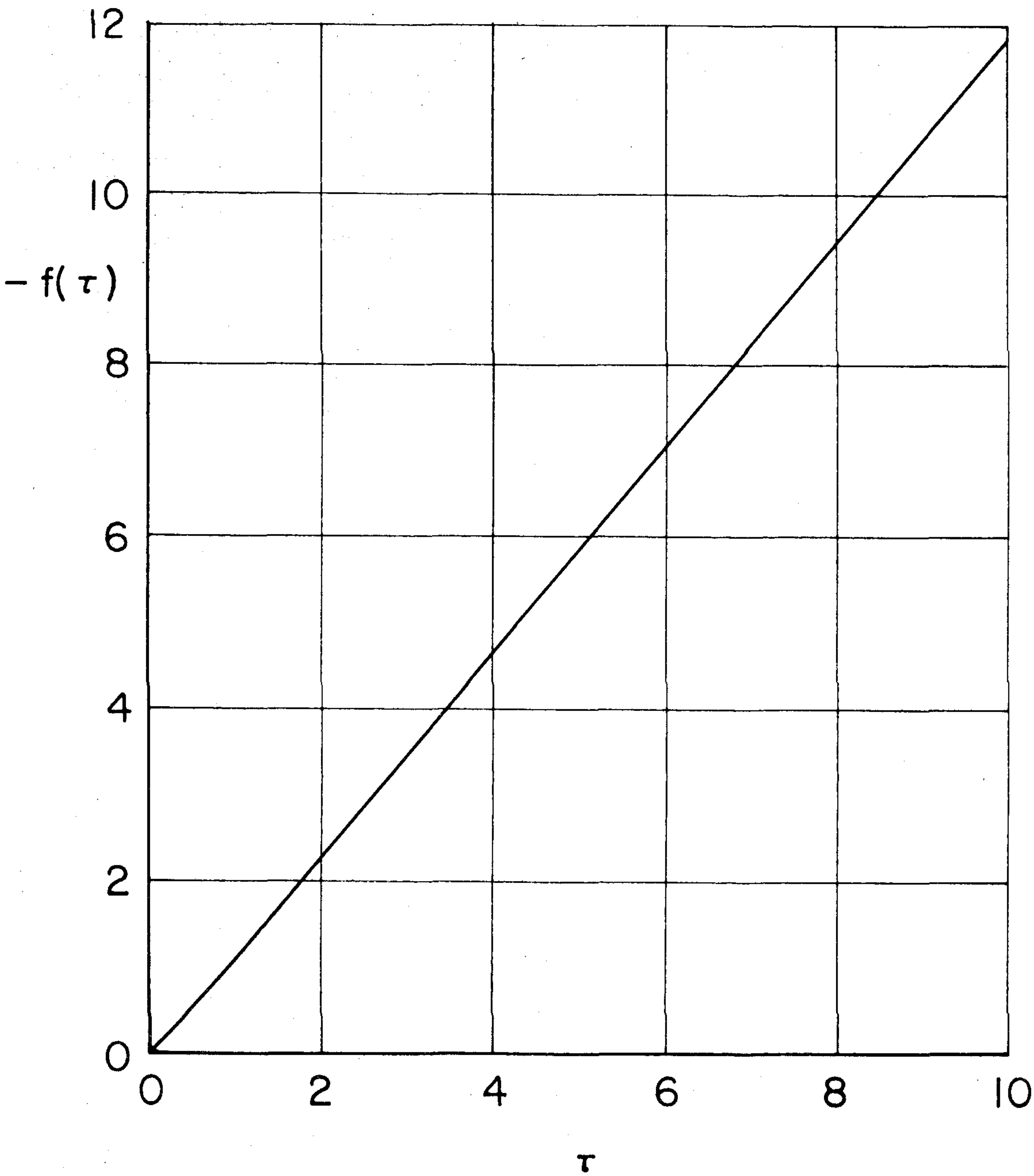


Figure 19 Coefficient of M_n^2 in expansion of m in powers of M_n^2 for flame.

Output \ Input	Vorticity	Entropy	Pressure
Vorticity (normal)	O(1)	O(1)	O(1)
Vorticity (lateral)	O(1)	O(1)	$O(1/M_n)$
Pressure (sound)	$O(M_n^2)$	$O(M_n^2)$	O(1)
Entropy	$O(M_n^2)$	O(1)	O(1)

Figure 20. Orders of magnitude of various outputs from flame-disturbance interaction, if input is of order 1.

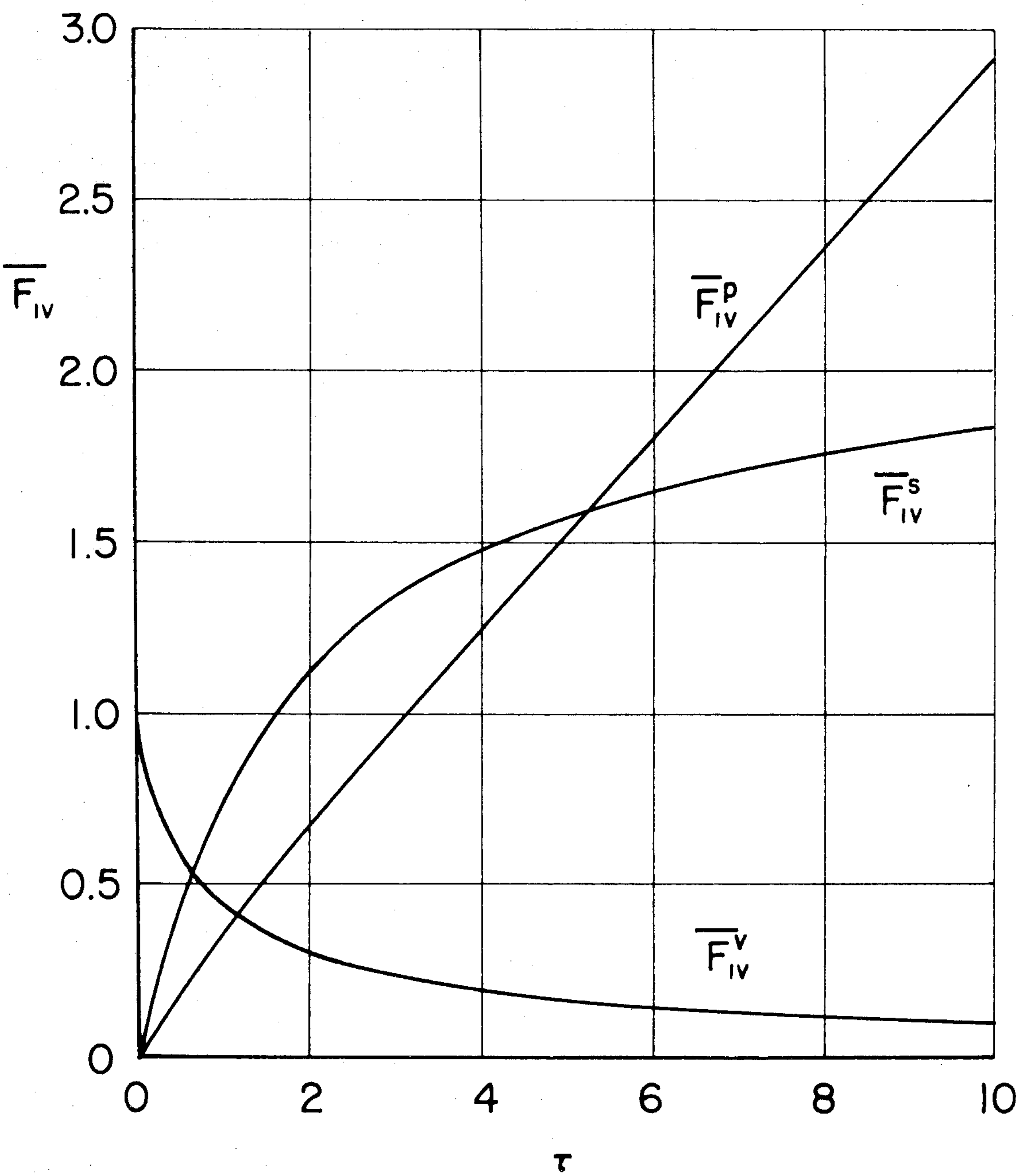


Figure 21 Average transfer functions for downstream normal vorticity (shear velocity normal to flame) resulting from interaction of flame with pressure, vorticity, and entropy disturbances.

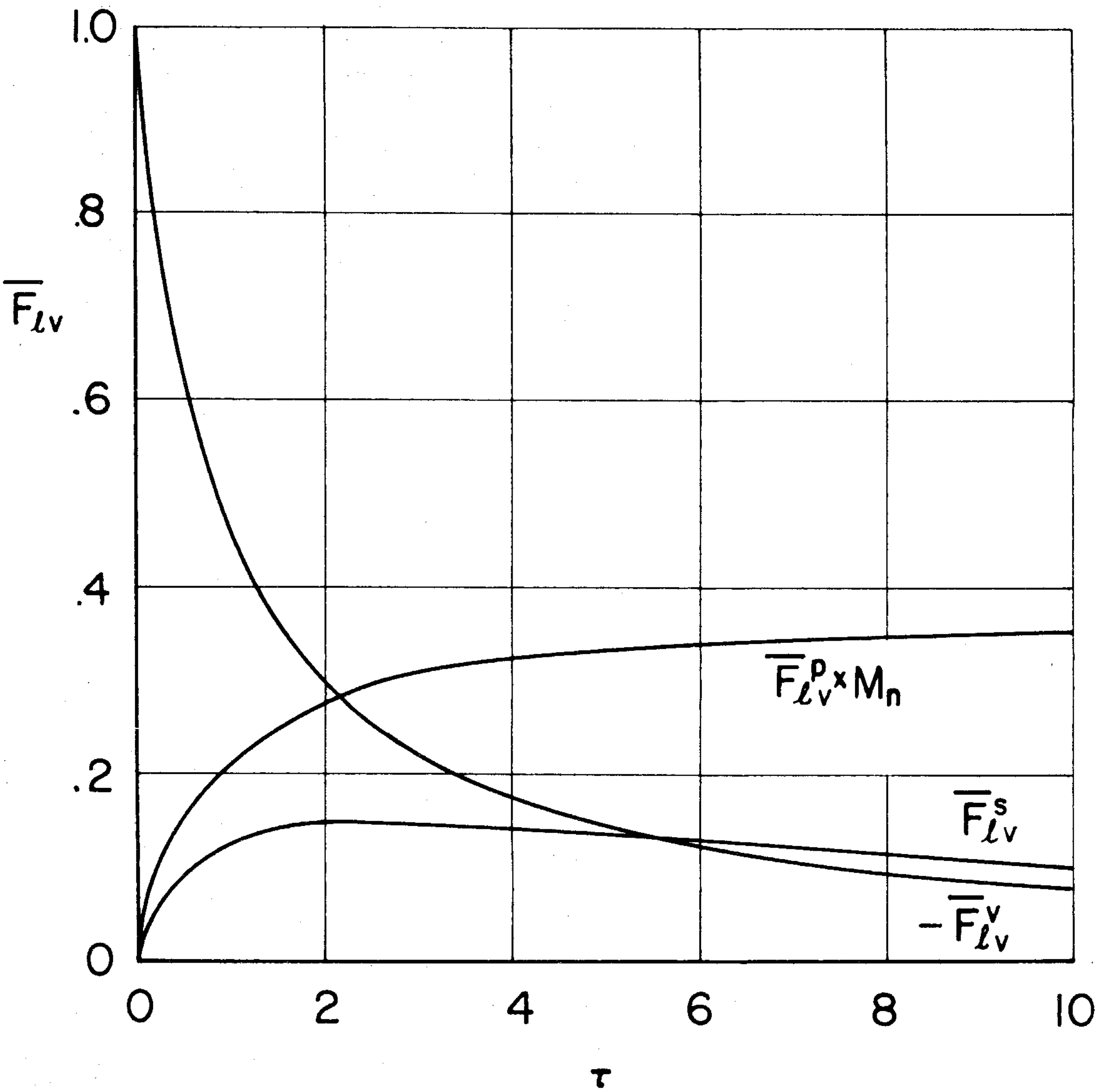


Figure 22. Average transfer functions for downstream lateral vorticity (shear velocity parallel to flame) resulting from interaction of flame with pressure, vorticity, and entropy disturbances.

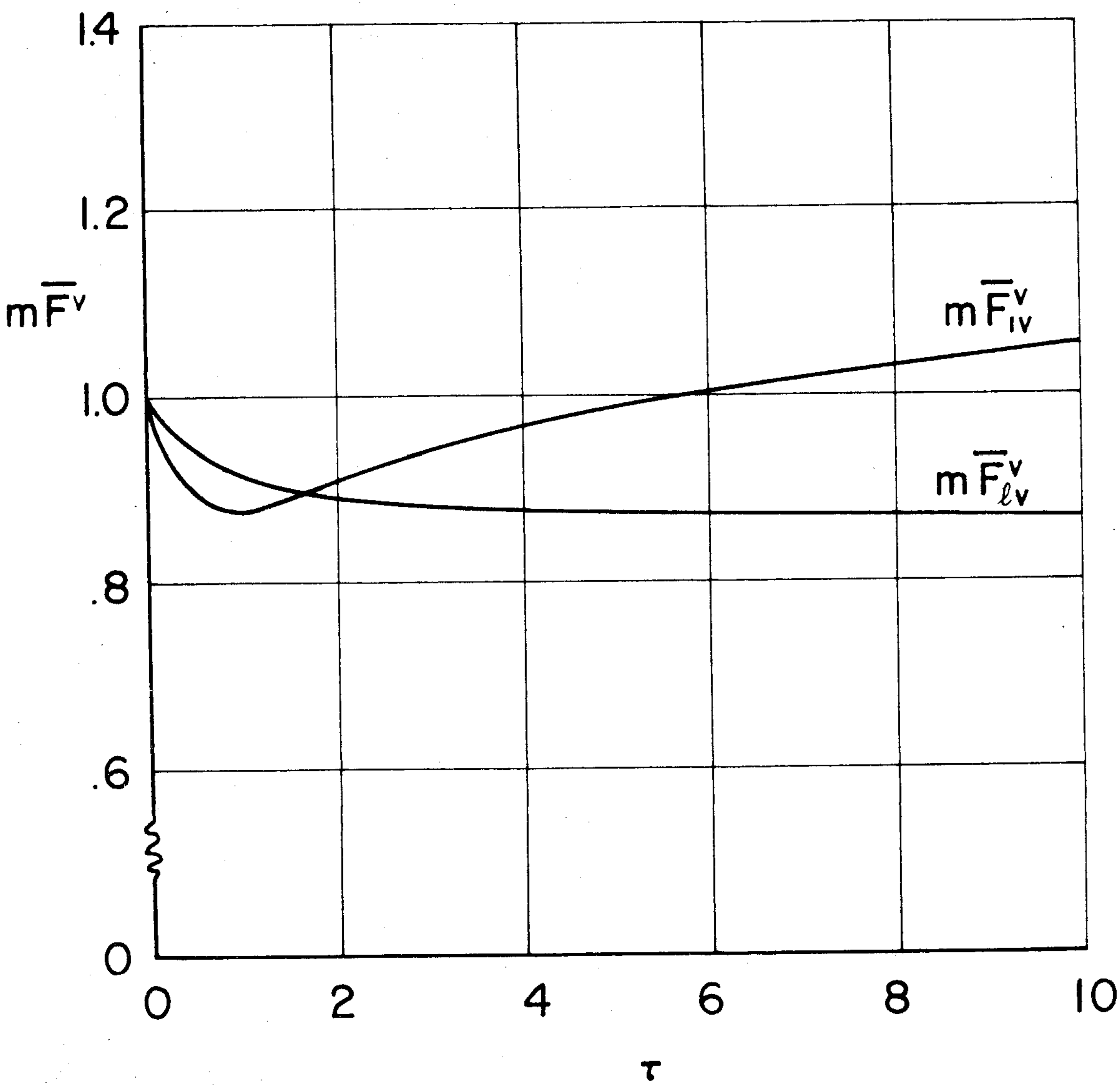


Figure 23 Average transfer functions for downstream vorticity components (shear velocity normal and parallel to flame) resulting from interaction of flame with vorticity disturbance, multiplied by velocity ratio to show relative absolute magnitudes of upstream and downstream vorticity.

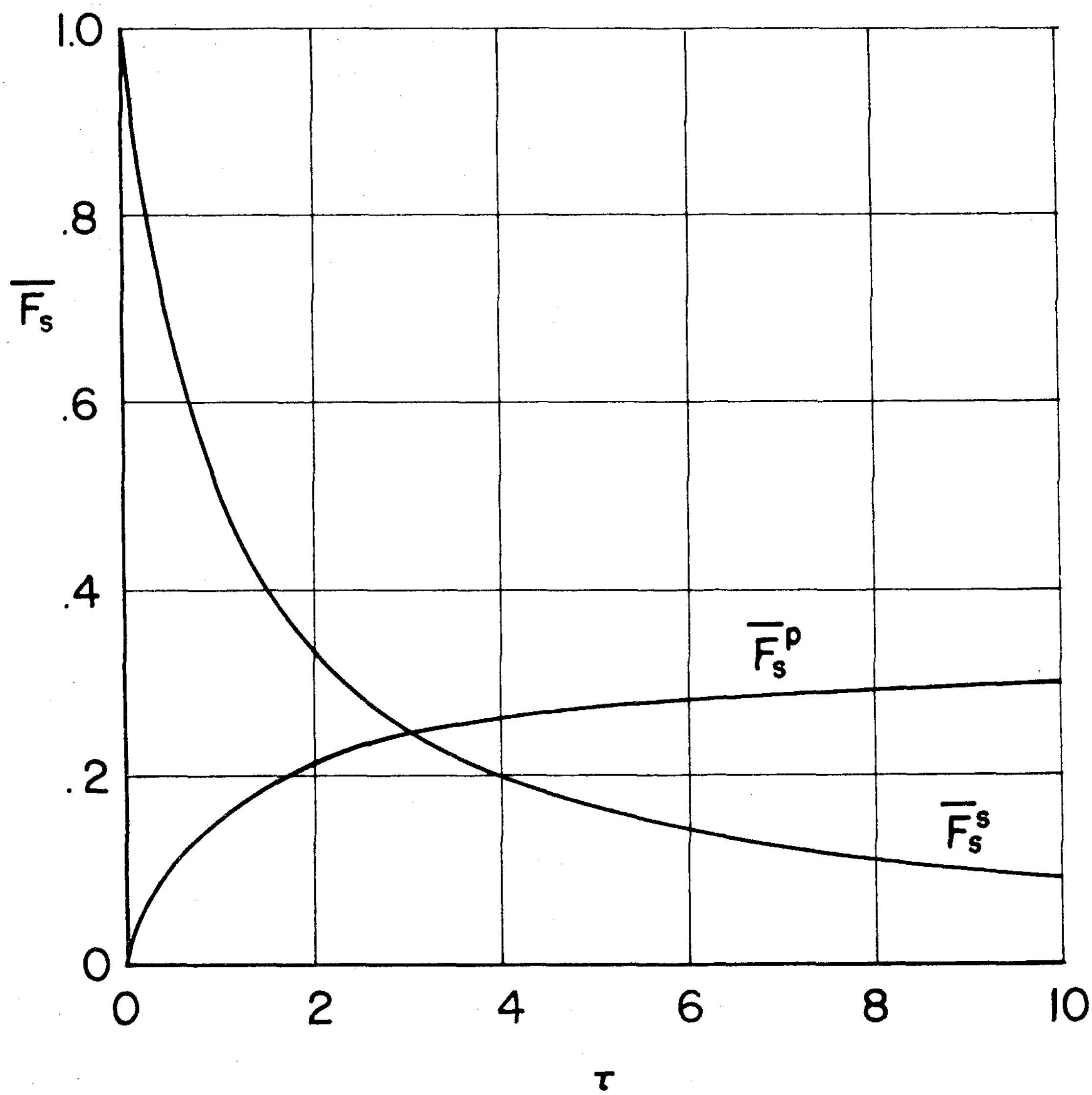


Figure 24 Average transfer functions for downstream entropy disturbance resulting from interaction of flame with pressure and entropy disturbances.

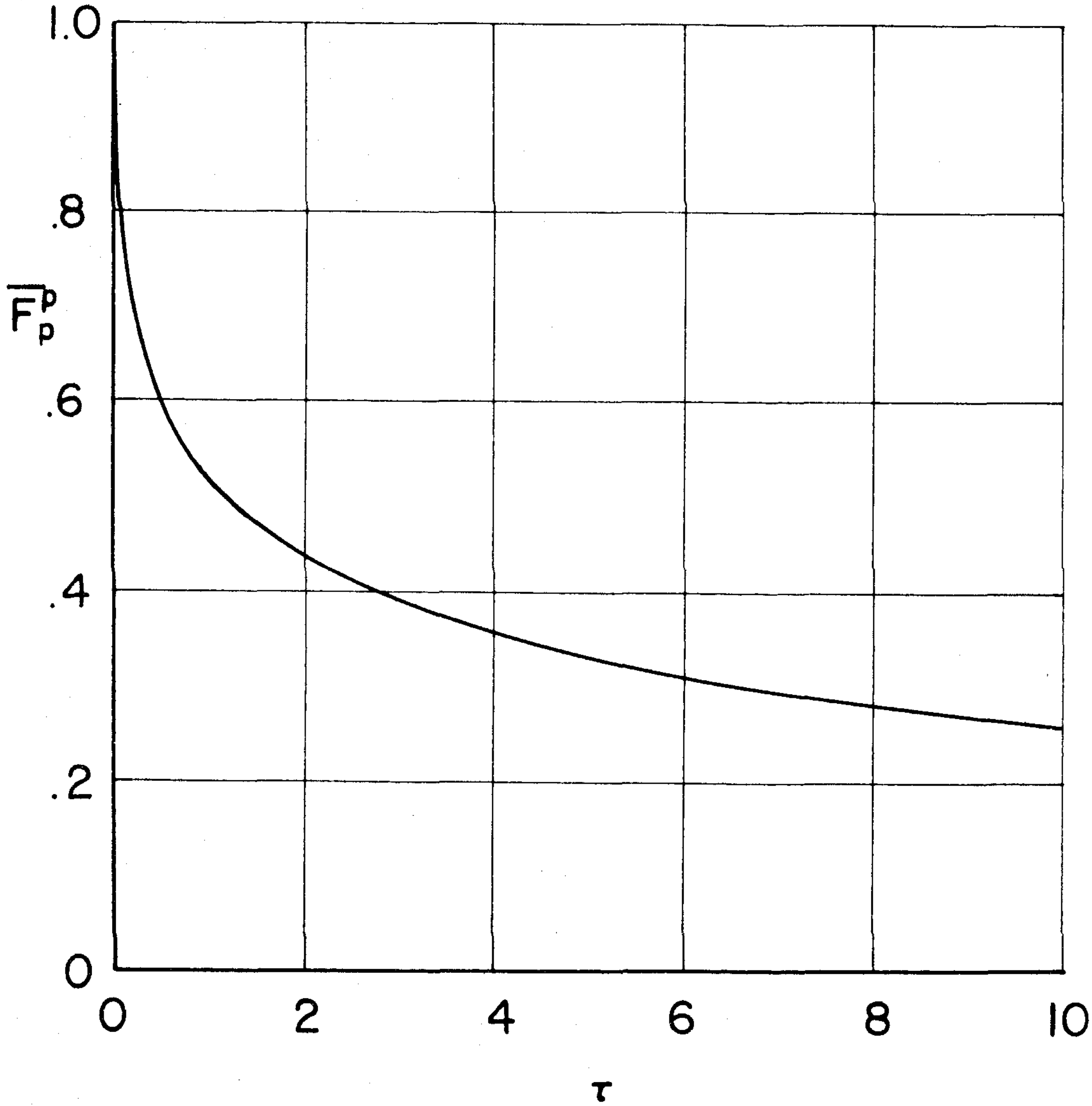


Figure 25 Average transfer function for sound pressure transmitted by flame when sound waves impinge on upstream face of flame.

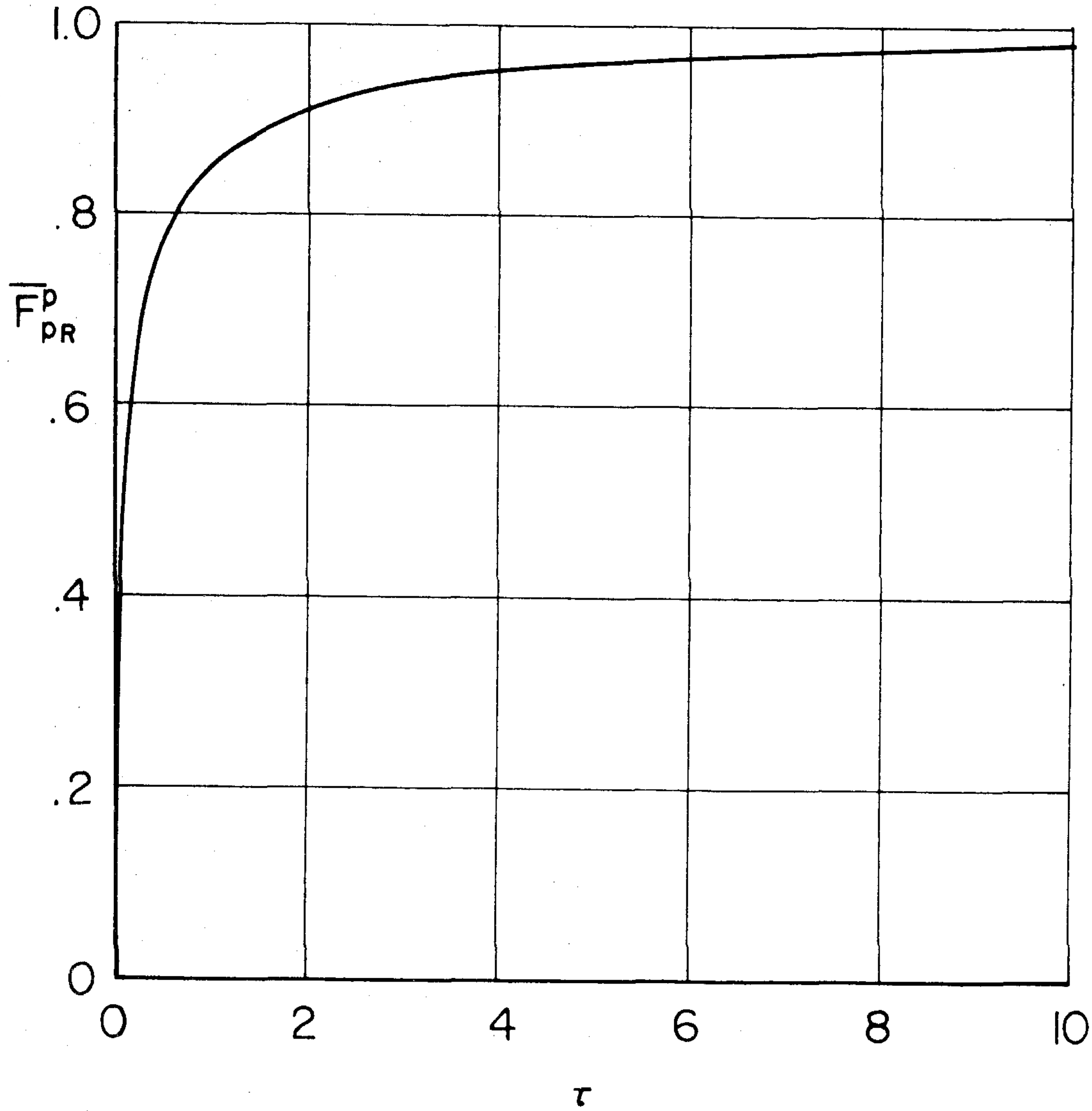


Figure 2[illegible] Average transfer function for sound pressure reflected by flame when sound waves impinge on upstream face of flame.

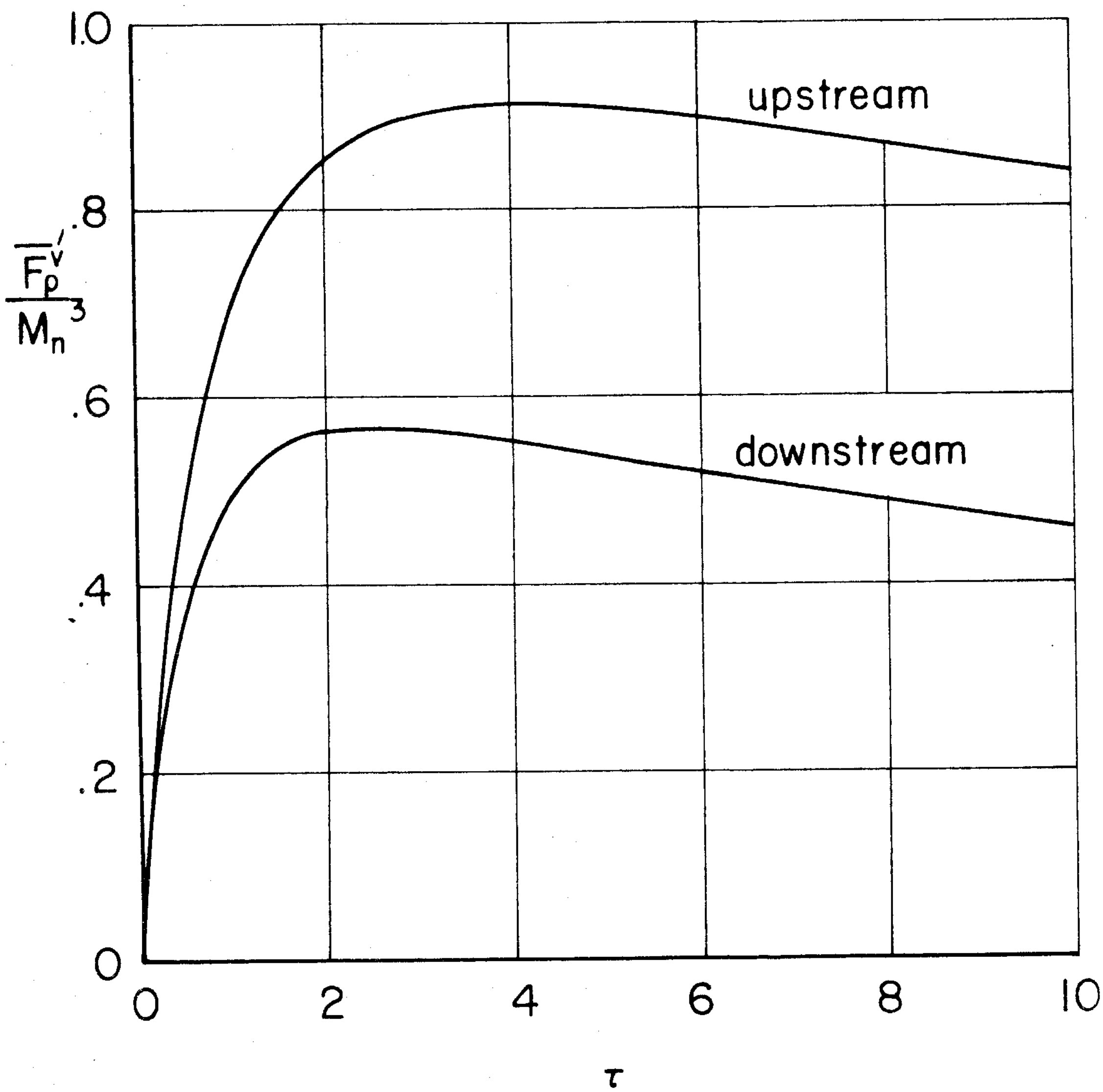


Figure 27 Average transfer functions for upstream and downstream sound pressure resulting from interaction of flame with vorticity disturbance.

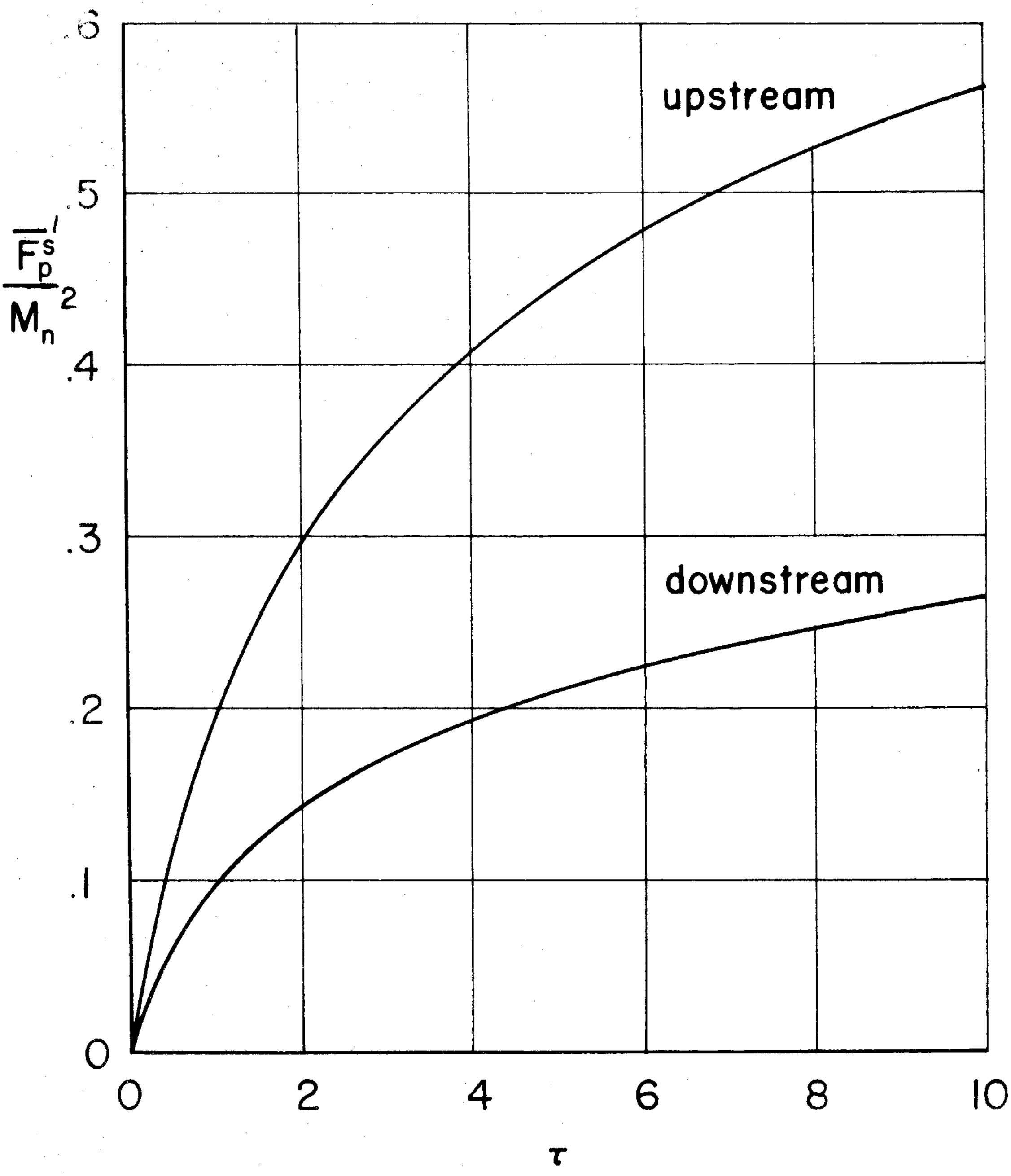


Figure 28 Average transfer functions for upstream and downstream sound pressure resulting from interaction of flame with entropy disturbance.

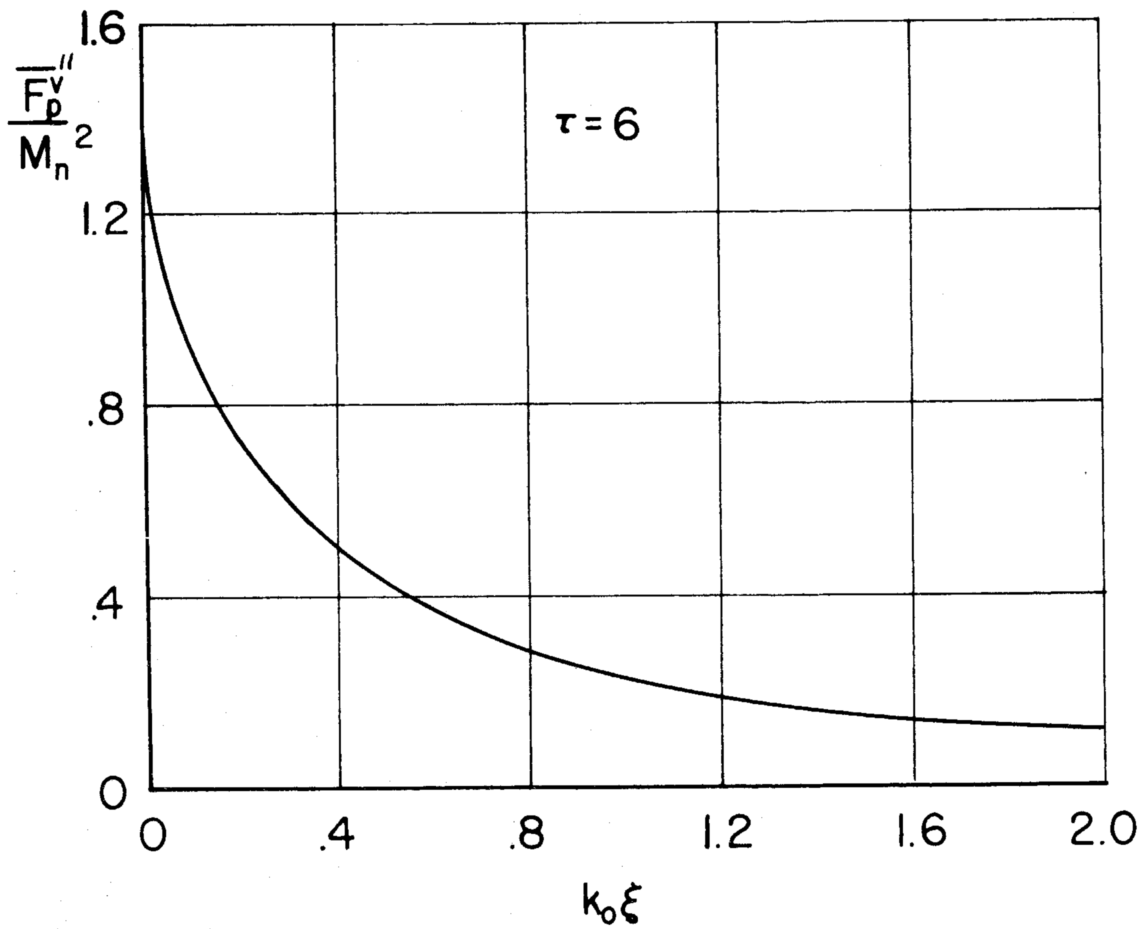


Figure 29 Average transfer function for attenuated downstream pressure field resulting from interaction of flame with vorticity disturbance (for $\tau = 6$).

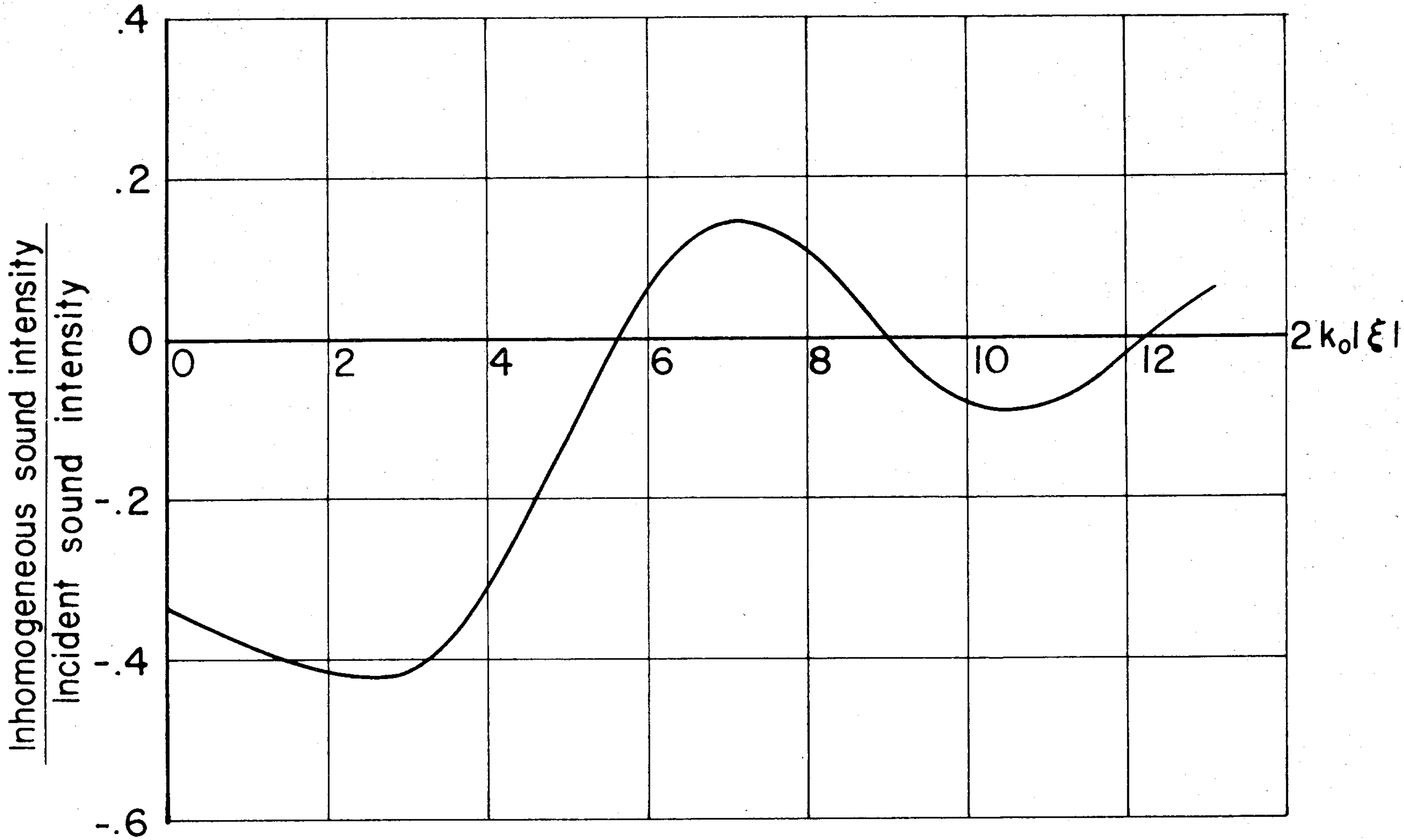


Figure 30 Inhomogeneous upstream sound pressure resulting from impingement of sound waves, of wave number k_o, and uniformly distributed in wave angle, on upstream face of flame.