# Exact bosonization in all dimensions: the duality between fermionic and bosonic phases of matter

Thesis by

Yu-An Chen

In Partial Fulfillment of the Requirements for the
Degree of
Doctor of Philosophy

CALIFORNIA INSTITUTE OF TECHNOLOGY
Pasadena, California

2020
Defended May 25, 2020

Yu-An Chen
ORCID: 0000-0002-8810-9355

# ACKNOWLEDGEMENTS

My time at Caltech has been exciting and rewarding. It is a pleasure to work with so many talented and friendly people.

First, I want to thank my advisor Prof. Anton Kapustin. His deep mathematical knowledge and clear physical intuition always fascinate me. I wouldn't have been able to build my solid background in both physics and mathematics without his guidance. I thank him for teaching the way of thinking on dealing with the most abstract and hardest problems, especially as a pioneer in our field. To complete my Ph.D. thesis, I needed to explore and combine different fields. My ability was nurtured with his helpful advising. He also gave me a lot of freedom to study everything I am interested in. With his support, I was able to be an intern in the Google Quantum AI team, which developed the latest quantum computers. The best decision I have ever made at Caltech was to choose him as my advisor.

I am also influenced by many other professors at Caltech. I am grateful to Prof. Xie Chen for her advice on teaching, research, and life. I was lucky to be a TA in her course. During the quarter, I learned how to organize a course and to give excellent lectures. She shared her opinion on my future directions as well, and this helped my career. She is very kind and makes me feel like she is my "elder sister". I am inspired by Prof. Alexei Kitaev. By taking his classes and talking with him during lunch, I learned many things from him. I admire his vision of physics and his continually pioneering works. It is also my honor to have many conversations with Prof. Jason Alicea and Prof. Olexei Motrunich. They gave me my first guidance in the field of condensed matter theory.

My Ph.D. life wouldn't have been enjoyable without meeting many students and post-docs at Caltech. I first thank Hsiao-Yi Chen for many discussions and for helping me get through my dark first year. I thank group members Alex Turzillo and Minyoung You for collaboration. As the latest one to join this small group, I am greatly enlightened by them. I also thank Tyler Ellison and Nathanan Tantivasadakarn for the wonderful collaborating. The project represents the most important work I have completed in my last two years. Finally, I had stimulating discussions with many people, and I appreciate their help during my Ph.D.: Boqiang Shen, Ying-Hsuan Lin, Po-Shen Hsin, Bowen Yang, Zitao Wang, Baoyi Chen, Yunxuan Li, Kuan-Yu Lin, Hung-I Yang, Jayden Ooi, Junyu Liu, Wilbur Shirley, Pouria Dadras, Djordje

Radicevic, Wenjie Ji, Yu-Ping Lin, and Su-Kuan Chu.

Lastly, I thank my parents for their enormous support and love for me. They make me pursue my dream without any pressure.

# ABSTRACT

We describe an $n$-dimensional ($n \geq 2$) analog of the Jordan-Wigner transformation, which maps an arbitrary fermionic system to Pauli matrices while preserving the locality of the Hamiltonian. When the space is simply-connected, this bosonization gives a duality between any fermionic system in arbitrary $n$ spatial dimensions and a new class of $(n-1)$-form $\mathbb{Z}_2$ gauge theories in $n$ dimensions with a modified Gauss's law. We describe several examples of 2d bosonization, including free fermions on square and honeycomb lattices and the Hubbard model, and 3d bosonization, including a solvable $\mathbb{Z}_2$ lattice gauge theory with Dirac cones in the spectrum. This bosonization formalism has an explicit dependence on the second Stiefel-Whitney class and a choice of spin structure on the manifold, a key feature for defining fermions. A new formula for Stiefel-Whitney homology classes on lattices is derived. We also derive the Euclidean actions for the corresponding lattice gauge theories from the bosonization. The topological actions contain Chern-Simons terms for $(2+1)$D or Steenrod Square terms for general dimensions. Finally, we apply the bosonization to construct various bosonic or fermionic symmetry-protected-topological (SPT) phases. It has been shown that supercohomology fermionic SPT phases are dual to bosonic higher-group SPT phases.

Keywords: bosonization, lattice gauge theory, spin systems, SPT phases

Thesis Supervisor: Anton Kapustin

# PUBLISHED CONTENT AND CONTRIBUTIONS

[1] Yu-An Chen. Exact bosonization in arbitrary dimensions, 2019. URL `https://arxiv.org/abs/1911.00017`. Yu-An Chen finished this paper alone.

[2] Yu-An Chen and Anton Kapustin. Bosonization in three spatial dimensions and a 2-form gauge theory. *Phys. Rev. B*, 100:245127, Dec 2019. doi: 10.1103/PhysRevB.100.245127. URL `https://link.aps.org/doi/10.1103/PhysRevB.100.245127`. Yu-An Chen developed the explicit formula for 3d bosonization, calculated examples, derived the spacetime action, drew figures, and participated in the writing of the manuscript.

[3] Yu-An Chen, Anton Kapustin, and Djordje Radicevic. Exact bosonization in two spatial dimensions and a new class of lattice gauge theories. *Annals of Physics*, 393:234 – 253, 2018. ISSN 0003-4916. doi: https://doi.org/10.1016/j.aop.2018.03.024. URL `http://www.sciencedirect.com/science/article/pii/S0003491618300873`. Yu-An Chen developed the explicit formula for 2d bosonization, derived the spacetime action, drew figures, and participated in the writing of the manuscript.

[4] Yu-An Chen, Anton Kapustin, Alex Turzillo, and Minyoung You. Free and interacting short-range entangled phases of fermions: Beyond the tenfold way. *Phys. Rev. B*, 100:195128, Nov 2019. doi: 10.1103/PhysRevB.100.195128. URL `https://link.aps.org/doi/10.1103/PhysRevB.100.195128`. Yu-An Chen calculated the indexes for low dimensional models and participated in the writing of the manuscript.

# TABLE OF CONTENTS


Acknowledgements . . . . . . . . . . . . . . . . . . . . . . . . . . . . iii
Abstract . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . v
Published Content and Contributions . . . . . . . . . . . . . . . . . . . vi
Bibliography . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . vi
Table of Contents . . . . . . . . . . . . . . . . . . . . . . . . . . . . vi
Chapter I: Introduction . . . . . . . . . . . . . . . . . . . . . . . . . 1

**I Bosonization 8**

Chapter II: Bosonization on two-dimensional lattices . . . . . . . . . . . . . 9
2.1 Square lattice . . . . . . . . . . . . . . . . . . . . . . . . . . . 9
2.2 Triangulation . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
2.3 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 17
2.4 Euclidean 3D gauge theories and their fermionic duals . . . . . . . . . 23
Chapter III: Bosonization on three-dimensional lattices . . . . . . . . . . . . 27
3.1 Cubic lattices . . . . . . . . . . . . . . . . . . . . . . . . . . . . 27
3.2 Triangulation . . . . . . . . . . . . . . . . . . . . . . . . . . . . 29
3.3 Examples . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 34
3.4 Euclidean 3+1D gauge theories with fermionic duals . . . . . . . . . . 36
3.5 Gauging fermion parity . . . . . . . . . . . . . . . . . . . . . . . . 42
Chapter IV: Exact bosonization in $n$ dimensions . . . . . . . . . . . . . . . 45
4.1 Triangulation . . . . . . . . . . . . . . . . . . . . . . . . . . . . 45
4.2 Modified Gauss's law and Euclidean action . . . . . . . . . . . . . . 47

**II Constructions of SPT phases 52**

Chapter V: (2+1)D SPT constructions . . . . . . . . . . . . . . . . . . . . 53
5.1 Review of group cohomology bSPT constructions . . . . . . . . . . . 53
5.2 Correspondence between bSPT and supercohomology fSPT . . . . . 54
Chapter VI: (3+1)D constructions . . . . . . . . . . . . . . . . . . . . . 59
6.1 2-group, 2-group extension, and 2-gauge theory . . . . . . . . . . . 59
6.2 Auxiliary "2-group" bSPT . . . . . . . . . . . . . . . . . . . . . . 62
6.3 Fermionic SPT . . . . . . . . . . . . . . . . . . . . . . . . . . . . 66
Chapter VII: Summary . . . . . . . . . . . . . . . . . . . . . . . . . . . 69
Appendix A: Chains and cochains notations and (higher) cup products on a triangulation and a cubic lattice . . . . . . . . . . . . . . . . . . . . . . 70
Appendix B: A formula for Stiefel-Whitney homology classes . . . . . . . . . 75
Appendix C: Identity for fermionic algebra . . . . . . . . . . . . . . . . . 78
Appendix D: Gauging the symmetries . . . . . . . . . . . . . . . . . . . . 81

Appendix E: Derivation of $\zeta$ (first descendant of $\rho$) . . . . . . . . . . . . . . 84
Bibliography . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 86

*Chapter 1*

# INTRODUCTION

The first part of this thesis is bosonization in all dimensions. It is well known that the Jordan-Wigner transformation establishes an equivalence between the quantum Ising chain in a transverse magnetic field and a system of free spinless fermions on a 1d lattice. This equivalence is very useful and is the quickest way to solve the 2D Ising model.[1] The use of the Jordan-Wigner transformation is not limited to the quantum Ising chain. It establishes a very general kinematic equivalence between 1d fermionic systems and 1d spin chains with a $\mathbb{Z}_2$ spin symmetry. One can regard it as a very special isomorphism between the algebra of fermionic observables with trivial fermion parity and the $\mathbb{Z}_2$-even subalgebra of the algebra of observables in a spin chain. Its distinguishing feature is that it maps local observables[2] with trivial fermion parity on the fermionic side to local observables that commute with the total spin parity

$$(-1)^{S^z} = \prod_i (-1)^{S_i^z}$$

on the bosonic side ($S_i^z = 0, 1$). Thus any local Hamiltonian for a 1d fermionic system can be mapped to a local spin chain Hamiltonian which preserves $S^z$ modulo 2. In this sense, the Jordan-Wigner transformation is local.

There are several ways to generalize the Jordan-Wigner transformation to 2d lattice systems. One obvious approach is to take a square lattice, represent it as a 1d system by picking a lattice path that snakes through the whole lattice and visits each site once, and apply the 1d Jordan-Wigner transformation. This leads to a bosonization map which maps some, but not all, local observables with a trivial fermion parity to local observables with trivial $S^z$. The lack of 2d locality causes problems since even very simple fermionic Hamiltonians are mapped to non-local spin Hamiltonians and vice versa. But there are interesting exceptions [1], the Kitaev honeycomb model [2] being among them [3–5]. Another approach to 2d bosonization is to use flux attachment [6, 7]: a fermion is represented by a boson interacting with a Chern-Simons gauge field. Related ideas in the continuum have been the focus of much recent interest [8–19]. However, it is hard to make this precise on the lattice, due

[1] 2D here means "two Euclidean space-time dimensions."

[2] That is, observables that act nontrivially on a finite number of lattice sites.

to well-known difficulties with defining a lattice Chern-Simons theory. A popular strategy is to eliminate the Chern-Simons gauge field by solving its equations of motion, but this again leads to a non-local map.

A very interesting example of exact 2d bosonization, or rather fermionization, was presented by A. Kitaev in his paper on the honeycomb model [2]. At first it appears quite special, but in fact, it provides a method for mapping an arbitrary system of Majorana fermions on a trivalent lattice to a system of bosonic spins on the same lattice. The spin Hilbert space is not a tensor product over all sites, but rather a subspace defined by a set of commuting constraints. There is one constraint for each face of the lattice, indicating that the dual bosonic system is a gauge theory. But it is a very unusual gauge theory since the gauge field is a composite of spins.

The starting point of this paper is to describe a 2d analog of the Jordan-Wigner transformation, which obeys locality, and to give some examples of 2d bosonization. We will show that any 2d fermionic system on a lattice can be mapped to a system of bosons. On a topologically trivial space, this map is an equivalence, and every local fermionic Hamiltonian is mapped to a local spin Hamiltonian. The main novelty compared to the 1d case is that the bosonic system is a $\mathbb{Z}_2$ gauge theory. This means that the Hilbert space is not a tensor product of local Hilbert spaces, but a subspace in such a tensor product defined by a set of commuting local constraints. They can be interpreted as Gauss law constraints.

Our bosonization procedure shares some similarities both with the flux-attachment approach and with Kitaev's approach. It follows the same strategy as the flux-attachment approach but uses a lattice $\mathbb{Z}_2$ gauge field in place of a $U(1)$ Chern-Simons gauge field. There is no problem writing down a Chern-Simons-like term for a $\mathbb{Z}_2$ gauge field. An additional benefit is that we do not need to introduce separate bosonic degrees of freedom to which the flux is attached: we make use only of degrees of freedom that are already present in the gauge field. Our bosonization procedure is completely general and local, just like in Kitaev's approach, but there are a couple of differences too: (1) the fermions are complex rather than Majorana, so that the fermionic Hilbert space is naturally a tensor product over all sites; (2) the bosonic variables live on edges rather than on sites, and the gauge field is fundamental rather than composite.

The connection between gauge symmetry and bosonization rests on the observation made in [20] that 2d bosonization should map fermionic systems to bosonic systems with a global 1-form $\mathbb{Z}_2$ symmetry and a suitable 't Hooft anomaly. On a lattice, global 1-form symmetries can exist only in gauge theories. The proposal of [20] was made concrete in [21] for topological systems (that is, spin-TQFTs), but it was implicit in that paper that the same strategy should apply for general fermionic systems on a lattice. In this thesis, we make this completely explicit. Namely, we show that on a simply-connected space one can isomorphically map the bosonic subalgebra of the algebra of local fermionic observables to the algebra of local gauge-invariant observables in a suitable $\mathbb{Z}_2$ lattice gauge theory. The bosonization map is not canonical and depends on some additional choices which depend on the type of lattice. We discuss two kinds of lattices in $n$-dimension: the $n$-dimensional cubic lattice and an arbitrary triangulation endowed with a branching structure. In either case, the fermions live on the $n$-cells, i.e., fermions at faces for 2d lattices.

Gauss law constraints for gauge theories dual to fermionic systems are not standard. Their meaning becomes clearer if we discretize time and consider the corresponding Euclidean lattice partition function. It turns out that the unusual Gauss law arises from a Chern-Simons-like term $i\pi \int A \cup \delta A$ in the spacetime action. These terms necessarily break invariance under the cubic symmetry (if one starts from a 2d square lattice).

This approach can be generalized to 3d [22]. Every fermionic lattice system in 3d is dual to a $\mathbb{Z}_2$ 2-form gauge theory with an unusual Gauss law. Here "2-form gauge theory" means that the $\mathbb{Z}_2$ variables live on faces (2-simplices), while the parameters of the gauge symmetry live on edges (1-simplices). 2-form gauge theories in 3+1D have local flux excitations, and the unusual Gauss law ensures that these excitations are fermions. This Gauss's law can be described by the "Steenrod square" topological action $i\pi \int B \cup B + B \cup_1 \delta B$. The form of the modified Gauss law was first observed in [20]: a bosonization of fermionic systems in $n$ dimensions must have a global $(n-1)$-form $\mathbb{Z}_2$ symmetry with a particular 't Hooft anomaly. The standard Gauss law leads to a trivial 't Hooft anomaly, so bosonization requires us to modify it in a particular way.

We further extend the result to arbitrary $n$ dimensions. We show that every fermionic lattice system in $n$-dimension is dual to a $\mathbb{Z}_2$ $(n-1)$-form gauge theory with a modified Gauss law. Our bosonization map is kinematic and local in the same sense as the Jordan-Wigner map: every local observable on the fermionic side, including the Hamiltonian density, is mapped to a local gauge-invariant observable on the $\mathbb{Z}_2$ gauge theory side. In the Euclidean picture, we show explicitly that our bosonization map is equivalent to introducing the topological term in the action:

$$S_{\text{top}} = i\pi \int_Y (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1}), \tag{1.1}$$

where $A_{n-1}$ is $(n-1)$-form gauge fields, a $(n-1)$-cochain $A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2)$, and $Y$ is $(n+1)$-dimensional spacetime manifold. When $A_{n-1}$ is closed (a cocycle), this term reduces to the Steenrod square operator [23]. This "Steenrod square" term appears in the construction of fermionic symmetry-protected-topological phases [24].

The second part of the thesis is about constructions of symmetry protected topological (SPT) phases. A major goal in condensed matter is the classification of phases given a certain symmetry. For quantum phases of matter, there are a class of gapped symmetric phases which have a unique ground state on a closed manifold known as symmetry protected topological (SPT) phases [25, 26]. In studying these phases, there have been two main programs that have evolved in tandem. The first is how to classify such phases given a symmetry group $G$. There are two main classes of phases. One is where the constituents are purely bosonic (bSPTs), and those that have physical fermions as excitations (fSPTs). For an onsite and unitary symmetry group, interacting bSPTs and fSPTs, are now believed to be classified by oriented cobordism [27] and spin cobordism [28] respectively. These classifications can also be obtained from a generalized cohomology theory [29–31].

The second is to construct Hamiltonians which realizes these phases. For bosonic SPTs, a successful class of models have been constructed using group cohomology[32]. Outside of this class, though they are predicted to exist using field theory arguments[30, 33] only a few phases "beyond" group cohomology have been constructed as exactly solvable lattice models[34–36]. For fermionic SPTs, a large class of models can be realized via free fermion models[37–39]. However, there are also certain interacting models that have no free-fermion counterpart[40].

In this thesis, we consider a large class of models that realize interacting fSPTs known as group supercohomology SPTs [41]. Using input data that characterize these models, we construct exactly solvable Hamiltonians that realize these phases and also construct their symmetric gapped boundaries, which exhibit topological order. The construction of exactly solvable Hamiltonians for fSPTs in (2+1)D [42, 43] and (3+1)D will be shown explicitly in this paper. The construction exploits a duality which relates the ground state of the fSPT to the ground state of a certain *auxillary* bSPT. In particular, gauging the $\mathbb{Z}_2$ $(n-2)$-form symmetry of this auxiliary bSPT (in $n$ spatial dimensions) gives rise to a $n-1$-form gauge theory with emergent fermions. We then use the boson-fermion duality developed in Part I to obtain the wavefunction and Hamiltonian for the fSPT in $n$ spatial dimensions. This picture is summarized in Figure 1.1.

We remark that although there are previous works which construct exactly-solvable fSPT phases, some of which are outside supercohomology [21, 41, 44–49], our construction of fSPT phases gives an expression for the disentangler for the ground state wavefunction. Specifically, in the absence of symmetry, we are able to explicitly trivialize the fSPT wavefunction using a Finite Depth Quantum Circuit (FDQC) which commutes with the symmetry as a whole. Furthermore, the group structure (or stacking rule) of the fSPT phases can be obtained explicitly by a composition of disentangling unitaries. Having an explicit disentangler also allows us to construct a Hamiltonian realizing the phase where every eigenstate is itself an SPT, and therefore allowing the Hamiltonian to be many-body localized by introducing disorder.

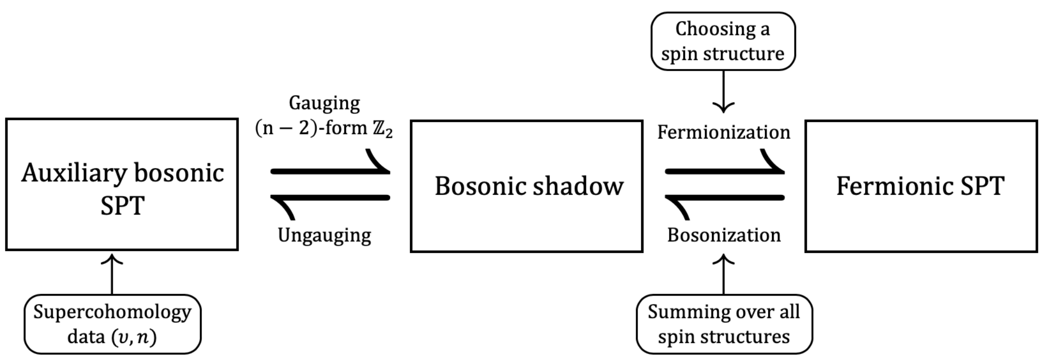



Figure 1.1: To construct a $G_f = G \times \mathbb{Z}_2^f$ supercohomology SPT model in $n$ spatial dimensions, we start with a model for a particular $(n-1)$-group SPT phase determined by the supercohomology data $(\rho, \nu)$. Next, we gauge the $\mathbb{Z}_2$ $(n-2)$-form symmetry of the $(n-1)$-group to build the shadow model. We then condense the fermion in the shadow model, or apply the fermionization duality, to obtain a model for the supercohomology SPT phase corresponding to $(\rho, \nu)$.

This thesis is organized as follows. In Part I, we construct the 2d bosonization map on the kinematic level, i.e. on the level of the algebras of observables in Section 2. We give several examples of bosonization for concrete fermionic systems, such as free fermions and the Hubbard model. Conversely, we describe the fermionization of the simplest lattice gauge theories with a non-standard Gauss law, which are dual to free fermionic theories and thus are integrable. The Euclidean partition function for the gauge theories is derived in Section 2.4. In Section 3, we explicitly construct the 3d bosonization in both cubic lattice and general triangulation, which utilizes the mathematical tools in algebraic topology: higher cup products. As in the 2d case, many examples and the spacetime action are discussed. The complete generalization for the bosonization in all dimensions is derived in Section 4. The formalism comes from comparing the formula between 2d and 3d bosonization, which can be extended to their general form directly.

In Part II, we begin by reviewing the construction of (2+1)D bSPTs [32] and supercohomology fSPTs [42] in Section 5. In Section 5.2, we construct an exactly solvable lattice Hamiltonian and the corresponding groundstate wavefunction of a fermionic SPT protected by an onsite finite unitary group $G$. We achieve this by using the supercohomology data to construct an auxiliary bSPT protected by a symmetry $\tilde{G}$, which is an extension of the $G$ symmetry by a $\mathbb{Z}_2$ symmetry. The fSPT is obtained by first gauging the $\mathbb{Z}_2$ symmetry followed by fermionization. The ground state wavefunction has an explicit disentangling circuit, which trivializes the SPT phase in the absence of symmetry. We use the circuit to derive the stacking rule for supercohomology phases. In Section 6, similar scheme applies to (3+1)D fSPTs. The only difference is that the auxiliary bSPT is now protected by a 2-group symmetry[50], which is an extension of a 0-form $G$ symmetry by a 1-form $\mathbb{Z}_2$ symmetry. In Section 6.1, we review the basic properties for 2-group, 2-group extension, and 2-gauge theory. In Section 6.2, we use these concepts to construct a "2-group" bSPT. After gauging the 1-form symmetry, it becomes a $\mathbb{Z}_2$ gauge theory, which is fermionizable. In Section 6.3, the fSPT wavefunction and its Hamiltonian are derived.

**Notations and Coventions**

This subsection introduces the notations and conventions adopted for this thesis. The more details for algebraic topology and the definition of (higher) cup products is included in Appendix A. In this paper, we will always work with an arbitrary triangulation of a simply-connected $n$-dimensional manifold $M_n$ equipped with a branching structure (orientations on edges without forming a loop in any triangle). The vertices, edges, faces, and tetrahedra are denoted $v, e, f, t$, respectively. The general $d$-simplex is denoted as $\Delta_d$. We can label the vertices of $\Delta_d$ as $0, 1, 2, \ldots, d$ such that the directions of edges are from the small number to the larger number. We denote this $d$-simplex as $\Delta_d = \langle 01 \ldots d\rangle$. Its boundaries are $(d-1)$-simplices $\langle 0, \ldots, \hat{i}, \ldots, d\rangle$ for $i = 0, 1, \ldots, d$, where $\hat{i}$ means $i$ is omitted. A formal sum of

$d$-simplices modulo 2 forms an element of the chain $C_d(M_n, \mathbb{Z}_2)$.

For every $v$, we define its dual 0-cochain $\boldsymbol{v}$, which takes value 1 on $v$, and 0 otherwise, i.e. $\boldsymbol{v}(v') = \delta_{v,v'}$. Similarly, $\mathbf{e}$ is an 1-cochain $\boldsymbol{e}(e') = \delta_{e,e'}$, and so forth, i.e. $\boldsymbol{\Delta}_d$ being a $d$-cochain $\boldsymbol{\Delta}_d(\Delta'_d) = \delta_{\Delta_d,\Delta'_d}$. All dual cochains will be denoted in bold. An evaluation of a cochain $\boldsymbol{c}$ on a chain $c'$ will sometimes be denoted $\int_{c'} \boldsymbol{c}$. When the integration range is not written, $\boldsymbol{c}$ is assumed to be the top dimension and $\int \boldsymbol{c} \equiv \int_{M_n} \boldsymbol{c}$. A $d$-cochain $\boldsymbol{c}_d \in C^d(M_n, \mathbb{Z}_2)$ can be identified as $\mathbb{Z}_2$ fields living on each $d$-simplex $\Delta_d$, with the value $\boldsymbol{c}_d(\Delta_d)$. The cup product $\cup$ of a $p$-cochain $\alpha_p$ and a $q$-cochain $\beta_q$ is a $(p+q)$-cochain defined as

$$\begin{aligned}
[\alpha_p \cup \beta_q](\langle 0, 1, \dots, p+q \rangle) &= \alpha_p(\langle 01 \dots p \rangle)\beta_q(\langle p, p+1, \dots, p+q \rangle) \\
&= \alpha_p(0 \sim p)\beta_q(p \sim p+q).
\end{aligned} \tag{1.2}$$

The definition of the higher cup product [20, 23] is

$$\begin{aligned}
&[\alpha_p \cup_a \beta_q](0, 1, \cdots, p+q-a) = \\
&\sum_{0 \le i_0 < i_1 < \cdots < i_a \le p+q-a} \alpha_p(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \cdots) \times \beta_q(i_0 \sim i_1, i_2 \sim i_3, \cdots),
\end{aligned} \tag{1.3}$$

where $i \sim j$ represents the integers from $i$ to $j$, i.e. $i, i+1, \dots, j$, and $\{i_0, i_1, \dots, i_a\}$ are chosen such that the arguments of $\alpha_p$ and $\beta_q$ contain $p+1$ and $q+1$ numbers separately.

The boundary operator is denoted by $\partial$. For an $n$-simplex $\Delta_n$, $\partial\Delta_n$ consists of all boundary $(n-1)$-simplices of $\Delta_n$. The coboundary operator is denoted by $\delta$ (not to be confused with the Kronecker delta previously). On a 0-cochain $\boldsymbol{v}$, $\delta\boldsymbol{v}$ is an 1-cochain acting on edges, and is 1 if $\partial e$ contains $v$ and 0 otherwise:

$$\delta\boldsymbol{v}(e) = \boldsymbol{v}(\partial e) = \delta_{v,\partial e}.$$

It is similar for simplices in any dimension.

Finally, $\Delta^1_n \supset \Delta^2_{n'}$ or $\Delta^2_{n'} \subset \Delta^1_n$ means that the simplex $\Delta^1_n$ contains $\Delta^2_{n'}$ as a subsimplex. A general rule of thumb is that the subset symbol always points to one higher dimension.

# Part I

# Bosonization

*Chapter 2*

# BOSONIZATION ON TWO-DIMENSIONAL LATTICES

## 2.1 Square lattice

We first introduce our bosonization method on an infinite 2d square lattice. Suppose that we have a model with fermions living at the centers of faces. Let us describe the generators and relations in the algebra of local observables with trivial fermion parity (the even fermionic algebra for short).

On each face $f$ we have a single fermionic creation operator $c_f$ and a single fermionic annihilation operator $c_f^\dagger$ with the usual anticommutation relations. The fermionic parity operator on face $f$ is $P_f = (-1)^{c_f^\dagger c_f}$. It is a "$\mathbb{Z}_2$ operator" (i.e. it squares to 1). All operators $P_f$ commute with each other. The even fermionic algebra is generated by these operators and the operators $c_f^\dagger c_{f'}$, $c_f c_{f'}$, and their Hermitean conjugates, where $f$ and $f'$ are two faces which share an edge. Overall, we get four generators for each edge and one generator for each face.

In fact, one can make do with a single generator for each edge and a single generator for each face, provided we choose a consistent orientation of all faces and arbitrary orientations of all edges. We introduce Majorana fermions

$$\gamma_f = c_f + c_f^\dagger, \quad \gamma'_f = (c_f - c_f^\dagger)/i. \tag{2.1}$$

The algebra of Majorana fermions is

$$\{\gamma_f, \gamma_{f'}\} = \{\gamma'_f, \gamma'_{f'}\} = 2\delta_{f,f'}, \ \{\gamma_f, \gamma'_{f'}\} = 0, \tag{2.2}$$

where $\{A, B\} = AB - BA$ is the anti-commutator. Then the operators

$$P_f = -i\gamma_f\gamma'_f$$

and

$$S_e = i\gamma_{L(e)}\gamma'_{R(e)}$$

are $\mathbb{Z}_2$ operators and generate the even algebra. Here $L(e)$ and $R(e)$ are faces to the left and to the right of the edge $e$ with respect to the chosen orientations.[1] We will

[1] That is, $L(e)$ is the face which induces the same orientation on $e$ as the given orientation of $e$, while $R(e)$ is the face which induces the opposite orientation.

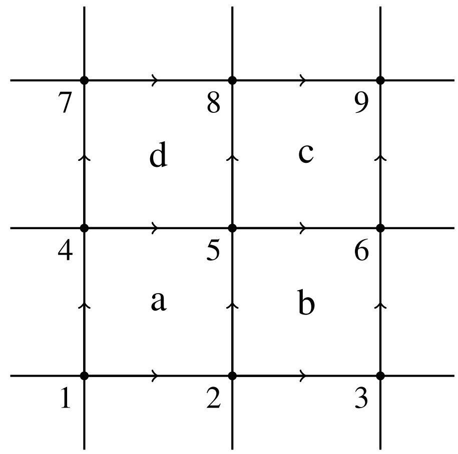



Figure 2.1: Bosonization on a square lattice requires constraints on vertices.

refer to $S_e$ as the hopping operator for edge $e$. It anticommutes with $P_f$ if $f = L(e)$ or $f = R(e)$ and commutes with all other $P_f$.

Other relations depend on the choice of orientations. We will choose the usual (counterclockwise) orientation of the plane and point all horizontal edges to the east, and all vertical edges to the north; see Fig. 2.1. Then it is easy to see that $S_e$ and $S_{e'}$ may fail to commute only if $e$ and $e'$ share a point and are perpendicular. If $e$ and $e'$ share a point and are perpendicular, then in the notation of Fig. 2.1 we have

$$[S_{56}, S_{58}] = [S_{25}, S_{45}] = 0, \quad \{S_{25}, S_{56}\} = \{S_{58}, S_{45}\} = 0. \tag{2.3}$$

In other words, $S_e$ and $S_{e'}$ anticommute if $e$ and $e'$ issue from the same vertex and point either east and south, or north and west. They commute in all other cases.

Additional relations emerge if we consider the product of four hopping operators corresponding to all edges issuing from a vertex. This corresponds to an operator taking a fermion full circle around the vertex. The resulting operator commutes with $P_f$ for all $f$ and thus must be some function of these operators. Indeed, a short calculation shows that

$$\begin{aligned} S_{58}S_{56}S_{25}S_{45} &= (i\gamma_d\gamma'_c)(i\gamma_c\gamma'_b)(i\gamma_a\gamma'_b)(i\gamma_d\gamma'_a) \\ &= (i\gamma'_a\gamma_a)(i\gamma'_c\gamma_c) \\ &= P_aP_c. \end{aligned} \tag{2.4}$$

It is clear intuitively and can be shown rigorously (see Appendix for a sketch of a proof) that these are all relations between our chosen generators if the lattice is infinite, or if it is finite but topologically trivial.

The dual description will consist of bosonic spins living on edges of the same lattice. The operators acting on each edge $e$ are Pauli matrices $\sigma_e^x$, $\sigma_e^y$, and $\sigma_e^z$. To reduce notation clutter, we denote them $X_e$, $Y_e$, and $Z_e$.

$$X_e = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ Y_e = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ Z_e = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{2.5}$$

This is the usual operator algebra of the toric code.

We have two kinds of edges: edges oriented east and edges oriented north. If $e$ is oriented east (resp. north), let $r(e)$ be the edge which points north (resp. east) and ends where $e$ begins. In the notation of Fig. 2.1, $r(e_{56}) = e_{25}$, $r(e_{58}) = e_{45}$. It will be useful to define the composite operator

$$U_e = X_e Z_{r(e)}. \tag{2.6}$$

In the toric code language, $U_e$ is the operator moving the $\epsilon$-particle across edge $e$. We also define the "flux operator" at each face $f$ to be

$$W_f = \prod_{e \subset f} Z_e. \tag{2.7}$$

Our bosonization map is defined as follows:

1. We identify the fermionic states $|P_f = 1\rangle$ and $|P_f = -1\rangle$ with bosonic states for which $W_f = 1$ and $W_f = -1$, respectively. This amounts to dualizing

$$P_f = -i\gamma_f \gamma'_f \longleftrightarrow W_f. \tag{2.8}$$

2. The fermionic hopping operator $S_e$ is identified with $U_e$ defined above,

$$S_e = i\gamma_{L(e)}\gamma'_{R(e)} \longleftrightarrow U_e. \tag{2.9}$$

All operator relations discussed above are preserved under this map. The only exception is the relation (2.4), which is absent on the bosonic side. Instead, the product $S_{58}S_{56}S_{25}S_{45}$ maps to

$$U_{58}U_{56}U_{25}U_{45} = W_{f_a} \prod_{e \supset v_5} X_e. \tag{2.10}$$

To get an algebra homomorphism, we must impose a constraint on the bosonic variables at vertex 5, namely

$$W_{f_c} \prod_{e \supset v_5} X_e = 1. \tag{2.11}$$

For a general vertex $v$, the constraint is

$$W_{\mathrm{NE}(v)} \prod_{e \supset v} X_e = 1, \tag{2.12}$$

where $\mathrm{NE}(v)$ is the face northeast of $v$.

We interpret this as a Gauss law for the bosonic system. The presence of the Gauss law means that we are dealing with a gauge theory. Since the constraint at each vertex is a $\mathbb{Z}_2$ operator, this is a $\mathbb{Z}_2$ gauge theory. The algebra of gauge-invariant observables on the bosonic side (i.e. the algebra of local observables commuting with all Gauss law constraints) is generated by operators $U_e$ and $W_f$, and there are no further relations between them apart from those which exist between $S_e$ and $P_f$. Thus the above map is an isomorphism and defines a 2d version of the Jordan-Wigner transformation.

The constraint (2.12) couples the electric charge at a vertex $v$ to the magnetic flux at face $\mathrm{NE}(v)$. Thus our modified Gauss law implements charge-flux attachment, and it is not surprising that operators $U_e$ which move the flux behave as fermionic bilinears.

Note also that the total fermion number operator[2]

$$F = \sum_f \frac{1}{2}(1 + i\gamma_f \gamma'_f)$$

is mapped to the net magnetic flux

$$\sum_f \frac{1}{2}(1 - W_f).$$

While the fermion number operator is ultra-local (it is a sum of operators each of which acts nontrivially only on fermions at a particular site), its bosonized version is not ultra-local.

## 2.2 Triangulation

The bosonization method described above also works for any triangulation $T$ with a branching structure.[3] The main idea of this approach was previously described in

[2]Not to be confused with the fermion parity operator $\prod_f P_f$.

[3]A branching structure on a triangulation is an orientation for every edge such that for every face the oriented edges do not form an oriented loop. A branching structure specifies an ordering of vertices of every face: on each face $f$ there will be exactly one vertex (denoted $f_0$) with two edges of $f$ oriented away from the vertex, one vertex ($f_1$) with one edge of $f$ entering it and one leaving it, and another vertex ($f_2$) with two edges of $f$ oriented towards it.

[21]. We will review this material first and then describe a general way to perform bosonization on a 2d triangulation.

We assume again that we are given a global orientation and for an edge $e$ define $L(e)$ and $R(e)$ to be the faces to the left and to the right of $e$, just as for the square lattice. On a face $f$ we have fermionic operators $c_f, c_f^\dagger$, or equivalently a pair of Majorana fermions $\gamma_f, \gamma'_f$. They are generators of a Clifford algebra. The fermion parity on face $f$ is $P_f = -i\gamma_f\gamma'_f$. A $\mathbb{Z}_2$ fermionic hopping operator on an edge $e$ can again be defined by $S_e = i\gamma_{L(e)}\gamma'_{R(e)}$.

The even fermionic algebra is generated by $P_f$ and $S_e$ for all faces and edges. The relations between them can also be described. Obviously, these operators are $\mathbb{Z}_2$, and $S_e$ anticommutes with $P_f$ whenever $e \subset f$ and commutes with it otherwise. The operators $S_e$ and $S_{e'}$ sometimes commute and sometimes anticommute. To describe the commutation rule more precisely, it is convenient to use the cup product on $\mathbb{Z}_2$ 1-cochains. Recall that a $\mathbb{Z}_2$ $p$-cochain is a $\mathbb{Z}_2$-valued function on $p$-simplices of the triangulation. In our case, $p$ can be 0, 1, or 2, corresponding to functions on vertices, edges, and faces respectively. The cup product of two 1-cochains is a 2-cochain defined as follows:

$$(\alpha \cup \beta)(\langle 012\rangle) = \alpha(\langle 01\rangle)\beta(\langle 12\rangle).$$

Here $\alpha$ and $\beta$ are arbitrary 1-cochains with values in $\mathbb{Z}_2$, and 0, 1, and 2 are vertices of a face $\langle 012\rangle$, ordered in accordance with the branching structure. $\langle 01\rangle$ (resp. $\langle 12\rangle$) is the edge from 0 to 1 (resp. from 1 to 2). The cup product is not commutative (or supercommutative, which is the same thing since we are working modulo 2). Let $\boldsymbol{e}$ be the 1-cochain which takes value 1 on the edge $e$ and value 0 on all other edges. Then the commutation rule for $S_e$ and $S'_e$ is

$$S_e S_{e'} = (-1)^{\int \boldsymbol{e}\cup\boldsymbol{e}' + \boldsymbol{e}'\cup\boldsymbol{e}} S_{e'} S_e. \tag{2.13}$$

Here the integral of a 2-cochain is simply the sum of its values on all faces of the space manifold $M$:

$$\int \boldsymbol{e}\cup\boldsymbol{e}' + \boldsymbol{e}'\cup\boldsymbol{e} = \sum_{f\in M} (\boldsymbol{e}\cup\boldsymbol{e}' + \boldsymbol{e}'\cup\boldsymbol{e}). \tag{2.14}$$

In other words, if $e$ and $e'$ are distinct edges, $S_e$ and $S_{e'}$ anticommute if $e$ and $e'$ belong the same face $f = \langle 012\rangle$ and their union contains edges $\langle 01\rangle$ and $\langle 12\rangle$ of that face, $\{e, e'\} = \{\langle 01\rangle, \langle 12\rangle\}$. They commute otherwise.

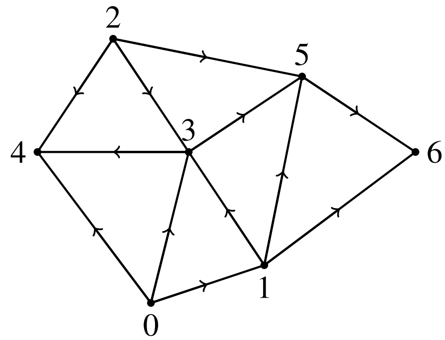



Figure 2.2: A branching structure on a general triangulation.

There is also a relation for each vertex $v$, analogous to (2.4), which reads

$$\prod_{e \supset v} S_e \sim \prod_{f \supset I_v^{02}} P_f, \tag{2.15}$$

where $I_v^{02}$ is the set of those faces $f = \langle 012 \rangle$ for which $v$ is either 0 or 2. The product of hopping terms equal to the product of fermion parity on these faces up to a sign. Its explicit form will be derived later.

To reproduce these relations in a bosonic model, we again introduce a spin variable for every edge and let $X_e, Y_e, Z_e$ be the corresponding Pauli matrices. We let

$$W_f = \prod_{e \subset f} Z_e, \tag{2.16}$$

as before. This operator measures flux through face $f$. We anticipate that the bosonic model will be a gauge theory, and thus the algebra of gauge-invariant observables will be generated by $W_f$ and $\mathbb{Z}_2$ operators $U_e$ which anticommute with $W_f$ if $e \subset f$ and commute with $W_f$ otherwise. We also anticipate that in order for $U_e$ to behave as fermion hopping operators, we must implement charge-flux attachment. Our convention will be that if $W_f = -1$ for some face $f = \langle 012 \rangle$, then electric charge will be sitting at the vertex 0. Then the flux hopping operator will take the form

$$U_e = X_e \prod_{f \in \{L(e), R(e)\}} Z_{f_{01}}^{\boldsymbol{e}(f_{12})}, \tag{2.17}$$

where $f_{ij}$ denotes the edge of $f = \langle ijk \rangle$ connecting vertices $f_i$ and $f_j$. Eq. (2.17) means that $U_e$ implements the motion of the magnetic flux across edge $e$ accompanied by the electric charge moving along edges $f_{01}$ of those faces for which $e = f_{12}$. For example, in Figure 2.2, we have $U_{35} = X_{35} Z_{23} Z_{13}$, $U_{13} = X_{13} Z_{01}$, and $U_{03} = X_{03}$.

We can simplify Eq. (2.17) using the cup product notation:

$$U_e = X_e(\prod_{e'} Z_{e'}^{\int \boldsymbol{e}'\cup\boldsymbol{e}}). \tag{2.18}$$

With this form, we can easily check that the operators $U_e$ satisfy the same commutation relations as $S_e$.

$$\begin{aligned} U_{e_1}U_{e_2} &= X_{e_1}(\prod_{e_1'} Z_{e_1'}^{\int \boldsymbol{e}_1'\cup\boldsymbol{e}_1})X_{e_2}(\prod_{e_2'} Z_{e_2'}^{\int \boldsymbol{e}_2'\cup\boldsymbol{e}_2}) \\ &= (-1)^{\int \boldsymbol{e}_2\cup\boldsymbol{e}_1+\boldsymbol{e}_1\cup\boldsymbol{e}_2} X_{e_2}(\prod_{e_2'} Z_{e_2'}^{\int \boldsymbol{e}_2'\cup\boldsymbol{e}_2})X_{e_1}(\prod_{e_1'} Z_{e_1'}^{\int \boldsymbol{e}_1'\cup\boldsymbol{e}_1}) \\ &= (-1)^{\int \boldsymbol{e}_2\cup\boldsymbol{e}_1+\boldsymbol{e}_1\cup\boldsymbol{e}_2} U_{e_2}U_{e_1}. \end{aligned} \tag{2.19}$$

One can check that if we impose a Gauss law of the form

$$\prod_{e\supset v} X_e = \prod_{f\in I^0(v)} W_f, \tag{2.20}$$

where $I^0(v)$ is the set of faces such that $v = v_0$ for that face, then $U_e$ satisfy a relation very similar to that of $S_e$:

$$\prod_{e\supset v} U_e \sim \prod_{f\supset I_v^{02}} W_f. \tag{2.21}$$

The equality holds up to a $v$-dependent sign.

To be precise about the sign, around each vertex $v$, we formulate the fermionic identity (2.15) precisely:

$$(-1)^{\int_{w_2} \boldsymbol{v}} S_{\delta\boldsymbol{v}} \prod_f P_f^{\int \boldsymbol{v}\cup\boldsymbol{f}+\boldsymbol{f}\cup\boldsymbol{v}} = 1, \tag{2.22}$$

where $\prod_f P_f^{\int \boldsymbol{v}\cup\boldsymbol{f}+\boldsymbol{f}\cup\boldsymbol{v}} = \prod_{f\supset I_v^{02}} P_f$ by the definition of the cup product and $S_{\delta\boldsymbol{v}}$ is the product of hopping terms on edges around $v$, similar to (2.15). To avoid sign ambiguity due to the ordering of the product, we defined the hopping term for 1-cochains. In general, for any 1-cochains $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$,

$$S_{\lambda+\lambda'} \equiv (-1)^{\int \lambda\cup\lambda'} S_{\lambda'}S_\lambda. \tag{2.23}$$

For example, $S_{e_1+e_2} = (-1)^{\int \boldsymbol{e}_2\cup\boldsymbol{e}_1} S_{e_1}S_{e_2} = (-1)^{\int \boldsymbol{e}_1\cup\boldsymbol{e}_2} S_{e_2}S_{e_1}$, which is independent from the ordering between $e_1$ and $e_2$. The sign is related to $w_2 \in C_0(M_2, \mathbb{Z}_2)$, which is the 0-chain (a formal sum of vertices) which is Poincaré dual to the second

Stiefel–Whitney cohomology class $\boldsymbol{w}_2(M_2)$. The explicit expression of $w_2$ is derived in Appendix B:

$$w_2 = \sum_v v + \sum_{f=\langle 012\rangle\in-\text{triangle}} \langle 1\rangle, \tag{2.24}$$

which is the formal sum (mod 2) of all vertices and vertex 1 for each "−"-oriented triangle $\langle 012\rangle$. The sign $(-1)^{\int_{w_2}\boldsymbol{v}}$ is defined by

$$\int_{w_2} \boldsymbol{v} = \boldsymbol{v}(w_2) = \begin{cases} 1, & \text{if } w_2 \text{ contains } v \\ 0, & \text{if } w_2 \text{ doesn't contain } v \end{cases} \tag{2.25}$$

The second Stiefel-Whitney class is the obstruction to a spin structure. The fermion can only be define on a manifold which admits spin structure $E \in C_1(M_2, \mathbb{Z}_2)$ such that $\partial E = w_2$.One can interpret the 1-chain $E$ as a lattice representation of a spin structure. Indeed, in the context of Riemannian geometry it is well-known that the 2nd Stiefel-Whitney class $\boldsymbol{w}_2(M) \in H^2(M, \mathbb{Z}_2)$ is an obstruction for defining a lift of the structure group of the tangent bundle from $SO(n)$ to $Spin(n)$. Thus any trivialization of this class leads to a lift of the structure group to $Spin(n)$ and enables one to define spinors. Since $E$ is a trivialization of the homology 0-cycle Poincaré-dual to $\boldsymbol{w}_2(M)$, a choice of $E$ is equivalent to a choice of a trivialization of $\boldsymbol{w}_2(M)$ and thus can be thought of as implicitly defining a spin structure. It is remarkable that although we are dealing with spinless fermions, a choice of spin structure is still required in order to construct the bosonization map.

This fermionic identity (2.22) is proved in Appendix C, including higher dimensional versions. Under the mapping $(-1)^{\int_E \boldsymbol{e}} S_e \to U_e$ and $P_f \to W_f$, it gives gauge constraints for bosonic operators:

$$\begin{aligned} G_v &= U_{\delta \boldsymbol{v}} \prod_f W_f^{\int \boldsymbol{v}\cup \boldsymbol{f} + \boldsymbol{f}\cup \boldsymbol{v}} \\ &= \prod_{e\supset v} X_e (\prod_{e'} Z_{e'}^{\int \delta\boldsymbol{v}\cup \boldsymbol{e}'}), \end{aligned} \tag{2.26}$$

where we have used the property $U_\lambda = \prod_e X_e^{\lambda(e)} \prod_{e'} Z_{e'}^{\int \boldsymbol{e}'\cup\lambda}$, which can be derived from the definition:

$$U_{\lambda+\lambda'} \equiv (-1)^{\int \lambda\cup\lambda'} U_{\lambda'} U_\lambda. \tag{2.27}$$

Therefore, our complete bosonization map is

$$
\begin{aligned}
W_f = \prod_{e \subset f} Z_e &\longleftrightarrow P_f = -i\gamma_f \gamma'_f, \\
U_e = X_e (\prod_{e'} Z_{e'}^{\int \boldsymbol{e}' \cup \boldsymbol{e}}) &\longleftrightarrow (-1)^{\int_E \boldsymbol{e}} S_e = (-1)^{\int_E \boldsymbol{e}} i\gamma_{L(e)}\gamma'_{R(e)}, \\
G_v = \prod_{e \supset v} X_e (\prod_{e'} Z_{e'}^{\int \delta \boldsymbol{v} \cup \boldsymbol{e}'}) &\longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{v}} S_{\delta \boldsymbol{v}} \prod_f P_f^{\int \boldsymbol{v} \cup \boldsymbol{f} + \boldsymbol{f} \cup \boldsymbol{v}} = 1, \\
\prod_f W_f = 1 &\longleftrightarrow \prod_f P_f
\end{aligned}
\tag{2.28}
$$

On the bosonic side, the gauge constraint $G_v = 1$ is required. On the fermionic side, total parity must be even.

## 2.3 Examples

### Spinless fermion on a square lattice

As a first example of the bosonization map, consider the theory of complex fermions on a square lattice with nearest-neighbor hopping and an on-site chemical potential $\mu$. The Hamiltonian is

$$
H = t \sum_e (c^\dagger_{L(e)} c_{R(e)} + c^\dagger_{R(e)} c_{L(e)}) + \mu \sum_f c^\dagger_f c_f. \tag{2.29}
$$

To apply our bosonization procedure, we first express (2.29) in terms of Majorana operators,

$$
\begin{aligned}
H &= \frac{t}{2} \sum_e (i\gamma_{L(e)}\gamma'_{R(e)} + i\gamma_{R(e)}\gamma'_{L(e)}) + \frac{\mu}{2} \sum_f (1 + i\gamma_f \gamma'_f) \\
&= \frac{t}{2} \sum_e \Big( i\gamma_{L(e)}\gamma'_{R(e)} - i(\gamma_{L(e)}\gamma'_{R(e)})(-i\gamma_{L(e)}\gamma'_{L(e)})(-i\gamma_{R(e)}\gamma'_{R(e)}) \Big) \\
&\quad + \frac{\mu}{2} \sum_f (1 + i\gamma_f \gamma'_f).
\end{aligned}
\tag{2.30}
$$

The bosonized Hamiltonian that follows from (3.2) and (3.3) is a $\mathbb{Z}_2$ gauge theory with Hamiltonian

$$
H = \frac{t}{2} \sum_e X_e Z_{r(e)} (1 - W_{L(e)} W_{R(e)}) + \frac{\mu}{2} \sum_f (1 - W_f) \tag{2.31}
$$

and a gauge constraint $(\prod_{e \supset v} X_e) W_{\text{NE}(v)} = 1$ on each vertex.

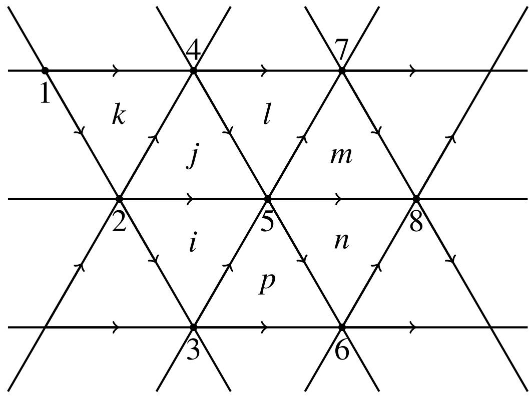



Figure 2.3: A branching structure on triangular lattice.

**Spinless fermion on a honeycomb lattice**

Next, consider fermions living on the faces of a triangular lattice (or on the vertices of a honeycomb lattice), shown on Fig. 2.3. We consider again the nearest neighbor hopping Hamiltonian

$$H = t\sum_{e}(c^\dagger_{L(e)}c_{R(e)} + c^\dagger_{R(e)}c_{L(e)}) + \mu\sum_{f} c^\dagger_f c_f. \tag{2.32}$$

The hopping operators map as

$$S_e = i\gamma_{L(e)}\gamma'_{R(e)} \longleftrightarrow (-1)^{\int_E e}U_e \tag{2.33}$$

for a suitably chosen sign $E$. The sign is chosen so that the vertex relations between $S_e$ and $U_e$ are identical. For the branching structure shown in Fig. 2.3 one can choose $E = 0$ ($E$ contains no edge at all), so that the bosonization map is simply

$$S_e \longleftrightarrow U_e. \tag{2.34}$$

With this choice, for the explicitly denoted edges on Fig. 2.3 the operators $U_e$ defined by (2.17) are

$$\begin{aligned} U_{58} &= X_{58}, \\ U_{57} &= X_{57}Z_{45}, \\ U_{56} &= X_{56}Z_{35}, \end{aligned} \tag{2.35}$$

and other edges are defined by translation. The bosonized Hamiltonian is

$$H = \frac{t}{2}\sum_e U_e(1 - W_{L(e)}W_{R(e)}) + \frac{\mu}{2}\sum_f (1 - W_f) \tag{2.36}$$

with gauge constraint $(\prod_{e\supset v} X_e)W_{\mathrm{NE}(v)}W_{\mathrm{SE}(v)} = 1$ (i.e. $(\prod_{e\supset v_5} X_e)W_m W_n = 1$) on each vertex.

It is well known that on the honeycomb lattice the Hamiltonian (2.32) gives rise to a dispersion law which has two Dirac points in the Brillouin zone. This is highly non-obvious for the equivalent gauge theory Hamiltonian (2.36).

Any fermionic operator with vanishing net fermion parity can be written in terms of $S_e$ and $P_f$ and thus have a bosonic counterpart. We start from simple examples. To bosonize $c_k^\dagger c_l$ from Fig. 2.3, we express it via Majorana operators as

$$c_k^\dagger c_l = \frac{1}{4}(\gamma_k\gamma_l + \gamma_k'\gamma_l' + i\gamma_k\gamma_l' + i\gamma_l\gamma_k'), \tag{2.37}$$

and then map these Majorana operators in the usual way,

$$\begin{aligned}
\gamma_k\gamma_l &= (i\gamma_k\gamma_j')(i\gamma_l\gamma_j') \longleftrightarrow U_{24}U_{45},\\
\gamma_k\gamma_l' &= i(\gamma_k\gamma_l)(-i\gamma_l\gamma_l') \longleftrightarrow iU_{24}U_{45}W_l,\\
\gamma_l\gamma_k' &= (-i)(\gamma_k\gamma_l)(-i\gamma_k\gamma_k') \longleftrightarrow -iU_{24}U_{45}W_k,\\
\gamma_k'\gamma_l' &= (-i)(\gamma_l\gamma_k')(-i\gamma_l\gamma_l') \longleftrightarrow -U_{24}U_{45}W_kW_l.
\end{aligned} \tag{2.38}$$

This way we obtain

$$c_k^\dagger c_l = \frac{1}{4}U_{24}U_{45}(1+W_k)(1-W_l). \tag{2.39}$$

Next, consider the operator $c_i^\dagger c_l = \frac{1}{4}(\gamma_i\gamma_l + \gamma_i'\gamma_l' + i\gamma_i\gamma_l' + i\gamma_i\gamma_l')$. Its first term is

$$\gamma_i\gamma_l = (i\gamma_j\gamma_i')(i\gamma_l\gamma_j')(-i\gamma_j\gamma_j')(-i\gamma_i\gamma_i')) \longleftrightarrow U_{25}U_{45}W_jW_i, \tag{2.40}$$

and the other terms can be computed the same way, giving

$$\begin{aligned}
c_i^\dagger c_l &= \frac{1}{4}U_{25}U_{45}W_jW_i(1+W_i)(1-W_l)\\
&= \frac{1}{4}U_{25}U_{45}W_j(1+W_i)(1-W_l).
\end{aligned} \tag{2.41}$$

Generalizing from (2.39) and (2.41), the rule for bosonization of a fermion bilinear $c_a^\dagger c_b$ can be stated as follows. First choose an arbitrary path from face $a$ to face $b$. Start with $(1+W_a)(1-W_b)/4$, and follow the path. When the path passes through a face $f$ by crossing two edges with different orientations, we need to multiply by $W_f$. Then, for each edge $e$ the path crosses, we multiply by $U_e$. For example, following the path $m \to l \to k \to i$, we can write down

$$c_i^\dagger c_m = \frac{1}{4}U_{25}U_{45}U_{57}W_j(1+W_i)(1-W_m). \tag{2.42}$$

If we use another path $m \to n \to p \to i$, it becomes

$$c_i^\dagger c_m = \frac{1}{4}U_{35}U_{56}U_{58}W_n(1+W_i)(1-W_m). \tag{2.43}$$

The above two formulas only differ by a gauge transformation.

**The Hubbard model on a square lattice**

The Hamiltonian of Hubbard model (with fermions on faces of a square lattice) is

$$H = t\sum_{e,\sigma}(c^\dagger_{L(e),\sigma}c_{R(e),\sigma} + h.c.) + U\sum_f n_{f\uparrow}n_{f\downarrow}, \tag{2.44}$$

where $\sigma \in \{\uparrow,\downarrow\}$ and $n_f = c^\dagger_f c_f$. It can be viewed as two copies of the nearest neighbor hopping Hamiltonian with an interaction on each face. Similar to (2.36), the bosonized system is a $\mathbb{Z}_2\times\mathbb{Z}_2$ gauge theory on the dual lattice with a Hamiltonian

$$H = \frac{t}{2}\sum_{e,\sigma} X^\sigma_e Z^\sigma_{r(e)}(1 - W^\sigma_{L(e)}W^\sigma_{R(e)}) + \frac{U}{4}\sum_f (1 - W^\uparrow_f)(1 - W^\downarrow_f) \tag{2.45}$$

with gauge constraints $(\prod_{e\supset v} X^\sigma_e)W^\sigma_{\mathrm{NE}(v)}W^\sigma_{\mathrm{SE}(v)} = 1$ for $\sigma =\uparrow,\downarrow$ at each vertex. On each edge, there are two species of spins labeled by $\uparrow$ and $\downarrow$.

Note that the $SU(2)$ spin symmetry is not manifest in this bosonized description. There exists a version of our bosonization procedure where the $SU(2)$ symmetry is manifest. In that description, one of the $\mathbb{Z}_2$ gauge fields is replaced with a bosonic spin which lives on the vertices of the dual lattice. The $SU(2)$ symmetry acts only on this spin variable.

**Some soluble 2+1D lattice gauge theories**

We have seen a couple of examples where a simple theory of free fermions on a lattice can be rewritten as a rather complicated $\mathbb{Z}_2$ lattice gauge theory on the dual lattice. Conversely, one can start with some simple $\mathbb{Z}_2$ gauge theory and ask if it can be rewritten as a theory of free fermions.

The standard 2+1D $\mathbb{Z}_2$ lattice gauge theory introduced by F. Wegner [51] can be written in the Hamiltonian form as follows [52]. There is a spin on every edge $e$, with Pauli matrices $X_e, Y_e, Z_e$. The physical Hilbert space is a subspace of the tensor product space defined by the Gauss law constraints

$$\prod_{e\supset v} X_e = 1, \quad \forall v. \tag{2.46}$$

The Hamiltonian is

$$H = g^2\sum_e X_e + \frac{1}{g^2}\sum_f W_f, \tag{2.47}$$

where $W_f$ is given by Eq. (2.16), as usual. This theory is not integrable and is related by Kramers-Wannier duality to the 3D Ising model.

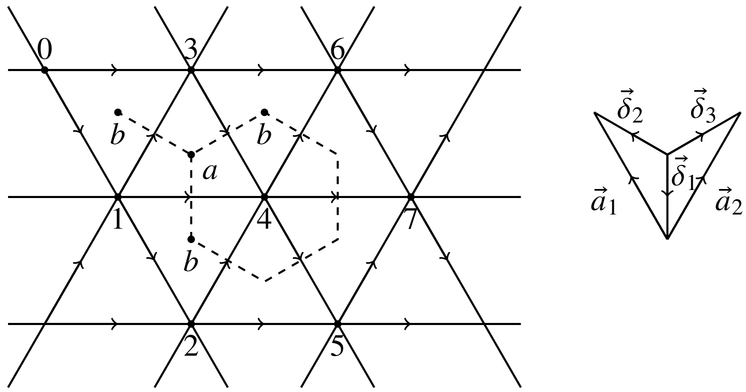



Figure 2.4: Fermions at the center of faces form a honeycomb lattice. The vectors are defined as $\vec{\delta}_1 = (0, -\frac{\sqrt{3}}{3})$, $\vec{\delta}_2 = (-\frac{1}{2}, \frac{\sqrt{3}}{6})$, $\vec{\delta}_3 = (\frac{1}{2}, \frac{\sqrt{3}}{6})$ and $\vec{a}_1 = \vec{\delta}_2 - \vec{\delta}_1 = (-\frac{1}{2}, \frac{\sqrt{3}}{2})$, $\vec{a}_2 = \vec{\delta}_3 - \vec{\delta}_1 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$.

To get a $\mathbb{Z}_2$ gauge theory which is dual to a fermionic theory, we need to replace Eq. (2.46) with the modified Gauss law (2.12) on a square lattice, or with (2.20) on a general triangulation. The second (potential) term in Eq. (2.47) is still gauge-invariant, but the first (kinetic) term is not. To fix this problem we simply replace each $X_e$ with $U_e = X_e Z_{r(e)}$, which is gauge-invariant by construction, and let

$$H' = g^2 \sum_e U_e + \frac{1}{g^2} \sum_f W_f. \tag{2.48}$$

Since $W_f$ maps to $-i\gamma_f \gamma'_f$, and $U_e$ maps to $i\gamma_{L(e)}\gamma'_{R(e)}$, the fermionic dual of this gauge theory is a theory of free fermions.

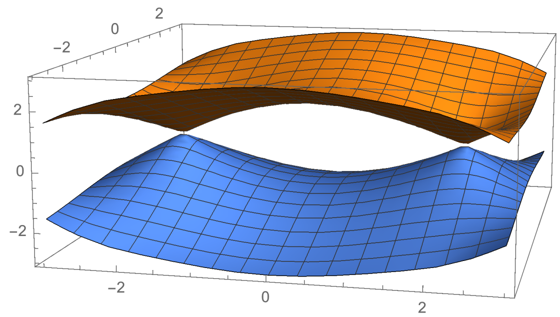
To analyze this fermionic theory in more detail, let us specialize to the case of a regular triangular lattice (Fig. 2.4), so that fermions live on the vertices of a regular honeycomb lattice. By bosonization map (2.33), the Hamiltonian (2.48) is (up to a constant) equivalent to

$$H'_f = t \sum_e (c_{L(e)} c_{R(e)} - c^\dagger_{L(e)} c^\dagger_{R(e)} + c^\dagger_{L(e)} c_{R(e)} + c^\dagger_{R(e)} c_{L(e)}) + \mu \sum_f c^\dagger_f c_f, \tag{2.49}$$

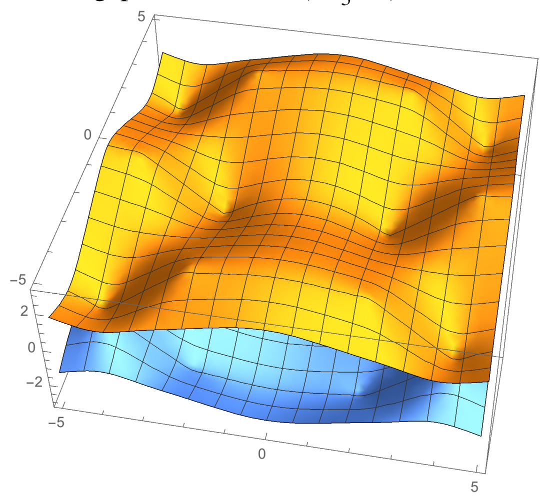
where $t = g^2$ and $\mu = 2/g^2$. After the usual Fourier transform $c_{\vec{x}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} c_{\vec{k}}$, the Hamiltonian becomes

$$H'_f = \sum_{\vec{k}} (\Delta_k \, c_{\vec{k},a} c_{-\vec{k},b} + \text{h.c.}) + \sum_{\vec{k}} (\epsilon_{\vec{k}} \, c^\dagger_{\vec{k},a} c_{\vec{k},b} + \text{h.c.}) + \mu \sum_{\vec{k}} (c^\dagger_{\vec{k},a} c_{\vec{k},a} + c^\dagger_{\vec{k},b} c_{\vec{k},b}),$$

where $\Delta_{\vec{k}} = t(e^{-i\vec{k}\cdot\vec{\delta}_1} - e^{-i\vec{k}\cdot\vec{\delta}_2} - e^{-i\vec{k}\cdot\vec{\delta}_3})$ and $\epsilon_{\vec{k}} = t(e^{i\vec{k}\cdot\vec{\delta}_1} + e^{i\vec{k}\cdot\vec{\delta}_2} + e^{i\vec{k}\cdot\vec{\delta}_3})$. We can




(a) For $\mu/t = 2$, the band gap closes at $\vec{k} = (\pm\frac{2\pi}{3}, 0)$, which form two Dirac cones.




(b) Another viewpoint from the top. Two Dirac cones lie in the first Brillouin zone (the hexagon).

Figure 2.5: (Color online) Band structure of $H_{\text{BdG}}$(equivalent to $H'_f$).

write this using the Bogoliubov-de-Gennes (BdG) formalism as

$$H'_f = \frac{1}{2}\sum_{\vec{k}} \Psi^{\dagger}_{\vec{k}} H_{\text{BDG}}(\vec{k})\Psi_{\vec{k}} \tag{2.50}$$

with

$$H_{\text{BDG}}(\vec{k}) = \begin{bmatrix} \mu & -\Delta^*_{\vec{k}} & \epsilon_{\vec{k}} & 0 \\ -\Delta_{\vec{k}} & -\mu & 0 & -\epsilon_{\vec{k}} \\ \epsilon^*_{\vec{k}} & 0 & \mu & -\Delta^*_{\vec{k}} \\ 0 & -\epsilon^*_{\vec{k}} & -\Delta_{\vec{k}} & -\mu \end{bmatrix}, \quad \Psi_{\vec{k}} = \begin{bmatrix} c_{\vec{k},a} \\ c^{\dagger}_{-\vec{k},b} \\ c_{\vec{k},b} \\ c^{\dagger}_{-\vec{k},a} \end{bmatrix}. \tag{2.51}$$

The eigenvalues are $E(\vec{k}) = \pm\sqrt{|\Delta_{\vec{k}}|^2 + (|\epsilon_{\vec{k}}| + \mu)^2}, \pm\sqrt{|\Delta_{\vec{k}}|^2 + (|\epsilon_{\vec{k}}| - \mu)^2}$. The gap closes at $k = (\pm\frac{2\pi}{3}, 0)$ and $\mu/t = 2$ $(g = 1)$. The spectrum is shown in Fig. 2.5.

## 2.4 Euclidean 3D gauge theories and their fermionic duals

In this section, we will derive the Euclidean 3D actions for gauge theories dual to some fermionic systems. Before considering the nearest neighbor hopping Hamiltonian (2.30), let us first look at the simpler Majorana hopping Hamiltonian on the square lattice,

$$H = -A \sum_e i\gamma_{L(e)}\gamma'_{R(e)} - B\sum_f (-i\gamma_f\gamma'_f), \tag{2.52}$$

whose bosonic dual is a gauge theory with a Hamiltonian

$$H = -A \sum_e X_e Z_{r(e)} - B \sum_f W_f. \tag{2.53}$$

Without loss of generality, we can assume $A > 0$. The Gauss law constraint is

$$G_v \equiv \left(\prod_{e\supset v} X_e\right) \prod_{e'\subset \mathrm{NE}(v)} Z_{e'} = 1. \tag{2.54}$$

The partition function is

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, T^M, \tag{2.55}$$

where $T$ is the transfer matrix defined as

$$T = \left(\prod_v \delta_{G'_v,1}\right) e^{-\delta\tau H}. \tag{2.56}$$

The prime on $G_v$ means that it acts to bra on the left, which will be clear in later calculations. The first factor projects to the gauge-invariant sector of the Hilbert space. We can rewrite it using a $\mathbb{Z}_2$ Lagrange multiplier field $\lambda_v$ as

$$\delta_{G'_v,1} = \frac{1}{2}(1+G'_v) = \frac{1}{2}\sum_{\lambda_v=\pm1} (-1)^{\frac{1-\lambda_v}{2}\sum_{e\supset v}\frac{1-X_e}{2}} (-1)^{\frac{1-\lambda_v}{2}\sum_{e'\subset\mathrm{NE}(v)}\frac{1-Z_{e'}}{2}}. \tag{2.57}$$

Let us define $|m(\tau)\rangle = |\{S_e\}\rangle$ as the configuration of spins (in the $Z_e$ basis). To evaluate the matrix element $\langle m'(\tau+\delta\tau)|T|m(\tau)\rangle$, we insert the "decomposition of unity" in terms of a full basis of $X_e$ (momentum) eigenstates, using the identity

$$\langle S^{z\prime}|f(\sigma^x,\sigma^z)|S^z\rangle = \frac{1}{2}\sum_{S^x=\pm1} f(S^x,S^z)(-1)^{\frac{1-S^x}{4}(2-S^{z\prime}-S^z)}, \tag{2.58}$$

where we assume $\sigma^x$ is always left to $\sigma^z$ in $f(\sigma^x,\sigma^z)$. The matrix element is

$$\begin{aligned}
&\langle m'(\tau+\delta\tau)|T|m(\tau)\rangle \\
&\propto \sum_{\{\lambda_v\}}\left[\prod_v (-1)^{\frac{1-\lambda_v}{4}\sum_{e'\subset\mathrm{NE}(v)}(1-S^{z\prime}_{e'})}\, e^{B\delta\tau\prod_{e'\subset\mathrm{NE}(v)} S^z_{e'}}\right]\times \\
&\quad\times\left[\prod_e \sum_{S^x_e=\pm1}(-1)^{\frac{1-S^x_e}{4}[2-S^{z\prime}_e-S^z_e+\sum_{v'\subset e}(1-\lambda_{v'})]}\, e^{A\delta\tau S^x_e S^z_{r(e)}}\right].
\end{aligned} \tag{2.59}$$

The next order of business is to integrate out the intermediate momentum fields, i.e. to perform the sum over the $S^x$ in the second bracket. This bracket equals $\prod_e (e^{A\delta\tau S^z_{r(e)}} + e^{-A\delta\tau S^z_{r(e)}} S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'})$. To simplify it, we need to consider two cases: $S^z_{r(e)} = 1$ and $S^z_{r(e)} = -1$. First, for $S^z_{r(e)} = 1$, we can simply write

$$\begin{aligned} e^{A\delta\tau S^z_{r(e)}} + e^{-A\delta\tau S^z_{r(e)}} S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'} &= e^{A\delta\tau} + e^{-A\delta\tau} S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'} \\ &= C e^{J S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'}}, \end{aligned} \tag{2.60}$$

where $C^2 = 2\sinh(2A\delta\tau)$ and $\tanh J = e^{-2A\delta\tau}$. For the other case $S^z_{r(e)} = -1$,

$$\begin{aligned} & e^{A\delta\tau S^z_{r(e)}} + e^{-A\delta\tau S^z_{r(e)}} S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'} \\ &= C e^{J S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'}} (-1)^{\frac{1}{2}[2 - S^z_e{}' - S^z_e + \sum_{v' \subset e}(1-\lambda_{v'})]}. \end{aligned} \tag{2.61}$$

We can combine (2.60) and (2.61) into the single equation

$$\begin{aligned} & e^{A\delta\tau S^z_{r(e)}} + e^{-A\delta\tau S^z_{r(e)}} S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'} \\ &= C e^{J S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'}} \ (-1)^{\frac{1}{4}[2 - S^z_e{}' - S^z_e + \sum_{v' \subset e}(1-\lambda_{v'})](1 - S^z_{r(e)})}. \end{aligned} \tag{2.62}$$

We can now substitute (2.62) back to (3.51) and write the matrix element in the suggestive form

$$\begin{aligned} & \langle m'(\tau + \delta\tau)|T|m(\tau)\rangle \\ & \propto \sum_{\{\lambda_v\}} \prod_{v,e} e^{K \prod_{e' \subset \mathrm{NE}(v)} S^z_{e'} + J S^z_e{}' S^z_e \prod_{v' \subset e} \lambda_{v'}} \ (-1)^{\frac{1}{2}(1-\lambda_v) \sum_{e' \subset \mathrm{NE}(v)} \frac{1}{2}(1 - S^z_{e'}{}')} \\ & \quad \times (-1)^{\frac{1}{2}(1 - S^z_{r(e)}) \left[ \frac{1}{2}(1 - S^z_e{}') + \frac{1}{2}(1 - S^z_e) + \sum_{v' \subset e} \frac{1}{2}(1-\lambda_{v'}) \right]}, \end{aligned} \tag{2.63}$$

where $K \equiv B\delta\tau$.

We can interpret $\lambda_v$ as gauge fields on temporal links. Therefore, the first, exponential term can be thought of as the exponential of the anisotropic Wegner action [51, 52]

$$\sum_f J_f \prod_{e \supset f} S_e, \tag{2.64}$$

where $J_f$ is different for spatial and temporal faces of the 3d lattice. The rest can be thought of as a topological factor in the partition function which gives the correct anomaly factors of $-1$ for fermionic statistics. Let $a_i \in C^0(L, \mathbb{Z}_2)$ be the 0-cochain on the $i$-th layer with value $\frac{1}{2}(1 - \lambda_v)$ on vertex $v$. We regard it as the $\mathbb{Z}_2$ gauge field on temporal links between the $i$-th and $(i+1)$-th layers. Let $\alpha_i \in C^1(L, \mathbb{Z}_2)$ be the

1-form on the $i$-th layer that represents the values of $S^z_e$ on the $i$-th layer. Then we can express the "topological" factors in the last line of (2.63) as

$$\begin{aligned}&\prod_{v,e}(-1)^{\frac{1-\lambda_v}{2}\sum_{e'\subset \mathrm{NE}(v)}\frac{1}{2}(1-S^{z}_{e'}{}')}(-1)^{\frac{1}{2}(1-S^z_{r(e)})[\frac{1}{2}(1-S^{z}_{e}{}')+\frac{1}{2}(1-S^z_e)+\sum_{v'\subset e}\frac{1}{2}(1-\lambda_{v'})]}\\ &\quad\equiv(-1)^{a_i(\Delta(\delta\alpha_{i+1}))}(-1)^{\alpha_i(\Delta(\alpha_{i+1}+\alpha_i+\delta a_i))},\end{aligned} \tag{2.65}$$

where $\Delta(x)$ is the Poincaré dual of $x$ at relative position $(-\frac{1}{2},-\frac{1}{2})$ (i.e. $\Delta(\delta_{\mathrm{NE}(v)})=v$ and $\Delta(\delta_e)=r(e)$). This expression is invariant (up to boundary terms) under the gauge transformation $a_i\to a_i+f_i+f_{i+1}$ and $\alpha_i\to\alpha_i+\delta f_i$.

If we put the Hamiltonian (2.52) on a general triangulation instead of a square lattice, its bosonic dual is

$$H=-A\sum_e U_e-B\sum_f W_f. \tag{2.66}$$

Its partition function is

$$Z=\sum_{S^z,\lambda}e^{-S_{\mathrm{topo}}}e^{K\sum_{f_s}S^z_{f_s}+J\sum_{f_\tau}S^z_{f_\tau}}, \tag{2.67}$$

where $f_s$ and $f_\tau$ are faces of spatial and temporal types, $S^z_f\equiv\prod_{e\subset f}S^z_e$, and

$$e^{-S_{\mathrm{topo}}}=(-1)^{\sum_i[\int a_i\cup\delta\alpha_{i+1}+\int\alpha_i\cup(\alpha_i+\alpha_{i+1}+\delta a_i)]}. \tag{2.68}$$

Notice that (2.68) is analogous to Chern-Simon action on a general triangulation of 3d manifold

$$S_{\mathrm{CS}}=i\pi\int a\cup\delta a. \tag{2.69}$$

This kind of topological term results in charge-flux attachment and generates fermionic degrees of freedom.

Now, let us go back to the usual fermionic hopping Hamiltonian on a general triangulation

$$H=-2A\sum_e(c^\dagger_{L(e)}c_{R(e)}+c^\dagger_{R(e)}c_{L(e)})+2B\sum_f c^\dagger_f c_f, \tag{2.70}$$

and its bosonic dual (up to some constant)

$$H=-A\sum_e U_e(1-W_{L(e)}W_{R(e)})-B\sum_f W_f \tag{2.71}$$

with gauge constraints on vertices $(\prod_{e\supset v}X_e)(\prod_f W_f^{\int\delta_v\cup\delta_f})=1$. The only difference from the Majorana hopping Hamiltonian is the factor $(1-W_{L(e)}W_{R(e)})$. With some careful calculations, one can show the partition function is

$$
\begin{aligned}
Z = \sum_{S^z,\lambda} & e^{-S_{\text{topo}}} e^{K \sum_{f_S} \prod_{e\subset f_S} S^z_e + J \sum_{f_\tau} \prod_{e'\subset f_\tau} S^z_{e'}} \\
& \times e^{-\frac{J-\ln 2}{2} \sum_{e_S} [1+(\prod_{e\subset L(e_S)} S^z_e)(\prod_{e\subset R(e_S)} S^z_e)]} \\
& \times e^{-l \sum_{e_S} [1+(\prod_{e\subset L(e_S)} S^z_e)(\prod_{e\subset R(e_S)} S^z_e)](1-S^z_{e_S}{}' S^z_{e_S} \prod_{v\subset e_S} \lambda_v)},
\end{aligned} \tag{2.72}
$$

where $l$ is taken to be infinity and $e_s$ is a edge on a spatial slice. Taking $l \to \infty$ imposes additional gauge constraints to the previous lattice gauge theory (2.67), and the topological term is not affected.

*Chapter 3*

# BOSONIZATION ON THREE-DIMENSIONAL LATTICES

## 3.1 Cubic lattices

Next, we introduce our bosonization method on an infinite 3d cubic lattice. Suppose that we have a model with fermions living at the centers of cubes. Let us describe the generators and relations in the algebra of local observables with trivial fermion parity (the even fermionic algebra for short).

On each cube $t$ we have a single fermionic creation operator $c_t$ and a single fermionic annihilation operator $c_t^\dagger$ with the usual anticommutation relations. The fermionic parity operator on cube $t$ is $P_t = (-1)^{c_t^\dagger c_t}$. All operators $P_t$ commute with each other. We work in the Majorana basis

$$\gamma_t = c_t + c_t^\dagger, \quad \gamma'_t = (c_t - c_t^\dagger)/i. \tag{3.1}$$

The even fermionic algebra is generated by $i\gamma_{L(f)}\gamma'_{R(f)}$ and $-i\gamma_t\gamma'_t$ where each face is assigned an orientation from cube $L(f)$ to cube $R(f)$.

To illustrate the definition of these operators, we draw the dual lattice of the original lattice. In Fig. 3.1, fermions live on vertices and the orientations of each dual edge (face of the original lattice) are taken to be along $+x$, $+y$, and $+z$ directions. The Majorana hopping operator is defined by $S_f = i\gamma_{L(f)}\gamma'_{R(f)}$, where $L(f)$ and $R(f)$ are source and sink (starting and ending points) of dual edge $f$ in the dual lattice. $S_{f_i}$ and $S_{f_j}$ anti-commute only when both dual edges $f_i$ and $f_j$ start from the same point or both end at the same point.

The dual bosonic system has $\mathbb{Z}_2$ spins living on faces of the original lattice (or equivalently, on edges of the dual lattice). To define bosonic hopping operators $U_f$, we need to choose a framing for each edge of the dual lattice, i.e. a small shift of each dual edge along some orthogonal direction. We also assume that when projected on some generic plane (such as the plane of the page) a shifted dual edge intersects all dual edges transversally. For example, in Fig. 3.1 such a framing is indicated by red, green, and blue lines (for dual edges along $x$, $y$ and $z$ directions, respectively), and the shift of the dual edge 1 intersects dual edges 3 and 4 [1]. Now

[1]There are many choices of framing, and accordingly many versions of the bosonization map. By construction, they are related by automorphisms of the algebra of observables.

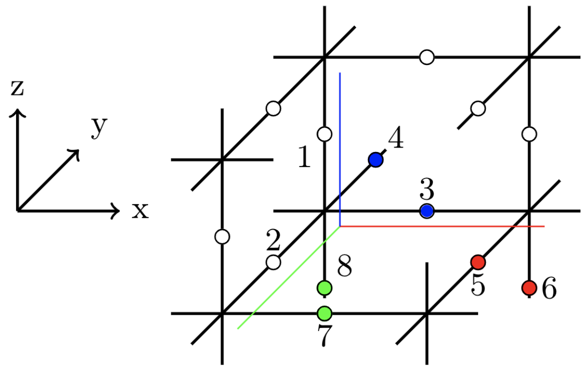



Figure 3.1: (Color online) For edges in the dual lattice, the "framing" is defined by green, red, and blue edges, which is a small shift of duel edges [53]. Given a dual edge $f$, the operator $U_f$ is defined as $X_f$ times $Z_{f'}$ for those $f'$ which intersect the framing of $f$ when projected to the plane (i.e. $U_{f_1} = X_1 Z_3 Z_4$, $U_{f_2} = X_2 Z_7 Z_8$, and $U_{f_3} = X_3 Z_5 Z_6$ ).

we define $U_f$ as a product of $X_f$ with all $Z_{f'}$ such that $f'$ intersects the framing of $f$ when projected to the plane of the page. For example, the hopping operator for the dual edge 1 is $U_1 = X_1 Z_3 Z_4$. Notice that $U_1$, $U_3$, and $U_4$ anti-commute with each other and $U_3$, $U_5$, and $U_6$ anti-commute with each other, while $U_2$ and $U_3$ commute, and $U_1$ and $U_8$ commute.

One can check that $S_f$ and $U_f$ have the same commutation relations. Therefore, the bosonization map in 3D can be defined as follows:

1. For any cube $t$ let $W_t \equiv \prod_{f\subset t} Z_f$. We identify the fermionic states $|P_t = 1\rangle$ and $|P_t = -1\rangle$ with bosonic states for which $W_t = 1$ and $W_t = -1$, respectively. Thus
$$P_t = -i\gamma_t\gamma_t' \longleftrightarrow W_t. \tag{3.2}$$

2. The fermionic hopping operator $S_f$ is identified with $U_f$ defined above:
$$S_f = i\gamma_{L(f)}\gamma'_{R(f)} \longleftrightarrow U_f. \tag{3.3}$$

As in 2d, the bosonic operators satisfy some constraints. In Fig. 3.2, we calculate the product of $S_f$ around the red square on the dual lattice:

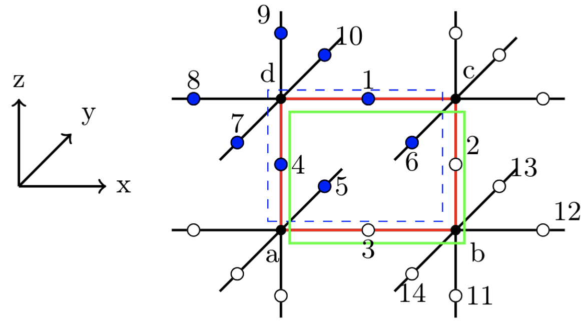



Figure 3.2: (Color online) The framing of the hopping term defined previously is indicated by the green square, while the gauge constraint involves the $Z$ operators in the opposite framing (blue dashed square).

$$
\begin{aligned}
&S_{f_1}S_{f_2}S_{f_3}S_{f_4} \\
=&(i\gamma_d\gamma'_c)(i\gamma_b\gamma'_c)(i\gamma_a\gamma'_b)(i\gamma_a\gamma'_d) \\
=&-(-i\gamma_b\gamma'_b)(-i\gamma_d\gamma'_d) \\
=&-P_bP_d \longleftrightarrow -W_bW_d.
\end{aligned}
\tag{3.4}
$$

Its bosonic dual defined by (3.3) is the product of the corresponding operators $U_f$. Their definition involves a framing of the red square given by the green square:

$$
\begin{aligned}
&U_{f_1}U_{f_2}U_{f_3}U_{f_4} \\
=&(X_1Z_2Z_6)(X_2Z_{12}Z_{13})(X_3Z_{11}Z_{14})(X_4Z_3Z_5) \\
=&-X_1X_2X_3X_4Z_5Z_6(Z_2Z_3Z_{11}Z_{12}Z_{13}Z_{14}) \\
=&-X_1X_2X_3X_4Z_5Z_6W_b.
\end{aligned}
\tag{3.5}
$$

Comparing (3.4) and (3.5), we get the constraint

$$
\begin{aligned}
1 =&X_1X_2X_3X_4Z_5Z_6W_d \\
=&X_1X_2X_3X_4Z_1Z_4Z_5Z_6Z_7Z_8Z_9Z_{10}
\end{aligned}
\tag{3.6}
$$

The operators $Z$'s are the edges crossed by dashed square in Fig. 3.2. The framing for gauge constraints is opposite to the framing used to define hopping operators. We have a gauge constraint for each face of dual lattice. In terms of the original lattice, there is one gauge constraint for each edge. All these constraints commute and thus define a $\mathbb{Z}_2$ 2-form gauge theory with an unusual Gauss law.

### 3.2 Triangulation

The bosonization method described above also works for any triangulation. For an arbitrary triangulation $T$ of a 3d manifold $M_3$, we choose a branching structure.

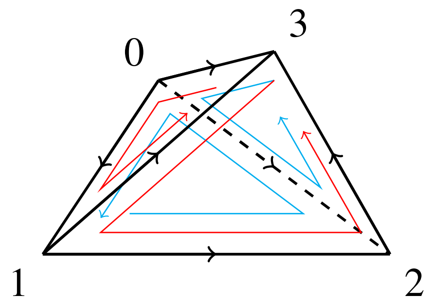



Figure 3.3: (Color online) A branching structure on a tetrahedron. The orientation of each face is determined by the right-hand rule. We defined this as the "+" tetrahedron, the directions of faces 012 and 023 are inward (blue) while the directions of faces 123 and 013 are outward (red). The directions of faces are reversed in the "−" tetrahedron (mirror image of this tetrahedron).

A branching structure is a choice of an orientation on each edge such that there is no oriented loop on any triangle. One simple way is to label vertices by different numbers and assign the direction of an edge from the vertex with smaller number to the vertex with larger number (see Fig. 3.3). Each tetrahedron has two inward faces and two outward faces (by right-hand rule). We place fermions at the centers of tetrahedra. Each tetrahedron $t$ contains Majorana operators $\gamma_t$ and $\gamma'_t$. We define the fermionic hopping operator on each face $f$ as

$$S_f = i\gamma_{L(f)}\gamma'_{R(f)}, \tag{3.7}$$

where $L(f)/R(f)$ is the tetrahedron with $f$ as a outward/inward face. Notice that $S_f$ and $S_{f'}$ anti-commute only when $f$ and $f'$ share a tetrahedron with both $f$ and $f'$ inward or outward. To express this property mathematically, we introduce (higher) cup product used in algebraic topology. The definition and properties of the (higher) cup products are described in Section A. If $\beta_1$ and $\beta_2$ are 2-cochains, then

$$\beta_1 \cup_1 \beta_2(0123) = \beta_1(023)\beta_2(012) + \beta_1(013)\beta_2(123). \tag{3.8}$$

Therefore, the commutation relations can be expressed as

$$S_f S_{f'} = (-1)^{\int \boldsymbol{f}\cup_1 \boldsymbol{f}' + \boldsymbol{f}'\cup_1 \boldsymbol{f}} S_{f'} S_f, \tag{3.9}$$

where the notation $\boldsymbol{f} \in C^2(T, \mathbb{Z}_2)$ denotes the 2-cochain with value 1 on face $f$ and 0 otherwise, and the integral represents the sum over all tetrahedra. The even fermionic algebra is generated by the operators $S_f$ for all faces and the fermionic parity operators $P_t$ for all tetrahedra.

The dual bosonic variables are $\mathbb{Z}_2$ spins which live on faces of the triangulation. As before, the flux operator

$$W_t = \prod_{f \supset t} X_f$$

corresponds to $P_t$ under the bosonization map.

Next we need to find bosonic operators $U_f$ which have the same commutation relation as fermionic operators $S_f$. We should define $U_f$ as $X_f$ times $Z_{f'}$ for some faces $f'$ which share a tetrahedron with $f$ and have the same orientation with respect to the tetrahedron. One way to define $U_f$ is

$$U_f = X_f \prod_{t \in \{L(f), R(f)\}} Z_{t_{023}}^{\boldsymbol{f}(t_{012})} Z_{t_{013}}^{\boldsymbol{f}(t_{123})} = X_f \prod_{f'} Z_{f'}^{\int \boldsymbol{f}' \cup_1 \boldsymbol{f}}. \tag{3.10}$$

$U_f$ satisfy the commutation relation

$$U_f U_{f'} = (-1)^{\int \boldsymbol{f} \cup_1 \boldsymbol{f}' + \boldsymbol{f}' \cup_1 \boldsymbol{f}} U_{f'} U_f \tag{3.11}$$

which is the same as (3.9).

The final step is to determine the constraints on bosonic variables. There is one such constraint for each edge $e$. In the product $\prod_{f \supset e} S_f$, the only surviving terms are $-i\gamma_t \gamma_t'$ with one face going inward and one face going outward of $t$. Therefore, the product can be written as

$$\prod_{f \supset e} S_f \sim \prod_{t | e = t_{01}, t_{03}, t_{12}, t_{23}} P_t, \tag{3.12}$$

where $\sim$ means that it is equal up to a sign, which will be treated carefully in the next paragraph. This produce the gauge constraints for bosonic operators:

$$\prod_{f \supset e} U_f \sim \prod_{t | e = t_{01}, t_{03}, t_{12}, t_{23}} W_t. \tag{3.13}$$

For a tetrahedron $t$ containing an edge $e$ with adjacent faces $f_1$ and $f_2$, consider the following product which gives $W_t$ for $e = t_{01}, t_{03}, t_{12}, t_{23}$ and 1 otherwise:

$$\begin{aligned} &Z_{f_1} Z_{f_2} \prod_{f' \subset t} Z_{f'}^{(\boldsymbol{f}_1 + \boldsymbol{f}_2) \cup_1 \boldsymbol{f}' + \boldsymbol{f}' \cup_1 (\boldsymbol{f}_1 + \boldsymbol{f}_2)} \\ &= \begin{cases} W_t, & \text{if } e = t_{01}, t_{03}, t_{12}, t_{23} \\ 1, & \text{otherwise} \end{cases} \end{aligned} \tag{3.14}$$

Substituting this into (3.13), we have

$$\prod_{f \supset e} U_f \sim \prod_{f \supset e} \prod_{f'} Z_{f'}^{\int \boldsymbol{f}' \cup_1 \boldsymbol{f} + \boldsymbol{f} \cup_1 \boldsymbol{f}'} = \prod_{f'} Z_{f'}^{\int \boldsymbol{f}' \cup_1 \delta \boldsymbol{e} + \delta \boldsymbol{e} \cup_1 \boldsymbol{f}'}. \tag{3.15}$$

On the other hand, the product $\prod_{f\supset e} U_f$ is

$$\prod_{f\supset e} U_f \sim \prod_{f\supset e} X_f \prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \boldsymbol{f}} = (\prod_{f\supset e} X_f)\prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \delta \boldsymbol{e}}. \tag{3.16}$$

Identifying (3.15) and (3.16) gives

$$(\prod_{f\supset e} X_f)\prod_{f'} Z_{f'}^{\int \delta \boldsymbol{e}\cup_1 \boldsymbol{f}'} = 1. \tag{3.17}$$

This is the modified Gauss law (gauge constraint) on each edge $e$. One can check that constraints for different edges $e_1$ and $e_2$ commute since

$$\begin{aligned}\int (\delta \boldsymbol{e}_1 \cup_1 \delta \boldsymbol{e}_2 + \delta \boldsymbol{e}_2 \cup_1 \delta \boldsymbol{e}_1) &= \\ &= \int (\boldsymbol{e}_1 \cup \delta \boldsymbol{e}_2 + \delta \boldsymbol{e}_2 \cup \boldsymbol{e}_1 + \boldsymbol{e}_2 \cup \delta \boldsymbol{e}_1 + \delta \boldsymbol{e}_1 \cup \boldsymbol{e}_2) = 0,\end{aligned} \tag{3.18}$$

where we have used the property $\int \delta \boldsymbol{e}_1 \cup_1 \delta \boldsymbol{e}_2 = \int (\boldsymbol{e}_1 \cup \delta \boldsymbol{e}_2 + \delta \boldsymbol{e}_2 \cup \boldsymbol{e}_1)$.

To be more precise about the signs in (3.15) and (3.16), we give the definition of $S_\beta$ for a 2-cochain $\beta \in C^2(T, \mathbb{Z}_2)$:

$$S_\beta S_{\beta'} = (-1)^{\int \beta' \cup_1 \beta} S_{\beta+\beta'}. \tag{3.19}$$

We can also define $U_\beta$ in the same way. It can be checked that

$$U_\beta = \prod_f X_f^{\beta(f)} \prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \beta}. \tag{3.20}$$

For example, we have $U_{\delta e} = (\prod_{f\supset e} X_f)\prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \delta \boldsymbol{e}}$. The explicit formula for the identity in fermionic evn algebra is

$$(-1)^{\int_{w_2} \boldsymbol{e}} S_{\delta \boldsymbol{e}} \prod_t P_t^{\int \boldsymbol{e}\cup_1 \boldsymbol{t} + \boldsymbol{t}\cup_1 \boldsymbol{e}} = 1 \tag{3.21}$$

which is derived in Appendix C. The 1-chain $w_2$ consists of all edges of the triangulation, together with the (02) edge for all "+" tetrahedra and the (13) edge for all "−" tetrahedra:

$$w_2 = \sum_e e + \sum_{t\in +\text{tetrahedra}} t_{02} + \sum_{t\in -\text{tetrahedra}} t_{13}. \tag{3.22}$$

This is exactly the 1-chain which is Poincaré-dual to the second Stiefel-Whitney class. It is a 1-cycle modulo 2 (that is, its boundary is trivial when regarded as a 0-chain with coefficients in $\mathbb{Z}_2$). If the topological space corresponding to the triangulation is simply-connected, then any 1-cycle is a boundary of some 2-cycle [2]. Thus we can define

$$S_\beta^E = (-1)^{\int_E \beta} S_\beta, \tag{3.23}$$

where $E$ is a 2-chain such that $\partial E = w_2$. Such a $E$ is not unique, but any two choices differ by a 2-cycle.

We see that if we identify $S_f^E$ and $U_f$, then the bosonic variables must satisfy a gauge constraint (3.17). The 3d bosonization map can be summarized as follows:

$$\begin{gathered}
W_t = \prod_{f\subset t} Z_f \longleftrightarrow P_t = -i\gamma_t\gamma_t', \\
U_f = X_f(\prod_{f'} Z_{f'}^{\int f'\cup_1 f}) \longleftrightarrow (-1)^{\int_E f} S_f = (-1)^{\int_E f} i\gamma_{L(f)}\gamma'_{R(f)}, \\
G_e = \prod_{f\supset e} X_f(\prod_{f'} Z_{f'}^{\int \delta e\cup_1 f'}) \longleftrightarrow (-1)^{\int_{w_2} e} S_{\delta e} \prod_t P_t^{\int e\cup_1 t + t\cup_1 e} = 1, \\
\prod_t W_t = 1 \longleftrightarrow \prod_t P_t,
\end{gathered} \tag{3.24}$$

where $w_2$ is defined in (3.22) and $E$ satisfies $\partial E = w_2$.

The modified Gauss law looks complicated, but it can be written down more concisely if we describe the spin configurations by a 2-cochain $B \in C^2(T, \mathbb{Z}_2)$. Our convention is that $B(f) = 1$ if $Z_f = -1$ and $B(f) = 0$ if $Z_f = 1$. Thus the unconstrained Hilbert space is spanned by vectors $|B\rangle$ for all $B$. A 2-form gauge transformation has a 1-cochain $\Lambda$ as a parameter and acts by $B \mapsto B + \delta\Lambda$. For a general $\Lambda$, the Gauss law constraint is given by

$$\left(\prod_{f\in\delta\Lambda} X_f\right)\left(\prod_{f'} Z_{f'}^{\int \delta\Lambda\cup_1 f'}\right)(-1)^{\int \Lambda\cup\delta\Lambda} = 1. \tag{3.25}$$

[2]Actually, one can show that the second Stiefel-Whitney class of any oriented 3-manifold is trivial, and therefore the above 1-cycle is a boundary even in the non-simply-connected case. Our bosonization procedure works also for non-simply-connected spaces. The only difference is that apart from local Gauss law constraints one also needs to impose non-local constraints, one for each non-trivial class in $H_1(X, \mathbb{Z}_2)$. For example, for a 3-torus one would need to impose three constraints, one for each direction $x, y, z$. These constraints on the bosonic side are needed to reproduce the freedom to impose either periodic or anti-periodic boundary conditions on the fermions. But in this paper we limit ourselves to topologically-trivial (simply-connected) spaces.

This formula is proved by $\int \boldsymbol{e} \cup \delta \boldsymbol{e} = 0$ and induction:

$$
\begin{aligned}
&(\prod_{f_1 \in \delta\Lambda_1} X_{f_1})(\prod_{f_1'} Z_{f_1'}^{\delta\Lambda_1 \cup_1 f_1'})(-1)^{\int \Lambda_1 \cup \delta\Lambda_1}(\prod_{f_2 \in \delta\Lambda_2} X_{f_2})(\prod_{f_2'} Z_{f_2'}^{\delta\Lambda_2 \cup_1 f_2'})(-1)^{\int \Lambda_2 \cup \delta\Lambda_2} \\
&= \left(\prod_{f \in \delta(\Lambda_1+\Lambda_2)} X_f\right)\left(\prod_{f'} Z_{f'}^{\delta(\Lambda_1+\Lambda_2)\cup_1 f'}\right)(-1)^{\int \Lambda_1 \cup \delta\Lambda_1 + \Lambda_2 \cup \delta\Lambda_2}(-1)^{\int \delta\Lambda_1 \cup_1 \delta\Lambda_2} \\
&= \left(\prod_{f \in \delta(\Lambda_1+\Lambda_2)} X_f\right)\left(\prod_{f'} Z_{f'}^{\delta(\Lambda_1+\Lambda_2)\cup_1 f'}\right)(-1)^{\int (\Lambda_1+\Lambda_2) \cup \delta(\Lambda_1+\Lambda_2)},
\end{aligned}
\tag{3.26}
$$

where we use the identity $\int \delta\Lambda_1 \cup_1 \delta\Lambda_2 = \int \Lambda_1 \cup \delta\Lambda_2 + \delta\Lambda_2 \cup \Lambda_1$ in the last equality. Eq. (3.25) can be concisely written as

$$
\left(\prod_{f \in \delta\Lambda} X_f\right)(-1)^{\int \Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B} = 1, \tag{3.27}
$$

where $B$ is an arbitrary 2-cochain with values in $\mathbb{Z}_2$.

Consider now the following 2-form gauge theory defined on a general triangulated 4D manifold $Y$:

$$
S(B) = \sum_t |\delta B(t)| + i\pi \int_Y (B \cup B + B \cup_1 \delta B). \tag{3.28}
$$

Here $B \in C^2(Y, \mathbb{Z}_2)$ is a $\mathbb{Z}_2$ field living at each face, and the gauge symmetry acts by $B \to B + \delta\Lambda$ for an arbitrary 1-cochain $\Lambda$. The second term is the Steenrod square topological action [23], which is used in [24] to construct fermionic topological phases. The action is gauge-invariant up to a boundary term:

$$
S(B + \delta\Lambda) - S(B) = \int_{\partial Y} (\Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B). \tag{3.29}
$$

This boundary term determines the Gauss law for the wave-function $\Psi(B)$ on the spatial slice $X = \partial Y$:

$$
\Psi(B + \delta\Lambda) = (-1)^{\omega(\Lambda, B)} \Psi(B), \tag{3.30}
$$

where $\omega(\Lambda, B) = \int_X (\Lambda \cup \delta\Lambda + \delta\Lambda \cup_1 B)$. The Gauss law is the same as the gauge constraint (3.27). In Section 3.4, we use this observation to construct a 4D lattice action for particular Hamiltonian gauge theories with the modified Gauss law.

### 3.3 Examples

In this section, we describe examples for the bosonization map in (3+1)D.

**Soluble 3+1D lattice gauge theories**

The standard Gauss law for a 2-form $\mathbb{Z}_2$ gauge theory is $\prod_{f\supset e} X_f = 1$. Such a bosonic gauge theory is dual to a theory of bosonic spins living on the vertices of the dual lattice. In particular, the quantum Ising model can be described by a $\mathbb{Z}_2$ 2-form gauge theory with the Hamiltonian

$$H_{Ising} = g^2 \sum_f X_f + \frac{1}{g^2} \sum_t W_t. \tag{3.31}$$

This model is not soluble.

If we impose the modified Gauss law (3.6) instead, the simplest analogous gauge-invariant Hamiltonian is

$$H_b = g^2 \sum_f U_f + \frac{1}{g^2} \sum_t W_t. \tag{3.32}$$

The first and second term can be thought of as the kinetic and potential energies, respectively. This is dual to the fermionic Hamiltonian

$$\begin{aligned} H_f =& t \sum_f \Big( c_{L(f)} c_{R(f)} - c^\dagger_{L(f)} c^\dagger_{R(f)} \\ & + c^\dagger_{L(f)} c_{R(f)} + c^\dagger_{R(f)} c_{L(f)} \Big) + \mu \sum_t c^\dagger_t c_t, \end{aligned} \tag{3.33}$$

where $t = g^2$ and $\mu = \frac{2}{g^2}$. The fermionic Hamiltonian is free and thus soluble. By Fourier transform $c_{\vec{x}} = \frac{1}{\sqrt{N}} \sum_{\vec{k}} e^{i\vec{k}\cdot\vec{x}} c_{\vec{k}}$, the fermionic Hamiltonian becomes

$$H_f = \sum_{\vec{k}} \epsilon_{\vec{k}} c^\dagger_{\vec{k}} c_{\vec{k}} + \sum_{\vec{k}} (\Delta_{\vec{k}} c_{\vec{k}} c_{-\vec{k}} + \text{h.c.}) \tag{3.34}$$

with $\epsilon_{\vec{k}} = \mu + 2t(\cos k_x + \cos k_y + \cos k_z)$ and $\Delta_{\vec{k}} = t(e^{-ik_x} + e^{-ik_y} + e^{-ik_z})$. The Hamiltonian (3.34) can be written in the Bogoliubov-de-Gennes (BdG) formalism as

$$H_f = \frac{1}{2} \sum_{\vec{k}} \Psi^\dagger_{\vec{k}} H_{\text{BDG}}(\vec{k}) \Psi_{\vec{k}} \tag{3.35}$$

with

$$H_{\text{BDG}}(\vec{k}) = \begin{bmatrix} \epsilon_{\vec{k}} & -\Delta^*_{\vec{k}} \\ -\Delta_{\vec{k}} & -\epsilon_{\vec{k}} \end{bmatrix}, \quad \Psi_{\vec{k}} = \begin{bmatrix} c_{\vec{k}} \\ c^\dagger_{-\vec{k}} \end{bmatrix}. \tag{3.36}$$

The spectrum is

$$\begin{aligned} & E^2 = \\ & t^2(3 + 2\cos(k_x - k_y) + 2\cos(k_x - k_z) + 2\cos(k_y - k_z)) \\ & + [\mu + 2t(\cos k_x + \cos k_y + \cos k_z)]^2. \end{aligned} \tag{3.37}$$

Notice that for $\mu = 0$ the gap closes for $\vec{k} = (q, q + \frac{2\pi}{3}, q + \frac{4\pi}{3})$ for arbitrary $q$.

**Bosonic model with Dirac cones**

Using the bosonization map (3.2) and (3.3), we can construct an equivalent bosonic model for any arbitrary fermionic model. For instance, Ref. [54] constructs a fermionic model on a cubic lattice with Dirac cones:

$$H = -t \sum_{\vec{r}} (s_x(\vec{r}) c^\dagger_{\vec{r}+\hat{x}} c_{\vec{r}} + s_y(\vec{r}) c^\dagger_{\vec{r}+\hat{y}} c_{\vec{r}} + s_z(\vec{r}) c^\dagger_{\vec{r}+\hat{z}} c_{\vec{r}} + \text{h.c.}) \tag{3.38}$$

with $s_x(\vec{r})$, $s_y(\vec{r})$, and $s_z(\vec{r})$ defined as

$$\begin{aligned} s_x(i,j,k) &= 1 \\ s_y(i,j,k) &= (-1)^i \\ s_z(i,j,k) &= (-1)^{i+j}. \end{aligned} \tag{3.39}$$

It is a model with nearest neighbor hopping. The spectrum is

$$E = \pm 2t\sqrt{\cos^2 k_x + \cos^2 k_y + \cos^2 k_z} \tag{3.40}$$

with two Dirac cones at $\vec{k} = (\pi/2, \pi/2, \pi/2)$ and $\vec{k} = (3\pi/2, \pi/2, \pi/2)$. Applying the bosonization map, the corresponding bosonic Hamiltonian is

$$\begin{aligned} H = &-\frac{t}{2}\sum_{f_x} s_x(L(f_x)) U_{f_x} (1 - W_{L(f_x)} W_{R(f_x)}) \\ &-\frac{t}{2}\sum_{f_y} s_y(L(f_y)) U_{f_y} (1 - W_{L(f_y)} W_{R(f_y)}) \\ &-\frac{t}{2}\sum_{f_z} s_z(L(f_z)) U_{f_z} (1 - W_{L(f_z)} W_{R(f_z)}), \end{aligned} \tag{3.41}$$

where $f_x$, $f_y$, $f_z$ refer to faces normal to $x$, $y$, $z$-directions, with gauge constraints (3.6). On the bosonic side, it is very nontrivial to see that the model describes Dirac cones.

## 3.4 Euclidean 3+1D gauge theories with fermionic duals

In the previous section we constructed a 3d bosonization map which works on the kinematic level (that is, is independent of the Hamiltonian). In this section we apply it to some specific models of free fermions and describe the corresponding dual gauge theories. We then construct Euclidean formulations of these gauge theories. We will make use of cup products, and thus will assume that the 3d space is triangulated, as in section 3.2. Accordingly, the (3+1)D lattice will be the product of the 3d triangulation and discrete time. As explained in the Appendix, (higher) cup products can also be defined on the 3d cubic lattice, thus similar considerations can be used to find the Euclidean formulation of gauge theories constructed in Section 3.1.

Consider the simplest gauge-invariant Hamiltonian compatible with the modified Gauss law:

$$H = -A \sum_f U_f - B \sum_t W_t. \tag{3.42}$$

The gauge constraint is

$$G_e \equiv \left( \prod_{f \supset e} X_f \right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'} = 1. \tag{3.43}$$

The partition function can be calculated by transfer matrix method:

$$\mathcal{Z} = \mathrm{Tr}\ e^{-\beta H} = \mathrm{Tr}\ T^M, \tag{3.44}$$

where $T$ is the transfer matrix defined as

$$T = \left( \prod_e \delta_{G_e,1} \right) e^{-\delta\tau H}, \tag{3.45}$$

where $\delta\tau \equiv \beta/M$, and $M \gg 1$ is a large positive integer. The first factor arises from the gauge constraints on the Hilbert space. For calculation purposes, we can rewrite it as

$$\begin{aligned} \delta_{G_e,1} &= \frac{1}{2}(1 + G_e) \\ &= \frac{1}{2} \sum_{\lambda_e = \pm 1} (-1)^{\frac{1-\lambda_e}{2} \sum_{f' \in \mathrm{NE}(e)} \frac{1-Z_{f'}}{2}} \cdot (-1)^{\frac{1-\lambda_e}{2} \sum_{f \supset e} \frac{1-X_f}{2}} \end{aligned} \tag{3.46}$$

with $\mathrm{NE}(e) \equiv \{f | \int \delta e \cup_1 f = 1\}$. Here $\lambda_e$ is the Lagrange multiplier on each edge $e$ of the spatial manifold $M$ and will be consider as $\mathbb{Z}_2$ fields living on "temporal" faces later. To calculate the partition function (3.44), the completed bases are inserted between $T$. Define $|m(\tau)\rangle = |\{S_f\}\rangle$ as the configuration of spins (in $Z_f$ basis). Then the matrix elements of $T$ are

$$\begin{aligned} &\langle m'(\tau + \delta\tau)|T|m(\tau)\rangle \\ &= \langle m'(\tau + \delta\tau)| \left( \prod_v \delta_{G_e,1} \right) e^{-\delta\tau H} |m(\tau)\rangle. \end{aligned} \tag{3.47}$$

Next we need to use an identity

$$\begin{aligned}&\langle S^{z\prime}|f(\sigma^x,\sigma^z)|S^z\rangle\\&=\frac{1}{2}\sum_{S^x=\pm1}f(S^x,S^z)(-1)^{\frac{1-S^x}{2}(\frac{1-S^{z\prime}}{2}+\frac{1-S^z}{2})},\end{aligned}\tag{3.48}$$

where we assume that $\sigma^x$ is to the right of $\sigma^z$ in the function $f(\sigma^x,\sigma^z)$. Plugging this into (3.47), we get

$$\begin{aligned}&\langle m'(\tau+\delta\tau)|\left(\prod_e\delta_{G_e,1}\right)e^{-\delta\tau H}|m(\tau)\rangle\\&=\langle m'(\tau+\delta\tau)|\left(\prod_e\delta_{G_e,1}\right)\left(\prod_f\sum_{S^x_f=\pm1}|S^x_f\rangle\langle S^x_f|\right)e^{-\delta\tau H}|m(\tau)\rangle\\&\sim\left[\sum_{\lambda_e=\pm1}(-1)^{\frac{1-\lambda_e}{2}\left(\sum_{f_2\supset e}\frac{1-S^x_{f_2}}{2}+\sum_{f_3\in\mathrm{NE}(e)}\frac{1-S^{z\ \prime}_{f_3}}{2}\right)}(-1)^{\sum_{\lambda_e,\lambda_{e'}=-1}\int e\cup\delta e'}\right]\\&\left[\prod_f\sum_{S^x_f=\pm1}(-1)^{\frac{1-S^x_f}{2}\left(\frac{1-S^{z\ \prime}_f}{2}+\frac{1-S^z_f}{2}\right)}e^{A\delta\tau S^x_e\prod_{f_1\in\Delta(f)}S^z_{f_1}}\right]\left(\prod_t e^{B\delta\tau\prod_{f_4\subset t}S^z_{f_4}}\right),\end{aligned}\tag{3.49}$$

where $\Delta(f)\equiv\{f'|\int \boldsymbol{f'}\cup_1\boldsymbol{f}=1\}$ and the term $(-1)^{\sum_{\lambda_e,\lambda_{e'}=-1}\int \boldsymbol{e}\cup\delta\boldsymbol{e'}}$ comes from pushing all $X_f$ operators to the right, which is the same as the last factor of Eq. (3.25). This term can be expressed as

$$i\pi\sum_i\int a_i\cup\delta a_i\tag{3.50}$$

if we define $a_i\in C^1(M,\mathbb{Z}_2)$ as 1-cochain on the spacetial manifold $M$ ($i^{th}$ layer) with $a_i(e)=1$ for $\lambda_e=-1$ and $a_i(e)=0$ for $\lambda_e=1$. We can interpret $a_i$ at the $i^{th}$ layer as a 2-cochain which lives on the “temporal” faces between the $i^{th}$ and $(i+1)^{th}$ layers.

After extracting this factor, the remaining terms are

$$\begin{aligned}
&\sum_{\lambda_e=\pm 1}\left(\prod_e (-1)^{\frac{1-\lambda_e}{2}\sum_{f_3\subset \mathrm{NE}(e)}\frac{1-S^z_{f_3}{}'}{2}}\right)\left(\prod_t e^{B\delta\tau\prod_{f_4\subset t}S^z_{f_4}}\right)\\
&\quad\left[\prod_f \sum_{S^x_f=\pm 1}(-1)^{\frac{1-S^x_f}{2}\left(\frac{1-S^z_f{}'}{2}+\frac{1-S^z_f}{2}+\sum_{e\subset f}\frac{1-\lambda_e}{2}\right)}\cdot e^{A\delta\tau S^x_e\prod_{f_1\in\Delta(f)}S^z_{f_1}}\right]\\
&=\sum_{\lambda_e=\pm 1}\left(\prod_e (-1)^{\frac{1-\lambda_e}{2}\sum_{f_3\subset \mathrm{NE}(e)}\frac{1-S^z_{f_3}{}'}{2}}\right)\left(\prod_t e^{B\delta\tau\prod_{f_4\subset t}S^z_{f_4}}\right)\\
&\quad\left[\prod_e (e^{A\delta\tau\prod_{f_1\in\Delta(f)}S^z_{f_1}}+e^{-A\delta\tau\prod_{f_1\in\Delta(f)}S^z_{f_1}}S^z_f{}'S^z_f\prod_{e\subset f}\lambda_e)\right]\\
&\sim\sum_{\lambda_e=\pm 1}\left(\prod_e (-1)^{\frac{1-\lambda_e}{2}\sum_{f_3\subset \mathrm{NE}(e)}\frac{1-S^z_{f_3}{}'}{2}}\right)\left(\prod_t e^{B\delta\tau\prod_{f_4\subset t}S^z_{f_4}}\right)\\
&\quad\left[\prod_f e^{JS^z_f{}'S^z_f\prod_{e\subset f}\lambda_e}(-1)^{\left(\sum_{f_1\in\Delta(f)}\frac{1-S^z_{f_1}}{2}\right)\left(\frac{1-S^z_f{}'}{2}+\frac{1-S^z_f}{2}+\sum_{e\subset f}\frac{1-\lambda_e}{2}\right)}\right]\\
&=\sum_{\lambda_e=\pm 1}(-1)^{\sum_e\left(\frac{1-\lambda_e}{2}\sum_{f_3\subset \mathrm{NE}(e)}\frac{1-S^z_{f_3}{}'}{2}\right)+\sum_f\left(\sum_{f_1\in\Delta(f)}\frac{1-S^z_{f_1}}{2}\right)\left(\frac{1-S^z_f{}'}{2}+\frac{1-S^z_f}{2}+\sum_{e_1\subset f}\frac{1-\lambda_{e_1}}{2}\right)}\\
&\quad e^{J\sum_f S^z_f{}'S^z_f\prod_{e_1\subset f}\lambda_{e_1}+B\delta\tau\sum_t\prod_{f_4\subset t}S^z_{f_4}},
\end{aligned}\tag{3.51}$$

where $\tanh J = e^{-2A\delta\tau}$. The last line is the usual action for a 4D $\mathbb{Z}_2$ gauge theory except for some sign factors. We regard these factors as coming from a topological action $S_{\text{top}}$. From the penultimate line in (3.51), we see that $S_{\text{top}}$ contains

$$\begin{aligned}
&i\pi\left[\sum_e \frac{1-\lambda_e}{2}\sum_{f_3\subset \mathrm{NE}(e)}\frac{1-S^z_{f_3}{}'}{2}+\right.\\
&\left.\quad\sum_f\left(\sum_{f_1\in\Delta(f)}\frac{1-S^z_{f_1}}{2}\right)\cdot\left(\frac{1-S^z_f{}'}{2}+\frac{1-S^z_f}{2}+\sum_{e_1\subset f}\frac{1-\lambda_{e_1}}{2}\right)\right].
\end{aligned}\tag{3.52}$$

The first term is

$$\begin{aligned}
&\sum_e \frac{1-\lambda_e}{2}\sum_{f\supset e}\sum_{f\cup_1 f_3=1}\frac{1-S^z_{f_3}{}'}{2}\\
&=\sum_f\Big(\sum_{e\subset f}\frac{1-\lambda_e}{2}\Big)\Big(\sum_{f\cup_1 f_3=1}\frac{1-S^z_{f_3}{}'}{2}\Big)
\end{aligned}\tag{3.53}$$

which is equal to $\int \delta a_i \cup_1 b_{i+1}$ if we define $b_i$ as a 2-cochain on the $i$th layer with $b_i(f) = \frac{1-S_f}{2}$. The second term is

$$\sum_f \Big( \sum_{f_1 | \int f_1 \cup_1 f = 1} \frac{1 - S^z_{f_1}}{2} \Big) \cdot \Big( \frac{1 - S^{z\,\prime}_f}{2} + \frac{1 - S^z_f}{2} + \sum_{e_1 \subset f} \frac{1 - \lambda_{e_1}}{2} \Big) \tag{3.54}$$

which is $\int b_i \cup_1 (b_i + b_{i+1} + \delta a_i)$. Collecting all terms in (3.50), (3.53), and (3.54), we get

$$\begin{aligned} S_{\text{top}}(\{a_i\}, \{b_i\}) =& i\pi \sum_i \int a_i \cup \delta a_i + \delta a_i \cup_1 b_{i+1} \\ & + b_i \cup_1 (b_i + b_{i+1} + \delta a_i). \end{aligned} \tag{3.55}$$

The usual term $e^{J \sum_f S^{z\,\prime}_f S^z_f \prod_{e_1 \subset f} \lambda_{e_1} + B\delta\tau \sum_T \prod_{f_4 \subset T} S^z_{f_4}}$ can be written as the exponential of (up to an unimportant constant)

$$\begin{aligned} & S_{\text{4D gauge}}(\{a_i\}, \{b_i\}) \\ &= \sum_i \Big( -2J \sum_f |b_i(f) + b_{i+1}(f) + \delta a_i(f)| \\ & \qquad - 2B\delta\tau \sum_t |\delta b_i(t)| \Big), \end{aligned} \tag{3.56}$$

where $|\cdots|$ gives the argument's parity 0 or 1. Combining (3.55) and (3.56), the Euclidean action becomes (up to an additive constant)

$$S(\{a_i\}, \{b_i\}) = S_{\text{top}}(\{a_i\}, \{b_i\}) + S_{\text{4D gauge}}(\{a_i\}, \{b_i\}), \tag{3.57}$$

which is analogous to generalized Steenrod square action:

$$i\pi \int_Y (B \cup B + B \cup_1 \delta B). \tag{3.58}$$

We will verify that these two actions produce the same boundary term under gauge transformation.

This action is gauge-invariant (up to boundary terms) under gauge transformations

$$b_i \to b_i + \delta\lambda_i, \;\; a_i \to a_i + \delta\mu_i + \lambda_i + \lambda_{i+1}, \tag{3.59}$$

where $\lambda_i$ are arbitrary 1-cochains and $\mu_i$ are arbitrary 0-cochains. Indeed, the change in the action is

$$
\begin{aligned}
\frac{\Delta S_{\text{top}}}{(i\pi)} = &\sum_i \int (a_i + \cancelto{0}{\delta\mu_i} + \lambda_i + \lambda_{i+1}) \cup (\delta\lambda_i + \delta\lambda_{i+1}) \\
&+ (\cancelto{0}{\delta\mu_i} + \lambda_i + \lambda_{i+1}) \cup \delta a_i + (\delta\lambda_i + \delta\lambda_{i+1}) \cup_1 b_{i+1} + \delta a_i \cup_1 \delta\lambda_{i+1} \\
&+ (\delta\lambda_i + \cancelto{0}{\delta\lambda_{i+1})} \cup_1 \delta\lambda_{i+1} + \delta\lambda_i \cup_1 (b_i + b_{i+1} + \delta a_i) \\
= &\sum_i \int a_i \cup (\delta\lambda_i + \delta\lambda_{i+1}) + (\lambda_i + \lambda_{i+1}) \cup (\delta\lambda_i + \delta\lambda_{i+1}) \\
&+ (\lambda_i + \lambda_{i+1}) \cup \delta a_i + a_i \cup \delta\lambda_{i+1} + \delta\lambda_{i+1} \cup a_i + \lambda_i \cup \delta\lambda_{i+1} + \delta\lambda_{i+1} \cup \lambda_i \\
&+ a_i \cup \delta\lambda_i + \delta\lambda_i \cup a_i \\
= &\sum_i \int \lambda_i \cup \delta\lambda_i + \lambda_{i+1} \cup \delta\lambda_{i+1} = 0,
\end{aligned}
\tag{3.60}
$$

where the terms with the same colors cancel out. In the above computation we assumed periodic time, so that there are no boundary terms. If we do not identify time periodically, the variation is a boundary term

$$
\int (\lambda_0 \cup \lambda_0 + \delta\lambda_0 \cup_1 b_0) + (\lambda_N \cup \lambda_N + \delta\lambda_N \cup_1 b_N), \tag{3.61}
$$

which is the same as the boundary term (3.29) in the previous section.

We can also check that the action is invariant under a 2-form global symmetry

$$
B \to B + \beta, \tag{3.62}
$$

where a closed 2-cochain $\beta$ can be represented by 2-cochains $\beta_i$ (one for each time slice) and 1-cochains $\alpha_i$ satisfying $\beta_i + \beta_{i+1} + \delta\alpha_i = 0$. Using a gauge transformation (4.30) with

$$
\lambda_i = \sum_{j=0}^{i-1} \alpha_j, \quad \mu_i = 0 \tag{3.63}
$$

for $i = 0, 1, \ldots, N-1$, we can see that $\beta_i' = \beta_0$, which is independent of $i$, and $\alpha_{N-1}' = \sum_{j=0}^{N-1} \alpha_j$ with other $\alpha_i' = 0$. Notice that $\alpha_{N-1}'$ is closed since $\beta_i' = \beta_{i+1}'$.

Under this 2-form symmetry transformation $\beta'$, the action changes by

$$\begin{aligned}\frac{\Delta S_{\text{top}}}{(i\pi)} &= \int \cancelto{0}{\alpha'_{N-1} \cup \delta a_{N-1}} + \sum_i \delta a_i \cup_1 \beta_0 \\ &\quad + \beta_0 \cup_1 (\cancelto{0}{\sum_i b_i + b_{i+1}}) + \sum_i \beta_0 \cup_1 \delta a_i \\ &= \sum_i \int a_i \cup \beta_0 + \beta_0 \cup a_i + \beta_0 \cup a_i + a_i \cup \beta_0 = 0.\end{aligned} \tag{3.64}$$

Thus the action is invariant under a global 2-form symmetry, as expected.

### 3.5 Gauging fermion parity

We have shown that a lattice fermionic system in 3d is dual to a bosonic spin system with the Gauss law constraints. In this section we show how to get rid of the constraints at the expense of coupling fermions to a $\mathbb{Z}_2$ gauge field.

Our bosonization map is

$$\begin{aligned} -i\gamma_t\gamma'_t &\longleftrightarrow W_t \equiv \prod_{f\subset t} Z_f \\ (-1)^{\int_E f}(i\gamma_{L(f)}\gamma'_{R(f)}) &\longleftrightarrow U_f \equiv X_f \prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \boldsymbol{f}} \end{aligned} \tag{3.65}$$

with gauge constraints

$$\left(\prod_{f\supset e} X_f\right)\prod_{f'} Z_{f'}^{\int \delta \boldsymbol{e}\cup_1 \boldsymbol{f}'} = 1. \tag{3.66}$$

Now, we introduce new $\mathbb{Z}_2$ fields (spins), with operators $\tilde{X}$, $\tilde{Y}$, and $\tilde{Z}$, which live on faces and couple to fermions via a Gauss law constraint

$$(-1)^{F_t} = \prod_{f\subset t} \tilde{Z}_f. \tag{3.67}$$

The fermionic hopping operator must be modified to

$$S_f^E = (-1)^{\int_E f}(i\gamma_{L(f)}\gamma'_{R(f)})\tilde{X}_f \tag{3.68}$$

in order to commute with the Gauss law constraint (3.67). The bosonization map becomes

$$\begin{aligned} -i\gamma_t\gamma'_t = \prod_{f\subset t}\tilde{Z}_f &\longleftrightarrow W_t \equiv \prod_{f\subset t} Z_f \\ (-1)^{\int_E f}(i\gamma_{L(f)}\gamma'_{R(f)})\tilde{X}_f &\longleftrightarrow U_f \equiv X_f \prod_{f'} Z_{f'}^{\int \boldsymbol{f}'\cup_1 \boldsymbol{f}} \end{aligned} \tag{3.69}$$

and, similarly, the gauge constraints become

$$\prod_{f\supset e} \tilde{X}_f \longleftrightarrow \left(\prod_{f\supset e} X_f\right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'}. \tag{3.70}$$

The equations (3.69) and (3.70) define a bosonization map for fermions coupled to a dynamical $\mathbb{Z}_2$ gauge field. In this case, their is no constraint on the bosonic variables.

We can apply this modified boson/fermion map to a $\mathbb{Z}_2$ version of the Levin-Wen rotor model [53] on general triangulation:

$$H = -\sum_t Q_t - \sum_e B_e \tag{3.71}$$

with

$$\begin{aligned} Q_t &= \prod_{f\subset t} Z_f \\ B_e &= \prod_{f\supset e} \left( X_f \prod_{f'} Z_{f'}^{\int f \cup_1 f'} \right) \\ &= \left(\prod_{f\supset e} X_f\right) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'}. \end{aligned} \tag{3.72}$$

Since $Q_t$ and $B_e$ are just $W_t$ and $(\prod_{f\supset e} X_f) \prod_{f'} Z_{f'}^{\int \delta e \cup_1 f'}$, the above bosonic model is equivalent to a model of a $\mathbb{Z}_2$ gauge field coupled to fermions and a Hamiltonian

$$H = -\sum_t \prod_{f\subset t} \tilde{Z}_f - \sum_e \prod_{f\supset e} \tilde{X}_f. \tag{3.73}$$

The fermions are static, since the above Hamiltonian does not include fermionic hopping terms. The only interaction between the fermions and the gauge field is via the Gauss law constraint

$$\prod_{f\subset t} \tilde{Z}_f = (-1)^{F_t}. \tag{3.74}$$

Thus a complicated model bosonic model is mapped to a simple $\mathbb{Z}_2$ lattice gauge theory coupled to static fermions.

As another application of the modified bosonization map, consider again the bosonic gauge theory on a cubic lattice with the Hamiltonian (3.41)

$$H = -\frac{t}{2} \sum_{i=x,y,z} \sum_{f_i} s_i(L(f_i)) U_{f_i} (1 - W_{L(f_i)} W_{R(f_i)}) \tag{3.75}$$

and a gauge constraint (3.6). This constrained model is dual to a model of free fermions with Dirac cones. After coupling the fermions to a $\mathbb{Z}_2$ gauge field and applying the modified map, we find that the bosonic model (3.75) without any gauge constraints is equivalent to a fermionic model with the Hamiltonian

$$
\begin{aligned}
H = -t \sum_{\vec{r}} \Big( & s_x(\vec{r})\tilde{X}_x(\vec{r}) c^\dagger_{\vec{r}+\hat{x}} c_{\vec{r}} + s_y(\vec{r})\tilde{X}_y(\vec{r}) c^\dagger_{\vec{r}+\hat{y}} c_{\vec{r}} \\
& + s_z(\vec{r})\tilde{X}_z(\vec{r}) c^\dagger_{\vec{r}+\hat{z}} c_{\vec{r}} + \text{h.c.} \Big)
\end{aligned} \tag{3.76}
$$

with $(-1)^{c_t^\dagger c_t} = \prod_{f \subset t} \tilde{Z}_f$. The operators $\tilde{W}_e \equiv \prod_{f \supset e} \tilde{X}_f$ commute with the Hamiltonian, so we can project the Hilbert space into sectors with fixed $\tilde{W}_e$ ($\tilde{W}_e$ is arbitrary $\pm 1$ as long as it satisfies $\prod_{e \supset v} \tilde{W}_e = 1$). In the sector $\tilde{W}_e = 1$ for all $e$, the Hamiltonian (3.76) returns to (3.38). The model of unconstrained spins with the Hamiltonian (3.75) thus can be regarded as a 3d analog of Kitaev's honeycomb model.

*Chapter 4*

# EXACT BOSONIZATION IN $n$ DIMENSIONS

## 4.1 Triangulation

From the 2d and 3d formulae (2.28) and (3.24), it is very natural to conjecture the $n$-dimensional boson-fermion duality. The fermions live at the center $n$-simplices, i.e. $\gamma_{\Delta_n}, \gamma'_{\Delta_n}$ for each $\Delta_n$. The Pauli matrices live on $(n-1)$-simplices, i.e. $X_{\Delta_{n-1}}$ and $Z_{\Delta_{n-1}}$ for each $\Delta_{n-1}$. The $n$-dimensional boson-fermion duality should be

$$
\begin{aligned}
&W_{\Delta_n} \equiv \prod_{\Delta_{n-1}\subset\Delta_n} Z_{\Delta_{n-1}} \longleftrightarrow P_t = -i\gamma_{\Delta_n}\gamma'_{\Delta_n}, \\
&U_{\Delta_{n-1}} \equiv X_{\Delta_{n-1}}\left(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \boldsymbol{\Delta_{n-1}}' \cup_{n-2} \boldsymbol{\Delta_{n-1}}}\right) \\
&\longleftrightarrow (-1)^{\int_E \boldsymbol{\Delta_{n-1}}} S_{\Delta_{n-1}} = (-1)^{\int_E \boldsymbol{\Delta_{n-1}}} i\gamma_{L(\Delta_{n-1})}\gamma'_{R(\Delta_{n-1})}, \\
&G_{\Delta_{n-2}} \equiv \prod_{\Delta_{n-1}\supset\Delta_{n-2}} X_{\Delta_{n-1}}\left(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta\boldsymbol{\Delta_{n-2}} \cup_{n-2} \boldsymbol{\Delta_{n-1}}'}\right) \\
&\longleftrightarrow (-1)^{\int_{w_2} \boldsymbol{\Delta_{n-2}}} S_{\delta\boldsymbol{\Delta}_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \boldsymbol{\Delta_{n-2}} \cup_{n-2} \boldsymbol{\Delta_n} + \boldsymbol{\Delta_n} \cup_{n-2} \boldsymbol{\Delta_{n-2}}} = 1, \\
&\prod_{\Delta_n} W_{\Delta_n} = 1 \longleftrightarrow \prod_{\Delta_n} P_{\Delta_n},
\end{aligned}
\tag{4.1}
$$

where $w_2 \in C_{n-2}(M_n, \mathbb{Z}_2)$ is the chain representative of the second Stiefel–Whitney class, $E \in C_{n-1}(M_n, \mathbb{Z}_2)$ denotes a choice of spin structure ($\partial E = w_2$), and for general $(n-1)$-cochain $\boldsymbol{\lambda}_{n-1}$ and $\boldsymbol{\lambda}'_{n-1}$, the product of $S$ operators is defined as

$$
S_{\lambda_{n-1}+\lambda'_{n-1}} \equiv (-1)^{\int \lambda_{n-1}\cup_{n-2}\lambda'_{n-1}} S_{\lambda'_{n-1}} S_{\lambda_{n-1}}. \tag{4.2}
$$

This $n$-dimensional boson-fermion duality (4.1) is the most crucial result of this paper, which will be proved by the end of this section.

### Commutation relations

Consider an $n$-simplex $\Delta_n = \langle 012\ldots n\rangle$. Its boundary contains all $(n-1)$-simplex $(\partial\Delta_n)^i = \langle 0\ldots\hat{i}\ldots n\rangle$ where $\hat{i}$ means the vertex $i$ is omitted. We define the orientation of $(\partial\Delta_n)^i$ as $O((\partial\Delta_n)^i) = (-1)^i$. For "+"-oriented $\Delta_n$, if $O((\partial\Delta_n)^i) = 1$, the boundary $(\partial\Delta_n)^i$ is outward, and if $O((\partial\Delta_n)^i) = -1$, the boundary $(\partial\Delta_n)^i$ is inward. For "−"-oriented $\Delta_n$, the inward and outward boundaries are opposite. $S_{\Delta_{n-1}}$ and $S_{\Delta'_{n-1}}$ anti-commute only when $\Delta_{n-1}$ and $\Delta'_{n-1}$ are both inward or both outward boundaries of some $n$-simplex, i.e. $\Delta_{n-1}, \Delta'_{n-1} \in \partial\Delta_n$. We are going to prove that this is equivalent to

$$S_{\Delta_{n-1}} S_{\Delta'_{n-1}} = (-1)^{\int \boldsymbol{\Delta_{n-1}} \cup_{n-2} \boldsymbol{\Delta'_{n-1}} + \boldsymbol{\Delta'_{n-1}} \cup_{n-2} \boldsymbol{\Delta_{n-1}}} S_{\Delta'_{n-1}} S_{\Delta_{n-1}}. \tag{4.3}$$

From the definition of the higher cup product (A.2), we have

$$\begin{aligned}
&[\boldsymbol{\Delta_{n-1}} \cup_{n-2} \boldsymbol{\Delta'_{n-1}}](0, 1, \cdots, n) \\
=& \sum_{0 \le i_0 < i_1 < \cdots < i_{n-2} \le n} \boldsymbol{\Delta_{n-1}}(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \cdots) \boldsymbol{\Delta'_{n-1}}(i_0 \sim i_1, i_2 \sim i_3, \cdots) \\
=& \sum_{0 \le j_1 < j_2 \le n | j_1, j_2 \in \text{even}} \boldsymbol{\Delta_{n-1}}(\langle 0 \dots \hat{j}_2 \dots n \rangle) \boldsymbol{\Delta'_{n-1}}(\langle 0 \dots \hat{j}_1 \dots n \rangle) \\
&+ \sum_{0 \le k_1 < k_2 \le n | k_1, k_2 \in \text{odd}} \boldsymbol{\Delta_{n-1}}(\langle 0 \dots \hat{k}_1 \dots n \rangle) \boldsymbol{\Delta'_{n-1}}(\langle 0 \dots \hat{k}_2 \dots n \rangle).
\end{aligned} \tag{4.4}$$

The $\cup_{n-2}$ only contains the product of boundaries $\Delta^i_{n-1}$ with the same orientation (inward or outward) and each pair of $\Delta^i_{n-1}, \Delta^{i'}_{n-1}$ with the same orientation appears exactly once. Therefore, the $\cup_{n-2}$ expression in (4.3) captures the commutation relations of fermionic hopping operators $S_{\Delta_{n-1}}$. It is easy to check that bosonic operators $U_{\Delta_{n-1}}$ satisfy the same commutation relations:

$$U_{\Delta_{n-1}} U_{\Delta'_{n-1}} = (-1)^{\int \boldsymbol{\Delta_{n-1}} \cup_{n-2} \boldsymbol{\Delta'_{n-1}} + \boldsymbol{\Delta'_{n-1}} \cup_{n-2} \boldsymbol{\Delta_{n-1}}} U_{\Delta'_{n-1}} U_{\Delta_{n-1}}. \tag{4.5}$$

Therefore, $\{S_{\Delta_{n-1}}, P_{\Delta_n}\}$ and $\{U_{\Delta_{n-1}}, W_{\Delta_n}\}$ in (4.1) have the same commutation relations.

**Gauge constraints**

In Appendix C, the identity for fermionic operators is derived:

$$(-1)^{\int_{w_2} \boldsymbol{\Delta}_{n-2}} S_{\delta\Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \boldsymbol{\Delta}_{n-2} \cup_{n-2} \boldsymbol{\Delta}_n + \boldsymbol{\Delta}_n \cup_{n-2} \boldsymbol{\Delta}_{n-2}} = 1. \tag{4.6}$$

We can modify the sign of $S_{\Delta_{n-1}}$ as

$$S^E_{\Delta_{n-1}} \equiv (-1)^{\int_E \boldsymbol{\Delta}_{n-1}} S_{\Delta_{n-1}}, \tag{4.7}$$

where $E \in C_{n-1}(M_n, \mathbb{Z}_2)$ is a choice of spin structure satisfying $\partial E = w_2$. In these modified operators, the constraint on the fermionic operator becomes

$$S^E_{\delta\Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \boldsymbol{\Delta}_{n-2} \cup_{n-2} \boldsymbol{\Delta}_n + \boldsymbol{\Delta}_n \cup_{n-2} \boldsymbol{\Delta}_{n-2}} = 1, \tag{4.8}$$

which is mapped to

$$\begin{aligned} G_{\Delta_{n-2}} =& U_{\delta\Delta_{n-2}} \prod_{\Delta_n} W_{\Delta_n}^{\int \boldsymbol{\Delta}_{n-2}\cup_{n-2}\boldsymbol{\Delta}_n + \boldsymbol{\Delta}_n \cup_{n-2} \boldsymbol{\Delta}_{n-2}} \\ =& \prod_{\Delta_{n-1}\supset\Delta_{n-2}} X_{\Delta_{n-1}} \Big( \prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta\boldsymbol{\Delta}_{n-2}\cup_{n-2}\boldsymbol{\Delta}_{n-1}'} \Big). \end{aligned} \tag{4.9}$$

We need to impose this gauge constraint $G_{\Delta_{n-2}} = 1$ on bosonic operators for every $(n-2)$-simplex $\Delta_{n-2}$.

We also need to impose the even total parity constraint for fermions

$$\prod_{\Delta_n} P_{\Delta_n} = 1 \tag{4.10}$$

since it is mapped to the bosonic operator $\prod_{\Delta_n} W_{\Delta_n} = 1$. After imposing the gauge constraints, the $n$-dimensional boson-fermion duality (4.1) is completed.

## 4.2 Modified Gauss's law and Euclidean action

### Gauss's law as boundary anomaly

First, we consider the standard $\mathbb{Z}_2$ lattice gauge theory on the $n$-dimensional manifold $M_n$:

$$H^0 = -A \sum_{\Delta_{n-1}} X_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \tag{4.11}$$

with the gauge constraint (Gauss's law)

$$G^0_{\Delta_{n-2}} = \prod_{\Delta_{n-1}\supset\Delta_{n-2}} X_{\Delta_{n-2}} = 1. \tag{4.12}$$

It is well-kwown that its Euclidean theory is $(n+1)$-dimensional Ising model (with some choice of $A$ and $B$) [55]:

$$S_{\text{Ising}}(A_{n-1}) = -J \sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)|, \tag{4.13}$$

where $A \in C^{n-1}(Y, \mathbb{Z}_2)$ is a $(n-1)$-cochain on the spacetime manifold $Y$. In this case, $S_{\text{Ising}}$ is invariant under the gauge transformation $A_{n-1} \to A_{n-1} + \delta\Lambda_{n-2}$ for arbitrary $(n-2)$-cochain $\Lambda_{n-2} \in C^{n-2}(Y, \mathbb{Z}_2)$. Therefore, $S_{\text{Ising}}$ has no boundary anomaly under the standard Gauss's law.

Now, we propose a new class of $\mathbb{Z}_2$ lattice gauge theory:

$$H = -A \sum_{\Delta_{n-1}} U_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \tag{4.14}$$

with the modified Gauss's law (gauge constraints) at $(n-2)$-simplices

$$G_{\Delta_{n-2}} = \prod_{\Delta_{n-1}\supset\Delta_{n-2}} X_{\Delta_{n-1}} \Big( \prod_{\Delta_{n-1}{}'} Z_{\Delta_{n-1}{}'}^{\int \delta \boldsymbol{\Delta}_{n-2} \cup_{n-2} \boldsymbol{\Delta}_{n-1}{}'} \Big) = 1. \tag{4.15}$$

This model describes a free fermion system, since it is dual to

$$\begin{aligned} H_f &= -A \sum_{\Delta_{n-1}} (-1)^{\int_E \Delta_{n-1}} i\gamma_{L(\Delta_{n-1})}\gamma'_{R(\Delta_{n-1})} - B\sum_{\Delta_n}(-i\gamma_{\Delta_n}\gamma'_{\Delta_n}) \\ &= -A \sum_{\Delta_{n-1}} S^E_{\Delta_{n-1}} - B \sum_{\Delta_n} P_{\Delta_n}. \end{aligned} \tag{4.16}$$

The modified Gauss's law (4.15) on a $(n-2)$-simplex $\Delta_{n-2}$, or equivalently on the dual $(n-2)$-cochain $\boldsymbol{\Delta}_{n-2}$, can be generalized to an arbitrary $(n-2)$-cochain $\lambda_{n-2} = \sum_i \boldsymbol{\Delta}^i_{n-2}$, the Gauss's law is

$$\begin{aligned} G_{\lambda_{n-2}} &= \prod_i G_{\Delta^i_{n-2}} \\ &= \Big( \prod_{\boldsymbol{\Delta}_{n-1}\in\delta\lambda_{n-2}} X_{\Delta_{n-1}} \Big) \Big( \prod_{\Delta_{n-1}{}'} Z_{\Delta_{n-1}{}'}^{\int \delta\lambda_{n-2}\cup_{n-2}\boldsymbol{\Delta}_{n-1}{}'} \Big) (-1)^{\int \lambda_{n-2}\cup_{n-4}\lambda_{n-2} + \lambda_{n-2}\cup_{n-3}\delta\lambda_{n-2}} \\ &= 1, \end{aligned} \tag{4.17}$$

where the sign comes from anti-commutation of $X$ and $Z$ on the same simplex. The derivation uses the following property of higher cup products:

$$A \cup_a B + B \cup_a A = A \cup_{a+1} \delta B + \delta A \cup_{a+1} B + \delta(A \cup_{a+1} B). \tag{4.18}$$

Consider now the following $(n-1)$-form gauge theory defined on a general triangulated $(n+1)$-dimensional manifold $Y$:

$$S(A_{n-1}) = -\sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)| + i\pi \int_Y (A_{n-1}\cup_{n-3} A_{n-1} + A_{n-1}\cup_{n-2}\delta A_{n-1}), \tag{4.19}$$

where $A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2)$, and the gauge symmetry acts by $A_{n-1} \longrightarrow A_{n-1} + \delta\Lambda_{n-2}$ for $\Lambda_{n-2} \in C^{n-2}(Y, \mathbb{Z}_2)$. The second term is the generalized Steenrod square term defined in [24]. The action is gauge-invariant up to a boundary term:

$$\begin{aligned} & S(A_{n-1} + \delta\Lambda_{n-2}) - S(A_{n-1}) \\ =& i\pi \int_{\partial Y} (\Lambda_{n-2}\cup_{n-4}\Lambda_{n-2} + \Lambda_{n-2}\cup_{n-3}\delta\Lambda_{n-2} + \delta\Lambda_{n-2}\cup_{n-2} A_{n-1}) \\ =& i\pi \int_{\partial Y} (\Lambda \cup_{n-4} \Lambda + \Lambda \cup_{n-3} \delta\Lambda + \delta\Lambda \cup_{n-2} A), \end{aligned} \tag{4.20}$$

where we have omited the subscript of $A_{n-1}$ and $\Lambda_{n-2}$ for simplicity. This boundary term determines the Gauss law for the wave-function $\Psi(A)$ on the spatial slice $M = \partial Y$:

$$\Psi(A + \delta\Lambda) = (-1)^{\omega(\Lambda,A)}\Psi(A), \tag{4.21}$$

where $\omega(\Lambda, A) = \int_M (\Lambda \cup_{n-4} \Lambda + \Lambda \cup_{n-3} \delta\Lambda + \delta\Lambda \cup_{n-2} A)$. The Gauss law is the same as the gauge constraint (4.17) if we identify $Z_{\Delta_{n-1}}$ as $(-1)^{A_{n-1}(\Delta_{n-1})}$ and $X_{\Delta_{n-1}}$ acts as the transformation $A_{n-1} \to A_{n-1} + \mathbf{\Delta}_{n-1}$. The modified Gauss's law (4.15) represents the boundary anomaly of topological action (4.19) as we claimed.

In the following section, we derive the Euclidean action of the modified $\mathbb{Z}_2$ lattice gauge theory (4.14) explicitly, which is analogous to (4.19).

**Euclidean path integral of lattice gauge theories**

Start with the Hamiltonian of modified $\mathbb{Z}_2$ lattice gauge theory:

$$\begin{aligned} H &= -A \sum_{\Delta_{n-1}} U_{\Delta_{n-1}} - B \sum_{\Delta_n} W_{\Delta_n} \\ &= -A \sum_{\Delta_{n-1}} X_{\Delta_{n-1}} \Big(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \boldsymbol{\Delta_{n-1}}' \cup_{n-2} \boldsymbol{\Delta_{n-1}}}\Big) - B \sum_{\Delta_n} \prod_{\Delta_{n-1} \subset \Delta_n} Z_{\Delta_{n-1}} \end{aligned} \tag{4.22}$$

with gauge constraints

$$G_{\Delta_{n-2}} = \prod_{\Delta_{n-1} \supset \Delta_{n-2}} X_{\Delta_{n-1}} \Big(\prod_{\Delta_{n-1}'} Z_{\Delta_{n-1}'}^{\int \delta\boldsymbol{\Delta_{n-2}} \cup_{n-2} \boldsymbol{\Delta_{n-1}}'}\Big) = 1. \tag{4.23}$$

The partition function is:

$$\mathcal{Z} = \mathrm{Tr}\, e^{-\beta H} = \mathrm{Tr}\, T^M, \tag{4.24}$$

where we use Trotter-Suzuki decomposition in imaginary time direction and $T$ is the transfer matrix defined as

$$T = \left(\prod_{\Delta_{n-2}} \delta_{G_{\Delta_{n-2}},1}\right) e^{-\delta\tau H}. \tag{4.25}$$

The first factor arises from the gauge constraints on the Hilbert space. The spacetime manifold consists of many time slices labelled by layers $\{i\}$. In the $i$th layer, we insert a complete basis (in Pauli matrix $Z_{\Delta_{n-1}}$): $b_{n-1}^i \in C^{n-1}(M_n, \mathbb{Z}_2)$ ($\mathbb{Z}_2$ fields on each $\Delta_{n-1}$ of the spatial manifold $M_n$ such that $Z_{\Delta_{n-1}} = (-1)^{b_{n-1}^i(\Delta_{n-1})}$). The transfer matrix $T$ between the $i$th layer and the $(i+1)$th layer contains gauge constraints on every spatial $(n-2)$-simplex $\Delta_{n-2}$:

$$\delta_{G_{\Delta_{n-2}},1} = \frac{1 + G_{\Delta_{n-2}}}{2} = \frac{1}{2} \sum_{a_{n-2}^{i+1/2}=0,1} (G_{\Delta_{n-2}})^{a_{n-2}^{i+1/2}}, \tag{4.26}$$

where we introduce the Lagrangian multiplier $a_{n-2}^{i+1/2} \in C^{n-2}(M_n, \mathbb{Z}_2)$ ($\mathbb{Z}_2$ fields on each $\Delta_{n-2}$ of the spatial manifold $M_n$). Notice that $a_{n-2}^{i+1/2}$ defined between two time slices lives on the spatial $(n-2)$-simplex $\Delta_{n-2}$, which can be interpreted as the spacetime $(n-1)$-simplex between the two layers. From the same calculation in Section 3.4, we have

$$\mathcal{Z} = \sum_{\{\{a_{n-2}^{i+1/2}\},\{b_{n-1}^i\}\}} \exp([S_{\text{Ising}} + S_{\text{top}}](\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\})), \tag{4.27}$$

where

$$\begin{aligned} &S_{\text{Ising}}(\{\{a_{n-2}^{i+1/2}\}, \{a_{n-1}^i\}\}) \\ &= \sum_i \left( -J_s \sum_{\Delta_n} |\delta b_{n-1}^i(\Delta_n)| - J_\tau \sum_{\Delta_{n-1}} | \left[ b_{n-1}^i + b_{n-1}^{i+1} + \delta a_{n-2}^{i+1/2} \right] (\Delta_{n-1})| \right) \end{aligned} \tag{4.28}$$

and

$$\begin{aligned} &S_{\text{top}}(\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\}) \\ &= i\pi \sum_i \int_{M_n} a_{n-2}^{i+1/2} \cup_{n-4} a_{n-2}^{i+1/2} + a_{n-2}^{i+1/2} \cup_{n-3} \delta a_{n-2}^{i+1/2} \\ &\qquad + \delta a_{n-2}^{i+1/2} \cup_{n-2} b_{n-1}^{i+1} + b_{n-1}^i \cup_{n-2} (b_{n-1}^i + b_{n-1}^{i+1} + \delta a_{n-2}^{i+1/2}). \end{aligned} \tag{4.29}$$

Here $J_s, J_\tau$ are constants depending on $A, B, \delta\tau$ in the original Hamiltonian and we assume $J_s = J_\tau = J$ for simplicity. $|\cdots|$ gives the argument's parity 0 or 1. The gauge transformations act as

$$\begin{aligned} a_{n-1}^i &\to a_{n-1}^i + \delta\lambda^i, \\ a_{n-2}^{i+1/2} &\to a_{n-2}^{i+1/2} + \delta\mu^i + \lambda^i + \lambda^{i+1}, \end{aligned} \tag{4.30}$$

where $\lambda^i$ are arbitrary $(n-2)$-cochains and $\mu^i$ are arbitrary $(n-3)$-cochains.

If we interpret $a_{n-2}^{i+1/2}$ as spacetime $(n-1)$-cochains, we can rewrite

$$\{\{a_{n-2}^{i+1/2}\}, \{b_{n-1}^i\}\} \to A_{n-1} \in C^{n-1}(Y, \mathbb{Z}_2), \tag{4.31}$$

which is $\mathbb{Z}_2$ fields on $(n-1)$-simplices in spacetime manifold $Y$. It is natural to write $S_{\text{Ising}}$ in (4.28) as

$$S_{\text{Ising}} = -\sum_{\Delta_n \subset Y} |\delta A_{n-1}(\Delta_n)|. \tag{4.32}$$

The spacetime manifold $Y = M_n \times [-\infty, 0]$ (spatial and temporal parts) is not a triangulation, since we only triangularize the spatial manifold $M_n$ under the discretized time. The (higher) cup products are not well-defined in $Y$. However, we can still write an expression

$$S_{\text{top}} = i\pi \int_{Y'} (A_{n-1} \cup_{n-3} A_{n-1} + A_{n-1} \cup_{n-2} \delta A_{n-1}) \tag{4.33}$$

in $(n{+}1)$-dimensional triangulation $Y'$ such that $Y'$ is a refinement of $Y$. We can check that (4.29) and (4.33) produce the same boundary term under gauge transformations.

# Part II

# Constructions of SPT phases

*Chapter 5*

# (2+1)D SPT CONSTRUCTIONS

In this chapter, we construct the supercohomology fSPT and its bosonic dual in (2+1)D. The main result is shown in Fig. 5.1. To construct the supercohomology fSPT protected by $G \times \mathbb{Z}_2^f$, we first consider another bSPT protected by $\tilde{G} = G \rtimes \mathbb{Z}_2$. $\tilde{G}$ is called $\mathbb{Z}_2$ extension of $G$ and will be defined explicitly in Section 5.2. We will show that the bSPT becomes a $\mathbb{Z}_2$ gauge theory after gauging the $\mathbb{Z}_2$ symmetry. This $\mathbb{Z}_2$ gauge theory is equivalent to a fermionic theory according to the exact bosonization developed in Part I. Therefore, from the well-known construction of bSPT, we are able to derive the finite-depth local unitary quantum circuit for supercohomology fSPT phases.

This section is started with the reviewing of the well-known bosonic SPT constructions. We will show the correspondence between bSPT phases and fSPT phases explicitly.

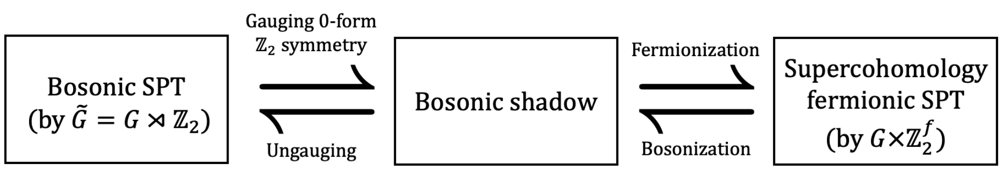



Figure 5.1: To construct a $G_f = G\times\mathbb{Z}_2^f$ supercohomology SPT model in 2d, we start with a model for a particular bosonic SPT phase determined by the supercohomology data $(\rho, \nu)$. The symmetry of this bosonic SPT is $\tilde{G} = G \rtimes \mathbb{Z}_2$ ($G$ extended by $ZZ_2$ according to 2-cocycle $n \in H^2(BG, \mathbb{Z}_2)$). Next, we gauge the $\mathbb{Z}_2$ subgroup of $\tilde{G}$ to build the shadow model, which is a $\mathbb{Z}_2$ lattice gauge theory. We then condense the fermion in the shadow model, or apply the fermionization duality, to obtain a model for the supercohomology SPT phase corresponding to $(\rho, \nu)$.

## 5.1 Review of group cohomology bSPT constructions

The bosonic SPT phases are partially classified by group cohomology [32]. The $d$-dimensional interacting bosonic SPT phases with symmetry $G$ is characterized by $\nu \in H^{d+1}(BG, \mathbb{R}/\mathbb{Z})$. Given the $G$-spins living at vertices, the SPT state can be written as

$$\begin{aligned}|\Psi_{SPT}\rangle &= \sum_{\{g_v\}} \prod_{\langle v_1 v_2 \cdots v_{d+1}\rangle} \exp(2\pi i \nu(1, g_{v_1}, g_{v_2} \cdots g_{v_{d+1}}))|\{g_v\}\rangle \\ &\equiv U \sum_{\{g_v\}} |\{g_v\}\rangle,\end{aligned} \tag{5.1}$$

where the product is over all spatial $d$-simplexes. The Hamiltonian is simply

$$H_{SPT} = U(-\sum_v P_v)U^\dagger, \tag{5.2}$$

where $P_v$ is the projector to the state $\sum_g |g\rangle$ at a vertex $v$. We can also use the inhomogeneous form of the cocycle:

$$\nu(g_0, g_1, \cdots, g_d, g_{d+1}) \equiv \omega(g_0^{-1} g_1, g_1^{-1} g_2, \cdots g_d^{-1} g_{d+1}). \tag{5.3}$$

We denote the configuration of $G$-spins $\{g_v\}$ as $\Phi$ and the SPT state (5.1) becomes

$$|\Psi_{SPT}\rangle = \sum_\Phi \exp(\int_{CM} \omega(d\Phi))|\Phi\rangle, \tag{5.4}$$

where $d\Phi$ is defined in Fig 5.2 and the integral is over spacetime manifold $CM$ (the "cone" of the spatial manifold $M$).

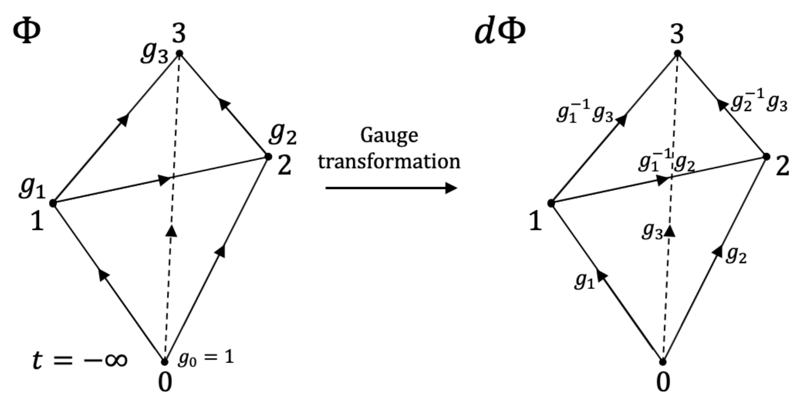



Figure 5.2: The configuration of $G$-spins and the gauging map.

## 5.2 Correspondence between bSPT and supercohomology fSPT

Let us start with the (2+1)D case with trivial product symmetry $G \times Z_2^f$ first [42]. fSPTs have supercohomology data $(\nu, n)$ with $\nu \in C^3(BG, \mathbb{R}/\mathbb{Z})$, $n \in H^2(BG, \mathbb{Z}_2)$. They satisfy Gu-Wen equations:

$$\begin{aligned} \delta\nu &= \frac{1}{2} n \cup n \\ \delta n &= 0. \end{aligned} \tag{5.5}$$

The 2-cocycle $n \in H^2(BG, \mathbb{Z}_2)$ gives the central extension

$$0 \to \mathbb{Z}_2 \to \tilde{G} \to G \to 1. \tag{5.6}$$

The group elements in $\tilde{G}$ are $g^a$ with label $a = 0, 1$ and the group law is

$$g_1^{a_1} g_2^{a_2} = (g_1 g_2)^{a_1 + a_2 + n(1, g_1, g_1 g_2)}. \tag{5.7}$$

We can define a cocycle in $H^3(BG', \mathbb{R}/\mathbb{Z})$ by

$$\alpha_3 = \nu + \frac{1}{2} n \cup \epsilon_1, \tag{5.8}$$

where $\epsilon_1(g_1^{a_1}, g_2^{a_2}) \equiv a_1 + a_2 + n(1, g_1, g_2)$ is a 1-cochain in $G'$ satisfying $\delta\epsilon_1 = n$. One can check that this definition is homogeneous, i.e. $\epsilon_1(h'g_i', h'g_j') = \epsilon(g_i', g_j')$. In inhomogeneous form of $\epsilon_1$, the above definition for $\epsilon_1$ is

$$\begin{aligned} \epsilon_1((g_i^{(a_i)})^{-1} g_j^{(a_j)}) &= \epsilon_1((g_i^{-1} g_j)^{(a_i + a_j + n(1, g_i, g_j))}) \\ &\equiv a_i + a_j + n(1, g_i, g_j), \end{aligned} \tag{5.9}$$

which is equivalent to $\epsilon_1(g^{(a)}) = a$ (we have assumed $n$ is normal, i.e. $n(1, 1, g) = 0$). By this $\alpha_3$, we can construct an auxiliary bosonic SPT (with $\tilde{G}$ symmetry) as

$$\begin{aligned} &|\Psi_b\rangle \\ &= \sum_{\{g_v\},\{a_v\}} \prod_{f = \langle pqr\rangle} e^{2\pi i \alpha_3(1, g_p^{a_p}, g_q^{a_q}, g_r^{a_r}) O_{pqr}} |\{g_v\}, \{a_v\}\rangle \\ &= \sum_{\{g_v\},\{a_v\}} \prod_{f = \langle pqr\rangle} \Big[ e^{2\pi i \nu(1, g_p, g_q, g_r) O_{pqr}} \times \\ &\qquad (-1)^{n(1, g_p, g_q)(a_q + a_r + n(1, g_q, g_r))} \Big] |\{g_v\}, \{a_v\}\rangle, \end{aligned} \tag{5.10}$$

where $f = \langle pqr\rangle$ is a 2-simplex (face) and $O_{pqr}$ denotes the orientation of the simplex $\langle pqr\rangle$ . This state is invariant under multiplication of constant $h^0$ on all vertices, i.e $g_v^{a_v} \to (hg_v)^{a_v + n(1, h, hg_v)}$. In other words, the $|\Psi_{SPT}\rangle$ is invariant under the symmetry action:

$$|\{g_v\}, \{a_v\}\rangle \to |\{hg_v\}, \{a_v + n(1, h, hg_v)\}\rangle. \tag{5.11}$$

The next step is to gauge the 0-form symmetry on $\{a_v\}$. This is a duality mapping $a_v$ degrees of freedom on vertices to those living on edges. The procedure for gauging the 0-form symmetry or higher-form symmetry is reviewed in Appendix D. Since each configuration $\{a_v\}$ can be represented by the cochain $\boldsymbol{a}_v$, the gauging map is defined as

$$\Gamma(|\boldsymbol{a}_v\rangle) = |\delta \boldsymbol{a}_v\rangle, \tag{5.12}$$

where $\delta \boldsymbol{a}_v$ are $\mathbb{Z}_2$ fields living on edges, i.e. $\delta \boldsymbol{a}_v(e_{ij}) = a_i + a_j$, shown in Fig. D.1 in Appendix D. The bosonic shadow wavefunction is

$$\begin{aligned}
|\Psi_s\rangle &= \Gamma(|\Psi_b\rangle) \\
&= \sum_{\{g_v\},\{a_v\}} \prod_{f=\langle pqr\rangle} \Big( e^{2\pi i \nu(1,g_p,g_q,g_r) O_{pqr}} \times \\
&\qquad (-1)^{n(1,g_p,g_q)(a_q+a_r+n(1,g_q,g_r))} \Big) |\{g_v\},\{\delta a_v\}\rangle \\
&= \sum_{\{g_v\},\{a_v\}} \prod_{\langle pqr\rangle} \Big( e^{2\pi i \nu(1,g_p,g_q,g_r) O_{pqr}} (-1)^{\bar{n}_{pq}(a_q+a_r+\bar{n}_{qr})} \Big) \\
&\qquad |\{g_v\},\{a_i + a_j + n(1,g_i,g_j)\}\rangle',
\end{aligned} \tag{5.13}$$

where in the last line, we defined a new basis of states $|\{g_v\},\{a_i + a_j + n(1,g_i,g_j)\}\rangle' \equiv |\{g_v\},\{a_i + a_j\}\rangle$ and $\bar{n} \in C^1(M,\mathbb{Z}_2)$ is a 1-cochain:

$$\bar{n}(e_{ij}) = \bar{n}_{ij} = n(1,g_i,g_j). \tag{5.14}$$

We can introduce the variables $b_e \equiv a_i + a_j + n(1,g_i,g_j)$ ($\mathbb{Z}_2$ fields living on edges) and Pauli operators $Z_e \equiv (-1)^{b_e}$ which measures the $b_e$ variables. Doing so, the symmetry action (5.11) becomes

$$V(h): \ |\{g_v\},\{b_e\}\rangle' \to |\{hg_v\},\{b_e\}\rangle' \tag{5.15}$$

which acts only on the vertex variables $g_v$. The action on $b_e$ cancels because of the cocycle condition of $n$. The bosonic shadow state can be written as

$$\begin{aligned}
|\Psi_s\rangle = &\sum_{\substack{\{g_v\},\{b_e\} \\ \delta b_e(\langle ijk\rangle) = n(g_i,g_j,g_k)}} \prod_{\langle pqr\rangle} \Big[ e^{2\pi i \nu(1,g_p,g_q,g_r) O_{pqr}} \times \\
&\qquad (-1)^{\bar{n}_{pq} b_{qr}} \Big] |\{g_v\},\{b_e\}\rangle',
\end{aligned} \tag{5.16}$$

where we have assumed that the state lives on a simply connected manifold $M$ for simplicity. If $M$ has noncontractible loops, there will be additional conditions on the holonomy of $b_e$ fields. In this gauged state, we notice that there is an additional symmetry:

$$|\{g_v\},\{b_e\}\rangle' \to (-1)^{\int \bar{n}\cup\lambda_e}|\{g_v\},\{b_e\}+\lambda\rangle' \tag{5.17}$$

for any closed 1-cochain $\lambda \in C^1(M,\mathbb{Z}_2)$ where we identify $\mathbb{Z}_2$ fields $\{b_e\}$ as an 1-cochain in $C^1(M,\mathbb{Z}_2)$. If we choose $\lambda_e = \delta\boldsymbol{v}$ for a vertex $v$. This symmetry is

$$\begin{aligned}|\{g_v\},\{b_e\}\rangle' \to &(-1)^{\int n\cup\boldsymbol{v}}|\{g_v\},\{b_e\}+\delta\boldsymbol{v}\rangle' \\ &= (\prod_{e'} Z_{e'}^{\int \delta\boldsymbol{e}'\cup\boldsymbol{v}})\prod_{e\supset v} X_e|\{g_v\},\{b_e\}\rangle' \\ &= \tilde{G}_v|\{g_v\},\{b_e\}\rangle',\end{aligned} \tag{5.18}$$

where we have used $\delta b_e = n$ and the Pauli matrices $X_e$ and $Z_e$ act on the second entry $\{b_e\}$, i.e.

$$\begin{aligned}X_e|\{g_v\},\{b_e\}\rangle' &= |\{g_v\},\{b_e\}+\boldsymbol{e}\rangle' \\ Z_e|\{g_v\},\{b_e\}\rangle' &= (-1)^{b_e}|\{g_v\},\{b_e\}\rangle'\end{aligned} \tag{5.19}$$

with $\tilde{G}_v = (\prod_{e'} Z_{e'}^{\int \boldsymbol{e}'\cup\delta\boldsymbol{v}})\prod_{e\supset v} X_e$. In other words, the bosonic wavefunction is gauge-invariant under

$$\tilde{G}_v|\Psi_s\rangle = |\Psi_s\rangle \tag{5.20}$$

and we can apply our fermion-boson duality (2.28) (with a slightly different convention) to get the dual fermionic wavefunction. More explicitly, we further simplify the state $|\Psi_s\rangle$ by expressing the state as Pauli matrices acting on the groundstate of toric code:

$$\begin{aligned}&|\Psi_s\rangle \\ &= \sum_{\{g_v\},\{a_v\}}\prod_{\langle pqr\rangle}\left(e^{2\pi i\nu(1,g_p,g_q,g_r)O_{pqr}} Z_{qr}^{\bar{n}_{pq}}\right) \\ &\qquad |\{g_v\},\{\delta a_v\}+\bar{n}\rangle' \\ &= \sum_{\{g_v\},\{a_v\}}\prod_{\langle pqr\rangle}\left(e^{2\pi i\nu(1,g_p,g_q,g_r)O_{pqr}}\right) \\ &\qquad \prod_{e'}\left(Z_{e'}^{\int \bar{n}\cup\boldsymbol{e}'}\right)\prod_{e} X_e^{\bar{n}(e)}|\{g_v\},\{\delta a_v\}\rangle' \\ &= \sum_{\{g_v\},\{a_v\}}\prod_{\langle pqr\rangle}\left(e^{2\pi i\nu(1,g_p,g_q,g_r)O_{pqr}}\right)\tilde{U}_{\bar{n}}|\{g_v\},\{\delta a_v\}\rangle',\end{aligned} \tag{5.21}$$

where $\bar{n}$ is the 1-cochain defined in (5.14), and $|\{g_v\},\{\delta a_v\}+\bar{n}\rangle'$ is the shorthand notation of $|\{g_v\},\{a_i+a_j+n(1,g_i,g_j)\}\rangle'$. The bosonic hopping operator $\tilde{U}_e$ is defined as

$$\tilde{U}_e = (\prod_{e'} Z_{e'}^{\int \boldsymbol{e} \cup \boldsymbol{e'}}) X_e \tag{5.22}$$

or more generally for any 1-cochain $\lambda \in C^1(M, \mathbb{Z}_2)$:

$$\tilde{U}_\lambda = (\prod_{e'} Z_{e'}^{\int \lambda \cup \boldsymbol{e'}}) \prod_{e|\lambda(e)=1} X_e. \tag{5.23}$$

(5.23) can be derived from (5.22) by

$$\tilde{U}_{\lambda+\lambda'} \equiv (-1)^{\int \lambda \cup \lambda'} \tilde{U}_{\lambda'} \tilde{U}_\lambda. \tag{5.24}$$

$|\Psi_s\rangle$ in (5.21) is still invariant under the symmetry action (5.15) since we just reorganize the state.

The final step is to apply the fermionization map (2.28):

$$\begin{aligned} &|\Psi_s\rangle \longleftrightarrow |\Psi_f\rangle \\ &= \sum_{\{g_v\}} \prod_{\langle pqr \rangle} \left( e^{2\pi i \nu(1, g_p, g_q, g_r) O_{pqr}} \right) S^E_{\tilde{n}} |\{g_v\}, \{vac\}\rangle_f, \end{aligned} \tag{5.25}$$

where $S^E_e = (-1)^{\int_E \boldsymbol{e}} i \gamma_{L(e)} \gamma'_{R(e)}$ and its definition on general 1-cochain is:

$$S^E_{\lambda+\lambda'} \equiv (-1)^{\int \lambda \cup \lambda'} S^E_{\lambda'} S^E_\lambda, \tag{5.26}$$

where $\lambda,\ \lambda' \in C^1(M, \mathbb{Z}_2)$. The toric code groundstate ($W_f = 1\ \forall f$) is mapped to the vacuum state ($P_f = 1\ \forall f$). The symmetry of this fermionic SPT states $|\Psi_f\rangle$ is

$$|\{g_v\}, \{P_f\}\rangle_f \to |\{h g_v\}, \{P_f\}\rangle_f \tag{5.27}$$

since the duality map is defined as $|\{g_v\}, \{b_e\}\rangle' \to |\{g_v\}, \{P_f = \delta b_e\}\rangle_f$ (up to a sign) and the symmetry action on the bosonic state is simply $|\{g_v\}, \{b_e\}\rangle' \to |\{h g_v\}, \{b_e\}\rangle'$ by (5.15).

*Chapter 6*

# (3+1)D CONSTRUCTIONS

In (2+1)D, the supercohomology fSPT with symmetry group $G$ corresponds to a bSPT with with symmetry group $\tilde{G}$, where $\tilde{G}$ is a $\mathbb{Z}_2$ extension of $G$ by 2-cocycle $n \in H^2(BG, \mathbb{Z}_2)$. For (3+1)D, the structure becomes more complicated. We will show that every supercohomology fSPT with symmetry group $G$ corresponds to a "2-group" bSPT. The "2-group" symmetry contains both 0-form $G$ symmetry and 1-form $\mathbb{Z}_2$ symmetry, with nontrivial coupling between them. The 2-group is formed by the group $G$ extend by $\mathbb{Z}_2$ 1-form symmetry, according the 3-cocycle $H^3(BG, \mathbb{Z}_2)$. The main result for the duality of 3d bSPT and fSPT phases is shown in Fig. 6.1. In this section, we will first introduce the concepts of 2-group, 2-group extension, and 2-gauge theory. We will use these concepts to construct 2-group bSPT in 3d. After gauging the 1-form $\mathbb{Z}_2$ symmetry of this 2-group, the bSPT becomes a $\mathbb{Z}_2$ gauge theory. By the bosonization in Part I, this $\mathbb{Z}_2$ gauge theory is equivalent a fermionic theory, which is the desired supercohomology fSPT in 3d.

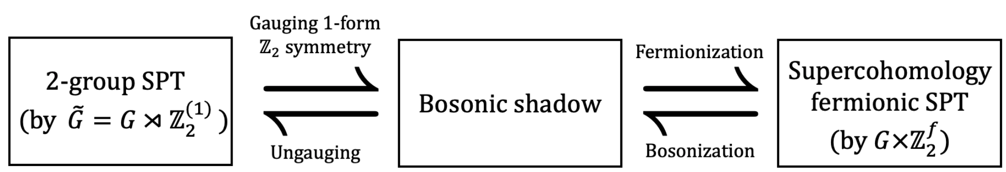



Figure 6.1: To construct a $G_f = G \times \mathbb{Z}_2^f$ supercohomology SPT model in 3d, we start with a model for a particular 2-group SPT phase determined by the supercohomology data $(\rho, \nu)$. Next, we gauge the $\mathbb{Z}_2$ 1-form symmetry of the 2-group to build the shadow model, which is a $\mathbb{Z}_2$ lattice gauge theory. We then condense the fermion in the shadow model, or apply the fermionization duality, to obtain a model for the supercohomology SPT phase corresponding to $(\rho, \nu)$.

## 6.1 2-group, 2-group extension, and 2-gauge theory

### 2-group and 2-group extension

A 2-group $\mathbb{G} = (G, A, t, \alpha)$ is defined as below. $G$ and $A$ are groups, $t : A \to G$ is a group homomorphism, and $\alpha : G \to \mathrm{Aut}(A)$ is an action of $G$ on $A$ such that it satisfies:

$$t(\alpha(g)(a)) = gt(a)g^{-1}, \;\; \alpha(t(a))(a') = aa'a^{-1}. \tag{6.1}$$

Given a $G$-module $A$ (group $G$ with action $\alpha(g)$ on a group $A$), we consider a double extension:

$$1 \to A \xrightarrow{i} A' \xrightarrow{t'} G' \xrightarrow{\pi} G \to 1, \tag{6.2}$$

where $(G', A', t', \alpha')$ is a 2-group and $\alpha'$ induces the action $\alpha$ from $G$ to $A$. By the property of exact sequence, we have $G = t'$ and $A = \ker t'$. It's known that the equivalence classes of the double extension are labeled by elements in $H^3(BG, A)$. We can illustrate how they are related. First, consider a section $s : G \to G'$ and it satisfies the group law projectively:

$$s(g)s(h) = f(g,h)s(gh) \tag{6.3}$$

with $f : G \times G \to \ker \pi$. By associative property, it must satisfy

$$[s(g)f(h,k)s(g)^{-1}]f(g,hk) = f(g,h)f(gh,k). \tag{6.4}$$

Since we have $t' = \ker \pi$, we can lift $f$ to $F : G \times G \to A'$ and (6.4) is satisfied projectively:

$$[\alpha'(s(g))(F(h,k))]F(g,hk) = i(\gamma(g,h,k))F(g,h)F(gh,k), \tag{6.5}$$

where $\gamma \in Z^3(BG, A)$ is a 3-cocycle. It can be shown that the cohomology class of $\gamma$ labels the equivalence class of double extension (6.2).

**2-gauge theory on lattice**

We now define 2-gauge theory, using data $(G, A, \alpha, \gamma)$ decribed in the previous section. For simplicity, we consider $A = \mathbb{Z}_2$ and therefore only trivial $\alpha$ action ($G$ on $\mathbb{Z}_2$) exists. The field configuration is an assigenment of an element $g_e \in G$ to each edge and of an element $b_f \in \mathbb{Z}_2$ to each triangle. A branching structure on the triangulation is chosen. We denote the configuration to be $(\bar{g}, \bar{b})$ with $\bar{g} \equiv \{g_e\}$ and $\bar{b} \equiv \{b_f\}$. $(\bar{g}, \bar{b})$ needs to satisfy some constraints. First, on each triangle $\Delta_{012}$, $\bar{g}$ satisfies:

$$g_{01}g_{12} = g_{02}. \tag{6.6}$$

Second, on each tetrahedron, $(\bar{g}, \bar{B})$ has the following relation:

$$\delta b = b_{012} + b_{013} + b_{023} + b_{123} = \gamma(g_{01}, g_{12}, g_{23}). \tag{6.7}$$

For the convenience, we will use homogeneous notation $\rho$ for cocycle later, i.e. $\rho(1, g, gh, ghk) = \gamma(g, h, k)$.

The 0-form gauge transformation by $\bar{h} = \{h_v\}$ ($h_v \in G$ at each vertex $v$) is

$$\begin{aligned} \bar{g} &\to \bar{g}^h : g_{ij}^h = h_i^{-1} g_{ij} h_j \\ \bar{b} &\to \bar{b}^h : b_{ijk}^h = b_{ijk} + \zeta(g_{ij}, g_{jk}, h_i, h_j, h_k) \end{aligned} \tag{6.8}$$

with $\zeta$ is $\mathbb{Z}_2$-valued function satisfying

$$\delta\zeta(\bar{g}, \bar{h}) = \rho(\bar{g}^h) - \rho(\bar{g}). \tag{6.9}$$

The solution for $\zeta$ always exists since we can consider $\rho$ as the label for Dijkgraaf-Witten theory and gauge transformation of $G$ fields doesn't change the cohomology class of $\rho$. Our choice is $\zeta$ is (derived in appendix E):

$$\begin{aligned} &\zeta(g_{12}, g_{23}, h_1, h_2, h_3) \\ =&\rho(1, g_{12}, g_{12}g_{23}, g_{12}g_{23}h_3) + \rho(1, g_{12}, g_{12}h_2, g_{12}g_{23}h_3) \\ &+ \rho(1, h_1, g_{12}h_2, g_{12}g_{23}h_3). \end{aligned} \tag{6.10}$$

The 1-form symmetry depends on 1-form $\mathbb{Z}_2$-valued cochain $\bar{\lambda} = \{\lambda_e\}$. The transformation is as usual

$$\begin{aligned} g_e &\to g_e \\ b_f &\to b_f + \delta\lambda_e. \end{aligned} \tag{6.11}$$

The theory defined above is called 2-gauge theory.

The classifying space of a 2-group $\mathbb{G}$, denoted as $B\mathbb{G}$, is a 2-gauge theory. It can be described by a $\Delta$-complex structure. $B\mathbb{G}$ contains one vertex and edges labelled by $g \in G$. Its 2-simplices $\langle 012\rangle$ are labeled by $(g_{01}, g_{12}, g_{02}, b_{012})$ such that $g_{01}g_{12} = g_{02}$ and $b_{012} = 0, 1$. Its 3-simplices $\langle 0123\rangle$ contains boundary 2-simplices $\langle 012\rangle$, $\langle 013\rangle$, $\langle 023\rangle$, and $\langle 123\rangle$ such that

$$\begin{aligned} \gamma(g_{01}, g_{12}, g_{23}) =&\rho(1, g_{01}, g_{01}g_{12}, g_{01}g_{12}g_{23}) \\ =&b_{012} + b_{013} + b_{023} + b_{123} \pmod 2. \end{aligned} \tag{6.12}$$

For $n \geq 4$, we glue the $n$-simplex to any $(n-1)$-cycle. More explicitly, we glue the boundary of a $n$-simplex $\langle 01 \dots n\rangle$ with $(n-1)$-simplices $\langle 0 \dots \hat{i} \dots n\rangle$ for $i = 0, 1, \dots, n$, where $\hat{i}$ means that $i$ is omitted. According to Dijkgraaf and Witten, topological gauge theories with gauge group $G$ in $(d+1)$ spacetime dimensions are classified by $H^{d+1}(BG, U(1))$. This was generalized to 2-group gauge theories by Kapustin and Thorngren [50]: the 2-gauge theories with 2-group $\mathbb{G}$ in $(d+1)$ spaectime dimensions is classified by $H^{d+1}(B\mathbb{G}, U(1))$.

### 6.2 Auxiliary “2-group” bSPT

In this section, we construct the supercohomology fermionic SPT phases in (3+1)-dimensions from bosonic 2-group SPT phases. Here we give a quick introduction to 2-group SPTs. A 2-group $\mathbb{G} = (G, \mathbb{Z}_2, \gamma)$ is labelled by two usual groups $G$, $\mathbb{Z}_2$, and a 3-cocycle $\gamma \in H^3(BG, \mathbb{Z}_2)$. A 2-gauge theory on a lattice has $G$ elements $g_e \in G$ on edges $e$ with flat condition $g_{01}g_{12} = g_{02}$ on each face $\langle ijk\rangle$ and $\mathbb{Z}_2$ fields $b_f = \{0, 1\}$ on faces $f$ such that on each tetrahedron $\langle 0123\rangle$

$$\gamma(g_{01}, g_{12}, g_{23}) = \delta b_f(\langle 0123\rangle), \tag{6.13}$$

which is the most crucial condition for 2-gauge theories.

We will construct a 2-group SPT wavefunction for a 2-group $\mathbb{G}$ and given $\alpha_4 \in H^4(B\mathbb{G}, U(1))$. On a spatial slice $M$, the “matter field” $\Phi$ contains $\{g_v\}$ (elements of $G$) on vertices and $\{a_e\}$ (elements of $\mathbb{Z}_2$) on edges. We consider $\Phi = (\{g_v\}, \{a_e\})$ as a gauge transformation operator (6.8) and (6.11) with $\{h_i\} = \{g_v\}$ and $\{\lambda_e\} = \{a_e\}$, and apply this gauge transformation on the trivial configuration in 2-gauge theory on the spacetime manifold $CM$ (the "cone" of the spatial manifold $M$), with $g_e = 1\ \forall e$ and $b_f = 0\ \forall f$ [50]. We obtain a configuration $d\Phi$ in the 2-gauge theory, as shown as Fig. 6.2. We will evaluate the 2-group cocycle $\alpha_4 \in H^4(B\mathbb{G}, U(1))$ on this $d\Phi$.

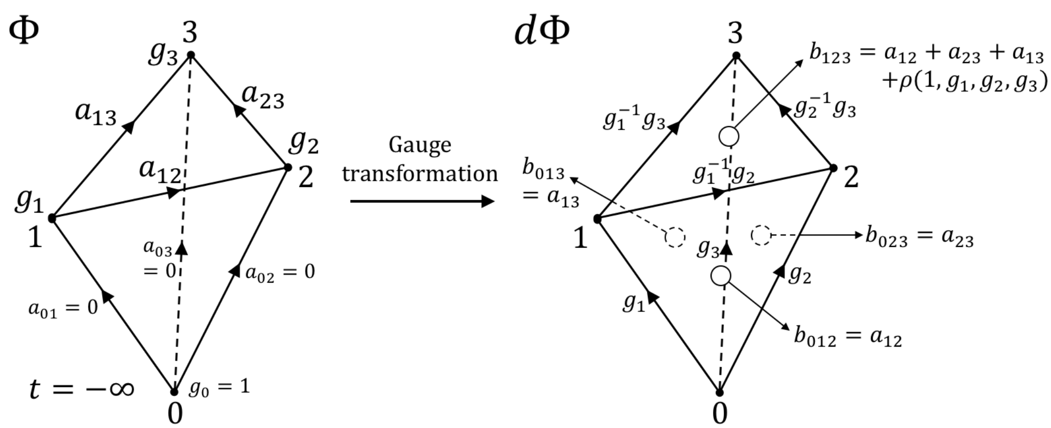



Figure 6.2: We have used $\zeta(1, 1, 1, g, h) = 0$, which is equivalent to normalized $\rho$, i.e. $\rho(1, 1, g, h) = 0$ [50].

The global symmetry transformation rules $(h, \lambda) \in (G, Z^1(M, \mathbb{Z}_2))$ for matter fields $\Phi$ on spatial manifold $M$ can be written as

$$\begin{aligned} g_v &\to h g_v \\ a_{ij} &\to a_{ij} + \lambda_{ij} + \kappa_h(g_i, g_j), \end{aligned} \tag{6.14}$$

where 1-cochain $\kappa$ is the second descendant of $\gamma$ satisfying

$$\delta\kappa_h = \zeta(1, 1, hg_i, hg_j, hg_k) - \zeta(1, 1, g_i, g_j, g_k). \tag{6.15}$$

It can be checked that $d\Phi$ is invariant under this symmetry transformation. The global symmetry contains constant 0-form transformation by $h$ and closed 1-form (i.e. $\delta\lambda = 0$) transformation by $\lambda$.

It is convenient to work with the homogeneous cocycle $\rho$, i.e. $\rho(1, g, gh, ghk) = \gamma(g, h, k)$ with $\rho(hg_1, hg_2, hg_3, hg_4) = \rho(g_1, g_2, g_3, g_4)$. With the choice of $\zeta$ in (6.10), we have

$$\zeta(1, 1, g_i, g_j, g_k) = \rho(1, g_i, g_j, g_k). \tag{6.16}$$

We can further define

$$\kappa_h(g_1, g_2) \equiv \rho(1, h^{-1}, g_1, g_2). \tag{6.17}$$

This $\kappa_h$ satisfies (6.15) by the cocycle condition of $\rho$.

We can evaluate the value of $\alpha_4$ on spacetime manifold $CM$ in an identical way as Chen-Liu-Gu-Wen's bosonic SPT phase [32]. The groundstate wavefunction on $M$ can be written as

$$|\alpha_4\rangle = \sum_{\Phi} \exp\left[2\pi i \int_{CM} \alpha_4(d\Phi)\right]|\Phi\rangle. \tag{6.18}$$

From [46], $\alpha_4 \in H^4(B\mathbb{G}, U(1))$ can be constructed from Gu-Wen supercohomology data $(\nu, \rho)$ with $\nu \in C^4(BG, U(1))$ and $\rho \in Z^3(BG, \mathbb{Z}_2)$ satisfying $\delta\nu = \frac{1}{2}\rho \cup_1 \rho$:

$$\alpha_4 = \nu + \frac{1}{2}\rho \cup_1 \epsilon_2 + \frac{1}{2}\epsilon_2 \cup \epsilon_2, \tag{6.19}$$

where $\epsilon_2$ is the 2-cochain in $C^2(M, \mathbb{Z}_2)$ satisfying $\delta\epsilon_2 = \rho$. To simplify future expressions, we introduce cochains $\bar{\nu} \in C^4(M, \mathbb{R}/\mathbb{Z})$ and $\bar{\rho} \in C^3(M, \mathbb{Z}/2)$

$$\bar{\nu}(1234) = \bar{\nu}(\Delta_{1234}) \equiv \nu(1, g_1, g_2, g_3, g_4), \tag{6.20}$$

$$\bar{\rho}(123) = \bar{\rho}(\Delta_{1234}) \equiv \rho(1, g_1, g_2, g_3). \tag{6.21}$$

It is important to note that these two cochains cannot be pulled-back from some cochain on $BG$.

With this, we choose the cochain $\epsilon_2$ to be

$$\epsilon_2(\Delta_{012}) = a_{12} \tag{6.22}$$

$$\epsilon_2(\Delta_{123}) = \delta a(\Delta_{123}) + \bar{\rho}(g_1, g_2, g_3) \tag{6.23}$$

for time-like faces $\Delta_{012}$ and spacelike faces $\Delta_{123}$, respectively[1].

We can then use this $\alpha_4$ to construct the 2-group SPT wavefunction as the following:

$$
\begin{aligned}
\Psi_{\alpha_4}(\Phi = (\{g_v\},\{a_e\})) &= \exp\left[2\pi i \int_{CM} \alpha_4(d\Phi)\right] \\
= \prod_{t=\langle 1234\rangle} & e^{2\pi i\, \nu(01234) O_t} (-1)^{\epsilon_2(012)\epsilon_2(234)} \\
& (-1)^{\rho(0234)\epsilon_2(012)+\rho(0134)\epsilon_2(123)+\rho(0124)\epsilon_2(234)}.
\end{aligned} \tag{6.24}
$$

Inserting $\epsilon_2$ from (6.23), the final form of the auxillary bosonic SPT in terms of supercohomology data ($\nu$ and $\rho$) is

$$
\begin{aligned}
|\Psi_b\rangle = &\sum_{\{g_v\},\{a_e\}} \prod_{t=\langle 1234\rangle} e^{2\pi i\, \bar{\nu}(1234) O_t} (-1)^{a(12)\delta a(234)} \\
&\times (-1)^{\bar{\rho}(134)(\delta a(123)+\bar{\rho}(123))+\bar{\rho}(124)(\delta a(234)+\bar{\rho}(234))} |\{g_v\},\{a_e\}\rangle.
\end{aligned} \tag{6.25}
$$

**Shadow theory**

The wavefunction we have written down contains the 1-form $\mathbb{Z}_2$ symmetry. This guarantees that when we apply the gauging map $\Gamma$ defined in (D.7), the gauged wavefunction will be the ground state the twisted toric code enriched by $G$.

$$
\begin{aligned}
&|\Psi_s\rangle = \Gamma(|\Psi_b\rangle) \\
&= \sum_{\{g_v\},\{a_e\}} \prod_{t=\langle 1234\rangle} e^{2\pi i\, \bar{\nu}(1234) O_t} (-1)^{a(12)\delta a(234)} \\
&\times (-1)^{\bar{\rho}(134)(\delta a(123)+\bar{\rho}(123))+\bar{\rho}(124)(\delta a(234)+\bar{\rho}(234))} |\{g_v\},\{\delta a_e\}\rangle.
\end{aligned} \tag{6.26}
$$

Next, we perform a basis transformation $|\{g_v\},\{\delta a_e\}\rangle \equiv |\{g_v\},\{\delta a_e\}+\bar{\rho}\rangle'$. Furthermore, we define Pauli matrices which acts on faces $f$ (the second entry of states) as

$$
\begin{aligned}
X_f|\{g_v\},\{b_f\}\rangle' &= |\{g_v\},\{b_f\}+\boldsymbol{f}\rangle' \\
Z_f|\{g_v\},\{b_f\}\rangle' &= (-1)^{b_f}|\{g_v\},\{b_f\}\rangle'.
\end{aligned} \tag{6.27}
$$

[1] This choice is consistent with choosing the cocycle $\rho$ such that $\rho(1,1,g_1,g_2) = 0$ (in our notations, the vertex 0 represents the $t=-\infty$ point and the group element at this vertex is $1 \in G$

The bosonic shadow state (6.26) becomes:

$$
\begin{aligned}
&|\Psi_s\rangle \\
&= \sum_{\{g_v\},\{a_e\}} \prod_{t=\langle 1234\rangle} e^{2\pi i\, \bar{\nu}(1234) O_t} (-1)^{a(12)\delta a(234)} \\
&\quad Z_{123}^{\bar{\rho}(134)} Z_{234}^{\bar{\rho}(124)} |\{g_v\}, \{\delta a_e\} + \bar{\rho}\rangle' \\
&= \sum_{\{g_v\},\{a_e\}} \prod_{t=\langle 1234\rangle} \left(e^{2\pi i\, \bar{\nu}(1234) O_t}\right) \\
&\quad \prod_{f'} \left(Z_{f'}^{\int \bar{\rho}\cup_1 f'}\right) \prod_{f} \left(X_f^{\bar{\rho}(f)}\right) (-1)^{\int a_e \cup \delta a_e} |\{g_v\}, \{\delta a_e\}\rangle' \\
&= \sum_{\{g_v\},\{a_e\}} \prod_{t=\langle 1234\rangle} \left(e^{2\pi i\, \bar{\nu}(1234) O_t}\right) \\
&\qquad \tilde{U}_{\bar{\rho}} (-1)^{\int \boldsymbol{a}_e \cup \delta \boldsymbol{a}_e} |\{g_v\}, \{\delta a_e\}\rangle',
\end{aligned}
\tag{6.28}
$$

where the bosonic hopping operator $\tilde{U}_f$ is:

$$
\tilde{U}_f = (\prod_{f'} Z_{f'}^{\int \boldsymbol{f} \cup_1 \boldsymbol{f}'}) X_f \tag{6.29}
$$

or more generally for any 2-cochain $\boldsymbol{\beta} \in C^2(M, \mathbb{Z}_2)$:

$$
\tilde{U}_\beta = (\prod_{f'} Z_{f'}^{\int \boldsymbol{\beta} \cup_1 \boldsymbol{f}'}) \prod_{f | \boldsymbol{\beta}(f)=1} X_f. \tag{6.30}
$$

(6.30) can be derived from (6.29) by

$$
\tilde{U}_{\beta+\beta'} \equiv (-1)^{\int \boldsymbol{\beta} \cup_1 \boldsymbol{\beta}'} \tilde{U}_{\beta'} \tilde{U}_\beta. \tag{6.31}
$$

Physically, emergent fermions in the twisted toric code are decorated on to junctions of $G$-domain walls according to $\bar{\rho}$. Hence, $\tilde{U}_{\bar{\rho}}$ describes the hopping of such fermions as the domain walls fluctuate.

The state (6.28) is invariant under the symmetry action:

$$
V(h):\ |\{g_v\}, \{b_f\}\rangle' \to |\{hg_v\}, \{b_f\}\rangle' \tag{6.32}
$$

which is derived from (6.14) and definition of states $|\cdots\rangle'$. The next step is to fermionize the bosonic shadow state (6.28).

## 6.3 Fermionic SPT

### Fermionization of the bosonic shadow wavefunction

Consider a $G$-spin paramagnet and a decoupled copy of the twisted toric code:

$$H_s^0 = -\sum_v P_v - \sum_e \tilde{G}_e - \sum_t W_t. \tag{6.33}$$

Here $P_v$ at each vertex $v$ is the projector onto the symmetric state $\frac{1}{\sqrt{|G|}} \sum_{g_v} |g_v\rangle$:

$$P_v = \frac{1}{|G|} \sum_{g_v, g_v'} |g_v'\rangle\langle g_v|. \tag{6.34}$$

$\tilde{G}_e$ is defined as

$$\tilde{G}_e = (\prod_{f'} Z_{f'}^{\int \boldsymbol{f}' \cup_1 \delta \boldsymbol{e}}) \prod_{f \supset e} X_f, \tag{6.35}$$

which commutes with $\tilde{U}_f$ and $W_t$. $\tilde{G}_e$ acts trivially on the state

$$|\Psi_s^0\rangle = \sum_{\{g_v\},\{a_e\}} (-1)^{\int \boldsymbol{a}_e \cup \delta \boldsymbol{a}_e} |\{g_v\}, \{\delta a_e\}\rangle', \tag{6.36}$$

i.e. $\tilde{G}_e|\Psi_s^0\rangle = |\Psi_s^0\rangle$. In other words, $|\Psi_s^0\rangle$ is the groundstate of (6.33). Therefore, $\tilde{G}_e$ acts trivially on (6.28) (because $\tilde{G}_e$ commutes with $\tilde{U}_f$):

$$\tilde{G}_e|\Psi_s\rangle = |\Psi_s\rangle. \tag{6.37}$$

We can now apply the 3d boson-fermion duality (3.24) to $|\Psi_s\rangle$ in (6.28):

$$\begin{aligned} &|\Psi_b\rangle \longleftrightarrow |\Psi_f\rangle \\ &= \sum_{\{g_v\}} \prod_{t=\langle 1234\rangle} \left(e^{2\pi i\, \nu(01234) O_t}\right) S_{\bar{\rho}}^E |\{g_v\}, \{vac\}\rangle_f, \end{aligned} \tag{6.38}$$

where $S_f^E = (-1)^{\int_E f} i\gamma_{L(f)}\gamma'_{R(f)}$ and its definition on general 2-cochain is

$$S_{\beta+\beta'}^E \equiv (-1)^{\int \beta \cup \beta'} S_{\beta'}^E S_\beta^E, \tag{6.39}$$

where $\beta,\ \beta' \in C^2(M, \mathbb{Z}_2)$. The groundstate of twisted toric code has been mapped to the vacuum state. The symmetry of this fermionic SPT states $|\Psi_f\rangle$ is

$$|\{g_v\}, \{P_t\}\rangle_f \to |\{hg_v\}, \{P_t\}\rangle_f \tag{6.40}$$

since the duality map is defined as $|\{g_v\}, \{b_f\}\rangle' \to |\{g_v\}, \{P_t = \delta b_f\}\rangle_f$ (up to a sign) and the symmetry action on the bosonic state is simply $|\{g_v\}, \{b_f\}\rangle' \to |\{hg_v\}, \{b_f\}\rangle'$ by (6.32).

We are going to derive the parent Hamiltonian corresponding to the fermionic SPT state (6.38). We start on bosonic side. The bosonic shadow state $|\Psi_s\rangle$ in (6.28) can be expressed by acting finite-depth local unitary quantum circuit $\hat{U}_s$, which will be defined shortly, on the twisted toric code groundstate $|\Psi_s^0\rangle$ in (6.36):

$$|\Psi_s\rangle = \hat{U}_s|\Psi_s^0\rangle. \tag{6.41}$$

Since $|\Psi_s^0\rangle$ is the groundstate of $H_s^0$ in (6.33), the parent Hamiltonian of $|\Psi_s\rangle$ is

$$H_s = \hat{U}_s H_s^0 \hat{U}_s^\dagger. \tag{6.42}$$

From (6.28), the quantum circuit $\hat{U}_s$ can be defined as

$$\begin{aligned}\hat{U}_s \equiv & \prod_{t=\langle 1234\rangle} \left(\exp(2\pi i\, \nu(01234) O_t) Z_{123}^{\bar{\rho}(134)} Z_{234}^{\bar{\rho}(124)}\right) \\ & \prod_{f} \left(X_f^{\bar{\rho}(f)}\right) \prod_{t=\langle 1234\rangle} \left(W_t^{\bar{\rho}(123)+\bar{\rho}(134)}\right) \\ = & \prod_{t=\langle 1234\rangle} \left(\exp(2\pi i\, \nu(01234) O_t)\right) \tilde{U}_{\bar{\rho}} \prod_t \left(W_t^{\int \bar{\rho}\cup_2 t}\right),\end{aligned} \tag{6.43}$$

where $\bar{\nu} \in C^3(M, \mathbb{R}/\mathbb{Z})$ is an operator ($\mathbb{R}/\mathbb{Z}$-valued function) acting on tetrahedrons $t = \langle 1234\rangle$ of the spatial manifold $M$: $\bar{\nu}(t) \equiv \nu(1, g_1, g_2, g_3, g_4)$. Notice that we have include the factor of $W_t$ before $\tilde{U}_{\bar{\rho}}$. When acting on the $|\Psi_b^0\rangle$, this factor becomes trivial since the groundstate of twisted toric code is a superposition of states with $W_t = 1\ \forall t$. However, this $W_t$ will be important when we show the symmetry property of this circuit.

Then, we can use our fermionization procedure to get the fermionic SPT Hamiltonian and wavefunction.

$$\begin{aligned}H_f^0 =& -\sum_v P_v - \sum_t (-i\gamma_t\gamma_t') \\ \hat{U}_f =& \prod_t \left(\exp(2\pi i\, \bar{\nu}(t) O_t)\right) S_{\bar{\rho}}^E \prod_t \left(P_t^{\int \bar{\rho}\cup_2 t}\right) \\ =& \left(\prod_t \exp(2\pi i\, \bar{\nu}(t) O_t)\right) \\ & \left((-1)^{\sum_{f<f'\in\bar{\rho}} \int f\cup_1 f'} \prod_{f\in\bar{\rho}} \left((-1)^{\int_E f} i\gamma_{L(f)}\gamma'_{R(f)}\right)\right) \\ & \left(\prod_t (-i\gamma_t\gamma_t')^{\int \bar{\rho}\cup_2 t}\right) \\ H_f =& \hat{U}_f H_f^0 \hat{U}_f^\dagger.\end{aligned} \tag{6.44}$$

As a reminder, the explicitly formula for $S^E_{\bar{\rho}}$ is described in (3.19). $f \in \bar{\rho}$ means that $\bar{\rho}(f) = 1$. The product $\prod_{f\in\bar{\rho}} S_f$ itself depends on the order of multiplying $S_f$ since the fermionic hopping operators don't commute with each other. The sign factor in front of $\prod_{f\in\bar{\rho}} S_f$ resolves this ambiguity of the multiplying order. $P_v$ is the projector to state $\frac{1}{\sqrt{|G|}} \sum_{g_v} |g_v\rangle$. $\gamma_t$ and $\gamma'_t$ are majorana fermion operators, forming a complex fermion at each tetrahedron $t$. The groundstate $|\Psi_f\rangle$ of $H_f$ is the groundstate $|\Psi^0_f\rangle$ of $H^0_f$ evolved by circuit $\hat{U}_f$. When we apply the quantum circuit $\hat{U}_f$ on $|\Psi^0_f\rangle$, we can ignore the last factor in $\hat{U}_f$ since it's 1. This is the (3+1)D fermionic SPT state and Hamiltonian. The circuit is symmetric:

$$V(g)\hat{U}_f V(g)^\dagger = \hat{U}_f \tag{6.45}$$

and the stacking law of two fermionic SPT phases is

$$\hat{U}_f(\nu_1, \rho_1)\hat{U}_f(\nu_2, \rho_2) = \hat{U}_f(\nu_1 + \nu_2 + \frac{1}{2}\rho_1 \cup_2 \rho_2, \rho_1 + \rho_2). \tag{6.46}$$

*Chapter 7*

# SUMMARY

In Part I, we have described a $n$-dimensional ($n \geq 2$) analog of the Jordan-Wigner transformation. We started by 2d square lattice and showed the bosonization formalism explicitly: an arbitrary fermionic system can be mapped to Pauli matrices while preserving the locality of the Hamiltonian. After introducing the (higher) cup product notations, the bosonization can be extended to arbitrary triangulation in all dimensions easily. For simply-connected space, this bosonization gives a duality between any fermionic system in arbitrary $n$ spatial dimensions and a new class of $(n-1)$-form $\mathbb{Z}_2$ gauge theories in $n$ dimensions with a modified Gauss's law (gauge constraint). Several examples of 2d bosonization, including free fermions on square and honeycomb lattices and the Hubbard model, and 3d bosonization, including a solvable $\mathbb{Z}_2$ lattice gauge theory with Dirac cones in the spectrum, have been discussed. The key property of this bosonization formalism is the explicit dependence on the second Stiefel-Whitney class and a choice of spin structure on the manifold, which is a feature for defining fermions. To establish this, we have derived a new formula for Stiefel-Whitney homology classes on lattices. We also derive the Euclidean actions for the corresponding lattice gauge theories from the bosonization. The topological actions contain Chern-Simons terms for $(2+1)$D or Steenrod Square terms for general dimensions.

In Part II, we apply the bosonization technique to construct various bosonic or fermionic SPT phases. We showed that for any supercohomolgy SPT with symmetry $G \times \mathbb{Z}_2^f$, we are able to find a special bosonic SPT which can be dualized to a fermionic model. The bosonic SPT is protected by a different symmetry $\tilde{G}$. In 2d, group $\tilde{G}$ is simply a $\mathbb{Z}_2$ extension of $G$. In 3d, $\tilde{G}$ becomes a 2-group: 0-form $G$ extended by 1-form $\mathbb{Z}_2$. The bosonization developed in Part II create a duality between any supercohomology fermionic SPT phase and a higher-group bosonic SPT phase.

*Appendix A*

# CHAINS AND COCHAINS NOTATIONS AND (HIGHER) CUP PRODUCTS ON A TRIANGULATION AND A CUBIC LATTICE

In this section, we review some concepts in algebraic topology and also introduce notations used in this paper. We will always work with an arbitrary triangulation of a simply-connected $n$-dimensional manifold $M_n$ equipped with a branching structure (orientations on edges without forming a loop in any triangle). The vertices, edges, faces, and tetrahedra are denoted $v, e, f, t$, respectively. A general $d$-simplex is denoted as $\Delta_d$. We can label the vertices of $\Delta_d$ as $0, 1, 2, \ldots, d$ such that the orientations of edges are from the smaller number to the larger number. We denote this $d$-simplex as $\Delta_d = \langle 01 \ldots d \rangle$. A finite linear combination of $d$-simplices with coefficients in $\mathbb{Z}_2 = \{0, 1\}$ is called a $d$-chain. $d$-chains form an abelian group denoted $C_d(M_n, \mathbb{Z}_2)$. A $\mathbb{Z}_2$-valued $d$-chain can be identified with a finite set of $d$-simplices (the set consisting of those simplices whose coefficients are nonzero). A $\mathbb{Z}_2$-valued $d$-cochain is a function from the set of $d$-simplices to $\mathbb{Z}_2$. The set of all $d$-cochains is an abelian group denoted $C^d(M_n, \mathbb{Z}_2)$, For example, a 0-cochain assigns 0 or 1 to all vertices, and a 1-cochain assigns 0 or 1 to all edges. A $d$-cochain can be evaluated on any $d$-chain by evaluating the cochain on each simplex of the $d$-chain and adding up the results modulo 2.

For every vertex $v$ we define its dual 0-cochain $\boldsymbol{v}$, which takes value 1 on $v$, and 0 otherwise, i.e. $\boldsymbol{v}(v') = \delta_{v,v'}$. Similarly, $\mathbf{e}$ is an 1-cochain $\boldsymbol{e}(e') = \delta_{e,e'}$, and so forth. All dual cochains will be denoted in bold. An evaluation of a cochain $\boldsymbol{c}$ on a chain $c'$ will sometimes be denoted $\int_{c'} \boldsymbol{c}$. For example, if $c'$ is a 2-chain and $\boldsymbol{c}$ is a 2-cochain, $\int_{c'} \boldsymbol{c} \equiv \boldsymbol{c}(c') = \sum_{f \in c'} \boldsymbol{c}(f)$. When the integration range is not written, $\boldsymbol{c}$ is assumed to be the top dimension and $\int \boldsymbol{c} \equiv \int_{M_n} \boldsymbol{c}$. A $d$-cochain $\boldsymbol{c}_d \in C^d(M_n, \mathbb{Z}_2)$ can be thought of as a collection of $\mathbb{Z}_2$-valued variables, one for each $d$-simplex $\Delta_d$, whose values given by $\boldsymbol{c}_d(\Delta_d)$. We will limit ourselves to the case of $\mathbb{Z}_2$-valued cochains, since this is all we need in this paper.

The boundary operator is denoted by $\partial$. For an $n$-simplex $\Delta_n$, $\partial\Delta_n$ consists of all boundary $(n-1)$-simplices of $\Delta_n$. The coboundary operator is denoted by $\delta$ (not to be confused with the Kronecker delta previously). For example, for a 0-cochain $\boldsymbol{v}$, $\delta\boldsymbol{v}$ is a function on the set of edges which takes value 1 on $e$ if $\partial e$ contains $v$, and takes value 0 otherwise:

$$\delta \boldsymbol{v}(e_{ij}) = \boldsymbol{v}(\partial e_{ij}) = \boldsymbol{v}(v_i + v_j) = \delta_{v,v_i} + \delta_{v,v_j}.$$

The notation $\Delta_n^1 \supset \Delta_{n'}^2$ or $\Delta_{n'}^2 \subset \Delta_n^1$ means that the simplex $\Delta_n^1$ contains $\Delta_{n'}^2$ as one of its faces. For example, a simplex $f = \langle 012 \rangle$ contains the edges $e = \langle 01 \rangle, \langle 02 \rangle, \langle 12 \rangle \subset f$.

In the case of a general triangulation, our bosonization procedure is based on the properties of the cup product $\cup$ and the higher cup product $\cup_1$. These mathematical operations have been defined by Steenrod [23] (see also Appendix B in [56] for a review) for an arbitrary simplicial complex, but not for a cubic lattice. Below we will define these operations on a triangulation and then describe a version which works for a cubic lattice.

On a lattice triangulation , the cup product $\cup$ of a $p$-cochain $\alpha_p$ and a $q$-cochain $\beta_q$ is a $(p+q)$-cochain defined as [57]:

$$\begin{aligned} &[\alpha_p \cup \beta_q](\langle 0, 1, \ldots, p+q \rangle) \\ &= \alpha_p(\langle 0, 1, \ldots, p \rangle)\beta_q(\langle p, p+1, \ldots, p+q \rangle) \\ &= \alpha_p(0 \sim p)\beta_q(p \sim p+q). \end{aligned} \tag{A.1}$$

The definition of the higher cup product [20, 23] is

$$\begin{aligned} &[\alpha_p \cup_a \beta_q](0, 1, \cdots, p+q-a) = \\ &\sum_{0 \le i_0 < i_1 < \cdots < i_a \le p+q-a} \alpha_p(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \cdots) \\ &\qquad \times \beta_q(i_0 \sim i_1, i_2 \sim i_3, \cdots), \end{aligned} \tag{A.2}$$

where $i \sim j$ represents the integers from $i$ to $j$, i.e. $i, i+1, \ldots, j$, and $\{i_0, i_1, \ldots, i_a\}$ are chosen such that the arguments of $\alpha_p$ and $\beta_q$ contain $p+1$ and $q+1$ numbers separately. For arbitrary $\mathbb{Z}_2$-cochains $A$ and $B$, the cup products satisfy this identity:

$$\begin{aligned} &A \cup_a B + B \cup_a A \\ &= \delta(A \cup_{a+1} B) + \delta A \cup_{a+1} B + A \cup_{a+1} \delta B \end{aligned} \tag{A.3}$$

with $\cup_0 \equiv \cup$ and $\cup_{-1} \equiv 0$.

To generalize these formulas to the cubic lattice, we first develop an intuition for the cup product $\cup$. On a triangle $\Delta_{012}$, the usual cup product for two 1-cochains $\lambda$ and $\lambda'$ is

$$\lambda \cup \lambda'(012) = \lambda(01)\lambda'(12). \tag{A.4}$$

We can think of it as starting from vertex 0, passing through edges 01 and 12 consecutively, and ending at vertex 2, all the while following the orientation of the edges. Following the same logic, it is intuitive to define the cup product on a square $\square_{0134}$ (the bottom face in Fig. A.1) as

$$\lambda \cup \lambda'(\square_{0134}) = \lambda(01)\lambda'(14) + \lambda(03)\lambda'(34). \tag{A.5}$$

The two terms come from two oriented paths from vertex 0 to vertex 4. If $\lambda$ and $\beta$

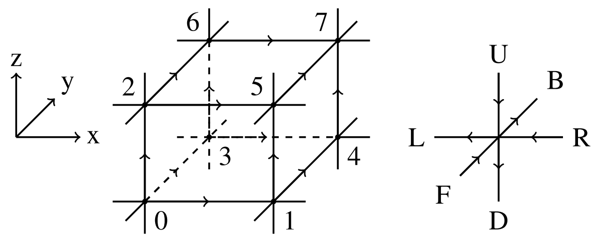



Figure A.1: There are six faces for each cube $c$. U,D,F,B,L,R stand for faces on direction "up","down","front","back","left","right". We assign the face U, F, R to be inward and D, B, L to be outward. The $\cup_1$ product on two 2-cochain is defined by $\beta \cup_1 \beta'(c) = \beta(L)\beta'(B) + \beta(L)\beta'(D) + \beta(B)\beta'(D) + \beta(U)\beta'(F) + \beta(U)\beta'(R) + \beta(F)\beta'(R)$

are a 1-cochain and a 2-cochain, the usual cup product is

$$\begin{aligned} \lambda \cup \beta(0123) &= \lambda(01)\beta(123) \\ \beta \cup \lambda(0123) &= \beta(012)\lambda(23). \end{aligned} \tag{A.6}$$

On the cubic lattice, the corresponding cup products are defined as follows:

$$\begin{aligned} &\lambda \cup \beta(c) \\ &= \lambda(01)\beta(\square_{1457}) + \lambda(02)\beta(\square_{2567}) + \lambda(03)\beta(\square_{3467}) \\ &\beta \cup \lambda(c) \\ &= \beta(\square_{0236})\lambda(67) + \beta(\square_{0125})\lambda(57) + \beta(\square_{0134})\lambda(47), \end{aligned} \tag{A.7}$$

where $c$ is a cube whose vertices are labeled in Fig. A.1. For a cup product involving 0-cochains $\alpha$, the definition is straightforward:

$$\begin{aligned} \alpha \cup \beta(\square_{0134}) &= \alpha(0)\beta(\square_{0134}) \\ \beta \cup \alpha(\square_{0134}) &= \beta(\square_{0134})\alpha(4) \\ \alpha \cup \lambda(01) &= \alpha(0)\lambda(01) \\ \lambda \cup \alpha(01) &= \lambda(01)\alpha(1). \end{aligned} \tag{A.8}$$

With the above definitions, it can be checked that the following equalities hold for cubic cochains of degrees 0, 1, and 2:

$$\begin{aligned} \boldsymbol{e}_1 \cup \delta \boldsymbol{e}_2 &= \delta \boldsymbol{e}_1 \cup \boldsymbol{e}_2 + \delta(\boldsymbol{e}_1 \cup \boldsymbol{e}_2) \\ \boldsymbol{v} \cup \delta \boldsymbol{f} &= \delta \boldsymbol{v} \cup \boldsymbol{f} + \delta(\boldsymbol{v} \cup \boldsymbol{f}). \end{aligned} \tag{A.9}$$

The next step is to define the $\cup_1$ product on the cubic lattice. It need not satisfy all the properties that $\cup_1$ has on a triangulation. The only properties of $\cup_1$ that we need are the anti-commutativity for faces with the same direction and the identity we used in (3.18), (3.26), and (3.60):

$$\int \boldsymbol{e}_1 \cup \delta \boldsymbol{e}_2 + \delta \boldsymbol{e}_2 \cup \boldsymbol{e}_1 = \int \delta \boldsymbol{e}_1 \cup_1 \delta \boldsymbol{e}_2 \pmod 2. \tag{A.10}$$

Therefore, we only need to define $\cup_1$ product for two 2-cochains so that it satisfies (A.10). Our convention for $\cup_1$ is shown in Fig. A.1:

$$\begin{aligned} \beta \cup_1 \beta'(c) = &\beta(L)\beta'(B) + \beta(L)\beta'(D) + \beta(B)\beta'(D) \\ &+ \beta(U)\beta'(F) + \beta(U)\beta'(R) + \beta(F)\beta'(R). \end{aligned} \tag{A.11}$$

Once the $\cup$ and $\cup_1$ products are defined on the cubic lattice, the bosonization procedure on a general triangulation can be applied to the cubic lattice. The formalism derived in Sec. 3.1 is just a special case of Sec. 3.2. In the derivation, (3.12) and (3.14) are modified as follows:

$$\begin{aligned} S_{\delta e} &= (-1)^{\sum_{f < f' \in \delta \boldsymbol{e}} \boldsymbol{f} \cup_1 \boldsymbol{f}'} \prod_{f \in \delta \boldsymbol{e}} S_f \\ &= \prod_{c | e \in \{01,14,02,47,67,26\}} W_c \end{aligned} \tag{A.12}$$

$$\begin{aligned} &Z_{f_1} Z_{f_2} \prod_{f' \subset c} Z_{f'}^{(\boldsymbol{f}_1+\boldsymbol{f}_2)\cup_1 \boldsymbol{f}' + \boldsymbol{f}' \cup_1 (\boldsymbol{f}_1+\boldsymbol{f}_2)} \\ &= \begin{cases} W_c, & \text{if } e \in \{01, 14, 02, 47, 67, 26\} \\ 0, & \text{otherwise} \end{cases} \end{aligned} \tag{A.13}$$

for faces $f_1$ and $f_2$ join at the edge $e$. We implicitly choose $w_2 = 0$. We can use the $\cup_1$ product defined above to reproduce the fermionic hopping terms defined by framing in Fig. 3.1. The hopping term defined by Eq. (3.10) is

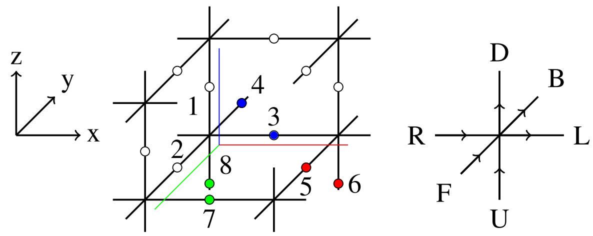



Figure A.2: (Color online) We rotate the axis U,D,F,B,R,L in Fig. A.1 to match the result in Fig. 3.1. Notice that the cube above is dual lattice and edges $1, 2, 3...$ in the dual lattice represent faces in the original lattice.

$$U_f = X_f \prod_{f'} Z_{f'}^{\int f' \cup_1 f}. \tag{A.14}$$

Fig. A.2 is dual to Fig. A.1. Therefore, faces in Fig. A.1 become edges in Fig. A.2. Consider the hopping term along dual edge 3. On the dual vertex to the right, it represents the face R. From terms $\beta(F)\beta'(R)$ and $\beta(U)\beta'(R)$, the hopping term contains $Z_5$ (from F) and $Z_6$ (from U). On the dual vertex to the left, it represents the face L. Since there is no $\beta(D)\beta'(L)$ or $\beta(B)\beta'(L)$ term, it contributes nothing. So we have

$$U_3 = X_3 Z_5 Z_6. \tag{A.15}$$

Similarly, for edge 2, the hopping term has $Z_7$ (from $\beta(L)\beta'(B)$) and $Z_8$ (from $\beta(U)\beta'(F)$)

$$U_2 = X_2 Z_7 Z_8. \tag{A.16}$$

For edge 1, the hopping term has $Z_3$ (from $\beta(L)\beta'(D)$) and $Z_4$ (from $\beta(B)\beta'(D)$)

$$U_1 = X_1 Z_3 Z_4. \tag{A.17}$$

We get the exact same hopping terms defined by "framing" in Fig. 3.1. We can interpret the choice of framing as a definition of $\cup_1$ product on the cubic lattice.

*Appendix B*

# A FORMULA FOR STIEFEL-WHITNEY HOMOLOGY CLASSES

In this section, we prove the following Lemma:

**Lemma 1.** *In n-dimension manifold with triangulation and branching structure, the homology class of $w_2$ can be represented by a $(n-2)$-chain $w_2 \in C_{n-2}(M_n, \mathbb{Z}_2)$:*

$$w_2 = \sum_{\Delta_{n-2}} c(\Delta_{n-2})\Delta_{n-2}, \tag{B.1}$$

*where*

$$\begin{aligned} c(\Delta_{n-2}) =& 1 + \sum_{\text{“}-\text{”-oriented } \Delta_n = \langle 0\ldots n\rangle} \sum_{j_1<j_2 | j_1, j_2 \in even} \mathbf{\Delta}_{n-2}(\langle 0 \cdots \hat{j_1} \cdots \hat{j_2} \cdots n\rangle) \\ &+ \sum_{\text{“}+\text{”-oriented } \Delta_n = \langle 0\ldots n\rangle} \sum_{k_1<k_2 | k_1, k_2 \in odd} \mathbf{\Delta}_{n-2}(\langle 0 \cdots \hat{k_1} \cdots \hat{k_2} \cdots n\rangle). \end{aligned} \tag{B.2}$$

First, let us recall the theorem proved in [58]. Let $s$ be a $p$-simplex, say $s = \langle v_0, v_1, \ldots, v_p\rangle$. Let $k$ be another simplex which has s as a face; i.e., $s \subset k$ ($s$ may be equal to $k$). Let

$$\begin{aligned} B_{-1} &= \text{set of vertices of } k \text{ less than } v_0, \\ B_0 &= \text{set of vertices of } k \text{ between } v_0 \text{ and } v_1, \\ B_m &= \text{set of vertices of } k \text{ between } v_m \text{ and } v_{m+1}, \\ B_p &= \text{set of vertices of } k \text{ greater than } v_p. \end{aligned} \tag{B.3}$$

We say that $s$ is regular in $k$, if $\#(B_m) = 0$ for every odd $m$. Let $\partial_p(k)$ denote the mod 2 chain which consists of all $p$-dimensional simplices $s$ in $k$ so that $s$ is regular in $k$. For example, $\langle 012\rangle$ and $\langle 023\rangle$ are regular in $\langle 0123\rangle$ and therefore $\partial_2(\langle 0123\rangle) = \langle 012\rangle + \langle 023\rangle$. The theorem is [58]:

**Theorem 1.** $\sum_{k|\dim k \geq (n-2)} \partial_{n-2}(k)$ *is a $(n-2)$-chain which represents $w_2$.*

In particular, for any $n'$-simplex $\Delta_{n'} = \langle 0 \ldots n'\rangle$, all $(n'-1)$-simplices regular in $\Delta_{n'}$ are

$$\langle 0 \ldots \hat{i} \ldots n\rangle \ \forall i \in \text{odd} \tag{B.4}$$

and all $(n'-2)$-simplices regular in $\Delta_{n'}$ are

$$\langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n\rangle \ \forall i \in \text{odd}, j \in \text{even}, i < j. \tag{B.5}$$

We now use this theorem to prove lemma 1.

*Proof of Lemma 1.* For every $(n-2)$-simplex $\Delta_{n-2}$, it is regular in itself. This contributes the 1 in the coefficient of $c(\Delta_{n-2})$ in (B.2).

For every $(n-1)$-simplex $\Delta_{n-1}$, it is a boundary of two $n$-simplices $\Delta_n^L$ and $\Delta_n^R$, with $\Delta_{n-1}$ being an outward boundary of $\Delta_n^L$ and an inward boundary of $\Delta_n^R$. We define that $\Delta_{n-1}$ belongs to $\Delta_n^R$ and the summation of dim $k = n-1,\ n$ in theorem 1 can be written as:

$$\begin{aligned}
&\sum_{\Delta_{n-1}} \partial_{n-2}(\Delta_{n-1}) + \sum_{\Delta_n} \partial_{n-2}(\Delta_n) \\
&= \sum_{\Delta_n} \left[ \partial_{n-2}(\Delta_n) + \sum_{\Delta_{n-1}\in\Delta_n|\Delta_{n-1} \text{ is inward}} \partial_{n-2}(\Delta_{n-1}) \right].
\end{aligned} \tag{B.6}$$

If $\Delta_n = \langle 0 \ldots n\rangle$ is "$+$"-oriented, the terms in the summation is

$$\begin{aligned}
&\partial_{n-2}(\langle 0 \ldots n\rangle) + \sum_{0\le i\le n|i\in\text{odd}} \partial_{n-2}(\langle 0 \ldots \hat{i} \ldots n\rangle) \\
&= \sum_{i,j|i<j,\ i\in\text{odd},\ j\in\text{even}} \langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n\rangle \\
&\quad + \sum_{0\le i\le n|i\in\text{odd}} \Big( \sum_{j<i|j\in\text{odd}} \langle 0 \ldots \hat{j} \ldots \hat{i} \ldots n\rangle + \sum_{j>i|j\in\text{even}} \langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n\rangle \Big) \\
&= \sum_{i,j|i<j,\ i\in\text{odd},\ j\in\text{odd}} \langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n\rangle,
\end{aligned} \tag{B.7}$$

where we have used the definition of regular simplex defined above. Similarly, we can derive that if $\Delta_n = \langle 0 \ldots n\rangle$ is "$-$"-oriented, the term is

$$\sum_{i,j|i<j,\ i\in\text{even},\ j\in\text{even}} \langle 0 \ldots \hat{i} \ldots \hat{j} \ldots n\rangle. \tag{B.8}$$

Combining (B.7) and (B.8) with the 1 from dim $k = n-2$ in theorem 1, we have

$$w_2 = \sum_{\Delta_{n-2}} c(\Delta_{n-2})\Delta_{n-2}, \tag{B.9}$$

where

$$\begin{aligned}
&c(\Delta_{n-2}) = \\
&1 + \sum_{\text{"}-\text{"-oriented } \Delta_n=\langle 0\ldots n\rangle} \ \sum_{j_1<j_2|j_1,j_2\in\text{even}} \boldsymbol{\Delta}_{n-2}(\langle 0 \cdots \hat{j}_1 \cdots \hat{j}_2 \cdots n\rangle) \\
&+ \sum_{\text{"+"-oriented } \Delta_n=\langle 0\ldots n\rangle} \ \sum_{k_1<k_2|k_1,k_2\in\text{odd}} \boldsymbol{\Delta}_{n-2}(\langle 0 \cdots \hat{k}_1 \cdots \hat{k}_2 \cdots n\rangle).
\end{aligned} \tag{B.10}$$

*Appendix C*

# IDENTITY FOR FERMIONIC ALGEBRA

In this section, we will derive the constraints on fermionic operators:

$$(-1)^{\int_{w_2} \Delta_{n-2}} S_{\delta\Delta_{n-2}} \prod_{\Delta_n} P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2}} = 1. \tag{C.1}$$

This follows directly from the following two lemmas.

**Lemma 2.** *The Majorana operators in $S_{\delta\Delta_{n-2}}$ cancel out with Majorana operators in $\prod_{\Delta_n} P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2}}$.*

**Lemma 3.** *The sign difference of $S_{\delta\Delta_{n-2}}$ and the product of $P_{\Delta_n}$ is $-(-1)^{\sum_{i=1}^{d} \int \mathbf{\Delta}_{n-1}^{i-1} \cup_{n-2} \mathbf{\Delta}_{n-1}^{i}}$ where we order $(n-1)$-simplices $\{\Delta_{n-1} | \Delta_{n-1} \supset \Delta_{n-2}\}$ counterclockwise as*

$$\mathbf{\Delta}_{n-1}^1, \mathbf{\Delta}_{n-1}^2, \ldots, \mathbf{\Delta}_{n-1}^{d-1}, \mathbf{\Delta}_{n-1}^{d} \equiv \mathbf{\Delta}_{n-1}^{0},$$

*as shown in Fig. C.2. This sign is a chain representative of 2nd Stiefel-Whitney class:*

$$-(-1)^{\sum_{i=1}^{d} \int \mathbf{\Delta}_{n-1}^{i-1} \cup_{n-2} \mathbf{\Delta}_{n-1}^{i}} = (-1)^{\int_{w_2} \Delta_{n-2}}. \tag{C.2}$$

*Proof of Lemma 2.* Let us denote $\Delta_n = \langle 01 \ldots n \rangle$ formed by $\Delta_{n-2}$ and two $(n-1)$-simplex $\Delta_{n-1}^L$ and $\Delta_{n-1}^R$, shown in Fig. C.1(a). We know that $S_{\delta\Delta_{n-2}}$ contains $\gamma_{\Delta_n}\gamma'_{\Delta_n}$ if and only if $\Delta_{n-1}^L, \Delta_{n-1}^R$ are one inward boundary and one outward boundary of $n$-simplex $\Delta_n$, as indicated in Fig. C.1(b) and (c).

For the product of $P_{\Delta_n}$, we simplify the integral as

$$\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2} = \int \delta\mathbf{\Delta}_{n-2} \cup_{n-1} \mathbf{\Delta}_n. \tag{C.3}$$

The contribution of $\Delta_n = \langle 01 \ldots n \rangle$ to (C.3) is

$$\begin{aligned}
&[(\mathbf{\Delta}_{n-1}^L + \mathbf{\Delta}_{n-1}^R) \cup_{n-1} \mathbf{\Delta}_{\boldsymbol{n}}](\langle 01 \ldots n \rangle) \\
=& \sum_{0 \le i_0 < i_1 < \cdots < i_{n-1} \le n} (\mathbf{\Delta}_{n-1}^L + \mathbf{\Delta}_{n-1}^R)(0 \sim i_0, i_1 \sim i_2, i_3 \sim i_4, \cdots) \mathbf{\Delta}_n (i_0 \sim i_1, i_2 \sim i_3, \cdots) \\
=& \sum_{0 \le j \le n | j \in \text{odd}} (\mathbf{\Delta}_{n-1}^L + \mathbf{\Delta}_{n-1}^R)(\langle 0 \ldots \hat{j} \ldots n \rangle) \mathbf{\Delta}_n (\langle 01 \ldots n \rangle) \\
=& \sum_{0 \le j \le n | j \in \text{odd}} (\mathbf{\Delta}_{n-1}^L + \mathbf{\Delta}_{n-1}^R)(\langle 0 \ldots \hat{j} \ldots n \rangle)
\end{aligned} \tag{C.4}$$

which is 1 if and only $\Delta^L_{n-1}, \Delta^R_{n-1}$ are one inward boundary and one outward boundary of the $n$-simplex $\Delta_n$. This shows that product of $P_{\Delta_n}$ contain $P_{\Delta_n} \sim \gamma_{\Delta_n}\gamma'_{\Delta_n}$ if and only if $\Delta^L_{n-1}, \Delta^R_{n-1}$ are one inward boundary and one outward boundary of the $n$-simplex $\Delta_n$. This cancels out with $S_{\delta\Delta_{n-2}}$ exactly. □

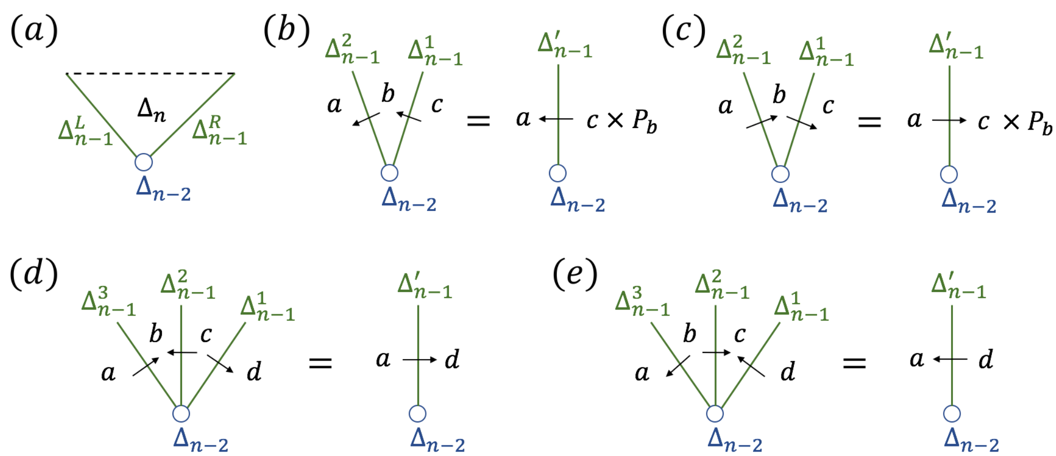



Figure C.1: (a) The $n$-simplex $\Delta_n$ is formed by $\Delta_{n-2}$ and two $(n-1)$-simplex $\Delta^L_{n-1}$ and $\Delta^R_{n-1}$. (b) The product of $S_{\Delta_{n-2}}$ is $(i\gamma_b\gamma'_a)(i\gamma_c\gamma'_b) = (i\gamma_c\gamma'_a)(-i\gamma_b\gamma'_b) = (i\gamma_c\gamma'_a)P_b$. (c) The product of $S_{\Delta_{n-2}}$ is $(i\gamma_a\gamma'_b)(i\gamma_b\gamma'_c) = (i\gamma_a\gamma'_c)(-i\gamma_b\gamma'_b) = (i\gamma_a\gamma'_c)P_b$. (d) The product of $S_{\Delta_{n-2}}$ is $(i\gamma_a\gamma'_b)(i\gamma_c\gamma'_b)(i\gamma_c\gamma'_d) = i\gamma_a\gamma'_d$. (e) The product of $S_{\Delta_{n-2}}$ is $(i\gamma_b\gamma'_a)(i\gamma_b\gamma'_c)(i\gamma_d\gamma'_c) = i\gamma_d\gamma'_a$.

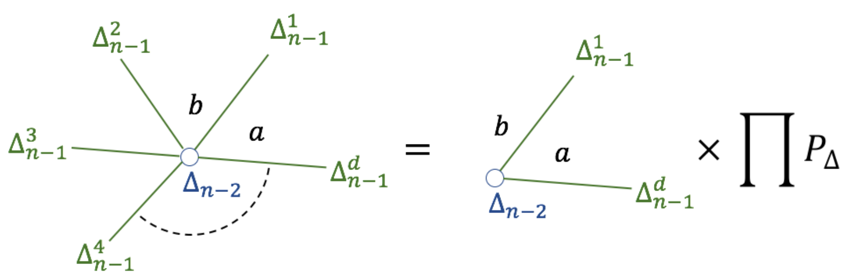



Figure C.2: By the operations defined in Fig. C.1, we can simplify the product $S_{\Delta^d_{n-1}} \cdots S_{\Delta^2_{n-1}} S_{\Delta^1_{n-1}} = S_{\Delta^d_{n-1}} S_{\Delta^1_{n-1}} \prod_{\Delta_n \neq a,b} P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2}}$.

*Proof of Lemma 3.* We compare the sign between

$$S_{\delta\Delta_{n-2}} = (-1)^{\sum_{\Delta_{n-1} < \Delta'_{n-1} | \Delta_{n-1}, \Delta'_{n-1} \supset \Delta_{n-2}} \mathbf{\Delta}_{n-1} \cup_{n-2} \mathbf{\Delta}'_{n-1}} \prod_{\Delta_{n-1} \supset \Delta_{n-2}} S_{\Delta_{n-1}} \tag{C.5}$$

and

$$\prod_{\Delta_n} P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2} \cup_{n-2} \mathbf{\Delta}_n + \mathbf{\Delta}_n \cup_{n-2} \mathbf{\Delta}_{n-2}}, \tag{C.6}$$

where we have used the definition of $S_{\lambda_{n-1}}$ in (4.2). As shown in Fig. C.2,

$$S_{\Delta^d_{n-1}}\cdots S_{\Delta^2_{n-1}}S_{\Delta^1_{n-1}} = S_{\Delta^d_{n-1}}S_{\Delta^1_{n-1}}\prod_{\Delta_n\neq a,b}P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2}\cup_{n-2}\mathbf{\Delta}_n+\mathbf{\Delta}_n\cup_{n-2}\mathbf{\Delta}_{n-2}}.$$

We can check that

$$S_{\Delta^d_{n-1}}S_{\Delta^1_{n-1}} = -(-1)^{\int \mathbf{\Delta}^1_n\cup_{n-2}\mathbf{\Delta}^d_n+\mathbf{\Delta}^d_n\cup_{n-2}\mathbf{\Delta}^1_n}\prod_{\Delta_n=a,b}P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2}\cup_{n-2}\mathbf{\Delta}_n+\mathbf{\Delta}_n\cup_{n-2}\mathbf{\Delta}_{n-2}},$$

and therefore

$$S_{\Delta^d_{n-1}}\cdots S_{\Delta^2_{n-1}}S_{\Delta^1_{n-1}} = -(-1)^{\int \mathbf{\Delta}^1_n\cup_{n-2}\mathbf{\Delta}^d_n+\mathbf{\Delta}^d_n\cup_{n-2}\mathbf{\Delta}^1_n}\prod_{\Delta_n}P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2}\cup_{n-2}\mathbf{\Delta}_n+\mathbf{\Delta}_n\cup_{n-2}\mathbf{\Delta}_{n-2}}.$$

Together with (C.5), we have

$$\begin{aligned}
&S_{\delta\Delta_{n-2}}\prod_{\Delta_n}P_{\Delta_n}^{\int \mathbf{\Delta}_{n-2}\cup_{n-2}\mathbf{\Delta}_n+\mathbf{\Delta}_n\cup_{n-2}\mathbf{\Delta}_{n-2}}\\
=&(-1)^{\int \mathbf{\Delta}^1_{n-1}\cup_{n-2}\mathbf{\Delta}^d_{n-1}+\sum_{i=2}^d\int \mathbf{\Delta}^{i-1}_{n-1}\cup_{n-2}\mathbf{\Delta}^i_{n-1}}\left(-(-1)^{\int \mathbf{\Delta}^1_n\cup_{n-2}\mathbf{\Delta}^d_n+\mathbf{\Delta}^d_n\cup_{n-2}\mathbf{\Delta}^1_n}\right)\\
=&-(-1)^{\sum_{i=1}^d\int \mathbf{\Delta}^{i-1}_{n-1}\cup_{n-2}\mathbf{\Delta}^i_{n-1}}.
\end{aligned}\tag{C.7}$$

From the definition of $\cup_{n-2}$ product (4.4),

$$\begin{aligned}
&\sum_{i=1}^d\int \mathbf{\Delta}^{i-1}_{n-1}\cup_{n-2}\mathbf{\Delta}^i_{n-1}\\
=&\sum_{i=1}^d\sum_{\Delta_n}\mathbf{\Delta}^{i-1}_{n-1}\cup_{n-2}\mathbf{\Delta}^i_{n-1}(\Delta_n)\\
=&\sum_{\text{“}-\text{”-oriented }\Delta_n=\langle 0\ldots n\rangle}\ \sum_{j_1<j_2|j_1,j_2\in\text{even}}\mathbf{\Delta}_{n-2}(\langle 0\cdots\hat{j}_1\cdots\hat{j}_2\cdots n\rangle)\\
&+\sum_{\text{“}+\text{”-oriented }\Delta_n=\langle 0\ldots n\rangle}\ \sum_{k_1<k_2|k_1,k_2\in\text{odd}}\mathbf{\Delta}_{n-2}(\langle 0\cdots\hat{k}_1\cdots\hat{k}_2\cdots n\rangle).
\end{aligned}\tag{C.8}$$

The distinct orientations of “$-$”-oriented $\Delta_n$ and “$+$”-oriented $\Delta_n$ in the summation come from the fact that $j_1, j_2$ and $k_1, k_2$ in (4.4) have opposite orders. Eq. (C.8) is related to $w_2$ by the following lemma 1 in appendix B. Therefore, we derive

$$-(-1)^{\sum_{i=1}^d\int \mathbf{\Delta}^{i-1}_{n-1}\cup_{n-2}\mathbf{\Delta}^i_{n-1}} = (-1)^{\int_{w_2}\mathbf{\Delta}_{n-2}}.\tag{C.9}$$

$\square$

*Appendix D*

# GAUGING THE SYMMETRIES

Let's review the gauging procedure described in [59]. In 2d, the gauging map is to map a state $|a, b\rangle$ on adjacent sites to a states $|a + b\rangle$ on the edge between this two sites (shown as D.1). The $X$ operator at the vertex $v$ will be mapped to the product

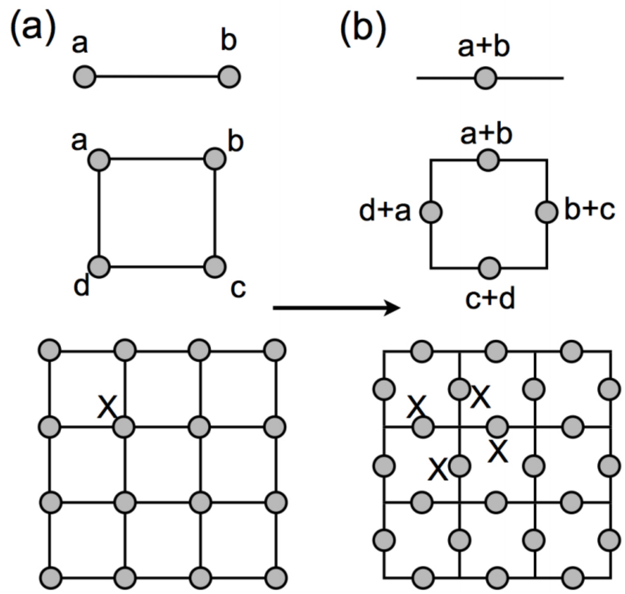



Figure D.1: The gauging map is a function from a state with $\mathbb{Z}_2$ fields (qubits) living on vextices to a state with $\mathbb{Z}_2$ fields (qubits) living on edges. We define the Hilbert spaces as $\mathcal{H}_0$ and $\mathcal{H}_1$ [59].

of $X$ operators on the edges connected to the vertex:

$$X_v \longrightarrow A_v^0 \equiv \prod_{e \supset v} X_e. \tag{D.1}$$

By noticing the summation of edges around a face is always 0, we can write down constraints in the Hilbert space $\mathcal{H}_1$ (mapped states)

$$W_f \equiv \prod_{e \subset f} Z_e = 1 \tag{D.2}$$

for all face $f$. As discussed in [59], the gauge map is a duality map between the following two subspaces:

$$\mathcal{H}_0^{sym} = \{|\psi\rangle \in \mathcal{H}_0 : S|\psi\rangle = |\psi\rangle\}, \tag{D.3}$$

where $S = \prod_v X_v$ and

$$\mathcal{H}_1^{sym} = \{|\psi\rangle \in \mathcal{H}_1 : B(\gamma)|\psi\rangle = |\psi\rangle\}, \tag{D.4}$$

where $\gamma$ represents an arbitrary colosed loop on a square lattice annd $B(\gamma) = \prod_{e\in\gamma} Z_e$. Consider the trivial system with $\mathbb{Z}_2$ symmetry:

$$H = -\sum_v X_v. \tag{D.5}$$

The ground state of this Hamiltonian will be mapped to the groundstate of

$$H = -\sum_v A_v^0 - \sum_f W_f \tag{D.6}$$

which is exactly 2d toric code.

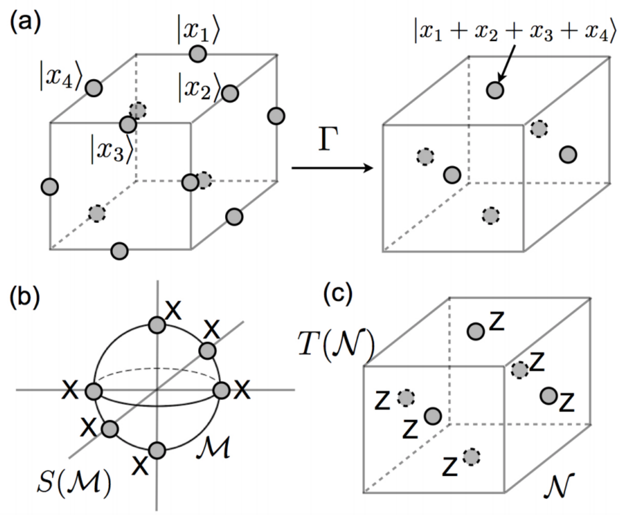



Figure D.2: Gauging 1-form symmetry [59].

For 3d lattice, [59] defined the gauging procedure in the similar way (shown in Fig. D.2). For any 1-cochain $a \in C^1(M, \mathbb{Z}_2)$ ($M$ is a manifold with triangulation), the gauging map $\Gamma^0 : \mathcal{H}_1 \to \mathcal{H}_2$ is defined as

$$\Gamma^0(|a\rangle) = |\delta a\rangle. \tag{D.7}$$

Similarly, the Hilbert space $\mathcal{H}_2$ has a constraint on each cube (tetrahedron) $t$

$$W_t \equiv \prod_{f\subset t} Z_f = 1. \tag{D.8}$$

This map $\Gamma^0$ induces a duality between the following two subspaces. In $\mathcal{H}_1$, the symmetric subspace is defined by

$$\mathcal{H}_1^{sym} = \{|\psi\rangle \in \mathcal{H}_1 : S(\mathcal{M})|\psi\rangle = |\psi\rangle \, \forall \mathcal{M}\}. \tag{D.9}$$

Here the 1-form symmetry operator is written as $S(\mathcal{M})$ for an arbitrary closed 2-manifold $\mathcal{M}$ with Pauli $X_e$ acting on edges intersected by $\mathcal{M}$. In $\mathcal{H}_2$, the gauge symmetric subspace is

$$\mathcal{H}_2^{sym} = \{|\phi\rangle \in \mathcal{H}_2 : T(\mathcal{N})|\phi\rangle = |\phi\rangle \forall \, \mathcal{N}\}, \tag{D.10}$$

where $\mathcal{N}$ is a closed 2-manifold consisting of faces of the lattice and $\mathcal{N}$ acts Pauli $Z_f$ on these faces. Note that closed 2-manifold $\mathcal{M}, \mathcal{N}$ live on dual lattice and direct lattice respectively.

The $X$ operator on a edge $e$ will be mapped to the product of $X$ operators on faces around the edge $e$:

$$X_e \longrightarrow A_e^0 \equiv \prod_{f \supset e} X_f, \tag{D.11}$$

where this mapping represents $\Gamma^0(X_e|a\rangle) = A_e^0|\delta a\rangle$ by definition.

Therefore, the trivial Hamiltonian

$$H = -\sum_e X_e \tag{D.12}$$

is mapped to

$$H = -\sum_e A_e^0 - \sum_t W_t. \tag{D.13}$$

This is (2,1)-toric code and doesn't have any fermionic excitation.

*Appendix E*

# DERIVATION OF $\zeta$ (FIRST DESCENDANT OF $\rho$)

In this section, we derive the choice of $\zeta$ in (6.10)

$$\begin{aligned}
&\zeta(g_{12}, g_{23}, h_1, h_2, h_3)\\
=&\rho(1, g_{12}, g_{12}g_{23}, g_{12}g_{23}h_3) + \rho(1, g_{12}, g_{12}h_2, g_{12}g_{23}h_3)\\
&+ \rho(1, h_1, g_{12}h_2, g_{12}g_{23}h_3).
\end{aligned} \tag{E.1}$$

Consider a tetrahedron $\langle 0123\rangle$ with group elements $g, h, k$ on edges $01, 12, 23$. The value of cocycle $\nu$ on this tetrahedron is expressed as $\nu(1, g, gh, ghk)$. First, if we now perform a gauge transformation on vertex 0 by group element $c_0$ (and identity element on all other vertices). $g, h, k$ becomes $c_0^{-1}g, h, k$ and the value of cocycle is

$$\begin{aligned}
&\rho(1, c_0^{-1}g, c_0^{-1}gh, c_0^{-1}ghk) = \rho(c_0, g, gh, ghk)\\
&= \rho(1, c_0, g, ghk) - \rho(1, c_0, g, gh)\\
&- \rho(1, c_0, gk, ghk) + \rho(1, g, gh, ghk).
\end{aligned} \tag{E.2}$$

From the last line, to satisfy (6.9), we can define the gauge transformation of $B_{ijk}$ by $c_0$ at the vertex $i$ of a face $\langle ijk\rangle$ as $\rho(1, c_0, g_{ij}, g_{jk})$ (or explicit $\zeta(g_{ij}, g_{jk}, c_0, 1, 1) = \rho(1, c_0, g_{ij}, g_{ij}g_{jk})$ ).

Secondly, we perform a gauge transformation on vertex 1 by group element $c_1$. $g, h, k$ becomes $gc_1, c_1^{-1}h, k$ and the value of the cocycle is

$$\begin{aligned}
&\rho(1, gc_1, gh, ghk)\\
=&\rho(1, g, gc_1, gh) - \rho(1, g, gc_1, ghk)\\
&+ \rho(g, gc_1, gh, ghk) + \rho(1, g, gh, ghk)\\
=&\rho(1, g, gc_1, gh) - \rho(1, g, gc_1, ghk)\\
&+ \rho(1, c_1, h, hk) + \rho(1, g, gh, ghk).
\end{aligned} \tag{E.3}$$

The first two terms in the last line indicate that the gauge transformation on the vertex $j$ of a face $\langle ijk\rangle$ is $\rho(1, g_{ij}, g_{ij}c_1, g_{jk})$ (or $\zeta(g_{ij}, g_{jk}, 1, c_1, 1) = \rho(1, g_{ij}, g_{ij}c_1, g_{jk})$) and the third term in the last line is consistent with the previous case.

Similarly for the gauge transformation on the vertex 2 by group element $c_2$, we have

$$\begin{aligned}
&\rho(1,g,ghc_2,ghk)\\
=&\rho(g,gh,ghc_2,ghk)-\rho(1,g,gh,ghc_2)\\
&+\rho(1,g,ghc_2,ghk)+\rho(1,g,gh,ghk)\\
=&\rho(1,h,hc_2,hk)-\rho(1,g,gh,ghc_2)\\
&+\rho(1,g,ghc_2,ghk)+\rho(1,g,gh,ghk).
\end{aligned} \tag{E.4}$$

The second term in the last line implies $\zeta(g_{ij},g_{jk},1,1,c_2)=\rho(1,g_{ij},g_{ij}g_{jk},g_{ij}g_{jk}c_2)$ and the other terms are consistent with previous cases.

We can check the gauge transformation on vertex 3 of the tetrahedron. However, it just gives us a consistence check. The previous three cases are complete gauge transformation.

Combining previous 3 cases, we apply gauge transformation $h_3$, $h_2$, and $h_1$ on vertices $3,2,1$ of a face $\langle 123\rangle$ in sequence. First, we add $\rho(1,g_{12},g_{12}g_{23},g_{12}g_{23}h_3)$ to $B_{123}$ and $g_{12},g_{23}\to g_{12},g_{23}h_3$. Second, we gain $\rho(1,g_{12},g_{12}h_2,g_{12}g_{23}h_3)$ and $g_{12},g_{23}h_3\to g_{12}h_2,h_2^{-1}g_{23}h_3$. Finally, we get $\rho(1,h_1,g_{12}h_2,g_{12}g_{23}h_3)$. Totally, the gauge transformation on $B_{123}$ is

$$\begin{aligned}
&\zeta(g_{12},g_{23},h_1,h_2,h_3)\\
&=\rho(1,g_{12},g_{12}g_{23},g_{12}g_{23}h_3)+\rho(1,g_{12},g_{12}h_2,g_{12}g_{23}h_3)\\
&+\rho(1,h_1,g_{12}h_2,g_{12}g_{23}h_3).
\end{aligned} \tag{E.5}$$

# BIBLIOGRAPHY


[1] Han-Dong Chen, Chen Fang, Jiangping Hu, and Hong Yao. Quantum phase transition in the quantum compass model. *Phys. Rev. B*, 75:144401, Apr 2007.

[2] Alexei Kitaev. Anyons in an exactly solved model and beyond. *Annals of Physics (New York)*, 321(1), 1 2006.

[3] Xiao-Yong Feng, Guang-Ming Zhang, and Tao Xiang. Topological characterization of quantum phase transitions in a spin-$1/2$ model. *Phys. Rev. Lett.*, 98:087204, Feb 2007.

[4] Han-Dong Chen and Jiangping Hu. Exact mapping between classical and topological orders in two-dimensional spin systems. *Phys. Rev. B*, 76:193101, Nov 2007.

[5] Han-Dong Chen and Zohar Nussinov. Exact results of the kitaev model on a hexagonal lattice: spin states, string and brane correlators, and anyonic excitations. *Journal of Physics A: Mathematical and Theoretical*, 41(7):075001, feb 2008.

[6] Eduardo Fradkin. Jordan-wigner transformation for quantum-spin systems in two dimensions and fractional statistics. *Phys. Rev. Lett.*, 63:322–325, Jul 1989.

[7] Y. R. Wang. Ground state of the two-dimensional antiferromagnetic heisenberg model studied using an extended wigner-jordon transformation. *Phys. Rev. B*, 43:3786–3789, Feb 1991.

[8] Ofer Aharony, Guy Gur-Ari, and Ran Yacoby. Correlation functions of large n chern-simons-matter theories and bosonization in three dimensions. *Journal of High Energy Physics*, 2012(12):28, 2012.

[9] Sachin Jain, Shiraz Minwalla, Tarun Sharma, Tomohisa Takimi, Spenta R. Wadia, and Shuichi Yokoyama. Phases of large n vector chern-simons theories on s2 ×s1. *Journal of High Energy Physics*, 2013(9):9, 2013.

[10] Dam Thanh Son. Is the composite fermion a dirac particle? *Phys. Rev. X*, 5:031027, Sep 2015.

[11] Đorde Rađičević. Disorder operators in chern-simons-fermion theories. *Journal of High Energy Physics*, 2016(3):131, 2016.

[12] Ofer Aharony. Baryons, monopoles and dualities in chern-simons-matter theories. *Journal of High Energy Physics*, 2016(2):93, 2016.

[13] Andreas Karch and David Tong. Particle-vortex duality from 3d bosonization. *Phys. Rev. X*, 6:031043, Sep 2016.

[14] Jeff Murugan and Horatiu Nastase. Particle-vortex duality in topological insulators and superconductors. *Journal of High Energy Physics*, 2017(5):159, 2017.

[15] Nathan Seiberg, T. Senthil, Chong Wang, and Edward Witten. A duality web in 2+1 dimensions and condensed matter physics. *Annals of Physics*, 374:395 – 433, 2016.

[16] Po-Shen Hsin and Nathan Seiberg. Level/rank duality and chern-simons-matter theories. *Journal of High Energy Physics*, 2016(9):95, 2016.

[17] Đorđe Radičević, David Tong, and Carl Turner. Non-abelian 3d bosonization and quantum hall states. *Journal of High Energy Physics*, 2016(12):67, 2016.

[18] Shamit Kachru, Michael Mulligan, Gonzalo Torroba, and Huajia Wang. Bosonization and mirror symmetry. *Phys. Rev. D*, 94:085009, Oct 2016.

[19] Jing-Yuan Chen, Jun Ho Son, Chao Wang, and S. Raghu. Exact boson-fermion duality on a 3d euclidean lattice. *Phys. Rev. Lett.*, 120:016602, Jan 2018.

[20] Davide Gaiotto and Anton Kapustin. Spin tqfts and fermionic phases of matter. *International Journal of Modern Physics A*, 31(28n29):1645044, 2016.

[21] Lakshya Bhardwaj, Davide Gaiotto, and Anton Kapustin. State sum constructions of spin-tfts and string net constructions of fermionic phases of matter. *Journal of High Energy Physics*, 2017(4):96, 2017.

[22] Yu-An Chen and Anton Kapustin. Bosonization in three spatial dimensions and a 2-form gauge theory. *Phys. Rev. B*, 100:245127, Dec 2019.

[23] N. E. Steenrod. Products of cocycles and extensions of mappings. *Annals of Mathematics*, 48(2):290–320, 1947.

[24] Tian Lan, Chenchang Zhu, and Xiao-Gang Wen. Fermion decoration construction of symmetry protected trivial orders for fermion systems with any symmetries $G_f$ and in any dimensions. *arXiv e-prints*, page arXiv:1809.01112, Sep 2018.

[25] Zheng-Cheng Gu and Xiao-Gang Wen. Tensor-entanglement-filtering renormalization approach and symmetry-protected topological order. *Phys. Rev. B*, 80:155131, Oct 2009.

[26] Xie Chen, Zheng-Xin Liu, and Xiao-Gang Wen. Two-dimensional symmetry-protected topological orders and their protected gapless edge excitations. *Phys. Rev. B*, 84:235141, Dec 2011.

[27] Anton Kapustin. Bosonic topological insulators and paramagnets: a view from cobordisms, 2014.

[28] Anton Kapustin, Ryan Thorngren, Alex Turzillo, and Zitao Wang. Fermionic symmetry protected topological phases and cobordisms. *Journal of High Energy Physics*, 2015(12):1–21, 2015.

[29] Xiao-Gang Wen. Construction of bosonic symmetry-protected-trivial states and their topological invariants via $g \times so(\infty)$ nonlinear $\sigma$ models. *Phys. Rev. B*, 91:205101, May 2015.

[30] Charles Zhaoxi Xiong. Minimalist approach to the classification of symmetry protected topological phases. *Journal of Physics A: Mathematical and Theoretical*, 51(44):445001, 2018.

[31] Davide Gaiotto and Theo Johnson-Freyd. Symmetry protected topological phases and generalized cohomology. *Journal of High Energy Physics*, 2019(5):7, 2019.

[32] Xie Chen, Zheng-Cheng Gu, Zheng-Xin Liu, and Xiao-Gang Wen. Symmetry protected topological orders and the group cohomology of their symmetry group. *Phys. Rev. B*, 87:155114, Apr 2013.

[33] Ashvin Vishwanath and T. Senthil. Physics of three-dimensional bosonic topological insulators: Surface-deconfined criticality and quantized magnetoelectric effect. *Phys. Rev. X*, 3:011016, Feb 2013.

[34] Kevin Walker and Zhenghan Wang. (3+ 1)-tqfts and topological insulators. *Frontiers of Physics*, 7(2):150–159, 2012.

[35] F. J. Burnell, Xie Chen, Lukasz Fidkowski, and Ashvin Vishwanath. Exactly soluble model of a three-dimensional symmetry-protected topological phase of bosons with surface topological order. *Phys. Rev. B*, 90:245122, Dec 2014.

[36] Lukasz Fidkowski, Jeongwan Haah, and Matthew B Hastings. An exactly solvable model for a $4 + 1d$ beyond-cohomology symmetry protected topological phase. *arXiv preprint arXiv:1912.05565*, 2019.

[37] Andreas P Schnyder, Shinsei Ryu, Akira Furusaki, and Andreas WW Ludwig. Classification of topological insulators and superconductors in three spatial dimensions. *Physical Review B*, 78(19):195125, 2008.

[38] Alexei Kitaev. Periodic table for topological insulators and superconductors. In *AIP conference proceedings*, volume 1134, pages 22–30. American Institute of Physics, 2009.

[39] Shinsei Ryu, Andreas P Schnyder, Akira Furusaki, and Andreas WW Ludwig. Topological insulators and superconductors: tenfold way and dimensional hierarchy. *New Journal of Physics*, 12(6):065010, 2010.

[40] Meng Cheng, Nathanan Tantivasadakarn, and Chenjie Wang. Loop braiding statistics and interacting fermionic symmetry-protected topological phases in three dimensions. *Phys. Rev. X*, 8:011054, Mar 2018.

[41] Zheng-Cheng Gu and Xiao-Gang Wen. Symmetry-protected topological orders for interacting fermions: Fermionic topological nonlinear $\sigma$ models and a special group supercohomology theory. *Phys. Rev. B*, 90:115141, Sep 2014.

[42] Tyler D. Ellison and Lukasz Fidkowski. Disentangling interacting symmetry-protected phases of fermions in two dimensions. *Phys. Rev. X*, 9:011016, Jan 2019.

[43] Nathanan Tantivasadakarn and Ashvin Vishwanath. Full commuting projector hamiltonians of interacting symmetry-protected topological phases of fermions. *Phys. Rev. B*, 98:165104, Oct 2018.

[44] Lukasz Fidkowski, Xie Chen, and Ashvin Vishwanath. Non-abelian topological order on the surface of a 3d topological superconductor from an exactly solved model. *Phys. Rev. X*, 3:041016, Nov 2013.

[45] Nicolas Tarantino and Lukasz Fidkowski. Discrete spin structures and commuting projector models for two-dimensional fermionic symmetry-protected topological phases. *Phys. Rev. B*, 94:115115, Sep 2016.

[46] Anton Kapustin and Ryan Thorngren. Fermionic spt phases in higher dimensions and bosonization. *Journal of High Energy Physics*, 2017(10):80, Oct 2017.

[47] Zitao Wang, Shang-Qiang Ning, and Xie Chen. Exactly solvable model for two-dimensional topological superconductors. *Phys. Rev. B*, 98:094502, Sep 2018.

[48] Qing-Rui Wang and Zheng-Cheng Gu. Construction and classification of symmetry protected topological phases in interacting fermion systems. *arXiv e-prints*, page arXiv:1811.00536, Nov 2018.

[49] Jun Ho Son and Jason Alicea. Commuting-projector hamiltonians for two-dimensional topological insulators: Edge physics and many-body invariants. *Phys. Rev. B*, 100:155107, Oct 2019.

[50] Anton Kapustin and Ryan Thorngren. Higher symmetry and gapped phases of gauge theories. *arXiv e-prints*, page arXiv:1309.4721, Sep 2013.

[51] F.J. Wegner. Duality in Generalized Ising Models and Phase Transitions Without Local Order Parameters. *J. Math. Phys.*, 12:2259–2272, 1971.

[52] John B. Kogut. An introduction to lattice gauge theory and spin systems. *Rev. Mod. Phys.*, 51:659–713, Oct 1979.

[53] Michael Levin and Xiao-Gang Wen. Quantum ether: Photons and electrons from a rotor model. *Phys. Rev. B*, 73:035122, Jan 2006.

[54] Michael Creutz. Emergent spin. *Annals of Physics*, 342:21 – 30, 2014.

[55] John B. Kogut. An introduction to lattice gauge theory and spin systems. *Rev. Mod. Phys.*, 51:659–713, Oct 1979.

[56] Anton Kapustin and Nathan Seiberg. Coupling a qft to a tqft and duality. *Journal of High Energy Physics*, 2014(4):1, Apr 2014.

[57] Allen Hatcher. *Algebraic topology*. Cambridge Univ. Press, Cambridge, 2000.

[58] Richard Z. Goldstein and Edward C. Turner. A formula for stiefel-whitney homology classes. *Proceedings of the American Mathematical Society*, 58(1):339–339, jan 1976.

[59] Beni Yoshida. Topological phases with generalized global symmetries. *Phys. Rev. B*, 93:155131, Apr 2016.